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The Cosmological Constant as a Thermodynamic Variable in 2d Dilaton Gravity

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Daniel Grumiller

durch

Jakob Salzer

Beheimgasse 28/10
1170 Wien

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Wien, 30.10.2013 Jakob Salzer

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The Cosmological Constant as a Thermodynamical Variable in 2d Dilaton Gravity

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Jakob Salzer

Abstract: In this thesis black hole thermodynamics with a varying cosmological constant is studied in two dimensional dilaton gravity. In the past, various mechanisms for a dynamical cosmological constant were proposed. If black hole thermodynamics is studied in the presence of a varying cosmological constant Λ the first law of black hole thermodynamics must take this into account by acquiring an additional term that is due to a change in Λ. This additional term amounts to a $V\,\mathrm{d}P$ term, since the cosmological constant can be regarded as negative pressure and its thermodynamically conjugate variable as a negative black hole volume. Here, the above is studied in the framework of 2d dilaton gravity. Black hole thermodynamics is obtained from the semi-classical approximation of the Euclidean path integral. This requires a well-defined action principle for 2d dilaton gravity with a varying cosmological constant. Following a review of black holes and black hole thermodynamics the Henneaux–Teitelboim mechanism for a varying cosmological constant is presented, and previous results regarding black hole thermodynamics with the cosmological constant as a thermodynamic variable are discussed. Subsequently, notion and importance of a well-defined variational principe are clarified. The framework of 2d dilaton gravity is introduced and several motivations for the study of it are discussed. The approach of [68] to black hole thermodynamics in dilaton gravity is reviewed. Finally, a novel action principle for 2d dilaton gravity with a variable cosmological constant is presented. The thermodynamic variable conjugate to Λ is shown to coincide with the volume of a 2d dilaton black hole presented in [62]. Previous results are reproduced for higher dimensional black holes, that can be attained from 2d dilaton gravity (e.g. Schwarzschild, BTZ).

Contents

1 Introduction

The search for quantum gravity is a natural consequence of the two revolutions physics saw in the first decades of the twentieth century. Quantum theory was conceived in an effort to understand the structure of atomic and subatomic particles. The result was a theory that was able to account for a vast amount of phenomena ranging from the chemical composition of stars and galaxies to more earthly technical applications. The price for this success was a profound change of our point of view of the way nature seems to work. The only other theory with a comparable amount of success was the theory of special and general relativity that came with a similar change of paradigm in its description of space and time as a manifold. The union of quantum mechanics and special relativity, quantum field theory, is the most successful physical theory up till now, with its most important output, the standard model of particle physics. The only thing missing in this model is gravity, described by the general theory of relativity. It is clear that the possibility that the two most successful theories of physics appear to be incompatible is disturbing.

This explains the effort put into the search for quantum gravity. Although it is not clear if a theory of quantum gravity is a sensible concept [25, 102], or even necessary [86], a vast amount of different approaches have been proposed. (For a slightly outdated but good history of quantum gravity see [118].) Whatever final form a theory of quantum gravity might take on, it appears that black holes provide important insights into its working, in the same way as the hydrogen atom did in the development of quantum mechanics. One of the first steps in this direction was the insight that black holes are thermodynamical systems with entropy and temperature. This realization led to an immense number of important ideas that helped to point out problems a final theory of quantum gravity should resolve (e.g. the information paradoxon) and concepts it should incorporate (e.g. the holographic principle).

The Cosmological Constant Problem is another important unresolved problem in both theoretical physics and cosmology, the relationship of which to a theory of quantum gravity is not clear. Perhaps a final theory of quantum gravity will solve both problems at one sweep (see the remarks at the end of this paragraph). The addition of a cosmological constant term to Einstein's equations was first considered by Einstein himself since the idea of an expanding universe predicted by his original equations

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab} \tag{1.1}$$

was disturbing to him. The additional cosmological constant

$$G_{ab} - \Lambda g_{ab} = 8\pi T_{ab} \tag{1.2}$$

allows for the solution of a static universe, appropriately called *Einstein static universe*. The requirement of a static universe fixes the relation between the scale factor of the metric a and the cosmological constant uniquely to $\Lambda = \frac{1}{a^2}$. Despite the fact that a static universe was ruled out due to Hubble's observation of galactic redshift, the static universe is unsatisfying also from a pure theoretical point of view: the Einstein static universe is an unstable solution and slightly different values of the parameters lead to an expanding or collapsing universe. Contrary to the original reason for including Λ in (1.1), today a small, positive cosmological constant is considered responsible for the accelerated expansion of the universe [112, 117]. Naive considerations based on quantum field theory— outlined in section 4.1—yield a much larger value of the cosmological constant. On the other hand, if the observed cosmological constant is interpreted as the sum of classical vacuum energy density ρ_{cl} and energy density induced from spontaneous symmetry breaking in the standard model ρ_{ind} one ends up with an extreme fine tuning problem as the two contributions have to cancel to an immense accuracy in order to explain the small observed value of Λ (cf. [122] for a review). Therefore, a number of mechanisms have been proposed in order to solve the cosmological constant problem, ranging from dark energy to reinterpretations and relaxation mechanisms for Λ (cf. [27, 96, 109] for recent reviews and [146] for Weinberg's seminal review). Other approaches suggest that the cosmological constant problem will be solved by a theory of quantum gravity. Since dilaton gravity allows a nonperturbative quantization it is best suited for the study of this proposal. Recent work showed indeed that the cosmological constant is determined by the requirement that the physical states should be annihilated by the quantized constraint operators. The cosmological constant is then a function of quantum numbers [58, 59]. Eventually, as a last resort remains the *anthropic principle*; a viable but highly unsatisfying option for most physicists.

In this work the consequences of a varying cosmological constant for black hole thermodynamics are studied. Although the cosmological constant seems to be positive, black hole thermodynamics in the presence of a positive cosmological constant, i.e. asymptotically de Sitter space, is not well understood. The reason for this is the presence of a cosmological horizon. An asymptotically de Sitter black hole spacetime yields two horizons with associated respective temperatures that do not coincide in general. Therefore, black hole thermodynamics in the presence of a cosmological constant is usually restricted to Anti-de Sitter spacetimes (AdS). Although probably not realized in nature, these spacetimes are of utter importance in various applications in theoretical physics. As mechanism for a varying cosmological constant we will mostly appeal to the Henneaux–Teitelboim mechanism [20, 21, 77–79, 131] but every other mechanism that results in a varying cosmological constant has the same consequences for black hole mechanics. Promoting Λ to a dynamical variable means that one must consider the variation of Λ in the first law of thermodynamics, where the variable conjugate to Λ is denoted by Θ. Since Λ can be regarded as a negative pressure when written on the right hand side of (1.2) one suspects that Θ might correspond to the negative volume of the black hole; it turns out that this is indeed the case. Since the volume of a black hole is usually regarded as ill-defined this opens up a nice possibility for the definition of a black hole volume as the variable conjugate to Λ. For spherically symmetric black holes one even recovers the naively expected result of the volume of a ball in Euclidean spacetime.

The above subject already has been treated in a number of works [37, 43–46, 89, 94, 95, 144]. Here, we choose the framework of *two dimensional dilaton gravity* (cf. [66] for a review). The complexity of Einstein gravity in four dimensions lies both in conceptual issues and technical difficulties. Lower dimensional gravity theories provide conceptual insights while reducing technical complications. Dilaton gravity is a theory of gravity in two dimensions, i.e. one time and one spatial dimensions, with the degrees of freedom described by the two dimensional metric g_{ab} and an additional scalar field X, the *dilaton field*. Dilaton gravity even allows the study of higher dimensional spacetimes that are effectively two dimensional, e.g. Schwarzschild in arbitrary dimensions. Dilaton gravity provides a natural framework for the discussion of a varying cosmological constant as Λ is easily implemented as the charge of a Maxwell field.

Since we are interested in black hole thermodynamics in the presence of a varying cosmological constant, we follow the most natural way to thermodynamics via the semi-classical approximation of the Euclidean path integral. The application of this approximation requires a well-defined action principle for dilaton gravity with varying cosmological constant. The cosmological constant determines the asymptotic behavior of a spacetime, and a varying cosmological constant means that the leading order of an asymptotic expansion of the metric may vary freely. This implies an unusual variational principle.

The outline of this work is as follows: In order to be largely self-contained, chapter 2 reviews important results regarding black hole solutions of Einstein's equation (1.1). Chapter 6.2.2 discusses the emergence of the notion of black hole thermodynamics. The four classical laws of black hole thermodynamics are studied in detail. The semiclassical concepts of *Bekenstein–Hawking entropy* and *Hawking temperature* are motivated and the Euclidean path integral approach to black hole thermodynamics is reviewed. In chapter 4 the cosmological constant problem is reviewed and the Henneaux–Teitelboim approach and various related mechanisms for the reduction of the cosmological constant are introduced. Finally, black hole thermodynamics with Λ regarded as a thermodynamic variable is discussed. In order to clarify the notion of a well-defined variational principle, chapter 5 contains a discussion of the variational principle and necessary boundary terms that need to be added to make the variational principle well-defined. An introduction to dilaton gravity is given in chapter 6. Three independent motivation for the study of dilaton gravity are discussed. The approach of [68] to dilaton black hole thermodynamics based on a well-defined variational principle is discussed in detail. Finally, chapter 7 contains the original contributions to this work. An action that gives rise to a well-defined variational principle for dilaton gravity with a variable cosmological constant is presented and two applications, Schwarzschild black holes and BTZ black holes, are discussed.

Throughout this thesis the convention (-+..+) is used when working in Lorentzian signature. Results obtained in Euclidean signature are interpreted in terms of Lorentzian signature. Natural units with $c = \hbar = k_B = 1$ are used with the exception of some important formulae where constants are restored. In chapters 6 and 7 we set $8\pi G_2 = 1$ to reduce clutter.

2 Black Hole Solutions

2.1 The Schwarzschild Solution

The first exact solution to Einstein's equations was found by Karl Schwarzschild only a few months after their publication [119], the Schwarzschild solution:

$$ds^2 = -(1 - \frac{2M}{r})\, \mathrm{d}t^2 + (1 - \frac{2M}{r})^{-1}\, \mathrm{d}r^2 + r^2\, \mathrm{d}\Omega_2^2, \tag{2.1}$$

where $\mathrm{d}\Omega_2^2 = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2$ denotes the metric on the round 2- sphere.

The Schwarzschild solution is a vacuum solution of Einstein's equations. It describes the exterior gravitational field of a static, spherically symmetric body of mass M, the chosen coordinates- the *Schwarzschild coordinates*- reflect these symmetries. The "time coordinate" t is measured along the timelike Killing vector field ξ^a, which generates the "time translation symmetry" of the spacetime. The existence of a timelike Killing vector field in a spacetime is equivalent to the notion of *stationarity*. Additionally, if a spacelike hypersurface exists that is everywhere orthogonal to the Killing vector field of the spacetime, i.e. one can choose coordinates without "$\mathrm{d}t\, \mathrm{d}x^i$" cross terms, the spacetime is called *static*. As mentioned above, the Schwarzschild solution is spherically symmetric, i.e. its isometry group contains $SO(3)$ as a subgroup. The metric on each orbit of this subgroup is proportional to the metric of a two sphere and is characterized by its area A or, equivalently, by $r = (\frac{A}{4\pi})^{1/2}$.

These considerations explain the overall structure of (2.1). Only the coefficients of $\mathrm{d}t^2$ and $\mathrm{d}r^2$ remain to be specified. These two functions are found straightforwardly by plugging the ansatz that respects the above symmetries into the vacuum equations

$$R_{ab} = 0. \tag{2.2}$$

By comparison with the Newtonian limit an appearing constant is identified with mass M of the central body.

As well known, the Schwarzschild metric becomes singular in both the limits $r \to 2M$ and $r \to 0$. The latter is an actual curvature singularity, where the *Kretschmann scalar* $R_{abcd}R^{abcd}$ diverges. The singularity at $r = 2M$ is a mere coordinate singularity, that may be circumvented by going to *Kruskal- Szekeres* coordinates [93, 127]:

$$ds^2 = \frac{32M^3 e^{-\frac{r}{2M}}}{r} dU dV + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.3}$$

where the coordinates T and V are defined as

$$UV = -\frac{r - 2M}{2M} e^{\frac{r}{2M}} \tag{2.4}$$

Clearly, in the above coordinates the metric is well defined at $r = 2M$.The range of the coordinates U and V is initially restricted to $U < 0$, $V > 0$. The range of the coordinates can be extended to positive U and negative V by analytic continuation. The resulting spacetime is *Kruskal spacetime*. Thus, the Schwarzschild solution is *extendible*, i.e. one

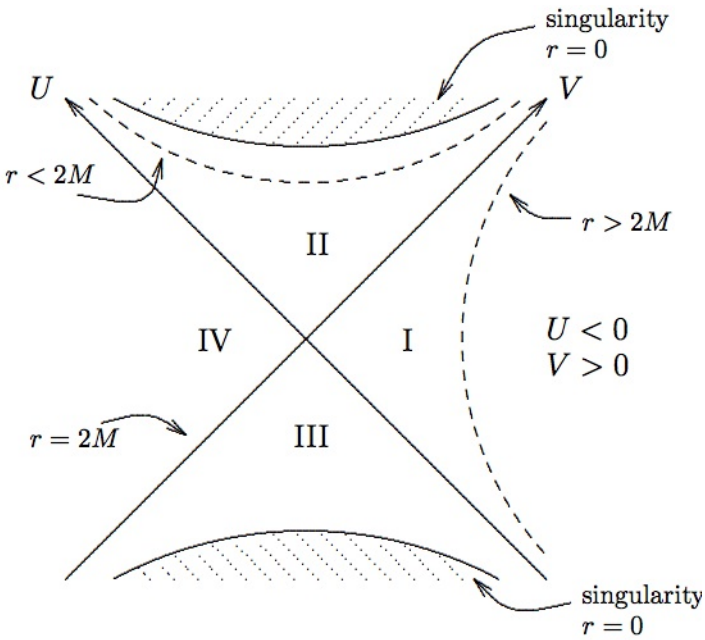

Figure 1. Kruskal spacetime (taken from [133]).

can find a new manifold $\mathcal{M}'$ with metric g' into which the original manifold $\mathcal{M}$ with metric g can be imbedded [75]. On the other hand, it turns out that Kruskal spacetime is inextendible, it is the *maximal extension* of Schwarzschild. The maximally extended Schwarzschild spacetime is shown in figure 1.

Region I describes the original asymptotically flat Schwarzschild spacetime. Every radially infalling observer eventually crosses the null surface $r = 2M$ to enter region II. Once entered, the observer inevitably ends up in the singularity $r = 0$ within finite proper time. Every light signal sent from within this region will also end up in the singularity. This is the *black hole* region. Region III is the "time reversed" counterpart of region II. The only way to enter this region is through the curvature singularity and every observer eventually leaves region III in finite proper time. This region is the *white hole*. Region IV is another asymptotically flat region identical to region I. In a realistic scenario of gravitational collapse to a black hole, regions III and IV are covered up by infalling matter.

The fact that the metric in Schwarzschild coordinates (2.1) is ill-defined at $r = 2M$, is due to a different reason. The coordinate t is measured along the timelike Killing vector field ξ^a, thus the square of the norm of ξ^a is given by

$$\xi^a \xi_a = 1 - \frac{2M}{r}. \tag{2.5}$$

This clearly vanishes at $r = 2M$, which defines a null hypersurface in spacetime. These are the defining properties of a *Killing horizon*. The Killing horizon coincides with the hypersurface $r = 2M$, which is the boundary of the black hole region II, the so called *event horizon*. That this is always the case for static black holes was shown in [29]. Whereas the Killing horizon is locally defined, the notion of an event horizon requires knowledge of the global properties of the respective spacetime. Crossing the event horizon, an observer does not notice anything unusual. A precise definition of an event horizon and the black hole region of a spacetime is presented in section (2.3).

2.2 Conformal Infinity and Black Holes

As seen above, a black hole is a "region of spacetime from which no signal can escape to infinity" (Penrose). The existence of a black hole therefore depends on the notion of infinity and the causal structure of spacetime. Since conformal transformations do not change the causal structure, a spacetime can be *conformally compactified*, i.e. infinity is added to the original physical spacetime $\mathcal{M}$, which yields a *unphysical spacetime* $\tilde{\mathcal{M}}$. That way, the treatment of asymptotic properties of a spacetime is considerably simplified. In addition, so called *Carter–Penrose diagrams* can be constructed, that clarify the causal structure of the respective physical spacetime. A rigorous definition of conformal compactification is presented in [139]. The following definition is sufficiently accurate for our needs.

Consider a spacetime $(\mathcal{M}, g)$. A conformally related metric on $\mathcal{M}$ is given by

$$\tilde{g} = \Omega^2 g. \tag{2.6}$$

For a conformal compactification, Ω is chosen in such a way that for points at infinity with respect to g, the conformal factor approaches 0. Thus, infinity is defined as the points where $\Omega = 0$. These points are not part of the original spacetime $\mathcal{M}$ but they may be added to $\mathcal{M}$ to yield the *unphysical* spacetime $\tilde{\mathcal{M}}$. $(\tilde{\mathcal{M}}, \tilde{g})$ is a conformal compactification of $(\mathcal{M}, g)$.

For example, consider Minkowski spacetime:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2. \tag{2.7}$$

A conformal compactification of Minkowski space is given by the metric

$$d\tilde{s}^2 = -4dpdq + \sin^2(p - q)d\Omega^2 \qquad -\frac{\pi}{2} \leq p \leq q \leq \frac{\pi}{2} \tag{2.8}$$

with $t - r = \tan p$, $t + r = \tan q$. The Carter–Penrose diagram obtained from this compactification is shown in figure 2. Five asymptotic regions can be distinguished in this diagram: *past timelike infinity* i^-; the three dimensional surface *past null infinity* $\mathscr{I}^-$; *spatial infinity* i^0; *future null infinity* $\mathscr{I}^+$; *future timelike infinity* i^+. All timelike geodesics, that can be infinitely extended, begin at i^- and end at i^+, whereas all infinitely extended spacelike geodesics begin and end at i^0. All null geodesics begin at $\mathscr{I}^-$ and end at $\mathscr{I}^+$.

The Carter–Penrose diagram of the maximally extended Schwarzschild spacetime obtained by conformally compactification is given in figure 3. The structure of this diagram is similar to the Kruskal diagram with the conformal infinite regions included. The fact that these regions look like Minkowski space is due to the asymptotic flatness of the Schwarzschild metric. In the framework of Penrose–Carter diagrams asymptotic flatness is recognized easily. Rigorous definitions of asymptotic flatness are presented in [75, 139].

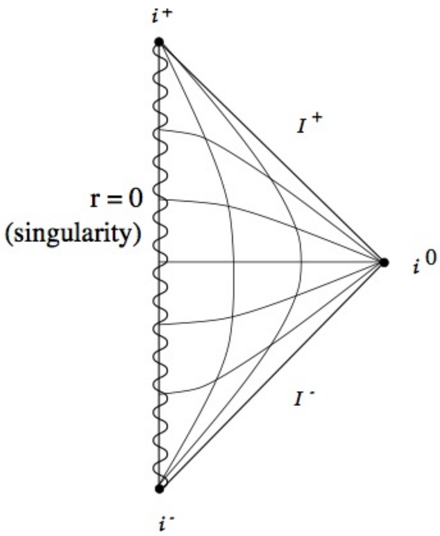

Figure 2. Carter Penrose diagram of Minkowski spacetime (taken from[26]).

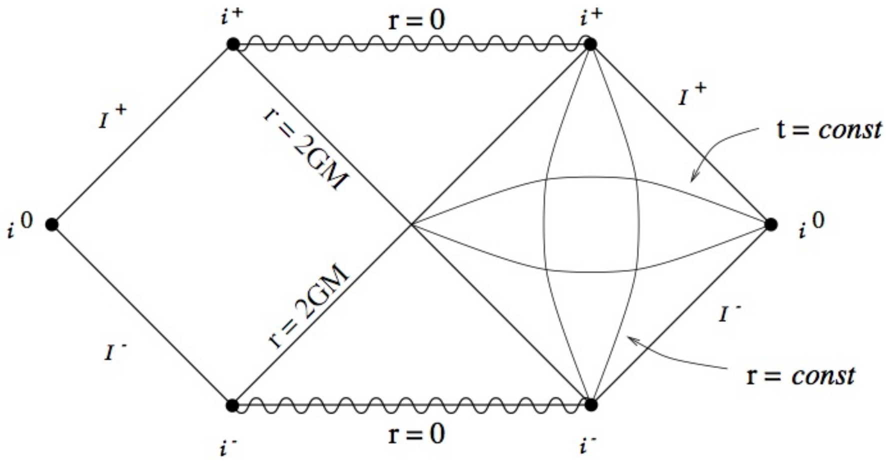

Figure 3. Carter Penrose of the maximally extended Schwarzschild spacetime (taken from [26]).

2.3 Horizons and the No-Hair Theorem

With the above definitions of conformal infinity the intuitive notion of an event horizon can be made precise: The *black hole region* B of an asymptotically flat spacetime $\mathcal{M}$ is the part of $\mathcal{M}$ that is not contained in the causal past of future null infinity $\mathscr{I}^+$, i.e. $B = \mathcal{M} - J^-(\mathscr{I}^+)$. The boundary of the black hole region B in $\mathcal{M}$ is called the *future event horizon*, $\mathscr{H}^+ = \dot{J}^-(\mathscr{I}^+) \cap \mathcal{M}$. In the same way, *white hole region* and *past event horizon* $\mathscr{H}^-$ are defined by replacing $J^-(\mathscr{I}^+)$ with $J^+(\mathscr{I}^-)$. In other words, from inside the event horizon photons or particles cannot escape to future infinity. Therefore, the event horizon causally separates two regions of spacetime and is thus always a null surface. As mentioned above, the notion of an event horizon is only globally defined.

A *Killing horizon* is a logically independent concept, however in the case of stationary black holes these concepts are related. A Killing horizon associated to a Killing vector k^a

is defined as "a null embedded hypersurface, invariant under the flow of a Killing vector k, which coincides with a connected component of the set $\mathscr{N}[k] := \{k^a k_a = 0, k \neq 0\}$" [34]. A quantity associated to a Killing horizon is the *surface gravity* κ, which describes the necessary acceleration of a test particle in order to remain static near the horizon. A Killing horizon with vanishing surface gravity is called *extremal*.

As mentioned above, event horizons and Killing horizons are related in black hole spacetimes. The *strong rigidity theorem* states that the event horizon of a stationary black hole is a Killing horizon [51, 75]. The Killing vector associated to the Killing horizon that coincides with the event horizon is not necessarily the Killing vector generating the time translation symmetry. If this is the case the Killing horizon is called *non-rotating*, otherwise *rotating*.

The rigidity theorem is an essential ingredient in the proof of the *uniqueness-* or *no hair-conjecture*, in the cases where it actually holds. This conjecture— due to Israel, Penrose and Wheeler—states that all stationary axisymmetric black hole solutions are isometric to the Kerr–Newman family (see section 2.4). Thus, they can be described in terms of a small set of parameters, namely mass, angular momentum, and charge. The most general form of this conjecture has been ruled out by the construction of various stable black holes with "hair", i.e. these solutions cannot be parametrized in terms of a small number of asymptotic flux integrals [47, 80, 81]. The most general statement of the no-hair conjecture that has been rigorously proven concerns Einstein–Maxwell theory assuming asymptotic flatness and analyticity of the metric provided that the event horizon is either non degenerate or rotating. A recent review and further references can be found in [34].

There are different notions of horizon other than Killing and event horizon, like apparent horizons, trapping horizons, dynamical horizons, etc. Since these concepts will not be relevant in the following discussion, the reader is referred to the literature [3, 139].

2.4 The Kerr–Newman family

The simplest black hole solution, the Schwarzschild metric was presented in section 2.1. The Schwarzschild metric can be generalized to include electromagnetic field. The corresponding solution of the Einstein–Maxwell equations is the *Reissner–Nordström* metric [108, 116] that describes the gravitational field of a spherical body of mass M and electrical charge Q

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_2^2 \tag{2.9}$$

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{2.10}$$

Again, the curvature singularity is at $r = 0$ while the roots of the timelike Killing vector— the Killing horizons—

$$r_\pm = M \pm \sqrt{M^2 - Q^2} \tag{2.11}$$

are mere coordinate singularities and can be gauged away by a suitable coordinate transformation. As seen above, the Reissner–Nordström black hole yields two horizons, an inner

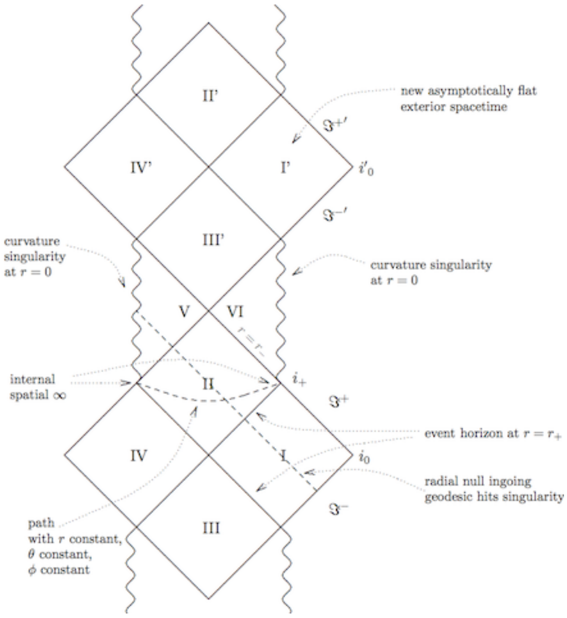

Figure 4. Carter Penrose of the maximally extended Reissner–Nordström spacetime (taken from [133]).

and an outer horizon, if $M > |Q|$. In the case $M < |Q|$, no event horizon is present, though the singularity at $r = 0$ still exists. Since the singularity is not shielded by an event horizon it is a *naked singularity*. In the limiting case $M = |Q|$, the black hole is called *extremal*, an infinitesimal perturbation of the black hole would produce a naked singularity.

The curvature singularity of Reissner–Nordström differs significantly from the singularity of Schwarzschild, insofar as it is a timelike singularity, as can be seen in figure 4). An observer can avoid the singularity by passing through the inner horizon to a different asymptotically flat region. Since all geodesics can be extended to infinity the spacetime is called *geodesically complete*.

Until now only static black hole solutions were considered and a long time these were the only known black hole solutions. Much later, stationary rotating black hole solutions were found by Kerr [91] and a generalization to stationary charged rotating black holes by Newman et al. [106]. In *Boyer–Lindquist* coordinates the Kerr–Newman metric and its vector potential are given by

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta\left(r^2 + a^2 - \Delta\right)}{\Sigma}dtd\phi + \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2. \quad (2.12)$$

$$A_a = -\frac{Qr}{\Sigma}\left[(dt)_a - a\sin^2\theta(d\phi)_a\right], \quad (2.13)$$

with

$$\Sigma = r^2 + a^2 \cos^2\theta \qquad (2.14)$$
$$\Delta = r^2 + a^2 + Q^2 - 2Mr,$$

M the total mass, Q the total electric charge, and $J = Ma$ the total angular momentum. Again, depending on the value of the three parameters three different cases can be distinguished: If $Q^2 + a^2 > M^2$, no event horizon shields the singularity, which is therefore a naked singularity. In the case $Q^2 + a^2 < M^2$, both inner and outer horizon exist at $r_\pm = M \pm (M^2 - a^2 - Q^2)^{1/2}$. If the parameters satisfy $Q^2 + a^2 = M^2$ the metric describes an extreme black hole.

The metric (2.12) exhibits the Killing vectors $\xi^a = (\frac{\partial}{\partial t})^a$, the asymptotic time translation Killing field, and $\psi^a = (\frac{\partial}{\partial \phi})^a$, the azimuthal Killing field. The norm of ξ^a is given by

$$\xi^a \xi_a = \frac{a^2 \sin^2\theta - \Delta}{\Sigma}. \qquad (2.15)$$

Thus, the time translation Killing field becomes spacelike in the region

$$r^2 + a^2 \cos^2\theta + Q^2 - 2Mr. \qquad (2.16)$$

A part of this region lies outside of r_+, as well. Therefore, an observer cannot remain static in the region

$$r_+ < r < M + (M^2 - Q^2 - a^2 \cos^2\theta)^{1/2}, \qquad (2.17)$$

called the *ergosphere*.

As mentioned above, the rigidity theorem requires the event horizon to be a Killing horizon. For the Kerr–Newman metric the event horizon is not generated by the time translation Killing field ξ^a but by the linear combination

$$\chi^a = \xi^a + \Omega_H \psi^a \qquad \Omega_H = \frac{a}{r_+^2 + a^2} \qquad (2.18)$$

The quantity Ω_H is interpreted as the angular velocity of the outer horizon.

Since the main part of this work will be focused almost exclusively on spherically symmetric black holes the reader is referred to the ample literature on the Kerr–Newman family.

3 Black Hole Thermodynamics

Thermodynamics is a powerful tool with a vast range of applicability since thermodynamics establishes relations between macroscopic variables like temperature, volume, pressure etc. without paying attention to the microscopic properties of the system. It was Boltzmann who showed that some of the microscopic properties of a system may be deduced from macroscopic variables which culminated in his famous formula

$$S = k_B \ln \Omega, \tag{3.1}$$

relating the macroscopic variable entropy S to Ω, the number of different microstates— or the phase space volume— of the system that are compatible with a given macrostate. This is the starting point for statistical mechanics, which calculates the macroscopic properties of a system from the underlying microscopic theory, thus establishing the theoretical foundation for the heuristically found laws of thermodynamics.

Now, the notion of black hole thermodynamics may sound preposterous, since one may ask for the microstates or phase space of a black hole justifying a thermodynamic approach. Nevertheless, it will be clarified how the notion of black hole thermodynamics comes about and what one can learn from it regarding the underlying theory.

3.1 The Area Theorem (The Second law of black hole mechanics)

The first hint towards black hole thermodynamics came from Hawking's area theorem [70]. The statement of the theorem follows [139] where the proof can be found, as well:

"*Let* $(\mathcal{M}, g_{ab})$ *be a strongly asymptotically predictable spacetime satisfying* $R_{ab}k^a k^b \geq 0$ *for every null vector* k^a. *Let* Σ_1 *and* Σ_2 *be spacelike Cauchy surfaces for the globally hyperbolic region* $\tilde{V}$ *with* $\Sigma_2 \subset I^+(\Sigma_1)$ *and let* $\mathcal{H}_1 = H \cap \Sigma_1, \mathcal{H}_2 = H \cap \Sigma_2$, *where* H *denotes the event horizon, i.e. the boundary of the black hole region of* $(\mathcal{M}, g_{ab})$. *Then the area of* $\mathcal{H}_2$ *is greater than or equal to the area of* $\mathcal{H}_1$."

Thus, using relatively mild conditions (weak energy condition, existence of Cauchy hypersurfaces, cosmic censorship hypothesis), we see that the area of a black hole horizon can never decrease. In other words

$$\delta A \geq 0. \tag{3.2}$$

This formulation is strongly reminiscent of the formulation of the second law of thermodynamics, namely

$$\delta S \geq 0, \tag{3.3}$$

the statement that entropy must increase in every process. The analogy goes even further if one remembers that from the second law of thermodynamics follows Carnot's theorem, that limits the efficiency of conversion of heat into work, as a corollary. Similar limits on the efficiency of conversions of mass into energy in black hole collisions can be derived from (3.2).

Consider two widely separated black holes with masses M_1 and M_2, respectively, such that the interaction between them can be neglected. Eventually, the dynamical evolution of this situation will result in a spacetime where the two black holes have coalesced to a single black hole that finally settles down to a Schwarzschild black hole of mass M_3. The energy emitted in this process is $M_1 + M_2 - M_3$, thus the efficiency of mass to energy conversion is

$$\eta = \frac{M_1 + M_2 - M_3}{M_1 + M_2} = 1 - \frac{M_3}{M_1 + M_2}. \tag{3.4}$$

Since the two black holes are initially widely apart we may assume that they are approximately stationary so that the respective horizon areas are $A_1 = 16\pi M_1^2$ and $A_2 = 16\pi M_2^2$. The area theorem asserts that the area of the final black hole obeys

$$A_3 \geq 16\pi(M_1^2 + M_2^2). \tag{3.5}$$

After the final black hole has settled down to a Schwarzschild solution its area is given by $A_3 = 16\pi M_3^2$. Thus, it follows that

$$M_3 \geq \sqrt{M_1^2 + M_2^2}. \tag{3.6}$$

Therefore, the efficiency of mass to energy conversion is limited by

$$\eta = 1 - \frac{M_3}{M_1 + M_2} \leq 1 - \frac{\sqrt{M_1^2 + M_2^2}}{M_1 + M_2} \leq 1 - \frac{1}{\sqrt{2}}. \tag{3.7}$$

Although the area theorem's resemblance to the second law of thermodynamics seems remarkable, one might have the impression that the similarities are just superficial. After all, the area theorem is a precise mathematical statement in the framework of general relativity whereas the second law of thermodynamics is an empirical statement valid for systems with a large number of degrees of freedom. However, in a famous paper Bardeen, Carter and Hawking [7] summarized the correspondence of the four laws of thermodynamics to counterparts in BH physics.

3.2 The Four Laws of Black Hole Mechanics

3.2.1 Surface Gravity and the Zeroth Law

As a first step, we will define a quantity called *surface gravity* κ that will prove to be important in the context of black hole thermodynamics. In Newtonian mechanics one associates surface gravity with the magnitude of the acceleration that a test particle undergoes in the field of a body, say the earth. In other words, surface gravity is the acceleration a test particle has to be accelerated with in order to remain static in the field of the body, near the body surface. We will see that this interpretation carries over to the general relativistic case. Our derivation will follow the outline of [141].

Let $\mathcal{H}$ denote a Killing horizon associated to the Killing vector χ^a, i.e. $\mathcal{H}$ is a null hypersurface with normal vector χ^a. By definition, the norm of the Killing vector vanishes on $\mathcal{H}$ and is thus trivially constant on $\mathcal{H}$. Therefore, the vector $\nabla^b(\chi^a\chi_a)$ is normal to $\mathcal{H}$. Since the vectors χ^a and $\nabla^b(\chi^a\chi_a)$ are both normal to $\mathcal{H}$ they must be proportional to each other. Hence, we define the surface gravity κ to be

$$\nabla^b(\chi^a\chi_a) = -2\kappa\chi^b. \tag{3.8}$$

The Lie derivative of this equation w.r.t to χ yields

$$\pounds_{\chi}\kappa = 0. \tag{3.9}$$

This means that κ is constant along the orbits of χ^a.

In the following we will derive a useful formula for calculating the surface gravity from the Killing vector χ^a. From Frobenius' theorem follows that a vector ξ is orthogonal to a foliation of hypersurfaces— or *hypersurface orthogonal*— if and only if the expression $\xi_{[a}\nabla_b\xi_{c]}$ vanishes on the hypersurface. The Killing vector χ^a is clearly hypersurface orthogonal, thus we have

$$\chi_{[a}\nabla_b\chi_{c]}\Big|_{\mathcal{H}} = 0. \tag{3.10}$$

Using Killing's equation $\nabla_a\chi_b + \nabla_b\chi_a = 0$ and contracting with $(\nabla^a\chi^b)$ one obtains

$$\chi_c(\nabla^a\chi^b)(\nabla_a\chi_b) = -2(\chi_a\nabla^a\chi^b)(\nabla_b\chi_c) = -2\kappa\chi^b\nabla_b\chi_c = -2\kappa^2\chi_c \tag{3.11}$$

after using (3.8) twice (where everything is evaluated on the horizon $\mathcal{H}$). Thus, we arrive at a simple formula for surface gravity κ:

$$\kappa^2 = -\frac{1}{2}(\nabla^a\chi^b)(\nabla_a\chi_b)\Big|_{\mathcal{H}}. \tag{3.12}$$

For the case of a static spherically symmetric black hole with the line element

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \tag{3.13}$$

the above formula (3.12) simplifies to

$$\kappa = \frac{1}{2}\partial_r f(r)\big|_{\mathcal{H}}, \tag{3.14}$$

as can be easily checked. As the simplest example we may calculate the surface gravity for a Schwarzschild black hole. Using (3.14) we arrive at $\kappa = \frac{1}{4M}$. This is the same result one would obtain in Newton's theory for the surface gravity of a body of mass M and radius $2M$. For general stationary charged black holes with angular momentum $J = aM$, mass M and charge Q, surface gravity is given by the expression

$$\kappa = \frac{(M^2 - a^2 - Q^2)^{\frac{1}{2}}}{2M(M + (M^2 - a^2 - Q^2)^{\frac{1}{2}}) - Q^2}. \tag{3.15}$$

Until now, we have only showed that κ is constant along the orbits of the Killing vector χ. Another remarkable property of κ is the fact that it is constant on the horizon. For a bifurcate Killing horizon this property can be shown easily: Consider the bifurcation sphere S of a Killing horizon $\mathcal{H}$ with tangent vector s^a. Differentiating equation (3.8) w.r.t. s^a yields

$$\kappa s^a\nabla_a\kappa = -\frac{1}{2}s^c(\nabla_c\nabla_a\chi_b)(\nabla^a\chi^b) = \frac{1}{2}s^c R^d{}_{abc}\chi_d\nabla^a\chi^b = 0. \tag{3.16}$$

Thus, κ is constant on the bifurcate sphere and since its derivative along the orbits of the generators of the Killing horizon is zero, it is constant on the Killing horizon.

The general proof that κ is constant on a Killing horizon is lengthy and will not be given here. In principle, there are two different ways to prove this result: One proof assumes that the black hole is static or stationary-axisymmetric together with a property called *t-ϕ orthogonality* [30, 114]; alternatively, Einstein's equations and the dominant energy condition [7] can be used.

Summarizing, we have seen that a quantity called surface gravity can be defined that may be interpreted as the force at infinity that must be exerted on a particle to hold it in place in the vicinity of the horizon. Remarkably, this quantity— although defined locally— is constant on the horizon, not only for static spacetimes but for stationary spacetimes, as well. This is the zeroth law of black hole mechanics:

The Zeroth Law: *The surface gravity, κ of a stationary black hole is constant over the event horizon* [7].

The suggestive naming is of course reminiscent of the zeroth law of thermodynamics, which states that the temperature is uniform for a body in thermodynamic equilibrium. At this point, the similarities between temperature and surface gravity seem to be quite superficial but the the following will show that the quantities are in fact intimately related.

3.2.2 The First Law

The first law of thermodynamics is a formulation of energy conservation. Different forms of energy may be transformed into each other, e.g. heat into work and internal energy, but energy is not created or destroyed. It turns out that the first law of black hole mechanics describes the change in the total energy of a black hole, the mass M in a similar way.

As pointed out in [141], two logically independent versions of this law exist: the *physical process version* and the *equilibrium version*. The physical process version starts with a black hole in equilibrium and changes its parameters in a physical process i.e. by throwing matter with infinitesimal mass, charge and angular momentum into the black hole. After the black hole has settled down one compares the parameters of the initial and final state.

In the equilibrium state version one is interested in black hole solutions whose parameters differ only by an infinitesimal amount. These two independent derivations of the first law have to agree with each other. Otherwise, some assumptions of the physical process version would turn out to be wrong, e.g. one has destroyed the black hole by adding infinitesimally small amounts of matter.

Since the rigorous calculations of these two versions are quite lengthy, we will derive a formula relating the parameters of a black hole to its mass— the *Smarr formula*— and use it to motivate the first law of thermodynamics. The below derivation of the first law follows [133].

The Komar integral for the angular momentum J of an axisymmetric-stationary spacetime is given by

$$J(V) = \frac{1}{16\pi G} \oint_{\partial\Sigma} dS_{ab} \nabla^a \psi^b \tag{3.17}$$

where we put $\xi^a = \psi^a$, the axial Killing field, and the coefficient was set to $c = 1$, which follows from comparison of $J(V)$ for a weak source with the definition of angular momentum. Gauss' law applied to $J(V)$ yields

$$\begin{aligned} J =& \frac{1}{8\pi G} \int_\Sigma dS_a \nabla_b \nabla^a \psi^b + \frac{1}{16\pi G} \oint_H dS_{ab} D^a \psi^b \\ =& \frac{1}{8\pi G} \int_\Sigma dS_a R^a{}_b \psi^b + J_H. \end{aligned} \tag{3.18}$$

Here the identity $\nabla_a \nabla_b \xi^c = R^c{}_{abd} \xi^d$ for Killing vectors ξ^a was used. With Einstein's equations one can rewrite the first term as

$$J = \int_\Sigma dS_a (T^a_{\ b} - \frac{1}{2} T \delta^a_{\ b}) \psi^b + J_H. \tag{3.19}$$

The spacetimes we are interested in are vacuum or electrovac solutions of Einstein's equations. The trace of T^{ab} vanishes for electromagnetic fields, therefore we may assume $T = 0$ and we have

$$J = \int_\Sigma dS_a T^a_{\ b}(F) \psi^b + J_H, \tag{3.20}$$

where $T(F)$ denotes the energy-momentum tensor of the Maxwell field. From the Komar integral for the total mass of spacetime we have

$$M = -\frac{1}{4\pi G} \int_\Sigma dS_a R^a{}_b \xi^b - \frac{1}{8\pi G} \oint_H dS_{ab} \nabla^a \xi^b \tag{3.21}$$

after applying Gauss' law. Here ξ^a denotes the stationary Killing vector. As mentioned in (2.18) the Killing vector χ^a normal to the horizon is given by $\chi^a = \xi^a + \Omega_H \psi^a$. With Einstein's equations and this relation (3.21) yields

$$M = \int_\Sigma dS_a (-2 T^a_{\ b} \xi^b + T k^a) - \frac{1}{8\pi G} \oint_H dS_{ab} (\nabla^a \chi^b - \Omega_H \nabla^a \psi^b). \tag{3.22}$$

For $T = 0$ and using the expression for the total angular momentum (3.20) this is

$$M = -2 \int_\Sigma dS_a T^a_{\ b}(F) \chi^b + 2\Omega_H J - \frac{1}{8\pi G} \oint_H dS_{ab} \nabla^a \chi^b. \tag{3.23}$$

The first term can be shown to equal $Q\Phi_H$ with Q the charge of the black hole and $\Phi_{\mathcal{H}}$ the co-rotating potential evaluated on the horizon. The area element dS_{ab} in the last integral is given by $dS_{ab} = (\chi_a n_b - \chi_b n_a) dA$ on $\mathcal{H}$ where n is another vector normal to the horizon obeying $\chi^a n_a = -1$. Thus, the last term in equation (3.23) can be evaluated to yield

$$-\frac{1}{8\pi G} \oint_H dS_{ab} \nabla^a \chi^b = \frac{\kappa}{4\pi G} A \tag{3.24}$$

where the zeroth law was used. Collecting all terms, we arrive at the Smarr formula [7, 30, 121],

$$M = \frac{\kappa A}{4\pi} + 2\Omega_H J + \Phi_H Q. \tag{3.25}$$

Now, in order to obtain the first law of black hole mechanics we have to evoke Euler's theorem of homogeneous functions, which states that if a function $f(x_1, ..n_x)$ obeys the relation $f(\alpha^{p_1}x_1, .., \alpha^{p_n}x_n) = \alpha^r f(x_1, .., x_n)$ then the partial derivatives satisfy

$$rf(x_1, ..x_n) = \sum_{i=1}^{n} p_i \frac{\partial f(x_1, .., x_n)}{\partial x_i} x_i. \tag{3.26}$$

Under an overall change in length scale L the physical quantities in (3.25) scale as $M \propto L^{d-2}$, $A \propto L^{d-1}$, $J \propto L^{d-1}$, $Q \propto L^{d-2}$. Euler's theorem thus implies

$$M(A, J, Q) = 2\left(\frac{\partial M}{\partial A}\right) A + 2\left(\frac{\partial M}{\partial J}\right) J + \left(\frac{\partial M}{\partial Q}\right) Q. \tag{3.27}$$

From comparison with (3.25) one can therefore read off the partial derivatives of M. From this follows the first law of black hole mechanics.

$$dM(A, J, Q) = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi_H dQ. \tag{3.28}$$

Now, this looks remarkably similar to the first law of thermodynamics. The second and third term correspond to the work terms one would expect for a charged rotating system in the first law of thermodynamics. With the zeroth law and the area theorem in mind one could interpret the first term as a TdS term. Obviously, the first law of black hole mechanics gives further evidence to the idea that the area of a black hole is related to entropy and the surface gravity is related to temperature. Classically, the black hole must have zero temperature since it emits no particles. But in the next section it will be shown that the surface gravity κ is in fact proportional to the actual temperature of the black hole.

3.2.3 The Third Law

For the third law of thermodynamics, two different formulations exist. The *Planck- Nernst* form of the third law states that $S \to 0$ as $T \to 0$. This statement is not true for the analogous quantities in black hole mechanics, since black holes with $\kappa = 0$ (*extremal black holes*) have nonzero horizon area A. As pointed out in [143] there is evidence that this formulation of the third law is not valid even for ordinary quantum systems. The violation of the third law in these systems is similar to the violation of the third law in black holes [142].

A different formulation of the third law of thermodynamics according to Nernst states that the temperature of a system cannot be reduced to zero in a finite number of steps. The analogue of this law— the surface gravity κ of a black cannot be reduced to zero in any finite process— holds in black hole mechanics as was shown in [82, 138]. In the case of a charged non-rotating black hole the argument is very simple: From equation (3.15) we see that a Reissner–Nordström black hole is extremal, i.e. $\kappa = 0$, if $M^2 = Q^2$. Thus, one starts with a nonextremal black hole with $|Q| < M$. In order to obtain an extremal black hole, charged particles are dropped into the black hole. But this works only up to a certain ratio $\frac{|Q_{end}|}{M} < 1$, since by then the repelling electromagnetic force is stronger than gravity and the particle does not end up in the black hole. In the same way, it can be shown that a particle dropped into a rotating black hole with enough angular momentum to make the black hole extremal, simply misses the black hole. Thus, following [7] we state:

The Third law: *It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.*

As already mentioned in section (3.1) the second law of black hole thermodynamics is Hawking's area law. Up to this point the relationship between the laws of thermodynamics and the laws of black hole mechanics is a mere analogy. Classically, black holes do not have temperature. But the similarities between black hole area A and entropy S should not be dismissed easily. After all, there are not many physical quantities that are forbidden to decrease with time. Furthermore, the role of energy is consequently played by mass in the laws of black hole mechanics. It is due to the work of Bekenstein and Hawking that the relationship between the laws of thermodynamics and the laws of black hole mechanics was made clear.

3.3 Generalized Second Law and Hawking Radiation

The *no-hair theorem* mentioned in previous section poses some problems. Consider a complex matter system that undergoes gravitational collapse to a black hole. The no-hair theorem asserts that the resulting black hole is indeed described by just a few physical quantities: mass, charge, angular momentum. But the initial system, e.g. a star, certainly needs a lot more parameters for a full description. Thus, phase space is dramatically reduced after gravitational collapse.

Another, very simple gedankenexperiment shows even more drastically that something is amiss in the interplay between black holes and the second law of thermodynamics: Consider a thermodynamical system with finite entropy that is dropped into the black hole. If the entropy is really lost in this process then the second law of thermodynamics is not valid anymore. It was Bekenstein who noticed that the second law of thermodynamics could be saved, if the black hole has an entropy itself, that is proportional to its area [9, 10]:

$$S_{BH} \propto A \tag{3.29}$$

with a coefficient of order one. As will be seen, this coefficient is $\frac{1}{4}$ for black holes in Einstein gravity, or with all constants restored

$$S_{BH} = \frac{k_B c^3 A}{4G\hbar}. \tag{3.30}$$

This is the Bekenstein–Hawking entropy. Bekenstein therefore proposed a *generalized entropy* as the sum of matter entropy S and the black hole entropy S_{BH},

$$S' = S + \sum_i \frac{k_B c^3 A_i}{4G\hbar}. \tag{3.31}$$

For this generalized entropy, the *generalized second law* of thermodynamics is valid:

$$\delta S' \geq 0. \tag{3.32}$$

The nature of the Bekenstein–Hawking entropy is intrinsically quantum as can be seen from the appearance of $\hbar$. It is closely linked to the semiclassical effect of *Hawking radiation*.

In classical general relativity a black hole is an object, from which no particles can escape. It came as a great surprise when it was shown that black holes emit particles with a thermal spectrum [71, 72]. Even though the derivation of Hawking radiation is not too difficult it requires techniques of quantum field theory in curved background which lie beyond the scope of this work. Therefore, we will settle for a motivation of Hawking radiation.

Although discovered later, the *Unruh effect* lies at the core of the Hawking effect [136]. The Unruh effect is the phenomenon that an accelerating observer with proper acceleration a in Minkowski space will perceive the Minkowski vacuum as a thermal bath of particles with temperature (with all constants restored)

$$T_U = \frac{\hbar a}{2\pi c k_B}. \tag{3.33}$$

The reason for this lies in the definition of the vacuum state. Following [36], it will be seen that two different observers do not agree on the definition of the vacuum in general. For simplicity, consider a massive scalar field with a set of solutions $\{f_i, f_i^*\}$ that are complete with respect to a suitably chosen inner product. The quantized field $\hat{\phi}(x)$ is expanded as

$$\hat{\phi}(x) = \sum_i \left(\hat{a}_i f_i(x) + \hat{a}_i^\dagger f_i^*(x) \right), \tag{3.34}$$

where a_i and $a_i^\dagger$ are creation and annihilator operators, respectively, that obey the well known algebra of ladder operators. The vacuum state $|0\rangle$ is defined as $a\,|0\rangle = 0$. Obviously, the choice of the f_i, called *positive frequency modes*, determines the vacuum state. Consider two complete sets of solutions $\{f_i^{(1)}, f_i^{(1)*}\}$ and $\{f_j^{(2)}, f_j^{(2)*}\}$. Since both sets are complete they may be expressed in terms of each other

$$f_j^{(2)} = \sum_i \left(A_{ij} f_i^{(1)} + B_{ij} f_j^{(1)*} \right), \tag{3.35}$$

$$f_j^{(2)*} = \sum_i \left(A_{ij}^* f_i^{(1)*} + B_{ij}^* f_j^{(1)} \right), \tag{3.36}$$

and vice versa. The scalar field can be expressed using either set of solutions

$$\hat{\phi}(x) = \sum_i \left(\hat{a}_i^{(1)} f_i^{(1)}(x) + \hat{a}_i^{(1)\dagger} f_i^{(1)*}(x) \right) = \sum_j \left(\hat{a}_j^{(2)} f_j^{(2)}(x) + \hat{a}_j^{(2)\dagger} f_j^{(2)*}(x) \right). \tag{3.37}$$

Thus, the two sets of ladder operators are related as

$$\hat{a}_i^{(1)} = \sum_j \left(A_{ij} \hat{a}_j^{(2)} + B_{ij}^* \hat{a}_j^{(2)\dagger} \right), \tag{3.38}$$

and similarly for the other operators. This transformation is a *Bogoliubov transformation*, which mixes creation and annihilation operators.

The two sets of solutions give rise to two different vacuum states, namely $|0_{(1)}\rangle$ and $|0_{(2)}\rangle$. The number operator $\hat{N}_i^{(1)} = \hat{a}_i^{(1)\dagger}\hat{a}_i^{(1)}$ gives 0 by construction when applied to the vacuum $|0_{(1)}\rangle$. But a short calculation shows that

$$\langle 0_{(2)}|N_i^{(1)}|0_{(2)}\rangle = \sum_j |B_{ij}|^2. \tag{3.39}$$

Thus, the vacuum of the set of solutions $\{f_j^{(2)}, f_j^{(2)*}\}$ is not necessarily a vacuum of the set $\{f_j^{(1)}, f_j^{(1)*}\}$. Minkowski space yields a natural choice for the positive frequency modes because of the global timelike Killing field $\xi^a = \frac{\partial}{\partial t}$. But there exists another timelike Killing field defined in the regions $|t| < z$ that plays the role of generating time translations for the accelerating observer. Therefore, these observers define different sets of positive frequency solutions which leads to the definition of different vacua and the measurement of a non-zero number of particles. The spectrum of particles measured by the accelerated observer can be shown to equal a thermal bath of temperature T_U.

The Hawking effect can be motivated easily using the Unruh effect (taken from [28]). For definiteness, consider an accelerated non-rotating observer hovering above a Schwarzschild black hole at a fixed distance r. If the observer is close to the horizon the acceleration a is very large and the timescale set by a^{-1} is much smaller than the Schwarzschild radius R_S. Therefore, spacetime curvature is effectively negligible over these length- and timescales and the static observer appears to be an accelerated observer in Minkowski spacetime, thus measuring an Unruh radiation at temperature $T_1 = \frac{a_1}{2\pi}$. The outgoing radiation will be redshifted by the norm of the timelike Killing vector $V = \sqrt{-\xi^a \xi_a}$, so that the ratio of temperatures measured by static observers at two different radii is given by $T_2/T_1 = V_1/V_2$. At infinity $V_2 = 1$ so that

$$T_\infty = \lim_{r_1 \to R_S} \frac{V_1 a_1}{2\pi} = \frac{\kappa}{2\pi}. \tag{3.40}$$

This is the Hawking temperature (with all constants restored)

$$T_H = \frac{\hbar\kappa}{2\pi c k_B} \tag{3.41}$$

of the black hole. An observer in the distance r_1 of the black holes will measure thermal radiation of temperature $T = \frac{T_H}{V_1}$. In a very crude manner Hawking radiation can be understood as spontaneous particle pair creation in the vicinity of the black hole. One of the particles has negative energy and will vanish behind the horizon to decrease the black hole's mass. The other particle will propagate to infinity and is interpreted as radiation originating from the black hole.

For a Schwarzschild black hole the Hawking temperature is $T_H \propto \frac{1}{8\pi GM}$, thus the temperature rises as the black hole's mass is radiated away. The timescale of this *black hole evaporation* can be estimated easily using Stefan's law

$$\frac{dE}{dt} \propto T^4 A. \tag{3.42}$$

For a Schwarzschild black hole one has: $E = M, A \propto M^2, T \propto M^{-1}$ and thus

$$\frac{dM}{dt} \propto -\frac{1}{M^2}. \tag{3.43}$$

Integration of this equation yields $\tau \propto M^3$ in Planck units. For a black hole of solar mass this is about 10^{53} times greater than the age of the universe.

The first law (3.28) can be rewritten using the above defined quantities of Bekenstein Hawking entropy $S_{BH} = \frac{A}{4}$ and Hawking temperature $T_H = \frac{\kappa}{2\pi}$ to yield

$$dM(S, J, Q) = T_H dS_{BH} + \Omega_H dJ + \Phi_H dQ. \tag{3.44}$$

This is no more a mere analogy to the first law of thermodynamics but the first law of thermodynamics applied to a black hole. T_H is the physical temperature of the black hole and S_{BH} is the physical entropy of the black hole. Boltzmann's law (3.1) asserts that the number of microstates compatible with a given macroscopic state is $\Omega \propto e^{A/4}$. Even for a solar mass black hole this is an enormous number. The origin of this black hole entropy is still unknown and the aim of intense research. A few lines of research shall be mentioned (see [11] for more references):

Black holes form in gravitational collapse from very complex objects like stars but are described by just a few numbers (at least those black holes that are included in the no-hair theorem). The black hole entropy is a remnant of the complexity of the black hole's origin and encodes the different ways that a black hole can be formed [103].

The black hole entropy has been linked to *entanglement entropy*. Entanglement entropy describes the correlation between two separated subsystems. In the case of black holes the subsystems are separated by the horizon, i.e. one subsystem is visible to an observer the other is hidden behind the horizon. Although the entropy turns out to be proportional to the area of the horizon a few problems remain (UV divergence of entanglement entropy, "species puzzle", etc.) See [123] for a recent review.

It was shown [83, 84] that a generalized form of the first law of black hole mechanics holds for any theory that can be derived from a diffeomorphism covariant Lagrangian, which can always be written in the form $L = L(g_{ab}; R_{abcd}, \nabla_a R_{bcde}, ..; \psi, \nabla_a \psi, ...)$, where ψ denote any matter fields present. It turns out that black hole entropy is the Noether charge of the diffeomorphism symmetry [140].

The proposition that sparked the most interest was the formulation of the *holographic principle* due to 't Hooft [128] and Susskind [126]. In its original form the holographic principle states that the number of degrees of freedom of every physical system is bounded by the boundary surface of the system measured in Planck units. The Bekenstein–Hawking entropy would then be the simplest example of this principle. In this form the holographic principle is ruled out. An alternative proposal that reduces to the above definition in most situations and circumvents its problems, is the *covariant entropy bound* proposed by Bousso [16]. For a review on the holographic principle see also [17].

A line of research where the holographic principle can be explicitly seen at work is the conjectured AdS/CFT correspondence, or more generally gauge/gravity duality. This duality states that a theory of gravity formulated in the bulk of spacetime is dual to a gauge theory on the conformal boundary of spacetime. The AdS/CFT correspondence is a particular realization of this duality, where the spacetime considered is Anti de Sitter space (AdS) and the gauge theory on the boundary is a conformal field theory (CFT). This duality was formulated for the first time in a now famous paper by Maldacena [97] as a correspondence between a N=4 Super-Yang–Mills theory in four dimensions on the boundary and a type IIb superstring theory in $AdS_5 \times S_5$. A first glimpse of this duality was already the realization that the asymptotic symmetry algebra of AdS_3 consists of two copies of the Virasoro algebra, the algebra of 2d CFT [19]. This result was later used to reproduce the Bekenstein–Hawking entropy for AdS_3 black holes by invoking the Cardy formula [124]. Gauge/gravity duality has been recently generalized and has produced a vast field of literature. The interested reader is referred to [2, 105, 113],

As can be seen from the last paragraph, AdS space plays an important role in the mentioned concepts. This is due to the interesting features of this spacetime, which will be presented in parts in the next section.

3.4 The Euclidean Path Integral and Black Hole Thermodynamics

Of all approaches to quantum gravity the Euclidean path integral is best suited to study thermodynamics. This technique was pioneered by Gibbons and Hawking [53, 74].

As known from quantum field theory, the path integral yields amplitudes for transitions between different physical states. In the case of gravity, the physical states are considered to be hypersurfaces S_1, S_2 equipped with respective metrics g_1, g_2 in spacetime

$$\langle S_1, g_1 | S_2, g_2 \rangle = \int \mathcal{D}[g, \Phi] e^{iI[g,\Phi]}, \tag{3.45}$$

where Φ denotes additional (matter) fields.

The action $I[g, \Phi]$ is real if Φ and g are both real and consequently, the path integral oscillates and does not converge. This behavior is known from ordinary quantum field theory where it is circumvented by Wick rotating the action, i.e. the time coordinate t is replaced by an imaginary coordinate τ. The same procedure solves the mentioned problem for the path integral (3.45). This redefinition of the time coordinate is equivalent to a change from Lorentzian signature $(-+++)$ to Euclidean signature $(++++)$ and replacement of the action $I[g, \Phi]$ with the Euclidean action $I_E[g, \Phi] = -iI[g, \Phi]$. If the Euclidean action were positive definite the integral (3.45) would converge. But this is not the case in Einstein gravity. This is explicitly seen, if the metric is conformally transformed $\tilde{g}_{ab} = \Omega^2 g_{ab}$. Under this transformation the action reads

$$I_E[\tilde{g}] = -\frac{1}{16\pi G} \int_{\mathcal{M}} \mathrm{d}^{d+1}x \sqrt{g} \Omega^2 R + 6\partial_a \Omega \partial^a \Omega - \frac{1}{8\pi G} \int_{\partial\mathcal{M}} \mathrm{d}^d \sqrt{\gamma} \Omega^2 K. \tag{3.46}$$

If a rapidly varying conformal factor Ω is chosen the action is unbounded from below. An ad-hoc prescription for the removal of this divergence presented in [54] is the complex rotation of this conformal mode. Other proposals stress the point that the conformal mode is no propagating degree of freedom and can therefore be removed in a Fadeev-Popov like manner. After evaluation of the path integral, the results should be analytically continued back to Lorentzian signature.[1]

The usefulness of the euclidean path integral in thermodynamic considerations is rooted in its close connection to the partition function. Consider the amplitude

$$\langle \Phi_2 | \exp\left(-iH(t_2 - t_1)\right) |\Phi_1\rangle \,, \tag{3.47}$$

with $\Phi_2 = \Phi_1$, i.e. a periodic field with periodicity $t_2 - t_1 = -i\beta$. Summation over a complete set of orthonormal states Φ_n with energy E_n leads to the partition function

$$\mathcal{Z} = \sum \exp\left(-\beta E_n\right), \tag{3.48}$$

with $\beta = T^{-1}$. The thermodynamic potential is then easily calculated from the partition function. For instance, for a system in the canonical ensemble, i.e. temperature and volume are fixed, the free energy $F(T, V)$ is

$$\ln \mathcal{Z} = -\beta F(T, V). \tag{3.49}$$

But the amplitude (3.47) can be calculated also using the Euclidean path integral, where the integration is over fields with periodicity β in the imaginary time coordinate. Similar considerations can be applied to the gravitational field g_{ab}. A set of boundary conditions is fixed and the path integral samples all possible metrics that satisfy the boundary conditions. The temperature is determined by the requirement that no conical defects should occur. This fixes the periodicity of the Euclidean imaginary time to $\beta = \frac{2\pi}{\kappa}$.

For the treatment of thermodynamics the saddlepoint or WKB approximation of the path integral is invoked. The dominant contributions to the path integral come from the classical solutions $\bar{g}, \bar{\phi}$ of the system, that extremize the action. Therefore, the partition function is expanded around these solutions

$$\mathcal{Z} = \exp\left(-I[\bar{g}, \bar{\Phi}]\right) \int \mathcal{D}\delta g \mathcal{D}\delta\Phi \exp\big(-\delta I[\bar{g}, \bar{\Phi}] - \delta^2 I[\bar{g}, \bar{\Phi}] + ...\big). \tag{3.50}$$

The variational principle demands that the first order variation of the action vanish on-shell. However, this is in general not the case, neither for Einstein gravity nor for other theories of gravity such as dilaton gravity. Furthermore, the on-shell action will diverge in most cases. A detailed discussion of these properties of the action is carried out in section 5. Thermodynamic stability requires that the second order variation is positive definite. If this is not the case then the partition function does not describe the appropriate thermodynamic potential of the system but decay amplitudes. As the specific heat of the Schwarzschild black hole is negative the above applies here. This problem can be avoided if the black hole is studied in a cavity of fixed radius and temperature [148].

[1]Needless to say, a Wick rotation singles out the time coordinate, which is contrary to the framework of general relativity. Presumably, a Wick rotation is well-defined only if the spacetime is endowed with a global timelike Killing field. In this work we are concerned with black hole solutions, which yield such a global notion of time.

In York's approach a cavity of fixed radius r, or area $A = 4\pi r^2$, and temperature T is given. If a black hole of mass M sits at the center of the cavity, this temperature is

$$T(r) = \frac{1}{8\pi M}(1 - \frac{2M}{r})^{-1/2}. \tag{3.51}$$

This is just the Hawking temperature of a Schwarzschild black hole redshifted by a Tolman factor. The classical solutions of Einstein's equations that obey these boundary conditions and can form in the cavity are given by the solutions for M of (3.51). In general, the solutions will be double valued. Depending on the quotient of r and T, either two black holes of masses M_1, M_2 with $M_1 < M_2$ or one black hole of mass $M_1 = M_2$ exist. It turns out that the lower mass solution M_1 is not stable.

The traditional way to eliminate the problems mentioned after (3.50) was the subtraction of the action evaluated for some reference spacetime— often the ground state— with the same boundary metric. The on-shell action is thus rendered finite. This procedure is somewhat analogous to the subtraction of the zero-point energy in quantum field theory. Despite yielding acceptable results in most cases, this technique is unsatisfactory to some extent. From a physical point of view, the introduction of a reference spacetime is contrary to the idea of background independence; from a mathematical point of view, the embedding of a given boundary metric into a reference spacetime is often not possible (e.g. for Taub–NUT spacetimes) . It would be preferable, if the counterterms were constructed from quantities that are intrinsic to the boundary. This method, originating in the AdS/CFT correspondence, is called *holographic renormalization* and is treated in section 5. For the moment we stick to the subtraction method employed by York.

The action is the usual Einstein–Hilbert action with a GHY boundary term and the flat Euclidean spacetime as the reference spacetime for which the subtracted action is evaluated:

$$I = I_{EH} + I_{GHY} - (I_{EH} + I_{GHY})\Big|_{flat}. \tag{3.52}$$

Since Schwarzschild is Ricci-flat, only the boundary terms contribute and the on-shell action is

$$I = 12\pi M^2 - 8\pi M r + \beta r. \tag{3.53}$$

Therefore, $\exp(-I)$ is the dominant contribution to the partition function and from basic relations from statistical mechanics follow

$$\langle E \rangle = -\frac{\partial \ln \mathcal{Z}}{\partial \beta} = r - r\left(1 - \frac{2M}{r}\right)^{1/2}, \tag{3.54}$$

$$S = \ln \mathcal{Z} - \beta \ln \mathcal{Z} = 4\pi M^2 = \pi r_h^2 \tag{3.55}$$

The first observation is that we recover the Bekenstein–Hawking entropy. This circumstance gave strong evidence that black hole thermodynamics was a meaningful concept since the Bekenstein–Hawking entropy was originally derived along independent lines of reasoning.

As a second interesting observation one can solve (3.54) for M

$$M = E - \frac{1}{2}\frac{E^2}{r}. \tag{3.56}$$

Thus, we see that the ADM mass M is just the thermal energy plus the gravitational self energy of the thermal energy. In the limit of vanishing cavity, i.e. $r \to \infty$, the two equal each other.

If one defines the surface pressure σ conjugate to the area A of the cavity as $\sigma = -\frac{\partial E}{\partial A}\Big|_S$ we obtain

$$\mathrm{d}E(S, A) = T\,\mathrm{d}S - \sigma\,\mathrm{d}A, \tag{3.57}$$

or in the limit $r \to \infty$

$$\mathrm{d}M = T\,\mathrm{d}S, \tag{3.58}$$

the first law of thermodynamics for a finite cavity, and for the removed cavity.

The calculation of the free energy from the above allows the consideration of phase transitions between hot flat space and the two black hole spacetimes of different mass, first proposed in [60]. Notice that this phase transition is not to be mistaken for a gravitational collapse. The two black holes of different mass represent the extrema of free energy, where the free energy of the lower mass black hole is always higher than the free energy of the higher mass black hole. Under certain conditions the free energy of the latter is negative, which implies that the black hole of mass M_2 can nucleate from hot flat space where the black hole of mass M_1 represents a potential barrier between the two stable configurations [148].

This concludes our review of black hole thermodynamics. In section 6 an approach, based on the above, will be applied to black hole thermodynamics in dilaton gravity. Since the original part of this work concerns black holes in AdS spacetime, the next section deals with the cosmological constant.

4 The cosmological constant

4.1 The cosmological constant problem

As mentioned in the introduction, the small value of the cosmological constant poses one of the greatest challenges in theoretical physics. The naively predicted value of the cosmological constant drastically deviates from the experimental measurements by approximately 120 orders of magnitude. A crude estimate of the magnitude of Λ based on field theoretic grounds is given as follows: A quantum field Φ is usually quantized as the sum over an infinite number of harmonic oscillators. The Hamiltonian H is then

$$H = \int \frac{d^3p}{(2\pi)^3}\omega_{\boldsymbol{p}}\frac{1}{2}\left(a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} - a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}}\right) = \int \frac{d^3p}{(2\pi)^3}\omega_{\boldsymbol{p}}\left(a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} + \frac{1}{2}\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}\right]\right), \tag{4.1}$$

in terms of creation and annihilation operators. The last term is proportional to $\delta(0)$ and divergent, which was expected since it corresponds to an infinite sum over all ground state energies $\omega_{\boldsymbol{p}}/2$. In quantum field theory one is only interested in the energy *differences* of various states. Therefore, this infinite zero point energy can be discarded by the introduction of *normal ordering*. For gravity the case is not so easy as it should couple to all energies, even vacuum fluctuations. The vacuum energy must be taken into account by introducing a vacuum energy momentum tensor $T^{(vac)}_{ab}$ of the form

$$T^{(vac)}_{ab} = -\rho_{vac}g_{ab}. \tag{4.2}$$

This form is dictated by the requirement that the vacuum be Lorentz invariant. Denoting energy density, pressure and velocity vector by ρ, p and u_a, respectively, this corresponds to the energy momentum tensor of a perfect fluid of the form

$$T_{ab} = (\rho + p)u_a u_b + pg_{ab}, \tag{4.3}$$

with $\rho_{vac} = -p_{vac}$. This is added to the matter energy momentum tensor $T^{(m)}$ in Einstein's equation as a separate contribution

$$G_{ab} = 8\pi G\left(T^{(m)} - \rho_{vac}g_{ab}\right). \tag{4.4}$$

Now, written on the left hand side and setting

$$\rho_{vac} = -p_{vac} = \frac{\Lambda}{8\pi G} \tag{4.5}$$

this is Einstein's equation with a cosmological constant. It is generally believed that quantum field theory is only valid up to some cutoff momentum p_{max}. If modes of higher energy than this are discarded the vacuum energy is approximately $\rho_{vac} \propto \hbar p^4_{max}$. If the cutoff scale is taken to be the Planck scale $m_P = 10^{19}GeV$ the vacuum energy density is $\rho_{vac} \approx 10^{120}\ GeV/m^3$. Experimentally, the vacuum energy density is expressed in terms of the critical density $\rho_{crit} = \frac{3H^2}{8\pi pG}$, where H is the Hubble parameter. The critical density is the energy density of a spatially flat universe for a given Hubble parameter. Current measurements of the density parameter $\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c}$ yield $\Omega_\Lambda = 0.73(3)$ [12]. With a measured value of ρ_c of about $\rho_c \approx 5.3GeV/m^3$ [12], the cosmological constants deviates about 120 orders of magnitude from its naively predicted value. This is known as the *cosmological constant problem*. It goes without saying that this discrepancy cries out for a solution. Consequently, various mechanisms have been developed to accommodate this conflict. Some of these are the subject of the next section.

4.2 A Changing Cosmological Constant

The starting point for the present work is the idea of considering the cosmological constant as a variable quantity presented in [77, 78]. As they point out, if Λ is to be regarded as a variable quantity it should be possible to vary it in the action. In the usual Einstein–Hilbert action

$$S = \frac{1}{16\pi G_4} \int \mathrm{d}^4x \sqrt{-g}\,(R - 2\Lambda) \tag{4.6}$$

this is not possible, as a variation of Λ would lead to $g = 0$, which is not possible. A simple mechanism that allows for this variation is the coupling of gravity to a completely antisymmetric gauge field $A_{\mu\nu\rho}$ with action

$$S = \frac{1}{16\pi G_4} \int \mathrm{d}^4x \sqrt{-g} \left(R + \frac{1}{4!2} F_{\mu\nu\lambda\rho} F^{\mu\nu\lambda\rho} \right), \tag{4.7}$$

where $F_{\mu\nu\lambda\rho} = \partial_{[\mu} A_{\nu\lambda\rho]}$. The solution of $F_{\mu\nu\lambda\rho}$ has the form

$$F_{\mu\nu\lambda\rho} = \frac{2\sqrt{3}}{L} \sqrt{-g}^{1/2} \epsilon_{\mu\nu\lambda\rho}, \tag{4.8}$$

where L is an integration of constant and additional factors were introduced for convenience. Inserted back into (4.7), we obtain (4.6) with $\Lambda = \frac{3}{L^2}$. In this framework, the variation of the cosmological constant is on the same footing as the variation of an electric charge. The addition of another field that can vary leads to a different variational principle. Henneaux and Teitelboim treated the variational problem of the above action (4.7) in the Hamiltonian framework in detail.

Further work in [21] showed that the above implementation of a cosmological constant could be used to account for a dynamical neutralization. The totally antisymmetric tensor field gives rise to the spontaneous creation of closed membranes where the value of the cosmological constant is inside smaller than outside. The action principle in D dimensions relevant for this model reads

$$\begin{aligned} S = &- m \int \mathrm{d}^d\xi \sqrt{-g^{(d)}} + \frac{e}{d!} \int \mathrm{d}^d\xi A_{\mu_1,\dots,\mu_d} \left(\frac{\partial z^{\mu_1}}{\partial \xi^{a_1}} \cdots \frac{\partial z^{\mu_d}}{\partial \xi^{a_d}} \right) \epsilon^{a_1\dots a_d} \\ &- \frac{1}{2D!} \int \mathrm{d}^D x \sqrt{-g} F_{\mu_1\dots\mu_D} F^{\mu_1\dots\mu_D} + \frac{1}{d!} \int \mathrm{d}^D x \partial_{\mu_1} \left(\sqrt{-g} F^{\mu_1\dots\mu_D} A_{\mu_2\dots\mu_D} \right) + S_{grav}(\lambda). \end{aligned} \tag{4.9}$$

The d-dimensional membrane, with coordinates $\xi^a, d = 0, ..., d-1$ on the membrane, is embedded in the D=d+1 dimensional spacetime using the embedding function $x^\mu = z^\mu(\xi^a)$. The induced metric on the membrane is denoted as $g^{(d)}_{ab}$. Additionally, the membrane has the mass m per unit volume, and e defines the strength of the coupling.

The total derivative is included in order to guarantee a well defined variational principle and $S_{grav}(\lambda)$ is the usual action for gravity with a cosmological constant λ. This cosmological constant is generated through other processes not considered in this model. The most general form of the totally antisymmetric field $F_{\mu_1,\ldots,\mu_D}$ is

$$F_{\mu_1,\ldots,\mu_D} = E\sqrt{-g}\epsilon_{\mu_1,\ldots,\mu_D}. \tag{4.10}$$

When inserted into the field equation for the gauge field $A_{\mu_1,..,\mu_d}$ one obtains

$$\epsilon^{\mu_1,\ldots,\mu_D}(\partial_{\mu_1}E) = -e\int \mathrm{d}^d\xi\delta^{(D)}(x-z(\xi))\left(\frac{\partial z^{\mu_1}}{\partial\xi^{a_1}}\cdots\frac{\partial z^{\mu_d}}{\partial\xi^{a_d}}\right)\epsilon^{a_1\ldots a_d}. \tag{4.11}$$

This means that E is constant away from the membrane and jumps to a different value when crossing the membrane.

Ignoring the effects of gravity for the moment, in 2 dimensions this corresponds to spontaneous pair creation in an external electric field. The electric field between these particles is E_i while the electric field outside is denoted by E_o. The difference is

$$E_i - E_o = -|e|\mathrm{sgn}E_o. \tag{4.12}$$

Since the particles accelerate outwards the region with a lower field strength grows.This process can be understood as instanton tunneling between two classical configurations. In a semiclassical approximation the probability for this transition is given by

$$P \sim \exp\left(-(S_E(instanton) - S_E(background))/\hbar\right), \tag{4.13}$$

where S_E denotes the classical Euclidean action evaluated on some field configuration. The calculation is carried out in detail in [21] with the result

$$P \sim \exp\left(-\frac{\pi m^2}{\hbar|eE_{on}|}\right), \tag{4.14}$$

where E_{on} is the average of E_i and E_o. This is —not surprisingly— the same result as for Schwinger pair creation in an external electric field [120]. Adding gravity to this picture, the value of the cosmological constant inside and outside the membrane, respectively, is given by

$$\Lambda_i = \lambda + \frac{1}{2}k(E_i)^2 \qquad \Lambda_o = \lambda + \frac{1}{2}k(E_o)^2, \tag{4.15}$$

where k denotes the factor appearing in Einstein's equations, i.e. $k = 8\pi G$ in $D = 4$. Again, one can calculate the probability for spontaneous bubble creation. In the case that the initial spacetime is flat or AdS, $\Lambda_o \leq 0$, membranes are created only for specific initial values Λ_o, E_o satisfying the condition

$$|eE_0| \geq \frac{e^2}{2} + \frac{kdm^2}{2(d-1)} + m\sqrt{-\frac{2d\Lambda_o}{d-1}}. \tag{4.16}$$

These bubbles always reduce the cosmological constant, i.e. $\Lambda_o > \Lambda_i$. If the initial geometry is de Sitter, membrane creation can both increase and decrease the cosmological constant, but it can be shown that the latter is much more likely to occur. Brown and Teitelboim used this fact to explain for the nearly vanishing value of the cosmological constant. Unfortunately, the timescales to decrease Λ from some initial de Sitter spacetime to the nearly flat geometry observed today is of order $\exp\left(10^{120}\right)$ in Planck units. Additionally, in order to explain the current small value of the cosmological constant, the membrane charges must be extremely small compared to other microphysical scales.

The above mechanism garnered a large amount of related ideas. A well-known work [18], proposed the quantization of E based on arguments originating from string theory. They showed that the addition of multiple four fields with different coupling constants allowed for richer and potentially more realistic dynamics since the charges of the membranes need not be small anymore.

As mentioned above, pair creation in 2 dimensions or membrane creation in higher dimensions requires tunneling through a potential barrier related to instantons. A different mechanism for membrane creation in de Sitter space was proposed in [55]. Instead of tunneling through the potential barrier, a thermal fluctuation provides the necessary energy for membrane creation. Under specific circumstances a pair of membranes is created, where the outer membrane expands and the inner membrane collapses to form a black hole, in the space between the membranes the value of the cosmological constant is lowered. The mass of the black hole created when the cosmological constant is changed from Λ_+ to Λ_-, is given by

$$M = \frac{1}{6} \frac{\Lambda_+ - \Lambda_- + 3\mu^2}{\left(\frac{\Lambda_+}{3} + \mu^2\right)^{3/2}}, \tag{4.17}$$

where μ is the tension of the membrane (cf. [56] for details). One can think of this process as a transformation of unlocalized dark energy, represented by the cosmological constant, into localized matter in the form of black holes [56] .

Apart from the above, a dynamical cosmological constant has implications that are directly related to black hole physics. If the cosmological constant is treated on the same footing as the charge of a gauge field or angular momentum its variation should be included in the first law of thermodynamics. Since black hole thermodynamics is not well understood for de Sitter spacetimes due to the appearance of the cosmological horizon, only black holes in AdS space are treated in the following.

4.2.1 The Cosmological Constant as a Thermodynamic Variable

In section 3.2.2 the first law of thermodynamics was derived making use of the Smarr formula (3.25) and Euler's theorem (3.26). Here we use a similar argument. Consider a Schwarzschild black hole in AdS_{d+1} of mass M and horizon area A (charge Q and angular momentum J do not change the argument). If we include the cosmological constant Λ as a thermodynamic variable we have $M = M(A, \Lambda)$. Under a change of length scale L the cosmological constant changes as $\Lambda \propto L^{-2}$, which can be derived from the action (4.6), whereas M and A scale as described below equation (3.26). The application of Euler's theorem yields

$$(d-2)M = (d-1)\left(\frac{\partial M}{\partial A}\right)A - 2\left(\frac{\partial M}{\partial \Lambda}\right)\Lambda \tag{4.18}$$

$$= (d-1)\frac{\kappa}{8\pi G}A - 2\Theta\Lambda, \tag{4.19}$$

where we have used results from 3.2.2 and defined $\Theta = \frac{\partial M}{\partial \Lambda}$. Since one should include the scaling properties of all quantities appearing in M, the cosmological constant must be included in the Smarr formula. In the following, the results of [89]—where a geometric expression for Θ was obtained using the Hamiltonian perturbation theory of [134, 135]—are reviewed in detail, since they constitute an important result for this work. The starting point is the usual Hamiltonian split of spacetime into a family of spacelike surfaces Σ with timelike normal vector n^a and $n^a n_a = -1$. The spacetime metric g_{ab} is therefore split as

$$g_{ab} = -n_a n_b + s_{ab}, \qquad s_{ab}n^a = 0 \tag{4.20}$$

where s_{ab} is the induced metric on Σ. The Hamiltonian and momentum constraints,

$$H = -2Gabn^a n^b \qquad H_a = -2G_{bc}n^b sc_a, \tag{4.21}$$

respectively, read

$$H = -2\Lambda \qquad H_a = 0, \tag{4.22}$$

if a cosmological constant is included and the stress energy tensor $T_{ab} = 0$. The Hamiltonian density $\mathcal{H}$ for evolution along the vector field $\xi^a = Fn^a + \beta^a$ is

$$\mathcal{H} = \sqrt{s}(F(H + 2\Lambda) + \beta^a H_a), \tag{4.23}$$

where F and β are the usual lapse and shift functions. The canonical variables are s_{ab} and its conjugate momentum π_{ab}.

Consider the solution $s^{(0)}_{ab}$, $\pi^{(ab)}_{(0)}$ with cosmological constant $\Lambda_{(0)}$ and Killing vector ξ^a, along which the system is evolved. Furthermore, we define

$$\xi^a = \nabla_b \omega^{ab}, \tag{4.24}$$

where the quantity ω^{ab} is called the Killing potential [8, 88]. This potential is not unique, as a different potential $\omega'^{ab} = \omega^{ab} + \lambda^{ab}$ yields the same Killing vector, provided that $\nabla_a \lambda^{ab} = 0$. A small perturbation around this solution reads $s_{ab} = s^{(0)}_{ab} + h_{ab}$, $\pi^{ab} = \pi^{ab}_{(0)} + p^{ab}$ and $\Lambda = \Lambda + \Lambda_{(0)}$. The linearized constraint operators δH and δH_a can be written as a total derivative

$$F\delta H + \beta^a \delta H_a = -D_c B^c, \tag{4.25}$$

where D_c is the covariant derivative compatible with $s^{(0)}_{ab}$ and the vector field B^c is

$$B^a = F(D^a h - D_b h^{ab}) + hD^a F + h^{ab}D_b F + \frac{1}{\sqrt{s}}\beta^b\left(\pi^{cd}_{(0)}h_{cd}s^{(0)a}_b - 2\pi^{ac}_{(0)}h_{bc} - 2p^a_b\right), \tag{4.26}$$

where h denotes the trace of the perturbation h_{ab}. For the above constraints this reduces to

$$D_c B^b = 2F\delta\Lambda, \tag{4.27}$$

if the perturbations are solutions of the linearized Einstein equations. The right hand side can be rewritten as a total derivative using the Killing potential and the expression is rewritten using Gauß' law as

$$d\int_{\partial\Sigma} \mathrm{d}a_c \left(B^c + 2\omega^{cd} n_d \delta\Lambda\right) = 0. \tag{4.28}$$

The unperturbed solution is taken to be a static asymptotically AdS black hole. The perturbations around this solution should satisfy the same boundary conditions. The Killing vector ξ^a, along which the system is evolved, is taken to be the generator of the horizon, that approaches $\left(\frac{\partial}{\partial t}\right)^a$ asymptotically. The boundary $\partial\Sigma$ in the above integral consists of a sphere at infinity $\partial\Sigma_\infty$ and the bifurcation two sphere at the horizon $\partial\Sigma_h$. Thus, the above integral can be rewritten as

$$I_\infty - I_h = 0. \tag{4.29}$$

The calculation of these quantities yields [89, 125, 135]

$$I_\infty = -16\pi\delta M - 2\left(\int_{\partial\Sigma_\infty} \mathrm{d}S_{ab}(\omega^{ab} - \omega^{ab}_{AdS})\right)\delta\Lambda \tag{4.30}$$

$$I_h = -2\kappa\delta A - 2\left(\int_{\partial\Sigma_h} \mathrm{d}S_{ab}\omega^{ab}\right). \tag{4.31}$$

Here, the Killing potential of pure AdS is taken to be

$$\omega^{rt}_{AdS} = -\omega^{tr}_{AdS} = \frac{r}{d}. \tag{4.32}$$

Inserted in equation (4.29) this is

$$\delta M = \frac{\kappa}{8\pi}\delta A + \frac{\Theta}{8\pi}\delta\Lambda, \tag{4.33}$$

with Θ given by the expression

$$\Theta = -\left[\int_{\partial\Sigma_\infty} \mathrm{d}S_{ab}(\omega^{ab} - \omega^{ab}_{AdS}) - \int_{\partial\Sigma_h} \mathrm{d}S_{ab}\omega^{ab}\right]. \tag{4.34}$$

This is the first law of black hole thermodynamics with a varying cosmological constant.

The above expression for Θ implies a simple physical interpretation for the variable conjugate to Λ. If the slicing of spacetime is orthogonal to the Killing vector field $\xi^a = Fn^a$, we have $\sqrt{-g^{d+1}} = F\sqrt{g^d}$. Thus, we have the relation

$$\int_{\partial\Sigma} \mathrm{d}S_{ab}\omega^{ab} = \int_\Sigma \mathrm{d}^d x\sqrt{g^d} n_b \xi^b = -\int_\Sigma \mathrm{d}^d x\sqrt{g^{d+1}}. \tag{4.35}$$

This is the infinite volume between the black hole horizon and infinity. From this quantity the infinite volume of AdS space is subtracted. The remaining finite volume is the volume excluded from spacetime due to the presence of the black hole. This provides a nice definition for the volume of a black hole. Inside the event horizon the radial coordinate is timelike, thus the naive definition of a volume by integrating from $r = 0$ to $r = r_h$ makes no sense. Nonetheless, the above considerations suggest the definition

$$V_{BH} = -\Theta \tag{4.36}$$

for a *thermodynamic volume* of a black hole in AdS space.

From comparison with the stress energy tensor of a perfect fluid in Einstein's equation one can associate the pressure

$$P = -\frac{\Lambda}{8\pi} \tag{4.37}$$

with the cosmological constant. Together with the above relations this leads to the intriguing form

$$\mathrm{d}M(S,P) = T\,\mathrm{d}S + V\,\mathrm{d}P. \tag{4.38}$$

The dependence on the state variables S and P is characteristic for the enthalpy commonly denoted by $H = H(T,P) = E + PV$. Thus, the mass of a black hole in AdS space is identified with enthalpy rather than with internal energy as usually done in black hole thermodynamics (cf. section 6.2.2). In [89] the above result is interpreted in the way that the definition of the AdS mass via a surface integral at infinity requires a subtraction of the infinite amount of energy contained in AdS between the black hole horizon and infinity in order to give a finite quantity. Since energy density and pressure in the cosmological constant differ by a sign, the addition of a PV term cancels the contribution of ρV. In other words [45], the mass of the black hole contains a contribution due to the negative energy density of AdS space. A black hole with volume V contains energy $\epsilon V = -PV$, thus the internal energy is $E = M - PV$.

Consider the line element of AdS Schwarzschild in d+1 dimensions

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + f(r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_{d-2}^2 \qquad f(r) = 1 - \frac{\mu}{r^{d-2}} - \frac{2\Lambda}{d(d-1)}r^2, \tag{4.39}$$

where the mass parameter μ is related to the mass of spacetime M via

$$M = \frac{(d-1)\mathcal{A}_{d-1}\mu}{16\pi}, \tag{4.40}$$

and A_{d-1} is the angle subtended by the S_{d-1} sphere. (The AdS Schwarzschild black holes [15] with different topology, where 1 in $f(r)$ is replaced by $k = 0, -1$ will not be discussed in the remainder of this work). In 3+1 dimensions we obtain

$$M = \frac{r_h}{2}\left(1 - \frac{\Lambda}{3}r_h^2\right) \tag{4.41}$$

Pressure and entropy are given by (4.37) and Bekenstein–Hawking

$$S = \pi r_h^2, \tag{4.42}$$

respectively. We apply the result (4.38) to AdS Schwarzschild in 3+1 dimensions. This yields

$$T = \frac{\partial M}{\partial S}\Big|_P = \frac{1 - \Lambda r_h^2}{4\pi r_h}, \tag{4.43}$$

$$V = \frac{\partial M}{\partial P}\Big|_S = \frac{4\pi r_h^3}{3}, \tag{4.44}$$

which is the correct temperature T for AdS Schwarzschild and the volume V that one would expect naively. This is easily shown to be true for all AdS Schwarzschild black holes. One obtains for the volume

$$V = \frac{\partial M}{\partial P}\Big|_S = \frac{A_{d-1}}{d} r_h^d, \tag{4.45}$$

which is the correct relation for the volume of a ball in d- dimensional Euclidean spacetime. (Analogous results hold for BTZ black holes in 2+1 dimensions cf. [45].) For definiteness, the following discussion is restricted to black holes in 3+1 dimensions with straightforward generalization to higher dimensions.

A Legendre transform of (4.41) with respect to P yields the internal energy

$$E(S, V) = \frac{r_h}{2}. \tag{4.46}$$

Here, a problem of the current approach is evident: the two thermodynamic variables V and S are not independent, since both are functions of r_h and can therefore expressed in terms of each other. Therefore, the Legendre transform is not invertible and the temperature obtained from the internal energy is not correct. This problem is not present if one considers rotating black holes in AdS space. Then the thermodynamic volume depends on r_h and angular momentum J which allows variations of V with S fixed and vice versa. In rotating black hole spacetimes one can distinguish between the thermodynamic volume V, obtained as the conjugate variable to P, and the naive geometrical volume V'.

An important quantity for thermodynamical stability is heat capacity at constant pressure and volume, respectively. These are easily calculated to

$$C_V = \frac{T}{\frac{\partial T}{\partial S}\Big|_V} = 0 \tag{4.47}$$

$$C_P = \frac{T}{\frac{\partial T}{\partial S}\Big|_P} = 2S\left(\frac{8PS + 1}{8PS - 1}\right). \tag{4.48}$$

We require $C_P > 0$, if the black hole is to be stable in AdS space. From this follows

$$|\Lambda| > \frac{1}{r_h^2}, \tag{4.49}$$

which implies a minimal temperature

$$T_{min} = \frac{1}{2\pi}\sqrt{-\Lambda} = \sqrt{\frac{2P}{\pi}} \tag{4.50}$$

The issue of stability in AdS space was treated in detail in the seminal work by Hawking and Page [73] where they showed that no black hole can form below T_{min} and the free energy of pure AdS space is lower for $T < T_{HP}$, where the Hawking–Page temperature is given by

$$T_{HP} = \frac{1}{\pi}\sqrt{-\frac{\Lambda}{3}} = \sqrt{\frac{8P}{3\pi}}. \tag{4.51}$$

We now turn to discuss rotating asymptotically AdS black holes. The mass of a charged rotating asymptotically AdS black hole in 3+1 dimensions expressed in terms of S, J, Q, P is [45]

$$H(S,P,J,Q) = M = \frac{1}{2}\sqrt{\frac{\left(S + \pi Q^2 + \frac{8PS^2}{3}\right) + 4\pi^2\left(1 + \frac{8PS}{3}\right)J^2}{\pi S}}. \tag{4.52}$$

The volume that follows from this enthalpy is

$$V = \frac{\partial H}{\partial P}\Big|_{S,J,Q} = \frac{2}{3\pi H}\left(S\left(S + \pi Q^2 + \frac{8PS^2}{3}\right) + 2\pi^2 J^2\right). \tag{4.53}$$

Obviously, V and S are now independent functions. In the limit $J \to 0$ this reduces to

$$V = \frac{4}{3}\frac{S^{3/2}}{\sqrt{\pi}} = \frac{4\pi}{3}r_h^3 \tag{4.54}$$

and V and S are not independent anymore. The fact that thermodynamic volume and naive geometric volume, denoted V and V' in the following, equal each other seems to be an artifact of the $J = 0$ approximation.

An analogue of the Penrose process, where energy is extracted from a rotating black hole, was calculated in [43]. Work can be extracted in an isentropic, isobaric process, if the angular momentum is reduced from a finite value to zero. The efficiency of this process is

$$\eta = \frac{E(J) - E(0)}{H(J)} \tag{4.55}$$

The greatest efficiency is obtained for extremal black holes

$$\eta = \frac{4\pi\ell^2 + 3S}{2(2\pi\ell^2 + 3S)} - \frac{\pi\ell^2}{(\pi\ell^2 + S)\sqrt{2\pi\ell^2 + 3S}}, \tag{4.56}$$

where we used the AdS radius ℓ. In the limit $\ell \to \infty$ the flat space result (3.7) is reproduced. The efficiency is always greater in AdS space with a maximum efficiency of $\eta = 0.58$. The reason for this is the greater value of J for extremal black holes in AdS compared with flat space. For charged rotating black holes the maximal efficiency can be pushed up to $\eta = 0.75$ compared to $\eta = 0.5$ for flat space.

The behavior of charged black holes in AdS space was compared to a liquid gas system governed by a van der Waals equation in [32]. The similarities were rather of mathematical than of physical nature as different quantities had to be identified, e.g. charge Q with the fluid temperature or inverse temperature β with fluid pressure. If one considers a phase space extended by P and V these discrepancies vanish [94]. The equation of state can be

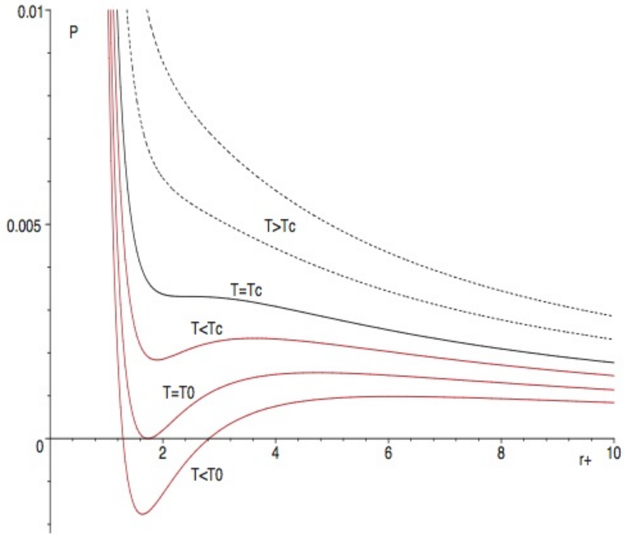

Figure 5. Temperature decreases from top to bottom. T_c denotes the critical isotherm. The dashed lines correspond to ideal gas behavior (taken from [94]).

written as $P = P(V, T)$ and shows a behavior very similar to a van der Waals equation (cf. Figure 5). The critical exponents, which determine the behavior near the critical point (T_c, V_c), turn out to equal those of a van der Waals fluid. This is another indication that pressure and volume should be included in the first law of black hole thermodynamics.

As another application of the thermodynamic volume, the status of the isoperimetric inequality in black hole physics can be investigated. The isoperimetric inequality is the well known statement in Euclidean geometry that the ball has the largest volume for a given boundary area. In general, in d space dimensions

$$R \leq 1 \qquad R = \left(\frac{d\,V}{\mathcal{A}_{d-1}}\right)^{\frac{1}{d}} \left(\frac{\mathcal{A}_{d-1}}{A}\right)^{\frac{1}{d-1}}, \tag{4.57}$$

with equality if the domain is a ball. This equality was tested for a number of black hole spacetimes in [37]. If the volume is taken to be the geometric volume V' than all Kerr-AdS black holes satisfy the isoperimetric inequality $R \leq 1$. On the other hand, if the thermodynamic volume V is used to test the inequality it turns out that it is violated in all cases studied. This led the authors of [37] to conjecture the *reverse isoperimetric inequality*, that states that all black hole in AdS space satisfy $R \geq 1$ with equality for Schwarzschild-AdS. In other words, given a fixed thermodynamic volume the horizon entropy is maximized for AdS- Schwarzschild. Similar calculations were performed for black holes in de Sitter spacetime [46]. The authors showed that the thermodynamic volume between cosmological and event horizon equals the geometric volume. All studied black holes satisfy the reverse isoperimetric inequality provided either the thermodynamic volume of the cosmological horizon or the thermodynamic volume of the event horizon is used. If the volume between the two horizons is used, black holes in de Sitter space satisfy the usual isoperimetric inequality.

The aim of this work is the study of 2d dilaton gravity with the cosmological constant treated as a varying quantity based on a well defined action principle. In order to have a well posed action principle a few conditions have to be satisfied, which might require additional terms to be added to the action. The study of these boundary terms is the subject of the next section.

5 The Variational Principle and Boundary Terms

The variational principle is one of the most fundamental and ubiquitous concepts of physics. Consequently, one expects a physically relevant theory to give rise to a well-defined variational principle. The usual starting point for a variational principle in the Lagrangian framework is the definition of an action functional I as the integral over some Lagrangian density $\mathcal{L}$

$$I[\Phi, \partial\Phi] = \int_{\mathcal{M}} \mathrm{d}^d x \mathcal{L}(\Phi, \partial\Phi), \tag{5.1}$$

where Φ denotes the field content of the theory considered. The classical solutions of the system are then found by considering small variations in the fields and demanding that the action be stationary. Boundary terms are usually eliminated by the requirement that the fields vanish sufficiently fast. Thus, setting to zero the first variation of the action

$$\delta I[\Phi, \partial\Phi] = \int_{\mathcal{M}} \mathrm{d}^d x A_\Phi(\Phi, \partial\Phi)\delta\Phi = 0 \qquad \Rightarrow A_\Phi(\Phi, \partial\Phi) = 0 \tag{5.2}$$

yields the equations of motion $A_\Phi(\Phi, \delta\Phi) = 0$.

This procedure is arguably well justified in the case that one is only interested in the local bulk behavior of the theory, which is governed by the equations of motion. Boundary conditions are then taken into account when solving the equations of motion.

However, it is obvious that the above approach has its drawbacks. First and foremost, due to boundary terms, that appear by partial integration, the first variation of the action might not be zero. Secondly, in the application of the variational principle one should spell out clearly which quantities are to be held fixed at the boundary (Dirichlet-, Neumann-, Robin-boundary conditions etc.) and which quantities are allowed to fluctuate. The number of boundary data required to render the variational principle well-posed equals the degrees of freedom of the theory. This is not clear in the above procedure. Additionally, particularly with regard to the path integral in semi-classical approximation, one demands that the leading contribution, i.e. the on-shell action, be finite. This is also not guaranteed in an approach that relies only on the bulk Lagrangian without additional counterterms.

In summary, an action principle will be called well-posed if it obeys the following criteria:

1. A set of boundary data with appropriate boundary conditions (Dirichlet, Neumann, mixed etc.) is defined.

2. The first variation of the action vanishes on-shell for all variations of the fields that preserve the boundary conditions.

3. The leading contribution to the semi-classical approximation, i.e. the on-shell action, is finite.

4. The second order variation of the action is positive definite.

In general, a given bulk action will not satisfy the above conditions. These drawbacks can be remedied, if the action is supplemented with boundary terms. These terms do not change the physical bulk content since the equations of motion remain unchanged. In the case of black hole thermodynamics, the last problem can be solved as was seen in section 3.4.

In the context of general relativity, the need to supplement the Einstein–Hilbert (EH) action with a boundary term, in order to satisfy the first condition, was first observed by York [147]. Further work in this direction was later carried out by Gibbons and Hawking [53]. The resulting boundary terms is the well-known Gibbons–Hawking–York (GHY) boundary term.

But the addition of a GHY boundary term to the action does not guarantee that the second condition is fulfilled. In the Hamiltonian framework, the corresponding counterterm was first considered by Arnowitt, Deser and Misner [41, 42] in the context of their calculations of the ADM mass, momentum and angular momentum. Regge and Teitelboim [115] later showed that this boundary term is essential if the Hamiltonian is required to be stationary under asymptotically flat boundary conditions.

In the Lagrangian formalism the appropriate counterterms — commonly called *holographic counterterms*— were considered only much later by various authors for diverse boundary conditions (cf. 5.2.2 for references).

In the following, the roles of the two different classes of counterterms will be clarified with an example from classical mechanics before turning to the form of the counterterms in dilaton gravity.

5.1 Gibbons–Hawking–York boundary term

Though fairly well known, we present the reasoning behind the introduction of the GHY boundary term in order to point out the difference to the holographic counterterms.

Consider a simple classical mechanical system with a Lagrangian given by

$$S = -\frac{1}{2}\int_{t_i}^{t_f} \mathrm{d}t q\ddot{q} \tag{5.3}$$

Variation of the action leads to

$$\delta S = \frac{1}{2}(\dot{q}\delta q - q\delta\dot{q})\Big|_{t_i}^{t_f} - \frac{1}{2}\int_{t_i}^{t_f} \mathrm{d}t\ \ddot{q}\delta q, \tag{5.4}$$

which shows that this is just a disguised action for the free particle $\ddot{q} = 0$. Notice that the equation of motion is just of second order, although the Lagrangian is also of second order. The reason for this is that the second order term is introduced via the addition of a total derivative to the Lagrangian of a free particle

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{q}q) = -\frac{1}{2}q\ddot{q} \tag{5.5}$$

Nonetheless, if we are interested in a well-defined variational principle the boundary term should vanish after we have chosen suitable boundary conditions, that allow for solutions of the equation of motion. If we were to fix the data $q(t_i), q(t_f), \dot{q}(t_i), \dot{q}(t_f)$ we would provide four conditions for a second order equation, which would render the equation unsolvable in most cases. The boundary condition $-\frac{1}{2}q^2\delta\left(\frac{\dot{q}}{q}\right)\Big|_{t_i}^{t_f} = 0$ for example, would annihilate the boundary term but allows for no solutions to the equation $\ddot{q} = 0$. Actually, one can show that no combination of q and $\dot{q}$ exists that allows for solutions and annihilates the boundary term [48]. The only viable solution is the addition of a boundary term of the form $\frac{1}{2}(q\dot{q})|_{t_i}^{t_f}$ to the action. This solves both problems since the variation δS reads

$$\delta S = \dot{q}\delta q\Big|_{t_i}^{t_f} - \frac{1}{2}\int_{t_i}^{t_f} \mathrm{d}t\ \ddot{q}\delta q. \tag{5.6}$$

Upon requiring Dirichlet boundary conditions $\delta q\Big|_{t_i}^{t_f} = 0$, the boundary term clearly vanishes and the equation of motion yields a solution.

The above example is similar to the problems encountered in the variation of the EH-action.

$$S_{EH} = -\frac{1}{16\pi G_d}\int_{\mathcal{M}} \mathrm{d}^d x\sqrt{|g|}\ R \tag{5.7}$$

The curvature scalar R contains second order derivatives of the metric that stem from the addition of a total derivative. Surprisingly, Einstein [49] first used the action

$$S = \frac{1}{16\pi}\int_{\mathcal{M}} \mathrm{d}^d x\sqrt{|g|}H = \frac{1}{16\pi}\int_{\mathcal{M}} \mathrm{d}^d x\sqrt{|g|}g^{\alpha\beta}(\Gamma^{\nu}_{\mu\alpha}\Gamma^{\mu}_{\nu\beta} - \Gamma^{\mu}_{\mu\nu}\Gamma^{\nu}_{\alpha\beta}), \tag{5.8}$$

which is of first order in $g_{\mu\nu}$ and differs from (5.7) by a total derivative

$$H = R - \nabla_\alpha(g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu} - g^{\alpha\mu}\Gamma^{\nu}_{\mu\nu}). \tag{5.9}$$

Thus, the case is similar to the classical example presented above and a well defined action principle requires the addition of a boundary term as can be verified by a straightforward calculation.

In the case of pure Einstein gravity, the action supplemented with a GHY boundary term reads

$$I_{EH+GHY} = -\frac{1}{16\pi G_d}\int_{\mathcal{M}} \mathrm{d}^d x\sqrt{|g|}\ R - \frac{1}{8\pi G_d}\int_{\partial\mathcal{M}} \mathrm{d}^{d-1}x\sqrt{|\gamma|}K, \tag{5.10}$$

where γ denotes the induced metric on the boundary and K the *extrinsic curvature* or *second fundamental form.* Similarly, one can easily convince oneself that the dilaton action (6.1) studied in the next section needs a related boundary term.

After this digression concerning the GHY boundary term that renders the action principle well-posed we turn to the discussion of the second kind of counterterms mentioned above. Only when supplemented with these counterterms, the first variation of the action vanishes on-shell for all variations compatible with the boundary conditions. Furthermore, the on-shell action remains finite.

5.2 Holographic counterterms

5.2.1 Hamilton–Jacobi counterterm: a toy example

In the same way as done for the GHY boundary term, we will introduce the notion of holographic counterterms with an example from classical mechanics taken from [63].

Consider the action

$$I[q] = \int_{t_i}^{t_f} \mathrm{d}t \left(\frac{\dot{q}}{2} - \frac{1}{q^2} \right). \tag{5.11}$$

This action describes a particle in the potential $V(q) = \frac{1}{q^2}$. For sufficiently large q the potential $V(q)$ is negligible and the particle propagates freely. This implies that the appropriate boundary conditions for $t_f \to \infty$ are $q|_{t_i}^{t_f} = \infty$, i.e. eventually the particle escapes to infinity. Setting the initial time $t_i = 0$, a variation of the action leads to

$$\delta I[q] = \dot{q}\delta q|^{t_f=\infty} - \dot{q}\delta q|^{t_i=0} + \int_0^\infty \mathrm{d}t \left(\frac{2}{q^3} - \ddot{q} \right) \delta q. \tag{5.12}$$

Now, two shortcomings of the above action become clear. Firstly, the on-shell action is not finite. This follows from the fact that the particle is essentially free in the limit $t_f \to \infty$, i.e. $\dot{q} = v$ and therefore

$$I[q] \sim \frac{v^2}{2} \int_0^\infty \mathrm{d}t \to \infty. \tag{5.13}$$

Additionally, the first variation of the on-shell action does not vanish for all variations that preserve the boundary conditions. A natural boundary condition at t_i would be $q(t_i = 0) = q_i$. A variation that preserves this boundary condition is obviously $\delta q|^{t_i} = 0$, thus the second term in (5.12) vanishes. Now, what variations preserve the natural boundary condition $q|_{t_i}^{t_f} = \infty$ at $t_f \to \infty$? Obviously, any finite variation $\delta q|^{t_i} = finite$ is allowed. Therefore, the first term in (5.12) does not vanish and we have

$$\delta I[q]\big|_{e.o.m} \neq 0. \tag{5.14}$$

Different methods to tackle these problems have been proposed and we will address these in the following but for this example we choose the method of Hamilton–Jacobi counterterms, which will be the procedure chosen for the main part of this work.

The action (5.11) is supplemented with a boundary term S that depends on quantities intrinsic to the boundary only

$$\Gamma[q] = I[q] - S(q,t)\big|_0^{t_f}. \tag{5.15}$$

A variation of this new action evaluated on shell leads to

$$\delta\Gamma\big|_{e.o.m.} = \left(\dot{q} - \frac{\partial S}{\partial q} \right) \delta q\big|_0^{t_f} = \left(p - \frac{\partial S}{\partial q} \right) \delta q\big|^{t_f}. \tag{5.16}$$

The boundary term vanishes if the function S satisfies

$$\frac{\partial S}{\partial q} = p \tag{5.17}$$

asymptotically. If one identifies the function S with *Hamilton's principal function* this is automatically satisfied. The boundary counterterm S is then obtained as the solution of the *Hamilton–Jacobi equation*

$$H\left(q, \frac{\partial S}{\partial q}\right) + \frac{\partial S}{\partial t} = 0. \tag{5.18}$$

The full solution of this nonlinear PDE (cf. [63]) yields the asymptotic expansion $S = \frac{q^2}{2t} + \mathcal{O}\left(\frac{t}{q^2}\right)$. This counterterm solves the two problems mentioned above. Asymptotically, $\dot{q} = v$ and therefore

$$I[q] \sim \frac{v^2}{2}\int_0^{tf} \mathrm{d}t - \frac{v^2}{2}t_f + \mathcal{O}(1) = \mathcal{O}(1), \tag{5.19}$$

the on shell action is finite. The second problem is solved simultaneously since (5.16) is now

$$\delta\Gamma\big|_{e.o.m.} = \left(\dot{q} - \frac{\partial S}{\partial q}\right)\delta q\big|^{t_f} = \left(\dot{q} - \frac{q}{t} + \mathcal{O}(1/t^2)\right)\big|^{t_f} = \mathcal{O}(1/t)\delta q\big|^{t_f} = 0, \tag{5.20}$$

because $\dot{q} - \frac{q}{t} = \mathcal{O}(1/t)$ and δq is finite.

5.2.2 The general framework

The above example shows that holographic counterterms should ideally solve two problems simultaneously: the divergence of the on-shell action and the vanishing of the first variation of the action. The former was acknowledged quite early and remedied with the technique of background subtraction accepting the deficits mentioned in 3.4. A systematic approach to remove the divergences from the on-shell action— and derived quantities such as the Brown-York stress tensor [22]— was developed in the light of the AdS/CFT correspondence [13, 14, 40, 50, 92]. An outline of this standard formalism is presented in appendix A.1. A number of different approaches to holographic renormalization have been proposed subsequently, including the Hamilton–Jacobi or dBVV approach [39, 101] and the related work [110]. More recent contributions to this field include [4, 76]— though these will not be treated here. A short presentation of the Hamilton–Jacobi method of holographic renormalization is given in appendix A.2.

The latter problem was first treated rigorously in the Hamiltonian formalism in the seminal work [115]. The starting point for their argumentation is the usual Hamiltonian of general relativity

$$H_0 = \int_\Sigma \mathrm{d}^3x\{N(x)\mathcal{H}(x) + N^i(x)\mathcal{H}_i(x)\}. \tag{5.21}$$

with some suitable slicing of spacetime $\Sigma \times \mathbb{R}$ Here N denotes the lapse function, N^i the shift function, $\mathcal{H}$ the Hamiltonian constraint and $\mathcal{H}_i$ the momentum constraint. They argued forcibly that this Hamiltonian should be supplemented by the surface integral

$$E[g_{ij}] = \int_{\partial\Sigma} \mathrm{d}^2x(\nabla_i g_{ik} - \nabla_k g_{ii}) \tag{5.22}$$

to the ends that not only the ADM energy is recovered— as was the point of view back then—, but also the correct equations of motion are obtained. If the Hamiltonian of a general field theory is varied

$$\delta(Hamiltonian) = \int_\Sigma d^3x\{A^{ij}\delta g_{ij} + B_{ij}\delta\pi^{ij}\}, \tag{5.23}$$

then Hamilton's equations read

$$\dot{g}_{ij} = \frac{\delta(Hamiltonian)}{\delta g_{ij}} = A^{ij} \tag{5.24}$$

$$\dot{\pi}_{ij} = -\frac{\delta(Hamiltonian)}{\delta\pi^{ij}} = -B_{ij} \tag{5.25}$$

But a straightforward variation of (5.21) yields not only the required functional derivatives of g_{ij} and π^{ij}, but also two boundary terms. For open asymptotically flat spaces only one boundary term remains. This term is given by $-E[g_{ij}]$. Thus, only the combination

$$H = H_0 + E[g_{ij}] \tag{5.26}$$

yields well defined functional derivatives in asymptotically flat spacetimes.

On account of the constraint equations

$$\mathcal{H} \approx 0 \qquad \mathcal{H}_i \approx 0, \tag{5.27}$$

— where Dirac's symbol $\approx$ denoting weakly vanishing is used— the Hamiltonian reduces to

$$H \approx E, \tag{5.28}$$

the ADM energy.

The connection between counterterms that render the on-shell action finite and counterterms necessary for the vanishing of the first variation of the action was first established in [111]— to the best of my knowledge. Subsequently, suitable counterterms have been found for various boundary conditions and theories (cf. [38, 67, 68, 98–100, 137] and references therein) with emphasis on either one of the problems solved depending on the context.

In the next section the ideas presented in this section are applied to dilaton gravity in two dimension, where a boundary term is needed for a finite action and a well defined variational principle.

6 Dilaton Gravity

As mentioned in the introduction, lower dimensional theories of gravity provide testing grounds for theories of classical and quantum gravity. Although current research interest in lower dimensional gravity is more focused on 2+1 dimensional theories of gravity, consistent theories of gravity can be formulated in 1+1 dimensions as well. The content of this section is the theory of 2d dilaton gravity. This theory is defined conventionally in terms of the bulk action

$$I_{bulk} = -\frac{1}{16\pi G_2}\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\left(XR - U(X)(\nabla X)^2 - 2V(X)\right), \tag{6.1}$$

where G_2 denotes Newton's constant in two dimensions, R the Ricci scalar of the two-dimensional manifold and X the dilaton field. The two functions $U(X)$ and $V(X)$, regarded as kinetic and potential functions, respectively, depend on the specific model.

As pointed out in the last section, boundary terms have to be added to (6.1) for a well-defined variational principle. This is the subject of section 6.2.1. In the following we will motivate the form of the action (6.1) by studying three different models that give rise to specific dilaton actions.

6.1 Three Different Ways to Dilaton Gravity

The straightforward generalization of General Relativity to 1+1 dimensions exhibits no local degrees of freedom. For Euclidean, compact manifolds without boundary, the Einstein–Hilbert action in 2 dimensions yields the Euler characteristic $\chi(\mathcal{M})$ of the spacetime $\mathcal{M}$ by the Gauß–Bonnet theorem

$$\int_{\mathcal{M}} d^2xR = 4\pi\chi(\mathcal{M}). \tag{6.2}$$

Therefore, additional structure is necessary for a theory of gravity in 1+1 dimensions. In the case of dilaton gravity, a scalar field— the *dilaton field*— is introduced as another dynamical field. Several independent ways of motivating dilaton gravity exist; some of these are outlined in the following.

6.1.1 Dilaton Gravity as the limit of $2+\epsilon$ dimensional gravity

One way to circumvent the above limitations of gravity in 2 dimensions, is the analysis of gravity in $2+\epsilon$ dimensions in the limit $\epsilon \to 0$

$$\lim_{\epsilon\to 0} I_{2+\epsilon} = \lim_{\epsilon\to 0} -\frac{1}{\kappa_{2+\epsilon}}\int d^{2+\epsilon}x\sqrt{g}R, \tag{6.3}$$

where $\kappa_{2+\epsilon}$ is Newton's constant in $2+\epsilon$ dimensions.

These theories were considered in [33, 52]. It was later shown by Weinberg that Einstein gravity in $2+\epsilon$ dimensions is *asymptotically safe* [145]. The concept of asymptotic safety will play no significant role in the main part of this work, but since asymptotic safety in quantum gravity is of interest in its own right, a few explanations are in order.

The non-renormalizability of general relativity in a perturbative sense is a well established fact, e.g. in 4 dimensions non-renormalizable divergences appear already at one-loop level for gravity coupled to matter and at two-loop level for pure Einstein gravity [57, 129]. Thus, perturbation theory around the Gaussian (trivial) fixed point of the renormalization group of general relativity—i.e. vanishing Newton's constant—leads to UV divergencies that cannot be canceled by a finite number of counterterms, when the UV cutoff is removed. In contrast, gauge theories common to the standard model of particle physics are known to be perturbatively renormalizable. For example, the renormalization group flow of QCD yields a trivial fixed point, consequently the UV cutoff can be removed safely and the theory is renormalizable and asymptotically free, i.e. the coupling constant vanishes for high momenta. Therefore, QCD is called *asymptotically safe*. QED on the other hand is not asymptotically free as it contains a Landau pole. Nonetheless, since QED is not expected to be correct at arbitrary high energy, the Landau pole is not considered to pose relevant problem [145].

But a trivial fixed point of the renormalization group flow is not a necessary condition for asymptotic safety. Nontrivial fixed points of the renormalization group flow, i.e. at some finite finite value of the coupling constants $g_i = g_i^*$, are also possible. Since there is no reason for the g_i^* to be small, perturbation theory might not be applicable. Nonetheless, the theory is asymptotically safe in a nonperturbative sense. This is still a viable solution for a quantum theory of gravity.

The number of free parameters of an asymptotically safe theory depends on the dimensionality of the *unstable manifold* of the fixed point (called *UV critical surface* in [145], a term used distinctly in modern terminology). Qualitatively speaking, the unstable manifold consists of all points in coupling space that move away from the fixed point as one lowers the cutoff from infinity (cf. [35] for a concise treatment of the *Kadanoff- Wilson* approach to renormalization). It is obvious from the above considerations that a unstable manifold of small but finite dimensionality would be most desirable.

This is the case for Einstein gravity in $2+\epsilon$ dimensions [145]. The theory yields a nontrivial fixed point with an unstable manifold of dimension 1. Unfortunately, the limit $\epsilon \to 2$ is not viable since ϵ is required to be small. Nevertheless, insights from various other gauge theory suggest that the nontrivial fixed point in $2+\epsilon$ dimensions could be a remnant of a nontrivial fixed point in higher dimensions. Therefore, asymptotical safety of gravity in $2+\epsilon$ dimensions opens up the possibility that a non-perturbative quantisation of gravity—independent of other approaches to quantum gravity— might be successful (cf. [107] for a review).

Taking into account one-loop effects and scaling Newton's constant as $\kappa_{2+\epsilon} \propto \epsilon$, the limiting action of (6.3) was obtained in [65]. The calculation is sketched in the following.

The first important observation is the changing number of degrees of freedom in the limit $\epsilon \to 0$ since the number of graviton modes in Einstein gravity in D dimensions is given by $\frac{D(D-3)}{2}$. A changing number of degrees of freedom would be an unpleasant feature. This shortcoming can be circumvented by restricting the action (6.3) to spherical symmetric geometries where the number of graviton modes is zero in every dimension. Owing to the fact that the number of Killing vectors implied by spherical symmetry—$(D-2)(D-1)/2$—

vanishes in two dimensions, the above restriction contains no loss of generality.

After inserting a spherical symmetric ansatz for the line element (see 6.1.3 for details),

$$ds^2 = g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta + \frac{1}{\lambda} X^{\frac{2}{D-2}}\,\mathrm{d}\Omega^2_{S_{D-2}} \tag{6.4}$$

an integration over the angular part leads to the action

$$I^\epsilon_{\mathrm{2dg}} = -\frac{1}{\kappa}\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\left(XR - \frac{1-\epsilon}{\epsilon X}(\nabla X)^2 - \lambda\epsilon(1-\epsilon)X^{1-2/\epsilon}\right), \tag{6.5}$$

where κ is a coupling constant given by

$$\kappa = \kappa_{2+\epsilon}\lambda^{\epsilon/2}\frac{\Gamma(\frac{1}{2}+\frac{\epsilon}{2})}{2\pi^{\frac{1}{2}+\frac{\epsilon}{2}}} \tag{6.6}$$

and λ is a parameter of dimension of inverse length squared. Though the limit $\epsilon \to 0$ is still not well defined, a dualized version of (6.5) is obtained by making use of the duality presented in [64]

$$\tilde{I}^\epsilon_{\mathrm{2dg}} = -\frac{1}{\tilde{\kappa}}\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\left(\tilde{X}R - 2a(1-\epsilon)\tilde{X}^{-\epsilon}\right). \tag{6.7}$$

These two action are equivalent in the sense that they yield the same solutions of the respective equations of motion. The limit $\epsilon \to 0$ of (6.7) can be taken straightforwardly in order to arrive at

$$\tilde{I}_{\mathrm{C}} := \lim_{\epsilon\to 0}\tilde{I}^\epsilon_{\mathrm{2dg}} = -\frac{1}{\tilde{\kappa}}\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\big(\tilde{X}R - 2a\big). \tag{6.8}$$

Due to its involutive character [64], applying the duality once more we are led to the desired limiting action of (6.5)

$$I_{\mathrm{L}} = -\frac{1}{\hat{\kappa}}\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\big(XR - (\nabla X)^2 - \lambda e^{-2X}\big). \tag{6.9}$$

This is the well-known *Liouville action*, which is a special case of (6.1) with $U(X)$ taken to be constant and $V(X)$ exponential. While originally studied by Joseph Liouville in association with the uniformization of Riemann surfaces, the Liouville action found a vast amount of applications in recent years due to its natural appearance in the context of non-critical bosonic string theory. A detailed treatment of Liouville theory is beyond the scope of this work and the interested reader is referred to the literature (cf. e.g. [104] and the references therein).

For our purpose, it is sufficient to notice that the action (6.9)— as a particular example of the dilaton action (6.1)— describes the limit of Einstein gravity to 2 dimensions.

6.1.2 Dilaton Gravity and String Theory

Strings propagating in background fields, i.e. curved target space, are described by the *non–linear sigma model*

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\left(\sqrt{-h}g_{\mu\nu}h^{ij}\partial_i X^\mu\partial_j X^\nu + \epsilon^{ij}B_{\mu\nu}\partial_i X^\mu\partial_j X^\nu + \alpha'\sqrt{h}R^{(2)}\Phi\right). \tag{6.10}$$

Here, h_{ij} is the metric on the string world sheet, that is parametrized by the coordinates σ, $g_{\mu\nu}$ the metric on the D- dimensional target space, X^μ the string coordinate functions, $B_{\mu\nu}$ the Kalb–Ramond field related to the torsion of spacetime, $R^{(2)}$ the Ricci scalar on the world sheet,Φ the dilaton field and α' the string tension. The three fields, $g_{\mu\nu}$, $B_{\mu\nu}$, Φ, constitute the three massless states of the closed bosonic string. Loosely speaking, these massless bosonic states can condensate to non-vanishing background values, thus constituting a curved target space geometry, that the string can interact with (for a detailed exposition see e.g. [24, 113, 132])

The crucial feature of the action (6.10) is its lack of Weyl invariance. In order to restore this essential symmetry of string theory, the trace of the energy momentum tensor

$$2\pi T^m_m = \beta^\Phi \sqrt{h} R^{(2)} + \beta^g_{\mu\nu}\sqrt{h} h^{ij}\partial_i X^\mu \partial_j X^\nu + \beta^B_{\mu\nu}\epsilon^{ij}\partial_i X^\mu \partial_j X^\nu, \tag{6.11}$$

with $\beta^\Phi, \beta^g_{\mu\nu}, \beta^B_{\mu\nu}$ defined below, must vanish. Thus, Weyl invariance is recovered if the beta–functions associated with $g_{\mu\nu}$, $B_{\mu\nu}$, Φ — playing the role of coupling constants in (6.10)— vanish. The beta functions can be calculated perturbatively in α' to yield [24]

$$\frac{\beta^\Phi}{\alpha'} = \frac{D-26}{48\pi^2\alpha'} + \frac{1}{16\pi^2}\left(4(\nabla\Phi)^2 - 4\nabla^2\Phi - R + \frac{1}{12}H^2\right) + O(\alpha'), \tag{6.12}$$

$$\beta^g_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}H_\mu^{\lambda\sigma}H_{\nu\lambda\sigma} + 2\nabla_\mu\nabla_\nu\Phi + O(\alpha'), \tag{6.13}$$

$$\beta^B_{\mu\nu} = \nabla_\lambda H^\lambda_{\mu\nu} - 2(\nabla_\lambda\Phi)H^\lambda_{\mu\nu} + O(\alpha'), \tag{6.14}$$

where $H_{\lambda\mu\nu} = 3\nabla_{[\lambda}B_{\mu\nu]}$. Since we are not interested in a theory of gravity with non–vanishing torsion we set $B_{\mu\nu} = 0$ in the following. It can be shown that the beta–functions β^Φ and $\beta^g_{\mu\nu}$ are equivalent to the equations of motion obtained from varying the target space action

$$L^{(dil)} = \int d^D x\sqrt{g}e^{-2\Phi}\left[R + 4(\nabla\Phi)^2 + \frac{D-26}{3\alpha'}\right]. \tag{6.15}$$

Therefore, this action is an effective low energy description of string theory. With the identification $X = e^{-2\Phi}$ and the spacetime dimension $D = 2$, the action (6.15) is equivalent to the generalized dilaton action (6.1) with appropriately chosen functions $U(X)$ and $V(X)$. Thus, dilaton gravity provides a low energy description of string theory.

6.1.3 Dilaton Gravity as Spherically Reduced Einstein Gravity

Dimensional reduction of general relativity provides a different, independent approach to dilaton gravity. Ignoring boundary terms for the moment — we will return to this issue in the next section— the Einstein–Hilbert action in d+1 dimensions with cosmological constant Λ is

$$I_{d+1} = -\frac{1}{16\pi G_{d+1}}\int_{\mathcal{M}} d^{d+1}x\sqrt{g_{d+1}}\left(R_{d+1} - 2\Lambda\right). \tag{6.16}$$

Following the conventions of [68], solutions of interest in dilaton gravity can be separated into a two dimensional metric $(g_2)_{\mu\nu}$ and the metric of a d-1 sphere

$$ds^2 = (g_2)_{\mu\nu}dx^\mu dx^\nu + (G_{d+1})^{\frac{2}{d-1}}\varphi(r)^2 d\Omega^2_{d-1}, \tag{6.17}$$

where G_{d+1} denotes Newton's constant in $d+1$ dimensions and $\varphi(r)$ is a dimensionless field proportional to the respective powers of r. Plugging the split metric (6.17) into the Einstein–Hilbert action (6.16), a lengthy but straightforward calculation (found in e.g. [61] in Appendix C) yields the generalized dilaton action (6.1) with

$$U(X) = -\left(\frac{d-2}{d-1}\right)\frac{1}{X} \qquad V(X) = -\frac{1}{2}(d-1)(d-2)\Upsilon^{\frac{2}{d-1}}X^{\frac{d-3}{d-1}} + \Lambda X, \tag{6.18}$$

where the dilaton field X is defined as

$$X(r) = \Upsilon G_{d+1}\varphi(r)^{d-1} \qquad \Upsilon := \frac{\mathcal{A}_{d-1}}{8\pi G_{d+1}}. \tag{6.19}$$

Here, $\mathcal{A}_{d-1}$ denotes the volume of the d–1 sphere.

Consequently, all spherically symmetric higher dimensional spacetimes can be treated in the framework of 2d dilaton gravity. These include among others Schwarzschild and Reissner- Nordstrøm. Furthermore, using a suitable compactification, non-rotating BTZ black holes can be treated in 2d dilaton gravity [1]. A generalization to rotating BTZ black holes was presented in [68].

The above list of theories that give rise to actions of the form (6.1) is far from exhaustive. Other well known examples that can be described in the framework of the above dilaton action include the CGHS model [23], Jackiw–Teitelboim model [85, 130], Katanaev–Volovich model [90] (cf. [66, 68] for a collection of models and associated functions $U(X)$ and $V(X)$).

6.2 Dilaton Thermodynamics

6.2.1 The correct action

As pointed out in section 3.4, quantities of interest in black hole thermodynamics are most easily computed in the semi-classical approximation to the euclidean path integral of gravity. It should be clear from the discussion in section 5 that the semi-classical approximation is well-defined only if the action exhibits the mentioned properties. In the following, it will be shown explicitly that the dilaton action needs a counterterm that can be obtained using the Hamilton–Jacobi formalism discussed in 5.2.1. This subsection follows closely the discussion carried out in [68] wherefrom all results are taken. The signature of spacetime is taken to be Euclidean, since we are focusing on thermodynamics. Furthermore, to reduce clutter, we will set $8\pi G_2 = 1$.

The action of 2d dilaton gravity supplemented with a GHY boundary term reads

$$I_{bulk+GHY} = -\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\left(XR - U(X)(\nabla X)^2 - 2V(X)\right) - \int_{\partial\mathcal{M}} \mathrm{d}x\sqrt{\gamma}XK. \tag{6.20}$$

The equations of motion of this action take the form

$$U(X)\nabla_\mu X\nabla_\nu X - \frac{1}{2}g_{\mu\nu}U(X)(\nabla X)^2 - g_{\mu\nu}V(X) - \nabla_\mu\nabla_\nu X - g_{\mu\nu}\nabla^2 X = 0 \tag{6.21}$$

$$R + \partial_X U(X)(\nabla X)^2 + 2U(X)\nabla^2 X - 2\partial_X V(X) = 0. \tag{6.22}$$

A gauge is chosen where the solutions take the form

$$X = X(r) \qquad ds^2 = \xi(r)d\tau^2 + \frac{1}{\xi(r)}dr^2 \tag{6.23}$$

with

$$\partial_r X = e^{-Q(X)} \tag{6.24}$$

$$\xi(X) = w(X)e^{Q(X)}\left(1 - \frac{2M}{w(X)}\right), \tag{6.25}$$

with the functions $Q(X)$ and $w(X)$ defined as

$$Q(X) := Q_0 + \int^X d\tilde{X}U(\tilde{X}) \tag{6.26}$$

$$w(X) := w_0 - 2\int^X d\tilde{X}V(\tilde{X})e^{Q(\tilde{X})}. \tag{6.27}$$

By an appropriate choice of w_0 the constant M can be restricted to nonnegative values, $M \geq 0$. If the norm of the Killing vector ∂_t, $\sqrt{\xi(X)}$, vanishes at $X = X_h$, the dilaton field exhibits a 'Killing horizon' at X_h. In most cases the Killing norm with $M = 0$ will be denoted as ξ_0 or as the 'ground state'.

In the models we will consider, the dilaton field X is positive in the interval

$$X_h \leq X \leq \infty, \tag{6.28}$$

where X_h denotes the Killing horizon at the largest value of X, if the Killing norm exhibits more than one root. In the limit $X \to \infty$ the function $w(X)$ diverges, in general. Thus, the asymptotic behavior of the Killing norm is described by the ground state, as is easily seen from (6.25). In order to avoid conical defects at the horizon, the Euclidean time is assumed to be periodic $\tau \sim \tau + \beta$, which determines the inverse temperature

$$\beta = T^{-1} = \left.\frac{4\pi}{\partial_r \xi}\right|_{r_h} = \left.\frac{4\pi}{w'(X)}\right|_{X_h}. \tag{6.29}$$

The proper local temperature $T(X)$ differs from β^{-1} by a redshift ('Tolman') factor

$$T(X) = \frac{1}{\sqrt{\xi(X)}}\beta^{-1}. \tag{6.30}$$

In order to study the thermodynamics of this model, the Euclidean path integral with appropriate boundary conditions is expanded around the classical solutions of the equations of motion. In the semi-classical limit the dominant contributions are the classical solutions and the path integral is given by

$$\mathcal{Z} \sim \exp\left(-I\left[g_{cl}, X_{cl}\right]\right) \int \mathcal{D}\delta g \mathcal{D}\delta X \exp\left(-\frac{1}{2}\delta^2 I\left[g_{cl}, X_{cl}; \delta g, \delta X\right]\right). \tag{6.31}$$

The above relation is correct provided the action satisfies condition 2 of the conditions enumerated in section 5. A short computation shows that this is not the case for the action (6.20).

A variation of the metric and the dilaton fields leads to the following boundary term

$$\delta I_{bulk+GHY}\Big|_{e.o.m} = \int \mathrm{d}\tau \left[-\frac{1}{2}\partial_r X \delta\xi + \left(U(X)\xi(X)\partial_r X - \frac{1}{2}\partial_r \xi \right) \delta X \right], \tag{6.32}$$

where we have placed the boundary at some regulating isosurface X_{reg}. Eventually, we are interested in the limit $X_{reg} \to \infty$. It will be enough to concentrate on the term originating from the variation of ξ. The coefficient of this term is proportional to $e^{-Q(X)}$ as is clear from (6.24). The asymptotic behavior of this term depends on the specific model; it is not clear that this term vanishes for $r_c \to \infty$. It is now that the discussion of the simple mechanical model in section 5.2.1 pays off. There we examined the asymptotic solution in order to clarify which variations of the metric are compatible with the boundary conditions. Since the function $w(X)$ diverges in general, the asymptotic solution will be $\xi_0 = w(X)e^{Q(X)}$. A metric with a small variation in the second term of (6.25) will yield the same asymptotic behavior. Therefore, small variations of δM are compatible with the boundary conditions and allowed variations of the metric. In other words, a variation of the metric $\delta\xi$ can be decomposed as $\delta\xi = e^{Q(X)}w(X)\delta\xi_0 - 2e^{Q(X)}\delta\xi_1$, with $\delta\xi_1 = \delta M$. The leading term defines the asymptotic behavior and is kept fixed, $\delta\xi_0 = 0$, while the subleading term can vary freely, $\delta\xi = -2\delta M e^{Q(X)}$. Compare this to the analog mechanical system: there we allowed for variations in the subleading terms as well since they were compatible with the boundary conditions.

The variation of the action is therefore

$$\delta I_{bulk+GHY}\Big|_{e.o.m,\delta X=0} = \int \mathrm{d}\tau \delta M \tag{6.33}$$

and nonzero.

The action (6.20) violates condition 3 as well. The on-shell action can be calculated to yield

$$I_{reg} = \beta \left(2M - w(X_{reg}) - 2\pi X_h T \right), \tag{6.34}$$

again evaluated at some regulated boundary X_{reg}. In the limit $X_{reg} \to \infty$ this diverges because of the second term. These shortcomings of the action cry for a remedy along the lines of holographic counterterms described in the previous section. In [68] the necessary counterterm was calculated using the method of Hamilton–Jacobi counterterms (see appendix A.2).

In contrast to the example discussed in appendix A.2, the Hamilton–Jacobi equation can be solved exactly in this case. The Hamiltonian constraint is calculated in a standard way to yield

$$\mathcal{H} = 2\pi_X \gamma_{ab}\pi^{ab} + 2U(X)(\gamma_{ab}\pi^{ab})^2 + V(X) = 0. \tag{6.35}$$

Here π^{ab} denotes the momentum w.r.t γ_{ab} and π_X the momentum associated with X. From the expansion of the counterterm action in appendix A.2, only the first term (A.20) is non-zero in the present case of a one-dimensional boundary.

$$L_{CT}(\gamma, X) = \sqrt{\gamma} W(X). \tag{6.36}$$

Thus, the functional differential equation reduces to an ordinary linear differential equation with the solution

$$W(X) = -\sqrt{e^{-Q(X)}w(X)}. \tag{6.37}$$

Alternatively, this boundary counterterm can be calculated from the requirement of local supersymmetry as pointed out in [69] (see the last paragraph of appendix A). Thus, we are led to the final form of the action

$$\begin{aligned}\Gamma =& I_{bulk+GHY} - I_{CT} = \\ &-\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\left(XR - U(X)(\nabla X)^2 - 2V(X)\right) - \int_{\partial\mathcal{M}} \mathrm{d}x\sqrt{\gamma}XK \\ &+ \int_{\partial\mathcal{M}} \mathrm{d}x\sqrt{\gamma}\sqrt{e^{-Q(X)}w(X)}.\end{aligned} \tag{6.38}$$

The on-shell action reads

$$\Gamma_{reg} = \beta\left(w(X_{reg})\sqrt{1-\frac{2M}{w(X_{reg})}} - w(X_{reg}) + 2M - 2\pi X_H T\right). \tag{6.39}$$

In the limit $X_{reg} \to \infty$ it is finite

$$\lim_{X_{reg}\to\infty}\Gamma_{reg} = \beta(M - 2\pi X_H T). \tag{6.40}$$

6.2.2 Black Hole Thermodynamics

Despite the efforts to arrive at an action with a well-posed action principle that is finite on-shell, the semi-classical approach to the Euclidean path integral still is not well defined. The above action violates condition 4 of section 5 as well. This is intimately related to the stability of the thermodynamic system studied. This was discussed in section 3.4 for the case of the Schwarzschild black hole. Here we will deal in the same way with this issue, pioneered by York [148], by putting the thermodynamic system in a cavity. This corresponds to restricting the dilaton to values $X \leq X_c$, where $X = X(r_c)$ is the locus of the cavity[2].

The heat bath surrounding the cavity fixes inverse temperature β and dilaton charge D_c as boundary conditions for the path integral.

The nonnegative solutions of

$$\beta_c = \sqrt{\xi(X_c, M)}\beta(M), \tag{6.41}$$

denote the possible values of M and thus the classical solutions consistent with the boundary conditions. Helmholtz- free energy F may be obtained from the action via

$$F_c(T_c, X_c) = T_c\Gamma(T_c, X_c). \tag{6.42}$$

[2]More generally, instead of holding the "volume" of the cavity fixed, one can demand that a particular charge $D(X)$ be fixed at the boundary. Since every sufficiently regular function can be used to construct a conserved charge, 2d dilaton gravity exhibits an infinite number of conserved charges. Here, we just chose $D(X) = X$ and demand that X_c be fixed. Other charges can provide useful for different applications.

With the above definitions this leads to

$$F_c(T_c, X_c) = -2\pi X_h T_c + e^{-Q_c}\left(\sqrt{\xi_0} - \sqrt{\xi_c}\right), \tag{6.43}$$

where a subscript c denotes evaluation at constant dilaton charge X_c. Notice that, although X_{reg} and X_c are treated in a similar way mathematically, these quantities are qualitatively different. The charge X_c comes with a concrete physical interpretation as denoting the cavity into which the black hole is placed. The regulator X_{reg} on the other hand is just a mathematical tool that allows one to work with finite quantities, but the limit $X_{reg} \to \infty$ is always implied.

A differential change of the free energy, dF, can be written as

$$\mathrm{d}F(T_c, X_c) = -2\pi X_h \,\mathrm{d}T_c - \psi_c \,\mathrm{d}X_c, \tag{6.44}$$

where ψ_c denotes the dilaton chemical potential associated to the charge X_c:

$$\psi_c = -\frac{1}{2}U_c e^{-Q_c}\left(\sqrt{\xi_c} - \sqrt{\xi_0}\right) + \frac{1}{2}w'_c\left(\frac{1}{\sqrt{\xi_c}} - \frac{1}{\sqrt{\xi_0}}\right). \tag{6.45}$$

The entropy is given by

$$S = 2\pi X_h \tag{6.46}$$

This result is independent of the cut-off and the specific model under consideration. It is a local property of the horizon, just as one would expect for an entropy. Recalling expression (6.19) for spherical reduced dilaton gravity, we see that this is precisely

$$S = \frac{A_h}{4G_{d+1}}, \tag{6.47}$$

the Bekenstein–Hawking area law.

Having seen that

$$\frac{\partial F_c(T_c, X_c)}{\partial T_c}\bigg|_{X_c} = -S \tag{6.48}$$

we can Legendre transform the free energy w.r.t. to T_c to obtain the internal energy

$$E_c(S, X_c) = F_c(T_x, X_c) + T_c S, \tag{6.49}$$

$$E_c = e^{-Q_c}\left(\sqrt{\xi_0} - \sqrt{\xi_c}\right). \tag{6.50}$$

The internal energy obeys the first law of black hole thermodynamics

$$\mathrm{d}E_c = T_c \,\mathrm{d}S - \psi_c \,\mathrm{d}D_c. \tag{6.51}$$

This agrees with the energy calculated from the Brown-York stress tensor

$$T^{ab} = \frac{2}{\sqrt{\gamma}}\frac{\delta\Gamma}{\delta\gamma^{ab}}, \tag{6.52}$$

contracted with the Killing vector $u_a = \delta_a^\tau$.

$$T^{ab}u_a u_b = E_c \tag{6.53}$$

The conserved charge associated with the timelike Killing vector, the "mass" M, is obtained from (6.52) as

$$M = \lim_{X_c \to \infty} \sqrt{\xi_c} T^{ab} u_a u_b, \tag{6.54}$$

where one should think of the limit $X_c \to \infty$ as integration at infinity and $\sqrt{\xi_c}$ as lapse function in the integral. From this we see that energy and mass are related by

$$\lim_{X_c \to \infty} E_c = \lim_{X_c \to \infty} \frac{M}{\sqrt{\xi_c}}. \tag{6.55}$$

6.2.3 Charged black holes

The above results are generalized easily to charged black holes by adding the Maxwell term

$$I_M = \int_{\mathcal{M}} \mathrm{d}^2 x \sqrt{g} f(X) F_{\mu\nu} F^{\mu\nu} \tag{6.56}$$

to the action (6.38). The function $f(X)$ describes the coupling of the abelian field strength $F_{\mu\nu}$ to the dilaton field. Maxwell's equations are then solved by

$$F_{\mu\nu} = \frac{q}{4 f(X)} \epsilon_{\mu\nu}, \tag{6.57}$$

where q denotes the electric charge and the factor $\frac{1}{4}$ is chosen for convenience.

In principle, it is possible that adding the Maxwell term requires the introduction of an additional boundary counterterm. This will be the case in the next section when the cosmological constant is included. Another model where the calculation of this counterterm was carried out explicitly can be found in [31].

The equations of motion for the action (6.38) with an additional Maxwell field are still solved by equation (6.23) but the Killing norm is changed to

$$\xi(X) = e^{Q(X)} \left(w(X) - 2M + \frac{1}{4} q^2 h(X) \right). \tag{6.58}$$

The function $h(X)$ is defined as

$$h(X) = \int^X d\tilde{X} \frac{e^{Q(\tilde{X})}}{f(\tilde{X})}. \tag{6.59}$$

In a gauge where $A_r = 0$ the gauge potential is given by

$$A_\tau(X) = -\frac{q}{4} \left(h(X) - h(X_h) \right) + A_\tau(X_h). \tag{6.60}$$

Thus, the proper electrostatic potential relative to the horizon is

$$\Phi(X) = \frac{A_\tau(X) - A_\tau(X_h)}{\sqrt{\xi(X)}}. \tag{6.61}$$

With these definitions the logarithm of the partition function may be calculated. The result is not the Helmholtz free energy $F_c(T_c, D_c, q)$ but rather the Legendre transformation of F_c

with respect to the potential Φ_c. The inverse Legendre transformation leads to the same expression for F_c as before

$$F_c(T_c, D_c, q) = -2\pi X_h T_c + e^{-Q_c}\left(\sqrt{\xi_0} - \sqrt{\xi_c}\right) \tag{6.62}$$

and consequently to the same internal energy (6.50). The dependence on q remains in the Killing norm $\sqrt{\xi(X)}$ and the locus of the horizon X_h. Notice that one may add an arbitrary number of Maxwell fields this way.

7 The Cosmological Constant as a Thermodynamic Variable in 2d Dilaton Gravity

7.1 Implementing the Cosmological Constant

As a first step we have to examine in what way a cosmological constant is implemented in 2d dilaton gravity. The addition of a constant term to the Lagrangian will not be enough. On-shell the Ricci scalar should equal the cosmological constant Λ for pure dS or AdS, therefore the cosmological constant couples linearly to the dilaton field. Thus, the bulk action reads

$$I_{bulk} = -\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\left(XR - U(X)(\nabla X)^2 - 2V(X) - 2\Lambda X\right). \tag{7.1}$$

If we are interested in a state-dependent cosmological constant it should be incorporated in the action in a way similar to the Henneaux–Teitelboim model discussed in section 4.2. In 2d dilaton gravity this is accomplished by adding a Maxwell field with a specific coupling. If we choose $f(X) = \frac{1}{X}$ for the coupling function the Maxwell term is

$$I_M = \int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}X\frac{q^2}{8}. \tag{7.2}$$

The dilaton field couples linearly to both, the Ricci scalar and the cosmological constant, if we set

$$\frac{q^2}{8} = -\Lambda. \tag{7.3}$$

We are interested in asymptotic AdS spacetimes, only. Therefore, the minus sign was made explicit in (7.3). The cosmological constant emerges as a thermodynamic variable naturally this way, since we may regard it as the charge of a Maxwell field with the peculiar coupling $f(X) = 1/X$. As mentioned above, the addition of a Maxwell field could require the introduction of an additional boundary term. This depends on the asymptotic behavior of the field. If the Maxwell field vanishes faster than $w(X)$ the boundary counterterm already included is enough. In our case the Maxwell field assumes the role of the cosmological constant and governs the asymptotic behavior of the solution. Therefore, another boundary term will be necessary. This boundary term will be presented in the next section, for the moment we work with the action (6.38), keeping in mind that it is not the correct one. Nonetheless, this action yields the results expected from the discussion in section (4.2). The reason for this is seen in the following section.

The internal energy is given by

$$E_c(S, D_c, q) = e^{-Q_c}\left(\sqrt{\xi_0} - \sqrt{\xi_c}\right). \tag{7.4}$$

Here and in the following the term ξ_0 denotes the ground state Killing norm, i.e. the Killing norm (6.58) with $M = 0$. Observe that the internal energy depends on q now[3]. We can

[3]The labelling of the various potentials is ambiguous in the following. If one thinks of q as a normal Maxwell charge $E_c(S, D_c, q)$ is indeed the internal energy. On the other hand, if we think of q as representing the cosmological constant, and thus pressure P, $E_c(S, D_c, q)$ is called enthalpy. In the following, we stick to the former label as long as we are working in the dilaton context, and use the latter naming, if we change to a higher dimensional interpretation.

regard this either as an ordinary charge with the coupling $f(X) = 1/X$ or as including Λ as a state variable. It is this way that the cosmological constant enters as a state variable naturally.

By making use of the relations

$$\mathrm{d}\xi_c = U_c \xi_c \,\mathrm{d}D_c + e^{Q_c}\left(w_c' \,\mathrm{d}D_c - 2\,\mathrm{d}M + \frac{q}{2} h_c \,\mathrm{d}D_c + \frac{q^2}{4} h_c' \,\mathrm{d}D_c\right) \tag{7.5}$$

$$\mathrm{d}M = 2\pi T \,\mathrm{d}X_h + \frac{q}{4} h_h \,\mathrm{d}X_h, \tag{7.6}$$

where prime denotes differentiation with respect to X and the subscript h evaluation at X_h, we arrive at the expression

$$\mathrm{d}E_c = -\psi_c \,\mathrm{d}D_c - \left(\frac{h_c}{\sqrt{\xi_0}} - \frac{h_c}{\sqrt{\xi_c}} + \frac{h_h}{\sqrt{\xi_c}}\right) \mathrm{d}\Lambda + \frac{T}{\sqrt{\xi_c}} \,\mathrm{d}S, \tag{7.7}$$

with $T = \beta^{-1}$ and ψ_c, the dilaton chemical potential, given by

$$\psi_c = -\frac{1}{2} U_c e^{-Q_c}\left(\sqrt{\xi_c} - \sqrt{\xi_0}\right) + \left(\frac{1}{2} w_c' - \Lambda h_c'\right)\left(\frac{1}{\sqrt{\xi_c}} - \frac{1}{\sqrt{\xi_0}}\right). \tag{7.8}$$

This is the first law of black hole thermodynamics in dilaton gravity with the cosmological constant treated as thermodynamic variable. For vanishing cosmological constant, (7.7) reduces to the first law derived in [68]. We stress again that— in the framework of dilaton gravity— treating the cosmological constant as a thermodynamical variable requires no further assumptions, since it is just a charge with a specific coupling to the dilaton field. The first law for gravity with a generic Maxwell field essentially looks the same. The only difference lies in the definition of the coupling function and the resulting function $h(X)$. For the chosen coupling the function $h(X)$ is

$$h(X) = \int^X d\tilde{X} \tilde{X} e^{Q(\tilde{X})}. \tag{7.9}$$

Notice that this is precisely the definition for the volume of a 2d black hole presented in [62]— up to a constant. Thus, the thermodynamic variable conjugate to the cosmological constant, usually denoted by Θ, is proportional to the difference between the volume of the cavity $h(X_c)$ and the volume of the black hole $h(X_h)$ rescaled by a Tolman factor

$$\Theta = -\left(\frac{h_c}{\sqrt{\xi_0}} - \frac{h_c}{\sqrt{\xi_c}} + \frac{h_h}{\sqrt{\xi_c}}\right). \tag{7.10}$$

In the asymptotic limit, i.e. with the cavity walls removed, the first two terms cancel and (7.10) yields

$$\lim_{w_c \to \infty} \Theta = -\frac{h_h}{\sqrt{\xi_c}}, \tag{7.11}$$

thus, only the volume of the black hole remains.

This is essentially the same result as in section 4.2.1. The thermodynamic variable conjugate to the cosmological constant is proportional to the volume of the black hole.

The difference between (7.10) and the corresponding quantities derived in the references of section 4.2.1, is the appearance of Tolman factors. The reason for this is that we used the local quantity E_c rather than the conserved charge M to derive this result. The two quantities are related by (6.55), which explains the Tolman factor. We may even generalize equation (7.7) for systems with an additional (electric) charge, like Reissner Nordström or BTZ black holes. This introduces just an additional term in the Killing norm

$$\xi(X) = e^{Q(X)} \left(w(X) - 2M + \frac{q_\Lambda^2}{4} h_\Lambda(X) + \frac{q^2}{4} h_q(X) \right). \tag{7.12}$$

Here, the subscripts Λ and q denote whether the function $h(X)$ is associated to the coupling function $f(X)$ of the cosmological constant, (7.9), or the coupling function of the charge, which is left unspecified provided that the asymptotic behavior is still governed by the cosmological constant. As before, we want the ground state to be asymptotic AdS, therefore we choose $\xi_0(X)$ to be the state with $M = q = 0$. The first law is thus generalized to

$$\mathrm{d}E_c = -\psi_c \,\mathrm{d}D_c - \left(\frac{h_{\Lambda,c}}{\sqrt{\xi_0}} - \frac{h_{\Lambda,c}}{\sqrt{\xi_c}} + \frac{h_{\Lambda,h}}{\sqrt{\xi_c}} \right) \mathrm{d}\Lambda + \frac{q}{4} \left(-\frac{h_{q,c}}{\sqrt{\xi_c}} + \frac{h_{q,h}}{\sqrt{\xi_c}} \right) \mathrm{d}q + \frac{T}{\sqrt{\xi_c}} \mathrm{d}S \tag{7.13}$$

As before, the subscripts h and c denote evaluation at horizon and cavity wall, respectively. The dilaton chemical potential changes to

$$\psi_c = -\frac{1}{2} U_c e^{-Q_c} \left(\sqrt{\xi_c} - \sqrt{\xi_0} \right) + \left(\frac{1}{2} w'_{\Lambda,c} - \Lambda h'_{\Lambda,c} \right) \left(\frac{1}{\sqrt{\xi_c}} - \frac{1}{\sqrt{\xi_0}} \right) - \frac{1}{\sqrt{\xi_c}} \frac{q^2}{8} h'_{q,c} \tag{7.14}$$

For vanishing additional charge q both expressions reduce to (7.7) and (7.8).

7.1.1 Example 1:AdS-Schwarzschild in 3+1 dimensions

The application of the above procedure to AdS-Schwarzschild is now straightforward. With the functions defined in section 6.1.3 we arrive at

$$\mathrm{d}E_c = -\psi_c \,\mathrm{d}D_c - \frac{1}{3}(2G_4)^{-\frac{1}{2}} \left(\frac{X_c^{\frac{3}{2}}}{\sqrt{\xi_0}} - \frac{X_c^{\frac{3}{2}}}{\sqrt{\xi_c}} + \frac{X_h^{\frac{3}{2}}}{\sqrt{\xi_c}} \right) \mathrm{d}\Lambda + \frac{1}{\sqrt{\xi_c}} T dS. \tag{7.15}$$

The dilaton field is related to the radial coordinate r of 3+1 dimensional AdS- Schwarzschild via

$$X(r) = \frac{1}{2G_4} r^2. \tag{7.16}$$

Thus, the known relations for temperature and surface pressure follow (see [87]).

With Λ replaced with P the above reads

$$dE_c = -\psi_c dD_c + \frac{4\pi}{3} \left(\frac{r_c^3}{\sqrt{\xi_0}} - \frac{r_c^3}{\sqrt{\xi_c}} + \frac{r_h^3}{\sqrt{\xi_c}} \right) dP + T_c dS. \tag{7.17}$$

If the cavity is removed only the last term in the parenthesis remains, which corresponds to the volume of a ball in three dimensional Euclidean space. Using relation (6.55), cavity and Tolman factors are removed and we obtain

$$dM(P, S) = V dP + T dS \tag{7.18}$$

Thus, we reproduce the result that mass is the enthalpy of a black hole in AdS spacetime. This result was already discussed in section 4.2.1.

7.1.2 Example 2: BTZ- Black holes

BTZ black holes are solutions of Einstein gravity in 3 dimensions with a negative cosmological constant [5, 6]. A nonzero angular momentum of BTZ can be dealt with by introducing an additional charge $q = \frac{J}{2}$ with a specific coupling. In the following we make use of the BTZ reduction presented in [68] with some minor changes since we want to obtain the first law of BTZ black holes using the general result (7.13). The following functions are used to model BTZ in dilaton gravity:

$$Q(X) = 0 \qquad w(X) = 0 \qquad h_\Lambda(X) = \frac{1}{2}X^2 \qquad h_J(X) = \frac{4}{X^2}. \tag{7.19}$$

In order to obtain the same conventions usually found in other works, the normalization of Newton's constant was changed from $G_3 = \frac{1}{4}$—as the conventions used so far would require— to $G_3 = \frac{1}{8}$. If one takes into account an additional factor 2 for the internal energy on behalf of the different normalization, inserting the functions (7.19) yields the first law for BTZ black holes with the cosmological constant as thermodynamic variable:

$$dE_c = -\psi_c dD_c - \Omega_c dJ + T_c dS - 2\left(\frac{h_{\Lambda,c}}{\sqrt{\xi_0}} - \frac{h_{\Lambda,c}}{\sqrt{\xi_c}} + \frac{h_{\Lambda,h}}{\sqrt{\xi_c}}\right) d\Lambda, \tag{7.20}$$

with

$$T = \frac{r_+^2 - r_-^2}{2\pi r_+ \ell^2} \tag{7.21}$$

$$\Omega_c = \frac{J}{2\sqrt{\xi_c}}\left(\frac{1}{r_+^2} - \frac{1}{X_c^2}\right) \tag{7.22}$$

$$\psi_c = \frac{E_c}{\sqrt{\xi_c}\ell} - \frac{J^2}{2X_c^3\sqrt{\xi_c}}, \tag{7.23}$$

where we have introduced the AdS radius ℓ. The locus of the horizon X_h was taken to be at r_+. Again, we express the cosmological constant in terms of the associated pressure given by

$$P = -\frac{\Lambda}{8\pi G_3}. \tag{7.24}$$

In our conventions this is $P = -\frac{\Lambda}{\pi}$. Thus, the first law reads

$$dE_c = -\psi_c dD_c - \Omega_c dJ + T_c dS + \left(\frac{r_c^2\pi}{\sqrt{\xi_0}} - \frac{r_c^2\pi}{\sqrt{\xi_c}} + \frac{r_+^2\pi}{\sqrt{\xi_c}}\right) dP. \tag{7.25}$$

Asymptotically, the first two terms in the parentheses cancel and the thermodynamic variable conjugate to the pressure is just the area of a circle with a Tolman factor. This is the 2 dimensional analogue of the volume of a spherically symmetric black hole. If we get rid of the Tolman factors by using relation (6.55) we recover the result found in [45].

7.2 The correct action principle

Under close inspection, the above discussion turns out to be flawed. The internal energy (7.4), that was used to derive the above results, stems from the action (6.38). This action principle was derived on the assumption that the asymptotic behavior of (6.58) is governed by the function $w(X)$ and the function $h(X)$, associated with the Maxwell field, contributes only subleading terms. In our approach the Maxwell field plays the role of the cosmological constant and should therefore dominate asymptotically. This implies a curious variational principle. For definiteness, compare the variation of the boundary metric $\gamma_{\tau\tau} = \xi(X)$ for AdS-Schwarzschild in 3+1 dimensions

$$\delta\gamma_{\tau\tau} = -\frac{\Lambda}{3}r^2 + 1 - \frac{2}{r}\delta M. \tag{7.26}$$

with the variation of the boundary metric for AdS-Schwarzschild if Λ is introduced as a Maxwell charge

$$\delta\gamma_{\tau\tau} = \frac{q}{12}r^2\delta q + 1 - \frac{2}{r}\delta M. \tag{7.27}$$

The leading order term and the subsubleading term of the boundary metric are allowed to fluctuate while the subleading term is fixed. This requires a modification of the action.

An action that respects the above changes in the variational principle is given by

$$\begin{aligned}\Gamma = &-\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\left(XR - U(X)(\nabla X)^2 - 2V(X)\right) + \int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}f(X)F^{\mu\nu}F_{\mu\nu}\\ &-\int_{\partial\mathcal{M}} \mathrm{d}x\sqrt{\gamma}XK + (-4-b_2)\int_{\partial\mathcal{M}} \mathrm{d}x\sqrt{\gamma}n_\mu f A_\nu F^{\mu\nu} + b_2\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\nabla_\mu(fF^{\mu\nu}A_\nu)\\ &+\int_{\partial\mathcal{M}} \mathrm{d}x\sqrt{\gamma}\mathrm{e}^{-Q(X)}\sqrt{\mathrm{e}^{Q(X)}\left(w(X) + 2F_{\mu\nu}F^{\mu\nu}f(X)^2h(X)\right)},\end{aligned} \tag{7.28}$$

where b_2 is an arbitrary constant. This one parameter family of actions yields a well defined variational principle and a finite on-shell action for variations where Λ can fluctuate freely. In the limit $F_{\mu\nu} \to 0$ it reduces to (6.38). The term in the third line is a new counterterm based on (6.37). This Born–Infeld like term

$$\int_{\partial\mathcal{M}} \mathrm{d}x\sqrt{\gamma}\mathrm{e}^{-Q(X)}\sqrt{\mathrm{e}^{Q(X)}\left(w(X) + 2F_{\mu\nu}F^{\mu\nu}f(X)^2h(X)\right)} \tag{7.29}$$

reduces on-shell to

$$\int_{\partial\mathcal{M}} \mathrm{d}x\sqrt{\gamma}\mathrm{e}^{-Q(X)}\sqrt{\xi_0}, \tag{7.30}$$

where ξ_0 is the Killing norm with M=0. One might be tempted to use (7.30) as a counterterm instead of (7.29) but this is not possible as one would introduce the constant of motion q directly into the action, and a constant of motion can never be part of the definition of an action principle.

If (7.28) is varied the contributions from the second and third term in the second line cancel the δA_μ variation from the Maxwell term and leave only variations proportional to $\delta F_{\mu\nu}$. Therefore, these terms convert the variational problem of A_μ from Dirichlet boundary conditions to Neumann boundary conditions.

The addition of the total derivative term

$$b_2\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\nabla_\mu(fF^{\mu\nu}A_\nu) \tag{7.31}$$

seems odd, especially as b_2 could be set to 0 to make it vanish.[4] Although the variational principle based on (7.28) is well defined and the on-shell action is finite, one does reproduce Bekenstein–Hawking entropy only if $b_2 = -4$. The explicit appearance of the gauge field A_μ seems to break gauge invariance of the action, but if the gauge transformations are required to obey the same boundary conditions as the field, in particular show the same periodicity β in the Euclidean time, the action is gauge invariant.

Consequently, the correct action that yields a well defined variational principle, a finite on-shell action and reproduces Bekenstein–Hawking is

$$\begin{aligned}\Gamma = &-\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\left(XR - U(X)(\nabla X)^2 - 2V(X)\right) + \int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}f(X)F^{\mu\nu}F_{\mu\nu} \\ &- \int_{\partial\mathcal{M}} \mathrm{d}x\sqrt{\gamma}XK - 4\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\nabla_\mu(fF^{\mu\nu}A_\nu) \\ &+ \int_{\partial\mathcal{M}} \mathrm{d}x\sqrt{\gamma}\mathrm{e}^{-Q(X)}\sqrt{\mathrm{e}^{Q(X)}\left(w(X) + 2F_{\mu\nu}F^{\mu\nu}f(X)^2h(X)\right)}. \end{aligned} \tag{7.32}$$

This is our main result.

The on-shell action is given by

$$F_c(T_c, D_c, q) = -2\pi X_h T_c + e^{-Q_c}\left(\sqrt{\xi_0} - \sqrt{\xi_c}\right), \tag{7.33}$$

the usual free energy, and is finite.[5] This is the reason that the naive approach of the last section gave the expected results.

In the following, we show explicitly that the above action yields a well-defined variational principle. Varying the above action with respect to F, g, X yields

$$\delta\Gamma = \int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}(\mathcal{E}_g^{\mu\nu}\delta g_{\mu\nu} + \mathcal{E}^{\mu\nu}\delta F_{\mu\nu} + \mathcal{E}_X\delta X) + \int_{\partial\mathcal{M}} \mathrm{d}x\sqrt{\gamma}(\pi_g^{\mu\nu}\delta g_{\mu\nu} + \pi_F^{\mu\nu}\delta F_{\mu\nu} + \pi_X\delta X) \tag{7.34}$$

The bulk variations yield the correct equations of motion while the momenta are given by

$$\begin{aligned}\pi_g^{\alpha\beta} = &-\frac{1}{2}n^\mu\nabla_\mu X\gamma^{\alpha\beta} + \frac{1}{2}\gamma^{\alpha\beta}\mathrm{e}^{-Q(X)}\sqrt{\xi_{BI}} \\ &- 4\left(\frac{1}{2}n_\lambda F^{\lambda\nu}A_\nu f g^{\alpha\beta} - n^\alpha F^{\beta\nu}A_\nu f - n_\mu F^{\mu\beta}fA^\alpha\right) - \frac{2F^{\alpha\sigma}F^\beta_{\ \sigma}f^2h}{\sqrt{\xi_{BI}}}\end{aligned} \tag{7.35}$$

$$\pi_F^{\mu\nu} = \left(\frac{2F^{\mu\nu}f^2h}{\sqrt{\xi_{BI}}} - 4n^\mu A^\nu f\right) \tag{7.36}$$

[4]The topology of the manifold $\mathcal{M}$ is that of a disk with circumference β and radial coordinate going from the horizon r_h at the center to the cut-off r_c. One might argue that Stoke's theorem on this manifold applied to the bulk total derivative term yields the boundary term in the second line. But this is not the case, otherwise the addition of the bulk total derivative term would be pointless. The reason for this is that the normal vector n^μ vanishes at r_h. If one applies Stoke's theorem with a disk of radius ϵ around r_h removed, an additional contribution at r_h is obtained in the limit $\epsilon \to 0$. Since this term was added in order to arrive at the Bekenstein–Hawking entropy law, one can interpret the above in the way that point like charges sit at the horizon that are responsible for the correct entropy. These charges do not change the equations of motion in the bulk.

[5]We call this potential "free energy", if q is regarded as a charge. On the other hand, we saw in the last section that $e^{-Q_c}\left(\sqrt{\xi_0} - \sqrt{\xi_c}\right)$ is the AdS mass and consequently equivalent to enthalpy, if q^2 is interpreted as cosmological constant. In order to avoid ambiguities, we stick with the former interpretation.

$$\pi_X = \Big(U(X) n^\mu \nabla_\mu X - K - 4 n_\mu A_\nu F^{\mu\nu} f' + \frac{-U(w + 2F^2 f^2 h) + (w' + 4F^2 f' f h + 2F^2 f^2 h')}{2\sqrt{\xi_{BI}}} \Big), \tag{7.37}$$

where we have denoted $\sqrt{\xi_{BI}} = \sqrt{\mathrm{e}^{Q(X)}\left(w(X) + 2F_{\mu\nu}F^{\mu\nu} f(X)^2 h(X)\right)}$ to reduce clutter. An immediate concern is the appearance of terms proportional to δg_{rr} in the second line of (7.35). However, it turns out that the terms in the second line cancel, since the normal vector n^μ has an r component only, and we are left with

$$\pi_g^{\alpha\beta} = -\frac{1}{2} n^\mu \nabla_\mu X \gamma^{\alpha\beta} + \frac{1}{2}\gamma^{\alpha\beta} \mathrm{e}^{-Q(X)} \sqrt{\xi_{BI}}. \tag{7.38}$$

In order to have a well defined variational principle, the boundary terms of (7.34) must vanish on-shell, if the regulator is removed. When evaluated on the solutions of the equations of motion, these terms read

$$\delta_g \Gamma = \int_{\partial\mathcal{M}} \mathrm{d}x \frac{1}{2} \mathrm{e}^{-Q(X)} \left(\frac{\sqrt{\xi_0}}{\sqrt{\xi}} - 1 \right) \delta\xi \tag{7.39}$$

$$\delta_X \Gamma = \int_{\partial\mathcal{M}} \mathrm{d}x \left(\frac{1}{2} U \xi e^{-Q} (1 - \frac{\sqrt{\xi_0}}{\sqrt{\xi}}) + \frac{1}{2}(w' + \frac{q^2}{4} h')(\frac{\sqrt{\xi}}{\sqrt{\xi_0}} - 1) + \frac{q^2}{4} \frac{f'}{f} h (\frac{\sqrt{\xi}}{\sqrt{\xi_0}} - 1) \right) \delta X \tag{7.40}$$

$$\delta_F \Gamma = \int_{\partial\mathcal{M}} \mathrm{d}x\, q h(X) f(X) \left(1 - \frac{\sqrt{\xi}}{\sqrt{\xi_0}} \right) \delta F_{r\tau} \tag{7.41}$$

All quantities are understood to be evaluated at a regulator X_{reg} with $X_{reg} \to \infty$ implied at the end of the calculations.

These terms have to vanish asymptotically but this is not the case if we consider independent variations of γ, F and X. This can be understood, if one remembers that a variation of $F_{\mu\nu}$ amounts to a variation of q, which governs the asymptotic behavior of $g_{\mu\nu}$ and cannot be neglected in the variation of g, accordingly. The variations have to satisfy the constraint

$$\delta\xi = \left(U\xi + \mathrm{e}^{Q(X)} (w' + \frac{q^2}{4} h') \right) \delta X - 2\mathrm{e}^{Q(X)} \delta M + \frac{q}{2} \mathrm{e}^{Q(X)} h \delta q. \tag{7.42}$$

Now we may use this constraint to rewrite the variation $\delta_g \Gamma$:

$$\begin{aligned} \delta_g \Gamma &= \int_{\partial\mathcal{M}} \mathrm{d}x \frac{1}{2} \mathrm{e}^{-Q(X)} \Big(\frac{\sqrt{\xi_0}}{\sqrt{\xi}} - 1 \Big) \Big(\Big(U\xi + \mathrm{e}^{Q(X)} (w' + \frac{q^2}{4} h') \Big) \delta X - 2\mathrm{e}^{Q(X)} \delta M + \frac{q}{2} \mathrm{e}^{Q(X)} h \delta q \Big) \\ &= \int_{\partial\mathcal{M}} \mathrm{d}x \frac{q}{4} h \Big(\frac{\sqrt{\xi_0}}{\sqrt{\xi}} - 1 \Big) \delta q + \frac{1}{2} \Big(\frac{\sqrt{\xi_0}}{\sqrt{\xi}} - 1 \Big) \Big(U \mathrm{e}^{-Q(X)} \xi + (w' + \frac{q^2}{4} h') \Big) \delta X - \Big(\frac{\sqrt{\xi_0}}{\sqrt{\xi}} - 1 \Big) \delta M \end{aligned} \tag{7.43}$$

We see immediately that the last term vanishes: the variation δM is an infinitesimal constant and $\frac{\sqrt{\xi_0}}{\sqrt{\xi}}$ approaches 1 for $X_{reg} \to \infty$. The two remaining variations, $\delta_X \Gamma$ and

$\delta_F\Gamma$ acquire contributions from $\delta_g\Gamma$. Including this new contribution, $\delta_F\Gamma$ reads

$$\begin{aligned}\delta_F\Gamma &= \int_{\partial\mathcal{M}} \mathrm{d}x\, qh(X)f(X)\left(1-\frac{\sqrt{\xi}}{\sqrt{\xi_0}}\right)\delta F_{r\tau} + \frac{q}{4}h\left(\frac{\sqrt{\xi_0}}{\sqrt{\xi}}-1\right)\delta q \\ &= \int_{\partial\mathcal{M}} \mathrm{d}x \frac{q}{4}h\left(\frac{\sqrt{\xi_0}}{\sqrt{\xi}}-1\right)\delta q - \frac{q}{4}h\left(1-\frac{\sqrt{\xi}}{\sqrt{\xi_0}}\right)\delta q \\ &= \int_{\partial\mathcal{M}} \mathrm{d}x \frac{q}{4}h\left(\frac{\sqrt{\xi_0}}{\sqrt{\xi}}+\frac{\sqrt{\xi}}{\sqrt{\xi_0}}-2\right)\delta q \end{aligned} \tag{7.44}$$

where we have used $\delta F_{r\tau} = -\frac{\delta q}{4}\frac{1}{f}$. In order to show that this term vanishes asymptotically, we have to evaluate the parenthesis. The term $\sqrt{\frac{\xi}{\xi_0}}$ is expanded as

$$\sqrt{\frac{\xi}{\xi_0}} = \sqrt{1-\frac{2M}{w(X)+\frac{q^2}{4}h(X)}} = 1 - \frac{M}{w(X)+\frac{q^2}{4}h(X)} + \mathcal{O}(\frac{1}{h^2}), \tag{7.45}$$

since the asymptotic behavior is governed by $h(X)$. Similarly we have

$$\sqrt{\frac{\xi_0}{\xi}} = \frac{1}{\sqrt{\frac{\xi}{\xi_0}}} = 1 + \frac{M}{w(X)+\frac{q^2}{4}h(X)} + \mathcal{O}(\frac{1}{h^2}) \tag{7.46}$$

Therefore, the first order terms of the expansions cancel and we are left with

$$\delta_F\Gamma = \int_{\partial\mathcal{M}} \mathrm{d}x \frac{q}{4}h\left(\mathcal{O}(\frac{1}{h^2})\right)\delta q \xrightarrow[X_{reg}\to\infty]{} 0, \tag{7.47}$$

as required for a well-defined variational principle.

The $\delta_X\Gamma$ variation remains. With the contributions from $\delta_g\Gamma$ we have

$$\delta_X\Gamma = \int_{\partial\mathcal{M}} \mathrm{d}x \frac{1}{2}(w' + \frac{q^2}{4}h')\left(\frac{\sqrt{\xi_0}}{\sqrt{\xi}}+\frac{\sqrt{\xi}}{\sqrt{\xi_0}}-2\right)\delta X + \frac{q^2}{4}\frac{f'}{f}h(\frac{\sqrt{\xi}}{\sqrt{\xi_0}}-1)\delta X \tag{7.48}$$

since the terms proportional to U cancel. The first term vanishes by the same argument as above, since h^2 grows certainly faster than $h'\delta X$. The parenthesis in the last term yields

$$\left(\frac{\sqrt{\xi}}{\sqrt{\xi_0}}-1\right) = -\frac{M}{\frac{q^2}{4}+w} + \mathcal{O}(\frac{1}{h^2}) \tag{7.49}$$

Therefore we are left with

$$\delta_X\Gamma \to \int_{\partial\mathcal{M}} \mathrm{d}x \left(-\frac{f'}{f}M\right)\delta X = \int_{\partial\mathcal{M}} \mathrm{d}x M\frac{\delta X}{X}. \tag{7.50}$$

If we apply the usual Dirichlet boundary conditions on the dilaton field, i.e. $\delta X_0 = 0$, where X_0 is the coefficient of the leading order term of X, then $\frac{\delta X}{X} \to 0$ and therefore

$$\delta_X\Gamma \xrightarrow[X_{reg}\to\infty]{} 0. \tag{7.51}$$

In summary, we have shown that $\delta\Gamma = 0$ for all variations $\delta\gamma_{\mu\nu}, \delta F_{\mu\nu}, \delta X$ that satisfy the constraint (7.42).

With the variations with respect to $\delta\gamma_{ab}$ we arrive again at the same expression for the Brown–York stress tensor and consequently at the same internal energy

$$E_c(S, X_c, q) = \mathrm{e}^{-Q(X_c)} \left(\sqrt{\xi_0} - \sqrt{\xi_c}\right) \tag{7.52}$$

A Legendre transformation of (7.33) leads to the same result.

The results from section 4.2.1 that concern those black holes attainable from 2d dilaton gravity can be reproduced using the above action (7.32). We have shown this explicitly for AdS-Schwarzschild and BTZ in the last section. But our result appears to be more general to apply to other intrinsically 2d models as well. The study of these in the above framework is left for further work.

8 Conclusions

In this work we studied the consequences of a varying cosmological constant for black hole thermodynamics. A review of previous ideas and results was presented. The starting point for most considerations in this direction is the Henneaux–Teitelboim model where the cosmological constant is replaced by a totally antisymmetric $d+1$- field strength. Brown and Teitelboim showed that this model exhibits the possibility of membrane creation that effectively reduces the cosmological constant in the interior of the membrane. Other competing mechanisms for an effective neutralization of the cosmological constant were presented in section 4.2.1.

Consequently, these processes should be taken into account in the first law of black hole thermodynamics. It was shown that the thermodynamically variable conjugate to the cosmological constant corresponds to the negative volume of the black hole.

The framework of 2d dilaton gravity was introduced, and the importance of a well-defined variational principle was clarified. In 2d dilaton gravity a varying cosmological constant can be implemented in a Henneaux–Teitelboim like manner. The $(d+1)$- field corresponds here to a Maxwell field. The cosmological constant behaves therefore like a charge with an unusual coupling to gravity. An action principle based on the action (7.32) was presented. It was shown that the action yields a finite on-shell action and a well-defined variational principle provided that the variations satisfy the constraint (7.42).

The thermodynamically conjugate variable to Λ in 2d dilaton gravity turned out to coincide with the volume of a two dimensional black hole presented in [62]. AdS-Schwarzschild and BTZ black holes were studied as two particular examples of higher dimensional space-times attainable from 2d dilaton gravity and previous results were recovered.

A few points remain to be clarified. Is there a particular interpretation for the bulk total derivative term (7.31), other than the necessity to arrive at a finite on-shell action and Bekenstein–Hawking area law? Can the constraint (7.42) be replaced by a set of different boundary conditions? The action (7.32) gives the expected results for AdS-Schwarzschild and BTZ and appears to be quite general, but does it actually hold for all dilaton models, i.e. arbitrary function $U(X)$, $V(X)$, or have we unwittingly excluded some models? Does the above action work for other models where the asymptotic behavior is dominated by a Maxwell field? These questions need a thorough treatment that is left for further work.

A Holographic Renormalization

A.1 The Standard Approach

In this appendix a short introduction to holographic renormalization is presented.

The conformal metric of AdS_{d+1} is taken as

$$ds^2 = G_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{\rho}g_{ij}(x,\rho)\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{A.1}$$

The conformal boundary lies at $\rho = 0$. The metric at the boundary yields an expansion of the form

$$g(x,\rho) = g_{(0)} + ... + \rho^{d/2}g_{(d)} + h_{(d)}\rho^{d/2}\log\rho, \tag{A.2}$$

where the last term is present for even d only. This is a suitable ansatz for Einstein gravity. Other theories of gravity may require the introduction of fractional powers of ρ or logarithmic terms. Ignoring potential indices, a generic field $\mathcal{F}(x,\rho)$ can be expanded asymptotically in the form

$$\mathcal{F}(x,\rho) = \rho^m(f_{(0)}(x) + f_{(2)}(x)\rho + ... + \rho^n(f_{(2n)}(x) + \tilde{f}_{(2n)}(x)\log\rho + ...)). \tag{A.3}$$

In the AdS/CFT correspondence the leading term $f_{(0)}(x)$ plays the role of a source for the boundary theory. The coefficients $f_{(2)}(x), f_{(4)}(x), ..., f_{(2n-2)}(x), \tilde{f}_{(2n)}(x)$ are uniquely determined by the equations of motion in terms of the boundary value $f_{(0)}(x)$, while the coefficient $f_{(2n)}(x)$ remains undetermined. The procedure continues to evaluate the on-shell action for some finite cut-off $\rho \geq \epsilon$. The divergences of the boundary term are ordered according to their respective degree of divergence. In general, only a finite number of terms diverge.

$$S_{reg}[f_{(0)};\epsilon] = \int_{\rho=\epsilon}\mathrm{d}^dx\sqrt{g_{(0)}}[\epsilon^{-\nu}a_{(0)} + \epsilon^{-(\nu+1)}a_{(2)} + ... - \log\epsilon\, a_{(2\nu)} + \mathcal{O}(1)], \tag{A.4}$$

where the functions $a_{(k)}$ are local function of $f_{(0)}$. Consequently, the counterterm action $S_{ct}[\mathcal{F}(x,\epsilon);\epsilon]$ is the divergent part of S_{reg}. Since the counterterm action ought to be a functional of quantities inherent to the boundary only, the series (A.3) is inverted. Then the relation $f_{(0)} = f_{(0)}(\mathcal{F}(x,\epsilon))$ is obtained and consequently $a_{(k)} = a_{(k)}(\mathcal{F}(x,\epsilon))$. The action (A.4) is then a functional of $\mathcal{F}(x,\epsilon)$ only. Thus, the renormalized action is

$$S_{ren}[f_{(0)}] = \lim_{\epsilon\to 0}(S_{reg}[f_{(0)}] + S_{ct}[\mathcal{F}(x,\epsilon);\epsilon]). \tag{A.5}$$

As an important example we will apply the above procedure to AdS_3 [40]. The first step is the computation of the asymptotic expansion of g_{ij}. As mentioned above, the coefficients are uniquely determined by the bulk equations of motion. For a split of the metric $G_{\mu\nu}$ of the form (A.1), Einstein's equations read

$$\rho\left[2g'' - 2g'g^{-1}g' + \mathrm{Tr}(g^{-1}g')g'\right] + \mathrm{Ric(g) - (d-2)g' - Tr(g^{-1}g')g} = 0 \tag{A.6}$$

$$\nabla_i\mathrm{Tr}(g^{-1}g') - \nabla^j g'_{ij} = 0 \tag{A.7}$$

$$\mathrm{Tr}(g^{-1}g'') - \frac{1}{2}\mathrm{Tr}(g^{-1}g'g^{-1}g') = 0. \tag{A.8}$$

This system of equations can be solved order by order by differentiating the equation w.r.t ρ and setting $\rho = 0$ afterwards. In the case $d = 2$ the coefficient

$$g_{(2)ij} = \frac{1}{2}(Rg_{(0)ij} + t_{ij}) \qquad h_{(2)ij} = 0 \tag{A.9}$$

is obtained, where t_{ij} is some symmetric tensor that satisfies $\nabla^i t_{ij} = 0, \mathrm{Tr}\, t_{ij} = -R$. Following the procedure outlined above, the EH action with a GHY boundary term restricted to $\rho \geq \epsilon$ is evaluated for the solution (A.3) which leads to [40]

$$S_{reg} = \frac{1}{16\pi G_{d+1}} \int \mathrm{d}^d x \Big[\int_\epsilon \mathrm{d}\rho \frac{d}{\rho^d d/2 + 1} \sqrt{\mathrm{det} g(x,\rho)}$$
$$+ \frac{1}{\rho^{d/2}}(-2d\sqrt{\mathrm{det} g(x,\rho)} + 4\rho\partial_\rho \sqrt{\mathrm{det} g(x,\rho)})\Big|_{\rho=\epsilon}\Big] \tag{A.10}$$
$$= \frac{l}{16\pi G_3} \int \mathrm{d}^2 x \sqrt{\mathrm{det} g_{(0)}} \left(\epsilon^{-1} a_{(0)} - a_{(2)} \mathrm{log}\epsilon\right) + \mathcal{O}(1), \tag{A.11}$$

where we have used the AdS radius l and the coefficients $a_{(0)}, a_{(2)}$ read

$$a_{(0)} = 2(1-d) = -1 \qquad a_{(2)} = \mathrm{Tr} g_{(2)}. \tag{A.12}$$

Expressed in terms of quantities intrinsic to the boundary, i.e. the induced metric $\gamma = \frac{g_{(0)}}{\rho}$, the latter coefficient reads $a_{(2)} = \frac{1}{2}R[\gamma]$. Thus we arrive at the renormalized action S_{ren}

$$S_{ren} = \lim_{\epsilon \to 0} \left(S_{reg} - \frac{1}{16\pi G_3} \int_{\rho=\epsilon} \mathrm{d}^2 x \sqrt{\gamma}(-1 - \frac{1}{2}R[\gamma] \log \epsilon) \right) \tag{A.13}$$

This renormalized action can be used to obtain the renormalized Brown–York stress tensor via the usual relation. This reduces to

$$\langle T_{ij} \rangle = \frac{2l}{16\pi G_3}(g_{(2)ij} - g_{(0)ij}\mathrm{Tr} g_{(2)}) \tag{A.14}$$

in 3 dimensions [40]. Inserting the above coefficients and taking the trace, we arrive at

$$\langle T \rangle = -\frac{c}{24\pi}R \qquad c = \frac{3l}{2G_3}, \tag{A.15}$$

the Brown-Henneaux result for the conformal anomaly of the dual boundary theory [19].

A.2 Holographic Renormalization: Hamilton–Jacobi approach

The starting point of the Hamilton–Jacobi approach is the Hamiltonian formulation of gravity. The "time" vector field t^a, along which the fields evolve, is replaced by the normal vector of the boundary with induced metric γ_{ab}. Thus, the Hamiltonian describes a radial evolution rather than a time evolution. In the following, additional dynamical fields of the theory are denoted by ϕ. The dynamics are governed by the constraint equations

$$\mathcal{H}(\gamma, \pi_\gamma, \phi, \pi_\phi) \approx 0 \qquad \mathcal{H}_i(\gamma, \pi_\gamma, \phi, \pi_\phi) \approx 0 \tag{A.16}$$

with $\mathcal{H}$ denoting the Hamiltonian constraint and $\mathcal{H}_i$ the momentum constraint. The momentum associated with the respective field is obtained from the on-shell action I as a functional derivative

$$\pi_\phi = \frac{1}{\sqrt{\gamma}} \frac{\delta I}{\delta \phi} \qquad \pi^{ab} = \frac{1}{\sqrt{\gamma}} \frac{\delta I}{\delta \gamma_{ab}}. \tag{A.17}$$

The Hamilton–Jacobi equation is now the Hamilton constraint $\mathcal{H}$ with the momenta replaced by the respective functional derivatives

$$\mathcal{H}\left[\gamma_{ab}, \frac{\delta I}{\delta \gamma_{ab}}, \phi, \frac{\delta I}{\delta \phi}\right] = 0 \tag{A.18}$$

In general, the search for an exact solution of this nonlinear functional differential equation is beyond hope. Therefore, the procedure continues by splitting up the action as

$$I = \Gamma + I_{ct}, \tag{A.19}$$

where Γ contains the finite part and I_{ct} the power law divergences. The theory under consideration dictates now an ansatz for the counterterm action I_{ct}, that depends on the field content and the dimension d. Usually, this counterterm can be written as an expansion in the inverse metric. For the case of gravity coupled to a single scalar field ϕ the first terms of the expansion read

$$I_{ct[0]} = \int \mathrm{d}^d x \sqrt{\gamma} W(\phi) \tag{A.20}$$

$$I_{ct[2]} = \int \mathrm{d}^d x \sqrt{\gamma} \left[\frac{1}{2} C(\phi) g^{ij} \partial_i \phi \partial_j \phi + D(\phi) R\right], \tag{A.21}$$

$$I_{ct[4]} = \int \mathrm{d}^d x \sqrt{\gamma} \left[E(\phi) R^2 + F(\phi) R^{ab} R_{ab} + ...\right] \tag{A.22}$$

where R denotes the Ricci scalar of the boundary and $W(\phi), D(\phi), D(\phi), E(\phi), F(\phi)$ denote arbitrary functions of the scalar field. Notice that the term $I_{ct[2k]}$ contains exactly k inverse metrics or $2k$ derivatives of the metric.The number of divergent terms can be estimated by power counting using the asymptotic expansion of the metric (A.2). The momentum (A.17) is split up according to (A.19)

$$\pi = \pi_\Gamma + P. \tag{A.23}$$

Using the above relations, the Hamiltonian constraint is split up in a similar form

$$\mathcal{H} = \mathcal{H}_{[0]} + \mathcal{H}_{[2]} + \mathcal{H}_{[4]} + ... + \mathcal{H}_\Gamma = 0 \tag{A.24}$$

A certain number of the expressions $\mathcal{H}_{[2k]}$, that can be determined by power counting, dominate over the term $\mathcal{H}_\Gamma$ asymptotically. This leads to a system of descent equations for the undetermined functions in the counterterm ansatz, that needs to be solved— at least asymptotically. In general, the system breaks down if $2k = d$, leaving behind some remainder $\mathcal{H}_{rem}$, that is associated with a logarithmic divergence in the action and a conformal anomaly in the boundary field theory [101]. The renormalized action S_{ren} is then obtained by

$$S_{ren} = \lim_{\rho \to 0} \left(I - I_{ct} + \rho \int \mathrm{d}^d x \sqrt{g} \mathcal{H}_{rem} \right) , \tag{A.25}$$

where we subtracted the logarithmic divergence associated with $\mathcal{H}_{ren}$. What remains to be shown is that all divergences of I can be canceled in the way presented (cf. [101]).

In [69], another remarkable approach was presented that was shown to work in lower dimensional gravity and, in particular in dilaton gravity in 2 dimensions. The authors showed that the requirement of local supersymmetry introduced the necessary boundary counterterms that otherwise have to be calculated using the more tedious techniques of holographic renormalization. In contrast to other approaches, no boundary conditions need to be specified. The deeper reason for the success of this procedure is not clear. Furthermore, it is unclear whether this procedure works in higher dimensions.

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