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Towards Lifshitz holography in 3-dimensional higher spin gravity

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durch

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Unterschrift

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Einleitung/Kurzfassung

Zum momentanen Zeitpunkt sind vier fundamentale Wechselwirkungen bekannt. Drei davon, die elektromagnetische, die schwache und die starke Wechselwirkung sind im Standardmodell der Elementarteilchenphysik vereinigt, wohingegen die Gravitation unabhängig behandelt wird. Ein fundamentales Problem der Physik ist die Inkompatibilität des quantenmechanischen Standardmodells mit der klassischen allgemeinen Relativitätstheorie. Diese Problematik kann darauf zurückgeführt werden, dass quantenmechanischen Observablen Wahrscheinlichkeiten zugeordnet sind, wohingegen klassische Größen stets definitive Werte haben.

Neue Ideen kamen als das Holographische Prinzip entdeckt wurde. Es besagt, dass die Information, welche für die Beschreibung einer fundamentalen Theorie notwendig ist, nicht proportional zum Volumen, sondern zu seiner Oberfläche ist. Dies ist ähnlich zu einem Hologramm, welches ebenfalls die Information für einen scheinbar dreidimensionalen Körper auf einer zweidimensionalen Oberfläche abspeichert.

Die mögliche holographische Theorie mit Dimension $D-1$ am Rand einer D dimensionalen Theorie kann meist eingeschränkt werden, wenn die asymptotische Symmetriealgebra bekannt ist. In dieser Arbeit wird die asymptotische Symmetriealgebra für asymptotische Lifshitz Raumzeiten für allgemeine Relativitätstheorie mit negativer kosmologischer Kontante gekoppelt an ein Eichfeld mit Spin 3 untersucht. Das asymptotische Verhalten von Lifshitz Raumzeiten kann als Modell für bestimmte Systeme kondensierter Materie mit Phasenübergängen am kritischen Punkt gesehen werden.

In dieser Arbeit werden zwei unterschiedliche konsistente Randbedingungen definiert, welche asymptotisch Lifshitz Raumzeiten ergeben. Die asymptotische Symmetriealgebra ergibt die eine $\mathcal{W}_3 \times \mathcal{W}_3$ Algebra, wohingegen die zweiten gegebenen Randbedingungen noch nicht eindeutig mit einer bestimmten Algebra identifiziert werden konnten. Wir vermuten, dass es sich um eine $\mathcal{W}_3^{(2)} \times \mathcal{W}_3^{(2)}$ Algebra handelt.

Abstract

We analyze asymptotically Lifshitz spacetimes with a dynamical critical exponent of $z = 2$ in spin-3 gravity in the principal embedding. Two different sets of consistent boundary conditions with conserved charges are presented. For the stricter set of boundary conditions the asymptotic symmetry algebra is $\mathcal{W}_3 \times \mathcal{W}_3$ with the central charges $c = \frac{3l}{2G_N}$. The variations of the looser set of boundary conditions are presented and we conjecture that the asymptotic symmetries in this case are given by a $\mathcal{W}_3^{(2)} \times \mathcal{W}_3^{(2)}$ algebra.

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Contents

1. Introduction

In this introduction we first want to motivate why a quantum theory of gravity is needed. We further give a short overview of gravity seen as an effective field theory and give minimum requirements for a valid theory of quantum gravity. We later show how this leads to the holographic principle which is one of the motivations of this work. We further discuss one of the most prominent realizations of holography, the AdS/CFT correspondence.

1.1 Quantum gravity

We currently know of four fundamental forces. Three of them, the electromagnetic, the weak and the strong interaction are incorporated in the Standard Model of particle physics whereas gravitation is not. Classical general relativity describes gravitational effects from cosmological down to sub-millimeter scales [1,2]. The other forces in the form of quantum theories are known to be valid at even smaller scales [3]. Together they explain a wide area of experimental results and at least in principle all of our everyday phenomena.

On the other hand there are reasons to believe that there must be a more fundamental framework in which quantum mechanics and classical general relativity are unified in a theory of quantum gravity [4–6][1]. One of the reasons to believe in a unification is that classical and quantum theories are known to be incompatible at a fundamental level. The observables of a classical theory are given by definite values, whereas quantum theories provide probabilities for the outcome of measurements. In the case of Einstein's equation

$$G_{\mu\nu} = 8\pi \hat{T}_{\mu\nu} \tag{1.1}$$

the left hand side is a classical tensor whereas the right hand side is given by a quantum operator. Superpositions of the form

$$|\psi\rangle = 1/\sqrt{2}\,(|\text{Mass at A}\rangle + |\text{Mass at B}\rangle) \tag{1.2}$$

would have no quantum mechanical counterpart on the left hand side of equation (1.1)[2].

It is widely believed that the true, fundamental theory unifies all fundamental forces. Here again the classical or the quantum theory needs to be adapted. While it is well established that a fundamental theory should be quantum mechanical another logical possibility is to alter the description of quantum phenomena (see for example [8]). Even though the arguments are not conclusive it seems reasonable to search for a quantum version of general relativity.

[1] Most of this section is motivated by and oriented on [4,5].

[2] The semi-classical interpretation using $G_{\mu\nu} = 8\pi \langle\psi|T_{\mu\nu}|\psi\rangle$ seems to be experimentally excluded [7].

There are various approaches to quantize theories [3, 9].In the case of gravity additionally to technical difficulties problems of conceptual nature exist, connected to the lack of understanding what it means to quantize spacetime itself.

A fruitful approach is the attempt to build the theory as a quantum field theory of the fluctuations $h_{\mu\nu}$ around some background $\widetilde{g}_{\mu\nu}$ i.e., the metric $g_{\mu\nu}$ gets split $g_{\mu\nu} = \widetilde{g}_{\mu\nu} + h_{\mu\nu}$. Diffeomorphism invariance is preserved and interesting quantities are calculated perturbatively in $h_{\mu\nu}$. After the development of the Feynman rules it became clear that general relativity is non-renormalizable [10]. In the modern point of view all quantum field theories are seen as effective field theories and as such they are only valid up to a specific scale [3, 9]. Seen as an effective field theory the Einstein-Hilbert Lagrangian consists of the leading two terms in the series

$$\mathcal{L} = \sqrt{-g}(M_\Lambda^4 + M_P^2 R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} + \ldots). \qquad (1.3)$$

In the regime where the energy of interest is much smaller than the modified Planck mass ($E \ll M_P$) the terms quadratic in curvature are suppressed by $(E/M_P)^2$. One is able to calculate these leading quantum corrections to the Newtonian potential energy [11, 12]

$$V(r) = -\frac{G_N m M}{r}\left(1 + 3\frac{G_N(m+M)}{rc^2} + \frac{41}{10\pi}\frac{G_N \hbar}{r^2 c^3} + \ldots\right). \qquad (1.4)$$

These are in principle measurable deviations from the classical potential which show low energy effects of quantum gravity.

As mentioned above general relativity as an effective field theory is only valid at most up to the Planck scale. This situation is analogous to the Fermi theory of weak interaction where we also know that it is only valid up to the electroweak scale. At higher energies the theory is UV-completed by the electroweak interaction. A possible UV-completion of general relativity could be string theory.

The current lack of observational or experimental evidence which could help to improve or exclude theoretical models makes the search for a unified theory difficult. Additional to experimental search [13] there are some minimum theoretical requirements which a theory of quantum gravity should fulfill:

Classical limit General relativity has passed all observational and experimental tests so far. Therefore the "zeroth test" of any quantum gravity theory is to reproduce general relativity in the classical limit. In the case of loop quantum theory this is still an open problem [14] and it is not clear that the classical limit is general relativity.

Hawking radiation Another robust prediction is the existence and spectrum of black hole Hawking radiation. In a classical world nothing could exit a black hole but since Hawking radiation is a quantum effect it is possible. It has been verified by various (semi-classical) calculations and is therefore a prediction that any theory of quantum gravity should reproduce. Hawking radiation is perfect black-body radiation with a for the black hole characteristic temperature. This associated temperature is called Hawking temperature and is given by

$$T_H = \frac{\kappa}{2\pi}. \tag{1.5}$$

κ is the surface gravity at the event horizon. In the case of the 3+1 dimensional Schwarzschild black hole $\kappa = \frac{1}{4M}$.

Black hole entropy Why one should associate entropy to a black hole can be motivated by the tension between the second law of thermodynamics and the no-hair theorem. On the one hand, entropy should never decrease, but on the other hand, any stationary black hole in 3+1 dimensions is characterized by just three parameters which are the mass M, charge Q and angular momentum J. So a lot of different start configurations end up in the same final state with entropy $S = 0$. A resolution is to associate to any black hole the Bekenstein-Hawking entropy

$$S_{\mathrm{BH}} = \frac{A_{\mathrm{horizon}}}{4} \tag{1.6}$$

where A_{horizon} is the area of the event horizon. Since entropy is a measure for the number of degrees of freedom, a quantum theory of gravity should be able to explain and describe the microscopic origin of them.

String theory reproduces (1.6) for a large class of extremal and near-extremal black holes [15]. Loop quantum gravity is able to calculate the entropy for various classes of black holes, including the Schwarzschild black hole, but fails to predict the proportionality factor [14].

Logarithmic corrections Logarithmic corrections to the black hole entropy of extremal black holes calculated by Euclidean gravity methods in terms of low energy data agree perfectly with the microscopic string theory calculations. This motivates to apply the former method to more general black holes and thus constrain the underlying theory [16]. In loop quantum theory there are logarithmic corrections to pure gravity Schwarzschild black holes that differ in sign and magnitude to the one Euclidean gravity predicts

$$S_{\mathrm{EG}} = S_{\mathrm{BH}} + (\frac{212}{45} - 3) \ln M. \tag{1.7}$$

For string theory there exists no microscopic calculation for the Schwarzschild black hole yet.

1.2 The holographic principle

If we look at the entropy of black holes and the regions outside of them separately the second law of thermodynamics can be violated. Throwing a cup into a black hole reduces the external entropy whereas Hawking radiation reduces the entropy of the black hole. To remain with a monotonically increasing entropy Bekenstein suggested [17–19] the generalized second law

$$dS_{\text{total}} \geq 0 \tag{1.8}$$

with $S_{\text{total}} = S_{\text{BH}} + S_{\text{matter}}$.

To be sure that the decrease of S_{matter} results in an increase of S_{total}, matter has to obey the Bekenstein bound

$$S_{\text{matter}} \leq 2\pi E R. \tag{1.9}$$

E is the mass-energy of the matter system and R is the radius of the smallest sphere around it. Now even a very dense cup of coffee which enters the black hole decreases S_{matter} such that the increase in S_{BH} results in an increase in S_{total}. One is able to derive the spherical entropy bound with the condition that the matter should be stable $2M \leq R$ which leads us to

$$S_{\text{matter}} \leq 2\pi M R \leq \pi R^2 = \frac{A}{4}. \tag{1.10}$$

Now this generalized thermodynamics has very surprising consequences known as the "holographic principle"[3]. It rests on the analysis of how many degrees of freedom $\mathfrak{dof}$ a fundamental theory has in a region with volume V and boundary area A. The degrees of freedom are defined to be the logarithm of the dimension of its quantum mechanical Hilbert space $\mathcal{H}$

$$\mathfrak{dof} = \ln \dim(\mathcal{H}). \tag{1.11}$$

We start with the estimation of $\mathfrak{dof}$ for a local field theory on a classical background spacetime. On a superficial level the answer would be $\mathfrak{dof} = \infty$ because there are infinitely many points in any volume. Furthermore the harmonic oscillator at each point has an infinite dimensional Hilbert space.

At closer inspection we recognize that there is a minimal length that can be resolved, the Planck length l_{P}. Additionally there is also a maximal energy that can be stored in such a Planck cell l_{P}^3 which is of the order of one Planck mass. So we can assume that the Hilbert space of one oscillator has a maximal dimension of n.

[3] The following analysis of the holographic principle is highly influenced by [20]. Throughout the description it is assumed that the spacetime is asymptotically flat and the gravitation is weak enough for the quantities A and V to make sense.

The total number of independent quantum states in a volume is V (in Planck units) so we get

$$\dim\mathcal{H} \sim n^V \to \mathfrak{dof} \gtrsim V. \tag{1.12}$$

Another way to calculate the degrees of freedom is by using thermodynamics. For an isolated thermodynamic system the number of possible states for specified macroscopic variables is given by

$$\dim\mathcal{H} = e^S. \tag{1.13}$$

where S is the entropy of the system. This number is restricted by the spherical entropy bound (1.10) which is saturated for a black hole. So the maximal number of states is $e^{A/4}$ with the degrees of freedom bound by

$$\mathfrak{dof} \leq \frac{A}{4}. \tag{1.14}$$

This disagreement with local field theory, e.g., (1.12) can be traced back to the fact that the ultra-violet cutoff by the Plank mass just restricts the formation of black holes on small scales. Violating the inequality (1.14) would result in gravitational collapse to a black hole.

If we assume that our fundamental theory is unitary and as such preserves information we conclude that any configuration that could end up as a black hole had to respect (1.14) from the start because otherwise there would be information loss.

So trusting in black hole thermodynamics and unitarity 't Hooft [21] and Susskind [22] proposed the holographic principle [20]:

"*A region with boundary of area A is fully described by no more than A/4 degrees of freedom, or about 1 bit of information per Planck area. A fundamental theory, unlike local field theory, should incorporate this counterintuitive result.*"

1.3 AdS/CFT correspondence

The holographic principle is realized in string theory in the form of the AdS/CFT correspondence. Maldacena [23] proposed that nonperturbative string theory in an asymptotically Anti-de Sitter (AdS) background is dual to a flat spacetime conformal field theory (CFT) in one dimension lower. A concrete example is type IIB string theory on asymptotically $\mathrm{AdS}_5 \times \mathbf{S}^5$ plus some appropriate boundary conditions, which is dual to a $3+1$ dimensional supersymmetric Yang-Mills theory in flat space. The conformal boundary of the bulk $\mathrm{AdS}_5 \times \mathbf{S}^5$ is $3+1$ dimensional and this is where the boundary theory "lives" [24]. Since the 4 dimensional boundary theory describes all the physics of the theory of the 5 dimensional bulk we have a dimensional reduction in agreement with the holographic principle [25].

To have a meaningful correspondence we further need to connect the interesting quantities of the bulk and the boundary. The bulk fields of the string theory ϕ get evaluated at their boundary values ϕ_0. The boundary CFT does not have particle states or an S-matrix so the physical relevant quantities are given by correlation functions of gauge invariant operators $\mathcal{O}$. The proposal [26, 27] for the connection between them is

$$Z_{\text{string}}[\phi_0] = \langle e^{\int_{\partial M} \phi_0 \mathcal{O}} \rangle_{\text{CFT}} \,. \tag{1.15}$$

The left hand side is the full partition function of string theory and the right hand side is the generating function of correlation functions of the field theory.

Various quantities of the bulk and the boundary theory have been compared. They matched in perfect agreement with the correspondence. For more details, other realizations and an extensive review of the results see [24].

2. Higher spin gauge theories

In this section we provide a non-exhaustive overview of higher spin gauge theories and the progress that has been made in the past. We will see that they provide examples for many features that were discussed in the previous section. Focus will be laid on three dimensional higher spin theories which provide a great balance between analytic tractability and intrinsic complexity. Finally higher spin gravity with the gauge algebra $sl(N,\mathbb{R}) \oplus sl(N,\mathbb{R})$ will be introduced.

2.1 Higher spin gauge theories and holography

All known physical fields in quantum theories can be classified by just the mass m and the angular momentum s about the center of mass [9]. They can further be divided in massive and massless particles with fulfill $p^2 = -m^2$ and $p^2 = 0$ respectively.

The equations for non interacting massless particles of integer spin in $3+1$ dimensions on a flat background were found by Frondsdal [28][4]. For $s = 0, 1, 2$ they reduce to the well known Klein Gordon equation, Maxwell equation and linearized general relativity.

The local degrees of freedom for a massless field of integer spin s in $D \geq 4$ flat spacetime is given by[5]

$$\frac{(D+2s-4)(D+s-5)!}{(D-4)!s!}. \tag{2.1}$$

It is interesting to note that in four spacetime dimensions the local degrees of freedom are 2, independently from the spin. This in agreement with the polarizations of the photon and the graviton.

It is comparably easy to write down free higher spin fields. But coupling these for $s > 2$ to gravity leads to various no-go theorems (for a review see [30]). Fradkin and Vasiliev [31] showed that consistent higher spin gauge theories involving gravity need to be defined on a curved background. They were first formulated by Vasiliev [32]. This theories involve an infinite tower of massless fields and can be constructed as the simplest example on (A)dS spaces.

One interesting aspect of higher spin gauge fields is that they might be connected to string theory in the tensionless limit in which the massive excitations of string theory become massless. It is conjectured that string theory is a broken phase of a higher spin gauge theory. For more details see [33] and references therein.

Furthermore the holographic principle finds a realization in the form of the proposal made by Klebanov and Polyakov [34] and Sezgin and Sundell [35,36]. They conjectured that there exists a duality in the large N limit of the critical 3-dimensional

[4]We will restrict our explanations for the sake of simplicity on integer spin.

[5]This is connected to the fact that the stability subgroup of the Poincare group is $ISO(D-2)$. The local degrees of freedom can then be determined from the dimension of the unitary irreducible representation of this group [29].

$O(N)$ model and the minimal bosonic higher spin theory in AdS_4. This holographic proposal got supported by calculations of Giombi and Yin [37] and the current status is reviewed in [38].

We will now focus on $2+1$ dimensions where the situation changes significantly. Gauge fields with "spin"[6] $s > 1$ posses no local degrees of freedom anymore. This makes theories in $2+1$ dimensions interesting in various aspects. While there is still enough structure to be nontrivial the technical difficulties that arise in the higher dimensional cases are often circumvented.

This is already seen in the famous result by Brown and Henneaux [39] which can be seen as a precursor of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. They showed that three dimensional Einstein-Hilbert gravity with a negative cosmological constant and Brown-Henneaux boundary conditions leads to asymptotic symmetries given by the conformal group in two dimensions. Interestingly there appears a nontrivial central charge in the algebra of the canonical generators.

This charge appears again in the analysis of another unexpected result, the BTZ black hole [40]. Even though there are no local degrees of freedom in three dimensional gravity, black holes are possible. Using the central charge it was shown that it is possible to calculate the asymptotic density of states and the entropy [41]. So a microscopic interpretation for the states of the black hole is possible and the holographic principle is realized.

To add interacting fields with spin $s > 2$, in contrast to the higher dimensional case, no infinite number of higher spin fields are needed [42]. The Brown Henneaux analysis has been generalized to higher spin fields [43–45]. In the case of the coupling of a spin 3 field to gravity the asymptotic symmetries are given by $\mathcal{W}_3 \times \mathcal{W}_3$ algebras (see Appendix A.3). Fields of spin $s = 3, 4, \ldots, N$ coupled to gravity are given by a Chern-Simons theory with gauge algebra $sl(n, \mathbb{R}) \oplus sl(n, \mathbb{R})$ (see section 2.2) and have in the case of an AdS_3 background the asymptotic symmetries $\mathcal{W}_N \times \mathcal{W}_N$. Using the higher spin algebras $hs[\lambda] \oplus hs[\lambda]$ as gauge algebra we get gravity coupled to spin fields $s = 3, 4, \ldots, \infty$ and again for AdS_3 asymptotic symmetries $\mathcal{W}_\infty[\lambda] \times \mathcal{W}_\infty[\lambda]$ (for details please see the cited references above).

Another aspect that is advantageous in $D = 3$ is that the dual to AdS_3 is given by CFT_2. Two dimensional conformal field theories are well understood and offer a high degree of analytic control. It was proposed by Gaberdiel and Gopakumar [46] that the $hs[\lambda]$ theory on AdS_3 is dual to the large-N limit of $\mathcal{W}_N$ minimal models on the CFT side. As a hint for the validity of this proposal can be seen that this limit on the CFT side leads, like in the bulk theory, also to a $\mathcal{W}_\infty$ algebra. The duality is reviewed in [47].

Since the BTZ black hole can also be generalized in higher spin gauge theories

[6] "Spin" in $D = 3$ refers to the transformation properties of the field under Lorentz transformations and is unrelated to the considerations given in footnote 5.

new questions concerning gauge invariant characterization and black hole thermodynamics arise (for a review of the proposed answers see [48]).

2.2 $sl(N,\mathbb{R}) \oplus sl(N,\mathbb{R})$ higher spin gravity

Much of the progress in $2+1$ dimensions is due to the fact that an action for the interacting fields is known [49]. This is in contrast to the higher dimensional case which is only known as a formulation at the level of field equations.

Higher spin gravity (HS gravity) is given as a $sl(N,\mathbb{R}) \oplus sl(N,\mathbb{R})$ Chern-Simons theory with the Lie algebra valued one forms A and $\bar{A}$[7]

$$\boxed{I[A,\bar{A}] = I_{CS,\pm}[A] - I_{CS,\pm}[\bar{A}]} \tag{2.2}$$

where

$$I_{CS,\pm}[A] = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) + B_{\pm}[A]. \tag{2.3}$$

The trace Tr is with respect to the $sl(N,\mathbb{R})$ algebra and $B_{\pm}[A]$ denotes a possible boundary term. For $N = 2$ standard three dimensional gravity with a negative cosmological constant is recovered [50, 51]. For $N > 2$ HS gravity is gravity coupled to symmetric tensor fields of rank $s = 3, 4, \ldots, N$. The $sl(2,\mathbb{R})$ part of the $sl(N,\mathbb{R})$ theory should match with the Einstein-Hilbert action. To guarantee that in the case of the fundamental (N-dimensional) representation of the algebra and the principal embedding we have to set [43]

$$k = \frac{l}{8G_N \mathrm{Tr}_N(L_0 L_0)} \tag{2.4}$$

where the trace of the L_0 generator of $sl(2,\mathbb{R})$ can be given in general by $\mathrm{Tr}_N(L_0 L_0) = \frac{N(N^2-1)}{12}$ [52].

The action is written for gauge fields A and $\bar{A}$. When we want to recover a spin n field in its "metric formulation" we use the relation

$$\phi_{\mu_1\mu_2\ldots\mu_n} \sim \mathrm{Tr}(e_{(\mu_1} e_{\mu_2} \ldots e_{\mu_n)}) \tag{2.5}$$

where the brackets around the indices on the right hand denote symmetrization. In the case of $s > 3$ this is already not unique anymore. For example for $s = 4$ we have the options $\mathrm{Tr}(e^4)$ and $\mathrm{Tr}(e^2)\mathrm{Tr}(e^2)$ [44]. The (zu)vielbein e and the spin connection ω are given by

$$e = \frac{l}{2}(A - \bar{A}) \quad \text{and} \quad \omega = \frac{1}{2}(A + \bar{A}) \tag{2.6}$$

where l is the AdS radius. In our analysis the manifold is assumed to have cylindrical topology $\mathcal{M} = \mathbb{R}\times\mathcal{D}$ and the boundary $\partial\mathcal{M}$ has the topology $\partial\mathcal{M} = \mathbb{R}\times S_1$ (figure 1). The coordinates are denoted by $x^\mu = (t, r, x)$.

[7] $\bar{A}$ is not the complex conjugate of A. The bar denotes an independent field.

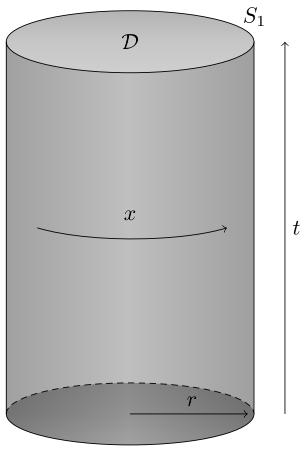

Figure 1: Topology of the spacetime

2.3 Variational principle

We will now focus on $I_{CS,\pm}[A]$ since the analysis for $\overline{A}$ follows by replacing $A \to \overline{A}$. Chern-Simons gauge theories are well documented, see for example [43, 51, 53–55] and references therein.

Since the gauge field A is a $sl(N,\mathbb{R})$-valued one-form we can write it as $A = A^a T_a = A_\mu dx^\mu = A^a_\mu T_a dx^\mu$. The T_a span the basis of the Lie algebra with the structure constants given by $[T_a, T_b] = f^c{}_{ab} T_c$. The trace Tr is with respect to this algebra and $g_{ab} = \mathrm{Tr}(T_a T_b)$ is the Cartan–Killing metric[8].

To have a kinetic energy

$$\int_{\mathcal{M}} \mathrm{Tr}(A \wedge dA) = g_{ab} \int_{\mathcal{M}} A^a \wedge dA^b \tag{2.7}$$

for all components of the gauge fields we need the metric g_{ab} to be non-degenerate. This is fulfilled in the case of semisimple Lie algebras and thus in our theory.

Varying now the action $I_{CS,\pm}[A]$ leads to

$$\delta I_{CS,\pm}[A] = \frac{k}{2\pi} \int_{\mathcal{M}} \mathrm{Tr}(F \wedge \delta A) - \frac{k}{4\pi} \int_{\partial\mathcal{M}} \mathrm{Tr}(A \wedge \delta A) + \delta B_\pm[A], \tag{2.8}$$

where $F = dA + A \wedge A$. Demanding $\delta I_{CS,\pm}[A] = 0$ leads to the equations of motion (EOM)

$$F = dA + A \wedge A = 0 \tag{2.9}$$

which can be written more explicit as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0. \tag{2.10}$$

[8] Not to be confused with the spacetime metric $g_{\mu\nu}$.

Going on-shell after varying the action leads to a boundary term

$$\delta I_{CS,\pm}[A]\Big|_{\text{EOM}} = -\frac{k}{4\pi}\int_{\partial\mathcal{M}} \text{Tr}(A\wedge\delta A) + \delta B_\pm[A] \tag{2.11}$$

$$= \frac{k}{4\pi}\int_{\partial\mathcal{M}} \text{Tr}(\delta A_x A_t - A_x \delta A_t) dx\wedge dt + \delta B_\pm[A]. \tag{2.12}$$

One option for a well defined variational principle is to start without an additional boundary term $B_\pm[A]$ and demand one of the following three conditions

$$A_x\Big|_{\partial\mathcal{M}} = 0 \qquad\qquad A_t\Big|_{\partial\mathcal{M}} = 0 \qquad\qquad A_x\Big|_{\partial\mathcal{M}} \propto A_t\Big|_{\partial\mathcal{M}}. \tag{2.13}$$

Another option which leaves us with less restrictions is, to define the boundary term as

$$B_\pm[A] = \pm\frac{k}{4\pi}\int_{\partial\mathcal{M}} \text{Tr}(A_x A_t) dx\wedge dt, \tag{2.14}$$

which gives in the case of a plus sign

$$\delta I_{CS,+}[A]\Big|_{\text{EOM}} = \frac{k}{2\pi}\int_{\partial\mathcal{M}} \text{Tr}(\delta A_x A_t) dx\wedge dt. \tag{2.15}$$

So we get the following options for a well defined variational principle

$$\delta I_{CS,+}[A]\Big|_{\text{EOM}} = 0 \quad\Longrightarrow\quad \delta A_x\Big|_{\partial\mathcal{M}} = 0 \quad\text{or}\quad A_t\Big|_{\partial\mathcal{M}} = 0 \tag{2.16}$$

$$\delta I_{CS,-}[A]\Big|_{\text{EOM}} = 0 \quad\Longrightarrow\quad \delta A_t\Big|_{\partial\mathcal{M}} = 0 \quad\text{or}\quad A_x\Big|_{\partial\mathcal{M}} = 0. \tag{2.17}$$

3. Lifshitz higher spin holography

In this section we will first give examples of holography besides AdS. Then the spacetime that will be further discussed will be defined and the symmetries recovered.

Next we follow the idea of Gary, Grumiller and Rashkov [56] and the algorithm suggested in [57]. Our goal is to find the asymptotic symmetry algebra and its central charges since this gives us hints about the dual field theory. To do that we first identify the bulk theory and the variational principle and further generate the background. We impose suitable boundary conditions where we want to find boundary preserving gauge transformations which lead to nontrivial, state-dependent, finite, conserved and integrable charges.

Two such boundary conditions were found. The stricter set leads to a $\mathcal{W}^3 \times \mathcal{W}^3$ algebra whereas the looser set is still work in progress.

3.1 Non-AdS holography

In many applications it is necessary to generalize holography to spacetimes more general than asymptotic AdS. Examples are [56]:

Null warped AdS spacetimes which arise in proposed holographic duals of non-relativistic CFTs describing cold atoms [58, 59]

Schrödinger spacetimes, which generalize null warped AdS by introducing an arbitrary scaling exponent [60]

Lifshitz spacetimes, which arise in gravity duals of Lifshitz-like fixed points [61] and also have a scaling exponent parametrizing spacetime anisotropy

AdS/log CFT [62], which requires a relaxation of the Brown–Henneaux boundary conditions [63–65]

Flat space holography for asymptotically flat spaces [66–72]

A variational principle for 3-dimensional higher spin gravity that accommodates spacetimes like asymptotically $\text{AdS}_2 \times \mathbb{R}$, $\mathbb{H}_2 \times \mathbb{R}$, Schrödinger, Lifshitz or warped AdS spacetimes was proposed and the connections that generate this backgrounds presented [56].

For the case of $\mathbb{H}_2 \times \mathbb{R}$ realized in $sl(3, \mathbb{R})$ HS gravity in the non-principal embedding the asymptotic symmetry algebra turned out to be $\mathcal{W}_3^{(2)} \oplus \hat{\mathfrak{u}}(1)$ [57].

During the preparation of this work Gutperle, Hijano and Samani [73] presented a work concerning asymptotically Lifshitz black holes in higher spin gravity.

3.2 Lifshitz spacetime

The Lifshitz background [61] is the proposed gravity dual to specific condensed matter systems with phase transitions governed by fixed points which exhibit an anisotropic scale invariance between spatial and temporal scaling

$$t \to \lambda^z t \qquad \vec{x} \to \lambda \vec{x} \qquad (r \to r/\lambda) \qquad z \neq 1 \tag{3.1}$$

with the scaling exponent $z \in \mathbb{R}$. As an example the Lifshitz field theory [74]

$$S[\phi] = \int d^2x\, dt \Big((\partial_t \phi)^2 - \kappa (\Delta \phi)^2 \Big) \tag{3.2}$$

is invariant under the scaling (3.1) when $z = 2$.

Motivated by such field theories which are invariant under this anisotropic scaling, time and space translations, spatial rotations, spatial parity, and time-reversal one gets the Lifshitz spacetime metric Lif_D^z

$$ds^2_{\mathrm{Lif}_D^z} = l^2 \left(-r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 d\vec{x}^2 \right) \tag{3.3}$$

where $d\vec{x}^2 = dx_1^2 + \ldots dx_{D-2}^2$ and l is the scale for the radius of curvature of the geometry. For the $z = 1$ the metric is that of Poincaré patch AdS_D. One way to obtain the Lifshitz spacetime is to couple general relativity to some appropriate matter content. For an example see [61] where Lif_4^z is realized using general relativity coupled to a 1-form and a 2-form field.

We will now restrict our discussion to $D = 3$ to obtain the spacetime metric

$$ds^2_{\mathrm{Lif}_3^z} = l^2 \left(-r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 dx^2 \right). \tag{3.4}$$

Furthermore we will set $z = 2$ and substitute by $r = e^{\rho/2}$ which leads us to the metric

$$ds^2_{\mathrm{Lif}_3^2} = l^2 \left(-e^{4\rho} dt^2 + d\rho^2 + e^{2\rho} dx^2 \right) \tag{3.5}$$

which will be used for the remaining work.

If we substitute $r = e^{4\rho}$ in equation (3.4) and set $z = 1/2$ we get the metric

$$ds^2_{\mathrm{Lif}_3^{1/2}} = l^2 \left(-e^{2\rho} dt^2 + 4 d\rho^2 + e^{4\rho} dx^2 \right). \tag{3.6}$$

We observe that the following analysis can also be used for $z = 1/2$ when we exchange $x \leftrightarrow t$ and the constant coefficients accordingly.

3.3 Symmetries of Lifshitz spacetimes Lif_3^z

We want to find the Killing vector fields $\xi = \xi^\mu(r,x,t)\partial_\mu$ and the symmetry algebra for Lif_3^z (for the special case $z=1$, i.e., AdS_3 see appendix B). So we want to solve

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu}\partial_\mu \xi^\lambda + g_{\mu\lambda}\partial_\nu \xi^\lambda = 0 \tag{3.7}$$

for ξ. Substitution of the metric (3.4) leads to the following system of equations

$$r\partial_r \xi^r(r,x,t) - \xi^r(r,x,t) = 0 \tag{3.8a}$$
$$\partial_x \xi^r(r,x,t) + \partial_r \xi^x(r,x,t) = 0 \tag{3.8b}$$
$$-r^{2-2z}\partial_r \xi^t(r,x,t) + \partial_t \xi^r(r,x,t) = 0 \tag{3.8c}$$
$$-r\partial_x \xi^x(r,x,t) + \xi^r(r,x,t) = 0 \tag{3.8d}$$
$$-r^{2-2z}\partial_x \xi^t(r,x,t) + \partial_t \xi^x(r,x,t) = 0 \tag{3.8e}$$
$$z\xi^r(r,x,t) - r\partial_t \xi^t(r,x,t) = 0. \tag{3.8f}$$

Equation (3.8a) is solved by

$$\xi^r(r,x,t) = r\xi^r(x,t) \tag{3.9}$$

which when inserted into equation (3.8d) leads to

$$\xi^r(x,t) = \partial_x \xi^x(r,x,t) \tag{3.10}$$

with the solution

$$\xi^x(r,x,t) = \int^x \xi^r(y,t)\,dy + \alpha(r,t). \tag{3.11}$$

Inserting now into (3.8c) gives the equation

$$\partial_t \xi^r(x,t) = r^{1-2z}\partial_r \xi^t(r,x,t) \tag{3.12}$$

with the solution

$$\xi^t(r,x,t) = \frac{r^{2z}\partial_t \xi^r(x,t)}{2z} + \beta(x,t). \tag{3.13}$$

We take now equation (3.8b)

$$r\partial_x \xi^r(x,t) + \partial_r \alpha(r,t) = 0 \tag{3.14}$$

which is solved by

$$\xi^r(x,t) = x\gamma(t) + \delta(t) \tag{3.15}$$
$$\alpha(r,t) = -\frac{1}{2}r^2\gamma(t) + \epsilon(t). \tag{3.16}$$

Equation (3.8e) now has the form

$$2r^{2-2z}\partial_x \beta(x,t) + \left(r^2\left(z^{-1}+1\right) - x^2\right)\gamma'(t) = 2\left(x\delta'(t) + \epsilon'(t)\right). \tag{3.17}$$

Looking at the coefficients we recognize that

$$\partial_x\beta(x,t) = 0 \to \beta(x,t) = \phi(t) \tag{3.18}$$
$$\gamma'(t) = 0 \to \gamma(t) = \gamma_0 \tag{3.19}$$
$$\delta'(t) = 0 \to \delta(t) = c \tag{3.20}$$
$$\epsilon'(t) = 0 \to \epsilon(t) = a. \tag{3.21}$$

Inserting in the last equation (3.8f) leads to

$$-\phi'(t) + cz + zx\gamma_0 = 0 \tag{3.22}$$

where we see that γ_0 and $-\phi'(t) + cz$ have to vanish independently

$$\gamma_0 = 0 \tag{3.23}$$
$$\phi(t) = czt + b. \tag{3.24}$$

So $\mathcal{L}_\xi g_{\mu\nu} = 0$ is fulfilled by arbitrary linear combinations of the following three vector fields

$$P = \partial_x \tag{3.25}$$
$$H = \partial_t \tag{3.26}$$
$$D = zt\partial_t + r\partial_r + x\partial_x \tag{3.27}$$

We recognize the generators of space translation P, time translation H and anisotropic dilatation D which lead to the Lie algebra $\mathfrak{lif}_3^z$

$$[P,H] = 0 \tag{3.28}$$
$$[P,D] = P \tag{3.29}$$
$$[H,D] = zH \tag{3.30}$$

Let us now analyze this algebra [75]. Since there is a nontrivial ideal $\mathfrak{i}$ spanned by $\{P,H\}$ this algebra is neither simple nor semi-simple.

The lower central series[9] $\mathcal{D}_k\mathfrak{lif}_3^z$ is spanned by $\{P,H\}$ for all $k \in \mathbb{N}_{>0}$ so the algebra is not nilpotent.

The derived series[10] $\mathcal{D}^k\mathfrak{lif}_3^z$ is

- $\mathcal{D}^1\mathfrak{lif}_3^z$ which is spanned by $\{P,H\}$
- $\mathcal{D}^2\mathfrak{lif}_3^z = 0$

[9] Defined inductively by $\mathcal{D}_1\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ and $\mathcal{D}_k\mathfrak{g} = [\mathfrak{g},\mathcal{D}_{k-1}\mathfrak{g}]$.

[10] Defined inductively by $\mathcal{D}^1\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ and $\mathcal{D}^k\mathfrak{g} = [\mathcal{D}^{k-1}\mathfrak{g},\mathcal{D}^{k-1}\mathfrak{g}]$.

which means that the algebra is solvable.

Three-dimensional solvable Lie algebras are classified see, e.g., [76] and citations therein. Our algebra is isomorphic to the Lie algebra L^3_a

$$[x_1, x_2] = 0 \tag{3.31}$$

$$[x_3, x_1] = x_2 \tag{3.32}$$

$$[x_3, x_2] = ax_1 + x_2 \tag{3.33}$$

in that article. The isomorphism is

$$D = -(z+1)x_3 \tag{3.34}$$

$$P = a(z+1)x_1 + x_2 \tag{3.35}$$

$$H = (a(z+1)+1)x_1 - x_2 \tag{3.36}$$

under the condition that

$$a = -\frac{z}{(z+1)^2} \quad \text{with} \quad z \neq -1 \tag{3.37}$$

3.4 Bulk theory and variational principle

The bulk action for our theory it the one for $sl(3,\mathbb{R}) \oplus sl(3,\mathbb{R})$ (see section 2) with a boundary term [56]

$$I[A, \overline{A}] = I_{CS,-}[A] - I_{CS,-}[\overline{A}] \tag{3.38}$$

where

$$I_{CS,-}[A] = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) - \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{Tr}(A_x A_t) dx \wedge dt. \tag{3.39}$$

To have a well defined variational principle we have to fulfill the flatness conditions for connections A and $\overline{A}$

$$F = dA + A \wedge A = 0 \qquad \bar{F} = d\overline{A} + \overline{A} \wedge \overline{A} = 0 \tag{3.40}$$

and the boundary conditions

$$\delta A_t\Big|_{\partial\mathcal{M}} = 0 \qquad \delta\overline{A}_t\Big|_{\partial\mathcal{M}} = 0. \tag{3.41}$$

Using no boundary term would lead to more restrictive boundary conditions (see section 2.3). Using $I_{CS,+}[A]$ or $I_{CS,+}[\overline{A}]$ would lead, when the less restrictive boundary conditions are chosen, by equation (3.65) or (3.66) to trivial charges.

We use the principal embedding of $sl(2,\mathbb{R})$ into $sl(3,\mathbb{R})$. The explicit $sl(3,\mathbb{R})$ representation is given in appendix A.

With $e = \frac{l}{2}(A - \overline{A})$ we define the spacetime metric as

$$g_{\mu\nu} = \frac{1}{2}\mathrm{Tr}(e_\mu e_\nu) \tag{3.42}$$

and the spin 3 field as

$$\phi_{\mu\nu\rho} = \frac{1}{3!}\mathrm{Tr}(e_{(\mu}e_\nu e_{\rho)}). \tag{3.43}$$

3.5 Background generation and general fluctuations

We decompose the connection as in [40, 43, 57]

$$A = b^{-1}db + b^{-1}\big(\hat{a}^{(0)} + a^{(0)} + a^{(1)}\big)b \tag{3.44}$$

$$\overline{A} = bdb^{-1} + b\big(\hat{\bar{a}}^{(0)} + \bar{a}^{(0)} + \bar{a}^{(1)}\big)b^{-1} \tag{3.45}$$

with $b = e^{\rho L_0}$. $\hat{a}^{(0)} = \hat{a}^{(0)}(t,x)$ has to generate the background asymptotically and has to fulfill the EOM asymptotically. $a^{(0)} = a^{(0)}(t,x)$ are the leading state-dependent fluctuations which are compatible with the boundary conditions of the action. In principle they do not have to fulfill the EOM asymptotically but it is convenient and without loss of generality to impose them. The sub-leading terms are always $a^{(1)} = \bar{a}^{(1)} = o(1)$.

First we construct the Lifshitz background. With the defined algebra this can be done with

$$\hat{a}^{(0)} = L_1 dx + \frac{4}{9} W_2 dt \tag{3.46}$$

$$\hat{\bar{a}}^{(0)} = L_{-1} dx + W_{-2} dt \tag{3.47}$$

which leads when inserted into our definition of the metric (3.42) to

$$l^2\left(-e^{4\rho}dt^2 + d\rho^2 + e^{2\rho}dx^2\right) \tag{3.48}$$

which is $ds^2_{\mathrm{Lif}^2_3}$. Since we want to fulfill the boundary conditions (3.41) our ansatz for the state-dependent fluctuations is

$$a_t^{(0)} = 0 \tag{3.49a}$$

$$a_\rho^{(0)} = 0 \tag{3.49b}$$

$$a_x^{(0)} = \sum_{n=-1}^{1} \alpha^{L_n}(t,x)\, L_n + \sum_{n=-2}^{2} \alpha^{W_n}(t,x)\, W_n \tag{3.49c}$$

and

$$\bar{a}_t^{(0)} = 0 \tag{3.50a}$$

$$\bar{a}_\rho^{(0)} = 0 \tag{3.50b}$$

$$\bar{a}_x^{(0)} = \sum_{n=-1}^{1} \bar{\alpha}^{L_n}(t,x)\, L_n + \sum_{n=-2}^{2} \bar{\alpha}^{W_n}(t,x)\, W_n. \tag{3.50c}$$

To be asymptotic Lifshitz we set $\alpha^{L_1} = 0$ and use the EOM to get

$$a_x^{(0)} = \big(l^0(x) + 4tw^{-2}(x)\big)L_0 + l^{-1}(x)L_{-1} + \left(-\frac{8}{9}tl^0(x) - \frac{16}{9}t^2w^{-2}(x) + w^2(x)\right)W_2 + \left(w^1(x) - \frac{16}{9}tl^{-1}(x)\right)W_1 + w^0(x)W_0 + w^{-2}(x)W_{-2}. \tag{3.51}$$

For the barred sector we set $\bar{\alpha}^{L_{-1}} = 0$ and get on-shell

$$\begin{aligned}\bar{a}_x^{(0)} &= \bar{l}^1(x)L_1 + \left(\bar{l}^0(x) - 9t\overline{w}^2(x)\right) L_0 + \overline{w}^2(x)W_2 + \overline{w}^0(x)W_0 \\ &\quad + \left(4t\bar{l}^1(x) + \overline{w}^{-1}(x)\right) W_{-1} + \left(-9\overline{w}^2(x)t^2 + 2\bar{l}^0(x)t + \overline{w}^{-2}(x)\right) W_{-2}.\end{aligned} \tag{3.52}$$

Further restrictions need to be set to get integrable charges. We will now discuss two separate cases where boundary preserving gauge transformations and nontrivial, finite, conserved, integrable charges were found.

3.6 $\mathcal{W}_3 \times \mathcal{W}_3$ boundary conditions

For the stricter set of boundary conditions (strict BC) we set l^0, w^2, w^1, w^0 of equation (3.51) and $\bar{l}^0$, $\overline{w}^0$, $\overline{w}^{-1}$,$\overline{w}^{-2}$ of equation (3.52) to zero. We rescale and rename the remaining fluctuations with hindsight and omit the x dependence

$$a_x^{(0)} = -\frac{8\pi}{9k}t\mathcal{W}\,L_0 + \frac{\pi}{2k}\mathcal{L}\,L_{-1} + \frac{32\pi}{81k}t^2\mathcal{W}\,W_2 - \frac{8\pi}{9k}t\mathcal{L}\,W_1 - \frac{2\pi}{9k}\mathcal{W}\,W_{-2} \tag{3.53}$$

$$\bar{a}_x^{(0)} = \frac{\pi}{2k}\overline{\mathcal{L}}\,L_1 + \frac{2\pi}{k}t\overline{\mathcal{W}}\,L_0 - \frac{2\pi}{9k}\overline{\mathcal{W}}\,W_2 + \frac{2\pi}{k}t\overline{\mathcal{L}}\,W_{-1} + \frac{2\pi}{k}t^2\overline{\mathcal{W}}\,W_{-2} \tag{3.54}$$

Finite gauge transformations are given by

$$A \to A' = g^{-1}(A+d)g. \tag{3.55}$$

and lead using $g = 1 + \epsilon$ to infinitesimal gauge transformations

$$\delta_\epsilon A = D\epsilon = d\epsilon + [A, \epsilon]. \tag{3.56}$$

In component form this is given by

$$\delta_\epsilon A_\mu^a = \partial_\mu \epsilon^a + f^a_{\ bc}\ A_\mu^b\ \epsilon^c \tag{3.57}$$

with the gauge parameter $\epsilon = \epsilon^a T_a$. To find the asymptotic symmetry algebra we have to find the boundary preserving gauge transformations that fulfill

$$\delta_\epsilon A_\mu^a = \partial_\mu \epsilon^a + f^a_{\ bc} A_\mu^b \epsilon^c = O(A_\mu^{(0)} + A_\mu^{(1)})^a. \tag{3.58}$$

We split now the gauge parameter

$$\epsilon = b^{-1}(\epsilon^{(0)} + \epsilon^{(1)})b \tag{3.59}$$

were $\epsilon^{(0)}$ is ρ-independent and $\epsilon^{(1)}$ is subleading.

The $\delta_\epsilon A_\rho^a$ equation of (3.58) is by construction fulfilled. So the remaining equations we have to fulfill are given by

$$\partial_i \epsilon^{(0)a} + f^a_{\ bc}(\hat{a}_i^{(0)} + a_i^{(0)})^b \epsilon^{(0)c} = O(a_i^{(0)})^a \quad \text{for } i = (t, x). \tag{3.60}$$

The strict BC are preserved by gauge transformations of the form

$$\epsilon = b^{-1}\left(\epsilon^{(0)} + o\left(1\right)\right)\, b \tag{3.61}$$
$$\bar{\epsilon} = b \quad \left(\bar{\epsilon}^{(0)} + o\left(1\right)\right)\, b^{-1}. \tag{3.62}$$

We define $\epsilon^{(0)} = \varepsilon^{T_a} T_a$, $\bar{\epsilon}^{(0)} = \bar{\varepsilon}^{T_a} T_a$ and the two free functions $\epsilon_{\mathcal{L}} = \epsilon_{\mathcal{L}}(x)$ and $\epsilon_{\mathcal{W}} = \epsilon_{\mathcal{W}}(x)$ to get the following boundary preserving gauge transformations for the unbarred sector

$$\begin{aligned}
\varepsilon^{L_1} &= \epsilon_{\mathcal{L}} - \frac{\pi}{3k} t\mathcal{L}'\epsilon_{\mathcal{W}} - \frac{5\pi}{6k} t\mathcal{L}\epsilon'_{\mathcal{W}} - \frac{1}{6} t\epsilon'''_{\mathcal{W}} \\
\varepsilon^{L_0} &= -\frac{8\pi}{9k} t\mathcal{W}\epsilon_{\mathcal{L}} - \epsilon'_{\mathcal{L}} + \frac{\pi^2}{k^2} t\mathcal{L}^2\epsilon_{\mathcal{W}} + \frac{\pi}{3k} t\mathcal{L}''\epsilon_{\mathcal{W}} + \frac{7\pi}{6k} t\mathcal{L}'\epsilon'_{\mathcal{W}} + \frac{4\pi}{3k} t\mathcal{L}\epsilon''_{\mathcal{W}} + \frac{1}{6} t\epsilon''''_{\mathcal{W}} \\
\varepsilon^{L_{-1}} &= \frac{\pi}{2k}\mathcal{L}\epsilon_{\mathcal{L}} + \frac{1}{2}\epsilon''_{\mathcal{L}} + \frac{\pi}{k}\mathcal{W}\epsilon_{\mathcal{W}} \\
\varepsilon^{W_2} &= \frac{32\pi}{81k} t^2\mathcal{W}\epsilon_{\mathcal{L}} + \frac{8}{9} t\epsilon'_{\mathcal{L}} + \epsilon_{\mathcal{W}} - \frac{4\pi^2}{9k^2} t^2\mathcal{L}^2\epsilon_{\mathcal{W}} \\
&\qquad - \frac{4\pi}{27k} t^2\mathcal{L}''\epsilon_{\mathcal{W}} - \frac{14\pi}{27k} t^2\mathcal{L}'\epsilon'_{\mathcal{W}} - \frac{16\pi}{27k} t^2\mathcal{L}\epsilon''_{\mathcal{W}} - \frac{2}{27} t^2\epsilon''''_{\mathcal{W}} \\
\varepsilon^{W_1} &= -\frac{8\pi}{9k} t\mathcal{L}\epsilon_{\mathcal{L}} - \frac{16\pi}{9k} t\mathcal{W}\epsilon_{\mathcal{W}} - \epsilon'_{\mathcal{W}} - \frac{8t}{9}\epsilon''_{\mathcal{L}} \\
\varepsilon^{W_0} &= \frac{\pi}{k}\mathcal{L}\epsilon_{\mathcal{W}} + \frac{1}{2}\epsilon''_{\mathcal{W}} \\
\varepsilon^{W_{-1}} &= -\frac{\pi}{3k}\mathcal{L}'\epsilon_{\mathcal{W}} - \frac{5\pi}{6k}\mathcal{L}\epsilon'_{\mathcal{W}} - \frac{1}{6}\epsilon'''_{\mathcal{W}} \\
\varepsilon^{W_{-2}} &= -\frac{2\pi}{9k}\mathcal{W}\epsilon_{\mathcal{L}} + \frac{\pi^2}{4k^2}\mathcal{L}^2\epsilon_{\mathcal{W}} + \frac{\pi}{12k}\mathcal{L}''\epsilon_{\mathcal{W}} + \frac{7\pi}{24k}\mathcal{L}'\epsilon'_{\mathcal{W}} + \frac{\pi}{3k}\mathcal{L}\epsilon''_{\mathcal{W}} + \frac{1}{24}\epsilon''''_{\mathcal{W}}.
\end{aligned} \tag{3.63}$$

For the barred sector we have the two free functions $\bar{\epsilon}_{\mathcal{L}} = \bar{\epsilon}_{\mathcal{L}}(x)$ and $\bar{\epsilon}_{\mathcal{W}} = \bar{\epsilon}_{\mathcal{W}}(x)$ and

$$\begin{aligned}
\bar{\varepsilon}^{L_1} &= \frac{\pi}{2k}\bar{\mathcal{L}}\bar{\epsilon}_{\mathcal{L}} + \frac{1}{2}\bar{\epsilon}''_{\mathcal{L}} + \frac{\pi}{k}\overline{\mathcal{W}}\bar{\epsilon}_{\mathcal{W}} \\
\bar{\varepsilon}^{L_0} &= \frac{2\pi}{k} t\overline{\mathcal{W}}\bar{\epsilon}_{\mathcal{L}} + \bar{\epsilon}'_{\mathcal{L}} - \frac{9\pi^2}{4k^2} t\bar{\mathcal{L}}^2\bar{\epsilon}_{\mathcal{W}} - \frac{3\pi}{4k} t\bar{\mathcal{L}}''\bar{\epsilon}_{\mathcal{W}} - \frac{21\pi}{8k} t\bar{\mathcal{L}}'\bar{\epsilon}'_{\mathcal{W}} - \frac{3\pi}{k} t\bar{\mathcal{L}}\bar{\epsilon}''_{\mathcal{W}} - \frac{3}{8} t\bar{\epsilon}''''_{\mathcal{W}} \\
\bar{\varepsilon}^{L_{-1}} &= \bar{\epsilon}_{\mathcal{L}} - \frac{3\pi}{4k} t\bar{\mathcal{L}}'\bar{\epsilon}_{\mathcal{W}} - \frac{15\pi}{8k} t\bar{\mathcal{L}}\bar{\epsilon}'_{\mathcal{W}} - \frac{3}{8} t\bar{\epsilon}'''_{\mathcal{W}} \\
\bar{\varepsilon}^{W_2} &= -\frac{2\pi}{9k}\overline{\mathcal{W}}\bar{\epsilon}_{\mathcal{L}} + \frac{\pi^2}{4k^2}\bar{\mathcal{L}}^2\bar{\epsilon}_{\mathcal{W}} + \frac{\pi}{12k}\bar{\mathcal{L}}''\bar{\epsilon}_{\mathcal{W}} + \frac{7\pi}{24k}\bar{\mathcal{L}}'\bar{\epsilon}'_{\mathcal{W}} + \frac{\pi}{3k}\bar{\mathcal{L}}\bar{\epsilon}''_{\mathcal{W}} + \frac{1}{24}\bar{\epsilon}''''_{\mathcal{W}} \\
\bar{\varepsilon}^{W_1} &= \frac{\pi}{3k}\bar{\epsilon}_{\mathcal{W}}\bar{\mathcal{L}}' + \frac{5\pi}{6k}\bar{\mathcal{L}}\bar{\epsilon}'_{\mathcal{W}} + \frac{1}{6}\bar{\epsilon}'''_{\mathcal{W}} \\
\bar{\varepsilon}^{W_0} &= \frac{\pi}{k}\bar{\mathcal{L}}\bar{\epsilon}_{\mathcal{W}} + \frac{1}{2}\bar{\epsilon}''_{\mathcal{W}} \\
\bar{\varepsilon}^{W_{-1}} &= \frac{2\pi}{k} t\bar{\mathcal{L}}\bar{\epsilon}_{\mathcal{L}} + 2t\bar{\epsilon}''_{\mathcal{L}} + \frac{4\pi}{k} t\overline{\mathcal{W}}\bar{\epsilon}_{\mathcal{W}} + \bar{\epsilon}'_{\mathcal{W}}
\end{aligned}$$

$$\bar{\varepsilon}^{W_{-2}} = \frac{2\pi}{k}t^2\overline{\mathcal{W}}\bar{\epsilon}_{\mathcal{L}} + 2t\bar{\epsilon}'_{\mathcal{L}} + \bar{\epsilon}_{\mathcal{W}} - \frac{9\pi^2}{4k^2}t^2\overline{\mathcal{L}}^2\bar{\epsilon}_{\mathcal{W}} - \frac{3\pi}{4k}t^2\overline{\mathcal{L}}''\bar{\epsilon}_{\mathcal{W}} - \frac{21\pi}{8k}t^2\overline{\mathcal{L}}'\bar{\epsilon}'_{\mathcal{W}} - \frac{3\pi}{k}t^2\overline{\mathcal{L}}\bar{\epsilon}''_{\mathcal{W}} - \frac{3}{8}t^2\bar{\epsilon}''''_{\mathcal{W}}. \tag{3.64}$$

The variation of the canonical asymptotic charges are in general given by [40][11] (for the canonical analysis of Chern-Simons theories see appendix C)

$$\delta\mathcal{Q} = -\frac{k}{2\pi}\oint \mathrm{Tr}(\epsilon^{(0)}\delta a_x^{(0)}dx) \tag{3.65}$$

$$\delta\overline{\mathcal{Q}} = -\frac{k}{2\pi}\oint \mathrm{Tr}(\bar{\epsilon}^{(0)}\delta\bar{a}_x^{(0)}dx). \tag{3.66}$$

This surface term needs to be added to the gauge generators to guarantee functional differentiability. Since $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ do not vanish weakly they are improper gauge transformations [77]. This means they are global symmetries which transform physically distinguishable states into each other. This is in contrast to gauge generators which vanish weakly and transform between physically equivalent configurations.

In the case of the strict BC the nontrivial, state-dependent, finite, conserved and integrable charges are given by

$$\mathcal{Q} = \oint dx\,(\mathcal{L}\epsilon_{\mathcal{L}} + \mathcal{W}\epsilon_{\mathcal{W}}) \tag{3.67}$$

$$\overline{\mathcal{Q}} = \oint dx\,\left(\overline{\mathcal{L}}\bar{\epsilon}_{\mathcal{L}} + \overline{\mathcal{W}}\bar{\epsilon}_{\mathcal{W}}\right). \tag{3.68}$$

The variations are

$$\delta_{\epsilon_{\mathcal{L}}}\mathcal{L} = \mathcal{L}'\epsilon_{\mathcal{L}} + 2\mathcal{L}\epsilon'_{\mathcal{L}} + \frac{k}{\pi}\epsilon'''_{\mathcal{L}} \tag{3.69a}$$

$$\delta_{\epsilon_{\mathcal{L}}}\mathcal{W} = \mathcal{W}'\epsilon_{\mathcal{L}} + 3\mathcal{W}\epsilon'_{\mathcal{L}} \tag{3.69b}$$

$$\delta_{\epsilon_{\mathcal{W}}}\mathcal{L} = 2\mathcal{W}'\epsilon_{\mathcal{W}} + 3\mathcal{W}\epsilon'_{\mathcal{W}} \tag{3.69c}$$

$$\delta_{\epsilon_{\mathcal{W}}}\mathcal{W} = -\chi\left[\left(\frac{16\pi}{k}\mathcal{L}\mathcal{L}' + 2\mathcal{L}'''\right)\epsilon_{\mathcal{W}} + \left(\frac{16\pi}{k}\mathcal{L}^2 + 9\mathcal{L}''\right)\epsilon'_{\mathcal{W}} + 15\mathcal{L}'\epsilon''_{\mathcal{W}} + 10\mathcal{L}\epsilon'''_{\mathcal{W}} + \frac{k}{\pi}\epsilon^{(5)}_{\mathcal{W}}\right], \tag{3.69d}$$

and identically, in the barred sector with $\epsilon \to \bar{\epsilon}$, $\mathcal{L} \to \overline{\mathcal{L}}$ and $\mathcal{W} \to \overline{\mathcal{W}}$. Even though $\chi = \frac{3}{16}$ in our current analysis it is without loss of generality to define it as $\chi \in \mathbb{R}_{>0}$.

The variations (3.69) are equivalent to the variations for asymptotically AdS $sl(3,\mathbb{R})$ HS-gravity, which makes the further analysis equivalent to Campoleoni et

[11] Please denote that we follow the conventions of [40]. The gauge generators have a relative minus sign with respect to the ones given in appendix C

al. [43, Section 4.2][12] and Henneaux and Rey [45]. The shortcut (see e.g. [43, 45])

$$\delta_\epsilon Q[\epsilon_\bullet] = \{Q[\epsilon], Q[\epsilon_\bullet]\} \tag{3.70}$$

$$\delta_\epsilon \bullet = \{Q[\epsilon], \bullet\} \tag{3.71}$$

leads to the Poisson structure of (3.69), which is that of a classical $\mathcal{W}_3$-algebra with a classical central charge of $c = 24k$. Using (2.4) we find that

$$c = 24k = \frac{3l}{2G_N} \tag{3.72}$$

which shows that our c is equal to the one of pure gravity [39]. In terms of Fourier modes the $\mathcal{W}_3$-algebra is given by [43, 45]

$$i\{\mathcal{L}_p, \mathcal{L}_q\} = (p-q)\,\mathcal{L}_{p+q} + \frac{c}{12}(p^3 - p)\,\delta_{p+q,0}\,, \tag{3.73a}$$

$$i\{\mathcal{L}_p, \mathcal{W}_q\} = (2p-q)\,\mathcal{W}_{p+q}\,, \tag{3.73b}$$

$$\begin{aligned} i\{\mathcal{W}_p, \mathcal{W}_q\} = \chi\Big[&(p-q)(2p^2+2q^2-pq-8)\,\mathcal{L}_{p+q} + \frac{96}{c}(p-q)\,\Lambda_{p+q} \\ &+ \frac{c}{12}\,p(p^2-1)(p^2-4)\,\delta_{p+q,0}\Big]\,, \end{aligned} \tag{3.73c}$$

where

$$\Lambda_p \equiv \sum_{q\in\mathbb{Z}} \mathcal{L}_{p-q}\mathcal{L}_q\,. \tag{3.74}$$

The quantum version of this algebra [78, 79] for, e.g., $\chi = \frac{1}{30}$ is given when we substitute

$$i\{\ ,\ \} \to [\ ,\ ] \tag{3.75}$$

$$\frac{16}{5c}(p-q)\,\Lambda_{p+q} \to \frac{16}{22+5c}(p-q)\,\Lambda_{p+q} \tag{3.76}$$

$$\Lambda_p \to \Lambda_p - \frac{3}{10}(p+3)(p+2)\mathcal{L}_p. \tag{3.77}$$

3.7 Loose boundary conditions

A second set of boundary preserving gauge transformations was found. For the looser boundary conditions we set l^0, w^2 of equation (3.51) and $\bar{l}^{\,0}$,$\overline{w}^{-2}$ of equation (3.52)

[12] It is equal for $\chi = -\frac{\sigma}{3}$. Note that different normalization of the Cartan–Killing metric leads to a rescaled k.

to zero. We rescale and rename again with hindsight and omit the x dependence:

$$a_x^{(0)} = \frac{8\pi}{9k} t\mathcal{W}_{-2} L_0 - \frac{\pi}{2k}\mathcal{L}_{-1} L_{-1} - \frac{32\pi}{81k} t^2 \mathcal{W}_{-2} W_2 + \left(\frac{8\pi}{9k} t\mathcal{L}_{-1} - \frac{8\pi}{9k}\mathcal{W}_1\right) W_1 + \frac{4\pi}{3k}\mathcal{W}_0 W_0 + \frac{2\pi}{9k}\mathcal{W}_{-2} W_{-2} \tag{3.78}$$

$$\bar{a}_x^{(0)} = -\frac{\pi}{2k}\bar{\mathcal{L}}_1 L_1 - \frac{2\pi}{k} t\overline{\mathcal{W}}_2 L_0 + \frac{2\pi}{9k}\overline{\mathcal{W}}_2 W_2 + \frac{4\pi}{3k}\overline{\mathcal{W}}_0 W_0 + \left(-\frac{2\pi}{k} t\bar{\mathcal{L}}_1 - \frac{8\pi}{9k}\overline{\mathcal{W}}_{-1}\right) W_{-1} - \frac{2\pi}{k} t^2 \overline{\mathcal{W}}_2 W_{-2} \tag{3.79}$$

This boundary conditions can be combined with the boundary conditions of section 3.6 to get mixed boundary conditions for the barred or unbarred sector.

Once again it is possible to determine the boundary condition preserving gauge transformations ϵ and $\bar{\epsilon}$, leading to the finite, integrable, conserved charges

$$\mathcal{Q} = -\oint dx \left(\mathcal{L}_{-1}\epsilon_{\mathcal{L}_{-1}} + \mathcal{W}_0 \epsilon_{\mathcal{W}_0} + \mathcal{W}_1 \epsilon_{W_1} + \mathcal{W}_{-2}\epsilon_{\mathcal{W}_{-2}}\right) \tag{3.80a}$$

$$\overline{\mathcal{Q}} = -\oint dx \left(\bar{\mathcal{L}}_1 \bar{\epsilon}_{\mathcal{L}_1} + \overline{\mathcal{W}}_2 \bar{\epsilon}_{\mathcal{W}_2} + \overline{\mathcal{W}}_0 \bar{\epsilon}_{\mathcal{W}_0} + \overline{\mathcal{W}}_{-1}\bar{\epsilon}_{\mathcal{W}_{-1}}\right). \tag{3.80b}$$

The variations of the asymptotic charges are given by

$$\begin{aligned} \delta_{\epsilon_{\mathcal{L}_{-1}}}\mathcal{L}_{-1} = &\left(\mathcal{L}'_{-1} + \frac{16\pi^3}{9k^3}\mathcal{W}_0\mathcal{W}_1^2\mathcal{W}'_0 + \frac{16\pi^3}{9k^3}\mathcal{W}_0^2\mathcal{W}_1\mathcal{W}'_1\right)\epsilon_{\mathcal{L}_{-1}} \\ &+ \left(-\frac{8\pi}{3k}\mathcal{W}'_0\mathcal{W}'_1 - \frac{4\pi}{3k}\mathcal{W}_1\mathcal{W}''_0 - \frac{4\pi}{3k}\mathcal{W}_0\mathcal{W}''_1\right)\epsilon_{\mathcal{L}_{-1}} \\ &+ \left(2\mathcal{L}_{-1} + \frac{16\pi^3}{9k^3}\mathcal{W}_0^2\mathcal{W}_1^2 - \frac{8\pi}{3k}\mathcal{W}_1\mathcal{W}'_0 - \frac{8\pi}{3k}\mathcal{W}_0\mathcal{W}'_1\right)\epsilon'_{\mathcal{L}_{-1}} - \frac{k}{\pi}\epsilon^{(3)}_{\mathcal{L}_{-1}} \end{aligned} \tag{3.80c}$$

$$\delta_{\epsilon_{\mathcal{L}_{-1}}}\mathcal{W}_1 = \left(3\mathcal{W}_0 - \frac{4\pi^2}{3k^2}\mathcal{W}_0\mathcal{W}_1^2 + \mathcal{W}'_1\right)\epsilon_{\mathcal{L}_{-1}} \tag{3.80d}$$

$$\delta_{\epsilon_{\mathcal{L}_{-1}}}\mathcal{W}_0 = -\left(\frac{4\pi^2}{3k^2}\mathcal{W}_1^2\mathcal{W}'_0 + \frac{4\pi^2}{3k^2}\mathcal{W}_0\mathcal{W}_1\mathcal{W}'_1\right)\epsilon_{\mathcal{L}_{-1}} - \frac{4\pi^2}{3k^2}\mathcal{W}_0\mathcal{W}_1^2\epsilon'_{\mathcal{L}_{-1}} - \mathcal{W}_1\epsilon''_{\mathcal{L}_{-1}} \tag{3.80e}$$

$$\begin{aligned} \delta_{\epsilon_{\mathcal{L}_{-1}}}\mathcal{W}_{-2} = &\left(\frac{8\pi^2}{3k^2}\mathcal{W}_0\mathcal{W}_1\mathcal{W}_{-2} + \frac{3\pi}{2k}\mathcal{W}_0\mathcal{L}'_{-1} + \frac{3\pi}{2k}\mathcal{L}_{-1}\mathcal{W}'_0 - \frac{2\pi^2}{k^2}\mathcal{W}_1\mathcal{W}_0'^2\right)\epsilon_{\mathcal{L}_{-1}} \\ &- \left(\frac{6\pi^2}{k^2}\mathcal{W}_0\mathcal{W}'_0\mathcal{W}'_1 - \mathcal{W}'_{-2} + \frac{2\pi^2}{k^2}\mathcal{W}_0\mathcal{W}_1\mathcal{W}''_0 + \frac{2\pi^2}{k^2}\mathcal{W}_0^2\mathcal{W}''_1\right)\epsilon_{\mathcal{L}_{-1}} \\ &+ \left(\frac{3\pi}{2k}\mathcal{L}_{-1}\mathcal{W}_0 + 3\mathcal{W}_{-2} - \frac{6\pi^2}{k^2}\mathcal{W}_0\mathcal{W}_1\mathcal{W}'_0 - \frac{4\pi^2}{k^2}\mathcal{W}_0^2\mathcal{W}'_1\right)\epsilon'_{\mathcal{L}_{-1}} \\ &+ \left(-\frac{2\pi^2}{k^2}\mathcal{W}_0^2\mathcal{W}_1 - \frac{3}{2}\mathcal{W}'_0\right)\epsilon''_{\mathcal{L}_{-1}} - \frac{3}{2}\mathcal{W}_0{\epsilon_{\mathcal{L}_{-1}}}^{(3)} \end{aligned} \tag{3.80f}$$

$$\delta_{\epsilon_{\mathcal{W}_1}}\mathcal{L}_{-1} = \left(-3\mathcal{W}_0 + \frac{4\pi^2}{3k^2}\mathcal{W}_0\mathcal{W}_1^2 - \mathcal{W}_1'\right)\epsilon_{\mathcal{W}_1} \tag{3.80g}$$

$$\delta_{\epsilon_{\mathcal{W}_1}}\mathcal{W}_1 = 0 \tag{3.80h}$$

$$\delta_{\epsilon_{\mathcal{W}_1}}\mathcal{W}_0 = \left(\frac{9k}{4\pi} - \frac{\pi}{k}\mathcal{W}_1^2\right)\epsilon_{\mathcal{W}_1} \tag{3.80i}$$

$$\delta_{\epsilon_{\mathcal{W}_1}}\mathcal{W}_{-2} = \left(\frac{9}{4}\mathcal{L}_{-1} - \frac{3\pi}{2k}\mathcal{W}_1\mathcal{W}_0' - \frac{3\pi}{2k}\mathcal{W}_0\mathcal{W}_1'\right)\epsilon_{\mathcal{W}_1} - \frac{3\pi}{2k}\mathcal{W}_0\mathcal{W}_1\epsilon_{\mathcal{W}_1}' - \frac{9k}{8\pi}\epsilon_{\mathcal{W}_1}'' \tag{3.80j}$$

$$\begin{aligned}\delta_{\epsilon_{\mathcal{W}_0}}\mathcal{L}_{-1} = &\left(-\frac{4\pi^2}{3k^2}\mathcal{W}_0\mathcal{W}_1\mathcal{W}_1' + \mathcal{W}_1''\right)\epsilon_{\mathcal{W}_0}\\ &+ \left(-\frac{4\pi^2}{3k^2}\mathcal{W}_0\mathcal{W}_1^2 + 2\mathcal{W}_1'\right)\epsilon_{\mathcal{W}_0}' + \mathcal{W}_1\epsilon_{\mathcal{W}_0}''\end{aligned} \tag{3.80k}$$

$$\delta_{\epsilon_{\mathcal{W}_0}}\mathcal{W}_1 = \left(-\frac{9k}{4\pi} + \frac{\pi}{k}\mathcal{W}_1^2\right)\epsilon_{\mathcal{W}_0} \tag{3.80l}$$

$$\delta_{\epsilon_{\mathcal{W}_0}}\mathcal{W}_0 = \frac{\pi}{k}\mathcal{W}_1\mathcal{W}_1'\epsilon_{\mathcal{W}_0} + \left(\frac{3k}{4\pi} + \frac{\pi}{k}\mathcal{W}_1^2\right)\epsilon_{\mathcal{W}_0}' \tag{3.80m}$$

$$\begin{aligned}\delta_{\epsilon_{\mathcal{W}_0}}\mathcal{W}_{-2} = &\left(-\frac{2\pi}{k}\mathcal{W}_1\mathcal{W}_{-2} - \frac{9}{8}\mathcal{L}_{-1}' + \frac{3\pi}{2k}\mathcal{W}_0'\mathcal{W}_1' + \frac{3\pi}{2k}\mathcal{W}_0\mathcal{W}_1''\right)\epsilon_{\mathcal{W}_0}\\ &- \left(\frac{9}{8}\mathcal{L}_{-1} - \frac{3\pi}{2k}\mathcal{W}_1\mathcal{W}_0' - \frac{3\pi}{k}\mathcal{W}_0\mathcal{W}_1'\right)\epsilon_{\mathcal{W}_0}' + \frac{3\pi}{2k}\mathcal{W}_0\mathcal{W}_1\epsilon_{\mathcal{W}_0}''\end{aligned} \tag{3.80n}$$

$$\begin{aligned}\delta_{\epsilon_{\mathcal{W}_{-2}}}\mathcal{L}_{-1} = &\left(-\frac{8\pi^2}{3k^2}\mathcal{W}_0\mathcal{W}_1\mathcal{W}_{-2} + 2\mathcal{W}_{-2}'\right)\epsilon_{\mathcal{W}_{-2}}\\ &+ \left(\frac{3\pi}{2k}\mathcal{L}_{-1}\mathcal{W}_0 + 3\mathcal{W}_{-2} + \frac{2\pi^2}{k^2}\mathcal{W}_0\mathcal{W}_1\mathcal{W}_0' - \frac{3}{2}\mathcal{W}_0''\right)\epsilon_{\mathcal{W}_{-2}}'\\ &+ \left(\frac{2\pi^2}{k^2}\mathcal{W}_0^2\mathcal{W}_1 - 3\mathcal{W}_0'\right)\epsilon_{\mathcal{W}_{-2}}'' - \frac{3}{2}\mathcal{W}_0\epsilon_{\mathcal{W}_{-2}}^{(3)}\end{aligned} \tag{3.80o}$$

$$\delta_{\epsilon_{\mathcal{W}_{-2}}}\mathcal{W}_1 = -\frac{9}{4}\mathcal{L}_{-1}\epsilon_{\mathcal{W}_{-2}} - \frac{3\pi}{2k}\mathcal{W}_0\mathcal{W}_1\epsilon_{\mathcal{W}_{-2}}' + \frac{9k}{8\pi}\epsilon_{\mathcal{W}_{-2}}'' \tag{3.80p}$$

$$\delta_{\epsilon_{\mathcal{W}_{-2}}}\mathcal{W}_0 = \frac{2\pi}{k}\mathcal{W}_1\mathcal{W}_{-2}\epsilon_{\mathcal{W}_{-2}} - \left(\frac{9}{8}\mathcal{L}_{-1} + \frac{3\pi}{2k}\mathcal{W}_1\mathcal{W}_0'\right)\epsilon_{\mathcal{W}_{-2}}' - \frac{3\pi}{2k}\mathcal{W}_0\mathcal{W}_1\epsilon_{\mathcal{W}_{-2}}'' \tag{3.80q}$$

$$\begin{aligned}\delta_{\epsilon_{\mathcal{W}_{-2}}}\mathcal{W}_{-2} = &\left(\frac{3\pi}{k}\mathcal{W}_{-2}\mathcal{W}_0' + \frac{3\pi}{k}\mathcal{W}_0\mathcal{W}_{-2}'\right)\epsilon_{\mathcal{W}_{-2}}\\ &+ \left(\frac{6\pi}{k}\mathcal{W}_0\mathcal{W}_{-2} - \frac{9\pi}{4k}\mathcal{W}_0'^2 - \frac{9\pi}{4k}\mathcal{W}_0\mathcal{W}_0''\right)\epsilon_{\mathcal{W}_{-2}}'\\ &- \frac{27\pi}{4k}\mathcal{W}_0\mathcal{W}_0'\epsilon_{\mathcal{W}_{-2}}'' - \frac{9\pi}{4k}\mathcal{W}_0^2\epsilon_{\mathcal{W}_{-2}}^{(3)}\end{aligned} \tag{3.80r}$$

and similar expressions for the barred sector. This algebra is still work in progress but we suspect that a suitable charge redefinition could lead to an asymptotic symmetry algebra $\mathcal{W}_3^{(2)} \times \mathcal{W}_3^{(2)}$.

4. Conclusion

We studied the asymptotic symmetries of Lifshitz spacetimes in $2+1$ dimensions with an anisotropic scaling of $z=2$ in $sl(3,\mathbb{R}) \oplus sl(3,\mathbb{R})$ higher spin gravity. Due to technical simplicity the Chern-Simons formulation has been used but the analysis is equivalent to Einstein-Hilbert gravity with a negative cosmological constant coupled to a spin 3 field.

Two different boundary conditions which fulfill a well defined variational principle and lead to conserved charges were presented.

The stricter boundary conditions lead to a $\mathcal{W}_3 \times \mathcal{W}_3$ algebra with a central charge which is equivalent to the one of pure gravity found by Brown-Henneaux [39] and asymptotically AdS in $sl(N,\mathbb{R}) \oplus sl(N,\mathbb{R})$ higher spin gravity [43–45]. Since the asymptotic symmetry algebra is equivalent to the one of $sl(3,\mathbb{R})$ higher spin gravity in asymptotically AdS it seems interesting to explore the exact relationship to the asymptotic Lifshitz theory we proposed.

The second boundary conditions which have been found are less restrictive on the fluctuations. The variations of the asymptotic charges are presented but the associtation with a specific algebra is still work in progress. Based on the structure unravelled so far we conjecture that it is a $\mathcal{W}_3^{(2)} \times \mathcal{W}_3^{(2)}$ algebra, i.e., two copies of the Polyakov–Bershadsky algebra.

An interesting aspect is that the given analysis can be used to derive one half of the asymptotic symmetry group of Schrödinger spacetimes [56] of the same scaling exponent. This is due to the fact that the barred sector of Schrödinger spacetimes is generated similar to the one of Lifshitz.

It will further be interesting to generalize this analysis to more general gauge algebras, i.e., $sl(N,\mathbb{R}) \otimes sl(N,\mathbb{R})$ higher spin gravity and a general scaling exponent.

Another aspect that should be discussed is the relation of this work to the work of Gutperle et al. [73]. It will be interesting to see if the given boundary conditions also admit asymptotically Lifshitz black holes and how they fit in the discussion given by Gutperle et al.

A. Conventions and algebra

A.1 Conventions

Throughout this work Planck units where used:

$$G_N = c = \hbar = k_B = 1. \tag{A.1}$$

If readability could have been improved and there was no risk of confusion one or more of the constants were restored. Furthermore the mostly plus convention for the metric is adopted. The Riemann tensor and Ricci tensor are given by

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \qquad R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}. \tag{A.2}$$

For the symmetrization of tensors denoted by parentheses no normalization factor is intended, for example $T_{(ab)} = T_{ab} + T_{ba}$.

A.2 $sl(3,\mathbb{R})$ algebra

The Lie algebra of $sl(3,\mathbb{R})$ is given by

$$[L_n, L_m] = (n-m)L_{n+m} \tag{A.3a}$$
$$[L_n, W_m] = (2n-m)W_{n+m} \tag{A.3b}$$
$$[W_n, W_m] = f(n,m)L_{n+m} \tag{A.3c}$$

with the non-zero entries of $f(n,m) = -f(m,n)$ given by

$$f(0,1) = -\frac{9}{8} \qquad f(0,-1) = \frac{9}{8} \tag{A.4a}$$
$$f(1,-1) = \frac{9}{8} \qquad f(1,-2) = -\frac{9}{4} \tag{A.4b}$$
$$f(2,-1) = -\frac{9}{4} \qquad f(2,-2) = -9. \tag{A.4c}$$

With the following matrix representation of $sl(3,\mathbb{R})$:

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \end{pmatrix} \qquad L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad L_{-1} = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \tag{A.5a}$$

$$W_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \qquad W_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{3}{2\sqrt{2}} & 0 & 0 \\ 0 & \frac{3}{2\sqrt{2}} & 0 \end{pmatrix} \qquad W_0 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{A.5b}$$

$$W_{-1} = \begin{pmatrix} 0 & \frac{3}{2\sqrt{2}} & 0 \\ 0 & 0 & -\frac{3}{2\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix} \qquad W_{-2} = \begin{pmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.5c}$$

The Cartan–Killing metric is given by

$$g_{ab} = \mathrm{Tr}(T_a T_b) = \left(\begin{array}{c|cccccccc} & L_{-1} & L_0 & L_1 & W_{-2} & W_{-1} & W_0 & W_1 & W_2 \\ \hline L_{-1} & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ L_0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ L_1 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ W_{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \\ W_{-1} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{9}{4} & 0 \\ W_0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 \\ W_1 & 0 & 0 & 0 & 0 & -\frac{9}{4} & 0 & 0 & 0 \\ W_2 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 \end{array}\right) \tag{A.6}$$

where $T = (L_{-1}, L_0, L_1, W_{-2}, \ldots, W_2)$. Using

$$e^{xA} B e^{-xA} = B + [A,B]x + [A,[A,B]]\frac{x^2}{2!} + [A,[A,[A,B]]]\frac{x^3}{3!} + \ldots \tag{A.7}$$

we get the following useful formulas ($b = e^{L_0 \rho}$)

$$b^{-1}\, L_n\, b = L_n\, e^{\rho n} \qquad b\, L_n\, b^{-1} = L_n\, e^{-\rho n} \tag{A.8}$$

$$b^{-1}\, W_n\, b = W_n\, e^{\rho n} \qquad b\, W_n\, b^{-1} = W_n\, e^{-\rho n}. \tag{A.9}$$

A.3 Some algebra definitions

The definitions are taken from [79–81].

Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\tilde{c}}{12} m(m^2-1)\delta_{m+n,0} \quad m,n \in \mathbb{Z} \tag{A.10}$$

Where $\tilde{c}$ is the central charge, i.e., an element of $\mathbb{Z}$ and $[L_n, \tilde{c}] = 0$. With $\tilde{c} = 0$ the algebra is called the Witt algebra. The Witt algebra for $m, n = -1, 0, 1$ is $sl(2, \mathbb{R})$.

Untwisted affine algebra $\hat{\mathfrak{g}}$ (=current algebra) associated with a compact finite-dimensional Lie algebra $\mathfrak{g}$ is defined by

$$[T^a_m, T^b_n] = f^{ab}{}_c T^c_{m+n} + \tilde{k}\, m\, \delta^{ab}\, \delta_{m+n,0} \quad m,n \in \mathbb{Z} \quad a,b,c \in 1 \ldots |\mathfrak{g}| \tag{A.11}$$

where $f^{ab}{}_c$ are structure constants associated with $\mathfrak{g}$. So the generators T^a_0 define a subalgebra of $\hat{\mathfrak{g}}$ which is equivalent to $\mathfrak{g}$. The full algebra is the "affinization" of this finite dimensional subalgebra. $\tilde{k}$ is the central element known as level of the current algebra.

$\hat{\mathfrak{u}}(1)$ is an example of an affine algebra where the associated Lie algebra is the abelian algebra $\mathfrak{u}(1)$

$$[T_m, T_n] = \tilde{k}\, m\, \delta_{m+n,0} \quad m,n \in \mathbb{Z} \tag{A.12}$$

Primary field of weight h_ϕ transforms under $x \to x + \epsilon(x)$ as

$$\delta_\epsilon \phi = h_\phi \epsilon' \phi + \epsilon \phi' \tag{A.13}$$

Algebra of primary field ϕ with conformal dimension or weight h_ϕ ,which is equivalent to the spin and the scaling dimension if $\bar{h}_\phi = 0$, is

$$[L_m, \phi_n] = ((h_\phi - 1)m - n)\phi_{m+n} \quad m, n \in \mathbb{Z}. \tag{A.14}$$

If that only holds for $m = -1, 0, +1$, as it is for example the case for the Virasoro algebra, then the field ϕ is called a $sl(2, \mathbb{R})$ primary.

$\mathcal{W}_3$ (=$\mathcal{W}(2,3)$) consists of an L_n Virasoro algebra and W_n Virasoro primaries of conformal dimension $h_W = 3$ and $[W_m, W_n]$ as given in [79] equation (3.10,11).

B. AdS$_3$ symmetries

For the sake of completeness we look at the case where $z = 1$ i.e., AdS_3. So we take equation (3.4) and set $l = 1$

$$ds^2 = -\frac{dt^2}{r^2} + \frac{dr^2 + dx^2}{r^2} \tag{B.1}$$

We proceed analog to section 3.3 and end up with the analog to equation (3.17)

$$2\partial_x \beta(x,t) + \left(2r^2 - x^2\right)\gamma'(t) = 2\left(x\delta'(t) + \epsilon'(t)\right). \tag{B.2}$$

Since the $\gamma(t)$ term is the only one with a r coefficient it needs to be constant

$$\gamma(t) = e. \tag{B.3}$$

There are two equations left that need to be fulfilled

$$\beta^{(1,0)}(x,t) = x\delta'(t) + \epsilon'(t) \tag{B.4a}$$

$$\beta^{(0,1)}(x,t) = -\frac{r^2}{2}\delta''(t) + \delta(t) + ex. \tag{B.4b}$$

The only term with an r coefficient in equation (B.4b) is the $\delta''(t)$ term which means it has to vanish. This leads to

$$\delta(t) = ft + d \tag{B.5}$$

which then transforms equation (B.4b) to

$$\beta^{(0,1)}(x,t) = ft + d + ex \tag{B.6}$$

which is solved by

$$\beta(x,t) = h(x) + \frac{ft^2}{2} + dt + ext. \tag{B.7}$$

The equation (B.4a) now reads

$$h'(x) + et = fx + \epsilon'(t) \tag{B.8}$$

and is solved by

$$h(x) = \frac{fx^2}{2} + cx + a \tag{B.9}$$

$$\epsilon(t) = \frac{et^2}{2} + ct + b. \tag{B.10}$$

So we have generator of time translation H, spatial translation P, boost L, dilatation D and the 2 special conformal transformations K_1 and K_2

$$H = \partial_t \qquad\qquad P = \partial_x \tag{B.11}$$

$$L = x\partial_t + t\partial_x \qquad\qquad D = t\partial_t + r\partial_r + x\partial_x \tag{B.12}$$

$$K_1 = \frac{1}{2}(t^2 + r^2 + x^2)\partial_t + tr\partial_r + tx\partial_x \qquad K_2 = tx\partial_t + rx\partial_r + \frac{1}{2}(t^2 - r^2 + x^2)\partial_x \tag{B.13}$$

which generate the algebra $so(2,2)$. With $P_i = (H, P)$ the non commuting elements are

$$[P_i, D] = P_i \tag{B.14}$$

$$[K_i, D] = -K_i \tag{B.15}$$

$$[P_i, K_j] = \delta_{ij} D + |\epsilon_{ij}| L \tag{B.16}$$

$$[P_i, L] = |\epsilon_{ij}| P_j \tag{B.17}$$

$$[K_i, L] = -|\epsilon_{ij}| K_j \tag{B.18}$$

This algebra is isomorphic to $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$ with the explicit isomorphism

$$L_1 = \frac{1}{\sqrt{2}}(H + P) \qquad\qquad \bar{L}_1 = \frac{1}{\sqrt{2}}(H - P) \tag{B.19}$$

$$L_0 = \frac{1}{2}(D + L) \qquad\qquad \bar{L}_0 = \frac{1}{2}(D - L) \tag{B.20}$$

$$L_{-1} = \frac{1}{\sqrt{2}}(K_1 + K_2) \qquad\qquad \bar{L}_{-1} = \frac{1}{\sqrt{2}}(K_1 - K_2) \tag{B.21}$$

with

$$[L_n, L_m] = (n - m)L_{n+m} \qquad\qquad \left[\bar{L}_n, \bar{L}_m\right] = (n - m)\bar{L}_{n+m} \tag{B.22}$$

$$[L_n, \bar{L}_m] = 0 \tag{B.23}$$

Since $so(2,1) \simeq sl(2,\mathbb{R})$ with the metric $\eta_{\mu\nu} = (-1, 1, 1)$ which also lowers and raises indices and $\epsilon_{012} = 1$ we have the isomorphism

$$J_0 = \frac{1}{2}(L_1 + L_{-1}) \tag{B.24}$$

$$J_1 = \frac{1}{2}(-L_1 + L_{-1}) \qquad\qquad [J_\mu, J_\nu] = \epsilon_{\mu\nu\rho} J^\rho \tag{B.25}$$

$$J_2 = L_0 \tag{B.26}$$

So we have the isomorphic algebras $so(2,2) \simeq sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R}) \simeq so(2,1) \oplus so(2,1)$.

C. Review of canonical analysis

For the readers convenience this appendix is copied from [57], with one correction before equation (C.7).

In order to proceed with the canonical analysis it is convenient to use a $2+1$ decomposition of the action (2.3) [40, 54].

$$I_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int_{\mathbb{R}} \mathrm{d}t \int_{\mathcal{D}} \mathrm{d}^2 x \, \epsilon^{ij} g_{ab} \left(\dot{A}_i^a A_j^b + A_0^a F_{ij}^b \right) , \tag{C.1}$$

with $F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + f^a{}_{bc} A_i^b A_j^c$, $A = A^a T_a$, $g_{ab} = \mathrm{Tr}(T_a T_b)$, $[T_a, T_b] = f^c{}_{ab} T_c$, $\epsilon^{ij} = \epsilon^{tij}$, dot denotes ∂_t, and we dropped boundary terms. Calculating the canonical momenta $\pi_a^\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu^a}$ corresponding to the canonical variables A_μ^a generates primary constraints ϕ_a^μ.

$$\phi_a^0 := \pi_a^0 \approx 0 \qquad \phi_a^i := \pi_a^i - \frac{k}{4\pi} \epsilon^{ij} g_{ab} A_j^b \approx 0 \tag{C.2}$$

The Poisson bracket has its canonical form, $\{A_\mu^a(\mathbf{x}), \pi_b^\nu(\mathbf{y})\} = \delta_b^a \, \delta_\mu^\nu \, \delta^2(\mathbf{x}-\mathbf{y})$. The canonical Hamiltonian density, up to boundary terms, is given by

$$\mathcal{H} = -\frac{k}{4\pi} \epsilon^{ij} g_{ab} A_0^a F_{ij}^b \, . \tag{C.3}$$

The total Hamiltonian is then given as $\mathcal{H}_T = \mathcal{H} + u_\mu^a \phi_a^\mu$, where u_μ^a are Lagrange multipliers. Conservation of the primary constraints, $\dot{\phi}_a^\mu = \{\phi_a^\mu, \mathcal{H}_T\} \approx 0$, leads to the following secondary constraints

$$\mathcal{K}_a \equiv -\frac{k}{4\pi} \epsilon^{ij} g_{ab} F_{ij}^b \approx 0 \, , \qquad D_i A_0^a - u_i^a \approx 0 \, , \tag{C.4}$$

with the covariant derivative $D_i X^a = \partial_i X^a + f^a{}_{bc} A_i^b X^c$. Defining $\bar{\mathcal{K}}_a = \mathcal{K}_a - D_i \phi_a^i$ the total Hamiltonian can be expressed as a sum over constraints.

$$\mathcal{H}_T = A_0^a \, \bar{\mathcal{K}}_a + u_0^a \, \phi_a^0 \tag{C.5}$$

The non-vanishing Poisson brackets between the constraints lead to the following algebra.

$$\{\phi_a^i(\mathbf{x}), \phi_b^j(\mathbf{y})\} = -\frac{k}{2\pi} \epsilon^{ij} g_{ab} \, \delta^2(\mathbf{x}-\mathbf{y}) \tag{C.6a}$$

$$\{\phi_a^i(\mathbf{x}), \bar{\mathcal{K}}_b(\mathbf{y})\} = -f_{ab}{}^c \phi_c^i \, \delta^2(\mathbf{x}-\mathbf{y}) \tag{C.6b}$$

$$\{\bar{\mathcal{K}}_a(\mathbf{x}), \bar{\mathcal{K}}_b(\mathbf{y})\} = -f_{ab}{}^c \bar{\mathcal{K}}_c \, \delta^2(\mathbf{x}-\mathbf{y}) \tag{C.6c}$$

Thus ϕ_a^0 and $\bar{\mathcal{K}}_a$ are first class constraints and ϕ_a^i are second class constraints. The second class constraints are eliminated by introducing the Dirac bracket (denoted again by $\{,\}$), which turns out to be identical to the Poisson bracket, except for the relation $\{A_i^a(\mathbf{x}), A_j^b(\mathbf{y})\} = \frac{2\pi}{k}\, g^{ab}\epsilon_{ij}\, \delta^2(\mathbf{x}-\mathbf{y})$.

As next step we construct the canonical generators of gauge transformations using Castellani's algorithm. They are given by $G = \dot{\epsilon}(t)G_1 + \epsilon(t)G_0$, where the constraints G_0 and G_1 have to fulfill the relations $G_1 = C_{\text{PFC}}$, $G_0 + \{G_1, \mathcal{H}_T\} = C_{\text{PFC}}$, $\{G_0, \mathcal{H}_T\} = C_{\text{PFC}}$. Here C_{PFC} denotes a primary first class constraint. These relations are fulfilled for $G_1 = \pi_a^0$ and $G_0 = \bar{\mathcal{K}}_a - f_{ab}{}^c A_0^b \pi_c^0$. The smeared generator of gauge transformations then reads

$$\bar{\mathcal{G}}[\epsilon] = \int_{\mathcal{D}} \mathrm{d}^2 x \left(D_0 \epsilon^a \pi_a^0 + \epsilon^a \bar{\mathcal{K}}_a \right) . \tag{C.7}$$

The generator $\bar{\mathcal{G}}$ is not yet functionally differentiable.

$$\delta\bar{\mathcal{G}}[\epsilon] = \text{regular} - \int_{\mathcal{D}} \mathrm{d}^2 x\, \partial_i \Big(\frac{k}{4\pi}\, \epsilon^{ij} g_{ab} \epsilon^a\, \delta A_j^b + \epsilon^a\, \delta\pi_a^i \Big) \tag{C.8}$$

The first term is the bulk variation of the generator (C.7). The second term is a boundary term and spoils functional differentiability. In order to fix this one adds a suitable boundary term $\mathcal{Q}$ to the canonical generator (C.7) such that the variation of this additional boundary term cancels exactly the boundary term in (C.8).

$$\delta\mathcal{G}[\epsilon] = \delta\bar{\mathcal{G}}[\epsilon] + \delta\mathcal{Q}[\epsilon] \tag{C.9}$$

with

$$\delta\mathcal{Q}[\epsilon] = \int_{\mathcal{D}} \mathrm{d}^2 x\, \partial_i \Big(\frac{k}{4\pi}\, \epsilon^{ij} g_{ab} \epsilon^a \delta A_j^b + \epsilon^a \delta\pi_a^i \Big) . \tag{C.10}$$

Using the Stokes theorem and the fact that in the reduced phase space the constraint ϕ_a^i strongly equals to zero, the variation of the boundary charge simplifies to

$$\delta\mathcal{Q}[\epsilon] = \frac{k}{2\pi} \oint_{\partial\mathcal{D}} \mathrm{d}\varphi\, g_{ab} \epsilon^a \delta A_\varphi^b . \tag{C.11}$$

D. Differential forms

This is a short collection of useful definitions and relations concerning differential forms which proved to be useful in the process of this work [55].

A differential form of order p also called p-form is a totally antisymmetric tensor of type $(0,p)$.

The wedge product $\wedge$ of p one-forms is defined by the totally antisymmetric tensor product

$$dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} = \sum_{P \in S_p} sgn(P)\; dx^{\mu_{P(1)}} \otimes dx^{\mu_{P(2)}} \otimes \cdots \otimes dx^{\mu_{P(p)}} \tag{D.1}$$

where P is an element of S_p, the symmetric group of order p and $sgn(P)$ is $+1$ for even and -1 for odd permutations. So we can write a p-form α as

$$\alpha = \frac{1}{p!}\alpha_{\mu_1\mu_2\cdots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} \tag{D.2}$$

which when using the antisymmetry reduces to

$$\alpha = \alpha_{\mu_1\mu_2\cdots\mu_p} dx^{\mu_1} \otimes dx^{\mu_2} \otimes \cdots \otimes dx^{\mu_p}. \tag{D.3}$$

For the p-form α and the b-form β the definition of the exterior product between forms $\wedge$ leads to an $p+b$-form and can be written as

$$\alpha \wedge \beta = \frac{1}{p!b!}\alpha_{\mu_1\mu_2\cdots\mu_p}\beta_{\mu_{p+1}\mu_{p+2}\cdots\mu_{p+b}} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_{p+b}}. \tag{D.4}$$

The exterior derivative d is defined by

$$d\alpha = \frac{1}{p!}\partial_\rho \alpha_{\mu_1\mu_2\cdots\mu_p} dx^\rho \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}. \tag{D.5}$$

Some other useful relations

$$\alpha \wedge \beta = (-1)^{pb}\beta \wedge \alpha \tag{D.6}$$

$$\alpha \wedge \alpha = 0 \quad \text{if } p \text{ is odd} \tag{D.7}$$

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) \tag{D.8}$$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \tag{D.9}$$

$$d^2 = 0 \tag{D.10}$$

References

[1] C. M. Will, "The Confrontation between general relativity and experiment," *Living Rev.Rel.* **9** (2006) 3, gr-qc/0510072.

[2] C. Hoyle, D. Kapner, B. R. Heckel, E. Adelberger, J. Gundlach, *et al.*, "Sub-millimeter tests of the gravitational inverse-square law," *Phys.Rev.* **D70** (2004) 042004, hep-ph/0405262.

[3] A. Zee, *Quantum field theory in a nutshell.* Universities Press, 2005.

[4] R. M. Wald, *General relativity.* University of Chicago press, 2010.

[5] S. Carlip, "Quantum gravity: A Progress report," *Rept.Prog.Phys.* **64** (2001) 885, gr-qc/0108040.

[6] S. Carlip, "Is Quantum Gravity Necessary?," *Class.Quant.Grav.* **25** (2008) 154010, 0803.3456.

[7] D. N. Page and C. Geilker, "Indirect Evidence for Quantum Gravity," *Phys.Rev.Lett.* **47** (1981) 979–982.

[8] R. Penrose, "On gravity's role in quantum state reduction," *Gen.Rel.Grav.* **28** (1996) 581–600.

[9] S. Weinberg, *The quantum theory of fields*, vol. 1. Cambridge university press, 1996.

[10] M. H. Goroff and A. Sagnotti, "The Ultraviolet Behavior of Einstein Gravity," *Nucl.Phys.* **B266** (1986) 709.

[11] J. F. Donoghue, "General relativity as an effective field theory: The leading quantum corrections," *Phys.Rev.* **D50** (1994) 3874–3888, gr-qc/9405057.

[12] N. Bjerrum-Bohr, J. F. Donoghue, and B. R. Holstein, "Quantum gravitational corrections to the nonrelativistic scattering potential of two masses," *Phys.Rev.* **D67** (2003) 084033, hep-th/0211072.

[13] S. Hossenfelder, "Experimental Search for Quantum Gravity," 1010.3420.

[14] C. Rovelli, "Loop quantum gravity," *Living Reviews in Relativity* **11** (2008), no. 5,.

[15] A. W. Peet, "TASI lectures on black holes in string theory," hep-th/0008241.

[16] A. Sen, "Logarithmic Corrections to Schwarzschild and Other Non-extremal Black Hole Entropy in Different Dimensions," 1205.0971.

[17] J. Bekenstein, "Black holes and the second law," *Lett.Nuovo Cim.* **4** (1972) 737–740.

[18] J. D. Bekenstein, "Black holes and entropy," *Phys.Rev.* **D7** (1973) 2333–2346.

[19] J. D. Bekenstein, "Generalized second law of thermodynamics in black hole physics," *Phys.Rev.* **D9** (1974) 3292–3300.

[20] R. Bousso, "The Holographic principle," *Rev.Mod.Phys.* **74** (2002) 825–874, hep-th/0203101.

[21] G. 't Hooft, "Dimensional reduction in quantum gravity," gr-qc/9310026.

[22] L. Susskind, "The World as a hologram," *J.Math.Phys.* **36** (1995) 6377–6396, hep-th/9409089.

[23] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Adv.Theor.Math.Phys.* **2** (1998) 231–252, hep-th/9711200.

[24] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," *Phys.Rept.* **323** (2000) 183–386, hep-th/9905111.

[25] L. Susskind and E. Witten, "The Holographic bound in anti-de Sitter space," hep-th/9805114.

[26] E. Witten, "Anti-de Sitter space and holography," *Adv.Theor.Math.Phys.* **2** (1998) 253–291, hep-th/9802150.

[27] S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," *Phys.Lett.* **B428** (1998) 105–114, hep-th/9802109.

[28] C. Fronsdal, "Massless Fields with Integer Spin," *Phys.Rev.* **D18** (1978) 3624.

[29] X. Bekaert and N. Boulanger, "The Unitary representations of the Poincare group in any spacetime dimension," hep-th/0611263.

[30] X. Bekaert, N. Boulanger, and P. Sundell, "How higher-spin gravity surpasses the spin two barrier: no-go theorems versus yes-go examples," *Rev.Mod.Phys.* **84** (2012) 987–1009, 1007.0435.

[31] E. Fradkin and M. A. Vasiliev, "On the Gravitational Interaction of Massless Higher Spin Fields," *Phys.Lett.* **B189** (1987) 89–95.

[32] M. A. Vasiliev, "More on equations of motion for interacting massless fields of all spins in (3+1)-dimensions," *Phys.Lett.* **B285** (1992) 225–234.

[33] A. Sagnotti, "Notes on Strings and Higher Spins," *J.Phys.* **A46** (2013) 214006, 1112.4285.

[34] I. Klebanov and A. Polyakov, "AdS dual of the critical O(N) vector model," *Phys.Lett.* **B550** (2002) 213–219, hep-th/0210114.

[35] E. Sezgin and P. Sundell, "Massless higher spins and holography," *Nucl.Phys.* **B644** (2002) 303–370, hep-th/0205131.

[36] E. Sezgin and P. Sundell, "Holography in 4D (super) higher spin theories and a test via cubic scalar couplings," *JHEP* **0507** (2005) 044, hep-th/0305040.

[37] S. Giombi and X. Yin, "Higher Spin Gauge Theory and Holography: The Three-Point Functions," *JHEP* **1009** (2010) 115, 0912.3462.

[38] S. Giombi and X. Yin, "The Higher Spin/Vector Model Duality," *J.Phys.* **A46** (2013) 214003, 1208.4036.

[39] J. D. Brown and M. Henneaux, "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity," *Commun.Math.Phys.* **104** (1986) 207–226.

[40] M. Banados, "Global charges in Chern-Simons field theory and the (2+1) black hole," *Phys.Rev.* **D52** (1996) 5816, hep-th/9405171.

[41] A. Strominger, "Black hole entropy from near horizon microstates," *JHEP* **9802** (1998) 009, hep-th/9712251.

[42] C. Aragone and S. Deser, "Consistency Problems of Hypergravity," *Phys.Lett.* **B86** (1979) 161.

[43] A. Campoleoni, S. Fredenhagen, S. Pfenninger, and S. Theisen, "Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields," *JHEP* **1011** (2010) 007, 1008.4744.

[44] A. Campoleoni, S. Fredenhagen, and S. Pfenninger, "Asymptotic W-symmetries in three-dimensional higher-spin gauge theories," *JHEP* **1109** (2011) 113, 1107.0290.

[45] M. Henneaux and S.-J. Rey, "Nonlinear $W_{infinity}$ as Asymptotic Symmetry of Three-Dimensional Higher Spin Anti-de Sitter Gravity," *JHEP* **1012** (2010) 007, 1008.4579.

[46] M. R. Gaberdiel and R. Gopakumar, "An AdS_3 Dual for Minimal Model CFTs," *Phys.Rev.* **D83** (2011) 066007, 1011.2986.

[47] M. R. Gaberdiel and R. Gopakumar, "Minimal Model Holography," *J.Phys.* **A46** (2013) 214002, 1207.6697.

[48] M. Ammon, M. Gutperle, P. Kraus, and E. Perlmutter, "Black holes in three dimensional higher spin gravity: A review," *J.Phys.* **A46** (2013) 214001, 1208.5182.

[49] M. Blencowe, "A Consistent Interacting Massless Higher Spin Field Theory in $D = (2+1)$," *Class.Quant.Grav.* **6** (1989) 443.

[50] A. Achucarro and P. Townsend, "A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories," *Phys.Lett.* **B180** (1986) 89.

[51] E. Witten, "(2+1)-Dimensional Gravity as an Exactly Soluble System," *Nucl.Phys.* **B311** (1988) 46.

[52] J. de Boer and J. I. Jottar, "Entanglement Entropy and Higher Spin Holography in AdS_3," 1306.4347.

[53] J. C. Baez and J. P. Muniain, *Gauge fields, knots and gravity*, vol. 6. World Scientific Singapore, 1994.

[54] M. Blagojevic, *Gravitation and gauge symmetries*. CRC Press, 2010.

[55] M. Nakahara, *Geometry, topology, and physics*. CRC Press, 2003.

[56] M. Gary, D. Grumiller, and R. Rashkov, "Towards non-AdS holography in 3-dimensional higher spin gravity," *JHEP* **1203** (2012) 022, 1201.0013.

[57] H. Afshar, M. Gary, D. Grumiller, R. Rashkov, and M. Riegler, "Non-AdS holography in 3-dimensional higher spin gravity - General recipe and example," *JHEP* **1211** (2012) 099, 1209.2860.

[58] D. Son, "Toward an AdS/cold atoms correspondence: A Geometric realization of the Schrodinger symmetry," *Phys.Rev.* **D78** (2008) 046003, 0804.3972.

[59] K. Balasubramanian and J. McGreevy, "Gravity duals for non-relativistic CFTs," *Phys.Rev.Lett.* **101** (2008) 061601, 0804.4053.

[60] A. Adams, K. Balasubramanian, and J. McGreevy, "Hot Spacetimes for Cold Atoms," *JHEP* **0811** (2008) 059, 0807.1111.

[61] S. Kachru, X. Liu, and M. Mulligan, "Gravity Duals of Lifshitz-like Fixed Points," *Phys.Rev.* **D78** (2008) 106005, 0808.1725.

[62] D. Grumiller and N. Johansson, "Instability in cosmological topologically massive gravity at the chiral point," *JHEP* **0807** (2008) 134, 0805.2610.

[63] D. Grumiller and N. Johansson, "Consistent boundary conditions for cosmological topologically massive gravity at the chiral point," *Int.J.Mod.Phys.* **D17** (2009) 2367–2372, 0808.2575.

[64] M. Henneaux, C. Martinez, and R. Troncoso, "Asymptotically anti-de Sitter spacetimes in topologically massive gravity," *Phys.Rev.* **D79** (2009) 081502, 0901.2874.

[65] A. Maloney, W. Song, and A. Strominger, "Chiral Gravity, Log Gravity and Extremal CFT," *Phys.Rev.* **D81** (2010) 064007, 0903.4573.

[66] L. Susskind, "Holography in the flat space limit," hep-th/9901079.

[67] G. Arcioni and C. Dappiaggi, "Holography in asymptotically flat space-times and the BMS group," *Class.Quant.Grav.* **21** (2004) 5655, hep-th/0312186.

[68] G. Arcioni and C. Dappiaggi, "Holography and BMS field theory," *AIP Conf.Proc.* **751** (2005) 176–178, hep-th/0409313.

[69] H. Afshar, A. Bagchi, R. Fareghbal, D. Grumiller, and J. Rosseel, "Higher spin theory in 3-dimensional flat space," *Phys.Rev.Lett.* **111** (2013) 121603, 1307.4768.

[70] H. A. Gonzalez, J. Matulich, M. Pino, and R. Troncoso, "Asymptotically flat spacetimes in three-dimensional higher spin gravity," *JHEP* **1309** (2013) 016, 1307.5651.

[71] A. Bagchi, S. Detournay, and D. Grumiller, "Flat-Space Chiral Gravity," *Phys.Rev.Lett.* **109** (2012) 151301, 1208.1658.

[72] A. Bagchi, S. Detournay, D. Grumiller, and J. Simon, "Cosmic evolution from phase transition of 3-dimensional flat space," *Phys.Rev.Lett.* **111** (2013) 181301, 1305.2919.

[73] M. Gutperle, E. Hijano, and J. Samani, "Lifshitz black holes in higher spin gravity," 1310.0837.

[74] G. Grinstein, "Anisotropic sine-gordon model and infinite-order phase transitions in three dimensions," *Phys. Rev. B* **23** (May, 1981) 4615–4630.

[75] W. Fulton and J. Harris, *Representation theory: a first course*, vol. 129. Springer, 1991.

[76] W. de Graaf, "Classification of solvable Lie algebras.," *Exp. Math.* **14** (2005), no. 1, 15–25.

[77] R. Benguria, P. Cordero, and C. Teitelboim, "Aspects of the Hamiltonian Dynamics of Interacting Gravitational Gauge and Higgs Fields with Applications to Spherical Symmetry," *Nucl.Phys.* **B122** (1977) 61.

[78] A. Zamolodchikov, "Infinite Additional Symmetries in Two-Dimensional Conformal Quantum Field Theory," *Theor.Math.Phys.* **65** (1985) 1205–1213.

[79] P. Bouwknegt and K. Schoutens, "W symmetry in conformal field theory," *Phys.Rept.* **223** (1993) 183–276, hep-th/9210010.

[80] P. Francesco, P. Mathieu, and D. Senechal, *Conformal field theory*. Springer New York, 1997.

[81] R. Blumenhagen and E. Plauschinn, "Introduction to conformal field theory," *Lect.Notes Phys.* **779** (2009) 1–256.