DIPLOMARBEIT

Stochastic Navier-Stokes Equations

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Contents

Introduction

The classical theory for stochastic partial differential equations depends strongly on the works of Ito at the beginning of the twentieth century, where he developed a framework within which integration of stochastic processes with respect to semi martingales can be defined. This allows us to rewrite any stochastic evolution equation to an integral equation, which not only makes the mathematical treatment of these equations much easier, but, in the first place, it also provides us with a formal definition of solutions to those equations. These solutions are called strong solutions. However, this definition is often not enough in more complicated situations, since, on the one hand, not every stochastic partial differential equation can be written as an evolution equation, and, on the other hand, for many stochastic evolution equations it has turned out to be very difficult to show existence of strong solutions, and therefore proofs of existence of strong solutions only exist in some special cases. Thus, other definitions of solutions are needed, either to weaken the definition of Ito in order to make existence proofs possible, or to provide us with a framework that can be applied apart from evolution equations. In many situations it is necessary to develop specialised frameworks that can only be applied to a rather small class of stochastic equations, or that may work only for a single equation. Important situations where the standard theory breaks down, and specialised theories are needed, are equations that appear in fluid dynamics.

This thesis deals with the stochastic Navier-Stokes equations in both the incompressible and the compressible case. The equations are considered with stochastic initial conditions and stochastic forces acting on the fluid. The domain for the fluid flow will always be a fixed bounded Lipschitz domain, and throughout this thesis we assume homogeneous Dirichlet boundary conditions. Thus, neither the domain for the fluid flow, nor the boundary conditions, will be subject to stochastic perturbation. The second, respectively the third, part of the thesis deals with the incompressible, respectively the compressible, Navier-Stokes equations. Following the ideas described in the previous paragraph, our goal in both situations is, roughly speaking, to present a definition of solutions to the SPDE systems and afterwards to prove existence of those solutions. In both cases, uniqueness is an open problem.

The second part of the thesis considers the incompressible Navier-Stokes equations. The definitions of strong, weak and martingale solutions to stochastic evolution equations are presented in section 5. Although the existence of strong or weak solutions to the incompressible Navier-Stokes system is an open problem, those definitions will be given for sake of completeness. The concept of martingale solutions differs essentially from those of strong or weak solutions, since solutions in the sense of the former are probability measures on the path space, while solutions in the sense of the latter two concepts are stochastic processes. To motivate the concept of martingale solutions, it will be shown that the probability measure on the path space induced by a weak solution is always a martingale solution, as it was proved in [14]. Following the presentation of [16], it will then be shown that the incompressible Navier-Stokes equations admit a formulation as a stochastic evolution equation in the space of divergence free vector fields.

In section 6, we prove the existence of martingale solutions, following again [16]. The proof relies on the existence of martingale solutions for finite dimensional Hilbert space, a result that can be obtained using Ito's calculus, and on Galerkin's approximation scheme.

As it was shown in [16], an abstract Markov selection theorem can be applied to the present situation to prove the existence of almost sure martingale solutions to the Markov problem associated with the incompressible Navier-Stokes equations. The original proof for this selection theorem from [13] is presented in section 7, together with the application to the current situation.

In the third part of the present thesis, the compressible Navier-Stokes equations are studied. At the beginning of section 8, a brief discussion of the deterministic compressible Navier-Stokes equations it provided. The concept of finite-energy weak solutions, as introduced in [11], is presented, as well as the main theorem proofed in [11], claiming the existence of those solutions for bounded and measurable forces. Afterwards, following [9], the extension of this concept to the stochastic Navier-Stokes system is discussed. Roughly speaking, a stochastic process will be called a finite-energy weak solution to the stochastic compressible Navier-Stokes system, if it solves the system path-wise. In other words, a stochastic process is called a solution, if for almost every fixed ω in the state space the following holds: The function that arises from fixing ω in the solution process is a finite-energy weak solution to the deterministic Navier-Stokes equations, driven by the force that arises from fixing ω in the force process. The major obstacle is that in all interesting situations, the forces obtained in this way are defined as the time derivative of a nowhere differentiable function. Thus, we need to deal with distributional forces and the existence theorem from [11] can not be applied.

The heart of the third part is section 9, where the compressible Navier-Stokes system governed by distributional forces is discussed, following the presentation of [9]. It will be shown that the system posses a finite-energy weak solution. The proof is based on the correspondent result for measurable forces and on approximation.

The proof of existence of solutions to the stochastic Navier-Stokes system, which is content of the first part of section 10, makes use of the measurable selection theorem proved in [17]. In the final subsection of this thesis, the developed theory is applied to Levy processes as presented in [9].

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Part I

Preliminaries

1 Analysis

1.1 Frequently used notations

- $\mathbb{N} := \{1, 2, 3, ...\}$.
- $\mathbb{N}_0 := \{0, 1, 2, 3...\}$.
- The symbols $c, c_1, c_2, ..., C, C_1, C_2, ..., \hat{c}, \tilde{c}...$ denote generic constants. They have different values in different parts of the text.
- We use the notation $const(A, B, ...)$ to denote a constant only depending on $A, B, ...$.
- For $p \in [1, \infty]$ we set

$$p' := \begin{cases} \infty & \text{if } p = 1 \\ \frac{p}{p-1} & \text{if } 1 < p < \infty \\ 1 & \text{if } p = \infty. \end{cases}$$

- If M is a set and $N \subseteq M$, then $\mathbb{I}_N$ denotes the characteristic function.
- Let M be any set and let A be an expression on M. We define

$$[A] := \{x \in M \mid A(x)\}.$$

  For instance, if $f : M \to \mathbb{R}$, we have

$$[f < c] = \{x \in M \mid f(x) < c\}.$$

- If $f : M \to N$ and $X \subseteq M$ then $f|_X : X \to N$ denotes the restriction of f to X. If $A \subseteq N^M$, then we define

$$A|_X := \{f|_X \mid f \in A\}.$$

- We denote the power set of M by 2^M.
- We denote the Lebesgue measure of a Borel measurable subset $A \subseteq \mathbb{R}^n$ by $\text{meas}A$.
- In Lebesgue spaces and in Lebesgue integrals, we denote by $\mathrm{d}t$, $\mathrm{d}s$ or $\mathrm{d}r$ the Lebesgue measure on $\mathbb{R}$ and by $\mathrm{d}x$, $\mathrm{d}y$ or $\mathrm{d}z$ the Lebesgue measure on $\mathbb{R}^N$ for $N > 1$. For instance, if $\mathcal{A}$ is the Borel σ-algebra on $\mathbb{R}$, then $L^1(\mathbb{R}, \mathcal{A}, \mathrm{d}t)$ denotes the space of Lebesgue integrable functions and $\int f \,\mathrm{d}t$ denotes the Lebesgue integral of f.

- If (M, d) is a metric space, we denote the open ball with radius R and center $x \in M$ by $B_R(x)$. If $A \subseteq M$, then we define

$$B_R(A) := \bigcup_{x \in A} B_R(x).$$

- If not otherwise stated, we always use the implicit summation convention, i.e. we sum up over all indices which appear precisely two times in a term. For instance, we have

$$|u|^2 = u_i u_i \quad \text{and} \quad \operatorname{div} u = \frac{\partial u_i}{\partial x_i}.$$

1.2 Functional Analysis

We will frequently use the following two theorems about product topologies. The proof can be found e.g. in [31], Theorem 1.3.1, respectively in [18], Theorem 1.12.

Theorem 1.1. *(Tychonoff) Let $(X_i)_{i \in I}$ be a family of compact topological spaces. Then $\prod_{i \in I} X_i$ is compact in the product topology.*

Theorem 1.2. *Let $(X_n)_{n \in \mathbb{N}}$ be a family of sequentially compact topological spaces. Then $\prod_{n \in \mathbb{N}} X_n$ is sequentially compact in the product topology.*

1.3 Banach spaces and Hilbert spaces

Unless otherwise stated, all Banach and Hilbert spaces in this thesis are real-valued spaces.

Let X be a vector space. We write $Y \leqslant X$ iff Y is a subspace of X. Let Z be a vector space and $X, Y \leqslant Z$. Suppose X respectively Y carry norms $\|\cdot\|_X$ respectively $\|\cdot\|_Y$. Then, the spaces $X + Y$ and $X \cap Y$ carry the norms

$$\|z\|_{X+Y} := \inf\{\|x\|_X + \|y\|_Y \,|\, x + y = z\}$$

and

$$\|z\|_{X \cap Y} := \|z\|_X + \|z\|_Y.$$

Let X be a Banach space. We denote dual space by

$$X' := \{x' : X \to \mathbb{R} \,|\, x' \text{ is linear and bounded}\}.$$

For $x' \in X'$ and $x \in X$ we denote the duality product by

$$\langle x', x \rangle :=_{X'} \langle x', x \rangle_X := x'(x).$$

We denote by $\iota_X : X \to X''$ the canonical embedding. For $Y \leqslant X'$ we denote by $\sigma(X, Y)$ the weak topology on X with respect to Y, i.e. $\sigma(X, Y)$ is the initial topology with respect to the set Y of functionals on X.

For a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ we write

$$x_n \overset{Y}{\rightharpoonup} x \quad \text{iff} \quad x_n \to x \text{ in } \sigma(\text{X,Y}).$$

We call $\sigma(X, X')$ *the weak topology on* X and $\sigma(X', \iota_X(X))$ *the weak-star topology on* X' and we write

$$x_n \rightharpoonup x \quad \text{iff} \quad x_n \overset{X'}{\rightharpoonup} x$$

and for a sequence $(x'_n)_{n\in\mathbb{N}} \subseteq X'$ we write

$$x'_n \overset{*}{\rightharpoonup} x' \quad \text{iff} \quad x'_n \overset{\iota_X(X)}{\rightharpoonup} x'.$$

We denote by X_w the topological vector spaces X endowed with the weak topology.

If Y is a further Banach space, then we write $Y \hookrightarrow X$ to denote that we can identify Y with some subspace of X, i.e. there is a canonical embedding $\iota_{Y\to X} : Y \to X$. If H is a Hilbert space, then we denote the scalar product of $x, y \in H$ by

$$\langle x, y\rangle_H$$

or simply by $\langle x, y\rangle$ if no confusion can arise. If $H = \mathbb{R}^n$, we use the notation $x \cdot y := \langle x, y\rangle_{\mathbb{R}^n} = x_i y_i$ for the scalar product.

Theorem 1.3. *Let X be a separable Banach space and let $A \subseteq X'$ be a bounded subset. Then, A equipped with the weak-star topology on X' is metrizable.*

Proof. Define the metric

$$d^*(x', y') := \sum_{k=1}^{\infty} \frac{1}{2^k \|a_k\|_X} |\langle x' - y', a_k\rangle|,$$

where $\{a_k\}_{k\in\mathbb{N}} \subseteq X$ is dense. Clearly, we have $x'_n \to x'$ in d^* if $x'_n \overset{*}{\rightharpoonup} x'$. Lemma 1.13 yields the inverse implication. □

Corollary 1.4. *If X' is separable and $A \subseteq X$ is bounded, then A equipped with the weak topology is metrizable.*

1.4 Linear operators

Let X and Y be Banach spaces.

Definition 1.5. We denote by $L(X, Y)$ the space of linear and bounded operators $A : X \to Y$. The space carries the operator norm $\|\cdot\|_{L(X,Y)}$.

We denote by $L_S(X, Y)$ the space $L(X, Y)$ equipped with the strong operator topology i.e. the initial topology with respect to the family of mappings $A \mapsto Ax$ from $L(X, Y)$ to Y for $x \in X$. In particular, $A_n \to A$ in $\mathcal{T}_S(X, Y)$ if $A_n x \to Ax$ in Y for all $x \in X$.

We denote by $L_W(X,Y)$ the space $L(X,Y)$ equipped with the weak operator topology, i.e. the initial topology with respect to the family of mappings $A \mapsto_{Y'} < y', Ax >_Y$ from $L(X,Y)$ to $\mathbb{R}$ for $x \in X$ and $y' \in Y'$.

Definition 1.6. Let H and U be Hilbert spaces and let $A \in L(H,U)$. Then,

$$A|_{(\ker A)^\perp} : (\ker A)^\perp \to U$$

is injective. The *pseudo inverse of* A is defined by

$$A^{-1} := A|_{(\ker A)^\perp}^{-1} : \operatorname{ran} A \to (\ker A)^\perp$$

1.5 Function spaces

If X is a topological space and Y is a Banach space, we define

$$C(X,Y) := \{f : X \to Y \mid f \text{ is continuous and bounded}\}$$

The space $C(X,Y)$ carries the supremum norm $\|\cdot\|_{C(X,Y)}$. If no confusion about the involved spaces can arise, we sometimes write $\|\cdot\|_\infty := \|\cdot\|_{C(X,Y)}$.

Let $k, n \in \mathbb{N}$, let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $A \subseteq \mathbb{R}^n$ be an arbitrary set. We define the spaces

$$\begin{aligned} C^0(A,\mathbb{R}^k) &:= \{f : A \to \mathbb{R}^k \mid f \text{ is continuous}\} \\ C^p(\Omega,\mathbb{R}^k) &:= \{f \in C^0(\Omega,\mathbb{R}^k) \mid f' \in C^{p-1}(\Omega,\mathbb{R}^k)\}, \quad \text{for } p \geqslant 1 \\ C^\infty(\Omega,\mathbb{R}^k) &:= \bigcap_{p\in\mathbb{N}} C^p(\Omega,\mathbb{R}^k) \end{aligned}$$

and for $m \in \mathbb{N} \cup \{\infty\}$

$$C^m(A,\mathbb{R}^k) := \{f : A \to \mathbb{R}^k \mid \exists U \supseteq A \text{ open and } g \in C^m(U,\mathbb{R}^k) \text{ such that } f = g|_A\}.$$

Furthermore, we define

$$\begin{aligned} C_b^m(A,\mathbb{R}^k) &:= \{f \in C^m(A,\mathbb{R}^k) \mid D^\alpha f \text{ is bounded for all } |\alpha| \leqslant m\} \\ C_c^m(A,\mathbb{R}^k) &:= \{f \in C^m(A,\mathbb{R}^k) \mid \operatorname{supp} f \subseteq A \text{ is compact}\}. \end{aligned}$$

In particular, we have

$$C(A,\mathbb{R}^k) = C_b^0(A,\mathbb{R}^k).$$

In order to simplify notations we set

$$C^m(A) := C^m(A,\mathbb{R}).$$

The spaces $C_b^m(A)$ and $C_c^m(A)$ are defined analogously. The spaces $C_b^m(A,\mathbb{R}^k)$ for $m \in \mathbb{N}_0$ are Banach spaces with respect to the norm

$$\|f\|_{C_b^m(A,\mathbb{R}^k)} := \sum_{|\alpha|<m} \|D^\alpha f\|_{C(A,\mathbb{R}^k)}.$$

For $m \in \mathbb{N} \cup \{\infty\}$ and $A \subseteq \mathbb{R}^n$, we define the space of divergence free vector fields

$$\mathcal{D}^m(A) := \{f \in C^m(A, \mathbb{R}^n) \mid \operatorname{div} f = 0\}.$$

The spaces $\mathcal{D}_b^m(A)$ and $\mathcal{D}_c^m(A)$ are defined analogously.

Let $I \subseteq \mathbb{R}$ be an interval and let E be a Banach space. We call a function $f : I \to E$ *cádlág* iff f is right continuous and has left limits (i.e. $\lim_{s\to t-} f(s)$ exists in E for all $t \in I\backslash\{\min I\}$). The set of cádlág functions is denoted by $D(I, E)$.

Let X be a topological space and Y be a Banach space. We define the the space

$$C_w(X, Y) := \{f : X \to Y \mid \langle f(\cdot), y'\rangle \in C_b(X) \text{ for all } y' \in Y'\},$$

i.e. the space of all continuous functions $f : X \to Y$, where Y carries the weak topology, such that $\|\langle f(\cdot), y'\rangle\|_\infty < \infty$ for all $y' \in Y'$. We say

$$f_n \to f \text{ in } C_w(X, Y)$$

iff

$$\langle f_n(\cdot), y'\rangle \to \langle f(\cdot), y'\rangle \text{ in } C_b(X)$$

for all $y' \in Y'$.

If X is any Banach space such that $C_c^\infty(A, \mathbb{R}^n) \hookrightarrow X$ for some $A \subseteq \mathbb{R}^n$, then we denote by

$$X_\sigma := \overline{\{f \in C^\infty(A, \mathbb{R}^n) \mid \operatorname{div} f = 0\} \cap X}^{\|\cdot\|_X}.$$

In particular, we have $\mathcal{D}_b^m(A) = C_{b,\sigma}^m(A)$ for $m \in \mathbb{N}$.

1.6 The set of compact subsets

Definition 1.7. Let (X, d) be a metric space. We denote by $\mathcal{C}(X)$ the space of compact non-empty subsets of X. We define a metric $d_\mathcal{C}$ on this space by

$$d_\mathcal{C}(K, H) := \inf\{\epsilon > 0 \,:\, K \subseteq B_\epsilon(H) \text{ and } H \subseteq B_\epsilon(K)\}$$

for all compact sets K, $H \in \mathcal{C}(X)$. The space $\mathcal{C}(X)$ is endowed with the Borel σ-algebra of this metric.

The next lemmata can be proved directly. For the proofs see [29], section 12.1.

Lemma 1.8. *If X is separable, then $\mathcal{C}(X)$ is separable.*

Lemma 1.9. *For $B \in \mathcal{B}(X)$ let*

$$\tau(B) := \{K \in \mathcal{C}(X) \,|\, K \subseteq B\},$$

and define

$$\mathcal{E} := \{\tau(U) \,|\, U \subseteq X \text{ is open}\}$$

and

$$\overline{\mathcal{E}} := \{\tau(A) \,|\, A \subseteq X \text{ is closed}\}$$

Then, each element of $\mathcal{E}$ is open in $\mathcal{C}(X)$ and each element of $\overline{\mathcal{E}}$ is closed in $\mathcal{C}(X)$. Furthermore, both $\mathcal{E}$ and $\overline{\mathcal{E}}$ are generators for the Borel σ-algebra $\mathcal{B}(\mathcal{C}(X))$.

Lemma 1.10. *Let (X, d) be a separable metric space and let (E, Ω) be a measurable space. Let $f : E \to X$ and $g : E \to \mathcal{C}(X)$ be measurable maps. Then the set $\{x \in E \,|\, f(x) \in g(x)\}$ is a measurable subset of E.*

1.7 Properties of the weak and the weak-* topology

We list some results about the weak topology. Let X be a Banach space.

Lemma 1.11. *(Banach-Alaoglu) Bounded sets in X' are pre-compact in the weak-* topology.*

The proof can be found in any introductory book about functional analysis, i.e. [31], Theorem 5.4.1. Since the weak-star topology on bounded sets is metrizable, bounded sets in X' are also sequentially pre-compact.

Lemma 1.12. *Let $x_k \rightharpoonup x$ in X (respectively $x_k \overset{*}{\rightharpoonup} x$ in X'). Then, the sequence $\{x_k\}_{k\in\mathbb{N}}$ is bounded and*

$$\|x\| \leqslant \liminf_{k\to\infty} \|x_k\|.$$

Proof. In both cases (i.e. the weak resp. the weak-star convergence), the boundedness of the sequence is a direct consequence from Banach-Steinhaus' Theorem. If $x_k \rightharpoonup x$ in X, then the estimate follows from

$$\|x\|_X = \sup_{\|x'\|_{X'}=1} |\langle x', x\rangle| = \sup_{\|x'\|_{X'}=1} \lim_{k\to\infty} |\langle x', x_k\rangle| \leqslant \liminf_{k\to\infty} \sup_{\|x'\|_{X'}=1} |\langle x', x_k\rangle| = \liminf_{k\to\infty} \|x_k\|_X.$$

If $x_k \overset{*}{\rightharpoonup} x$, the statement follows by a similar calculation. □

Lemma 1.13. *Let $\{x_{n,k}\}_{(n,k)\in\mathbb{N}^2} \subseteq X$ such that $x_{n,k} \to \varphi_k$ for any fixed k and $x_{n,k} \rightharpoonup \psi_n$ for any fixed n. Suppose that*

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \|x_{n,k} - \varphi_k\|_X = 0.$$

Then, there is $x \in X$ such that $\varphi_k \rightharpoonup x$, $\psi_k \to x$ and $x_{k,k} \rightharpoonup x$.

If $x_{n,k} \to \psi_n$ strong in X for all fixed n, then $\varphi_k \to x$ and $x_{k,k} \to x$.

Proof. Let $\epsilon > 0$. There is $N \in \mathbb{N}$ such that

$$\|\psi_n - \psi_m\|_X \leqslant \liminf_{k\to\infty} \|x_{n,k} - x_{m,k}\|_X \leqslant \liminf_{k\to\infty} \|x_{n,k} - \varphi_k\|_X + \|x_{m,k} - \varphi_k\|_X \leqslant \epsilon$$

for all $n, m > N$. Thus, $\psi_n \to x$ for some $x \in X$. We get for all $x' \in X' \backslash \{0\}$ and any $k, n \in \mathbb{N}$

$$|\langle x - \varphi_k, x' \rangle| \leqslant \|x - \psi_n\|_X \|x'\|_{X'} + |\langle \psi_n - x_{n,k}, x' \rangle| + \|x_{n,k} - \varphi_k\|_X \|x'\|_{X'}.$$

Choose $n \in \mathbb{N}$ such that $\|x - \psi_n\|_X < \epsilon \frac{1}{\|x'\|_{X'}}$ and $\sup_{k\in\mathbb{N}} \|x_{n,k} - \varphi_k\|_X < \epsilon \frac{1}{\|x'\|_{X'}}$. Now, choose $K \in \mathbb{N}$ such that $|\langle \psi_n - x_{n,k}, x' \rangle| < \epsilon$ for all $k > K$. Then, the right hand side is $< 3\epsilon$ for all $k > K$. Thus, we have $\varphi_k \rightharpoonup x$. Furthermore, we have

$$|\langle x - x_{k,k}, x' \rangle| \leqslant \|x_{k,k} - x_{n,k}\|_X \|x'\|_{X'} + \|x_{n,k} - \varphi_k\|_X \|x'\|_{X'} + |\langle \varphi_k - x, x' \rangle|,$$

where n is chosen as above. Then, $\|x_{k,k} - x_{n,k}\|_X \leqslant \|x_{k,k} - \varphi_k\|_X + \|x_{n,k} - \varphi_k\|_X \leqslant 2\epsilon$ and $\|x_{n,k} - \varphi_k\|_X \leqslant \epsilon$. By the above there is $K \in \mathbb{N}$ such that for all $k \geqslant K$ we have $|\langle \varphi_k - x, x' \rangle| \leqslant \epsilon$. This shows $x_{k,k} \rightharpoonup x$.

If $x_{n,k} \to \psi_n$ strong in X, then have

$$\|x - \varphi_k\|_X \leqslant \|x - \psi_n\|_X + \|\psi_n - x_{n,k}\|_X + \|x_{n,k} - \varphi_k\|_X,$$

where the right hand side can be estimated as above. The assertion $x_{k,k} \to x$ can be shown similar. □

Lemma 1.14. *Let $x_n \rightharpoonup x$ in X and $x'_n \to x'$ in X'. Then the iterated limit exists, and we have*

$$\lim_{n\to\infty} \lim_{m\to\infty} \langle x'_n, x_m \rangle = \lim_{m\to\infty} \lim_{n\to\infty} \langle x'_n, x_m \rangle.$$

Proof. Since the sequence x_n is bounded in X and $x'_n \to x'$ uniformly on bounded sets, we have

$$\lim_{n\to\infty} \sup_{m\in\mathbb{N}} |\langle x'_n - x', x_m \rangle| = 0.$$

Lemma 1.13 now yields the desired conclusion. □

Lemma 1.15. *(weak lower semicontinuity of convex functions) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, $f_n \rightharpoonup f$ in $L^1(\Omega)$ and let $F : \mathbb{R} \to \mathbb{R}$ be a convex function. Then*

$$\int F(f)\,\mathrm{d}x \leqslant \liminf_{n\to\infty} \int F(f_n)\,\mathrm{d}x.$$

For the proof see [21], Lemma 3.5.

Lemma 1.16. *Let $(\mathscr{S}, \mathcal{A}, \mu)$ be a complete, σ-finite measure space, let $F : \mathbb{R} \to \mathbb{R}$ be a strictly convex function and let $f_n, f : \mathscr{S} \to \mathbb{R}$ be measurable. Suppose that*

$$f_n \rightharpoonup f$$
$$F(f_n) \rightharpoonup F(f_n)$$

in $L^1(\Omega)$. Then $f_n \to f$ in $L^1(\Omega)$.

The proof can be found in [30], Theorem 2.

Lemma 1.17. *Let $T > 0$ and $1 \leqslant p, q, r, s \leqslant \infty$ such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s} = 1.$$

Let $f_n \rightharpoonup f$ in $L^p(0,T;L^r(\Omega))$ and $g_n \rightharpoonup g$ in $L^q(0,T;L^s(\Omega))$. Assume that

$$\|\frac{\partial f_n}{\partial t}\|_{L^1(0,T;W^{-m,1}(\Omega))} \leqslant c$$

for some constants $m \geqslant 0$ and $c > 0$ independent of n, and

$$\lim_{|\xi|\to 0} \sup_{n\in\mathbb{N}} \|g_n(\cdot + \xi, r) - g_n\|_{L^q(0,T;L^s(\Omega))} = 0.$$

Then, $f_n g_n \to fg$ in $\mathcal{D}'((0,T) \times \Omega)$. Furthermore, the weak convergence can be replaced with the weak- convergence, if some of the exponents are infinite.*

The proof can be found in [24], Lemma 5.1.

2 Partial Differential Equations

2.1 Distributions

Let $\Omega \subseteq \mathbb{R}^n$ be a measurable set.

Definition 2.1. We denote the set of test functions by

$$\mathcal{D}(\Omega) := C_c^\infty(\Omega).$$

Definition 2.2. We say that $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$ iff

(1) there exists a compact set $K \subseteq \Omega$ such that $\operatorname{supp} \varphi_n \subseteq K$ for all n;

(2) for all $\alpha \in \mathbb{N}_0^n$ we have

$$\lim_{n\to\infty} \|D^\alpha(\varphi_n - \varphi)\|_\infty = 0.$$

Definition 2.3. We denote the set of distributions by

$$\mathcal{D}'(\Omega) := \left\{ u : \mathcal{D}(\Omega) \to \mathbb{R} \,\middle|\, u \text{ is linear and } \varphi_n \to \varphi \text{ in } \mathcal{D}(\Omega) \text{ implies } u(\varphi_n) \to u(\varphi) \right\}.$$

For $\varphi \in \mathcal{D}(\Omega)$ and $u \in \mathcal{D}'(\Omega)$ we denote the duality product by

$$\langle u, \varphi \rangle := u(\varphi).$$

We define the derivative of distributions by

$$\langle D^\alpha u, \varphi \rangle := (-1)^{|\alpha|} \langle u, D^\alpha \varphi \rangle.$$

We say $u_n \to u$ in $\mathcal{D}'(\Omega)$ iff $\langle u_n, \varphi \rangle \to \langle u, \varphi \rangle$ for all $\varphi \in \mathcal{D}(\Omega)$.

2.2 Sobolev Spaces

All results about Sobolev spaces can be found in [8], chapter 5, if not otherwise stated.

Definition 2.4. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, let $k \in \mathbb{N}$ and $p \in \mathbb{N} \cup \{\infty\}$. We define the Sobolev spaces

$$W^{k,p}(\Omega, \mathbb{R}^m) := \{u \in L^p(\Omega, \mathbb{R}^m) \mid D^\alpha u \in L^p(\Omega, \mathbb{R}^m) \text{ for all multi-indices } \alpha \text{ with } |\alpha| \leqslant k\}$$

and the norm

$$\|f\|_{W^{k,p}(\Omega;\mathbb{R}^m)} := \begin{cases} \left(\sum_{|\alpha| \leqslant k} \|D^\alpha f\|^p_{L^p(\Omega,\mathbb{R}^m)} \right)^{\frac{1}{p}} & \text{if } p < \infty \\ \max_{|\alpha| \leqslant k} \|D^\alpha f\|_{L^\infty(\Omega,\mathbb{R}^m)} & \text{if } p = \infty. \end{cases}$$

Then we have

$$W^{k,p}(\Omega, \mathbb{R}^m) = \overline{C^\infty(\Omega, \mathbb{R}^m) \cap W^{k,p}(\Omega, \mathbb{R}^m)}^{\|\cdot\|_{W^{k,p}(\Omega,\mathbb{R}^m)}}$$

for any $p < \infty$. The spaces $W^{k,2}(\Omega, \mathbb{R}^m)$ are Hilbert spaces, endowed with the scalar product

$$\langle f, g \rangle^2_{W^{k,2}(\Omega,\mathbb{R}^m)} := \sum_{|\alpha| \leqslant k} \langle D^\alpha f, D^\alpha g \rangle^2_{L^2(\Omega,\mathbb{R}^m)}.$$

Theorem 2.5. *(Trace Operator) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. There exists a unique bounded linear operator $T : W^{k,p}(\Omega, \mathbb{R}^m) \to L^p(\partial\Omega, \mathbb{R}^m)$ such that*

$$T(u) = u|_{\partial\Omega}$$

for all $u \in C^\infty(\Omega, \mathbb{R}^m) \cap W^{k,p}(\Omega, \mathbb{R}^m)$.

We define the space

$$W^{k,p}_0(\Omega; \mathbb{R}^m) := \{u \in W^{k,p}(\Omega; \mathbb{R}^n) \mid T(u) = 0\}.$$

Lemma 2.6. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Then we have*

$$W^{k,p}(\Omega; \mathbb{R}^m) = \overline{C^\infty(\overline{\Omega}; \mathbb{R}^m)}^{\|\cdot\|_{W^{k,p}(\Omega;\mathbb{R}^m)}},$$
$$W^{k,p}_0(\Omega; \mathbb{R}^m) = \overline{C^\infty_c(\Omega; \mathbb{R}^m)}^{\|\cdot\|_{W^{k,p}(\Omega;\mathbb{R}^m)}}.$$

Theorem 2.7. *(Sobolev embedding) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain and let $1 \leqslant p, q < \infty$ and*

$k, m \in \mathbb{N}_0$ such that $k < m$. Then, the embedding

$$W^{m,p}(\Omega) \hookrightarrow W^{k,q}(\Omega)$$

is continuous, if $m - \frac{n}{p} \geqslant k - \frac{n}{q}$, and compact, if $m - \frac{n}{p} > k - \frac{n}{q}$. The embedding

$$W^{m,p}(\Omega) \hookrightarrow C^k(\overline{\Omega})$$

is compact, if $m - \frac{n}{p} > k$.

Theorem 2.8. *(Poincare inequality) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain and let $1 \leqslant p < \infty$. Then,*

$$\|f\|_{W_0^{1,p}(\Omega)} \leqslant const(\Omega, p)\|\nabla f\|_{L^p(\Omega)}.$$

For the proof of the next Lemma see [10], Theorem 10.18.

Lemma 2.9. *(div-curl lemma) Let $1 < p < \infty$. There exists a constant c such that for all $u \in W^{1,p}(\mathbb{R}^N, \mathbb{R}^N)$ we have*

$$\|\nabla u\|_{L^p(\mathbb{R}^N;\mathbb{R}^{N\times N})} \leqslant c(\|\operatorname{div} u\|_{L^p(\mathbb{R}^N)} + \|\operatorname{curl} u\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)})$$

Definition 2.10. For $1 \leqslant k < \infty$ and $1 < p < \infty$ we define the space

$$W^{-k,p}(\Omega) := W_0^{k,p}(\Omega)'.$$

We have

$$\{D^\alpha f \mid f \in L^p(\Omega) \text{ and } |\alpha| \leqslant k\} \subseteq W^{-k,p}(\Omega).$$

Lemma 2.12 in the next subsection can be used to show the next immediate corollary:

Corollary 2.11. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain and let $1 \leqslant p, q < \infty$ and $k, m \in \mathbb{N}_0$ such that $k < m$. Then, the embedding*

$$W^{-m,p}(\Omega) \hookrightarrow W^{-k,q}(\Omega)$$

is continuous if $k - \frac{n}{q'} \geqslant m - \frac{n}{p'}$.

2.3 Gelfand triples

Lemma 2.12. *Let $Y \hookrightarrow X$ continuously and densely. Then $X' \hookrightarrow Y'$ continuously. If X is reflexive, then $X' \hookrightarrow Y'$ densely.*

The proof can be found in [32], Problem 18.6.

Definition 2.13. Let X be a separable and reflexive Banach space and let H be a separable Hilbert space. Suppose $X \hookrightarrow H$ continuously and densely. Then, $H \overset{\sim}{=} H' \hookrightarrow X'$ continuously and densely and we call the triple

$$X \hookrightarrow H \hookrightarrow X'$$

a *Gelfand triple* (or *evolution triple*).

Lemma 2.14. *Let $X \hookrightarrow H \hookrightarrow X'$ be a Gelfand triple, such that the embedding $X \hookrightarrow H$ is compact. Then, there exists an orthonormal basis $\{b_n\}_{n\in\mathbb{N}} \subseteq X$ of H such that*

$$\|\mathbb{P}_n x\|_{X'} \leqslant \|x\|_{X'}$$

for all $x \in X'$, where $\mathbb{P}_n$ is the projection

$$\mathbb{P}_n x := \sum_{i=1}^{n} {}_{X'}\langle x, b_i\rangle_X b_i.$$

For the proof see [16], Lemma 4.4.

2.4 Auxiliary Equations

2.4.1 Bogovskii Operator

Theorem 2.15. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain and let $p, r \in (1, \infty)$. There exists a bounded linear operator*

$$\mathcal{B} : \{f \in L^p(\Omega) \,:\, \int_\Omega f \,\mathrm{d}x = 0\} \to W_0^{1,p}(\Omega, \mathbb{R}^3)$$

such that $v = \mathcal{B}[f]$ solves the problem

$$\begin{aligned} \operatorname{div} v &= f \ \ \textit{in } \Omega \\ v &= 0 \ \ \textit{on } \partial\Omega. \end{aligned}$$

Furthermore, if $\operatorname{div} g = f$ for some $g \in W^{1,r}(\Omega, \mathbb{R}^3)$ with $g \cdot \overrightarrow{\nu} = 0$ on $\partial\Omega$, then

$$\|\mathcal{B}[f]\|_{L^r(\Omega)} \leqslant const(p, r, \Omega)\|g\|_{L^r(\Omega)}.$$

For the proof see [15], Theorem 3.3.

Definition 2.16. The operator constructed in Theorem 2.15 is called *Bogovskii operator.*

2.4.2 L^p multiplier

Definition 2.17. Let $N \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$ and let $m : \mathbb{R}^N \to \mathbb{R}$ be a measurable function. We denote by T_m the formal Operator on $L^p(\mathbb{R}^N)$ defined by

$$T_m := \begin{cases} D(T_m)\big(\subseteq L^p(\mathbb{R}^N)\big) \to L^p(\mathbb{R}^N) \\ v \mapsto \mathcal{F}^{-1}(m\mathcal{F}(v)) \end{cases}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transformation and the inverse Fourier transformation on $\mathbb{R}^N$. The operator T_m is called *multiplier operator* and m is the *symbol* of T_m. If T is a multiplier operator with symbol m, we write

$$T \approx m(\xi).$$

The function m is called an $L^p(\mathbb{R}^N)$-*multiplier* if $T_m : D(T_m)(\subseteq L^p(\mathbb{R}^N)) \to L^p(\mathbb{R}^N)$ is a densely defined, bounded operator. The unique linear and bounded extension $T_m : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is then called $L^p(\mathbb{R}^N)$-*multiplier operator.*

Let $\Omega \subseteq \mathbb{R}^N$ be a domain and denote for $v \in L^p(\Omega)$ by $[v]_{\mathbb{R}^N} \in L^p(\mathbb{R}^N)$ the function that agrees with v in Ω and is prolonged by zero on $\mathbb{R}^N \backslash \Omega$. If T_m is a multiplier operator, then we denote by the same symbol T_m the operator $T_m : D_\Omega(T_m)\big(\subseteq L^p(\Omega)\big) \to L^p(\Omega)$, defined by

$$T_m(v) := T_m([v]_{\mathbb{R}^N})|_\Omega,$$

where $D_\Omega(T_m) := \{v \in L^p(\Omega) \,|\, [v]_{\mathbb{R}^N} \in D(T_m)\}$.

Lemma 2.18. *If T and S are multiplier operators, then $TS = ST$ whenever both sides are defined.*

Proof. Follows directly from the definition. □

Theorem 2.19. *(Mikhiln multiplier theorem) Let $m \in L^\infty(\mathbb{R}^N)$ be smooth except possibly at the origin and suppose m satisfies*

$$|x|^k |\nabla^k m| \in L^\infty(\mathbb{R}^N; \mathbb{R}^{(N^k)})$$

for all $0 \leqslant k \leqslant \frac{N}{2} + 1$. Then m is an $L^p(\mathbb{R}^N)$ multiplier for all $1 < p < \infty$.

The original proof of this Theorem can be found in [25].

Definition 2.20. We introduce the following *pseudo differential operators* on $\mathbb{R}^N$ used in later sections:

- the *double Riesz transform* $\mathcal{R} =(\mathcal{R}_{j,k})_{j,k=1,\dots,N}$, where

$$\mathcal{R}_{j,k} \approx \frac{\xi_j \xi_k}{|\xi|^2},\ j,k = 1, \dots, N$$

- the *inverse divergence* $\mathcal{A} = (\mathcal{A}_j)_{j=1,\dots,N}$, where

$$\mathcal{A}_j \approx -\frac{i\xi_j}{|\xi|^2},\ j = 1, \dots, N$$

- the *inverse Laplacian* $\triangle^{-1}$, where

$$\triangle^{-1} \approx -\frac{1}{|\xi|^2}$$

Remark 2.21. Note, that we have

$$\begin{aligned}
\mathcal{A}_j &= \partial_j \triangle^{-1} \\
\mathcal{A} &= \nabla \triangle^{-1} \\
\mathcal{R} &= \nabla\nabla\triangle^{-1} = \nabla\mathcal{A}
\end{aligned}$$

and

$$\triangle\triangle^{-1}[v] = \operatorname{div}(\mathcal{A}[v]) = v.$$

Theorem 2.22. *The double Riesz transform is an $L^p(\mathbb{R}^N)$-multiplier operator for any $1 < p < \infty$.*

Proof. This follows immediately from the Mikhlin multiplier Theorem. $\square$

The operators $\mathcal{A}_k$ are not $L^p(\mathbb{R}^N)$-multiplier operators for any $p \in [1, \infty]$, but one can show the following result:

Theorem 2.23. *The operators $\mathcal{A}_k$ are bounded linear operators*

$$\mathcal{A}_k : L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$$

and

$$\mathcal{A}_k : L^p(\mathbb{R}^N) \to L^{\frac{Np}{N-p}}(\mathbb{R}^N)$$

for all $1 < p < N$. Furthermore, we have for all $1 \leqslant i, j, k \leqslant N$

$$\begin{aligned}
\int_\Omega \mathcal{A}_k[u] v \,\mathrm{d}x &= -\int_\Omega u \mathcal{A}_k[v] \,\mathrm{d}x \\
\int_\Omega \mathcal{R}_{i,j}[u] v \,\mathrm{d}x &= \int_\Omega u \mathcal{R}_{i,j}[v] \,\mathrm{d}x
\end{aligned}$$

whenever both sides are defined.

This can be proved by using basic properties of the Fourier transform, see [10], Theorem 10.26, and the formulae at the end of section 10.16. Now the following is easy:

Corollary 2.24. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. Then, the inverse divergence is a compact operator*

$$\mathcal{A}_i : L^p(\Omega) \to C(\overline{\Omega})$$

and a continuous operator

$$\mathcal{A}_i : L^q(\Omega) \to W^{1,q}(\Omega)$$

for any $p > N$ and $q > 1$ and all $1 \leqslant i \leqslant N$.

Proof. For $q < N$, the mapping $\mathcal{A}_k : L^q(\Omega) \to L^{\frac{Nq}{N-q}}(\Omega)$ is continuous, and since $\frac{Nq}{N-q} > q$, the embedding $L^{\frac{Nq}{N-q}}(\Omega) \hookrightarrow L^q(\Omega)$ is also continuous. Therefore,

$$\|\mathcal{A}_i[v]\|_{L^q(\Omega)} \leqslant c\|v\|_{L^q(\Omega)}.$$

On the other hand, for any $q \geqslant N$, there is $r \in [\frac{N}{2}, N)$ such that $\frac{Nr}{N-r} = q$. Due to the continuity of the embedding $L^q(\Omega) \hookrightarrow L^r(\Omega)$ and the continuity of the mapping $A_j : L^r(\Omega) \to L^q(\Omega)$, we conclude again

$$\|\mathcal{A}_i[v]\|_{L^q(\Omega)} \leqslant c\|v\|_{L^q(\Omega)}.$$

Furthermore, by the continuity of the double Riesz transform, we get

$$\|\partial_j \mathcal{A}_i[v]\|_{L^q(\Omega)} = \|\mathcal{R}_{i,j}[v]\|_{L^q(\Omega)} \leqslant c\|v\|_{L^q(\Omega)}.$$

Consequently, the second statement follows. The first statement now follows from the compactness of the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ for $p > N$. □

Finally, we state a crucial result about commutators involving the double Riesz transform. For the proof see [10], Theorem 10.27.

Theorem 2.25. *Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} =: \frac{1}{r} < 1$ and let $v_n \rightharpoonup v$ in $L^p(\mathbb{R}^3)$ and $u_n \rightharpoonup u$ in $L^q(\mathbb{R}^3)$. Then we have*

$$v_n \mathcal{R}_{i,j}[u_n] - u_n \mathcal{R}_{i,j}[v_n] \rightharpoonup v\mathcal{R}_{i,j}[u] - u\mathcal{R}_{i,j}[v] \ \text{ in } L^r(\mathbb{R}^3)$$

for all $1 \leqslant i, j \leqslant 3$.

2.4.3 On the regularization of solutions of transport equations

Definition 2.26. Let $\xi \in C_c^\infty(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} \xi \, \mathrm{d}x = 1$, $\operatorname{supp}(\xi) \subseteq B_1(0)$ and $\xi \geqslant 0$. Then, the family $(\xi_\epsilon)_{\epsilon \in (0,1]}$, defined by $\xi_\epsilon := \frac{1}{\epsilon^N} \xi(\frac{\cdot}{\epsilon})$, is called *smoothing sequence.*

The next Lemma follows immediately; see e.g. [19], Lemma 13.3.10 for the case $p = 1$.

Lemma 2.27. *Let $(\xi_\epsilon)_{\epsilon \in (0,1]}$ be a smoothing sequence and let $v \in L^p(\mathbb{R}^N)$ for $1 \leqslant p < \infty$. Then*

$$v * \xi_\epsilon \to 0 \text{ in } L^p(\mathbb{R}^N)$$

as $\epsilon \to 0$.

The proof for the next Lemma can be found in [23], Lemma 2.3.

Lemma 2.28. *Let (ξ_ϵ) be a smoothing sequence, $v \in W^{1,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$, $g \in L^\beta(\mathbb{R}^N)$ with $1 \leqslant \alpha, \beta \leqslant \infty$ and $\frac{1}{\gamma} := \frac{1}{a} + \frac{1}{\beta} \leqslant 1$, where we set $\frac{1}{\infty} := 0$. Then, we have*

$$A_\epsilon := \| \operatorname{div}(vg) * \xi_\epsilon - \operatorname{div}(v(g * \xi_\epsilon)) \|_{L^\gamma(\mathbb{R}^N)} \leqslant C \|v\|_{W^{1,\alpha}(\mathbb{R}^N;\mathbb{R}^N)} \|g\|_{L^\beta(\mathbb{R}^N)}$$

for some $C \geqslant 0$ independent of ϵ, v and g. Furthermore, if $\gamma < \infty$, then $\lim_{\epsilon \to 0} A_\epsilon = 0$.

We need the following immediate corollary:

Lemma 2.29. *Let $\gamma > 1$, $f \in L^1((0,T) \times \Omega)$ and let $(u, \rho) \in L^2(0,T; W^{1,2}(\mathbb{R}^3; \mathbb{R}^3)) \times L^\infty(0,T; L^\gamma(\mathbb{R}^3))$ be a solution of the transport equation*

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = f$$

in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$. Furthermore, let (ξ_ϵ) be a smoothing sequence and define $\tilde{\rho}_\epsilon := \xi_\epsilon * \rho$. Then,

$$\frac{\partial \tilde{\rho}_\epsilon}{\partial t} + \operatorname{div}(\tilde{\rho}_\epsilon u) = f * \xi_\epsilon + r_\epsilon \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^3)$$

with $r_\epsilon \to 0$ in $L^2(0,T; L^\alpha(\mathbb{R}^3))$ as $\epsilon \to 0$, where $\alpha := \frac{2\gamma}{\gamma+2}$.

Proof. We have

$$\frac{\partial \tilde{\rho}_\epsilon}{\partial t} + \operatorname{div}(\rho u) * \xi_\epsilon = \frac{\partial \rho}{\partial t} * \xi_\epsilon + \operatorname{div}(\rho u) * \xi_\epsilon = f * \xi_\epsilon$$

and consequently

$$r_\epsilon = \operatorname{div}(\tilde{\rho}_\epsilon u) - \operatorname{div}(\rho u) * \xi_\epsilon.$$

By the preceding Lemma, the right hand side tends to zero in $L^\alpha(\mathbb{R}^3)$ for a.e. fixed $t \in (0,T)$. Since

$$\|r_\epsilon\|_{L^2(0,T;L^\alpha(\mathbb{R}^3))} = \|\operatorname{div}(\tilde{\rho}_\epsilon u) - \operatorname{div}(\rho u) * \xi_\epsilon\|_{L^2(0,T;L^\alpha(\mathbb{R}^3))} \leqslant C\|v\|_{W^{1,2}(\mathbb{R}^3;\mathbb{R}^3)}\|\rho\|_{L^\gamma(\mathbb{R}^3)},$$

Lebesgue's theorem yields the desired conclusion.

□

3 Measure theory

3.1 General measure theory

Definition 3.1. Let Ω be a set. If $E \subseteq 2^\Omega$ is a system of subsets of Ω, we denote by $\mathcal{A}_\sigma(E)$ the smallest σ-algebra $\mathcal{A}$ on Ω such that $E \subseteq \mathcal{A}$. If $(\Omega_i, \mathcal{A}_i)_{i \in I}$ is a family of measurable spaces and $X_i : \Omega \to \Omega_i$ a family of functions, we denote by $\mathcal{A}_\sigma(\{X_i \mid i \in I\})$ the smallest σ-algebra $\mathcal{A}$ on Ω such that X_i is $\mathcal{A}/\mathcal{A}_i$ measurable for all $i \in I$.

If $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ are two measurable spaces, then we denote by $L^0(\Omega, \mathcal{A}; \Omega', \mathcal{A}')$ the set of all $\mathcal{A}\backslash\mathcal{A}'$ measurable functions $f : \Omega \to \Omega'$.

Lemma 3.2. *(Coincidence criterion) Let $(\Omega, \mathcal{A})$ be a measurable space, let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures on $(\Omega, \mathcal{A})$ and let $\mathcal{E} \subseteq \mathcal{A}$ be a generator of $\mathcal{A}$ which is closed under finite intersection. Assume $\mathbb{P}(A) = \mathbb{Q}(A)$ for all $A \in \mathcal{E}$. Then $\mathbb{P} = \mathbb{Q}$.*

For the proof see [4], Lemma 1.9.4.

Let X be a topological space. We denote by $\mathcal{B}(X)$ the Borel σ-algebra on X.

Theorem 3.3. *Let X and Y be topological spaces and assume that Y has a countable base. Then, $\mathcal{B}(X \times Y) = \mathcal{B}(X) \times \mathcal{B}(Y)$.*

For the proof see [5], Lemma 6.4.2.

Definition 3.4. Let (Ω, A) be a measurable space. Then we denote by $\mathscr{P}(\Omega, A)$ the set of all probability measures on (Ω, A). We sometimes simply write $\mathscr{P}(\Omega)$ if no confusion can occur; if Ω carries a topology, this notation usually indicates that Ω is equipped with the Borel σ-algebra, if not otherwise stated.

Definition 3.5. *(weak convergence of measures)* Let Ω be a topological space, let $\mathcal{A}$ be the Borel σ-algebra and let $\{\mu_n\}_{n\in\mathbb{N}} \subseteq \mathscr{P}(\Omega, \mathcal{A})$. We write $\mu_n \rightharpoonup \mu$ in $\mathscr{P}(\Omega)$ iff

$$\int f \, \mathrm{d}\mu_n \to \int f \, \mathrm{d}\mu$$

for all bounded continuous functions $f : \Omega \to \mathbb{R}$.

If Ω carries a topology, then the set $\mathscr{P}(\Omega, \mathcal{A})$ is always endowed with the topology of weak convergence of measures.

Theorem 3.6. *If (Ω, d) is a (separable and/or complete) metric space, then the topology of weak convergence of measures is metrizable by a (separable and/or complete) metric.*

For the proof see [5], Theorem 8.3.2.

Lemma 3.7. *Let Ω be a separable metric space, let $C \subseteq \mathscr{P}(\Omega)$ be a convex and weakly closed subset and let $(\Omega', \mathcal{A}', \mathbb{P}')$ be a probability space. Assume $f : \Omega' \to C$ is $\mathcal{A}'/\mathcal{B}(\mathscr{P}(\Omega))$ measurable. Then*

$$\int_{\Omega} f(\omega)(\cdot)\mathbb{P}'(\,\mathrm{d}\omega) \in C.$$

The proof can be found in [16], Lemma 7.2.

Definition 3.8. Let (Ω, d) be a metric space. We call a subset $X \subseteq \mathscr{P}(\Omega)$ *tight*, iff for all $\epsilon > 0$, there is a compact set $K \subseteq \Omega$ such that $\mathbb{P}(\Omega \backslash K) < \epsilon$ for all $\mathbb{P} \in X$.

Theorem 3.9. *(Prokorhov's theorem) Let (Ω, d) be a complete and separable metric space. A set $K \subseteq \mathscr{P}(\Omega)$ is tight if and only if K is pre-compact with respect to the weak convergence of measures.*

For the proof see [5], Theorem 8.6.2.

Theorem 3.10. *(Skorohod's theorem) Let E be a separable topological space and let $\mathbb{P}_n, \mathbb{P} \in \mathscr{P}(E, \mathcal{B}(E))$ for all $n \in \mathbb{N}$. Suppose that $\mathbb{P}_n \rightharpoonup \mathbb{P}$. Then, there exist a probability space $(\mathcal{O}, \mathcal{F}, \mathbb{Q})$ and random variables $X_n : \mathcal{O} \to \Omega$ and $X : \mathcal{O} \to \Omega$ such that $\mathbb{Q} \circ X_n^{-1} = \mathbb{P}_n$, $\mathbb{Q} \circ X^{-1} = \mathbb{P}$ and $X_n \to X$ a.e. in $\mathcal{O}$.*

For the proof see [3], Theorem 6.7.

3.2 Lebesgue-Bochner spaces

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let E be a separable Banach space. Integration of measurable functions with values in separable Banach spaces can be formally defined as in the real-valued case. For the proofs of the Theorems see [26], chapter 1 and 2.

Definition 3.11. For a simple function

$$f(\omega) := \sum_{i=1}^{n} f_i \mathbb{I}_{A_i}(\omega),$$

where $n \in \mathbb{N}$, $f_i \in E$ and $A_i \in \mathcal{A}$, the *Lebesgue-Bochner integral* is defined by

$$\int f \,\mathrm{d}\mu := \sum_{i=1}^{n} f_i \mu(A_i).$$

Analogously to the real-valued case, we have the following result:

Theorem 3.12. *Let $f : \Omega \to E$ be $\mathcal{A}/\mathcal{B}(E)$ measurable. Then, there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of simple functions, such that*

$$f_n \to f \qquad \mu - a.e.\ in\ \Omega$$

and

$$\|f_n\|_E \leqslant \|f\|_E \quad \mu - a.e. \text{ in } \Omega.$$

Remark 3.13. The preceding theorem does not hold for non-separable Banach spaces.

Definition 3.14. Let $f : \Omega \to E$ be $\mathcal{A}/\mathcal{B}(E)$ measurable and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of simple functions enjoying the properties described in the above theorem. The *Lebesgue-Bochner integral* of f is defined as

$$\int f \,\mathrm{d}\mu := \lim_{n\to\infty} \int f_n \,\mathrm{d}\mu$$

if the limit exists. In this case, we call f *Lebesgue-Bochner integrable.* We set

$$\int_A f \,\mathrm{d}\mu := \int \mathbb{I}_A f \,\mathrm{d}\mu.$$

The Lebesgue-Bochner integral is well defined:

Theorem 3.15. *The definition of the Lebesgue-Bochner integral does not depend on the approximating sequence.*

Definition 3.16. The *Lebesgue-Bochner space* $L^p(\Omega, \mathcal{A}, \mu; E)$ is the space of all (equivalence classes of μ-a.e. identical) $\mathcal{A}/\mathcal{B}(E)$ measurable functions $f : \Omega \to E$ such that

$$\|f\|_{L^p(\Omega,\mathcal{A},\mu;E)} := \left(\int \|f\|_E^p \,\mathrm{d}\mu \right)^{\frac{1}{p}} < \infty.$$

iff $p < \infty$ and

$$\|f\|_{L^\infty(\Omega,\mathcal{A},\mu;E)} := \operatorname{ess\,sup} \|f\|_E < \infty$$

iff $p = \infty$. In order to simply notation, we set $L^p(\Omega, \mu; E) := L^p(\Omega, \mathcal{A}, \mu; E)$ if no confusion can arise.

We list a few important theorems of Lebesgue-Bochner spaces:

Theorem 3.17. *The Lebesgue-Bochner space $L^1(\Omega, \mathcal{A}, \mu; E)$ is precisely the set of all (equivalence classes of) Lebesgue-Bochner integrable functions.*

Remark 3.18. Obviously, the Lebesgue-Bochner integral does not depend on μ-null sets, thus we can define $\int f \,\mathrm{d}\mu$ for $f \in L^1(\Omega, \mathcal{A}, \mu; E)$.

Theorem 3.19. *The Lebesgue-Bochner spaces are Banach spaces. For $1 < p < \infty$ the spaces are reflexive. For $1 \leqslant p < \infty$ the spaces are separable and we have*

$$(L^p(\Omega, \mu; E))' = L^{p'}(\Omega, \mu; E').$$

If $\mu(\Omega) < \infty$, then $L^q(\Omega, \mathcal{A}, \mu; E) \subseteq L^p(\Omega, \mathcal{A}, \mu; E)$ whenever $1 \leqslant p < q \leqslant \infty$. Finally, if H is a separable Hilbert space, then the space $L^2(\Omega, \mathcal{A}, \mu; H)$ is a Hilbert space with respect to the scalar product

$$\langle f, g\rangle_{L^2(\Omega,\mathcal{A},\mu;H)} := \int \langle f, g\rangle_H \,\mathrm{d}\mu$$

Definition 3.20. Let Ω be a topological space, let $\mathcal{A} := \mathcal{B}(\Omega)$ and μ be a measure on $(\Omega, \mathcal{A})$. Then we define

$$L^p_{loc}(\Omega, \mathcal{A}, \mu; E) := \{f \in L^0(\Omega, \mathcal{A}; E, \mathcal{B}(E)) \mid f\mathbb{I}_C \in L^p(\Omega, \mathcal{A}, \mu; E) \text{ for all compact } C \subseteq \Omega\}.$$

We say $f_n \to f$ in $L^p_{loc}(\Omega, \mathcal{A}, \mu; E)$ iff $\int_C \|f_n - f\|^p_E \, \mathrm{d}\mu \to 0$ for all compact $C \subseteq \Omega$.

3.3 Conditional expectation

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let E be a separable Banach space.

Theorem 3.21. *Let $\mathbb{P} \in \mathscr{P}(\Omega, \mathcal{A})$ be a probability measure, let $\mathcal{F} \subseteq \mathcal{A}$ be sub-σ-algebra and let $f \in L^1(\Omega, \mathcal{A}, \mathbb{P}; E)$. Then, there exists a unique $g \in L^1(\Omega, \mathcal{F}, \mathbb{P}|_{\mathcal{F}}; E)$ such that*

$$\int_C f \, \mathrm{d}\mathbb{P} = \int_C g \, \mathrm{d}\mathbb{P}$$

for all $C \in \mathcal{F}$.

For the proof see [26], chapter 11.

Definition 3.22. We call the function $g \in L^1(\Omega, \mathcal{F}, \mathbb{P}|_{\mathcal{F}}; E)$ defined in the last theorem the *conditional expectation of f with respect to $\mathcal{F}$* and we use the notation

$$\mathbb{E}[f|\mathcal{F}] := \mathbb{E}^{\mathbb{P}}[f|\mathcal{F}] := g.$$

Definition 3.23. Let $\mathbb{P} \in \mathscr{P}(\Omega, \mathcal{A})$ be a probability measure and let $\mathcal{F} \subseteq \mathcal{A}$ be sub-σ-algebra. A *version of the conditional probability distribution of $\mathbb{P}$ with respect to $\mathcal{F}$* is a mapping $(\omega, A) \mapsto \mathbb{P}(A|\mathcal{F})(\omega)$ for $A \in \mathcal{A}$ and $\omega \in \Omega$, such that

$$\mathbb{P}(A|\mathcal{F}) = \mathbb{E}^{\mathbb{P}}[\mathbb{I}_A|\mathcal{F}] \quad \mathbb{P}\text{- a.e. in}\, \Omega$$

for all $A \in \mathcal{A}$.

A version $\mathbb{P}(\cdot|\mathcal{F})(\cdot) : \Omega \times \mathcal{A} \to [0, 1]$ of the conditional probability distribution of $\mathbb{P}$ with respect to $\mathcal{F}$ is called *regular conditional probability distribution (r.c.p.d.) of $\mathbb{P}$ with respect to $\mathcal{F}$ iff*

(1) $\mathbb{P}(A|\mathcal{F})(\cdot)$ is $\mathcal{F}/\mathcal{B}([0, 1])$ measurable for each $A \in \mathcal{A}$;

(2) $\mathbb{P}(\cdot|\mathcal{F})(\omega) \in \mathscr{P}(\Omega, \mathcal{A})$ for each $\omega \in \Omega$.

Theorem 3.24. *Let Ω be a polish space. Then, for any sub-σ-algebra $\mathcal{F} \subseteq \mathcal{B}(\Omega)$ and any $\mathbb{P} \in \mathscr{P}(\Omega)$, there exists a r.c.p.d. of $\mathbb{P}$ with respect to $\mathcal{F}$. Furthermore, if $\mathbb{P}(\cdot|\mathcal{F})(\cdot)$ and $\mathbb{P}\widetilde{(\cdot|\mathcal{F})}(\cdot)$ are two r.c.p.d., then there is a $\mathbb{P}$-null set $N \in \mathcal{F}$ such that*

$$\mathbb{P}(A|\mathcal{F})(\omega) = \mathbb{P}\widetilde{(A|\mathcal{F})}(\omega)$$

for any $A \in \mathcal{B}(\Omega)$ and $\omega \in \Omega \backslash N$.

For the proof see [5], Corollary 10.4.6. From now on, if Ω is a polish space, the expression $\mathbb{P}(\cdot|\mathcal{F})$ always refers to a regular conditional probability distribution. Regular conditional probabilities are helpful in calculations:

Theorem 3.25. *Let Ω be measurable space, let $\mathcal{A} := \mathcal{B}(\Omega)$ and let $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{A}$ be sub-σ-algebras. Assume that some $\mathbb{P} \in \mathscr{P}(\Omega, \mathcal{A})$ admits a r.c.p.d. with respect to both sub-σ-algebras $\mathcal{F}$ and $\mathcal{G}$. Then*

$$\mathbb{P}(\cdot|\mathcal{F}) = \int_\Omega \mathbb{P}(\cdot\,|\,\mathcal{G})(\omega)\,\mathbb{P}(\,\mathrm{d}\omega\,|\,\mathcal{F}) \;\; \mathbb{P}\text{-}a.e.,$$

and for any $f \in L^1(\Omega, \mathcal{A}, \mathbb{P}; E)$, where E is a separable Banach space, we have

$$\mathbb{E}[f\,|\,\mathcal{F}] = \int_\Omega f(x)\,\mathbb{P}(\,\mathrm{d}x|\mathcal{F}) \;\; \mathbb{P}\text{-}a.e.$$

For the proof see [5], Proposition 10.4.18.

Furthermore, the next two results will be helpful in different parts of the text:

Theorem 3.26. *Let $(\Omega, \mathcal{A})$ be a measurable space, let $\mathcal{F} \subseteq \mathcal{A}$ be a countable generated sub-σ-algebra, i.e. $\mathcal{F} = \mathcal{A}_\sigma(\mathcal{D})$ for some countable $\mathcal{D} \subseteq \mathcal{A}$, and let $\mathbb{P}(\cdot|\mathcal{F})(\cdot)$ be a r.c.p.d.. Define the set*

$$K(\omega) := \bigcap\{A \in \mathcal{F}\,|\,\omega \in A\}$$

for any $\omega \in \Omega$. Then, $K(\omega) \in \mathcal{F}$, and there is a $\mathbb{P}$-null set $N \in \mathcal{F}$, such that

$$\mathbb{P}(K(\omega)|\mathcal{F})(\omega) = 1$$

for all $\omega \in \Omega\backslash N$.

For the proof see [29], Theorem 1.1.8.

Lemma 3.27. *Let $(\Omega, \mathcal{A})$ be a measurable space, let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures on $(\Omega, \mathcal{A})$ and let $\mathcal{G} \subseteq \mathcal{A}$ be a sub-σ-algebra. Assume $\mathbb{P}|_\mathcal{G} = \mathbb{Q}|_\mathcal{G}$ and $\mathbb{P}(\cdot|\mathcal{G}) = \mathbb{Q}(\cdot|\mathcal{G})$, where both are r.c.p.d. Then $\mathbb{P} = \mathbb{Q}$.*

Proof. For any $A \in \mathcal{A}$ we have

$$\mathbb{P}(A) = \int \mathbb{P}(A|\mathcal{G})\,\mathrm{d}\mathbb{P}|_\mathcal{G} = \int \mathbb{Q}(A|\mathcal{G})\,\mathrm{d}\mathbb{Q}|_\mathcal{G} = \mathbb{Q}(A).$$

□

3.4 Inequalities

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let E be a separable Banach space. To simply notations, denote in this section $L^p := L^p(\Omega, \mathcal{A}, \mu; E)$. We have the following inequalities:

Theorem 3.28. *(Hölder's inequality) Let $p \in [1, \infty]$ and $q := p'$. Then, we have*

$$\|fg\|_{L^1} \leqslant \|f\|_{L^p}\|g\|_{L^q}.$$

Corollary 3.29. *Let $p = q' \in [1, \infty]$ and $r \in (1, \infty)$. Then, we have*

$$\|fg\|_{L^r} \leqslant \|f\|_{L^{pr}}\|g\|_{L^{qr}}.$$

Proof. By Hölder's inequality we have

$$\|f^r g^r\|_{L^1} \leqslant \|f^r\|_{L^p}\|g^r\|_{L^q} = \|f\|^r_{L^{pr}}\|g\|^r_{L^{qr}}.$$

□

Corollary 3.30. *Let $\gamma \in (1,\infty)$, $\alpha > 0$ and $p = q' \in (1,\infty)$. Assume $\alpha\gamma p \geqslant 1$ and $(1-\alpha)\gamma q \geqslant 1$. Then, we have*

$$\|f\|_{L^\gamma} \leqslant \|f\|^{\alpha}_{L^{\alpha\gamma p}}\|f\|^{(1-\alpha)}_{L^{(1-\alpha)\gamma q}}.$$

Proof. By the preceding Corollary, we have

$$\|f\|_{L^\gamma} = \|f^\alpha f^{(1-\alpha)}\|_{L^\gamma} \leqslant \|f^\alpha\|_{L^{\gamma p}}\|f^{(1-\alpha)}\|_{L^{\gamma q}} = \|f\|^{\alpha}_{L^{\alpha\gamma p}}\|g\|^{(1-\alpha)}_{L^{(1-\alpha)\gamma q}}.$$

□

Finally, the following Theorem is often helpful:

Theorem 3.31. *Let μ be a finite measure and let $p > 1$. Suppose that $\|f_n\|_{L^p} \leqslant c$ for some c independent of n, and $f_n \to f$ in measure with respect to μ. Then, $f \in L^p$ and $f_n \to f$ in L^s for any $s < p$.*

Proof. Let $Q_{\epsilon,n} := [\|f_n - f\|_E > \epsilon]$ for any $n \in \mathbb{N}$ and $\epsilon > 0$.

To show $f \in L^p$, assume conversely that $f \notin L^p$. Let $A_n := [n-1 \leqslant |f| < n]$. Since μ is finite, we know that[1] $(|f|-2)^+ \notin L^p$, and thus,

$$\sum_{n=2}^{\infty} \mu(A_n)(n-2)^p = \infty.$$

Now, fix $R > 0$ and choose $N \in \mathbb{N}$ such that

$$\sum_{n=1}^{N} \mu(A_n)(n-2)^p > 2R.$$

Let

$$s := \min\{\mu(A_n) \,|\, 1 \leqslant n \leqslant N,\, \mu(A_n) > 0\}$$

and choose k_0 such that for any $k > k_0$

$$\mu(Q_{1,k}) = \mu([|f_k - f| > 1]) < \frac{s}{2}.$$

[1]For any real-valued function h, we denote $h^+ := \max\{h, 0\}$.

Now, we have

$$\begin{aligned}\int |f_k|^p \,\mathrm{d}\mu &\geqslant \sum_{n=1}^{N} \int_{A_n} |f_k|^p \,\mathrm{d}\mu \\ &\geqslant \sum_{n=1}^{N} \int_{A_n \cap Q_{1,k}^c} |f_k|^p \,\mathrm{d}\mu \\ &\geqslant \sum_{n=1}^{N} \int_{A_n \cap Q_{1,k}^c} ((|f|-1)^+)^p \,\mathrm{d}\mu \\ &\geqslant \sum_{n=1}^{N} \frac{1}{2}\mu(A_n)((n-2)^+)^p \\ &\geqslant R,\end{aligned}$$

in contradiction to $\|f_k\|_{L^p} \leqslant c$. We conclude that $f \in L^p$.

Now, fix $\epsilon > 0$, and choose n_0 such that

$$\mu(Q_{\epsilon,n}) < \epsilon$$

for all $n \geqslant n_0$. Then, we have

$$\begin{aligned}(\int \|f_n - f\|_E^s \,\mathrm{d}\mu)^{\frac{p}{s}} &\leqslant (\mu(\Omega)\epsilon^s + \int_{Q_{\epsilon,n}} \|f_n - f\|_E^s \,\mathrm{d}\mu)^{\frac{p}{s}} \\ &\leqslant 2^{\frac{p}{s}-1}\Big(\mu(\Omega)^{\frac{p}{s}}\epsilon^p + \mu(Q_{\epsilon,n})^{\frac{p}{s}-1}\int_{Q_{\epsilon,n}} \|f_n - f\|_E^p \,\mathrm{d}\mu\Big) \\ &\leqslant 2^{\frac{p}{s}-1}(\mu(\Omega)^{\frac{p}{s}}\epsilon^p + \epsilon^{\frac{p}{s}-1}(c + \|f\|_{L^p})),\end{aligned}$$

where the right hand side tends to zero as $\epsilon \to 0$. □

3.5 Measurable selections

We need the following result about the existence of measurable selections for multivalued mappings. The proof can be found in [17], Theorem 1.5.

Theorem 3.32. *(Kuratowski-Ryll-Nardzewski theorem) Let $(X, \mathcal{A})$ we any measurable space and let (Y, d) be a complete and separable metric space. Let $F : X \to 2^Y$. Assume that $F(x) \subseteq Y$ is closed and non-empty for any $x \in X$ and assume*

$$\{x \in X \mid F(x) \cap U \neq \varnothing\} \in \mathcal{A}$$

for all open sets $U \subseteq Y$. Then, there exists a measurable selection $\overline{F}$ of F, i.e. a measurable mapping $\overline{F} : X \to Y$ such that $\overline{F}(x) \in F(x)$ for all $x \in X$.

We need the following immediate corollary:

Corollary 3.33. *Let X and Y be a complete metric spaces, and let $F : X \to 2^Y$ such that $F(x)$ is closed and non-empty for any $x \in X$. Assume that $\operatorname{graph} F := \{(x, y) \in X \times Y \mid y \in F(x)\}$ is closed in $X \times Y$. Then there exists a Borel measurable selection of F.*

Proof. Let $U \subseteq Y$ be an open set and let $A := X \times U \in \mathcal{B}(X) \times \mathcal{B}(Y)$. Since F posses a closed graph, we have graph $F \in \mathcal{B}(X \times Y)$. By Lemma 3.3 we have $\mathcal{B}(X \times Y) = \mathcal{B}(X) \times \mathcal{B}(Y)$, thus $A \cap \text{graph F} \in \mathcal{B}(X) \times \mathcal{B}(Y)$. By writing

$$\{x \in X \mid F(x) \cap U \neq \varnothing\} = \{x \in X \mid \exists y \in Y \text{ such that } (x, y) \in \text{graph } F \cap A\},$$

we conclude $\{x \in X \mid F(x) \cap U \neq \varnothing\} \in \mathcal{B}(X)$. Theorem 3.32 now yields the desired conclusion. □

4 Stochastic processes

4.1 General stochastic processes

Let $(\mathcal{O}, \mathcal{F}, \mathbb{P})$ be a probability space, let $(E, \mathcal{A})$ be a measurable space and let $I \subseteq \mathbb{R}$.

Definition 4.1. A family $(\mathcal{F}_t)_{t\in I}$ of σ-algebras is called a *filtration* of $(\mathcal{O}, \mathcal{F}, \mathbb{P})$ iff $\mathcal{F}_t \subseteq \mathcal{F}_s \subseteq \mathcal{F}$ for all $s, t \in I$ with $s \geqslant t$. The tuple $(\mathcal{O}, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, \mathbb{P})$ is called *filtered probability space.*

Definition 4.2. The *predictable σ-algebra $\mathcal{P}$* (with respect to $(\mathcal{F}_t)_{t\in I}$) is the σ-algebra on $I \times \mathcal{O}$ defined by

$$\mathcal{P} := \mathcal{A}_\sigma\Big(\big\{\, ((s,t] \cap I) \times A \mid s < t,\, A \in \mathcal{F}_s\big\} \cup \mathcal{P}_0\Big)$$

where

$$\mathcal{P}_0 := \begin{cases} \varnothing & \text{if } \nexists \min I, \\ \Big\{\{\min I\} \times A \mid A \in \mathcal{F}_{\min I}\Big\} & \text{if } \exists \min I. \end{cases}$$

Definition 4.3. An E-valued stochastic process is a family of random variables $(X_t)_{t\in I}$ such that X_t is $\mathcal{F}/\mathcal{A}$ measurable for all $t \in I$.

Definition 4.4. The process X is called *adapted to* $(\mathcal{F}_t)_{t\in I}$ iff X_t is $\mathcal{F}_t/\mathcal{A}$ measurable for all $t \in I$, *progressively measurable with respect to* $(\mathcal{F}_t)_{t\in I}$ iff the mapping $X|_{I\cap(-\infty,t]} : (s, \omega) \mapsto X_s(\omega)$ is $\mathcal{B}(I \cap (-\infty, t]) \otimes \mathcal{F}_t$ measurable for all $t \in I$ and *predictable* if the mapping $X : I \times \mathcal{O} \to E$ is measurable with respect to the predictable σ-algebra.

The next Lemma follows immediately from the Definition.

Lemma 4.5. *Every predictable process is progressively measurable. Every progressively measurable process is adapted.*

Definition 4.6. The filtration $(\mathcal{F}_t)_{t\in I}$ given by

$$\mathcal{F}_t := \mathcal{A}_\sigma(\{X_s \mid s \in I,\, s \leqslant t\})$$

is called the *canonical filtration for* X.

Definition 4.7. If E is a Banach space, then we call an E-valued process $(X_t)_{t\in I}$ *simple*, if there are random variables $\widehat{X_i} : \mathcal{O} \to E$, $i = 0, \ldots, n$, such that $\widehat{X_i}$ takes only a finite number of different values for all $1 \leqslant i \leqslant n$, and $t_1 < ... < t_n$ with $t_i \in I$ for $i = 1, ..., n$, such that

$$X_t = \begin{cases} \sum_{i=1}^n \widehat{X}_i \mathbb{I}_{(t_{i-1},t_i] \cap I}(t) & \text{if } \nexists \min I, \\ \\ \sum_{i=1}^n \widehat{X}_i \mathbb{I}_{(t_{i-1},t_i] \cap I}(t) + \widehat{X}_0 \mathbb{I}_{\{\min I\}}(t) & \text{if } \exists \min I. \end{cases}$$

for all $t \in I$.

The following theorem follows immediately:

Theorem 4.8. *Let E be a separable Banach space. Then, the set $\mathcal{E}$ of simple processes is dense in $L^p(I \times \mathcal{O}, \mathcal{P}, \mathrm{d}t \otimes \mathbb{P}; E)$ for any $1 \leqslant p < \infty$.*

Proof. In this proof, we call a set $A \in \mathcal{P}$ simple, if $A = ((s,t] \cap I) \times B$ for some $s, t \in I$ and $B \in \mathcal{F}_s$, or, in the case that I posses a minimum, $A = \{\min I\} \times B$ for some $B \in \mathcal{F}_{\min I}$. Let $\mathcal{E}$ be the set of simple subsets. The set of processes of the form $X_t(\omega) = \sum_{i=1}^n \alpha_i \mathbb{I}_{A_i}(t, \omega)$ where $\alpha_i \in E$ and $A_i \in \mathcal{P}$ is dense in $L^p(I \times \mathcal{O}, \mathcal{P}, \mathrm{d}t \otimes \mathbb{P}; E)$. The process X is simple if A_i is simple for each $1 \leqslant i \leqslant n$. The proof is finished when we can show that for each $A \in \mathcal{P}$ and $\epsilon > 0$, there are simple sets B_i, $i = 1, ..., n$ such that $\mathrm{d}t \otimes \mathbb{P}(A \Delta (\bigcup_i B_i)) < \epsilon$. But the set of all those sets A form a Dynkin system $\mathcal{D}$ such that $\mathcal{E} \subseteq \mathcal{D}$. Since $\mathcal{E}$ is a generator of $\mathcal{F}$ which is closed under finite intersection, we have

$$\mathcal{F} = \mathcal{A}_\sigma(\mathcal{E}) = \mathcal{D}(\mathcal{E}) \subseteq \mathcal{D} \subseteq \mathcal{F}.$$

□

Definition 4.9. For each $\omega \in \Omega$, the mapping $t \mapsto X_t(\omega)$ is called a *path* of X. Let $F \subseteq E$. The process X is called *concentrated on the paths with values in* F, iff for a.e. $\omega \in \Omega$

$$X_t(\omega) \in F \text{ for all } t \in I.$$

If E is a topological space, then the process X is called *continuous* if for *a.e.* $\omega \in \Omega$ the path $t \mapsto X_t(\omega)$ is continuous. Right-, left-, Höldercontinuous, ... are defined analogously.

The proof for the next Lemma can be found e.g. in [20] Proposition 1.12.

Lemma 4.10. *Let E be a topological space equipped with the Borel algebra. Then, every right (or left) continuous and adapted process is progressively measurable.*

Definition 4.11. Let E be a Banach space. We call X *(square) integrable* iff X_t is (square) integrable for all $t \in I$.

Definition 4.12. For $t_1, \ldots, t_n \in I$, the probability measures $\mathbb{Q}_{t_1, \ldots, t_n} \in \mathscr{P}(E^n)$, defined by

$$\mathbb{Q}_{t_1, \ldots, t_n}(B_1 \times ... \times B_n) = \mathbb{P}(X_{t_i} \in B_i \text{ for all } 1 \leqslant i \leqslant n)$$

for all $B_i \in \mathcal{A}$, are called *finite dimensional distributions of* X. A stochastic process $Y : I \times \widetilde{\mathcal{O}} \to E$, where $\widetilde{\mathcal{O}}$ is some further probability space, is called a *version* of X iff X and Y have the same finite dimensional distributions.

4.2 Path space

Definition 4.13. The space E^I is called *path space* (for X). The *standard σ-algebra $\mathcal{G}$ on the path space* is the σ-algebra generated by sets of the form (so-called cylindrical sets)

$$\{f \in E^I \mid f(t) \in B\}$$

where $t \in I$ and $B \in \mathcal{A}$.

Definition 4.14. The probability measure $\mathbb{Q} := \mathbb{P} \circ X^{-1} \in \mathscr{P}(E^I)$ is called *distribution of* X.

Remark 4.15. A probability measure on the path space $\mathbb{Q} \in \mathscr{P}(E^I, \mathcal{G})$, is a distribution of X, if and only if for all $n \in \mathbb{N}$, $t_1, \ldots, t_n \in I$ and $B_i \in \mathcal{A}$ we have

$$\mathbb{Q}(\{u \in E^I \mid u(t_i) \in B_i \text{ for all } 1 \leqslant i \leqslant n\}) = \mathbb{P}(X_{t_i} \in B_i \text{ for all } 1 \leqslant i \leqslant n),$$

since the system of sets of this form are a generator of $\mathcal{G}$ which is closed under finite intersection. Thus, two stochastic process X and Y have the same distribution if and only if they are version of each other (Daniel-Kolmogorov theorem).

Definition 4.16. The *canonical process* $(\xi_t)_{t\in I}$ *of the path space* is given by

$$\xi_t(u) := u(t), \;\; t \in I, \, u \in E^I.$$

Definition 4.17. The *canonical filtration* $(\mathcal{G}_t)_{t\in I}$ on $(E^I, \mathcal{G})$ is given by

$$\mathcal{G}_t := \mathcal{A}_\sigma(\{\xi_s \mid s \in I, \, s \leqslant t\}),$$

i.e the canonical filtration on $(E^I, \mathcal{G})$ is the canonical filtration of ξ. In particular, the canonical process is adapted to the canonical filtration.

Remark 4.18. A probability measure $\mathbb{Q} \in \mathscr{P}(E^I)$ is a distribution of X iff the canonical process ξ and X are versions of each other.

Lemma 4.19. *Let $B \in \mathcal{G}_t$. Then* $\{u \in E^I \mid u|_{I\cap(-\infty,t]} \in B|_{I\cap(-\infty,t]}\} = B$.

Proof. The system

$$\mathcal{F} := \left\{G \in \mathcal{G} \,\middle|\, \{u \in E^I \mid u|_{I\cap(-\infty,t]} \in G|_{I\cap(-\infty,t]}\} = G\right\}$$

is a σ-algebra. Let $s \in I \cap (-\infty, t]$ and $A \in \mathcal{A}$. Then we have

$$\xi_s^{-1}(A) = \{u \in E^I \mid u(s) \in A\} \in \mathcal{F},$$

i.e. ξ_s is $\mathcal{F}/\mathcal{A}$ measurable. Thus, $\mathcal{G}_t \subseteq \mathcal{F}$. ∎

Definition 4.20. Let $I = (T, \infty)$ for some $T \in \mathbb{R}$. Then, for any $t \geqslant 0$, we define the *shift operator* $\Psi_t : E^I \to E^I$ by

$$\Psi_t(u)(s) := u(t + s).$$

Note, that images of $\mathcal{G}$-measurable sets are $\mathcal{G}$-measurable. In particular, we use the same symbol to denote the corresponding image operator $\Psi_t : \mathcal{G} \to \mathcal{G}$ defined by $\Psi_t(A) := \{\Psi_t(u) \mid u \in A\}$.

If $(X_t)_{t\in I}$ is a stochastic process with paths in $\mathcal{M} \subseteq E^I$, we use the same notation and definitions on the space $\mathcal{M}$. In particular we also call $\mathcal{M}$ the path space. The canonical filtration on $\mathcal{M}$ consists of the trace σ-algebras of the σ-algebras of the canonical filtration of E^I.

Let $F \subseteq E$. We denote by $\mathscr{P}_F(\mathcal{M})$ the set of all probability measures on $\mathcal{M}$ such that the canonical process is concentrated on the paths with values in F.

4.3 The space $C([0,\infty); E)$ as a path space

The most important path space in this thesis is the space of continuous functions from the interval $[0,\infty)$ to a separable Banach space E endowed with the Borel algebra. Everything described in this section holds for any other interval analogously. We usually denote this space by the symbol $\mathcal{U} := C([0,\infty), E)$.

We have the following result:

Theorem 4.21. *The standard σ-algebra $\mathcal{G}$ on $C([0,\infty), E)$ is the Borel σ-algebra. The system $\mathcal{E} \subseteq \mathcal{G}$ consisting of sets of the form*

$$\{u \in C([0,\infty), E) \mid u(t_i) \in A_i \ \textit{for all } 1 \leqslant i \leqslant n\}$$

where $0 \leqslant t_i$ and $A_i \subseteq E$ open, is a generator of $\mathcal{G}$ which is stable under finite intersections. The system $\mathcal{E}_t \subseteq \mathcal{G}_t$ consisting of sets of the form

$$\{u \in C([0,\infty), E) \mid u(t_i) \in A_i \ \textit{for all } 1 \leqslant i \leqslant n\}$$

where $0 \leqslant t_i \leqslant t$ and $A_i \subseteq E$ open, is a generator of $\mathcal{G}_t$ which is stable under finite intersections.

Proof. Denote by $\mathcal{G}$ the standard σ-algebra defined in the previous section. Let $\epsilon > 0$ and $u \in C([0,\infty), E)$. For every $q \in \mathbb{Q} \cap [0,\infty)$ we have $\{v \in C([0,\infty), E) \mid u(q) \in B_\epsilon(u(q))\} \in \mathcal{G}$ and therefore

$$B_\epsilon(u) = \bigcap_{q\in\mathbb{Q}\cap[0,\infty)} \bigcup_{\delta\in\mathbb{Q}\cap(0,\epsilon)} \{v \in C([0,\infty), E) \mid v(q) \in B_\delta(u(q))\} \in \mathcal{G}.$$

This shows $\mathcal{B}(C([0,\infty), E)) \subseteq \mathcal{G}$.

On the other hand, for any fixed $t \geqslant 0$, the mapping $\varphi : C([0,\infty), E) \to E$, defined by $\varphi(u) = u(t)$, is $\mathcal{B}(C([0,\infty), E))/\mathcal{B}(E)$ measurable, i.e. we have $\mathcal{G} \subseteq \mathcal{B}(C([0,\infty), E))$.

The first part of the above proof also shows that $\mathcal{E}$ is a generator of $\mathcal{G}$. The assertion about $\mathcal{G}_t$ follows similar. □

Sometimes, it will be important to consider only the space of continuous functions on the interval $[t,\infty)$ for some $t > 0$; we usually use the notation $\mathcal{U}^t := C([t,\infty), E)$ for this space. Furthermore, we use the notation $\mathcal{G}^t := \mathcal{B}(\mathcal{U}^t)$ for the standard σ-algebra and $(\mathcal{G}^t_s)_{s\geqslant t}$ for the canonical filtration. Let $\mathbb{P} \in \mathscr{P}(\mathcal{U})$ be any probability measure. By Theorem 3.26, the r.c.p.d. with respect to $\mathcal{G}_t$ satisfies the property

$$\mathbb{P}(\{v \mid u|_{[0,t]} = v|_{[0,t]}\} \mid \mathcal{G}_t)(u) = 1.$$

Thus, we can consider $\mathbb{P}(\cdot|\mathcal{G}_t)(u)$ as a probability measure on $\mathcal{U}^t$, defined by

$$\mathbb{P}(A|\mathcal{G}_t)(u) := \mathbb{P}(\{v \in \mathcal{U} \mid \exists w \in A : \ w = v|_{[t,\infty)}\} \mid \mathcal{G}_t)(u)$$

for any $A \in \mathcal{B}(\mathcal{U}^t)$. We will make use of this in different parts of the thesis.

We need three results about the space $\mathcal{U}$. The proofs can be found in [27], Lemma 4.3, Lemma 8.2, and Lemma 8.3.

Definition 4.22. Let E be a Banach space. For $q \geqslant 1$, we denote by $\mathcal{A}^q(E)$ the set of all functionals

$$\mathcal{Z} : E \to [0, \infty]$$

such that

(1) $\mathcal{Z}(x) = 0$ iff $x = 0$;

(2) $\mathcal{Z}$ is lower semi-continuous;

(3) $\mathcal{Z}(\alpha x) \leqslant \alpha^q \mathcal{Z}(x)$ for all $\alpha > 1$ and $x \in E$;

(4) $\mathcal{Z}^{-1}([0, 1])$ is relatively compact in E.

Lemma 4.23. *Let $X \hookrightarrow H \hookrightarrow X'$ be a Gelfand triple such that $X \hookrightarrow H$ compactly. Let $\{\mathbb{P}_n\}_{n\in\mathbb{N}} \subseteq \mathscr{P}_H(\mathcal{U})$ be a sequence of probability measures. Assume for some $\beta > 0$, $q \geqslant 2$ and $\mathcal{Z} \in \mathcal{A}^q(H)$ we have*

$$\sup_{n\in\mathbb{N}} \mathbb{E}^{\mathbb{P}_n}\left[\sup_{t\in[0,T]} \|\xi(t)\|_{\mathbb{H}} + \sup_{0\leqslant s<t\leqslant T} \frac{\|\xi(t) - \xi(s)\|_{\mathbb{X}'}}{|t - s|^\beta} + \int_0^T \mathcal{Z}(\xi(s))\,\mathrm{d}s\right] < \infty$$

for all $T > 0$. Then, $\mathbb{P}_n(L^q_{loc}([0, \infty), X') \cap \mathcal{U}) = 1$ and $\{\mathbb{P}_n\}_{n\in\mathbb{N}} \subseteq \mathscr{P}(L^q_{loc}([0, \infty), X') \cap \mathcal{U})$ is tight.

Theorem 4.24. *Let $D := \{(s, t) \in [0, \infty)^2 \mid s \leqslant t\}$ and let $(X_{s,t})_{(s,t)\in D}$ and $(Y_{s,t})_{(s,t)\in D}$ be families of $\mathcal{G}\backslash\mathcal{B}(\mathbb{R}^+)$ measurable random variables from $\mathcal{U}$ to $\mathbb{R}^+$, let $\mathbb{P} \in \mathscr{P}(\mathcal{U})$ and $r \geqslant 0$.*

Assume that for any fixed $s \geqslant 0$ the mapping $t \mapsto X_{s,t}$ is $\mathbb{P}$-a.s. increasing and $t \mapsto Y_{s,t}$ is $\mathbb{P}$-a.s. right continuous and $Y_{s,t}$ is $\mathcal{G}_s$-measurable for any $t \geqslant s$.

Assume further that for any $t \geqslant s \geqslant r \geqslant 0$ we have $X_{\cdot,t}\,,\,,Y_{\cdot,t} \in L^1((0, t) \times \Omega; \mathrm{d}t \otimes \mathbb{P})$ and $X_{s,t}(u) = X_{s-r,t-r}(\Psi_r u)$ and $Y_{s,t}(u) = Y_{s-r,t-r}(\Psi_r u)$

Then, the following are equivalent

(1) *There is a Lebesgue null set $T \subseteq (r, \infty)$ such that for any $s \in (r, \infty)\backslash T$ and $t \geqslant s$ we have*

$$\mathbb{E}^{\mathbb{P}}[X_{s,t} \mid \mathcal{F}_s] \leqslant Y_{s,t}.$$

(2) *For[2] $\mathbb{P}|_{\mathcal{F}_r} -$ a.e. $u \in \mathcal{U}$, there is Lebesgue null set $T_u \subseteq (0, \infty)$ such that for any $s \in (0, \infty)\backslash T_u$ and $t \geqslant s$ we have*

$$\mathbb{E}^{\mathbb{P}(\cdot|\mathcal{F}_r)\circ\Psi_r^{-1}}[X_{s,t} \mid \mathcal{F}_s] \leqslant Y_{s,t}.$$

[2] *The expression $\mathbb{P}|_{\mathcal{F}_r} -$ a.e. $u \in \mathcal{U}$ means that we can choose the exceptional null set in $\mathcal{F}_r$.*

Theorem 4.25. *Let $(X_t)_{t\geqslant 0}$ and $(Y_t)_{t\geqslant 0}$ be $(\mathcal{G}_t)_{t\geqslant 0}$-adapted, real-valued, integrable processes on $(\mathcal{U},\mathcal{G},\mathbb{P})$ such that $X_t(u) = X_{t-r}(\Psi_r u)$ and $Y_t(u) = Y_{t-r}(\Psi_r u)$ for all $t \geqslant r \geqslant 0$. Then, the following are equivalent:*

(1) *$(X_t)_{t\geqslant r}$ is a continuous martingale with square variation process $(Y_t)_{t\geqslant r}$ with respect to $\mathbb{P}$.*

(2) *For $\mathbb{P}|_{\mathcal{F}_r}$ − a.e. $u \in \mathcal{U}$, the process $(X_t)_{t\geqslant 0}$ is a continuous martingale with square variation process $(Y_t)_{t\geqslant 0}$ with respect to $\mathbb{P}(\cdot|\mathcal{F}_r)(u) \circ \Psi_r^{-1}$.*

4.4 Markov processes

Let $(\mathcal{O},\mathcal{F},\mathbb{P})$ be a probability space, $(\mathcal{F}_t)_{t\in I}$ a filtration and let E be a separable Banach space.

Definition 4.26. An adapted E-valued stochastic process $(X_t)_{t\in I}$ is called *Markov process* if $X_t \in L^1(\mathcal{O},\mathcal{F},\mathbb{P})$ for all $t \in I$ and

$$\mathbb{E}[X_t \,|\, \mathcal{F}_s] = \mathbb{E}[X_t \,|\, \mathcal{A}_\sigma(X_s)] \tag{4.1}$$

for all $s,t \in I$ with $s \leqslant t$.

Lemma 4.27. *An adapted, E-valued, integrable stochastic process $(X_t)_{t\in I}$ is a Markov process iff*

$$\mathbb{P}(X_t \in A \,|\, \mathcal{F}_s) = \mathbb{P}(X_t \in A \,|\, \mathcal{A}_\sigma(X_s)) \tag{4.2}$$

for all $s,t \in I$ with $s \leqslant t$ and $A \in \mathcal{B}(E)$, iff

$$\omega \mapsto \mathbb{P}(X_t \in A \,|\, \mathcal{F}_s)(\omega) \tag{4.3}$$

is $\mathcal{A}_\sigma(X_s)/\mathcal{B}(\mathbb{R})$ measurable for all $s,t \in I$ with $s \leqslant t$ and $A \in \mathcal{B}(E)$.

Proof. **Step 1.** Let $(X_t)_{t\in I}$ be a Markov process and let $s,t \in I$ with $s \leqslant t$ and $A \in \mathcal{B}(E)$. Since $\mathbb{I}_A$ is $\mathcal{B}(E)/\mathcal{B}(\mathbb{R})$ measurable, (4.1) implies

$$\mathbb{E}[\mathbb{I}_A(X_t) \,|\, \mathcal{F}_s] = \mathbb{E}[\mathbb{I}_A(X_t) \,|\, \mathcal{A}_\sigma(X_s)]. \tag{4.4}$$

This is equivalent to (4.2). Assume conversely that (4.4) holds. Then, by linearity we have

$$\mathbb{E}[\varphi(X_t) \,|\, \mathcal{F}_s] = \mathbb{E}[\varphi(X_t) \,|\, \mathcal{A}_\sigma(X_s)]. \tag{4.5}$$

for all simple functions φ and by approximating id with simple functions, we get (4.1) by Daniell's continuity of the conditional expectation.

Step 2. Since $\mathcal{A}_\sigma(X_s) \subseteq \mathcal{F}_s$ we have

$$\mathbb{P}(\mathbb{P}(X_t \in A \,|\, \mathcal{F}_s) \,|\, \mathcal{A}_\sigma(X_s)) = \mathbb{P}(X_t \in A \,|\, \mathcal{A}_\sigma(X_s))$$

and we obviously have

$$\mathbb{P}(\mathbb{P}(X_t \in A \,|\, \mathcal{F}_s) \,|\, \mathcal{A}_\sigma(X_s)) = \mathbb{P}(X_t \in A \,|\, \mathcal{F}_s)$$

iff $\omega \mapsto \mathbb{P}(X_t \in A \,|\, \mathcal{F}_s)(\omega)$ is $\mathcal{A}_\sigma(X_s)/\mathcal{B}(\mathbb{R})$ measurable ∎

4.5 Martingales

Let $(\mathcal{O}, \mathcal{F}, \mathbb{P})$ be a probability space, $(\mathcal{F}_t)_{t\in I}$ a filtration and let E be a separable Banach space.

Definition 4.28. An adapted E-valued stochastic process $(X_t)_{t\in I}$ is called *martingale*, if $X_t \in L^1(\mathcal{O}, \mathcal{F}, \mathbb{P})$ for all $t \in I$ and

$$\mathbb{E}[X_t \,|\, \mathcal{F}_s] = X_s$$

for all $s, t \in I$ with $t \geqslant s$.

Definition 4.29. A mapping $\tau : \mathcal{O} \to I$ is called *stopping time*, if

$$[\tau \leqslant t] \in \mathcal{F}_t$$

for all $t \in I$.

Definition 4.30. The process X is called a *local martingale,* if there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$, such that $\tau_n \to \infty$ a.e. in $\mathcal{O}$ and $(X_{(\tau_n \wedge t)})_{t\in\mathbb{I}}$ is a martingale for all $n \in \mathbb{N}$.

Theorem 4.31. *(Burkholder-Davis-Gundy inequality or BDG inequality) For any $1 \leqslant p < \infty$ there are constants c_1 and c_2 such that for all $\mathbb{R}$-valued, continuous, local martingales X with $X_0 = 0$ a.e. and $X_t \in L^p(\mathcal{O}, \mathcal{F}, \mathbb{P})$ for all $t \in I$, we have*

$$c_1\mathbb{E}[\langle X\rangle_t^{\frac{p}{2}}] \leqslant \mathbb{E}[\sup_{s\leqslant t} |X_s|^p] \leqslant c_2\mathbb{E}[\langle X\rangle_t{}^{\frac{p}{2}}].$$

For the proof see [20], Theorem 3.28. For future use, we remark that in the case $p = 1$, a possible choice for the right constant c_2 is $c_2 = 4\sqrt{2}$.

Corollary 4.32. *(BDG inequality for conditional expectation) Let $\mathcal{U} := C([0,\infty); E)$ for some separable Banach space E, let $\mathbb{P} \in \mathscr{P}(\mathcal{U})$ and let $r > 0$. Moreover, let $(X_t)_{t\geqslant 0}$ be an $\mathbb{R}$-valued, continuous, square integrable martingale X with $X_r = 0$ a.e. and assume that X satisfies $X(t,u) = X(t-s, \Psi_s u)$ for all $s > 0$ and $\mathbb{P}$-a.e. $u \in \mathcal{U}$. Then, we have*

$$\mathbb{E}^{\mathbb{P}}[\sup_{r\leqslant s\leqslant t} |X_s| \,|\, \mathcal{F}_r] \leqslant 4\sqrt{2}\mathbb{E}^{\mathbb{P}}[\langle X\rangle_t^{\frac{1}{2}} \,|\, \mathcal{F}_r].$$

Proof. We have by Theorem 4.25 and the BDG-inequality

$$\begin{aligned}
\mathbb{E}[\sup_{r\leqslant s\leqslant t} |X_s| \,|\, \mathcal{F}_r] &= \mathbb{E}^{\mathbb{P}(\cdot|\mathcal{F}_r)}[\sup_{r\leqslant s\leqslant t} |X_s|] \\
&= \mathbb{E}^{\mathbb{P}(\cdot|\mathcal{F}_r)\circ\Psi_r^{-1}}[\sup_{0\leqslant s\leqslant t-r} |X_s|] \\
&\leqslant 4\sqrt{2}\mathbb{E}^{\mathbb{P}(\cdot|\mathcal{F}_r)\circ\Psi_r^{-1}}[\langle X\rangle_{t-r}^{\frac{1}{2}}] \\
&= 4\sqrt{2}\mathbb{E}^{\mathbb{P}}[\langle X\rangle_t^{\frac{1}{2}} |\mathcal{F}_r]
\end{aligned}$$

□

4.6 Stochastic integration in finite Dimensions

Let $(\mathcal{O}, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ be a filtered probability space, let $n, m \in \mathbb{N}$ and let $I = [0, \infty)$.

Definition 4.33. Let H be an $\mathbb{R}^{n\times m}$-valued simple process. Then, for any $\mathbb{R}^m$-valued adapted stochastic process $(X_t)_{t\geqslant 0}$, *the stochastic integral of H with respect to X* is defined by

$$\int_0^t H_s \,\mathrm{d}X_s := \sum_{i=1}^k \widehat{H_i}(X_{t_i\wedge t} - X_{t_{i-1}\wedge t})$$

for all $t \geqslant 0$, if

$$H_t = \widehat{H_0}\mathbb{I}_0(t) + \sum_{i=1}^k \widehat{H_i}\mathbb{I}_{(t_{i-1},t_i]}(t)$$

Definition 4.34. An $\mathbb{R}^m$-valued adapted stochastic process X is called *total semimartingale*, if X is cádlág and adapted, and whenever H is a simple process and $\{(H^n_t)_{t\geqslant 0}\}_{n\in\mathbb{N}}$ is a sequence of simple processes, such that $H^n \to H$ in $L^\infty((0,T)\times\mathcal{O}, \mathrm{d}t\otimes\mathbb{P})$ for any $T > 0$, then

$$\int_0^T H^n_s \,\mathrm{d}X_s \to \int_0^T H_s \,\mathrm{d}X_s \text{ in probaility.}$$

In other words, the last assertion is equivalent to the fact that $\int_0^T \cdot\, \mathrm{d}X_s : \mathcal{S} \to L^0$ is a continuous operator for any $T \geqslant 0$, where the space $\mathcal{S}$ of simple processes is endowed with the topology of $L^\infty((0,T)\times\mathcal{O}, \mathbb{R}^{n\times m}))$, and the space of random variables $L^0 = L^0(\mathcal{O};\mathbb{R}^m)$ is endowed with the topology of convergence in probability.

Definition 4.35. An $\mathbb{R}^m$-valued adapted stochastic process X is called *semimartingale*, iff $(X_{t\wedge T})_{t\geqslant 0}$ is a total semimartingale for any $T > 0$.

Definition 4.36. Let H^n and H be stochastic processes for $n \in \mathbb{N}$. We say $H^n \to H$ *uniformly on compacts in probability,* or short *u.c.p.*, iff for any $T \geqslant 0$ we have

$$\sup_{0\leqslant t\leqslant T} |H^n_t - H_t| \to 0 \text{ in probability}$$

as $n \to \infty$.

Theorem 4.37. *For any semimartingale X, the mapping $H \mapsto \int_0^t H_s \,\mathrm{d}X_s$ is a bounded linear mapping if both sides are endowed with the u.c.p. topology. Moreover, the stochastic integral can be uniquely extended to the set of continuous stochastic processes by continuity. More precisely, for any $t \geqslant 0$, the stochastic integral with respect to X is a continuous map*

$$\int_0^t \cdot\, \mathrm{d}X := \begin{cases} \mathcal{C} \to \mathcal{D} \\ H \mapsto \int_0^t H_s \,\mathrm{d}X_s \end{cases}$$

where $\mathcal{C}$ respectively $\mathcal{D}$ denote the sets of continuous respectively cádlág processes, both endowed with the u.c.p topology.

For the proof see [28], Theorem 11.

Lemma 4.38. *Any adapted continuous local martingale is a semimartingale.*

For the proof see [28], Corollary 2.

Lemma 4.39. *Any adapted cádlág process of (path wise a.e.) finite variation on compact intervals is a semimartingale.*

For the proof see [28], Theorem 7.

Lemma 4.40. *The stochastic integral is continuous if and only if X is continuous a.e.*

For the proof see [28], Theorem 13.

Definition 4.41. Let $(X_t)_{t\geqslant 0}$ and $(Y_t)_{t\geqslant 0}$ be continuous, real-valued semimartingales. The *(predictable)*[3] *quadratic covariation* $\langle X, Y\rangle = (\langle X, Y\rangle_t)_{t\geqslant 0}$ is defined by

$$\langle X, Y\rangle_t := X_t Y_t - \int_0^t X \,\mathrm{d}Y - \int_0^t Y \,\mathrm{d}X.$$

The *(predictable) quadratic variation* of X is defined by

$$\langle X\rangle_t := \langle X, X\rangle_t.$$

Lemma 4.42. *The quadratic covariation process is an adapted, continuous, increasing semimartingale.*

For the proof see [28], Theorem 22.

Lemma 4.43. *Let X be a continuous, real-valued semimartingale of (path wise a.e.) finite variation. Then, $\langle X\rangle = const$, i.e. $\langle X\rangle_t = X_0^2$ for all $t \geqslant 0$.*

This is an immediate consequence from [28], Theorem 17.

Lemma 4.44. *Let X and Y be continuous, real-valued semimartingales, and assume $\langle X\rangle_t = const$, i.e. $\langle X\rangle_t = \langle X\rangle_0 = X_0^2$ for all $t \geqslant 0$. Then, $\langle X, Y\rangle = const$, i.e. $\langle X, Y\rangle_t = X_0 Y_0$ for all $t \geqslant 0$.*

This is an immediate consequence from [28], Theorem 25.

Lemma 4.45. *If $(X_t)_{t\geqslant 0}$ is a continuous, $\mathbb{R}$ -valued (local) martingale, then the quadratic variation is the unique adapted, continuous, increasing process $(\langle X\rangle_t)_{t\geqslant 0}$ starting at zero a.s., such that $|X|^2 - \langle X\rangle$ is a (local) martingale.*

For the proof see [28], Theorem 27. The importance of quadratic covariation is due to Ito's formula, which can be interpreted as a generalization of the fundamental theorem of calculus to stochastic processes.

Theorem 4.46. *(Ito's Formula) Let $X = (X^i)_{i=1}^n$ be an $\mathbb{R}^n$-valued continuous semimartingale and $f \in C^2(\mathbb{R}^n)$. Then, $f(X)$ is a semimartingale and the following formula holds for all $t \geqslant 0$:*

$$f(X_t) = f(X_0) + \int_0^t \frac{\partial f}{\partial x_i}(X_s)\,\mathrm{d}X_s^i + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)\,\mathrm{d}\langle X^i, X^j\rangle_s.$$

[3] Formally, the Definition below defines the predictable quadratic covariation and not the quadratic covariation. But these two definitions only differ if one of the involved processes is not continuous. Since in this thesis, (predictable) quadratic covariation is only used in a case when X and Y are both continuous, we restrict the definition to continuous processes. Then, the process defined below is actually both, the quadratic covariation and the predictable quadratic covariation of X and Y. Thus, from now on, this process will be called the quadratic covariation.

For the proof see [28], Theorem 32.

Definition 4.47. Let X be an $\mathbb{R}^n$-valued semimartingale. The *quadratic covariation operator* is the unique $\mathbb{R}^{n\times n}$-valued process $\langle\langle X\rangle\rangle = (\langle\langle X\rangle\rangle_t)_{t\geqslant 0}$ such that $\langle\langle X\rangle\rangle_t$ is symmetric and

$$u^T \langle\langle X\rangle\rangle_t u = \langle X \cdot u\rangle_t$$

for all $t \geqslant 0$.

4.7 Brownian Motion

The proof for all theorems about (cylindrical) Brownian motions and integration with respect to (cylindrical) Brownian motions can be found in [27], chapter 2 and appendix B.

Let $(\mathcal{O}, \mathcal{F}, \mathbb{P})$ be a probability space and let E be a separable Banach space.

Definition 4.48. A random variable $X : \mathcal{O} \to E$ is called *Gaussian,* if there is $x \in E$ and a positive and symmetric operator $Q \in L(E', E)$ such that

$$\hat{X}(x') := \int_{\mathcal{O}} \exp(i\langle x', X\rangle)\,\mathrm{d}\mathbb{P} = \exp(i\langle x', x\rangle - \frac{1}{2}\langle x', Qx'\rangle)$$

for all $x' \in E'$. Then, x and the operator Q are called *the mean* and *the covariance operator* of X.

Theorem 4.49. *Mean and covariance of a Gaussian random variable are uniquely determined. The covariance operator is the unique positive and symmetric operator $Q \in L(E', E)$ such that*

$$\langle x', Qx'\rangle = \mathbb{E}[\langle x', X\rangle^2]$$

for all $x' \in E'$.

Theorem 4.50. *A random variable $X : \mathcal{O} \to E$ is Gaussian iff $\langle x', X\rangle$ is an $\mathbb{R}$-valued Gaussian random variable for all $x' \in E'$.*

Definition 4.51. Let $(\mathcal{F}_t)_{t\in I}$ be a filtration and $Q \in L(E', E)$ be symmetric and positive. An adapted stochastic process $(W_t)_{t\geqslant 0}$ is called *Brownian motion with covariation Q* if

(1) $W(0) = 0$ a.s;

(2) $W(t) - W(s)$ is Gaussian with zero mean and covariance operator $(t-s)Q$ for all $t \geqslant s \geqslant 0$;

(3) $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $t \geqslant s \geqslant 0$;

(4) W is continuous.

Theorem 4.52. *(Lévy's martingale characterization theorem) An adapted $\mathbb{R}$-valued stochastic process $(W_t)_{t\geqslant 0}$ is a Brownian motion if and only if $W(0) = 0$ a.s. and W is a continuous martingale with quadratic variation $< W >_t = t$.*

Not every positive and symmetric operator $Q \in L(E', E)$ is the covariance operator of some Gaussian random variable. In Hilbert spaces, the set of all those operators Q can be explicitly identified:

Definition 4.53. Let U and H be separable Hilbert spaces and let $T \in L(U,H)$. Then, T is called *nuclear operator* or *of finite trace*, if there exists sequences $\{a_k\}_{k\in\mathbb{N}} \subseteq H$ and $\{b_k\}_{k\in\mathbb{N}} \subseteq U$ such that

$$Tx = \sum_{k=1}^{\infty} \langle b_k, x\rangle_U \, a_k$$

for all $x \in U$, and

$$\sum_{k\in\mathbb{N}} \|a_k\|_H \|b_k\|_U < \infty.$$

We denote the set of all nuclear operators by $L_1(U,H)$ and we set $L_1(U) := L_1(U,U)$.

Lemma 4.54. *Any nuclear operator is compact.*

Theorem 4.55. *Let $Q \in L(H)$ be symmetric and positive. Then, Q is the covariance operator of some H-valued Gaussian random variable if and only if Q is of finite trace.*

Finally, we have the following representation theorem for Brownian motions:

Theorem 4.56. *(Representation theorem) Let $(W_t)_{t\geqslant 0}$ be a Brownian motion in H with covariance operator Q. Let $(b_n)_{n\in\mathbb{N}}$ be an orthonormal basis consisting of eigenvectors of Q and let $(\lambda_n)_{n\in\mathbb{N}}$ be the corresponding sequence of eigenvalues. Then*

$$W_t = \sum_{n\in\mathbb{N}} b_n \langle W_t, b_n\rangle$$

where the sum converges in H and the processes $(\langle W_t, b_n\rangle)_{t\geqslant 0}$ are independent, real-valued Brownian motions with covariance λ_n. Conversely, if $(\beta_n)_{n\in\mathbb{N}}$ is a sequence of independent real-valued Brownian motions with identity covariance, $(\lambda_n)_{n\in\mathbb{N}} \in l^2(\mathbb{N},\mathbb{R})$ and $(b_n)_{n\in\mathbb{N}}$ is a orthonormal basis of H, then

$$W_t := \sum_{n\in\mathbb{N}} \lambda_n \beta_n b_n$$

defines a Brownian motion, where the covariance operator is given by

$$Qx = \sum_{n\in\mathbb{N}} \lambda_n \langle x, b_n\rangle_H.$$

4.8 Cylindrical Brownian motions

Let H and U be separable Hilbert spaces and let $Q \in L(H)$ be symmetric and positive.

Definition 4.57. *(Hilbert-Schmidt operators)* Let $(b_k)_{k\in\mathbb{N}}$ be an orthonormal basis of U. The space of *Hilbert-Schmidt operators* from U to H is defined by

$$L_2(U,H) := \{A \in L(U,H) \mid \sum_{k\in\mathbb{N}} \|Ab_k\|_H^2 < \infty\}.$$

The space $L_2(U,H)$ carries the scalar product

$$\langle A, B\rangle_{L_2(U,H)} := \sum_{k\in\mathbb{N}} \langle Ab_k, Bb_k\rangle_H.$$

We denote $L_2(H) := L_2(H,H)$.

Theorem 4.58. *The definition of $L_2(U,H)$ and the value of $\langle A, B\rangle_{L_2(U,H)}$ are independent of the choice of the orthonormal basis $(b_k)_{k\in\mathbb{N}}$, the space $L_2(U,H)$ is a Hilbert space and we have $\|A\|_{L(U,H)} \leqslant \|A\|_{L_2(U,H)}$ for all $A \in L_2(U,H)$.*

Theorem 4.59. *For $A \in L(U,H)$, we have $A \in L_2(U,H)$ iff $A^* \in L_2(H,U)$, and in this case $\|A\|_{L_2(U,H)} = \|A^*\|_{L_2(H,U)}$.*

The next Lemma states an important connection between nuclear and Hilbert-Schmidt operator:

Lemma 4.60. *Let $A \in L_2(H_1,H_2)$ and $B \in L_2(H_2,H_3)$, where H_i are separable Hilbert spaces for $i = 1,2,3$. Then, $BA \in L_1(H_1,H_3)$.*

Definition 4.61. A *Hilbert-Schmidt embedding from H to U* is an injective mapping $\iota \in L_2(H,U)$.

Hilbert-Schmidt embeddings always exist: Let $(b_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H, let $(c_k)_{k\in\mathbb{N}}$ be an orthonormal basis of U and let $(\alpha_k)_{k\in\mathbb{N}} \in l^2(\mathbb{N};\mathbb{R}^+)$. Then

$$\iota(x) := \sum_{k=1}^{\infty} \alpha_k \langle x, b_k\rangle c_k$$

for $x \in H$ defines a Hilbert-Schmidt embedding from H to U.

Let

$$H_0 := \operatorname{ran} Q^{\frac{1}{2}}$$

be equipped with the scalar product

$$\langle x, y\rangle_{H_0} := \langle Q^{-\frac{1}{2}}x, Q^{-\frac{1}{2}}y\rangle_H,$$

where $Q^{-\frac{1}{2}}$ denotes the pseudo inverse. We have the following result.

Lemma 4.62. *The space H_0 is a Hilbert space.*

Let $J : H_0 \to U$ be a Hilbert-Schmidt embedding. Then, $\widetilde{Q} := JJ^* \in L_1(U)$ and $\widetilde{Q}$ is a positive and symmetric operator. Thus, there exists a Brownian motion $(W_t)_{t\geqslant 0}$ in U with covariance $\widetilde{Q}$. A simple calculation shows the following.

Lemma 4.63. *The mapping $J : H_0 \to ran\ \widetilde{Q}^{\frac{1}{2}}$ is an isometry.*

Definition 4.64. A Brownian motion $(W_t)_{t\geqslant 0}$ in U with covariance $\widetilde{Q}$ is called *cylindrical Brownian motion* (in H) with covariance operator Q.

We end this subsection by introducing the following concept:

Definition 4.65. Let U and $\widetilde{U}$ be separable Hilbert spaces and let $J : U \to \widetilde{U}$ be a Hilbert-Schmidt embedding. A *Brownian stochastic basis* (or simply *Brownian basis*) *for* $(U, \widetilde{U}, J)$ is a tuple

$$(\mathcal{O}, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P}, (W_t)_{t \geqslant 0})$$

such that $(\mathcal{O}, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ is a filtered probability space and $(W_t)_{t \geqslant 0}$ is an adapted a Brownian motion in $\widetilde{U}$ with covariance operator JJ^*.

We call $(\mathcal{O}, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P}, (W_t)_{t \geqslant 0})$ a *Brownian stochastic basis* (or simply *Brownian basis*) *for* U if there exists $\widetilde{U}$ and J such that $(\mathcal{O}, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P}, (W_t)_{t \geqslant 0})$ is a Brownian stochastic basis for $(U, \widetilde{U}, J)$.

Remark 4.66. Let $(\mathcal{O}, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P}, (W_t)_{t \geqslant 0})$ be Brownian basis for $(U, \widetilde{U}, J)$. Then $(W_t)_{t \geqslant 0}$ is a cylindrical Brownian motion in U with identity covariance and corresponding Hilbert-Schmidt embedding J.

4.9 Integration with respect to Brownian motions

Let H and U be separable Hilbert spaces, let $Q \in L_1(U)$ be symmetric and positive and let $(W_t)_{t \geqslant 0}$ be a Brownian motion with covariance Q. Denote by $U_0 := \operatorname{ran} Q^{\frac{1}{2}}$, equipped with the scalar product defined in the preceding subsection.

Denote by $\mathcal{E}$ the space of all simple $L_2(U, H)$-valued processes $(\Phi_t)_{t \geqslant 0}$,

$$\Phi_t := \widehat{\Phi_0} \mathbb{I}_{\{0\}}(t) + \sum_{i=1}^{n} \widehat{\Phi_i} \mathbb{I}_{(t_{i-1}, t_i]}(t)$$

for all $t \geqslant 0$.

Definition 4.67. The *stochastic integral of* Φ *with respect to* W is defined as

$$\int_0^t \Phi_t \, \mathrm{d}W_s := \sum_{i=1}^{n} \widehat{\Phi_i} (W_{t_i \wedge t} - W_{t_{i-1} \wedge t}).$$

Lemma 4.68. *(Ito's isometry) Let* $\Phi_t \in \mathcal{E}$. *Then, for any* $t \geqslant 0$, *we have*

$$\| \int_0^t \Phi_s \, \mathrm{d}W_s \|_{L^2(\mathcal{O}, \mathcal{F}_t, \mathbb{P}; H)} = \| \Phi \|_{L^2([0,t] \times \mathcal{O}, \mathcal{B}([0,t]) \otimes \mathcal{F}_t, \, \mathrm{d}t \otimes \mathbb{P}; L_2(U_0, H))}.$$

Since Theorem 4.8 implies that

$$\overline{\mathcal{E}}^{\| \cdot \|_{L^2([0,t] \times \mathcal{O}, \mathcal{B}([0,t]) \otimes \mathcal{F}_t, \, \mathrm{d}t \otimes \mathbb{P}; L_2(U_0, H))}} = L^2([0,t] \times \mathcal{O}, \mathcal{P}_t, \mathrm{d}t \otimes \mathbb{P}; L_2(U_0, H)) := N_W^2(0, T; H)$$

where $\mathcal{P}_t$ denotes the predictable σ-algebra on $[0,t] \times \mathcal{O}$, the following definition makes sense.

Definition 4.69. The stochastic integral is extended to an isometrical mapping

$$\int_0^t \cdot \, \mathrm{d}W \; : \; N_W^2(0, T; H) \to L^2(\mathcal{O}, \mathcal{F}_t, \mathbb{P}; H).$$

4.10 Integration with respect to cylindrical Brownian motions

Let H and U be separable Hilbert spaces, let $Q \in L(U)$ be symmetric and positive, let $U_0 := \operatorname{ran} Q^{\frac{1}{2}}$ and let $(W_t)_{t \geqslant 0}$ be a cylindrical Brownian motion with covariance Q and corresponding Hilbert-Schmidt embedding $J : U_0 \to \widetilde{U}$. Let $\widetilde{Q} := JJ^* \in L_1(\widetilde{U})$ and $\widetilde{U_0} := \operatorname{ran} \widetilde{Q}^{\frac{1}{2}}$. Then, $J : U_0 \to \widetilde{U_0}$ is an isometry. An easy calculation shows that

$$X \in L^2([0,t] \times \mathcal{O}, \mathcal{P}_t, \,\mathrm{d}t \otimes \mathbb{P}; L_2(U_0, H)) := N_W^2(0,T;H)$$

if and only if

$$X \circ J^{-1} \in L^2([0,t] \times \mathcal{O}, \mathcal{P}_t, \mathrm{d}t \otimes \mathbb{P}; L_2(\widetilde{U_0}, H)).$$

Definition 4.70. Let $X \in N_W^2(0,T;H)$. Then, the *integral of X with respect to W* is defined as

$$\int_0^t X_s \,\mathrm{d}W_s := \int_0^t X_s \circ J^{-1} \,\mathrm{d}W_s.$$

Theorem 4.71. *The stochastic integral is a continuous, square integrable martingale, and Ito's isometry*

$$\| \int_0^t \Phi_s \,\mathrm{d}W_s \|_{L^2(\mathcal{O}, \mathcal{F}_t, \mathbb{P}; H)} = \|\Phi\|_{L^2([0,t] \times \mathcal{O}, \mathcal{B}([0,t]) \otimes \mathcal{F}_t, dt \otimes \mathbb{P}; L_2(U_0, H))}.$$

holds.

Theorem 4.72. *The quadratic covariation of a stochastic integral is given by*

$$\langle \int_0^{\cdot} X_s \,\mathrm{d}W_s \rangle_t = \int_0^t \|X_s\|^2_{L_2(U_0,H)} \,\mathrm{d}s.$$

4.11 Lévy processes

Let E be a separable Banach space.

Definition 4.73. We call an E-valued stochastic process $(X_t)_{t \geqslant 0}$ a *Lévy process* if

(1) $X(0) = 0$ a.s.;

(2) *(independent increments)* for all $0 \leqslant t_0 < \ldots < t_n$ the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent;

(3) *(stationary increments)* for all $s, t \geqslant 0$, the distribution of $X_{s+t} - X_s$ does not depend on s;

(4) X is continuous in probability (i.e. $\lim_{s \to t} X_s = X_t$ in probability for all $t \geqslant 0$);

(5) X is cádlág.

An important example of Lévy process are Brownian motions:

Theorem 4.74. *An E-valued stochastic process X is a Brownian motion if and only if X is a continuous Lévy process.*

Part II

Incompressible Equations

5 Introduction and Framework

5.1 Introduction

Let $(\mathcal{O}, \mathcal{G}, \mathbb{P})$ be a probability space. We consider the incompressible Navier-Stokes equations on a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^3$ driven by a stochastic Noise w, that can be formally written in the form

$$\operatorname{div} u = 0, \tag{5.1}$$

$$\mathrm{d}u = \big(\nu \triangle u - (u \cdot \nabla) u + \nabla p\big)\,\mathrm{d}t + f(x, u)\,\mathrm{d}t + \mathrm{d}(w(t, x, u)), \tag{5.2}$$

subject to the no-slip boundary condition

$$u|_{\partial\Omega} = 0, \tag{5.3}$$

and the initial condition

$$u(0) = u_0. \tag{5.4}$$

Here, the unknown functions $u : [0, \infty) \times \Omega \times \mathcal{O} \to \mathbb{R}^3$ and $p : [0, \infty) \times \Omega \times \mathcal{O} \to \mathbb{R}$ represent the random velocity and the random pressure of the fluid, the function $f : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ represents the deterministic force and the stochastic process $w : [0, \infty) \times \Omega \times \mathbb{R}^3 \times \mathcal{O} \to \mathbb{R}^3$ represents the random noise. This thesis studies the equations in the case that w is a Brownian motion, see Section 5.4.

The incompressible Navier-Stokes equations can be written as a stochastic evolution equation in the space of divergence-free vector fields. Thus, we develop an abstract framework for stochastic evolution equations in the next subsection. The subsequent subsection provides a summary of different types of solutions to abstract stochastic evolution equations. Subsection 5.4 describes how the Navier-Stokes equations can be formulated in this abstract framework. Afterwards, we focus on martingale solutions (see Definition 5.3) and show the existence of a martingale solution for arbitrary initial conditions u_0 in section 6. In section 7 abstract Markov and pre-Markov families are introduced and an abstract Markov-selection theorem is proved. Afterwards, the existence of a.s. martingale Markov solutions (see Definition 5.8) is an rather immediate consequence of this theorem.

5.2 Stochastic Evolution Equations

All definitions and results are formulated for separable Banach spaces (respectively separable Hilbert spaces); everything holds for finite dimensional spaces analogously.

Let $\mathbb{X} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{X}'$ be an evolution triple and let $(\mathcal{O}, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)_{n\in\mathbb{N}}, (W(t))_{t\geqslant 0})$ be a Brownian stochastic basis for $\mathbb{U}$ (see Definition 4.65), where $\mathbb{U}$ is a further separable Hilbert space. We assume $\mathbb{X} \hookrightarrow \mathbb{H}$ compactly.

Let A and B be measurable operators

$$A : \mathbb{H} \to \mathbb{X}', \tag{5.5}$$

$$B : \mathbb{H} \to L_2(\mathbb{U}, \mathbb{H}), \tag{5.6}$$

where all spaces are endowed with their respective Borel algebra. We consider the abstract stochastic evolution equation

$$du(t) = A(u(t))\,\mathrm{d}t + B(u(t))\,\mathrm{d}W_t, \tag{5.7}$$

$$u(0) = u_0, \tag{5.8}$$

for $t \geqslant 0$ and $u_0 \in \mathbb{H}$.

We always assume that the operators A and B satisfy the following three assumptions **(A1)**-**(A3)** :

(A1) (Demi-Continuity) We have

$$A : \mathbb{H} \to \mathbb{X}'_w$$

continuously, and

$$B^* : \mathbb{H} \to L_S(\mathbb{H}, \mathbb{U})$$

continuously. Here, $B^*(x) \in L_2(\mathbb{H}, \mathbb{U})$ denotes the adjoint operator of $B(x)$.

(A2) (Coercivity) There exists a constant $\kappa_1 \geqslant 0$ and $\mathcal{Z} \in \mathcal{A}^q(\mathbb{H})$ for some $q \geqslant 2$ such that

$$\langle A(x), x \rangle \leqslant -\mathcal{Z}(x) + \kappa_1(1 + \|x\|_{\mathbb{H}}^2)$$

for all $x \in \mathbb{X}$, where $\mathcal{A}^q(\mathbb{H})$ is defined in Definition 4.22.

(A3) (Growth condition) There exists a constant $\kappa_2 > 0$ and $\tilde{\gamma} \geqslant \gamma > 1$ such that

$$\|A(x)\|_{\mathbb{X}'}^{\gamma} \leqslant \kappa_2(1 + \|x\|_{\mathbb{H}}^{\tilde{\gamma}} + \mathcal{Z}(x)),$$

$$\|B(x)\|_{L_2(\mathbb{U},\mathbb{H})}^2 \leqslant \kappa_2(1 + \|x\|_{\mathbb{H}}^2),$$

for all $x \in \mathbb{X}$, where $\mathcal{Z}$ is as in **(A2)**.

5.3 Concepts of solutions

A huge amount of the theory of stochastic evolution equations we know today is due to, or was inspired by, Ito's works at the beginning of the twentieth century. One of Ito's most important achievements was the development of a framework within which stochastic integration can be performed. The theory of integration of stochastic processes, in turn, leads to the natural concept of *strong solutions*. The first definition below formally introduces this concept. But, as the existence of strong solutions to the incompressible Navier-Stokes equations is still an open problem today, this will be done solely for completeness. A major difficulty in the present situation is that the Navier-Stokes equations do not satisfy any kind monotonicity conditions. On the other hand, some kind of monotonicity condition on the operators A and B is usually essential to prove existence of strong solutions.

Introducing weak and martingale solutions, the two subsequent definitions weaken the concept of strong solutions considerably. The concept of martingale solutions is the weakest of the three and it is also of different nature: while strong and weak solutions are stochastic processes, martingale solutions are probability measures on the path space. As shown below, the probability distribution on the path space induced by a weak solution is always a martingale solution. In this sense, martingale solutions weaken the concept of weak solutions once more.

Finally, any of these three definitions can be extended to the corresponding Markov problem. The goal of part two of the thesis in hand is to show the existence of almost sure martingale solutions to the Markov problem associated with (5.7), (5.8).

For the rest of this section, we let $\mathcal{U} := C([0,\infty),\mathbb{X}')$. We consider $\mathcal{U}$ as a path space and use the notations and definitions from the preliminaries. The next three definitions introduce strong, weak and martingale solutions.

Definition 5.1. Let $(\mathcal{O},\mathcal{F},(\mathcal{F}_t)_{t\geqslant 0},\mathbb{P},(W_t)_{t\geqslant 0})$ be a given Brownian basis for $\mathbb{U}$ and let $u_0 \in \mathbb{H}$. We call an $(\mathcal{F}_t)_{t\geqslant 0}$-predictable stochastic process $u : [0,\infty)\times\mathcal{O} \to \mathbb{H}$ a *strong solution of (5.7), (5.8)* iff

(1) $u(.,\omega) \in C([0,\infty);\mathbb{X}')$ for *a.e.* $\omega \in \Omega$;

(2) $u(0,\omega) = u_0$ for *a.e.* $\omega \in \Omega$;

(3) $A(u) \in L^1((0,T)\times\mathcal{O}, \mathrm{dt}\otimes\mathbb{P};\mathbb{X}')$ for all $T > 0$;

(4) $B(u) \in L^2((0,T)\times\mathcal{O}, \mathrm{dt}\otimes\mathbb{P};L_2(\mathbb{U},\mathbb{H}))$ for all $T > 0$;

(5) We have for all $t \geqslant 0$

$$u(t) = u(0) + \int_0^t A(u(s))\,\mathrm{d}s + \int_0^t B(u(s))\,\mathrm{d}W_s.$$

Definition 5.2. Let $u_0 \in \mathbb{H}$. A *weak solution of (5.7), (5.8)* consists of a Brownian stochastic basis $(\mathcal{O},\mathcal{F},(\mathcal{F}_t)_{t\geqslant 0},\mathbb{P},(W_t)_{t\geqslant 0})$ and an $(\mathcal{F}_t)_{t\geqslant 0}$-predictable stochastic process $u : [0,\infty)\times\mathcal{O} \to \mathbb{H}$ such that (1) - (5) from the preceding Definition 5.1 are satisfied.

Often, we simply call u a *weak solution of (5.7), (5.8)* and $(\mathcal{O},\mathcal{F},(\mathcal{F}_t)_{t\geqslant 0},\mathbb{P},(W_t)_{t\geqslant 0})$ the *Brownian basis for the solution u.*

Definition 5.3. Let $u_0 \in \mathbb{H}$. A probability measure $\mathbb{P} \in \mathscr{P}(\mathcal{U})$ is called a *martingale solution of (5.7), (5.8)* iff

(1) The canonical process $\xi : [0,\infty)\times\mathcal{U} \to \mathbb{X}'$ is $\mathbb{P}-$concentrated on the paths with values in $\mathbb{H}$;

(2) $u(0) = u_0$ for $\mathbb{P} - a.e.\ u \in \mathcal{U}$;

(3) $A(\xi) \in L^1((0,T)\times\mathcal{U}, \mathrm{dt}\otimes\mathbb{P};\mathbb{X}')$ for all $T > 0$;

(4) $B(\xi) \in L^2((0,T)\times\mathcal{U}, \mathrm{dt}\otimes\mathbb{P};L_2(\mathbb{U},\mathbb{H}))$ for all $T > 0$;

(5) There exists a dense subspace $\mathbb{Y} \leqslant \mathbb{X}$ such that for all $x \in \mathbb{Y}$ the process

$$M_x(t,u) := \langle u(t), x\rangle - \langle u(0), x\rangle - \int_0^t \langle A(u(s)), x\rangle \,\mathrm{d}s$$

is a continuous, square integrable $(\mathcal{G}_t)$-martingale with respect to $\mathbb{P}$, whose quadratic variation process is given by

$$\langle M_x\rangle(t,u) = \int_0^t \|B^*(u(s))(x)\|_{\mathbb{U}}^2 \,\mathrm{d}s$$

where $B^*(u)$ denotes the adjoin operator of $B(u)$.

Obviously, any strong solution is also a weak solution. A little more work is needed to show that the probability measure on the path space induced by an arbitrary weak solution is always a martingale solution. The proof is given below.

The next Lemma provides a natural probability space for weak solutions:

Lemma 5.4. *Let $\mathbb{V}$ be any separable Hilbert space and let*

$$\mathcal{V} := C([0,\infty); \mathbb{V}).$$

If there exists a weak solution, then there exists an identically distributed weak solution of the form

$$\left[u, (\mathcal{U} \times \mathcal{V}, \mathcal{G} \otimes \mathcal{F}, (\mathcal{G}_t \otimes \mathcal{F}_t)_{t\geqslant 0}, \mathbb{P}, (W_t)_{t\geqslant 0})\right]$$

where $\mathcal{G}$ and $\mathcal{G}_t$, respectively $\mathcal{F}$ and $\mathcal{F}_t$, are the Borel $\sigma-$Algebra and the natural filtration on $\mathcal{U}$, respectively $\mathcal{V}$, $\mathbb{P}$ is a probability measure on $(\mathcal{U} \times \mathcal{V}, \mathcal{G} \otimes \mathcal{F})$, the process u is the canonical process on the first component, i.e.

$$u(t,v,w) := \xi(t,v) = v(t) \qquad t \geqslant 0,\ v \in \mathcal{U},\ w \in \mathcal{V}$$

and W is the canonical process on the second component, i.e.

$$W(t,v,w) := \xi(t,w) = w(t) \qquad t \geqslant 0,\ v \in \mathcal{U},\ w \in \mathcal{V}$$

Proof. Since there exists a weak solution, there is a separable Hilbert space $\widetilde{\mathbb{U}}$ and $J : \mathbb{U} \to \widetilde{\mathbb{U}}$ such that $(\overline{\mathcal{O}}, \overline{\mathcal{F}}, (\overline{\mathcal{F}_t})_{t\geqslant 0}, \overline{\mathbb{P}}, (\overline{W_t})_{t\geqslant 0})$ is a Brownian stochastic basis for $(\mathbb{U}, \widetilde{\mathbb{U}}, J)$ and

$$[\overline{u}, (\overline{\mathcal{O}}, \overline{\mathcal{F}}, (\overline{\mathcal{F}_t})_{t\geqslant 0}, \overline{\mathbb{P}}, (\overline{W_t})_{t\geqslant 0})]$$

is a weak solution. Let $\varphi : \widetilde{\mathbb{U}} \to \mathbb{V}$ be an isometrical isomorphism. Then, $J_{\mathbb{V}} := \varphi \circ J$ is a Hilbert-Schmidt embedding from $\mathbb{U}$ to $\mathbb{V}$ and $(W_t^{\mathbb{V}})_{t\geqslant 0} := (\varphi(W_t))_{t\geqslant 0}$ is a Brownian motion in $\mathbb{V}$ with covariation operator $\varphi J(\varphi J)^*$. Consequently, $(\overline{\mathcal{O}}, \overline{\mathcal{F}}, (\overline{\mathcal{F}_t})_{t\geqslant 0}, \overline{\mathbb{P}}, (\overline{W_t^{\mathbb{V}}})_{t\geqslant 0})$ is a Brownian basis for $(\mathbb{U}, \mathbb{V}, J_{\mathbb{V}})$ and $[\overline{u}, (\overline{\mathcal{O}}, \overline{\mathcal{F}}, (\overline{\mathcal{F}_t})_{t\geqslant 0}, \overline{\mathbb{P}}, (\overline{W_t^{\mathbb{V}}})_{t\geqslant 0})]$ is a weak solution. Thus, we can assume without loss of generality that $\widetilde{\mathbb{U}} = \mathbb{V}$ and $W_t = W_t^{\mathbb{V}}$.

We can regard $(\overline{u}, \overline{W})$ as a random variable with values in $(\mathcal{U} \times \mathcal{V}, \mathcal{G} \otimes \mathcal{F})$, thus we can define

$$\mathbb{P} := \overline{\mathbb{P}} \circ (\overline{u}, \overline{W})^{-1}.$$

Then, the combined processes (u, W) and $(\overline{u}, \overline{W})$ are equally distributed, i.e.

$$\overline{\mathbb{P}} \circ (\overline{u} \circ \overline{W})^{-1} = \mathbb{P} \circ (u, W)^{-1}.$$

Furthermore, $(W_t)_{t \geqslant 0}$ is a cylindrical Brownian motion in $\mathbb{U}$ and u and W are adapted processes. Now it follows immediately that $\left[u, (\mathcal{U} \times \mathcal{V}, \mathcal{G} \otimes \mathcal{F}, (\mathcal{G}_t \otimes \mathcal{F}_t)_{t \geqslant 0}, \mathbb{P}, (W_t)_{t \geqslant 0})\right]$ is a weak solution. $\square$

The next three Definitions introduce the concept of Markov-solutions.

Definition 5.5. Let $\mathcal{B} = (\mathcal{O}, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P}, (W_t)_{t \geqslant 0})$ be a Brownian basis for $\mathbb{U}$ and let $(u_x)_{x \in \mathbb{H}}$ be a family of stochastic processes, such that for all $x \in \mathbb{H}$ the process u_x is a strong solution of (5.7), (5.8) with initial value $u_0 = x$. Then, the family $(u_x)_{x \in \mathbb{H}}$ is called *almost sure strong solution to the Markov problem associated with (5.7),(5.8)*, or short *a.s. strong Markov solution,* iff

(1) for each $0 \leqslant t_1 < t_2 < \ldots < t_n$ and all $A_i \in \mathcal{B}(\mathbb{X}')$, $i = 1, \ldots, n$, the mapping

$$x \mapsto \mathbb{P} \circ (u_x(t_1), \ldots, u_x(t_n))^{-1}(A_1 \times \ldots \times A_n)$$

is $\mathcal{B}(\mathbb{H})/\mathcal{B}(\mathbb{R})$ measurable;

(2) for all $x \in \mathbb{H}$ there exists a Lebesgue null set $T_x \subseteq (0, \infty)$ such that for all $t \in (0, \infty) \backslash T_x$ and any $t_1, t_2, \ldots, t_n \geqslant t$ we have

$$\mathbb{P}(\cdot | \mathcal{F}_t) \circ \big(u_x(t_1), u_x(t_2), \ldots, u_x(t_n)\big)^{-1} = \mathbb{P} \circ \big(u_{u_x(t)}(t_1 - t), u_{u_x(t)}(t_2 - t), \ldots, u_{u_x(t)}(t_n - t)\big)^{-1}.$$

With minor modifications, this definition can be extended to weak solutions:

Definition 5.6. Let $(u_x)_{x \in \mathbb{H}}$ be a family of stochastic processes and let

$$(\mathcal{B}^x)_{x \in \mathbb{H}} = (\mathcal{O}^x, \mathcal{F}^x, (\mathcal{F}^x_t)_{t \geqslant 0}, \mathbb{P}^x, (W^x_t)_{t \geqslant 0})_{x \in \mathbb{H}}$$

be a family of Brownian bases for $\mathbb{U}$, such that for all $x \in \mathbb{H}$ the process u_x is a weak solution of (5.7), (5.8) with initial value $u_0 = x$ and Brownian basis $\mathcal{B}^x$. Then, the family $(u_x)_{x \in \mathbb{H}}$ is called an *almost sure weak solution to the Markov problem associated with (5.7),(5.8)*, or short *a.s weak Markov solution,* iff

(1) for each $0 \leqslant t_1 < t_2 < \ldots < t_n$ and all $A_i \in \mathcal{B}(\mathbb{X}')$, $i = 1, \ldots, n$, the mapping

$$x \mapsto \mathbb{P}^x \circ (u_x(t_1), \ldots, u_x(t_n))^{-1}(A_1 \times \ldots \times A_n)$$

is $\mathcal{B}(\mathbb{H})/\mathcal{B}(\mathbb{R})$ measurable;

(2) for all $x \in \mathbb{H}$ there exists a Lebesgue null set $T_x \subseteq (0, \infty)$ such that for all $t \in (0, \infty) \backslash T_x$ and any $t_1, t_2, \ldots, t_n \geqslant t$ we have

$$\mathbb{P}^x(\cdot | \mathcal{F}^x_t) \circ \big(u_x(t_1), u_x(t_2), \ldots, u_x(t_n)\big)^{-1} = \mathbb{P}^{u_x(t)} \circ \big(u_{u_x(t)}(t_1 - t), u_{u_x(t)}(t_2 - t), \ldots, u_{u_x(t)}(t_n - t)\big)^{-1}.$$

Remark 5.7. Let $(u_x)_{x\in\mathbb{H}}$ be an a.s. weak Markov solution. The above definition implies that each u_x is a Markov process: on the one hand, since the paths of u_x are continuous, it is sufficient to show the Markov property only for a dense subset $I \subseteq (0,\infty)$; in particular, we can assume $I = (0,\infty)\backslash T_x$. On the other hand, since u_x is $(\mathcal{F}_t^x)_{t\geqslant 0}$ adapted, it is sufficient to show that for each $t \in I$, $s > t$ and $A \in \mathcal{B}(\mathbb{H})$, the random variable $\mathbb{P}^x(\cdot|\mathcal{F}_t^x) \circ \big(u_x(s)\big)^{-1}(A)$ is $\mathcal{A}_\sigma(u_x(t))/\mathcal{B}(\mathbb{R})$ measurable. But this is obvious by (2) of Definition 5.6, since

$$\omega \mapsto u_x(t)(\omega)$$

is $\mathscr{A}_\sigma(u_x(t))/\mathcal{B}(\mathbb{H})$ measurable and

$$y \mapsto \mathbb{P}^y \circ u_y(s-t)^{-1}(A)$$

is $\mathcal{B}(\mathbb{H})/\mathcal{B}(\mathbb{R})$ measurable.

We can also formulate the idea of the Markov problem in terms of martingale solutions. Again, the connection to the previous definitions is shown below.

Definition 5.8. Let $(\mathbb{Q}_x)_{x\in\mathbb{H}} \subseteq \mathscr{P}(\mathcal{U})$ be a family of probability measures such that $\mathbb{Q}_x$ is a martingale solution of (5.7), (5.8) with initial value $u_0 = x$. Then, the family $(\mathbb{Q}_x)_{x\in\mathbb{H}}$ is called *almost sure martingale solution to the Markov problem associated with* (5.7), (5.8), or short *a.s. martingale Markov solution*, iff

(1) for each $A \in \mathcal{G} = \mathcal{B}(C(0,T;\mathbb{X}'))$ the mapping

$$x \mapsto \mathbb{Q}_x(A)$$

is $\mathcal{B}(\mathbb{H})/\mathcal{B}(\mathbb{R})$ measurable;

(2) for all $x \in \mathbb{H}$ there exists a Lebesgue null set $T_x \subseteq (0,\infty)$ such that for all $t \in (0,\infty)\backslash T_x$ and $v \in \mathcal{U}$ we have

$$\mathbb{Q}_x(\cdot|\mathcal{G}_t)(v) = \mathbb{Q}_{v(t)} \circ \Psi_t,$$

where both sides are considered as probability measures on $\mathcal{U}^t = C([t,\infty),\mathbb{X}')$ (see section 4.3) and where $\Psi_t : \mathcal{U}^t \to \mathcal{U}$ denotes the shift operator

$$\Psi_t(v)(s) = v(s+t).$$

The next two theorems summarize the connection between weak and martingale solutions and the connections between the corresponding Markov problems.

Theorem 5.9. *Let $u_0 \in \mathbb{H}$ and let $(u_t)_{t\geqslant 0}$ be a weak solution with initial value u_0 and Brownian basis $(\mathcal{O},\mathcal{F},(\mathcal{F}_t)_{t\geqslant 0},\mathbb{P},(W_t)_{t\geqslant 0})$. Then,*

$$\mathbb{Q} := \mathbb{P} \circ u^{-1} \in \mathscr{P}(\mathcal{U})$$

is a martingale solution with initial value u_0.

Assume conversely that the noise is state independent, i.e. assume that $B(x) = B$ for some $B \in L_2(\mathbb{U}, \mathbb{H})$ and all $x \in \mathbb{H}$, and assume $\ker B = \{0\}$. Let $\mathbb{Q} \in \mathscr{P}(\mathcal{U})$ be a martingale solution with initial value u_0. Then, the canonical process ξ_t is a weak solution with initial value u_0.

Proof. **Step 1.** The canonical process $\xi : [0,\infty) \times \mathcal{U} \to \mathbb{X}'$ has the same distribution under $\mathbb{Q}$ as u under $\mathbb{P}$, i.e. we have

$$\mathbb{Q} \circ \xi^{-1} = \mathbb{P} \circ u^{-1}.$$

Since u is concentrated on the paths with values in $\mathbb{H}$, the process ξ is too. Next, we have

$$\mathbb{Q}(\{v \in \mathcal{U} \,:\, v(0) = u_0\}) = \mathbb{P}\big[u(0) = u_0\big] = 1.$$

The integrability conditions on A and B follow immediately. For the last point of Definition 5.3 choose $\mathbb{Y} = \mathbb{X}$ and let $x \in \mathbb{X}$ arbitrary. Define $M_t(v) := M_x(t, v)$, then Definition 5.2 implies

$$M_t(u(\cdot,\omega)) = M_x(t, u(\cdot,\omega)) = \bigg(\int_0^t \langle B(u(s)), x\rangle \,\mathrm{d}W_s\bigg)(\omega)$$

for $a.e.\ \omega \in \mathcal{O}$. By this equation, M_t is defined $\mathbb{Q}-a.e.$ in $\mathcal{U}$. We have $\big(v \mapsto M_t(v)\big) \in L^2(\mathcal{U}, \mathcal{G}_t, \mathbb{Q}; \mathbb{R})$ for all $t \geqslant 0$, since by Ito's isometry we have

$$\mathbb{E}^{\mathbb{Q}}[M_t(\cdot)^2] = \mathbb{E}^{\mathbb{P}}[M_x(t,\xi)^2] \overset{Ito}{=} \|\langle B(u(\cdot,\cdot)), x\rangle\|^2_{L_2((0,T)\times\mathcal{O}, dt\otimes\mathbb{P};\, L_2(\mathbb{U},\mathbb{R}))} \overset{Def. 5.2, (4)}{<} \infty.$$

Furthermore, we have $\big(t \mapsto M_t(v)\big) \in C([0,\infty))$ for $\mathbb{Q}-a.e.\ v \in \mathcal{U}$. We show that $(M_t)_{t\geqslant 0}$ is a martingale with respect to $(\mathcal{U}, \mathcal{G}, (\mathcal{G}_t)_{t\geqslant 0}, \mathbb{Q})$, i.e. we need to show

$$\mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{G}_s](v) = M_s(v) \qquad \text{for } \mathbb{Q}-a.e.\ v \in \mathcal{U}$$

for all $t \geqslant s$. From the definition of $\mathbb{Q}$ it is clear, that it is sufficient to show

$$\mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{G}_s](u(\cdot,\omega)) = M_s(u(\cdot,\omega)) \qquad \text{for } \mathbb{P}-a.e.\ \omega \in \mathcal{O}$$

From the theory of stochastic integration with respect to a cylindrical Brownian motion, we know that

$$N_t(\omega) := M_t(u(\cdot,\omega))$$

is martingale with respect to $(\mathcal{O}, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, i.e. we have

$$\mathbb{E}^{\mathbb{P}}[M_t(u)|\mathcal{F}_s](\omega) = M_s(u(\cdot,\omega)). \tag{5.9}$$

Consequently, we are left to show

$$\mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{G}_s] \circ u(\cdot,\omega) = \mathbb{E}^{\mathbb{P}}[M_t \circ u|\mathcal{F}_s](\omega) \qquad \text{for } \mathbb{P}-a.e.\ \omega \in \Omega.$$

If we consider both sides as a function in ω, then both sides are mappings from $\mathcal{O}$ to $\mathbb{R}$. Furthermore, if we consider u as a mapping $u : \mathcal{O} \to \mathcal{U}$, $\omega \mapsto u(\cdot,\omega)$, then both sides are $u^{-1}(\mathcal{G}_s)/\mathcal{B}(\mathbb{R})$ measurable, since both sides are compositions of a $\mathcal{G}_s/\mathcal{B}(\mathbb{R})$-measurable mapping and the $u^{-1}(\mathcal{G}_s)/\mathcal{G}_s$-measurable mapping u; for the right hand side, this follows from (5.9). Let $B \in \mathcal{G}_s$ and $A := u^{-1}(B)$, then we have

$$\begin{aligned}\int_A \mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{G}_s] \circ u \,\mathrm{d}\mathbb{P} &= \int_B \mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{G}_s] \,\mathrm{d}\mathbb{Q} \\ &= \int_B M_t \,\mathrm{d}\mathbb{Q} \\ &= \int_A M_t \circ u \,\mathrm{d}\mathbb{P} \\ &= \int_A \mathbb{E}^{\mathbb{P}}[M_t \circ u|\mathcal{F}_s] \,\mathrm{d}\mathbb{P}.\end{aligned}$$

Thus, we have shown that $(M_t)_{t\geqslant 0}$ is a continuous, square integrable martingale with respect to $\mathbb{Q}$.

Finally, define the process $(R_t)_{t\geqslant 0}$ by

$$R_t(v) := \int_0^t \|B^*(v(s))(x)\|_{\mathbb{U}}^2 \,\mathrm{d}s = \int_0^t \|\langle B(v(s))(\cdot), x\rangle\|_{L_2(\mathbb{U},\mathbb{R})}^2 \,\mathrm{d}s.$$

The proof is finished, when we can show that $(R_t)_{t\geqslant 0}$ is the quadratic variation process of $(M_t)_{t\geqslant 0}$. Obviously, $(R_t)_{t\geqslant 0}$ is a continuous, non-decreasing, $(\mathcal{G}_t)$-adapted process starting at zero a.e. with respect to $\mathbb{Q}$. Thus, we are left to show that $(|M_t|^2 - R_t)_{t\geqslant 0}$ is a martingale with respect to $\mathbb{Q}$.

The theory of stochastic integration implies that $\omega \mapsto R_t(u(\cdot,\omega))$ is the quadratic variation process of $\omega \mapsto M_t(u(\cdot,\omega))$ with respect to $\mathbb{P}$, and consequently

$$\omega \mapsto \Big(|M_t(u(\cdot,\omega))|^2 - R_t(u(\cdot,\omega))\Big)$$

is a martingale with respect to $(\mathcal{O}, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$. Exactly as above, we deduce that

$$v \mapsto \Big(|M_t(v)|^2 - R_t(v)\Big)$$

is a martingale with respect to $(\mathcal{U}, \mathcal{G}, (\mathcal{G}_t)_{t\geqslant 0}, \mathbb{Q})$.

Step 2. The first four points of Definition 5.2 are easy to check. For the last point, let $x \in \mathbb{Y}$. We have

$$\langle M_x\rangle(t) = t\sigma^2$$

with

$$\sigma = \|B^* x\|_U,$$

and by Lévy's martingale characterisation theorem it follows that $(M_x(t))_{t\geqslant 0}$ is a Brownian motion. Now choose an orthonormal basis $(b_n)_{n\in\mathbb{N}}$ of $\mathbb{H}$ with $b_n \in \mathbb{Y}$ for all $n \in \mathbb{N}$ and define

$$W_t := \sum_{n\in\mathbb{N}} b_n M_{b_n}(t).$$

Then W_t is a Brownian motion on $(\mathcal{U}, \mathcal{G}, (\mathcal{G}_t)_{t\geqslant 0}, \mathbb{Q})$ with values in $\mathbb{H}$ and covariance operator BB^*. To construct the desired Brownian basis we choose $\widetilde{\mathbb{U}} = \mathbb{H}$ and $J = B$. Then $(W_t)_{t\geqslant 0}$ is a cylindrical Brownian motion in $\mathbb{U}$ with identity covariance and thus $(\mathcal{U}, \mathcal{G}, (\mathcal{G}_t)_{t\geqslant 0}, \mathbb{Q}, (W_t)_{t\geqslant 0})$ is a Brownian basis for $(\mathbb{U}, \mathbb{H}, B)$. Now, we have

$$M_x(t) = \sum_{n\in\mathbb{N}} \langle b_n, x\rangle M_{b_n}(t) = \langle W(t), x\rangle = \langle B \circ B^{-1} W(t), x\rangle = \langle \int_0^t B \,\mathrm{d}W_s, x\rangle$$

for all $x \in \mathbb{Y}$ and consequently

$$\langle u(t), x\rangle = \langle u(0), x\rangle + \langle \int_0^t A(u(s))\,\mathrm{d}s, x\rangle + \langle \int_0^t B(u(s))\,\mathrm{d}W_s, x\rangle.$$

Since $\mathbb{Y}$ is dense in $\mathbb{X}$, this shows

$$u(t) = u(0) + \int_0^t A(u(s))\,\mathrm{d}s + \int_0^t B(u(s))\,\mathrm{d}W_s.$$

□

Theorem 5.10. *Let $(u_x)_{x\in\mathbb{H}}$ be an a.s. weak Markov solution, and denote the Brownian basis for u_x by $\mathcal{B}^x := (\mathcal{O}^x, \mathcal{F}^x, (\mathcal{F}_t^x)_{t\geqslant 0}, \mathbb{P}^x, (W_t^x)_{t\geqslant 0})$. Then, the family of probability measures $(\mathbb{Q}_x)_{x\in\mathbb{H}}$ where $\mathbb{Q}_x := \mathbb{P}^x \circ u_x^{-1}$ is an a.s. martingale Markov solution.*

Proof. Let $\mathcal{E}$ be the set of all $\Gamma \in \mathcal{G}$ of the form $\Gamma = \bigcap_{i=1}^n \Gamma_i$ with

$$\Gamma_i = \{v \in \mathcal{U} \,:\, v(t_i) \in A_i\}$$

where $n \in \mathbb{N}$, $0 < t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n$ and $A_i \in \mathcal{B}(\mathbb{X}')$. Then, $\mathcal{E}$ is a generator of $\mathcal{G}$ and $\mathcal{E}$ is closed under finite intersection.

For all $A \in \mathcal{E}$ the mapping

$$x \mapsto \mathbb{Q}_x(A) = \mathbb{P}^x \circ u_x^{-1}(A)$$

is measurable by Definition 5.6. By Lemma 5.11 below, the above mapping is measurable for all $A \in \mathcal{G}$.

To prove point (2), we assume without loss of generality that all weak solutions $[u_x, \mathcal{B}^x]$ are defined as in Lemma 5.4. Let $\mathcal{E}^t$ be the set of all $\Gamma \in \mathcal{G}^t = \mathcal{B}(\mathcal{U}^t)$ of the form $\Gamma = \bigcap_{i=1}^n \Gamma_i$ with

$$\Gamma_i = \{v \in \mathcal{U} \,:\, v(t_i) \in A_i\}$$

where $n \in \mathbb{N}$, $t < t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n$ and $A_i \in \mathcal{B}(\mathbb{X}')$. It is enough to show that we have

$$\mathbb{Q}_x(\Gamma|\mathcal{G}_t)(v) = \mathbb{Q}_{v(t)} \circ \Psi_t(\Gamma) \tag{5.10}$$

for all $\Gamma \in \mathcal{E}^t$. We have for a.e. $v \in \mathcal{U}$ by the Definition of a.s. weak Markov solutions

$$\begin{aligned}\mathbb{Q}_x(\Gamma|\mathcal{G}_t)(v) &= \mathbb{P}^x((u_x(t_1), ..., u_x(t_n))^{-1}(A_1 \times ... \times A_n)|\mathcal{F}_t^x)(v)\\ &= \mathbb{P}^{v(t)} \circ (u_{v(t)}(t_1 - t), ..., u_{v(t)}(t_n - t))^{-1}(A_1 \times ... \times A_n)\\ &= \mathbb{Q}_{v(t)}(\{w \in \mathcal{U} : w(t_i - t) \in A_i \quad \text{for all } 1 \leqslant i \leqslant n\})\\ &= \mathbb{Q}_{v(t)} \circ \Psi_t(\Gamma).\end{aligned}$$

□

Lemma 5.11. *Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be measurable spaces, and let $(\mathbb{P}_\omega)_{\omega\in\Omega}$ be a family of probability measures on $(\Omega', \mathcal{A}')$. Assume that the mapping $\omega \mapsto \mathbb{P}_\omega(A)$ is $\mathcal{A}/\mathcal{B}(\mathbb{R})$ measurable for all $A \in \mathcal{E}$, where $\mathcal{E} \subseteq \mathcal{A}'$ is a generator of $\mathcal{A}'$ which is closed under finite intersection. Then, $\omega \mapsto \mathbb{P}_\omega(A)$ is $\mathcal{A}/\mathcal{B}(\mathbb{R})$ measurable for all $A \in \mathcal{A}'$.*

Proof. Let

$$\mathcal{D} := \{A \in \mathcal{A}' \,|\, \omega \mapsto \mathbb{P}_\omega(A) \text{ is } \mathcal{A}/\mathcal{B}(\mathbb{R}) \text{ measurable}\}.$$

Then, $\mathcal{D}$ is a Dynkin system and $\mathcal{E} \subseteq \mathcal{D}$. Thus, we have

$$\mathcal{A}' = \mathcal{A}_\sigma(\mathcal{E}) = \mathcal{D}(\mathcal{E}) \subseteq \mathcal{D} \subseteq \mathcal{A}',$$

where $\mathcal{D}(\mathcal{E})$ denotes the Dynkin system generated by $\mathcal{E}$. □

Finally, we state an existence result about martingale solutions to stochastic evolution equations in finite dimensions. For the proof see e.g. [16], Theorem 9.3.

Theorem 5.12. *Let A and B satisfy (A1)-(A3) and assume $\dim \mathbb{H} < \infty$. Then, there exists a martingale solution to equation (5.7) for any initial value $u_0 \in \mathbb{H}$.*

5.4 Abstract Framework for the Navier-Stokes Equations

We use the notation from the previous subsection. From now on, we will restrict our attention to stochastic perturbation dw in (5.2) of the form $dw_t = B(u(t))dW_t$ for some cylindrical Brownian motion $(W_t)_{t\geqslant 0}$. To formulate the Navier-Stokes system (5.1)-(5.4) in the framework of an abstract stochastic evolution equation, let

$$\mathbb{H} := L^2_{0,\sigma}(\Omega, \mathbb{R}^3) := \overline{\mathcal{D}_c^\infty(\Omega)}^{L^2(\Omega;\mathbb{R}^3)},$$

and

$$\mathbb{X} := W^{3,2}_{0,\sigma}(\Omega, \mathbb{R}^3).$$

Denote by $\mathbb{P} : L^2(\Omega, \mathbb{R}^3) \to L^2(\Omega, \mathbb{R}^3)$ the orthogonal projection with $\operatorname{ran} \mathbb{P} = \mathbb{H}$, and let

$$\begin{aligned}A_1(u) &:= \nu\mathbb{P}(\triangle u)\\ A_2(u) &:= \mathbb{P}((u \cdot \nabla)u)\end{aligned}$$

for $u \in \mathcal{D}_c^\infty(\Omega)$, and

$$A_3(u) := \mathbb{P}(f(\cdot, u(\cdot)))$$

for $u \in D(A_3)(\subseteq \mathbb{H})$. Then we get the following result:

Lemma 5.13. *The operators A_1 and A_2 are local uniformly continuous mappings from $\mathbb{H} \cap \mathcal{D}_c^\infty(\Omega)$ to $\mathbb{X}$' satisfying*

$$\|A_i(u)\|_{\mathbb{X}'} \leqslant c\|u\|_{\mathbb{H}}^2 \tag{5.11}$$

for $i = 1, 2$. In particular, we can extend them to continuous operators from $\mathbb{H}$ to $\mathbb{X}'$.

Proof. We prove the statement about A_2, the statement about A_1 follows similar. Let $B := \{w \in \mathcal{D}_c^\infty(\Omega) : \|w\|_{\mathbb{X}} \leqslant 1\}$. By continuity of the Sobolev embedding $W^{3,2}(\Omega) \hookrightarrow C_b^1(\Omega)$ we have

$$\begin{aligned}
\|A_2(u) - A_2(v)\|_{\mathbb{X}'} &= \sup_{w \in B} |{}_{\mathbb{X}'}\langle A_2(u) - A_2(v), w\rangle_{\mathbb{X}}| \\
&= \sup_{w \in B} |\int_\Omega \operatorname{div}(u \otimes u - v \otimes v) w \,\mathrm{d}x| \\
&= \sup_{w \in B} |\int_\Omega (u \otimes u - v \otimes v) \cdot \nabla w \,\mathrm{d}x| \\
&\leqslant \sup_{w \in B} \|\nabla w\|_{C(\Omega)} \|u \otimes u - v \otimes v\|_{L^1(\Omega)} \\
&\leqslant c(\|u\|_{\mathbb{H}} + \|v\|_{\mathbb{H}})\|u - v\|_{\mathbb{H}}.
\end{aligned}$$

This shows that A_2 is local uniformly continuous. To see (5.11), let $v = 0$. □

Assume that for some $\lambda > 0$ and $g \in L^2(\Omega)$ we have

$$|f(x, u)| \leqslant \lambda |u| + g(x) \tag{5.12}$$

for all $(x, u) \in \Omega \times \mathbb{R}^3$. Then we have also $A_3 : \mathbb{H} \to \mathbb{X}'$ and consequently

$$A := A_1 - A_2 + A_3 : \mathbb{H} \to \mathbb{X}'.$$

To motivate the definition of a solution for the stochastic incompressible Navier-Stokes equations, it is convenient to have a look at the deterministic equations first. Thus, consider the deterministic Navier-Stokes equations:

$$\operatorname{div}(u) = 0 \qquad \text{in } (0, T) \times \Omega \tag{5.13}$$

$$\frac{\partial u}{\partial t} = \nu \triangle u - (u \cdot \nabla)u + \nabla p + f \qquad \text{in } (0, T) \times \Omega \tag{5.14}$$

$$u = 0 \qquad \text{on } (0, T) \times \partial\Omega \tag{5.15}$$

and consider a smooth solution (u, p). We will construct a weak formulation of these equations in the following way: First, for smooth solutions, the equation of continuity (i.e. $\operatorname{div}(u) = 0$) together with the boundary condition are equivalent to $u(t) \in \mathbb{H}$ for any $t \in [0,T]$. Secondly, by integrating the second equation from 0 to $t \in [0,T]$ and then applying the projection operator $\mathbb{P}$, we get

$$u(t) = u(0) + \int_0^t A(u)\,\mathrm{d}\tau. \tag{5.16}$$

Note, that the pressure has disappeared, since $\mathbb{P}(\nabla p) = 0$ for any smooth function p. Combining these two steps, we call a function $u \in C([0,T],\mathbb{H})$ a *weak solution* of (5.13)-(5.15) iff (5.16) is satisfied.

Back to the stochastic equations: by the thoughts above it seems plausible to restrict ourselves to noises w_t with values in $\mathbb{H}$ for any $t \geqslant 0$. Furthermore, for simplicity, we will restrict ourselves to the case when the noise w_t is a Brownian motion[4]. To be concrete, we assume that there exists a further separable Hilbert space $\mathbb{U}$ and a Brownian basis $(\mathcal{O},\mathcal{F},\mathbb{P},(\mathcal{F}_n)_{n\in\mathbb{N}},(W(t))_{t\geqslant 0})$ for $\mathbb{U}$, such that $w_t = B(u)W_t$ for some measurable operator $B : \mathbb{H} \to L_2(\mathbb{U},\mathbb{H})$ satisfying assumption (A1)-(A3).

The above considerations make it seem reasonable to call an $\mathbb{H}$-valued stochastic process u a solution[5] to (5.1), (5.2) iff

$$u(t) = u(0) + \int_0^t A(u)\,\mathrm{d}\tau + \int_0^t B(u)\,\mathrm{d}W_\tau. \tag{5.17}$$

Since any strong solution to the abstract stochastic evolution equation satisfies (5.17), we can rewrite the Navier-Stokes equations to an abstract stochastic evolution equation:

$$du(t) = A(u(t))\,\mathrm{d}t + B(u(t))\,\mathrm{d}W_t, \tag{5.18}$$

$$u(0) = u_0, \tag{5.19}$$

where $u_0 \in \mathbb{H}$.

Proposition 5.14. *The operators A and B satisfy assumptions (A1)-(A3).*

Proof. **(A1)** The operators A_1 and A_2 are continuous from $\mathbb{H}$ to $\mathbb{X}'$ and consequently demicontinuous. Now let $u_n \to u$ in $\mathbb{H}$. Then

$$\|A_3(u_n)\|_{\mathbb{X}'} \leqslant c_1\|f(.,u_n(.))\|_{L^2(\Omega)} \leqslant c_2(\|u_n\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}) \leqslant c_3,$$

and consequently, by reflexivity of $\mathbb{X}$, the sequence $A_3(u_n)$ is sequentially pre-compact in $\mathbb{X}'_w$. Let v be any accumulation point of $(A_3(u_n))_{n\in\mathbb{N}}$ in $\mathbb{X}'_w$ and choose a subsequence u_{n_k} such that $A_3(u_{n_k}) \rightharpoonup v$ in $\mathbb{X}'$. We can assume $u_{n_k} \to u$ a.e. in Ω, passing to another subsequence if necessary. Then, $A_3(u_{n_k}) \to A_3(u)$ a.e. in Ω and by the Dominated convergence theorem $A_3(u_{n_k}) \to A_3(u)$ in $\mathbb{H}$. This shows $v = A_3(u)$, and consequently, the sequence $A_3(u_n)$ has only a single accumulation point in $\mathbb{X}'_w$. Thus, we have $A_3(u_n) \rightharpoonup A_3(u)$.

(A2) For $u \in \mathbb{H}$ define

$$\mathcal{Z}(u) := \begin{cases} \|\nabla u\|^2_{L^2(\Omega,\mathbb{R}^3)} & \text{for } u \in W^{1,2}_{0,\sigma}(\Omega,\mathbb{R}^3) \\ +\infty & \text{otherwise.} \end{cases}$$

[4]A generalization of the results presented in section 6 and 7 to Lévy processes was published in [7].
[5]This definition is only a motivation; we will never use (5.17) as an actual Definition for a solution.

We show that $\mathcal{Z} \in \mathcal{A}^2(\mathbb{H})$, where $\mathcal{A}^2(\mathbb{H})$ is defined in Definition 4.22. Let $y_n \to y$ in $\mathbb{H}$. We show

$$\mathcal{Z}(y) \leqslant \liminf_{n\to\infty} \mathcal{Z}(y_n).$$

We can assume $\sup_n \mathcal{Z}(y_n) < \infty$ and consequently $\|y_n\|_{W_0^{1,2}(\Omega)} \leqslant c$. Therefore, at least for a subsequence, we have $y_n \rightharpoonup y$ in $W_0^{1,2}(\Omega)$. By Poincare's inequality, the mapping $u \mapsto (\mathcal{Z}(u))^{\frac{1}{2}}$ defines an equivalent norm on $W_0^{1,2}(\Omega)$. Since norms are weak lower-semicontinuous, this yields the desired conclusion.

Obviously, we have $\mathcal{Z}(u) = 0$ iff $u = 0$ and

$$\mathcal{Z}(cu) = c^2 \mathcal{Z}(u)$$

for all $c > 0$. Finally,

$$\mathcal{Z}^{-1}([0,1]) \subseteq \mathbb{H}$$

is pre-compact by compactness of Sobolev embedding $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$. This shows $\mathcal{Z} \in \mathcal{A}^2(\mathbb{H})$.

Now let $u \in \mathbb{X}$. Then we have

$$_{\mathbb{X}'}\langle A_1(u), u\rangle_{\mathbb{X}} = -\mathcal{Z}(u),$$

$$_{\mathbb{X}'}\langle A_2(u), u\rangle_{\mathbb{X}} = \int_\Omega u^i u^j \partial_i u^j \,\mathrm{d}x = \frac{1}{2}\int_\Omega u^i \partial_i (u^j u^j)\,\mathrm{d}x = -\frac{1}{2}\int_\Omega \partial_i u^i (u^j u^j)\,\mathrm{d}x = 0,$$

and

$$_{\mathbb{X}'}\langle A_3(u), u\rangle_{\mathbb{X}} = \langle f(\cdot, u(\cdot)), u\rangle_{\mathbb{H}} \leqslant C(\|u\|_{\mathbb{H}}^2 + 1).$$

This shows (**A2**).

(A3) By Lemma 5.13 and (5.12) we have

$$\|A(u)\|_{\mathbb{X}'} \leqslant c(\|u\|_{\mathbb{H}}^2 + 1).$$

□

6 Existence of martingale solutions

In this section we show the existence of martingale solutions in the sense of Definition 5.3 for any initial value $x \in \mathbb{H}$. Fix an arbitrary orthonormal basis $\{b_n\} \subseteq \mathbb{X}$ of $\mathbb{H}$ enjoying the properties from Lemma 2.14. Let

$$\overline{\mathbb{Y}} := \text{span}\,\{b_n \,|\, n \in \mathbb{N}\}.$$

Then, we have $\overline{\mathbb{Y}} \leqslant \mathbb{X}$ and $\overline{\mathbb{Y}}$ is dense in $\mathbb{X}$. For all $p \in [2, \infty)$ we define functions $k_p \in C^\infty([0,\infty); \mathbb{R}^+)$ by[6]

$$k_p(t) := (4 + \gamma_p t) \exp(3\gamma_p t) \tag{6.1}$$

with $\gamma_p := \max\{2p\kappa_1 + (2p(p-1) + 64p^4)\kappa_2, 1\}$, where κ_1 and κ_2 are as in **(A2)-(A3)**, and we define the lower semi-continuous functionals on $\mathbb{H}$:

$$\mathcal{Z}_p(u) := \|u\|_{\mathbb{H}}^{p-2} \mathcal{Z}(u) \tag{6.2}$$

where $\mathcal{Z}$ is as in **(A2)-(A3)**.

Definition 6.1. Let $x \in \mathbb{H}$. We denote by $\mathscr{S}_{A,B,\overline{\mathbb{Y}}}(x) \subseteq \mathscr{P}(\mathcal{U})$ the set of all martingale solutions $\mathbb{P}$ of (5.7) with initial value x, which satisfy (5) in Definition 5.3 with $\mathbb{Y} = \overline{\mathbb{Y}}$, and which satisfy the following condition:

For any $p \in [2, \infty)$ there exists a Lebesgue null-set $T_p \subseteq (0, \infty)$ such that[7]

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{r \in [s,t]} \|\xi(r)\|_{\mathbb{H}}^p + \int_s^t \mathcal{Z}_p(\xi(r))\, \mathrm{d}r \middle| \mathcal{G}_s \right] \leqslant k_p(t-s) \cdot (\|\xi(s)\|_{\mathbb{H}}^p + 1) \tag{6.3}$$

for all $s \in (0,\infty) \backslash T_p$ and $t \geqslant s$, where ξ denotes the canonical process on $\mathcal{U}$.

We simply write $\mathscr{S}(x) := \mathscr{S}_{A,B,\overline{\mathbb{Y}}}(x)$ if no confusion about the involved operators A and B and the space $\overline{\mathbb{Y}}$ can arise.

Remark 6.2. Condition (6.3) and the growth condition **(A3)** immediately imply points (3) and (4) from Definition 5.3.

We show that $\mathscr{S}(x) \neq \varnothing$ for all $x \in \mathbb{H}$.

6.1 Weak stability of the set of solutions

Definition 6.3. We denote by

$$\mathcal{U}_q := \mathcal{U} \cap L^q_{loc}([0,\infty); \mathbb{H}) = C([0,\infty); \mathbb{X}') \cap L^q_{loc}([0,\infty); \mathbb{H})$$

for any $1 \leqslant q \leqslant \infty$.

Lemma 6.4. *For any x in $\mathbb{H}$ and any $\mathbb{Q} \in \mathscr{S}(x)$, we have $\mathbb{Q}(\mathcal{U}_q) = 1$ for all $1 \leqslant q \leqslant \infty$.*

Proof. Condition (6.3) implies that

$$\mathbb{Q}(\{u \in \mathcal{U} \,:\, \operatorname*{ess\,sup}_{r \in [s,t]} \|u(r)\|_{\mathbb{H}} = \infty\}) = 0$$

for a.e. $t \geqslant s \geqslant 0$, i.e. $\mathbb{Q}(\mathcal{U}_\infty) = 1$. Since $\mathcal{U}_\infty \subseteq \mathcal{U}_q$ we have $\mathbb{Q}(\mathcal{U}_q) = 1$ for all $1 \leqslant q \leqslant \infty$. $\square$

Lemma 6.5. *We have $\mathscr{P}(\mathcal{U}_\infty) \subseteq \mathscr{P}_{\mathbb{H}}(\mathcal{U})$.*

[6] See end of proof of proposition 6.7.

[7] The function $\omega \mapsto \sup_{r \in [s,t]} \|\xi(r)\|_{\mathbb{H}}^p(\omega)$ is measurable: Let $K := \{x \in X \mid \|x\|_{\mathbb{H}} = 1\}$, $r \in [s,t]$ and let $r_n \to r$ with $r_n \in [s,t] \cap \mathbb{Q}$. Then, we have $\|\xi(r)\|_{\mathbb{H}} = \sup_{x \in K} {}_{X'}\langle \xi(r), x \rangle_X = \sup_{x \in K} \lim_{n \in \mathbb{N}} \langle \xi(r_n), x \rangle \leqslant \lim_n \sup_x \langle \xi(r_n), x \rangle = \lim_n \|\xi(r_n)\|_{\mathbb{H}}$. Thus, it is enough to take the supremum over the countable set $[s,t] \cap \mathbb{Q}$.

Proof. Let $u \in L^\infty_{loc}(0,\infty;\mathbb{H})$ and $t \in (0,\infty)$. There exists a sequence $t_n \to t$ such that $u(t_n) \in \mathbb{H}$. Since $\|u(t_n)\|_{\mathbb{H}} < c < \infty$ for all $n \in \mathbb{N}$ we have

$$u(t_n) \rightharpoonup h \qquad \text{in } \mathbb{H}$$

for some subsequence and some $h \in \mathbb{H}$. By continuity we have $u(t_n) \to u(t)$ in $\mathbb{X}'$ and consequently $u(t) = h$ and therefore $u(t) \in \mathbb{H}$. Thus, $u(t) \in \mathbb{H}$ for all $t \geqslant 0$. ☐

Proposition 6.6. *Let $A_n : \mathbb{H} \to \mathbb{X}'$ and $B_n : \mathbb{H} \to L_2(\mathbb{U},\mathbb{H})$ satisfy the conditions **(A1)-(A3)** with the same constants κ_1, κ_2, γ, $\tilde{\gamma}$ and q as A and B. Assume that*

$$A_n(h_n) \rightharpoonup A(h) \text{ in } \sigma(\mathbb{X}', \overline{\mathbb{Y}}) \tag{6.4}$$

and

$$B_n^*(h_n)|_{\overline{\mathbb{Y}}} \to B^*(h)|_{\overline{\mathbb{Y}}} \text{ in } L_S(\overline{\mathbb{Y}}; \mathbb{U}) \tag{6.5}$$

*whenever $h_n \to h$ in $\mathbb{H}$. Let $x_n \to x$ in $\mathbb{H}$ and $\mathbb{Q}_n \in \mathscr{S}_{A_n,B_n,\overline{\mathbb{Y}}}(x_n)$ and assume $\mathbb{Q}_n \rightharpoonup \mathbb{Q}$ in $\mathscr{P}(\mathcal{U}_q)$, where $q \geqslant 2$ is as in **(A2)**. Then, $\mathbb{Q} \in \mathscr{S}_{A,B,\overline{\mathbb{Y}}}(x)$. In particular, the set*

$$\{(x,\mu) \mid x \in \mathbb{H} \text{ and } \mu \in \mathscr{S}_{A,B,\overline{\mathbb{Y}}}(x)\}$$

is closed in $\mathbb{H} \times \mathscr{P}(\mathcal{U}_q)$.

Proof. **Step 1.** We have to show that $\mathbb{Q}$ satisfies Definition 5.3 and (6.3) with initial condition x. By Skorohod's theorem, there exists a probability space $(\mathcal{O},\mathcal{F},\mathbb{P})$ and $\mathcal{U}_q$ valued random variables $(Y_n)_{n\in\mathbb{N}}$ and Y such that

$$\mathbb{P} \circ Y_n^{-1} = \mathbb{Q}_n, \quad \mathbb{P} \circ Y^{-1} = \mathbb{Q}$$

and

$$Y_n \to Y \text{ in } \mathcal{U}_q \qquad \mathbb{P} - a.e.$$

Step 2. Since $\mathbb{Q} \in \mathscr{P}(\mathcal{U}_\infty)$ we have $\mathbb{Q} \in \mathscr{P}_{\mathbb{H}}(\mathcal{U})$, thus (1) from Definition 5.3 is satisfied. Next, we have

$$\mathbb{Q}([\xi(0) = x]) = \mathbb{P}([Y(0) = x]) \geqslant \mathbb{P}(\bigcap_{n\in\mathbb{N}} [Y_n(0) = x]) \geqslant 1 - \sum_{n\in\mathbb{N}} \mathbb{Q}_n([\xi(0) \neq x]) = 1,$$

thus (2) of Definition 5.3 is satisfied.

Step 3. We show (6.3). Fix $p \geqslant 2$ and define for $0 \leqslant s < t$

$$\tau_p(s,t;u) := \sup_{r\in[s,t]} \|u(r)\|_{\mathbb{H}}^p + \int_s^t \mathcal{Z}_p(u(r))\,\mathrm{d}r$$

where $\mathcal{Z}_p$ is as in (6.3). We show that $\tau_p(s,t;\cdot) : \mathcal{U}_q \to [0,\infty]$ is lower semi-continuous. Let $u_n \to u$ in $\mathcal{U}_q$ and assume without loss of generality that $K := \liminf_{k\to\infty} \tau_p(s,t;u_k) < \infty$. By passing to a subsequence, we can assume that $\tau_p(s,t;u_n) \to K$ as $n \to \infty$. Since $u_n|_{[s,t]} \to u|_{[s,t]}$ in $L^q(s,t;\mathbb{H})$ as $n \to \infty$, we can assume $u_n(r) \to u(r)$ in $\mathbb{H}$ for $a.e.\, r \in [s,t]$, passing to a subsequence once more if necessary. Since $\mathcal{Z}$ is lower semicontinuous, we have by Fatou's lemma[8]

$$\begin{aligned}
\liminf_{n\to\infty} \tau_p(s,t;u_n) &\geqslant \liminf_{n\to\infty} \sup_{r\in[s,t]} \|u_n(r)\|_{\mathbb{H}}^p + \liminf_{n\to\infty} \int_s^t \|u_n(r)\|_{\mathbb{H}}^{p-2} \mathcal{Z}(u_n(r))\,\mathrm{d}r \\
&\geqslant \sup_{r\in[s,t]} \liminf_{n\to\infty} \|u_n(r)\|_{\mathbb{H}}^p + \int_s^t \liminf_{n\to\infty} \|u_n(r)\|_{\mathbb{H}}^{p-2} \liminf_{n\to\infty} \mathcal{Z}(u_n(r))\,\mathrm{d}r \\
&\geqslant \sup_{r\in[s,t]} \|u(r)\|_{\mathbb{H}}^p + \int_s^t \|u(r)\|_{\mathbb{H}}^{p-2} \mathcal{Z}(u(r))\,\mathrm{d}r \\
&= \tau_p(s,t;u).
\end{aligned}$$

Thus, $\tau_p(s,t,\cdot)$ is lower semicontinuous.

Fix $T \geqslant 0$. We have $Y_n \to Y$ in $L^q(0,T;\mathbb{H})$ pointwise $\mathbb{P}$-a.e. in $\mathcal{O}$. Consequently, we have $Y_n \to Y$ in $L^q(0,T;\mathbb{H})$ in measure with respect to $\mathbb{P}$. By (6.3) we have for any $p \geqslant 1$

$$\|Y_n\|_{L^p(\mathcal{O},\mathcal{F},\mathbb{P};L^q(0,T;\mathbb{H}))}^p = \mathbb{E}^{\mathbb{P}}[(\int_0^T \|Y_n(s)\|_{\mathbb{H}}^q\,\mathrm{d}s)^{\frac{p}{q}}] \leqslant T^{\frac{p}{q}} \mathbb{E}^{\mathbb{P}}[\sup_{0\leqslant s\leqslant T} \|Y_n(s)\|_{\mathbb{H}}^p] \leqslant const(T) \tag{6.6}$$

Thus, Theorem 3.31 yields

$$Y_n \to Y \text{ in } L^p(\mathcal{O},\mathcal{F},\mathbb{P};L^q(0,T;\mathbb{H})) \cong L^q(0,T;L^p(\mathcal{O},\mathcal{F},\mathbb{P};\mathbb{H}))$$

for any $p \geqslant 1$.

Now, fix $p \geqslant 2$. By passing to subsequence, we obtain a Lebesgue null set $N_0 \subseteq (0,\infty)$ such that

$$\mathbb{E}[\|Y_n(s) - Y(s)\|_{\mathbb{H}}^p] \to 0 \tag{6.7}$$

for all $s \in (0,\infty)\backslash N_0$. Let N_n be the exceptional set in condition (6.3) for $\mathbb{Q}_n$ and let $N := \bigcup_{n\geqslant 0} N_n$. Fix $s \in (0,\infty)\backslash N$ and $t \geqslant s$. We need to show that

$$\mathbb{E}^{\mathbb{Q}}[\tau_p(s,t;\xi)|\mathcal{G}_s] \leqslant k_p(t-s)(\|\xi(s)\|_{\mathbb{H}}^p + 1),$$

which is equivalent to the relation

$$\mathbb{E}^{\mathbb{Q}}[\tau_p(s,t;\xi)g(\xi)] \leqslant k_p(t-s)\mathbb{E}^{\mathbb{Q}}[(\|\xi(s)\|_{\mathbb{H}}^p + 1)g(\xi)]$$

for any $\mathcal{G}_s$-measurable, bounded and continuous function $g : \mathcal{U}_q \to [0,\infty)$. Since $g(Y_n) \to g(Y)$ $\mathbb{P}$-a.e. in $\mathcal{O}$,

[8] With the same argumentation as in Footnote 7 on page 54, it is enough to take to supremum $\sup_{r\in[s,t]} \|u_n(r)\|_{\mathbb{H}}^p$ only over a countable and dens subset I of $[s,t]$; in particular $\sup_{r\in[s,t]} \|u_n(r)\|_{\mathbb{H}}^p$ is measurable. Furthermore, we can choose $I \subseteq [s,t]$ such that $u_n(r) \to u(r)$ in $\mathbb{H}$ for any $r \in I$. This shows $\sup_{r\in[s,t]} \liminf_{n\to\infty} \|u_n(r)\|_{\mathbb{H}}^p = \sup_{r\in[s,t]} \|u(r)\|_{\mathbb{H}}^p$.

the boundedness of g implies $g(Y_n) \to g(Y)$ in $L^r(\mathcal{O}, \mathbb{P})$ for all $1 \leqslant r < \infty$. Now we get

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[\tau_p(s,t;\xi)g(\xi)] &= \mathbb{E}^{\mathbb{P}}[\tau_p(s,t;Y)g(Y)] \\
&\leqslant \liminf_{n\to\infty} \mathbb{E}^{\mathbb{P}}[\tau_p(s,t;Y_n)g(Y_n)] \\
&= \liminf_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n}[\tau_p(s,t;\xi)g(\xi)] \\
&\leqslant k_p(t-s) \liminf_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n}[(\|\xi(s)\|_{\mathbb{H}}^p + 1)g(\xi)] \\
&= k_p(t-s) \liminf_{n\to\infty} \mathbb{E}^{\mathbb{P}}[(\|Y_n(s)\|_{\mathbb{H}}^p + 1)g(Y_n)].
\end{aligned}$$

By (6.7), we can choose a subsequence such that $Y_n(s) \to Y(s)$ $\mathbb{P}$-a.e. in $\mathcal{O}$ and $\|Y_n(s)\|_{\mathbb{H}} \leqslant \varphi$ for some $\varphi \in L^p(\mathcal{O}, \mathbb{P})$ and all $n \in \mathbb{N}$. Thus, $(\|Y_n(s)\|_{\mathbb{H}}^p + 1)g(Y_n) \leqslant (\varphi^p + 1)\|g\|_\infty \in L^1(\mathcal{O}, \mathbb{P})$, and therefore $(\|Y_n(s)\|_{\mathbb{H}}^p + 1)g(Y_n) \to (\|Y(s)\|_{\mathbb{H}}^p + 1)g(Y)$ in $L^1(\mathcal{O}, \mathbb{P})$ by Lebesgue's theorem.

Thus,

$$\begin{aligned}
k_p(t-s) \liminf_{n\to\infty} \mathbb{E}^{\mathbb{P}}[(\|Y_n(s)\|_{\mathbb{H}}^p + 1)g(Y_n)] &= k_p(t-s)\mathbb{E}^{\mathbb{P}}[(\|Y(s)\|_{\mathbb{H}}^p + 1)g(Y)] \\
&= k_p(t-s)\mathbb{E}^{\mathbb{Q}}[(\|\xi(s)\|_{\mathbb{H}}^p + 1)g(\xi)]
\end{aligned}$$

Step 4. Fix $y \in \overline{\mathbb{Y}}$. We show that

$$M_y(t,u) = \langle u(t), y\rangle - \langle u(0), y\rangle - \int_0^t \langle A(u(s)), y\rangle \,\mathrm{d}s \tag{6.8}$$

is a $(\mathcal{G}_t)$-martingale. For $R > 0$ let $\chi_R \in C_c^\infty(\mathbb{R})$ be any function satisfying $\chi_R(t) = t$ for all $|t| \leqslant R$, $\chi_R(t) = 0$ for all $|t| \geqslant 2R$, $0 \leqslant \chi_R(t) \leqslant 2t$ for $t > 0$ and $0 \geqslant \chi_R(t) \geqslant 2t$ for $t < 0$. For $n \in \mathbb{N}$ and $R > 0$ define

$$\lambda_R(t,u) := \chi_R(\langle u(t), y\rangle)$$

$$\lambda(t,u) := \langle u(t), y\rangle$$

and

$$\begin{aligned}
\mu_R^n(t,u) &:= \int_0^t \chi_R(\langle A_n(u(s)), y\rangle) \,\mathrm{d}s, \\
\mu_R(t,u) &:= \int_0^t \chi_R(\langle A(u(s)), y\rangle) \,\mathrm{d}s, \\
\mu^n(t,u) &:= \int_0^t \langle A_n(u(s)), y\rangle \,\mathrm{d}s, \\
\mu(t,u) &:= \int_0^t \langle A(u(s)), y\rangle \,\mathrm{d}s,
\end{aligned}$$

for all $t \geqslant 0$ and $u \in \mathcal{U}_q$. We show

$$\lambda(t, Y_n) \to \lambda(t, Y) \text{ in } L^1(\mathcal{O}, \mathcal{F}, \mathbb{P}) \text{ as } n \to \infty \tag{6.9}$$

and

$$\mu^n(t, Y_n) \to \mu(t, Y) \text{ in } L^1(\mathcal{O}, \mathcal{F}, \mathbb{P}) \text{ as } n \to \infty. \tag{6.10}$$

Since for all $t \geqslant 0$ we have $Y_n(t, \cdot) \to Y(t, \cdot)$ in $\mathbb{X}'$, $\mathbb{P}-$a.e. in $\mathcal{O}$, we also have $\lambda_R(t, Y_n) \to \lambda_R(t, Y)$ in $\mathbb{R}$, $\mathbb{P}$-a.e. in $\mathcal{O}$. Since λ_R is bounded, Lebesgue's theorem yields

$$\lim_{n\to\infty} \lambda_R(t, Y_n) = \lambda_R(t, Y) \text{ in } L^1(\mathcal{O}, \mathcal{F}, \mathbb{P}). \tag{6.11}$$

Furthermore, we have

$$\begin{aligned}\sup_{n\in\mathbb{N}} \mathbb{E}^{\mathbb{P}}|\lambda(t, Y_n) - \lambda_R(t, Y_n)| &\leqslant \sup_{n\in\mathbb{N}} \mathbb{E}^{\mathbb{P}}[|\lambda(t, Y_n)|\mathbb{I}_{[|\lambda(t,Y_n)|>R]}] \\ &\leqslant \|y\|_{\mathbb{H}} \sup_{n\in\mathbb{N}} \mathbb{E}^{\mathbb{P}}[\|Y_n(t)\|_{\mathbb{H}}^2]^{\frac{1}{2}} \sup_{n\in\mathbb{N}} \mathbb{E}^{\mathbb{P}}[\mathbb{I}_{[|\lambda(t,Y_n)|>R]}]^{\frac{1}{2}}.\end{aligned}$$

By (6.3) for $\mathbb{Q}_n$, we get on the one hand

$$\sup_{n\in\mathbb{N}} \mathbb{E}^{\mathbb{P}}[\|Y_n(t)\|_{\mathbb{H}}^2] \leqslant \sup_{n\in\mathbb{N}} k_2(t)(\|x_n\|_{\mathbb{H}}^2 + 1) \leqslant const(t),$$

and on the other hand

$$\begin{aligned}\sup_{n\in\mathbb{N}} \mathbb{E}^{\mathbb{P}}[\mathbb{I}_{[|\lambda(t,Y_n)|>R]}] &= \sup_{n\in\mathbb{N}} \mathbb{P}[|\lambda(t, Y_n)| > R] \leqslant \sup_{n\in\mathbb{N}} \frac{\mathbb{E}^{\mathbb{P}}[|\lambda(t, Y_n)|^2]}{R^2} \\ &\leqslant \|y\|_{\mathbb{H}}^2 \sup_{n\in\mathbb{N}} \frac{\mathbb{E}^{\mathbb{P}}[\|Y_n(t)\|_{\mathbb{H}}^2]}{R^2} \leqslant \frac{1}{R^2} const(t).\end{aligned}$$

Consequently, we have

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \|\lambda_R(t, Y_n) - \lambda(t, Y_n)\|_{L^1(\mathcal{O},\mathcal{F},\mathbb{P})} = 0. \tag{6.12}$$

In the same way, we deduce $\lim_{R\to\infty} \lambda_R(t, Y) = \lambda(t, Y)$ in $L^1(\mathcal{O}, \mathcal{F}, \mathbb{P})$, since (6.3) holds for $\mathbb{Q}$ by the preceding step. Now, Lemma 1.13 implies (6.9).

We show that

$$\lim_{n\to\infty} \mu_R^n(t, Y_n) = \mu_R(t, Y) \text{ in } L^1(\mathcal{O}, \mathcal{F}, \mathbb{P}). \tag{6.13}$$

for any $t \geqslant 0$. Let $N \subseteq \mathcal{O}$ be the $\mathbb{P}$-null set such that $Y_n(\omega) \to Y(\omega)$ in $\mathcal{U}_q$ for all $\omega \in \mathcal{O}\backslash N$. Let $\omega \in \mathcal{O}\backslash N$. We show that

$$\mu_R^n(t, Y_n(\omega)) \to \mu_R(t, Y(\omega)) \tag{6.14}$$

as $n \to \infty$. This holds if and only if every subsequence (n_k) contains a subsequence (n_{k_j}) such that $\mu_R^{n_{k_j}}(t, Y_{n_{k_j}}(\omega)) \to \mu_R(t, Y(\omega))$. Since $Y_n(\omega) \to Y(\omega)$ in $\mathcal{U}_q$ implies $Y_n(\omega) \to Y(\omega)$ in $L^q(0, t; \mathbb{H})$, any subsequence $Y_{n_k}(\omega)$ contains a subsequence $Y_{n_{k_j}}(\omega)$ such that $Y_{n_{k_j}}(\omega)(s) \to Y(\omega)(s)$ in $\mathbb{H}$ for *a.e.* $s \in (0, t)$. By (6.4) we have $\chi_R(\langle A_{n_{k_j}}(Y_{n_{k_j}}(\omega)(s)), y\rangle) \to \chi_R(\langle A(Y(\omega)(s)), y\rangle)$ for a.e. $s \in (0, t)$. Lebesgue's convergence theorem now yields $\chi_R(\langle A_{n_{k_j}}(Y_{n_{k_j}}(\omega)(\cdot)), y\rangle) \to \chi_R(\langle A(Y(\omega)(\cdot)), y\rangle)$ in $L^1(0, t)$, and thus $\mu_R^{n_{k_j}}(t, Y_{n_{k_j}}(\omega)) \to$

$\mu_R(t,Y(\omega))$. Therefore, (6.14) holds. Now, Lebesgue's theorem yields (6.13).

Next, we show that

$$\lim_{R\to\infty}\sup_{n\in\mathbb{N}}\|\mu_R^n(t,Y_n)-\mu^n(t,Y_n)\|_{L^1(\mathcal{O},\mathcal{F},\mathbb{P})}=0. \tag{6.15}$$

By the definition of χ_R and Hölder's inequality, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}|\mu^n(t,Y_n)-\mu_R^n(t,Y_n)|\\
&\leqslant \mathbb{E}^{\mathbb{P}}[\int_0^t|\langle A_n(Y_n(s)),y\rangle|\mathbb{I}_{[|\langle A_n(Y_n(s)),y\rangle|\geqslant R]}\,\mathrm{d}s]\\
&\leqslant \|y\|_{\mathbb{X}}\ \|A_n(Y_n)\|_{L^\gamma((0,t)\times\mathcal{O},\,\mathrm{d}s\otimes\mathbb{P};\mathbb{X}')}\ \|\mathbb{I}_{[|\langle A_n(Y_n(s)),y\rangle|\geqslant R]}\|_{L^{\gamma'}((0,t)\times\mathcal{O},\,\mathrm{d}s\otimes\mathbb{P})}.
\end{aligned}$$

We can estimate the terms on the right hand side in the following way: By the growth condition (A3) and (6.3), we have

$$\begin{aligned}
\|A_n(Y_n)\|^\gamma_{L^\gamma((0,t)\times\mathcal{O},\,\mathrm{d}s\otimes\mathbb{P};\mathbb{X}')} &\leqslant \kappa_2\mathbb{E}^{\mathbb{Q}_n}[\int_0^t\Big(1+\|\xi(s)\|_{\mathbb{H}}^{\tilde{\gamma}}+\mathcal{Z}_2(\xi(s))\Big)\,\mathrm{d}s]\\
&\leqslant \kappa_2\Big(t+t\,k_{\tilde{\gamma}}(t)(\|x_n\|_{\mathbb{H}}^{\tilde{\gamma}}+1)+k_2(t)(\|x_n\|_{\mathbb{H}}^2+1)\Big)\\
&\leqslant const(t).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|\mathbb{I}_{[|\langle A_n(Y_n(s)),y\rangle|\geqslant R]}\|^{\gamma'}_{L^{\gamma'}((0,t)\times\mathcal{O},\,\mathrm{d}s\otimes\mathbb{P})} &= \int_0^t\mathbb{P}[|\langle A_n(Y_n(s)),y\rangle|\geqslant R]\,\mathrm{d}s\\
&\leqslant \int_0^t\frac{1}{R}\mathbb{E}[\|A_n(Y_n(s))\|_{\mathbb{X}'}\|y\|_{\mathbb{X}}]\,\mathrm{d}s\\
&\leqslant \frac{1}{R}\|A_n(Y_n)\|_{L^\gamma((0,t)\times\mathcal{O},\,\mathrm{d}s\otimes\mathbb{P};\mathbb{X}')}\|y\|^\gamma_{L^{\gamma'}((0,t)\times\mathcal{O},\,\mathrm{d}s\otimes\mathbb{P};\mathbb{X})}\\
&\leqslant \frac{1}{R}const(t).
\end{aligned}$$

Thus, we have

$$\mathbb{E}^{\mathbb{P}}|\mu^n(t,Y_n)-\mu_R^n(t,Y_n)|\leqslant c\,R^{-\frac{1}{\gamma'}},$$

where the constant c is independent of R and n. This shows (6.15). A similar calculation shows

$$\lim_{R\to\infty}\mu_R(t,Y)=\mu(t,Y)\text{ in }L^1(\mathcal{O},\mathcal{F},\mathbb{P}). \tag{6.16}$$

Combining (6.13), (6.15) and (6.16), Lemma 1.13 implies (6.10).

Consequently, by definition of M_y, we have

$$M_y^n(t,Y_n)\to M_y(t,Y)\ \text{ in }L^1(\mathcal{O},\mathcal{F},\mathbb{P})\text{ as }n\to\infty, \tag{6.17}$$

where M_y^n denotes the process defined in (6.8) with A replaced by A_n. Let $t \geqslant s \geqslant 0$ and let $g : \mathcal{U}_q \to \mathbb{R}$ be any $\mathcal{G}_s$-measurable, continuous and bounded function. Lebesgue's theorem implies $g(Y_n) \to g(Y)$ in $L^p(\mathcal{O}, \mathcal{F}, \mathbb{P})$ as $n \to \infty$ for any $p \in [1, \infty)$, and by the uniform boundedness of $g(Y_n)$, we also have this convergence in the weak-star topology of $L^\infty(\mathcal{O}, \mathcal{F}, \mathbb{P})$. Thus,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[(M_y(t) - M_y(s))g] &= \mathbb{E}^{\mathbb{P}}[(M_y(t, Y) - M_y(s, Y))g(Y)] \\
&= \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}[(M_y^n(t, Y_n) - M_y^n(s, Y_n))g(Y_n)] \\
&= \lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n}[(M_y^n(t, \xi) - M_y^n(s, \xi))g(\xi)] \\
&= 0,
\end{aligned}$$

where the last equality follows from Definition 5.3 for $\mathbb{Q}_n$. This relation is equivalent to

$$\mathbb{E}^{\mathbb{Q}}[M_y(t)|\mathcal{G}_s] = M_y(s).$$

Step 5. We show that $M_y(t)$ is square integrable with respect to $\mathbb{Q}$. By BDG's inequality and the growth condition (A3), we have for any $p \geqslant 1$

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[M_y^n(t, Y_n)^{2p}] &\overset{BDG}{\leqslant} c_1 \mathbb{E}^{\mathbb{P}}[(\int_0^t \|B_n^*(Y_n(s))(y)\|_{\mathbb{U}}^2 \,\mathrm{d}s)^p] \\
&\leqslant c_2 \|y\|_{\mathbb{H}}^{2p} \mathbb{E}^{\mathbb{P}}[\int_0^t \|B_n(Y_n(s))\|_{L_2(\mathbb{U},\mathbb{H})}^{2p} \,\mathrm{d}s] \\
&\overset{(A3)}{\leqslant} c_2 \|y\|_{\mathbb{H}}^{2p} \mathbb{E}^{\mathbb{P}}[\int_0^t (\kappa_2(1 + \|Y_n(s)\|_{\mathbb{H}}^2))^p \,\mathrm{d}s] \\
&\leqslant c_3 \int_0^t 1 + \mathbb{E}^{\mathbb{P}}[\|Y_n(s)\|_{\mathbb{H}}^{2p}] \,\mathrm{d}s,
\end{aligned}$$

where c_1, c_2 and c_3 are independent of n. Relation (6.3) for $\mathbb{Q}_n$ yields

$$\mathbb{E}^{\mathbb{P}}[\|Y_n(s)\|_{\mathbb{H}}^{2p}] \leqslant k_{2p}(t)(\|Y_n(0)\|_{\mathbb{H}}^{2p} + 1) \leqslant c \tag{6.18}$$

where c is independent of $n \in \mathbb{N}$ and $s \leqslant t$. Consequently, $M_y^n(t, Y_n)$ is uniformly bounded in $L^{2p}(\mathcal{O}, \mathcal{F}, \mathbb{P})$ for any $p \geqslant 1$. Now, (6.17) and Theorem 3.31 imply

$$M_y^n(t, Y_n) \to M_y(t, Y) \text{ in } L^2(\mathcal{O}, \mathcal{F}, \mathbb{P}) \text{ as } n \to \infty. \tag{6.19}$$

Step 6. Fix $y \in \overline{\mathbb{Y}}$. We show that

$$\mathfrak{N}_y(t, u) := M_y(t, u)^2 - \int_0^t \|B^*(u(s))(y)\|_{\mathbb{U}}^2 \,\mathrm{d}s \tag{6.20}$$

is a martingale with respect to $\mathbb{Q}$. Define

$$\Phi_n(s,\omega) := B_n^*(Y_n(s,\omega))(y)$$
$$\Phi(s,\omega) := B^*(Y(s,\omega))(y).$$

Let $T > 0$ and $N \subseteq \mathcal{O}$ be the $\mathbb{P}$-null set such that $Y_n(\cdot,\omega) \to Y(\cdot,\omega)$ in $L^q(0,T;\mathbb{H})$ as $n \to \infty$ for all $\omega \notin N$, and fix $\omega \in \mathcal{O}\backslash N$. We show that

$$\Phi_n(\cdot,\omega) \to \Phi(\cdot,\omega) \text{ in } \mathbb{U} \text{ in measure with respect to } \mathrm{d}t|_{[0,T]}. \tag{6.21}$$

This holds if and only if any subsequence (Φ_{n_k}) contains a subsequence $(\Phi_{n_{k_j}})$ such that $\Phi_{n_{k_j}}(\cdot,\omega) \to \Phi(\cdot,\omega)$ in $\mathbb{U}$ pointwise a.e. in $(0,T)$. Since $Y_n(\cdot,\omega) \to Y(\cdot,\omega)$ in $L^q(0,T;\mathbb{H})$, for any subsequence (n_k) there is a subsequence (n_{k_j}) such that $Y_{n_{k_j}}(\cdot,\omega) \to Y(\cdot,\omega)$ in $\mathbb{H}$, pointwise a.e. in $(0,T)$. By (6.5) we have $\Phi_{n_{k_j}}(s,\omega) \to \Phi(s,\omega)$ in $\mathbb{U}$ for a.e. $s \in (0,T)$. Thus, (6.21) holds. Furthermore, by the growth condition **(A3)** we have

$$\begin{aligned}\|\Phi_n(\cdot,\omega)\|^q_{L^q(0,T;\mathbb{U})} &= \int_0^T \|B_n^*(Y_n(s,\omega))(y)\|^q_{\mathbb{U}}\,\mathrm{d}s \leqslant \kappa_2^q\|y\|^q_{\mathbb{H}}\int_0^T (1+\|Y_n(s,\omega)\|^2_{\mathbb{H}})^{\frac{q}{2}}\,\mathrm{d}s\\ &\leqslant c(1+\|Y_n(\cdot,\omega)\|^q_{L^q(0,T;\mathbb{H})}) \leqslant const(\omega).\end{aligned}$$

Thus, Theorem 3.31 yields

$$\Phi_n(\cdot,\omega) \to \Phi(\cdot,\omega) \text{ in } L^r(0,T;\mathbb{U}) \tag{6.22}$$

for all $r < q$ and $\mathbb{P}$-a.e. $\omega \in \mathcal{O}$.

By the growth condition (A3) and (6.18), we have $\omega \mapsto \Phi_n(s,\omega)$ bounded in $L^p(\mathcal{O},\mathcal{F},\mathbb{P};\mathbb{U})$ uniformly in n and s for all $p \geqslant 1$, and in particular, we have

$$\|\Phi_n\|_{L^p((0,T)\times\mathcal{O},\,\mathrm{d}t\otimes\mathcal{F};\mathbb{U})} \leqslant c. \tag{6.23}$$

Relations (6.22) and (6.23) imply the following relations:

$$\Big(\omega \mapsto \Phi_n(\cdot,\omega)\Big) \to \Big(\omega \mapsto \Phi(\cdot,\omega)\Big) \text{ in } L^r(0,T;\mathbb{U}) \text{ in measure w.r.t. } \mathbb{P},$$
$$\|\Phi_n\|_{L^p(\mathcal{O},\mathbb{P};L^r(0,T;\mathbb{U}))} \leqslant c.$$

Thus, Theorem 3.31 yields

$$\Phi_n \to \Phi \text{ in } L^p(\mathcal{O},\mathbb{P};L^r(0,T;\mathbb{U}))$$

for any $p \geqslant 1$, and in particular, we have

$$\Phi_n \to \Phi \text{ in measure w.r.t. } \mathrm{d}t\otimes\mathbb{P}. \tag{6.24}$$

Thus, by applying Theorem 3.31 once more, relations (6.23) and (6.24) imply

$$\Phi_n \to \Phi \text{ in } L^2((0,T)\times\mathcal{O},\, \mathrm{d}t\otimes\mathbb{P};\mathbb{U})$$

In particular, we have

$$\mathfrak{N}_y^n(t,Y_n) \to \mathfrak{N}_y(t,Y) \ \text{ in } L^1(\mathcal{O},\mathcal{F},\mathbb{P})$$

for all $t \geqslant 0$, where $\mathfrak{N}_y^n$ denotes the process defined in (6.20) with B replaced by B_n. Now, for $t > s$, by calculating $\mathbb{E}^{\mathbb{Q}}[(\mathfrak{N}_y(t)-\mathfrak{N}_y(s))g]$ for all continuous, bounded, $\mathcal{G}_s$-measurable functions g, exactly as in step 5, we deduce that $\mathfrak{N}_y$ is a martingale with respect to $\mathbb{Q}$.

Condition (5) from Definition 5.3 follows now by combining steps 4, 5 and 6. □

6.2 Existence of solutions

Proposition 6.7. *Assume* $\dim\mathbb{H} < \infty$*. Then, for any* $x \in \mathbb{H}$ *the set* $\mathscr{S}(x)$ *is non-empty.*

Proof. Without loss of generality, we assume $\mathbb{H} = \mathbb{X} = \mathbb{R}^N \cong \mathbb{X}'$ and all spaces are equipped with the euclidean norm, denoted by $|x| = \|x\|_{\mathbb{H}} = \|x\|_{\mathbb{X}} = \|x\|_{\mathbb{X}'}$ for $x \in \mathbb{R}^N$. By Theorem 5.12, there exists a martingale solution $\mathbb{Q}$ with initial condition x. Thus, our task is to show that $\mathbb{Q}$ satisfies condition (6.3). We set $T_{\mathbb{Q}} := \varnothing$.

Define

$$M(t,u) := \sum_{i=1}^{N} M_{e_i}(t,u)e_i$$

where e_i denotes the i-th unit vector and M_{e_i} is defined in Definition 5.3, (5). Now, we have

$$\xi(t) = x + \int_0^t A(\xi(r))\,\mathrm{d}r + M(t,\xi)$$

and $(M(t))_{t\geqslant 0}$ is a continuous, $\mathbb{R}^N$-valued, $(\mathcal{G}_t)$-martingale with respect to $\mathbb{Q}$. By polarization, the covariation operator process $\langle\!\langle M\rangle\!\rangle$ in $\mathbb{R}^{N\times N}$ is uniquely determined by symmetry and the values $x^T\langle\!\langle M\rangle\!\rangle x = \langle M_x\rangle$ for all $x \in \mathbb{R}^N$. Thus, the covariation operator process of M is given by

$$\langle\!\langle M\rangle\!\rangle(t,u) = \int_0^t B(u(s))B^*(u(s))\,\mathrm{d}s.$$

Since $r \mapsto A(\xi(r))$ is continuous, the process $(\int_0^t A(\xi(r))\,\mathrm{d}r)_{t\geqslant 0}$ is continuous and of finite variation. Thus, Lemma (4.39) implies that $(\int_0^t A^i(\xi(r))\,\mathrm{d}r)_{t\geqslant 0}$ is a semimartingale, and since the set of semimartingales is a vector space, the process $(\xi(t))_{t\geqslant 0}$ is also a semimartingale.

We use Ito's formula on the function $F : \mathbb{R}^N \to \mathbb{R}$, $F(x) = |x|^p$ for $p \geqslant 2$. Using Lemmata (4.43) and

(4.44), we obtain after a lengthy but straight forward calculation the following formula:

$$
\begin{gathered}
|\xi(t)|^p = \\
|\xi(0)|^p + p\int_0^t \xi^i(r)|\xi(r)|^{p-2}\,\mathrm{d}\xi^i(r) + p(p-2)\int_0^t \xi^i(r)\xi^j(r)|\xi(r)|^{p-4}d\langle \xi^i,\xi^j\rangle(r) \\
+p\int_0^t |\xi(r)|^{p-2}d\langle \xi^i(r),\xi^i(r)\rangle = \\
|\xi(0)|^p + p\int_0^t |\xi(r)|^{p-2}\langle A(\xi(r),\xi(r)\rangle\,\mathrm{d}r + p\int_0^t |\xi(r)|^{p-2}\|B(\xi(r))\|^2_{L_2(\mathbb{U},\mathbb{R}^N)}\,\mathrm{d}r \\
+p(p-2)\int_0^t |\xi(r)|^{p-4}\|B^*(\xi(r))(\xi(r))\|^2_{\mathbb{U}}\,\mathrm{d}r + M^{(p)}(t,\xi)
\end{gathered}
$$

where $M^{(p)}$ is a continuous, $\mathbb{R}$-valued, $(\mathcal{G}_t)$-martingale with respect to $\mathbb{Q}$, whose quadratic variation process can be estimated by

$$
\langle M^{(p)}\rangle(t,u) - \langle M^{(p)}\rangle(s,u) \leqslant p^2\int_s^t |u(r)|^{2p-2}\|B(u(r))\|^2_{L_2(\mathbb{U},\mathbb{R}^N)}\,\mathrm{d}r.
$$

Fix $s \geqslant 0$ and $p \geqslant 2$, and let

$$
g(t) := \mathbb{E}^{\mathbb{Q}}[\sup_{r\in[s,t]}|\xi(r)|^p|\mathcal{G}_s]
$$

for $t \geqslant s$. Using the estimate

$$
x^p + x^{p-2} \leqslant 2(x^p+1) \tag{6.25}
$$

for all $x \geqslant 0$, we have by BDG's inequality for conditional expectation (Lemma 4.32), growth condition (A3) and Young's inequality

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\Big[\sup_{r\in[s,t]}|M^{(p)}(r,\xi)-M^{(p)}(s,\xi)|\Big|\mathcal{G}_s\Big] &\overset{BDG}{\leqslant} C_1\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_s^t|\xi(r)|^{2p-2}\|B(\xi(r))\|^2_{L_2(\mathbb{U},\mathbb{R}^N)}\,\mathrm{d}r\Big)^{\frac{1}{2}}\Big|\mathcal{G}_s\Big] \\
&\overset{(A3)}{\leqslant} \sqrt{\kappa_2}C_1\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_s^t|\xi(r)|^{2p-2}(|\xi(r)|^2+1)\,\mathrm{d}r\Big)^{\frac{1}{2}}\Big|\mathcal{G}_s\Big] \\
&\overset{(6.25)}{\leqslant} C_2\mathbb{E}^{\mathbb{Q}}[\sup_{r\in[s,t]}|\xi(r)|^{\frac{p}{2}}\cdot\Big(\int_s^t(|\xi(r)|^p+1)\,\mathrm{d}r\Big)^{\frac{1}{2}}|\mathcal{G}_s] \\
&\overset{Young}{\leqslant} \frac{1}{2}g(t)+\frac{1}{2}C_2^2\mathbb{E}^{\mathbb{Q}}[\int_s^t(|\xi(r)|^p+1)\,\mathrm{d}r|\mathcal{G}_s] \\
&\leqslant \frac{1}{2}g(t)+\frac{1}{2}C_2^2\int_s^t(g(r)+1)\,\mathrm{d}r,
\end{aligned}
$$

where $C_1 = 4\sqrt{2}p^2$ and $C_2 = 2\sqrt{\kappa_2}C_1 = 8\sqrt{2\kappa_2}p^2$.

Furthermore, we have

$$\begin{aligned}
\|\xi(\tau)\|_{\mathbb{H}}^{p} \quad &= \quad \|\xi(s)\|_{\mathbb{H}}{}^{p} + p\int_s^\tau \|\xi(r)\|_{\mathbb{H}}{}^{p-2}\langle A(\xi(r),\xi(r)\rangle\,\mathrm{d}r + p\int_s^\tau \|\xi(r)\|_{\mathbb{H}}{}^{p-2}\|B(\xi(r))\|^2_{L_2(\mathbb{U},\mathbb{R}^N)}\,\mathrm{d}r \\
&\quad +p(p-2)\int_s^\tau \|\xi(r)\|_{\mathbb{H}}{}^{p-4}\|B^*(\xi(r))(\xi(r))\|^2_{\mathbb{U}}\,\mathrm{d}r + M^{(p)}(\tau,\xi) - M^{(p)}(s,\xi) \\
&\overset{(A2),(A3)}{\leqslant} \|\xi(s)\|_{\mathbb{H}}^{p} + p\int_s^\tau \|\xi(r)\|_{\mathbb{H}}^{p-2}\Big(-\mathscr{Z}(\xi(r)) + (\kappa_1+\kappa_2)(1+\|\xi(r)\|_{\mathbb{H}}^2)\Big)\,\mathrm{d}r \\
&\quad +p(p-2)\kappa_2\int_s^\tau \|\xi(r)\|_{\mathbb{H}}^{p-2}\Big(1+\|\xi(r)\|_{\mathbb{H}}^2\Big)\,\mathrm{d}r + M^{(p)}(\tau,\xi) - M^{(p)}(s,\xi) \\
&\overset{(6.25)}{\leqslant} \|\xi(s)\|_{\mathbb{H}}^{p} - p\int_s^\tau \mathscr{Z}_p(\xi(r))\,\mathrm{d}r + C_3\int_s^\tau (\|\xi(r)\|_{\mathbb{H}}^{p}+1)\,\mathrm{d}r + M^{(p)}(\tau,\xi) - M^{(p)}(s,\xi),
\end{aligned}$$

where $C_3 = 2p\kappa_1 + 2p(p-1)\kappa_2$.

Combining the last two relations, taking first the supremum over $\tau \in [s,t]$ in the last equation and then the conditional expectation with respect to $\mathcal{G}_s$, we get

$$g(t) \leqslant \|\xi(s)\|_{\mathbb{H}}^{p} - p\mathbb{E}^{\mathbb{Q}}\Big[\int_s^t \mathscr{Z}_p(\xi(r))\,\mathrm{d}r|\mathcal{G}_s\Big] + \frac{1}{2}g(t) + \gamma_p\int_s^t (g(r)+1)\,\mathrm{d}r \tag{6.26}$$

where γ_p is as in (6.1). Consequently, we have

$$g(t) \leqslant 2\|\xi(s)\|_{\mathbb{H}}^{p} + 2\gamma_p\int_s^t (g(r)+1)\,\mathrm{d}r,$$

and, by applying Grownwall's inequality, we get

$$g(t) \leqslant \exp(2\gamma_p(t-s))(2\|\xi(s)\|_{\mathbb{H}}^{p} + 2\gamma_p(t-s)) \leqslant 2\exp(3\gamma_p(t-s))(\|\xi(s)\|_{\mathbb{H}}^{p}+1)$$

and this combined with (6.26) yields

$$\begin{aligned}
g(t) + \mathbb{E}^{\mathbb{Q}}\Big[\int_s^t \mathscr{Z}_p(\xi(r))\,\mathrm{d}r|\mathcal{G}_s\Big] \quad &\leqslant \quad g(t) + p\mathbb{E}^{\mathbb{Q}}\Big[\int_s^t \mathscr{Z}_p(\xi(r))\,\mathrm{d}r|\mathcal{G}_s\Big] \\
&\leqslant \quad \frac{1}{2}g(t) + \gamma_p\int_s^t (g(r)+1)\,\mathrm{d}r + \|\xi(s)\|_{\mathbb{H}}^{p} \\
&\leqslant \quad 2(\|\xi(s)\|_{\mathbb{H}}^{p}+1)\exp(3\gamma_p(t-s)) + \frac{2}{3}(\|\xi(s)\|_{\mathbb{H}}^{p}+1)(\exp(3\gamma_p(t-s))-1) \\
&\quad\quad +\gamma_p(t-s) + \|\xi(s)\|_{\mathbb{H}}^{p} \\
&\leqslant \quad (4+\gamma_p(t-s))\exp(3\gamma_p(t-s))(\|\xi(s)\|_{\mathbb{H}}^{p}+1) \\
&\leqslant \quad k_p(t-s)(\|\xi(s)\|_{\mathbb{H}}^{p}+1).
\end{aligned}$$

Thus, (6.3) holds. □

Finally, we can show the existence of solutions:

Theorem 6.8. *Let $\mathbb{H}$ be a separable Hilbert space and let A and B satisfy the assumption (A1)-(A3). Then, for any $x \in \mathbb{H}$, the set $\mathscr{S}(x)$ is non-empty. In particular, there exists a martingale solution to equation (5.7).*

Proof. We use Galerkin's approximation. Let

$$\mathbb{H}_n := \operatorname{span}\{b_i \,:\, i = 1, \dots, n\} \subseteq \mathbb{X}$$

where b_n is the orthonormal basis of $\mathbb{H}$ defined at the beginning of this section. Let

$$\mathcal{P}_n : \mathbb{H} \to \mathbb{H}$$

be the orthogonal projection onto $\mathbb{H}_n$. We set $x_n := \mathcal{P}_n(x)$ and we define the operators

$$A_n := \mathcal{P}_n \circ A$$

and

$$B_n := \mathcal{P}_n \circ B.$$

Then, the operators A_n and B_n satisfy assumptions (A1)-(A3) with the same constants κ_1, κ_2, γ, $\tilde{\gamma}$ and q as A and B. Consider the equation

$$du(t) = A_n(u(t))\,\mathrm{d}t + B_n(u(t))\,\mathrm{d}W_t, \tag{6.27}$$
$$u(0) = x_n. \tag{6.28}$$

in $\mathbb{H}_n$. Since $\dim \mathbb{H}_n < \infty$, there exists a martingale solution $\mathbb{Q}_n$ satisfying (6.3), i.e. $\mathbb{Q}_n \in \mathscr{S}_{A_n,B_n,\mathbb{H}_n}(x_n)$. The measures $\mathbb{Q}_n$ are probability measures on $C([0,\infty);\mathbb{H}_n)$, but we can assume $\mathbb{Q}_n \in \mathscr{P}(\mathcal{U}_q)$ by letting $\mathbb{Q}_n(A) := \mathbb{Q}_n(A \cap C([0,\infty);\mathbb{H}_n))$ for $A \in \mathcal{G} = \mathcal{B}(C([0,\infty);\mathbb{X}'))$. Since $\{b_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathbb{H}$, we have for all $m > n$

$$M_{b_m}(t,u) = \langle u(t), b_m\rangle - \langle u(0), b_m\rangle - \int_0^t \langle A(u(s)), b_m\rangle\,\mathrm{d}s = 0$$

and since $\ker B_n^*(u(s)) = (\mathrm{ran} B_n(u(s)))^\perp \supseteq \mathbb{H}_n^\perp$, we have $B_n^*(u(s))(b_m) = 0$ and thus

$$\langle M_{b_m}\rangle(t,u) = 0 = \int_0^t \|B_n^*(u(s))(b_m)\|_{\mathbb{U}}^2\,\mathrm{d}s.$$

Thus, (5) from Definition 5.3 holds for $\mathbb{Q}_n$ with $\mathbb{Y} = \overline{\mathbb{Y}}$. Thus, we have $\mathbb{Q}_n \in \mathscr{S}_{A_n,B_n,\overline{\mathbb{Y}}}(x_n)$.

We show that $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ is tight in $\mathscr{P}(\mathcal{U}_q)$. By (6.3) we have

$$\mathbb{E}^{\mathbb{Q}_n}[\sup_{t\in[0,T]} \|\xi(t)\|_{\mathbb{H}} + \int_0^T \mathcal{Z}(\xi(r))\,\mathrm{d}r] \leqslant c_1 k_2(T)(x_n^2 + 1) \leqslant c_2$$

where c_1 and c_2 are independent of n. Thus, by Lemma 4.23, it is sufficient to show that for some β, $c > 0$ independent of n we have

$$\mathbb{E}^{\mathbb{Q}_n}[\sup_{0\leqslant s<t\leqslant T} \frac{\|\xi(t) - \xi(s)\|_{\mathbb{X}'}}{|t-s|^\beta}] \leqslant c. \tag{6.29}$$

We set

$$M_n(t,u) := \sum_{i=1}^{n} M_{b_i}(t,u)b_i,$$

then, similar to the proof of Proposition 6.7, we have

$$\xi(t) = x_n + \int_0^t A_n(\xi(r))\,\mathrm{d}r + M_n(t,\xi) \qquad \mathbb{Q}_n - a.e. \tag{6.30}$$

for all $n \in \mathbb{N}$. We estimate the terms on the right hand side.

We show

$$\mathbb{E}^{\mathbb{Q}_n}\left[\sup_{0\leqslant s<t\leqslant T} \frac{\|\int_s^t A_n(\xi(r))\,\mathrm{d}r\|_{\mathbb{X}'}}{|t-s|^{\frac{\gamma-1}{\gamma}}}\right] \leqslant c \tag{6.31}$$

for some $c \geqslant 0$ independent of n. By Jensen's inequality, we have

$$\mathbb{E}^{\mathbb{Q}_n}\left[\sup_{0\leqslant s<t\leqslant T} \frac{1}{(t-s)^{\gamma-1}}\|\int_s^t A_n(\xi(r))\,\mathrm{d}r\|_{\mathbb{X}'}^{\gamma}\right] \leqslant \mathbb{E}^{\mathbb{Q}_n}\left[\int_0^T \|A_n(\xi(r))\|_{\mathbb{X}'}^{\gamma}\,\mathrm{d}r\right]$$

and by (A3) and (6.3) we get

$$\mathbb{E}^{\mathbb{Q}_n}\left[\int_0^T \|A(\xi(r))\|_{\mathbb{X}'}^{\gamma}\,\mathrm{d}r\right] \leqslant \kappa_2 \mathbb{E}^{\mathbb{Q}_n}\left[\int_0^T \left(1 + \|\xi(r)\|_{\mathbb{H}}^{\tilde{\gamma}} + \mathcal{Z}(\xi(r))\right)\mathrm{d}r\right] \leqslant const(T,\tilde{\gamma}).$$

Thus, (6.31) follows. Next, we show

$$\mathbb{E}^{\mathbb{Q}_n}[\sup_{0\leqslant s<t\leqslant T} \frac{\|M_n(t,\xi) - M_n(s,\xi)\|_{\mathbb{X}'}}{|t-s|^{\frac{1}{2}}}] \leqslant const(T). \tag{6.32}$$

Exactly as in the proof of Proposition 6.7 we deduce that

$$\langle\!\langle M_n \rangle\!\rangle(t,u) = \int_0^t B_n(u(s))B_n^*(u(s))\,\mathrm{d}s \qquad \mathbb{Q}_n - a.e.$$

Let $Q := \{(t,s) \in [0,T]^2 \mid s < t\}$ and let $f(t,s,u) := \frac{\|M_n(t,u)-M_n(s,u)\|_{\mathbb{X}'}^2}{|t-s|}$. Then, we have for all $q \geqslant 1$

$$\|f\|_{L^q(Q\times\mathcal{U},\,\mathrm{d}t\otimes\mathrm{d}s\otimes\mathrm{d}\mathbb{Q}_n)} \leqslant c\left(\int\int_Q \mathbb{E}^{\mathbb{Q}_n}\left[\frac{\|M(t,\xi)-M_n(s,\xi)\|_{\mathbb{H}}{}^{2q}}{|t-s|^q}\right]\mathrm{d}t\,\mathrm{d}s\right)^{\frac{1}{q}}.$$

We have for all $0 \leqslant s < t \leqslant T$ and $n \in \mathbb{N}$

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_n}\Big[\|M_n(t,\xi) - M_n(s,\xi)\|_{\mathbb{H}}^{2q}\Big] &\overset{BDG}{\leqslant} const(q)\mathbb{E}^{\mathbb{Q}_n}\Big[\Big(\int_s^t \|B(\xi(r))\|_{L_2(\mathbb{U},\mathbb{H})}^2\,\mathrm{d}r\Big)^q\Big] \\
&\overset{Jensen}{\leqslant} const(q)|t-s|^{q-1}\int_s^t \mathbb{E}^{\mathbb{Q}_n}\Big[\|B(\xi(r))\|_{L_2(\mathbb{U},\mathbb{H})}^{2q}\Big]\,\mathrm{d}r \\
&\overset{(A3)}{\leqslant} const(q)|t-s|^{q-1}\int_s^t \mathbb{E}^{\mathbb{Q}_n}\Big[\|\xi(r)\|_{\mathbb{H}}^{2q} + 1\Big]\,\mathrm{d}r \\
&\overset{(6.3)}{\leqslant} const(q,T)|t-s|^q(\|x_n\|_{\mathbb{H}}^{2q} + 1)
\end{aligned}$$

and therefore

$$\mathbb{E}^{\mathbb{Q}_n}\Big[\frac{\|M(t,\xi) - M_n(s,\xi)\|_{\mathbb{X}'}{}^{2q}}{|t-s|^q}\Big] \leqslant c(\alpha^q + 1)$$

where $\alpha := \sup_{n\in\mathbb{N}} \|x_n\|_{\mathbb{H}}^2$. Thus, we have

$$\|f\|_{L^q(Q\times\mathcal{U},\,\mathrm{d}t\otimes\mathrm{d}s\otimes\mathrm{d}\mathbb{Q})} \leqslant c$$

where c is independent of n and q. Since $\|f\|_{L^q(Q\times\mathcal{U},\,\mathrm{d}t\otimes\mathrm{d}s\otimes\mathrm{d}\mathbb{Q})} \to \|f\|_{L^\infty(Q\times\mathcal{U},\,\mathrm{d}t\otimes\mathrm{d}s\otimes\mathrm{d}\mathbb{Q})}$ as $q \to \infty$ we have

$$\operatorname*{ess\,sup}_{(t,s,u)\in Q\times\mathcal{U}} |f(t,s,u)| < c$$

and therefore

$$\mathbb{E}^{\mathbb{Q}_n}\Big[\sup_{0\leqslant s<t\leqslant T}\frac{\|M_n(t,\xi) - M_n(s,\xi)\|_{\mathbb{X}'}}{|t-s|^{\frac{1}{2}}}\Big] \leqslant \operatorname*{ess\,sup}_{(t,s,u)\in Q\times\mathcal{U}} |f(t,s,u)|^{\frac{1}{2}} < \infty.$$

The estimates (6.31) and (6.32) together with relation (6.30) imply (6.29). Thus, $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ is tight in $\mathscr{P}(\mathcal{U}_q)$ and, for a suitable subsequence, we have

$$\mathbb{Q}_n \rightharpoonup \mathbb{Q} \qquad \text{in } \mathscr{P}(\mathcal{U}_q).$$

Proposition 6.6 now yields $\mathbb{Q} \in \mathscr{S}(x)$, provided we can show (6.4) and (6.5). Let $h_n \to h$ in $\mathbb{H}$. Since $\mathcal{P}_n$ is $\mathbb{H}$-self adjoin and $\mathcal{P}_n y = y$ for all $n \geqslant N_y$ for some $N_y \in \mathbb{N}$, we have

$$\lim_{n\to\infty}\langle A_n(h_n), y\rangle = \lim_{n\to\infty}\langle A(h_n), \mathcal{P}_n y\rangle = \lim_{n\to\infty}\langle A(h_n), y\rangle = \langle A(h), y\rangle.$$

Finally, we have $B_n^*(h_n)y = B^*(h_n)\mathcal{P}_n y = B^*(h_n)y$ for all $n \geqslant N_y$ and by the demicontinuity of B, i.e. assumption **(A1)**, relation (6.5) holds. □

7 Solution to the Markov Problem

The key tool to show the existence of a.s. martingale Markov solutions is the notation of Markov families and pre-Markov families. Especially the latter is a rather abstract definition, but we will show in the last section of this part, that the set $\mathscr{S}(x)$ constructed in the previous section is in fact a pre-Markov family.

Then it is clear from the definition that the proof of existence of a.s. martingale Markov solutions is finished, provided we can show that any pre-Markov family admits a Markov selection. This abstract Markov selection Theorem was proved in [13].

7.1 Abstract Markov and pre-Markov families

Let $\mathbb{X}$ and $\mathbb{Y}$ be separable Banach spaces such that $\mathbb{Y} \hookrightarrow \mathbb{X}$ continuously and densely and let $\mathcal{U} := C([0,\infty); \mathbb{X})$. We consider $\mathcal{U}$ as path space and use the notations from the preliminaries.

Definition 7.1. A family $(\mathbb{P}_y)_{y\in\mathbb{Y}} \subseteq \mathscr{P}_{\mathbb{Y}}(\mathcal{U})$ is called *almost sure Markov family* iff

(1) for each $A \in \mathcal{G}$ the mapping

$$y \mapsto \mathbb{P}_y(A)$$

is $\mathcal{B}(\mathbb{Y})/\mathcal{B}(\mathbb{R})$ measurable;

(2) for each $y \in \mathbb{Y}$ there exists a Lebesgue null set $T_y \subseteq (0,\infty)$ such that for all $t \in (0,\infty)\backslash T_y$ and $s \geqslant t$ we have

$$\mathbb{P}_y(\cdot|\mathcal{G}_t)(u) = \mathbb{P}_{u(t)} \circ \Psi_t,$$

where both sides are considered as probability measures on $\mathcal{U}^t$, see section 4.3.

Remark 7.2. A family $(\mathbb{Q}_x)_{x\in\mathbb{H}}$ of probability measures, where $\mathbb{Q}_x$ is a martingale solutions of equation (5.7) with initial value x, is an a.s. martingale Markov solution iff it is an a.s. Markov family.

In order to introduce pre-Markov families, we start with the following lemma.

Lemma 7.3. *Let $t > 0$, $\mathbb{P} \in \mathscr{P}(\mathcal{U})$ and let $Q : \mathcal{U} \to \mathscr{P}(\mathcal{U})$ (resp. $Q : \mathcal{U} \to \mathscr{P}(\mathcal{U}^t)$) be a map such that*

(1) *for any $A \in \mathcal{G}$ (resp. $A \in \mathcal{G}^t$), the mapping*

$$u \mapsto Q(u)(A)$$

is $\mathcal{G}_t/\mathcal{B}(\mathbb{R})$ measurable;

(2) *for $\mathbb{P}$-a.e. $u \in \mathcal{U}$ we have*

$$Q(u)(\{v \in \mathcal{U} \,|\, u|_{[0,t]} = v|_{[0,t]}\}) = 1,$$

resp.

$$Q(u)(\{v \in \mathcal{U}^t \,|\, u(t) = v(t)\}) = 1.$$

Then, there is a unique probability measure $\mathbb{P} \otimes_t Q \in \mathscr{P}(\mathcal{U})$ such that

$$\mathbb{P} \otimes_t Q|_{\mathcal{G}_t} = \mathbb{P}|_{\mathcal{G}_t} \tag{7.1}$$

and the mapping $(u, A) \mapsto Q(u)(A)$ is a r.c.p.d. of $\mathbb{P} \otimes_t Q$ with respect to $\mathcal{G}_t$, i.e.

$$(\mathbb{P} \otimes_t Q)(\cdot|\mathcal{G}_t)(u) = Q(u) \tag{7.2}$$

for $\mathbb{P} \otimes_t Q$-a.e. $u \in \mathcal{U}$, where the left hand side is considered as a probability measure on $\mathcal{U}$ (resp. $\mathcal{U}^t$).

Proof. We show the version for $Q : \mathcal{U} \to \mathscr{P}(\mathcal{U})$, the other version follows similar. The mapping $u \mapsto Q(u)(A)$ is $\mathcal{G}_t/\mathcal{B}(\mathbb{R})$ measurable by assumption. Thus, we can define

$$\mathbb{P} \otimes_t Q(A) := \mathbb{E}^{\mathbb{P}}[Q(\cdot)(A)].$$

It is routine to check that $\mathbb{P} \otimes_t Q$ defines a probability measure on $(\mathcal{U}, \mathcal{G})$. If $A \in \mathcal{G}_t$, then $u \in A$ if and only if $\{v \in \mathcal{U} \mid u|_{[0,t]} = v|_{[0,t]}\} \subseteq A$ by Lemma 4.19. Thus, $Q_u(A) = \mathbb{I}_A(u)$ for $\mathbb{P}$-a.e. $u \in \mathcal{U}$, and consequently we get

$$\mathbb{E}^{\mathbb{P}}[Q(\cdot)(A)] = \mathbb{E}^{\mathbb{P}}[\mathbb{I}_A] = \mathbb{P}(A),$$

i.e. (7.1) holds. For $A \in \mathcal{G}$ and $C \in \mathcal{G}_t$ we have

$$\begin{aligned}\int_C \mathbb{P} \otimes_t Q(A|\mathcal{G}_t) \, \mathrm{d}\mathbb{P} \otimes_t Q &= \mathbb{P} \otimes_t Q(A \cap C) \\ &= \mathbb{E}^{\mathbb{P}}[Q(\cdot)(A \cap C)] \\ &= \int_C Q(u)(A)\mathbb{P}(\,\mathrm{d}u) \\ &= \int_C Q(u)(A)\mathbb{P} \otimes_t Q(\,\mathrm{d}u)\end{aligned}$$

where the last equality holds, since $u \mapsto Q(u)(A)$ is $\mathcal{G}_t/\mathcal{B}(\mathbb{R})$ measurable and $\mathbb{P}|_{\mathcal{G}_t} = \mathbb{P} \otimes_t Q|_{\mathcal{G}_t}$. Thus, we know that $Q(u)$ is a conditional probability measure of $\mathbb{P} \otimes_t Q$ with respect to $\mathcal{G}_t$. Since $Q(u) \in \mathscr{P}(\mathcal{U})$ for any $u \in \mathcal{U}$ and $u \mapsto Q(u)(A)$ is $\mathcal{G}_t$ measurable by assumption for any $A \in \mathcal{G}$, Q is regular. We conclude that (7.2) holds.

Finally, the uniqueness follows from Lemma 3.27. □

Definition 7.4. A family $\{\Theta_y\}_{y \in \mathbb{Y}} \subseteq \mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U}))$ is called an *almost sure pre-Markov family* iff for each $y \in \mathbb{Y}$ and $\mathbb{P} \in \Theta_y$, there exists a Lebesgue null set $T_{y,\mathbb{P}} \subseteq (0, \infty)$ such that for all $t \in (0, \infty) \backslash T_{y,\mathbb{P}}$ there is a $\mathbb{P}-$null set $N \in \mathcal{G}_t$, such that

(1) The mapping

$$y \mapsto \Theta_y$$

is $\mathcal{B}(\mathbb{Y})/\mathcal{B}(\mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U})))$ measurable;

(2) We have $\mathbb{P}(\{v \in \mathcal{U} \mid v(0) = y\}) = 1$ for all $y \in \mathbb{Y}$ and all $\mathbb{P} \in \Theta_y$;

(3) *(Disintegration)* For all $u \in \mathcal{U} \backslash N$

$$u(t) \in \mathbb{Y}$$

and there is a regular version $\mathbb{P}(\cdot|\mathcal{G}_t)(\cdot)$ of the conditional probability distribution, such that

$$\mathbb{P}(\cdot|\mathcal{G}_t)(u) \circ \Psi_t^{-1} \in \Theta_{u(t)},$$

where the isomorphism $\Psi_t^{-1} : \mathcal{U} \to \mathcal{U}^t$ is given by $\Psi_t^{-1}(u)(s) = u(s-t)$.

(4) *(Reconstruction)* Let

$$Q : \mathcal{U} \to \mathscr{P}(\mathcal{U}^t)$$

be any mapping such that the following conditions are satisfied:

(a) For any $u \in \mathcal{U} \backslash N$ we have $u(t) \in \mathbb{Y}$, $Q(u) \in \mathscr{P}_{\mathbb{Y}}(\mathcal{U})$ and

$$Q(u) \circ \Psi_t^{-1} \in \Theta_{u(t)};$$

(b) For any $A \in \mathcal{G}^t$, the mapping

$$u \mapsto Q(u)(A)$$

is $\mathcal{G}_t/\mathcal{B}(\mathbb{R})$ measurable;

(c) For any $u \in \mathcal{U} \backslash N$

$$Q(u)(\{v \in \mathcal{U} \,:\, u(t) = v(t)\}) = 1.$$

Then

$$\mathbb{P} \otimes_t Q \in \Theta_y.$$

Remark 7.5. In the context of the disintegration property, the notation $\mathbb{P}(\cdot|\mathcal{G}_t)(u)$ always denotes a r.c.p.d. enjoying the properties in (3) in the preceding Definition.

Definition 7.6. We call a pre-Markov family $(\Theta_y)_{y \in Y}$ *regular* iff Θ_y is convex for all $y \in \mathbb{Y}$.

7.2 Abstract Markov Selections

In this subsection we prove that a regular pre-Markov family $(\Theta_y)_{y \in \mathbb{Y}}$ admits a Markov selection, i.e. there exists a Markov family $(\mathbb{Q}_y)_{y \in \mathbb{Y}}$ such that $\mathbb{Q}_y \in \Theta_y$ for all $y \in \mathbb{Y}$. Until the end of this subsection, fix a regular pre-Markov family $(\Theta_y)_{y \in \mathbb{Y}}$.

Definition 7.7. Let $f \in C_b(\mathbb{X}, \mathbb{R})$ and $\lambda > 0$. We define the following functions:

$$\begin{aligned}
L_{f,\lambda}(t,u) &:= \int_t^\infty e^{-\lambda s} f(u(s))\,\mathrm{d}s && t \geqslant 0,\, u \in \mathcal{U} \\
J_{f,\lambda}(\mathbb{P}) &:= \mathbb{E}^{\mathbb{P}}[L_{f,\lambda}(0,\cdot)] && \mathbb{P} \in \mathscr{P}(\mathcal{U}) \\
M_{f,\lambda,\Theta}(y) &:= \sup_{\mathbb{P} \in \Theta_y} J_{f,\lambda}(\mathbb{P}) && y \in \mathbb{Y} \\
\Xi_{f,\lambda,\Theta}(y) &:= \{\mathbb{P} \in \Theta_y \mid J_{f,\lambda}(\mathbb{P}) = M_{f,\lambda,\Theta}(y)\} && y \in \mathbb{Y}
\end{aligned}$$

Proposition 7.8. *For all $f \in C_b(\mathbb{X})$ and $\lambda > 0$, the family $(\Xi_{f,\lambda,\Theta}(y))_{y \in \mathbb{Y}}$ is a regular pre-Markov family.*

Proof. **Step 1.** $J_{f,\lambda}$ is linear[9] and continuous with respect to the weak topology. Since Θ_y is a convex and weakly compact subset of $\mathscr{P}_{\mathbb{Y}}(\mathcal{U})$, the set $\Xi_{f,\lambda,\Theta}(y)$ is non-empty, convex and weakly compact. In particular, we have $\Xi_{f,\lambda,\Theta}(y) \in \mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U}))$. Finally, (2) from Definition 7.4 is satisfied, since $\Xi_{f,\lambda,\Theta}(y) \subseteq \Theta_y$.

Step 2. We show that $y \mapsto \Xi_{f,\lambda,\Theta}(y)$ is $\mathcal{B}(\mathbb{Y})/\mathcal{B}(\mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U})))$ measurable. Per definition, that mapping

$$y \mapsto \Theta_y$$

is $\mathcal{B}(\mathbb{Y})/\mathcal{B}(\mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U})))$ measurable. Thus, we need to show that

$$K \mapsto \varphi(K) := \{\mathbb{P} \in K \mid J_{f,\lambda}(\mathbb{P}) = \sup_{\mathbb{Q} \in K} J_{f,\lambda}(\mathbb{Q})\}$$

is $\mathcal{B}(\mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U})))/\mathcal{B}(\mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U})))$ measurable. Let $U \subseteq \mathscr{P}_{\mathbb{Y}}(\mathcal{U})$ be an open set. It is sufficient to show that

$$\varphi^{-1}\left(\left\{K \in \mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U})) \,\middle|\, K \subseteq U\right\}\right) \in \mathcal{B}(\mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U}))),$$

since by Lemma 1.9 the system of sets of this form is a generator for $\mathcal{B}(\mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U})))$. We have

$$\varphi^{-1}\left(\left\{K \in \mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U})) \,\middle|\, K \subseteq U\right\}\right) = \left\{K \,\middle|\, K \subseteq U\right\} \cup \left\{K \,\middle|\, K \cap U^c \neq \varnothing \text{ and } \sup_{\mathbb{P} \in K \cap U^c} J_{f,\lambda}(\mathbb{P}) < \sup_{\mathbb{P} \in K} J_{f,\lambda}(\mathbb{P})\right\}.$$

The set $\{K \mid K \subseteq U\} \subseteq \mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U}))$ is open. Next, we have

$$\{K \in \mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U})) \mid \sup_{\mathbb{P} \in K} J_{f,\lambda}(\mathbb{P}) < s\} = \bigcup_{\epsilon > 0} \{K \mid K \subseteq \{\mathbb{P} \mid J_{f,\lambda}(\mathbb{P}) < s - \epsilon\}\}$$

and since by continuity of $J_{f,\lambda}$, the set $\{\mathbb{P} \mid J_{f,\lambda}(\mathbb{P}) < s - \epsilon\}$ is open in $\mathscr{P}_{\mathbb{Y}}(\mathcal{U})$, the right hand side is open in $\mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U}))$. Thus,

$$K \mapsto \sup_{\mathbb{P} \in K} J_{f,\lambda}(\mathbb{P}) \tag{7.3}$$

[9] To be more precise: the set $\mathscr{P}_{\mathbb{Y}}(\mathcal{U})$ is a convex and weakly closed subset of the space $\mathcal{M}(\mathcal{U})$, where $\mathcal{M}(\mathcal{U})$ denotes the set of all signed real-valued finite measures μ on $(\mathcal{U}, \mathcal{G})$. We can prolong the functional $J_{f,,\lambda}$ to $\mathcal{M}(\mathcal{U})$ by letting $J_{f,\lambda}(\mu) := \int L_{f,\lambda}(0,\cdot)\,\mathrm{d}\mu$. Now $J_{f,\lambda}$ is linear and weakly continuous.

is continuous and in particular $\mathcal{B}(\mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U})))/\mathcal{B}(\mathbb{R})$ measurable. We have

$$\left\{K \in \mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U})) \,\middle|\, K \cap U^c \neq \varnothing\right\} = \left\{K \,\middle|\, K \subseteq U\right\}^c \in \mathcal{B}(\mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U})))$$

and it is easy to see, that $K \mapsto K \cap U^c$ is $\mathcal{B}(\mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U})))/\mathcal{B}(\mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U})))$ measurable on $\{K \in \mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U})) \,|\, K \cap U^c \neq \varnothing\}$. Therefore, the mapping

$$K \mapsto \sup_{\mathbb{P} \in K \cap U^c} J_{f,\lambda}(\mathbb{P})$$

is measurable. Thus, we have

$$\left\{K \,\middle|\, K \cap U^c \neq \varnothing \text{ and } \sup_{\mathbb{P} \in K \cap U^c} J_{f,\lambda}(\mathbb{P}) < \sup_{\mathbb{P} \in K} J_{f,\lambda}(\mathbb{P})\right\} \in \mathcal{B}(\mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U}))).$$

Step 3. We prove the disintegration property of Definition 7.4. Fix $y \in \mathbb{Y}$, $\mathbb{P} \in \Xi_{f,\lambda,\Theta}(y)$ and $t \in (0,\infty)\backslash T_{y,\mathbb{P}}$, where $T_{y,\mathbb{P}}$ and N denote the corresponding exceptional null sets in the definition of the disintegration property of Θ_y. Let[10]

$$N_2 := \{u \in N^c \,|\, \mathbb{P}(\Psi_t^{-1}(\cdot)|\mathcal{G}_t)(u) \notin \Xi_{f,\lambda,\Theta}(u(t))\}.$$

By the disintegration property for Θ we have $N \in \mathcal{G}_t$ and $\mathbb{P}(N) = 0$. By Lemma 1.10 we have $N_2 \in \mathcal{G}_t$. The disintegration property follows, provided we can show $\mathbb{P}(N_2) = 0$.

First, we apply Theorem 3.32 to the mapping $y \mapsto \Xi_{f,\lambda,\Theta}$. Since $\mathscr{P}(\mathcal{U})$ is a complete and separable metric space and $\Xi_{f,\lambda,\Theta} \subseteq \mathscr{P}_{\mathbb{Y}}(\mathcal{U})$ is non-empty and compact, and therefore closed, for all $y \in \mathbb{Y}$, we are left to show that

$$A(U) := \left\{y \in \mathbb{Y} \,\middle|\, \Xi_{f,\lambda,\Theta}(y) \cap U \neq \varnothing\right\} \in \mathcal{B}(\mathbb{Y}) \tag{7.4}$$

for all open sets $U \subseteq \mathscr{P}_{\mathbb{Y}}(\mathcal{U})$. But this is an easy consequence from Lemma 1.9 and the measurability of the mapping $y \mapsto \Xi_{f,\lambda,\Theta}(y)$ since

$$A(U) = \left\{y \in \mathbb{Y} \,\middle|\, \Xi_{f,\lambda,\Theta}(y) \nsubseteq U^c\right\} = \left\{y \in \mathbb{Y} \,\middle|\, \Xi_{f,\lambda,\Theta}(y) \subseteq U^c\right\}^c = \Xi_{f,\lambda,\Theta}^{-1}(\{K \in \mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U})) \,|\, K \subseteq U^c\})^c.$$

Therefore (7.4) holds. Thus, there is a measurable selection $\eta : \mathbb{Y} \to \mathscr{P}_{\mathbb{Y}}(\mathcal{U})$ such that $\eta_y \in \Xi_{f,\lambda,\Theta}(y)$ for all $y \in \mathbb{Y}$.

For $B \in \mathcal{G}$ and $u \in \mathcal{U}$ let $\tau_u(B) := \{v \in B \,|\, v|_{[0,t]} = u|_{[0,t]}\}$. Then, it is routine to check that $A \mapsto \eta_{u(t)} \circ \Psi_t \circ \tau_u(A)$ defines a probability measure on $(\mathcal{U}, \mathcal{G})$. We define the probability measures

$$Q(u) := \begin{cases} \mathbb{P}(\cdot|\mathcal{G}_t)(u) & \text{for } u \notin N \cup N_2 \\ \eta_{u(t)} \circ \Psi_t \circ \tau_u & \text{for } u \in N_2 \\ \delta_u & \text{for } u \in N, \end{cases}$$

for all $u \in \mathcal{U}$, where δ_u denotes the Dirac distribution with pole u. We show that Q satisfies the conditions

[10] $\mathbb{P}(\cdot|\mathcal{G}_t)$ denotes a r.c.p.d. such that (3) of Definition 7.4 holds for Θ.

of the reconstruction property for Θ with $\mathbb{P}$ and t and with the exceptional $\mathbb{P}$-null set N. Condition (a) for $\mathbb{P}(\cdot|\mathcal{G}_t)(u)$ follows from the disintegration property for Θ. Furthermore, direct calculations show

$$\eta_{u(t)} \circ \Psi_t \circ \tau_u \circ \Psi_t^{-1} = \eta_{u(t)} \in \Xi_{f,\lambda,\Theta}(u(t)) \subseteq \Theta_{u(t)}$$

and

$$\eta_{u(t)} \circ \Psi_t \circ \tau_u \in \mathscr{P}_{\mathbb{Y}}(\mathcal{U}).$$

Thus, (a) is satisfied on N_2. Since $N \in \mathcal{G}_t$ and $N_2 \in \mathcal{G}_t$, it is sufficient to show condition (b) on the three sets $(N \cup N_2)^c$, N_2 and N separately. Obviously, (b) holds on $(N \cup N_2)^c$ and on N. To show (b) on N_2, by Lemma 5.11 it is enough to show the measurability for all $A \in \mathcal{E}$, where $\mathcal{E}$ is the generator of $\mathcal{G}$ from Theorem 4.21. For $v \in \mathcal{U}$ let

$$\tilde{v}(s) := \begin{cases} u(s) & \text{for } s \leqslant t \\ v(s) & \text{for } t < s. \end{cases}$$

Let $A \in \mathcal{E}$ and assume that $u|_{[0,t]} \in A|_{[0,t]}$. Then, for any $v \in A$ with $v(t) = u(t)$, we have $\tilde{v} \in A$. Since $\eta_{u(t)}(\{v \in \mathcal{U} \mid v(0) \neq u(t)\}) = 0$ by (2) from Definition 7.4, we get

$$\begin{aligned} \eta_{u(t)} \circ \Psi_t \circ \tau_u(A) &= \eta_{u(t)}(\{v \in \mathcal{U} \mid \exists w \in \tau_u(A) : w(s) = v(s-t) \text{ for all } s \geqslant t \text{ and } w(t) = u(t)\}) \\ &= \eta_{u(t)}(\{v \in \mathcal{U} \mid \exists w \in \tau_u(A) : w(s) = v(s-t) \text{ for all } s \geqslant t \text{ and } w(s) = u(s) \text{ for } s \leqslant t\}) \\ &= \eta_{u(t)}(\{v \in \mathcal{U} \mid \exists w \in A : w(s) = v(s-t) \text{ for all } s \geqslant t \text{ and } w(s) = u(s) \text{ for } s \leqslant t\}) \\ &= \eta_{u(t)}(\{v \in \mathcal{U} \mid \exists w \in A : w(s) = v(s-t) \text{ for all } s \geqslant t\}) \\ &= \eta_{u(t)}(\Psi_t(A)). \end{aligned}$$

On the other hand, if $A \in \mathcal{E}$ with $u|_{[0,t]} \notin A|_{[0,t]}$, then we have

$$\eta_{u(t)} \circ \Psi_t \circ \tau_u(A) = \eta_{u(t)} \circ \Psi_t(\varnothing) = 0.$$

Now, we have

$$M := \{u \in N_2 \mid u|_{[0,t]} \in A|_{[0,t]}\} \in \mathcal{G}_t,$$

and $u \mapsto \eta_{u(t)}(\Psi_t(A))$ is $\mathcal{G}_t\backslash\mathcal{B}(\mathbb{R})$ measurable on M, since $u \mapsto u(t)$ is $\mathcal{G}_t\backslash\mathcal{B}(\mathbb{Y})$ measurable and $y \mapsto \eta_y(\Psi_t(A))$ is measurable, since η is a measurable selection. Thus (b) is satisfied on N_2. Finally, (c) is obviously satisfied in all three cases.

Thus, we have $\mathbb{P} \otimes_t Q \in \Theta_y$ and we get

$$
\begin{aligned}
J_{f,\lambda}(\mathbb{P}) = M_{f,\lambda,\Theta}(y) \quad &\geqslant \quad J_{f,\lambda}(\mathbb{P} \otimes_t Q) \\
&\overset{(7.1)}{=} \quad \mathbb{E}^{\mathbb{P}}[\int_0^t e^{-\lambda s} f(\xi(s))\,\mathrm{d}s] + \mathbb{E}^{\mathbb{P}\otimes_t Q}[L_{f,\lambda}(t,\cdot)] \\
&= \quad J_{f,\lambda}(\mathbb{P}) - \mathbb{E}^{\mathbb{P}}[L_{f,\lambda}(t,\cdot)] + \mathbb{E}^{\mathbb{P}\otimes_t Q}[L_{f,\lambda}(t,\cdot)] \\
&= \quad J_{f,\lambda}(\mathbb{P}) - \int_{\mathcal{U}}\int_{\mathcal{U}} L_{f,\lambda}(t,u)\mathbb{P}(\mathrm{d}u|\mathcal{G}_t)(v)\mathbb{P}(\mathrm{d}v) \\
&\quad\quad + \int_{\mathcal{U}}\int_{\mathcal{U}} L_{f,\lambda}(t,u)(\mathbb{P}\otimes_t Q)(\mathrm{d}u|\mathcal{G}_t)(v)\mathbb{P}(\mathrm{d}v) \\
&\overset{(7.2)}{=} \quad J_{f,\lambda}(\mathbb{P}) - \int_{\mathcal{U}}\int_{\mathcal{U}} L_{f,\lambda}(t,u)\mathbb{P}(\mathrm{d}u|\mathcal{G}_t)(v)\mathbb{P}(\mathrm{d}v) \\
&\quad\quad + \int_{\mathcal{U}}\int_{\mathcal{U}} L_{f,\lambda}(t,u)\ Q(v)(\mathrm{d}u)\mathbb{P}(\mathrm{d}v).
\end{aligned}
$$

where ξ denotes the canonical process. Since by definition of Q we have $\mathbb{P}(\mathrm{d}u|\mathcal{G}_t)(v) = Q(v)(\mathrm{d}u)$ for $v \notin N \cup N_2$ and $\mathbb{P}(N) = 0$, this yields

$$
0 \geqslant \int_{N_2}\left(\int_{\mathcal{U}} L_{f,\lambda}(t,u)\ (\eta_{v(t)} \circ \Psi_t \circ \tau_v)(\mathrm{d}u)\right)\mathbb{P}(\mathrm{d}v) - \int_{N_2}\left(\int_{\mathcal{U}} L_{f,\lambda}(t,u)\mathbb{P}(\mathrm{d}u|\mathcal{G}_t)(v)\right)\mathbb{P}(\mathrm{d}v).
$$

We have

$$
\begin{aligned}
\int_{\mathcal{U}} L_{f,\lambda}(t,u)\ (\eta_{v(t)} \circ \Psi_t \circ \tau_v)(\mathrm{d}u) &= e^{-\lambda t}\int_{\{w\in\mathcal{U}\,|\,w|_{[0,t]}=v|_{[0,t]}\}} L_{f,\lambda}(0,\Psi_t u)\ (\eta_{v(t)} \circ \Psi_t)(\mathrm{d}u) \\
&= e^{-\lambda t}\int_{\{w\in\mathcal{U}\,|\,w(0)=v(t)\}} L_{f,\lambda}(0,u)\ \eta_{v(t)}(\mathrm{d}u) \\
&= e^{-\lambda t}\int_{\mathcal{U}} L_{f,\lambda}(0,u)\ \eta_{v(t)}(\mathrm{d}u),
\end{aligned}
$$

for any $v \in N_2$, where the first equality holds, since

$$
\eta_{v(t)} \circ \Psi_t \circ \tau_v(\{w \in \mathcal{U}\,|\,w|_{[0,t]} = v|_{[0,t]}\}^c) = 0
$$

and

$$
\eta_{v(t)} \circ \Psi_t \circ \tau_v(A) = \eta_{v(t)} \circ \Psi_t(A)
$$

for any measurable $A \subseteq \{w \in \mathcal{U}\,|\,w|_{[0,t]} = v|_{[0,t]}\}$, and where the last equality holds, since $\eta_{v(t)}(\{w \in$

$\mathcal{U} \,|\, w(0) = v(t)\}) = 1$ by (2) from Definition 7.4. Now, we get

$$\begin{aligned} 0 \geqslant e^{-\lambda t} \int_{N_2} \Big(\int_{\mathcal{U}} L_{f,\lambda}(0,u)\ \eta_{v(t)}(\mathrm{d}u) - \int_{\mathcal{U}} L_{f,\lambda}(0,\Psi_t u) \mathbb{P}(\mathrm{d}u|\mathcal{G}_t)(v) \Big) \mathbb{P}(\mathrm{d}v) \\ = e^{-\lambda t} \int_{N_2} \Big(J_{f,\lambda}(\eta_{v(t)}) - J_{f,\lambda}(\mathbb{P}(\Psi_t^{-1}(\cdot)|\mathcal{G}_t)(v)) \Big) \mathbb{P}(\mathrm{d}v). \end{aligned}$$

On the other hand, since by Definition $v \in N_2$ implies $\mathbb{P}(\Psi_t^{-1}(\cdot)|\mathcal{G}_t)(v) \in \Theta_{v(t)} \backslash \Xi_{f,\lambda,\Theta}(v(t))$, we have

$$J_{f,\lambda}(\mathbb{P}(\Psi_t^{-1}(\cdot)|\mathcal{G}_t)(v)) < M_{f,\lambda,\Theta}(v(t)) = J_{f,\lambda}(\eta_{v(t)}).$$

The last two inequalities yield

$$\mathbb{P}(N_2) = 0.$$

Step 4. We show the reconstruction property of Definition 7.4. Let $y \in \mathbb{Y}$, $\mathbb{P} \in \Xi_{f,\lambda,\Theta}(y)$, $t \in (0,\infty) \backslash T_{y,\mathbb{P}}$ and $Q : \mathcal{U} \to \mathscr{P}(\mathcal{U}^t)$ be a mapping satisfying the assumptions (4a)-(4c) from Definition 7.4 with the family $(\Xi_{f,\lambda,\Theta}(y))_{y \in \mathbb{Y}}$, and denote by N the corresponding null set. We have $\mathbb{P} \otimes_t Q \in \Theta_y$ by the reconstruction property for $(\Theta_y)_{y \in \mathbb{Y}}$. Now, we get

$$\begin{aligned} J_{f,\lambda}(\mathbb{P} \otimes_t Q) - \mathbb{E}^{\mathbb{P}}[\int_0^t e^{-\lambda s} f(\xi(s))\, \mathrm{d}s] &\overset{(7.1)}{=} \mathbb{E}^{\mathbb{P} \otimes_t Q}[L_{f,\lambda}(t,\cdot)] \\ &= \mathbb{E}^{\mathbb{P} \otimes_t Q|_{\mathcal{G}_t}} \Big[\mathbb{E}^{\mathbb{P} \otimes_t Q}[L_{f,\lambda}(t,\cdot)|\mathcal{G}_t] \Big] \\ &\overset{(7.1),(7.2)}{=} \mathbb{E}^{\mathbb{P}} \Big[\mathbb{E}^{Q(\cdot)}[L_{f,\lambda}(t,\cdot)] \Big] \\ &= e^{-\lambda t} \int_{\mathcal{U}} J_{f,\lambda}(Q(u) \circ \Psi_t^{-1}) \mathbb{P}(\mathrm{d}u) \\ &\overset{Def.\,7.4,(3)+(4)}{=} e^{-\lambda t} \int_{\mathcal{U}} J_{f,\lambda}(\mathbb{P}(\cdot|\mathcal{G}_t)(u) \circ \Psi_t^{-1}) \mathbb{P}(\mathrm{d}u) \\ &= \int_{\mathcal{U}} \Big(\int_{\mathcal{U}} L_{f,\lambda}(t,u) \mathbb{P}(\mathrm{d}u|\mathcal{G}_t)(v) \Big) \mathbb{P}(\mathrm{d}v) \\ &= \mathbb{E}^{\mathbb{P}}[L_{f,\lambda}(t,\cdot)] \\ &= J_{f,\lambda}(\mathbb{P}) - \mathbb{E}^{\mathbb{P}}[\int_0^t e^{-\lambda s} f(\xi(s))\, \mathrm{d}s] \end{aligned}$$

where the fifth equality follows from the fact that on the one hand we have $Q(u) \circ \Psi_t^{-1} \in \Xi_{f,\lambda,\Theta}(u(t))$, due to Definition 7.4, (4), and on the other hand $\mathbb{P}(\cdot|\mathcal{G}_t)(u) \circ \Psi_t^{-1} \in \Xi_{f,\lambda,\Theta}(u(t))$, due to Definition 7.4, (3), and thus

$$J_{f,\lambda}(Q(u) \circ \Psi_t^{-1}) = M_{f,\lambda,\Theta}(u(t)) = J_{f,\lambda}(\mathbb{P}(\cdot|\mathcal{G}_t)(u) \circ \Psi_t^{-1}).$$

Consequently, we have

$$J_{f,\lambda}(\mathbb{P} \otimes_t Q) = J_{f,\lambda}(\mathbb{P}) = M_{f,\lambda,\Theta}(y).$$

and therefore

$$\mathbb{P} \otimes_t Q \in \Xi_{f,\lambda,\Theta}(y).$$

$\square$

We can now prove the following crucial Theorem:

Theorem 7.9. *Let $(\Theta_y)_{y\in\mathbb{Y}}$ be a regular pre-Markov family. Then, there exists a Markov selection $(\mathbb{Q}_y)_{y\in\mathbb{Y}}$ for Θ, i.e. $(\mathbb{Q}_y)_{y\in\mathbb{Y}}$ is a Markov family such that $\mathbb{Q}_y \in \Theta_y$ for all $y \in \mathbb{Y}$.*

Proof. **Step 1.** Let $\Upsilon \subseteq C_b(\mathbb{X};\mathbb{R})$ be countable and dense, and let (λ_n, f_n) be an enumeration of $(\mathbb{Q} \cap (0,\infty)) \times \Upsilon$. For each $y \in \mathbb{Y}$ let $\Xi_0(y) := \Theta_y$ and define inductively

$$\Xi_{n+1}(y) := \Xi_{f_n,\lambda_n,\Xi_n}(y)$$

where $\Xi_{f_n,\lambda_n,\Xi_n}(y)$ is defined in Definition 7.7. By Proposition 7.8, the families $(\Xi_{n+1}(y))_{y\in\mathbb{Y}}$ are regular a.s. pre-Markov families. Let

$$\Xi_\infty(y) := \bigcap_{n\in\mathbb{N}} \Xi_n(y).$$

Step 2. We show that $(\Xi_\infty(y))_{y\in\mathbb{Y}}$ is a regular a.s. pre-Markov family. First, since $\Xi_\infty(y)$ is a closed subset of a compact set, it is compact, and since $(\Xi_n(y))_{n\in\mathbb{N}}$ enjoys the finite intersection property, $\Xi_\infty(y)$ is not empty. Thus, we have $\Xi_\infty(y) \in \mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U}))$. For any $y \in \mathbb{Y}$ and $\epsilon > 0$ let $K_\epsilon := B_\epsilon(\Xi_\infty(y))^c$, where B_ϵ denotes the ball with radius ϵ with respect to the metric $d_{\mathcal{C}}$. For any fixed $\epsilon > 0$ and $y \in \mathbb{Y}$, the sequence $(\Xi_n(y) \cap K_\epsilon)_{n\in\mathbb{N}}$ is a family of closed sets with empty intersection. Thus, for some $N \in \mathbb{N}$, we have $\bigcap_{n\leqslant N} \Xi_n(y) \cap K_\epsilon = \varnothing$ and consequently $d_{\mathcal{C}}(\Xi_n(y), \Xi_\infty(y)) < \epsilon$ for $n > N$. It follows that $\Xi_n \to \Xi_\infty$ in $\mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U}))$ pointwise in $\mathbb{Y}$, and thus, the mapping $y \mapsto \Xi_\infty(y)$ is measurable.

The disintegration property follows since $\mathbb{P} \in \Xi_\infty(y)$ implies $\mathbb{P} \in \Xi_n(y)$ for all $n \in \mathbb{N}$. Thus we have $\mathbb{P}(\Psi_t^{-1}(\cdot)|\mathcal{G}_t)(u) \in \Xi_n(y(t))$ for $t \in (0,\infty)\backslash T_{y,\mathbb{P},n}$ and $u \in \mathcal{U}\backslash N_n$, for some null sets $T_{y,\mathbb{P},n}$ and N_n, and thus $\mathbb{P}(\Psi_t^{-1}(\cdot)|\mathcal{G}_t)(u) \in \Xi_\infty(y(t))$ for $t \in (0,\infty)\backslash \bigcup_{n\in\mathbb{N}} T_{y,\mathbb{P},n}$ and $u \in \mathcal{U}\backslash \bigcup_{n\in\mathbb{N}} N_n$. The reconstruction property follows similarly. Thus, $\Xi_\infty(y)$ is an a.s. pre-Markov family. As the intersection of convex sets, $\Xi_\infty(y)$ is convex. Thus, $(\Xi_\infty(y))_{y\in Y}$ is a regular a.s. pre-Markov family.

Step 3. Fix $y \in \mathbb{Y}$ and probability measures $\mathbb{P}, \mathbb{Q} \in \Xi_\infty(y)$ and let $T_{y,\mathbb{P}}$ and $T_{y,\mathbb{Q}}$ be the respective null-sets in Definition 7.4. Let $T := T_{y,\mathbb{P}} \cup T_{y,\mathbb{Q}}$. Fix $0 \leqslant t_1 < ... < t_n \in (0,\infty)\backslash T$, let $\mathcal{F} := \mathcal{A}_\sigma(\{\xi(t_i) : 1 \leqslant i \leqslant n\})$ and assume $\mathbb{P}|_{\mathcal{F}} = \mathbb{Q}|_{\mathcal{F}}$. We show that for some $A \in \mathcal{F}$ with $\mathbb{P}(A) = \mathbb{Q}(A) = 0$ we have

$$\begin{aligned} \mathbb{P}(\cdot|\mathcal{F})(u) \circ \Psi_{t_n}^{-1} \in \Xi_\infty(u(t_n)) \\ \mathbb{Q}(\cdot|\mathcal{F})(u) \circ \Psi_{t_n}^{-1} \in \Xi_\infty(u(t_n)) \end{aligned} \tag{7.5}$$

for all $u \notin A$. [11]

By the disintegration property, there is a $\mathbb{P}-$null set $N_{\mathbb{P}} \subseteq \mathcal{U}$ such that

$$u(t_n) \in \mathbb{Y} \tag{7.6}$$

[11] By the second part of Theorem 3.24, this does not depend on the choice of the r.c.p.d.'s, although the exceptional set A may depend on the choice.

and

$$\mathbb{P}(\cdot|\mathcal{G}_{t_n})(u) \circ \Psi_{t_n}^{-1} \in \Xi_\infty(u(t_n)). \tag{7.7}$$

for all $u \notin N_\mathbb{P}$

Since $\mathcal{F} \subseteq \mathcal{G}_{t_n}$, there is a $\mathbb{P}$-null set $B_\mathbb{P}^1 \in \mathcal{F}$ such that

$$\mathbb{P}(\cdot|\mathcal{F})(u) = \int_\mathcal{U} \mathbb{P}(\cdot|\mathcal{G}_{t_n})(v)\mathbb{P}(\mathrm{d}v|\mathcal{F})(u)$$

for all $u \notin B_\mathbb{P}^1$.

Since $\int_\mathcal{U} \mathbb{P}(N_\mathbb{P}|\mathcal{F})(u)\mathbb{P}(\mathrm{d}u) = \mathbb{P}(N_\mathbb{P}) = 0$, there is a $\mathbb{P}$-null set $B_\mathbb{P}^2 \in \mathcal{F}$ such that

$$\mathbb{P}(N_\mathbb{P}|\mathcal{F})(u) = 0$$

for all $u \notin B_\mathbb{P}^2$.

Finally, Lemma 3.26 yields a $\mathbb{P}$-null set $B_\mathbb{P}^3 \in \mathcal{F}$ such that $\mathbb{P}(\{v : v(t_n) \neq u(t_n)\}|\mathcal{F})(u) = 0$ for any $u \notin B_\mathbb{P}^3$. Let $B_\mathbb{P} := B_\mathbb{P}^1 \cup B_\mathbb{P}^2 \cup B_\mathbb{P}^3$.

For $u \notin B_\mathbb{P}$ we get

$$\begin{aligned}\mathbb{P}(\cdot|\mathcal{F})(u) \circ \Psi_{t_n}^{-1} &= \int_{N_\mathbb{P}^c} \mathbb{P}(\cdot|\mathcal{G}_{t_n})(v) \circ \Psi_{t_n}^{-1}\mathbb{P}(\mathrm{d}v|\mathcal{F})(u) \\ &= \int_{N_\mathbb{P}^c \cap \{v : v(t_n) = u(t_n)\}} \mathbb{P}(\cdot|\mathcal{G}_{t_n})(v) \circ \Psi_{t_n}^{-1}\mathbb{P}(\mathrm{d}v|\mathcal{F})(u)\end{aligned}$$

If $u \in B_\mathbb{P}^c \cap \{v : v(t_n) \notin \mathbb{Y}\}$, then this implies $\mathbb{P}(\cdot|\mathcal{F})(u) \circ \Psi_{t_n}^{-1} = 0$ since (7.6) implies $N_\mathbb{P}^c \cap \{v : v(t_n) = u(t_n)\} = \varnothing$. But since $\mathbb{P}(\cdot|\mathcal{F})(\cdot)$ is a r.c.d.p., we conclude $B_\mathbb{P}^c \cap \{v : v(t_n) \notin \mathbb{Y}\} = \varnothing$.

If $u \in B_\mathbb{P}^c \cap \{v : v(t_n) \in \mathbb{Y}\}$, then we have for all $v \in N_\mathbb{P}^c \cap \{v : v(t_n) = u(t_n)\}$

$$\mathbb{P}(\cdot|\mathcal{G}_{t_n})(v) \circ \Psi_{t_n}^{-1} \in \Xi_\infty(v(t_n)) = \Xi_\infty(u(t_n)).$$

Since for $u \notin B_\mathbb{P}$ we have

$$\mathbb{P}(\cdot|\mathcal{F})(u) = \int_\mathcal{U} \mathbb{P}(\cdot|\mathcal{G}_{t_n})(v)\mathbb{P}(\mathrm{d}v|\mathcal{F})(u),$$

Lemma 3.7 implies

$$\mathbb{P}(\cdot|\mathcal{F})(u) \circ \Psi_{t_n}^{-1} \in \Xi_\infty(u(t_n)).$$

We can repeat the above argumentation to obtain the corresponding result for $\mathbb{Q}$ with some $\mathbb{Q}$-null sets $N_\mathbb{Q} \in \mathcal{G}$ and $B_\mathbb{Q} \in \mathcal{F}$. Let $A := B_\mathbb{P} \cup B_\mathbb{Q}$. Since $\mathbb{P}$ and $\mathbb{Q}$ agree on $\mathcal{F}$, we have

$$\mathbb{P}(A) = \mathbb{Q}(A) = 0,$$

and (7.5) follows for $u \in A^c$.

Step 4. We show that for all $y \in \mathbb{Y}$, all probability measures $\mathbb{P}, \mathbb{Q} \in \Xi_\infty(y)$ and all bounded measurable functions $f : \mathbb{X} \to \mathbb{R}$ we have

$$\mathbb{E}^{\mathbb{P}}[f(\xi(t))] = \mathbb{E}^{\mathbb{Q}}[f(\xi(t))] \tag{7.8}$$

for all $t \geqslant 0$. By Definition of $\Xi_n(y)$ and $\Xi_\infty(y)$ we have

$$\int_0^\infty e^{-qt}\mathbb{E}^{\mathbb{P}}\Big[\varphi(\xi(t))\Big]\,\mathrm{d}t = \int_0^\infty e^{-qt}\mathbb{E}^{\mathbb{P}}\Big[\varphi(\xi(t))\Big]\,\mathrm{d}t$$

for all $q \in \mathbb{Q} \cap (0,\infty)$ and $\varphi \in \Upsilon$. The uniqueness of the Laplace transform implies

$$\mathbb{E}^{\mathbb{P}}[\varphi(\xi(t))] = \mathbb{E}^{\mathbb{Q}}[\varphi(\xi(t))]$$

and by approximation we obtain (7.8).

Step 5. Fix $y \in \mathbb{Y}$. We show that $\Xi_\infty(y)$ contains only one element. Let $\mathbb{P}, \mathbb{Q} \in \Xi_\infty(y)$. It is sufficient to show that the canonical process has the same finite dimensional distribution under $\mathbb{P}$ and $\mathbb{Q}$. Thus, we need to show that $\mathbb{P}^{-1} \circ (\xi(t_1), ..., \xi(t_n)) = \mathbb{Q}^{-1} \circ (\xi(t_1), ..., \xi(t_n))$ for any $n \in \mathbb{N}$ and $0 \leqslant t_1 < ... < t_n$. By continuity, it is enough to show this in the case $t_i \in B$ for all $1 \leqslant i \leqslant n$, where B is dense in $(0,\infty)$; in particular, we can assume $t_i \notin T = T_{y,\mathbb{P}} \cup T_{y,\mathbb{Q}}$. The assertion follows when we can show that

$$\mathbb{E}^{\mathbb{P}}[\prod_{i=1}^n f_i(\xi(t_i))] = \mathbb{E}^{\mathbb{Q}}[\prod_{i=1}^n f_i(\xi(t_i))] \tag{7.9}$$

for any bounded measurable functions $f_i : \mathbb{X} \to \mathbb{R}$.

By the preceding step, (7.9) holds for $n = 1$.

Assume that (7.9) holds for all $t_1 < ... < t_n$ for some $n \in \mathbb{N}$ and let $0 \leqslant t_1 < ... < t_{n+1}$ such that $t_i \notin T$ and let $\mathcal{F} := \mathcal{A}_\sigma(\{\xi(t_i) \,|\, 1 \leqslant i \leqslant n\})$. Then we have $\mathbb{P}|_{\mathcal{F}} = \mathbb{Q}|_{\mathcal{F}}$. Let A be the corresponding $\mathbb{P}$-null and $\mathbb{Q}$-null set from step 3. We need to show that

$$\mathbb{E}^{\mathbb{P}}\Big[\prod_{i=1}^n f_i(\xi(t_i))\mathbb{E}^{\mathbb{P}}[f_{n+1}(\xi(t_{n+1}))|\mathcal{F}]\Big] = \mathbb{E}^{\mathbb{Q}}\Big[\prod_{i=1}^n f_i(\xi(t_i))\mathbb{E}^{\mathbb{Q}}[f_{n+1}(\xi(t_{n+1}))|\mathcal{F}]\Big]$$

i.e.

$$\int_{\mathcal{U}}\prod_{i=1}^n f_i(u(t_i))\Big(\int_{\mathcal{U}} f_{n+1}(v(t_{n+1}))\mathbb{P}(\mathrm{d}v|\mathcal{F})(u)\Big)\mathbb{P}(\mathrm{d}u) = \int_{\mathcal{U}}\prod_{i=1}^n f_i(u(t_i))\Big(\int_{\mathcal{U}} f_{n+1}(v(t_{n+1}))\mathbb{Q}(\mathrm{d}v|\mathcal{F})(u)\Big)\mathbb{Q}(\mathrm{d}u).$$

Thus, it is sufficient to show that

$$\int_{\mathcal{U}} f_{n+1}(v(t_{n+1}))\mathbb{P}(\mathrm{d}v|\mathcal{F})(u) = \int_{\mathcal{U}} f_{n+1}(v(t_{n+1}))\mathbb{Q}(\mathrm{d}v|\mathcal{F})(u)$$

for all $u \notin A$.

Now Step 4, with $\mathbb{P}$ and $\mathbb{Q}$ replaced by $\mathbb{P}(\cdot|\mathcal{F})(u) \circ \Psi_{t_n}^{-1}$ and $\mathbb{Q}(\cdot|\mathcal{F})(u) \circ \Psi_{t_n}^{-1}$, yields the desired conclusion, since

$$\int_{\mathcal{U}} f_{n+1}(v(t_{n+1}))\mathbb{P}(\mathrm{d}v|\mathcal{F})(u) = \mathbb{E}^{\mathbb{P}(\cdot|\mathcal{F})(u)\circ\Psi_{t_n}^{-1}}[f_{n+1}(v(t_{n+1}-t_n))].$$

Step 6. Conclusion. Let $\mathbb{Q}_y$ be the unique element of $\Xi_\infty(y)$. Since $d_{\mathscr{P}_{\mathbb{Y}}(\mathcal{U})}(\mathbb{P},\mathbb{Q}) = d_{\mathcal{C}}(\{\mathbb{P}\},\{\mathbb{Q}\})$, we have $\mathscr{P}_{\mathbb{Y}}(\mathcal{U}) \hookrightarrow \mathcal{C}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U}))$, where the embedding is isometrical, and thus, the mapping $y \mapsto \mathbb{Q}_y$ is $\mathcal{B}(\mathbb{Y})/\mathcal{B}(\mathscr{P}_{\mathbb{Y}}(\mathcal{U}))$ measurable. Since for all $A \in \mathcal{G}$ the mapping $\mathbb{Q} \mapsto \mathbb{Q}(A)$ is weakly continuous and therefore measurable, the mapping $y \mapsto \mathbb{Q}_y(A)$ is $\mathcal{B}(\mathbb{Y})/\mathcal{B}(\mathbb{R})$ measurable.

Point (2) of Definition 7.1 follows immediately from the disintegration property for Ξ_∞. ☐

7.3 Martingale solutions to the Markov problem

We return to the notations of Section 6. In particular, we have the Gelfand triple $\mathbb{X} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{X}'$, where the embeddings are compact, and a fixed countable dimensional and dense subspace $\overline{\mathbb{Y}} \leqslant \mathbb{X}$. From Proposition 6.8 we know that $\mathscr{S}(x)$ is non-empty for all $x \in \mathbb{H}$. We show that the family $(\mathscr{S}(x))_{x\in\mathbb{H}}$ is a regular a.s. pre-Markov family, then Theorem 7.9 implies the existence of a Markov selection and thus the existence of an a.s. martingale solution to the Markov Problem associated with (5.7).

Proposition 7.10. *Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{H}$ with $x_n \to x$ in $\mathbb{H}$ and let $\mathbb{P}_n \in \mathscr{S}(x_n)$. Then there exists a subsequence such that $\mathbb{P}_{n_k} \rightharpoonup \mathbb{P}$ in $\mathscr{P}(\mathcal{U})$ for some $\mathbb{P} \in \mathscr{S}(x)$.*

Proof. **Step 1.** Weak convergence in $\mathscr{P}(\mathcal{U}_q)$ implies weak convergence in $\mathscr{P}(\mathcal{U})$. Thus, we need only to show that $(\mathbb{P}_n)_{n\in\mathbb{N}}$ is tight in $\mathscr{P}(\mathcal{U}_q)$, where q is as in (A2), since then $\mathbb{P}_{n_k} \rightharpoonup \mathbb{P}$ in $\mathscr{P}(\mathcal{U}_q)$ for some subsequence, and $\mathbb{P} \in \mathscr{S}(x)$ by Proposition 6.6. Let $(b_n)_{n\in\mathbb{N}}$ be the orthonormal basis of $\mathbb{H}$ defined at the beginning of the previous section and let $\mathcal{P}_n$ be the corresponding orthogonal projection onto $\operatorname{span}\{b_1, \dots, b_n\}$. For all $n \in \mathbb{N}$ let

$$M_n(t,u) := \sum_{k=1}^{\infty} M_{b_k}(t,u)b_k - x_n$$

where M_{b_k} are as in Definition 5.3.

Step 2. We show that M_n is a continuous, $\mathbb{H}$-valued $(\mathcal{G}_t)_{t\geqslant 0}$-martingale starting at zero a.e. with respect to $\mathbb{P}_n$, whose covariation operator process is given by

$$\langle\!\langle M_n \rangle\!\rangle(t,u) = \int_0^t B(u(s))B^*(u(s))\,\mathrm{d}s. \tag{7.10}$$

Let $M_n^m(t,u) := \sum_{k=1}^m M_{b_k}(t,u)b_k - \mathcal{P}_m x_n$. Then M_n^m is a continuous, $\mathbb{H}$-valued $(\mathcal{G}_t)$-martingale with respect to $\mathbb{P}_n$. By applying the BDG's inequality, we get

$$\begin{aligned}\mathbb{E}^{\mathbb{P}_n}[\sup_{t\in[0,T]} \|M_n^m(t,\xi) - M_n^l(t,\xi)\|_{\mathbb{H}}^2] &\overset{BDG}{\leqslant} C\int_0^T \mathbb{E}^{\mathbb{P}_n}[\sum_{k=l}^m \|B^*(\xi(s))(b_k)\|_{\mathbb{U}}^2]\,\mathrm{d}s \\ &= C\|\sum_{k=l}^m \|B^*(\xi)(b_k)\|_{\mathbb{U}}^2\|_{L^1((0,T)\times\mathcal{U};\,\mathrm{d}t\otimes\mathbb{P}_n)}\end{aligned}$$

and since

$$\sum_{k=l}^{m} \|B^*(\xi)(b_k)\|_{\mathbb{U}}^2 \leqslant \sum_{k=1}^{\infty} \|B^*(\xi)(b_k)\|_{\mathbb{U}}^2 = \|B^*(\xi)\|_{L_2(\mathbb{H},\mathbb{U})}^2 \in L^1((0,T) \times \mathcal{U}; \mathrm{d}t \otimes \mathbb{P}_n)$$

and $\sum_{k=l}^{m} \|B^*(\xi)(b_k)\|_{\mathbb{U}}^2 \to 0$ as $l, m \to \infty$ pointwise in $(0,T) \times \mathcal{U}$, Lebesgue's theorem yields

$$\| \sum_{k=l}^{m} \|B^*(\xi)(b_k)\|_{\mathbb{U}}^2 \|_{L^1((0,T)\times\mathcal{U};\, \mathrm{d}t\otimes\mathbb{P}_n)} \to 0$$

as $l, m \to \infty$. Thus, $(M_n^m)_{m\in\mathbb{N}}$ is a fundamental sequence in $L^2(\mathcal{U}, \mathbb{P}_n; C([0,T];\mathbb{H}))$, and since $M_n^m \to M_n$ pointwise in $(0,T) \times \mathcal{U}$, we also have this convergence in $L^2(\mathcal{U}, \mathbb{P}_n; C([0,T];\mathbb{H}))$. Consequently, M_n is a continuous, $\mathbb{H}$-valued $(\mathcal{G}_t)_{t\geqslant 0}$-martingale with respect to $\mathbb{P}_n$. Furthermore, $\int_0^t B(u(s))B^*(u(s))\,\mathrm{d}s$ is a bounded and symmetric operator and

$$\begin{aligned}
\left\langle \left(\int_0^t B(u(s))B^*(u(s))\,\mathrm{d}s \right) b_m, b_m \right\rangle &= \int_0^t \|B^*(u(s))b_m\|_{\mathbb{U}}^2\,\mathrm{d}s \\
&\overset{Def.\,5.3}{=} \langle M_{b_m} \rangle (t,u) \\
&= \left\langle \left(\langle\!\langle M_n \rangle\!\rangle (t,u) \right) b_m, b_m \right\rangle.
\end{aligned}$$

Since these properties characterise the covariance operator uniquely, we have proved (7.10).

Step 3. Definition 5.3, (5), and the preceding step imply

$$\langle \xi(t), b_m \rangle = \langle x_n + \int_0^t A(\xi(s))\,\mathrm{d}s + M_n(t,\xi), b_m \rangle \qquad \mathbb{P}_n - a.e.$$

for all $m \in \mathbb{N}$, and consequently we have

$$\xi(t) = x_n + \int_0^t A(\xi(s))\,\mathrm{d}s + M_n(t,\xi) \qquad \mathbb{P}_n - a.e.$$

Exactly as (6.31) in the proof of Proposition 6.8, we deduce that

$$\mathbb{E}^{\mathbb{Q}_n}\left[\sup_{0\leqslant s<t\leqslant T} \frac{\|\int_s^t A_n(\xi(r))\,\mathrm{d}r\|_{\mathbb{X}'}}{|t-s|^{\frac{\gamma-1}{\gamma}}} \right] \leqslant c$$

and exactly as (6.32) in the proof of Proposition 6.8, we deduce that

$$\mathbb{E}^{\mathbb{Q}_n}[\sup_{0\leqslant s<t\leqslant T} \frac{\|M_n(t,\xi) - M_n(s,\xi)\|_{\mathbb{X}'}}{|t-s|^{\frac{1}{2}}}] \leqslant const(T,\beta)$$

Indeed, the only difference between the current situation and Proposition 6.8 is that now the dimension of $\mathbb{H}$ is allowed to be infinite, but this does not effect the proof. Exactly as in Proposition 6.8, we deduce that $(\mathbb{P}_n)_{n\in\mathbb{N}}$ is tight. □

Theorem 7.11. *Let A and B satisfy **(A1)-(A3)**. Then there exists an a.s. martingale solution to the Markov problem associated with (5.7).*

Proof. We show that the family $(\mathscr{S}(x))_{x\in\mathbb{H}}$ is a regular a.s. pre-Markov family. Then Theorem 7.9 yields the existence of a Markov selection, and it follows directly by definition that any Markov selection of $(\mathscr{S}(x))_{x\in\mathbb{H}}$ is an a.s. martingale Markov solution.

Step 1. Lemma 6.5 implies $\mathscr{S}(x) \subseteq \mathscr{P}_{\mathbb{H}}(\mathcal{U})$. The compactness of $\mathscr{S}(x)$ in $\mathscr{P}(\mathcal{U})$ follows immediately from Proposition 7.10 by letting $x_n = x$ for all $n \in \mathbb{N}$. Thus, we have $\mathscr{S}(x) \in \mathcal{C}(\mathscr{P}_{\mathbb{H}}(\mathcal{U}))$ for all $x \in \mathbb{H}$. We show that the mapping $x \to \mathscr{S}(x)$ is $\mathcal{B}(\mathbb{Y})/\mathcal{B}(\mathcal{C}(\mathscr{P}_{\mathbb{H}}(\mathcal{U}))$ measurable. This holds, provided we can show that

$$\mathscr{S}^{-1}(\{K \in \mathcal{C}(\mathscr{P}_{\mathbb{H}}(\mathcal{U})) \,|\, K \subseteq U\}) = \{x \in \mathbb{H} \,|\, \mathscr{S}(x) \subseteq U\}$$

is open in $\mathbb{H}$ for any open $U \subseteq \mathscr{P}_{\mathbb{H}}(\mathcal{U})$, or equivalently that $\{x \in \mathbb{H} \,|\, \mathscr{S}(x) \cap U^c \neq \varnothing\}$ is closed. Thus, assume $x_n \in \mathbb{H}$ such that $\mathscr{S}(x_n) \cap U^c \neq \varnothing$ for $n \in \mathbb{N}$, and let $x_n \to x$ in $\mathbb{H}$. Choose $\mathbb{P}_n \in \mathscr{S}(x_n) \cap U^c$. Then, at least for some subsequence, we have $\mathbb{P}_n \rightharpoonup \mathbb{P}$ in $\mathscr{P}_{\mathbb{H}}(\mathcal{U})$ for some $\mathbb{P} \in \mathscr{S}(x)$ by Proposition 7.10, and $\mathbb{P} \in U^c$, since U^c is closed. Consequently, $\mathscr{S}(x) \cap U^c$ is non-empty, and this yields the desired conclusion. Thus, (1) of Definition 7.4 is satisfied. Definition 7.4, (2), follows from Definition 5.3, (2), since any $\mathbb{P} \in \mathscr{S}(x)$ is a martingale solution with initial value x.

For the remaining steps (i.e. the disintegration and the reconstruction property) let $x \in \mathbb{H}$ and $\mathbb{Q} \in \mathscr{S}(x)$. We set $T_{x,\mathbb{Q}} := T_{\mathbb{Q}}$, where $T_{\mathbb{Q}}$ denotes the exceptional null set in Definition 6.1 of $\mathscr{S}(x)$. Fix $t \in (0,\infty)\backslash T_{x,\mathbb{Q}}$.

Step 2. We show that the family $(\mathscr{S}(x))_{x\in\mathbb{H}}$ satisfies the disintegration property. We show that there is a $\mathbb{Q}$-null set $N \in \mathcal{G}_t$ such that $\mathbb{Q}(\Psi_t^{-1}(\cdot)|\mathcal{G}_t)(u) \in \mathscr{S}(u(t))$ for all $u \in \mathcal{U}\backslash N$, i.e. we need to show (1), (2) and (5) from Definition 5.3 and (6.3), since (3) and (4) from Definition 5.3 are immediate consequences from (6.3).

Definition 5.3, (1) holds, since we have

$$\int_{\mathcal{U}} \mathbb{Q}(\Psi_t^{-1}(\{v \,|\, v(s) \in \mathbb{H} \text{ for all } s \geqslant 0\})|\mathcal{G}_t)(u)\mathbb{Q}(\mathrm{d}u) = \mathbb{Q}(\{v \,|\, v(s) \in \mathbb{H} \text{ for all } s \geqslant 0\}) = 1$$

and consequently, there is a $\mathbb{Q}$-null set $N_1 \in \mathcal{G}_t$ such that

$$\mathbb{Q}(\Psi_t^{-1}(\{v \,|\, v(s) \in \mathbb{H} \text{ for all } s \geqslant 0\})|\mathcal{G}_t)(u) = 1,$$

i.e. ξ is $\mathbb{Q}(\Psi_t^{-1}(\cdot)|\mathcal{G}_t)(u)$-concentrated on the paths with values in $\mathbb{H}$, for all $u \notin N_1$.

Definition 5.3, (2), follows, since by Lemma 3.26 there is a $\mathbb{Q}$-null set $N_2 \in \mathcal{G}_t$, such that

$$\mathbb{Q}(\Psi_t^{-1}(\{v \in \mathcal{U} \,|\, v(0) = u(t)\})|\mathcal{G}_t)(u) = \mathbb{Q}(\xi(t) = u(t)|\mathcal{G}_t)(u) = 1$$

for any $u \notin N_2$.

Definition 5.3, (5), at least for all $u \in \mathcal{U}\backslash N_3$, where $N_3 \in \mathcal{G}_t$ is some $\mathbb{Q}$-null set, follows immediately from Theorem 4.25.

Finally, we show (6.3). We apply Theorem 4.24 to

$$X_{a,b} := \sup_{r\in[a,b]} \|\xi(r)\|_{\mathbb{H}}^p + \int_a^b \mathcal{Z}_p(\xi(r))\,\mathrm{d}r$$

and

$$Y_{a,b} := k_p(b-a) \cdot (\|\xi(a)\|_{\mathbb{H}}^p + 1),$$

for $(a,b) \in D := \{(\alpha,\beta) \in [0,\infty) \mid \alpha \leqslant \beta\}$. It is easy to check that the conditions for X and Y are satisfied with respect to $\mathbb{Q}$, and consequently there is a $\mathbb{Q}$-null set $N_4 \in \mathcal{G}_t$ such that (6.3) holds for $u \notin N_4$.

Finally, $N := N_1 \cup N_2 \cup N_3 \cup N_4 \in \mathcal{G}_t$ is the desired null set.

Step 3. We show that the family $(\mathscr{S}(x))_{x \in \mathbb{H}}$ satisfies the reconstruction property. Let $Q : \mathcal{U} \to \mathscr{P}(\mathcal{U})$ be a mapping satisfying (4a)-(4c) from Definition 7.4. We need to show that $\mathbb{Q} \otimes_t Q \in \mathscr{S}(x)$.

Definition 5.3, (1) holds, since

$$\mathbb{Q} \otimes_t Q(\{v \mid v(s) \in \mathbb{H} \text{ for all } s \geqslant 0\}) \overset{(7.2)}{=} \int_{\mathcal{U}} Q(u)(\{v \mid v(s) \in \mathbb{H} \text{ for all } s \geqslant 0\}) \mathbb{Q} \otimes_t Q(\mathrm{d}u) = 1.$$

Definition 5.3, (2) holds, since $\mathbb{Q} \otimes_t Q|_{\mathcal{G}_t} = \mathbb{Q}|_{\mathcal{G}_t}$.

We show Definition 5.3, (5). Since $\mathbb{Q}|_{\mathcal{G}_t} = \mathbb{Q} \otimes_t Q|_{\mathcal{G}_t}$, (5) holds up to time t. By (a) from the reconstruction property we know that $Q(u) \circ \Psi_t^{-1}$ satisfies (5). Since by Definition of $\mathbb{Q} \otimes_t Q$ we have $\mathbb{Q} \otimes_t Q(\cdot|\mathcal{G}_t) = Q(u)$, we know that (5) holds for $\mathbb{Q} \otimes_t Q(\cdot|\mathcal{G}_t) \circ \Psi_t^{-1}$. By Theorem 4.25, (2) $\Rightarrow$ (1) we know that (5) holds for $\mathbb{Q} \otimes_t Q$ at least on (t,∞). We conclude that (5) holds on $(0,\infty)$.

Finally, we show (6.3). For $s \leqslant t$, (6.3) holds since $\mathbb{Q}|_{\mathcal{G}_t} = \mathbb{Q} \otimes_t Q|_{\mathcal{G}_t}$. By (a) from the reconstruction property we know that $Q(u) \circ \Psi_t^{-1}$ satisfies (6.3). By Theorem 4.24, (2) $\Rightarrow$ (1) we know that (6.3) holds for $\mathbb{Q} \otimes_t Q(\cdot)$ if $s \geqslant t$. $\square$

Part III

Compressible Equations

8 Introduction

8.1 Deterministic equations

We consider the deterministic compressible Navier-Stokes equations driven by a bounded external force $f \in L^\infty((0,T) \times \Omega; \mathbb{R}^3)$:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0, \tag{8.1}$$

$$\frac{\partial(\rho u)}{\partial t} + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - \operatorname{div} \mathscr{S} = \rho f, \tag{8.2}$$

together with the no-slip boundary condition

$$u|_{\partial\Omega} = 0 \tag{8.3}$$

and initial conditions for the density and the momentum

$$\rho(0, .) = \rho_0, \tag{8.4}$$

$$(\rho u)(0, .) = q_0. \tag{8.5}$$

The symbol $\mathscr{S}$ denotes the viscous stress tensor

$$\mathscr{S} = \nu(\nabla u + \nabla u^T - \frac{2}{3} \operatorname{div} u\mathbb{I}) + \eta \operatorname{div} u\mathbb{I}, \quad \nu > 0,\ \eta \geqslant 0, \tag{8.6}$$

$\rho : [0,T] \times \Omega \to \mathbb{R}$ the density, $u : [0,T] \times \Omega \to \mathbb{R}^3$ the velocity and $p = p(\rho)$ the pressure. We always assume for the pressure p the following conditions:

$$p \in C[0,\infty) \cap C^2(0,\infty),\ p(0) = 0,\ p'(\rho) > 0 \text{ for } \rho > 0, \text{ and } \lim_{\rho\to\infty} \frac{p'(\rho)}{\rho^{\gamma-1}} = \rho_\infty \tag{8.7}$$

for some $\gamma > \frac{3}{2}$ and $\rho_\infty > 0$.

As in the incompressible case, existence or uniqueness of smooth solutions is not known in general. Instead, one can show the existence of so-called finite-energy weak solutions. The term „finite-energy" refers to the fact that the solution satisfies an energy inequality: the change of total energy

$$E(t) := \int_\Omega \frac{1}{2}\rho|u|^2 + Q(\rho)\,\mathrm{d}x, \tag{8.8}$$

where $\frac{1}{2}\rho|u|^2$ is the kinetic energy density and

$$Q(\rho) := \rho \int_1^\rho \frac{p(z)}{z^2}\,\mathrm{d}z$$

is the internal energy density for isentropic fluid flows, is bounded from above by the power resulting from external force and dissipation

$$\int_\Omega pf \cdot u - \mathscr{S}(\nabla u) : \nabla u \, \mathrm{d}x,$$

see [10], Section 1, or [6] for physical details. This is a-priori not clear for weak solutions; as shown below, smooth solutions always satisfy an energy equality. Next, we give the precise definition of those solutions, the motivation for this definition can be found below.

Definition 8.1. We call a pair (u, ρ) *finite-energy weak solution* of (8.1)-(8.5), iff

(1) $\rho \in L^\infty(0, T; L^\gamma(\Omega))$, $\rho \geqslant 0$;

(2) $u \in L^2(0, T; W_0^{1,2}(\Omega, \mathbb{R}^3))$;

(3) The equation of continuity (8.1) holds in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$, where u and ρ are prolonged by zero on $\mathbb{R}^3 \backslash \Omega$;

(4) The momentum equation (8.2) holds in $\mathcal{D}'((0,T) \times \Omega)$;

(5) The renormalized equation of continuity

$$\frac{\partial(b(\rho))}{\partial t} + \operatorname{div}(b(\rho)u) + (b'(\rho)\rho - b(\rho)) \operatorname{div}(u) = 0 \tag{8.9}$$

holds in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$ for all

$$b \in C^1(\mathbb{R}) \text{ with } b'(z) = 0 \text{ for all } z \geqslant M \text{ (where the constant } M \geqslant 0 \text{ depends on } b), \tag{8.10}$$

where ρ and u are again prolonged by zero outside Ω;

(6) $\rho \in C_w([0,T], L^\gamma(\Omega))$ and $(pu) \in C_w([0,T], L^{\frac{2\gamma}{\gamma+1}}(\Omega, \mathbb{R}^3))$ and the initial conditions (8.4) and (8.5) hold in these spaces;

(7) The energy is locally integrable, i.e. $E \in L^1_{loc}([0,T))$, and the energy inequality

$$\frac{\partial}{\partial t} E(t) + \int_\Omega \mathscr{S}(\nabla u) : \nabla u \, \mathrm{d}x \leqslant \int_\Omega pf \cdot u \, \mathrm{d}x. \tag{8.11}$$

holds in $\mathcal{D}'([0,T))$.

Remark 8.2. The choice of the spaces in (6) in the preceding Definition can be motivated in the following way: Let (ρ, u) be any pair of functions satisfying all conditions in the above Definition except possibly (6). Then it can be shown (see [11]), that this already implies $\rho \in C_w([0,T], L^\gamma(\Omega))$ and $(pu) \in C_w([0,T], L^{\frac{2\gamma}{\gamma+1}}(\Omega, \mathbb{R}^3))$.

Remark 8.3. If (ρ, u) is a solution in the sense of Definition 8.1, then $\rho \in C([0,T], L^1(\Omega))$. This can be proved via regularization, see first part of Section 9.6.

Remark 8.4. If the equation of continuity holds in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$, then we have the additional boundary condition

$$(\rho u)\cdot\vec{\nu}=0 \text{ on } \partial\Omega.$$

Furthermore, a simple calculation shows that the total mass $\int_\Omega \rho \,\mathrm{d}x$ is constant in time. This does not follow from the equation of continuity in $\mathcal{D}'((0,T)\times\Omega)$, if ρ is not known to be square integrable.

Remark 8.5. If (ρ,u) is a solution in the sense of Definition 8.1, then (8.9) holds for all $b\in C^1([0,\infty))$ satisfying

$$b'(\rho)\rho \leqslant c(1+\rho^{\frac{\gamma}{2}}).$$

This follows immediately by applying Lebesgue's Theorem to the sequence $b_n(\rho)$ and $b_n'(\rho)\rho$, where b_n is any sequence satisfying (8.10), such that $|b_n|\leqslant|b|$, $|b_n'|\leqslant|b'|$, $b_n\to b$ and $b_n'\to b'$, both pointwise in $[0,\infty)$.

In the following, we motivate the preceding definition. We start by motivating the renormalized equation of continuity: Let $b:\mathbb{R}\to\mathbb{R}$ satisfy (8.10) and assume that (ρ,u) is a smooth solution of the compressible Navier-Stokes System. Then, by multiplying the equation of continuity (8.1) by $b'(\rho)$ we get

$$\begin{aligned}
0 &= \frac{\partial\rho}{\partial t}b'(\rho)+\rho\operatorname{div}(u)b'(\rho)+\nabla b(\rho)\cdot u\\
&= \frac{\partial(b(\rho))}{\partial t}+\rho\operatorname{div}(u)b'(\rho)+\operatorname{div}(b(\rho)u)-b(\rho)\operatorname{div}(u)\\
&= \frac{\partial(b(\rho))}{\partial t}+\operatorname{div}(b(\rho)u)+(b'(\rho)\rho-b(\rho))\operatorname{div}(u),
\end{aligned}$$

which is (8.9). Thus, smooth solutions satisfy the renormalized equation of continuity. In fact, the renormalized equation of continuity can be deduced from the equation of continuity as soon as $\rho\in L^\infty((0,T)\times\Omega)$, see [10], Section 10.18.

We use the same strategy to motivate the energy inequality, i.e. we show that it is satisfied by any smooth solution. Thus, assume (u,ρ) is a smooth solution of the Navier-Stokes equations and consider the momentum equation (8.2), which, by direct calculation, yields for $i=1,2,3$

$$\frac{\partial(\rho u_i)}{\partial t}+\operatorname{div}(\rho u_i u)+p_i(\rho)-\nu\triangle u_i-(\eta+\frac{1}{3}\nu)\frac{\partial\operatorname{div}u}{\partial x_i}=\rho f_i.$$

By multiplying this equation by u_i, summing up over $i=1,2,3$ and integrating over Ω we deduce

$$\int_\Omega \rho f\cdot u\,\mathrm{d}x=\int_\Omega \frac{\partial(\rho u)}{\partial t}\cdot u+\operatorname{div}(\rho u_i u)u_i+\nabla p(\rho)\cdot u-\nu u_i\triangle u_i-(\eta+\frac{1}{3}\nu)\frac{\partial\operatorname{div}u}{\partial x_i}u_i\,\mathrm{d}x,$$

where the implicit summation convention is used. Using the equation of continuity, we obtain by a direct calculation the following three formulas: first we have

$$\int_\Omega \frac{\partial(\rho u)}{\partial t}\cdot u+\operatorname{div}(\rho u_i u)u_i-\nu u_i\triangle u_i\,\mathrm{d}x=\int_\Omega \frac{1}{2}\frac{\partial(\rho|u|^2)}{\partial t}-\frac{1}{2}\frac{\partial\rho}{\partial t}|u|^2-\frac{1}{2}\operatorname{div}(\rho u)|u|^2-\nu|\nabla u|^2\,\mathrm{d}x,$$

secondly, a little bit lengthy but straight forward calculation shows

$$\int_\Omega \nabla p(\rho)\cdot u \,\mathrm{d}x = \int_\Omega \frac{\partial Q(\rho)}{\partial t} + \operatorname{div}(u\rho Q'(\rho))\,\mathrm{d}x = \int_\Omega \frac{\partial Q(\rho)}{\partial t}\,\mathrm{d}x,$$

and finally we have

$$\int_\Omega \frac{\partial \operatorname{div} u}{\partial x_i} u_i \,\mathrm{d}x = \int_\Omega (\operatorname{div}(u \operatorname{div} u) + \operatorname{div}(u)^2)\,\mathrm{d}x = \int_\Omega \operatorname{div}(u)^2\,\mathrm{d}x.$$

Thus, we get

$$\begin{aligned}\int_\Omega \rho f\cdot u\,\mathrm{d}x =& \int_\Omega \frac{1}{2}\frac{\partial(\rho|u|^2)}{\partial t} + \frac{\partial Q(\rho)}{\partial t}\,\mathrm{d}x - \int_\Omega \frac{1}{2}\Big(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u)\Big)|u|^2 + \nu|\nabla u|^2 + (\eta + \frac{1}{3}\nu)\operatorname{div}(u)^2\,\mathrm{d}x\\ =& \frac{\partial}{\partial t}E(t) + \int_\Omega \nu|\nabla u|^2 + (\eta+\frac{1}{3}\nu)\operatorname{div}(u)^2\,\mathrm{d}x.\end{aligned}$$

Again by direct calculation, we deduce

$$\frac{\partial}{\partial t}E(t) + \int_\Omega \mathscr{S}(\nabla u) : \nabla u \,\mathrm{d}x = \int_\Omega \rho f\cdot u\,\mathrm{d}x$$

for any smooth solution. Consequently, smooth solutions even satisfy the energy equality.

The next Lemma about Q will be helpful later.

Lemma 8.6. *Suppose that the pressure p satisfies (8.7), then the function $\rho \mapsto Q(\rho)$ is a continuously differentiable function mapping $[0,\infty)$ onto $[0,\infty)$ and satisfying $Q(0) = 0$ and $Q'(\rho) > 0$. Furthermore, there exists constants c_1, c_2, d_1, $d_2 \geqslant 0$ such that*

$$c_1 p(\rho) - d_1 \leqslant Q(\rho) \leqslant c_2 p(\rho) + d_2$$

Proof. Q is obviously continuously differentiable and $p(z) > 0$ for $z > 0$ yields $Q'(z) > 0$. We get

$$\lim_{\rho\to 0} Q(\rho) = \lim_{\rho\to 0} \frac{\int_1^\rho \frac{p(z)}{z^2}\,\mathrm{d}z}{\frac{1}{\rho}} = -p(0) = 0.$$

To see the last statement let $z_0 > 1$ be such that for all $\rho \geqslant z_0$ we have $p'(\rho) \in (\frac{1}{2}p_\infty\rho^{\gamma-1}, \frac{3}{2}p_\infty\rho^{\gamma-1})$. Now by continuity we have $Q(\rho) \leqslant K$ for all $\rho \leqslant z_0$ for some $K > 0$. For $\rho > z_0$ we get

$$\begin{aligned}Q(\rho) &\leqslant \rho\int_1^{z_0} \frac{p'(z)}{z}\,\mathrm{d}z + \rho\int_{z_0}^\rho \frac{3}{2}p_\infty z^{\gamma-2}\,\mathrm{d}z\\ &= \rho\int_1^{z_0}\frac{p'(z)}{z}\,\mathrm{d}z + \frac{3}{2}\frac{p_\infty}{\gamma-1}(\rho^\gamma - \rho z_0^{\gamma-1}).\\ &\leqslant \tilde{d}_2 + \rho^\gamma \tilde{c}_2\\ &\leqslant d_2 + p(\rho)c_2.\end{aligned}$$

A similar calculation shows the other inequality.

□

We have the following existence result of weak solutions to the deterministic compressible Navier-Stokes equations:

Theorem 8.7. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain and let $f \in L^\infty((0,T) \times \Omega)$. Assume that the pressure p satisfies (8.7) for some $\gamma > \frac{3}{2}$ and assume that the initial condition satisfy:*

$$\rho_0 \in L^\gamma(\Omega),\ \rho_0 \geqslant 0,\ q \in L^1(\Omega, \mathbb{R}^3),\ \frac{|q|^2}{\rho_0} \in L^1(\Omega). \tag{8.12}$$

Then, for any $T > 0$, there exists a finite-energy weak solution (ρ, u) such that $\rho \in C([0,T], L^1(\Omega))$ and the total mass is constant in time, i.e.

$$\int_\Omega \rho(t)\,\mathrm{d}x = M. \tag{8.13}$$

The original proof of this theorem in [11] requires some smoothness of the boundary $\partial\Omega$, which were relaxed to Lipschitz domains in [22].

8.2 Stochastic equations

In the whole part let $(\mathcal{O}, \mathcal{B}, \mathbb{P})$ be some topological probability space, i.e. $\mathcal{O}$ is a topological space, $\mathcal{B}$ is the Borel algebra on $\mathcal{O}$ and $\mathbb{P}$ is a regular probability measure. The compressible Navier-Stokes equations driven by a stochastic noise w can formally be written as

$$\mathrm{d}\rho + \operatorname{div}(\rho u)\,\mathrm{d}t = 0 \tag{8.14}$$

$$\mathrm{d}(\rho u) + \big(\operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - \operatorname{div} \mathscr{S}\big)\,\mathrm{d}t = \rho\,\mathrm{d}w \tag{8.15}$$

where the unknown random variables $\rho : [0,T] \times \Omega \times \mathcal{O} \to \mathbb{R}$ and $u : [0,T] \times \Omega \times \mathcal{O} \to \mathbb{R}^3$ are the density and the velocity, the given random variable $w : [0,T] \times \Omega \times \mathcal{O} \to \mathbb{R}^3$ is the stochastic noise and $\mathscr{S}$ denotes the viscous stress defined in (8.6). We always assume that the pressure $p = p(\rho)$ satisfies (8.7) for a certain $\gamma > \frac{3}{2}$. We also assume the initial and boundary conditions (8.3)-(8.5), where the initial conditions (8.4) and (8.5) are random variables.

The compressible case is more complicated then the incompressible case and the mathematical theory of these equations is far from being complete. In particular, the only successful approach so far to get solutions to the compressible stochastic Navier-Stokes equations was done in [9]. The basic idea is to consider the equations path wise, i.e. we fix any $\omega \in \mathcal{O}$, and show the existence of a weak solution. Then, we obtain for a.e. ω a non-empty set of solutions. In the second step we show that there exists a measurable selection and thus a random variable (ρ, u) satisfying that Navier-Stokes system for *a.e.* $\omega \in \mathcal{O}$.

The major problem is that in most relevant situations the paths of the noise process are not differentiable, or even not continuous, with respect to the time variable (important examples are Lévy processes, see section

10.2). In those situations, Theorem 8.7 can not be applied directly, because the resulting force $f = \partial_t w$ is not a function any more. In order to solve this problem we start with a slightly different definition of a weak solution then Definition 8.1, which, as shown below, is in fact equivalent to Definition 8.1 in the case when w is bounded and differentiable with respect to t.

Definition 8.8. We call a pair (ρ, u) a *finite-energy weak solution* of (8.14), (8.15), iff

(1) $\rho(\cdot,\cdot,\omega) \in C([0,T], L^1(\Omega)) \cap C_w([0,T], L^\gamma(\Omega))$ and $\rho \geqslant 0$ for $a.e.\ \omega \in \mathcal{O}$;

(2) $u(\cdot,\cdot,\omega) \in L^2(0,T; W_0^{1,2}(\Omega,\mathbb{R}^3))$ for $a.e.\ \omega \in \mathcal{O}$;

(3) $\rho u(\cdot,\cdot,\omega) \in L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega,\mathbb{R}^3))$ and $\rho(u-w)(\cdot,\cdot,\omega) \in C_w([0,T], L^{\frac{2\gamma}{\gamma+1}}(\Omega,\mathbb{R}^3))$ for $a.e.\ \omega \in \mathcal{O}$;

(4) the following weak formulation of the renormalized equation of continuity

$$\begin{aligned} &\int_0^T \int_\Omega \big(\rho + b(\rho)\big) \frac{\partial \varphi}{\partial t} + \big(\rho + b(\rho)\big) u \cdot \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_\Omega \big(b'(\rho)\rho - b(\rho)\big) \operatorname{div} u\ \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \big(\rho_0 + b(\rho_0)\big)\varphi(0,\cdot)\,\mathrm{d}x \end{aligned} \tag{8.16}$$

holds for all b satisfying (8.10), $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$ and $a.e.\ \omega \in \mathcal{O}$;

(5) the following weak formulation of the momentum equation

$$\begin{aligned} &\int_0^T \int_\Omega \rho(u-w) \cdot \frac{\partial \varphi}{\partial t} + \rho u \otimes u : \nabla \varphi + p(\rho) \operatorname{div} \varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_\Omega \mathscr{S}(\nabla u) : \nabla \varphi + \rho u \cdot \nabla (w \cdot \varphi)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega q \cdot \varphi(0,\cdot)\,\mathrm{d}x \end{aligned} \tag{8.17}$$

holds for all $\varphi \in C_c^\infty([0,T) \times \Omega, \mathbb{R}^3)$ and $a.e.\ \omega \in \mathcal{O}$;

(6) the following weak formulation of the energy inequality

$$\begin{aligned} &- \int_0^T \frac{\partial \psi}{\partial t} \int_\Omega \overline{E}(t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \psi \int_\Omega \mathscr{S}(\nabla u) : \nabla u \,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \psi(0) \int_\Omega \frac{1}{2}\frac{|q|^2}{\rho_0} + Q(\rho_0)\,\mathrm{d}x \\ &+ \int_0^T \psi \int_\Omega \mathscr{S}(\nabla u) : \nabla w - \rho u \otimes u : \nabla w - p(\rho)\operatorname{div} w + \frac{1}{2}\rho u \nabla |w|^2 \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{8.18}$$

where

$$\overline{E}(t) := \int_\Omega \frac{1}{2}\rho|u-w|^2(t) + Q(\rho(t))\,\mathrm{d}x$$

holds for all $\psi \in C_c^\infty([0,T))$, $\psi \geqslant 0$ and $a.e.\ \omega \in \mathcal{O}$.

Remark 8.9. The statement of Remark 8.5 still holds for Definition 8.8.

Remark 8.10. Definition 8.8 for finite-energy weak solutions to the stochastic Navier-Stokes equations can also be seen as a modified definition for finite-energy weak solutions to the deterministic Navier-Stokes equations (8.1)-(8.5), by identifying the noise w, the initial conditions (ρ_0, q) and the solution (ρ, u) with the respective constant random variables on $\mathcal{O}$. From now on, we will do so without further comment.

The motivation for Definition 8.8 is contained in the following Theorem:

Theorem 8.11. *Let $f \in L^\infty(0, T, W^{1,\infty}(\Omega))$ and define*

$$w(t, x) := \int_0^t f(s, x)\,\mathrm{d}s,$$

i.e. $f = \partial_t w$ and $w(0, \cdot) = 0$. Then, (ρ, u) is a solution in the sense of Definition 8.1 if and only if it is a solution in the sense of Definition 8.8.

Proof. Let (ρ, u) be a solution in the sense of Definition 8.1. Points (1) and (2) of Definition 8.8 follow immediately. The assertion about $\rho u(\cdot, \cdot, \omega)$ in (3) of Definition 8.8 follows by (6) from Definition 8.1. To see the condition about $\rho(u - w)$ in (3) of Definition 8.8, by (6) from Definition 8.1 we have left to show that

$$\rho w \in C_w([0, T], L^{\frac{2\gamma}{\gamma+1}}(\Omega, \mathbb{R}^3)).$$

By Sobolev embedding we have $w \in C([0, T] \times \Omega, \mathbb{R}^3)$, and since $\rho \in C([0, T], L^1(\Omega))$, the assertion follows.

We show (4) from Definition 8.8. Points (3) and (5) of Definition 8.1 are equivalent to

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} \big(\rho + b(\rho)\big)\frac{\partial \varphi}{\partial t} + \big(\rho + b(\rho)\big)u \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}^3} \big(b'(\rho)\rho - b(\rho)\big)\operatorname{div} u\ \varphi \,\mathrm{d}x\,\mathrm{d}y - \int_{\mathbb{R}^3} \big(\rho_0 + b(\rho_0)\big)\varphi(0, \cdot)\,\mathrm{d}x \end{aligned}$$

for all $\varphi \in C_c^\infty([0, T) \times \mathbb{R}^3)$. Since u and ρ are prolonged by zero on $\mathbb{R}^3 \backslash \Omega$ and since $\{\varphi|_{[0,T)\times\Omega} : \varphi \in C_c^\infty([0, T) \times \mathbb{R}^3)\} = C_c^\infty([0, T) \times \overline{\Omega})$, this is equivalent to (8.16).

We show the momentum equation. Point (4) in Definition 8.1 is equivalent to:

$$\begin{aligned} &\int_0^T \int_\Omega (\rho u \cdot \frac{\partial \varphi}{\partial t} + \rho u \otimes u : \nabla\varphi + p(\rho)\operatorname{div}\varphi)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_\Omega \mathscr{S}(\nabla u) : \nabla\varphi - \rho\frac{\partial w}{\partial t} \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega q_0 \cdot \varphi(0, \cdot)\,\mathrm{d}x \end{aligned} \tag{8.19}$$

for all $\varphi \in C_c^\infty([0, T) \times \Omega; \mathbb{R}^3)$. Using the identity

$$\begin{aligned} \int_0^T \int_\Omega \rho\frac{\partial w}{\partial t} \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_\Omega \rho\frac{\partial(w \cdot \varphi)}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega \rho w \cdot \frac{\partial \varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T \int_\Omega \rho u \cdot \nabla(w \cdot \varphi)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega \rho w \cdot \frac{\partial \varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{8.20}$$

for all $\varphi \in C_c^\infty([0, T) \times \Omega; \mathbb{R}^3)$, we deduce (8.17).

Finally, we show the energy inequality. The energy inequality in (7) in Definition 8.1 is equivalent to

$$
\begin{aligned}
-\int_0^T \int_\Omega \frac{1}{2}\rho|u|^2 + Q(\rho)\,\mathrm{d}x \frac{\partial\psi}{\partial t}\,\mathrm{d}t + \int_0^T \int_\Omega \mathscr{S}(\nabla u) : \nabla u \,\mathrm{d}x\psi\,\mathrm{d}t& \\
\leqslant \psi(0)\int_\Omega \frac{1}{2}\frac{|q|^2}{\rho_0} + Q(\rho_0)\,\mathrm{d}x + \int_0^T \int_\Omega \rho\frac{\partial w}{\partial t}\cdot u\,\mathrm{d}x\psi\,\mathrm{d}t&
\end{aligned}
$$

for all $\psi \in C_c^\infty([0,T))$, $\psi \geqslant 0$. By approximation in the space $W^{1,2}((0,T)\times\Omega)$ we deduce that (8.19) and (8.20) hold with $\varphi = w\psi$ as test function for any $\psi \in C_c^\infty([0,T))$. Consequently,

$$
\begin{aligned}
\int_0^T \int_\Omega \left(\rho u \cdot \frac{\partial w}{\partial t} + \rho u \otimes u : \nabla w + p(\rho)\operatorname{div} w - \mathscr{S}(\nabla u) : \nabla w\right)\mathrm{d}x\psi\,\mathrm{d}t& \\
= -\int_0^T \int_\Omega \rho u\cdot w\,\mathrm{d}x\frac{\partial\psi}{\partial t}\,\mathrm{d}t - \int_0^T \int_\Omega \rho\frac{\partial w}{\partial t}\cdot w\psi\,\mathrm{d}x\,\mathrm{d}t& \\
= -\int_0^T \int_\Omega \rho u\cdot w\,\mathrm{d}x\frac{\partial\psi}{\partial t}\,\mathrm{d}t - \int_0^T \int_\Omega \frac{1}{2}\rho u\cdot\nabla(w\cdot\varphi)\,\mathrm{d}x\,\mathrm{d}t&
\end{aligned}
$$

and these relations combined yield (8.18).

The other direction follows by inverting the arguments. □

9 Equations driven by irregular force

9.1 Introduction and convergence results

In this section we show the existence of solutions to the deterministic compressible Navier-Stokes equations driven by noises w with low regularity with respect to the time variable. The main result of this section is the following Theorem:

Theorem 9.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Suppose that the pressure p satisfies (8.7) and the initial values satisfy (8.12). Suppose further that $w \in L^\infty(0,T;W_0^{1,\infty}(\Omega,\mathbb{R}^3))$. Then the Navier-Stokes system admits a weak solution in the sense of Definition 8.8.*

To prove this Theorem, we approximate the irregular noise w by smooth functions w_n in such a way, that at least a subsequence of the corresponding sequence of weak solutions converges to a weak solution of the problem driven by w. Now, as shown below, the energy inequality yields uniform boundedness on the sequence of solutions, provided the sequence w_n is bounded in $L^\infty(0,T;W_0^{1,\infty}(\Omega,\mathbb{R}^3))$; consequently, we will restrict our attention to functions w in this space. From now on, to the end of the proof of Theorem 9.1, fix $w \in L^\infty(0,T;W_0^{1,\infty}(\Omega,\mathbb{R}^3))$ and a sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq C^\infty([0,T]\times\Omega,\mathbb{R}^3)$ satisfying

$$
\|w_n\|_{L^\infty(0,T;W_0^{1,\infty}(\Omega))} \leqslant R, \tag{9.1}
$$

$$
w_n \to w \ \text{ in } L^1(0,T;W^{1,1}(\Omega,\mathbb{R})). \tag{9.2}
$$

for some constant $R > 0$. We can choose a subsequence such that $w_n \to w$ and $\nabla w_n \to \nabla w$ a.e. in $(0,T)\times\Omega$, and by the Lebesgue's theorem, we get

$$w_n \to w \ \text{ in } L^q(0,T;W^{1,s}(\Omega,\mathbb{R}))$$

for all $1 \leqslant q,\ s, < \infty$. Next, let

$$\beta := \begin{cases} \gamma & \text{if } \gamma \geqslant 3, \\ \frac{3\gamma}{2\gamma-3} & \text{if } \gamma < 3. \end{cases}$$

Define a modified pressure by

$$p_n(z) := p(z) + \delta_n z^\beta,$$

where $\delta_n := \frac{1}{n}$, and let Q_n be the corresponding function in the energy inequality with p replaced by p_n. Then, p_n satisfies the assumptions (8.7) with γ replaced by β. We note for future use that we have $\beta \geqslant \gamma$ and

$$\frac{6\gamma}{7\gamma-6} \leqslant \frac{2\beta}{\beta+2}.$$

Let c_1, c_2, d_1, d_2 be the constants from Lemma 8.6 for p. A simple calculation shows

$$\begin{aligned} Q_n(\rho) &= \rho \int_1^\rho \frac{p(z)}{z^2} + \delta_n z^{\beta-2}\,\mathrm{d}z \\ &= Q(p) + \rho\delta_n \frac{1}{\beta-1}(\rho^{\beta-1}-1) \\ &\leqslant \max\{c_2,\ \frac{1}{\beta-1}\}(p(\rho)+\delta_n\rho^\beta) + d_2 \\ &= \tilde{c_2} p_n(\rho) + d_2, \end{aligned}$$

where the constants $\tilde{c_2}$ and d_2 do not depend on n. A similar calculation shows the other inequality. Thus, replacing γ by β, the functions p_n and Q_n satisfy Lemma 8.6 with constants that do not depend on n.

Moreover, fix initial conditions ρ_0 and q satisfying 8.12 and let (u_n, ρ_n) be any sequence of weak solutions of the compressible Navier-Stokes system with the pressure p_n, driven by the noise w_n, in the sense of Definition 8.8, whose existence is guaranteed by Theorem 8.7 and Theorem 8.11. We begin by deriving uniform bounds on the sequence of solutions and consequently existence of a weakly convergent subsequence.

Lemma 9.2. *We have, at least for a suitable subsequence,*

$$\rho_n \to \rho \qquad \text{in } C_w([0,T];L^\gamma(\Omega)), \tag{9.3}$$

$$u_n \rightharpoonup u \qquad \text{in } L^2(0,T;W_0^{1,2}(\Omega,\mathbb{R}^3)). \tag{9.4}$$

Furthermore, for all $b \in C_b^1(\mathbb{R})$ *satisfying (8.10), there is a function* $\overline{b(\rho)} \in C_w([0,T];L^p(\Omega))$ *for all* $p < \infty$*,* *such that*

$$b(\rho_n) \rightharpoonup \overline{b(\rho)} \qquad \text{in } C_w([0,T];L^p(\Omega)) \text{ for all } 1 \leqslant p < \infty., \tag{9.5}$$

$$b(\rho_n)u_n \rightharpoonup \overline{b(\rho)}u \qquad \text{in } L^2((0,T)\times\Omega), \tag{9.6}$$

passing again to a subsequence if necessary. As a consequence, the following convergences hold for a suitable subsequence

$$\rho_n u_n \to \rho u \qquad \text{in } L^q(0,T;W^{-1,2}(\Omega,\mathbb{R}^3)), \tag{9.7}$$

$$\rho_n u_n \rightharpoonup \rho u \qquad \text{in } L^2(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega,\mathbb{R}^3)), \tag{9.8}$$

$$\rho_n w_n \to \rho w \qquad \text{in } L^q(0,T;W^{-1,r}(\Omega,\mathbb{R}^3)) \tag{9.9}$$

$$\rho_n(u_n - w_n) \to \rho(u-w) \qquad \text{in } C_w([0,T];L^{\frac{2\gamma}{\gamma+1}}(\Omega,\mathbb{R}^3)), \tag{9.10}$$

$$\rho_n u_n \otimes u_n \rightharpoonup \rho u \otimes u \qquad \text{in } L^2(0,T;L^{\frac{6\gamma}{4\gamma+3}}(\Omega,\mathbb{R}^{3\times 3})), \tag{9.11}$$

for any $1 \leqslant q < \infty$ *and some* $r < 2$*. In particular,* (ρ, u)*, prolonged by zero outside* Ω*, solve*

$$\begin{aligned} \frac{\partial\rho}{\partial t} + \operatorname{div}(\rho u) &= 0 \\ u|_{\partial\Omega} &= 0 \\ \rho(0) &= \rho_0 \end{aligned} \tag{9.12}$$

in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$*.*

Proof. **Step 1.** We show that the energy

$$E_n(t) := \Big(\int_\Omega \frac{1}{2}\rho_n|u_n|^2 + Q_n(\rho_n)\,\mathrm{d}x\Big)(t)$$

is uniformly bounded in n and a.e. $t \in (0,T)$. The energy inequality yields for a.e. $t \in (0,T)$

$$\begin{aligned} \overline{E_n}(t) &:= \Big(\int_\Omega \frac{1}{2}\rho_n|u_n - w_n|^2 + Q_n(\rho_n)\,\mathrm{d}x\Big)(t) \\ &\leqslant b + \int_0^t\int_\Omega \mathscr{S}(\nabla u_n) : (\nabla w_n - \nabla u_n) - \rho_n u_n \otimes u_n : \nabla w_n - p_n(\rho_n)\operatorname{div}(w_n) + \frac{1}{2}\rho_n u_n \nabla|w_n|^2\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

where $b := \sup_{n\in\mathbb{N}} \int_\Omega \frac{1}{2}\frac{|q|^2}{\rho_0} + Q_n(\rho_0)\,\mathrm{d}x = \int_\Omega \frac{1}{2}\frac{|q|^2}{\rho_0} + Q_1(\rho_0)\,\mathrm{d}x$. We can estimate the terms on the right hand side in the following way:

(1) We have

$$\int_0^t\int_\Omega \mathscr{S}(\nabla u_n) : \nabla u_n\,\mathrm{d}x\,\mathrm{d}t \geqslant \nu\int_0^t \|\nabla u_n\|^2_{L^2(\Omega)}\,\mathrm{d}x$$

and by (9.1)

$$\int_0^t \int_\Omega \mathscr{S}(\nabla u_n) : \nabla w_n \,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^t \|\mathscr{S}(\nabla u_n)\|_{L^1(\Omega)} \|\nabla w_n\|_{L^\infty(\Omega)} \,\mathrm{d}t$$
$$\leqslant const(R,\Omega) \int_0^t \|\mathscr{S}(\nabla u_n)\|_{L^2(\Omega)} \,\mathrm{d}t$$

and consequently

$$\int_0^t \int_\Omega \mathscr{S}(\nabla u_n) : (\nabla w_n - \nabla u_n) \,\mathrm{d}x\,\mathrm{d}t \leqslant const(R,\Omega).$$

(2) By (9.1) we have

$$|\int_0^t \int_\Omega \rho_n u_n \otimes u_n : \nabla w_n \,\mathrm{d}x\,\mathrm{d}t| \leqslant const(R,\Omega) \int_0^t \int_\Omega \rho_n |u_n|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

(3) By Lemma 8.6 we have

$$|\int_0^t \int_\Omega p_n(\rho_n) \operatorname{div} w_n \,\mathrm{d}x| \leqslant const(p,R,\Omega) + const(p,R,\Omega) \int_0^t \int_\Omega Q_n(\rho_n) \,\mathrm{d}x\,\mathrm{d}t.$$

(4) Since the total mass is constant in time, we have

$$\|\rho_n(t,\cdot)\|_{L^1(\Omega)} = M \text{ for all } t \in [0,T]$$

and by the estimate

$$\rho_n |u_n| \leqslant \begin{cases} \rho_n & \text{for } |u_n| \leqslant 1 \\ \rho_n |u_n|^2 & \text{for } |u_n| > 1 \end{cases}$$

we get

$$\int_0^t \int_\Omega \frac{1}{2} \rho_n u_n \nabla |w_n|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant const(R)(\int_0^t \int_\Omega \rho_n + \rho_n |u_n|^2 \,\mathrm{d}x\,\mathrm{d}t),$$
$$\leqslant const(R,M)(1 + \int_0^t \int_\Omega \rho_n |u_n|^2 \,\mathrm{d}x\,\mathrm{d}t).$$

Summing up these results, we obtain

$$\overline{E_n}(t) \leqslant b + \int_0^t c_1 + c_2 E_n(s) \,\mathrm{d}s$$

where the constants c_1 and c_2 do not depend on n or t.

Finally, we have

$$\int_\Omega \frac{1}{2}\rho_n|u_n - w_n|^2(t)\,\mathrm{d}x = \frac{1}{2}\int_\Omega \rho_n|u_n|^2(t) - 2\rho_n u_n\cdot w_n(t) + \rho_n|w_n|^2(t)\,\mathrm{d}x$$
$$\geqslant const(M,R) + const(M,R)\int_\Omega \rho_n|u_n|^2(t)\,\mathrm{d}x.$$

for all $t \in (0,T)$ and therefore

$$E_n(t) \leqslant c_3 + c_4\overline{E_n}(t) \leqslant c_5 + \int_0^t c_6 + c_7 E_n(t)\,\mathrm{d}t$$

where the constants to not depend on n or t.

Now, Grownwall's inequality can be used to obtain

$$\operatorname*{ess\,sup}_{t\in(0,T)} E_n(t) \leqslant c \tag{9.13}$$

where c is independent of n.

Step 2. We show (9.3). By (9.13) we have

$$\|\sqrt{\rho_n}u_n\|_{L^\infty(0,T;L^2(\Omega,\mathbb{R}^3))} \leqslant c_1, \tag{9.14}$$
$$\|Q_n(\rho_n)\|_{L^\infty(0,T;L^1(\Omega))} \leqslant c_2, \tag{9.15}$$

where the latter yields

$$\|\rho_n\|_{L^\infty(0,T;L^\gamma(\Omega))} \leqslant c_3, \tag{9.16}$$

where all constants are independent of n. Thus, for a suitable subsequence, we have

$$\rho_n \overset{*}{\rightharpoonup} \rho \text{ in } L^\infty(0,T;L^\gamma(\Omega)). \tag{9.17}$$

To finish the proof of (9.3) let $\varphi \in C_c^\infty(\Omega)$ and let

$$\Psi_n(t) := \int_\Omega \rho_n(t)\varphi\,\mathrm{d}x.$$

Then, Ψ_n is continuous by (1) of Definition 8.8. We use Ascoli's theorem to show that $\{\Psi_n\}_{n\in\mathbb{N}}$ is pre-compact in $C([0,T];\mathbb{R})$. We have

$$|\Psi_n(t)| \leqslant \|\rho_n\|_{L^\infty(0,T;L^\gamma(\Omega))}\|\varphi\|_{L^{\gamma'}(\Omega)} \leqslant c$$

and this shows the uniform boundedness. To see that $\{\Psi_n\}$ is equicontinuous, we use the equation of continuity (8.16) to deduce

$$
\begin{aligned}
|\int_\Omega (\rho_n(t)-\rho_n(s))\varphi \,\mathrm{d}x| = |\int_s^t \int_\Omega \rho_n u_n \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}\tau| \\
&\leqslant \int_s^t \|\sqrt{\rho_n} u_n\|_{L^2(\Omega,\mathbb{R}^3)} \|\sqrt{\rho_n}\|_{L^{2\gamma}(\Omega)} \|\nabla\varphi\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega,\mathbb{R}^3)} \,\mathrm{d}\tau \\
&\leqslant c|t-s|,
\end{aligned}
$$

where the constant c is independent of n, t and s. This shows that $\{\Psi_n\}$ is equicontinuous, and consequently pre-compact in $C([0,T];\mathbb{R})$. Furthermore, if $\Psi_{n_k} \to \Psi$ in $C([0,T])$ for some subsequence, then we have

$$
\Psi(t) = \int_\Omega \rho(t)\varphi \,\mathrm{d}x
$$

by (9.17), i.e. Ψ_n has only a single accumulation point in $C([0,T])$, and consequently

$$
\Psi_n \to \int_\Omega \rho(\cdot)\varphi \,\mathrm{d}x \text{ in } C([0,T]).
$$

Now, let $\varphi \in L^{\gamma'}(\Omega)$ and $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq C_c^\infty(\Omega)$ such that $\varphi_n \to \varphi$ in $L^{\gamma'}(\Omega)$. Then, the above yields

$$
\int_\Omega \rho_n\varphi_k \,\mathrm{d}x \to \int_\Omega \rho\varphi_k \,\mathrm{d}x \text{ in } C([0,T])
$$

for any fixed $k \in \mathbb{N}$. Moreover, we have

$$
\int_\Omega |\rho_n(t)\varphi_k - \rho_n(t)\varphi| \,\mathrm{d}x \leqslant \|\rho_n(t)\|_{L^\gamma(\Omega)} \|\varphi_k - \varphi\|_{L^{\gamma'}(\Omega)} \leqslant c\|\varphi_k-\varphi\|_{L^{\gamma'}(\Omega)}
$$

where c is independent of n and t, and therefore

$$
\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \| \int_\Omega \rho_n\varphi_k \,\mathrm{d}x - \int_\Omega \rho_n\varphi \,\mathrm{d}x\|_{C([0,T])} = 0.
$$

Now, Lemma 1.13 yields

$$
\int_\Omega \rho_n(\cdot)\varphi \,\mathrm{d}x \to \int_\Omega \rho(\cdot)\varphi \,\mathrm{d}x \text{ in } C([0,T])
$$

for all $\varphi \in L^{\gamma'}(\Omega)$, and this finishes the proof of (9.3).

Step 3. To show (9.4), we estimate the terms in the energy inequality similar as above to deduce

$$
\begin{aligned}
\|u_n\|_{L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))} &\leqslant c_1 \int_0^t \int_\Omega \mathscr{S}(\nabla u_n) : \nabla u_n \,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant c_1 \Big(\overline{E_n}(t) + b + \int_0^t \int_\Omega \mathscr{S}(\nabla u_n) : \nabla w_n - \rho_n u_n \otimes u_n : \nabla w_n \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega -p(\rho_n)\,\mathrm{div}(w_n) + \frac{1}{2}\rho_n u_n \nabla |w_n|^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}t \Big) \\
&\leqslant c_2
\end{aligned}
$$

where the constants do not depend on n. By passing to a subsequence, we have (9.4).

Step 4. We show (9.5). Let

$$X := \overline{\{b \in C_b^1(\mathbb{R}) \,:\, b \text{ satisfies } (8.10)\}}^{C_b^1(\mathbb{R})}$$

For the whole step fix $1 < p < \infty$ and $q := p'$. For $b \in X$ and $\varphi \in C_c^\infty(\Omega)$ define

$$\Psi_k[b,\varphi](t) := \Big(\int_\Omega b(\rho_k)\varphi \,\mathrm{d}x\Big)(t).$$

We use Ascoli's theorem to show that the sequence $(\Psi_k[b,\varphi])_{k\in\mathbb{N}}$ is pre-compact in $C([0,T];\mathbb{R})$. Obviously, we have $|\Psi_k[b,\varphi](t)| \leqslant c$ for some c independent of k and t. We can use the renormalized equation of continuity to deduce for $0 \leqslant s < t \leqslant T$

$$\begin{aligned}
\Big|\int_\Omega \big(b(\rho_n(t)) - b(\rho_n(s))\big)\varphi \,\mathrm{d}x\Big| &\leqslant \Big|\int_s^t \int_\Omega b(\rho_n)u_n\nabla\varphi + \big(b'(\rho_n)\rho_n - b(\rho_n)\big)\operatorname{div} u_n \;\varphi \,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\leqslant \int_s^t \|b(\rho_n)\|_{L^2}\|u_n\|_{L^2}\|\nabla\varphi\|_{L^\infty} + \|b'(\rho_n)\rho_n - b(\rho_n)\|_{L^2}\|\operatorname{div} u_n\|_{L^2}\|\varphi\|_{L^\infty}\,\mathrm{d}t \\
&\leqslant c(t-s)
\end{aligned}$$

where the constant c does not depend on t, s or n, and this shows the equicontinuity of $\Psi_k[b,\varphi]$.

Next, we choose countable and dense subsets $\mathfrak{X} \subseteq X$ and $\mathfrak{Y} \subseteq L^q(\Omega)$. Since for all $b \in \mathfrak{X}$, the sequence $b(\rho_k)$ is bounded uniformly in $L^\infty(0,T;L^q(\Omega))$, it is sequentially pre-compact in the weak star topology, and by Theorem 1.2, there exists a subsequence independent of $b \in \mathfrak{X}$, such that

$$b(\rho_{n_k}) \overset{*}{\rightharpoonup} \overline{b(\rho)} \text{ in } L^\infty(0,T;L^q(\Omega)) \tag{9.18}$$

for all $b \in \mathfrak{X}$. We pass to this subsequence.

Since $\{\Psi_k[b,\varphi]\}_{k\in\mathbb{N}}$ is sequentially pre-compact in $C([0,T])$ for any fixed $b \in \mathfrak{X}$ and $\varphi \in \mathfrak{Y}$, by Theorem 1.2 there exists a further subsequence (again independently of b and φ) such that

$$\Psi_k[b,\varphi] \to \Psi[b,\varphi] \text{ in } C([0,T]) \tag{9.19}$$

for all $(b,\varphi) \in \mathfrak{X} \times \mathfrak{Y}$ and for some functions $\Psi[b,\varphi] \in C([0,T])$. We pass to this subsequence.

Combining (9.18) and (9.19), we deduce

$$\Psi[b,\varphi](t) = \int_\Omega \overline{b(\rho)}(t)\varphi \,\mathrm{d}x$$

for all $b \in \mathfrak{X}$ and $\varphi \in \mathfrak{Y}$, i.e. we have

$$\int_\Omega b(\rho_k)\varphi \,\mathrm{d}x \to \int_\Omega \overline{b(\rho)}\varphi \,\mathrm{d}x \text{ in } C([0,T])$$

By interpolation and Lemma 1.13, we infer that the last relation holds for all $\varphi \in L^q(\Omega)$, exactly as in step 2, and consequently we have

$$b(\rho_k) \to \overline{b(\rho)} \ \text{ in } \ C_w([0,T]; L^p(\Omega)), \ \text{ as } k \to \infty$$

for all $b \in \mathfrak{X}$.

Finally, let $b \in X$ and $b_n \to b$ in $C_b^1(\mathbb{R})$ with $\{b_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{X}$. The estimate

$$\begin{aligned}
\sup_{t\in[0,T]} \|\overline{b_n(\rho(t))} - \overline{b_m(\rho(t))}\|_{L^p(\Omega)} &\leqslant \sup_{t\in[0,T]} \liminf_{k\to\infty} \|b_n(\rho_k(t)) - b_m(\rho_k(t))\|_{L^p(\Omega)} \\
&\leqslant \operatorname{meas}(\Omega)^{\frac{1}{p}} \|b_k - b_m\|_{L^\infty(\Omega)}
\end{aligned}$$

shows that $\overline{b_n(\rho)}$ converges strong in $L^\infty(0,T;L^p(\Omega))$. We denote the limit by $\overline{b(\rho)}$. Then we have for any $\varphi \in L^q(\Omega)$

$$\begin{aligned}
\int_\Omega \overline{b(\rho)}\varphi \,\mathrm{d}x &= \lim_{n\to\infty} \int_\Omega \overline{b_n(\rho)}\varphi \,\mathrm{d}x \\
&= \lim_{n\to\infty} \lim_{k\to\infty} \int_\Omega b_n(\rho_k)\varphi \,\mathrm{d}x \\
&= \lim_{k\to\infty} \int_\Omega b(\rho_k)\varphi \,\mathrm{d}x
\end{aligned}$$

where all limits are in $C([0,T])$ and where the swap of the limits in n and k is allowed, because the limit in n is uniformly in k. This finishes the proof of (9.5).

Step 5. We show (9.6) by applying Lemma 1.17 to $f_n = b(\rho_n)$ and $g_n = u_n$. By the preceding steps, we have $b(\rho_n) \rightharpoonup \overline{b(\rho)}$ and $u_n \rightharpoonup u$, both in $L^2((0,T)\times\Omega)$. We show that $\partial_t b(\rho_n)$ is bounded in $L^1(0,T;W^{-5,1}(\Omega))$. Let

$$K := \{\varphi \in C_c^\infty((0,T)\times\Omega) \mid \|\varphi\|_{L^2(0,T;W_0^{3,1}(\Omega))} \leqslant 1\}.$$

By the renormalized equation of continuity (8.16), we get

$$\begin{aligned}
\|\frac{\partial b(\rho_n)}{\partial t}\|_{L^2(0,T;W^{-3,1}(\Omega))} &= \sup_{\varphi\in K} |\int_0^T \int_\Omega b(\rho_n)\frac{\partial\varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t| \\
&= \sup_{\varphi\in K} |\int_0^T \int_\Omega -b(\rho_n)u_n\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega (b'(\rho_n)\rho_n - b(\rho_n)) \operatorname{div} u_n\varphi\,\mathrm{d}x\,\mathrm{d}t|
\end{aligned}$$

The first line on the right hand side is bounded in n since u_n is bounded in $L^2((0,T)\times\Omega)$ and since by Sobolev embedding, we have $\|\nabla\varphi\|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \leqslant c$ for all $\varphi \in K$ and some c independent of φ. A similar argument shows that the second line is bounded. Since $\|\cdot\|_{L^1(0,T;W^{-3,1}(\Omega))} \leqslant C\|\cdot\|_{L^2(0,T;W^{-3,1}(\Omega))}$, we get

$$\|\frac{\partial b(\rho_n)}{\partial t}\|_{L^1(0,T;W^{-3,1}(\Omega))} \leqslant c.$$

Next, we show

$$\lim_{|\xi|\to 0} \sup_{n\in\mathbb{N}} \|u_n(\cdot+\xi,\cdot)-u_n\|_{L^2((0,T)\times\Omega,\mathbb{R}^3)} = 0. \tag{9.20}$$

Let $\hat{\xi} := \frac{1}{|\xi|}\xi$. Then, we have the following estimate for all $n \in \mathbb{N}$ and $r \in (0,T)$

$$\begin{aligned}
\|u_n(\cdot+\xi,r)-u_n(\cdot,r)\|^2_{L^2(\Omega,\mathbb{R}^3)} &= \int_\Omega \Big|\int_0^{|\xi|} \nabla u_n(\cdot+\lambda\hat{\xi},r)\hat{\xi}\,\mathrm{d}\lambda\Big|^2\,\mathrm{d}x \\
&\leqslant \int_\Omega \int_0^{|\xi|} |\nabla u_n(\cdot+\lambda\hat{\xi},r)|^2\,\mathrm{d}\lambda\,\mathrm{d}x \\
&= \int_0^{|\xi|} \|\nabla u_n(\cdot+\lambda\hat{\xi},r)\|^2_{L^2(\Omega,\mathbb{R}^{3\times 3})}\,\mathrm{d}x \\
&\leqslant |\xi|\|u_n(\cdot,r)\|^2_{W^{1,2}(\Omega,\mathbb{R}^3)}.
\end{aligned}$$

Thus,

$$\|u_n(\cdot+\xi,\cdot)-u_n\|^2_{L^2((0,T)\times\Omega,\mathbb{R}^3)} \leqslant |\xi|\|u_n\|_{L^2(0,T;W^{1,2}(\Omega,\mathbb{R}^3))} \leqslant c|\xi|$$

where c is independent of $n \in \mathbb{N}$. This shows (9.20). Now, Lemma 1.17 yields

$$b(\rho_n)u_n \to \overline{b(\rho)u} \text{ in } \mathcal{D}'((0,T)\times\Omega). \tag{9.21}$$

For all $b \in C_b^1(\mathbb{R})$ satisfying (8.10) we have

$$\|b(\rho_n)u_n\|_{L^2((0,T)\times\Omega)} \leqslant c,$$

and thus $b(\rho_n)u_n$ is weakly pre-compact in $L^2((0,T)\times\Omega)$. Since by relation (9.21) every $L^2_w((0,T)\times\Omega)$ accumulation point agrees with $\overline{b(\rho)u}$, this shows (9.6).

Step 6. We show (9.8). Let $b_k \in C_b^1(\mathbb{R})$ such that $b_k(x) = x$ for all $x \leqslant k$, $b_k(x) = 2k$ for all $x \geqslant 3k$ and b_k concave. Then, we have for all $1 \leqslant \alpha < \gamma$, $\beta := (\frac{\gamma}{\alpha})'$ and all fixed $t \in (0,T)$

$$\begin{aligned}
\|b_k(\rho_n(t))-\rho_n(t)\|_{L^\alpha(\Omega)} &\leqslant \int_{[\rho_n(t)>k]} \rho_n(t)^\alpha\,\mathrm{d}x \\
&\leqslant \|\rho_n(t)^\alpha\|_{L^{\frac{\gamma}{\alpha}}(\Omega)} \|\mathbb{I}_{[\rho_n(t)\geqslant k]}\|_{L^\beta(\Omega)} \\
&\leqslant \|\rho_n(t)\|^\alpha_{L^\gamma(\Omega)} \mathrm{meas}([p_n(t)\geqslant k])^{\frac{1}{\beta}}.
\end{aligned}$$

Since $\mathrm{meas}([\rho_n(t)\geqslant k]) \leqslant \frac{1}{k}\|p_n(t)\|_{L^\gamma(\Omega)}$, the right hand side is tends to zero uniformly in n and t as $k\to\infty$, i.e. we have

$$\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \|b_k(\rho_n)-\rho_n\|_{L^\infty(0,T;L^\alpha(\Omega))} = 0, \tag{9.22}$$

and in particular we have

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}}\|b_k(\rho_n)-\rho_n\|_{L^\alpha((0,T)\times\Omega)}=0. \tag{9.23}$$

Since by the preceding step, we have

$$\lim_{n\to\infty} b_k(\rho_n)=\overline{b_k(\rho)} \text{ in } L^\alpha_w((0,T)\times\Omega),$$

Lemma 1.13 yields for all $1\leqslant\alpha<\gamma$

$$\lim_{k\to\infty}\overline{b_k(\rho)}=\rho \text{ in } L^\alpha((0,T)\times\Omega), \tag{9.24}$$

and since

$$\|\overline{b_k(\rho)}\|_{L^\infty(0,T;L^\gamma(\Omega))}\leqslant\liminf_{k\to\infty}\|b_k(\rho)\|_{L^\infty(0,T;L^\gamma(\Omega))}\leqslant\liminf_{k\to\infty}\|\rho\|_{L^\infty(0,T;L^\gamma(\Omega))}\leqslant c,$$

we also have

$$\overline{b_k(\rho)}\overset{*}{\rightharpoonup}\rho \text{ in } L^\infty(0,T;L^\alpha(\Omega)) \tag{9.25}$$

for any $1\leqslant\alpha<\gamma$.

We use Lemma 1.13 once more on the sequence $b_k(\rho_n)u_n$ in the Banach space $L^1((0,T)\times\Omega)$. First, the preceding step implies that

$$\lim_{n\to\infty} b_k(\rho_n)u_n=\overline{b_k(\rho)}u \text{ in } L^1_w((0,T)\times\Omega)$$

and by choosing $\alpha=6'=\frac{6}{5}<\gamma$ in (9.25), we deduce

$$\lim_{k\to\infty}\overline{b_k(\rho)}u=\rho u \text{ in } L^1((0,T)\times\Omega)$$

since $u\in L^2(0,T;L^6(\Omega))$ by Sobolev embedding. Finally, we have for all $\frac{6}{5}<\alpha<\gamma$

$$\begin{aligned}\|b_k(\rho_n)u_n-\rho_nu_n\|_{L^1((0,T)\times\Omega)}&\leqslant c\|b_k(\rho_n)u_n-\rho_nu_n\|_{L^2(0,T;L^{\frac{6\alpha}{\alpha+6}}(\Omega))}\\&\leqslant c\|b_k(\rho_n)-\rho_n\|_{L^2(0,T;L^\alpha(\Omega))}\|u_n\|_{L^2(0,T;L^6(\Omega))}\end{aligned}$$

where by (9.22) the right hand side tends to zero uniformly in n as $k\to\infty$. The last three relations are enough for Lemma 1.13 to yield

$$\lim_{n\to\infty}\rho_nu_n=\rho u \text{ in } L^1_w((0,T)\times\Omega).$$

We now infer (9.8) by the estimate

$$\|\rho_nu_n\|_{L^2(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega))}\leqslant\|\sqrt{\rho_n}\|_{L^2(0,T;L^{2\gamma}(\Omega))}\|\sqrt{\rho_n}u_n\|_{L^2((0,T)\times\Omega)},$$

where the right hand side is bounded in n by (9.14) and (9.16).

Step 7. We show (9.10). First, we have

$$\rho_n(u_n - w_n) \to \rho(u-w) \text{ in } \mathcal{D}'((0,T)\times\Omega) \tag{9.26}$$

by the preceding step and the strong convergence of w_n.

Now, let $\varphi \in C_c^\infty(\Omega)$ and

$$\Psi_n(t) := \int_\Omega \rho_n(t)(u_n(t) - w_n(t))\varphi \,\mathrm{d}x$$

We use Ascoli's theorem to prove that $\{\Psi_n\}$ is pre-compact in $C([0,T])$. By (9.26), the sequence $\{\Psi_n(t)\}_{n\in\mathbb{N}}$ is uniformly bounded for any fixed $t \in [0,T]$. Now, the momentum equation (8.17) yields

$$\begin{aligned}\Psi_n(t) - \Psi_n(s) = \int_s^t \int_\Omega &- \rho_n u_n \otimes u_n : \nabla\varphi - p(\rho_n)\operatorname{div}\varphi \\ &+ \mathscr{S}(\nabla u_n) : \nabla\varphi + \rho_n u_n \cdot \nabla(w_n \cdot \varphi)\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

an since we have

$$\begin{aligned}
\|\rho_n u_n \otimes u_n\|_{L^1(\Omega,\mathbb{R}^{3\times3})} &\leqslant c_1 && \text{by (9.14)},\\
\|p(\rho_n)\|_{L^1(\Omega)} &\leqslant c_2 && \text{by (9.3) and (8.7)},\\
\|\mathscr{S}(\nabla u_n)\|_{L^2(\Omega,\mathbb{R}^{3\times3})} &\leqslant c_3 && \text{by (9.4)},\\
\|\rho_n u_n\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega,\mathbb{R}^3)} &\leqslant c_4 && \text{by (9.3) and (9.14)},
\end{aligned}$$

we get

$$|\Psi_n(t) - \Psi_n(s)| \leqslant c|t-s|$$

where the constant c is independent of n, t and s. Consequently,

$$\Psi_n \to \int_\Omega \rho(u-w)\varphi \,\mathrm{d}x$$

uniformly on $[0,T]$. By approximation, exactly as in step 2, Lemma 1.13 yields that

$$\int_\Omega \rho_n(u_n - w_n)\varphi \,\mathrm{d}x \to \int_\Omega \rho(u-w)\varphi \,\mathrm{d}x$$

uniformly on $[0,T]$ for all $\varphi \in (L^{\frac{2\gamma}{\gamma+1}}(\Omega))'$.

Step 8. In order to show (9.7) and (9.9), we first show for a.e. $t \in (0,T)$

$$(\rho_n u_n)(t) \rightharpoonup (\rho u)(t) \text{ in } L^{\frac{2\gamma}{\gamma+1}}(\Omega), \tag{9.27}$$

$$(\rho_n w_n)(t) \rightharpoonup (\rho w)(t) \text{ in } L^{\frac{2\gamma}{\gamma+1}}(\Omega). \tag{9.28}$$

To see this, fix $\varphi \in (L^{\frac{2\gamma}{\gamma+1}}(\Omega))' = L^{\frac{2\gamma}{\gamma-1}}(\Omega)$. By (9.10) we conclude for any $t \in (0,T)$

$$\int_\Omega (\rho_n(u_n - w_n))(t)\varphi \,\mathrm{d}x \to \int_\Omega (\rho(u-w))(t)\varphi \,\mathrm{d}x. \tag{9.29}$$

Since $w_n(t) \to w(t)$ pointwise in Ω for *a.e.* $t \in (0,T)$, the Dominated convergence theorem yields for a.e. $t \in (0,T)$

$$w_n(t)\varphi \to w(t)\varphi \text{ in } L^{\gamma'}(\Omega),$$

and further

$$\int_\Omega (\rho_n w_n)(t)\varphi \,\mathrm{d}x \to \int_\Omega (\rho w)(t)\varphi \,\mathrm{d}x.$$

Thus, (9.28) holds and, together with (9.29), this implies (9.27).

Since $\frac{2\gamma}{\gamma+1} > \frac{6}{5}$, the embedding $L^{\frac{2\gamma}{\gamma+1}}(\Omega) \hookrightarrow W^{-1,2}(\Omega)$ is compact, and therefore

$$(\rho_n u_n)(t) \to (\rho u)(t) \text{ in } W^{-1,2}(\Omega)$$

for a.e. $t \in (0,T)$, and since

$$\|(\rho_n u_n)(t)\|_{W^{-1,2}(\Omega)} \leqslant \|(\rho_n u_n)(t)\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega)} \leqslant \|\rho_n(t)\|_{L^\gamma(\Omega)}\|u_n(t)\|_{L^2(\Omega)} \leqslant c,$$

where c is independent of t, Lebesgue's theorem yields (9.7). The same argumentation shows relation (9.9), where we choose $r < 2$ such that $L^{\frac{2\gamma}{\gamma+1}}(\Omega) \hookrightarrow W^{-1,r}(\Omega)$ is still compact, i.e.

$$r > \frac{6\gamma}{5\gamma - 3}.$$

Step 9. We show (9.11). First, we have

$$\|\rho_n u_n \otimes u_n\|_{L^2(0,T;L^{\frac{6\gamma}{4\gamma+3}}(\Omega,\mathbb{R}^{3\times 3}))} \leqslant \|\sqrt{\rho_n}\|_{L^\infty(0,T;L^{2\gamma}(\Omega))}\|\sqrt{\rho_n}u_n\|_{L^\infty(0,T;L^2(\Omega,\mathbb{R}^3))}\|u_n\|_{L^2(0,T;L^6(\Omega,\mathbb{R}^3))},$$

where the right hand side is bounded by (9.14), (9.16) and the continuity of the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$. Thus we can pass to a weakly convergent subsequence. To identify the limit, we show that $\rho_n u_n \otimes u_n \to \rho u \otimes u$ in $\mathcal{D}'((0,T)\times\Omega)$. By applying Lemma 1.17 to the sequence $f_n := \rho_n u_n^i$ and $g_n := u_n^j$ for any $1 \leqslant i,j \leqslant 3$, where $u_n^i \in L^2(0,T;W^{1,2}(\Omega))$ denotes the i-th component of u_n. We know already that both sequences are weakly convergent:

$$f_n \rightharpoonup \rho u^i \text{ in } L^2(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega)) \text{ by (9.8)}$$
$$g_n \rightharpoonup u^j \text{ in } L^2(0,T;L^6(\Omega)) \text{ by (9.4)}.$$

We have proved in step 4 that the sequence g_n satisfies the properties needed in Lemma 1.17. Finally, we can show that $\frac{\partial f_n}{\partial t}$ is bounded in $L^2(0,T;W^{-3,1}(\Omega))$. Let $K := \{\varphi \in C_c^\infty((0,T)\times\Omega) \,|\, \|\varphi\|_{L^2(0,T;W_0^{-3,1}(\Omega))}\}$, then the momentum equation yields

$$
\begin{aligned}
\|\frac{\partial f_n}{\partial t}\|_{L^2(0,T;W^{-3,1}(\Omega))} &= \sup_{\varphi\in K} |\int_0^T \int_\Omega \rho_n u_n \frac{\partial \varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t| \\
&= \sup_{\varphi\in K} \left| \int_0^T \int_\Omega (-\rho_n u_n \otimes u_n : \nabla\varphi - p(\rho_n)\operatorname{div}\varphi)\,\mathrm{d}x\,\mathrm{d}t + \right. \\
&\qquad \left. \int_0^T \int_\Omega \mathscr{S}(\nabla u_n) : \nabla\varphi - \rho_n \frac{\partial w_n}{\partial t}\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t \right| \\
&\leqslant c
\end{aligned}
$$

Note, that the estimation of the right hand side makes no trouble in view of the estimates already obtained and the uniform boundedness of $\frac{\partial \rho_n}{\partial t}$ in $L^2(0,T;W^{-3,1}(\Omega))$; the latter can be proved exactly as in step 4 by replacing $b(\rho_n)$ with ρ_n and replacing the renormalized equation of continuity with the not renormalized equation.

Step 10. Finally, the relations inferred in this theorem are enough to pass to the limit in the distributional formulation of the (not renormalized) equation of continuity to deduce that (9.12) holds. Moreover, the boundary condition follows from $u(t) \in W_0^{1,2}(\Omega,\mathbb{R}^3)$ and the initial condition $\rho(0) = \rho_0$ follows from $\rho_0 = \rho_n(0) \to \rho(0)$ in $L^\gamma_w(\Omega)$. ☐

9.2 Estimation of the pressure

In this section we show that the pressure $p_n(\rho_n)$ is bounded in some $L^\alpha((0,T)\times\Omega)$ and consequently there exists a weakly convergent subsequence. Then, the crucial point is to identify the weak limit of $p_n(\rho_n)$ as $p(\rho)$, to which the next three subsections are devoted.

Proposition 9.3. *There exists α, $c > 0$ independent of n such that for all $n \in \mathbb{N}$ we have*

$$\|p_n(\rho_n)\rho_n^\alpha\|_{L^1((0,T)\times\Omega)} \leqslant c.$$

Consequently, $p_n(\rho_n)$ is bounded in $L^q((0,T)\times\Omega)$ and therefore, passing to a subsequence if necessary,

$$p_n(\rho_n) \rightharpoonup \overline{p(\rho)} \ \text{ in } L^q((0,T)\times\Omega), \tag{9.30}$$

where $q = 1 + \frac{\alpha}{\gamma}$.

Proof. Let $(\xi_\epsilon)_{\epsilon\in(0,1]}$ be any fixed smoothing sequence and define

$$\rho_n^\epsilon := \xi_\epsilon * \rho_n.$$

Furthermore, fix $\psi \in C_c^\infty(0,T)$ with $0 \leqslant \psi \leqslant 1$ and $b \in C^1(\mathbb{R})$ satisfying $b(0) = 0$ and $b(z) = z^\alpha$ for all $z \geqslant 1$, where $\alpha > 0$ is arbitrary, satisfying

$$\alpha < \min\{\frac{\gamma}{2}, \frac{2\gamma-3}{3}, \frac{\gamma-1}{2}\}. \tag{9.31}$$

Since $\rho_n(t)$ is bounded in $L^\gamma(\Omega)$ uniformly in n and t, we have $\rho_n^\epsilon(t)$ is bounded in $L^\gamma(\Omega)$ uniformly in n, t and ϵ. Consequently, $b(\rho_n^\epsilon(t))$ is bounded in $L^{\frac{\gamma}{\alpha}}(\Omega)$ uniformly in n, t and ϵ. In particular, b is bounded in $L^p(\Omega)$ for

$$p \in \{2, \frac{2\gamma}{\gamma-1}, \frac{3\gamma}{2\gamma-3}\}$$

uniformly in n, t and ϵ. Moreover, by Remark 8.5, the renormalized equation of continuity holds for b.

We consider test functions of the form

$$\varphi(t,x) = \psi(t)\mathcal{B}\Big[\langle b(\rho_n^\epsilon)\rangle\Big]$$

in the equations (8.16), (8.17), where we denote for $f \in L^1(\Omega)$

$$\langle f\rangle := f - \frac{1}{\mathrm{meas}(\Omega)}\int_\Omega f\,\mathrm{d}x,$$

and where the Bogovskii operator $\mathcal{B}$ is defined in Theorem 2.15. Using Lemma 2.29 we obtain, after a lengthy but straightforward calculation, the following formula (see also [12]):

$$\begin{gathered}
\int_0^T\int_\Omega \psi p_n(\rho_n)b(\rho_n^\epsilon)\,\mathrm{d}x\,\mathrm{d}t = \\
\int_0^T \psi\int_\Omega p_n(\rho_n)\,\mathrm{d}x\int_\Omega b(\rho_n^\epsilon)\,\mathrm{d}x\,\mathrm{d}t + \\
(\eta + \frac{1}{3}\nu)\int_0^T\int_\Omega \psi b(\rho_n^\epsilon)\,\mathrm{div}\,u_n\,\mathrm{d}x\,\mathrm{d}t - \\
\int_0^T\int_\Omega \psi_t\rho_n(u_n - w_n)\cdot\mathcal{B}[\langle b(\rho_n^\epsilon)\rangle]\,\mathrm{d}x\,\mathrm{d}t + \\
\nu\int_0^T\int_\Omega \psi\nabla u_n : \nabla\mathcal{B}[\langle b(\rho_n^\epsilon)\rangle]\,\mathrm{d}x\,\mathrm{d}t - \\
\int_0^T\int_\Omega \psi\rho_n u_n\otimes u_n : \mathcal{B}[\langle b(\rho_n^\epsilon)\rangle]\,\mathrm{d}x\,\mathrm{d}t + \\
\int_0^T\int_\Omega \psi\rho_n(u_n - w_n)\cdot\mathcal{B}[\langle\big(b(\rho_n^\epsilon) - b'(\rho_n^\epsilon)\rho_n^\epsilon\big)\,\mathrm{div}\,u_n\rangle]\,\mathrm{d}x\,\mathrm{d}t + \\
\int_0^T\int_\Omega \psi\rho_n(u_n - w_n)\cdot\mathcal{B}[\langle r_n^\epsilon b'(\rho_n^\epsilon)\rangle]\,\mathrm{d}x\,\mathrm{d}t \\
\int_0^T\int_\Omega \psi\rho_n(u_n - w_n)\cdot\mathcal{B}[\langle\mathrm{div}\,\big(b(\rho_n^\epsilon)u_n\big)\rangle]\,\mathrm{d}x\,\mathrm{d}t - \\
\int_0^T\int_\Omega \rho_n u_n\cdot\nabla(w_n\cdot\varphi)\,\mathrm{d}x\,\mathrm{d}t
\end{gathered}$$

Where $r_n^\epsilon \to 0$ in $L^2(0,T;L^{\frac{2\beta}{\beta+2}}(\Omega))$ as $\epsilon \to 0$ for any fixed n. We can estimate the terms on the right hand side in the following way:

(1) By (9.15) and Lemma 8.6, the first integral on the right hand side is bounded uniformly in n, ϵ and ψ, where the latter follows from $0 \leqslant \psi \leqslant 1$.

(2) Next, we have

$$\Big|\int_0^T\int_\Omega \psi b(\rho_n^\epsilon)\operatorname{div} u_n \,\mathrm{d}x\,\mathrm{d}t\Big| \leqslant c\|\operatorname{div} u_n\|_{L^2((0,T)\times\Omega)}\|b(\rho_n^\epsilon)\|_{L^2((0,T)\times\Omega)},$$

where the right hand side is again bounded uniformly of n, ϵ and ψ.

(3) By virtue of Hölder's inequality, Theorem 2.15 and Sobolev embedding, we can estimate the third integral as follows:

$$\begin{aligned}
&\Big|\int_0^T\int_\Omega \frac{\partial\psi}{\partial t}\rho_n(u_n-w_n)\cdot\mathcal{B}[\langle b(\rho_n^\epsilon)\rangle]\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\leqslant c_1\int_0^T \Big|\frac{\partial\psi}{\partial t}\Big|\|\rho_n(u_n-w_n)\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega)}\|\mathcal{B}[\langle b(\rho_n^\epsilon)\rangle]\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)}\,\mathrm{d}t \\
&\leqslant c_1\int_0^T \Big|\frac{\partial\psi}{\partial t}\Big|\Big(\|\rho_n\|_{L^\gamma(\Omega)}^{\frac12}\|\sqrt{\rho_n}u_n\|_{L^2(\Omega)}+\|\rho_n\|_{L^\gamma(\Omega)}\|w_n\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega,\mathbb{R}^3)}\Big)\|b(\rho_n^\epsilon)\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)}\,\mathrm{d}t.
\end{aligned}$$

Now, by using (9.3), (9.14) and (9.31) we can estimate the latter by

$$c_1\int_0^T \Big|\frac{\partial\psi}{\partial t}\Big|\Big(\|\rho_n\|_{L^\gamma(\Omega)}^{\frac12}\|\sqrt{\rho_n}u_n\|_{L^2(\Omega)}+\|\rho_n\|_{L^\gamma(\Omega)}\|w_n\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega,\mathbb{R}^3)}\Big)\|b(\rho_n^\epsilon)\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)}\,\mathrm{d}t \leqslant c_2\int_0^T\Big|\frac{\partial\psi}{\partial t}\Big|\,\mathrm{d}t+c_3$$

(4) Similarly, we can estimate the fourth term by

$$\Big|\int_0^T\int_\Omega \psi\nabla u_n:\nabla\mathcal{B}[\langle b(\rho_n^\epsilon)\rangle]\,\mathrm{d}x\,\mathrm{d}t\Big| \leqslant c\|\nabla u_n\|_{L^2((0,T)\times\Omega)}\|b(\rho_n^\epsilon)\|_{L^2((0,\mathrm{T})\times\Omega)}$$

where the latter is bounded uniformly in n, ϵ and ψ.

(5) Again by Hölder's inequality, the fifth term can be estimated to

$$\Big|\int_0^T\int_\Omega \psi\rho_n u_n\otimes u_n:\mathcal{B}[\langle b(\rho_n^\epsilon)\rangle]\,\mathrm{d}x\,\mathrm{d}t\Big| \leqslant c\int_0^T \|\rho_n\|_{L^\gamma(\Omega)}\|u_n\|_{L^6(\Omega)}^2\|b(\rho_n^\epsilon)\|_{L^q(\Omega)}\,\mathrm{d}t$$

where $q=\frac{3\gamma}{2\gamma-3}$. By (9.3), (9.4) and Sobolev embedding, this is again bounded uniformly in n, ϵ and ψ.

(6) To estimate the sixth term define

$$r=\frac{6\gamma}{5\gamma-6},\ q=\max\{1,\frac{6\gamma}{7\gamma-6}\},\ p=\max\{2,\frac{3\gamma}{2\gamma-3}\}.$$

Then we have

$$\begin{aligned}
&\left|\int_0^T \int_\Omega \psi\rho_n(u_n-w_n)\cdot \mathcal{B}[\langle\big(b(\rho_n^\epsilon)-b'(\rho_n^\epsilon)\rho_n^\epsilon\big)\operatorname{div} u_n\rangle]\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\leqslant c\int_0^T \|\rho_n\|_{L^\gamma(\Omega)}\|u_n-w_n\|_{L^6(\Omega)}\|\mathcal{B}[\langle\big(b(\rho_n^\epsilon)-b'(\rho_n^\epsilon)\rho_n^\epsilon\big)\operatorname{div} u_n\rangle]\|_{L^r(\Omega)}\,\mathrm{d}t
\end{aligned} \tag{9.32}$$

Now, by continuity of the Sobolev embedding $W^{1,q}(\Omega)\hookrightarrow L^r(\Omega)$ we have

$$\begin{aligned}
\|\mathcal{B}[\langle\big(b(\rho_n^\epsilon)-b'(\rho_n^\epsilon)\rho_n^\epsilon\big)\operatorname{div} u_n\rangle]\|_{L^r(\Omega)} &\leqslant c_1\|\mathcal{B}[\langle\big(b(\rho_n^\epsilon)-b'(\rho_n^\epsilon)\rho_n^\epsilon\big)\operatorname{div} u_n\rangle]\|_{W^{1,q}(\Omega)} \\
&\leqslant c_2\|\big(b(\rho_n^\epsilon)-b'(\rho_n^\epsilon)\rho_n^\epsilon\big)\operatorname{div} u_n\|_{L^q(\Omega)} \\
&\leqslant c_3\|b(\rho_n^\epsilon)-b'(\rho_n^\epsilon)\rho_n^\epsilon\|_{L^p(\Omega)}\|\operatorname{div} u_n\|_{L^2(\Omega)}
\end{aligned}$$

And consequently, by Sobolev embedding, (9.3), (9.4) and the assumptions on b and α, the right hand side of (9.32) is bounded uniformly in n,ϵ and ψ. Note, that $b'(z)z=\alpha b(z)$ for all $z\geqslant 1$, and thus, $b'(\rho_n^\epsilon)\rho_n^\epsilon$ is bounded in $L^p((0,T)\times\Omega)$ uniformly in n and ϵ.

(7) Similar, the seventh term can be estimated to

$$\begin{aligned}
\left|\int_0^T \int_\Omega \psi\rho_n(u_n-w_n)\cdot\mathcal{B}[\langle r_\epsilon b'(\rho_n^\epsilon)\rangle]\,\mathrm{d}x\,\mathrm{d}t\right| &\leqslant c\int_0^T \|\rho_n\|_{L^\gamma(\Omega)}\|u_n-w_n\|_{L^6(\Omega)}\|r_\epsilon\|_{L^q}\,\mathrm{d}t \\
&\leqslant \tilde{c}\|r_\epsilon\|_{L^2(0,T;L^q(\Omega))},
\end{aligned}$$

with

$$q=\frac{6\gamma}{7\gamma-6}\leqslant\frac{2\beta}{\beta+2}.$$

(8) By Hölder's inequality and the continuity of the Sobolev embedding $W^{1,q}(\Omega)\hookrightarrow L^r(\Omega)$, where q and r are as in (6), we have

$$\begin{aligned}
\left|\int_0^T \int_\Omega \psi\rho_n(u_n-w_n)\cdot\mathcal{B}[\langle\operatorname{div}\big(b(\rho_n^\epsilon)u_n\big)\rangle]\,\mathrm{d}x\,\mathrm{d}t\right| &\leqslant c\int_0^T \|\rho_n\|_{L^\gamma(\Omega)}\|u_n-w_n\|_{L^6(\Omega)}\|b(\rho_n^\epsilon)u_n\|_{L^r(\Omega)}\,\mathrm{d}s \\
&\leqslant c\int_0^T \|\rho_n\|_{L^\gamma(\Omega)}\|u_n-w_n\|_{L^6(\Omega)}\|u_n\|_{L^6(\Omega)}\|b(\rho_n^\epsilon)\|_{L^p(\Omega)}\,\mathrm{d}s
\end{aligned}$$

where p is as in (6), and by (9.3), (9.4), (9.1), the assumption on α and Sobolev embedding, the right hand side is again bounded uniformly in n, ϵ and ψ.

(9) Finally, we have

$$\int_0^T \int_\Omega \rho_n u_n\cdot\nabla(w_n\cdot\varphi)\,\mathrm{d}x\,\mathrm{d}t\leqslant c\int_0^T \|\rho_n\|_{L^\gamma(\Omega)}^{\frac{1}{2}}\|\sqrt{\rho_n}u_n\|_{L^2(\Omega)}\|w_n\varphi\|_{W^{1,\frac{2\gamma}{\gamma-1}}(\Omega)}\,\mathrm{d}t$$

and the right hand side is bounded uniformly in n and ϵ.

Summing up these results, we obtain

$$\int_0^T \psi \int_\Omega p(\rho_n) b(\rho_n^\epsilon) \, \mathrm{d}x \, \mathrm{d}t \leqslant c(1 + \|r_n^\epsilon\|_{L^2(0,T;L^s(\Omega))} + \int_0^T |\frac{\partial \psi}{\partial t}| \, \mathrm{d}t)$$

with

$$s = \frac{2\beta}{\beta + 2},$$

where the constant c is independent of n and ϵ.

Now, choose a sequence $\psi_n \in C_c^\infty(0,T)$ with $\psi_n \to 1$ in $L^1(0,T)$ and $\int_0^T |\frac{\partial \psi}{\partial t}| \, \mathrm{d}t \leqslant K$ for some constant K to obtain

$$\int_0^T \int_\Omega p(\rho_n) b(\rho_n^\epsilon) \, \mathrm{d}x \, \mathrm{d}t \leqslant \tilde{c}(1 + \|r_n^\epsilon\|_{L^2(0,T;L^s(\Omega))})$$

And finally letting $\epsilon \to 0$ and using Fatou's lemma, we conclude

$$\int_0^T \int_\Omega p(\rho_n) b(\rho_n) \, \mathrm{d}x \, \mathrm{d}t \leqslant \hat{c}$$

for some $\hat{c}$ independent of n.

□

Remark 9.4. The estimate (7) in the preceding proof is the reason for introducing the artificial pressure term $\delta_n \rho^\beta$.

For later use, we show the following immediate consequences:

Corollary 9.5. *We have*

$$p(\rho_n) \rightharpoonup \overline{p(\rho)} \ \text{ in } L^q((0,T) \times \Omega), \tag{9.33}$$

where $q > 1$ is as in the preceding Proposition.

Proof. Since $p(\rho_n) = p_n(\rho_n) - \frac{1}{n}\rho_n^\beta$, it is enough to show that

$$\frac{1}{n}\rho_n^\beta \to 0 \text{ in } L^1((0,T) \times \Omega).$$

Let $f_n := n^{-\frac{1}{\beta}} \rho_n$. Then, we have

$$\|n^{\frac{1}{\beta}} f_n\|_{L^1((0,T)\times\Omega)} = \|\rho_n\|_{L^1((0,T)\times\Omega)} \leqslant c$$

and consequently, $f_n \to 0$ in $L^1((0,T) \times \Omega)$, and in particular, we have this convergence in measure with respect to the Lebesgue measure. Since

$$\|f_n\|_{L^{\beta q}((0,T)\times\Omega)} \leqslant c$$

for some $q > 1$ by the preceding Proposition, Theorem 3.31 implies $f_n \to 0$ in $L^\beta((0,T)\times\Omega)$. $\square$

Corollary 9.6. *The functions (ρ, u) solve the following equation:*

$$\begin{aligned}&\int_0^T\int_\Omega \rho(u-w)\cdot\frac{\partial\varphi}{\partial t} + \rho u\otimes u : \nabla\varphi + \overline{p(\rho)}\operatorname{div}\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T\int_\Omega \mathscr{S}(\nabla u):\nabla\varphi + \rho u\cdot\nabla(w\cdot\varphi)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega q\cdot\varphi(0,\cdot)\,\mathrm{d}x\end{aligned} \tag{9.34}$$

for all $\varphi \in C_c^\infty([0,T)\times\Omega;\mathbb{R}^3)$.

Proof. Relations (9.3) - (9.11) and Proposition 9.3 together with (9.2) are sufficient to pass to the limit in the weak formulation of the momentum equation (8.17) and consequently we obtain (9.34). $\square$

9.3 Stability of the effective viscous flux

With (9.34) in mind, the major task left is to show that the weak limit $\overline{p(\rho)}$ is in fact equal to $p(\rho)$. In a first step we prove the so-called weak stability of the effective viscous flux:

Proposition 9.7. *We have for all $b \in C_b^1(\mathbb{R})$ satisfying (8.10)*

$$\overline{p(\rho)b(\rho)} - (\frac{4}{3}\nu+\eta)\overline{b(\rho)\operatorname{div}u} = \overline{p(\rho)}\;\overline{b(\rho)} - (\frac{4}{3}\nu+\eta)\overline{b(\rho)}\operatorname{div}u \tag{9.35}$$

where $\overline{p(\rho)}$ denotes the weak limit of $p(\rho_n)$ in $L^q((0,T)\times\Omega)$ for some $q>1$ as in Theorem 9.3 and

$$p_n(\rho_n)b(\rho_n) \rightharpoonup \overline{p(\rho)b(\rho)} \qquad \text{in } L^\alpha((0,T)\times\Omega), \tag{9.36}$$

$$b(\rho_n)\operatorname{div}u_n \to \overline{b(\rho)\operatorname{div}u} \qquad \text{in } L^2((0,T)\times\Omega), \tag{9.37}$$

$$b(\rho_n) \rightharpoonup \overline{b(\rho)} \qquad \text{in } C_w([0,T];L^p(\Omega)) \text{ for all } 1<p<\infty. \tag{9.38}$$

for a certain $\alpha > 1$, again passing to a subsequence if necessary.

We start with the following Lemma:

Lemma 9.8. *There exists a subsequence of (ρ_n, u_n) such that (9.36)-(9.38) and*

$$b(\rho_n)u_n \rightharpoonup \overline{b(\rho)}u \qquad \text{in } L^2((0,T)\times\Omega) \tag{9.39}$$

$$(b'(\rho_n)\rho_n - b(\rho_n))\operatorname{div}u_n \rightharpoonup \overline{(b'(\rho)\rho - b(\rho))\operatorname{div}u} \qquad \text{in } L^2((0,T)\times\Omega) \tag{9.40}$$

hold for all $b \in C_b^1(\mathbb{R})$ satisfying (8.10).

Proof. Relations (9.38) and (9.39) have been proved in Lemma 9.2. Relations (9.36), (9.37) and (9.40) for all $b \in C_b^1(\mathbb{R})$ satisfying (8.10), can be shown similar to Step 4 in the proof of Lemma 9.2: Let

$$X := \overline{\{b \in C_b^1(\mathbb{R}) \, : \, b \text{ satisfies } (8.10)\}}^{C_b^1(\mathbb{R})}$$

and choose a countable and dense subset $Y \subseteq X$. The usual estimates and Theorem 1.2 yield that there is a subsequence such that all three relations hold for all $b \in Y$. For an arbitrary $b \in X$, approximation yields the desired conclusion.

□

We now pass to the subsequence described in the Lemma above. To prove Proposition 9.7, we use test functions of the form

$$\varphi_n(t,x) = \psi(t)\vartheta(x)\mathcal{A}[\xi b(\rho_n)]$$

where $\psi \in C_c^\infty(0,T)$, $\vartheta, \xi \in C_c^\infty(\Omega)$, $b \in C^\infty[0,\infty)$ satisfying (8.10) and the inverse divergence $\mathcal{A}$ is defined in Definition 2.20, in the momentum equation (8.17), and test functions of the form

$$\varphi(t,x) = \psi(t)\vartheta(x)\mathcal{A}[\xi\overline{b(\rho)}]$$

in the limit equation (9.34). We need the following Lemmata:

Lemma 9.9. *We have for all* $1 < p, q < \infty$*:*

$$\begin{aligned} \mathcal{A}[\xi b(\rho_n)] &\overset{*}{\rightharpoonup} \mathcal{A}[\xi\overline{b(\rho)}] && \text{in } L^\infty(0,T;W^{1,q}(\Omega,\mathbb{R}^3)), \\ \mathcal{A}[\xi b(\rho_n)] &\to \mathcal{A}[\xi\overline{b(\rho)}] && \text{in } L^p(0,T;C(\Omega,\mathbb{R}^3)), \\ \mathcal{A}[(b'(\rho_n)\rho_n - b(\rho_n))\operatorname{div} u_n] &\rightharpoonup \mathcal{A}[\overline{(b'(\rho)\rho - b(\rho))\operatorname{div} u}] && \text{in } L^2(0,T;W^{1,2}(\Omega,\mathbb{R}^3)), \end{aligned}$$

passing to a subsequence if necessary.

Proof. Since $\mathcal{A} : L^p(\Omega) \to C(\Omega;\mathbb{R}^3)$ is a compact operator for all $p \geqslant 3$ and $\xi b(\rho_n)(t) \rightharpoonup \xi\overline{b(\rho)}(t)$ in $L^p(\Omega)$ for all $t \in [0,T]$ we get $\mathcal{A}[\xi b(\rho_n(t))] \to \mathcal{A}[\xi\overline{b(\rho)}(t)]$ in $C(\Omega,\mathbb{R}^3)$ and Lebesgue's theorem yields the second relation.

By Corollary 2.24 we have $\mathcal{A} : L^q(\Omega) \to W^{1,q}(\Omega)$ continuously and therefore

$$\|\mathcal{A}[\xi b(\rho_n)]\|_{L^\infty(0,T;W^{1,q}(\Omega,\mathbb{R}^3))} \leqslant \|\xi b(\rho_n)\|_{L^\infty(0,T;L^q(\Omega,\mathbb{R}^3))} \leqslant c$$

for all $q > 1$ and this shows the first relation.

The statement about $\mathcal{A}[(b'(\rho_n)\rho_n - b(\rho_n))\operatorname{div} u_n]$ follows similar. □

Lemma 9.10. *We have*

$$\partial_t\overline{b(\rho)} + \operatorname{div}(\overline{b(\rho)}u) + \overline{(b'(\rho)\rho - b(\rho))\operatorname{div} u} = 0 \qquad \text{in } \mathcal{D}'([0,T) \times \mathbb{R}^3) \tag{9.41}$$

for all $b \in C^1([0,\infty))$ satisfying (8.10).

Proof. This follows immediately by passing to the limit in the renormalized equation of continuity (8.16). $\square$

Lemma 9.11. *The limit equation (9.34) and the momentum equation (8.17) are satisfied respectively with φ and φ_n as test functions.*

Proof. To prove the statement concerning φ_n, it is sufficient to show $\varphi_n \in W^{1,2}((0,T)\times\Omega)$. By Theorem 2.24 we have $\varphi_n \in L^\infty(0,T;W^{1,2}(\Omega))$, thus it is left to show $\partial_t\mathcal{A}[\xi b(\rho_n)] \in L^2((0,T)\times\Omega,\mathbb{R}^3)$. For $g = (g^i)_{i=1,2,3} \in C_c^\infty((0,T)\times\Omega;\mathbb{R}^3)$ we have

$$\langle \partial_t\mathcal{A}[\xi b(\rho_n)], g\rangle = -\int_0^T\int_\Omega \mathcal{A}[\xi b(\rho_n)]\cdot\partial_t g\,\mathrm{d}x\,\mathrm{d}t$$

where $\langle .,.\rangle : \mathcal{D}'\times\mathcal{D}\to\mathbb{R}$ denotes the duality product for distributions.

We have $\partial_t\mathcal{A}[v] = \mathcal{A}[\partial_t v]$ and $\partial_{x_i}\mathcal{A}[v] = \mathcal{A}[\partial_{x_i}v]$ whenever $v \in C^1((0,T)\times\Omega)$. Indeed, the first statement follows immediately from the differentiability of integrals with respect to a parameter, the second follows from the fact, that differentiation is a multiplier operator ($\partial_{x_j}\approx -i\xi_j$) and thus commutates with $\mathcal{A}$. Thus, by Theorem 2.23 and by using the renormalized equation of continuity (8.16), we get

$$\begin{aligned}
-\int_0^T\int_\Omega \mathcal{A}[\xi b(\rho_n)]\cdot\partial_t g\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega b(\rho_n)\partial_t\big(\xi\mathcal{A}_i(g^i)\big)\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_0^T\int_\Omega \xi\big(b'(\rho_n)\rho_n - b(\rho_n)\operatorname{div}u_n\big)\,\mathcal{A}_i(g^i)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad -\int_0^T\int_\Omega b(\rho_n)u_n\cdot(\xi\nabla\mathcal{A}_i(g^i) + \nabla\xi\,\mathcal{A}_i(g^i))\,\mathrm{d}x\,\mathrm{d}t\\
&= -\int_0^T\int_\Omega \mathcal{A}\Big[\xi\big(b'(\rho_n)\rho_n - b(\rho_n)\operatorname{div}u_n\big)\Big]\cdot g\,\mathrm{d}x\,\mathrm{d}t\\
&\quad +\int_0^T\int_\Omega \operatorname{div}(\mathcal{A}[b(\rho_n)u_n\xi])\cdot g + \mathcal{A}[b(\rho_n)u_n\cdot\nabla\xi]\cdot g\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

Because of $\operatorname{div}(\mathcal{A}[b(\rho_n)u_n\xi]) = b(\rho_n)u_n\xi$, the last integral shows

$$\partial_t\mathcal{A}[\xi b(\rho_n)] = \mathcal{A}\Big[\xi\big(b'(\rho_n)\rho_n - b(\rho_n)\operatorname{div}u_n\big)\Big] - b(\rho_n)u_n\xi - \mathcal{A}[b(\rho_n)u_n\cdot\nabla\xi] \in L^2((0,T)\times\Omega) \tag{9.42}$$

and this shows $\varphi_n \in W^{1,2}((0,T)\times\Omega)$.

Finally, we show the statement concerning φ. By the previous Lemma we have

$$\mathcal{A}[\xi\overline{b(\rho)}] \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)).$$

Furthermore, we can estimate the right hand side of (9.42) in the usual way to deduce that $\partial_t\mathcal{A}[\xi b(\rho_n)]$ is bounded in $L^2((0,T)\times\Omega)$. Since $\mathcal{A}[\xi b(\rho_n)]\to\mathcal{A}[\xi\overline{b(\rho)}]$ in $\mathcal{D}'((0,T)\times\Omega)$ we have $\partial_t\mathcal{A}[\xi b(\rho_n)]\to\partial_t\mathcal{A}[\xi\overline{b(\rho)}]$

in $\mathcal{D}'((0,T)\times\Omega)$, and this is enough to infer that any accumulation point of $\partial_t\mathcal{A}[\xi b(\rho_n)]$ in the weak-$L^2((0,T)\times\Omega)$ topology agrees with $\partial_t\mathcal{A}[\xi\overline{b(\rho)}]$. We deduce $\partial_t\mathcal{A}[\xi b(\rho_n)] \rightharpoonup \partial_t\mathcal{A}[\xi\overline{b(\rho)}]$ in $L^2((0,T)\times\Omega)$. In particular we get $\partial_t\mathcal{A}[\xi\overline{b(\rho)}] \in L^2((0,T)\times\Omega)$. $\square$

Finally, we can prove Theorem 9.7.

Proof of Theorem 9.7. **Step 1.** We use

$$\varphi_n(t,x) = \psi(t)\vartheta(x)\mathcal{A}[\xi b(\rho_n)]$$

as test functions in the momentum equation (8.17)

$$\begin{aligned}\int_0^T\int_\Omega \rho_n(u_n-w_n)\cdot\frac{\partial\varphi_n}{\partial t} + \rho_n u_n\otimes u_n : \nabla\varphi_n + p_n(\rho_n)\operatorname{div}\varphi_n \,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T\int_\Omega \mathscr{S}(\nabla u_n):\nabla\varphi_n + \rho_n u_n\cdot\nabla(w_n\cdot\varphi_n)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{9.43}$$

Similar, we use

$$\varphi(t,x) = \psi(t)\vartheta(x)\mathcal{A}[\xi\overline{b(\rho)}]$$

as a test function in the limit equation (9.34)

$$\begin{aligned}\int_0^T\int_\Omega \rho(u-w)\cdot\frac{\partial\varphi}{\partial t} + \rho u\otimes u : \nabla\varphi + \overline{p(\rho)}\operatorname{div}\varphi \,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T\int_\Omega \mathscr{S}(\nabla u):\nabla\varphi + \rho u\cdot\nabla(w\cdot\varphi)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{9.44}$$

We show, that in fact most of the terms in the momentum equation converge to their counterparts in the limit equation. Then we infer that the remaining parts in the momentum equation converge to the remaining parts in limit equation, and this will be precisely the statement of the theorem.

Step 2. By straight forward calculations, we obtain the following formulas:

$$\begin{aligned}\int_0^T\int_\Omega \rho_n u_n\cdot\frac{\partial\varphi_n}{\partial t}\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \vartheta\frac{\partial\psi}{\partial t}\rho_n u_n\cdot\mathcal{A}[\xi b(\rho_n)] - \vartheta\psi\rho_n u_n\cdot\partial_i\mathcal{A}[\xi b(\rho_n)u_n^i]\,\mathrm{d}x\,\mathrm{d}t + \\ &\quad\int_0^T\int_\Omega -\vartheta\psi\rho_n u_n\cdot\big(\mathcal{A}[b(\rho_n)u_n\cdot\nabla\xi] + \mathcal{A}[(b'(\rho_n)\rho_n - b(\rho_n))\operatorname{div}u_n]\big)\,\mathrm{d}x\,\mathrm{d}t \\ &=: \sum_{i=1}^4 T_i^1 \\ \int_0^T\int_\Omega \rho u\cdot\frac{\partial\varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \vartheta\frac{\partial\psi}{\partial t}\rho u\cdot\mathcal{A}[\xi\overline{b(\rho)}] - \vartheta\psi\rho u\cdot\partial_i\mathcal{A}[\xi\overline{b(\rho)u^i}]\,\mathrm{d}x\,\mathrm{d}t + \\ &\quad\int_0^T\int_\Omega -\vartheta\psi\rho u\cdot\big(\mathcal{A}[\overline{b(\rho)}u\cdot\nabla\xi] + \mathcal{A}[\overline{(b'(\rho)\rho - b(\rho))\operatorname{div}u}]\big)\,\mathrm{d}x\,\mathrm{d}t \\ &=: \sum_{i=1}^4 \overline{T_i^1}\end{aligned}$$

By Lemma 9.9 and (9.7) we have $T_k^1 \to \overline{T_k^1}$ for $k \in \{1,4\}$. By Lemma 9.8 and Corollary 2.24 we have $T_3^1 \to \overline{T_3^1}$. Thus, only the behaviour of the term T_2^1 for $n \to \infty$ remains unclear.

$$
\begin{aligned}
\int_0^T \int_\Omega \rho_n w_n \cdot \frac{\partial \varphi_n}{\partial t} \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_\Omega \vartheta \rho_n w_n \cdot \big(\frac{\partial \psi}{\partial t} \mathcal{A}[\xi b(\rho_n)] + \psi \partial_i \mathcal{A}[\xi b(\rho_n) u_n^i] \big) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \int_0^T \int_\Omega \vartheta \psi \rho_n w_n \cdot \big(- \mathcal{A}[b(\rho_n) u_n \cdot \nabla \xi] + \mathcal{A}[(b'(\rho_n)\rho_n - b(\rho_n)) \operatorname{div} u_n] \big) \,\mathrm{d}x\,\mathrm{d}t \\
&=: \sum_{i=1}^4 T_i^2
\end{aligned}
$$

$$
\begin{aligned}
\int_0^T \int_\Omega \rho w \cdot \frac{\partial \varphi}{\partial t} \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_\Omega \vartheta \rho w \cdot \left(\frac{\partial \psi}{\partial t} \mathcal{A}[\xi \overline{b(\rho)}] + \psi \partial_i \mathcal{A}[\xi \overline{b(\rho)} u^i] \right) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \int_0^T \int_\Omega \vartheta \psi \rho w \cdot \left(- \mathcal{A}[\overline{b(\rho)} u \cdot \nabla \xi] + \mathcal{A}[\overline{(b'(\rho)\rho - b(\rho)) \operatorname{div} u}] \right) \,\mathrm{d}x\,\mathrm{d}t \\
&=: \sum_{i=1}^4 \overline{T_i^2}
\end{aligned}
$$

By Lemma 9.9 and (9.9) we have $T_k^2 \to \overline{T_k^2}$ for $k \in \{1,4\}$. By Lemma 9.8 and Corollary 2.24 we have $T_3^2 \to \overline{T_3^2}$. Thus, the behaviour of the term T_2^2 for $n \to \infty$ remains unclear.

$$
\begin{aligned}
\int_0^T \int_\Omega \rho_n u_n \otimes u_n : \nabla \varphi_n \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_\Omega \psi \vartheta \rho_n u_n \otimes u_n : \nabla \mathcal{A}[\xi b(\rho_n)] \,\mathrm{d}x\,\mathrm{d}t + \\
&\quad \int_0^T \int_\Omega \psi \rho_n (u_n \otimes u_n) : (\mathcal{A}[\xi b(\rho_n)] \otimes \nabla \vartheta) \,\mathrm{d}x\,\mathrm{d}t \\
&=: \sum_{i=1}^2 T_i^3
\end{aligned}
$$

$$
\begin{aligned}
\int_0^T \int_\Omega \rho u \otimes u : \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_\Omega \psi \vartheta \rho u \otimes u : \nabla \mathcal{A}[\xi \overline{b(\rho)}] \,\mathrm{d}x\,\mathrm{d}t + \\
&\quad \int_0^T \int_\Omega \psi \rho (u \otimes u) : (\mathcal{A}[\xi \overline{b(\rho)}] \otimes \nabla \vartheta) \,\mathrm{d}x\,\mathrm{d}t \\
&=: \sum_{i=1}^2 \overline{T_i^3}
\end{aligned}
$$

By Lemma 9.9 and (9.11) we have $T_2^3 \to \overline{T_2^3}$ as $n \to \infty$. Thus, the behaviour of the term T_1^3 for $n \to \infty$

remains unclear.

$$\begin{aligned}\int_0^T\int_\Omega p_n(\rho_n)\operatorname{div}\varphi_n\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \psi p_n(\rho_n)\Big(\vartheta\xi b(\rho_n)+\nabla\vartheta\cdot\mathcal{A}[\xi b(\rho_n)]\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &=:\sum_{i=1}^2 T_i^4\end{aligned}$$

$$\begin{aligned}\int_0^T\int_\Omega \overline{p(\rho)}\operatorname{div}\varphi\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \psi\overline{p(\rho)}\Big(\vartheta\xi\overline{b(\rho)}+\nabla\vartheta\cdot\mathcal{A}[\xi\overline{b(\rho)}]\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &=:\sum_{i=1}^2 \overline{T_i^4}\end{aligned}$$

By (9.30) and Lemma 9.9 we have $T_2^4 \to \overline{T_2^4}$ as $n \to \infty$. Thus, the behaviour of the term T_1^4 for $n \to \infty$ remains unclear.

$$\begin{aligned}\int_0^T\int_\Omega \mathscr{S}(\nabla u_n):\nabla\varphi_n\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \mathscr{S}(\nabla u_n):(\psi\nabla\vartheta\otimes\mathcal{A}[\xi b(\rho_n)])\,\mathrm{d}x\,\mathrm{d}t+\\ &\quad\int_0^T\int_\Omega \psi\vartheta\xi(\eta+\frac{4}{3}\nu)\operatorname{div}u_n\,b(\rho_n)\,\mathrm{d}x\,\mathrm{d}t+\\ &\quad 2\int_0^T\int_\Omega \nu\psi(\nabla\nabla\vartheta):(u_n\otimes\mathcal{A}[\xi b(\rho_n)])\,\mathrm{d}x\,\mathrm{d}t+\\ &\quad 2\int_0^T\int_\Omega \nu\psi\operatorname{div}u_n\,\nabla\vartheta\cdot\mathcal{A}[\xi b(\rho_n)]\,\mathrm{d}x\,\mathrm{d}t+\\ &\quad 2\int_0^T\int_\Omega \nu\psi\xi b(\rho_n)\nabla\vartheta\cdot u_n\,\mathrm{d}x\,\mathrm{d}t\\ &=\sum_{i=1}^5 T_i^5\end{aligned}$$

$$\begin{aligned}\int_0^T\int_\Omega \mathscr{S}(\nabla u):\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \mathscr{S}(\nabla u):(\psi\nabla\vartheta\otimes\mathcal{A}[\xi\overline{b(\rho)}])\,\mathrm{d}x\,\mathrm{d}t+\\ &\quad\int_0^T\int_\Omega \psi\vartheta\xi(\eta+\frac{4}{3}\nu)\operatorname{div}u\,\overline{b(\rho)}\,\mathrm{d}x\,\mathrm{d}t+\\ &\quad 2\int_0^T\int_\Omega \nu\psi(\nabla\nabla\vartheta):(u\otimes\mathcal{A}[\xi\overline{b(\rho)}])\,\mathrm{d}x\,\mathrm{d}t+\\ &\quad 2\int_0^T\int_\Omega \nu\psi\operatorname{div}u\,\nabla\vartheta\cdot\mathcal{A}[\xi\overline{b(\rho)}]\,\mathrm{d}x\,\mathrm{d}t+\\ &\quad 2\int_0^T\int_\Omega \nu\psi\xi\overline{b(\rho)}\nabla\vartheta\cdot u\,\mathrm{d}x\,\mathrm{d}t\\ &=\sum_{i=1}^5 \overline{T_i^5}\end{aligned}$$

By (9.4) we have $\mathscr{S}(\nabla u_n) \rightharpoonup \mathscr{S}(\nabla u)$, $\operatorname{div} u_n \rightharpoonup \operatorname{div} u$ and $u_n \rightharpoonup u$ in $L^2((0,T)\times\Omega)$. These relations together with Lemma 9.9 imply $T_k^5 \to \overline{T_k^5}$ as $n \to \infty$ for $k \in \{1,3,4\}$. Relation (9.39) immediately implies $T_5^5 \to \overline{T_5^5}$ as $n \to \infty$. Thus, the behaviour of the term T_2^5 for $n \to \infty$ remains unclear. Note at this point that the statement of Proposition 9.7 is precisely

$$T_1^4 - T_2^5 \to \overline{T_1^4} - \overline{T_2^5} \text{ as } n \to \infty.$$

Finally, we have:

$$\begin{aligned}
\int_0^T \int_\Omega \rho_n u_n \cdot \nabla(w_n \cdot \varphi_n) \,\mathrm{d}x \,\mathrm{d}t &= \int_0^T \int_\Omega \psi\vartheta\rho_n u_n \cdot \big(w_n \cdot \nabla\mathcal{A}[\xi b(\rho_n)]\big) \,\mathrm{d}x \,\mathrm{d}t + \\
&\quad \int_0^T \int_\Omega \psi\rho u_n \cdot \big(\mathcal{A}[\xi b(\rho_n)] \cdot \nabla(\vartheta w_n)\big) \,\mathrm{d}x \,\mathrm{d}t \\
&=: \sum_{i=1}^2 T_i^6 \\
\int_0^T \int_\Omega \rho u \cdot \nabla(w \cdot \varphi) \,\mathrm{d}x \,\mathrm{d}t &= \int_0^T \int_\Omega \psi\vartheta\rho u \cdot \big(w \cdot \nabla\mathcal{A}[\xi b(\rho)]\big) \,\mathrm{d}x \,\mathrm{d}t + \\
&\quad \int_0^T \int_\Omega \psi\rho u \cdot \big(\mathcal{A}[\xi b(\rho)] \cdot \nabla(\vartheta w)\big) \,\mathrm{d}x \,\mathrm{d}t \\
&=: \sum_{i=1}^2 \overline{T_i^6}
\end{aligned}$$

Step 3. We show that $T_2^6 \to \overline{T_2^6}$. Let $s, q \in (1, \infty)$ satisfy

$$\frac{1}{s} + \frac{1}{q} = \frac{\gamma - 1}{2\gamma}.$$

From Lemma 9.9 we infer

$$\mathcal{A}[\xi b(\rho_n)](t) \to \mathcal{A}[\xi \overline{b(\rho)}](t) \text{ in } L^s(\Omega)$$

for all $a.e.\, t \in (0, T)$. Since

$$\nabla w_n \to \nabla w \text{ a.e. in } (0, T) \times \Omega$$

we have

$$\nabla(w_n\vartheta)(t) \to \nabla(w\vartheta)(t) \text{ a.e. in } \Omega$$

for a.e. $t \in (0, T)$, and by Lebesgue's theorem we infer for a.e. $t \in (0, T)$

$$\nabla(w_n\vartheta)(t) \to \nabla(w\vartheta)(t) \text{ in } L^q(\Omega).$$

This gives rise to

$$\big(\mathcal{A}[\xi b(\rho_n)]\nabla(w_n\vartheta)\big)(t) \to \big(\mathcal{A}[\xi\overline{b(\rho)}]\nabla(w\vartheta)\big)(t) \text{ in } L^{\frac{2\gamma}{\gamma-1}}(\Omega).$$

for a.e. $t \in (0,T)$. Since

$$\|\big(\mathcal{A}[\xi b(\rho_n)]\nabla(w_n\vartheta)\big)(t)\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)} \leqslant \|\mathcal{A}[\xi b(\rho_n)](t)\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)}\|\nabla(w_n\vartheta)(t)\|_{L^\infty(\Omega)} \leqslant c$$

for a.e. $t \in (0,T)$, Lebesgue's theorem yields

$$\mathcal{A}[\xi b(\rho_n)]\nabla(w_n\vartheta) \to \mathcal{A}[\xi\overline{b(\rho)}]\nabla(w\vartheta) \qquad \text{in } L^2(0,T;L^{\frac{2\gamma}{\gamma-1}}(\Omega))$$

and since by Lemma 9.2

$$\rho_n u_n \rightharpoonup \rho u \qquad \text{in } L^2(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega))$$

we deduce $T_2^6 \to \overline{T_2^6}$ as $n \to \infty$.

Step 4. We show that $T_2^1 - T_1^3 \to \overline{T_2^1} - \overline{T_1^3}$ as $n \to \infty$. Thus, we have to show

$$\lim_{n\to\infty}\int_0^T \psi \int_\Omega \rho_n\vartheta\big(u_n^j\mathcal{R}_{i,j}[\xi b(\rho_n)u_n^i] - u_n^i u_n^j\mathcal{R}_{i,j}[\xi b(\rho_n)]\big)\,\mathrm{d}x\,\mathrm{d}t = \\ \int_0^T \psi \int_\Omega \rho\vartheta\psi\big(u^j\mathcal{R}_{i,j}[\xi\overline{b(\rho)}u^i] - u^i u^j\mathcal{R}_{i,j}[\xi\overline{b(\rho)}]\big)\,\mathrm{d}x\,\mathrm{d}t,$$

where the double Riesz operators $\mathcal{R}_{i,j}$ are defined in Definition 2.20. We have

$$\int_\Omega \rho_n\vartheta u_n^j\mathcal{R}_{i,j}[\xi b(\rho_n)u_n^i]\,\mathrm{d}x = \int_\Omega \mathcal{R}_{i,j}[\rho_n\vartheta u_n^j]\xi b(\rho_n)u_n^i\,\mathrm{d}x \\ \int_\Omega \rho\vartheta u^j\mathcal{R}_{i,j}[\xi\overline{b(\rho)}u^i]\,\mathrm{d}x = \int_\Omega \mathcal{R}_{i,j}[\rho\vartheta u^j]\xi\overline{b(\rho)}u^i\,\mathrm{d}x.$$

and consequently, we have to show

$$\lim_{n\to\infty}\int_0^T \psi \int_\Omega u_n^i\big(\mathcal{R}_{i,j}[\vartheta\rho_n u_n^j]\xi b(\rho_n) - \vartheta\rho_n u_n^j\mathcal{R}_{i,j}[\xi b(\rho_n)]\big)\,\mathrm{d}x\,\mathrm{d}t = \\ \int_0^T \psi \int_\Omega u^i\big(\mathcal{R}_{i,j}[\vartheta\rho u^j]\xi b(\rho) - \vartheta\rho u^j\mathcal{R}_{i,j}[\xi b(\rho)]\big)\,\mathrm{d}x\,\mathrm{d}t \tag{9.45}$$

By (9.27) we have for a.e. $t \in (0,T)$

$$(\vartheta\rho_n u_n^j)(t) \rightharpoonup (\vartheta\rho u^j)(t) \text{ in } L^{\frac{2\gamma}{\gamma+1}}(\Omega),$$

and by (9.38) we have for a.e. $t \in (0,T)$

$$\xi b(\rho_n(t)) \rightharpoonup \xi\overline{b(\rho(t))} \text{ in } L^p(\Omega)$$

for all $1 < p < \infty$. By virtue of Lemma 2.25 we have for a.e. $t \in (0,T)$

$$Z_n(t) := \big(\mathcal{R}_{i,j}[\vartheta\rho_n u_n^j]\xi b(\rho_n) - \vartheta\rho_n u_n^j\mathcal{R}_{i,j}[\xi b(\rho_n)]\big)(t) \rightharpoonup Z(t) := \big(\mathcal{R}_{i,j}[\vartheta\rho u^j]\xi b(\rho) - \vartheta\rho u^j\mathcal{R}_{i,j}[\xi b(\rho)]\big)(t)$$

in $L^p(\Omega)$ for all $p < \frac{2\gamma}{\gamma+1}$. Using the usual bounds on ρ_n and u_n and the continuity of the double Riesz operator $\mathcal{R}_{i,j}$ we deduce that Z_n is bounded in $L^p(\Omega)$ for all $p < \frac{2\gamma}{\gamma+1}$ uniformly in $t \in (0,T)$ and n.

Now, choose p such that $\frac{6}{5} < p < \frac{2\gamma}{\gamma+1}$; this is possible, since $\gamma > \frac{3}{2}$. From the above results and the compactness of the Sobolev embedding $L^p(\Omega) \hookrightarrow W^{-1,2}(\Omega)$, we deduce for a.e. $t \in (0,T)$

$$Z_n(t) \to Z(t) \text{ in } W^{-1,2}(\Omega).$$

and consequently, by Lebesgue's Theorem,

$$Z_n \to Z \text{ in } L^2(0,T;W^{-1,2}(\Omega)).$$

This, together with (9.4) gives rise to (9.45).

Step 5. We show $T_2^2 + T_1^6 \to \overline{T_2^2} + \overline{T_1^6}$. Thus, we have to show

$$\begin{aligned}\lim_{n\to\infty} \int_0^T \psi \int_\Omega \vartheta \rho_n w_n \cdot \left(\nabla \mathcal{A}_i[\xi b(\rho_n) u_n^i] - \nabla \mathcal{A}[\xi b(\rho_n)] u_n\right) \mathrm{d}x \, \mathrm{d}t = \\ \int_0^T \psi \int_\Omega \vartheta \rho w \cdot \left(\nabla \mathcal{A}_i[\xi \overline{b(\rho)} u^i] - \nabla \mathcal{A}[\xi \overline{b(\rho)}] u\right) \mathrm{d}x \, \mathrm{d}t\end{aligned} \tag{9.46}$$

We can use Lemma 2.9 to deduce that

$$W_n := \nabla \mathcal{A}_i[\xi b(\rho_n) u_n^i] - \nabla \mathcal{A}[\xi b(\rho_n)] \cdot u_n$$

is bounded in $L^2(0,T;W^{1,q}(\mathbb{R}^3,\mathbb{R}^3))$ for any $1 < q < 2$. Indeed, a direct calculation shows

$$\begin{aligned}\operatorname{div}(W_n) &= \mathcal{R}_{i,j}[\xi b(\rho_n)] \partial_j u_n^i - \xi b(\rho_n) \operatorname{div} u_n, \\ \operatorname{curl}(W_n) &= 0.\end{aligned}$$

By the continuity of the Riesz operator $\mathcal{R}_{i,j}$ we have $\mathcal{R}_{i,j}[\xi b(\rho_n)]$ bounded in $L^s(\mathbb{R})$ for all $1 < s < \infty$ uniformly in $t \in (0,T)$. Consequently, $\mathcal{R}_{i,j}[\xi b(\rho_n)] \partial_j u_n^i$ is bounded in $L^q(\mathbb{R})$ uniformly in $t \in (0,T)$ for $1 < q < 2$. Since $\xi b(\rho_n) \operatorname{div} u_n$ is bounded in $L^2(0,T;L^q(\mathbb{R}))$ for $1 < q < 2$, Lemma 2.9 yields that W_n is bounded in $L^2(0,T;W^{1,q}(\mathbb{R}^3,\mathbb{R}^3))$. Thus, we have

$$W_n \rightharpoonup \nabla \mathcal{A}_i[\xi \overline{b(\rho)} u^i] - \nabla \mathcal{A}[\xi \overline{b(\rho)}] u \text{ in } L^2(0,\mathrm{T};W^{1,q}(\mathbb{R}^3,\mathbb{R}^3)),$$

since by continuity of the double Riesz transform this convergence holds in $L^q_w((0,T) \times \Omega)$ for any $1 \leqslant q < 2$. This, together with (9.9), shows (9.46).

Step 6. To sum up the above results, we have proved

$$\sum_{i=1}^4 T_i^1 + \sum_{i=1}^4 T_i^2 + \sum_{i=1}^2 T_i^3 + T_2^4 - \sum_{\substack{i=1\\ i\neq 2}}^5 T_i^5 - \sum_{i=1}^2 T_i^6 \to \sum_{i=1}^4 \overline{T_i^1} + \sum_{i=1}^4 \overline{T_i^2} + \sum_{i=1}^2 \overline{T_i^3} + \overline{T_2^4} + \sum_{\substack{i=1\\ i\neq 2}}^5 \overline{T_i^5} + \sum_{i=1}^2 \overline{T_i^6}$$

and, equations (9.43) and (9.44) in mind, we deduce

$$T_1^4 - T_2^5 \to \overline{T_1^4} - \overline{T_2^5} \text{ as } n \to \infty.$$

We have proved Theorem 9.7. □

9.4 The renormalized solution

The goal of this subsection is to show the following result:

Proposition 9.12. *The limit functions u and ρ solve the renormalized equation of continuity (8.16).*

Since Proposition 9.1 states that

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) &= 0 \\ u|_{\partial\Omega} &= 0 \\ \rho(0) &= \rho_0 \end{aligned}$$

in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$, our task is to show

$$\begin{aligned} &\int_0^T \int_\Omega b(\rho)\frac{\partial \varphi}{\partial t} + b(\rho)u\cdot\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\ = &\int_0^T \int_\Omega \big(b'(\rho)\rho - b(\rho)\big)\operatorname{div} u\ \varphi\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega b(\rho_0)\varphi(0,\cdot)\,\mathrm{d}x \end{aligned} \tag{9.47}$$

for all $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$ and all b satisfying (8.10).

To show this, we introduce the following family of cut-off functions

$$\Gamma_k(z) := k\Gamma(\frac{z}{k}) \tag{9.48}$$

where $\Gamma \in C^\infty(\mathbb{R})$ is an arbitrary concave function satisfying

$$\Gamma(z) = z \text{ for } z \leqslant 1 \text{and } \Gamma(z) = 2 \text{ for } z \geqslant 3.$$

We can use Proposition 9.7 to prove the following Lemma:

Lemma 9.13. *There is a constant c independent of k such that*

$$\limsup_{n\to\infty} \|\Gamma_k(\rho_n) - \Gamma_k(\rho)\|_{L^{\gamma+1}((0,T)\times\Omega)} \leqslant c$$

for all $k \in \mathbb{N}$.

Proof. Since $\Gamma(z) \leqslant z$, we have

$$c_1(p(z) - p(y))(\Gamma_k(z) - \Gamma_k(y)) + c_2 \geqslant |\Gamma_k(z) - \Gamma_k(y)|^{\gamma+1}$$

for all $y, z \geqslant 0$ and some constants $c_1, c_2 > 0$ independent of y, z and k. Furthermore, since $z \mapsto p(z)$ is convex and $z \mapsto \Gamma_k(z)$ is concave, we have

$$\int_0^T \int_\Omega (\overline{p(\rho)} - p(\rho))(\Gamma_k(\rho) - \overline{\Gamma_k(\rho)})\,\mathrm{d}x\,\mathrm{d}t \geqslant 0.$$

Therefore, we get

$$\begin{aligned}
&\limsup_{n\to\infty} \|\Gamma_k(\rho_n) - \Gamma_k(\rho)\|_{L^{\gamma+1}((0,T)\times\Omega)}^{\gamma+1} \\
&\leqslant c_1 \limsup_{n\to\infty} \int_0^T \int_\Omega (p(\rho_n) - p(\rho))(\Gamma_k(\rho_n) - \Gamma_k(\rho))\,\mathrm{d}x\,\mathrm{d}t + c_2 + c_1 \int_0^T \int_\Omega (\overline{p(\rho)} - p(\rho))(\Gamma_k(\rho) - \overline{\Gamma_k(\rho)})\,\mathrm{d}x\,\mathrm{d}t \\
&= c_1 \lim_{n\to\infty} \int_0^T \int_\Omega p(\rho_n)\Gamma_k(\rho_n) - \overline{p(\rho)}\,\overline{\Gamma_k(\rho)}\,\mathrm{d}x\,\mathrm{d}t + c_2 \\
&\leqslant c_1 \lim_{n\to\infty} \int_0^T \int_\Omega p_n(\rho_n)\Gamma_k(\rho_n) - \overline{p(\rho)}\,\overline{\Gamma_k(\rho)}\,\mathrm{d}x\,\mathrm{d}t + c_2.
\end{aligned}$$

By virtue of Proposition 9.7, we obtain

$$\begin{aligned}
&c_1 \lim_{n\to\infty} \int_0^T \int_\Omega p_n(\rho_n)\Gamma_k(\rho_n) - \overline{p(\rho)}\,\overline{\Gamma_k(\rho)}\,\mathrm{d}x\,\mathrm{d}t + c_2 \\
&= c_1(\frac{4}{3}\nu + \eta) \lim_{n\to\infty} \int_0^T \int_\Omega \operatorname{div} u_n \Gamma_k(\rho_n) - \operatorname{div} u \overline{\Gamma_k(\rho)}\,\mathrm{d}x\,\mathrm{d}t + c_2 \\
&= c_1(\frac{4}{3}\nu + \eta) \lim_{n\to\infty} \int_0^T \int_\Omega (\Gamma_k(\rho_n) - \Gamma_k(\rho) + \Gamma_k(\rho) - \overline{\Gamma_k(\rho)}) \operatorname{div} u_n\,\mathrm{d}x\,\mathrm{d}t + c_2 \\
&\leqslant 2c_1(\frac{4}{3}\nu + \eta) \sup_{n\in\mathbb{N}} \|\operatorname{div} u_n\|_{L^2((0,T)\times\Omega)} \limsup_{n\to\infty} \|\Gamma_k(\rho_n) - \Gamma_k(\rho)\|_{L^2((0,T)\times\Omega)} + c_2.
\end{aligned}$$

Since $\sup_{n\in\mathbb{N}} \|\operatorname{div} u_n\|_{L^2((0,T)\times\Omega)} \leqslant c$ and $\|\cdot\|_{L^2((0,T)\times\Omega)} \leqslant c\|\cdot\|_{L^{\gamma+1}((0,T)\times\Omega)}$, this yields the desired conclusion. □

Furthermore, we can show strong convergence of the sequence $\overline{\Gamma_k(\rho)}$:

Lemma 9.14. *We have*

$$\|\Gamma_k(\rho_n) - \rho_n\|_{L^q((0,T)\times\Omega)} \leqslant c \cdot k^{q-\gamma}$$

for any $q \geqslant 1$ and any fixed $k \in \mathbb{N}$. Moreover, we have

$$\overline{\Gamma_k(\rho)} \to \rho \ \text{ in } L^p((0,T)\times\Omega), \ \text{ as } k \to \infty$$

for all $1 \leqslant p < \gamma$.

Proof. Since (9.38) implies $\Gamma_k(\rho_n) \rightharpoonup \overline{\Gamma_k(\rho)}$ in $L^p((0,T)\times\Omega)$, we have

$$\|\overline{\Gamma_k(\rho)} - \rho\|_{L^p((0,T)\times\Omega)} \leqslant \liminf_{n\to\infty} \|\Gamma_k(\rho_n) - \rho_n\|_{L^p((0,T)\times\Omega)},$$

and since $\Gamma_k(z) = z$ for all $z \leqslant k$ and $z \leqslant \Gamma_k(z) \leqslant 2z$ for all $z \geqslant k$, we have the following estimates:

$$\begin{aligned}
\|\Gamma_k(\rho_n) - \rho_n\|^p_{L^p((0,T)\times\Omega)} &\leqslant k^p \int_0^T \int_\Omega |\frac{1}{k}\rho_n|^p \mathbb{I}_{[\frac{1}{k}\rho_n \geqslant 1]} \,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant k^p \int_0^T \int_\Omega |\frac{1}{k}\rho_n|^\gamma \mathbb{I}_{[\frac{1}{k}\rho_n \geqslant 1]} \,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant k^{p-\gamma} \|\rho_n\|^\gamma_{L^\gamma((0,T)\times\Omega)}. \\
&\leqslant k^{p-\gamma} \cdot c
\end{aligned}$$

for some c independent of k. This show the first assertion. If $\gamma > p$, then the right hand side tends to zero uniformly in n as $k \to \infty$, and this shows the second assertion.

□

Proof of Proposition 9.12. By Lemma 9.10 we have

$$\partial_t \overline{\Gamma_k(\rho)} + \operatorname{div}(\overline{\Gamma_k(\rho)} u) + \overline{(\Gamma_k'(\rho)\rho - \Gamma_k(\rho)) \operatorname{div} u} = 0 \text{ in } \mathcal{D}'([0,T)\times\mathbb{R}^3)$$

Let $(\xi_\epsilon)_{\epsilon\in(0,1]}$ be a smoothing sequence. By Lemma 2.29 we get

$$\partial_t \xi_\epsilon * \overline{\Gamma_k(\rho)} + \operatorname{div}(\xi_\epsilon * \overline{\Gamma_k(\rho)}\, u) + \xi_\epsilon * (\overline{(\Gamma_k'(\rho)\rho - \Gamma_k(\rho)) \operatorname{div} u}) = r_\epsilon^k \text{ in } \mathcal{D}'([0,T)\times\mathbb{R}^3) \tag{9.49}$$

where for any fixed k we have $r_\epsilon^k \to 0$ in $L^2(0,T;L^q(\Omega))$ for any $q < 2$ as $\epsilon \to 0$. Now, fix any function b satisfying (8.10). Since $\xi_\epsilon * \overline{\Gamma_k(\rho)}(t) \in C_b^\infty(\mathbb{R}^3)$ for all $t \in [0,T]$ and $u \in L^2(0,T;W_0^{1,2}(\Omega))$, we have $\operatorname{div}(\xi_\epsilon * \overline{\Gamma_k(\rho)}\, u) \in L^2((0,T)\times\Omega)$. The terms $\xi_\epsilon * (\overline{(\Gamma_k'(\rho)\rho - \Gamma_k(\rho)) \operatorname{div} u})$ and r_ϵ^k are obviously integrable functions on $(0,T)\times\Omega$. Thus, $\partial_t \xi_\epsilon * \overline{\Gamma_k(\rho)} \in L^1((0,T)\times\Omega)$, and therefore $\xi_\epsilon * \overline{\Gamma_k(\rho)} \in W^{1,1}((0,T)\times\Omega)$. Consequently, equation (9.49) holds in the sense of weak derivatives and in particular we are allowed to multiply the above equation with $b'(\xi_\epsilon * \overline{\Gamma_k(\rho)})$. A straight forward calculation yields

$$\begin{gathered}
\partial_t b(\xi_\epsilon * \overline{\Gamma_k(\rho)}) + \operatorname{div}(b(\xi_\epsilon * \overline{\Gamma_k(\rho)})u) + \Big(b'(\xi_\epsilon * \overline{\Gamma_k(\rho)})(\xi_\epsilon * \overline{\Gamma_k(\rho)}) - b(\xi_\epsilon * \overline{\Gamma_k(\rho)})\Big) \operatorname{div} u = \\
- b'(\xi_\epsilon * \overline{\Gamma_k(\rho)})(\xi_\epsilon * \overline{(\Gamma_k'(\rho)\rho - \Gamma_k(\rho)) \operatorname{div} u}) + b'(\xi_\epsilon * \overline{\Gamma_k(\rho)}) r_\epsilon^k
\end{gathered} \tag{9.50}$$

in $\mathcal{D}'([0,T)\times\mathbb{R}^3)$ and by letting $\epsilon \to 0$ we get

$$\begin{gathered}
\partial_t b(\overline{\Gamma_k(\rho)}) + \operatorname{div}(b(\overline{\Gamma_k(\rho)})u) + \big(b'(\overline{\Gamma_k(\rho)})\overline{\Gamma_k(\rho)} - b(\overline{\Gamma_k(\rho)})\big) \operatorname{div} u = \\
b'(\overline{\Gamma_k(\rho)})\overline{(\Gamma_k'(\rho)\rho - \Gamma_k(\rho)) \operatorname{div} u})
\end{gathered} \tag{9.51}$$

in $\mathcal{D}'([0,T)\times\mathbb{R}^3)$.

By the strong convergence of $\overline{\Gamma_k(\rho)}$ in $L^p((0,T)\times\Omega)$, the left hand side of (9.51) tends to

$$\partial_t b(\rho) + \operatorname{div}(b(\rho)u) + \big(b'(\rho)\rho - b(\rho)\big)\operatorname{div} u$$

in $\mathcal{D}'([0,T)\times\mathbb{R}^3)$ as $k\to\infty$. Thus, relation (9.51) yields the desired conclusion, provided we can show

$$b'(\overline{\Gamma_k(\rho)})\overline{(\Gamma_k'(\rho)\rho - \Gamma_k(\rho))\operatorname{div} u)} \to 0 \text{ in } L^1((0,T)\times\Omega) \tag{9.52}$$

as $k\to\infty$.

Let $M\geqslant 0$ be such that $b'(z)=0$ for all $z\geqslant M$. Denote

$$Q_k := \big[\overline{T_k(\rho)} \leqslant M\big] \subseteq (0,T)\times\Omega.$$

Then we have

$$\begin{aligned}\int_0^T\int_\Omega &|b'(\overline{\Gamma_k(\rho)})\overline{(\Gamma_k'(\rho)\rho - \Gamma_k(\rho))\operatorname{div} u)}|\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant \|b'\|_{L^\infty(0,\infty)}\int\int_{Q_k}|\overline{(\Gamma_k'(\rho)\rho - \Gamma_k(\rho))\operatorname{div} u}|\,\mathrm{d}x\,\mathrm{d}t \qquad (9.53)\\ &\leqslant \|b'\|_{L^\infty(0,\infty)}\sup_{n\in\mathbb{N}}\|u_n\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))}\liminf_{n\to\infty}\|\Gamma_k'(\rho_n)\rho_n - \Gamma_k(\rho_n)\|_{L^2(Q_k)}.\end{aligned}$$

Corollary 3.30 implies that

$$\begin{aligned}&\|\Gamma_k'(\rho_n)\rho_n - \Gamma_k(\rho_n)\|_{L^2(Q_k)}.\\ &\leqslant \|\Gamma_k'(\rho_n)\rho_n - \Gamma_k(\rho_n)\|^{\alpha}_{L^1(Q_k)}.\|\Gamma_k'(\rho_n)\rho_n - \Gamma_k(\rho_n)\|^{(1-\alpha)}_{L^{\gamma+1}(Q_k)}\end{aligned}$$

where $\alpha = \frac{\gamma-1}{2\gamma}$.

We can estimate the right hand side of this inequality in the following way: On the one hand, we have

$$\begin{aligned}\liminf_{n\to\infty}\|\Gamma_k'(\rho_n)\rho_n - \Gamma_k(\rho_n)\|_{L^1(Q_k)}. &\leqslant \liminf_{n\to\infty}\big(\|(\Gamma_k'(\rho_n)-1)\rho_n\|_{L^1(Q_k)} + \|\rho_n - \Gamma_k(\rho_n)\|_{L^1(Q_k)}\big) \qquad (9.54)\\ &\leqslant \liminf_{n\to\infty}\big(\|\Gamma_k'(\rho_n)-1\|_{L^{\gamma'}(Q_k)}\|\rho_n\|_{L^\gamma(Q_k)} + \|\rho_n - \Gamma_k(\rho_n)\|_{L^1(Q_k)}\big)\\ &\leqslant \liminf_{n\to\infty}(\|\rho_n\|_{L^\gamma((0,T)\times\Omega)}+1)k^{1-\gamma}\\ &\leqslant c\cdot k^{1-\gamma}\end{aligned}$$

where the estimate

$$\|\Gamma_k'(\rho_n)-1\|_{L^{\gamma'}(Q_k)} \leqslant k^{1-\gamma}$$

can be proved exactly as the estimate $\|\rho_n - \Gamma_k(\rho_n)\|_{L^1(Q_k)} \leqslant k^{1-\gamma}$ in Lemma 9.14.

On the other hand, since $\Gamma_k'(z)z \leqslant \Gamma_k(z)$, we have

$$
\begin{aligned}
&\|\Gamma_k'(\rho_n)\rho_n - \Gamma_k(\rho_n)\|_{L^{\gamma+1}(Q_k)} \\
&\leqslant \|\Gamma_k(\rho_n)\|_{L^{\gamma+1}(Q_k)} \\
&\leqslant \|\Gamma_k(\rho_n) - \Gamma_k(\rho)\|_{L^{\gamma+1}(Q_k)} + \|\Gamma_k(\rho) - \overline{\Gamma_k(\rho)}\|_{L^{\gamma+1}(Q_k)} + \|\overline{\Gamma_k(\rho)}\|_{L^{\gamma+1}(Q_k)} \\
&\leqslant \|\Gamma_k(\rho_n) - \Gamma_k(\rho)\|_{L^{\gamma+1}(Q_k)} + \|\Gamma_k(\rho) - \overline{\Gamma_k(\rho)}\|_{L^{\gamma+1}(Q_k)} + M \mathrm{meas}(\Omega)^{\frac{1}{\gamma+1}},
\end{aligned}
$$

where the last inequality follows from the Definition of Q_k. By Lemma 9.13, we get

$$
\liminf_{n\to\infty} \|\Gamma_k(\rho_n) - \Gamma_k(\rho)\|_{L^{\gamma+1}(Q_k)} + \|\Gamma_k(\rho) - \overline{\Gamma_k(\rho)}\|_{L^{\gamma+1}(Q_k)} \leqslant 2 \limsup_{n\to\infty} \|\Gamma_k(\rho_n) - \Gamma_k(\rho)\|_{L^{\gamma+1}(Q_k)} \leqslant c
$$

Thus, we have

$$
\liminf_{n\to\infty} \|\Gamma_k'(\rho_n)\rho_n - \Gamma_k(\rho_n)\|_{L^{\gamma+1}(Q_k)} \leqslant c \tag{9.55}
$$

where c is independent of k.

Relations (9.54) and (9.55) together with (9.53) imply (9.52). This completes the proof of Proposition 9.12. □

9.5 Strong convergence of the density

In this section we can finally show the following crucial result:

Proposition 9.15. *For a suitable subsequence, we have*

$$
\rho_n \to \rho \textit{ in } L^1((0,T)\times\Omega)
$$

for $n \to \infty$.

Proof. We begin by introducing a family of functions

$$
L_k(z) := \begin{cases} z\log(z) & \text{for } 0 \leqslant z \leqslant k \\ z\log(k) + z\displaystyle\int_k^z \frac{\Gamma_k(s)}{s^2}\,\mathrm{d}s & \text{for } z \geqslant k \end{cases}
$$

where the functions $\Gamma_k(z) = k\Gamma(\frac{z}{k})$ are defined in (9.48). Then, we have $L_k \in C^\infty([0,\infty))$ and L_k is convex, since

$$
L_k''(z) = \frac{1}{z} > 0, \text{ for } 0 \leqslant z \leqslant k, \text{ and } L_k''(z) = \frac{\Gamma_k'(z)}{z} > 0, \text{ for } k \leqslant z.
$$

Furthermore, since $L_k'(z) = \log(k) + \frac{\Gamma_k(3k)}{3k}$ for all $z > 3k$, we can write L_k in the form

$$
L_k(z) = \beta_k z + b_k(z) \tag{9.56}
$$

where $\beta_k \in \mathbb{R}$ and b_k satisfies (8.10). Finally, since $0 \leqslant \Gamma_k(s) \leqslant s$, it follows directly from the Definition that $\exp(-1) \leqslant L_k(z) \leqslant z\log(z)$.

Define the double sequence

$$F_{n,k} := L_k(\rho_n) - L_k(\rho)$$

for $n, k \in \mathbb{N}$.

Step 1. $\lim_{n\to\infty} F_{n,k}$. By virtue of (9.3) and (9.38) we have

$$\lim_{n\to\infty} L_k(\rho_n) \to \overline{L_k(\rho)} \text{ in } C_w([0,T]; L^\gamma(\Omega)). \tag{9.57}$$

Thus, we have

$$\lim_{n\to\infty} F_{n,k} = \overline{L_k(\rho)} - L_k(\rho)$$

Step 2. $\lim_{k\to\infty}\lim_{n\to\infty} F_{n,k}$. Since (ρ_n, u_n) and (ρ, u) are renormalized solutions, we have

$$\begin{aligned}\partial_t L_k(\rho_n) + \operatorname{div}(L_k(\rho_n)u_n) + \Gamma_k(\rho_n)\operatorname{div} u_n &= 0\\ \partial_t L_k(\rho) + \operatorname{div}(L_k(\rho)u) + \Gamma_k(\rho)\operatorname{div} u &= 0\end{aligned}$$

in $\mathcal{D}'([0,T) \times \mathbb{R}^3)$.

Passing to the limit $n \to \infty$ in the difference of the weak formulations of these two equations, we obtain

$$\begin{aligned}&\int_\Omega \left(\overline{L_k(\rho)} - L_k(\rho)\right)(t)\varphi \,\mathrm{d}x\\ &= \int_0^t\int_\Omega \left(\overline{L_k(\rho)u} - L_k(\rho)u\right)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^t\int_\Omega\left(\Gamma_k(\rho)\operatorname{div} u - \overline{\Gamma_k(\rho)\operatorname{div} u}\right)\varphi\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for all $\varphi \in \mathcal{D}(\overline{\Omega})$ and all $t \in [0,T]$, where $\overline{L_k(\rho)u}$ and $\overline{\Gamma_k(\rho)\operatorname{div} u}$ denote the weak limits as in Proposition 9.7. The particular choice of $\varphi \equiv 1$ yields

$$\begin{aligned}&\int_\Omega (\overline{L_k(\rho)} - L_k(\rho))(t)\,\mathrm{d}x\\ &= \int_0^t\int_\Omega \Gamma_k(\rho)\operatorname{div} u\,\mathrm{d}x\,\mathrm{d}t - \int_0^t\int_\Omega \overline{\Gamma_k(\rho)\operatorname{div} u}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

By Proposition 9.7 we get

$$\begin{aligned}&(\frac{4}{3}\nu+\eta)\int_0^t\int_\Omega \overline{\Gamma_k(\rho)\operatorname{div} u}\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^t\int_\Omega \overline{\Gamma_k(\rho)p(\rho)} - \overline{\Gamma_k(\rho)}\;\overline{p(\rho)}\,\mathrm{d}x\,\mathrm{d}t + (\frac{4}{3}\nu+\eta)\int_0^t\int_\Omega \overline{\Gamma_k(\rho)}\;\operatorname{div} u\,\mathrm{d}x\,\mathrm{d}t\\ &\geqslant (\frac{4}{3}\nu+\eta)\int_0^t\int_\Omega \overline{\Gamma_k(\rho)}\;\operatorname{div} u\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where the last inequality holds, since

$$\begin{aligned}\int_0^t \int_\Omega \overline{\Gamma_k(\rho)p(\rho)} - \overline{\Gamma_k(\rho)}\ \overline{p(\rho)}\,\mathrm{d}x\,\mathrm{d}t &= \lim_{n\to\infty}\int_0^t\int_\Omega (p_n(\rho_n) - \overline{p(\rho)})\Gamma_k(\rho_n)\,\mathrm{d}x\,\mathrm{d}t \\ &\geqslant \lim_{n\to\infty}\int_0^t\int_\Omega (p(\rho_n) - \overline{p(\rho)})\Gamma_k(\rho_n)\,\mathrm{d}x\,\mathrm{d}t \\ &\geqslant 0\end{aligned}$$

Therefore, we have

$$\int_\Omega (\overline{L_k(\rho)} - L_k(\rho))(t)\,\mathrm{d}x \leqslant \int_0^t\int_\Omega (\Gamma_k(\rho) - \overline{\Gamma_k(\rho)})\,\mathrm{div}\,u\,\mathrm{d}x\,\mathrm{d}t. \tag{9.58}$$

By Definition of Γ_k we have $\Gamma_k(\rho) \to \rho$ point wise in $(0,T)\times\Omega$ and by Lemma 9.14 we have $\overline{\Gamma_k(\rho)} \to \rho$ point wise in $(0,T)\times\Omega$, where the latter holds at least for some subsequence. Since by Lemma 9.13 we have

$$\|\Gamma_k(\rho) - \overline{\Gamma_k(\rho)}\|_{L^{\gamma+1}((0,T)\times\Omega)} \leqslant c,$$

and since $\gamma+1>2$, Lemma 3.31 yields $\Gamma_k(\rho) - \overline{\Gamma_k(\rho)} \to 0$ in $L^2((0,T)\times\Omega)$, and thus, the right hand side of (9.58) tends to zero as $k\to\infty$.

On the other hand, since L_k are convex functions, we have

$$\int_\Omega (\overline{L_k(\rho)} - L_k(\rho))(t)\,\mathrm{d}x \geqslant 0$$

and consequently

$$(\overline{L_k(\rho)} - L_k(\rho))(t) \to 0 \ \text{ in } L^1(\Omega) \ \text{ as } k\to\infty \tag{9.59}$$

for all $t\in[0,T]$. Thus, we have proved

$$\lim_{k\to\infty}\lim_{n\to\infty} F_{n,k}(t) = 0 \ \text{ in } L^1(\Omega)$$

□

Step 3. Convergence $\lim_{k\to\infty} F_{n,k}$. Since $L_k(z)\to z\log(z)$ for all $z\geqslant 0$, we have $F_{n,k}\to \rho_n\log(\rho_n) - \rho\log(\rho)$ point wise in $(0,T)\times\Omega$ for any fixed n as $k\to\infty$. Furthermore, we have $\big(\rho_n\log(\rho_n)\big)(t)$ bounded in $L^\alpha(\Omega)$ for all $1\leqslant\alpha<\gamma$ uniformly in n and *a.e.* $t\in(0,T)$, and consequently, $L_k(\rho_n)(t)$ is bounded in $L^\alpha(\Omega)$ uniformly in k, n and *a.e.* $t\in(0,T)$. In particular, Lebesgue's theorem yields

$$\lim_{k\to\infty} F_{n,k} = (\rho_n\log(\rho_n) - \rho\log(\rho))(t) \ \text{ in } L^\alpha(\Omega)$$

for any $1\leqslant\alpha<\gamma$.

Step 4. Convergence $\lim_{n\to\infty}\lim_{k\to\infty} F_{n,k}$. Since $\rho_n\log(\rho_n) - \rho\log(\rho)$ is uniformly bounded in $L^\alpha(\Omega)$

for any $1 \leqslant \alpha < \gamma$, we have

$$\lim_{n\to\infty}\lim_{k\to\infty} F_{n,k}(t) = \big(\overline{\rho\log(\rho)} - \rho\log(\rho)\big)(t) \text{ weakly in } L^\alpha(\Omega)$$

for some function $\overline{\rho\log(\rho)} \in L^\alpha(\Omega)$, passing to a subsequence if necessary.

Step 5. Our goal is to show $\overline{\rho\log(\rho)} = \rho\log(\rho)$. To show this, we can use Lemma 1.13 on the Banach space $L^1(\Omega)$. Thus, we have to show that

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}} \|F_{n,k} - \rho_n\log(\rho_n) - \rho\log(\rho)\|_{L^1(\Omega)}(t) = 0. \tag{9.60}$$

Since

$$z \mapsto |L_k(z) - z\log(z)| = z\log(z) - L_k(z)$$

is convex, we have

$$\|L_k(\rho_n) - \rho_n\log\rho_n\|_{L^1(\Omega)} \geqslant \|L_k(\rho) - \rho\log\rho\|_{L^1(\Omega)}.$$

Thus, it is sufficient to show

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}} \|L_k(\rho_n) - \rho_n\log(\rho_n)\|_{L^1(\Omega)}(t) = 0. \tag{9.61}$$

Let

$$P_k^n := \big[\rho_n \geqslant k\big].$$

Let $1 < \beta < \gamma$ and let $z_0 \geqslant 1$ be such that $\log(z) \leqslant z^{\beta-1}$ for all $z \geqslant z_0$ and let $k \geqslant z_0$. Since $L_k(z) = z\log z$ for all $z \leqslant k$ and $L_k(z) \leqslant z\log z$ for all $z > k$, we have

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \|L_k(\rho_n) - \rho_n\log(\rho_n)\|_{L^1(\Omega)}(t) &\leqslant \sup_{n\in\mathbb{N}} \int_{P_k^n} \rho_n\log\rho_n \,\mathrm{d}x, \\
&\leqslant \sup_{n\in\mathbb{N}} \int_{P_k^n} \rho_n^\beta \,\mathrm{d}x \\
&\leqslant \sup_{n\in\mathbb{N}} k^{\beta-\gamma} \int_\Omega \rho_n^\gamma \,\mathrm{d}x \\
&\leqslant c\, k^{\beta-\gamma}
\end{aligned}$$

where $c > 0$ is independent of n, and this shows the desired conclusion. Thus, by virtue of Lemma 1.13 we get $\overline{\rho\log(\rho)} = \rho\log(\rho)$.

Since $z \mapsto z\log(z)$ is strictly convex, Lemma 1.16 implies

$$\rho_n(t) \to \rho(t) \qquad \text{in } L^1(\Omega)$$

for all $t \in (0,T)$, and since $\rho_n(t)$ is bounded in $L^1(\Omega)$ uniformly in n and t, Lebesgue's theorem yields

$$\rho_n \to \rho \qquad \text{in } L^1((0,T)\times\Omega).$$

9.6 Conclusion

We can finish the proof of Theorem 9.1. To show $\rho \in C([0,T];L^1(\Omega))$, choose a smoothing sequence $(\xi_\epsilon)_{\epsilon\in(0,1]}$ and define $\rho_\epsilon := \xi_\epsilon * \rho$. Then ρ_ϵ is smooth in the space variables for all ϵ. By Lemma 2.29 we have

$$\frac{\partial \tilde{\rho}_\epsilon}{\partial t} + \operatorname{div}(\tilde{\rho}_\epsilon u) = r_\epsilon \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^3)$$

with $r_\epsilon \to 0$ in $L^1((0,T)\times\Omega)$. Now, we have for $\epsilon, \delta \in (0,1]$

$$\frac{\partial}{\partial t}|\rho_\epsilon - \rho_\delta| + \operatorname{div}(u|\rho_\epsilon - \rho_\delta|) = \operatorname{sgn}(\rho_\epsilon - \rho_\delta)(r_\epsilon - r_\delta) \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^3)$$

and by using $\varphi = \varphi(t) \in C_c^\infty(0,T)$ as test function we deduce

$$\frac{\partial}{\partial t}\int_\Omega |\rho_\epsilon - \rho_\delta| \,\mathrm{d}x = \int_\Omega \operatorname{sgn}(\rho_\epsilon - \rho_\delta)(r_\epsilon - r_\delta)\,\mathrm{d}x \quad \text{in } \mathcal{D}'(0,T).$$

Since $\rho_\epsilon(0) = \rho_\delta(0)$ we get

$$\sup_{t\in[0,T]}\int_\Omega |\rho_\epsilon - \rho_\delta|\,\mathrm{d}x \leqslant \int_0^T\int_\Omega |r_\epsilon - r_\delta|\,\mathrm{d}x\,\mathrm{d}t,$$

where the right hand side tends to 0 as $\epsilon, \delta \to \infty$ and consequently $(\rho_\epsilon)_{\epsilon\in(0,1]}$ is a fundamental sequence in $C([0,T];L^1(\Omega))$. Since $\rho_\epsilon \to \rho$ in $L^1((0,T)\times\Omega)$ we have $\rho_\epsilon \to \rho$ in $C([0,T];L^1(\Omega))$ and in particular we have $\rho \in C([0,T];L^1(\Omega))$.

In view of Lemma 9.2, we have proven that the limit functions (ρ, u) satisfy (1)-(3) from Definition 8.8.

By Lemma 9.12, the limit functions satisfy (4) from Definition 8.8.

Passing to a subsequence for the last time, we can assume $\rho_n \to \rho$ *a.e.* in $(0,T)\times\Omega$. Consequently, we have $p(\rho_n) \to p(\rho)$ in $L^1(\Omega)$ and consequently $\overline{p(\rho)} = p(\rho)$. The limit equation (9.34) then gives rise to (5) from Definition 8.8.

Finally, since $Q_n \to Q$ locally uniformly on $[0,\infty)$ and $\rho_n \to \rho$ a.e. in $(0,T)\times\Omega$, we have $Q_n(\rho_n) \to Q(\rho)$ a.e. $(0,T)\times\Omega$. Since by Lemma 8.6 and Proposition 9.3, we have $Q_n(\rho_n)$ bounded in $L^q((0,T)\times\Omega)$, Theorem 3.31 yields $Q_n(\rho_n) \to Q(\rho)$ in $L^r((0,T)\times\Omega)$ for all $r < q$. Now, it is routine to check that all terms in the weak formulation of the energy inequality converge to their respective counterparts.

Theorem 9.1 has been proved.

10 Equations driven by stochastic force

10.1 General results

We return to the stochastic Navier-Stokes equations. Recall that the initial data (ρ_0, q) and the noise w are random variables on some regular topological probability space $(\mathcal{O}, \mathcal{B}, \mathbb{P})$. Assume that the initial data and the noise are of the form

$$\begin{aligned}(\rho_0, q) &: \mathcal{O} \to \mathcal{C}\\ w &: \mathcal{O} \to \mathcal{W}\end{aligned}$$

for some spaces $\mathcal{C}$ and $\mathcal{W}$ and the solution (ρ, u) is of the form

$$(\rho, u) : \mathcal{O} \to \mathcal{X}$$

for a suitable space $\mathcal{X}$. We start by giving an explicit description of those function spaces in such a way that on the one hand Theorem 9.1 yields the existence of solutions $u(.,.,\omega)$ for a.e. $\omega \in \mathcal{O}$ and consequently a mapping (see proof below)

$$\mathcal{M} : \mathcal{C} \times \mathcal{W} \to 2^{\mathcal{X}},$$

where $2^{\mathcal{X}}$ denotes the power set of $\mathcal{X}$, mapping (deterministic) initial data and (deterministic) noise to the non-empty set of solutions. On the other hand, we choose the spaces $\mathcal{C}$, $\mathcal{W}$ and $\mathcal{X}$ in such a way that Corollary 3.33 yields a measurable selection of $\mathcal{M}$.

We motivate and define the spaces for the initial data. First observe, that by writing

$$q = \sqrt{\rho_0}\frac{q}{\sqrt{\rho_0}}$$

the conditions (8.12) imply

$$q \in L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^3).$$

Thus, in order for Theorem 9.1 to yield solutions, we need the function space for the initial value (ρ_0, q) to satisfy

$$\mathcal{C} \subseteq L^{\gamma}(\Omega) \times L^{\frac{2\gamma}{\gamma+1}}(\Omega).$$

For Corollary 3.33 to yield a measurable selection, we need the multivalued mapping $\mathcal{M}$ to have a closed graph; in particular assume a sequence of initial conditions (ρ_0^n, q^n) that converge in $\mathcal{C}$ to (ρ_0, q) and assume for each member of this sequence a solution (ρ^n, u^n). Then the sequence of solutions has to converge in $\mathcal{X}$ to a solution with the initial conditions (ρ_0, q). Recalling the first part of the proof of Lemma 9.2 (see also below) it is necessary to assume that the initial Energy is bounded, i.e. we assume there exists some constant $\tilde{E}$ such that

$$E(\rho_0, q) := \int_{\Omega} \frac{1}{2}\frac{|q|^2}{\rho_0} + Q(\rho_0)\,\mathrm{d}x \leqslant \tilde{E}$$

for all $(\rho_0, q) \in \mathcal{C}$. Finally, we assume the natural condition that the total mass does not depend on $\omega \in \mathcal{O}$, i.e. we assume there exists some constant $M > 0$ such that

$$\int_\Omega \rho_0 \,\mathrm{d}x = M$$

for all $(\rho_0, q) \in \mathcal{C}$, and, of course, the assumption that ρ_0 is non-negative. Therefore, we arrive at the following definition:

$$\mathcal{C} := \{(\rho_0, q) \in L^\gamma(\Omega) \times L^{\frac{2\gamma}{\gamma+1}}(\Omega) \,|\, (8.12) \text{ is satisfied}, \int_\Omega \rho_0 \,\mathrm{d}x = M,\ E(\rho_0, q) \leqslant \tilde{E}\}. \tag{10.1}$$

Note that $\mathcal{C}$ is a convex and closed subset of $L^\gamma(\Omega) \times L^{\frac{2\gamma}{\gamma+1}}(\Omega)$.

Next, we introduce the space $\mathcal{W}$ for the noise w. In accordance to Theorem 9.1, we assume

$$\mathcal{W} \subseteq L^\infty(0, T; W_0^{1,\infty}(\Omega, \mathbb{R}^3)). \tag{10.2}$$

As shown below, it is not necessary that $\mathcal{W}$ inherits the norm of $L^\infty(0, T; W_0^{1,\infty}(\Omega, \mathbb{R}^3))$; in particular, for the multi-valued mapping $\mathcal{M}$ to be closed, it would be sufficient to assume that $\mathcal{W}$ carries a topology $\mathcal{T}$, which is finer than the weak-star topology of this space, and which has the property that convergence in $\mathcal{T}$ implies the existence of some a.e. convergent subsequence. But on the other hand, we need $\mathcal{W}$ to be a complete and separable metric space. Thus, we suppose that $\mathcal{W}$ carries a complete and separable metric $d_\mathcal{W}$ such that

$$w_n \to w \text{ in } d_w \text{ implies } \begin{cases} w_n \overset{*}{\rightharpoonup} w \text{ in } L^\infty(0, T; W_0^{1,\infty}(\Omega, \mathbb{R}^3)), \\ \exists \text{ a subsequene, such that } w_{n_k} \to w \text{ a.e. in } (0, T) \times \Omega. \end{cases} \tag{10.3}$$

Example 10.1. For a Banach space $E \neq \{0\}$ the space $L^\infty(0, T; E)$ is never separable, but the space $C([0, T]; E)$ is separable iff E is separable. Thus, if the process w is continuous with respect to the time variable, a possible choice for $\mathcal{W}$ is to start with some separable Banach space E continuously embedded into $W_0^{1,\infty}(\Omega; \mathbb{R}^3)$, say, for example, $E = W_0^{k,p}(\Omega; \mathbb{R}^3)$ where $1 < k, p < \infty$ and $k - \frac{3}{p} > 1$, and consider $C([0, T]; E)$. In the next section we shall see that one can even consider the space $D([0, T]; E)$ of cádlág functions.

Finally, we introduce the Space $\mathcal{X}$ for the solution (ρ, u). This is easy, because in accordance to Definition 8.8 we can define $\mathcal{X}$ as the Banach space

$$\mathcal{X} := C([0, T]; L^1(\Omega)) \times L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^3)). \tag{10.4}$$

We are now ready to state and prove the main result of part 2 of this theses:

Theorem 10.2. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain and let $\mathcal{C}$, $\mathcal{X}$ and $\mathcal{W}$ be the spaces defined above, and suppose $\mathcal{W}$ carries an arbitrary complete and separable metric satisfying (10.3). Suppose further that the pressure satisfies (8.7). Let*

$$(\rho_0, q) : \mathcal{O} \to \mathcal{C}$$

and

$$w : \mathcal{O} \to \mathcal{W}$$

be random variables, where all spaces carry their respective Borel algebras. Then, there exists a random variable

$$(\rho, u) : \mathcal{O} \to \mathcal{X}$$

satisfying the Navier-Stokes system in the sense of Definition 8.8.

Proof. We define the multi valued mapping

$$\mathcal{M} : \begin{cases} \mathcal{C} \times \mathcal{W} \to 2^{\mathcal{X}} \\ (\widetilde{\rho}_0, \widetilde{q}, \widetilde{w}) \mapsto \{(\widetilde{\rho}, \widetilde{u}) \in \mathcal{X} \,|\, (\widetilde{\rho}, \widetilde{u}) \text{is a solution}\} \end{cases}$$

assigning to each triple $(\widetilde{\rho}_0, \widetilde{q}, \widetilde{w}) \in \mathcal{C} \times \mathcal{W}$ the set of solution in the sense of Definition 8.8, where $2^{\mathcal{X}}$ denotes the power set. By Theorem 9.1 the set $\mathcal{M}(\widetilde{\rho}_0, \widetilde{q}, \widetilde{w})$ is non-empty for all $(\widetilde{\rho}_0, \widetilde{q}, \widetilde{w}) \in \mathcal{C} \times \mathcal{W}$. We show the existence of a measurable selection $\overline{\mathcal{M}}$ of $\mathcal{M}$, i.e. a measurable mapping

$$\overline{\mathcal{M}} : \mathcal{C} \times \mathcal{W} \to \mathcal{X}$$

such that $\overline{\mathcal{M}}(\widetilde{\rho}_0, \widetilde{q}, \widetilde{w}) \in \mathcal{M}(\widetilde{\rho}_0, \widetilde{q}, \widetilde{w})$. Corollary 3.33 yields such a measurable selection, provided we can show that $\mathcal{M}$ posses a closed graph. Therefore, let $(\rho_0^n, q^n, w^n, \rho^n, u^n)_{n\in\mathbb{N}} \subseteq \mathcal{C} \times \mathcal{W} \times \mathcal{X}$ be a sequence such that $(\rho^n, u^n) \in \mathcal{M}(\rho_0^n, q^n, w^n)$ for all n and

$$(\rho_0^n, q^n, w^n, \rho^n, u^n) \to (\widetilde{\rho}_0, \widetilde{q}, \widetilde{w}, \widetilde{\rho}, \widetilde{u}) \text{ in } \mathcal{C} \times \mathcal{W} \times \mathcal{X}.$$

Our task is to show, that $(\widetilde{\rho}, \widetilde{u})$ is a solution of the Navier-Stokes equation with initial condition $(\widetilde{\rho}_0, \widetilde{q})$ and noise $\widetilde{w}$. By passing to a subsequence, we can assume all five convergences pointwise a.e. in Ω respectively a.e. in $(0,T) \times \Omega$. Moreover, we can assume $\nabla u_n \to \nabla u$ and $\nabla w_n \to \nabla w$ a.e. in $(0,T) \times \Omega$. Our assumptions on $\mathcal{W}$, $\mathcal{C}$ and $\mathcal{X}$ are precisely what we need to repeat almost verbatim the the proof of Lemma 9.2. Indeed, there are only three differences between Lemma 9.2 and the present situation, which could cause trouble: first, in the present situation, the initial data may depend on n, secondly, w is not as regular as in Lemma 9.2, and finally, we do not have the artificial pressure term (represented through the δ quantities). But the facts that the initial data (ρ_0^n, q^n) is bounded in $L^\gamma(\Omega) \times L^{\frac{2\gamma}{\gamma+1}}(\Omega)$ and the initial energy is bounded are clearly enough to overcome the first problem, and neither the regularity of w nor the artificial pressure term have been use in the proof. In particular, we deduce that ρ^n is bounded in $L^\infty(0,T; L^\gamma(\Omega))$. Now, this is clearly enough to infer that all quantities in the weak formulation of the equation of continuity (8.16), the equation of momentum (8.17) and the energy inequality (8.18) converge to their respective counterparts. Thus, we have shown that $(\widetilde{\rho}, \widetilde{u})$ is a solution, and this yields the claimed closeness of the graph of $\mathcal{M}$ and consequently the existence of a measurable selection $\overline{\mathcal{M}}$.

Finally, if (ρ_0, q, w) is a $\mathcal{C} \times \mathcal{W}$-valued random variable, we can define the desired solution:

$$(\rho, u) := \overline{\mathcal{M}} \circ (\rho_0, q, w) \,:\, \mathcal{O} \to \mathcal{X}.$$

□

10.2 Application: Lévy processes

In this section we apply Theorem 10.2 to Lévy processes. The hardest part, i.e. the choice of $\mathcal{W}$, can be done due to the following preliminaries.

Let E be an separable Banach space. We consider the Space of cádlág functions $D([0,T];E)$. Since cádlág functions are always bounded, one may consider the uniform norm on this space, but $D([0,T];E)$ is not separable with this norm. One possibility to overcome this problem is to consider the Skorokhod metric on the space $D([0,T];E)$. The idea of this metric is as follows. Two functions $x, y \in D([0,T];E)$ are „near to each other" in the uniform metric, if the graph of x can be carried onto the graph of y by a uniformly small perturbation in the space coordinates (i.e. in E) while the time coordinate is kept fixed. The Skorokhod metric on the other hand also allows uniformly small perturbations in the time variable t. To make this idea concrete, we introduce the set Λ of strictly increasing, bijective and continuous mappings form $[0,T]$ to $[0,T]$. In particular we have $\lambda(0) = 0$ and $\lambda(T) = T$ for all $\lambda \in \Lambda$. Note that $(\Lambda, \circ)$ is a group. Now we define for $x, y \in D([0,T];E)$ the metric

$$d(x,y) := \inf\{\epsilon > 0 \,:\, \exists \lambda \in \Lambda \, \text{such that } |\lambda(t) - t| < \epsilon \text{ and } \|x(t) - y(\lambda(t))\|_E < \epsilon \text{ for all } t \in [0,T]\}.$$

One can easily check, that this defines a metric on $D([0,T];E)$.

We sketch the proof that d is a separable metric: One can consider the (countable) set of those cádlág functions of the form

$$\sum_{k=1}^{K} \xi_k \mathbb{I}_{[a_{k-1}, a_k)}$$

with $K \in \mathbb{N}$, $a_k \in \mathbb{Q}$ and $\xi_k \in F$ for some fixed countable and dense subset $F \subseteq E$. Since each $x \in D([0,T];E)$ has only finitely many jumps „higher" then δ for all $\delta > 0$, the set of those functions is dense with respect to the Skorokhod metric.

The problem is that this metric is not complete. But there is a complete metric d_0 such that d and d_0 induce the same topology . We define for $\lambda \in \Lambda$

$$L(\lambda) := \sup_{s \neq t} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right|,$$

and for $x, y \in D([0,T];E)$

$$d_0(x,y) := \inf \left\{ \epsilon > 0 \,\middle|\, \exists \lambda \in \Lambda \text{ such that } L(\lambda) < \epsilon \text{ and } \|x(t) - y(\lambda(t))\|_E < \epsilon \text{ for all } t \in [0,T] \right\}.$$

Note, that the case $L(\lambda) = \infty$ is allowed; those $\lambda \in \Lambda$ do not play a role in the definition of d_0. On the other hand, if $\lambda : [0,1] \to [0,1]$ such that $\lambda(0) = 0$ and $\lambda(1) = 1$, then $L(\lambda) < \infty$ implies $\lambda \in \Lambda$.

Lemma 10.3. *The metrics d_0 and d induce the same topology. In particular, d_0 is separable.*

Proof. Denote the induced topologies by $\mathcal{T}$ and $\mathcal{T}_0$.

Step 1. We show $\mathcal{T} \subseteq \mathcal{T}_0$. Let $d_0(x,y) < \epsilon < \frac{1}{4}$. Then we have for some $\lambda \in \Lambda$

$$\log(1-2\epsilon) < -\epsilon < \log\frac{\lambda(t)}{t} < \epsilon < \log(1+2\epsilon)$$

and consequently $|\log\frac{\lambda(t)}{T} - \log\frac{t}{T}| < 2\epsilon$. Since $\frac{\lambda(t)}{T} < 1$ and $\frac{t}{T} < 1$ and $\frac{\partial}{\partial x}\log(x) > 1$ for $x \in [0,1]$, this implies $|\lambda(t) - t| < 2\epsilon T$ and therefore

$$d(x,y) \leqslant 2T d_0(x,y)$$

if $d_0(x,y) < \frac{1}{4}$. Consequently, we have $\mathcal{T} \subseteq \mathcal{T}_0$.

Step 2. We show $\mathcal{T}_0 \subseteq \mathcal{T}$. Assume $d(x,y) < \epsilon^2 < \frac{1}{16}$. For $I \subseteq [0,T]$ and $z \in D([0,T];E)$ define

$$S_z(I) := \sup\{\|z(s) - z(t)\|_E \mid s,t \in I\}$$

and for $\delta > 0$ define

$$T_z(\delta) := \inf\left\{ \max_{1\leqslant i\leqslant r} S_z([t_{i-1},t_i)) \,\middle|\, r \in \mathbb{N},\ 0 = t_0 < ... < t_r = T \ \text{ and } \ t_i - t_{i-1} > \delta \,\text{for all } \ i\right\}.$$

We show that there is some $\lambda \in \Lambda$ such that

$$\|x(t) - y(\lambda(t))\|_E \leqslant T_x(\epsilon) + \epsilon \tag{10.5}$$

and

$$L(\lambda) \leqslant 4\epsilon. \tag{10.6}$$

for all $t \in [0,T]$. Choose $0 = t_0 < ... < t_r = T$ such that $S_x([t_{i-1},t_i)) < T_x(\epsilon) + \epsilon$ and $t_i - t_{i-1} > \epsilon$ for all $1 \leqslant i \leqslant r$, and choose $\mu \in \Lambda$ such that $\sup_{0\leqslant t\leqslant T}\|x(t) - y(\mu(t))\|_E < \epsilon^2$ and $\sup_{0\leqslant t\leqslant T}|\mu(t) - t| < \epsilon^2$. Define λ to agree with μ at t_i for all $1 \leqslant i \leqslant r$ and to be linear between them. Then $\lambda \in \Lambda$ and therefore $\mu^{-1}\circ\lambda \in \Lambda$, and since $\mu^{-1}\circ\lambda(t_i) = t_i$, we have $[t_{i-1},t_i) = (\mu^{-1}\circ\lambda)([t_{i-1},t_i))$. Thus, we get

$$\begin{aligned}\|x(t) - y(\lambda(t))\|_E &\leqslant \|x(t) - x(\mu^{-1}\circ\lambda(t))\|_E + \|x(\mu^{-1}\circ\lambda(t)) - y(\lambda(t))\|_E \\ &= \|x(t) - x(\mu^{-1}\circ\lambda(t))\|_E + \|x(\lambda(t)) - y(\mu\circ\lambda(t))\|_E \\ &\leqslant T_x(\epsilon) + \epsilon^2 < T_x(\epsilon) + \epsilon.\end{aligned}$$

and thus, (10.5) holds.

Since $\lambda(t_i) = \mu(t_i)$ we have

$$|(\lambda(t_i) - \lambda(t_{i-1})) - (t_i - t_{i-1})| < 2\epsilon^2 < 2\epsilon(t_i - t_{i-1}).$$

Because λ is linear between $t_{i-1}$and t_i this relation holds for all $s,t \in [0,1]$:

$$|(\lambda(t) - \lambda(s)) - (t-s)| < 2\epsilon(t-s),$$

and consequently

$$\log(1-2\epsilon) \leqslant \log \frac{\lambda(t)-\lambda(s)}{t-s} \leqslant \log(1+2\epsilon)$$

and since $\epsilon < \frac{1}{4}$ we deduce $L(\lambda) < 4\epsilon$, and thus (10.6) holds.

Relations (10.5) and (10.6) imply $d_0(x,y) \leqslant 4\epsilon + T_x(\epsilon)$. Now fix $x \in D([0,T];E)$ an let $B_\epsilon^{d_0}(x)$ be the open ball with center x and radius ϵ with respect to d_0. Choose $\delta < \frac{1}{4}$ such that $4\delta + T_x(\delta) < \epsilon$. Then we have $B_\delta^d(x) \subseteq B_\epsilon^{d_0}(x)$ and consequently $\mathcal{T}_0 \subseteq \mathcal{T}$. □

Lemma 10.4. *The metric d_0 is complete.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a fundamental sequence and assume without loss of generality $d_0(x_n, x_{n+1}) < 2^{-n}$. Choose $\mu_n \in \Lambda$ such that

$$\sup_t \|x_n(t) - x_{n+1}(\mu_n(t))\|_E < \frac{1}{2^n}$$

and

$$L(\mu_n) < \frac{1}{2^n}.$$

Define $\nu_n^m \in \Lambda$ by

$$\nu_n^m := \mu_{n+m+1} \circ \mu_{n+m} \circ ... \circ \mu_{n+1} \circ \mu_n.$$

Then we have

$$\begin{aligned}\|\nu_n^{m+1} - \nu_n^m\|_{C([0,T])} =& \sup_t |\mu_{n+m+1}(t) - t| = T \sup_t |\frac{\mu_{n+m+1}(t)}{T} - \frac{t}{T}| \leqslant T \sup_t |\log \frac{\mu_{n+m+1}(t)}{T} - \log \frac{t}{T}| \\ =& T \sup_t |\log \mu_{n+m+1}(t) - \log t| \leqslant T \cdot L(\mu_{n+m+1}) < \frac{T}{2^{n+m+1}}\end{aligned}$$

and therefore we have

$$\nu_n^m \to \lambda_n \text{ in } C([0,T]).$$

for some $\lambda_n \in C([0,T])$.

The limit λ_n satisfies $\lambda_n(0) = 0$ and $\lambda_n(T) = T$. Since $L(\lambda \circ \mu) \leqslant L(\lambda) + L(\mu)$ for all λ, $\mu \in \Lambda$, we have for all $s,t \in [0,T]$, $s \neq t$

$$\left| \log \frac{\nu_n^m(t) - \nu_n^m(s)}{t-s} \right| \leqslant L(\nu_n^m) \leqslant L(\mu_n) + ... + L(\mu_{n+m}) \leqslant \frac{1}{2^{n-1}}.$$

By letting $m \to \infty$ be deduce $L(\lambda_n) < \frac{1}{2^{n-1}} < \infty$ an therefore $\lambda_n \in \Lambda$. Since $\lambda_n = \lambda_{n+1} \circ \mu_n$we have

$$\sup_t \|y(\lambda_n^{-1}(t)) - y_{n+1}(\lambda_{n+1}^{-1}(t))\|_E = \sup_t \|y_n(s) - y_{n+1}(\mu_n(s))\|_E < \frac{1}{2^n}.$$

Consequently, $x_n \circ \lambda_n^{-1} \to x$ uniformly in $[0,T]$ for some $x \in D([0,T];E)$, an together with $L(\lambda_n) \to 0$ we have $d_0(x_n, x) \to 0$. $\square$

With this preliminaries we return to the problem of showing the existence of solutions to the Navier-Stokes equation with Lévy noise. In view of Theorem 10.2 we define

$$E := W_0^{k,p}(\Omega)$$

where $1 < k, p < \infty$ and

$$k - \frac{3}{p} > 1.$$

Thus, E is a separable Banach space and we have the continuous and compact embedding

$$E \hookrightarrow W_0^{1,\infty}(\Omega).$$

Now, let the space $\mathcal{W}$ be defined by

$$\mathcal{W} := D([0,T]; E)$$

equipped with the metric d_0 constructed above. One can easily check, that $\mathcal{W}$ satisfies the conditions (10.2) and (10.3).

Therefore, Theorem 10.2 yields

Corollary 10.5. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain. Let the pressure p satisfy (8.7), let the initial condition be a $\mathcal{C}$-valued random variable and let w be an E-valued Lévy process. Then the stochastic Navier-Stokes equations admit a solution (ρ, u) in the sense of Definition 8.8.*

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