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DIPLOMARBEIT

In-Hadron Condensates in an Analytical Salpeter Approach

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I hereby declare that this thesis is my original work and it has been written by me in its entirety. I have acknowledged all the sources of information which have been used in the thesis.

Master thesis (2018)

Institut für Hochenergiephysik der Österreichischen Akademie der Wissenschaften

Abstract

Bound states in a quantum field theory can be described by the Bethe-Salpeter equation [1]. This equation has been the subject of many studies and with this work, we want to build on these studies in order to investigate the pion. The later is the lightest hadron composed of a quark and an antiquark. We connect our calculations in the Bethe-Salpeter formalism to in-hadron condensates and examine the Gell-Mann-Oakes-Renner relation.

The idea of in-hadron condensates is that quark fields do not have a vacuum expectation value all over spacetime, instead their condensate is spatially confined within a hadron [2–5]. This concept could solve a problem concerning the cosmological constant [3], which describes the vacuum energy density and is—within QCD—calculated to be forty orders of magnitude greater than experimentally measured. If the quark condensate were actually constrained to exist within a hadron, then it would not be taken into account for the vacuum energy density, but rather already be included in the matter energy density.

The Bethe-Salpeter equation follows from the assumption that bound states are poles in Green's functions. These poles, which are at the energy corresponding to the rest mass of a bound state, can be directly observed as resonances in experimental high energy physics. Under certain assumptions, like instantaneous interactions and equal quark masses for the up- and down-quarks, one can simplify the Bethe-Salpeter equation to the Salpeter equation. The solution of this Salpeter equation is the Salpeter amplitude, which is the analogue to the wave function in regular quantum mechanics. Since we aim to describe the pion, we demand a certain transformation under parity and charge conjugation transformation. As a result, only certain components of the Salpeter amplitude remain and can be determined. After transforming the Salpeter equation to an eigenvalue problem, these components appear as eigenvectors and the eigenvalue is the mass of the corresponding bound state, i.e. of a pion.

In this thesis, we aim to explore in-hadron condensates in connection with the Gell-Mann-Oakes-Renner relation [6]. This relation connects the "current" quark mass as it appears in the Lagrangian to the mass of a bound state and its decay constant. Rigorously, this relation is valid in the so-called chiral limit where quarks are assumed to be massless. Furthermore, in the Gell-Mann-Oakes-Renner relation, the *vacuum* expectation value of the constituent quarks enters the equation. We would like to investigate what happens when we replace the vacuum condensate with the in-hadron condensate.

Kurzfassung

Gebundene Zustände können in einer Quantenfeldtheorie durch die Bethe-Salpeter-Gleichung [1] beschrieben werden. Diese Gleichung war Gegenstand vieler Untersuchungen und in dieser Arbeit wollen wir, auf diesen Studien aufbauend, das Pion analysieren. Dieses ist das leichteste Hadron, bestehend aus einem Quark und einem Antiquark. Wir verbinden unsere Berechnungen im Bethe-Salpeter-Formalismus zu in-Hadron-Kondensaten und untersuchen die Gell-Mann-Oakes-Renner-Relation.

Die Idee von in-Hadron-Kondensaten ist, dass Quarks keinen Vakuumerwartungswert überall in der Raumzeit haben, stattdessen ist deren Kondensat innerhalb eines Hadrons räumlich beschränkt [2–5]. Dieses Konzept könnte ein Problem bezüglich der kosmologischen Konstante lösen [3]. Die kosmologische Konstante beschreibt die Vakuumenergiedichte und ist—im Rahmen der QCD—um vierzig Größenordnungen größer als experimentell gemessen. Falls das Quarkkondensat tatsächlich innerhalb eines Hadrons eingeschlossen wäre, würde es nicht zur Berechnung der Vakuumenergiedichte beitragen, sondern zur Massenenergiedichte.

Die Bethe-Salpeter-Gleichung folgt aus dem Ansatz, dass gebundene Zustände Pole in Greenfunktionen sind. Diese Pole, bei denen die Energie der Greenfunktion der Ruhemasse eines gebundenen Zustandes entspricht, können direkt als Resonanzen in experimenteller Hochenergiephysik beobachtet werden. Unter gewissen Annahmen, wie instantaner Wechselwirkung und gleichen Quarkmassen für die *up*- und *down*-Quarks, kann man die Bethe-Salpeter-Gleichung zur Salpeter-Gleichung vereinfachen. Die Lösung der Salpeter-Gleichung ist die Salpeter-Amplitude, die das Analogon zur Wellenfunkion in gewöhnlicher Quantenmechanik ist. Da wir das Pion beschreiben wollen, verlangen wir eine bestimmte Transformationseigenschaft unter Parität und Ladungskonjugation. Daraus resultierend verbleiben nur gewisse Komponenten der Salpeter-Amplitude und können daraufhin bestimmt werden. Nachdem die Salpeter-Gleichung in ein Eigenwertproblem umgeschrieben wurde, können diese Komponenten als Eigenvektoren identifiziert werden und der Eigenwert ist die Masse des beschribenen gebundenen Zustandes, also die des Pions.

In dieser Arbeit wollen wir in-Hadron-Kondensate in Verbindung mit der Gell-Mann-Oakes-Renner-Relation [6] erforschen. Diese Relation verbindet die "*current*" Quarkmasse, die in der Lagrangedichte aufscheint, mit der Masse eines gebundenen Zustands und dessen Zerfallskonstante. Streng genommen ist diese Relation im so genannten chiralen Limes gültig, in dem Quarks als masselos angenommen werden. In der Gell-Mann-Oakes-Renner-Relation steckt außerdem der *Vakuum*erwartungswert der Komponentenquarks. Wir wollen den Fall untersuchen, dass wir stattdessen das in-Hadron-Kondensat verwenden.

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Wien, 11. Februar 2018 Stephan Hübsch

Meiner Familie und Marie-Therese.

Contents

Chapter 1

Introduction

The Bethe-Salpeter formalism is a framework in quantum field theory to describe bound states of quantum fields. In this thesis, we want to investigate the pion, which is the lightest hadron, using this framework. By calculating the pion's Bethe-Salpeter amplitude, we aim to investigate the generalized Gell-Mann-Oakes-Renner relation, an equation that combines the mass of a bound state, its decay constant and its in-hadron condensate.

It was first suggested by Casher and Susskind [2] that spontaneous breaking of the chiral symmetry is connected to hadronic wavefunctions rather than to the vacuum. In Ref. [3], Brodsky and Shrock argue that the idea of in-hadron condensates can help solve an ongoing debate about the cosmological constant. In cosmology, the Friedmann equations describe the expansion of our universe in the context of general relativity. The first Friedmann equation (which can be derived from the (00)-term in the Einstein field equations) can be expressed in terms of energy densities Ω_i. There are three different kinds of energy densities: matter Ω_m, radiation Ω_γ and the vacuum itself Ω_Λ. Experimental data suggests $\Omega_\Lambda \approx 0.76$ [3], whereas QCD predicts an order of magnitude of 10^{45}. The crucial assumption is that QCD condensates are constants in space-time everywhere in our universe. However, if we consider *in-hadron* condensates, they are spatially confined to only emerge within hadrons. Thus QCD condensates ought to contribute to Ω_m instead.

The outline of this thesis is as follows. In the next section, we briefly review in-hadron condensates. We will arrive at a generalized Gell-Mann-Oakes-Renner relation, which involves in-hadron condensates that can be calculated using Bethe-Salpeter amplitudes. In Chapter 2, we start by defining the Bethe-Salpeter amplitude and deriving the Bethe-Salpeter equation. Under certain assumptions, we can further simplify this equation to the instantaneous Bethe-Salpeter equation and the Salpeter equation. Chapter 3 shows how to solve the Salpeter equation. We first discuss the interaction potential we are taking under consideration. This potential is given by a scalar function $V(r)$, describing the interaction in coordinate space, and the Dirac structure $\Gamma \otimes \Gamma$, encoding interactions in the Dirac spinor space. For the scalar function $V(r)$, we use an analytically known, exact expression [7] and for the Dirac structure, we employ the Böhm-Joos-Krammer (BJK) kernel structure [8]. Furthermore, we expand our wave functions in a series of basis functions. Depending on how many basis functions we include, we obtain a matrix equation of a certain size N. This takes on the form of an eigenvalue equation, where the mass of the bound state appears as eigenvalue and the eigenvectors will be used to calculate the pion's decay constant and in-hadron condensate. In Chapter 4, we show our results. First, we present the mass of the the pion, its decay constant and its in-hadron condensate plotted over the constituent quark mass. We finish by comparing different matrix sizes.

1.1 In-hadron Condensates

Following Refs. [9, 10], we will now briefly review in-hadron condensates in order to set a starting point for this thesis and to establish our notation. We will use natural units where $\hbar = c = 1$ and a Minkowski metric $\eta = \mathrm{diag}\,(+,-,-,-)$. Additional notation is summarized in Appendix C.1. The starting point is the Ward-Takahashi identity concerning $SU(N_f)_A$ transformations, the so-called axial-vector Ward-Takahashi identity,

$$\mathrm{i}P_\mu {\Gamma_A}^{\mu,a}(k,P) \overset{A.1}{=} 2m{\Gamma_P}^a(k,P) - S^{-1}(k_+)\gamma^5 \mathrm{t}^a - \gamma^5 \mathrm{t}^a S^{-1}(k_-), \tag{1.1}$$

with Γ_A and Γ_P being amputated three-point Green functions (three-point vertices) where two quarks interact with an axial-vector or pseudoscalar current of momentum P, respectively, and $S(k_\pm)$ denotes the dressed quark propagator ($k_\pm = k \pm P/2$). $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ is the fifth Dirac Gamma-matrix and t^a are the generators of $SU(N_f)$. Ward-Takahashi identities are relations between correlation functions and follow from symmetries underlying a quantum field theory.

We now make an ansatz for every vertex[1] appearing in Eq. (1.1) (See Ref. [11] and Appendix A.2),

$$\begin{aligned}
{\Gamma_A}^{\mu,a}(k,P) &= \mathrm{t}^a\gamma^5 \left[\gamma^\mu F_A^{(1)} + \not{k}k^\mu F_A^{(2)} - [\gamma^\mu,\not{k}]F_A^{(3)}\right] \\
&\quad + {\widetilde{\Gamma}_A}{}^{\mu,a}(k,P) + \frac{r_A P^\mu}{P^2 - M_\pi^2}{\Gamma_\pi}^a(k,P), &(1.2a)\\
{\Gamma_\pi}^a(k,P) &= 2\mathrm{t}^a\mathrm{i}\gamma^5 \left[F_\pi^{(1)} + \not{P}F_\pi^{(2)} + \not{k}k\cdot P F_\pi^{(3)} + [\not{k},\not{P}]F_\pi^{(4)}\right], &(1.2b)\\
\mathrm{i}{\Gamma_P}^a(k,P) &= \mathrm{t}^a\mathrm{i}\gamma^5 \left[F_P^{(1)} + \not{P}F_P^{(2)} + \not{k}k\cdot P F_P^{(3)} + [\not{k},\not{P}]F_P^{(4)}\right] \\
&\quad + \frac{r_P}{P^2 - M_\pi^2}{\Gamma_\pi}^a(k,P), &(1.2c)
\end{aligned}$$

where M_π is the mass of a pion and $\widetilde{\Gamma}$ contains all regular terms when $P^2 \to M_\pi^2$. These equations are based on how an axial-vector (or pseudoscalar) vertex should behave under C and P transformations. Each term is built up by a Dirac structure using gamma matrices and a function F_X, which contains the momentum dependencies. We use these vertices in the axial-vector Ward-Takahashi identity Equation (1.1) to get a relation between the residues,

$$-\, r_A M_\pi^2 = 2m r_P, \tag{1.3}$$

which we will calculate in the following.

The axial-vector residue, r_A. We start with the Dyson equation for a three-point vertex, coupling an axial-vector current to two quarks. The first term is the bare vertex and the second one contains all possible interactions:

$${\Gamma_A}^{\mu,a}(k,P) \overset{B.1}{=} \mathrm{t}^a\gamma^\mu\gamma^5 + \int\limits_{q^\mu} K(k,q,P){G_A}^{\mu,a}(q,P). \tag{1.4}$$

G_A is the axial-vector Green's function, which is related to the axial-vector vertex by $G = S\Gamma S$. We replace the interaction kernel K with the quark-antiquark scattering amplitude M. The amplitude M describes the interaction by means of the kernel K

[1] The additional factor of 2 in Γ_π is introduced in order to stay consistent with Ref. [10]

occurring once, twice, three times etc. This allows us to eliminate the axial-vector Green's function on the right-hand side,

$$\Gamma_{A}{}^{\mu,a}_{\alpha\beta}(k,P) \overset{B.2}{=} \mathrm{t}^a \gamma^5_{\alpha\delta}\gamma^\mu_{\delta\beta} + \int\limits_{q^\mu} M_{\alpha\beta\delta\epsilon}(k,q,P) S_{\delta\omega}(q_+) S_{\rho\epsilon}(q_-) \mathrm{t}^a \gamma^5_{\omega\xi}\gamma^\mu_{\xi\rho}. \tag{1.5}$$

Next we make an ansatz for the scattering amplitude M. We assume it consists of a term where a pion appears near the pion pole $P^2 \to M_\pi^2$ and all other terms that are regular near the pion pole. We use Equation (1.5), the ansatz for the axial-vector vertex Γ_A (1.2a) and this ansatz for the scattering amplitude M,

$$M_{\alpha\beta\delta\epsilon}(k,q,P) \overset{B.3}{=} \frac{\Gamma_\pi{}^c_{\alpha\beta}(k,P)\bar{\Gamma}^{\mathsf{T}}_\pi{}^c_{\delta\epsilon}(q,-P)}{P^2 - M_\pi^2} + \mathrm{Reg}_{\alpha\beta\delta\epsilon}(k,q,P), \tag{1.6}$$

to arrive at an identity for the axial-vector residue r_A,

$$r_A P^\mu \delta^{ac} \overset{B.4}{=} \int\limits_{q^\mu} \mathrm{Tr}_{F,D}\big\{\mathrm{t}^a S(q_+)\Gamma_\pi{}^c S(q_-)\gamma^\mu\gamma^5\big\}. \tag{1.7}$$

Next, we make an ansatz for an axial current's vacuum polarization, Π_A, in two ways: The first is a Dyson equation and the second one again separates regular terms from terms near the pion pole,

$$\Pi_A{}^{\mu\nu,ab}(P^2) \overset{B.5}{=} (\eta^{\mu\nu}P^2 - P^\mu P^\nu)\delta^{ab} + g^2 \int\limits_{q^\mu} \mathrm{Tr}_{F,D}\big\{\mathrm{t}^a\gamma^\mu\gamma^5 G_A{}^{\nu,b}(q,P)\big\}, \tag{1.8}$$

$$\Pi_A{}^{\mu\nu,ab}(P^2) \overset{B.6}{=} (\eta^{\mu\nu}P^2 - P^\mu P^\nu)\delta^{ab}\left[\widetilde{\Pi}_{A,\mathrm{reg}}(P^2) + \frac{g^2 f_\pi^2}{P^2 - M_\pi^2}\right]. \tag{1.9}$$

We use these two expressions for the vacuum polarization together with Equations (1.2a) and (1.7) to arrive at:

$$r_A \overset{B.7}{=} \mathrm{i} f_\pi. \tag{1.10}$$

The pseudoscalar residue, r_P. For the pseudoscalar residue, the first step is to again rewrite the Dyson equation for a pseudoscalar three-point vertex in terms of a scattering amplitude M,

$$\Gamma_P{}^a_{\alpha\beta}(k,P) \overset{B.2}{=} \mathrm{t}^a \mathrm{i}\gamma^5_{\alpha\beta} + \int\limits_{q^\mu} M_{\alpha\beta\delta\epsilon}(k,q,P) S_{\delta\omega}(q_+) S_{\rho\epsilon}(q_-) \mathrm{t}^a \mathrm{i}\gamma^5_{\omega\rho}, \tag{1.11}$$

then we make use of the pseudoscalar vertex' ansatz in Equation (1.2c) as well as the pion contribution in the scattering amplitude M in Equation (1.6) to identify the pseudoscalar residue with the in-hadron condensate of the — in our case — pion:

$$\mathbb{C}_\pi \overset{B.8}{=} \mathrm{i} r_P. \tag{1.12}$$

Finally, we can express the relation between the residues (1.3) as:

$$f_\pi M_\pi^2 = 2m\mathbb{C}_\pi, \tag{1.13}$$

which is a generalized Gell-Mann-Oakes-Renner relation.

Chapter 2

Model

In this chapter, we introduce our notation by reviewing the Bethe-Salpeter formalism. In Section 2.1, we show that bound states appear as poles in Green functions. We will define Bethe-Salpeter amplitudes and investigate the region near the pion pole. In Section 2.2, we derive the Dyson equation for a 4-point Green function, which we use in Section 2.3 to arrive at the homogenous Bethe-Salpeter equation. In this thesis, we are interested in the homogenous Bethe-Salpeter equation, because it describes bound states. In Section 2.4, we assume instantaneous interactions between the constituent particles such that our interactions get reduced to three dimensions. Finally, in Section 2.5, we assume the constituent particles to be freely propagating quarks of equal mass, in order to arrive at the most simple form of the Salpeter equation. We make use of a certain interaction kernel and close this chapter with the equations that we aim to solve in Chapter 3.

2.1 Poles in Green functions

Consider a fermionic 4-point Green function defined as

$$G^{(4)}(x_1, x_2; y_1, y_2) = \langle 0| \mathsf{T}\psi(x_1)\overline{\psi}(x_2)\overline{\psi}(y_1)\psi(y_2) |0\rangle\,, \tag{2.1}$$

where $\psi(x_i)$ and $\overline{\psi}(x_i) = \psi^\dagger(x_i)\gamma^0$ are fermionic field operators, corresponding to the annihilation and creation of a fermionic particle (e.g. a quark), respectively. $x_i = (t, \mathbf{x}_i)$ is the four-position vector, T represents time-ordering and $|0\rangle$ is the vacuum ground state. We assume that the fermions' positions can be split up into a center-of-mass movement and a relative movement, according to Appendix A.3. The available states in our Hilbert space are the vacuum $|0\rangle$, the one-particle states $|k\rangle$ with momentum $\mathbf{k}$ (and energy $\omega_\mathbf{k} = +\sqrt{\mathbf{k}^2 + m^2}$) and multi-particle states $|k, n\rangle$ with center-of-mass momentum $\mathbf{k}$ and other (discrete and/or continuous) quantum numbers n. We insert the completeness relation [12],

$$1 = |0\rangle\langle 0| + \int \widetilde{\mathrm{d}k}\, |k\rangle\langle k| + \sum\!\!\!\!\!\!\int_n \widetilde{\mathrm{d}k}\, |k,n\rangle\langle k,n|\,, \qquad \widetilde{\mathrm{d}k} \equiv \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\omega_\mathbf{k}} \tag{2.2}$$

into the 4-point Green function, Equation (2.1). In the scope of this work, we are interested in mesons, which consist of a quark anti-quark pair. Therefore, we will only consider the multi-particle terms in Equation (2.2) that can be brought to the vacuum's quantum numbers via one ψ and one $\overline{\psi}$ such that the time-ordered product simplifies to [13],

$$\begin{aligned} G^{(4)}_{\text{meson}}(x_1, x_2; y_1, y_2) = \sum\!\!\!\!\!\!\int_n \widetilde{\mathrm{d}k}\, \langle 0| \psi_{x_1}\overline{\psi}_{x_2} |k,n\rangle\langle k,n| \overline{\psi}_{y_1}\psi_{y_2} |0\rangle \\ \times\, \Theta(\min(x_1^0, x_2^0) - \max(y_1^0, y_2^0)), \end{aligned} \tag{2.3}$$

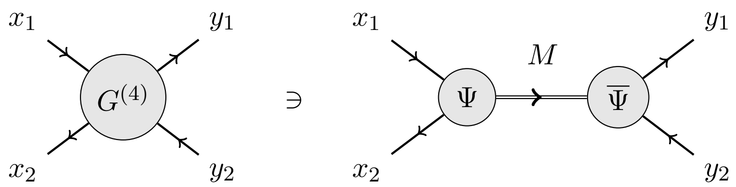


FIGURE 2.1: The 4-point Green function contains a term where a meson is created and annihilated via the Bethe-Salpeter amplitudes.

where $\Theta(x)$ is the Heaviside unit step function, which will be adressed in Equations (2.8) to (2.11). We introduce the Bethe-Salpeter amplitude Ψ and its conjugate via,

$$\begin{aligned}\Psi(x_1, x_2; \mathbf{k}, n) &= \langle 0|\,\mathsf{T}\psi(x_1)\overline{\psi}(x_2)\,|k, n\rangle\,,\\ \overline{\Psi}(x_1, x_2; \mathbf{k}, n) &= \langle k, n|\,\mathsf{T}\overline{\psi}(x_1)\psi(x_2)\,|0\rangle\,,\end{aligned} \tag{2.4}$$

and will consider the meson state from this point on as a one-particle state with three-momentum $\mathbf{P}_M$ (i.e. $|k, n\rangle \to |M\rangle$ and $(\mathbf{k}, n) \to (\mathbf{P}_M)$),

$$\Psi(x_1, x_2; \mathbf{P}_M) = \langle 0|\,\mathsf{T}\psi(x_1)\overline{\psi}(x_2)\,|M\rangle\,, \tag{2.5}$$

see Figure 2.1 for a visual representation. Using the center-of-mass definitions given in Appendix A.3, we can denote Equation (2.5) differently, using how the translation operator acts on a scalar field, $\varphi(x + a) = \exp(\mathrm{i}\hat{P}a)\varphi(x)\exp(-\mathrm{i}\hat{P}a)$,

$$\begin{aligned}\Psi(x_1, x_2; \mathbf{P}_M) &= \langle 0|\,\mathsf{T}\psi(X + \eta_2 x)\overline{\psi}(X - \eta_1 x)\,|M\rangle\\ &= \langle 0|\,\mathsf{T}\mathrm{e}^{\mathrm{i}\hat{P}\cdot X}\psi(\eta_2 x)\mathrm{e}^{-\mathrm{i}\hat{P}\cdot X}\mathrm{e}^{\mathrm{i}\hat{P}\cdot X}\overline{\psi}(-\eta_1 x)\mathrm{e}^{-\mathrm{i}\hat{P}\cdot X}\,|M\rangle\\ &= \langle 0|\,\mathsf{T}\psi(\eta_2 x)\overline{\psi}(-\eta_1 x)\,|M\rangle\,\mathrm{e}^{-\mathrm{i}P_M\cdot X}\\ &=: \Psi(x; \mathbf{P}_M)\mathrm{e}^{-\mathrm{i}P_M\cdot X},\end{aligned}$$

where $\hat{P}^\mu$ is the four-momentum operator acting as the generator for spacetime translations and we used that the meson state is an eigenvector of this momentum operator with eigenvalues P_M^μ. With the definitions of a Fourier transformation,

$$f(p) = \int_{x^\mu} f(p)\,\mathrm{e}^{\mathrm{i}p\cdot x}, \qquad f(x) = \int_{p^\mu} f(x)\,\mathrm{e}^{-\mathrm{i}p\cdot x}, \tag{2.6}$$

where we denote momentum space integrals as $\int_{p^\mu} \equiv \int \mathrm{d}^4p(2\pi)^{-4}$ and coordinate space integrals as $\int_{x^\mu} \equiv \int \mathrm{d}^4x$ for brevity. The amplitudes are transformed to momentum space via

$$\begin{aligned}\Psi(x; \mathbf{P}_M) &= \int_{p^\mu} \mathrm{e}^{-\mathrm{i}px}\,\Psi(p, \mathbf{P}_M),\\ \overline{\Psi}(y; \mathbf{P}_M) &= \int_{q^\mu} \mathrm{e}^{\mathrm{i}qy}\,\overline{\Psi}(q, \mathbf{P}_M).\end{aligned} \tag{2.7}$$

In the following, we address the Heaviside step function in Equation (2.3), ensuring time-ordering in the original Green function. With the help of the identities,

$$\max(a, b) = \frac{a + b}{2} + \frac{|a - b|}{2}, \qquad \min(a, b) = \frac{a + b}{2} - \frac{|a - b|}{2}, \tag{2.8}$$

the argument of the Heaviside step function can be expressed as (see Appendix A.3 for the conventions on $x_{1,2}, x, X$),

$$\min(x_1^0, x_2^0) - \max(y_1^0, y_2^0) = X^0 - Y^0 - \frac{1}{2}|x^0| - \frac{1}{2}|y^0| + \frac{1}{2}(\eta_2 - \eta_1)(x^0 - y^0). \tag{2.9}$$

We use the following integral representation of the Heaviside step function [14, 15],

$$\Theta(x) = \frac{\mathrm{i}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}z \frac{\mathrm{e}^{-\mathrm{i}xz}}{z + \mathrm{i}\epsilon}, \quad \epsilon \to 0^+, \tag{2.10}$$

so that we can write the step function as,

$$\Theta(\min(x_1^0, x_2^0) - \max(y_1^0, y_2^0)) = \frac{\mathrm{i}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}z \frac{\mathrm{e}^{-\mathrm{i}z(X^0 - Y^0 + \xi^0)}}{z + \mathrm{i}\epsilon}, \tag{2.11}$$

where we set $\xi^0 := -\frac{1}{2}|x^0| - \frac{1}{2}|y^0| + \frac{1}{2}(\eta_2 - \eta_1)(x^0 - y^0)$ for brevity. Let us insert Equation (2.11) into the meson 4-point Green function (2.3),

$$\begin{aligned} G^{(4)}_{\text{meson}}(x_1, x_2; y_1, y_2) = \frac{\mathrm{i}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}z \int \widetilde{\mathrm{d}P_M} \int_{p^\mu} \int_{q^\mu} \frac{\Psi(p, \mathbf{P}_M)\overline{\Psi}(q, \mathbf{P}_M)}{z + \mathrm{i}\epsilon} \\ \times \mathrm{e}^{-\mathrm{i}P_M(X-Y)} \mathrm{e}^{-\mathrm{i}p\cdot x} \mathrm{e}^{\mathrm{i}q\cdot y} \mathrm{e}^{-\mathrm{i}z(X^0 - Y^0 + \xi^0)}. \end{aligned} \tag{2.12}$$

For the last step, we transform this Green function to momentum space,

$$G^{(4)}_{\text{meson}}(k, k'; K) = \int_{x^\mu} \int_{y^\mu} \int_{(X-Y)^\mu} \mathrm{e}^{\mathrm{i}k\cdot x} \mathrm{e}^{-\mathrm{i}k'\cdot y} \mathrm{e}^{\mathrm{i}K\cdot(X-Y)} G^{(4)}_{\text{meson}}(x, y; X - Y), \tag{2.13}$$

where we chose the three translational invariant coordinates to be x, y and $X - Y$. If we only look at the $\mathrm{d}\big(X^0 - Y^0\big)$ integration, we pick up the following arguments,

$$\int_{X^0 - Y^0} \mathrm{e}^{-\mathrm{i}(X^0 - Y^0)\,(P_M^0 + z - K^0)} = 2\pi\, \delta\big(z - (K^0 - P_M^0)\big). \tag{2.14}$$

This shows that after the z integration, a pole emerges, when the total energy K^0 transferred through a Green function equals the energy P_M^0 of a bound state in the same Hilbert space. This means, we can drop the label "meson", since all other terms arising from Equation (2.2) are regular when $K^0 \to P_M^0$. We are left with,

$$\begin{aligned} G^{(4)}(k, k'; K) = \mathrm{i} \int_{x^\mu} \int_{y^\mu} \int_{\mathbf{X}-\mathbf{Y}} \int \widetilde{\mathrm{d}P_M} \int_{p^\mu} \int_{q^\mu} \frac{\Psi(p, \mathbf{P}_M)\overline{\Psi}(q, \mathbf{P}_M)}{K^0 - P_M^0 + \mathrm{i}\epsilon} \mathrm{e}^{\mathrm{i}(k\cdot x - k'\cdot y - p\cdot x + q\cdot y)} \\ \times \mathrm{e}^{-\mathrm{i}\mathbf{P}_M(\mathbf{X}-\mathbf{Y})} \mathrm{e}^{-\mathrm{i}\xi^0(K^0 - P_M^0)} \mathrm{e}^{\mathrm{i}\mathbf{K}(\mathbf{X}-\mathbf{Y})}. \end{aligned}$$

Next, we perform the $\mathrm{d}(\mathbf{X} - \mathbf{Y})$ integration, which leads to a delta function $\delta^{(3)}(\mathbf{K} - \mathbf{P}_M)$. The factors of (2π) cancel the ones of the tilded integration measure, and integrating over $\mathbf{P}_M$ yields,

$$G^{(4)}(k, k'; K) = \mathrm{i} \int_{x^\mu} \int_{y^\mu} \int_{p^\mu} \int_{q^\mu} \frac{1}{2P_M^0} \frac{\Psi(p, \mathbf{K})\overline{\Psi}(q, \mathbf{K})}{K^0 - P_M^0 + \mathrm{i}\epsilon} \mathrm{e}^{\mathrm{i}x\cdot(k-p)} \mathrm{e}^{\mathrm{i}y\cdot(q-k')} \mathrm{e}^{-\mathrm{i}\xi^0(K^0 - P_M^0)}.$$

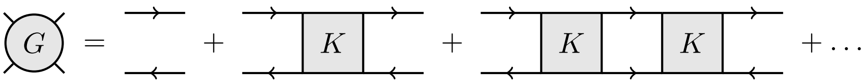
At this point, the d^4x and d^4y integrations cannot be performed to obtain Dirac delta functions, due to the presence of ξ^0, which contains x^0 and y^0. But, if we take the limit $K^0 \to P_M^0$, that is only consider momenta with energies near a bound-state energy, the exponential factor containing ξ^0 drops out,

$$\begin{aligned}\lim_{K^0 \to P_M^0} G^{(4)}(k,k';K) &= \mathrm{i} \int_{x^\mu}\int_{y^\mu}\int_{p^\mu}\int_{q^\mu} \frac{1}{2P_M^0} \frac{\Psi(p,\mathbf{K})\overline{\Psi}(q,\mathbf{K})}{\mathrm{i}\epsilon} \mathrm{e}^{\mathrm{i}x\cdot(k-p)}\mathrm{e}^{\mathrm{i}y\cdot(q-k')} \\ &= \mathrm{i} \int_{p^\mu}\int_{q^\mu} \frac{1}{2P_M^0} \frac{\Psi(p,\mathbf{K})\overline{\Psi}(q,\mathbf{K})}{\mathrm{i}\epsilon} (2\pi)^8 \delta^{(4)}(p-k)\delta^{(4)}(q-k') \\ &= \frac{\Psi(k,\mathbf{K})\overline{\Psi}(k',\mathbf{K})}{2P_M^0\epsilon}. \end{aligned} \tag{2.15}$$

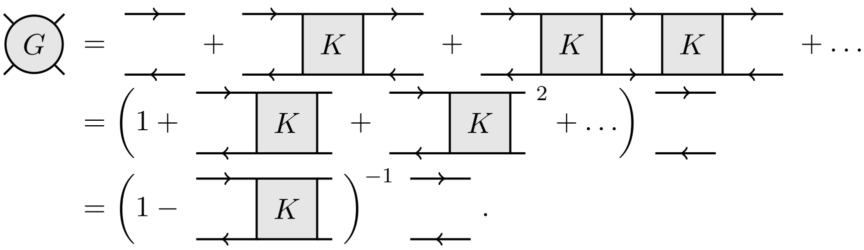
Equation (2.15) shows, that for energies near a bound state energy, the 4-point Green function can be represented by a diverging term, containing the Bethe-Salpeter amplitude and its conjugate.

2.2 Dyson equation for the 4-point Green function

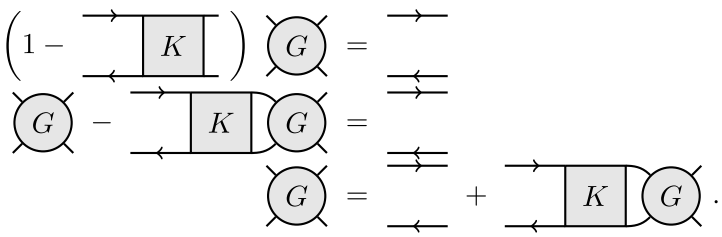
The Dyson equation emerges after sorting the occurring diagrams of a given process. For a 2-to-2 scattering process, one can arrange the diagrams in such a way that they, at first, do not interact at all. The next possibility is to interact once, via all one-particle irreducible (1PI) diagrams, abbreviated by a scattering kernel $\mathrm{i}K$. Then they can interact twice and so on.



Here, lines with arrows represent full propagators. The above equation has the form of a geometric series. This fact can be used to formally manipulate the equation in a way to get a recursive equation,



Multiplying the expression in brackets from the left yields,



(2.16)

Equation (2.16) is the Dyson equation for a 4-point Green function. We use this equation in the next section to derive the Bethe-Salpeter equation.

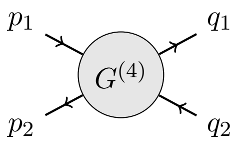
$$G^{(4)}(p,q;P) =$$ 

FIGURE 2.2: 4-point Green function. Here, $p = p_1 - (-p_2)$, $q = q_1 - (-q_2)$ and $P = p_1 - p_2 + q_1 - q_2$.

2.3 Bethe-Salpeter Equation

The Dyson equation for a 4-point Green function reads (see Section 2.2 for a derivation),

$$G^{(4)}(p,q;P) = S_1(p_1)S_2(-p_2)\delta^{(4)}(p-q) + S_1(p_1)S_2(-p_2)\int_{\ell^\mu} \mathrm{i}K(p,\ell;P)G^{(4)}(\ell,q;P),$$

where $G^{(4)}(p,q;P)$ is the 4-point Green function, $S(p)$ is the dressed quark propagator carrying momentum p and $\mathrm{i}K(p,\ell;P)$ is an interaction kernel, containing infinitely many interactions. We choose our notation in a way, that the first (second) argument of a 4-point Green function corresponds to the difference of the left-hand (right-hand) side momenta and the third argument corresponds to the total transferred momentum, as seen in Figure 2.2. If we only want to consider a bound state $|M\rangle$, we let $P^0 \to P^0_M$, insert the main result of the previous section, Equation (2.15), and look at the dominating pole term contributions of the Dyson-Schwinger equation,

$$\frac{\Psi(p,\mathbf{P})\overline{\Psi}(q,\mathbf{P})}{2P^0_M\epsilon} = S_1(p_1)S_2(-p_2)\int_{\ell^\mu} \mathrm{i}K(p,\ell;P)\frac{\Psi(\ell,\mathbf{P})\overline{\Psi}(q,\mathbf{P})}{2P^0_M\epsilon}. \tag{2.17}$$

On the right side, the conjugate Bethe-Salpeter amplitude $\overline{\Psi}$ does not depend on the integration variable ℓ, such that we can multiply the equation with its inverse from the right. Renaming $\ell \to q$ and cancelling remaining factors yields the homogenous Bethe-Salpeter equation for the Bethe-Salpeter amplitude Ψ,

$$\Psi(p,\mathbf{P}) = S_1(p_1)S_2(-p_2)\int_{q^\mu} \mathrm{i}K(p,q;P)\Psi(q,\mathbf{P}). \tag{2.18}$$

This equation describes bound states, since in order to arrive at this expression, we took the limit of only considering energies near a bound-state pole, thus omitting the term with only propagators.

2.4 Instantaneous Bethe-Salpeter equation

The instantaneous Bethe-Salpeter formalism [16] assumes the interaction kernel $\mathrm{i}K$ to only depend on spatial components of the relative momenta p and q. This means, replacing

$$K(p,q,P) \to \hat{K}(\mathbf{p},\mathbf{q},P). \tag{2.19}$$

We will later on switch to the bound state's rest frame $\mathbf{P} = \mathbf{0}$, hence from here on, we suppress the P-dependence for notational simplicity. In this instantaneous formalism, one defines a reduced Bethe-Salpeter amplitude, the so-called Salpeter amplitude,

$$\Phi(\mathbf{p}) = \int_{p^0} \Psi(p), \tag{2.20}$$

as well as a new interaction term,

$$I(\mathbf{p}) = \int\limits_{\mathbf{q}} \hat{K}(\mathbf{p},\mathbf{q})\Phi(\mathbf{q}). \tag{2.21}$$

With these new expressions, we take the homogenous Bethe-Salpeter equation (2.18), integrate over p^0 and rewrite the instantaneous Bethe-Salpeter equation,

$$\begin{aligned}
\int\limits_{p^0} \Psi(p) &= \int\limits_{p^0} S_1(p_1)S_2(-p_2) \int\limits_{q^\mu} \mathrm{i}\hat{K}(\mathbf{p},\mathbf{q})\Psi(q) \\
\Phi(\mathbf{p}) &= \mathrm{i} \int\limits_{p^0} S_1(p_1)S_2(-p_2) \int\limits_{\mathbf{q}} \hat{K}(\mathbf{p},\mathbf{q})\Phi(\mathbf{q}) \\
\Phi(\mathbf{p}) &= \mathrm{i} \int\limits_{p^0} S_1(p_1)I(\mathbf{p})S_2(-p_2).
\end{aligned} \tag{2.22}$$

We moved the propagators to the left and right end because of their suppressed matrix structure (see Appendix A.4 for details).

2.5 Salpeter equation

The Salpeter equaiton follows from assuming that the quarks have a constant mass m_q instead of a momentum-dependent mass function. The free fermion propagator with a constant constituent mass m_q is given by,

$$S(p) = \frac{\mathrm{i}}{\not{p} - m_q + \mathrm{i}\epsilon}. \tag{2.23}$$

Within the instantaneous Bethe-Salpeter formalism, we assume that the quarks propagate as free particles. For each (anti-)quark, $i = \{1, 2\}$, in the meson, we define the single-particle energy,

$$E_i(\mathbf{p}_i) = \sqrt{\mathbf{p}_i^2 + m_{q,i}^2}, \tag{2.24}$$

and a generalized Hamiltonian,

$$H_i(\mathbf{p}_i) = \gamma^0\big[\boldsymbol{\gamma}\cdot\mathbf{p}_i + m_{q,i}\big], \tag{2.25}$$

where γ^μ are the Dirac gamma matrices satisfying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\}_{\alpha\beta} = 2\eta^{\mu\nu}\delta_{\alpha\beta}$. These definitions of energy and Hamiltonian satisfy the following relations,

$$H_i^2(\mathbf{p}_i) = E_i^2(\mathbf{p}_i), \qquad H_i(\mathbf{p}_i)\gamma^0 = \gamma^0 H_i(-\mathbf{p}_i), \tag{2.26}$$

and help define energy projection operators,

$$\Lambda_i^\pm(\mathbf{p}_i) = \frac{1}{2E_i(\mathbf{p}_i)}\big(E_i(\mathbf{p}_i) \pm H_i(\mathbf{p}_i)\big), \tag{2.27}$$

which obey the defining relations of projection operators, idempotence $\Lambda^\pm\Lambda^\pm = \Lambda^\pm$ and being able to be written as a direct sum $\Lambda^\pm + \Lambda^\mp = 1$ with $\Lambda^\pm\Lambda^\mp = 0$. By using the following identity

$$\not{p} + m_q \overset{B.9}{=} \Big[\big[p^0 + E(\mathbf{p})\big]\Lambda^+(\mathbf{p}) + \big[p^0 - E(\mathbf{p})\big]\Lambda^-(\mathbf{p})\Big]\gamma^0, \tag{2.28}$$

we can write the fermion propagator as

$$S(p) \overset{B.10}{=} \mathrm{i}\left(\frac{\Lambda^+(\mathbf{p})}{p^0 - E(\mathbf{p}) + \mathrm{i}\epsilon} + \frac{\Lambda^-(\mathbf{p})}{p^0 + E(\mathbf{p}) - \mathrm{i}\epsilon}\right)\gamma^0. \tag{2.29}$$

With the help of Equation (2.29) we can evaluate the integral in Equation (2.22),

$$\begin{aligned}\int_{p^0} S_1(p_1)I(\mathbf{p})S_2(-p_2) &\overset{B.11}{=} \\ -\mathrm{i}\left(\frac{\Lambda_1^+(\mathbf{p}_1)\gamma^0 I(\mathbf{p})\Lambda_2^-(-\mathbf{p}_2)\gamma^0}{P^0 - E_1 - E_2} - \frac{\Lambda_1^-(\mathbf{p}_1)\gamma^0 I(\mathbf{p})\Lambda_2^+(-\mathbf{p}_2)\gamma^0}{P^0 + E_1 + E_2}\right)&,\end{aligned} \tag{2.30}$$

where we used contour integration. Using this result in Equation (2.22), we arrive at the instantaneous Bethe-Salpeter equation for fermion-antifermion bound states,

$$\Phi(\mathbf{p}) = \left(\frac{\Lambda_1^+(\mathbf{p}_1)\gamma^0 I(\mathbf{p})\Lambda_2^-(-\mathbf{p}_2)\gamma^0}{P^0 - E_1 - E_2} - \frac{\Lambda_1^-(\mathbf{p}_1)\gamma^0 I(\mathbf{p})\Lambda_2^+(-\mathbf{p}_2)\gamma^0}{P^0 + E_1 + E_2}\right).$$

In the subspace spanned by the energy projection operators $\Lambda^{\pm}_{1,2}$, one can define components of the Salpeter amplitude Φ,

$$\Phi^{\pm\pm} := \Lambda_1^{\pm}\Phi\Lambda_2^{\pm}, \qquad \text{with } \Phi^{++} + \Phi^{+-} + \Phi^{-+} + \Phi^{--} = \Phi, \tag{2.31}$$

and investigate their properties with respect to the instantaneous Bethe-Salpeter equation. When applying $\Lambda_1^{\pm}$ from the left and $\Lambda_2^{\pm}$ from the right, that is only looking at the components Φ^{++} and Φ^{--}, the right-hand side of the instantaneous Bethe-Salpeter equation vanishes and we are left with,

$$\Phi^{++} = 0 = \Phi^{--}. \tag{2.32}$$

If we use the definition of I, Equation (2.21), that is write the Salpeter equation in terms of the kernel $\hat{K}$, we arrive at [17],

$$\begin{aligned}\Phi(\mathbf{p}) = \int_{\mathbf{q}} \Bigg(&\frac{\Lambda_1^+(\mathbf{p}_1)\gamma^0 \hat{K}(\mathbf{p},\mathbf{q})\Phi(\mathbf{q})\gamma^0\Lambda_2^-(\mathbf{p}_2)}{P^0 - E_1 - E_2} \\ &- \frac{\Lambda_1^-(\mathbf{p}_1)\gamma^0 \hat{K}(\mathbf{p},\mathbf{q})\Phi(\mathbf{q})\gamma^0\Lambda_2^+(\mathbf{p}_2)}{P^0 + E_1 + E_2}\Bigg).\end{aligned} \tag{2.33}$$

Next, we want to discuss how the interaction kernel $\hat{K}$ acts on the Salpeter amplitude.

Interaction Kernel. We start by making an ansatz for the interaction kernel via a tensor product of elements on the basis of Dirac matrices,

$$\hat{K}_{\alpha\beta\gamma\delta}(\mathbf{p},\mathbf{q}) = \sum_{\Gamma,\Gamma'} V_{\Gamma\Gamma'}(\mathbf{p},\mathbf{q})\Gamma_{\alpha\gamma}\otimes\Gamma'_{\delta\beta}, \tag{2.34}$$

and if we further assume identical coupling of quark and anti-quark $\delta_{\Gamma\Gamma'}$, we can simplify this to,

$$\hat{K}_{\alpha\beta\gamma\delta}(\mathbf{p},\mathbf{q}) = \sum_{\Gamma} V_{\Gamma\Gamma}(\mathbf{p},\mathbf{q})\Gamma_{\alpha\gamma}\otimes\Gamma_{\delta\beta}, \tag{2.35}$$

where $\Gamma \in \{1, \mathrm{i}\gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]\}$ and lowercase Greek letters are spinor indices. The kernel is said to have a $\Gamma \otimes \Gamma$ Lorentz structure. It acts on the Salpeter amplitude in the following way,

$$\hat{K}\Phi \doteq \hat{K}_{\alpha\beta\gamma\delta}\Phi_{\gamma\delta} = \sum_{\Gamma} V_{\Gamma\Gamma}\,\Gamma_{\alpha\gamma}\Phi_{\gamma\delta}\Gamma_{\delta\beta} \doteq \sum_{\Gamma} V_{\Gamma\Gamma}\,\Gamma\Phi\Gamma. \tag{2.36}$$

By requiring symmetry under Fierz transformations (see [18] and Appendix A.5), we may choose the interaction kernel to be the so-called Böhm-Joos-Krammer (BJK) kernel [8],

$$\Gamma \otimes \Gamma = \frac{1}{2}\left(\gamma^\mu \otimes \gamma_\mu + \gamma^5 \otimes \gamma^5 - 1 \otimes 1\right). \tag{2.37}$$

Structure of the Salpeter amplitude. In this work, $\Phi(\mathbf{p})$ is supposed to describe pseudoscalar mesons, like the pion or the kaon. These mesons have a well-defined CP-transformation with eigenvalues $PC = -+$. Under the assumption of equal quark masses $m_{q,1} = m_{q,2}$, an expansion over the basis of Dirac matrices shows that only two components of the Salpeter amplitude are relevant (see Refs. [19, 20] and Appendix A.6), namely terms proportional to φ_1 and φ_2 in

$$\Phi^a(\mathbf{p}) = \left[\varphi_1(\mathbf{p})\frac{H(\mathbf{p})}{E(\mathbf{p})} + \varphi_2(\mathbf{p})\right]\gamma^5 \mathbf{t}^a. \tag{2.38}$$

We change to the bound-state's center-of-mass system, that is $P^0 = M$, $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}$, which leads to $E_i(\mathbf{p}_i) = E(\mathbf{p})$. Using this decomposition, and that

$$\varphi_{1,2}(\mathbf{p}) = Y_{00}(\vartheta, \phi)\varphi_{1,2}(p) = \frac{1}{\sqrt{4\pi}}\varphi_{1,2}(p), \tag{2.39}$$

the Salpeter equation (2.33) finally yields the two equations,

$$M\varphi_1(p) = 2E(p)\varphi_2(p) + 2\int_{q_+} \frac{q^2}{(2\pi)^2} V(p, q)\varphi_2(\mathrm{q}), \tag{2.40a}$$

$$M\varphi_2(p) = 2E(p)\varphi_1(p), \qquad p = |\mathbf{p}|. \tag{2.40b}$$

We will further analyze Equations (2.40a) and (2.40b) in the next chapter. Note that from here on, $p = |\mathbf{p}|$ denotes the length of a three-vector, instead of a four-vector p^μ.

Chapter 3

Methods

In the previous chapter, we derived the Salpeter equation, which describes bound states in QCD, and wrote down a special case, valid for pseudoscalar mesonic bound states under the assumption of equal quark masses. This equation depends on two components of the pseudoscalar Salpeter amplitude, φ_1 and φ_2. In this chapter, we use recent results for the interaction potential in order to solve these equations.

In Section 3.1, we present the Salpeter equation we want to solve and discuss the interacting potential. In Section 3.2, we expand our wave functions in a set of basis functions, thus arriving at an eigenvalue problem. In order to perform calculations, we simplify certain expressions. In Section 3.4, we describe how to calculate the in-hadron condensate and the decay constant, using the eigenfunctions of the eigenvalue problem.

3.1 Ansatz

Let us restate the equations we want to solve,

$$M\varphi_1(p) = 2E(p)\varphi_2(p) + 2\int\limits_{q_+} \frac{q^2}{(2\pi)^2} V(p,q)\varphi_2(q), \tag{3.1a}$$

$$M\varphi_2(p) = 2E(p)\varphi_1(p). \tag{3.1b}$$

Here, M is the bound-state's rest mass, $\varphi_{1,2}(p)$ are the components of a pseudoscalar Salpeter amplitude in Equation (2.38), $E(p)$ is the single particle energy in Equation (2.24) and $V(p,q)$ is a function describing the interaction kernel according to Equation (2.34). The second equation is of algebraic nature, therefore we can solve it for $\varphi_1(p)$. Inserting this result into the first equation yields

$$M^2\varphi_2(p) = 4E^2(p)\varphi_2(p) + \frac{8}{\pi}\frac{E(p)}{p}\int\limits_{q_+} q\,\varphi_2(q)\int\limits_{r_+} \sin(pr)\sin(qr)V(r). \tag{3.2}$$

The interaction potential $V(r)$ in position space is defined in terms of its Fourier-Bessel transform $V(p,q)$,

$$V(p,q) = \frac{8\pi}{pq}\int\limits_{r_+} \sin(pr)\sin(qr)V(r). \tag{3.3}$$

The analytical form of $V(r)$ has been calculated for vanishing bound state mass $M = 0$ [7]. This was done by first fitting a quark mass function to certain data points. This mass function was then used to build a Bethe-Salpeter amplitude, which in turn was modified to a Salpeter amplitude and its components. These components of the Salpeter amplitude, valid for $M = 0$, helped reverse-engineering the potential from the Salpeter equation. This means, using $V(r)$ should lead to a vanishing bound-state mass. The analytic expressions read

$$
\begin{aligned}
V(r) &= -T(r)/\varphi(r),\\
T(r) &= \sqrt{\frac{\pi^{3/2}\Gamma(2\gamma)}{\Gamma(2\gamma-3/2)}}\frac{2^{1-\gamma}}{\sin(\gamma\pi)\Gamma(\gamma)r^{5/2}}\Big[2(1-\gamma)(br)^{\gamma}I_{\gamma-3/2}(br)-(br)^{1+\gamma}I_{\gamma-1/2}(br)\\
&\quad+(br)^{\gamma}\mathbf{L}_{3/2-\gamma}(br)+(br)^{1+\gamma}\mathbf{L}_{5/2-\gamma}(br)+\frac{\sin(\gamma\pi)}{2^{3/2-\gamma}\pi^{3/2}}\Gamma(\gamma-2)(br)^{5/2}\Big],\\
\varphi(r) &= \sqrt{\frac{\Gamma(2\gamma)}{\sqrt{\pi}\Gamma(2\gamma-3/2)}}\frac{2^{2-\gamma}b^{\gamma}}{\Gamma(\gamma)}r^{\gamma-3/2}K_{3/2-\gamma}(br),
\end{aligned}
$$

where $\Gamma(x)$ is the Gamma function [14, 15],

$$\Gamma(x) = \int_{t_+} t^{x-1}\mathrm{e}^{-t}, \tag{3.4}$$

$I_a(x)$ and $K_a(x)$ are the modified Bessel functions of first and second kind, respectively, of order a,

$$I_a(x) = (\tfrac{1}{2}x)^a \sum_{k=0}^{\infty}\frac{(\frac{1}{2}x)^{2k}}{k!\,\Gamma(a+k+1)}, \qquad K_a(x) = \frac{\pi}{2}\frac{I_{-a}(x)-I_a(x)}{\sin(a\,\pi)}, \tag{3.5}$$

and $\mathbf{L}_a(x)$ is the modified Struve function of order a,

$$\mathbf{L}_a(x) = (\tfrac{1}{2}x)^{a+1}\sum_{k=0}^{\infty}\frac{(\frac{1}{2}x)^{2k}}{\Gamma(k+3/2)\Gamma(k+a+3/2)}. \tag{3.6}$$

In Ref. [7], Lucha and Schöberl present values for the parameters b and γ that enter the potential:

$$b = 1.693\,34\ \text{GeV}, \qquad \gamma = 6.492\,92. \tag{3.7}$$

We would like to solve Equation (3.2). This is equivalent to an eigenvalue problem: the second component of the Salpeter amplitude, $\varphi_2(p)$, is an eigenfunction and the mass of the pseudoscalar bound state, M, is the eigenvalue. In the next section, we will manipulate Equation (3.2) in order to arrive at a matrix equation.

3.2 Eigenvalue Problem

We expand $\varphi_2(p)$ in a basis of orthonormal states $|\phi_i^{(l)}\rangle$, defined by their position space representation,

$$\phi_i^{(l)}(r) = \langle r|\phi_i^{(l)}\rangle = \sqrt{\frac{(2\mu)^{2l+3}i!}{\Gamma(2l+i+3)}}\,r^l\exp(-\mu r)L_i^{(2l+2)}(2\mu r), \tag{3.8}$$

where $\Gamma(x)$ is again the gamma function and $L_a^{(b)}(x)$ are the associated Laguerre polynomials [14, 15]. Dimensional analysis requires us to introduce a parameter μ with one mass dimension. They specific choice of μ will we addressed in Appendix A.7. We will call the amount of basis functions N, i.e. we are dealing with $N \times N$ matrices. The basis functions obey the following normalization with weight function $w(r) = r^2$:

$$\int_{r_+} w(r)\phi_i^{(l)}(r)\phi_j^{(l)}(r) = \delta_{ij}. \tag{3.9}$$

The basis in momentum space $\widetilde{\phi}_i^{(l)}(p)$ is obtained via a Fourier-Bessel transformation,

$$\begin{aligned} \phi_i^{(l)}(r) &= \mathrm{i}^l \sqrt{\frac{2}{\pi}} \int_{p_+} p^2 j_l(pr)\widetilde{\phi}_i^{(l)}(p), \\ \widetilde{\phi}_i^{(l)}(p) &= (-\mathrm{i})^l \sqrt{\frac{2}{\pi}} \int_{r_+} r^2 j_l(pr)\phi_i^{(l)}(r). \end{aligned} \tag{3.10}$$

In this work we will make use of the case $l = 0$, so that the basis functions in position space read

$$\phi_i^{(0)}(r) = \sqrt{\frac{(2\mu)^3 i!}{\Gamma(i+3)}} \exp(-\mu r) L_i^{(2)}(2\mu r), \tag{3.11}$$

and in momentum space they are given by

$$\begin{aligned} \widetilde{\phi}_i^{(0)}(p) = \sqrt{\frac{i!}{\mu\pi\Gamma(i+3)}} \frac{4}{p} \sum_{t=0}^{i} (-2)^t (t+1) \binom{i+2}{i-t} \times \\ \left(1 + \frac{p^2}{\mu^2}\right)^{-(t+2)/2} \sin\left((t+2)\arctan\left(\frac{p}{\mu}\right)\right). \end{aligned} \tag{3.12}$$

Now we use the momentum space basis functions $\widetilde{\phi}_i^{(0)}(p)$ to write $\varphi_2(p)$ as a linear combination with coefficients v_j:

$$\varphi_2(p) = v_j \widetilde{\phi}_j^{(0)}(p). \tag{3.13}$$

Making use of the orthonormal properties of our basis functions, we mutliply with the weight function, p^2, and a basis function, $\widetilde{\phi}_i^{(0)}(p)$, in order to extract M^2 on the left-hand side by integrating over p,

$$\begin{aligned} M^2 \underbrace{\int_{p_+} p^2 \widetilde{\phi}_i^{(0)}(p)\widetilde{\phi}_j^{(0)}(p)}_{\delta_{ij}} v_j = \underbrace{4\int_{p_+} p^2 E^2(p)\widetilde{\phi}_i^{(0)}(p)\widetilde{\phi}_j^{(0)}(p)}_{\mathcal{A}_{ij}} v_j \\ + \underbrace{\frac{8}{\pi}\int_{p_+}\int_{q_+}\int_{r_+} pE(p)\widetilde{\phi}_i^{(0)}(p) q\widetilde{\phi}_j^{(0)}(q)\sin(pr)\sin(qr)V(r)}_{\mathcal{B}_{ij}} v_j. \end{aligned} \tag{3.14}$$

We arrive at an eigenvalue equation with eigenvalue M^2 and eigenfunctions v_i,

$$\mathcal{M}_{ij} v_j \equiv \big(\mathcal{A} + \mathcal{B}\big)_{ij} v_j = M^2 v_i. \tag{3.15}$$

Let us look at the matrices $\mathcal{A}$ and $\mathcal{B}$ and how we can simplify them.

The matrix $\mathcal{A}$. We use the following abbreviation for an integral over the basis functions,

$$I_{ij}^{(n)} \equiv \int_{k_+} k^{2+n}\,\phi_i^{(0)}(k)\,\phi_j^{(0)}(k)\;, \quad n = 0, 1, 2, \ldots \tag{3.16}$$

which has an analytic expression [21]

$$\begin{aligned} I_{ij}^{(n)} = &\frac{4\,\mu^n}{\pi\sqrt{(i+1)\,(i+2)\,(j+1)\,(j+2)}} \\ &\times \sum_{r=0}^{i}\sum_{s=0}^{j}(-2)^{r+s}\binom{i+2}{i-r}\binom{j+2}{j-s}(r+1)\,(s+1) \\ &\left[\sum_{k=0}^{|r-s|}\binom{|r-s|}{k}\frac{\Gamma(\frac{1}{2}\,(k+n+1))\,\Gamma(\frac{1}{2}\,(r+s+3+|r-s|-n-k))}{\Gamma(\frac{1}{2}\,(r+s+4+|r-s|))}\cos\left(\frac{k\,\pi}{2}\right)\right. \\ &\left. -\sum_{k=0}^{r+s+4}\binom{r+s+4}{k}\frac{\Gamma(\frac{1}{2}\,(k+n+1))\,\Gamma(\frac{1}{2}\,(2\,r+2\,s+7-n-k))}{\Gamma(r+s+4)}\cos\left(\frac{k\,\pi}{2}\right)\right]. \end{aligned}$$

With this, we write $E^2(p) = p^2 + m_q^2$ such that we can use $I_{ij}^{(2)}$ and $I_{ij}^{(0)} = \delta_{ij}$ and express $\mathcal{A}_{ij}$ as:

$$\mathcal{A}_{ij} = 4I_{ij}^{(2)} + 4m_q^2\delta_{ij}, \tag{3.17}$$

where m_q is the constant constituent quark mass.

The matrix $\mathcal{B}$. In $\mathcal{B}_{ij}$ we can evaluate the q integration using the Fourier-Bessel transformation (3.10),

$$\int_{q_+} q\widetilde{\phi}_j^{(0)}(q)\sin(qr) = r\int_{q_+} q^2\widetilde{\phi}_j^{(0)}(q)\frac{\sin(qr)}{qr} = r\sqrt{\frac{\pi}{2}}\phi_j^{(0)}(r), \quad j_0(x) = \sin(x)/x. \tag{3.18}$$

In order to perform the p integration, we expand $E(p)\widetilde{\phi}_i^{(0)}(p)$ in a linear combination of basis elements,

$$E(p)\widetilde{\phi}_i^{(0)}(p) = b_{ik}\widetilde{\phi}_k^{(0)}(p), \tag{3.19}$$

such that the coefficients b_{ik} are given by

$$b_{ik} = \int_{p_+} p^2 E(p)\widetilde{\phi}_i^{(0)}(p)\widetilde{\phi}_k^{(0)}(p).$$

Now, we can solve the p integration again using the Fourier-Bessel transformation

$$b_{ik}\int_{p_+} p\widetilde{\phi}_k^{(0)}(p)\sin(pr) = rb_{ik}\int_{p_+} p^2\widetilde{\phi}_k^{(0)}(p)\frac{\sin(pr)}{pr} = rb_{ik}\sqrt{\frac{\pi}{2}}\phi_k^{(0)}(r). \tag{3.20}$$

This leaves us with the third integration over r in the matrix $\mathcal{B}_{ij}$:

$$\mathcal{B}_{ij} = 4b_{ik}\int_{r_+} r^2\phi_k^{(0)}(r)\phi_j^{(0)}(r)V(r).$$

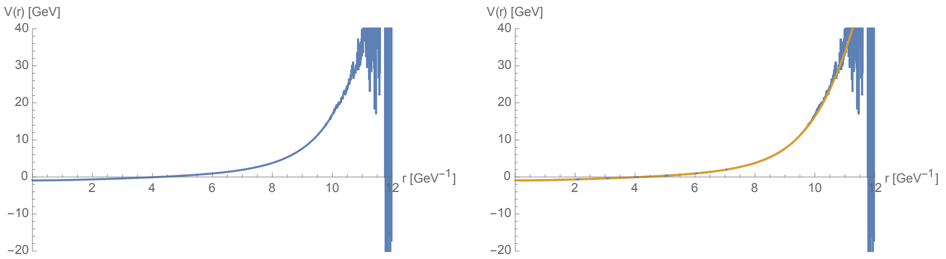


FIGURE 3.1: Left: The potential $V(r)$ is affected by numerical errors for large values of its argument. Right: Our fit of the potential.

Using the definition of $\phi_i^{(0)}(r)$, we can express $\mathcal{B}_{ij}$ using Laguerre polynomials as

$$\mathcal{B}_{ij} = 4b_{ik}\sqrt{\frac{(2\mu)^3 k!}{\Gamma(k+3)}}\sqrt{\frac{(2\mu)^3 j!}{\Gamma(j+3)}}\int_{r_+} r^2 \mathrm{e}^{-2\mu r} L_k^{(2)}(2\mu r) L_j^{(2)}(2\mu r) V(r)$$

Now, we can write down our eigenvalue equation as follows:

$$\left[4\underbrace{I_{ij}^{(2)}}_{\propto\mu^2} + 4m_q^2\delta_{ij} + 32b_{ik}\mu^3\sqrt{\frac{k!j!}{(k+2)!(j+2)!}}\int_{r_+} r^2\mathrm{e}^{-2\mu r} L_k^{(2)}(2\mu r)L_j^{(2)}(2\mu r)V(r)\right]v_j = M^2 v_i.$$

In the next section we will take a closer look at the potential $V(r)$ and how to perform the last remaining integration.

3.2.1 Approximating the Potential

The potential $V(r)$ is a rather complicated function, so in order to handle it analytically, we approximate it *piecewise* in a power series up to fourth order:

$$V(r) = \sum_{h=0}^{7}\sum_{p=0}^{4} a_{hp} r^p \Theta(r-h)\Theta(h+1-r) + \sum_{p=0}^{4} a_{8p} r^p \Theta(r-8) \tag{3.21}$$

$$= \sum_{p=0}^{4}\begin{cases} a_{0p}r^p & 0 \le r < 1 \\ a_{1p}r^p & 1 \le r < 2 \\ \dots & \\ a_{8p}r^p & 8 \le r \end{cases}. \tag{3.22}$$

The coefficients are listed below, the matrixelement a_{ij} is given in units of $[\mathrm{GeV}^{j+1}]$:

$$a_{ij} = \begin{pmatrix} -0.926815 & 0.0000174443 & 0.0862116 & -0.000307632 & -0.00532234 \\ -0.99365 & 0.196217 & -0.115383 & 0.081671 & -0.0150562 \\ -1.51373 & 0.973044 & -0.499373 & 0.143089 & -0.0143649 \\ 1.02981 & -2.2813 & 1.08877 & -0.207713 & 0.0152381 \\ 1.63807 & -2.33096 & 0.889904 & -0.13698 & 0.00838439 \\ 2.75972 & -2.90901 & 0.963495 & -0.132943 & 0.00746306 \\ 5.31962 & -4.88215 & 1.5185 & -0.200919 & 0.0105356 \\ 12.6866 & -11.5004 & 3.44131 & -0.431006 & 0.0203911 \\ 37.5486 & -34.5959 & 9.73796 & -1.1168 & 0.0467693 \end{pmatrix}.$$

This fit is smooth and approximates the function well except for the region where we assume the wave functions to be exponentially small. The later region is therefore negligible.

3.2.2 Integrating the Potential

We would like to solve the last remaining integral in $\mathcal{B}$. As an abbreviation, we define the matrix $\mathcal{C}$:

$$\mathcal{C}_{kj} = \int\limits_{r_+} r^2 \mathrm{e}^{-2\mu r} L_k^{(2)}(2\mu r) L_j^{(2)}(2\mu r) V(r) \tag{3.23}$$

Using a certain representation of the associated Laguerre polynomials [14, 15],

$$L_a^{(b)}(x) = \sum_{t=0}^{a} (-1)^t \frac{(a+b)!}{(a-t)!(b+t)!t!} x^t, \tag{3.24}$$

we obtain

$$\begin{aligned} \mathcal{C}_{kj} &= \sum_{t_1=0}^{k} \sum_{t_2=0}^{j} (-1)^{t_1+t_2} \frac{(k+2)!(j+2)!}{(k-t_1)!(2+t_1)!t_1!(j-t_2)!(2+t_2)!t_2!} \int\limits_{r_+} r^2 \mathrm{e}^{-2\mu r} (2\mu r)^{t_1+t_2} V(r) \\ &= \sum_{t_1=0}^{k} \sum_{t_2=0}^{j} \underbrace{\frac{(-1)^{t_1+t_2}(k+2)!(j+2)!(2\mu)^{t_1+t_2}}{(k-t_1)!(2+t_1)!t_1!(j-t_2)!(2+t_2)!t_2!}}_{\mathbb{F}(t_1,t_2,k,j,\mu)} \int\limits_{r_+} r^2 \mathrm{e}^{-2\mu r} r^{t_1+t_2} V(r). \end{aligned} \tag{3.25}$$

With the piecewise approximation of the potential, we can write $\mathcal{C}$ as

$$\begin{aligned} \mathcal{C}_{kj} = \sum_{t_1 t_2 h p} \mathbb{F}(t_1, t_2, k, j, \mu) a_{hp} \int\limits_{h}^{h+1} \mathrm{d}r\, r^{2+t_1+t_2+p} \mathrm{e}^{-2\mu r} \\ + \sum_{t_1 t_2 p} \mathbb{F}(t_1, t_2, k, j, \mu) a_{8p} \int\limits_{8}^{\infty} \mathrm{d}r\, r^{2+t_1+t_2+p} \mathrm{e}^{-2\mu r}. \end{aligned} \tag{3.26}$$

First, we use that

$$\int\limits_{h}^{h+1} \mathrm{d}r\, r^{2+t_1+t_2+p} \mathrm{e}^{-2\mu r} = (2\mu)^{-(p+t_1+t_2+3)} \Big[\Gamma(p+t_1+t_2+3, 2h\mu) - \Gamma(p+t_1+t_2+3, 2(h+1)\mu)\Big]$$

where $\Gamma(a, b)$ is the upper incomplete Gamma function [14, 15],

$$\Gamma(a, b) = \int\limits_{b}^{\infty} t^{a-1} \mathrm{e}^{-t} \mathrm{d}t, \tag{3.27}$$

and that (for $\mu > 0$)

$$\int\limits_{8}^{\infty} \mathrm{d}r\, r^{2+t_1+t_2+p} \mathrm{e}^{-2\mu r} = 8^{p+t_1+t_2+3} E_{-2-p-t_1-t_2}(16\mu), \tag{3.28}$$

where $E_a(x)$ is the exponential integral function [14, 15]

$$E_a(x) = \int\limits_{1}^{\infty} t^{-a} \mathrm{e}^{-xt} \mathrm{d}t. \tag{3.29}$$

Finally, we have an analytic expression for $\mathcal{C}$, which reads

$$\begin{aligned}\mathcal{C}_{kj} = \sum_{t_1 t_2 h p} \mathbb{F}(t_1, t_2, k, j, \mu) a_{hp} (2\mu)^{-(p+t_1+t_2+3)} \Big[\Gamma(p + t_1 + t_2 + 3, 2h\mu) \\ - \Gamma(p + t_1 + t_2 + 3, 2(h+1)\mu) \Big] \\ + \sum_{t_1 t_2 p} \mathbb{F}(t_1, t_2, k, j, \mu) a_{8p} 8^{p+t_1+t_2+3} E_{-2-p-t_1-t_2}(16\mu) \end{aligned} \tag{3.30}$$

We have now successfully simplified our Bethe-Salpeter equation into an eigenvalue equation. The eigenvalues of $\mathcal{M}$ are M^2, the squared masses of the bound state. In Section 4.1, we will present our results for the mass M over a variable constituent mass m_q.

3.3 Normalization

Since the Salpeter equation is a homogenous equation, the solution of this equation, i.e. the Salpeter amplitude Φ, is known up to a multiplicative factor: $n_\Phi \Phi$. A normalization can be imposed by means of the inhomogenous Bethe-Salpeter equation, which reads [19,22,23]

$$P^\mu = \overline{\Psi} \left(\frac{\partial}{\partial P_\mu} (S_1 \otimes S_2) - S_1\, S_2 \left(\frac{\partial}{\partial P_\mu} K \right) S_1\, S_2 \right) \Psi. \tag{3.31}$$

In terms of the components of the Salpeter amplitude, this can be written as [21]

$$||\Phi||^2 = 4|n_\Phi|^2 N_c \int_{\mathbf{k}} \Big[\varphi_1^*(\mathbf{k}) \varphi_2(\mathbf{k}) + \varphi_2^*(\mathbf{k}) \varphi_1(\mathbf{k}) \Big]. \tag{3.32}$$

The factor N_c follows from $\mathrm{Tr}_c\{\mathbb{1}\} = N_c$ and will be set to $N_c = 3$. Using Equation (2.40b) we can express φ_1 in terms of φ_2 and use the basis expansion $\varphi_2(p) = v_j \widetilde{\phi}_j^{(0)}(p)$

$$||\Phi||^2 = 12|n_\Phi|^2 \int_{\mathbf{k}} \left[\frac{M}{2E(\mathbf{k})} \varphi_2(\mathbf{k}) \varphi_2(\mathbf{k}) + c.c. \right] \tag{3.33a}$$

$$= 12|n_\Phi|^2 \int_{k_+} \left[\frac{4\pi k^2}{(2\pi)^3} \frac{M}{2E(k)} \frac{1}{\sqrt{4\pi}} \varphi_2(k) \frac{1}{\sqrt{4\pi}} \varphi_2(k) + c.c. \right] \tag{3.33b}$$

$$= \frac{12|n_\Phi|^2 M}{(2\pi)^3} \int_{k_+} \left[\frac{k^2}{2\sqrt{k^2 + m_q^2}} v_i v_j \widetilde{\phi}_i^{(0)}(k) \widetilde{\phi}_j^{(0)}(k) + c.c. \right] \tag{3.33c}$$

$$= \frac{3|n_\Phi|^2 M}{2\pi^3} \int_{k_+} \left[\frac{k^2}{\sqrt{k^2 + m_q^2}} v_i v_j \widetilde{\phi}_i^{(0)}(k) \widetilde{\phi}_j^{(0)}(k) \right]. \tag{3.33d}$$

Equation (3.31) implies that the norm of the Salpeter amplitde $||\Phi||^2$ is related to the norm of the bound state vector $|\pi\rangle$ we introduced in Equation (2.3)

$$||\Phi||^2 = \frac{(2\pi)^3 2P^0}{(2\pi)^3} \quad \text{for} \quad \big\langle \pi(\mathbf{P}) \big| \pi(\mathbf{P}') \big\rangle = (2\pi)^3 2P^0 \delta^{(3)}(\mathbf{P} - \mathbf{P}'). \tag{3.34}$$

We choose a covariant normalization, such that the norm of the Salpeter amplitude should yield

$$||\Phi||^2 = 2P^0 = 2M.$$

With this, we are able to calculate the value of n_Φ, such that the normalization in Equation (3.31) is fulfilled. The normalization n_Φ is shown in Figure 3.2.

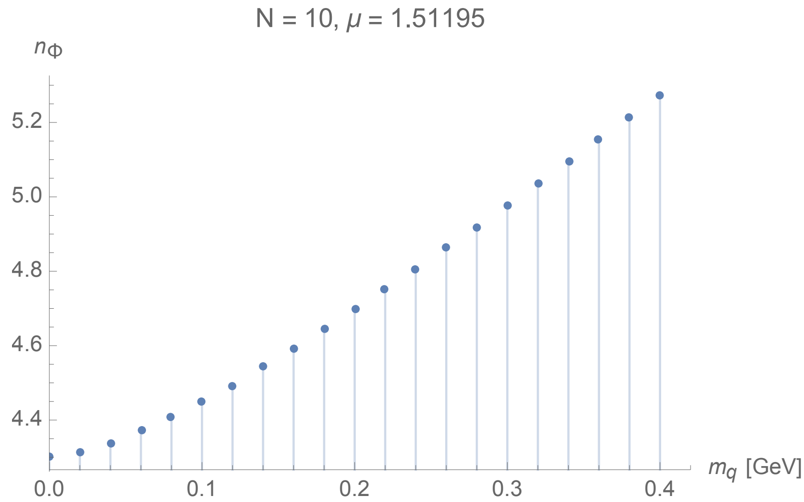


FIGURE 3.2: The normalization of the Salpeter amplitude, n_Φ, for various values of m_q at a matrix size of $N = 10$ and $\mu = 1.5$.

3.4 In-Hadron Condensate and Decay Constant

In this section, we will calculate the in-hadron condensate and the decay constant using the eigenvectors of our eigenvalue problem, defined in Equation (3.15). A pseudoscalar Salpeter amplitude[1] is given in terms of its components as

$$\Phi_P(\mathbf{p}) = \left[\varphi_1(\mathbf{p})\frac{H(\mathbf{p})}{E(\mathbf{p})} + \varphi_2(\mathbf{p})\right]\gamma^5. \tag{3.35}$$

The component φ_1 of this Salpeter amplitude can be expressed in terms of the second one via the analytic equation (3.1b) and the second component φ_2 is the one we expanded in a Laguerre basis in the previous chapter:

$$\varphi_1(\mathbf{p}) = \frac{M}{2E(\mathbf{p})}\varphi_2(\mathbf{p}), \qquad \varphi_2(\mathbf{p}) = \frac{1}{\sqrt{4\pi}}\varphi_2(p) = \frac{1}{\sqrt{4\pi}}v_j\widetilde{\phi}_j^{(0)}(p). \tag{3.36}$$

Inserting this yields the Salpeter amplitude

$$\Phi_P(\mathbf{p}) = \varphi_2(\mathbf{p})\left[\frac{M\gamma^0\boldsymbol{\gamma}\cdot\mathbf{p} + M\gamma^0 m_q}{2(\mathbf{p}^2 + m_q^2)} + 1\right]\gamma^5 \tag{3.37a}$$

$$=: \left[f_1^j(\mathbf{p})\gamma^0\gamma^j\gamma^5 + f_2(\mathbf{p})\gamma^0\gamma^5 + f_3(\mathbf{p})\gamma^5\right]. \tag{3.37b}$$

Here we used some abbreviations to emphasize the Dirac gamma matrix structure of Φ,

$$f_1^j(\mathbf{p}) = \frac{\varphi_2(\mathbf{p})M\mathbf{p}^j}{2(\mathbf{p}^2 + m_q^2)}, \quad f_2(\mathbf{p}) = \frac{\varphi_2(\mathbf{p})Mm_q}{2(\mathbf{p}^2 + m_q^2)}, \quad f_3(\mathbf{p}) = \varphi_2(\mathbf{p}). \tag{3.38}$$

In order to do the following calculations, we will need to evaluate expressions such as

$$\langle 0|\overline{\psi}(x)\mathcal{O}\psi(x)|\pi(\mathbf{P})\rangle, \tag{3.39}$$

[1] Since we assumed equal quark masses, $m_u = m_d$, we will ignore the isospin structure from here on. There is no longer a mass difference between the pions $\pi^\pm$ and π^0, due to the assumption of equal quark masses.

where $\mathcal{O}$ is some matrix in Dirac spinor space. If we pull this operator out of the bra-ket term, we can write this in terms of a Bethe-Salpeter amplitude,

$$\langle 0|\overline{\psi}(x)\mathcal{O}\psi(x)|\pi(\mathbf{P})\rangle = \langle 0|\overline{\psi}_\alpha(x)\mathcal{O}_{\alpha\beta}\psi_\beta(x)|\pi(\mathbf{P})\rangle \tag{3.40a}$$

$$= \mathcal{O}_{\alpha\beta}\langle 0|\overline{\psi}_\alpha(x)\psi_\beta(x)|\pi(\mathbf{P})\rangle \tag{3.40b}$$

$$= \mathcal{O}_{\alpha\beta}\int_{q^\mu} \mathrm{Tr}_c\,\Psi_{\pi,\beta\alpha}(q;P) \tag{3.40c}$$

$$= \int_{q^\mu} \mathrm{Tr}_D\,\mathrm{Tr}_c\{\mathcal{O}\Psi_\pi(q;P)\}. \tag{3.40d}$$

Next, we switch to the pion's rest frame and integrate over q^0 to arrive at the Salpeter amplitude (including the correct normalization, cf. Section 3.3), $n_\Phi\Phi$, which we then expand according to Equation (3.37b):

$$\langle 0|\overline{\psi}(x)\mathcal{O}\psi(x)|\pi(\mathbf{0})\rangle = \int_{q^\mu} \mathrm{Tr}_D\,\mathrm{Tr}_c\{\mathcal{O}\Psi_\pi(q;P)\} \tag{3.41a}$$

$$= n_\Phi \int_{\mathbf{q}} \mathrm{Tr}_D\,\mathrm{Tr}_c\{\mathcal{O}\Phi_\pi(q)\} \tag{3.41b}$$

$$= n_\Phi \underbrace{\mathrm{Tr}_c\{\mathbb{1}\}}_{N_c=3}\Bigg[\int_{\mathbf{q}} f_1^j(\mathbf{q})\,\mathrm{Tr}_D\{\mathcal{O}\gamma^0\gamma^j\gamma^5\}$$

$$+\int_{\mathbf{q}} f_2(\mathbf{q})\,\mathrm{Tr}_D\{\mathcal{O}\gamma^0\gamma^5\} + \int_{\mathbf{q}} f_3(\mathbf{q})\,\mathrm{Tr}_D\{\mathcal{O}\gamma^5\}\Bigg]. \tag{3.41c}$$

The in-hadron condensate $\mathbb{C}$. The condensate is given by this expression (See Ref. [10] and Equation (3.42)):

$$\langle 0|\overline{\psi}\mathrm{i}\gamma^5\psi|\pi(\mathbf{P})\rangle = \mathbb{C} \quad \leftrightarrow \quad \mathcal{O} = \mathrm{i}\gamma^5, \tag{3.42}$$

where we chose the normalization as in Equation (3.34). We use the following identities of Dirac gamma matrices,

$$\mathrm{Tr}_D\left\{\gamma^5\gamma^0\gamma^j\gamma^5\right\} = 0, \quad \mathrm{Tr}_D\left\{\gamma^5\gamma^0\gamma^5\right\} = 0, \quad \mathrm{Tr}_D\left\{\gamma^5\gamma^5\right\} = 4,$$

such that the only remaining term in Equation (3.41c) is the one with f_3. Therefore, the condensate is given by

$$\mathbb{C}_\pi = 12\,\mathrm{i}\,n_\Phi \int_{\mathbf{q}} f_3(\mathbf{q}) \tag{3.43a}$$

$$= 12\,\mathrm{i}\,n_\Phi \int_{\mathbf{q}} \varphi_2(\mathbf{q}) \tag{3.43b}$$

$$= 12\,\mathrm{i}\,n_\Phi \int_{q_+} \frac{4\pi q^2}{(2\pi)^3}\frac{1}{\sqrt{4\pi}}\varphi_2(q). \tag{3.43c}$$

This means, we can calculate the in-hadron condensate $\mathbb{C}$ using the eigenvectors of our eigenvalue equation:

$$\boxed{\mathbb{C}_\pi = 3\pi^{-3/2}\,\mathrm{i}\,n_\Phi v_j \int_{q_+} q^2\widetilde{\phi}_j^{(0)}(q).} \tag{3.44}$$

The results for the condensate are presented in Section 4.3.

Calculating the Decay Constant. The decay constant of a charged pseudoscalar meson is defined via [24–26]

$$\langle 0|{j_A}^\mu(x)|\pi(P)\rangle = \mathrm{i}P^\mu f_\pi \mathrm{e}^{-\mathrm{i}P\cdot x}, \tag{3.45}$$

where ${j_A}^\mu = \overline{\psi}\gamma^\mu\gamma^5\psi$ is an axial-vector current and we chose the normalization as in Equation (3.34). We will look at the case $\mu = 0$ and set $x^\mu = 0$, i.e. $\mathcal{O} = \gamma^0\gamma^5$. We use the following identities of Dirac gamma matrices,

$$\mathrm{Tr}_D\left\{\gamma^0\gamma^5\gamma^0\gamma^j\gamma^5\right\} = 0, \quad \mathrm{Tr}_D\left\{\gamma^0\gamma^5\gamma^0\gamma^5\right\} = -4, \quad \mathrm{Tr}_D\left\{\gamma^0\gamma^5\gamma^5\right\} = 0.$$

This means, that the only remaining term in Equation (3.41c) is the one containing f_2:

$$\mathrm{i}Mf_\pi = -12n_\Phi \int\limits_{\mathbf{q}} f_2(\mathbf{q}). \tag{3.46}$$

We now substitute f_2 according to Equation (3.38) and change to spherical coordinates:

$$\begin{aligned} Mf_\pi &= 12\,\mathrm{i}\,n_\Phi \int\limits_{\mathbf{q}} f_2(\mathbf{q}) && (3.47\text{a}) \\ &= 12\,\mathrm{i}\,n_\Phi \int\limits_{\mathbf{q}} \frac{\varphi_2(\mathbf{q})Mm_q}{2(\mathbf{q}^2+m_q^2)} && (3.47\text{b}) \\ &= 12\,\mathrm{i}\,Mn_\Phi \int\limits_{q_+} \frac{4\pi q^2}{(2\pi)^3}\frac{1}{\sqrt{4\pi}}\frac{\varphi_2(q)m_q}{2(q^2+m_q^2)} && (3.47\text{c}) \\ &= \frac{3}{2}\,\mathrm{i}\,\pi^{-3/2}Mm_qn_\Phi \int\limits_{q_+} q^2\frac{\varphi_2(q)}{q^2+m_q^2}. && (3.47\text{d}) \end{aligned}$$

Using the expansion in the Laguerre basis, we arrive at the result how to calculate the decay constant:

$$\boxed{f_\pi = \frac{3}{2}\,\mathrm{i}\,\pi^{-3/2}m_qn_\Phi v_j \int\limits_{q_+} \frac{q^2}{q^2+m_q^2}\widetilde{\phi}_j^{(0)}(q).} \tag{3.48}$$

Chapter 4

Results

We started from the definition of a Bethe-Salpeter amplitude and calculated the mass of a pseudoscalar bound state, as well as its decay constant and the in-hadron condensate. This was possible under the assumptions that the interactions happen instantaneously and that the quarks porpagate freely with a constant mass. In this chapter, we present our results as a function of the constituent quark mass. The only other free parameter is μ, which appeared in the Laguerre basis expansion due to dimensional analysis. In Section 4.1, we present the eigenvalue, i.e. the mass of the bound state, whereas Section 4.2 shows the eigenvectors. Section 4.3 presents the in-hadron condensate and the decay constant. Finally, in Section 4.4, we compare our results to the Gell-Mann-Oakes-Renner relation.

Appendix A.7 shows that the explicit choice of μ does indeed alter the results, although only in the case of a small matrix, i.e. few basis elements. For larger matrices, the choice of μ becomes more and more irrelevant.

4.1 Mass Eigenvalue M

The mass of the bound state appeared as eigenvalue in the matrix approximation to the Salpeter equation,

$$\mathcal{M}_{ij} v_j = M^2 v_i. \tag{4.1}$$

Since $\mathcal{M}$ is an $N \times N$ matrix, it can have up to N distinct eigenvalues. And because we are interested in the ground state, we only consider the smallest eigenvalue. Due to the fact that our problem is not self-adjoint, the eigenvalues M^2 are not guaranteed to be positive, or even real numbers. Figure 4.1 shows a linear relation between the mass of the bound state M and the quark constituent mass m_q. Even if we look at very small values of the constituent quark mass m_q, we see a linear relation, as shown in Figure 4.2, that is

$$\left.\frac{\mathrm{d}M}{\mathrm{d}m_q}\right|_{m_q=0} \neq 0. \tag{4.2}$$

4.2 Eigenvectors φ_2

The eigenvectors v_i together with the basis functions $\widetilde{\phi}_i$ build up the second component of the Salpeter amplitude, φ_2. Figure 4.3 presents the function φ_2 plotted over its argument, the absolute value of the three-momentum. It very closely matches Fig. 2 of Ref. [7]. This shows, that our transformation from the Salpeter equation to an eigenvalue equation did not alter the original equation.

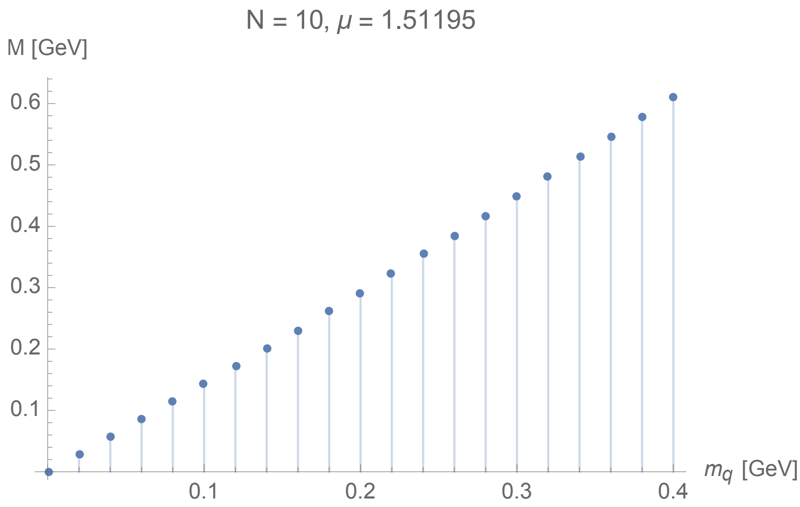


FIGURE 4.1: The mass of the pion as given by Equation (3.15) at $N = 10$, $\mu = 1.51195$.

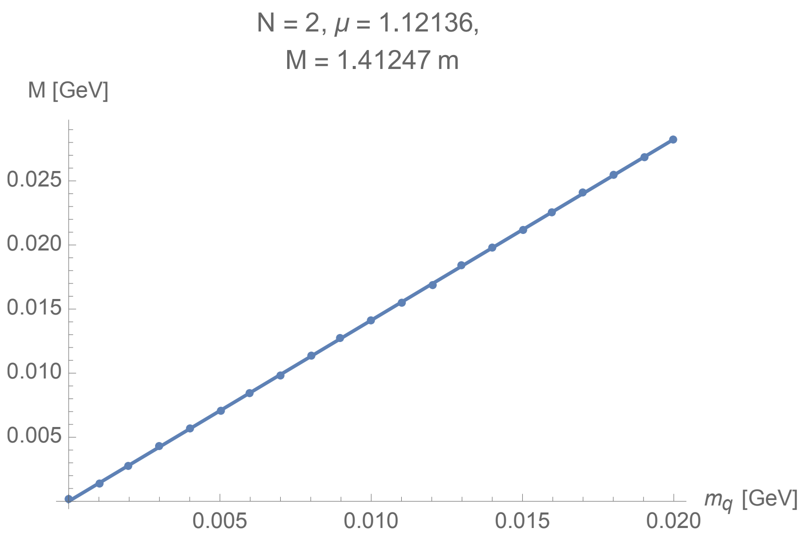


FIGURE 4.2: Even for very small m_q, the mass of the pion is linearly proportional to the constituent quark mass.

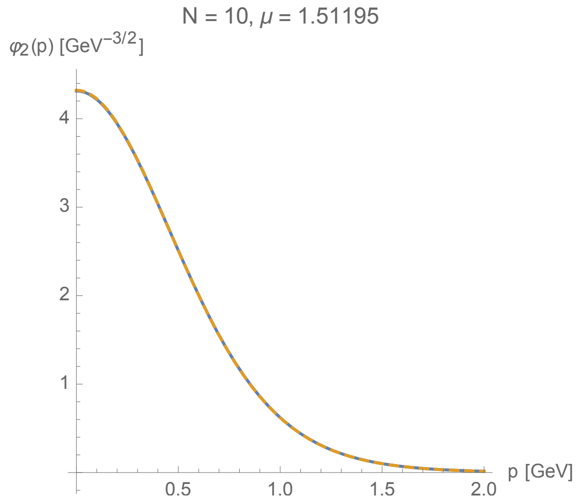


FIGURE 4.3: Blue: The second component for the Salpeter amplitude for $m_q = 0$, which appeared as eigenvector in Equation (3.2). Orange, dashed: Eq. 10 of Ref. [7] which was used to construct the potential $V(r)$.

4.3 In-Hadron Condensate and Decay Constant

We calculate the in-hadron condensate according to Equation (3.44),

$$\mathbb{C}_\pi = 3\pi^{-3/2}\,\mathrm{i}\, n_\Phi v_j \int\limits_{q+} q^2 \widetilde{\phi}_j^{(0)}(q). \tag{4.3}$$

Section 3.4 showed how to calculate the decay constant, again using the eigenvectors:

$$f_\pi = \frac{3}{2}\,\mathrm{i}\,\pi^{-3/2} m_q n_\Phi v_j \int\limits_{q+} \frac{q^2}{q^2 + m_q^2} \widetilde{\phi}_j^{(0)}(q). \tag{4.4}$$

Figure 4.4 shows the decay constant f_π over a range of values for the constituent mass m_q. The in-hadron condensate is shown in Figure 4.5.

It can be shown that in the chiral limit, the in-hadron condensate reduces to the vacuum condensate divided by the decay constant [27],

$$\mathbb{C}_{\text{hadron}} \to \frac{\langle \bar{q} q \rangle}{f_{\text{hadron}}}. \tag{4.5}$$

4.4 Comparing to the Gell-Mann-Oakes-Renner Relation

The generalized Gell-Mann-Oakes-Renner relation (1.13),

$$f_\pi M_\pi^2 = 2m\mathbb{C}_\pi, \tag{4.6}$$

contains the “current” quark mass m, which is the mass parameter in the Lagrangian. Until now, we calculated M, the decay constant f_π and the in-hadron condensate $\mathbb{C}_\pi$ for different values of the constituent quark mass m_q. This means, we can rearrange the equation such that

$$f_\pi M_\pi^2 = 2m\mathbb{C} \quad \to \quad \frac{f_\pi M_\pi^2}{2\mathbb{C}} = m,$$

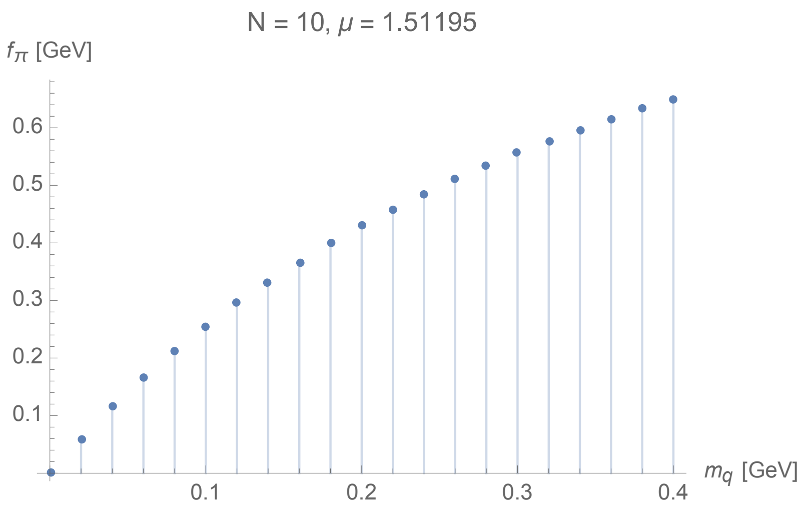


FIGURE 4.4: The pion's decay constant f_π, as it appeared in Equation (3.48).

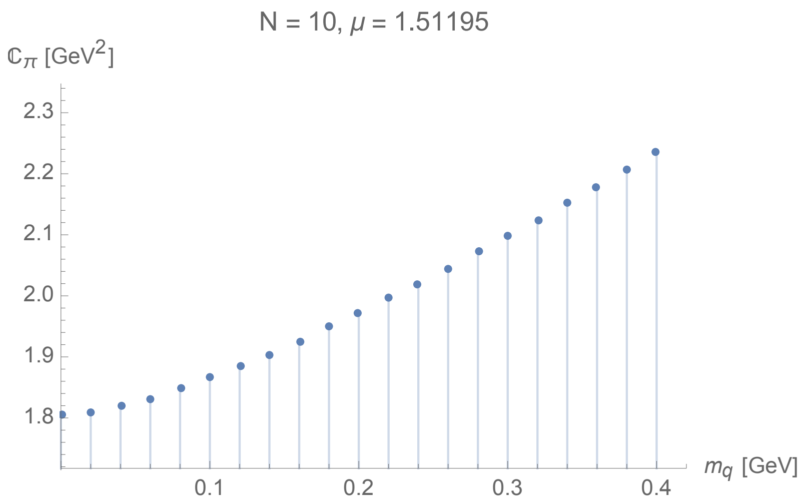


FIGURE 4.5: The pion's in-hadron condensate $\mathbb{C}_\pi$, as it appeared in Equation (3.44).

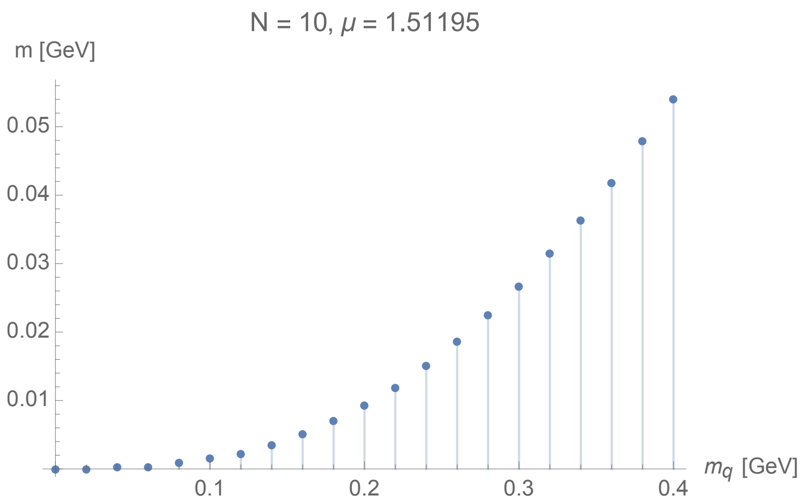


FIGURE 4.6: The current quark mass, calculated by means of rearranging the Gell-Mann-Oakes-Renner relation.

and we can investigate which value of m_q leads to a current quark mass of some MeV, which is in range of the u and d quarks. Figure 4.6 shows how the quark mass parameter in the Lagrangian should behave in order for the generalized Gell-Mann-Oakes-Renner relation to be fulfilled away from the chiral limit.

Chapter 5

Conclusion

In this thesis we started from the Bethe-Salpeter equation with the intent to describe bound states in a quantum field theoretical framework. We arrived at the Salpeter equation and successfully rewrote it into an eigenvalue equation. The components of the Salpeter amplitude are given as the eigenvectors, while the corresponding mass is given by the eigenvalue. This enabled us to calculate the in-hadron condensate and the decay constant using the eigenvectors.

We were able to show that using the potential derived in Ref. [7], the mass of the pion approaches zero as the mass of the quarks goes to zero, which was an assumption used to calculate the potential.

Within our framework, the generalized Gell-Mann-Oakes-Renner, i.e. replacing the vacuum condensate by the in-hadron condensate, is valid for vanishing quark masses m, since the product $f_\pi M_\pi$ goes to zero as m goes to zero, cf. Figure 4.4. For larger quark masses, the generalized Gell-Mann-Oakes-Renner relation can be fulfilled for certain constituent quark masses m_q. The mass of the pion, as well as its decay constant, appear larger than they are experimentally measured. A possible explanation for this is, that the potential used in this thesis was determined under the assumption of vanishing quark masses. However, in Ref. [7] the authors show that for larger quark masses, the potential grows wider. This allows for lower eigenvalues, which may explain the overestimation in our calculations.

There are three ways to extend this thesis. First, on could use bigger matrices at the cost of more laborious calculations. For larger N, the fact that our Hamiltonian is not self-adjoint should play a more and more inferior role, meaning that one should be able to find ground state solutions for almost any μ. Second, renormalization could be considered. Ref. [10] presented calculations including renormalization coefficients, whereas for this thesis we skipped the aspect of renormalization. Third, it might be possible to further investigate the Bethe-Salpeter equation without restraining oneself to either instantaneous interactions, equal quark masses or free propagation of the quarks. Also, a different spinor space interaction kernel than the BJK structure might lead to different results.

Appendix A

Detailed Calculations

A.1 Axial-Vector Ward-Takahashi Identity

Ward-Takahashi identities are useful relations of $n+1$-point and n-point Green functions. A general $n+1$-point Green function reads,

$$G_X(x, x_1, \ldots, x_n) = \langle 0| \,\mathsf{T} j(x)\psi(x_1)\ldots\overline{\psi}(x_n)\, |0\rangle\,, \tag{A.1}$$

where X denotes whether it is a (pseudo)scalar or (axial-)vector Green function and j is a current interacting with n quarks. For Chapter 1, we are interested in the 3-point function, where an isovector axial-vector current interacts with two quark fields ψ,

$$G_{A\,\alpha\beta}^{\mu,a}(x, x_1, x_2) = \langle 0| \,\mathsf{T} {j_A}^{\mu,a}(x)\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\, |0\rangle\,. \tag{A.2}$$

Following the calculations in [30], Ward-Takahashi identities follow from taking a derivative of a time-ordered product (we use $\Theta_{x>y}$ as a shorthand for $\Theta(x^0-y^0)$ and δ_{xy} for $\delta(x^0-y^0)$),

$$\begin{aligned}
\partial_x\big[\mathsf{T} j(x)\psi(y)\overline{\psi}(z)\big] &= \partial_x\big[\Theta_{x>y}\Theta_{y>z}j\psi\overline{\psi} + \Theta_{x>z}\Theta_{z>y}j\overline{\psi}\psi + \Theta_{y>x}\Theta_{x>z}\psi j\overline{\psi} \\
&\quad + \Theta_{y>z}\Theta_{z>x}\psi\overline{\psi}j + \Theta_{z>x}\Theta_{x>y}\overline{\psi}j\psi + \Theta_{z>y}\Theta_{y>x}\overline{\psi}\psi j\big] \\
&= \mathsf{T}(\partial_x j)\psi\overline{\psi} + \delta_{xy}\Theta_{y>z}j\psi\overline{\psi} + \delta_{xz}\Theta_{z>y}j\overline{\psi}\psi - \delta_{xy}\Theta_{x>z}\psi j\overline{\psi} \\
&\quad + \delta_{xz}\Theta_{y>x}\psi j\overline{\psi} - \delta_{xz}\Theta_{y>z}\psi\overline{\psi}j - \delta_{xz}\Theta_{x>y}\overline{\psi}j\psi \\
&\quad + \delta_{xy}\Theta_{z>x}\overline{\psi}j\psi - \delta_{xy}\Theta_{z>y}\overline{\psi}\psi j \\
&= \mathsf{T}(\partial_x j)\psi\overline{\psi} + \delta_{xy}\Theta_{y>z}\big[j,\psi\big]\overline{\psi} + \delta_{xy}\Theta_{z>y}\overline{\psi}\big[j,\psi\big] \\
&\quad + \delta_{xz}\Theta_{z>y}\big[j,\overline{\psi}\big]\psi + \delta_{xz}\Theta_{y>z}\psi\big[j,\overline{\psi}\big] \\
&= \mathsf{T}(\partial_x j)\psi\overline{\psi} + \delta_{xy}\mathsf{T}\big[j,\psi\big]\overline{\psi} + \delta_{xz}\mathsf{T}\psi\big[j,\overline{\psi}\big],
\end{aligned}$$

where we collected terms to restore the time ordering and used the identity of differentiating a Heaviside step function,

$$\partial^x_\mu\Theta(x^0-y^0) = \delta(x^0-y^0)\delta_{0\mu}.$$

Putting this into a vacuum expectation value and using Equation (A.2) yields,

$$\begin{aligned}
\partial^x_\mu {G_A}^{\mu,a}(x, x_1, x_2) &= \langle 0|\,\mathsf{T}\big(\partial_\mu {j_A}^{\mu,a}(x)\big)\psi(x_1)\overline{\psi}(x_2)\,|0\rangle \\
&\quad + \delta(x^0-x_1^0)\,\langle 0|\,\mathsf{T}\big[{j_A}^{0,a}(x),\psi(x_1)\big]\overline{\psi}(x_2)\,|0\rangle \\
&\quad + \delta(x^0-x_2^0)\,\langle 0|\,\mathsf{T}\psi(x_1)\big[{j_A}^{0,a}(x),\overline{\psi}(x_2)\big]\,|0\rangle\,.
\end{aligned} \tag{A.3}$$

We now need to calculate three expressions: the Noether current j, its divergence $\partial \cdot j$ and the commutator of the Noether current with a quark field $[j, \psi]$. We shall do this step by step:

1. For the Noether current, consider a quantum field theory with an action $S = \int \mathrm{d}^4x\, \mathcal{L}(\varphi, \partial\varphi)$ and an infinitesimal transformation,
$$\varphi \to \varphi' = \varphi + \delta\varphi,$$
for which the variation of the action reads,
$$\begin{aligned}\delta S &= \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial\varphi_i}\delta\varphi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\delta(\partial_\mu\varphi_i)\right]\\ &= \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial\varphi_i}\delta\varphi_i + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\delta\varphi_i\right) - \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\right)\delta\varphi_i\right]\\ &= \int \mathrm{d}^4x \left[\partial_\mu\Big(\underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\delta\varphi_i}_{=:\epsilon^a j^{\mu,a}}\Big) + \Big(\underbrace{\frac{\partial\mathcal{L}}{\partial\varphi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}}_{=0 \text{ for classical solutions}}\Big)\delta\varphi_i\right] \overset{!}{=} 0.\end{aligned}$$
Therefore, we can calculate the Noether current as follows,
$$j^{\mu,a} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\frac{\delta\varphi_i}{\delta\epsilon^a}, \tag{A.4}$$
whose divergence vanishes, if the transformation is a symmetry transformation. For the axial-vector Ward-Takahashi identity, we consider the Lie group $SU(N_f)_A$ with elements $\exp\big(-\mathrm{i}\gamma^5\epsilon^a\mathsf{t}^a\big)$. In order to calculate the corresponding Noether current, we need to know how the fields transform under this group. The quark field transforms infinitesimally as
$$\psi \to \psi' = \psi + \delta\psi = \psi - \mathrm{i}\gamma^5\epsilon^a\mathsf{t}^a\psi, \quad \text{similarly: } \delta\overline{\psi} = \mathrm{i}\overline{\psi}\gamma^5\epsilon^a\mathsf{t}^a, \tag{A.5}$$
and for the $SU(N_f)_A$ Noether current $j_A^{\mu,a}$, commonly denoted as $A^{\mu,a}$, we use Equation (A.4) to get
$$\begin{aligned}j_A{}^{\mu,a} &= \Bigg[\underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_f^\alpha)}}_{\mathrm{i}\overline{\psi}\gamma^\mu}\underbrace{\frac{\delta\psi_f^\alpha}{\delta\epsilon^a}}_{-\mathrm{i}\gamma^5\mathsf{t}^a\psi} + \underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi}_f^\alpha)}}_{0}\underbrace{\frac{\delta\overline{\psi}_f^\alpha}{\delta\epsilon^a}}_{\mathrm{i}\overline{\psi}\gamma^5\mathsf{t}^a}\Bigg]\\ &= \overline{\psi}\gamma^\mu\gamma^5\mathsf{t}^a\psi.\end{aligned} \tag{A.6}$$

2. The divergence of this current reads,
$$\begin{aligned}\partial_\mu j_A^{\mu,a} &= (\partial_\mu\overline{\psi})\gamma^\mu\gamma^5\mathsf{t}^a\psi + \overline{\psi}\gamma^\mu\gamma^5\mathsf{t}^a(\partial_\mu\overline{\psi})\\ &= \mathrm{i}\overline{\psi}\mathsf{M}\gamma^5\mathsf{t}^a\psi + \mathrm{i}g\overline{\psi}\not{A}\gamma^5\mathsf{t}^a\psi + \mathrm{i}\overline{\psi}\gamma^5\mathsf{t}^a\mathsf{M}\psi + \mathrm{i}g\overline{\psi}\gamma^5\mathsf{t}^a\not{A}\psi\\ &= \mathrm{i}\overline{\psi}\{\mathsf{M}, \mathsf{t}^a\}\gamma^5\psi\end{aligned} \tag{A.7}$$
which is known as the PCAC-relation (partially conserved axial-vector current). Here, we used the quarks' Euler-Lagrange equations (M is a diagonal matrix in flavor space, containing the quark masses in its entries),
$$\begin{aligned}[\mathrm{i}\not{D} - \mathsf{M}]\psi = 0 \quad &\to \quad \not{\partial}\psi = -\mathrm{i}\mathsf{M}\psi + \mathrm{i}g\not{A}\psi,\\ \overline{\psi}[\mathrm{i}\overleftarrow{\not{D}} + \mathsf{M}] = 0 \quad &\to \quad \overline{\psi}\overleftarrow{\not{\partial}} = \mathrm{i}\overline{\psi}\mathsf{M} - \mathrm{i}g\overline{\psi}\not{A},\end{aligned}$$
as well as the Clifford algebra $\{\gamma^\mu, \gamma^\nu\}_{\alpha\beta} = 2\eta^{\mu\nu}\delta_{\alpha\beta}$, $\{\gamma^\mu, \gamma^5\}_{\alpha\beta} = 0$.

3. For the commutator of an isovector, axial-vector current $j_A^{\mu,a} = \overline{\psi}\gamma^\mu\gamma^5 \mathsf{t}^a \psi$ with the quark fields $\psi, \overline{\psi}$, we make use of the following relation,

$$[AB, C] = A\{B, C\} - \{A, C\}B,$$

in order to use the equal-time anticommutation rule of fermion fields,

$$\{\psi_f^\alpha(t,\mathbf{x}), \psi_{f'}^{\dagger\beta}(t,\mathbf{y})\} = \delta_{\alpha\beta}\delta_{ff'}\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$
$$\{\psi_f^\alpha(t,\mathbf{x}), \psi_{f'}^{\beta}(t,\mathbf{y})\} = 0 = \{\psi_f^{\dagger\alpha}(t,\mathbf{x}), \psi_{f'}^{\dagger\beta}(t,\mathbf{y})\}.$$

For the zero-component of the current, this yields,

$$\begin{aligned}[j_A^{0,a}(x), \psi(x_1)] &= [\overline{\psi}(x)\gamma^0\gamma^5\mathsf{t}^a\psi(x), \psi(x_1)] \\ &= \overline{\psi}(x)\gamma^0\gamma^5\mathsf{t}^a \underbrace{\{\psi(x), \psi(x_1)\}}_{0} - \{\overline{\psi}(x)\gamma^0\gamma^5\mathsf{t}^a, \psi(x_1)\}\psi(x) \\ &= -\gamma^5\mathsf{t}^a\psi(x)\delta^{(3)}(\mathbf{x}-\mathbf{x}_1). \end{aligned} \tag{A.8}$$

Similarly,

$$[j_A^{0,a}(x), \overline{\psi}(x_2)] = -\overline{\psi}(x)\gamma^5\mathsf{t}^a\delta^{(3)}(\mathbf{x}-\mathbf{x}_1). \tag{A.9}$$

Let us now evaluate the Ward-Takahashi identity Equation (A.3) using Equations (A.6) to (A.9). We will assume equal quark masses ($\mathsf{M} = m\mathbb{1}$), such that the PCAC relation reads $\partial_\mu j_A^{\mu,a} = 2m\mathrm{i}\overline{\psi}\mathsf{t}^a\gamma^5\psi \equiv 2m j_P^a$ (with the pseudoscalar current $j_P^a = \overline{\psi}\mathsf{t}^a\mathrm{i}\gamma^5\psi$),

$$\begin{aligned}\partial_\mu^x {G_A}^{\mu,a}(x,x_1,x_2) &= \langle 0|\,\mathsf{T}\big(\partial_\mu j_A^{\mu,a}(x)\big)\psi(x_1)\overline{\psi}(x_2)\,|0\rangle \\ &\quad + \delta(x^0 - x_1^0)\,\langle 0|\,\mathsf{T}\big[j_A^{0,a}(x), \psi(x_1)\big]\overline{\psi}(x_2)\,|0\rangle \\ &\quad + \delta(x^0 - x_2^0)\,\langle 0|\,\mathsf{T}\psi(x_1)\big[j_A^{0,a}(x), \overline{\psi}(x_2)\big]\,|0\rangle \\ &= 2m\,\langle 0|\,\mathsf{T}j_P^a(x)\psi(x_1)\overline{\psi}(x_2)\,|0\rangle \\ &\quad - \delta^{(4)}(x-x_1)\gamma^5\mathsf{t}^a\,\langle 0|\,\mathsf{T}\psi(x)\overline{\psi}(x_2)\,|0\rangle \\ &\quad - \delta^{(4)}(x-x_2)\,\langle 0|\,\mathsf{T}\psi(x_1)\overline{\psi}(x)\,|0\rangle\,\gamma^5\mathsf{t}^a \\ &= 2m{G_P}^a(x,x_1,x_2) \\ &\quad - \delta^{(4)}(x-x_1)\gamma^5\mathsf{t}^a S(x,x_2) \\ &\quad - \delta^{(4)}(x-x_2)S(x_1,x)\gamma^5\mathsf{t}^a. \end{aligned}$$

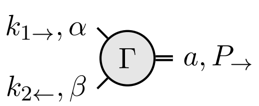
In momentum space, where P is the current's momentum and $k_\pm = k \pm P/2$ are the quarks' momenta, this equation reads,

$$\mathrm{i}P_\mu {G_A}^{\mu,a}(k,P) = 2m{G_P}^a(k,P) - \gamma^5\mathsf{t}^a S(k_-) - S(k_+)\gamma^5\mathsf{t}^a, \tag{A.10}$$

with 4-momentum conservation at every vertex. Multiplication of $S^{-1}(k_+)$ from the right and $S^{-1}(k_-)$ from the left yields an equation with the amputated vertices Γ instead of the Green functions G,

$$\mathrm{i}P_\mu {\Gamma_A}^{\mu,a}(k,P) = 2m{\Gamma_P}^a(k,P) - S^{-1}(k_+)\gamma^5\mathsf{t}^a - \gamma^5\mathsf{t}^a S^{-1}(k_-), \tag{A.11}$$

$$\Gamma_{A\,\alpha\beta}^{\mu,a}(k,P) \longleftrightarrow$$



. (A.12)

A.2 Ansatz for Three-Point Vertices

The three-point vertex we consider here connects a certain current to two quark fields. The nature of this current determines how the vertex transforms under discrete transformations like parity or charge conjugation. Following Refs. [11, 31], we need to make an ansatz for the pseudoscalar and axial-vector vertices.

At these vertices, an incoming quark with momentum k interacts with the current's momentum P and leaves as an outgoing quark of momentum $k' = k + P$. Therefore, the vertex can only depend on k and P. For a quark-scalar vertex, one can write down basis elements,

$$b_i \in \{\mathbb{1}, \not{k}, \not{P}, [\not{k}, \not{P}]\}, \tag{A.13}$$

and for a quark-vector vertex with an additional Minkowski index, the basis elements read,

$$b_i^\mu \in \{\gamma^\mu, k^\mu, P^\mu\} \times \{\mathbb{1}, \not{k}, \not{P}, [\not{k}, \not{P}]\}. \tag{A.14}$$

The vertices are now built using the basis elements and some functions, depending on k and P,

$$\begin{aligned} \Gamma_S &= \sum F_S^{(i)}(k,P)\,(b_i), & \Gamma_V^\mu &= \sum F_V^{(i)}(k,P)\,(b_i^\mu), \\ \Gamma_P &= \sum F_P^{(i)}(k,P)\,(\mathrm{i}\gamma^5 b_i), & \Gamma_A^\mu &= \sum F_A^{(i)}(k,P)\,(\gamma^5 b_i^\mu). \end{aligned} \tag{A.15}$$

Such vertices transform under parity as,

$$\begin{aligned} \Gamma(k,P) &= \eta_P \gamma^0 \Gamma(\Lambda_\mathsf{P} k, \Lambda_\mathsf{P} P)\gamma^0, \\ \Gamma^\mu(k,P) &= \eta_P \gamma^0 (\Lambda_\mathsf{P})^\mu{}_\nu \Gamma^\nu(\Lambda_\mathsf{P} k, \Lambda_\mathsf{P} P)\gamma^0, \end{aligned} \tag{A.16}$$

where η_P is the eigenvalue under a parity transformation and the transformation matrix reads $\Lambda_\mathsf{P} = \mathrm{diag}\,(1,-1,-1,-1)$. Under charge conjugation, this vertex transforms like,

$$\begin{aligned} \Gamma(k,P) &= \eta_C C \Gamma^\mathsf{T}(-k,P) C^{-1}, \\ \Gamma^\mu(k,P) &= \eta_C C (\Gamma^\mu)^\mathsf{T}(-k,P) C^{-1}, \end{aligned} \tag{A.17}$$

where η_C is the eigenvalue under a charge conjugation, A^T denotes a matrix transpose and the quark's momentum changes sign, since charge conjugation transforms a quark to its antiquark. For completeness, let us look at some P and C eigenvalues.
A pseudoscalar can be denoted as $J^{PC} = 0^{-+}$ (with spin J and P, C the eigenvalues under parity transformation and charge conjugation), hence we can make an ansatz,

$$\mathrm{i}\Gamma_P \propto \mathrm{i}\gamma^5 + \gamma^5 \not{P} + \gamma^5 \not{k} k \cdot P + [\not{k}, \not{P}] + \dots \tag{A.18}$$

Let us look at the second term in this ansatz and check for the right eigenvalues using Equation (A.16) (this should yield $\eta_P = -1$ and $\eta_C = +1$),

$$\begin{aligned} \gamma^5 \gamma^\mu P_\mu &= \eta_P \gamma^0 \gamma^5 \gamma^\mu \tilde{P}_\mu \gamma^0 \qquad (\tilde{P}^\mu = (\tilde{P}^0, \tilde{P}^i) = (P^0, -P^i)) \\ &= \eta_P \gamma^0 \gamma^5 \gamma^0 \tilde{P}_0 \gamma^0 - \eta_P \gamma^0 \gamma^5 \gamma^i \tilde{P}_i \gamma^0 \\ &= \eta_P \gamma^0 \gamma^5 \gamma^0 P_0 \gamma^0 + \eta_P \gamma^0 \gamma^5 \gamma^i P_i \gamma^0 \\ &= -\eta_P \gamma^5 \gamma^0 \gamma^0 \gamma^0 P_0 + \eta_P \gamma^5 \gamma^0 \gamma^0 \gamma^i P_i \\ &= -\eta_P \big(\gamma^5 \gamma^0 P_0 - \gamma^5 \gamma^i P_i\big) \\ &= -\eta_P \big(\gamma^5 \gamma^\mu P_\mu\big) \quad \rightarrow \quad \eta_P = -1. \end{aligned}$$

Now let us check Equation (A.17) using $C\gamma^\mu C^{-1} = -(\gamma^\mu)^T$ and $-C = C^{-1} = C^\mathsf{T}$,

$$\begin{aligned}
\gamma^5\gamma^\mu P_\mu &= \eta_C C\big(\gamma^5\gamma^\mu P_\mu\big)^\mathsf{T} C^{-1}\\
&= \eta_C C\Big(\mathrm{i}(\gamma^\mu)^\mathsf{T}(\gamma^3)^\mathsf{T}(\gamma^2)^\mathsf{T}(\gamma^1)^\mathsf{T}(\gamma^0)^\mathsf{T} P_\mu\Big)C^{-1}\\
&= \mathrm{i}\eta_C C(\gamma^\mu)^\mathsf{T}C^{-1}C(\gamma^3)^\mathsf{T}C^{-1}C(\gamma^2)^\mathsf{T}C^{-1}C(\gamma^1)^\mathsf{T}C^{-1}C(\gamma^0)^\mathsf{T}C^{-1}P_\mu\\
&= \mathrm{i}\eta_C(-1)^5\gamma^\mu\gamma^3\gamma^2\gamma^1\gamma^0 P_\mu\\
&= \eta_C(-1)^{11}\gamma^\mu\gamma^5 P_\mu\\
&= \eta_C\gamma^5\gamma^\mu P_\mu \quad\rightarrow\quad \eta_C = +1.
\end{aligned}$$

Similar calculations show that the other terms in the ansatz have the same eigenvalues. For the axial-vector vertex, we have to consider $J^{PC} = 1^{++}$, which leads to other terms,

$$(\Gamma_A)^\mu \propto \gamma^5\gamma^\mu + \gamma^5 \not{k} k^\mu + \gamma^5[\gamma^\mu, \not{k}] + \dots \tag{A.19}$$

In order to arrive at the axial-vector vertex, as seen in Chapter 1, we use the procedure to make a PC-appropriate ansatz only for terms linear in k. Every term linear in P is now contained in $\tilde{\Gamma}$, which is regular near a certain pole, and the pole itself. Since we want to investigate the pion in Chapter 1, we chose this pole to be the pion pole.

A.3 Center-of-Mass System

In a non-relativistic two-body center-of-mass system, the center-of-mass position is given by

$$\mathbf{R} = \frac{\Sigma\, m_i \mathbf{r}_i}{\Sigma\, m_i} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M} =: \eta_1\mathbf{r}_1 + \eta_2\mathbf{r}_2, \tag{A.20}$$

with $\eta_i = \frac{m_i}{M}$ and therefore, $\eta_1 + \eta_2 = 1$. Here, m_i and $\mathbf{r}_i$ are the mass and position of the i-th particle. The center-of-mass velocity is the time derivative of the center-of-mass position,

$$\mathbf{V} = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{M}. \tag{A.21}$$

The total momentum, as the sum of the individual momenta therefore reads,

$$\mathbf{P} = M\mathbf{V} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = \mathbf{p}_1 + \mathbf{p}_2. \tag{A.22}$$

In the rest frame of the second particle, the relative momentum between the two particles can be calculated as m_1 times the reduced velocity of the first particle, i.e. $\mathbf{v}_1$ minus the center-of-mass velocity,

$$\mathbf{p}_{12} \equiv \mathbf{p} = m_1(\mathbf{v}_1 - \mathbf{V}) = \eta_2\mathbf{p}_1 - \eta_1\mathbf{p}_2.$$

The system can be described by either the two absolute momenta $(\mathbf{p}_1, \mathbf{p}_2)$ or by the total and relative momenta $(\mathbf{P}, \mathbf{p})$. They are related via,

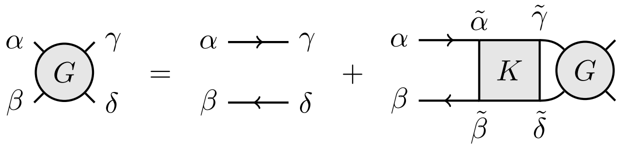
$$\begin{aligned}
\mathbf{P} &= \mathbf{p}_1 + \mathbf{p}_2, & \mathbf{p}_1 &= \eta_1\mathbf{P} + \mathbf{p},\\
\mathbf{p} &= \eta_2\mathbf{p}_1 - \eta_1\mathbf{p}_2, & \mathbf{p}_2 &= \eta_2\mathbf{P} - \mathbf{p},
\end{aligned}$$

whereas the position vectors are related by,

$$\begin{aligned}
\mathbf{R} &= \eta_1\mathbf{r}_1 + \eta_2\mathbf{r}_2, & \mathbf{r}_1 &= \mathbf{R} + \eta_2\mathbf{r},\\
\mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2, & \mathbf{r}_2 &= \mathbf{R} - \eta_1\mathbf{r}.
\end{aligned}$$

A.4 Index Structure of the Bethe-Salpeter Equation

The Bethe-Salpeter equation in its diagrammatic form reads,



, (A.23)

where greek letters represent the quarks' spinor indices. This represents the analytic expression,

$$G_{\alpha\beta\gamma\delta} = S_{\alpha\gamma}S_{\delta\beta} + S_{\alpha\tilde{\alpha}}S_{\tilde{\beta}\beta}\mathrm{i}K_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}G_{\tilde{\gamma}\tilde{\delta}\gamma\delta}. \tag{A.24}$$

For bound states, we omit the propagator-only term since there the quarks do not interact to form a bound state, and express the four-point Green function G by two Bethe-Salpeter amplitudes $G_{\alpha\beta\gamma\delta} = \Psi_{\alpha\beta}\overline{\Psi}_{\gamma\delta}$,

$$\Psi_{\alpha\beta}\overline{\Psi}_{\gamma\delta} = S_{\alpha\tilde{\alpha}}S_{\tilde{\beta}\beta}\mathrm{i}K_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}\Psi_{\tilde{\gamma}\tilde{\delta}}\overline{\Psi}_{\gamma\delta}. \tag{A.25}$$

We multiply by the inverse of $\overline{\Psi}_{\gamma\delta}$ and define $M_{\tilde{\alpha}\tilde{\beta}} = \mathrm{i}K_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}\Psi_{\tilde{\gamma}\tilde{\delta}}$,

$$\Psi_{\alpha\beta} = S_{\alpha\tilde{\alpha}}S_{\tilde{\beta}\beta}M_{\tilde{\alpha}\tilde{\beta}}, \tag{A.26}$$

this shows that, in order to be written as a matrix equation, we must rearrange terms to place pairing indices next to each other,

$$\Psi_{\alpha\beta} = S_{\alpha\tilde{\alpha}}M_{\tilde{\alpha}\tilde{\beta}}S_{\tilde{\beta}\beta}. \tag{A.27}$$

This result shows that the propagators are supposed to be at the left and right ends of the expression.

A.5 Fierz Transformations

Consider a spinor structure like,

$$(\overline{\psi}_1\Gamma\psi_2)(\overline{\psi}_3\Gamma\psi_4), \tag{A.28}$$

called a quadrilinear, where Γ is one of the 16 elements of the basis of 4×4-matrices of the Dirac bilinears, $\Gamma \in \{1, \mathrm{i}\gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}$. A Fierz transformation transforms one Dirac quadrilinear into a linear combination of other Dirac quadrilinears. This short presentation of Fierz transformations is based on Ref. [18] pp. 160–162. If we collect expressions like Equation (A.28) into a vector, it can be shown that it is equal to a linear combination of similar expressions with exchanged spinors,

$$\begin{pmatrix} S \\ P \\ V \\ A \\ T \end{pmatrix}_{12;34} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 4 & -4 & -2 & 2 & 0 \\ 4 & -4 & 2 & -2 & 0 \\ 6 & 6 & 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} S \\ P \\ V \\ A \\ T \end{pmatrix}_{14;32}, \tag{A.29}$$

where the index represents the spinor's ordering and the quadrilinears are given by,

$$S = 1 \otimes 1, \quad \rightarrow \quad S_{ij;kl} = (\overline{\psi}_i 1 \psi_j)(\overline{\psi}_k 1 \psi_l)$$

$$P = \gamma^5 \otimes \gamma^5,$$
$$V = \gamma^\mu \otimes \gamma_\mu,$$
$$A = \gamma^\mu\gamma^5 \otimes \gamma^5\gamma_\mu,$$
$$T = \tfrac{1}{2}\sigma^{\mu\nu} \otimes \sigma_{\mu\nu},$$

and the factor $\frac{1}{2}$ in the tensorial quadrilinear corrects for the over-counting when summing over both μ and ν. For instance, the vector Dirac quadrilinear can be written as,

$$(\overline{\psi}_1\gamma^\mu\psi_2)(\overline{\psi}_3\gamma_\mu\psi_4) = (\overline{\psi}_1\psi_4)(\overline{\psi}_3\psi_2) - (\overline{\psi}_1\gamma^5\psi_4)(\overline{\psi}_3\gamma^5\psi_2) - \frac{1}{2}(\overline{\psi}_1\gamma^\mu\psi_4)(\overline{\psi}_3\gamma_\mu\psi_2) + \frac{1}{2}(\overline{\psi}_1\gamma^\mu\gamma^5\psi_4)(\overline{\psi}_3\gamma^5\gamma_\mu\psi_2).$$

In this work, we demand symmetry under such Fierz transformations, that is, we choose $\Gamma \otimes \Gamma$ as an eigenvector of the above transformation. In particular, we choose the eigenvector [8],

$$\Gamma \otimes \Gamma = \frac{1}{2}\left(\gamma^\mu \otimes \gamma_\mu + \gamma^5 \otimes \gamma^5 - 1 \otimes 1\right). \quad \text{(A.30)}$$

The factor in front of the gamma matrices could in principle be absorbed into the potential, but is kept here in order to make it possible to compare our results to those of other works. We can check that it is indeed an eigenvector,

$$V\big|_{12;34} = (S - P - \frac{1}{2}V + \frac{1}{2}A)\big|_{14;32},$$
$$P\big|_{12;34} = \frac{1}{4}(S + P - V - A + T)\big|_{14;32},$$
$$S\big|_{12;34} = \frac{1}{4}(S + P + V + A + T)\big|_{14;32},$$

that is, $\frac{1}{2}(V + P - S)$ is an eigenvector with eigenvalue -1.

A.6 Structure of the Salpeter Amplitude

Following Ref. [19], a Salpeter amplitude χ can be expanded in the following basis of Dirac matrices,

$$\chi = \mathcal{L}_0 + \mathcal{L}_i\rho_i + \boldsymbol{\mathcal{N}}_0 \cdot \boldsymbol{\sigma} + \boldsymbol{\mathcal{N}}_i \cdot \rho_i\boldsymbol{\sigma}, \quad \text{(A.31)}$$

where the basis elements are given by [20],

$$\rho_1 = \gamma^5, \qquad \rho_2 = \mathrm{i}\gamma^5\gamma^0,$$
$$\rho_3 = \gamma^0, \qquad \boldsymbol{\sigma} = \boldsymbol{\gamma}\gamma^5\gamma^0.$$

According to the well-defined $PC = -+$ eigenvalues, only some terms survive,

$$\chi = \mathcal{L}_1\rho_1 + \mathcal{L}_2\rho_2 + \sin\theta L_2\hat{\mathbf{k}} \cdot \boldsymbol{\sigma} - \cos\phi L_1\hat{\mathbf{k}} \cdot \rho_3\boldsymbol{\sigma}. \quad \text{(A.32)}$$

Under the assumption of equal quark masses, we can insert the definitions given in [19] for the (co)sine functions[1] and apply trigonometric identities, so that we arrive at,

$$\chi = \left[L_1(\mathbf{p})\frac{H(\mathbf{p})}{E(\mathbf{p})} + L_2(\mathbf{p})\right]\gamma_0\gamma^5. \quad \text{(A.33)}$$

[1]The angles are in fact functions of the quarks masses, which means that the following statements are only valid for $m_1 = m_2$!

One important thing to notice is that in Ref. [19], the Salpeter amplitude of a bound state B was defined as,

$$\chi(\mathbf{p}) = \langle 0| \, \psi_1(\mathbf{p})\psi^\dagger(\mathbf{p}) \, |B\rangle \,, \tag{A.34}$$

where $\psi^\dagger$ was used instead of $\overline{\psi}$. This aspect (which is elaborated on in more detail in section 2.6 of [16]) accounts for the removal of γ^0 from Equation (A.33) to Equation (2.38).

A.7 The Choice of μ

The parameter μ appeared when we expanded the second component φ_2 of the Salpeter amplitude in a basis ϕ, which relied on Laguerre polynomials,

$$\phi_i^{(l)}(r) = \sqrt{\frac{(2\mu)^{2l+3} i!}{\Gamma(2l+i+3)}} \, r^l \exp(-\mu r) L_i^{(2l+2)}(2\mu r). \tag{A.35}$$

The parameter μ was introduced in order for the mass dimensions of μ and r to cancel within the Laguerre function. In principle, every value of μ leads to a different, but nonetheless valid basis. However, depending on the choice of μ, the mass M of the bound state would behave very differently, as seen in Figure A.1. We would expect the mass of the bound state to be zero when the constituent mass vanishes $M(m_q = 0) = 0$, since this is how the interacting potential was originally constructed, cf. Chapter 3. This behavior is based on the fact that we "only" use a finite amount of basis functions for our calculations. In the limit of $N \to \infty$, we would expect any value of μ to yield $M(m_q = 0) = 0$. And this is in fact what we found, as seen in Figure A.2. This means, that for large enough matrix sizes, the choice of the parameter μ becomes irrelevant.

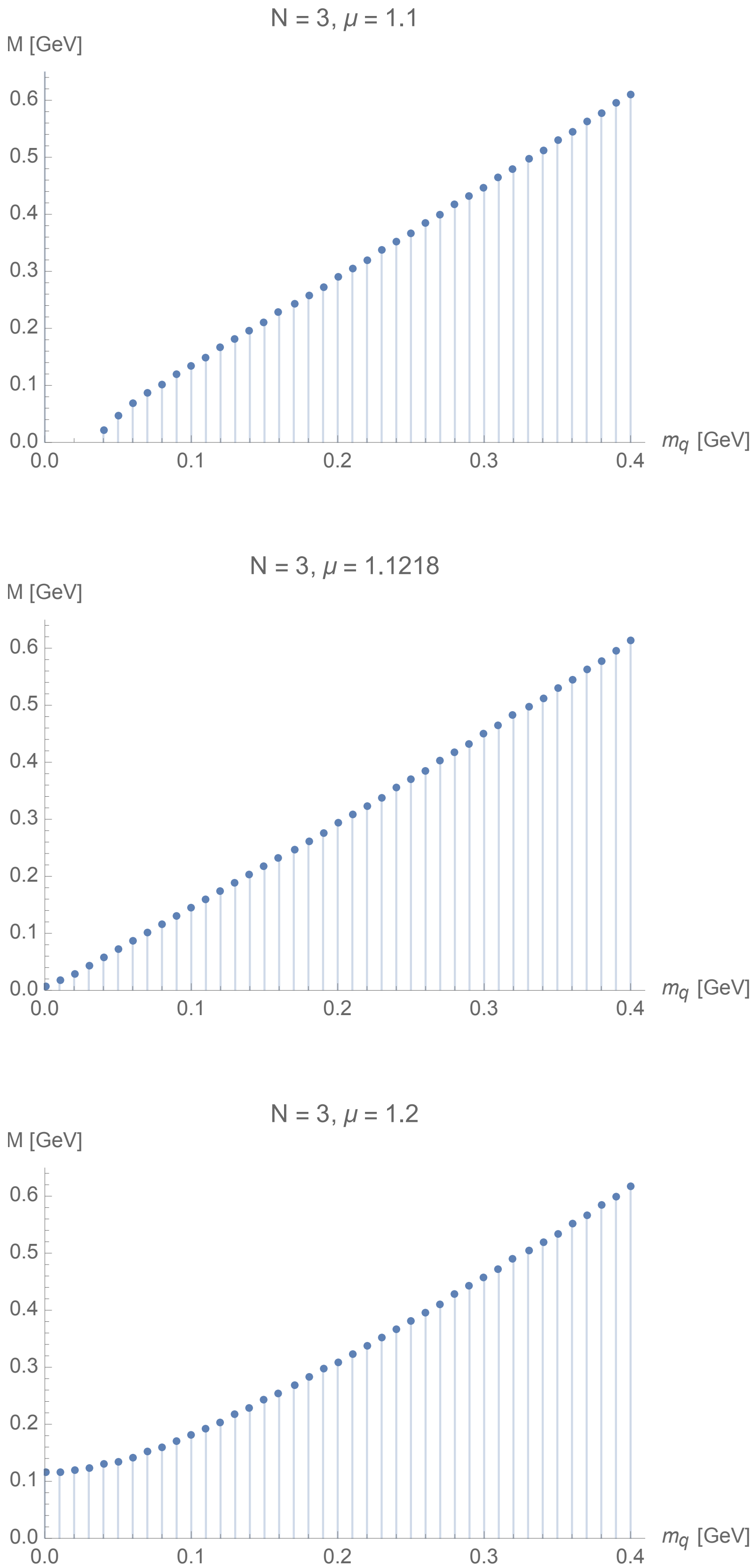


FIGURE A.1: The eigenvalues M at different values for $\mu = \{1.1, 1.1218, 1.2\}$ at $N = 3$ show how the choice of μ can influence the region of small m_q.

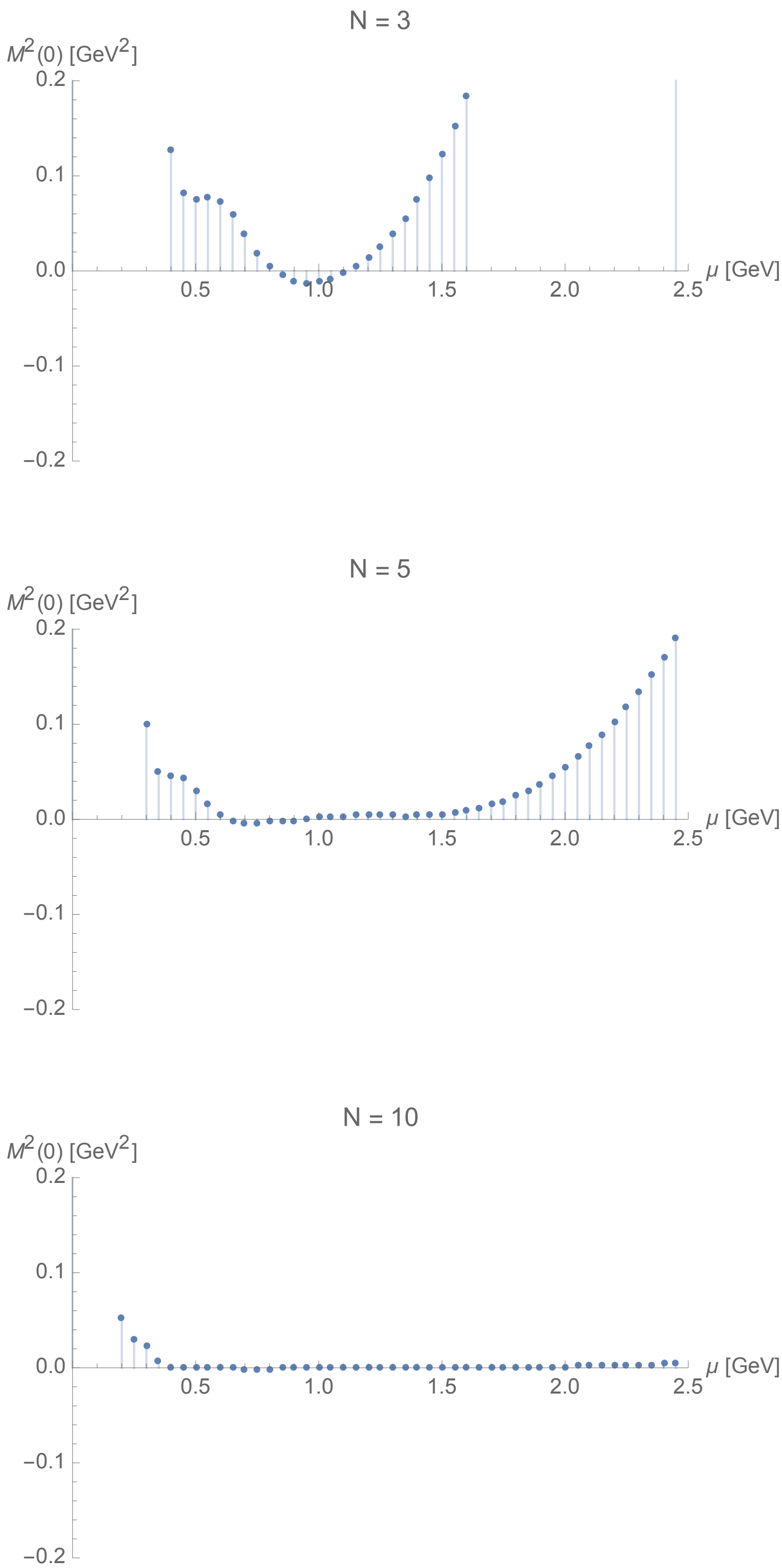


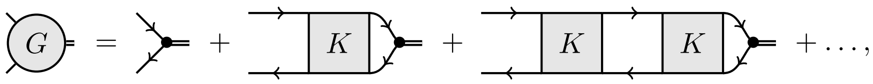
FIGURE A.2: $M^2(0)$ for different matrix sizes $N = \{3, 5, 10\}$. The bigger the matrix size, the broader the region where $M^2(0) = 0$. The first figure shows that $M^2(0) = 0$ happens somewhere between $\mu = 1.1$ and $\mu = 1.2$, which is exactly what Figure A.1 already showed.

Appendix B

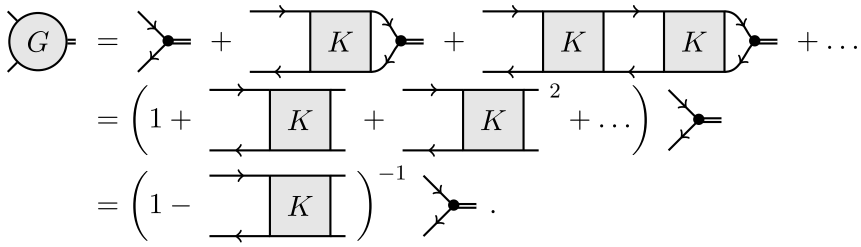
Proofs

B.1 Proof of Equation (1.4) – Dyson Equation for Three-Point Vertices

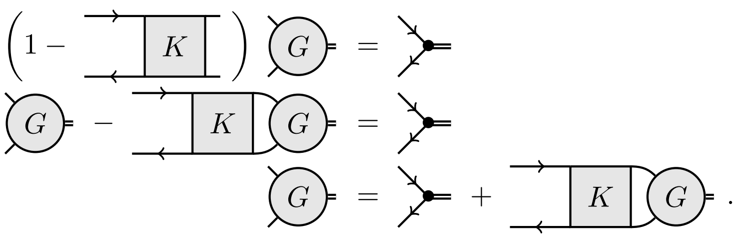
The Dyson equation emerges after sorting the occurring diagrams of a given process. For a 2-to-2 scattering process, one can arrange the diagrams in such a way that they, at first, do not interact at all. The next possibility is to interact once, via all one-particle irreducible (1PI) diagrams, abbreviated by a scattering kernel iK. Then they can interact twice and so on.



where lines with arrows represent full propagators (Note that $G = S\Gamma S$, so when we picture the Green's fuction G in a diagram, there are two propagators on the left-hand connections already included). The above equation has the form of a geometric series. This fact can be used to formally manipulate the equation in a way to get a recursive equation,

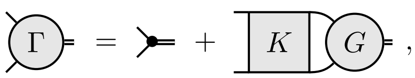


Multiplying the expression in brackets from the left yields,



(B.1)

Equation (B.1) is the Dyson equation for a 3-point Green function. In order to arrive at the equation for the truncated 3-point vertex, we remove the propagators on the left-side of each diagram,



(B.2)

which in analytic form reads,

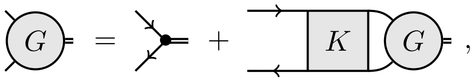
$$\Gamma_P{}^a(k,P) = \mathsf{t}^a \mathrm{i}\gamma^5 + \int\limits_{q^\mu} \mathrm{i}K(k,q,P)G_P{}^a(q,P), \tag{B.3}$$

$$\Gamma_A{}^{\mu,a}(k,P) = \mathsf{t}^a \gamma^\mu \gamma^5 + \int\limits_{q^\mu} \mathrm{i}K(k,q,P)G_A{}^{\mu,a}(q,P), \tag{B.4}$$

for a pseudoscalar and axial-vector current, respectively.

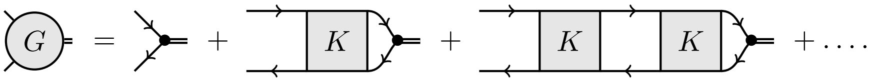
B.2 Proof of Equation (1.5) – Rewriting the Dyson Equation

Starting from the Dyson equation for Three-Point Green function,



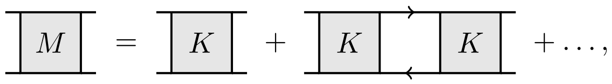
(B.5)

we iteratively use this expression to replace the Green function on the right-hand side,



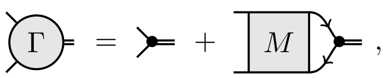
(B.6)

We define the quark-antiquark scattering amplitude M,



(B.7)

and can now write the equation for the truncated 3-point vertex as follows,



(B.8)

which in analytic form reads,

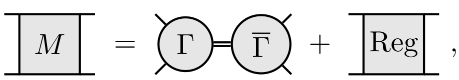
$$\Gamma_P{}^a_{\alpha\beta}(k,P) = \mathsf{t}^a \mathrm{i}\gamma^5_{\alpha\beta} + \int\limits_{q^\mu} M_{\alpha\beta\delta\epsilon}(k,q,P)S_{\delta\omega}(q_+)S_{\rho\epsilon}(q_-)\mathsf{t}^a \mathrm{i}\gamma^5_{\omega\rho}, \tag{B.9}$$

$$\Gamma_A{}^{\mu,a}_{\alpha\beta}(k,P) = \mathsf{t}^a \gamma^5_{\alpha\delta}\gamma^\mu_{\delta\beta} + \int\limits_{q^\mu} M_{\alpha\beta\delta\epsilon}(k,q,P)S_{\delta\omega}(q_+)S_{\rho\epsilon}(q_-)\mathsf{t}^a \gamma^5_{\omega\xi}\gamma^\mu_{\xi\rho}, \tag{B.10}$$

for a pseudoscalar and axial-vector current, respectively.

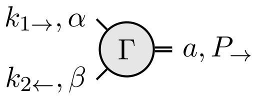
B.3 Proof of Equation (1.6) – Pion Contribution to the Scattering Amplitude M

Within all processes that can happen inside the 2-to-2 scattering amplitude M, there is a possibility that a pion forms as a bound state and later on decays again into two quarks. We saw in Chapter 2 that bound states in Green functions give rise to a pole term. This means, our ansatz consists of two three-point vertices, where two quarks can combine to a pseudoscalar current (a pion), and a pole term near the pion's mass. Diagrammatically, this means,

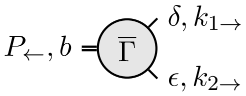


(B.11)

where [Reg] denotes all diagrams that are regular for $P^2 \to M_\pi^2$. The index structure looks as follows,



$$\leftrightarrow \Gamma^a_{\alpha\beta}(k,P) \tag{B.12}$$



$$\leftrightarrow \overline{\Gamma}^{\mathsf{T}}{}^b_{\delta\epsilon}(k,P), \tag{B.13}$$

where A^T denotes a matrix transpose, so that we can write,

$$M_{\alpha\beta\delta\epsilon}(k,q,P) = \frac{\Gamma_\pi{}^c_{\alpha\beta}(k,P)\overline{\Gamma}^{\mathsf{T}}_\pi{}^c_{\delta\epsilon}(q,-P)}{P^2 - M_\pi^2} + \mathrm{Reg}_{\alpha\beta\delta\epsilon}(k,q,P). \tag{B.14}$$

The unit residue on the pion pole follows from the canonical normalization of a Bethe-Salpeter amplitude [11].

B.4 Proof of Equation (1.7) – An Identity for the Axial-Vector Residue

We start with the Dyson equation for an axial-vector three-point vertex, Equation (1.5),

$$\Gamma_A{}^{\mu,a}_{\alpha\beta}(k,P) = \mathrm{t}^a \gamma^5_{\alpha\delta}\gamma^\mu_{\delta\beta} + \int\limits_q M_{\alpha\beta\delta\epsilon}(k,q,P) S_{\delta\omega}(q_+) S_{\rho\epsilon}(q_-) \mathrm{t}^a \gamma^5_{\omega\xi}\gamma^\mu_{\xi\rho}. \tag{B.15}$$

On the left-hand side, we use an ansatz for the axial-vector three-point vertex, Equation (1.2a),

$$\begin{aligned}\Gamma_A{}^{\mu,a}(k,P) = \mathrm{t}^a\gamma^5 \Big[\gamma^\mu F_A^{(1)} + \not{k} k^\mu F_A^{(2)} - [\gamma^\mu, \not{k}] F_A^{(3)}\Big] \\ + \widetilde{\Gamma}_A{}^{\mu,a}(k,P) + \frac{r_A P^\mu}{P^2 - M_\pi^2}\Gamma_\pi{}^a(k,P)\end{aligned} \tag{B.16}$$

and on the right-hand side, we use an ansatz for the scattering amplitude M, containing a pion contribution, Equation (1.6),

$$M_{\alpha\beta\delta\epsilon}(k,q,P) = \frac{\Gamma_\pi{}^c_{\alpha\beta}(k,P)\overline{\Gamma}^{\mathsf{T}}_\pi{}^c_{\delta\epsilon}(q,-P)}{P^2 - M_\pi^2} + \mathrm{Reg}_{\alpha\beta\delta\epsilon}(k,q,P). \tag{B.17}$$

In the limit of $P^2 \to M_\pi^2$, we discard all regular terms which leaves us with,

$$r_A P^\mu \Gamma_\pi{}^a_{\alpha\beta}(k,P) = \int\limits_q \Gamma_\pi{}^c_{\alpha\beta}(k,P)\overline{\Gamma}^{\mathsf{T}}_\pi{}^c_{\delta\epsilon}(q,-P) S_{\delta\omega}(q_+) S_{\rho\epsilon}(q_-)\mathrm{t}^a\gamma^5_{\omega\xi}\gamma^\mu_{\xi\rho}. \tag{B.18}$$

The first pion vertex on the right-hand side does not depend on the integration variable q, so we can multiply by its inverse to get,

$$r_A P^\mu \delta^{ac} = \int\limits_q \mathrm{Tr}_f\Big\{\overline{\Gamma}^{\mathsf{T}}_\pi{}^c_{\delta\epsilon}(q,-P) S_{\delta\omega}(q_+) S_{\rho\epsilon}(q_-)\mathrm{t}^a\gamma^5_{\omega\xi}\gamma^\mu_{\xi\rho}\Big\}. \tag{B.19}$$

Next, we look at the conjugate vertex, $\overline{\Gamma}$. It is connected to the original vertex through charge conjugation with positive eigenvalue (pions have $J^{PC} = 0^{-+}$),

$$\overline{\Gamma}^{\mathsf{T}\,c}_{\pi\,\delta\epsilon}(q, P) = C^{-1}_{\delta\alpha}\Gamma^{\,c}_{\pi\,\alpha\beta}(-q, P)C_{\beta\epsilon}, \tag{B.20}$$

and use this in the above equation,

$$r_A P^\mu \delta^{ac} = \int_q \mathrm{Tr}_f \Big\{ C^{-1}_{\delta\alpha}\Gamma^{\,c}_{\pi\,\alpha\beta}(-q,-P)C_{\beta\epsilon}S_{\delta\omega}(q_+)S_{\rho\epsilon}(q_-)\mathsf{t}^a\gamma^5_{\omega\xi}\gamma^\mu_{\xi\rho}\Big\}. \tag{B.21}$$

In the following, we will perform two changes of variable, first we let $P \to -P$, implying $q_+ \leftrightarrow q_-$,

$$-\, r_A P^\mu \delta^{ac} = \int_q \mathrm{Tr}_f \Big\{ C^{-1}_{\delta\alpha}\Gamma^{\,c}_{\pi\,\alpha\beta}(-q,P)C_{\beta\epsilon}S_{\delta\omega}(q_-)S_{\rho\epsilon}(q_+)\mathsf{t}^a\gamma^5_{\omega\xi}\gamma^\mu_{\xi\rho}\Big\}, \tag{B.22}$$

and $q \to -q$, which implies $q_+ \leftrightarrow -q_-$,

$$-\, r_A P^\mu \delta^{ac} = \int_q \mathrm{Tr}_f \Big\{ C^{-1}_{\delta\alpha}\Gamma^{\,c}_{\pi\,\alpha\beta}(q,P)C_{\beta\epsilon}S_{\delta\omega}(-q_+)S_{\rho\epsilon}(-q_-)\mathsf{t}^a\gamma^5_{\omega\xi}\gamma^\mu_{\xi\rho}\Big\}. \tag{B.23}$$

We will now briefly switch to matrix representation in Dirac space in order to perform the following calculations more efficiently. We make use of the following relations,

$$C^{-1}S(p)C = S^{\mathsf{T}}(-p), \quad C^{-1}\gamma^5 C = {\gamma^5}^{\mathsf{T}}, \quad C^{-1}\gamma^\mu C = -{\gamma^\mu}^{\mathsf{T}}, \tag{B.24}$$

with the charge conjugation matrix $-C = C^{\mathsf{T}} = C^{-1} = C^\dagger$. We reposition terms in order to place pairing indices next to each other and switch to matrix notation,

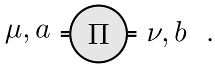
$$\begin{aligned}
-r_A P^\mu \delta^{ac} &= \int_q \mathrm{Tr}_f \Big\{ \mathsf{t}^a C^{-1}_{\delta\alpha}\Gamma^{\,c}_{\pi\,\alpha\beta}(q,P)C_{\beta\epsilon}S_{\rho\epsilon}(-q_-)\gamma^\mu_{\xi\rho}\gamma^5_{\omega\xi}S_{\delta\omega}(-q_+)\Big\} \\
&= \int_q \mathrm{Tr}_D\,\mathrm{Tr}_f \left\{ \mathsf{t}^a C^{-1}\Gamma_\pi{}^c C S^{\mathsf{T}}(-q_-){\gamma^\mu}^{\mathsf{T}}{\gamma^5}^{\mathsf{T}}S^{\mathsf{T}}(-q_+)\right\} \\
&= \int_q \mathrm{Tr}_D\,\mathrm{Tr}_f \left\{ \mathsf{t}^a C^{-1}\Gamma_\pi{}^c C S^{\mathsf{T}}(-q_-)C^{-1}C{\gamma^\mu}^{\mathsf{T}}C^{-1}C{\gamma^5}^{\mathsf{T}}C^{-1}CS^{\mathsf{T}}(-q_+)\right\} \\
&= \int_q \mathrm{Tr}_D\,\mathrm{Tr}_f \Big\{ \mathsf{t}^a \Gamma_\pi{}^c \underbrace{CS^{\mathsf{T}}(-q_-)C^{-1}}_{S(q_-)}\underbrace{C{\gamma^\mu}^{\mathsf{T}}C^{-1}}_{-\gamma^\mu}\underbrace{C{\gamma^5}^{\mathsf{T}}C^{-1}}_{\gamma^5}\underbrace{CS^{\mathsf{T}}(-q_+)C^{-1}}_{S(q_+)}\Big\} \\
&= -\int_q \mathrm{Tr}_D\,\mathrm{Tr}_f \left\{ \mathsf{t}^a \Gamma_\pi{}^c S(q_-)\gamma^\mu\gamma^5 S(q_+)\right\} \\
&= -\int_q \mathrm{Tr}_D\,\mathrm{Tr}_f \left\{ \mathsf{t}^a S(q_+)\Gamma_\pi{}^c S(q_-)\gamma^\mu\gamma^5\right\}.
\end{aligned} \tag{B.25}$$

This leaves us with this identity,

$$r_A P^\mu \delta^{ac} = \int_q \mathrm{Tr}_D\,\mathrm{Tr}_f \left\{ \mathsf{t}^a S(q_+)\Gamma_\pi{}^c S(q_-)\gamma^\mu\gamma^5\right\}. \tag{B.26}$$

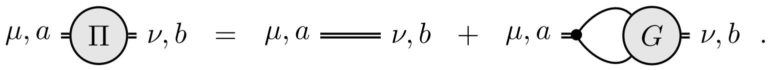
B.5 Proof of Equation (1.8) – Dyson Equation for an Axial-vector Current Vacuum Polarization

A vacuum polarization describes all possible processes that can happen during the propagation of a vector field. Here, we want to consider axial-vector currents interacting with a pion. We assume an ingoing current with Lorentz and flavor indices μ, a and an outgoing current with ν, b,



(B.27)

There are two possible types of processes. Either the current does not interact at all, i.e. it propagates freely, or it creates a quark-antiquark pair. This quark-antiquark pair can itself interact in any way, before finally combining to an axial-vector current again. A process, where two quarks enter and an axial-vector current is created is described by the axial-vector three point Green function ${G_A}^{\mu,a}$, see Appendix B.1. Therefore, the vacuum polarization can be split up into,



(B.28)

In analytic form, this equation reads,

$$ {\Pi_A}^{\mu\nu,ab}(P^2) = (\eta^{\mu\nu} P^2 - P^\mu P^\nu)\delta^{ab} + g^2 \int\limits_{q^\mu} \mathrm{Tr}\{\mathrm{t}^a \gamma^\mu \gamma^5 {G_A}^{\nu,b}(q,P)\}, \tag{B.29} $$

where g is the axial-vector current's coupling constant to quarks (the current-quark interaction happens twice, therefore we have g^2), the trace runs over flavor, Dirac and color indices (this results from the closed loop in the Feynman diagram) and the index structure of the first term is determined by the Ward identity,

$$ P_\mu \Pi^{\mu\nu} = 0. \tag{B.30} $$

B.6 Proof of Equation (1.9) – Pion Contribution to an Axial-vector Vacuum Polarization

Similar to Appendix B.3, we now investigate the pion contribution to the axial-vector current's vacuum polarization. Here, we consider an ingoing axial-vector current ${j_A}^{\mu,a}$ and an outgoing current ${j_A}^{\nu,b}$. We already saw in Section 2.1, that the formation of bound states leads to poles in Green functions. We want to describe an ingoing axial-vector current forming a pion, and later on continuing propagation as an axial-vector current. We can write this as,

$$ \langle 0 | \, {j_A}^{\mu,a} \, | \pi(P) \rangle \langle \pi(P) | \, {j_A}^{\nu,b} \, | 0 \rangle \,, \tag{B.31} $$

where $\langle 0 | \, {j_A}^{\mu,a} \, | \pi(P) \rangle = \mathrm{i} f_\pi P^\mu \mathrm{t}^a$ defines the pion's weak decay constant f_π [24]. Therefore, this contribution to an axial-vector current's vacuum polarization enter the equation as,

$$ {\Pi_A}^{\mu\nu,ab}(P^2) = (\eta^{\mu\nu} P^2 - P^\mu P^\nu)\delta^{ab} \left[\widetilde{\Pi}_{A,\mathrm{reg}}(P^2) + \frac{g^2 f_\pi^2}{P^2 - M_\pi^2} \right]. \tag{B.32} $$

Here, the axial-vector current's coupling to quarks enters quadratically, because the current has to interact twice with quarks in this process and $\widetilde{\Pi}_{A,\mathrm{reg}}$ are regular term near the pion pole. Again, the index structure is determined by the Ward identity $P_\mu \Pi^{\mu\nu} = 0$.

B.7 Proof of Equation (1.10) – Determining the Axial-Vector Residue

In this section we use the ansatz for the axial-vector vertex, Equation (1.2a), an identity for the axial-vector residue, Equation (1.7), and two representations of an axial-vector current's vacuum polarization (Dyson equation and pion contribution), Equations (1.8) and (1.9),

$$\begin{aligned} {\Gamma_A}^{\mu,a}(k,P) &= \mathrm{t}^a\gamma^5\left[\gamma^\mu F_A^{(1)} + \not{k}k^\mu F_A^{(2)} - [\gamma^\mu,\not{k}]F_A^{(3)}\right] \\ &\quad + {\widetilde{\Gamma}_A}{}^{\mu,a}(k,P) + \frac{r_A P^\mu}{P^2 - M_\pi^2}{\Gamma_\pi}^a(k,P), \\ r_A P^\mu \delta^{ac} &= \int\limits_{q^\mu} \mathrm{Tr}\{\mathrm{t}^a S(q_+){\Gamma_\pi}^c S(q_-)\gamma^\mu\gamma^5\}, \\ {\Pi_A}^{\mu\nu,ab}(P^2) &= (\eta^{\mu\nu}P^2 - P^\mu P^\nu)\delta^{ab} + g^2\int\limits_q \mathrm{Tr}\Big\{\mathrm{t}^a\gamma^\mu\gamma^5 {G_A}^{\nu,b}(q,P)\Big\}, \\ {\Pi_A}^{\mu\nu,ab}(P^2) &= (\eta^{\mu\nu}P^2 - P^\mu P^\nu)\delta^{ab}\left[\widetilde{\Pi}_{A,\mathrm{reg}}(P^2) + \frac{g_W^2 f_\pi^2}{P^2 - M_\pi^2}\right], \end{aligned}$$

to calculate the axial-vector residue. We start by equating the expressions for the vacuum polarization (note: $G = S\Gamma S$),

$$\begin{aligned} (\eta^{\mu\nu}P^2 - P^\mu P^\nu)\delta^{ab} + g^2\int\limits_q \mathrm{Tr}\Big\{\mathrm{t}^a\gamma^\mu\gamma^5 S(q_+){\Gamma_A}^{\nu,b}(q,P)S(q_-)\Big\} = \\ (\eta^{\mu\nu}P^2 - P^\mu P^\nu)\delta^{ab}\left[\widetilde{\Pi}_{A,\mathrm{reg}}(P^2) + \frac{g^2 f_\pi^2}{P^2 - M_\pi^2}\right]. \end{aligned} \tag{B.33}$$

We use the expression for the axial-vector vertex and only look at singular terms near $P^2 \to M_\pi^2$,

$$g^2 r_A P^\nu \int\limits_{q^\mu} \mathrm{Tr}\Big\{\mathrm{t}^a\gamma^\mu\gamma^5 S(q_+){\Gamma_P}^b(q,P)S(q_-)\Big\} = (\eta^{\mu\nu}P^2 - P^\mu P^\nu)\delta^{ab}g^2 f_\pi^2. \tag{B.34}$$

The term on the left-hand side corresponds exactly to our identity for the axial-vector residue, therefore,

$$g^2 r_A^2 P^\mu P^\nu \delta^{ab} = (\eta^{\mu\nu}P^2 - P^\mu P^\nu)\delta^{ab}g^2 f_\pi^2. \tag{B.35}$$

Finally, we use a special projection operator $(\eta_{\mu\nu}P^2 - 4P_\mu P_\nu)$ in order to isolate the axial-vector residue,

$$\begin{aligned} g^2 r_A^2(\eta_{\mu\nu}P^2 - 4P_\mu P_\nu)P^\mu P^\nu &= (\eta_{\mu\nu}P^2 - 4P_\mu P_\nu)(\eta^{\mu\nu}P^2 - P^\mu P^\nu)g^2 f_\pi^2 \\ r_A^2(-3P^4) &= (3P^4)f_\pi^2 \\ r_A &= \mathrm{i}f_\pi. \end{aligned} \tag{B.36}$$

B.8 Proof of Equation (1.12) – Defining the Quark Condensate

Starting from the Dyson equation for the pseudoscalar three-point vertex Equation (1.11),

$${\Gamma_P}^a_{\alpha\beta}(k,P) = \mathrm{t}^a\mathrm{i}\gamma^5_{\alpha\beta} + \int\limits_{q^\mu} M_{\alpha\beta\delta\epsilon}(k,q,P)S_{\delta\omega}(q_+)S_{\rho\epsilon}(q_-)\mathrm{t}^a\mathrm{i}\gamma^5_{\omega\rho}, \tag{B.37}$$

we use the pseudoscalar vertex Equation (1.2c),

$$\begin{aligned}\mathrm{i}\Gamma_P{}^a(k,P) = {}& \mathrm{t}^a \mathrm{i}\gamma^5 \left[F_P^{(1)} + \not{P} F_P^{(2)} + \not{k} k\cdot P F_P^{(3)} + [\not{k},\not{P}] F_P^{(4)}\right] \\ & + \frac{r_P}{P^2 - M_\pi^2}\Gamma_\pi{}^a(k,P),\end{aligned} \tag{B.38}$$

and the pion contribution to the scattering amplitude Equation (1.6),

$$M_{\alpha\beta\delta\epsilon}(k,q,P) = \frac{\Gamma_\pi{}^c_{\alpha\beta}(k,P)\overline{\Gamma}^{\mathsf{T}}_\pi{}^c_{\delta\epsilon}(q,-P)}{P^2 - M_\pi^2} + \mathrm{Reg}_{\alpha\beta\delta\epsilon}(k,q,P), \tag{B.39}$$

which, looking at singular terms only, yields,

$$-\,\mathrm{i}r_P \Gamma_\pi{}^a_{\alpha\beta}(k,P) = \int\limits_{q^\mu} \Gamma_\pi{}^c_{\alpha\beta}(k,P)\overline{\Gamma}^{\mathsf{T}}_\pi{}^c_{\delta\epsilon}(q,-P) S_{\delta\omega}(q_+) S_{\rho\epsilon}(q_-) \mathrm{t}^a \mathrm{i}\gamma^5_{\omega\rho}. \tag{B.40}$$

We multiply by the inverse of $\Gamma_\pi{}^a_{\alpha\beta}(k,P)$ and get,

$$-\,\mathrm{i}r_P \delta^{ac} = \int\limits_{q^\mu} \mathrm{Tr}_f\, \mathrm{Tr}_c \left\{\overline{\Gamma}^{\mathsf{T}}_\pi{}^c_{\delta\epsilon}(q,-P) S_{\delta\omega}(q_+) S_{\rho\epsilon}(q_-) \mathrm{t}^a \mathrm{i}\gamma^5_{\omega\rho}\right\}, \tag{B.41}$$

which in matrix notation reads,

$$-\,r_P \delta^{ac} = \int\limits_{q^\mu} \mathrm{Tr}\left\{\overline{\Gamma}^{\mathsf{T}}_\pi{}^c(q,-P) S(q_+) S(q_-) \mathrm{t}^a \gamma^5\right\}. \tag{B.42}$$

We already calculated this right-hand side expression in Appendix B.4, this leads to,

$$-\,r_P \delta^{ac} = \int\limits_{q^\mu} \mathrm{Tr}\left\{S(q_+)\Gamma_\pi{}^c(q,P) S(q_-) \mathrm{t}^a \gamma^5\right\}. \tag{B.43}$$

In analogy to [10], we identify an in-hadron condensate with the pseudoscalar residue,

$$\mathbb{C} := \mathrm{i}r_P = -\mathrm{i} \int\limits_{q^\mu} \mathrm{Tr}\left\{S(q_+)\Gamma_\pi{}^a(q,P) S(q_-) \mathrm{t}^a \gamma^5\right\} \tag{B.44}$$

B.9 Proof of Equation (2.28)

We will use the following definition of the energy projection operator and the Clifford algebra,

$$\Lambda_i^\pm(\mathbf{p}) = \frac{1}{2E_i(\mathbf{p})}\big(E_i(\mathbf{p}) \pm H_i(\mathbf{p})\big), \qquad \{\gamma^\mu, \gamma^\nu\}_{\alpha\beta} = 2\eta^{\mu\nu}\delta_{\alpha\beta}.$$

Then,

$$\begin{aligned}(2.28) &= \Big[\big(p^0 + E(\mathbf{p})\big)\Lambda^+(\mathbf{p}) + \big(p^0 - E(\mathbf{p})\big)\Lambda^-(\mathbf{p})\Big]\gamma^0 \\ &= \frac{(p^0 + E(\mathbf{p}))(E(\mathbf{p}) + H(\mathbf{p}))\gamma^0 + (p^0 - E(\mathbf{p}))(E(\mathbf{p}) - H(\mathbf{p}))\gamma^0}{2E(\mathbf{p})} \\ &= \frac{2\gamma^0 p^0 E(\mathbf{p}) + 2E(\mathbf{p})H(\mathbf{p})\gamma^0}{2E(\mathbf{p})}\end{aligned}$$

$$
\begin{aligned}
&= \gamma^0 p^0 + \gamma^0 H(-\mathbf{p}) \\
&= \gamma^0 p^0 + \gamma^0 \gamma^0 \big[-\boldsymbol{\gamma} \cdot \mathbf{p} + m_q \big] \\
&= \gamma^0 p^0 - \boldsymbol{\gamma} \cdot \mathbf{p} + m_q \\
&= \not{p} + m_q .
\end{aligned}
$$

□

B.10 Proof of Equation (2.29)

We take the free fermion propagator and manipulate it in the following way:

$$
\begin{aligned}
S(p) &= \frac{\mathrm{i}}{\not{p} - m_q + \mathrm{i}\epsilon} \\
&= \mathrm{i} \frac{\not{p} + m_q - \mathrm{i}\epsilon}{p^2 - m_q^2 + 2\mathrm{i}\epsilon + \epsilon^2} \\
&= \mathrm{i} \frac{\not{p} + m_q}{(p^0)^2 - (\mathbf{p}^2 + m_q^2 - \mathrm{i}\epsilon)} \\
&= \mathrm{i} \frac{\not{p} + m_q}{\big(p^0 + \sqrt{\mathbf{p}^2 + m_q^2 - \mathrm{i}\epsilon}\big)\big(p^0 - \sqrt{\mathbf{p}^2 + m_q^2 - \mathrm{i}\epsilon}\big)} \\
&= \mathrm{i} \frac{\not{p} + m_q}{(p^0 + E(\mathbf{p}) - \mathrm{i}\epsilon)(p^0 - E(\mathbf{p}) + \mathrm{i}\epsilon)},
\end{aligned}
$$

where we Taylor expanded the square root and omitted terms of order $\mathcal{O}(\epsilon^2)$. With the help of another identity, Equation (2.28),

$$
\not{p} + m_q = \Big[\big(p^0 + E(\mathbf{p})\big)\Lambda^+(\mathbf{p}) + \big(p^0 - E(\mathbf{p})\big)\Lambda^-(\mathbf{p}) \Big] \gamma^0,
$$

we can write the propagator as,

$$
\begin{aligned}
S(p) &= \mathrm{i} \frac{\big(p^0 + E(\mathbf{p})\big)\Lambda^+(\mathbf{p}) + \big(p^0 - E(\mathbf{p})\big)\Lambda^-(\mathbf{p})}{(p^0 + E(\mathbf{p}) - \mathrm{i}\epsilon)(p^0 - E(\mathbf{p}) + \mathrm{i}\epsilon)} \gamma^0 \\
&= \mathrm{i} \left(\frac{\Lambda^+(\mathbf{p})}{p^0 - E(\mathbf{p}) + \mathrm{i}\epsilon} + \frac{\Lambda^-(\mathbf{p})}{p^0 + E(\mathbf{p}) - \mathrm{i}\epsilon} \right) \gamma^0 .
\end{aligned}
$$

□

B.11 Proof of Equation (2.30)

We want to calculate the expression,

$$
\int_{-\infty}^{\infty} \mathrm{d}p^0 S_1(p_1) I(\mathbf{p}) S_2(-p_2), \tag{B.45}
$$

where S_i is the free fermion propagator for the i-th particle, represented in a certain way,

$$
S_i(k) = \mathrm{i} \left(\frac{\Lambda_i^+(\mathbf{k})}{k^0 - E_i(\mathbf{k}) + \mathrm{i}\epsilon} + \frac{\Lambda_i^-(\mathbf{k})}{k^0 + E_i(\mathbf{k}) - \mathrm{i}\epsilon} \right) \gamma^0, \tag{B.46}
$$

and I is the interaction kernel, depending only on the spatial components of the momentum. The momenta $p_{1,2}$ are given in a generalized way to 4 dimensions according to Appendix A.3 by $p_1 = \eta_1 P + p$ and $p_2 = \eta_2 P - p$. After multiplying the propagators, we have to solve four different cases of the following integral,

$$\int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{x - a - \mathrm{i}\epsilon_1} \frac{1}{x - b - \mathrm{i}\epsilon_2}, \tag{B.47}$$

where a, b are constant expressions with respect to the integration variable and there are four variations for $(\epsilon_1, \epsilon_2) = \{(\epsilon, \epsilon), (\epsilon, -\epsilon), (-\epsilon, \epsilon), (-\epsilon, -\epsilon)\}$ with $\epsilon \to 0^+$, depending on which parts of the propagator we are multiplying. Let us do an analytic continuation to the complex plane and close the integration contour in the upper or lower half plane. If we can show that the auxiliary path in the upper or lower half plane vanishes, we can use Cauchy's theorem and calculate the integral via residues. To integrate over a half circle, we substitute,

$$x = \lim_{R \to \infty} R\mathrm{e}^{\pm \mathrm{i}\varphi}, \quad \mathrm{d}x = \mathrm{i}R\mathrm{e}^{\mathrm{i}\varphi}\mathrm{d}\varphi, \quad \varphi \in [0, \pi], \tag{B.48}$$

so that Equation (B.47) gives,

$$\lim_{R \to \infty} \mathrm{i}R \int_0^{\pi} \mathrm{d}\varphi\, \mathrm{e}^{\pm \mathrm{i}\varphi} \frac{1}{R\mathrm{e}^{\pm \mathrm{i}\varphi} - a - \mathrm{i}\epsilon_1} \frac{1}{R\mathrm{e}^{\pm \mathrm{i}\varphi} - b - \mathrm{i}\epsilon_2} = 0. \tag{B.49}$$

The integral vanishes for $R \to \infty$, therefore we can use the residue theorem,

$$\oint_{\mathcal{C}} \mathrm{d}z f(z) = 2\pi\mathrm{i} \sum_i \mathrm{Res}_{z_i} f(z). \tag{B.50}$$

Since we are dealing with single poles, we calculate the residue at a point z_0 as follows,

$$\mathrm{Res}_{z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z). \tag{B.51}$$

Now let us consider the four cases we have:

1. For $\epsilon_1 = \epsilon_2 = \epsilon$, both poles are in the upper half plane, so we close the countour in the lower half plane. Since there is no residue, this term vanishes.

2. For $\epsilon_1 = -\epsilon_2 = \epsilon$, the pole at $x = a + \mathrm{i}\epsilon$ is within the contour and we calculate the residue to

$$\mathrm{Res}_{a+\mathrm{i}\epsilon} \frac{1}{x - a - \mathrm{i}\epsilon} \frac{1}{x - b + \mathrm{i}\epsilon} = \frac{1}{a + \mathrm{i}\epsilon - b + \mathrm{i}\epsilon} \xrightarrow{\epsilon \to 0} \frac{1}{a - b}. \tag{B.52}$$

3. For $\epsilon_1 = -\epsilon_2 = -\epsilon$, the pole at $x = b + \mathrm{i}\epsilon$ lies within the contour and we calculate the residue to

$$\mathrm{Res}_{b+\mathrm{i}\epsilon} \frac{1}{x - a + \mathrm{i}\epsilon} \frac{1}{x - b - \mathrm{i}\epsilon} = \frac{1}{b + \mathrm{i}\epsilon - a + \mathrm{i}\epsilon} \xrightarrow{\epsilon \to 0} \frac{1}{b - a}. \tag{B.53}$$

4. For $\epsilon_1 = \epsilon_2 = -\epsilon$, both poles are in the lower half plane, thus closing the contour in the upper half plane makes this term also vanish.

Equation (B.45) reads,

$$\int_{-\infty}^{\infty} \mathrm{d}p^0 \mathrm{i} \Bigg(\underbrace{\frac{\Lambda_1^+(\mathbf{p}_1)\gamma^0}{\eta_1 P^0 + p^0 - E_1(\mathbf{p}_1) + \mathrm{i}\epsilon}}_{\mathrm{I}} + \underbrace{\frac{\Lambda_1^-(\mathbf{p}_1)\gamma^0}{\eta_1 P^0 + p^0 + E_1(\mathbf{p}_1) - \mathrm{i}\epsilon}}_{\mathrm{II}} \Bigg) I(\mathbf{p})$$
$$\times \Bigg(\underbrace{\frac{\Lambda_2^+(-\mathbf{p}_2)\gamma^0}{-\eta_2 P^0 + p^0 - E_2(-\mathbf{p}_2) + \mathrm{i}\epsilon}}_{\mathrm{III}} + \underbrace{\frac{\Lambda_2^-(-\mathbf{p}_2)\gamma^0}{-\eta_2 P^0 + p^0 + E_2(-\mathbf{p}_2) - \mathrm{i}\epsilon}}_{\mathrm{IV}} \Bigg).$$

We use $\Lambda_i^{\pm}(\mathbf{p})\gamma^0 = \gamma^0 \Lambda_i^{\pm}(-\mathbf{p})$ and look at the resulting terms separately:

(I×III) corresponds to case 4, this term vanishes.
(I×IV) corresponds to case 3 with $a = E_1 - \eta_1 P^0$ and $b = \eta_2 P^0 - E_2$. This yields a residue

$$\mathrm{Res} = \frac{1}{P^0 - E_1 - E_2}. \tag{B.54}$$

(II×III) corresponds to case 2 with $a = -\eta_1 P^0 - E_1$ and $b = \eta_2 P^0 + E_2$. This yields a residue

$$\mathrm{Res} = -\frac{1}{P^0 + E_1 + E_2}. \tag{B.55}$$

(II×IV) corresponds to case 1, this term also vanishes.

Applying the residue theorem, Equation (B.50), this finally yields,

$$-2\pi\mathrm{i} \left(\frac{\Lambda_1^+(\mathbf{p}_1)\gamma^0 I(\mathbf{p}) \Lambda_2^-(-\mathbf{p}_2)\gamma^0}{P^0 - E_1 - E_2} - \frac{\Lambda_1^-(\mathbf{p}_1)\gamma^0 I(\mathbf{p}) \Lambda_2^+(-\mathbf{p}_2)\gamma^0}{P^0 + E_1 + E_2} \right).$$

□

Appendix C

Overview of Symbols

a_{ij}	Coefficients in the expansion of $V(r)$	(3.21)
$\mathcal{A}, \mathcal{B}, \mathcal{C}$	Abbreviations for matrices	(3.14), (3.23)
b, γ	Parameters in the potential $V(r)$	(3.7)
C	Charge conjugation matrix or its eigenvalue	(A.17)
$\mathbb{C}$	In-hadron condensate	(3.42)
$E_a(b)$	Exponential integral function	(3.29)
$E_i(\mathbf{p}_i)$	One-particle energy	(2.24)
f, f_π	Decay constant (of the pion)	(B.32)
$\mathbb{F}$	Abbreviation for a scalar function	(3.25)
$G^{(4)}$	4-point Green's function	(2.1)
H_i	One-particle Hamiltonian	(2.25)
I	Interaction term, integral over $\hat{K}\Phi$	(2.21)
$I_a(x)$	Modified Bessel function of the first kind	(3.5)
$I_{ij}^{(n)}$	Abbreviation for a certain integral	(3.16)
j_A	Axial-vector current	(A.6)
K	Scattering kernel, interaction term	p. 7
$\hat{K}$	Spatial scattering	(2.19)
$K_a(x)$	Modified Bessel function of the second kind	(3.5)
$L_a^b(x)$	Associated Laguerre polynomials	(3.24)
$\mathbf{L}_a(x)$	Modified Struve function of order a	(3.6)
M	Mass of a bound state (e.g. of a pion)	p. 11
$M_{\alpha\beta\gamma\delta}$	Quark-antiquark scattering amplitude	(B.7)
m_q	Constituent quark mass, i.e. $m_q \approx m_p/3$	p. 30
m	"Current" quark mass, i.e. the parameter in the QCD Lagrangian	(A.10)
n_Φ	Normalization of Salpeter amplitude	Section 3.3
$S(p)$	Fermion propagator	(2.23)
t^a	Generator of $SU(N)$	(A.5)
$V(r)$	Interaction potential	(3.3)
γ^μ	Dirac gamma matrices	p. 2.5
Γ	Basis element of 4×4 matrices $\Gamma \in \{1, \mathrm{i}\gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}$	(A.28)
$\Gamma(a)$	Gamma function	(3.4)
$\Gamma(a, b)$	Incomplete Gamma function	(3.27)
${\Gamma_P}^a$	Pseudoscalar three point vertex	(A.11)
${\Gamma_A}^{\mu a}$	Axial-vector three point vertex	(A.11)
$\Theta(x)$	Heaviside theta function	(2.10)

$\Lambda_i^{\pm}$	Energy projection operators	(2.27)
μ	Parameter appearing in the Laguerre basis expansion	(3.8)
Φ	Salpeter amplitude	(2.20)
$\varphi_{1,2}$	Components of a pseudoscalar Salpeter amplitude	(2.38)
$\widetilde{\phi}_i^{(l)}, \phi_i^{(l)}$	Basis elements in the expansion of $\varphi_2(\mathbf{p})$ and its Fourier transform	(3.8)
$\psi, \overline{\psi}$	Fermionic quantum fields, in particular quark fields	(A.1)
$\Psi, \overline{\Psi}$	Bethe-Salpeter amplitude of a bound state M	(2.4)

C.1 Further Notation

Units. We employ units where $c = \hbar = 1$ and choose a mostly-minus Minkowski metric $\eta = \mathrm{diag}\,(+,-,-,-)$.

Integration. Unless otherwise stated, we abbreviate position space (x, y, z) and momentum space (p, q, k) integration by

$$\int_x \equiv \int_{-\infty}^{\infty} \mathrm{d}x \qquad \int_{\mathbf{x}} \equiv \int_{-\infty}^{\infty} \mathrm{d}^3\mathbf{x} \qquad \int_{x^\mu} \equiv \int_{-\infty}^{\infty} \mathrm{d}^4x$$

$$\int_p \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \qquad \int_{\mathbf{p}} \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \qquad \int_{p^\mu} \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}^4p}{(2\pi)^4}$$

such that x^μ represents a Lorentz four-vector, $\mathbf{x}$ the three spatial components of x^μ and $x = \sqrt{\mathbf{x}\cdot\mathbf{x}}$ stands for the length of a three-vector. Occasionally we will make use of the abbreviation

$$\int_{\xi_+} \equiv \int_0^{\infty} \mathrm{d}\xi \tag{C.1}$$

to denote an integration over a non-negative region.

Trace. In this work, we perform traces in Dirac spinor space, $SU(N_f)$ flavor space and $SU(3)$ color space. We denote them by Tr_D, Tr_f and Tr_c, respectively. A trace without index means tracing over all three, Dirac spinor, flavor and color space: $\mathrm{Tr} = \mathrm{Tr}_D\,\mathrm{Tr}_f\,\mathrm{Tr}_c$.

In order to keep the main text free of lengthy claculations, we put most of them in to the appendix.

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