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Diplomarbeit

Gravity with null boundaries

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durch

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Kurzfassung

In dieser Arbeit untersuchen wir lichtartige Randsegmente einer gegebenen Raumzeit, welche über nicht-glatte Ecken miteinander verbunden sind. Wir präsentieren ein lichtartiges Analogon zu dem bekannten Gibbons-Hawking-York (GHY) Randterm [1, 2] und Eckterme, welche Lorentz-Winkel zwischen den Oberflächennormalen der Randsegmente enthalten. Dafür folgen wir grob den Behandlungen in [3–5] und erhalten ein verallgemeinertes Resultat. Als prototypisches Beispiel, welches lichtartige und zeitartige Randsegmente, verbunden über nicht-glatte Ecken enthällt, betrachten wir ein Bañados-Teitelboim-Zanelli (BTZ) Schwarzes Loch [6, 7]. Die BTZ Lösung erfüllt jedoch keine Dirichlet Randbedingungen und daher benötigen wir holographische Randterme um ein wohl-definiertes Variationsprinzip zu erhalten ($\delta I|_{EOM} = 0$). Wir zeigen, dass am asymptotischen (zeitartigen) Rand $r \to \infty$ es nicht möglich ist einen passenden kovarianten Randterm zu finden und daher konstruieren wir einen nicht-kovarianten Term, welcher zu verschwindender Variation der Wirkung führt. Dieser nicht-kovariante Term, welcher aus der radialen Komponente der Metrik besteht, kann als Dilatonfeld am Rand interpretiert werden. Des Weiteren zeigen wir, dass es nicht möglich ist passende Eckterme zur Wirkung zu addieren, sodass das Variationsproblem wohl-definiert ist. Als Lösung schlagen wir eine geeignete Fixierung der Boost-Eichung vor, wobei Ecken keinen Beitrag im Wirkungsintegral mehr geben. Die komplette Wirkung besteht dann aus dem Einstein-Hilbert Term plus lichtartigen GHY Termen am Ereignishorizont plus einem nicht-kovarianten Term am asymptotischen Rand. Um Berechnungen einfach zu halten arbeiten wir in $2 + 1$ Dimensionen.

Abstract

We study null boundary segments intersecting each other at non-smooth corners. As a result, a null analog to the common Gibbons-Hawking-York (GHY) counterterm [1,2] and corner counterterms, containing Lorentz angles between hypersurface normals, are presented. Considering that, we follow loosely the treatments in [3–5] and obtain a slightly more general result. As a prototype spacetime that exhibits null and timelike boundary segments connected via non-smooth corners, we investigate a Bañados-Teitelboim-Zanelli (BTZ) black hole (BH) [6,7]. The BTZ solution does not preserve Dirichlet boundary conditions (bcs) and therefore needs holographic counterterms added to the action such that we get a well-defined action principle ($\delta I|_{EOM} = 0$). We show that on the asymptotic (timelike) boundary segment $r \to \infty$ it is not possible to construct a suitable counterterm solely out of covariant quantities and therefore propose a non-covariant counterterm that can be interpreted as a dilaton-like scalar field on the boundary. Furthermore we show that we cannot find any counterterms to the corner contributions in the action integral such that the variation vanishes. We propose to fix the boost gauge in a suitable manner such that corners do not contribute at all. The full action is then given by the standard Einstein-Hilbert (EH) action plus GHY-like counterterms on the null segments and a non-covariant term on the asymptotic boundary. For the ease of the calculations we work in $2+1$ dimensions.

Acknowledgements

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Contents

1 Introduction

1.1 Holographic principle and AdS/CFT

Assumed to be a manifest property of quantum gravity the holographic principle was first proposed by t'Hooft and Susskind [8,9]. It states that the description of a $(d+1)$ dimensional spacetime is encoded on its d dimensional boundary or in more technical terms, a gravitational theory in $(d+1)$ dimensions is equivalent to a quantum field theory on its d dimensional boundary. Since a hologram is a three dimensional image stored on a two dimensional medium, the term "holographic principle" is an accurate expression indeed.

The development of this proposal was strongly influenced by black hole (BH) thermodynamics. Initially BHs were thought to have vanishing entropy but Bekenstein showed in [10], that BHs having no entropy would violate the second law of thermodynamics. In a famous Gedankenexperiment he stated that if one threw an object with a certain mass into a BH, the entropy in the outer region (outside the event horizon) would decrease, violating the second law of thermodynamics. Since an outside observer can only detect that the event horizon area grows, Bekenstein proposed that the entropy scales with the area rather than the volume, which was later confirmed by Hawking [11]. In natural units $c = \hbar = k = 1$, that will be used throughout this thesis, the so called Bekenstein-Hawking entropy is given by

$$S_{BH} = \frac{A}{4G}. \tag{1.1}$$

In statistical mechanics the entropy is proportional to the number of microstates, which in turn is proportional to the volume. The fact that for BHs, which are considered to be the connecting piece between quantum mechanics and gravity, the entropy scales with the area leads to the proposal that the holographic principle might be a manifest property of a quantized theory of gravity.

The first concrete realization was given by Maldacena in [12], who conjectured an equivalency between a type IIb superstring theory in an $AdS_5 \times S^5$ space and an

$N = 4$ super-Yang-Mills theory in four dimensions. This duality is of strong/weak type, meaning that the coupling parameters in the superstring theory and the dual super-Yang-Mills theory are inversely related.
To this day there has not been found a proof of this Anti-de-Sitter/Conformal field theory (AdS/CFT) correspondence. Nonetheless the conjectured relation [12–14]

$$\left\langle e^{\int j(x)\mathcal{O}(x)} \right\rangle_{\text{CFT}} = Z_{\text{gravity}} \Big[\varphi(x,z)|_{z\to 0} = j(x) \Big] \tag{1.2}$$

holds for numerous tests by calculating quantities on both sides and checking for agreement. The left hand side of (1.2) corresponds to the generating functional of the correlation functions in the CFT, where $\mathcal{O}(x)$ is a gauge invariant operator. The quantity $j(x)$ appears both on the CFT side as the source and as boundary conditions (bcs) for the field $\varphi(x)$ in the classical partition function Z_{gravity}, where the asymptotic boundary is reached in the limit $z \to 0$.
While being an interesting research topic AdS/CFT fails to describe our universe properly. AdS spacetimes describe universes with a negative cosmological constant Λ, while our universe is equipped with a (small) positive Λ. In most cases our universe can be approximated by $\Lambda = 0$, making flat space holography an interesting research topic as well.

1.2 Gravity in three dimensions

Gravitational theories in more than 2+1 spacetime dimensions possess a lot of technical difficulties, especially in the context of holographic analysis. Therefore it can be useful to work in lower dimensions in order to get more profound conceptual insights. There are various reasons to work in three spacetime dimensions:

- It is the lowest dimension in which Einstein gravity can be formulated. In two spacetime dimensions any metric fulfills $R_{ab} - \frac{1}{2} g_{ab} R = 0$ and one dimensional spacetimes do not possess any curvature at all.

- The curvature is only determined by the cosmological constant Λ. The Riemann tensor can be decomposed completely in terms of the Ricci tensor and scalar, $R_{abcd} = g_{ac}R_{bd} + g_{bd}R_{ac} - g_{ad}R_{bc} - g_{bc}R_{ab} - \frac{1}{2}R\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)$. The number of independent components of the Riemann tensor in arbitrary dimensions d are given by $\frac{d^2\left(d^2-1\right)}{12}$, while those of the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$ are given by $\frac{d(d+1)}{2}$. We see that for $d = 3$ those numbers coincide, which proofs that the Riemann curvature is completely determined by Λ, since $G_{ab} = -\Lambda g_{ab}$.

All solutions to Einstein gravity are therefore locally (A)dS or flat (assuming there is no matter content such that the energy momentum tensor vanishes).

- In contrast to earlier beliefs it has been shown by Bañados, Teitelboim and Zanelli that BHs do exist in three dimensions [6, 7]. More importantly those BHs possess a non-trivial horizon that is two dimensional.

- Even though Einstein gravity in $d = 3$ does not have any local degrees of freedom, meaning that there are no wave solutions to Einsteins equations, it possesses highly non-trivial global dynamics. It has been shown by Brown and Henneaux in [15] that the physical phase space of asymptotically AdS_3 spacetimes, preserving certain fall-off conditions, falls into representations of two copies of the Virasoro algebra, which is the symmetry algebra of CFTs in $d = 2$.

- In terms of the holographic dictionary, a theory of gravity in three dimensions, equipped with Brown-Henneaux bcs, is equivalent to a CFT in two dimensions. This is particularly helpful since the symmetry algebra of CFTs in two dimensions is infinite dimensional which makes them very tractable.

1.3 BTZ black holes

Initially it was believed that there could not exist any BH solutions in 2+1 dimensional Einstein gravity. One of the reasons was that, for vanishing cosmological constant Λ, all solutions are locally flat ($R_{abcd} = 0$). In 1992, Bañados, Teitelboim and Zanelli (BTZ) found a BH solution to Einstein gravity with a negative cosmological constant [6]. Its global geometry is shown in Fig. 1.1 and the line-element in Schwarzschild-like coordinates reads

$$ds^2 = -\frac{\left(r^2 - r_+^2\right)\left(r^2 - r_-^2\right)}{l^2 r^2} dt^2 + \frac{l^2 r^2}{\left(r^2 - r_+^2\right)\left(r^2 - r_-^2\right)} dr^2 + r^2 \left(d\phi - \frac{r_+ r_-}{l r^2} dt\right)^2, \tag{1.3}$$

where l is the AdS radius following the definition $\Lambda = -\frac{2}{l^2}$ and r_+, r_- are real constants with $r_+ \geq r_- \geq 0$. The BTZ solution became of great importance since it exhibits a lot of similarities to the four dimensional Kerr BH. This allows us to derive conceptual results relevant to our universe in the technically more simple framework of lower dimensional gravity.

The BTZ BH has an outer Killing horizon at $r = r_+$, which is also an event horizon. The corresponding Killing vector, whose norm vanishes on $r = r_+$ reads

$$k^a = \partial_t^a + \frac{r_-}{lr_+}\partial_\phi^a. \tag{1.4}$$

Since the line-element (1.3) is symmetric under exchange of the parameters r_+ and r_- we must also have a Killing horizon at $r = r_-$ for the Killing vector

$$k^a = \partial_t^a + \frac{r_+}{lr_-}\partial_\phi^a. \tag{1.5}$$

This inner horizon is a Cauchy horizon and vanishes for a non-rotating BH. General BTZ BHs rotate with an angular velocity $\Omega = \frac{r_-}{lr_+}$.
It has also been shown that, even though there is no Newtonian limit in 3d Einstein gravity, the BTZ BH is the endpoint of gravitational collapsing dust [16].
Since we work in just two dimensions of space, the area of the event horizon corresponds to the circumference and therefore the Bekenstein-Hawking entropy is given by

$$S_{BH} = \frac{\pi r_+}{2G}. \tag{1.6}$$

A year after the original BTZ paper [6] was published the authors also showed that a BTZ BH can be seen as an orbifold of global AdS [7]. In particular, this is achieved by identifying points by a discrete subgroup of $SO(2,2)$. What is more, Bañados presented in [17] a general solution to AdS Einstein Gravity preserving Brown-Henneaux bcs[1]

$$\begin{aligned} ds^2 = d\rho^2 + dx^+dx^-\left(e^{\frac{2\rho}{l}} + e^{-\frac{2\rho}{l}}\mathcal{L}^+(x^+)\mathcal{L}^-(x^-)\right) \\ + \left(dx^+\right)^2\mathcal{L}^+(x^+) + \left(dx^-\right)^2\mathcal{L}^-(x^-), \end{aligned} \tag{1.7}$$

where the light cone coordinates $x^\pm = t \pm l\phi$ are used. These types of spacetimes are called Bañados geometries and are fully determined by the two functions $\mathcal{L}^+(x^+)$ and $\mathcal{L}^-(x^-)$. The BTZ BH is obtained for constant

$$\mathcal{L}^+ = \frac{1}{4l^2}\left(r_+ - r_-\right)^2, \qquad \mathcal{L}^- = \frac{1}{4l^2}\left(r_+ + r_-\right)^2. \tag{1.8}$$

[1]Brown-Henneaux bcs [15] are a certain type of asymptotically AdS_3 bcs that will be discussed in more detail in the next section.

Since by its derivation any Bañados geometry is asymptotically AdS_3, the BTZ spacetime shares this property. A more detailed analysis of BTZ BHs can be found in Carlip's very comprehensive review [18].

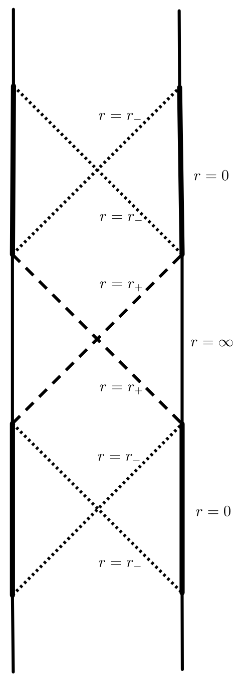

Figure 1.1: Carter-Penrose diagram of the BTZ BH. The normal lines correspond to the asymptotic boundary at $r \to \infty$, dashed lines to the event horizon at $r = r_+$, dotted lines to the inner horizon at $r = r_-$ and bold lines to the singularity at $r = 0$. The intersection of two dashed lines shows a bifurcation 1-sphere. Note that each point in the printed diagram corresponds to an S^1 in the actual Carter-Penrose diagram.

1.4 Space- and timelike hypersurfaces

This short section on hypersurfaces will follow in large extents the exceptional treatment in Poisson's toolkit [19]. The number of spacetime dimensions will be adapted to $2 + 1$ in order to match the topic of this thesis. A hypersurface $\mathcal{S}$ embedded in a three dimensional manifold $\mathcal{M}$, equipped with a metric g_{ab} can either be specified by a scalar field

$$\phi(x^a) = 0, \tag{1.9}$$

that puts a restriction on the coordinate vectors x^a or by a relation between induced coordinates on the hypersurface x^i, where the index runs only over two components, and the global coordinates

$$x^a = x^a(x^i). \tag{1.10}$$

The scalar field in (1.9) allows us to uniquely define a normal vector to a space- or timelike hypersurface[2]

$$n_a = \frac{\varepsilon \partial_a \phi}{\sqrt{|g^{ab}\partial_a\phi\partial_b\phi|}}, \tag{1.11}$$

normalized such that $n_a n^a = \varepsilon = \pm 1$, with the positive sign for timelike and the negative sign for spacelike hypersurfaces. In foresight to our later treatment of variational principles we follow [3–5] and demand that the hypersurface description does not change under variations

$$\delta\phi = 0. \tag{1.12}$$

This has the result that the variation of the normal vector does not change its orientation

$$\delta n_a = \delta\left(\frac{1}{\sqrt{|g^{ab}\partial_a\phi\partial_b\phi|}}\right)\varepsilon\partial_a\phi = \delta\left(\ln\frac{1}{\sqrt{|g^{ab}\partial_a\phi\partial_b\phi|}}\right)n_a. \tag{1.13}$$

Furthermore one can project out the normal direction in the metric g_{ab} to obtain the so called transverse or induced metric

$$h_{ab} = g_{ab} - \varepsilon n_a n_b, \tag{1.14}$$

where it follows per definition that $h_{ab}n^b = 0$. Using the parametric equation (1.10) one can also define vectors tangent to the hypersurface $\mathcal{S}$

$$e^a_i = \frac{dx^a}{dx^i}. \tag{1.15}$$

On the hypersurface the line-element can be expressed solely in terms of the induced coordinates

$$ds^2|_{\mathcal{S}} = g_{ab}dx^a dx^b = h_{ij}dx^i dx^j. \tag{1.16}$$

[2]Null hypersurfaces possess an ambiguity in the definition of a normal vector that leads to additional gauge freedom. They will be discussed in detail in Sec. 3.

Here $h_{ij} = g_{ab} e^a_i e^b_j$ denotes the so called first fundamental form. Note that this induced 2-metric transforms as a scalar regarding transformations of the global coordinates $x^a \to x'^a$. The inverse 2-metric follows the relation

$$h^{ij} = h^{ab} e^i_a e^j_b. \tag{1.17}$$

We see that the tangent vectors e^a_i work as transformation matrices that transform three dimensional bulk quantities into their corresponding quantities on the hypersurface. Since $e^a_i n_a = 0$ all directions normal to the hypersurface are projected out in this transformation. However, throughout this thesis we will mainly work in the global coordinates x^a and use the transverse metric h_{ab}.
Decomposing the metric into directions normal and tangent to the hypersurface allows for the introduction of new curvature quantities as well. The extrinsic curvature (or second fundamental form)

$$K_{ab} = h^c_a h^d_b \nabla_c n_d = \frac{1}{2} \left(\mathcal{L}_n h\right)_{ab} \tag{1.18}$$

measures the covariant rate of change of the normal vector projected onto the hypersurface. Note that it is symmetric in its indices and also orthogonal to n^a

$$K_{ab} = K_{ba}, \qquad K_{ab} n^b = 0. \tag{1.19}$$

In general one can also define a hypersurface-covariant derivative

$$\mathcal{D}_a T^{b_1 \ldots b_n}{}_{c_1 \ldots c_m} = h^r_a h^{b_1}_{q_1} \ldots h^{b_n}_{q_n} h^{p_1}_{c_1} \ldots h^{p_m}_{c_m} \nabla_r T^{q_1 \ldots q_n}{}_{p_1 \ldots p_2}, \tag{1.20}$$

which can be used to define intrinsic Riemann curvature $\mathcal{R}^c{}_{abd}$ on the hypersurface via

$$\left[\mathcal{D}_a, \mathcal{D}_b\right] v^c = \mathcal{R}^c{}_{abd} v^d. \tag{1.21}$$

2 Variational principle

Throughout this thesis we will demand that a consistent theory of gravity fulfills the principle of a stationary action

$$\delta I|_{EOM} = 0. \tag{2.1}$$

Historically, variational principles first came up when Fermat proposed that light rays always take the path of the least time. In a uniform medium this will of course be a straight line and equal to the shortest distance. When light travels through different media this simple principle leads to Snell's law of refraction.
A more general principle was developed by Maupertuis [20], who claimed that light does not travel along paths of least time nor paths of the shortest distance. He proposed that light rays follow the path of least action. However, he used a slightly different definition of this quantity. He defined the action as the sum of distances travelled by light, weighted by its velocity

$$S = \int v ds. \tag{2.2}$$

Just shortly after Maupertuis, Euler [21] independently developed a principle of least action to describe the motion of particles with a certain mass m and momentum $p = mv$

$$\delta S = \delta \left(\int p dq \right) = 0. \tag{2.3}$$

The common definition of the action today was strongly influenced by Lagrange and Hamilton. It is given by

$$S = \int L(x, \dot{x}, t) dt, \tag{2.4}$$

where $L(x, \dot{x}, t)$ denotes Lagrange's function that describes the time evolution of physical systems.

2.1 Mechanics

As an insightful and technically manageable example we first want to discuss the action principle in the context of a simple one dimensional theory, namely mechanics[1]. The Hamiltonian function for a system of N particles with coordinate q_i and momentum p_i for the i-th particle is given by

$$H(q_1,\ldots,q_N,p_1,\ldots,p_N) = \frac{p_i p_i}{2} + V(q_1,\ldots,q_N,p_1,\ldots,p_N), \tag{2.5}$$

where $V(q_1,\ldots,q_N,p_1,\ldots,p_N)$ is an arbitrary potential. The Lagrangian function is obtained by Legendre transformation which then leads to the action

$$I = \int_0^{t_f} dt\,(-q_i\dot{p}_i - H(q_1,\ldots,q_N,p_1,\ldots,p_N))\,, \tag{2.6}$$

where the dot denotes differentiation with respect to the time coordinate t. Without loss of generality we also set the initial time $t_i = 0$. Variation of the action (2.6) gives us the equations of motion (EOM) (which in this case are Hamilton's equations) and a boundary term

$$\begin{aligned} \delta I &= \int_0^{t_f} dt \left(-q_i\delta\dot{p}_i + \delta q_i\left(-\dot{p}_i - \frac{\partial V}{\partial q_i}\right) + \delta p_i\left(-p_i - \frac{\partial V}{\partial p_i}\right)\right) = \\ &= \int_0^{t_f} dt \left(\delta q_i\left(-\dot{p}_i - \frac{\partial V}{\partial q_i}\right) + \delta p_i\left(\dot{q}_i - p_i - \frac{\partial V}{\partial p_i}\right)\right) - q_i\delta p_i\Big|_0^{t_f}. \end{aligned} \tag{2.7}$$

It is common to demand Dirichlet bcs that fix the coordinates on the boundary but leave the momenta fluctuating

$$\delta q_i\Big|_0^{t_f} = 0, \quad \delta p_i\Big|_0^{t_f} \neq 0 \quad \forall i \in [1, N]. \tag{2.8}$$

We see that in order to get a well-defined Dirichlet boundary value problem we have to add a suitable boundary term to the action (2.6) such that we get rid of the last term in (2.7). The emphasis here lies on boundary, we do not want to add a bulk term since this would then lead to different EOM.

[1]This approach is inspired by D. Grumiller's lectures on BHs. A more detailed summary can be found in his lecture notes [22].

It can easily be checked that adding the term

$$I_{GHY} = q_i p_i \Big|_0^{t_f} \tag{2.9}$$

leads to a well-defined Dirichlet boundary value problem

$$\delta I|_{EOM} = \delta q_i p_i \Big|_0^{t_f}. \tag{2.10}$$

The subscript GHY in (2.9) refers to Gibbons-Hawking-York [1, 2], in analogy to the common boundary term used in gravity theories. We will introduce it in more detail in the next section. If t_f were finite we would now have obtained a well-defined variational principle.
A problem arises if we consider asymptotic boundaries $t_f \to \infty$. Depending on the potential $V(q_1, \ldots, q_N, p_1, \ldots, p_N)$ there could be systems for which some $q_i \to \infty$ on that boundary. One example is the half binding potential $V(q) = \frac{1}{q^2}$, discussed in [23]. If we take a look at Fig. 2.1 we see that a particle will roll down the potential and reach $q \to \infty$ at $t_f \to \infty$.

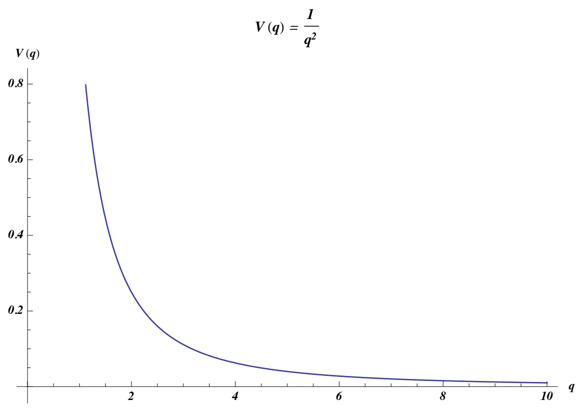

Figure 2.1: Plot of a half binding potential

As a result, it is not possible to impose Dirichlet bcs on the asymptotic boundary. The variation could in theory be finite

$$\lim_{t_f \to \infty} \delta q_i \Big|^{t_f} \neq 0. \tag{2.11}$$

This not only leads to an ill-defined action principle but also to a diverging on-shell action $I|_{EOM}$. Proceeding as before, we add an additional boundary term $S(q_1, \ldots, q_N, p_1, \ldots, p_N)\Big|^{t_f}$ on the boundary $t_f \to \infty$. This process is called holographic renormalization [24]. The term holographic refers to the fact that we are working on the boundary and renormalization because fixing the action principle also leads to a finite on-shell action. The full action that should have vanishing variation is given by

$$I_{\text{full}} = I + I_{GHY} + S(q_1, \ldots, q_N, p_1, \ldots, p_N)\Big|^{t_f}. \tag{2.12}$$

The variation of (2.12) gives

$$\delta I_{\text{full}} = \delta q_i \left(p_i - \frac{\partial S}{\partial q_i} \right)\Bigg|^{t_f}, \tag{2.13}$$

where we see that we get a well-defined variational principle for

$$p_i = \frac{\partial S}{\partial q_i} \qquad \text{on } t_f \to \infty. \tag{2.14}$$

Luckily the solution to this problem is already well known and corresponds to Hamilton's principle function which is the solution to the Hamilton-Jacobi equation

$$H\left(q_1, \ldots, q_N, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_N}, t\right) + \frac{\partial S}{\partial t} = 0. \tag{2.15}$$

2.2 General Relativity

In 1915, Hilbert showed independently from Einstein that the gravitational field equations (nowadays referred to as Einstein equations)

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0 \tag{2.16}$$

can also be derived from a principle of a stationary action [25]. The corresponding action governing the laws of Einstein gravity is given by

$$I_{EH} = \frac{1}{16\pi G} \int_{\mathcal{M}} d^3x \sqrt{-g} \, (R - 2\Lambda) \,, \tag{2.17}$$

where we adapted the number of spacetime dimensions to 3. In analogy to our example in mechanics this action only leads to a well-defined variational principle if we neglect boundary contributions.

The variation of the action (2.17) yields the EOM and a total derivative term[2]

$$
\begin{aligned}
\delta I_{EH} =& -\frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{-g}\left(R^{ab}-\frac{1}{2}Rg^{ab}+\Lambda g^{ab}\right)\delta g_{ab} \\
&+\frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{-g}\nabla_a\left(g^{bc}\delta\Gamma^a_{bc}-g^{ab}\delta\Gamma^c_{bc}\right) = \\
=& -\frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{-g}\left(R^{ab}-\frac{1}{2}Rg^{ab}+\Lambda g^{ab}\right)\delta g_{ab} \\
&+\frac{\varepsilon}{16\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{|h|}n_a\left(g^{bc}\delta\Gamma^a_{bc}-g^{ab}\delta\Gamma^c_{bc}\right).
\end{aligned}
\tag{2.18}
$$

The second term was rewritten using Stokes' theorem

$$
\int_{\mathcal{M}} d^3x\sqrt{-g}\nabla_a A^a = \oint_{\partial\mathcal{M}} d^2x\sqrt{|h|}\varepsilon n_a A^a, \tag{2.19}
$$

where h_{ab} is the transverse metric and n_a normal to the boundary $\partial\mathcal{M}$. The variation of the Christoffel symbols Γ^a_{bc} contains metric derivatives and is given by

$$
\delta\Gamma^a_{bc} = \frac{1}{2}g^{am}\left(\nabla_b\delta g_{mc}+\nabla_c\delta g_{mb}-\nabla_m\delta g_{bc}\right). \tag{2.20}
$$

Following the procedure introduced in Sec. 2.1, we try to add suitable boundary terms such that we obtain a well-defined variational principle, demanding Dirichlet bcs

$$
\begin{aligned}
\delta g_{ab}\big|_{\partial\mathcal{M}} &= 0 \\
n^c\nabla_c\delta g_{ab}\big|_{\partial\mathcal{M}} &\neq 0.
\end{aligned}
\tag{2.21}
$$

These are analogous to (2.8), where the metric corresponds to the (generalized) coordinates q_i and its fluctuating normal derivative to the momenta p_i. The additional boundary term that leads to a well-defined Dirichlet boundary value problem was first derived by Gibbons, Hawking [1] and independently York [2]

$$
I_{GHY} = \frac{\varepsilon}{8\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{|h|}K, \tag{2.22}
$$

where $K = K^{ab}g_{ab}$ denotes the contracted extrinsic curvature and ε corresponds to the definition (1.11).

[2] Assuming a space- or timelike boundary with corresponding $n_a n^a = \varepsilon = \pm 1$, as in (1.11).

The variation of the improved action $I = I_{EH} + I_{GHY}$ yields the Brown-York stress tensor [26] contracted with the metric variation

$$\delta I|_{EOM} = -\frac{1}{2}\int_{\partial\mathcal{M}} d^2x\sqrt{|h|}T^{ab}_{BY}\delta g_{ab} \tag{2.23}$$
$$T^{ab}_{BY} = \frac{1}{8\pi G}\left(K^{ab} - \varepsilon K h^{ab}\right). \tag{2.24}$$

Analogously to the mechanics example this action does still not lead to a well-defined variational principle in the presence of asymptotic boundaries. In general, the assumption that $g_{ab}\big|_{\partial\mathcal{M}} = 0$ seems unphysical. Usually $q_i\big|_{\partial\mathcal{M}} = 0$ would just correspond to a vacuum state but a vanishing metric would account for a singularity on the boundary, which is a rather unnatural event. Depending on the chosen bcs, a non-vanishing metric variation at the boundary,

$$g_{ab}\big|_{\partial\mathcal{M}} \to \infty \tag{2.25}$$
$$\delta g_{ab}\big|_{\partial\mathcal{M}} = \mathcal{O}(1) \tag{2.26}$$

is not unlikely and calls for the addition of a holographic counterterm $S\left(g_{ab}, \nabla_c g_{ab}\right)\big|_{\partial\mathcal{M}}$ to the action [24].
We already showed in Sec. 2.1 that holographic renormalization can be obtained by solving the Hamilton-Jacobi equation. Unfortunately, in $d = 3$ Einstein gravity, this is in the least cases exactly solvable. Instead, we try to find $S\left(g_{ab}, \nabla_c g_{ab}\right)\big|_{\partial\mathcal{M}}$ by making an Ansatz

$$S\left(g_{ab}, \nabla_c g_{ab}\right)\big|_{\partial\mathcal{M}} = \frac{1}{16\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{|h|}\left(\alpha + \beta K + \gamma\mathcal{R} + \text{higher derivatives}\right), \tag{2.27}$$

where Hamilton's principle function is expanded as a series of extrinsic and intrinsic curvature terms. The quantity $\mathcal{R} = \mathcal{R}^a{}_{bad}g^{bd}$ denotes the contracted intrinsic Ricci curvature and K the contracted extrinsic curvature (1.18). Higher derivative terms are $K_{ab}K^{ab}, K^2, \mathcal{R}_{ab}\mathcal{R}^{ab}, \mathcal{R}^2$ etc.

2.3 Brown-Henneaux boundary conditions

In their famous paper [15] from 1986 Brown and Henneaux found that the asymptotic symmetries of a certain type of asymptotic AdS_3 spacetimes fall into two copies of the Virasoro algebra, the symmetry algebra of CFTs in two dimensions. To perform a Hamiltonian analysis and derive the central charges they imposed certain fall-off conditions

$$\begin{aligned} g_{tt} &= -\frac{r^2}{l^2} + \mathcal{O}(1) \\ g_{rr} &= \frac{l^2}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right) \\ g_{\varphi\varphi} &= r^2 + \mathcal{O}\left(r^2\right) \\ g_{tr} &= \mathcal{O}\left(\frac{1}{r^3}\right) \\ g_{t\varphi} &= \mathcal{O}(1) \\ g_{r\varphi} &= \mathcal{O}\left(\frac{1}{r^3}\right), \end{aligned} \tag{2.28}$$

such that the metric asymptotes to AdS in the limit $r \to \infty$. It should be mentioned that l refers to the AdS radius following $\Lambda = -\frac{2}{l^2}$. These bcs are nowadays commonly referred to as Brown-Henneaux bcs. The metric variation preserving them can be written as

$$\begin{aligned} \delta g_{tt} &= \delta\gamma_{tt} + o(1) \\ \delta g_{rr} &= \delta\gamma_{rr}\frac{l^4}{r^4} + o\left(\frac{1}{r^2}\right) \\ \delta g_{\varphi\varphi} &= \delta\gamma_{\varphi\varphi} l^2 + o(1) \\ \delta g_{tr} &= \mathcal{O}\left(\frac{1}{r^3}\right) \\ \delta g_{t\varphi} &= \mathcal{O}(1) \\ \delta g_{r\varphi} &= \mathcal{O}\left(\frac{1}{r^3}\right), \end{aligned} \tag{2.29}$$

where factors of l have been introduced such that the fluctuations $\delta\gamma_{ab}$ are of the same dimension.

It can be more convenient to work in Gaussian normal coordinates and express (2.28) in a Fefferman-Graham expansion [27]

$$ds^2|_{aAdS_3} = d\rho^2 + \left(e^{\frac{2\rho}{l}}\gamma_{ij}^{(0)} + \gamma_{ij}^{(2)} + \dots\right) dx^i dx^j, \tag{2.30}$$

where the metric is given as a series in orders of $e^{\frac{\rho}{l}}$. The ellipsis refers to terms that vanish in the limit $\rho \to \infty$ and the indices i, j run over the time and angular coordinate. For Brown-Henneaux bcs the so called boundary metric $\gamma_{ij}^{(0)}$ is fixed while $\gamma_{ij}^{(2)}$ is free to fluctuate.

To check the variational principle for the action $I = I_{EH} + I_{GHY}$ we simply plug the bcs (2.29) into the variation (2.23) and obtain

$$\delta I|_{EOM} = -\frac{1}{16\pi G}\int_{\partial\mathcal{M}} d^2x\,(\delta\gamma_{tt} - \delta\gamma_{\varphi\varphi} + o(1))\,. \tag{2.31}$$

We note that the remaining integral is $\mathcal{O}(1)$ and therefore does not vanish in the limit $r \to \infty$. Furthermore the Brown-York stress tensor diverges in this limit since $T_{BY,tt} = T_{BY,\varphi\varphi} = \mathcal{O}(r^2)$. As already proposed in the previous section, instead of trying to find a solution to the Hamilton-Jacobi equation we can use a simple Ansatz where the holographic counterterm will only contain a constant

$$S\big|_{\partial\mathcal{M}} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{|h|}\alpha. \tag{2.32}$$

As a result, the variation of the full action $I_{\text{full}} = I_{EH} + I_{GHY} + S\big|_{\partial\mathcal{M}}$ becomes

$$\begin{aligned}\delta I_{\text{full}}\big|_{EOM} &= -\frac{1}{2}\int_{\partial\mathcal{M}} d^2x\sqrt{|h|}T^{ab}_{BY-\text{ren}}\delta g_{ab} = \\ &= -\frac{1}{16\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{|h|}\left(K^{ab} - h^{ab}K - \alpha h^{ab}\right)\delta g_{ab} = \end{aligned} \tag{2.33}$$

$$= -\frac{1}{16\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{|h|}\Big((1 + l\alpha)\,(\delta\gamma_{tt} - \delta\gamma_{\varphi\varphi}) + o(1)\Big), \tag{2.34}$$

where we see that we get a well-defined variational principle for $\alpha = -\frac{1}{l}$. Fixing this problem also has the nice feature that simultaneously we renormalize the Brown-York stress tensor, which in terms of the Fefferman-Graham expansion (2.30) is given by the finite expression

$$T_{BY-ren,ij} = -\frac{1}{8\pi Gl}\gamma_{ij}^{(2)}. \tag{2.35}$$

3 Null boundaries

In the previous sections we showed how to deal with space- and timelike boundaries embedded in a given spacetime and presented boundary counterterms that have to be added to the action in order to get a well-defined variational principle. Based on the hypersurface description $\phi(x^a) = 0$ we could define a unique normal vector and then build a transverse metric and corresponding curvature quantities. In this section we follow loosely [3–5, 19] but obtain more general results.

A null hypersurface exhibits a normal vector k^a that is by definition also tangent to the hypersurface since

$$k_a k^a = 0. \tag{3.1}$$

This leads to an important ambiguity that only arises for null hypersurfaces. Because of the fact that k_a has vanishing norm we cannot uniquely define its normalization

$$k_a = A \partial_a \phi. \tag{3.2}$$

This ambiguity will be referred to as the so called boost gauge freedom, where we follow the discussion by Hopfmüller and Freidel in [28]. This additional gauge parameter A is assumed to be state-dependent i.e. is allowed to fluctuate

$$\delta k_a = \delta A\, \partial_a \phi = \delta\left(\ln A\right) k_a = \alpha k_a. \tag{3.3}$$

In order to define a transverse metric we need an auxiliary null vector l^a that has a non-vanishing inner product with k^a. Throughout this thesis we will demand that l^a is normalized such that

$$l_a l_b g^{ab} = 0 \tag{3.4}$$

$$k_a l_b g^{ab} = -1. \tag{3.5}$$

An induced metric can be constructed that projects out directions given by k^a and l^a,

$$\gamma_{ab} = g_{ab} + k_a l_b + l_a k_b. \tag{3.6}$$

We note that k^a and l^a are both orthogonal to γ_{ab} i.e. have a vanishing inner product $\gamma_{ab}k^a = \gamma_{ab}l^b = 0$. Because of this property the induced metric (or the null hypersurface in general) is effectively codimension 2, which means that for our purposes it will be one dimensional.

3.1 Null-GHY counterterm

Following the general procedure, our first goal is to find a suitable boundary counterterm such that we get a well-defined Dirichlet boundary value problem preserving the bcs (2.21). First we adapt the variation of the Einstein-Hilbert action (2.18) to null hypersurfaces and obtain on-shell

$$\delta I_{EH}\big|_{EOM} = -\frac{1}{16\pi G}\int_{\partial\mathcal{M}} d\varphi d\lambda \sqrt{|\gamma|} k_a \left(g^{bc}\delta\Gamma^a_{bc} - g^{ab}\delta\Gamma^c_{bc}\right), \tag{3.7}$$

where k^a denotes the null normal, γ_{ab} the induced metric and λ generates null geodesics on the hypersurface. Following the conventions in [4], we demand that the auxiliary vectors l^a should always be outward pointing with respect to the given spacetime manifold. As a result, k^a is not necessarily future-directed and the sign in (3.7) is fixed to be negative. Note that these conventions differ significantly from [5], where k^a was demanded to be future-pointing and therefore the sign in (3.7) changes whether the null hypersurface lies in the timelike future or past of the enclosed spacetime.
The determinant γ is particularly simple in our case since the null boundary is effectively one dimensional and therefore $\gamma = \gamma_{\varphi\varphi}$[1]. The variation of the Christoffel symbols is given by (2.20).

[1]Assuming that the null directions defined by k^a and l^a do not include the angular coordinate.

Following the treatment in [3] we make use of the identities

$$\begin{aligned} \delta\left(\nabla_a \xi_b\right) &= \nabla_a \delta\xi_b - \delta\Gamma^c_{ab}\xi_c \\ \delta\left(\nabla_a \xi^a\right) &= \nabla_a \delta\xi^a + \delta\Gamma^a_{ab}\xi^b, \end{aligned} \tag{3.8}$$

that hold for arbitrary vectors ξ^a. This allows us to extract a total variation term that can be cancelled by adding a corresponding counterterm to the action

$$\begin{aligned} \delta I_{EH}\big|_{EOM} = &\frac{1}{8\pi G}\int_{\partial\mathcal{M}} d\varphi d\lambda \delta\left(\sqrt{|\gamma|}\nabla_a k^a\right) \\ &- \frac{1}{16\pi G}\int_{\partial\mathcal{M}} d\varphi d\lambda \sqrt{|\gamma|}\left(\nabla_a \delta k^a_\perp + \left(\nabla_a k_b - \nabla_c k^c \gamma_{ab}\right)\delta g^{ab}\right). \end{aligned} \tag{3.9}$$

We have also adobted Parattu et al.'s notation in [3], where $\delta k^a_\perp = \delta k^a + g^{ab}\delta k_b$. Note that up to now this derivation is completely equivalent to the space- and timelike case.

To get more physical insight into what actually happens at the boundary we express (3.9) in terms of boundary curvature quantities. Unlike in the space- and timelike case we cannot only define an extrinsic curvature Θ^{ab} but also surface gravity quantities κ and $\overline{\kappa}$

$$\Theta^{ab} = \gamma^a_c \gamma^b_d \nabla_c k_d \tag{3.10}$$

$$\kappa = -k^a l^b \nabla_a k_b \tag{3.11}$$

$$\overline{\kappa} = -l^a k^b \nabla_a k_b, \tag{3.12}$$

where we will refer to the latter as auxiliary surface gravity. Note that for null normals that are globally null

$$k^b \nabla_a k_b = \frac{1}{2}\nabla_a k^2 = 0, \tag{3.13}$$

and therefore $\overline{\kappa}$ vanishes. In principle we can (and will) also define null normals that only have vanishing norm at the boundary and therefore $\nabla_a k^2\big|_{\partial\mathcal{M}}$ could be non-zero leading to a non-neglectable auxiliary surface gravity. Using these definitions the boundary counterterm (and null analog to the GHY term) that cancels the total variation in (3.9) reads

$$I_{NGHY} = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} d\varphi d\lambda \sqrt{|\gamma|}\left(\Theta + \kappa + \overline{\kappa}\right), \tag{3.14}$$

where Θ denotes the contracted extrinsic curvature (3.10).

In foresight to finding a well-defined variational principle for a BTZ BH we also have to consider corners in our spacetime. Hence, we have to deal with the $\nabla_a \delta k_\perp^a$ term in (3.9). After some manipulation we get

$$\nabla_a \delta k_\perp^a = \nabla_a \delta k^a + \alpha \left(\Theta + \kappa + \overline{\kappa} \right) + k^a \nabla_a \alpha, \tag{3.15}$$

where we have used the property of null normals that their variation is proportional to the original covector and α is defined in (3.3). It will also turn out to be convenient to manipulate the stress tensor term,

$$\nabla_a k_b \delta g^{ab} = -\nabla^a k^b \delta \gamma_{ab} - \alpha \left(\kappa + \overline{\kappa} \right) + \nabla^a k^b \left(\delta l_a k_b + \delta l_b k_a \right), \tag{3.16}$$

such that the surface gravity terms cancel. In order to partially integrate the corner terms we have to remind ourselves that for null hypersurfaces k^a lies on the hypersurface, which is built up by null geodesics following the geodesic equation

$$k^a \nabla_a k_b = \kappa k_b. \tag{3.17}$$

These geodesics are generated by the parameter λ that is related to k^a via

$$dx^a = k^a d\lambda. \tag{3.18}$$

The contracted extrinsic curvature Θ can be interpreted as the so called null expansion since after some manipulation one can show that

$$\Theta = \gamma^{ab} \nabla_a k_b = \frac{1}{2} \gamma^{ab} \left(\mathcal{L}_k \gamma \right)_{ab} = \frac{1}{\sqrt{|\gamma|}} \partial_\lambda \sqrt{|\gamma|}, \tag{3.19}$$

where we made use of the fact that on the hypersurface the induced metric can be reduced to a codimension 2 metric γ^{AB},

$$\gamma^{ab} = \gamma^{AB} e_A^a e_B^b. \tag{3.20}$$

The transformation vectors e^a_A are tangent to curves on the hypersurface, which implies that $e^a_A k_a = 0$ and $\mathcal{L}_k e^a_A = 0$. The codimension 2 metric γ_{AB} acts as a scalar regarding transformations of the global coordinates $x^a \to x'^a$[2] and therefore

$$\mathcal{L}_k \left(\gamma_{AB}\right) = k^a \nabla_a \gamma_{AB} = \partial_\lambda \gamma_{AB}. \tag{3.21}$$

Inserting the identities (3.15) - (3.21) in (3.9) yields a corner term,

$$-\frac{1}{16\pi G}\int_{\partial\mathcal{M}} d\varphi d\lambda \sqrt{|\gamma|}\left(\alpha\Theta + k^a\nabla_a\alpha\right) = -\left(\frac{1}{16\pi G}\int_{\partial^2\mathcal{M}} d\varphi \sqrt{|\gamma|}\alpha\right)\Bigg|_{\lambda_i}^{\lambda_f}. \tag{3.22}$$

Putting all the pieces together and adding a suitable null-GHY counterterm, the variation of the action

$$\begin{aligned}
&\delta\left(I_{EH} + I_{NGHY}\right)\big|_{EOM} = \\
&= -\frac{1}{16\pi G}\int_{\partial\mathcal{M}} d\varphi d\lambda \sqrt{|\gamma|}\left(\nabla_a \delta k^a + \nabla^a k^b\left(\delta l_a k_b + \delta l_b k_a\right) - \left(\nabla^a k^b - \nabla_c k^c \gamma^{ab}\right)\delta\gamma_{ab}\right) \\
&\quad -\left(\frac{1}{16\pi G}\int_{\partial^2\mathcal{M}} d\varphi\sqrt{|\gamma|}\alpha\right)\Bigg|_{\lambda_i}^{\lambda_f},
\end{aligned} \tag{3.23}$$

takes a slightly more complicated form than for the non-null case. This is a generalization of the results presented in [3–5], where the authors made assumptions on the boost gauge ($\delta k^a = 0$) or did not allow for $\nabla_a k^2 \neq 0\big|_{\partial M}$.

[2]Regarding transformations of the induced boundary coordinates $x^A \to x'^A$, γ_{AB} transforms as a 2-tensor. [19]

4 Corner contributions

Until now we only worked on isolated boundary segments, that are not connected to each other. To give a complete description of a given spacetime one also has to take into account corner contribution that arise at the intersection of two such boundaries. A complete list of such corners would consist of all 6 possible combinations of spacelike, timelike and null boundary segments connected to each other. However, since space- and timelike boundaries basically differ by a factor -1 their derivation is analogous and we will only present the intersections of space-/timelike, null/spacelike and null/null boundaries.

4.1 Space-/timelike corner

This type of corner term is already well known and was first presented in [29, 30]. We assume that we have a manifold $\mathcal{M}$ equipped with a metric g_{ab} that describes our whole spacetime. Let the boundary of this spacetime consist of a timelike segment $\mathcal{T}$ and a spacelike segment $\mathcal{S}$ that intersect at a junction $\mathcal{C}$. To describe our boundaries we can build induced metrics using the corresponding normal vectors n^a and s^a

$$g_{ab}|_{\mathcal{S}} = h_{ab} - n_a n_b \tag{4.1}$$

$$g_{ab}|_{\mathcal{T}} = \sigma_{ab} + s_a s_b. \tag{4.2}$$

The junction $\mathcal{C}$ is embedded in both $\mathcal{S}$ and $\mathcal{T}$ and we can further decompose the metric

$$g_{ab}|_{\mathcal{C}} = \gamma_{ab} - n_a n_b + m_a m_b = \gamma_{ab} + s_a s_b - t_a t_b, \tag{4.3}$$

where γ_{ab} is the induced metric on the corner and the normal vectors are correspondingly space- and timelike, $n_a n^a = t_a t^a = -m_a m^a = -s_a s^a = 1$.

Varying the action

$$I = \frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{-g}\,(R-2\Lambda) + \frac{1}{8\pi G}\int_{\mathcal{T}} d^2x\sqrt{|h|}K_I - \frac{1}{8\pi G}\int_{\mathcal{S}} d^2x\sqrt{|\sigma|}K_{II}, \tag{4.4}$$

where K_I and K_{II} denote the contracted extrinsic curvatures on the respective boundary segments, leads to the following non-vanishing variation on $\mathcal{C}$

$$\delta I\big|_{\mathcal{C}} = -\frac{1}{16\pi G}\int_{\mathcal{C}} dx\sqrt{|\gamma|}\,(m_b n_a + t_b s_a)\,\delta g^{ab}. \tag{4.5}$$

For orthogonal boundaries these terms vanish and there would not be any need for a suitable corner counterterm in the action. In the general case though, one can show that

$$(m_b n_a + t_b s_a)\,\delta g^{ab} = 2\delta\eta, \tag{4.6}$$

where $\eta = \operatorname{arsinh}(n_a s^a)$ corresponds to the Lorentz angle at which the two boundary segments are intersecting. As a result we can extract a total variation

$$\delta I\big|_{\mathcal{C}} = -\frac{1}{8\pi G}\int_{\mathcal{C}} dx\,\delta\left(\sqrt{|\gamma|}\eta\right) + \frac{1}{16\pi G}\int_{\mathcal{C}} dx\sqrt{|\gamma|}\eta\gamma^{ab}\delta g_{ab}, \tag{4.7}$$

that leads us to the definition of the suitable corner counterterm

$$I_{\mathcal{C}} = \frac{1}{8\pi G}\int_{\mathcal{C}} dx\sqrt{|\gamma|}\eta. \tag{4.8}$$

4.2 Null/spacelike corner

In contrast to the well-known corner term we just presented, the research on intersections that involve null hypersurfaces has only recently started [4, 5].
In order to describe this situation properly we have to be careful with the way we define the direction of the null normal k^a. Throughout this thesis we adopt the convention of [4] that the k^a should always be defined such that l^a is outward pointing with respect to our spacetime manifold $\mathcal{M}$.

We discuss a case in which the spacelike boundary segment $\mathcal{S}$ intersects the null segment $\mathcal{N}$ at its future boundary $\mathcal{C}$[1]. Thus, the variation of the action

$$\begin{aligned} I = & \frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{-g}\,(R-2\Lambda) \\ & -\frac{1}{8\pi G}\int_{\mathcal{S}} d^2x\sqrt{|h|}K \\ & -\frac{1}{8\pi G}\int_{\mathcal{N}} d\varphi d\lambda\sqrt{|\gamma|}\,(\Theta+\kappa+\overline{\kappa})\,, \end{aligned} \tag{4.9}$$

leads us to the corner term

$$\delta I\big|_{\mathcal{C}} = -\frac{1}{16\pi G}\int_{\mathcal{C}} dx\sqrt{|\gamma|}\,\left(m_b n_a \delta g^{ab}+\alpha\right), \tag{4.10}$$

where n_a is the normal to $\mathcal{S}$, m_a is normal to $\mathcal{C}$ embedded in $\mathcal{S}$ and α follows the definition (3.3). To find a suitable counterterm, that will take a similar form to (4.7), we first state some relations that hold on $\mathcal{C}$

$$\begin{aligned} g_{ab}|_{\mathcal{C}} &= \gamma_{ab} - n_a n_b + m_a m_b \\ &= \gamma_{ab} - k_a l_b - l_a k_b \end{aligned} \tag{4.11}$$

$$k_a = (k\cdot n)\,(n_a - m_a) \tag{4.12}$$

$$l_a = \frac{1}{2\,(k\cdot n)}\,(n_a + m_a) \tag{4.13}$$

$$m_a n_b \delta g^{ab} = m_a \delta n^a \tag{4.14}$$

$$\alpha = k_a \delta l^a. \tag{4.15}$$

Furthermore we require that the metric variation on $\mathcal{C}$ embedded in $\mathcal{S}$ is equal to $\mathcal{C}$ embedded in $\mathcal{N}$, which gives us

$$-\delta n_{(a} n_{b)} + \delta m_{(a} m_{b)} = -\delta k_{(a} l_{b)} - \delta l_{(a} k_{b)} \tag{4.16}$$

$$-\delta n^a n_a + \delta m^a m_a = -\delta k^a l_a - \delta l^a k_a. \tag{4.17}$$

Using the relations (4.11) - (4.17) allows us to write

$$\alpha + m_b n_a \delta g^{ab} = 2\delta \ln{(k\cdot n)} + l_a \delta k^a, \tag{4.18}$$

where we can see that there is again a Lorentz angle appearing.

[1]Here future does not refer to the time direction but to the null parameter $\lambda=\lambda_f$ appearing in (3.22) that is defined via $dx^a = k^a d\lambda$.

We can now extract a total variation and end up with the final expression

$$\begin{aligned}\delta I|_{\mathcal{C}} = &-\frac{1}{8\pi G}\int_{\mathcal{C}} dx\delta\left(\sqrt{|\gamma|}\ln(k\cdot n)\right) \\ &+\frac{1}{16\pi G}\int_{\mathcal{C}} dx\sqrt{|\gamma|}\left(\ln(k\cdot n)\gamma^{ab}\delta g_{ab} - l_a\delta k^a\right),\end{aligned} \tag{4.19}$$

that leads us to the definition of the corner counterterm

$$I_{\mathcal{C}} = \frac{1}{8\pi G}\int_{\mathcal{C}} dx\sqrt{|\gamma|}\ln(k\cdot n). \tag{4.20}$$

Comparing this with [5], we see that our result (4.19) is slightly more general since we did not fix the boost gauge such that $\delta k^a = 0$. What is more, our derivation was done assuming no special coordinate system, in contrast to the adapted coordinates used in [5].

4.3 Null/null corner

Since we introduced the convention that the l^a vectors should always be outward pointing, we can distinguish between two types of corners [4]. The corners are either located at solely the initial or final values of the corresponding parameters $\lambda_{I/II}$, that are defined on the null boundary segments $\mathcal{N}_I$ and $\mathcal{N}_{II}$. To make this statement clear we examine 2d Minkowski space as an example spacetime that exhibits four types of null/null corners. After demanding that the l^a should be outward pointing with respect to the spacetime manifold, the direction of the k^a is thus fixed by $k_a l^a = -1$. The null normals k^a define a parameter λ via

$$dx^a\big|_X = k^a_X d\lambda_X, \tag{4.21}$$

where the subscript $X \in \{I, II, III, IV\}$ denotes the boundary segment. In Fig. 4.1 we see that all corners appear either at solely the initial or final values of the corresponding λ_X. For example timelike infinity i^+ is located at $\lambda_I = \lambda_{I,\text{final}}$ and $\lambda_{IV} = \lambda_{IV,\text{final}}$. Conversely spatial infinity i^0 lies at $\lambda_{IV} = \lambda_{IV,\text{initial}}$ and $\lambda_{III} = \lambda_{III,\text{initial}}$ etc.
These two cases only differ by an overall sign, hence we only discuss the case with solely initial corners.

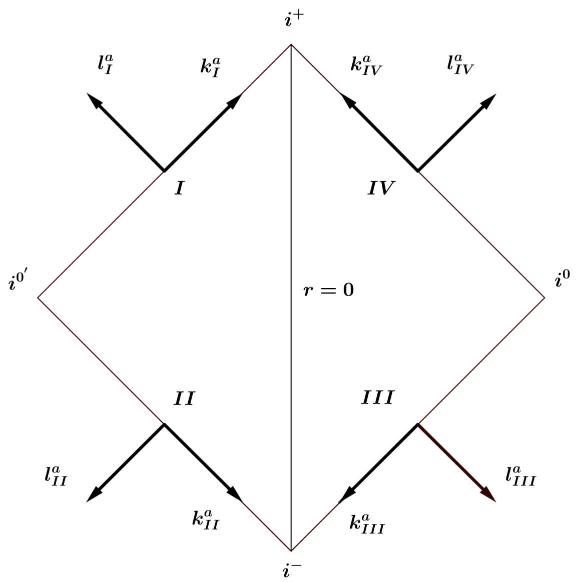

Figure 4.1: Carter-Penrose diagram of 2d Minkowski space. The null normals have been defined such that the auxiliary vectors l^a are outward pointing with respect to the spacetime manifold.

We start of with an action of the form

$$\begin{aligned} I = & \frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{-g}\,(R-2\Lambda) \\ & -\frac{1}{8\pi G}\int_{\mathcal{N}_I} d\varphi d\lambda\sqrt{|\gamma_I|}\,(\Theta_I+\kappa_I+\overline{\kappa_I}) \\ & -\frac{1}{8\pi G}\int_{\mathcal{N}_{II}} d\varphi d\lambda\sqrt{|\gamma_{II}|}\,(\Theta_{II}+\kappa_{II}+\overline{\kappa_{II}})\,, \end{aligned} \tag{4.22}$$

where the boundary counterterms follow the definitions in Sec. 3.1. Variation leads to a corner term

$$\delta I\big|_{\mathcal{C}} = \frac{1}{16\pi G}\int_{\mathcal{C}} dx\sqrt{|\gamma|}\,(\alpha_I+\alpha_{II})\,, \tag{4.23}$$

where α_I and α_{II} are defined following (3.3). Just like in Sec. 4.2 we follow loosely [4,5], meaning that we try to extract a variation of a Lorentz angle from this expression. After some manipulation we get the simple relation

$$\alpha_I+\alpha_{II} = \delta\ln\left(k_I\cdot k_{II}\right)+\frac{k_I^a k_{II}^b}{k_I\cdot k_{II}}\delta g_{ab}. \tag{4.24}$$

This allows us to extract a total variation such that we end up with

$$
\begin{aligned}
\delta I\big|_{\mathcal{C}} =& \frac{1}{16\pi G}\int_{\mathcal{C}} dx\delta\left(\sqrt{|\gamma|}\ln\left(k_I\cdot k_{II}\right)\right)\\
&+\frac{1}{16\pi G}\int_{\mathcal{C}} dx\sqrt{|\gamma|}\left(\frac{k_I^a k_{II}^b}{k_I\cdot k_{II}}-\frac{1}{2}\ln\left(k_I\cdot k_{II}\right)\gamma^{ab}\right)\delta g_{ab}.
\end{aligned}
\tag{4.25}
$$

The correct null/null corner counterterm is thus given by

$$
I_{\mathcal{C}} = -\frac{1}{16\pi G}\int_{\mathcal{C}} dx\sqrt{|\gamma|}\ln\left(k_I\cdot k_{II}\right). \tag{4.26}
$$

It should be pointed out that this is not precisely the same counterterm that the authors derived in [4,5], as our result differs by a factor $\frac{1}{2}$. However, this is not a disagreement and due to the fact that we did not fix the boost gauge. Demanding $\delta k^a = 0$ implies that the second term in (4.25) can be rewritten as a total variation

$$
\frac{k_I^a k_{II}^b}{k_I\cdot k_{II}}\delta g_{ab} = \delta\ln\left(k_I\cdot k_{II}\right), \tag{4.27}
$$

and leads to the counterterm in [4,5]

$$
I_{\mathcal{C}} = -\frac{1}{8\pi G}\int_{\mathcal{C}} dx\sqrt{|\gamma|}\ln\left(k_I\cdot k_{II}\right). \tag{4.28}
$$

5 The BTZ black hole as a prototypical example

As we already explained in Sec. 2, the GHY-like boundary and corner terms that we just derived do still not necessarily lead to a well-defined variational principle. Typically we also need to introduce holographic counterterms to get a vanishing variation of the action $\delta I|_{EOM} = 0$. We use a BTZ BH as an example since its spacetime exhibits null and timelike boundaries in the form of a bifurcate Killing horizon and an asymptotically (timelike) AdS_3 boundary. What is more, we make explicit comparisons with [31], where a near horizon limit of BTZ was used to define bcs in the context of soft hair [32]. The BTZ line element (1.3) lets us easily find the Killing horizon at $r = r_+$. Nonetheless we need other coordinates to describe the whole spacetime at the horizon.

5.1 Near-horizon boundary conditions

To start of we shift the horizon to $x = r - r_+ = 0$ and go over to a corotating frame, which will later help us to derive null coordinates. The corresponding coordinate transformations

$$x^2 := \frac{l^2}{r_+^2 - r_-^2}\left(r^2 - r_+^2\right), \qquad \phi := \varphi - \frac{r_-}{lr_+}t, \tag{5.1}$$

lead to a near horizon line element, that asymptotes to Rindler space in the limit $x \to 0$

$$ds^2 = -a^2x^2dt^2 + \frac{l^2}{l^2 + x^2}dx^2 + 2a\frac{r_-}{l}x^2dtd\phi + \left(r_+^2 + \frac{r_+^2 - r_-^2}{l^2}x^2\right)d\phi^2. \tag{5.2}$$

Note that we defined $a := \frac{r_+^2 - r_-^2}{l^2 r_+}$, which will later turn out to be useful because we will demand bcs such that $\delta a = 0$. The authors in [31] proceeded their calculations in the Chern-Simons (CS) formulation and also defined their bcs in this framework.

One of its advantages is that those bcs can be formulated to be independent from the radial coordinate. Hence, a well-defined variational principle was found that also holds on the boundaries $x = 0$ and $x \to \infty$.
To get more geometric insight we proceed in the metric formulation. Assuming a state-independent Rindler acceleration our bcs in this Rindler-type gauge are non-vanishing for the angular components only,

$$\delta g_{ab} = \begin{pmatrix} 0 & 0 & \frac{a\delta r_- x^2}{l} \\ 0 & 0 & 0 \\ \frac{a\delta r_- x^2}{l} & 0 & \frac{2\delta r_+ r_+ \left(l^2 + x^2\right) - 2\delta r_- r_- x^2}{l^2} \end{pmatrix}. \tag{5.3}$$

These bcs differ significantly from Brown-Henneaux bcs since $\delta g_{t\phi} = \mathcal{O}(x^2)$. This is due to the fact that we chose a corotating frame in order the access the horizon region $x = 0$ properly. Thus, this spacetime is not asymptotically AdS$_3$ in the sense of Brown-Henneaux, but rather asymptotes to a rotating AdS$_3$ in the limit $x \to \infty$. Before we proceed our calculation by choosing a Kruskal-like gauge in order to properly describe the horizon and more importantly the bifurcation 1-sphere, we check if we are able to add boundary counterterms to the action such that we get a well-defined variational principle on the $x \to \infty$ hypersurface. The most general Ansatz we propose is a counterterm of the form,

$$I_{\mathcal{B}} = \frac{1}{8\pi G} \int_{\mathcal{B}} d^2x \sqrt{|h|} \left((1+\alpha)\, K + \frac{\beta}{l} \right), \tag{5.4}$$

where we have allowed for an arbitrary prefactor in front of the GHY - term plus a constant cosmological boundary term. We do not take into account any higher order terms like K^2, $K_{ab}K^{ab}$ and $\mathcal{R}_{ab}\mathcal{R}^{ab}$ in the expansion (2.27) since they will only effectively contribute to higher orders in $\frac{1}{x}$. The contracted intrinsic curvature $\mathcal{R}$ is also not considered since it corresponds to the Euler characteristic and therefore has vanishing variation. As a result, we propose that the only possible covariant counterterms are completely determined by α and β.

Varying the total action $I = I_{EH} + I_{\mathcal{B}}$ leads to the boundary term

$$\begin{aligned}
&\delta\left(I_{EH}+I_{\mathcal{B}}\right)\Big|_{\mathcal{B}} = \\
&= -\frac{1}{16\pi G}\int_{\mathcal{B}} d^2x\sqrt{|h|}\left(\left(1+2\alpha\right)K^{ab}-\left(\left(1+\alpha\right)+\frac{\beta}{l}\right)h^{ab}-\alpha K n^a n^b\right)\delta g_{ab} \\
&\quad -\frac{\alpha}{16\pi G}\int_{\mathcal{B}} d^2x\sqrt{|h|}h^{ab}n^c\left(2\nabla_a\delta g_{bc}-\nabla_c\delta g_{ab}\right) = \\
&= -\frac{1}{16\pi G}\int_{\mathcal{B}} d^2x\left(\frac{2a\delta r_+ x^2}{l^2}\left(2\alpha+\beta+1\right)+a\delta r_+\left(2\alpha+\beta+2\right)+\mathcal{O}\left(\frac{1}{x}\right)\right).
\end{aligned} \tag{5.5}$$

We see that it is not possible to find a pair (α, β) such that (5.5) vanishes in the limit $x \to \infty$. We therefore propose to add a suitable non-covariant counterterm to obtain a well-defined variational principle. This result is rather surprising since the bcs used in [31] are equivalent to ours and lead to a well-defined variational principle in the CS-formulation, which leads us to suspect that this formulation might not be completely equivalent to the metric formulation when we also take boundaries into account. A suitable non-covariant term we could add is the metric component g_{xx} since it is $\mathcal{O}(\frac{1}{x^2})$ and state-independent. To clarify this argumentation we should point out that $\delta\sqrt{|h|} = \mathcal{O}(x^2)$ and therefore a counterterm proportional to g_{xx} will only affect the $\mathcal{O}(1)$ term in (5.5) and make our problem solvable. We use the Ansatz

$$I_{\mathcal{B}} = \frac{1}{8\pi G}\int_{\mathcal{B}} d^2x\sqrt{|h|}\left(\left(1+\alpha\right)K+\frac{\beta}{l}+\frac{\gamma}{l}g_{xx}\right), \tag{5.6}$$

where we chose γ as an arbitrary prefactor and divided by the AdS radius l to make it dimensionless. When we now vary the total action we get an expression that vanishes for the right choice of (α, β, γ)

$$\begin{aligned}
&\delta\left(I_{EH}+I_{\mathcal{B}}\right)\Big|_{\mathcal{B}} = \\
&= -\frac{1}{16\pi G}\int_{\mathcal{B}} d^2x\sqrt{|h|}\left(\left(1+2\alpha\right)K^{ab}-\left(\left(1+\alpha\right)+\frac{\beta}{l}+\frac{\gamma}{l}g_{xx}\right)h^{ab}-\alpha K n^a n^b\right)\delta g_{ab} \\
&\quad -\frac{\alpha}{16\pi G}\int_{\mathcal{B}} d^2x\sqrt{|h|}h^{ab}n^c\left(2\nabla_a\delta g_{bc}-\nabla_c\delta g_{bc}\right) = \\
&= -\frac{1}{16\pi G}\int_{\mathcal{B}} d^2x\left(\frac{2a\delta r_+ x^2}{l^2}\left(2\alpha+\beta+1\right)+a\delta r_+\left(2\alpha+\beta+2\gamma+2\right)+\mathcal{O}\left(\frac{1}{x}\right)\right).
\end{aligned} \tag{5.7}$$

We see that we get a well-defined variational principle for $\gamma = -\frac{1}{2}$ and a family of (α, β) defined by $2\alpha + \beta = -1$[1]. For the remainder of our calculations we work with $\alpha = 0$ and $\beta = -1$ since this leads us to the standard GHY-counterterm plus a holographic counterterm that includes non-covariant corrections. Thus, neglecting other boundary segments for now, the correct action is given by

$$I = I_{EH} + I_{GHY} - \frac{1}{8\pi G}\int_{\mathcal{B}} d^2x\sqrt{|h|}\left(\frac{1}{l} + \frac{1}{2l}g_{xx}\right). \tag{5.8}$$

Of course this additional, non-covariant counterterm leaves room for interpretation. One way to look at it could be that, since we are on an $x = \text{const.}$ hypersurface, the x-direction actually gets projected out and g_{xx} could be considered as a scalar field with respect to the boundary coordinates (t, ϕ). In a Kaluza-Klein-like split, where the electromagnetic vector potential $A_\mu = \frac{1}{g_{xx}}\begin{pmatrix} g_{xt} \\ g_{x\phi}\end{pmatrix}$ would be zero, this should lead to AdS Einstein equations on the boundary including a dilaton-like scalar field g_{xx}. We will not further discuss this matter though and proceed with our task to find a well-defined variational principle for the BTZ BH, including all boundary segments.

5.2 Horizon boundary conditions

To properly access the Killing horizon and the bifurcation 1-sphere we have to perform another coordinate transformation. In order to find suitable null coordinates we have to solve the geodesic equation

$$-a^2x^2\dot{t}^2 + \frac{l^2}{l^2 + x^2}\dot{x}^2 = 0, \tag{5.9}$$

where the dot denotes differentiation with respect to an affine parameter τ. Furthermore we demanded that the angular coordinate does not depend on τ and therefore does not give any contributions to the geodesic equation. Solving Eq. (5.9), we find that null geodesics are defined by

$$at = \ln\frac{x}{l + \sqrt{l^2 + x^2}}, \tag{5.10}$$

which allows us to define Kruskal-coordinates

$$U = \frac{lx}{l + \sqrt{l^2 + x^2}}e^{at}, \qquad V = \frac{lx}{l + \sqrt{l^2 + x^2}}e^{-at}. \tag{5.11}$$

[1]This result is equivalent to Eq. (14) in [33], where a transverse gauge for the fluctuations $\delta\gamma_{ab}$ was assumed.

Due to the fact the we assumed $\dot{\phi} = 0$ those coordinate transformations only involve state-independent quantities. As a result, the metric fluctuations δg_{ab}, formulated in the Rindler-like coordinates are equivalent to those formulated in Kruskal-like coordinates. The transformed line element and the bcs now read

$$
\begin{aligned}
ds^2 =& \frac{4l^4}{\left(UV - l^2\right)^2} dU dV + \frac{4l^3 r\_V}{\left(UV - l^2\right)^2} dU d\phi - \frac{4l^3 r\_U}{\left(UV - l^2\right)^2} dV d\phi \\
&+ \left(\frac{4l^2 UV \left(r_+^2 - r_-^2\right)}{\left(UV - l^2\right)^2} + r_+^2 \right) d\phi^2
\end{aligned}
\tag{5.12}
$$

$$
\delta g_{ab} = \begin{pmatrix} 0 & 0 & \frac{2\delta r_- l^3 V}{\left(l^2 - UV\right)^2} \\ 0 & 0 & -\frac{2\delta r_- l^3 U}{\left(l^2 - UV\right)^2} \\ \frac{2\delta r_- l^3 V}{\left(l^2 - UV\right)^2} & -\frac{2\delta r_- l^3 U}{\left(l^2 - UV\right)^2} & \frac{2\left(\delta r_+ r_+ \left(l^2 + UV\right)^2 - 4l^2 \delta r_- r_- UV\right)}{\left(l^2 - UV\right)^2} \end{pmatrix}. \tag{5.13}
$$

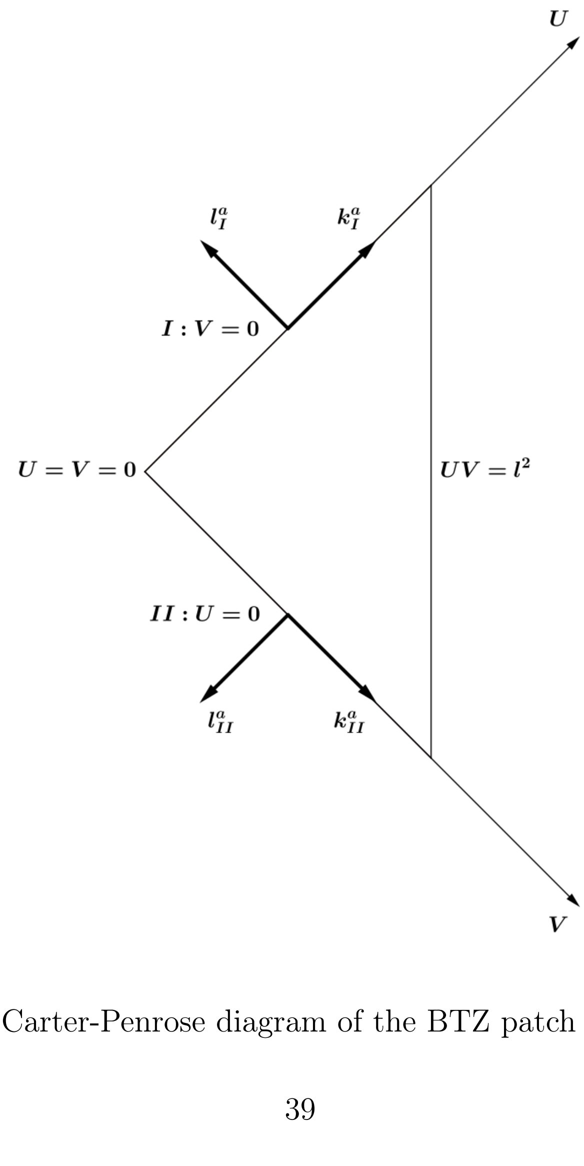

Figure 5.1: Carter-Penrose diagram of the BTZ patch $r_+ \leq r \leq \infty$.

In this coordinate system we observe that the BTZ patch in Fig. 5.1 has three different boundary segments connected via non-smooth corners:

- The ingoing ($U = 0$) Killing horizon
- The outgoing ($V = 0$) Killing horizon
- An asymptotic (timelike) boundary at $UV = l^2$, corresponding to $x \to \infty$
- The bifurcation 1-sphere at $U = V = 0$
- A null/timelike corner at $U = 0, V \to \infty$
- A null/timelike corner at $V = 0, U \to \infty$

5.2.1 Boundary segments

To distinguish between the two horizon segments we denote quantities defined on the $V = 0$ horizon with an index I and quantities on $U = 0$ with an index II. For now, we do not fix the boost gauge on those hypersurfaces yet and the null normals and auxiliary vectors take the form

$$\begin{aligned} k_{I,a} &= A\delta_a^V, & k_{II,a} &= B\delta_a^U \\ l_I^a &= -\frac{1}{A}\delta_V^a, & l_{II}^a &= -\frac{1}{B}\delta_U^a. \end{aligned} \tag{5.14}$$

In Sec. 3.1 we derived a boundary counterterm as natural analog to the GHY-counterterm. We check now if adding such a counterterm leads to a well-defined variational principle on the respective horizon segment. Neglecting corner terms for now, we observe that, after inserting k_a and l_a, the variation of the action, preserving the given bcs, vanishes on the horizons $U = 0$ and $V = 0$ without the need for holographic renormalization

$$\begin{aligned} &\delta\left(I_{EH} + I_{NGHY,I}\right)\big|_{EOM,\mathcal{H}_I} = \\ &= -\frac{1}{16\pi G}\int_{\mathcal{H}_I} d\phi d\lambda_I \sqrt{|\gamma_I|}\left(\nabla_a \delta k_I^a + \nabla^a k_I^b\left(\delta l_{I,a}k_{I,b} + \delta l_{I,b}k_{I,a}\right)\right) \\ &\quad + \frac{1}{16\pi G}\int_{\mathcal{H}_I} d\phi d\lambda_I \sqrt{|\gamma_I|}\left(\nabla^a k_I^b - \nabla_c k_I^c \gamma_I^{ab}\right)\delta\gamma_{I,ab} = \\ &= -\frac{1}{16\pi G}\int_{\mathcal{H}_I} d\phi d\lambda_I \left(\frac{\left(3r_-^2 + r_+^2\right)r_+\delta A - 2r_- A\left(\delta r_+ r_- - 2\delta r_- r_+\right)}{l^2 r^2} V + \mathcal{O}(V^2)\right) \end{aligned} \tag{5.15}$$

$$\delta\left(I_{EH}+I_{NGHY,II}\right)\big|_{EOM,\mathcal{H}_{II}}=$$

$$=-\frac{1}{16\pi G}\int_{\mathcal{H}_{II}}d\phi d\lambda_{II}\sqrt{|\gamma_{II}|}\left(\nabla_a\delta k^a_{II}+\nabla^a k^b_{II}\left(\delta l_{II,a}k_{II,b}+\delta l_{II,b}k_{II,a}\right)\right)$$
$$+\frac{1}{16\pi G}\int_{\mathcal{H}_{II}}d\phi d\lambda_{II}\sqrt{|\gamma_{II}|}\left(\nabla^a k^b_{II}-\nabla_c k^c_{II}\gamma^{ab}_{II}\right)\delta\gamma_{II,ab}=$$

$$=-\frac{1}{16\pi G}\int_{\mathcal{H}_{II}}d\phi d\lambda_{II}\left(\frac{\left(3r_-^2+r_+^2\right)r_+\delta B-2r_-B\left(\delta r_+r_--2\delta r_-r_+\right)}{l^2r^2}U+\mathcal{O}(U^2)\right). \tag{5.16}$$

We see that both integrals are at least of linear order in V or U and therefore lead to a well-defined variational principle on the horizon (neglecting corner terms). The last boundary segment we have to work on is the $UV=l^2$ element. The corresponding normal vector is given by

$$n^a=\frac{U\left(l^2-UV\right)}{2l^2\sqrt{UV}}\delta^a_U+\frac{V\left(l^2-UV\right)}{2l^2\sqrt{UV}}\delta^a_V, \tag{5.17}$$

and is normalized such that $n_an^a=1$. We already saw earlier in this section that in order to get a well-defined variational principle we have to add a suitable non-covariant counterterm as well as a cosmological boundary term. Since non-covariant terms, by definition, do not behave well under coordinate transformations, we cannot just transform the term introduced in (5.8). Instead, we will add a term proportional to $(g_{UV})^{-1}$, which is similar to g_{xx} in the sense that it is also state-independent and $\mathcal{O}(\frac{1}{x^2})$. Thus, we use an Ansatz of the form

$$I_{\mathcal{B}}=\frac{1}{8\pi G}\int_{\mathcal{B}}d^2x\sqrt{|h|}\left(\left(1+\alpha\right)K+\frac{\beta}{l}+\frac{\gamma}{l}(g_{UV})^{-1}\right). \tag{5.18}$$

Adding this to the Einstein-Hilbert action gives the variational principle

$$\delta\left(I_{EH}+I_{\mathcal{B}}\right)\big|_{\mathcal{B}}=$$

$$=-\frac{1}{16\pi G}\int_{\mathcal{B}}d^2x\sqrt{|h|}\left(\left(1+2\alpha\right)K^{ab}-\left(\left(1+\alpha\right)+\frac{\beta}{l}+\frac{\gamma}{l}g_{xx}\right)h^{ab}-\alpha Kn^an^b\right)\delta g_{ab}$$
$$-\frac{\alpha}{16\pi G}\int_{\mathcal{B}}d^2x\sqrt{|h|}h^{ab}n^c\left(2\nabla_a\delta g_{bc}-\nabla_c\delta g_{ab}\right)=$$

$$=-\frac{1}{16\pi G}\int_{\mathcal{B}}d^2x\left(\frac{2a\delta r_+x^2}{l^2}\left(2\alpha+\beta+1\right)+a\delta r_+\left(2\alpha+\beta+4\gamma+2\right)+\mathcal{O}\left(\frac{1}{x}\right)\right), \tag{5.19}$$

where $x^2 = \frac{4l^4UV}{(l^2-UV)^2} \to \infty$ at the boundary. We see that choosing $\gamma = -\frac{1}{4}$ and $2\alpha + \beta = -1$ sets the leading two terms to zero and therefore gives us well-defined action principle. Again, we set $\alpha = 0$ and $\beta = -1$. This way we get the standard GHY-counterterm plus holographic corrections. Still neglecting corner terms the holographically renormalized action for the total spacetime $\mathcal{M}$, including all three boundary segments is thus given by

$$\begin{aligned} I = I_{EH} + I_{NGHY,I} + I_{NGHY,II} + I_{GHY} + I_{hc} = \\ = \frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{-g}\left(R + \frac{2}{l^2}\right) \\ - \frac{1}{8\pi G}\int_{\mathcal{H}_I} d\phi d\lambda_I \sqrt{|\gamma_I|}\,(\Theta_I + \kappa_I + \overline{\kappa_I}) \\ - \frac{1}{8\pi G}\int_{\mathcal{H}_{II}} d\phi d\lambda_{II} \sqrt{|\gamma_{II}|}\,(\Theta_{II} + \kappa_{II} + \overline{\kappa_{II}}) \\ + \frac{1}{8\pi G}\int_{\mathcal{B}} d^2x\sqrt{|h|}K \\ - \frac{1}{8\pi G}\int_{\mathcal{B}} d^2x\sqrt{|h|}\left(\frac{1}{l} + \frac{1}{4l}\,(g_{UV})^{-1}\right). \end{aligned} \tag{5.20}$$

5.2.2 Corner contributions

As already mentioned earlier in this section, our prototype spacetime consists of three boundary segments with non-smooth intersections. Therefore we also check for a well-defined variational principle on these corners and, if necessary, add suitable corner counterterms to the action (5.20).

First we examine the $V = U = 0$ corner $\mathcal{C}_{I/II}$, which corresponds to the bifurcation 1-sphere and therefore is a null/null corner. The null normals (5.14) are defined such that the auxiliary vectors $l^a_{I/II}$ are outward pointing with respect to the spacetime manifold and as a result k^a_I is pointing in the positive U-direction and k^a_{II} in the positive V-direction. As we can also observe in Fig. 5.1, the corner is located at $\lambda_I = \lambda_{I,i}$ and $\lambda_{II} = \lambda_{II,i}$, where $\lambda_{I/II}$ follow the definition (3.18).

We showed in Sec. 4.3, that after adding the null-GHY terms on the horizon segments $\mathcal{H}_{I/II}$ the variation of the action at the corner is given by

$$\delta I|_{\mathcal{C}_{I/II}} = \frac{1}{16\pi G}\int_{\mathcal{C}_{I/II}} d\phi\sqrt{|\gamma|}\left(\alpha_I + \alpha_{II}\right), \tag{5.21}$$

where $\gamma = \gamma_{I,\phi\phi} = \gamma_{II,\phi\phi}$. Following the definition (3.3), α_I and α_{II} are determined by the null normals and auxiliary vectors that we defined in (5.14),

$$\alpha_I = k_{I,a}\delta l_I^a, \qquad \alpha_{II} = k_{II,a}\delta l_{II}^a. \tag{5.22}$$

After adding the corner term derived in (4.26) and inserting $l^a_{I/II}$, $k^a_{I/II}$ and γ_{ab}[2] the action variation on $U = V = 0$ becomes

$$\begin{aligned}\delta I_{\mathcal{C}_{I/II}} &= \frac{1}{16\pi G}\int_{\mathcal{C}} d\phi\sqrt{|\gamma|}\left(\frac{k_I^a k_{II}^b}{k_I\cdot k_{II}} - \frac{1}{2}\ln\left(k_I\cdot k_{II}\right)\gamma^{ab}\right)\delta g_{ab} = \\ &= -\frac{1}{16\pi G}\int_{\mathcal{C}_{I/II}} d\phi\left(\ln\left(\frac{AB}{2}\right)\delta r_+ + \mathcal{O}(U/V)\right).\end{aligned} \tag{5.23}$$

The factor of $\frac{1}{2}$ in the logarithm can either be removed by adding a constant holographic renormalization term of the form

$$I_{\mathcal{C}_{I/II,hr}} = -\frac{1}{16\pi G}\int_{\mathcal{C}_{I/II}} d\phi\sqrt{|\gamma|}\ln(2), \tag{5.24}$$

or by redefining the boost gauge parameters[3] A or B such that $\frac{AB}{2}\to AB$. Since the corner is one dimensional we do not have any possible curvature terms that we could add in order to get rid of the state-dependent term in the logarithm. Adding terms proportional to $k^a_{I/II}$ also just gives us terms depending on δA or δB again. As a consequence, we propose that the only way to get vanishing variation on this corner is to demand

$$B = \frac{1}{A} \tag{5.25}$$

such that we are left with only one additional boost gauge degree of freedom instead of two.

[2]At the corner the transverse metrics $\gamma_{I,ab}$ and $\gamma_{II,ab}$ are equivalent i.e. $\gamma_I^{ab}\delta g_{ab} = \gamma_{II}^{ab}\delta g_{ab} = \gamma^{\phi\phi}\delta g_{\phi\phi} =: \gamma^{ab}\delta g_{ab}$

[3]This does not correspond to gauge fixing nor does it result in any loss of generality. The additional (fixed) factor in front of A and/or B in (5.14) can, for example, be obtained by redefining the hypersurface equation $\phi(x^a) = 0 \to \tilde{\phi}(x^a) = \frac{1}{2}\phi(x^a) = 0$.

The other corners appearing in Fig. 5.1 are both an intersection of either the $U = 0$ or the $V = 0$ horizon $\mathcal{H}_{I/II}$ with the timelike $UV = l^2$ hypersurface $\mathcal{B}$. This corner can be described as the $\frac{U}{V} = \text{const.}$ boundary of a $UV = \text{const.}$ hypersurface, which corresponds to $t = \text{const.}$ and $x = \text{const.}$ in terms of Schwarzschild-like coordinates. Therefore the normal vector $m_a \sim \partial_a \left(\frac{U}{V}\right)$ is orthogonal to n^a and $\delta m_a \sim m_a$.
As a result, the corner contribution coming from the asymptotic boundary $\mathcal{B}$ vanishes since

$$m_a n_b \delta g^{ab} = \delta \left(m^a n_a\right) + \delta m_a n^a + \delta n_a m^a = 0. \tag{5.26}$$

The null hypersurfaces $\mathcal{H}_{I/II}$ intersect $\mathcal{B}$ at $\lambda_{I/II} = \lambda_{I/II,\text{final}}$ and, as already shown in (3.22), the variation of the action on those corners is thus given by

$$\delta I|_{\mathcal{C}_{I/\mathcal{B}}} = -\frac{1}{16\pi G}\int_{\mathcal{C}_{I/\mathcal{B}}} d\phi \sqrt{|\gamma_I|}\alpha_I \tag{5.27}$$

$$\delta I|_{\mathcal{C}_{II/\mathcal{B}}} = -\frac{1}{16\pi G}\int_{\mathcal{C}_{II/\mathcal{B}}} d\phi \sqrt{|\gamma_{II}|}\alpha_{II}. \tag{5.28}$$

Since we do not have to try to cancel the $m_a n_b \delta g^{ab}$ term anymore, we expand $\alpha_{I/II}$ using the Lorentz angle between k^a and an arbitrary vector ξ^a,

$$\alpha_{I/II} = \delta \ln \left(k_{I/II} \cdot \xi_{I/II}\right) - \frac{k_{I/II,a}\delta\xi^a_{I/II}}{k_{I/II} \cdot \xi_{I/II}}. \tag{5.29}$$

Given that $k_\phi = 0$ we can, without loss of generality, express ξ^a as a linear combination of the normal vectors n^a and m^a,

$$\xi^a_{I/II} = \mu_{I/II} n^a + \nu_{I/II} m^a, \tag{5.30}$$

where $\mu_{I/II}$ and $\nu_{I/II}$ are coefficients that we want to fix such that we get a well-defined action principle on the corners. The corresponding corner counterterms that we can add to the action (5.20) are thus given by

$$\begin{aligned} I_{\mathcal{C}_{I/\mathcal{B}}} &= \frac{1}{16\pi G}\int_{I/\mathcal{B}} d\phi \sqrt{|\gamma_I|} \ln\left(k_I \cdot \xi_I\right) \\ I_{\mathcal{C}_{II/\mathcal{B}}} &= \frac{1}{16\pi G}\int_{II/\mathcal{B}} d\phi \sqrt{|\gamma_{II}|} \ln\left(k_{II} \cdot \xi_{II}\right). \end{aligned} \tag{5.31}$$

We observe that, after adding those counterterms, the variation of the action cannot be renormalized for any choice of the coefficients $\mu_{I/II}$ and $\nu_{I/II}$

$$\begin{aligned}
\delta I\big|_{\mathcal{C}_{I/\mathcal{B}}} &= \frac{1}{16\pi G}\int_{\mathcal{C}_{I/\mathcal{B}}} d\phi\sqrt{|\gamma_I|}\left(\frac{k_{I,a}\delta\xi_I^a}{k_I\cdot\xi_I}+\frac{1}{2}\ln\left(k_I\cdot\xi_I\right)\gamma_I^{ab}\delta g_{ab}\right) = \\
&= \frac{1}{16\pi G}\int_{\mathcal{C}_{I/\mathcal{B}}} d\phi\frac{\mu_I r_+^2(\delta r_+ r_+ - \delta r_- r_-)\ln\left(\frac{le^{-at}\left(\nu_I\sqrt{r_+^2-r_-^2}+\mu_I r_+\right)A}{r_+ x}\right)}{lr_+\sqrt{r_+^2-r_-^2}\left(\nu_I\sqrt{r_+^2-r_-^2}+\mu_I r_+\right)}x \\
&+ \frac{1}{16\pi G}\int_{\mathcal{C}_{I/\mathcal{B}}} d\phi\frac{\nu_I(\delta r_+ r_+ - \delta r_- r_-)\ln\left(\frac{le^{-at}\left(\nu_I\sqrt{r_+^2-r_-^2}+\mu_I r_+\right)A}{r_+ x}\right)}{l\left(\nu_I\sqrt{r_+^2-r_-^2}+\mu_I r_+\right)}x \\
&+ \frac{1}{16\pi G}\int_{\mathcal{C}_{I/\mathcal{B}}} d\phi\frac{\nu_I r_-(\delta r_+ r_- - \delta r_- r_+)}{lr_+\left(\nu_I\sqrt{r_+^2-r_-^2}+\mu_I r_+\right)}x \\
&+ \frac{1}{16\pi G}\int_{\mathcal{C}_{I/\mathcal{B}}} d\phi\left(\frac{\delta r_- r_- - \delta r_+ r_+}{\sqrt{r_+^2-r_-^2}}+\mathcal{O}\left(\frac{1}{x}\right)\right),
\end{aligned} \tag{5.32}$$

$$\begin{aligned}
\delta I\big|_{\mathcal{C}_{II/\mathcal{B}}} &= \frac{1}{16\pi G}\int_{\mathcal{C}_{II/\mathcal{B}}} d\phi\sqrt{|\gamma_{II}|}\left(\frac{k_{II,a}\delta\xi_{II}^a}{k_{II}\cdot\xi_{II}}+\frac{1}{2}\ln\left(k_{II}\cdot\xi_{II}\right)\gamma_{II}^{ab}\delta g_{ab}\right) = \\
&= \frac{1}{16\pi G}\int_{\mathcal{C}_{II/\mathcal{B}}} d\phi\frac{\mu_{II} r_+^2(\delta r_+ r_+ - \delta r_- r_-)\ln\left(\frac{le^{at}\left(-\nu_{II}\sqrt{r_+^2-r_-^2}+\mu_{II} r_+\right)B}{r_+ x}\right)}{lr_+\sqrt{r_+^2-r_-^2}\left(-\nu_{II}\sqrt{r_+^2-r_-^2}+\mu_{II} r_+\right)}x \\
&- \frac{1}{16\pi G}\int_{\mathcal{C}_{II/\mathcal{B}}} d\phi\frac{\nu_{II}(\delta r_+ r_+ - \delta r_- r_-)\ln\left(\frac{le^{at}\left(-\nu_{II}\sqrt{r_+^2-r_-^2}+\mu_{II} r_+\right)B}{r_+ x}\right)}{l\left(-\nu_{II}\sqrt{r_+^2-r_-^2}+\mu_{II} r_+\right)}x \\
&- \frac{1}{16\pi G}\int_{\mathcal{C}_{II/\mathcal{B}}} d\phi\frac{\nu_{II} r_-(\delta r_+ r_- - \delta r_- r_+)}{lr_+\left(-\nu_{II}\sqrt{r_+^2-r_-^2}+\mu_{II} r_+\right)}x \\
&+ \frac{1}{16\pi G}\int_{\mathcal{C}_{II/\mathcal{B}}} d\phi\left(\frac{\delta r_- r_- - \delta r_+ r_+}{\sqrt{r_+^2-r_-^2}}+\mathcal{O}\left(\frac{1}{x}\right)\right).
\end{aligned} \tag{5.33}$$

Here we reintroduced the radial coordinate $x = \sqrt{\frac{4l^4UV}{(l^2-UV)^2}}$, which goes to infinity at this corner. We see that while we could find a solution $(\mu_{I/II}, \nu_{I/II})$ that leads to a vanishing $\mathcal{O}(x)$ term, the $\mathcal{O}(1)$ is not affected by our corner counterterms (5.31). That means, in order to get a well-defined variational principle, preserving the bcs (5.3), we propose to fix the boost gauge in a suitable manner. The most convenient gauge we can pick is setting $A = B = 1$. Then $\delta k_{I,a} = \delta k_{II,a} = 0$ and there are no corner contributions in the action integral at all. As a result, (5.20) is the total action that leads to a well-defined variational principle on the full spacetime manifold, including all boundary segments and, non-contributing, corners.

6 Conclusion

We explicitly calculated boundary counterterms for null boundaries and non-smooth corners. Doing so we followed loosely [3–5] and derived explicit expressions for the stress-tensor on the different boundary segments. We used a slightly more general framework since we did not assume that the null normals are null globally. Instead we loosened this assumption and only demanded that k^a is null on the corresponding boundary segment. We then wanted to construct a boundary counterterm for the soft hair bcs introduced in [31] and found that working in the metric formulation, it is not possible to find a suitable covariant counterterm on the asymptotic boundary such that we get a well-defined variational principle. This could be a hint that the CS-formulation might not be completely equivalent to the metric formulation in the presence of boundaries. As a solution, we proposed a non-covariant counterterm that involved a Kaluza-Klein-like scalar field, which could be interpreted as a dilaton-like field from the perspective on the boundary. What is more, we found Kruskal-like coordinates that allowed us to analyze the horizon region and the bifurcation 1-sphere. We found that, demanding the bcs (5.3), it is not possible to find corner counterterms to the action such that we get a well-defined variational principle. Instead we proposed to fix the boost gauge, that appears due to an ambiguity in the definition of null normals (5.14), in an adequate manner. The full action leading to a well-defined variational principle, assuming certain bcs and a boost gauge fixing such that $k_{I,a} = \delta_a^V$ and $k_{II,a} = \delta_a^U$, is then given by

$$
\begin{aligned}
&I = I_{EH} + I_{NGHY,I} + I_{NGHY,II} + I_{GHY} + I_{hc} = \\
&= \frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{-g}\left(R + \frac{2}{l^2}\right) \\
&- \frac{1}{8\pi G}\int_{\mathcal{H}_I} d\phi d\lambda_I \sqrt{|\gamma_I|}\left(\Theta_I + \kappa_I + \overline{\kappa_I}\right) - \frac{1}{8\pi G}\int_{\mathcal{H}_{II}} d\phi d\lambda_{II} \sqrt{|\gamma_{II}|}\left(\Theta_{II} + \kappa_{II} + \overline{\kappa_{II}}\right) \\
&+ \frac{1}{8\pi G}\int_{\mathcal{B}} d^2x\sqrt{|h|}\left(K - \frac{1}{l} - \frac{1}{4l}\left(g_{UV}\right)^{-1}\right).
\end{aligned}
\tag{6.1}
$$

A Explicit Expressions

A.1 Brown-Henneaux boundary conditions

The following identities have been used in calculations throughout Sec. 2.3:

$$n_a = \frac{1}{\sqrt{g^{rr}}}\delta^r_a = \left(\frac{l}{r} + \mathcal{O}\left(\frac{1}{r^3}\right)\right)\delta^r_a \tag{A.1}$$

$$n^a = g^{ab}n_b = \begin{pmatrix} \mathcal{O}\left(\frac{1}{r^4}\right) \\ \frac{r}{l} + \mathcal{O}\left(\frac{1}{r}\right) \\ \mathcal{O}\left(\frac{1}{r^4}\right) \end{pmatrix}^a \tag{A.2}$$

$$h_{ab} = g_{ab} - n_a n_b = \begin{pmatrix} -\frac{r^2}{l^2} + \mathcal{O}(1) & \mathcal{O}\left(\frac{1}{r^3}\right) & \mathcal{O}(1) \\ \mathcal{O}\left(\frac{1}{r^3}\right) & 0 & \mathcal{O}\left(\frac{1}{r^3}\right) \\ \mathcal{O}(1) & \mathcal{O}\left(\frac{1}{r^3}\right) & r^2 + \mathcal{O}(1) \end{pmatrix}_{ab} \tag{A.3}$$

$$h_{ij} = \begin{pmatrix} -\frac{r^2}{l^2} + \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & r^2 + \mathcal{O}(1) \end{pmatrix}_{ij} \tag{A.4}$$

$$h = \det h_{ij} = -\frac{r^4}{l^2} + \mathcal{O}\left(r^2\right) \tag{A.5}$$

$$\sqrt{-h} = \frac{r^2}{l} + \mathcal{O}(1) \tag{A.6}$$

$$K_{ab} = \frac{1}{2}\left(\mathcal{L}_n h\right)_{ab} = \begin{pmatrix} -\frac{r^2}{l^3} + \mathcal{O}(1) & \mathcal{O}\left(\frac{1}{r^3}\right) & \mathcal{O}\left(\frac{1}{r}\right) \\ \mathcal{O}\left(\frac{1}{r^3}\right) & \mathcal{O}\left(\frac{1}{r^8}\right) & \mathcal{O}\left(\frac{1}{r^3}\right) \\ \mathcal{O}\left(\frac{1}{r}\right) & \mathcal{O}\left(\frac{1}{r^3}\right) & \frac{r^2}{l} + \mathcal{O}(1) \end{pmatrix}_{ab} \tag{A.7}$$

$$K = K^{ab}g_{ab} = \frac{2}{l} + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{A.8}$$

$$\sqrt{-h}K^{ab}\delta g_{ab} = \delta\gamma_{\varphi\varphi} - \delta\gamma_{tt} + o(1) \tag{A.9}$$

$$\sqrt{-h}Kh^{ab}\delta g_{ab} = 2\left(\delta\gamma_{\varphi\varphi} - \delta\gamma_{tt}\right) + o(1) \tag{A.10}$$

$$\sqrt{-h}\alpha h^{ab}\delta g_{ab} = \alpha l\left(\delta\gamma_{\varphi\varphi} - \delta\gamma_{tt}\right) + o(1) \tag{A.11}$$

A.2 Near-horizon boundary conditions

The following identities have been used in calculations throughout Sec. 5.1:

$$n_a = \left(\frac{l}{x} + \mathcal{O}\left(\frac{1}{x^3}\right)\right)\delta_a^x \tag{A.12}$$

$$n^a = \left(\frac{x}{l} + \mathcal{O}\left(\frac{1}{x}\right)\right)\delta_x^a \tag{A.13}$$

$$h_{ab} = \begin{pmatrix} -a^2x^2 + \mathcal{O}\left(\frac{1}{x^2}\right) & 0 & \frac{ar_-x^2}{l} + \mathcal{O}\left(\frac{1}{x^2}\right) \\ 0 & \mathcal{O}\left(\frac{1}{x^3}\right) & 0 \\ \frac{ar_-x^2}{l} + \mathcal{O}\left(\frac{1}{x^2}\right) & 0 & \frac{(r_+^2-r_-^2)x^2}{l^2} + r_+^2 + \mathcal{O}\left(\frac{1}{x^2}\right) \end{pmatrix}_{ab} \tag{A.14}$$

$$h_{ij} = \begin{pmatrix} -a^2x^2 + \mathcal{O}\left(\frac{1}{x^2}\right) & \frac{ar_-x^2}{l} + \mathcal{O}\left(\frac{1}{x^2}\right) \\ \frac{ar_-x^2}{l} + \mathcal{O}\left(\frac{1}{x^2}\right) & \frac{(r_+^2-r_-^2)x^2}{l^2} + r_+^2 + \mathcal{O}\left(\frac{1}{x^2}\right) \end{pmatrix}_{ij} \tag{A.15}$$

$$h = \det h_{ij} = -\frac{a^2r_+^2x^4}{l^2} - a^2r_+^2x^2 + \mathcal{O}\left(\frac{1}{x^2}\right) \tag{A.16}$$

$$\sqrt{-h} = \frac{ar_+x^2}{l} + \frac{1}{2}alr_+ + \mathcal{O}\left(\frac{1}{x^2}\right) \tag{A.17}$$

$$K_{ab} = \begin{pmatrix} -\frac{a^2x^2}{l} - \frac{a^2l}{2} + \mathcal{O}\left(\frac{1}{x^2}\right) & 0 & \frac{ar_-x^2}{l^2} + \frac{ar_-}{2} + \mathcal{O}\left(\frac{1}{x^2}\right) \\ 0 & 0 & 0 \\ \frac{ar_-x^2}{l^2} + \frac{ar_-}{2} + \mathcal{O}\left(\frac{1}{x^2}\right) & 0 & \frac{(r_+^2-r_-^2)x^2}{l^3} + \frac{r_+^2-r_-^2}{2l} + \mathcal{O}\left(\frac{1}{x^2}\right) \end{pmatrix}_{ab} \tag{A.18}$$

$$K = K^{ab}g_{ab} = \frac{2}{l} + \mathcal{O}\left(\frac{1}{x^2}\right) \tag{A.19}$$

$$g_{xx} = \frac{l^2}{x^2} + \mathcal{O}\left(\frac{1}{x^4}\right) \tag{A.20}$$

$$\sqrt{-h}K^{ab}\delta g_{ab} = \frac{2a\delta r_+x^2}{l^2} \tag{A.21}$$

$$\sqrt{-h}h^{ab}\delta g_{ab} = \frac{2a\delta r_+x^2}{l} + a\delta r_+l + \mathcal{O}\left(\frac{1}{x^2}\right) \tag{A.22}$$

$$n^an^b\delta g_{ab} = 0 \tag{A.23}$$

$$\sqrt{-h}h^{ab}n^c\nabla_a\delta g_{bc} = -\frac{2a\delta r_+x^2}{l^2} \tag{A.24}$$

$$\sqrt{-h}h^{ab}n^c\nabla_c\delta g_{ab} = 0 \tag{A.25}$$

A.3 Horizon boundary conditions

The following identities have been used in calculations throughout Sec. 5.2:

$$k_{I,a} = A\delta_a^V \tag{A.26}$$

$$k_I^a = \begin{pmatrix} \frac{A}{2} + \mathcal{O}(V) \\ \mathcal{O}(V^2) \\ \mathcal{O}(V) \end{pmatrix}^a \tag{A.27}$$

$$l_{I,a} = \begin{pmatrix} -\frac{2}{A} + \mathcal{O}(V) \\ 0 \\ \frac{2r_-U}{lA} + \mathcal{O}(V) \end{pmatrix}_a \tag{A.28}$$

$$l_I^a = -\frac{1}{A}\delta_V^a \tag{A.29}$$

$$k_{II,a} = B\delta_a^U \tag{A.30}$$

$$k_{II}^a = \begin{pmatrix} \mathcal{O}(U^2) \\ \frac{B}{2} + \mathcal{O}(U) \\ \mathcal{O}(U) \end{pmatrix}^a \tag{A.31}$$

$$l_{II,a} = \begin{pmatrix} 0 \\ -\frac{2}{B} + \mathcal{O}(U) \\ \frac{2r_-V}{lB} + \mathcal{O}(U) \end{pmatrix}_a \tag{A.32}$$

$$l_{II}^a = -\frac{1}{B}\delta_U^a \tag{A.33}$$

$$\gamma_{I,ab} = \begin{pmatrix} 0 & 0 & \frac{2r_-V}{l} + \mathcal{O}(V^2) \\ 0 & 0 & 0 \\ \frac{2r_-V}{l} + \mathcal{O}(V^2) & 0 & r_+^2 + \frac{4(r_+^2 - r_-^2)UV}{l^2} + \mathcal{O}(V^2) \end{pmatrix}_{ab} \tag{A.34}$$

$$\gamma_{II,ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{2r_-U}{l} + \mathcal{O}(U^2) \\ 0 & -\frac{2r_-U}{l} + \mathcal{O}(U^2) & r_+^2 + \frac{4(r_+^2 - r_-^2)UV}{l^2} + \mathcal{O}(U^2) \end{pmatrix}_{ab} \tag{A.35}$$

$$n_a = \frac{l^2}{\sqrt{UV}\,(l^2 - UV)} \begin{pmatrix} V \\ U \\ 0 \end{pmatrix}_a \tag{A.36}$$

$$n^a = \frac{l^2 - UV}{2l^2\sqrt{UV}} \begin{pmatrix} U \\ V \\ 0 \end{pmatrix}^a \tag{A.37}$$

$$h_{ab} = \begin{pmatrix} -\frac{l^4 V}{U(l^2-UV)^2} & \frac{l^4}{(l^2-UV)^2} & \frac{2l^3 r_- V}{(l^2-UV)^2} \\ \frac{l^4}{(l^2-UV)^2} & -\frac{l^4 U}{V(l^2-UV)^2} & -\frac{2l^3 r_- U}{(l^2-UV)^2} \\ \frac{2l^3 r_- V}{(l^2-UV)^2} & -\frac{2l^3 r_- U}{(l^2-UV)^2} & \frac{r_+^2 l^4 + 2\left(r_+^2 - 2r_-^2\right) UVl^2 + r_+^2 U^2 V^2}{(l^2-UV)^2} \end{pmatrix}_{ab} \tag{A.38}$$

$$K_{ab} = \frac{1}{2}\left(\mathcal{L}_n h\right)_{ab} = \frac{l^2+UV}{\left(l^2-UV\right)^2} \begin{pmatrix} \frac{l^2 V}{2\sqrt{VU^3}} & \frac{l^2}{2\sqrt{UV}} & \frac{lr_- V}{\sqrt{UV}} \\ \frac{l^2}{2\sqrt{UV}} & -\frac{l^2 U}{2\sqrt{UV^3}} & -\frac{lr_- U}{\sqrt{UV}} \\ \frac{lr_- V}{\sqrt{UV}} & -\frac{lr_- U}{\sqrt{UV}} & 2\left(r_+^2 - r_-^2\right)\sqrt{UV} \end{pmatrix}_{ab} \tag{A.39}$$

$$\nabla_a k_{I,b} = \begin{pmatrix} 0 & 0 & \frac{2lr_- V^2 A}{l^4-U^2V^2} \\ 0 & -\frac{2UA}{l^2-UV} & -\frac{r_-\left(l^4+U^2V^2\right)A}{l^5-lU^2V^2} \\ \frac{2lr_- V^2 A}{l^4-U^2V^2} & -\frac{r_-\left(l^4+U^2V^2\right)A}{l^5-lU^2V^2} & \frac{\left(r_+^2-r_-^2\right)V\left(l^2+UV\right)A}{l^4-l^2UV} \end{pmatrix}_{ab} \tag{A.40}$$

$$\nabla_a k_{II,b} = \begin{pmatrix} -\frac{2VB}{l^2-UV} & 0 & \frac{r_-\left(l^4+U^2V^2\right)B}{l^5-lU^2V^2} \\ 0 & 0 & -\frac{2lr_- U^2 B}{l^4-U^2V^2} \\ \frac{r_-\left(l^4+U^2V^2\right)B}{l^5-lU^2V^2} & -\frac{2lr_- U^2 B}{l^4-U^2V^2} & \frac{\left(r_+^2-r_-^2\right)U\left(l^2+UV\right)B}{l^4-l^2UV} \end{pmatrix}_{ab} \tag{A.41}$$

$$\delta l_{I,a} k_{I,b} + \delta l_{I,b} k_{I,a} = \begin{pmatrix} 0 & \frac{2\delta A}{A} + \mathcal{O}(V) & 0 \\ \frac{2\delta A}{A} + \mathcal{O}(V) & 0 & \frac{2U(\delta r_- A - r_- \delta A)}{lA} + \mathcal{O}(V) \\ 0 & \frac{2U(\delta r_- A - r_- \delta A)}{lA} + \mathcal{O}(V) & 0 \end{pmatrix}_{ab} \tag{A.42}$$

$$\delta l_{II,a} k_{II,b} + \delta l_{II,b} k_{II,a} = \begin{pmatrix} 0 & \frac{2\delta B}{B} + \mathcal{O}(U) & -\frac{2V(\delta r_- B - r_- \delta B)}{lB} + \mathcal{O}(U) \\ \frac{2\delta B}{B} + \mathcal{O}(U) & 0 & 0 \\ -\frac{2V(\delta r_- B - r_- \delta B)}{lB} + \mathcal{O}(U) & 0 & 0 \end{pmatrix}_{ab} \tag{A.43}$$

$$\sqrt{\gamma_{I/II}} = r_+ + \mathcal{O}\left(U/V\right) \tag{A.44}$$

$$\nabla_a k_I^a = \frac{A\left(r_+^2 + r_-^2\right)V}{l^2 r_+^2} + \mathcal{O}\left(V^2\right) \tag{A.45}$$

$$\nabla_a k_{II}^a = \frac{B\left(r_+^2 + r_-^2\right)U}{l^2 r_+^2} + \mathcal{O}\left(U^2\right) \tag{A.46}$$

$$\nabla_a \delta k_I^a = \frac{2Ar_-\left(r_+\delta r_- - r_-\delta r_+\right) + r_+\left(r_+^2 - r_-^2\right)\delta A}{l^2 r_+^3} V + \mathcal{O}\left(V^2\right) \tag{A.47}$$

$$\nabla_a \delta k_{II}^a = \frac{2Br_-\left(r_+\delta r_- - r_-\delta r_+\right) + r_+\left(r_+^2 - r_-^2\right)\delta B}{l^2 r_+^3} U + \mathcal{O}\left(U^2\right) \tag{A.48}$$

$$k_{I,a} k_{II}^a = \frac{AB}{2} + \mathcal{O}\left(UV\right) \tag{A.49}$$

$$k_I^a k_{II}^b \delta g_{ab} = \frac{2r_-\left(r_+\delta r_- - r_-\delta r_+\right)AB}{l^2 r_+^3} UV + \mathcal{O}\left(U^2V^2\right) \tag{A.50}$$

$$\gamma_I^{ab}\delta g_{ab} = \gamma_{II}^{ab}\delta g_{ab} = \gamma_I^{ab}\delta\gamma_{I,ab} = \gamma_{II}^{ab}\delta\gamma_{II,ab} = \frac{2\delta r_+}{r_+} + \mathcal{O}\left(UV\right) \tag{A.51}$$

$$k_{I,a}\delta\xi^a_I = \frac{e^{-at}lr_- \left(r_-\delta r_+ - r_+\delta r_-\right)\nu_I A}{r_+^2\sqrt{r_+^2 - r_-^2}x} + \mathcal{O}\left(\frac{1}{x^2}\right) \tag{A.52}$$

$$k_{II,a}\delta\xi^a_{II} = -\frac{e^{at}lr_- \left(r_-\delta r_+ - r_+\delta r_-\right)\nu_{II} B}{r_+^2\sqrt{r_+^2 - r_-^2}x} + \mathcal{O}\left(\frac{1}{x^2}\right) \tag{A.53}$$

$$k_{I,a}\xi^a_I = \frac{e^{-at}l\left(r_+\mu_I + \sqrt{r_+^2 - r_-^2}\nu_I\right)A}{r_+ x} + \mathcal{O}\left(\frac{1}{x^2}\right) \tag{A.54}$$

$$k_{II,a}\xi^a_{II} = \frac{e^{at}l\left(r_+\mu_I - \sqrt{r_+^2 - r_-^2}\nu_I\right)B}{r_+ x} + \mathcal{O}\left(\frac{1}{x^2}\right) \tag{A.55}$$

$$h^{ab}\delta g_{ab} = \frac{2\delta r_+}{r_+} \tag{A.56}$$

$$K^{ab}\delta g_{ab} = \frac{2\delta r_+}{lr_+} + \mathcal{O}\left(\frac{1}{x^2}\right) \tag{A.57}$$

$$n^a n^b \delta g_{ab} = 0 \tag{A.58}$$

$$h^{ab}n^c\nabla_a\delta g_{bc} = -\frac{2\delta r_+}{lr_+} + \mathcal{O}\left(\frac{1}{x^2}\right) \tag{A.59}$$

$$h^{ab}n^c\nabla_c\delta g_{ab} = 0 \tag{A.60}$$

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