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Spherical Isoperimetric Inequalities

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Kurzfassung

Diese Arbeit enthält Beiträge zur Theorie konvexer Körper, das sind konvexe, kompakte Mengen, in Räumen konstanter Krümmung, insbesondere der Euklidischen Einheitssphäre.

Zuerst wird eine Definition des Schwerpunktkörpers auf der Sphäre gegeben, in der die geometrische Konstruktion ebenjener im flachen Raum imitiert wird. Grundlegende Eigenschaften dieses neuen Objekts, einschließlich dessen stochastische Approximation, werden untersucht, sowie eine isoperimetrische Ungleichung für den Polarkörper des sphärischen Schwerpunktkörpers bewiesen. Dies ist eine gemeinsame Arbeit mit Florian Besau, Peter Pivovarov und Franz Schuster.

Das zweite Thema dieser Arbeit ist eine randomisierte Version einer isoperimetrischen Ungleichung von Gao, Hug und Schneider im sphärischen Raum, welche besagt, dass die Wahrscheinlichkeit, dass ein sphärisch konvexer Körper mit gegebenem Volumen eine zufällig gewählte Groß-Hypersphäre schneidet, minimiert wird, wenn es sich bei diesem Körper um eine sphärische Kappe handelt. Zusammen mit Peter Pivovarov wird deren Ergebnis auf konvexe Hüllen zufällig gewählter Punkte erweitert. Es wird gezeigt, dass der Erwartungswert des obigen Funktionals minimal wird, wenn die Punkte auf sphärischen Kappen gleichverteilt sind.

Zum Schluss werden Durchschnitte und Vereinigungen endlich vieler geodätischer Kugeln mit fixem Radius im sphärischen, Euklidischen oder hyperbolischen Raum, deren Mittelpunkte zufällig gewählt werden, betrachtet. Es wird gezeigt, dass das erwartete Volumen einer solchen Menge zunimmt (im Falle eines Durchschnitts), respektive abnimmt (im Falle einer Vereinigung), wenn die Dichtefunktionen der Verteilungen der Mittelpunkte durch ihre Radialsymmetrisierungen ersetzt werden. Dadurch wird ein Resultat von Paouris und Pivovarov auf Räume konstanter Krümmung erweitert.

Abstract

This thesis contains contributions to the theory of convex bodies, that is, convex, compact sets, in spaces of constant curvature, in particular, the Euclidean unit sphere.

First, a definition of centroid bodies on the sphere is given by mimicking the geometric construction from flat space. Basic properties of this new object, including a stochastic approximation procedure, are established, and an isoperimetric inequality for the polar of the spherical centroid body is obtained. This is a joint work with Florian Besau, Peter Pivovarov, and Franz Schuster.

The second topic of this thesis is a randomized version of an isoperimetric inequality of Gao, Hug, and Schneider in spherical space, which says that the probability of a spherical convex body of given volume meeting a random great hypersphere is minimized, if the body is a spherical cap. Together with Peter Pivovarov, their result is extended to convex hulls of finitely many points drawn according to probability distributions. Uniform distributions on spherical caps are shown to be minimizers. As a corollary, a randomized Blaschke–Santaló inequality on the sphere is obtained.

Finally, intersections and unions of finitely many geodesic balls of given radius in spherical, Euclidean, or hyperbolic space, whose centers are chosen according to probability densities, are considered. It is shown that the expected volume of such sets is increasing (in the case of intersections), or decreasing (in the case of unions, respectively), if the density functions are replaced by their symmetric decreasing rearrangements. Thereby, a result of Paouris and Pivovarov is extended to spaces of constant curvature.

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Contents

CHAPTER 1

Introduction

One of the most fundamental inequalities in convex geometry, the Euclidean isoperimetric inequality, states that among all convex bodies, that is, convex, compact subsets, in Euclidean space of a given volume, precisely balls have the smallest surface area. Since a rigorous proof of this inequality was given in the 19th century, results of a similar type have appeared for many different geometric quantities, for example *Urysohn's inequality* for mean width, the *isodiametric inequality* for diameter, or the *Blaschke–Santaló inequality* concerning polar volume.

Although the notion of convex bodies easily extends to spherical space, only a small number of geometric inequalities are known in this setting. It was shown by Paul Lévy at the beginning of the 20th century that spherical caps minimize surface area among all spherical convex bodies of a given volume — a result known as the spherical isoperimetric inequality.

In recent years, a stochastic approach towards isoperimetric inequalities was developed by Grigoris Paouris and Peter Pivovarov [Pao17b]. In particular, they have shown that many of such inequalities admit stronger forms, in which the involved geometric objects are generated at random and probabilistic quantities, such as expected volume and surface area, are compared. Again, albeit the progress that has been made in Euclidean space, no examples of randomized isoperimetric inequalities in spherical space have been discovered so far.

The first contribution of this thesis is the introduction of centroid bodies in spherical space. For an origin-symmetric convex body K in n-dimensional Euclidean space, the centroids of the intersections of K with half-spaces through the origin form the surface of a convex set, its centroid body ΓK. In the case $n = 3$, this construction first explicitly appeared in a paper by Blaschke [Bla17], where he conjectured that the ratio of the volume of a body to that of its centroid body attains its maximum for ellipsoids. This conjecture was confirmed by Petty [Pet61] (who also coined the name centroid bodies), by reinterpreting Busemann's random simplex inequality [Bus53] as what would become known as the *Busemann–Petty centroid inequality.*

Since then, centroid bodies and their associated isoperimetric inequalities have been extended to the L^p-setting [Lut97], [Lut00], [Cam02] (see also [Hab09; Iva16; Lut86; Lut90; Lut10; Ngu18; Zhu12] for further generalizations), where they were shown to lead to functional affine invariant Sobolev-type inequalities (see [De 18; Had16; Had18; Had19; Ngu16]). Other applications of centroid bodies have been found in asymptotic geometric analysis (see e.g., [Bra14; Kla12; Mil89; Pao06]), geometric tomography (see e.g., [Gar06;

Iva17; Yas06b]), and integral geometry (see e.g., [Hab12; Lud05]), as well as recently even in Finsler geometry (see [Ber14]) and information theory (see [Pao12b]).

In a joint work with Florian Besau, Peter Pivovarov, and Franz Schuster [Bes19], we introduce a spherical analogue of the centroid body of a centrally-symmetric convex body in the Euclidean unit sphere, by mimicking Blaschke's geometric approach to centroid bodies in linear vector spaces. Combining our geometric definition with the gnomonic projection, naturally leads to centroid bodies (in the tangent linear space) with respect to a specific weight. These *weighted centroid bodies* will allow us to deduce several basic properties of spherical centroid bodies such as continuity in the Hausdorff metric and injectivity as well as the fact that, like in the linear setting, all spherical centroid bodies are C^2-smooth and have everywhere positive Gauß–Kronecker curvature. Our main results are a spherical version of the random approximation result for centroid bodies from [Pao12a] and an isoperimetric inequality for the polar of the spherical centroid body.

This project belongs to a line of research of recent origin, dealing with the question of which *affine* constructions and inequalities from linear vector spaces allow for generalizations to spaces of constant curvature (then no longer compatible with the affine group but rather the isometry group of the respective space). More results in the same spirit can be found in [Bes18a; Bes16a; Bes16b; Bes18b; Dan18; Yas06a].

The second focus of this thesis are randomized isoperimetric inequalities in spaces of constant curvature. In 2002, Fuchang Gao, Daniel Hug, and Rolf Schneider [Gao02] showed that among all convex bodies in spherical space of given volume, spherical caps minimize the total measure of great hyperspheres that meet the given set. In a joint work with Peter Pivovarov [Hac], we extend this result by replacing the convex body by the convex hull of finitely many points, drawn independently according to probability distributions, and show that on average a minimum is attained at uniform distributions on spherical caps. The proof is carried out entirely on the sphere, using two-point symmetrization and spherical rearrangements. Moreover, it works similarly in hyperbolic space. By letting the number of points tend to infinity and drawing from indicator functions on convex sets, we recover the result by Gao, Hug, and Schneider.

Inequalities for expected mean values have a long history in stochastic geometry and go back (at least) to Blaschke's resolution of Sylvester's four point problem [Bla17], and its numerous generalizations, e.g., arbitrary dimension [Bus53], [Gro74], [Cam99], compact sets and other intrinsic volumes [Pfi82], [Har03], continuous distributions [Pao12a] (see also [Sch14, Chapter 10]). Interest is driven in part by applications to high-dimensional probability, especially small-ball probabilities [Pao13], [Pao17b].

Finally, we consider unions and intersections of finitely many gedoesic balls in spherical, Euclidean or hyperbolic space of a fixed radius and centers chosen independently according to probability distributions. We prove that the expected volume of such a union or intersection is decreased or increased, respectively, if one replaces the densities of the random points by their symmetric decreasing rearrangements. This extends results of Paouris and Pivovarov [Pao17a] to curved geometries. Moreover, as the number of points tends to infinity we recover isoperimetric inequalities for outer parallel and r-dual sets from [Sch48], [Bez18a]. Unions and intersections of balls have already been studied in connection with the Kneser–Poulsen conjecture (see e.g. [Bez02; Bez04; Bez18b], and also [Bez07]). In this work, they yield yet another model of how randomness can be introduced in geometric inequalities.

Two-point symmetrization has been used as an analytical tool in spaces of constant curvature, especially in multiple integral rearrangement inequalities [Bae76], [Bur01], [Mor02], and also more recently in isoperimetric inequalities [Bez18a]. However, such techniques have not yet been fused with stochastic convex geometry in curved spaces. The results of Chapters 5 and 6 are a first step in this direction.

This thesis is organized as follows: The second chapter contains background information about convex geometry and the symmetrization techniques used therein. In the third chapter, we state important rearrangement inequalities that will provide the technical backbone of our results. Chapters four to six are devoted to our new results about spherical centroid bodies, randomized Urysohn-type inequalities and random ball polyhedra.

CHAPTER 2

Background

This chapter collects all necessary background material that is needed in the subsequent parts of this thesis. We first choose models for the spaces of constant curvature and fix some notation. Secondly, we review the basic notions of Euclidean and spherical convexity along with some important isoperimetric inequalities. Finally, we introduce two important symmetrization techniques. These will act as the main tools for obtaining the results of Chapters 4, 5, and 6.

2.1 Models and notation

Let $\mathbb{R}^{n+1}$ be $(n+1)$-dimensional Euclidean space endowed with the standard Euclidean scalar product $x \cdot y = x_1 y_1 + \cdots + x_{n+1} y_{n+1}$, Euclidean norm $\|x\| = \sqrt{x \cdot x}$, and the indefinite Minkowski product $\langle x, y \rangle := x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}$. We fix $e = (0, \dots, 0, 1)^T \in \mathbb{R}^{n+1}$ and define the n-dimensional

- Spherical space: $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} \,|\, x \cdot x = 1\}$,
- Euclidean space (at height 0): $\mathbb{R}^n_{e,0} := \{x \in \mathbb{R}^{n+1} \,|\, x \cdot e = 0\}$,
- Euclidean space (at height 1): $\mathbb{R}^n_{e,1} := \{x \in \mathbb{R}^{n+1} \,|\, x \cdot e = 1\}$,
- Hyperbolic space: $\mathbb{H}^n := \{x \in \mathbb{R}^{n+1} \,|\, x \cdot e > 0, \langle x, x \rangle = -1\}$.

With $\mathbb{M}^n$ we denote at once $\mathbb{S}^n$, $\mathbb{R}^n_{e,1}$, and $\mathbb{H}^n$, that is, statements involving $\mathbb{M}^n$ will be true in all three geometries. Choosing the affine subspace $\mathbb{R}^n_{e,1}$ as a representative for Euclidean space will result in more concise formulas. Topological interior int, closure cl, and boundary bd are always relative to $\mathbb{M}^n$. Let $x \in \mathbb{R}^{n+1}$. We will use the following abbreviations:

$$x^\perp := \{y \in \mathbb{R}^{n+1} \,|\, x \cdot y = 0\} \quad \text{and} \quad x^{\langle\perp\rangle} := \{y \in \mathbb{R}^{n+1} \,|\, \langle x, y \rangle = 0\}$$

Moreover, we set $H_x := x^\perp$ and write $H_x^+ := \{y \in \mathbb{R}^{n+1} \,|\, x \cdot y \geq 0\}$ and $H_x^- := -H_x^+$ for the associated halfspaces, and correspondingly use $H_x^{\langle\rangle}$, $H_x^{\langle+\rangle}$, and $H_x^{\langle-\rangle}$. When working with two-point symmetrization, we will just write H without subscript for a set of the form $H_x \cap \mathbb{M}^n$ or $H_x^{\langle\rangle} \cap \mathbb{M}^n$ and always take H^+ to be the halfspace that contains $e \in \mathbb{M}^n$. Especially in Chapter 4, we will also use the notation

$$\mathbb{S}_x := \mathbb{S}^n \cap x^\perp, \quad \mathbb{S}_x^+ := \mathbb{S}^n \cap H_x^+, \quad \mathbb{S}_x^- := -\mathbb{S}_x^+.$$

Lastly, we set $\mathbb{S}^{n-1} := \mathbb{S}_e$ and $\mathbb{R}^n := \mathbb{R}^n_{e,0}$, thus, $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$, as usual.

We will write vol_{n+1} for the standard Euclidean volume measure on $\mathbb{R}^{n+1}$. On all n-dimensional subgeometries, the volume measure will be the restriction of n-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$. Although, we introduce two different notations:

- $(\mathbb{M}^n, \lambda_n)$, whenever we do not distinguish between spherical, Euclidean, and hyperbolic geometry.
- $(\mathbb{R}^n, \mathrm{vol}_n)$ and $(\mathbb{S}^n, \sigma_n)$, whenever a statement concerns Euclidean or spherical space only (there will be no statements exclusively about hyperbolic space).

For integrals, we will abbreviate

$$dx := d\lambda_n(x), \quad dx := d\,\mathrm{vol}_n(x), \quad dx := d\sigma_n(x),$$

whenever the measure of integration is clear from the context, and also use measurable as short for λ_n-measurable. We write $d_{\mathbb{M}^n}(x, y)$ for the geodesic distance between $x, y \in \mathbb{M}^n$, that is,

$$d_{\mathbb{M}^n}(x,y) = \begin{cases} \arccos(x \cdot y), & \text{if } \mathbb{M}^n = \mathbb{S}^n, \\ \|x - y\|, & \text{if } \mathbb{M}^n = \mathbb{R}^n_{e,1}, \\ \operatorname{arcosh}\left(\langle x, y\rangle\right), & \text{if } \mathbb{M}^n = \mathbb{H}^n. \end{cases}$$

We denote the orthogonal and special orthogonal groups on $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$ by $\mathrm{O}(n+1)$, $\mathrm{O}(n)$, and $\mathrm{SO}(n+1)$, $\mathrm{SO}(n)$, respectively, and write $\mathrm{Isom}(\mathbb{M}^n)$ for the group of isometries on $\mathbb{M}^n$.

For $i \in \{0, \dots, n\}$, let $\mathcal{M}^n_i$ be the collection of i-dimensional totally geodesic submanifolds of $\mathbb{M}^n$, that is,

$$\mathcal{M}^n_i = \{E \cap \mathbb{M}^n \mid E \text{ is an } (i+1)\text{–dimensional subspace of } \mathbb{R}^{n+1}\}.$$

We denote the $\mathrm{Isom}(\mathbb{M}^n)$-invariant measure on $\mathcal{M}^n_i$ by μ^n_i, but again abbreviate integration as $dM := d\mu^n_i(M)$. The normalization is $\mu^n_i(\mathcal{M}^n_i) = 1$, if $\mathbb{M}^n = \mathbb{S}^n$, and such that $\mu^n_i(\{M \in \mathcal{M}^n_i \mid M \cap B_1(e) \neq \emptyset\}) = \kappa_{n-i}$, if $\mathbb{M}^n = \mathbb{R}^n_{e,1}$ or $\mathbb{H}^n$, where $B_1(e)$ is the geodesic ball of radius 1 around e, and κ_i is the i-dimensional Euclidean volume of the i-dimensional Euclidean unit ball.

To introduce polar coordinates on $\mathbb{M}^n$, we set

$$R^{\mathbb{M}} := \begin{cases} \pi, & \text{if } \mathbb{M}^n = \mathbb{S}^n, \\ \infty, & \text{if } \mathbb{M}^n = \mathbb{R}^n_{e,1} \text{ or } \mathbb{M}^n = \mathbb{H}^n. \end{cases}$$

For $t \in \left[0, R^{\mathbb{M}}\right]$, we define the functions

$$\operatorname{cs} t := \begin{cases} \cos t, & \text{if } \mathbb{M}^n = \mathbb{S}^n, \\ 1, & \text{if } \mathbb{M}^n = \mathbb{R}^n_{e,1}, \\ \cosh t, & \text{if } \mathbb{M}^n = \mathbb{H}^n, \end{cases} \qquad \operatorname{sn} t := \begin{cases} \sin t, & \text{if } \mathbb{M}^n = \mathbb{S}^n, \\ t, & \text{if } \mathbb{M}^n = \mathbb{R}^n_{e,1}, \\ \sinh t, & \text{if } \mathbb{M}^n = \mathbb{H}^n, \end{cases}$$

and write polar coordinates as $x(t,u) := e \operatorname{cs} t + u \operatorname{sn} t$, for $u \in \mathbb{S}^{n-1}$ and $t \in [0, R^{\mathbb{M}}]$. The following transformation formula holds in all three geometries:

$$\int_{\mathbb{M}^n} f(x)dx = \int_{\mathbb{S}^{n-1}} \int_0^{R^{\mathbb{M}}} f(x(t,u)) \operatorname{sn}^{n-1} t \, dtdu, \tag{2.1}$$

for an integrable function $f\colon \mathbb{M}^n \to \mathbb{R}$ (cf., e.g., [Sch48, §3], [San76, Section IV.17.3], or [Gal04, Sections 3.F and 3.H]). In polar coordinates, we have $d_{\mathbb{M}^n}(e, x(t,u)) = t$, for $u \in \mathbb{S}^{n-1}$ and $t \in [0, R^{\mathbb{M}}]$. Of course, polar coordinates on $\mathbb{R}^n = \mathbb{R}^n_{e,0}$ have the simpler expression $x(t,u) = tu$, for $u \in \mathbb{S}^{n-1}$ and $t \in [0, \infty)$. Sometimes, we also write r instead of t for the radial variable.

Finally, the words *increasing* and *decreasing* will always be used in the non-strict sense.

2.2 Convexity

We start with general facts about convex sets, that are true in spherical, Euclidean, and hyperbolic space, and will later point out distinct features. Let $K \subseteq \mathbb{M}^n$ (or $K \subseteq \mathbb{R}^n$). If for any $x, y \in K$, $x \neq -y$ in the case $\mathbb{M}^n = \mathbb{S}^n$, the shortest geodesic segment $[x,y]$ connecting x, y lies inside K, we call K *convex*. In Euclidean space, we have

$$[x,y] = \{(1-t)x + ty \,|\, 0 \leq t \leq 1\}.$$

Moreover, K is a *convex body* if it is convex, compact, and non-empty. We will write $\mathcal{K}(\mathbb{M}^n)$ for the set of all convex bodies in $\mathbb{M}^n$, and for a set $A \subseteq \mathbb{M}^n$, we let $\mathcal{K}(A)$ be the subset of convex bodies contained in A. The *convex hull* of a set A, denoted by $\operatorname{conv} A$, is the intersection of all convex sets containing A. In the case of $\mathbb{M}^n = \mathbb{S}^n$, we exclude the whole sphere from the set of convex bodies. A convex body $K \subseteq \mathbb{S}^n$ is called *proper*, if it is contained in an open hemisphere, that is, $K \subseteq \operatorname{int} \mathbb{S}^+_u$ for some $u \in \mathbb{S}^n$.

We will use the letter χ for the following function: For any set $A \subseteq \mathbb{M}^n$, we define

$$\chi(A) := \begin{cases} 1, & \text{if } A \neq \emptyset, \\ 0, & \text{if } A = \emptyset. \end{cases} \tag{2.2}$$

Note that on convex bodies in $\mathbb{R}^n$ and $\mathbb{H}^n$, and on *proper* convex bodies in $\mathbb{S}^n$, χ equals the Euler characteristic (which is usually denoted by χ in the literature). The Euler characteristic can be extended to more general sets in a way different from our definition. As we are interested in intersection probabilities of sets, we stick with (2.2).

We now define origin-symmetry in $\mathbb{M}^n$. For $x \in \mathbb{M}^n$ we denote its *geodesic reflection* about the origin $e \in \mathbb{M}^n$ by

$$x^e := -x + 2(x \cdot e)e,$$

that is, orthogonal reflection about $\operatorname{span}\{e\}$ in $\mathbb{R}^{n+1}$. A subset $A \subseteq \mathbb{M}^n$ is called *(centrally)-symmetric* with center $e \in \mathbb{M}^n$, if $A^e := \{x^e \,|\, x \in A\} = A$. Let $\mathcal{K}_c(\mathbb{M}^n)$ denote the subset of all centrally-symmetric convex bodies in $\mathbb{M}^n$. If $\mathbb{M}^n = \mathbb{S}^n$ and $K \in \mathcal{K}_c(\mathbb{S}^n)$ is centrally-symmetric, then $K \subseteq \mathbb{S}^+_e$. Moreover, if K is proper, we have $K \subseteq \operatorname{int} \mathbb{S}^+_e$.

Next, we describe the metric structure of $\mathcal{K}(\mathbb{M}^n)$. Let $A \subseteq \mathbb{M}^n$ be non-empty. We write

$$d_{\mathbb{M}^n}(x, A) := \inf\{d_{\mathbb{M}^n}(x, y) \,|\, y \in A\}$$

for the distance of x from A. For $r > 0$ and, we denote Bby

$$A_r := \{x \in \mathbb{M}^n \,|\, d_{\mathbb{M}^n}(x, A) \leq r\} \quad \text{and} \quad A^r := \{x \in \mathbb{M}^n \,|\, d_{\mathbb{M}^n}(x, y) \leq r \;\forall y \in A\},$$

the *outer parallel* and the *r-dual* (cf. [Bez18a]) set of A. Further, for $x \in \mathbb{M}^n$, $r > 0$ let $B_r(x) = \{x\}_r$ be the geodesic ball of radius r around x. Then one has

$$A_r = \bigcup_{x \in A} B_r(x) \quad \text{and} \quad A^r = \bigcap_{x \in A} B_r(x).$$

On $\mathbb{S}^n$, we also write $C_r(u) = B_r(u)$ for the spherical cap of radius r centered at $u \in \mathbb{S}^n$.

The *Hausdorff distance* between closed sets $A, B \subseteq \mathbb{M}^n$ is given by

$$\delta_{\mathbb{M}^n}(A,B) = \min\{0 \leq r \leq \mathbb{R}^{\mathbb{M}} \colon A \subseteq B_r \text{ and } B \subseteq A_r\}.$$

It induces a topology on $\mathcal{K}(\mathbb{M}^n)$ that we will examine more closely in the following paragraphs, where we consider Euclidean and spherical space separately.

In $\mathbb{R}^n$, there are different ways to associate functions to convex bodies: For $K \in \mathcal{K}(\mathbb{R}^n)$ the *support function* $h_K \colon \mathbb{R}^n \to \mathbb{R}$ is given by

$$h_K(x) = \sup\{x \cdot y \,|\, y \in K\}.$$

The function h_K is sublinear, that is, it satisfies $h_K(tx) = th_K(x)$ and $h_K(x + y) \leq h_K(x) + h_K(y)$ for all $x, y \in K$ and $t \geq 0$. If $u \in \mathbb{S}^{n-1}$, $h_K(u)$ is given by the smallest real number t, such that K is contained in $tu + H_u^-$. Moreover, every sublinear function on $\mathbb{R}^n$ is also the support functions of a convex body (see e.g. [Sch14]).

The *Minkowski sum* of two sets $A, B \subseteq \mathbb{R}^n$ is defined as

$$A + B := \{a + b \,|\, a \in A, b \in B\}.$$

A very useful feature of support functions is that $h_{K+L} = h_K + h_L$ for every $K, L \in \mathcal{K}(\mathbb{R}^n)$, and that $\delta_{\mathbb{R}^n}(K, L) = \sup\{|h_K(u) - h_L(u)| \colon u \in \mathbb{S}^{n-1}\}$.

One can also describe $K \in \mathcal{K}(\mathbb{R}^n)$ by giving its *radial function* $\rho_K(x) \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$, defined by

$$\rho_K(x) = \sup\{t > 0 \,|\, tx \in K\}.$$

More generally, radial functions can be used to describe star-shaped sets, that is, sets S that satisfy $[0, x] \subseteq S$, whenever $x \in S$. In Chapter 4, we will also use radial functions to describe unbounded sets, and hence allow the value $+\infty$.

There is a natural notion of duality between convex bodies in Euclidean space: For

$K \in \mathcal{K}(\mathbb{R}^n)$ containing the origin in its interior, the *polar body* of K is defined by

$$K^\circ = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}.$$

We have $h_{K^\circ}(u) = \rho_K(u)^{-1}$ for all $u \in \mathbb{S}^{n-1}$ and all $K \in \mathcal{K}(\mathbb{R}^n)$ that contain the origin in their interior. The isoperimetric inequality associated to polar bodies is the famous *Blaschke–Santaló inequality* [San49]:

$$\mathrm{vol}_n(K^\circ) \leq \mathrm{vol}_n(B_K^\circ),$$

for every centrally-symmetric convex body $K \in \mathcal{K}(\mathbb{R}^n)$ that contains the origin in its interior. Here, B_K is the Euclidean ball around the origin, whose volume equals that of K.

The volume of an outer parallel set in $\mathbb{R}^n$ has a simple expression, which is known as the *Steiner formula*: Let $K \in \mathcal{K}(\mathbb{R}^n)$ and $\varepsilon > 0$, then

$$\mathrm{vol}_n(K_\varepsilon) = \sum_{i=0}^{n} V_i(K)\kappa_{n-i}\varepsilon^{n-i},$$

where κ_{n-i} is the $(n-i)$-dimensional volume of the $(n-i)$-dimensional unit ball, and $V_i(K)$ is the ith *intrinsic volume* of K, a quantity that is invariant under translations and rotations of K. For example, V_n equals regular volume, V_{n-1} (up to a constant) surface area, and V_0 the Euler characteristic. More generally, we have the following integral-geometric formulas:

$$V_i(K) = \binom{n}{i} \frac{\kappa_n}{\kappa_i \kappa_{n-i}} \int_{\mathrm{AGr}_{n-i}^n} \chi(K \cap E)\, dE, \qquad \text{(Crofton)} \tag{2.3}$$

where AGr_{n-i}^n is the affine $(n-i)$-Grassmanian, that is, the set of all $(n-i)$-dimensional affine subspaces (i.e. $\mathcal{M}_{n-i}^n$) equipped with its motion-invariant measure, normalized as in Section 2.1, and

$$V_i(K) = \binom{n}{i} \frac{\kappa_n}{\kappa_i \kappa_{n-i}} \int_{\mathrm{Gr}_i^n} \mathrm{vol}_i(K|F)\, dF, \qquad \text{(Kubota)} \tag{2.4}$$

where Gr_i^n is the Grassmanian, that is, the collection of all i-dimensional affine subspaces equipped with its rotation-invariant probability measure, and $K|F$ is the orthogonal projection of K onto F (see e.g. [Sch08]).

For the intrinsic volumes, the isoperimetric inequalities are known (see e.g. [Sch14, Section 7.4]): they are minimized on balls,

$$V_i(K) \geq V_i(B_K), \tag{2.5}$$

for all $1 \leq i \leq n$, $K \in \mathcal{K}(\mathbb{R}^n)$, and where again B_K is a ball, whose volume is that of K.

These inequalities have been extended to the following random setting: Let $N \in \mathbb{N}$ and $X_1, \dots, X_N$ be independent random vectors on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, that are identically distributed uniformly in K, that is,

$$X_i \sim \frac{\mathbb{1}_K(x)}{\mathrm{vol}_n(K)}\, dx,$$

for all $1 \leq i \leq N$. In other words, $\mathbb{P}(X_i \in A) = \frac{\mathrm{vol}_n(K\cap A)}{\mathrm{vol}_n(K)}$ for $A \subseteq \mathbb{R}^n$ measurable. We define a random set $[K]_N$ as

$$[K]_N := \mathrm{conv}\{X_1, \dots, X_N\}.$$

Correspondingly, for independent random vectors $Z_1, \dots, Z_N$ identically distributed uniformly in B_K, that is,

$$Z_i \sim \frac{\mathbb{1}_{B_K}(x)}{\mathrm{vol}_n(B_K)}\, dx,$$

we set $[B_K]_N := \mathrm{conv}\{Z_1, \dots, Z_N\}$. Then the following inequalities hold:

$$\mathbb{E}V_i([K]_N) \geq \mathbb{E}V_i([B_K]_N), \tag{2.6}$$

for all $1 \leq i \leq n$. Here, as opposed to (2.5), the case $i = n$ is non-trivial and due to Groemer [Gro74]. For $i < n$, (2.6) was obtained by Pfiefer [Pfi82] (see also [Har03]).

If we let the number of points N go to infinity, we have that $[K]_N \to K$ almost surely in the Hausdorff metric, and hence that $\mathbb{E}V_i([K]_N) \to V_i(K)$. This means that in the limit (2.6) implies (2.5). For an overview over the various stochastic strengthenings of isoperimetric inequalities confer the survey of Paouris and Pivovarov [Pao17b].

Next, we recall facts from spherical geometry. However, the main part of this section is devoted to the gnomonic projection and spherical centroids as well as their interplay, for which we prove several auxiliary results needed in the next sections. As a general reference for this section we recommend [Bes16a], [Gla96], or [Sch08, Section 6.5].

For $0 \leq k \leq n$, a k-sphere $S \in \mathcal{M}^n_{n-1}$ is a k-dimensional great sub-sphere of $\mathbb{S}^n$, that is, the intersection of $\mathbb{S}^n$ with a $(k+1)$-dimensional subspace in $\mathbb{R}^{n+1}$. Clearly, every k-sphere is convex.

Also on $\mathbb{S}^n$, we have a similar notion of polarity: Let $K \in \mathcal{K}(\mathbb{S}^n)$, then its *(spherical) polar body* K^* is given by

$$K^* = \{x \in \mathbb{S}^n \,|\, x \cdot y \leq 0 \text{ for all } y \in K\}.$$

The spherical version of the Blaschke–Santaló inequality associated to the spherical polar body was obtained by Gao, Hug, Schneider in 2002 [Gao02]:

$$\sigma_n(K^*) \leq \sigma_n(C_K^*), \tag{2.7}$$

where C_K is a spherical cap, such that $\sigma_n(C_K) = \sigma_n(K)$.

The expression for the spherical volume of outer parallel sets of a proper spherical convex body $K \in \mathcal{K}(\mathbb{S}^n)$, known as the spherical Steiner formula, is more complicated, compared to Euclidean space:

$$\sigma_n(K_\varepsilon) = \omega_n V_n(K) + \sum_{i=0}^{n-1} f_i(\varepsilon)\omega_j\omega_{n-j-1}V_j(K), \tag{2.8}$$

where $\omega_i = \sigma_i(\mathbb{S}^i)$, $\omega_n V_n(K) = \sigma_n(K)$ and

$$f_i(\varepsilon) := \int_0^\varepsilon \cos^j t \sin^{n-j-1} t \, dt.$$

Early versions of the spherical Steiner formula for different classes of sets occur in works of Allendoerfer [All48] and Herglotz [Her43]; a proof for spherical convex bodies can be found in [Gla96].

Although the coefficients V_i appearing in (2.8), are invariant under rotations of the body K, one is reluctant to call them *the* spherical intrinsic volumes. The reason is that, unlike in $\mathbb{R}^n$, expressions as in the formulas of Crofton and Kubota yield different series of functionals $U_i, W_i \colon \mathcal{K}(\mathbb{S}^n) \to \mathbb{R}$: Set

$$U_i(K) := \int_{\mathcal{M}^n_{n-i}} \chi(K \cap S)\, dS,$$

and for the spherical projection onto an i-sphere, $K|S := S \cap \operatorname{pos}(K \cap L^\perp)$, where $L \in \mathrm{Gr}^n_{i+1}$ such that $S = L \cap \mathbb{S}^n$, and pos denotes the positive hull, $\operatorname{pos} A = \{tx \mid t \in \mathbb{R}^+, x \in A\}$, set

$$W_i(K) := \frac{1}{\omega_i}\int_{\mathcal{M}^n_i} \sigma_i(K|S)\, dS,$$

Between V_i, U_i, and W_i, we have the relations

$$\frac{1}{2}U_i(K) = \sum_{k=0}^{\lfloor \frac{n-i}{2} \rfloor} V_{i+2k}(K) \quad \text{and} \quad W_i(K) = \sum_{k=i}^{n} V_k(K),$$

see [Gla96]. Surprisingly, not many of the isoperimetric problems for these series of functionals are solved. There is *the* spherical isoperimetric inequality, saying that caps minimize spherical surface area V_{n-1} $(= U_{n-1}/2)$, by Lévy (see also [Sch48], [Ben84]):

$$V_{n-1}(K) \geq V_{n-1}(C_K),$$

where C_K is a spherical cap whose spherical volume is that of K. One usually formulates this inequality in terms of volumes of outer parallel sets:

$$\sigma_n(K_t) \geq \sigma_n((C_K)_t),$$

for $t > 0$. The second known result is due Gao, Hug and Schneider [Gao02] and concerns U_1. Since in Euclidean space, the isoperimetric inequality for the first intrinsic volume is named after Urysohn, the following inequality is often referred to as the spherical Urysohn inequality:

$$U_1(K) \geq U_1(C_K). \tag{2.9}$$

Different to the situation in $\mathbb{R}^n$, no random extensions of any of the spherical isoperimetric inequalities are known. There is one remarkable identity that is special to $\mathbb{S}^n$. It shows that inequalities (2.7) and (2.9) are equivalent:

Proposition 2.2.1. *Let $K \in \mathcal{K}(\mathbb{S}^n)$ be a convex body. Then*

$$\frac{U_1(K)}{\mu^n_{n-1}(\mathcal{M}^n_{n-1})} + \frac{2\sigma_n(K^*)}{\sigma_n(\mathbb{S}^n)} = 1.$$

Proof. If K is proper the proof can be found in [Gao02, eq. 20]. On the other hand, as soon as K contains antipodal points, we have $U_1(K) = \mu^n_{n-1}(\mathcal{M}^n_{n-1})$ and $\operatorname{int} K^* = \emptyset$. □

In Chapter 5, we will consider the functional U_1 simultaneously in all three geometries and write

$$U_1(K) := \int_{\mathcal{M}^n} \chi(K \cap M)\, dM, \tag{2.10}$$

for $K \in \mathcal{K}(\mathbb{M}^n)$. Note that since K is compact, the integral in (2.10) is always finite.

We move on now with more preparatory material, that we will need in the later chapters. The next lemma contains useful properties of the spherical Hausdorff metric.

Lemma 2.2.2. *For $m \in \mathbb{N}$, let $C_m, C \subseteq \mathbb{S}^n$ be closed and $K, L \in \mathcal{K}(\mathbb{S}^n)$ such that $\delta_{\mathbb{S}^n}(K, L) < \frac{\pi}{2}$. Then the following statements hold:*

(a) The sequence C_m converges to C in the spherical Hausdorff metric if and only if it does so in the Hausdorff metric of the ambient space $\mathbb{R}^{n+1}$;

(b) $\delta_{\mathbb{S}^n}(K,L) = \delta_{\mathbb{S}^n}(\operatorname{bd} K, \operatorname{bd} L)$;

(c) $\delta_{\mathbb{S}^n}(K,L) = \delta_{\mathbb{S}^n}(K^*, L^*)$.

Proof. Statement (a) is a consequence of the fact that $\|u - v\| \leq d_{\mathbb{S}^n}(u, v) \leq \frac{\pi}{2}\|u - v\|$ for all $u, v \in \mathbb{S}^n$, that is, of the equivalence of the spherical and the Euclidean distance in the ambient space.

In order to see (b), we use that for $d_{\mathbb{S}^n}(x, K) < \frac{\pi}{2}$, there exists a unique point $p(K, x)$ in K such that $d_{\mathbb{S}^n}(x, p(K, x)) \leq d_{\mathbb{S}^n}(x, y)$ for all $y \in K$ (see e.g. [Sch08, Section 6.5]). From this, (b) follows by the same argument as in the linear setting (see e.g. [Sch14, Lemma 1.8.1]).

Finally, a proof of (c) was, for example, given in [Gla96, Hilfssatz 2.2]. □

We turn now to one of the most important tools in spherical convexity, the gnomonic projection. First, recall that for $e \in \mathbb{S}^n$, the *gnomonic projection* with respect to e is given by

$$g_e : \operatorname{int} \mathbb{S}^+_e \to \mathbb{R}^n = \mathbb{R}^n_{e,0}, \qquad g_e(u) = \frac{u}{e \cdot u} - e.$$

In the following, we write $\varrho_e : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ for the orthogonal reflection about $\mathbb{R}^n_{e,0}$ in $\mathbb{R}^{n+1}$. Our next lemma contains several well known properties of the gnomonic projection, the proofs of which can be found, e.g., in [Bes16a] and [Gao02].

Lemma 2.2.3. *The gnomonic projection $g_e : \operatorname{int} \mathbb{S}^+_e \to \mathbb{R}^n$ has the following properties:*

(a) The map g_e is a bijection with inverse given by

$$g_e^{-1} : \mathbb{R}^n \to \operatorname{int} \mathbb{S}^+_e, \qquad g_e^{-1}(x) = \frac{x + e}{\|x + e\|}.$$

(b) If $S \subseteq \mathbb{S}^n$ is a k-sphere, $0 \le k \le n-1$, such that $S \cap \operatorname{int} \mathbb{S}^+_e$ is non-empty, then $g_e(S \cap \operatorname{int} \mathbb{S}^+_e)$ is a k-dimensional affine subspace of $\mathbb{R}^n$. Conversely, g_e^{-1} maps k-dimensional affine subspaces of $\mathbb{R}^n$ to k-spheres in $\operatorname{int} \mathbb{S}^+_e$.

(c) The map g_e induces a homeomorphism between $\mathcal{K}(\operatorname{int} \mathbb{S}^+_e)$ and $\mathcal{K}(\mathbb{R}^n)$.

(d) For every $u \in \operatorname{int} \mathbb{S}^+_e$, we have $g_e(u^e) = -g_e(u)$.

(e) For every $K \in \mathcal{K}(\operatorname{int} \mathbb{S}^+_e)$ containing e in its interior, $g_e(\varrho_e K^*) = g_e(K)^\circ$.

For our purposes it is important to know the push-forwards of certain measures on $\mathbb{S}^n$ under gnomonic projection. These are the content of the following lemma.

Lemma 2.2.4. *Let $g_e \colon \operatorname{int} \mathbb{S}^+_e \to \mathbb{R}^n$ be the gnomonic projection.*

(a) The push-forward $\widehat{\sigma}_n := g_e \# \sigma_n$ under g_e of spherical Lebesgue measure σ_n is absolutely continuous with density given by $\xi_n(x) = (1 + \|x\|^2)^{-\frac{n+1}{2}}$.

(b) For $u \in \operatorname{int} \mathbb{S}^+_e$, we have $u \cdot e = \phi(g_e(u))$, where $\phi(x) = (1 + \|x\|^2)^{-\frac{1}{2}}$.

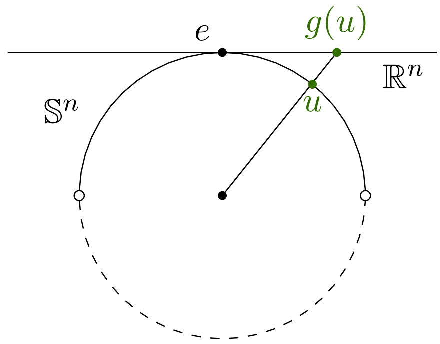

Figure 2.1: Gnomonic projection $g_e \colon \operatorname{int} \mathbb{S}^+_e \to \mathbb{R}^n$

(c) The push-forward $\widehat{\tau}_n := g_e \# \tau_n$ *under* g_e *of the absolutely continuous measure* τ_n *on* $\mathbb{S}^+_e$ *with density* $u \mapsto e \cdot u$ *is also absolutely continuous with density given by* $\psi_n(x) = (1 + \|x\|^2)^{-\frac{n+2}{2}}$.

Proof. In order to see (a), we need the Jacobian of the inverse g_e^{-1} at $x \in \mathbb{R}^n$. It was, for example, calculated in [Bes16b, Proposition 4.2] and is given by

$$Jg_e^{-1}(x) = (1 + \|x\|^2)^{-\frac{n+1}{2}} = \xi_n(x).$$

Thus, by the area formula (see, e.g., [Mag12, Theorem 8.1]), we have

$$\sigma_n(A) = \int_{g_e(A)} (1 + \|x\|^2)^{-\frac{n+1}{2}} \, dx,$$

for every Borel set $A \subseteq \operatorname{int} \mathbb{S}^+_e$, which proves (a).

Statement (b) follows from Pythagoras' theorem, since $\|g_e(u)\|^2 + 1 = (u \cdot e)^{-2}$, and, finally, combining (a) and (b) yields statement (c). □

Next, we discuss the notion of centroids for certain subsets of the unit sphere which we use in Chapter 4 (for other notions of centroids on $\mathbb{S}^n$, cf. [Gal93]).

Definition. *For* $\{u_1, \dots, u_N\} \subseteq \mathbb{S}^n$ *and a Borel subset* $A \subseteq \mathbb{S}^n$ *such that* $\sigma_n(A) > 0$, *we define their respective* spherical centroids *by*

$$c_s(u_1, \dots, u_N) := \frac{\sum_{i=1}^N u_i}{\left\|\sum_{i=1}^N u_i\right\|} \qquad \text{and} \qquad c_s(A) := \frac{\int_A u \, d\sigma_n(u)}{\left\|\int_A u \, d\sigma_n(u)\right\|}$$

whenever they exist, that is, whenever the denominators are non-zero.

While this definition of spherical centroids makes use of the vector space structure of the ambient space, it is well known that both $c_s(u_1, \dots, u_N)$ and $c_s(A)$ can also be defined (with more complicated formulas) intrinsically, that is, by making use only of the metric structure of the sphere (see e.g., Galperin [Gal93], where he also characterized c_s by a set of natural properties).

In order to carry out explicit computations later on, we combine now the gnomonic projection with spherical centroids. To this end, we need to consider centroids in $\mathbb{R}^n$ with respect to arbitrary densities.

Definition. *For* $\{x_1, \dots, x_N\} \subseteq \mathbb{R}^n$ *and a positive function* $f : \mathbb{R}^n \to \mathbb{R}^+$, *let*

$$c_f(x_1, \dots, x_N) := \frac{1}{\sum_{i=1}^N f(x_i)} \sum_{i=1}^N f(x_i) x_i. \tag{2.11}$$

For an absolutely continuous measure μ *on* $\mathbb{R}^n$ *and a bounded Borel subset* $A \subseteq \mathbb{R}^n$ *such* *that* $\mu(A) > 0$, *we define the* μ-centroid *of* A *by*

$$c_\mu(A) := \frac{1}{\mu(A)} \int_A x \, d\mu(x). \tag{2.12}$$

Our next lemma will be critical for the proof of the main results of Chapter 4. Here and in the following, we use again the notation from Lemma 2.2.4.

Lemma 2.2.5. *If* $\{u_1, \ldots, u_N\} \subseteq \operatorname{int} \mathbb{S}^+_e$, *then*

$$g_e(c_s(u_1, \ldots, u_N)) = c_\phi(g_e(u_1), \ldots, g_e(u_N)). \tag{2.13}$$

If $A \subseteq \operatorname{int} \mathbb{S}^+_e$ *is a Borel subset such that* $\sigma_n(A) > 0$, *then*

$$g_e(c_s(A)) = c_{\widehat{\tau}_n}(g_e(A)). \tag{2.14}$$

Proof. By Lemma 2.2.4 (b) and definition (2.11), relation (2.13) follows from

$$g_e\left(\sum_{i=1}^N u_i\right) = \frac{\sum_{i=1}^N u_i}{\sum_{i=1}^N u_i \cdot e} - e = \frac{\sum_{i=1}^N (u_i \cdot e)\left(\frac{u_i}{u_i \cdot e} - e\right)}{\sum_{i=1}^N u_i \cdot e} = \frac{1}{\sum_{i=1}^N \phi(x_i)} \sum_{i=1}^N \phi(x_i) x_i,$$

where $x_i = g_e(u_i)$, $1 \le i \le N$. In order to prove (2.13), we use again the area formula (see, e.g., [Mag12, Remark 8.3]) to obtain

$$g_e(c_s(A)) = \frac{\int_A u \, d\sigma_n(u)}{e \cdot \int_A u \, d\sigma_n(u)} - e = \frac{\int_{g_e(A)} g_e^{-1}(x) J g_e^{-1}(x) \, dx}{\int_{g_e(A)} e \cdot g_e^{-1}(x) J g_e^{-1}(x) \, dx} - e.$$

Since for $x \in \mathbb{R}^n$, we have $\|x + e\|^2 = 1 + \|x\|^2$, and by the proof of Lemma 2.2.4 (a) $J g_e^{-1}(x) = \xi_n(x)$, we conclude that

$$\begin{aligned} g_e(c_s(A)) = \frac{\int_{g_e(A)} (x+e) \xi_n(x) \, dx}{\int_{g_e(A)} e \cdot (x+e) \xi_n(x) \, dx} - e &= \frac{\int_{g_e(A)} (x+e) \, d\widehat{\tau}_n(x)}{\int_{g_e(A)} 1 \, d\widehat{\tau}_n(x)} - e \\ &= \frac{1}{\widehat{\tau}_n(g_e(A))} \int_{g_e(A)} x \, d\widehat{\tau}_n(x) = c_{\widehat{\tau}_n}(g_e(A)). \end{aligned}$$

□

We conclude this section by collecting a number of properties of spherical centroids for later reference.

Proposition 2.2.6. *Let* $\{u_1, \ldots, u_N\} \subseteq \mathbb{S}^n$ *and* $K_m, K \in \mathcal{K}(\mathbb{S}^n)$, $m \in \mathbb{N}$, *such that their spherical centroids exist. Then the map* c_s *has the following properties:*

(a) It is continuous, that is, if $u_{i,m} \to u_i$ *for* $1 \le i \le N$ *and* $K_m \to K$, $m \in \mathbb{N}$, *in the spherical Hausdorff metric, then*

$$c_s(u_{1,m}, \ldots, u_{N,m}) \to c_s(u_1, \ldots, u_N) \quad \text{and} \quad c_s(K_m) \to c_s(K).$$

(b) It is $\mathrm{O}(n+1)$*-equivariant, that is, for every* $\vartheta \in \mathrm{O}(n+1)$*, we have*

$$c_s(\vartheta u_1, \ldots, \vartheta u_N) = \vartheta c_s(u_1, \ldots, u_N) \quad \text{and} \quad c_s(\vartheta K) = \vartheta c_s(K).$$

(c) It is proper, that is, $c_s(K) \in \operatorname{int} K$*.*

(d) It is consistent, that is, if $U_1, \ldots, U_N$ *are independent random variables uniformly distributed in* K*, then*

$$c_s(U_1, \ldots, U_N) \to c_s(K)$$

almost surely as N *tends to infinity.*

Proof. Property (a) is trivial in the discrete case and follows for convex bodies from the continuity of spherical volume in the Hausdorff topology on convex bodies (see e.g. [Gla96, Hilfssatz 2.4]), since

$$\left| \int_{K_m} u \, d\sigma_n(u) - \int_K u \, d\sigma_n(u) \right| \leq \sigma_n(K_m \triangle K) \to 0.$$

Property (b) is also trivial in the discrete case and for convex bodies a simple consequence of the $\mathrm{O}(n+1)$-invariance of spherical Lebesgue measure and the transformation rule for integrals.

In order to see (c), note that $u \in \operatorname{int} K$ if and only if $w \cdot u < 0$ for all $w \in K^*$. Now since $w \cdot c_s(K) < 0$ for all $w \in K^*$, by definition, we obtain the desired property.

Finally, since $\sigma_n(\operatorname{bd} K) = 0$ and $\operatorname{int} K$ is proper, and (b), we may assume for the proof of (d) that $K \subseteq \operatorname{int} \mathbb{S}^+_e$. Then, by Lemma 2.2.4 (a), the random vectors $X_i := g_e(U_i)$, $1 \leq i \leq N$, are independent and identically distributed according to

$$\frac{\mathbb{1}_{g_e(K)}}{\widehat{\sigma}_n(g_e(K))} \widehat{\sigma}_n.$$

Moreover, by Lemma 2.2.5, we have

$$g_e(c_s(U_1, \ldots, U_N)) = \frac{1}{\sum_{i=1}^N \phi(X_i)} \sum_{i=1}^N \phi(X_i) X_i.$$

But, by the strong law of large numbers (see, e.g., [Dud02, Theorem 8.3.5]),

$$\sum_{i=1}^N \phi(X_i) \to \int_{g_e(K)} \phi(x) \, d\widehat{\sigma}_n(x) = \widehat{\tau}_n(g_e(K))$$

and

$$\sum_{i=1}^N \phi(X_i) X_i \to \int_{g_e(K)} \phi(x) x \, d\widehat{\sigma}_n(x) = \int_{g_e(K)} x \, d\widehat{\tau}_n(x)$$

almost surely as $N \to \infty$. Since, by the continuous mapping theorem, the product of almost surely convergent sequences of random variables converges almost surely to the product of their limits, we obtain from another application of Lemma 2.2.5,

$$g_e(c_s(U_1,\dots,U_N)) \to \frac{1}{\widehat{\tau}_n(g_e(K))}\int_{g_e(K)} x\, d\widehat{\tau}_n(x) = c_{\widehat{\tau}_n}(g_e(K)) = g_e(c_s(K)),$$

almost surely as $N \to \infty$, which, by Lemma 2.2.3 (c), yields property (d). □

2.3 Symmetrizations

Consider the following optimization problem: minimize geometric quantity A over all convex bodies $K \subseteq \mathbb{M}^n$ of given volume c. Whenever A is continuous and geodesic balls (of suitable radii) are conjectured to be extremal, one can hope to find a family of transformations, or *symmetrizations*, $(S_\lambda)_{\lambda\in\Lambda}$, such that each $S_\lambda\colon \mathcal{K}(\mathbb{M}^n) \to \mathcal{K}(\mathbb{M}^n)$ preserves volume, and decreases A. To conclude that balls are indeed minimizers, one then needs to find for each $K \in \mathcal{K}(\mathbb{M}^n)$ a sequence $(\lambda_m)_{m\in\mathbb{N}} \subseteq \Lambda$, such that the iterated symmetrizations $S_{\lambda_m} \circ \cdots S_{\lambda_1} K$ converge to a geodesic ball. Classical examples for A are surface area, the other instrinsic volumes (see (2.5)), or the diameter $\operatorname{diam} K = \sup\{\|x-y\|, x,y \in K\}$, where $K \in \mathcal{K}(\mathbb{R}^n)$. We describe now two important symmetrization techniques used in convexity: Steiner symmetrization in $\mathbb{R}^n$ and two-point symmetrization in $\mathbb{M}^n$.

We start with Steiner symmetrization (see, e.g., [Sch14, Section 10.3] or [Art15, Section 1.1.7]). Let $u \in \mathbb{S}^{n-1}$ be a unit vector and H its orthogonal complement in $\mathbb{R}^n$. The Steiner symmetral $S_u K$ of a bounded, measurable set $K \subseteq \mathbb{R}^n$ that has non-empty interior is defined is follows: For each line $G = \{x + \mathbb{R}u\}$, $x \in H$, orthogonal to H, that intersects K, $S_u K \cap G$ is a closed interval with midpoint in H, such that $\operatorname{vol}_1(S_u K \cap G) = \operatorname{vol}_1(K \cap G)$. In particular, if $K \in \mathcal{K}(\mathbb{R}^n)$, there exist $f, g\colon K|H \to \mathbb{R}$ such that f is concave, g is convex, and

$$K = \{x + tu\colon x \in K|H, t \in [g(x), f(x)].$$

In this case, $S_u K$ is given by

$$S_u K = \left\{x + tu\colon x \in K|H, t \in \left[-\frac{f(x)-g(x)}{2}, \frac{f(x)-g(x)}{2}\right]\right\}.$$

Here, $K|H$ denotes the orthogonal projection of K onto H.

By Fubini's theorem, $\operatorname{vol}_n(S_u K) = \operatorname{vol}_n(K)$ for bounded, measurable sets $K \subseteq \mathbb{R}^n$. Moreover, if $K \in \mathcal{K}(\mathbb{R}^n)$, so is $S_u K$. The following result states that balls can be reached by iterated Steiner symmetrizations. It can be found, e.g., in [Art15, Theorem 1.1.16].

Proposition 2.3.1. *Let $K \in \mathcal{K}(\mathbb{R}^n)$ be a convex body with non-empty interior. Then there exists a sequence of directions $u_m \in \mathbb{S}^{n-1}$, such that $S_{u_m} \circ \cdots \circ S_{u_1} K$ converges in the Hausdorff metric to a Euclidean ball of the same volume as K.*

Many classical geometric extremal problems can be solved by showing first that a geometric quantity of a convex body behaves monotonically whenever a Steiner symmetrization is

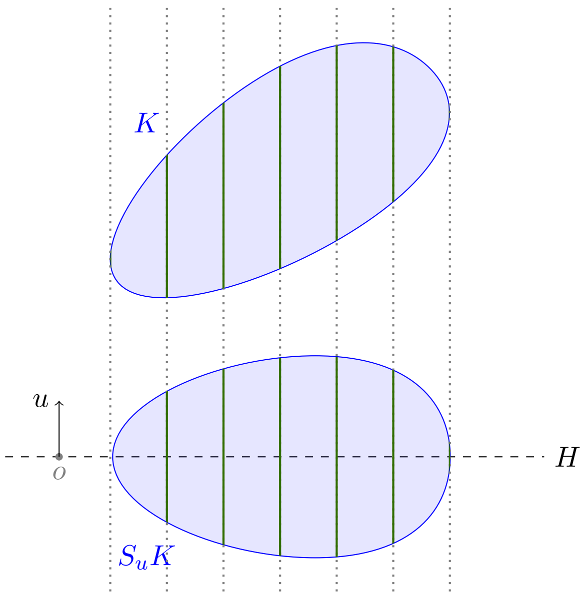

Figure 2.2: A Steiner symmetrization of K.

applied to the body, and then using the above proposition that exhibits balls as extremizers.

Lastly, Steiner symmetrization can be extended to functions, using the layer-cake formula: Let $f\colon \mathbb{R}^n \to \mathbb{R}^+$ be integrable, such that in particular, all sup-level sets $\{f > s\}$, $s \in \mathbb{R}^+$, are bounded and measurable, then

$$S_u f(x) := \int_0^\infty \mathbb{1}_{S_u\{f>s\}}(x)\, ds.$$

Clearly, $S_u F$ is integrable and $\|S_u f\|_{L^1(\mathbb{R}^n)} = \|f\|_{L^1(\mathbb{R}^n)}$.

Next, we consider two-point symmetrization, also known as polarization, in $\mathbb{M}^n$ (see, e.g., [Bae19, Section 1.7], [Bro00], or [Wol52]). Its ingredients are hyperplanes and orthogonal reflections: For every $M \in \mathcal{M}^n_{n-1}$, we can find a vector $u \in \mathbb{R}^{n+1}$, such that $M = u^\perp \cap \mathbb{M}^n$, where the orthogonal complement is taken either with respect to Euclidean or Minkowski scalar product in $\mathbb{R}^{n+1}$. A hyperplane $H \in \mathcal{M}^n_{n-1}$ divides $\mathbb{M}^n$ into two connected components, which we will call the closed halfspaces H^+ and H^- in such a way that always $e \in H^+$. The associated orthogonal reflection about H will be denoted by $\rho\colon \mathbb{M}^n \to \mathbb{M}^n$. If $H = u^\perp \cap \mathbb{M}^n$, for $u \in \mathbb{R}^{n+1}$, then ρ is given by

$$\rho x := \rho(x) = \begin{cases} x - 2\frac{x\cdot u}{u\cdot u}u, & \text{if } \mathbb{M}^n = \mathbb{S}^n, \\ x - 2\frac{x\cdot u}{1-(u\cdot e)^2}[u - (u\cdot e)e], & \text{if } \mathbb{M}^n = \mathbb{R}^n_{e,1}, \\ x - 2\frac{\langle x,u\rangle}{\langle u,u\rangle}u, & \text{if } \mathbb{M}^n = \mathbb{H}^n. \end{cases}$$

Lemma 2.3.2. *If $x, y \in H^+$ then $d_{\mathbb{M}^n}(x, y) \le d_{\mathbb{M}^n}(x, \rho(y))$.*

Proof. Indeed, let z be the intersection of the geodesic segment $[x,\rho y]$ with H. Then

$$d_{\mathbb{M}^n}(x,y) \leq d_{\mathbb{M}^n}(x,z) + d_{\mathbb{M}^n}(z,y) = d_{\mathbb{M}^n}(x,z) + d_{\mathbb{M}^n}(z,\rho y) = d_{\mathbb{M}^n}(x,\rho y)$$

by the triangle inequality. □

If we decompose a set $K \subseteq \mathbb{M}^n$ as

$$K = \underbrace{(K \cap \rho K)}_{K^{\mathrm{sym}}} \,\dot\cup\, \underbrace{\left[(K \cap H^+) \setminus K^{\mathrm{sym}}\right]}_{K^{\mathrm{fix}}} \,\dot\cup\, \underbrace{\left[(K \cap H^-) \setminus K^{\mathrm{sym}}\right]}_{K^{\mathrm{mov}}},$$

the *two-point symmetrization* $T = (H, \rho, T)$ with respect to H is given by

$$TK = \underbrace{(K \cap \rho K)}_{K^{\mathrm{sym}}} \,\dot\cup\, \underbrace{\left[(K \cap H^+) \setminus K^{\mathrm{sym}}\right]}_{K^{\mathrm{fix}}} \,\dot\cup\, \underbrace{\rho\left[(K_i \cap H^-) \setminus K^{\mathrm{sym}}\right]}_{\rho K^{\mathrm{mov}}}.$$

Note that all unions are disjoint up to sets of measure zero, which immediately shows that $\lambda_n(TK) = \lambda_n(K)$ for all measurable sets $K \subseteq \mathbb{M}^n$. Intuitively, T pushes as much mass as possible towards e (that is, into H^+) without double-covering points.

Proposition 2.3.3. *Let $K \subseteq \mathbb{M}^n$ be a compact set with non-empty interior. Then there exists a sequence of two-point symmetrizations $(T_m)_{m\in\mathbb{N}}$ such that $T_m \circ \cdots \circ T_1 K$ converges in the Hausdorff metric to a geodesic ball of the same volume as K.*

Proof. The full compactness argument is included in the proof of Proposition 3.3.3, so we give here only an idea of how one can obtain the statement. Let B_R be a geodesic ball around $e \in \mathbb{M}^n$ that contains K and $\mathfrak{T}$ be the set of all sets that can be reached from K by applying iterated two-point symmetrizations. Then every member of $\mathfrak{T}$ is contained in B_R, and thus $\operatorname{cl}\mathfrak{T}$ is compact in the Hausdorff metric. Let B_K be a geodesic ball around e whose volume is that of K. Then the function $L \mapsto \lambda_n(L \cap B_K)$ is continuous in the Hausdorff

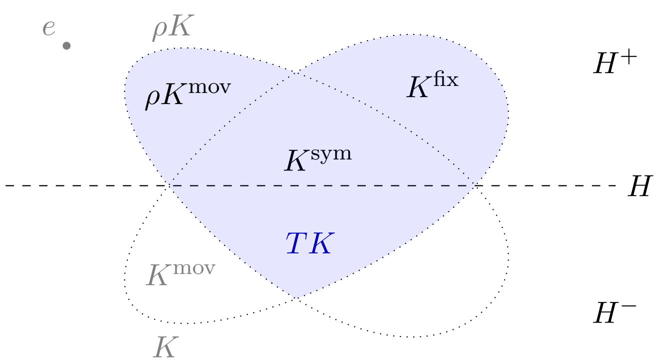

Figure 2.3: A two-point symmetrization of K.

metric and thus attains a maximum L' on $\operatorname{cl}\mathcal{T}$. Suppose $L' \neq B_K$, then there exist $r > 0$ and points $x, y \in \mathbb{M}^n$ such that $B_r(x) \subseteq L' \setminus B_K$ and $B_r(y) \subseteq B_K \setminus L'$. Now, apply to L' one more two-point symmetrization $T = (H, \rho, T)$ such that H is the orthogonal bisector of $[x, y]$ to arrive at a contradiction, since $\lambda_n(TL' \cap B_K) > \lambda_n(L' \cap B_K)$. □

The two-point symmetrization of a function $f\colon \mathbb{M}^n \to \mathbb{R}^+$ is given by

$$Tf(x) = \begin{cases} \max\{f(x), f(\rho x)\}, & \text{if } x \in H^+, \\ \min\{f(x), f(\rho x)\}, & \text{if } x \in H^-. \end{cases}$$

We have $T\mathbb{1}_K = \mathbb{1}_{TK}$ for any set $K \subseteq \mathbb{M}^n$. More generally,

$$\{Tf > s\} = T\{f > s\} \tag{2.15}$$

for all $s > 0$. To see this, write

$$\{Tf > s\} = \underbrace{\{x \in H^+\colon \max\{f(x), f(\rho x)\} > s\}}_{D_1} \cup \underbrace{\{x \in H^-\colon \min\{f(x), f(\rho x)\} > s\}}_{D_2},$$

$$T\{f > s\} = \underbrace{\{x\colon f(x) > s \wedge f(\rho x) > s\}}_{E_1} \cup \underbrace{\{x \in H^+\colon f(x) > s\}}_{E_2} \cup \underbrace{\{x \in H^+\colon f(\rho x) > s\}}_{E_3},$$

and note that $D_1 \subseteq E_2 \cup E_3$ and $D_2 \subseteq E_1$, and on the other hand $E_1 \subseteq D_1 \cup D_2$ and $E_2 \cup E_3 \subseteq D_1$.

For a continuous function $f \in C(\mathbb{M}^n)$ and $\delta > 0$, denote by

$$\omega(\delta, f) = \sup\{|f(x) - f(y)|\colon d_{\mathbb{M}^n}(x,y) < \delta, x,y \in \mathbb{M}^n\}$$

the *modulus of continuity* of f.

Lemma 2.3.4 ([Bae76]). *For every continuous function $f \in C(\mathbb{M}^n)$, $\delta > 0$, and every two-point symmetrization T, we have*

$$\omega(\delta, Tf) \leq \omega(\delta, f).$$

Proof. We reproduce the proof from [Bae76, Lemma 1]: For numbers $a_1, a_2, b_1, b_2 \in \mathbb{R}$ and $\alpha_1 = \max\{a_1, a_2\}$, $\alpha_2 = \min\{a_1, a_2\}$, $\beta_1 = \max\{b_1, b_2\}$, $\beta_2 = \min\{b_1, b_2\}$, we have

$$\max\{|\alpha_1 - \beta_1|, |\alpha_2 - \beta_2|\} \leq \max\{|a_1 - b_1|, |a_2 - b_2|\}.$$

Now, let $T = (H, \rho, T)$ and $x, y \in H^+$ or $x, y \in H^-$ with $d_{\mathbb{M}^n}(x,y) \leq \delta$. Then by the above inequality, and since $d_{\mathbb{M}^n}(\rho x,\rho y) = d_{\mathbb{M}^n}(x,y)$, we get

$$\begin{aligned} |Tf(x) - Tf(y)| \leq \max\{&|\max\{f(x),f(\rho x)\} - \max\{f(y),f(\rho y)\}|, \\ &|\min\{f(x),f(\rho x)\} - \min\{f(y),f(\rho y)\}|\} \\ \leq \max\{&|f(x) - f(y)|, |f(\rho x) - f(\rho y)|\} \leq \omega(\delta, f). \end{aligned}$$

If on the other hand, say, $x \in H^+$ and $y \in H^-$, we have

$$\begin{aligned}|Tf(x) - Tf(y)| \leq \max\{&|f(x) - f(y)|, |f(x) - f(\rho y)|,\\ &|f(\rho x) - f(y)|, |f(\rho x) - f(\rho y)|\} \leq \omega(\delta, f),\end{aligned}$$

since also $d_{\mathbb{M}^n}(x,\rho y) = d_{\mathbb{M}^n}(\rho x, y) \leq d_{\mathbb{M}^n}(x,y) \leq \delta$, by Lemma 2.3.2. □

We close this section with a lemma that is needed in Chapter 6.

Lemma 2.3.5. *Let $T = (H, \rho, T)$ be a two-point symmetrization, $K, S \subseteq \mathbb{M}^n$, such that S is symmetric with respect to H, that is, $\rho(S) = S$. Then we have*

$$T(K \cap S) = TK \cap S, \tag{2.16}$$
$$T(K \cup S) = TK \cup S. \tag{2.17}$$

Proof. Let $K = K^{\mathrm{sym}} \,\dot\cup\, K^{\mathrm{fix}} \,\dot\cup\, K^{\mathrm{mov}}$ as above. We compute:

$$\begin{aligned}(K \cap S)^{\mathrm{sym}} &= K \cap S \cap \rho K \cap \rho S = K \cap \rho K \cap S = K^{\mathrm{sym}} \cap S,\\ (K \cup S)^{\mathrm{sym}} &= (K \cup S) \cap \rho(K \cup S) = (K \cap \rho K) \cup S = K^{\mathrm{sym}} \cup S.\end{aligned}$$

Moreover, denoting by $A^c = \mathbb{M}^n \setminus A$ the complement of a set A in $\mathbb{M}^n$, we have

$$\begin{aligned}(K \cap S)^{\mathrm{fix}} &= (K \cap S \cap H^+) \setminus (K \cap S)^{\mathrm{sym}} = (K \cap H^+ \cap S) \setminus (K^{\mathrm{sym}} \cap S)\\ &= \big((K \cap H^+) \setminus K^{\mathrm{sym}}\big) \cap S = K^{\mathrm{fix}} \cap S,\\ (K \cup S)^{\mathrm{fix}} &= \big((K \cup S) \cap H^+\big) \setminus (K \cup S)^{\mathrm{sym}} = \big((K \cap H^+) \cup (S \cap H^+)\big) \cap (K^{\mathrm{sym}} \cup S)^c\\ &= K \cap H^+ \cap (K^{\mathrm{sym}})^c \cap S^c = K^{\mathrm{fix}} \cap S^c,\end{aligned}$$

and similarly $(K \cap S)^{\mathrm{mov}} = K^{\mathrm{mov}} \cap S$, $(K \cup S)^{\mathrm{mov}} = K^{\mathrm{mov}} \cap S^c$. Thus,

$$\begin{aligned}T(K \cap S) &= (K \cap S)^{\mathrm{sym}} \cup (K \cap S)^{\mathrm{fix}} \cup \rho\left((K \cap S)^{\mathrm{mov}}\right)\\ &= (K^{\mathrm{sym}} \cap S) \cup \left(K^{\mathrm{fix}} \cap S\right) \cup \rho\left(K^{\mathrm{mov}} \cap S\right)\\ &= (K^{\mathrm{sym}} \cup K^{\mathrm{fix}} \cup \rho K^{\mathrm{mov}}) \cap S = TK \cap S,\\ T(K \cup S) &= (K \cup S)^{\mathrm{sym}} \cup (K \cup S)^{\mathrm{fix}} \cup \rho\left((K \cup S)^{\mathrm{mov}}\right)\\ &= (K^{\mathrm{sym}} \cup S) \cup \left(K^{\mathrm{fix}} \cap S^c\right) \cup \rho\left(K^{\mathrm{mov}} \cap S^c\right)\\ &= S \cup K^{\mathrm{sym}} \cup \left((K^{\mathrm{fix}} \cup \rho K^{\mathrm{mov}}) \cap S^c\right)\\ &= S \cup \left((K^{\mathrm{sym}} \cup K^{\mathrm{fix}} \cup \rho K^{\mathrm{mov}}) \cap (K^{\mathrm{sym}} \cup S^c)\right)\\ &= (S \cup TK) \cap (S \cup K^{\mathrm{sym}} \cup S^c) = TK \cup S,\end{aligned}$$

which concludes the proof. □

CHAPTER 3

Rearrangement inequalities

In this chapter, we put together various integral inequalities concerning the rearrangements of functions. They provide the analytic foundation for the geometric inequalities appearing later in Chapters 4, 5, and 6. A good introduction to the subject is offered in the books of Lieb and Loss [Lie01] or Baernstein [Bae19].

3.1 Definition and basic properties

During this short presentation of rearrangements we mostly follow [Lie01, Chapter 3]. For a measurable set $A \subseteq \mathbb{M}^n$ that has finite volume $\lambda_n(A) < \infty$, we define its *symmetric rearrangement* $A^\star$ as the open ball around the origin $e \in \mathbb{M}^n$ such that

$$\lambda_n(A^\star) = \lambda_n(A).$$

We will extend this definition to positive functions by means of the layer-cake representation formula:

$$f(x) = \int_0^\infty \chi_{\{f>s\}}(x)\, ds.$$

Let $f\colon \mathbb{M}^n \to \mathbb{R}^+$, be a measurable function that vanishes at infinity, that is, satisfies $\lambda_n(\{f > s\}) < \infty$ for all $s > 0$ (in particular, integrable functions have this property). We define the *symmetric decreasing rearrangement* of f to be

$$f^\star(x) = \int_0^\infty \chi_{\{f>s\}^\star}(x)\, ds.$$

In particular, $\chi_A^\star = \chi_{A^\star}$. The following facts can be found, e.g., in [Lie01, Section 3.3]:

Facts. *The symmetric decreasing rearrangement has the following properties:*

- *It is non-negative, radially symmetric, and radially decreasing, that is,*

  $$f^\star(x) = f^\star(y),$$

  if $d_{\mathbb{M}^n}(x,e) = d_{\mathbb{M}^n}(y,e)$, *and*

  $$f^\star(x) \geq f^\star(y),$$

  if $d_{\mathbb{M}^n}(x,e) \leq d_{\mathbb{M}^n}(y,e)$.

- *The level sets of* $f^\star$ *are the symmetric rearrangements of the level sets of* f*, in particular,*

$$\lambda_n(\{f^\star > s\}) = \lambda_n(\{f > s\}),$$

- *If* $f \in L^p(\mathbb{M}^n)$ *for* $1 \leq p \leq \infty$*, then* $f^\star \in L^p(\mathbb{M}^n)$ *and*

$$\|f^\star\|_{L^p(\mathbb{M}^n)} = \|f\|_{L^p(\mathbb{M}^n)}.$$

for all $s > 0$*. In particular, the set of probability densities on* $\mathbb{M}^n$ *is closed under taking symmetric decreasing rearrangements.*

- *It is non-expansive, that is,*

$$\|f^\star - g^\star\|_{L^p(\mathbb{M}^n)} \leq \|f - g\|_{L^p(\mathbb{M}^n)},$$

for all $f, g \in L^p(\mathbb{M}^n)$*, where* $1 \leq p \leq \infty$*.*

In this chapter, we are concerned with integral inequalities involving rearrangements of functions of the following type:

Problem. *Let* $N \in \mathbb{N}$*. Find functions* $\Psi\colon (\mathbb{M}^n)^N \to \mathbb{R}^+$ *that satisfy*

$$\int_{(\mathbb{M}^n)^N} \Psi(x_1,\dots,x_N) \prod_{i=1}^N f_i(x_i)\,dx_i \leq \int_{(\mathbb{M}^n)^N} \Psi(x_1,\dots,x_N) \prod_{i=1}^N f_i^\star(x_i)\,dx_i, \tag{3.1}$$

for all integrable functions $f_1,\dots f_N\colon \mathbb{M}^n \to \mathbb{R}^+$*.*

The functions f_i can be seen as probability densities of independent random variables X_i. In this case, the integral in (3.1) equals the expectation of $\Psi(X_1,\dots,X_N)$ and the inequality tells us that this value is increasing, whenever the f_i are replaced with their symmetric decreasing rearrangements.

A strategy towards such inequalities is to employ symmetrization techniques that gradually approach the symmetric decreasing rearrangement, and show that inequality holds at each step of this approximation process. We will see two examples of this method in the subsequent sections.

3.2 Inequalities via Steiner symmetrization

It is shown in [Bra74], that if $f\colon \mathbb{R}^n \to \mathbb{R}^+$ is integrable, its symmetric decreasing rearrangement can be approximated in the L^1-norm by a sequence of Steiner symmetrizations of f, that is, there exist directions u_m, $m \in \mathbb{N}$, such that

$$\|S_{u_m} \circ \dots \circ S_{u_1} f - f^\star\|_{L^1(\mathbb{R}^n)} \to 0,$$

as $m \to \infty$. Using Christ's version [Chr84] of the classical Rogers/Brascamp–Lieb–Luttinger rearrangement inequality [Rog57], [Bra74], Paouris and Pivovarov [Pao12a] obtained the following theorem:

Theorem 3.2.1 ([Pao12a]). *Let $N \in \mathbb{N}$, $f_1, \dots f_N \colon \mathbb{R}^n \to \mathbb{R}^+$ be integrable and $\Psi \colon (\mathbb{R}^n)^N \to \mathbb{R}^+$ satisfy the following condition: For each $u \in \mathbb{S}^{n-1}$ and every $Y = \{y_1, \dots, y_N\} \subseteq u^\perp$ the function $\Psi_Y \colon \mathbb{R}^N \to \mathbb{R}^+$ defined by*

$$(t_1, \dots, t_N) \mapsto \Psi_Y(t) = \Psi(y_1 + t_1 u, \dots, y_N + t_n u)$$

is even and quasi-concave (which means its sup-level sets $\{\Psi_Y > s\}$ are convex for all $s > 0$). Then (3.1) holds, that is,

$$\int_{(\mathbb{R}^n)^N} \Psi(x_1, \dots, x_N) \prod_{i=1}^N f_i(x_i)\, dx_i \le \int_{(\mathbb{R}^n)^N} \Psi(x_1, \dots, x_N) \prod_{i=1}^N f_i^\star(x_i)\, dx_i,$$

In [Pao12a], [Cor15], and [Pao17b] it is then shown that various geometric functionals such as, for example,

$$\Psi(x_1, \dots, x_N) = -\operatorname{vol}_n\left(\operatorname{conv}\{\pm x_1, \dots, \pm x_N\}\right),$$

or

$$\Psi(x_1, \dots, x_N) = \operatorname{vol}_n\left(\operatorname{conv}\{\pm x_1, \dots, \pm x_N\}^\circ\right),$$

indeed satisfy the assumptions of Theorem 3.2.1. We state now a particular result of [Cor15], that we will need in Chapter 4. If $x_1, \dots, x_N$ are vectors in $\mathbb{R}^n$, we write $(x_1, \dots, x_N)$ for the $n \times N$-matrix that has columns $x_1, \dots, x_N$. Also, set $(x_1, \dots, x_N)C := \{(x_1, \dots, x_N)c \,|\, c \in C\}$ for any set $C \in \mathbb{R}^N$.

Theorem 3.2.2 ([Cor15]). *Let $N \in \mathbb{N}$, $f_1, \dots, f_N \colon \mathbb{R}^n \to \mathbb{R}^+$ bounded, integrable functions, and ν be a finite, absolutely continuous Borel measure on $\mathbb{R}^n$ that has a radially symmetric and radially decreasing density. Then*

$$\int_{(\mathbb{R}^n)^N} \nu\left(\left(\frac{1}{N}(x_1, \dots, x_N) B_\infty^N\right)^\circ\right) \prod_{i=1}^N f_i(x_i)\, dx_1 \dots dx_N$$
$$\le \int_{(\mathbb{R}^n)^N} \nu\left(\left(\frac{1}{N}(x_1, \dots, x_N) B_\infty^N\right)^\circ\right) \prod_{i=1}^N f_i^\star(x_i)\, dx_1 \dots dx_N.$$

Here, $B_\infty^N = \{t \in \mathbb{R}^N \,|\, -1 \le t_i \le 1$ for all $1 \le i \le N\}$ is the unit ball in ℓ_∞^N.

We note that the proof of Theorem 3.2.2 in [Cor15] was given for probability densities bounded by one. However, it goes through verbatim in the case of arbitrary bounded, integrable functions.

3.3 Inequalities via two-point symmetrization

In this section we review inequalities of the form (3.1) that arise from repeated two-point symmetrizations. Since two-point symmetrization is similar in spherical, Euclidean, and hyperbolic space, this method has the advantage of providing inequalities that hold in all three geometries.

Theorem 3.3.1 ([Mor02]). *Let $N \in \mathbb{N}$ and $f_1, \dots, f_N \colon \mathbb{M}^n \to \mathbb{R}^+$ be integrable, and $\Psi \colon (\mathbb{M}^n)^N \to \mathbb{R}^+$ of the form*

$$\Psi_\kappa(x_1, \dots, x_N) = \prod_{i<j} \kappa_{ij}(d_{\mathbb{M}^n}(x_i, x_j)), \tag{3.2}$$

where $\kappa_{ij} \colon \mathbb{R}^+ \to \mathbb{R}^+$ are decreasing functions for all $i < j$. Then (3.1) holds, that is

$$\int_{(\mathbb{M}^n)^N} \Psi_\kappa(x_1, \dots, x_N) \prod_{i=1}^N f_i(x_i)\, dx_i \le \int_{(\mathbb{M}^n)^N} \Psi_\kappa(x_1, \dots, x_N) \prod_{i=1}^N f_i^\star(x_i)\, dx_i.$$

A version of Theorem 3.3.1 has also appeared in [Bur01]. We remark that although the proof in [Mor02] is carried out in Euclidean space, it (as noted in the paper) goes through as well in spherical or hyperbolic space.

Since the functionals we consider in Chapters 5 and 6 are not of the form (3.2), Theorem 3.3.1 will not be applicable. On the other hand, we will provide new examples of functions $\Psi(x_1, \dots, x_N)$ that satisfy (3.1).

We now give a detailed explanation of the method described by Baernstein and Taylor in [Bae76] on how to obtain (3.1) (in the opposite direction, with $\ge$), once one knows that the integral in (3.1) decreases whenever the functions $f_1, \dots, f_N$ are replaced by their two-point symmetrals with respect to a common hyperplane. A good resource is also [Bae19, Chapter 2]. We start with a preparatory lemma that relates rearrangements with two-point symmetrizations.

Lemma 3.3.2 ([Bae76]). *Let $f \colon \mathbb{M}^n \to \mathbb{R}^+$ be integrable and $T = (H, \rho, T)$ any two-point symmetrization in $\mathbb{M}^n$. Then the following holds:*

(a) For $x \in H^+$ we have $f(x)f^\star(x) + f(\rho x)f^\star(\rho x) \le Tf(x)f^\star(x) + Tf(\rho x)f^\star(\rho x)$

(b) $\int_{\mathbb{M}^n} |f(x) - f^\star(x)|^2\, dx \ge \int_{\mathbb{M}^n} |Tf(x) - f^\star(x)|^2\, dx$

Proof. We give the argument of [Bae76, Lemma 1]: Let $a_1, a_2, b_1, b_2 \in \mathbb{R}$ be numbers with $b_1 \ge b_2$. Using notation from the proof of Lemma 2.3.4, the following holds:

$$a_1 b_1 + a_2 b_2 \le \alpha_1 b_1 + \alpha_2 b_2$$

If $x \in H^+$, we have $f^\star(x) \ge f^\star(\rho x)$ and thus

$$f(x)f^\star(x) + f(\rho x)f^\star(\rho x) \le Tf(x)f^\star(x) + Tf(\rho x)f^\star(\rho x),$$

which is (a). Integration over H^+ yields

$$\int_{\mathbb{M}^n} f(x)f^\star(x)\,dx = \int_{H^+} f(x)f^\star(x) + f(\rho x)f^\star(\rho x)\,dx$$
$$\leq \int_{H^+} Tf(x)f^\star(x) + Tf(\rho x)f^\star(\rho x)\,dx = \int_{\mathbb{M}^n} Tf(x)f^\star(x)\,dx,$$

hence, (b) follows from

$$\int_{\mathbb{M}^n} |f(x) - f^\star(x)|^2\,dx = \int_{\mathbb{M}^n} f(x)^2\,dx - \int_{\mathbb{M}^n} f(x)f^\star(x)\,dx + \int_{\mathbb{M}^n} f^\star(x)^2\,dx$$

and the fact that two-point symmetrization preserves L^2-norms. □

We now introduce the following notation for the expressions appearing in (3.1):

$$I_\Psi(f_1,\dots,f_N) := \int_{\mathbb{M}^n} \cdots \int_{\mathbb{M}^n} \Psi(x_1,\dots,x_N) \prod_{i=1}^N f_i(x_i)\,dx_1 \dots dx_N, \tag{3.3}$$

for a measurable function $\Psi\colon (\mathbb{M}^n)^N \to \mathbb{R}^+$ and bounded, integrable functions $f_i\colon \mathbb{M}^n \to \mathbb{R}^+$, $1 \leq i \leq N$. Sometimes, we will also consider the truncated functional

$$I_\Psi^R(f_1,\dots,f_N) := I_\Psi(\mathbb{1}_{B_R} f_1,\dots,\mathbb{1}_{B_R} f_N),$$

where B_R is the geodesic ball of radius $R > 0$ around the origin $e \in \mathbb{M}^n$. Clearly, in the case $\mathbb{M}^n = \mathbb{S}^n$, we have $I^R = I$ whenever $R \geq \pi$.

The next proposition is due to Baernstein and Taylor in the case $\mathbb{M}^n = \mathbb{S}^n$. We reproduce their proof and show that it works similarly in Euclidean or hyperbolic space.

Proposition 3.3.3 ([Bae76]). *Let $f_1,\dots,f_N\colon \mathbb{M}^n \to \mathbb{R}^+$ be bounded, integrable functions, and let $\Psi\colon (\mathbb{M}^n)^N \to \mathbb{R}^+$ be bounded and measurable. Furthermore, assume that*

$$I_\Psi(f_1,\dots,f_N) \geq I_\Psi(Tf_1,\dots,Tf_N),$$

for every two-point symmetrization T on $\mathbb{M}^n$. Then

$$I_\Psi(f_1,\dots,f_N) \geq I_\Psi(f_1^\star,\dots,f_N^\star).$$

Proof. We follow [Bae76, Section 2]. We start with the following facts:

(i) $\mathbb{1}_{B_R} f_i \to f_i$ in $L^1(\mathbb{M}^n)$ as $R \to \infty$,

(ii) $|I_\Psi(f_1,\dots,f_N)| \leq \|\Psi\|_\infty \prod_{i=1}^N \|f_i\|_{L^1(\mathbb{M}^n)}$,

(iii) the map $f \mapsto f^\star$ is continuous in $L^1(\mathbb{M}^n)$

(iv) there are sequences $(\phi_i^j)_{j\in\mathbb{N}}$ in $C_c(\mathbb{M}^n)$, $\operatorname{spt}\phi_i^j \subseteq \operatorname{spt} f_i$, such that $\phi_i^j \to f_i$ in $L^1(\mathbb{M}^n)$

Indeed, (i) follows from dominated convergence, (ii) is a standard estimate, (iii) follows from the non-expansivity of symmetric decreasing rearrangements (see [Lie01, Theorem 3.5]), and (iv) is an application of Urysohn's lemma.

By (i) and (ii), we can assume without loss of generality that $\operatorname{spt} f_i \subseteq B_R$, and by (ii), (iii) and (iv), we can restrict further to continuous functions f_i (supported in B_R) and denote this space by $C(B_R) \subseteq L^2(B_R)$. We define for $f \in C(B_R)$

$$\mathcal{S}(f) := \{F \in C(B_R) \colon \omega(\cdot, F) \leq \omega(\cdot, f) \text{ and } F^\star = f^\star\}.$$

We show now that $\mathcal{S}(f)$ is a compact subset of $L^2(B_R)$. Since f is uniformly continuous, for each $\epsilon > 0$ there exists $\delta > 0$, such that $\omega(\delta, F) \leq \omega(\delta, f) < \epsilon$, for all $F \in \mathcal{S}(f)$. Moreover, $\|F\|_\infty = \|F^\star\|_\infty = \|f^\star\|_\infty$. Hence, $\mathcal{S}(f)$ is a uniformly equicontinuous, uniformly bounded family of continuous functions, thus relatively compact in $(C(B_R), \|\cdot\|_\infty)$ by the Arzelá-Ascoli theorem. Since the map $f \mapsto f^\star$ is continuous also in the $(C(B_R), \|\cdot\|_\infty)$ topology (take $p \to \infty$ in [Lie01, Theorem 3.5]) and the sets $\{\omega(\delta, F) \leq \omega(\delta, f)\}$ and therefore

$$\{\omega(\cdot, F) \leq \omega(\cdot, f)\} = \bigcap_{\delta \in \mathbb{R}^+} \{\omega(\delta, F) \leq \omega(\delta, f)\},$$

are closed (by the triangle inequality), the set $\mathcal{S}(f)$ is compact in $(C(B_R), \|\cdot\|_\infty)$. It is also compact in $(C(B_R), \|\cdot\|_{L^2})$, since by $\|f\|_{L^2}^2 \leq \lambda_n(B_R)\|f\|_\infty^2$, the latter space has a coarser topology.

Next, by Lemma 2.3.4, $T\mathcal{S}(f) \subseteq \mathcal{S}(f)$ for every two-point symmetrization T. Since, by the Cauchy–Schwarz inequality, we have

$$|I_\Psi(f_1, \dots, f_N)| \leq \lambda_n(B_R)^N \|\Psi\|_\infty \prod_{i=1}^N \|f_i\|_{L^2(B_R)},$$

for fixed $f_1, \dots, f_N \in C(B_R)$, the set

$$\mathcal{P} := \{(F_1, \dots, F_N) \in \mathcal{S}(f_1) \times \dots \times \mathcal{S}(f_N) \colon I(f_1, \dots, f_N) \geq I(F_1, \dots, F_N)\}$$

is compact in $L^2(B_R) \times \dots \times L^2(B_R)$. By assumption, $\mathcal{P}$ is closed under simultaneous two-point symmetrizations $(F_1, \dots, F_N) \mapsto (TF_1, \dots, TF_N)$. We are done, if we can show that $\mathcal{P}$ contains $(f_1^\star, \dots, f_N^\star)$.

By compactness, there exist $(F_1^0, \dots, F_N^0) \in \mathcal{P}$ such that

$$\sum_{i=1}^N \|F_i^0 - f_i^\star\|_{L^2(B_R)} = \min \left\{ \sum_{i=1}^N \|F_i - f_i^\star\|_{L^2(B_R)} \colon (F_1, \dots, F_N) \in \mathcal{P} \right\}.$$

Without loss of generality, assume that $F_1^0 \neq f_1^\star$, that is, there exists $t > 0$ such that

$$E_1 := \{F_1^0 > t\} \neq \{f_1^\star > t\} =: E_2, \quad E_1, E_2 \subseteq B_R.$$

Since $\lambda_n(E_1) = \lambda_n(E_2)$, there exist $x \in \operatorname{int}(E_1 \setminus E_2)$ and $y \in \operatorname{int}(E_2 \setminus E_1)$. Let $H \in \mathcal{M}_{n-1}^n$

be the totally geodesic hypersurface that orthogonally bisects the geodesic segment $[x, y]$, ρ the reflection about H, and T the associated two-point symmetrization. Then we can find a small ball B around y, such that $B \subseteq (E_2 \setminus E_1)$ and $\rho B \subseteq (E_1 \setminus E_2)$.

For all $z \in B$, we then have $f_1^\star(z) > t \geq f_1^\star(\rho z)$ and $F_1^0(\rho z) > t \geq F_1^0(z)$, yielding

$$F_1^0(z) f_1^\star(z) + F_1^0(\rho z) f_1^\star(\rho z) < T F_1^0(z) f_1^\star(z) + T F_1^0(\rho z) f_1^\star(\rho z).$$

Since, by Lemma 3.3.2 (a), the same inequality holds with "$\leq$" for all $z \in B_R$, we get

$$\int_{B_R} F_1^0(z) f_1^\star(z)\, dz < \int_{B_R} T F_1^0(z) f_1^\star(z)\, dz,$$

and, since T preserves L^2-norms,

$$\begin{aligned}\int_{B_R} |F_1^0 - f_1^\star|^2 &= \int_{B_R} (F_1^0)^2 - 2F_1^0 f_1^\star + (f_1^\star)^2 \\ &> \int_{B_R} (TF_1^0)^2 - 2TF_1^0 f_1^\star + (f_1^\star)^2 = \int_{B_R} |TF_1^0 - f_1^\star|^2.\end{aligned}$$

Moreover, by 3.3.2 (b) we have

$$\int_{B_R} |F_i^0 - f_i^\star|^2 \geq \int_{B_R} |TF_i^0 - f_i^\star|^2$$

for $2 \leq i \leq N$, hence,

$$\sum_{i=1}^N \|F_i^0 - f_i^\star\|_{L^2(B_R)} > \sum_{i=1}^N \|TF_i^0 - f_i^\star\|_{L^2(B_R)}.$$

which, since $(TF_1^0, \ldots, TF_N^0) \in \mathcal{P}$, is a contradiction. □

3.4 From rotation invariance to balls

In some situations, one can improve (3.1) even further, by showing that among radially symmetric, radially decreasing densities, (multiples of) indicator functions of geodesic balls are extremizers. To do so, we give two instances of a bathtub-type argument. For the first one, we follow [Cor15, Lemma 4.3]:

Proposition 3.4.1. *Let $N \in \mathbb{N}$, $f_1, \ldots, f_N$ bounded, radially symmetric, and radially decreasing functions, $L_1, \ldots, L_N \in \mathcal{K}(\mathbb{R}^n)$ and $\Psi\colon (\mathbb{R}^n)^N \to \mathbb{R}^+$ be measurable, radially symmetric, and radially decreasing in each coordinate. Then*

$$\begin{aligned}&\int_{(\mathbb{R}^n)^N} \Psi(x_1, \ldots, x_N) \prod_{i=1}^N (\mathbb{1}_{L_i}(x_i) f_i(x_i))^\star \, dx_1 \cdots dx_N \\ &\qquad \leq \int_{(\mathbb{R}^n)^N} \Psi(x_1, \ldots, x_N) \prod_{i=1}^N \mathbb{1}_{B_i}(x_i) f_i(x_i)\, dx_1 \cdots dx_N,\end{aligned}$$

where B_i is a ball around the origin, such that $\int_{L_i} f_i(x)\,dx = \int_{B_i} f_i(x)\,dx$.

Proof. Since rearrangements preserve integrals, we have for $1 \le i \le N$,

$$\int_{\mathbb{R}^n} (\mathbb{1}_{B_i} f_i)(x)\,dx = \int_{\mathbb{R}^n} (\mathbb{1}_{L_i} f_i)(x)\,dx = \int_{\mathbb{R}^n} (\mathbb{1}_{L_i} f_i)^\star(x)\,dx.$$

Denote by $\overline{f_i}\colon \mathbb{R}^+ \to \mathbb{R}^+$ the decreasing function that satisfies $\overline{f_i}(\|x\|) = f_i(x)$ for all $x \in \mathbb{R}^n$, and let R_i be the radius of the ball B_i. Using polar coordinates $x(t,u)$, $t \in \mathbb{R}^+$, $u \in \mathbb{S}^{n-1}$, we see that

$$\int_0^{R_i} \overline{f_i}(t) t^{n-1}\,dt = \int_0^\infty \overline{(\mathbb{1}_{L_i} f_i)^\star}(t) t^{n-1}\,dt. \tag{3.4}$$

Now, define functions $\alpha_i : \mathbb{R}^+ \to \mathbb{R}$, $1 \le i \le N$, by

$$\alpha_i(t) := \left((\mathbb{1}_{[0,R_i]} \overline{f_i})(t) - \overline{(\mathbb{1}_{L_I} f_i)^\star}(t) \right) t^{n-1}.$$

Then, by (3.4) and the fact that $(\mathbb{1}_{L_i} f_i)^\star \le f_i$ (because f_i is radially symmetric and radially decreasing), the α_i have the following two properties:

$$\text{(i)} \int_0^\infty \alpha_i(t)\,dt = 0 \qquad \text{(ii)}\ \alpha_i(t) \begin{cases} \le 0 \text{ for } t > R_i, \\ \ge 0 \text{ for } t \le R_i. \end{cases}$$

Combining (i) and (ii), it follows that for any radially symmetric, and radially decreasing $F\colon \mathbb{R}^n \to \mathbb{R}^+$, we have

$$\int_0^\infty \overline{F}(t)\alpha_i(t)\,dr = \int_0^\infty \left(\overline{F}(t) - \overline{F}(R_i)\right) \alpha_i(t)\,dt \ge 0$$

or equivalently, by the definition of α_i, that

$$\int_0^\infty \overline{F(\mathbb{1}_{L_i} f_i)^\star}(t) t^{n-1}\,dt \le \int_0^{R_i} \overline{F f_i}(t) t^{n-1}\,dt.$$

Transferring back to cartesian coordinates, this inequality becomes

$$\int_{\mathbb{R}^n} F(x)(\mathbb{1}_{L_i}(x) f_i(x))^\star\,dx \le \int_{\mathbb{R}^n} F(x)\mathbb{1}_{B_i}(x) f_i(x)\,dx.$$

Now, given $\Psi\colon (\mathbb{R}^n)^N \to \mathbb{R}^+$ that is radially symmetric, and radially decreasing in each coordinate, we can apply the above inequality coordinatewise ($\Psi \prod_{i=1}^m \mathbb{1}_{B_i} f_i$ is still radially decreasing in each coordinate for all $1 \le m \le N$) to obtain the statement. □

Secondly, we argue as in [Pao12a, Lemma 3.5]:

Lemma 3.4.2. *Let* $f\colon [0, R^{\mathbb{M}}] \to \mathbb{R}^+$ *be bounded, measurable and assume that*

$$\int_0^{R^{\mathbb{M}}} f(t) \operatorname{sn}^{n-1} t \, dt < \infty.$$

Define $A \in [0, R^{\mathbb{M}}]$ *such that*

$$\int_0^{R^{\mathbb{M}}} f(t) \operatorname{sn}^{n-1} t \, dt = \int_0^A \|f\|_\infty \operatorname{sn}^{n-1} t \, dt.$$

Then for any increasing function $\phi\colon [0, R^{\mathbb{M}}] \to \mathbb{R}^+$,

$$\int_0^{R^{\mathbb{M}}} \phi(t) f(t) \operatorname{sn}^{n-1} t \, dt \geq \int_0^{R^{\mathbb{M}}} \phi(t) \|f\|_\infty \mathbb{1}_{[0,A]}(t) \operatorname{sn}^{n-1} t \, dt.$$

Proof. By the monotonicity of ϕ and the positivity of sn on $[0, R^{\mathbb{M}}]$ (see Section 2.1), we can estimate

$$\begin{aligned}
\int_0^{R^{\mathbb{M}}} \phi(t) f(t) \operatorname{sn}^{n-1} t \, dt &= \int_0^A \phi(t) f(t) \operatorname{sn}^{n-1} t \, dt + \int_A^{R^{\mathbb{M}}} \phi(t) f(t) \operatorname{sn}^{n-1} t \, dt \\
&\geq \int_0^A \phi(t) f(t) \operatorname{sn}^{n-1} t \, dt + \phi(A) \int_A^{R^{\mathbb{M}}} f(t) \operatorname{sn}^{n-1} t \, dt \\
&= \int_0^A \phi(t) f(t) \operatorname{sn}^{n-1} t \, dt + \phi(A) \int_0^A (\|f\|_\infty - f(t)) \operatorname{sn}^{n-1} t \, dt \\
&\geq \int_0^A \phi(t) f(t) \operatorname{sn}^{n-1} t \, dt + \int_0^A \phi(t) (\|f\|_\infty - f(t)) \operatorname{sn}^{n-1} t \, dt \\
&= \int_0^A \phi(t) \|f\|_\infty \operatorname{sn}^{n-1} t \, dt,
\end{aligned}$$

which gives the statement. □

Using polar coordinates $x(t,u) = e \operatorname{cs} t + u \operatorname{sn} t$, $t \in [0, R^{\mathbb{M}}]$, $u \in \mathbb{S}^{n-1}$ on $\mathbb{M}^n$ (cf. Section 2.1), we can formulate the next proposition. It is a variant of [Pao12a, Proposition 3.9].

Proposition 3.4.3. *Let* $N \in \mathbb{N}$, $f_1, \ldots, f_N \colon \mathbb{M}^n \to \mathbb{R}^+$ *be bounded, integrable functions, and let* $\Psi\colon (\mathbb{M}^n)^N \to \mathbb{R}^+$ *be bounded, measurable, such that the function*

$$\phi(t_1, \ldots, t_N) = \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} \Psi(x(t_1, u_1), \ldots, x(t_N, u_N)) \, du_1 \ldots du_N$$

is increasing in each coordinate. Then, using the notation from (3.3), *we have*

$$I_\Psi(f_1^\star, \ldots, f_N^\star) \geq I_\Psi(\|f_1\|_\infty \mathbb{1}_{B_1}, \ldots, \|f_N\|_\infty \mathbb{1}_{B_N}),$$

where B_i *is a geodesic ball around the origin* $e \in \mathbb{M}^n$ *such that* $\lambda_n(B_i) = \frac{\|f_i\|_{L^1(\mathbb{M}^n)}}{\|f_i\|_\infty}$.

Proof. We can assume without loss of generality, that already $f_i = f_i^\star$, since taking rearrangements neither changes L^1 nor L^∞ norms. Using (2.1), we obtain

$$I_\Psi(f_1,\dots,f_N) = \int_{\mathbb{M}^n}\dots\int_{\mathbb{M}^n}\Psi(x_1,\dots,x_N)\prod_{i=1}^N f_i(x_i)\,dx_1\dots dx_N,$$
$$= \int_{\mathbb{S}^{n-1}}\dots\int_{\mathbb{S}^{n-1}}\int_0^{R^{\mathbb{M}}}\dots\int_0^{R^{\mathbb{M}}}\Psi(x(t_1,u_1),\dots,x(t_N,u_N))$$
$$\times\prod_{i=1}^N f_i(x(t_i,u_i))\,\mathrm{sn}^{n-1}\,t_i\,dt_1\dots dt_N\,du_1\dots du_N$$

By the radial symmetry of f_i, we can write $\overline{f_i}(t_i) = f_i(x(t_i,u_i))$ to arrive at

$$I_\Psi(f_1,\dots,f_N) = \int_0^{R^{\mathbb{M}}}\dots\int_0^{R^{\mathbb{M}}}\phi(t_1,\dots,t_N)\prod_{i=1}^N\overline{f_i}(t_i)\,\mathrm{sn}^{n-1}\,t_i\,dt_1\dots dt_N.$$

Now, as in the proof of Proposition 3.4.1, applying Lemma 3.4.2 successively to each coordinate and noticing that

$$\lambda_n(B_i) = n\kappa_n\int_0^{A_i}\mathrm{sn}^{n-1}\,t\,dt = \frac{n\kappa_n}{\|f_i\|_\infty}\int_0^{R^{\mathbb{M}}}\tilde{f}_i(t)\,\mathrm{sn}^{n-1}\,t\,dt = \frac{\|f_i\|_{L^1(\mathbb{M}^n)}}{\|f_i\|_\infty},$$

where the $A_i \in \mathbb{R}^+$ come from the lemma, yields the statement. □

CHAPTER 4

Spherical centroid bodies

We introduce the spherical centroid body of a centrally-symmetric convex body in the Euclidean unit sphere. Two alternative definitions – one geometric, the other probabilistic in nature – are given and shown to lead to the same objects. The geometric approach is then used to establish a number of basic properties of spherical centroid bodies, while the probabilistic approach inspires the proof of a spherical analogue of the classical polar Busemann–Petty centroid inequality. The results in this chapter are published in joint work with Florian Besau, Peter Pivovarov, and Franz Schuster [Bes19].

4.1 Definition and basic properties

In Euclidean space, the boundary of the centroid body of an origin-symmetric convex body K is given by the (Euclidean) centroids of intersections of K with halfspaces in all directions (see e.g. [Gar06, Section 9.1] or [Sch14, Section 10.8]). We mimick this procedure on the unit sphere by replacing halfspaces with hemispheres and computing spherical centroids instead.

Definition. *For a convex body $K \subseteq \mathbb{S}^n$ which is centrally-symmetric with center $e \in \mathbb{S}^n$, we define its spherical centroid body $\mathit{\Gamma}_s K$ by*

$$\operatorname{bd} \mathit{\Gamma}_s K := \{c_s(K \cap \mathbb{S}^+_u) : u \in \mathbb{S}_e\}.$$

We will show that $\mathit{\Gamma}_s K$ is indeed a well defined proper spherically convex body which is centrally-symmetric with the same center as K.

In the linear setting, a probabilistic approach towards centroid bodies was first noted in [Pao12a], and can be described as follows. Given an origin-symmetric convex body $L \subseteq \mathbb{R}^n$ and $N \in \mathbb{N}$ independent random points $X_1, \dots, X_N$ distributed uniformly in L, define the (random) convex body

$$\mathit{\Gamma}(X_1, \dots, X_N) := \frac{1}{N} \sum_{i=1}^{N} [-X_i, X_i] = \operatorname{conv}\{c(\pm X_1, \dots, \pm X_N)\}, \tag{4.1}$$

where $[-X_i, X_i]$ denotes the line segment joining $\pm X_i$, the sum is the standard Minkowski addition, and $c(x_1, \dots, x_N)$ denotes the usual centroid of finitely many points in $\mathbb{R}^n$. The crucial observation from [Pao12a] is that $\mathit{\Gamma}(X_1, \dots, X_N)$ converges almost surely in the Hausdorff metric to the centroid body $\mathit{\Gamma} L$ as N tends to infinity.

Although there is no natural analogue of Minkowski addition on $\mathbb{S}^n$, both the convex hull and centroids of finite point sets do have natural counterparts. In order to mimic definition (4.1) on $\mathbb{S}^n$, we can therefore use the second equation in (4.1), but here we replace $-v$ with the geodesic reflection of v about $e \in \mathbb{S}^n$, that is, $v \mapsto v^e := -v + 2(v \cdot e)e$, and use the abbreviation $v^{(e)}$ for $\{v,v^e\}$.

Definition. *For a proper finite set $\{u_1, \dots, u_N\} \subseteq \mathbb{S}^n$ contained in $\operatorname{int} \mathbb{S}^+_e$, we define*

$$\Gamma_{s,e}(u_1, \dots, u_N) := \operatorname{conv}\left\{c_s\left(u_1^{(e)}, \dots, u_N^{(e)}\right)\right\}.$$

Since our main tool in this chapter will be gnomonic projection, we discuss in the first part of this section the definition and properties of weighted centroid bodies in linear vector spaces. The second part is devoted to spherical centroid bodies and their basic properties. We also establish a few auxiliary results required for the proof of Theorem 4.2.1.

We begin with a definition of weighted centroid bodies of arbitrary convex bodies in $\mathbb{R}^n$.

Definition. *For $\{x_1, \dots, x_N\} \subseteq \mathbb{R}^n$ and a positive function $f \colon \mathbb{R}^n \to \mathbb{R}^+$, define*

$$h(\Gamma_f(x_1, \dots, x_N), u) := \frac{1}{\sum_{i=1}^N f(x_i)} \sum_{i=1}^N |u \cdot f(x_i) x_i|. \tag{4.2}$$

For a finite Borel measure μ on $\mathbb{R}^n$ with positive density and $L \in \mathcal{K}(\mathbb{R}^n)$, define the μ-centroid body *of L by*

$$h(\Gamma_\mu L, u) := \frac{1}{\mu(L)} \int_L |u \cdot y| \, d\mu(y). \tag{4.3}$$

Note that, by our assumption on μ, $\Gamma_\mu L$ is an origin-symmetric convex body for every $L \in \mathcal{K}(\mathbb{R}^n)$. While $\Gamma_f(x_1, \dots, x_N)$ is, in general, always an origin-symmetric, compact, convex set, it has non-empty interior if and only if $\operatorname{span}\{x_1, \dots, x_N\} = \mathbb{R}^n$. It is also worth noting that when μ is taken to be Lebesgue measure, (4.3) defines Blaschke's classical centroid body (of the not necessarily origin-symmetric) body L. In the following, when $f \equiv 1$ in (4.2), we simply write $\Gamma(x_1, \dots, x_N)$ and use $h([-z,z],u) = |u \cdot z|$ for every $z \in \mathbb{R}^n$, to see that, in this case,

$$\Gamma(x_1, \dots, x_N) = \frac{1}{N} \sum_{i=1}^N [-x_i, x_i].$$

Our first goal is to relate weighted centroid bodies with weighted centroids. In the discrete case this is the content of the following lemma.

Lemma 4.1.1. *Let $\{x_1, \dots, x_N\} \subseteq \mathbb{R}^n$ be a finite subset and assume that $f \colon \mathbb{R}^n \to \mathbb{R}^+$ is even. Then*

$$\Gamma_f(x_1, \dots, x_N) = \operatorname{conv}\{c_f(\pm x_1, \dots, \pm x_N)\}.$$

Proof. Since for arbitrary $z_1, \dots, z_N \in \mathbb{R}^n$, we have

$$\sum_{i=1}^{N} [-z_i, z_i] = \operatorname{conv}\{\pm z_1 \pm \dots \pm z_N\},$$

we obtain

$$\Gamma_f(x_1, \dots, x_N) = \operatorname{conv}\left\{\frac{\pm f(x_1)x_1 \pm \dots \pm f(x_N)x_N}{\sum_{i=1}^N f(x_i)}\right\}.$$

But, since f is even, this is equal to $\operatorname{conv}\{c_f(\pm x_1, \dots, \pm x_N)\}$. □

In contrast to $\Gamma_f(x_1, \dots, x_N)$, which is, as a Minkowski sum of line segments, always a polytope, our next lemma shows that the boundary of $\Gamma_\mu L$ exhibits higher regularity. For the classical centroid body this was first proved by Petty [Pet61] and we follow his arguments closely (see also [Hua18, Theorem 1.2] for a recent variant). In order to state the result precisely, recall that a convex body L is said to be of class C^2_+ if the boundary of L is a C^2 submanifold of $\mathbb{R}^n$ with everywhere positive Gauß–Kronecker curvature.

Lemma 4.1.2. *Let μ be a finite Borel measure on $\mathbb{R}^n$ with positive bounded density and $L \in \mathcal{K}(\mathbb{R}^n)$. Then $\Gamma_\mu L$ is of class C^2_+. In particular, it is strictly convex.*

Proof. We first want to show that $h(\Gamma_\mu L, \cdot)$ is twice differentiable. To this end, we compute its directional derivative at $x \in \mathbb{R}^n$ in the direction $u \in \mathbb{S}^{n-1}$ by

$$\lim_{t \to 0^+} \frac{h(\Gamma_\mu L, x + tu) - h(\Gamma_\mu L, x)}{t} = \frac{1}{\mu(L)} \left(\int_{L \cap H_x^+} u \cdot y \, d\mu(y) - \int_{L \cap H_x^-} u \cdot y \, d\mu(y) \right).$$

Consequently, the gradient of $h(\Gamma_\mu L, \cdot)$ exists and is given by

$$\nabla h(\Gamma_\mu L, \cdot)(x) = \frac{1}{\mu(L)} \left(\int_{L \cap H_x^+} y \, d\mu(y) - \int_{L \cap H_x^-} y \, d\mu(y) \right).$$

In order to compute second derivatives at $\bar{x} \in \mathbb{R}^n$, we choose an orthonormal coordinate frame $\{e_1, \dots, e_n\}$ such that $\bar{x} = (0, \dots, 0, \bar{x}_n)^{\mathrm{T}}$, where $\bar{x}_n > 0$ (see, e.g., [Bus58, p. 57]). Since $\nabla h(\Gamma_\mu L, \cdot)(x)$ is 0-homogeneous in x, it follows that

$$\frac{\partial^2 h(\Gamma_\mu L, \cdot)}{\partial e_i \partial e_n}(\bar{x}) = 0$$

for $1 \le i \le n$. Letting $x = (0, \dots, 0, x_j, 0, \dots, 0, \bar{x}_n)^{\mathrm{T}}$ for $j < n$, we get for $i,j < n$,

$$\frac{\frac{\partial h(\Gamma_\mu L, \cdot)}{\partial e_i}(x) - \frac{\partial h(\Gamma_\mu L, \cdot)}{\partial e_i}(\bar{x})}{x_j} = \frac{2}{x_j \mu(L)} \left(\int_{L \cap H_x^+ \cap H_{\bar{x}}^-} y_i \, d\mu(y) - \int_{L \cap H_x^- \cap H_{\bar{x}}^+} y_i \, d\mu(y) \right). \tag{4.4}$$

In order to compute the limit $x_j \to 0$ in (4.4), we make the change of variables $y_1 = v_1$, …, $y_{n-1} = v_{n-1}$, and $y_n = v_j \tan v_n$. The Jacobian of this transformation is given by $J(v) = v_j \sec^2 v_n$. Note that it is negative on $L \cap H_x^- \cap H_{\bar{x}}^+$ when $x_j > 0$ and on $L \cap H_x^+ \cap H_{\bar{x}}^-$ when $x_j < 0$. Letting $\alpha := \arctan(|x_j|/\bar{x}_n)$,

$$H(v_n) := \begin{cases} y_n = (\tan v_n) y_j & \text{for } x_j < 0, \\ y_n = -(\tan v_n) y_j & \text{for } x_j > 0, \end{cases}$$

and $L^\pm(v_n) := L \cap H(v_n) \cap H_{e_j}^\pm$ for $v_n > 0$, the right hand side of (4.4) becomes

$$\frac{2}{\bar{x}_n \mu(L) \tan\alpha} \left(\int_0^\alpha \sec^2 v_n \, \varphi^+(v_n) \, dv_n + \int_0^\alpha \sec^2 v_n \, \varphi^-(v_n) \, dv_n \right), \tag{4.5}$$

where

$$\varphi^\pm(s) := \int_{L^\pm(s)} v_i v_j \, f_\mu(v_1, \dots, v_{n-1}, v_j \tan s) \, dv_1 \cdots dv_{n-1}$$

with f_μ being the density of μ. In order to compute the limit $\alpha \to 0$ in (4.5), we use that, by the mean value theorem, for every function ζ which is continuously differentiable near 0 such that $\zeta(0) = 0$ and every φ continuous near 0,

$$\lim_{\alpha \to 0} \frac{1}{\zeta(\alpha)} \int_0^\alpha \zeta'(s) \varphi(s) \, ds = \varphi(0).$$

Taking here $\zeta(s) := \tan s$, $\varphi = \varphi^\pm$, and letting $\alpha \to 0$ in (4.5) as well as changing back to the variables $y_1, \dots, y_{n-1}$, we obtain

$$\lim_{x_j \to 0} \frac{\frac{\partial h(\Gamma_\mu L, \cdot)}{\partial e_i}(x) - \frac{\partial h(\Gamma_\mu L, \cdot)}{\partial e_i}(\bar{x})}{x_j} = \frac{2}{\bar{x}_n \mu(L)} \int_{L \cap H_{\bar{x}}} y_i y_j \, f_\mu(y_1, \dots, y_{n-1}, 0) \, dy_1 \cdots dy_{n-1},$$

provided we can show that $\varphi^\pm$ is continuous near 0. But since $L \subseteq B_R$ for some Euclidean ball B_R of radius R in $\mathbb{R}^n$ and for every $s_0 \in (0, \epsilon)$, $L^\pm(s) \to L^\pm(s_0)$ in the Hausdorff metric in $\mathbb{R}^n$ as $s \to s_0$, it is not difficult to see that

$$|\varphi^\pm(s) - \varphi^\pm(s_0)| \to 0.$$

Letting $A := (h_{ij})_{i,j=1}^{n-1}$ denote the Hessian matrix of $h(\Gamma_\mu L, \cdot)$ at $\bar{x} \in \mathbb{R}^n$ (w.r.t. $\{e_1, \dots, e_{n-1}\}$), we can now conclude that for any $b \in \mathbb{R}^{n-1} \setminus \{0\}$,

$$b \cdot Ab = \frac{2}{\bar{x}_n \mu(L)} \int_{L \cap H_{\bar{x}}} (b \cdot y)^2 \, f_\mu(y_1, \dots, y_{n-1}, 0) \, dy_1 \cdots dy_{n-1} > 0,$$

that is, A is a positive-definite matrix. Since $\bar{x}$ was arbitrary, it is well known (cf. [Sch14, Section 2.5]), that this implies that $\Gamma_\mu L$ is of class C_+^2. □

The following lemma shows that the weighted centroid body with respect to an even

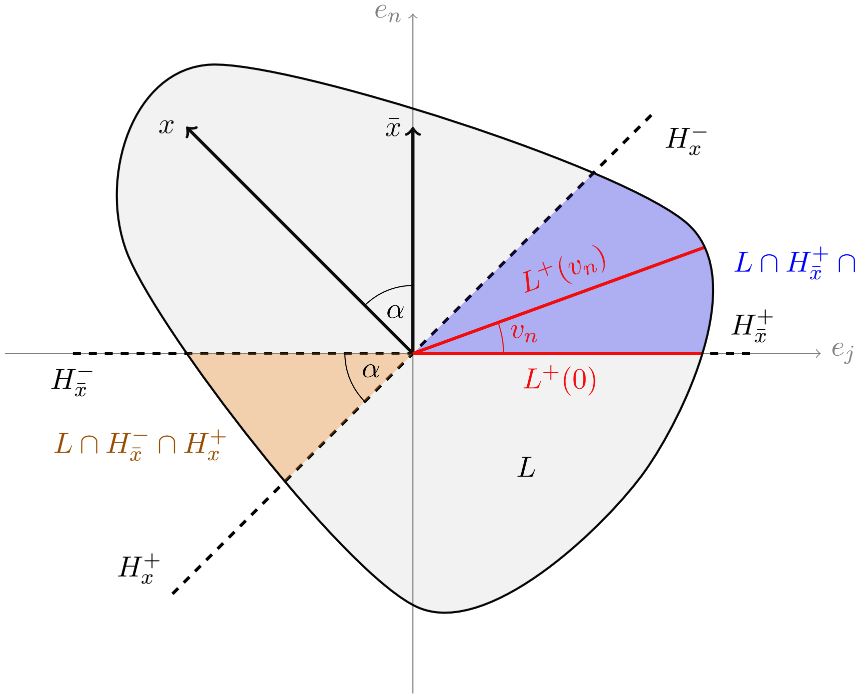

Figure 4.1: Sketch for the proof of Lemma 4.1.2

measure of an origin-symmetric convex body L can also be obtained geometrically.

Lemma 4.1.3. *Let μ be a finite even Borel measure on $\mathbb{R}^n$ with positive density and assume that $L \in \mathcal{K}(\mathbb{R}^n)$ is origin-symmetric. Then*

$$\operatorname{bd} \Gamma_\mu L = \{c_\mu(L \cap H_u^+) : u \in \mathbb{S}^{n-1}\}.$$

Proof. Since μ is even and L origin-symmetric, we have for $u \in \mathbb{S}^{n-1}$,

$$\begin{aligned} h(\Gamma_\mu L, u) &= \frac{1}{\mu(L)} \int_L |u \cdot y| \, d\mu(y) = \frac{2}{\mu(L)} \int_{L \cap H_u^+} u \cdot y \, d\mu(y) \\ &= u \cdot \frac{1}{\mu(L \cap H_u^+)} \int_{L \cap H_u^+} y \, d\mu(y) = u \cdot c_\mu(L \cap H_u^+). \end{aligned}$$

Thus, by the definition of support functions, $c_\mu(L \cap H_u^+) \in \operatorname{bd} \Gamma_\mu L$. Since $\Gamma_\mu L$ is strictly convex by Lemma 4.1.2, all boundary points are obtained in this way. □

We also note that if L in Lemma 4.1.3 is not origin-symmetric, then a similar computation shows that every boundary point of $\Gamma_\mu L$ is a convex combination of $c_\mu(L \cap H_u^+)$ and $-c_\mu(L \cap H_u^-)$ for $u \in \mathbb{S}^{n-1}$ (cf. [Gar06, Section 9.1]).

We now turn our focus towards spherical centroid bodies and first recall their definition:

For $K \in \mathcal{K}_c(\mathbb{S}^n)$ with center $e \in \mathbb{S}^n$, its spherical centroid body is defined by

$$\operatorname{bd} \mathit{\Gamma}_s K := \{c_s(K \cap \mathbb{S}^+_u) : u \in \mathbb{S}_e\}.$$

Note that since K has nonempty interior, $c_s(K \cap \mathbb{S}^+_u)$ exists for every $u \in \mathbb{S}_e$. Moreover, since $K \subseteq \mathbb{S}^+_e$, Proposition 2.2.6 (c) implies that $\operatorname{bd} \mathit{\Gamma}_s K$ is contained in $\operatorname{int} \mathbb{S}^+_e$ and, hence, we can consider its gnomonic projection.

Proposition 4.1.4. *Let $K \in \mathcal{K}_c(\mathbb{S}^n)$ have center $e \in \mathbb{S}^n$ and let $g_e : \operatorname{int} \mathbb{S}^+_e \to \mathbb{R}^n$ denote the gnomonic projection. Then*

$$g_e(\operatorname{bd} \mathit{\Gamma}_s K) = \operatorname{bd} \mathit{\Gamma}_{\widehat{\tau}}\, g_e(K \cap \operatorname{int} \mathbb{S}^+_e).$$

Proof. Let us first assume that K is proper, that is, $K = K \cap \operatorname{int} \mathbb{S}^+_e$. Then, by Lemma 2.2.5, we have

$$g_e(\operatorname{bd} \mathit{\Gamma}_s K) = \{c_{\widehat{\tau}}(g_e(K \cap \mathbb{S}^+_u)) \colon u \in \mathbb{S}_e\}.$$

But since $g_e(K \cap \mathbb{S}^+_u) = g_e(K) \cap H^+_u$ for every $u \in \mathbb{S}_e$, $\widehat{\tau}$ is even, and $g_e(K)$ is origin-symmetric, it follows from Lemma 4.1.3 that

$$g_e(\operatorname{bd} \mathit{\Gamma}_s K) = \{c_{\widehat{\tau}}(g_e(K) \cap H^+_u) \colon u \in \mathbb{S}_e\} = \operatorname{bd} \mathit{\Gamma}_{\widehat{\tau}}\, g_e(K).$$

Now, if K is not proper, then $g_e(K \cap \operatorname{int} \mathbb{S}^+_e)$ is a closed, convex, and origin-symmetric set in $\mathbb{R}^n_e$ with nonempty interior which is unbounded. However, since $\widehat{\tau}$ has finite first moments, that is, $\int_{\mathbb{R}^n} y_i \, d\widehat{\tau}(y) < \infty$ for all $1 \le i \le n$, (4.3) still makes sense and defines a convex body $\mathit{\Gamma}_{\widehat{\tau}}\, g_e(K \cap \operatorname{int} \mathbb{S}^+_e)$ in $\mathbb{R}^n_e$.

Moreover, since the density function ψ of $\widehat{\tau}$ is radially symmetric, radially decreasing, and satisfies

$$\int_{\mathbb{R}^{n-1}} y_i y_j \, \psi(y_1, \ldots, y_{n-1}, 0) \, dy_1 \cdots dy_{n-1} < \infty$$

for all $1 \le i,j \le n-1$, it is not difficult to show that Lemma 4.1.2 also holds for $\mathit{\Gamma}_{\widehat{\tau}}\, g_e(K \cap \operatorname{int} \mathbb{S}^+_e)$ (the key is to prove continuity near 0 of the respective functions $\varphi^{\pm}$ from (4.5)), and, therefore, so does Lemma 4.1.3. Consequently, since removing sets of measure zero does not affect centroid computations, the arguments from the first part of the proof yield the desired relation also for nonproper K. □

As a consequence of Proposition 4.1.4 and the properties of the gnomonic projection, we obtain that the spherical centroid body map is, in fact, well defined.

Corollary 4.1.5. *Let $K \in \mathcal{K}_c(\mathbb{S}^n)$ have center $e \in \mathbb{S}^n$ and let $g_e : \operatorname{int} \mathbb{S}^+_e \to \mathbb{R}^n$ denote the gnomonic projection. Then $\mathit{\Gamma}_s K \in \mathcal{K}_c(\mathbb{S}^n)$ is proper and has center e. Moreover,*

$$g_e(\mathit{\Gamma}_s K) = \mathit{\Gamma}_{\widehat{\tau}}\, g_e(K \cap \operatorname{int} \mathbb{S}^+_e).$$

Proof. Since $\mathit{\Gamma}_{\widehat{\tau}}L$ is an origin-symmetric convex body in $\mathbb{R}^n$ for any (possibly unbounded) closed, convex set $L \subseteq \mathbb{R}^n$ with nonempty interior, the statement follows from Proposition 4.1.4 and Lemma 2.2.3. □

With the following proposition we collect several basic properties of the spherical centroid body map.

Proposition 4.1.6. *The spherical centroid body map $\mathit{\Gamma}_s\colon \mathcal{K}_c(\mathbb{S}^n) \to \mathcal{K}_c(\mathbb{S}^n)$ has the following properties:*

(a) It is $\mathrm{O}(n+1)$-equivariant, that is, $\mathit{\Gamma}_s(\vartheta K) = \vartheta \mathit{\Gamma}_s K$ for all $\vartheta \in \mathrm{O}(n+1)$;

(b) It is continuous;

(c) It is injective on bodies of equal spherical volume;

(d) $\mathit{\Gamma}_s K$ is of class C^2_+ for every $K \in \mathcal{K}_c(\mathbb{S}^n)$.

Proof. Property (a) is an immediate consequence of Proposition 2.2.6 (b) and the definition of $\mathit{\Gamma}_s K$.

In order to prove (b), let $K_m, K \in \mathcal{K}_c(\mathbb{S}^n)$ such that $K_m \to K$ in the spherical Hausdorff metric. In addition, let us first also assume that all K_m and K have the same center $e \in \mathbb{S}^n$. In this case, it follows that $K_m \cap \mathbb{S}^+_u \to K \cap \mathbb{S}^+_u$ for all $u \in \mathbb{S}_e$ and thus, by Proposition 2.2.6 (a),

$$c_s(K_m \cap \mathbb{S}^+_u) \to c_s(K \cap \mathbb{S}^+_u). \tag{4.6}$$

Now note the following two consequences of (4.6):

(i) For every $v = c_s(K \cap \mathbb{S}^+_u) \in \operatorname{bd} \mathit{\Gamma}_s K$, there exists a sequence $v_m \in \operatorname{bd} \mathit{\Gamma}_s K_m$ such that $v_m \to v$, namely $v_m = c_s(K_m \cap \mathbb{S}^+_u)$.

(ii) For every convergent subsequence $v_{m_l} \to v$ with $v_{m_l} \in \operatorname{bd} \mathit{\Gamma}_s K_m$, we have $v \in \operatorname{bd} \mathit{\Gamma}_s K$. Indeed, since $v_{m_l} = c_s(\mathbb{S}^+_{u_{m_l}} \cap K_{m_l})$ for some $u_{m_l} \in \mathbb{S}_e$ and $\mathbb{S}_e$ is compact, we find a subsequence (which we again call u_{m_l}) such that $u_{m_l} \to u \in \mathbb{S}_e$ and, thus,

$$v = \lim_{l \to \infty} c_s\left(\mathbb{S}^+_{u_{m_l}} \cap K_{m_l}\right) = c_s(\mathbb{S}^+_u \cap K) \in \operatorname{bd} \mathit{\Gamma}_s K.$$

Moreover, the sequence $\operatorname{bd} \mathit{\Gamma}_s K_m \subset \mathbb{S}^n$ is bounded in $\mathbb{R}^{n+1}$.

It is well known (cf. [Sch14, p. 69]) that (i) and (ii) imply $\operatorname{bd} \mathit{\Gamma}_s K_m \to \operatorname{bd} \mathit{\Gamma}_s K$ in the Hausdorff metric in $\mathbb{R}^{n+1}$. Consequently, by Lemma 2.2.2 (a) and (b), $\mathit{\Gamma}_s K_m \to \mathit{\Gamma}_s K$ in the spherical Hausdorff metric.

It remains to settle the case where the bodies K_m have center $e_m \in \mathbb{S}^n$ and K has center $e \in \mathbb{S}^n$. Clearly, the convergence $K_m \to K$ implies that $e_m \to e$ as $m \to \infty$. In the following, we make use (twice) of the fact that if $\vartheta_m, \vartheta \in \mathrm{O}(n+1)$ and $K_m, K \in \mathcal{K}_c(\mathbb{S}^n)$, then $\vartheta_m \to \vartheta$ and $K_m \to K$ imply $\vartheta_m K_m \to \vartheta K$ in the spherical Hausdorff metric. To apply this, note that since $e_m \to e$, there exists a sequence $\vartheta_m \in \mathrm{O}(n+1)$ such that

$\vartheta_m e_m = e$ and $\vartheta_m \to \mathrm{Id}$. Hence, $\vartheta_m K_m \to K$ and, since all $\vartheta_m K_m$ have center e, property (a) and the first part of the proof of (b) imply

$$\vartheta_m \mathit{\Gamma}_s K_m = \mathit{\Gamma}_s(\vartheta_m K_m) \to \mathit{\Gamma}_s K.$$

Making use a second time of the above fact, now for $\vartheta_m^{-1} \to \mathrm{Id}$, yields $\mathit{\Gamma}_s K_m \to \mathit{\Gamma}_s K$ which completes the proof of (b).

In order to prove the injectivity property (c), let $K, L \in \mathcal{K}_c(\mathbb{S}^n)$ such that $\mathit{\Gamma}_s K = \mathit{\Gamma}_s L$ and assume w.l.o.g. that $\tau(K) \leq \tau(L)$. Since $\mathit{\Gamma}_s K$ and $\mathit{\Gamma}_s L$ have the same center as K and L, respectively, it follows that K and L have the same center, say $e \in \mathbb{S}^n$. Moreover, by using polar coordinates, we have

$$h(\mathit{\Gamma}_{\widehat{\tau}}\, g_e(K \cap \operatorname{int} \mathbb{S}^+_e),u) = \int_{\mathbb{S}^{n-1}} |u \cdot v| \, \frac{1}{\tau(K)} \int_0^{\rho_{g_e(K\cap \operatorname{int}\mathbb{S}^+_e)}(v)} \frac{r^n}{(1+r^2)^{\frac{n+2}{2}}}\, dr\, dv,$$

where $\rho_{g_e(K\cap \operatorname{int}\mathbb{S}^+_e)}$ denotes the (possibly infinite) radial function of $K \cap \operatorname{int} \mathbb{S}^+_e$. Hence, by our assumption that $\mathit{\Gamma}_s K = \mathit{\Gamma}_s L$, Corollary 4.1.5, and the injectivity of the spherical cosine transform on even functions (cf. [Gar06, Theorem C.2.1]), we conclude that

$$\frac{1}{\tau(K)} \int_0^{\rho_{g_e(K\cap \operatorname{int}\mathbb{S}^+_e)}(v)} \frac{r^n}{(1+r^2)^{\frac{n+2}{2}}}\, dr = \frac{1}{\tau(L)} \int_0^{\rho_{g_e(L\cap \operatorname{int}\mathbb{S}^+_e)}(v)} \frac{r^n}{(1+r^2)^{\frac{n+2}{2}}}\, dr$$

for all $v \in \mathbb{S}^{n-1}$. Thus, since $t \to \int_0^t r^n(1+r^2)^{-\frac{n+2}{2}}\, dr$ is strictly increasing, it follows that

$$\rho_{g_e(K\cap \operatorname{int}\mathbb{S}^+_e)}(v) \leq \rho_{g_e(L\cap \operatorname{int}\mathbb{S}^+_e)}(v),$$

for all $v \in \mathbb{S}^{n-1}$ or, equivalently, $K \subseteq L$. Hence, if K and L have equal spherical volume, they must coincide.

Finally, for the proof of (d) assume that $K \in \mathcal{K}_c(\mathbb{S}^n)$ has center $e \in \mathbb{S}^n$. Since the restriction of g_e to any spherical cap of radius $\alpha < \frac{\pi}{2}$ is a diffeomorphism onto some Euclidean ball in $\mathbb{R}^n$, the boundary of $\mathit{\Gamma}_s K$ is a C^2 submanifold by Lemma 4.1.2 (and its extension to unbounded convex sets discussed in the proof of Proposition 4.1.4). Moreover, it follows from [Bes16b, Lemma 4.4] that the spherical Gauß–Kronecker curvature of $\mathit{\Gamma}_s K$ at $u \in \mathbb{S}^n$ vanishes precisely when the one of $\mathit{\Gamma}_{\widehat{\tau}}\, g_e(K \cap \operatorname{int} \mathbb{S}^+_e)$ vanishes at $g_e(u) \in \mathbb{R}^n$. Hence, Lemma 4.1.2 and its extension complete the proof. □

Before we continue, we remark that, like Blaschke's classical centroid body map (see [Lut90]), it is not difficult to see that $\mathit{\Gamma}_s$ is *not* monotone under set inclusion.

In the last part of this section, we establish a couple of auxiliary results concerning discrete spherical centroid bodies defined above,

$$\mathit{\Gamma}_{s,e}(u_1,\ldots,u_N) = \operatorname{conv}\left\{c_s\left(u_1^{(e)},\ldots,u_N^{(e)}\right)\right\},$$

which are used in Theorem 4.2.1 to approximate spherical centroid bodies of convex bodies.

(Recall that here, $\{u_1, \dots, u_N\} \subseteq \operatorname{int} \mathbb{S}^+_e$ and $u^{(e)} = \{u, u^e\}$.)

Note that, by definition and Lemma 2.2.6, $\mathit{\Gamma}_{s,e}(u_1, \dots, u_N) \in \mathcal{K}_c(\mathbb{S}^n)$ has center e and is proper. Lemma 2.2.6 also implies that the map $\mathit{\Gamma}_{s,e}$ is continuous and $\mathrm{O}(n+1)$-equivariant. Moreover, as a direct consequence of Lemma 2.2.3 (d), Lemma 2.2.5, and Lemma 4.1.1 we obtain for its gnomonic projection the following.

Corollary 4.1.7. *For $e \in \mathbb{S}^n$, let $g_e : \operatorname{int} \mathbb{S}^+_e \to \mathbb{R}^n$ denote the gnomonic projection and $\{u_1, \dots, u_N\} \subseteq \operatorname{int} \mathbb{S}^+_e$. Then*

$$g_e(\mathit{\Gamma}_{s,e}(u_1, \dots, u_N)) = \mathit{\Gamma}_\phi(g(u_1), \dots, g(u_N)).$$

Recall that our definition of $\mathit{\Gamma}_{s,e}(u_1, \dots, u_N)$ was motivated by relation (4.1) for discrete centroid bodies in a linear vector space. However, in the linear setting, there is an alternative way to express these $\mathit{\Gamma}(x_1, \dots, x_N)$, for $x_1, \dots, x_N \in \mathbb{R}^n$, namely,

$$\mathit{\Gamma}(x_1, \dots, x_N) = \frac{1}{N} \sum_{i=1}^N [-x_i, x_i] = \{c(y_1, \dots, y_N) : y_i \in [-x_i, x_i], 1 \le i \le N\}.$$

By mimicking this approach on the sphere, we define, $\{u_1, \dots, u_N\} \subseteq \operatorname{int} \mathbb{S}^+_e$,

$$\widetilde{\mathit{\Gamma}}_{s,e}(u_1, \dots, u_N) := \{c_s(v_1, \dots, v_N) : v_i \in [u_i^e, u_i], 1 \le i \le N\},$$

where $[u_i^e, u_i]$ denotes the geodesic segment connecting u_i^e and the geodesic reflection of u_i about e. These new sets are, in general, not spherically convex. However, there is the following interesting relation between them and $\mathit{\Gamma}_{s,e}(u_1, \dots, u_N)$.

Proposition 4.1.8. *For $e \in \mathbb{S}^n$ and any $\{u_1, \dots, u_N\} \subseteq \operatorname{int} \mathbb{S}^+_e$, we have*

$$\mathit{\Gamma}_{s,e}(u_1, \dots, u_N) = \operatorname{conv} \widetilde{\mathit{\Gamma}}_{s,e}(u_1, \dots, u_N).$$

Proof. Let $g_e : \operatorname{int} \mathbb{S}^+_e \to \mathbb{R}^n$ denote gnomonic projection. Then, by Lemma 2.2.3, it suffices to prove that

$$g_e(\mathit{\Gamma}_{s,e}(u_1, \dots, u_N)) = \operatorname{conv} g_e(\widetilde{\mathit{\Gamma}}_{s,e}(u_1, \dots, u_N)).$$

But, by Lemma 4.1.7 and Lemma 2.2.5, this is equivalent to

$$\mathit{\Gamma}_\phi(x_1, \dots, x_N) = \operatorname{conv} \{c_\phi(y_1, \dots, y_N) : y_i \in [-x_i, x_i], 1 \le i \le N\}, \tag{4.7}$$

where $\phi(x) = (1 + \|x\|^2)^{-\frac{1}{2}}$ and $x_i = g_e(u_i)$, $1 \le i \le N$. In order to prove (4.7), note that, by Lemma 4.1.1,

$$\begin{aligned} \mathit{\Gamma}_\phi(x_1, \dots, x_N) &= \operatorname{conv} \{c_\phi(\pm x_1, \dots, \pm x_N)\} \\ &\subseteq \operatorname{conv} \{c_\phi(y_1, \dots, y_N) : y_i \in [-x_i, x_i], 1 \le i \le N\}. \end{aligned}$$

Thus, it only remains to prove the reverse inclusion. To this end, recall that for $z_1, \dots, z_N \in$

$\mathbb{R}^n$ and $v \in \mathbb{S}^n$,

$$h(\Gamma_\phi(z_1,\dots,z_N),v) = \frac{1}{\sum_{i=1}^N \phi(z_i)} \sum_{i=1}^N \phi(z_i)|v\cdot z_i|.$$

Using $\nabla\phi(x) = -\phi(x)^3 x$, a straightforward computation yields

$$\frac{d}{d\lambda}\bigg|_{\lambda=1} h(\Gamma_\phi(\lambda z_1,\dots,z_N),v) = \frac{\|z_1\|^2\phi(z_1)^3}{\left(\sum_{i=1}^N \phi(z_i)\right)^2} \sum_{i=1}^N \phi(z_i)\left[\frac{|v\cdot z_1|}{\|z_1\|^2} + |v\cdot z_i|\right] > 0.$$

Repeating this computation for $z_2,\dots,z_N$ shows that for any $v \in \mathbb{S}^{n-1}$, the function $(z_1,\dots,z_N) \mapsto h(\Gamma_\phi(z_1,\dots,z_N),\theta)$ is radially increasing in every coordinate. By applying this fact to each coordinate successively, we obtain for all $y_i \in [0,x_i]$, $1 \le i \le N$, and every $v \in \mathbb{S}^{n-1}$,

$$h(\Gamma_\phi(y_1,\dots,y_N),v) \le h(\Gamma_\phi(x_1,\dots,x_N),v),$$

that is, $\Gamma_\phi(y_1,\dots,y_N) \subseteq \Gamma_\phi(x_1,\dots,x_N)$. But, since both sets are origin-symmetric, this inclusion also holds for all $y_i \in [-x_i,x_i]$, $1 \le i \le N$. In particular,

$$c_\phi(y_1,\dots,y_N) \in \Gamma_\phi(y_1,\dots,y_N) \subseteq \Gamma_\phi(x_1,\dots,x_N).$$

Since $\Gamma_\phi(x_1,\dots,x_N)$ is convex, this concludes the proof. □

In the next section, we present Theorem 4.2.1, showing that the discrete centroid bodies $\Gamma_{s,e}(u_1,\dots,u_N)$ approximate $\Gamma_s K$, when $u_1,\dots,u_N$ are chosen randomly from K. By Proposition 4.1.8, the same holds true for the bodies $\operatorname{conv}\widetilde{\Gamma}_{s,e}(u_1,\dots,u_N)$. Our final result of this section is a critical ingredient in the proof of these facts and based on a variant of the proof of [Pao12a, Corollary 5.2].

Lemma 4.1.9. *Let μ,ν be finite, absolutely continuous Borel measures on $\mathbb{R}^n$ and let f denote the density of μ with respect to ν. Then, for $L \in \mathcal{K}(\mathbb{R}^n)$ and independent random vectors $X_1,\dots,X_N$ on $\mathbb{R}^n$, identically distributed according to $\frac{\mathbb{1}_L}{\nu(L)}\,d\nu$, we have*

$$\Gamma_f(X_1,\dots,X_N) \to \Gamma_\mu L$$

almost surely in the Hausdorff metric as $N \to \infty$.

Proof. By the strong law of large numbers (see, e.g., [Dud02, Theorem 8.3.5]), we have

$$\frac{1}{N}\sum_{i=1}^N f(X_i) \to \frac{1}{\nu(L)}\int_L f(x)\,d\nu(x) = \frac{\mu(L)}{\nu(L)}$$

and

$$\frac{1}{N}\sum_{i=1}^{N}|y\cdot X_i|f(X_i)\to\frac{1}{\nu(L)}\int_L|y\cdot x|f(x)\,d\nu(x)=\frac{1}{\nu(L)}\int_L|y\cdot x|\,d\mu(x),$$

almost surely for every $y\in\mathbb{R}^n$ as $N\to\infty$. Since the product of almost surely convergent sequences of random variables converges almost surely to the product of their respective limits, we conclude

$$\begin{aligned}h(\Gamma_f(X_1,\dots,X_N),y)&=\frac{1}{\sum_{i=1}^N f(X_i)}\sum_{i=1}^{N}|y\cdot X_i|f(X_i)\\&\to\frac{1}{\mu(L)}\int_L|y\cdot x|\,d\mu(x)=h(\Gamma_\mu(L),y),\end{aligned}$$

almost surely for every $y\in\mathbb{R}^n$. This proves the desired statement, since pointwise convergence of support functions is equivalent to the convergence of the respective bodies in the Hausdorff metric (see, e.g., [Sch14, p. 54]). □

Finally, we note that Lemma 4.1.9 holds true for any closed and unbounded convex set L in $\mathbb{R}^n$ as long as μ has finite first moments (so that $\Gamma_\mu L$ exists).

4.2 Statement of results

Our first result concerns random approximation of spherical centroid bodies by spherical polytopes.

Theorem 4.2.1. *Let $K\subseteq\mathbb{S}^n$ be a spherically convex body which is centrally-symmetric with center $e\in\mathbb{S}^n$. If $U_1,\dots,U_N$ are independent random unit vectors distributed uniformly in K, then*

$$\Gamma_{s,e}(U_1,\dots,U_N)\to\Gamma_s K$$

almost surely in the spherical Hausdorff metric as N tends to infinity.

Secondly, we obtain an isoperimetric inequality for the polar of the spherical centroid body. For a fixed $e\in\mathbb{S}^n$, denote by τ the absolutely continuous (w.r.t. to σ) measure on $\mathbb{S}^n$ with density $d\tau(u)=|e\cdot u|\,d\sigma(u)$.

Theorem 4.2.2. *If $K\subseteq\mathbb{S}^n$ is a spherically convex body which is centrally-symmetric with center $e\in\mathbb{S}^n$, then*

$$\sigma(\Gamma_s^*K)\le\sigma(\Gamma_s^*C_K^\tau),$$

where C_K^τ is the spherical cap centered at e such that $\tau(C_K^\tau)=\tau(K)$.

4.3 Proofs

The strategy of our proofs will be to establish a corresponding result in Euclidean space and apply gnomonic projection to transfer it to the sphere. We begin with Theorem 4.2.1.

Proof of Theorem 4.2.1. Let us first assume that K is proper, that is, $K \subsetneq \operatorname{int} \mathbb{S}^+_e$, and let again $g_e : \operatorname{int} \mathbb{S}^+_e \to \mathbb{R}^n$ denote the gnomonic projection. Putting $X_i := g_e(U_i)$, $1 \le i \le N$, the independence and uniform distribution of the U_i together with Lemma 2.2.4 (a), implies that the X_i are independent random vectors in $\mathbb{R}^n$, identically distributed according to

$$\frac{\mathbb{1}_{g_e(K)}}{\widehat{\sigma}(g_e(K))}\, d\widehat{\sigma}.$$

Moreover, by Lemma 2.2.4, we have

$$\phi(x) = (1+\|x\|^2)^{-\frac{1}{2}} = \frac{(1+\|x\|^2)^{-\frac{n+2}{2}}}{(1+\|x\|^2)^{-\frac{n+1}{2}}} = \frac{d\widehat{\tau}}{d\widehat{\sigma}}.$$

Hence, by Lemma 4.1.9, we obtain

$$\mathit{\Gamma}_\phi(X_1, \dots, X_N) \to \mathit{\Gamma}_{\widehat{\tau}}\,(g_e(K))\,,$$

almost surely in the Hausdorff metric as $N \to \infty$. Applying now g_e^{-1} and using Lemma 2.2.3 and Corollaries 4.1.5 and 4.1.7, we arrive at the desired statement

$$\mathit{\Gamma}_{s,e}(U_1, \dots, U_N) \to \mathit{\Gamma}_s K.$$

If K is not proper, then we still have $K \subseteq \mathbb{S}^+_e$, and, since $K \setminus \operatorname{int} \mathbb{S}^+_e$ is a set of measure zero, we may assume that $U_1, \dots, U_N$ lie in $\operatorname{int} \mathbb{S}^+_e$. Thus, as in the first part of the proof, it follows from Lemma 4.1.9 and the remark directly following it, that

$$\mathit{\Gamma}_\phi(X_1, \dots, X_N) \to \mathit{\Gamma}_{\widehat{\tau}}\big(g_e(K \cap \operatorname{int} \mathbb{S}^+_e)\big)\,,$$

where, as before, $X_i := g_e(U_i)$, $1 \le i \le N$. Applying g_e^{-1} and using Corollaries 4.1.5 and 4.1.7 yield again the desired result. □

As an immediate consequence of Theorem 4.2.1 and Proposition 4.1.8, we note the following:

Corollary 4.3.1. *Let $K \in \mathcal{K}_c(\mathbb{S}^n)$ have center $e \in \mathbb{S}^n$. If $U_1, \dots, U_N$ are independent random unit vectors distributed uniformly in K, then*

$$\operatorname{conv} \widetilde{\mathit{\Gamma}}_{s,e}(U_1, \dots, U_N) \to \mathit{\Gamma}_s K$$

almost surely in the spherical Hausdorff metric as $N \to \infty$.

We turn to the proof of Theorem 4.2.2 which is based on the following proposition of independent interest.

Proposition 4.3.2. *Let μ, ν be finite, absolutely continuous Borel measures on $\mathbb{R}^n$ such that their densities are radially symmetric and radially decreasing. Then, for an origin-symmetric convex body $L \in \mathcal{K}(\mathbb{R}^n)$, we have*

$$\nu(\mathit{\Gamma}^\circ_\mu L) \le \nu(\mathit{\Gamma}^\circ_\mu B^\mu_L),$$

where B_L^μ is a Euclidean ball around the origin, such that $\mu(B_L^\mu) = \mu(L)$.

Proof. Let $X_1, \dots, X_N$ and $Z_1, \dots, Z_N$ be two families of independent random vectors in $\mathbb{R}^n$ such that each family is identically distributed according to

$$\frac{\mathbb{1}_L(x)}{\mu(L)} d\mu(x) \qquad \text{and} \qquad \frac{\mathbb{1}_{B_L^\mu}(x)}{\mu(B_L^\mu)} d\mu(x),$$

respectively, and denote by $f_\mu \colon \mathbb{R}^n \to \mathbb{R}^+$ the density of μ. Using the notation introduced in the paragraph preceeding Theorem 3.2.2, we have

$$\Gamma(x_1, \dots, x_N) = \frac{1}{N} \sum_{i=1}^N [-x_i, x_i] = \frac{1}{N} (x_1, \dots, x_N) B_\infty^N.$$

Thus, we can invoke Theorem 3.2.2, with $f_i(x) = \frac{\mathbb{1}_L(x)}{\mu(L)} f_\mu(x)$, and Proposition 3.4.1, with $f_i = f_\mu$ and $L_i = L$, to obtain

$$\mathbb{E}\left[\nu\left(\Gamma^\circ(X_1, \dots, X_N)\right)\right] \leq \mathbb{E}\left[\nu\left(\Gamma^\circ(Z_1, \dots, Z_N)\right)\right]. \tag{4.8}$$

Now, by Proposition 4.1.9, we know that $\Gamma(X_1, \dots, X_N) \to \Gamma_\mu(L)$ almost surely in the Hausdorff metric as $N \to \infty$. Moreover, since taking the polar body and ν are continuous on origin-symmetric convex bodies in $\mathbb{R}^n$ (see, e.g., [Cor15, Lemma 5.2]), we also have that

$$\nu\left(\Gamma^\circ(X_1, \dots, X_N)\right) \to \nu\left(\Gamma_\mu^\circ L\right)$$

almost surely as $N \to \infty$. Since $\Gamma_\mu L$ has nonempty interior, there exists $r > 0$ such that, for N large enough, we have $rB_2^n \subseteq \Gamma(X_1, \dots, X_N)$ almost surely and, hence, $\nu\left(\Gamma^\circ(X_1, \dots, X_N)\right) \leq \nu\left(\frac{1}{r} B_2^n\right)$ almost surely. Therefore, by the theorem of dominated convergence, we conclude that

$$\mathbb{E}\left[\nu\left(\Gamma^\circ(X_1, \dots, X_N)\right)\right] \to \mathbb{E}\left[\nu\left(\Gamma_\mu^\circ L\right)\right] = \nu\left(\Gamma_\mu^\circ L\right)$$

and, by the same arguments,

$$\mathbb{E}\left[\nu\left(\Gamma^\circ(Z_1, \dots, Z_N)\right)\right] \to \mathbb{E}\left[\nu\left(\Gamma_\mu^\circ B_L^\mu\right)\right] = \nu\left(\Gamma_\mu^\circ B_L^\mu\right).$$

Thus, by letting $N \to \infty$ in (4.8), we obtain the desired inequality. □

We are now in a position to prove Theorem 4.2.2:

Proof of Theorem 4.2.2. Let us first assume that K is proper, that is, $K \subseteq \operatorname{int} \mathbb{S}_e^+$, and let again $g_e : \operatorname{int} \mathbb{S}_e^+ \to \mathbb{R}^n$ denote gnomonic projection. Since, by Lemma 2.2.4, the push-forwards $g_e \# \sigma =: \widehat{\sigma}$ and $g_e \# \tau =: \widehat{\tau}$ have radially symmetric and radially decreasing densities, an application of Proposition 4.3.2 to the origin-symmetric convex body $g_e(K)$

yields

$$\widehat{\sigma}\left(\Gamma_{\widehat{\tau}}^{\circ} g_e(K)\right) \leq \widehat{\sigma}\left(\Gamma_{\widehat{\tau}}^{\circ} B_{g_e(K)}^{\widehat{\tau}}\right) = \widehat{\sigma}\left(\Gamma_{\widehat{\tau}}^{\circ} g_e(C_K^{\tau})\right). \tag{4.9}$$

Thus, using Corollary 4.1.5, the desired inequality

$$\sigma(\Gamma_s^* K) \leq \sigma(\Gamma_s^* C_K^{\tau})$$

follows by applying g_e^{-1} to (4.9).

If $K \in \mathcal{K}_c(\mathbb{S}^n)$ is nonproper, then we still have $K \subseteq \mathbb{S}_e^+$, and we can choose a sequence $K_m \in \mathcal{K}_c(\mathbb{S}^n)$ of proper convex bodies with center e such that $K_m \to K$ in the spherical Hausdorff metric. Moreover, by the first part of the proof, we know that $\sigma(\Gamma_s^* K_m) \leq \sigma(\Gamma_s^* C_{K_m}^{\tau})$ for all $m \in \mathbb{N}$. But, since σ, τ, and Γ_s are continuous on $\mathcal{K}_c(\mathbb{S}^n)$ (the latter by Proposition 4.1.6 (b)), and taking the polar is continuous on proper bodies in $\mathcal{K}_c(\mathbb{S}^n)$ (recall that $\Gamma_s L$ is proper for all $L \in \mathcal{K}_c(\mathbb{S}^n)$ by Corollary 4.1.5), we obtain the desired inequality by letting $m \to \infty$. □

We conclude this chapter with three remarks concerning possible extensions and improvements of Theorem 4.2.2. We begin by discussing a version for not necessarily centrally-symmetric bodies. To this end let $K \in \mathcal{K}(\mathbb{S}^n)$ be proper and assume that $K \subseteq \operatorname{int} \mathbb{S}_w^+$ for some $w \in \mathbb{S}^n$ or, equivalently, that $w \in -\operatorname{int} K^*$. Moreover, let τ_w denote the absolutely continuous measure on $\mathbb{S}^n$ with density $d\tau_w(u) = |u \cdot w|\, d\sigma(u)$ and let $\widehat{\tau}_w := g_w \# \tau_w$ denote its push-forward under gnomonic projection $g_w : \operatorname{int} \mathbb{S}_w^+ \to \mathbb{R}_{w,0}^n$. If we define the spherical centroid body of K by

$$g_w(\Gamma_s K) = \Gamma_{\widehat{\tau}_w} g_w(K), \tag{4.10}$$

then the arguments leading up to Theorem 4.2.2 yield the inequality

$$\sigma(\Gamma_s^* K) \leq \sigma(\Gamma_s^* C_K^{\tau_w}).$$

However, we are reluctant to use (4.10) as definition for $\Gamma_s K$, since, on the one hand, it is not intrinsic and, on the other hand, there is the question what would be a natural choice for $w \in -\operatorname{int} K^*$? Of course, this choice should coincide with the center for centrally-symmetric bodies, like, for example, the centroid $c_s(K)$. We do not know whether $c_s(K) \in -\operatorname{int} K^*$ for every proper $K \in \mathcal{K}(\mathbb{S}^n)$, apart from special cases:

Lemma 4.3.3. *Let $K \in \mathcal{K}(\mathbb{S}^n)$. If $K \subseteq -K^*$, then $c_s(K) \in -K^*$.*

This may seem trivial, but if we are able to bound K from above by a self-dual body ($L = -L^*$), then $K \subseteq L = -L^* \subseteq -K^*$ follows. So for instance, if K is contained in a cap of radius $\frac{\pi}{4}$, or if K is contained in a self-dual spherical convex polytope (for instance the regular simplex of edge length $\frac{\pi}{2}$), then $K \subseteq -K^*$ and $c_s(K)$ is contained $-\operatorname{int} K^*$.

Lemma 4.3.4. *Let $K \in \mathcal{K}(\mathbb{S}^n)$. If $\operatorname{diam}(K) \leq \frac{\pi}{2}$, then $c_s(K) \in -\operatorname{int} K^*$.*

Proof. Since $\operatorname{diam}(K) = \max\{d_{\mathbb{S}^n}(u,v) \mid u,v \in K\}$, we have $u \cdot v \geq 0$ for all $u,v \in K$. This means that

$$u \in \bigcap_{v \in K} \mathbb{S}_v^+$$

for all $u \in K$. Since

$$K^* = \bigcap_{v \in K} \mathbb{S}_v^-$$

and $-\mathbb{S}_v^- = \mathbb{S}_v^+$, we get $K \subseteq -K^*$, in particular $c_s(K) \in -\operatorname{int} K^*$. □

Lemma 4.3.5. *If $n = 2$, then $c_s(K) \in -\operatorname{int} K^*$ for all $K \in \mathcal{K}(\mathbb{S}^2)$.*

Proof. Let $a \in \mathbb{R}^3$ be any vector, and $F(x) := a \times x$, where $\times$ denotes the cross product in $\mathbb{R}^3$. We apply Stokes' theorem to K as a surface in $\mathbb{R}^3$ and obtain

$$\int_K \operatorname{rot} F(u) \cdot u \, d\sigma(u) = \int_{\partial K} F(u(s)) \cdot T(s) \, ds.$$

Here, T is the tangential unit vector field along the positively oriented boundary curve of K, that is, K lies on the left of that curve. We compute $\operatorname{rot} F = a$, and since

$$(a \times u) \cdot T = \det(T, a, u) = \det(a, u, T) = a \cdot (u \times T),$$

we end up with

$$a \cdot \int_K u \, d\sigma(u) = a \cdot \frac{1}{2} \int_{\partial K} (u \times T)(s) \, ds,$$

for any $a \in \mathbb{R}^3$, that is,

$$\int_K u \, d\sigma(u) = \frac{1}{2} \int_{\partial K} \underbrace{(u \times T)(s)}_{n(s)} \, ds.$$

Here, $n(s)$ is the inward pointing normal vector to K at $s \in \partial K$, in particular, $-n(s) \in K^*$. Now, for $v \in K$, compute

$$c_s(K) \cdot v = \frac{\int_{\partial K} n(s) \cdot v \, ds}{\left\| \int_{\partial K} n(s) \cdot v \, ds \right\|} > 0,$$

since $-n(s) \cdot v \leq 0$ for all $s \in \partial K$ and $-n(s) \cdot v > 0$ on a set of positive measure. Hence, $v \in \operatorname{int} \mathbb{S}^+_{c_s(K)}$, and thus $c_s(K) \in -\operatorname{int} K^*$. □

Our second remark concerns a possible version of Theorem 4.2.2, where C_K^τ is replaced by C_K^σ, that is, the inequality

$$\sigma(\Gamma_s^* K) \leq \sigma(\Gamma_s^* C_K^\sigma), \tag{4.11}$$

where C_K^σ is a spherical cap such that $\sigma(C_K^\sigma) = \sigma(K)$. This would be a stronger isoperimetric inequality than Theorem 4.2.2, since $C_K^\tau \subseteq C_K^\sigma$ for every $K \in \mathcal{K}_c(\mathbb{S}^n)$ with center $e \in \mathbb{S}^n$ and equality holds if and only if K is already a cap centered at e.

A possible approach to establishing (4.11) is via a spherical analogue of inequality (4.8). More precisely, if $U_1, \dots, U_N$ and $V_1, \dots, V_N$ are independent random unit vectors uniformly distributed in K and C_K^σ, respectively, is it true that

$$\mathbb{E}\left[\sigma\left(\Gamma_{s,e}^*(U_1, \dots, U_N)\right)\right] \leq \mathbb{E}\left[\sigma\left(\Gamma_{s,e}^*(V_1, \dots, V_N)\right)\right]? \tag{4.12}$$

A combination of inequality (4.12) with Theorem 4.2.1 would then yield (4.11).

Finally, let us state the most interesting and probably hardest open problem concerning spherical centroid bodies – a spherical analogue of the Busemann–Petty centroid inequality:

Open Problem. If $K \in \mathcal{K}_c(\mathbb{S}^n)$, then

$$\sigma(\Gamma_s K) \geq \sigma(\Gamma_s C_K^\sigma). \tag{4.13}$$

Let us emphasize that inequality (4.13) would not only imply Theorem 4.2.2, by combining (4.13) with the spherical Blaschke–Santaló inequality from [Gao02], but the stronger inequality discussed in the above remark. Moreover, (4.13) would also imply the classical Busemann–Petty centroid inequality by considering spheres with radii going to infinity and rescaling.

CHAPTER 5

Randomized Urysohn–type inequalities

As a natural analog of Urysohn's inequality in Euclidean space, Gao, Hug, and Schneider [Gao02] showed in 2002 that in spherical or hyperbolic space, the total measure of totally geodesic hypersurfaces meeting a given convex body K is minimized when K is a geodesic ball. We present a random extension of this result by taking K to be the convex hull of finitely many points drawn according to a probability distribution and by showing that the minimum is attained for uniform distributions on geodesic balls. As a corollary, we obtain a randomized Blaschke–Santaló inequality on the sphere. The results in this chapter are joint work with Peter Pivovarov [Hac].

5.1 Statement of results

Let $\mathbb{M}^n$ be either spherical, Euclidean, or hyperbolic space, equipped with its isometry-invariant volume measure λ_n, and define

$$U_1(K) = \int_{\mathbb{M}^n_{n-1}} \chi(K \cap M)\, dM,$$

for $K \subseteq \mathbb{M}^n$ compact and convex as in (2.10). Our first main theorem then reads as follows:

Theorem 5.1.1. *Let $N \in \mathbb{N}$ and $f_1, \dots, f_N : \mathbb{M}^n \to \mathbb{R}^+$ bounded, integrable. Set*

$$I(f_1, \dots, f_N) = \int_{\mathbb{M}^n} \cdots \int_{\mathbb{M}^n} U_1(\operatorname{conv}\{x_1, \dots, x_N\}) \prod_{i=1}^N f_i(x_i)\, dx_1 \dots dx_N.$$

Then

$$I(f_1 \dots, f_N) \geq I(f_1^\star \dots, f_N^\star), \tag{5.1}$$

where $f_i^\star$ denotes the symmetric decreasing rearrangement of f_i, $1 \leq i \leq N$. Under the additional assumption that in the case $\mathbb{M}^n = \mathbb{S}^n$ the functions $f_1, \dots, f_N$ are supported in $\operatorname{int} \mathbb{S}^+_e$, *also*

$$I(f_1^\star \dots, f_N^\star) \geq I(\|f_1\|_\infty \mathbb{1}_{B_1}, \dots, \|f_N\|_\infty \mathbb{1}_{B_N}), \tag{5.2}$$

where the B_i are geodesic balls in $\mathbb{M}^n$ centered at $e \in \mathbb{M}^n$, satisfying

$$\lambda_n(B_i) = \frac{\|f_i\|_{L^1(\mathbb{M}^n)}}{\|f_i\|_\infty}.$$

The left hand sides in (5.1) *and* (5.2) *may be infinite in the cases* $\mathbb{M}^n = \mathbb{R}^n_{e,1}$ *or* $\mathbb{H}^n$.

Let $N \in \mathbb{N}$, $K \subseteq \mathbb{M}^n$ compact, and $X_1, \ldots, X_N$ be independent random points uniformly distributed in K. Then we denote by

$$[K]_N = \operatorname{conv}\{X_1, \ldots, X_N\}$$

the random set given by the convex hull of the N random points (see also Section 2.2). By plugging in indicator functions on compact sets in Theorem (5.1), we obtain the following inequality for the expected value of U_1 on $[K]_N$.

Corollary 5.1.2. *Let* $N \in \mathbb{N}$ *and* $K \subseteq \mathbb{M}^n$ *be compact. Then*

$$\mathbb{E}U_1([K]_N) \geq \mathbb{E}U_1([B_K]_N),$$

where B_K *is a geodesic ball satisfying* $\lambda_n(K) = \lambda_n(B_K)$.

If additionally K is convex, then $[K]_N \to K$ almost surely in the Hausdorff metric as N tends to infinity, and thus Corollary 5.1.2 recovers the inequality (2.9) by Gao, Hug, and Schneider.

As noted in [Gao02], there is a special relationship between U_1 and spherical polar duality (see Proposition 2.2.1). In this way, (5.1) can also be reinterpreted as a spherical Blaschke–Santaló inequality in stochastic form.

Theorem 5.1.3. *Let* $N \in \mathbb{N}$ *and* $f_1, \ldots, f_N \colon \mathbb{S}^n \to \mathbb{R}^+$ *bounded, integrable and assume that all* f_i, $1 \leq i \leq N$, *are supported in the hemisphere* $\mathbb{S}^+_e$.*Set*

$$\tilde{I}(f_1, \ldots, f_N) = \int_{\mathbb{S}^n} \cdots \int_{\mathbb{S}^n} \lambda_n(\operatorname{conv}\{x_1, \ldots, x_N\}^*) \prod_{i=1}^{N} f_i(x_i)\, dx_1 \ldots dx_N.$$

Then

$$\tilde{I}(f_1 \ldots, f_N) \leq \tilde{I}(f_1^\star \ldots, f_N^\star), \tag{5.3}$$

where $f_i^\star$ *denotes the symmetric decreasing rearrangement of* f_i, $1 \leq i \leq N$. *Under the additional assumption that the functions* $f_1, \ldots, f_N$ *are supported in* $\operatorname{int} \mathbb{S}^+_e$,

$$\tilde{I}(f_1^\star \ldots, f_N^\star) \leq \tilde{I}(\|f_1\|_\infty \mathbb{1}_{C_1}, \ldots, \|f_N\|_\infty \mathbb{1}_{C_N}),$$

where the C_i *are spherical caps centered at a common point, satisfying*

$$\lambda_n(C_i) = \frac{\|f_i\|_{L^1(\mathbb{S}^n)}}{\|f_i\|_\infty}.$$

Again, by plugging in indicator functions on compact sets in (5.3), we obtain as a corollary:

Corollary 5.1.4. *Let $N \in \mathbb{N}$ and $K \subseteq \mathbb{S}^n$ be compact. Then*

$$\mathbb{E}\lambda_n([K]_N^*) \leq \mathbb{E}\lambda_n([C_K]_N^*),$$

where C_K is a spherical cap satisfying $\lambda_n(K) = \lambda_n(C_K)$.

Following a strategy similar to Chapter 4, one can arrive at a symmetric version of Corollary 5.1.4, where $[K]_N$ is replaced by the convex hull of the random points X_i and their reflections about some fixed origin, by transferring a result of [Cor15] from Euclidean space to the sphere using gnomonic projection. However, here we take a different path and work directly on $\mathbb{M}^n$ to obtain Theorem 5.1.1 in full generality.

5.2 Proofs

The proof of Theorem 5.1.1 does not rely on the deterministic result by Gao, Hug, and Schneider. Rather, we first prove a rearrangement inequality for $I(f_1, \dots, f_N)$ that reduces the problem to radially decreasing densities. This is similar to the route taken in the Euclidean setting [Pao12a], [Pao17b], but we use two-point, rather than Steiner symmetrization.

We split the proof into two parts: first, we show how to pass from given functions to their symmetric decreasing rearrangements. In a second step, we further move from radially symmetric, decreasing functions to (multiples) of indicators of geodesic balls. For positive, bounded, and integrable functions $f_1, \dots, f_N$ on $\mathbb{M}^n$ write

$$I(f_1, \dots, f_N) = \int_{\mathbb{M}^n} \cdots \int_{\mathbb{M}^n} U_1(\operatorname{conv}\{x_1, \dots, x_N\}) \prod_{i=1}^N f_i(x_i)\, dx_1 \dots dx_N.$$

Proposition 5.2.1. *Let $f_1, \dots, f_N \colon \mathbb{M}^n \to \mathbb{R}^+$ bounded, integrable. Then*

$$I(f_1 \dots, f_N) \geq I(f_1^\star \dots, f_N^\star).$$

Proof. For bounded, measurable subsets $K_1, \dots, K_N \subseteq \mathbb{M}^n$ we set $I(K_1, \dots, K_N) := I(\mathbb{1}_{K_1}, \dots, \mathbb{1}_{K_N})$. Our first step is to show that $I(K_1, \dots, K_N) \geq I(TK_1, \dots, TK_N)$ for every two-point symmetrization (H, ρ, T). To this end, for $M \in \mathcal{M}_{n-1}^n$, let

$$I(K_1, \dots, K_N; M) := \int_{K_1} \cdots \int_{K_N} \chi(\operatorname{conv}\{x_1, \dots, x_N\} \cap M)\, dx_1 \dots, dx_N.$$

We want to investigate how the quantity $I(K_1, \dots, K_N; M) + I(K_1, \dots, K_N; \rho M)$ changes, when the K_i are replaced by TK_i. Note that we have

$$I(K_1, \dots, K_N; \rho M) = I(\rho K_1, \dots, \rho K_N; M)$$

by the ρ-invariance of χ. We begin by decomposing each K_i according to the symmetrization

$$K_i = \underbrace{(K_i \cap \rho K_i)}_{K_i^{\mathrm{sym}}} \dot\cup \underbrace{\left[(K_i \cap H^+) \setminus K_i^{\mathrm{sym}}\right]}_{K_i^{\mathrm{fix}}} \dot\cup \underbrace{\left[(K_i \cap H^-) \setminus K_i^{\mathrm{sym}}\right]}_{K_i^{\mathrm{mov}}},$$

that is, $TK_i = K_i^{\mathrm{sym}} \,\dot\cup\, K_i^{\mathrm{fix}} \,\dot\cup\, \rho K_i^{\mathrm{mov}}$. Now, let $(x_1, \dots, x_N) \in K_1 \times \cdots \times K_N$ and introduce the following labeling:

$$\begin{aligned}
\{a_1, \dots, a_{N_0}\} &:= \{x_i \,|\, x_i \in K_i^{\mathrm{sym}}, 1 \le i \le N\}, \\
\{b_1, \dots, b_{N_1}\} &:= \{x_i \,|\, x_i \in K_i^{\mathrm{fix}}, 1 \le i \le N\}, \\
\{c_1, \dots, c_{N_2}\} &:= \{x_i \,|\, x_i \in K_i^{\mathrm{mov}}, 1 \le i \le N\},
\end{aligned}$$

where $N_0 + N_1 + N_2 = N$. For brevity we will use the notation $\bar{x} := \rho x$ for $x \in \mathbb{M}^n$ and consider the tuples

$$\begin{aligned}
D_1 &:= (a_1, \dots, a_{N_0}, b_1, \dots, b_{N_1}, c_1, \dots, c_{N_2}) \in \times_{i=1}^N K_i, \\
D_2 &:= (\bar{a}_1, \dots, \bar{a}_{N_0}, b_1, \dots, b_{N_1}, c_1, \dots, c_{N_2}) \in \times_{i=1}^N K_i, \\
D_3 &:= (\bar{a}_1, \dots, \bar{a}_{N_0}, \bar{b}_1, \dots, \bar{b}_{N_1}, \bar{c}_1, \dots, \bar{c}_{N_2}) \in \times_{i=1}^N \rho K_i, \\
D_4 &:= (a_1, \dots, a_{N_0}, \bar{b}_1, \dots, \bar{b}_{N_1}, \bar{c}_1, \dots, \bar{c}_{N_2}) \in \times_{i=1}^N \rho K_i,
\end{aligned}$$

and

$$\begin{aligned}
E_1 &:= (a_1, \dots, a_{N_0}, b_1, \dots, b_{N_1}, \bar{c}_1, \dots, \bar{c}_{N_2}) \in \times_{i=1}^N TK_i, \\
E_2 &:= (\bar{a}_1, \dots, \bar{a}_{N_0}, b_1, \dots, b_{N_1}, \bar{c}_1, \dots, \bar{c}_{N_2}) \in \times_{i=1}^N TK_i, \\
E_3 &:= (\bar{a}_1, \dots, \bar{a}_{N_0}, \bar{b}_1, \dots, \bar{b}_{N_1}, c_1, \dots, c_{N_2}) \in \times_{i=1}^N \rho TK_i, \\
E_4 &:= (a_1, \dots, a_{N_0}, \bar{b}_1, \dots, \bar{b}_{N_1}, c_1, \dots, c_{N_2}) \in \times_{i=1}^N \rho TK_i.
\end{aligned}$$

Note that exchanging c_l with $\bar{c}_l$, $1 \le l \le N_2$, yields the mapping

$$D_1 \mapsto E_1, \quad D_2 \mapsto E_2, \quad D_3 \mapsto E_3, \quad D_4 \mapsto E_4, \tag{5.4}$$

whereas exchanging b_k with $\bar{b}_k$, $1 \le k \le N_1$ induces

$$D_1 \mapsto E_4, \quad D_2 \mapsto E_3, \quad D_3 \mapsto E_2, \quad D_4 \mapsto E_1. \tag{5.5}$$

We claim that

$$\sum_{i=1}^{4} \chi(\mathrm{conv}\{D_i\} \cap M) \ge \sum_{i=1}^{4} \chi(\mathrm{conv}\{E_i\} \cap M) \tag{5.6}$$

for almost all $M \in \mathcal{M}_{n-1}^n$, that is, we tacitly assume that $x_1, \dots, x_N$ do not lie on M. We will verify the claim by checking all possible positions of the points a_j, b_k, c_l relative to M. In doing so, we mean by a *pair* of points a point and its reflection about H, that is x and $\bar{x} = \rho x$. We call a pair of points x and $\bar{x}$ *split* if they lie on opposite sides of M.

- Case 1: *None of the pairs of b's are split.* By (5.5), the terms on both sides of (5.6) are just a permutation of each other, thus there is equality in (5.6).
- Case 2: *None of the pairs of c's are split.* By (5.4) and the same argument as in the

first case, we have equality in (5.6).

- Case 3: *There exist split pairs of b's and split pairs of c's.* Suppose that $\{b_k, \bar{b}_k\}$, $1 \leq k \leq N_1$ and $\{c_l, \bar{c}_l\}$, $1 \leq l \leq N_2$ are split. Since $b_k, \bar{c}_l \in H^+$ and $\bar{b}_k, c_l \in H^-$, the geodesic segments $[b_k, c_l]$ and $[\bar{b}_k, \bar{c}_l]$ intersect in H. As M divides $\mathbb{M}^n$ into two connected components, b_k and $\bar{c}_l$ must lie on one side of M, whereas $\bar{b}_k$ and c_l must lie on the other. Thus, the left hand side of (5.6) equals 4 and the inequality holds.

Integrating the pointwise inequality (5.6) over $K_1 \times \cdots \times K_N$ yields

$$\begin{aligned} 2I(K_1, \dots, K_N; M) + 2I(K_1, \dots, K_N; \rho M) \\ &= 2I(K_1, \dots, K_N; M) + 2I(\rho K_1, \dots, \rho K_N; M) \\ &\geq 2I(TK_1, \dots, TK_N; M) + 2I(\rho TK_1, \dots, \rho TK_N; M) \\ &= 2I(TK_1, \dots, TK_N; M) + 2I(TK_1, \dots, TK_N; \rho M), \end{aligned}$$

that is, the quantity $I(K_1, \dots, K_N; M) + I(K_1, \dots, K_N; \rho M)$ decreases whenever the sets $K_1, \dots, K_N$ are replaced by $TK_1, \dots, TK_N$.

Our next step is to use the layer-cake formula to generalize the previous inequality to functions. Let $f_1, \dots, f_N \colon \mathbb{M}^n \to \mathbb{R}^+$ be bounded, integrable functions and set

$$I(f_1, \dots, f_N; M) := \int_{\mathbb{M}^n} \dots \int_{\mathbb{M}^n} \chi(\mathrm{conv}\{x_1, \dots, x_N\} \cap M) \prod_{i=1}^{N} f_i(x_i)\, dx_1 \dots, dx_N.$$

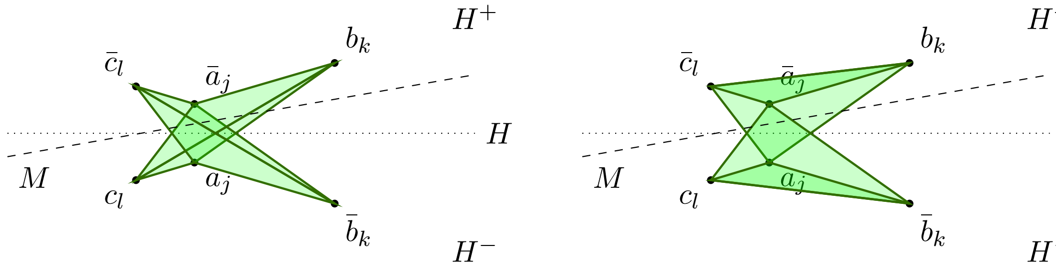

Figure 5.1: $\mathrm{conv}\{D_i\}$ (left) and $\mathrm{conv}\{E_i\}$ (right) for $i \in \{1,2,3,4\}$.

Indeed, we have that

$$\begin{aligned}
&I(f_1,\dots,f_N;M)+I(f_1,\dots,f_N;\rho M)=\\
&=\int_0^\infty\dots\int_0^\infty I(\{f_1>s_1\},\dots,\{f_N>s_N\};M)\\
&\qquad\qquad+I(\{f_1>s_1\},\dots,\{f_N>s_N\};\rho M)\,ds_1\dots ds_N\\
&\geq\int_0^\infty\dots\int_0^\infty I(T\{f_1>s_1\},\dots,T\{f_N>s_N\};M)\\
&\qquad\qquad+I(T\{f_1>s_1\},\dots,T\{f_N>s_N\};\rho M)\,ds_1\dots ds_N\\
&=\int_0^\infty\dots\int_0^\infty I(\{Tf_1>s_1\},\dots,\{Tf_N>s_N\};M)\\
&\qquad\qquad+I(\{Tf_1>s_1\},\dots,\{Tf_N>s_N\};\rho M)\,ds_1\dots ds_N\\
&=I(Tf_1,\dots,Tf_N;M)+I(Tf_1,\dots,Tf_N;\rho M).
\end{aligned}$$

Here, we used the layer-cake representation $f(x)=\int_0^\infty \mathbb{1}_{\{f>s\}}(x)\,ds$, identity (2.15), and the above inequality for sets. We can now apply Proposition 3.3.3 to the bounded function $\Psi(x_1,\dots,x_N)=\chi(\operatorname{conv}\{x_1,\dots,x_N\}\cap M)+\chi(\operatorname{conv}\{x_1,\dots,x_N\}\cap\rho M)$ to obtain

$$I(f_1,\dots,f_N;M)+I(f_1,\dots,f_N;\rho M)\geq I(f_1^\star,\dots,f_N^\star;M)+I(f_1^\star,\dots,f_N^\star;\rho M).$$

The proof of the inequality is now completed by integrating M over $\mathcal{M}_{n-1}^n$. □

Proposition 5.2.2. *Let $f_1,\dots,f_N\colon\mathbb{M}^n\to\mathbb{R}^+$ bounded, integrable, with $\operatorname{spt} f_i\subseteq\operatorname{int}\S_e^+$, $1\leq i\leq N$, in the case $\mathbb{M}^n=\mathbb{S}^n$. Then*

$$I(f_1^\star,\dots,f_N^\star)\geq I(\|f_1\|_\infty\mathbb{1}_{B_1},\dots,\|f_N\|_\infty\mathbb{1}_{B_N})$$

where B_i is a geodesic ball around e such that $\lambda_n(B_i)=\frac{\|f_i\|_{L^1(\mathbb{M}^n)}}{\|f_i\|_\infty}$.

Proof. We use polar coordinates around $e\in\mathbb{M}^n$ (see Section 2.1),

$$x(t,u)=e\operatorname{cs}t+u\operatorname{sn}t,\qquad t\in\left[0,R^{\mathbb{M}}\right],u\in\mathbb{S}^{n-1},$$

and appeal to Proposition 3.4.3. In doing so, we will justify monotonicity in each coordinate of the following function:

$$\begin{aligned}
\phi(t_1,\dots,t_N)=\int_{\mathbb{S}^{n-1}}\dots\int_{\mathbb{S}^{n-1}}&\chi\left(\operatorname{conv}\{(x(t_1,u_1),\dots,x(t_N,u_N))\}\cap M\right)\\
&+\chi\left(\operatorname{conv}\{(x(t_1,u_1),\dots,x(t_N,u_N))\}\cap M^e\right)\,du_1\dots du_N,
\end{aligned}$$

where $M\in\mathcal{M}_{n-1}^n$ is fixed and $x^e:=-x+(x\cdot e)e$ denotes the geodesic reflection of $x\in\mathbb{M}^n$ about e, that is, orthogonal reflection about $\operatorname{span}\{e\}$ in $\mathbb{R}^{n+1}$.

Without loss of generality, we assume that $e\notin M$, and show that ϕ is increasing in $t_1=:t$. We fix $t_2,\dots,t_N$ and $u_1,\dots,u_N$ and write $x(t):=x(t,u_1)$ and $x_i:=x(t_i,u_i)$,

$2 \le i \le N$. Define

$$\alpha_1(t) := \chi(\text{conv}\{x(t), x_2, \dots, x_N\} \cap M), \quad \alpha_2(t) := \chi(\text{conv}\{x(t)^e, x_2, \dots, x_N\} \cap M),$$
$$\alpha_3(t) := \chi(\text{conv}\{x(t), x_2^e, \dots, x_N^e\} \cap M), \quad \alpha_4(t) := \chi(\text{conv}\{x(t)^e, x_2^e, \dots, x_N^e\} \cap M),$$

and set $\alpha(t) := \alpha_1(t) + \alpha_2(t) + \alpha_3(t) + \alpha_4(t)$.

Note that we have $\alpha_1 = \alpha_4^e$ and $\alpha_2 = \alpha_3^e$. Our goal is to show that the function $\alpha\colon [0, R^{\mathbb{M}}] \to \{0,1,2,3,4\}$ is increasing in t. We set $X := \text{conv}\{x_2, \dots, x_N\}$ and consider the following cases:

- Case 1: $e \in X$, and thus, $e \in X^e$. For $s \le t$, we have $[e, x(s)] \subseteq [e, x(t)]$ as geodesic segments. Therefore

$$\alpha_1(s) = \chi(\text{conv}\{[e, x(s)] \cup X\} \cap M) \le \chi(\text{conv}\{[e,x(t)] \cup X\} \cap M) = \alpha_1(t),$$

  and similarly for $\alpha_2, \alpha_3, \alpha_4$, hence, $\alpha(s) \le \alpha(t)$.
- Case 2: $e \notin X$, and thus, $e \notin X^e$.
  - Case 2a: *M meets both X and X^e*. Here, $\alpha(t) \equiv 4$.
  - Case 2b: *M meets X but not X^e*. We first show that in this case e must lie on the same side of M as X^e. Assume the opposite, that is, e lies opposite of X^e. Since M meets X, there exist points of X on either side of M. Therefore, we find $y \in X$ lying opposite of e. But then $y^e \in X^e$, and thus the segment $[y, y^e]$ also lies opposite of e, which is a contradiction, as $e \in [y, y^e]$ (Here, we use that in the spherical case the functions $f_1, \dots, f_N$, and thus their rearrangements, $f_1^\star, \dots, f_N^\star$ are supported in $\text{int}\, \mathbb{S}_e^+$). Hence, for t small enough, we have

$$\alpha_1(t) = 1, \quad \alpha_2(t) = 1, \quad \alpha_3(t) = 0, \quad \alpha_4(t) = 0,$$

    that is, $\alpha(t) = 2$. As t increases, as soon as either $x(t)$ or $x(t)^e$ cross M, $\alpha_3(t) = 1$ or $\alpha_4(t) = 1$, that is, $\alpha(t) = 3$.
  - Case 2c: *M meets X^e but not X* is similar to Case 2b.

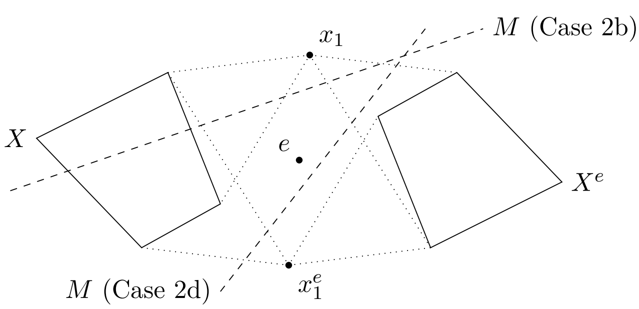

Figure 5.2: Different positions of the hypersurface M in Cases 2b and 2d.

– Case 2d: *M meets neither X nor* X^e. If X and X^e lie on opposite sides of M, then $\alpha(t) \equiv 2$ is constant, as $\alpha_1(t) + \alpha_3(t) \equiv 1$ and $\alpha_2(t) + \alpha_4(t) \equiv 1$ (except for at most one value of t, where x_1 or x_1^e might lie on M). If X and X^e lie on the same side of M, then so does e, and thus, $\alpha(t) = 0$ for small t, and $\alpha(t) = 2$, as soon as $x(t)$ or $x(t)^e$ cross M, since then $\alpha_1(t) = 1$, $\alpha(t)_3 = 1$ or $\alpha(t)_2 = 1$, $\alpha(t)_4 = 1$, respectively.

Integrating $u_1, \dots, u_N$ over $\mathbb{S}^{n-1} \times \cdots \times \mathbb{S}^{n-1}$ now yields that

$$2\phi(t, t_2, \dots, t_N) = \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} \alpha(t)\, du_1 \dots du_N$$

is increasing in t as well. Hence, an application of Proposition 3.4.3 gives

$$\begin{aligned} &I(f_1^\star, \dots, f_N^\star; M) + I(f_1^\star, \dots, f_N^\star; M^e) \\ &\quad \geq I(\|f_1\|_\infty \mathbb{1}_{B_1}, \dots, \|f_N\|_\infty \mathbb{1}_{B_N}; M) + I(\|f_1\|_\infty \mathbb{1}_{B_1}, \dots, \|f_N\|_\infty \mathbb{1}_{B_N}; M^e) \end{aligned}$$

Once again, integrating M over $\mathcal{M}_{n-1}^n$ concludes the proof. □

CHAPTER 6

Random ball–polyhedra

We present inequalities for the expected volume of the intersection or union of finitely many geodesic balls of given radius, whose centers are chosen according to probability densities. Namely, we show that extremizers are to be found among radially symmetric, radially decreasing density functions. We treat spherical, Euclidean, and hyperbolic space at once and thereby extend a result by Paouris and Pivovarov [Pao17a] to curved geometries.

6.1 Statement of results

Let $\mathbb{M}^n$ be either spherical, Euclidean, or hyperbolic space, equipped with its isometry-invariant volume measure λ_n. Our first theorem is about intersections of geodesic balls, sometimes referred to as ball polyhedra (see, e.g., [Bez07]):

Theorem 6.1.1. *Let $r > 0$, $N \in \mathbb{N}$ and $f_1, \ldots, f_N \colon \mathbb{M}^n \to \mathbb{R}^+$ bounded, integrable. Set*

$$I^r(f_1, \ldots, f_N) = \int_{\mathbb{M}^n} \cdots \int_{\mathbb{M}^n} \lambda_n \left(\bigcap_{i=1}^N B_r(x_i) \right) \prod_{i=1}^N f_i(x_i)\, dx_1 \ldots dx_N.$$

Then

$$I^r(f_1 \ldots, f_N) \leq I^r(f_1^\star \ldots, f_N^\star), \tag{6.1}$$

where $f_i^\star$ denotes the symmetric decreasing rearrangement of f_i, $1 \leq i \leq N$.

Let $r > 0$, $N \in \mathbb{N}$, $K \subseteq \mathbb{M}^n$ compact, and $X_1, \ldots, X_N$ be independent random points uniformly distributed in K. Then we denote by

$$\langle K \rangle_N^r = \bigcap_{i=1}^N B_r(X_i)$$

the random set given by the intersection of N balls of radius r, centered at the X_i. By plugging in indicator functions on compact sets in Theorem 6.1.1, we obtain the following inequality for the expected volume of $\langle K \rangle_N^r$.

Corollary 6.1.2. *Let $r > 0$, $N \in \mathbb{N}$ and $K \subseteq \mathbb{M}^n$ be compact. Then*

$$\mathbb{E}\lambda_n(\langle K \rangle_N^r) \leq \mathbb{E}\lambda_n(\langle B_K \rangle_N^r),$$

where B_K is a geodesic ball satisfying $\lambda_n(K) = \lambda_n(B_K)$.

If in addition K is convex, then since

$$\langle K\rangle_N^r \to K^r = \{x \in \mathbb{M}^n \,|\, d_{\mathbb{M}^n}(x,y) \le r \,\forall y \in K\}$$

almost surely in the Hausdorff metric as N tends to infinity, Corollary 6.1.2 recovers an isoperimetric inequality for K^r due to Schmidt [Sch48] and Bezdek [Bez18a]. Also note that for $x_1,\dots,x_N \in \mathbb{S}^n$, we have

$$\mathrm{conv}\{x_1,\dots,x_N\}^* = \bigcap_{i=1}^N B_{\frac{\pi}{2}}(-x_i),$$

thus, Corollary 6.1.2 yields another method to obtain Corollary 5.1.4. Our second result concerns unions of geodesic balls:

Theorem 6.1.3. *Let $r>0$, $N\in\mathbb{N}$ and $f_1,\dots,f_N\colon \mathbb{M}^n \to \mathbb{R}^+$ bounded, integrable. Set*

$$I_r(f_1,\dots,f_N) = \int_{\mathbb{M}^n}\cdots\int_{\mathbb{M}^n} \lambda_n\left(\bigcup_{i=1}^N B_r(x_i)\right)\prod_{i=1}^N f_i(x_i)\,dx_1\dots dx_N.$$

Then

$$I_r(f_1\dots,f_N) \ge I_r(f_1^\star\dots,f_N^\star), \tag{6.2}$$

where $f_i^\star$ denotes the symmetric decreasing rearrangement of f_i, $1\le i\le N$.

Let $X_1,\dots,X_N$ as above and write $\langle K\rangle_{r,N} = \bigcup_{i=1}^N B_r(X_i)$ for the random set given by the union of N balls of radius r, centered at the X_i. By taking indicator functions on compact sets in Theorem 6.1.3, we obtain as a corollary:

Corollary 6.1.4. *Let $r>0$, $N\in\mathbb{N}$ and $K\subseteq\mathbb{M}^n$ be compact. Then*

$$\mathbb{E}\lambda_n(\langle K\rangle_{r,N}) \ge \mathbb{E}\lambda_n(\langle B_K\rangle_{r,N}),$$

where B_K is a geodesic ball satisfying $\lambda_n(K)=\lambda_n(B_K)$.

Again, if in addition K is convex,

$$\langle K\rangle_{r,N} \to K_r = \{x\in\mathbb{M}^n \,|\, d_{\mathbb{M}^n}(x,K)\le r\}$$

almost surely in the Hausdorff metric as N tends to infinity, and thus Corollary 6.1.4 recovers the spherical isoperimetric inequality for outer parallel sets (see, e.g., [Sch48] or [Ben84]).

6.2 Proofs

We prove both Theorems 6.1.1 and 6.1.3 at the same time, since the arguments are almost identical.

Proof of Theorems 6.1.1 and 6.1.3. We start as in the proof of Theorem 5.1. For bounded, measurable subsets $K_1, \dots, K_N \subseteq \mathbb{M}^n$ we set $I^r(K_1, \dots, K_N) := I^r(\mathbb{1}_{K_1}, \dots, \mathbb{1}_{K_N})$ and $I_r(K_1, \dots, K_N) := I_r(\mathbb{1}_{K_1}, \dots, \mathbb{1}_{K_N})$. Our first step is to show that $I^r(K_1, \dots, K_N) \le I^r(TK_1, \dots, TK_N)$ and $I_r(K_1, \dots, K_N) \ge I_r(TK_1, \dots, TK_N)$ for every two-point symmetrization (H, ρ, T). We begin by decomposing each K_i according to the symmetrization

$$K_i = \underbrace{(K_i \cap \rho K_i)}_{K_i^{\mathrm{sym}}} \dot\cup \underbrace{\left[(K_i \cap H^+) \setminus K_i^{\mathrm{sym}}\right]}_{K_i^{\mathrm{fix}}} \dot\cup \underbrace{\left[(K_i \cap H^-) \setminus K_i^{\mathrm{sym}}\right]}_{K_i^{\mathrm{mov}}},$$

that is, $TK_i = K_i^{\mathrm{sym}} \,\dot\cup\, K_i^{\mathrm{fix}} \,\dot\cup\, \rho K_i^{\mathrm{mov}}$. Now, let $(x_1, \dots, x_N) \in K_1 \times \dots \times K_N$ and use again the labeling:

$$\begin{aligned} \{a_1, \dots, a_{N_0}\} &:= \{x_i \,|\, x_i \in K_i^{\mathrm{sym}}, 1 \le i \le N\} \\ \{b_1, \dots, b_{N_1}\} &:= \{x_i \,|\, x_i \in K_i^{\mathrm{fix}}, 1 \le i \le N\} \\ \{c_1, \dots, c_{N_2}\} &:= \{x_i \,|\, x_i \in K_i^{\mathrm{mov}}, 1 \le i \le N\}, \end{aligned}$$

where $N_0 + N_1 + N_2 = N$. With the notation $\bar{x} := \rho x$ for $x \in \mathbb{M}^n$ we consider the tuples

$$\begin{aligned} D_1 &:= (a_1, \dots, a_{N_0}, b_1, \dots, b_{N_1}, c_1, \dots, c_{N_2}) \in \times_{i=1}^N K_i, \\ D_2 &:= (\bar{a}_1, \dots, \bar{a}_{N_0}, b_1, \dots, b_{N_1}, c_1, \dots, c_{N_2}) \in \times_{i=1}^N K_i, \end{aligned}$$

and

$$\begin{aligned} E_1 &:= (a_1, \dots, a_{N_0}, b_1, \dots, b_{N_1}, \bar{c}_1, \dots, \bar{c}_{N_2}) \in \times_{i=1}^N TK_i, \\ E_2 &:= (\bar{a}_1, \dots, \bar{a}_{N_0}, b_1, \dots, b_{N_1}, \bar{c}_1, \dots, \bar{c}_{N_2}) \in \times_{i=1}^N TK_i. \end{aligned}$$

Note that exchanging c_l with $\bar{c}_l$, $1 \le l \le N_2$, that is, performing the symmetrization $K_i \mapsto TK_i$, yields the mapping

$$D_1 \mapsto E_1, \quad D_2 \mapsto E_2.$$

We claim that

$$\lambda_n\left((D_1)^r\right) + \lambda_n\left((D_2)^r\right) \le \lambda_n\left((E_1)^r\right) + \lambda_n\left((E_2)^r\right), \tag{6.3}$$

$$\lambda_n\left((D_1)_r\right) + \lambda_n\left((D_2)_r\right) \ge \lambda_n\left((E_1)_r\right) + \lambda_n\left((E_2)_r\right). \tag{6.4}$$

In order two prove this claim, we introduce the following abbreviations:

$$\mathsf{a} := \bigcap_{i=1}^{N_0} B_r(a_i), \qquad \mathsf{b} := \bigcap_{j=1}^{N_1} B_r(b_j), \qquad \mathsf{c} := \bigcap_{k=1}^{N_2} B_r(c_k),$$
$$\bar{\mathsf{a}} := \bigcap_{i=1}^{N_0} B_r(\bar{a}_i), \qquad \bar{\mathsf{b}} := \bigcap_{j=1}^{N_1} B_r(\bar{b}_j), \qquad \bar{\mathsf{c}} := \bigcap_{k=1}^{N_2} B_r(\bar{c}_k),$$
$$\mathsf{A} := \bigcup_{i=1}^{N_0} B_r(a_i), \qquad \mathsf{B} := \bigcup_{j=1}^{N_1} B_r(b_j), \qquad \mathsf{C} := \bigcup_{k=1}^{N_2} B_r(c_k),$$
$$\bar{\mathsf{A}} := \bigcup_{i=1}^{N_0} B_r(\bar{a}_i), \qquad \bar{\mathsf{B}} := \bigcup_{j=1}^{N_1} B_r(\bar{b}_j), \qquad \bar{\mathsf{C}} := \bigcup_{k=1}^{N_2} B_r(\bar{c}_k).$$

Using this notation, (6.3) and (6.4) read as

$$\lambda_n(\mathsf{a} \cap \mathsf{b} \cap \mathsf{c}) + \lambda_n(\bar{\mathsf{a}} \cap \mathsf{b} \cap \mathsf{c}) \leq \lambda_n(\mathsf{a} \cap \mathsf{b} \cap \bar{\mathsf{c}}) + \lambda_n(\bar{\mathsf{a}} \cap \mathsf{b} \cap \bar{\mathsf{c}}) \tag{6.5}$$
$$\lambda_n(\mathsf{A} \cup \mathsf{B} \cup \mathsf{C}) + \lambda_n(\bar{\mathsf{A}} \cup \mathsf{B} \cup \mathsf{C}) \geq \lambda_n(\mathsf{A} \cup \mathsf{B} \cup \bar{\mathsf{C}}) + \lambda_n(\bar{\mathsf{A}} \cup \mathsf{B} \cup \bar{\mathsf{C}}) \tag{6.6}$$

Setting furthermore

$$\mathsf{d} := \mathsf{b} \cap \mathsf{c}, \qquad \bar{\mathsf{d}} := \mathsf{b} \cap \bar{\mathsf{c}}, \qquad \mathsf{s}_1 := \mathsf{a} \cap \bar{\mathsf{a}}, \qquad \mathsf{s}_2 := \mathsf{a} \cup \bar{\mathsf{a}},$$
$$\mathsf{D} := \mathsf{B} \cup \mathsf{C}, \qquad \bar{\mathsf{D}} := \mathsf{B} \cup \bar{\mathsf{C}}, \qquad \mathsf{S}_1 := \mathsf{A} \cap \bar{\mathsf{A}}, \qquad \mathsf{S}_2 := \mathsf{A} \cup \bar{\mathsf{A}},$$

we can use the additivity of volume to rewrite (6.5) and (6.6) as

$$\lambda_n(\mathsf{d} \cap \mathsf{s}_1) + \lambda_n(\mathsf{d} \cap \mathsf{s}_2) \leq \lambda_n(\bar{\mathsf{d}} \cap \mathsf{s}_1) + \lambda_n(\bar{\mathsf{d}} \cap \mathsf{s}_2) \tag{6.7}$$
$$\lambda_n(\mathsf{D} \cup \mathsf{S}_1) + \lambda_n(\mathsf{D} \cup \mathsf{S}_2) \geq \lambda_n(\bar{\mathsf{D}} \cup \mathsf{S}_1) + \lambda_n(\bar{\mathsf{D}} \cup \mathsf{S}_2). \tag{6.8}$$

Note that $\mathsf{s}_i, \mathsf{S}_i$, $i = 1{,}2$ are symmetric with respect to orthogonal reflection about H, that is, $\rho(\mathsf{s}_i) = \mathsf{s}_i$, $\rho(\mathsf{S}_i) = \mathsf{S}_i$, $i = 1{,}2$.

The next two identities can be found in [Bez18a, Lemma 5] and [Ben84, Proposition 1.1]: Let $M \subseteq \mathbb{M}^n$ be any set and $r > 0$. If we write $M^r := \bigcap_{x\in M} B_r(x)$ and $M_r := \bigcup_{x\in M} B_r(x)$, then

$$T(M^r) \subseteq (TM)^r \qquad \text{(Bezdek)} \tag{6.9}$$
$$T(M_r) \supseteq (TM)_r \qquad \text{(Benyamini)} \tag{6.10}$$

Now, in (6.9) and (6.10), let

$$M := \{b_1, \dots, b_{N_1}, c_1, \dots, c_{N_2}\},$$

then

$$TM = \{b_1, \dots, b_{N_1}, \bar{c}_1, \dots, \bar{c}_{N_2}\},$$

and so $M^r = \mathsf{d}$, $(TM)^r = \bar{\mathsf{d}}$, $M_r = \mathsf{D}$, $(TM)_r = \bar{\mathsf{D}}$. Thus, (6.9) and (6.10) yield

$$T\mathsf{d} \subseteq \bar{\mathsf{d}} \quad \text{and} \quad T\mathsf{D} \supseteq \bar{\mathsf{D}}.$$

Intersecting and joining with the H-symmetric sets s_i and S_i, $i = 1,2$, and using (2.16) and (2.17), we obtain

$$T(\mathsf{d} \cap \mathsf{s}_i) = T\mathsf{d} \cap \mathsf{s}_i \subseteq \bar{\mathsf{d}} \cap \mathsf{s}_i \quad \text{and} \quad T(\mathsf{D} \cup \mathsf{S}_i) = T\mathsf{D} \cup \mathsf{S}_i \supseteq \bar{\mathsf{D}} \cup \mathsf{S}_i,$$

for $i = 1,2$. Since two-point symmetrization is volume preserving, taking volumes on both sides proves (6.7) and (6.8) and therefore also (6.3) and (6.4).

Integrating the pointwise inequalities (6.3) and (6.4) over $K_1 \times \dots \times K_N$ yields

$$2I^r(K_1, \dots, K_N) \leq 2I^r(TK_1, \dots, TK_N),$$
$$2I_r(K_1, \dots, K_N) \geq 2I_r(TK_1, \dots, TK_N),$$

that is, $I^r(K_1, \dots, K_N; M)$ increases, whereas $I_r(K_1, \dots, K_N; M)$ decreases whenever the sets $K_1, \dots, K_N$ are replaced by $TK_1, \dots, TK_N$.

Next, we use the layer-cake formula to generalize the previous inequality to functions. Let $f_1, \dots, f_N \colon \mathbb{M}^n \to \mathbb{R}^+$ be bounded integrable functions. Using $f(x) = \int_0^\infty \mathbb{1}_{\{f>s\}}(x)\, ds$, we have by the above inequality for sets and (2.15) that

$$\begin{aligned} I^r(f_1, \dots, f_N) &= \int_0^\infty \dots \int_0^\infty I^r(\{f_1 > s_1\}, \dots, \{f_N > s_N\})\, ds_1 \dots ds_N \\ &\leq \int_0^\infty \dots \int_0^\infty I^r(T\{f_1 > s_1\}, \dots, T\{f_N > s_N\})\, ds_1 \dots ds_N \\ &= \int_0^\infty \dots \int_0^\infty I^r(\{Tf_1 > s_1\}, \dots, \{Tf_N > s_N\})\, ds_1 \dots ds_N \\ &= I^r(Tf_1, \dots, Tf_N), \end{aligned}$$

and similarly

$$\begin{aligned} I_r(f_1, \dots, f_N) &= \int_0^\infty \dots \int_0^\infty I_r(\{f_1 > s_1\}, \dots, \{f_N > s_N\})\, ds_1 \dots ds_N \\ &\geq \int_0^\infty \dots \int_0^\infty I_r(T\{f_1 > s_1\}, \dots, T\{f_N > s_N\})\, ds_1 \dots ds_N \\ &= \int_0^\infty \dots \int_0^\infty I_r(\{Tf_1 > s_1\}, \dots, \{Tf_N > s_N\})\, ds_1 \dots ds_N \\ &= I_r(Tf_1, \dots, Tf_N). \end{aligned}$$

We can now finish the proof by applying Proposition 3.3.3 to the bounded functions

$$\Psi^r(x_1,\dots,x_N) = \lambda_n\left(\bigcap_{i=1}^{N} B_r(x_i)\right),$$
$$\Psi_r(x_1,\dots,x_N) = \lambda_n\left(\bigcup_{i=1}^{N} B_r(x_i)\right),$$

to obtain the desired inequalities. □

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