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The Cramér-Lundberg ruin model with a high dividend barrier

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unter der Anleitung von
Ao.Univ.Prof. Dr. Peter Grandits

durch

Martin Pleischl, MSc

Hasnerstraße 52/13
A-1160 Wien, Österreich

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Abstract

In the classical collective risk theory the Cramér-Lundberg model is used to model the surplus in a non-life insurance company. The Cramér-Lundberg inequality can be derived, giving an upper bound for the probability of ruin dependent on the initial surplus. Moreover, the usage of a major result from renewal theory (the Key Renewal Theorem) shows the probability of ruin to be asymptotically exponential as the initial surplus tends to infinity. However, ruin can only be avoided if the surplus increases to infinity.

The main goal in this Master thesis is to analyze a modified version that includes a dividend barrier in order to prevent this behavior. In this new model ruin occurs with probability one and it is interesting to know when. If the barrier tends to infinity, an asymptotic distribution for the time of ruin can be found. Depending on the barrier being attained or not, ruin happens on different time scales. If the barrier is reached, the surplus process performs a recurrent motion in the vicinity of the barrier and ruin takes place after a very long exponentially distributed time. Otherwise, ruin occurs quite soon and the time of ruin has the same distribution as in the classical model conditional on ruin occurring. As a next step, the proportion of time the surplus is below some given level can be derived by using some relations to queueing theory.

In case of exponentially distributed claims, the density of the time of ruin is found by numerical inversion of its Laplace transform which can be calculated explicitly. Finally, additional support for the found asymptotic formula is provided by a Monte Carlo simulation of the surplus process with Erlang distributed claims.

Contents

1 Introduction

The classical approach to model the surplus of a non-life insurance company goes back to the work of the Swedish actuary Filip Lundberg (1903). In the 1930s Harald Cramér incorporated Lundberg's work into the emerging theory of stochastic processes. The Cramér-Lundberg model was found. Its basic idea is to model the claims in the surplus process of an insurance as a compound Poisson process.
In 2000, Johan Irbäck published a paper considering a modified version of the classical Cramér-Lundberg model that incorporates a dividend barrier in order to prevent the surplus from tending to infinity [I]. The results are profound. This thesis is based on his paper and should serve as an extension of his work by adding lots of important details. In addition, an analysis of the classical Cramér-Lundberg model as well as an introduction to queueing theory are included. Results from these areas are used within different proofs of Irbäck's work. The thesis concludes with an example. In case of exponential claims explicit results can be derived, whereas for other distributions a Monte Carlo simulation should be helpful to demonstrate Irbäck's main result, i.e. an asymptotic law for the time of ruin.

2 Classical ruin theory

This thesis starts off with a chapter about the classical ruin theory in finance and thus is mainly dedicated to the Cramér-Lundberg model. Basically it follows the ideas of H. U. Gerber in Chapter 2 (Stochastic processes) and Chapter 8 (Ruin theory: Part 1) of his monograph about mathematical risk theory [G]. Additional sources are mentioned in the respective case.

2.1. Counting processes and the compound Poisson process

Definition 2.1.1. A **counting process** is a continuous-time stochastic process $\{N_t\}_{t\geq 0}$, with $N_0 = 0$, whose sample paths are step functions with jumps of size one.

N_t can be interpreted as number of times a certain event occurs between 0 and t. Such a process may be described conveniently by a sequence of positive random variables $\{I_k\}_{k\in\mathbb{N}}$ called **inter-arrival times** and the corresponding **arrival times**

$$A_0 := 0 \quad \text{and} \quad A_n := \sum_{k=1}^{n} I_k,\ n \in \mathbb{N}, \tag{2.1}$$

enabling the representation

$$N_t := \max(n \in \mathbb{N}_0 | A_n \leq t), \quad t \geq 0. \tag{2.2}$$

Note that $\{N_t\}_{t\geq 0}$ is a càdlàg process by definition (right continuous with left limits). Let $H_t := \{N_s | 0 \leq s \leq t\} = \{N_t, A_1, \dots, A_{N_t}\}$ denote the **history** of the counting process at time t. Assume, from now on, that for any time t and any history H_t the conditional distribution of A_{N_t+1} is absolutely continuous. Then

$$\lambda(t, H_t) := \lim_{\epsilon \searrow 0} \frac{\mathbb{P}[N_{t+\epsilon} - N_t = 1 | H_t]}{\epsilon} \tag{2.3}$$

is called the **intensity of frequency** at time t, given the history H_t. Intuitively, in the infinitesimal time interval from $(t, t+dt]$, either there will be a jump (with probability

$\lambda(t, H_t)dt$) or not (with probability $1 - \lambda(t, H_t)dt$).
For arbitrary time horizon $h > 0$, the law of total probability yields

$$\mathbb{P}[N_{t+h} > N_t | H_t] = \int_0^h \underbrace{\mathbb{P}[N_{t+s} = N_t | H_t]}_{\text{no jumps before time } t+s} \cdot \underbrace{\lambda(t+s, H_{t+s})}_{\text{jump at time } t+s} \, ds. \tag{2.4}$$

The substitution $\mathbb{P}[N_{t+h} > N_t | H_t] = 1 - \mathbb{P}[N_{t+h} = N_t | H_t]$ leads to the following differential equation for $\mathbb{P}[N_{t+h} = N_t | H_t]$:

$$d\, \mathbb{P}[N_{t+s} = N_t | H_t] = -\mathbb{P}[N_{t+s} = N_t | H_t] \cdot \lambda(t+s, H_{t+s}) \, ds, \quad s > 0. \tag{2.5}$$

Therefore

$$\ln\left(\mathbb{P}[N_{t+h} = N_t | H_t]\right) = \int_0^h \frac{d\, \mathbb{P}[N_{t+s} = N_t | H_t]}{\mathbb{P}[N_{t+s} = N_t | H_t]} = -\int_0^h \lambda(t+s, H_{t+s}) \, ds. \tag{2.6}$$

One ends up with the solution

$$\mathbb{P}[N_{t+h} = N_t | H_t] = e^{-\int_0^h \lambda(t+s, H_{t+s}) \, ds}. \tag{2.7}$$

Example 2.1.2. Assume that $\lambda(t, H_t) = \lambda > 0$ depends neither on time nor on history. In this case, formula 2.7 simplifies to

$$\mathbb{P}[N_{t+h} = N_t] = e^{-\lambda h}. \tag{2.8}$$

One can even show by induction the increments $N_{t+h} - N_t,\ h > 0,\ t \geq 0$, to follow a Poisson distribution $Poi(\lambda)$:

$$\mathbb{P}[N_{t+h} - N_t = k] = \frac{(\lambda h)^k}{k!} e^{-\lambda h},\ k \in \mathbb{N}_0. \tag{2.9}$$

For this purpose, rewrite the left hand side using the law of total probability ($k \in \mathbb{N}$):

$$\mathbb{P}[N_{t+h} - N_t = k] = \int_0^h \underbrace{\mathbb{P}[N_{t+s} - N_t = k-1]}_{k-1 \text{ jumps before } t+s} \cdot \underbrace{\lambda}_{k\text{-th jump}} \cdot \underbrace{\mathbb{P}[N_{t+h} = N_{t+s}]}_{\text{no more jumps until } t+h} \, ds \tag{2.10}$$

Applying the induction assumption together with formula 2.8 gives

$$\mathbb{P}[N_{t+h} - N_t = k] = \int_0^h \frac{(\lambda s)^{k-1}}{(k-1)!} e^{-\lambda s} \cdot \lambda \cdot e^{-\lambda(h-s)} ds = \frac{(\lambda h)^k}{k!} e^{-\lambda h}. \tag{2.11}$$

Due to the Poisson distributed increments, a counting process with constant intensity λ is called homogeneous **Poisson process**. Observe that this process has independent and stationary increments. If, on the other hand, a counting process has these two properties, it must have a constant intensity and therefore be a Poisson process. Hence, the Poisson process is the only counting process that has independent and stationary increments.

Definition 2.1.3. A counting process $\{N_t\}_{t\geq 0}$ is a homogeneous **Poisson process** with intensity $\lambda > 0$ if

1. $N_0 = 0$,
2. N_t has independent increments,
3. $N_t - N_s \sim Poi(\lambda(t-s)) \quad \forall\, 0 \leq s < t$.

The concept of Poisson processes can also be embedded in renewal theory in the following way (Asmussen [A]).

Definition 2.1.4. If the inter-arrival times $\{I_k\}_{k\in\mathbb{N}}$ are positive iid random variables with common distribution P, then the arrival times process $\{A_n\}_{n\in\mathbb{N}}$ is called a (pure) **renewal process**. Moreover, $\{A_n\}_{n\in\mathbb{N}}$ is called a delayed renewal process if instead of $A_0 = 0$ a.s. one has $A_0 = I_0$, where I_0 is a.s. positive, independent of $\{I_k\}_{k\in\mathbb{N}}$ but not necessarily equally distributed. The A_n are called the **renewals** or the **epochs** of the renewal process. The corresponding counting process $\{N_t\}_{t\geq 0}$ is named (pure or delayed) **renewal counting process**.

Consider a pure renewal counting process $\{N_t\}_{t\geq 0}$, where p is the corresponding density of the inter-arrival times I_k. Due to independence of the inter-arrival times, A_{N_t} summarizes the past history H_t and thus the intensity λ simplifies to

$$\begin{aligned}
\lambda(t, H_t) &= \lim_{\epsilon\searrow 0} \frac{\mathbb{P}[N_{t+\epsilon} - N_t = 1|H_t]}{\epsilon} = \lim_{\epsilon\searrow 0} \frac{\mathbb{P}[N_{t+\epsilon} - N_t = 1|A_{N_t}]}{\epsilon} = &(2.12)\\
&= \lim_{\epsilon\searrow 0} \frac{\mathbb{P}[t - A_{N_t} < I_{N_t+1} \leq t - A_{N_t} + \epsilon \mid I_{N_t+1} > t - A_{N_t}]}{\epsilon} = &(2.13)\\
&= \lim_{\epsilon\searrow 0} \frac{P(t - A_{N_t} + \epsilon) - P(t - A_{N_t})}{\epsilon(1 - P(t - A_{N_t}))} = \frac{p(t - A_{N_t})}{1 - P(t - A_{N_t})}. &(2.14)
\end{aligned}$$

This means the intensity of frequency depends only on the time elapsed since the last arrival.

In the special case where $I_k \sim Exp(\lambda)$, $\lambda > 0$, the intensity of frequency becomes

$$\lambda(t, H_t) = \frac{p(t - A_{N_t})}{1 - P(t - A_{N_t})} = \frac{\lambda e^{-\lambda(t - A_{N_t})}}{e^{-\lambda(t - A_{N_t})}} = \lambda, \tag{2.15}$$

which means that the renewal counting process then is a homogeneous Poisson process.

Lemma 2.1.5. *The renewals A_n, $n \in \mathbb{N}$, of a renewal process with exponentially distributed inter-arrival times are Gamma distributed. To be precisely,*

$$A_n = \sum_{k=1}^{n} I_k \sim Gam(n, \lambda) \quad \text{and} \quad \mathbb{P}[A_n \leq x] = 1 - e^{-\lambda x} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!}, \quad \text{for } x \geq 0. \tag{2.16}$$

Proof.
Due to independence of the exponentially distributed inter-arrival times, the moment generating function of A_n is

$$M_{A_n}(z) = \prod_{k=1}^{n} M_{I_k}(z) = \left(\frac{\lambda}{\lambda - z}\right)^n, \quad \text{for } z < \lambda, \tag{2.17}$$

which uniquely determines the distribution of A_n to be $Gam(n, \lambda)$. Since n is an integer this is equivalent to an Erlang distribution, whose distribution function can be computed easily by iterative partial integration of the Erlang/Gamma density:

$$\begin{aligned}
\mathbb{P}[A_n \leq x] &= \frac{\lambda^n}{(n-1)!} \int_0^x e^{-\lambda y} y^{n-1} dy = && (2.18)\\
&= \frac{\lambda^n}{(n-1)!} \left(-\frac{1}{\lambda} e^{-\lambda y} y^{n-1} \Big|_0^x + \frac{n-1}{\lambda} \int_0^x e^{-\lambda y} y^{n-2} dy \right) = && (2.19)\\
&= -\frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x} - \frac{(\lambda x)^{n-2}}{(n-2)!} e^{-\lambda x} - \ldots - \frac{(\lambda x)^2}{2} e^{-\lambda x} + \lambda^2 \int_0^x e^{-\lambda y} y dy = && (2.20)\\
&= 1 - e^{-\lambda x} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!} && (2.21)
\end{aligned}$$

□

Now, let $\{N_t\}_{t\geq 0}$ be a counting process and $\{C_n\}_{n\in\mathbb{N}}$ a sequence of iid random variables with common distribution F that are independent of the counting process. C_n should represent the size of the n-th jump. Then, the **aggregate jumps process** is defined as

$$S_t = \sum_{n=1}^{N_t} C_n, \ t \geq 0, \tag{2.22}$$

with the understanding that $S_t = 0$ if $N_t = 0$.

In risk theory, S_t models the aggregate claims until time t, with N_t denoting the number of claims until t and C_n the size of the n-th claim. The most popular choice for the counting process (or claim number process) is a homogeneous Poisson process, yielding the following important process.

Definition 2.1.6. An aggregate jumps process $\{S_t\}_{t\geq 0}$ where the number of claims is modeled as a Poisson process $\{N_t\}_{t\geq 0}$ with intensity $\lambda > 0$ (expected number of claims per unit of time) is called **compound Poisson process**.

Intuitively, in every time interval $(t, t + dt]$ there is either no claim (with probability $1 - \lambda dt$) or exactly one claim (with probability λdt). In the latter case the amount of the claim is a random variable that is independent of anything else and distributed according to distribution F. The corresponding distribution function of S_t is of the following form:

$$G(y,t) := \mathbb{P}[S_t \le y] = \sum_{n=0}^{\infty} \mathbb{P}[N_t = n] \cdot \mathbb{P}[S_t \le y | N_t = n] = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} F^{n\star}(y), \quad (2.23)$$

where $y \ge 0$ and $F^{n\star}$ represents the **n-fold convolution** defined by

$$F^{n\star}(y) := \begin{cases} \mathbb{1}_{[0,\infty)}(y) & \text{if } n = 0, \\ \int_{[0,y]} F(y-z) F^{(n-1)\star}(dz) & \text{if } n \in \mathbb{N}. \end{cases} \quad (2.24)$$

The moment generating function of a compound Poisson process S_t has the following form:

$$\begin{aligned} M_{S_t}(z) &= \mathbb{E}[e^{zS_t}] = \mathbb{E}[\mathbb{E}[e^{zS_t}|N_t]] = \mathbb{E}[M_C^{N_t}(z)] = & (2.25) \\ &= \mathbb{E}[e^{N_t \ln M_C(z)}] = M_{N_t}(\ln M_C(z)) = e^{\lambda t (M_C(z)-1)}, & (2.26) \end{aligned}$$

where M_C is the moment generating function of the claim sizes C_n and M_{N_t} the moment generating function of the number of claims N_t, respectively.
Hence one derives the expectation and variance of a compound Poisson process:

$$\mathbb{E}[S_t] = M'_{S_t}(0) = M'_{N_t}(\ln M_C(0)) \frac{M'_C(0)}{M_C(0)} = \mathbb{E}[N_t] \cdot \mathbb{E}[C_n] = \lambda t \mu \quad (2.27)$$

and

$$\begin{aligned} \mathbb{V}[S_t] &= M''_{S_t}(0) - \mathbb{E}[S_t]^2 = & (2.28) \\ &= M''_{N_t}(0) M'_C(0)^2 + M'_{N_t}(0)(M''_C(0) - M'_C(0)^2) - \mathbb{E}[S_t]^2 = & (2.29) \\ &= \mathbb{E}[N_t^2] \cdot \mathbb{E}[C_n]^2 + \mathbb{E}[N_t] \cdot \mathbb{V}[C_n] - \mathbb{E}[N_t]^2 \cdot \mathbb{E}[C_n]^2 = & (2.30) \\ &= \mathbb{E}[N_t] \cdot \mathbb{V}[C_n] + \mathbb{V}[N_t] \cdot \mathbb{E}[C_n]^2 = \lambda t \mu_2 & (2.31) \end{aligned}$$

where $\mu_k := \int_0^\infty y^k dF(y)$, $k \in \mathbb{N}$, is the k-th moment of the claim distribution and $\mu := \mu_1$ denotes the expected size of a claim.

From the properties of the Poisson process it follows that the compound Poisson process is a **Lévy process**, i.e. it has stationary and independent increments as well. In fact, the increment $S_{t+h} - S_t$ has a compound Poisson distribution with parameter λh and claim distribution F. Moreover, the corresponding **Lévy exponent** is

$$g(z) := \ln(M_{S_1}(z)) = \lambda(M_C(z) - 1). \quad (2.32)$$

2.2. Specification of the classical Cramér-Lundberg model

Consider an insurance company whose **surplus** at time $t \geq 0$ is modeled as the following risk process (the superscript c signals the classical form)

$$X_t^c := x + pt - S_t, \tag{2.33}$$

where $x > 0$ is the **initial surplus**, $p > 0$ is the **premium income rate** and S_t denotes aggregate claims up to time t. $\{S_t\}_{t\geq 0}$ is modeled as a compound Poisson process with intensity $\lambda > 0$ and distribution function F of claim amounts. The concern here is the event that ruin occurs, i.e. that the surplus X_t^c becomes negative for some t.

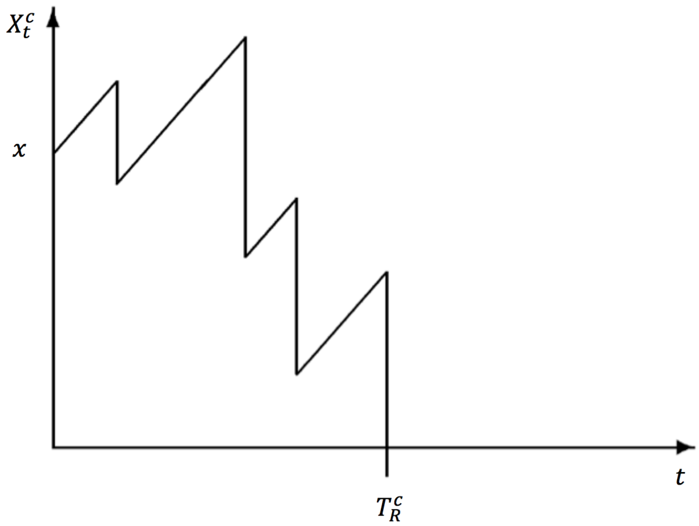


Figure 2.1.: Sample path of the surplus process $\{X_t^c\}_{t\geq 0}$ in the classical Cramér-Lundberg model

Definition 2.2.1. Let us define the **time of ruin** as

$$T_R^c := \inf(t \geq 0 | X_t^c < 0). \tag{2.34}$$

Then, the **probability of ruin**, with the understanding that $T_R^c = \infty$ if $X_t^c > 0$ for all t, is denoted as

$$\psi(x) := \mathbb{P}[T_R^c < \infty | X_0^c = x], \tag{2.35}$$

and the **probability of ruin before time t** as

$$\psi(x,t) := \mathbb{P}[T_R^c < t | X_0^c = x]. \tag{2.36}$$

Similarly, $U(x) := 1 - \psi(x)$ denotes the **probability of ultimate survival**, and $U(x,t) := 1 - \psi(x,t)$ the **probability of survival to time t**.
Usually it is assumed that

$$p > \lambda\mu. \tag{2.37}$$

This condition means that the premiums received per unit of time exceed the expected claim payments per unit of time, i.e. the risk process has positive drift. Using the formula for the expectation of a compound Poisson process in equation 2.27, one can find the expectation of the surplus at time t:

$$\mathbb{E}[X_t^c] = x + pt - \mathbb{E}[S_t] = x + (p - \lambda\mu)t \tag{2.38}$$

According to the strong law of large numbers,

$$\lim_{t\to\infty} \frac{X_t^c}{t} = p - \lambda\mu \quad a.s. \tag{2.39}$$

Thus the so-called **safety loading** (or security loading)

$$\Lambda := \frac{p}{\lambda\mu} - 1 \tag{2.40}$$

is assumed positive to ensure the ruin probability is less than one. In fact, this condition guarantees that, for any sample path, the surplus drifts ultimately to infinity. But the main question is whether it does this without ever becoming negative.

2.3. The adjustment coefficient and the Cramér-Lundberg inequality

Theorem 2.3.1. *The probability of survival to time t without initial surplus can be computed as*

$$U(0,t) = \frac{1}{pt}\int_0^{pt} G(y,t)dy, \tag{2.41}$$

where G is the distribution function of S_t from 2.23.
The probability of ultimate survival (or ruin) without initial surplus depends only on the relative security loading, but not on the specific form of the claim distribution. To be precisely,

$$U(0) = \frac{\Lambda}{1+\Lambda} \qquad \text{and} \qquad \psi(0) = \frac{1}{1+\Lambda}. \tag{2.42}$$

Proof.
Let I denote an infinitesimal interval $(x, x+dx)$, where $x \geq 0$. Then, using continuous-time versions of the duality principle and of a famous result from Dwass and Dinges (both without proof here) yields

$$\mathbb{P}[X_t^c \in I, X_u^c \geq 0 \text{ for } 0 \leq u \leq t] = \mathbb{P}[X_t^c \in I, X_u^c < X_t^c \text{ for } 0 \leq u < t] = \frac{x}{pt}\mathbb{P}[X_t^c \in I]. \tag{2.43}$$

By integrating over x from 0 to pt and substituting $S_t = pt - X_t^c$ and $y = pt - x$ one derives

$$\begin{aligned} U(0,t) &= \frac{1}{pt}\int_0^{pt} x\mathbb{P}[X_t^c \in I] = \frac{1}{pt}\int_0^{pt}(pt-y)\mathbb{P}[S_t \in (y, y+dy)] = &(2.44)\\ &= \frac{1}{pt}\left((pt-y)G(y,t)\Big|_0^{pt} + \int_0^{pt} G(y,t)dy\right) = \frac{1}{pt}\int_0^{pt} G(y,t)dy. &(2.45) \end{aligned}$$

Rewriting the above integral yields

$$\begin{aligned} U(0,t) &= \frac{1}{pt}\int_0^{pt}(1-(1-G(y,t)))dy = &(2.46)\\ &= 1 - \frac{1}{pt}\left(\int_0^{\infty}(1-G(y,t))dy - \int_{pt}^{\infty}(1-G(y,t))dy\right) = &(2.47)\\ &= 1 - \frac{\lambda\mu}{p} + \frac{1}{pt}\int_{pt}^{\infty}(1-G(y,t))dy &(2.48)\\ &= \frac{\Lambda}{1+\Lambda} + \frac{1}{pt}\int_{pt}^{\infty}(1-G(y,t))dy. &(2.49) \end{aligned}$$

Using Chebyshev's inequality and the explicit forms for expectation and variance of a compound Poisson process in formulas 2.27 and 2.31, one finds an upper bound for the integrand:

$$1 - G(y,t) \quad = \quad \mathbb{P}[S_t > y] \leq \mathbb{P}[|S_t - \lambda t\mu| \geq y - \lambda t\mu] \leq \frac{\lambda t\mu_2}{(y - \lambda t\mu)^2} \tag{2.50}$$

Substituting this inequality in equation 2.49 one obtains the estimate

$$U(0,t) \leq \frac{\Lambda}{1+\Lambda} + \frac{1}{pt}\int_{pt}^{\infty} \frac{\lambda t\mu_2}{(y-\lambda t\mu)^2}dy = \frac{\Lambda}{1+\Lambda} + \frac{1}{pt}\frac{\lambda\mu_2}{p-\lambda\mu}. \tag{2.51}$$

This implies that in the limit, $t \to \infty$, the integral in formula 2.49 is 0 and hence

$$U(0) = \frac{\Lambda}{1+\Lambda}. \tag{2.52}$$

Furthermore

$$\psi(0) = 1 - U(0) = 1 - \frac{\Lambda}{1+\Lambda} = \frac{1}{1+\Lambda}. \tag{2.53}$$

□

Lemma 2.3.2. *The probability of ruin $\psi(x)$ fulfills the following double integral equation:*

$$\psi(x) = \int_0^{\infty}\int_0^{\infty} \psi(x+pt-y)\lambda e^{-\lambda t} dt\, dF(y). \tag{2.54}$$

Proof.
This proof is borrowed from F. Hubalek [H].
Introduce $\psi_n(x)$, the **probability that ruin occurs with or before the n-th claim**, as

$$\psi_n(x) := \mathbb{P}[T_R^c \leq A_n | X_0^c = x], \quad n \in \mathbb{N}. \tag{2.55}$$

For every x, $\psi_n(x)$ is increasing in n with $\psi(x) = \lim_{n\to\infty} \psi_n(x)$.
By distinguishing according to time and amount of the first claim (law of total probability) and using the fact that I_k and C_k are all iid by definition, one computes:

$$\begin{aligned}
\psi_n(x) &= \mathbb{P}[\min_{k=1,\dots,n} X_{A_k}^c < 0 | X_0^c = x] = && (2.56)\\
&= \mathbb{P}[\min_{k=1,\dots,n} X_{I_1+\ldots+I_k}^c < 0 | X_0^c = x] = && (2.57)\\
&= \mathbb{P}[\min_{k=1,\dots,n} (x + p(I_1+\ldots+I_k) - (C_1+\ldots+C_k)) < 0] = && (2.58)\\
&= \int_0^{\infty}\int_0^{\infty} \mathbb{P}[\min_{k=1,\dots,n} (x + p(I_1+\ldots+I_k) - (C_1+\ldots+C_k)) < 0| && (2.59)\\
&\qquad |I_1 = t, C_1 = y] \cdot \lambda e^{-\lambda t} dt\, dF(y) = && (2.60)\\
&= \int_0^{\infty}\int_0^{\infty} \mathbb{P}[\min_{k=1,\dots,n-1} ((x+pt-y) + p(I_1+\ldots+I_k) - && (2.61)\\
&\qquad -(C_1+\ldots+C_k)) < 0] \cdot \lambda e^{-\lambda t} dt\, dF(y) = && (2.62)\\
&= \int_0^{\infty}\int_0^{\infty} \psi_{n-1}(x+pt-y)\lambda e^{-\lambda t} dt\, dF(y) && (2.63)
\end{aligned}$$

Now, letting $n \to \infty$ and using Lebesgue's monotone convergence theorem, one ends up with the final result:

$$\psi(x) = \int_0^\infty \int_0^\infty \psi(x + pt - y)\lambda e^{-\lambda t} dt \, dF(y). \tag{2.64}$$

□

Lemma 2.3.3. *The double integral equation from Lemma 2.3.2. can be transformed into the following defective renewal equation for the probability of ruin ψ (renewal theory is discussed in section 2.4):*

$$\psi(x) = \frac{\lambda}{p} \left(\int_x^\infty \bar{F}(y) dy + \int_0^x \psi(x - y)\bar{F}(y) dy \right) \tag{2.65}$$

where $\bar{F}(y) := 1 - F(y)$ is the tail distribution.

Proof.
Let us transform the double integral equation from Lemma 2.3.2.

$$\psi(z) = \int_0^\infty \int_0^\infty \psi(z + pt - y)\lambda e^{-\lambda t} dF(y) \, dt, \tag{2.66}$$

with the understanding that $\psi(z) = 1$ for $z < 0$. Performing the change of variable $s = z + pt$, this equation becomes

$$\psi(z) = \frac{\lambda}{p} \int_z^\infty e^{-\frac{\lambda}{p}(s-z)} \int_0^\infty \psi(s - y) dF(y) \, ds. \tag{2.67}$$

Taking the derivative, one obtains by using Leibniz's rule

$$\psi'(z) = \frac{\lambda}{p}\psi(z) - \frac{\lambda}{p} \int_0^\infty \psi(z - y) dF(y) = \tag{2.68}$$

$$= \frac{\lambda}{p} \left(\psi(z) - \int_0^z \psi(z - y) dF(y) - \bar{F}(z) \right) \tag{2.69}$$

Integrating over z from 0 to x, one finds

$$\psi(x) = \psi(0) + \frac{\lambda}{p} \left(\int_0^x \psi(z) dz - \int_0^x \int_0^z \psi(z - y) dF(y) \, dz - \int_0^x \bar{F}(z) dz \right) \tag{2.70}$$

Now one uses the explicit expression for the probability of ruin without initial surplus from Theorem 2.3.1. to calculate

$$\psi(0) = \frac{1}{1 + \Lambda} = \frac{\lambda \mu}{p} = \frac{\lambda}{p} \int_0^\infty \bar{F}(y) dy. \tag{2.71}$$

Furthermore, one can simplify the middle term in equation 2.70 by changing the order of integration, then integrating by parts and using Leibniz's rule as follows.

$$\begin{aligned}
\int_0^x \int_0^z \psi(z-y)dF(y)\, dz &= \int_0^x \int_y^x \psi(z-y)dz\, dF(y) = && (2.72)\\
&= \int_y^x \psi(z-y)dz\, F(y)\Big|_{y=0}^{x} - && (2.73)\\
&\quad - \int_0^x \frac{\partial}{\partial y}\left(\int_y^x \psi(z-y)dz\right) F(y)dy = && (2.74)\\
&= \int_0^x \left(\int_y^x \psi'(z-y)dz + \psi(0)\right) F(y)dy = && (2.75)\\
&= \int_0^x \psi(x-y)F(y)dy && (2.76)
\end{aligned}$$

Finally one ends up with the proposed form for $\psi(x)$:

$$\begin{aligned}
\psi(x) &= \frac{\lambda}{p}\left(\int_0^\infty \bar{F}(y)dy + \int_0^x \psi(z)dz - \int_0^x \psi(x-y)F(y)dy - \int_0^x \bar{F}(z)dz\right) = && (2.77)\\
&= \frac{\lambda}{p}\left(\int_x^\infty \bar{F}(y)dy + \int_0^x \psi(x-y)\bar{F}(y)dy\right) && (2.78)
\end{aligned}$$

□

Example 2.3.4. Assume an exponential claim amout distribution $C_n \sim Exp(a)$, with $a > 0$, and $F(x) = 1 - e^{-ax}$ for $x \geq 0$. Hence $f(x) = ae^{-ax}$ and $\mu = \frac{1}{a}$.
Let us consider again equation 2.69 for $\psi'(x)$ and perform the substitution $z = x - y$:

$$\begin{aligned}
\psi'(x) &= \frac{\lambda}{p}\left(\psi(x) - \int_0^x \psi(x-y)dF(y) - \bar{F}(x)\right) && (2.79)\\
\Longleftrightarrow \frac{p}{\lambda}\psi'(x) &= \psi(x) - \int_0^x \psi(z)f(x-z)dz - \bar{F}(x) && (2.80)\\
\Longleftrightarrow \frac{p}{\lambda}\psi'(x) &= \psi(x) - a\int_0^x \psi(z)e^{-a(x-z)}dz - e^{-ax} && (2.81)
\end{aligned}$$

Differentiation according to Leibniz's rule yields the equation

$$\begin{aligned}
\frac{p}{\lambda}\psi''(x) &= \psi'(x) - \psi(x)f(0) - \int_0^x \psi(z)f'(x-z)dz + f(x) && (2.82)\\
\Longleftrightarrow \frac{p}{\lambda}\psi''(x) &= \psi'(x) - a\psi(x) + a^2\int_0^x \psi(z)e^{-a(x-z)}dz + ae^{-ax} && (2.83)
\end{aligned}$$

Thus if one multiplies equation 2.81 by a and adds to equation 2.83, one eliminates the integral and obtains the following differential equation with constant coefficients:

$$\frac{p}{\lambda}\psi''(x) + \left(\frac{p}{\lambda\mu} - 1\right)\psi'(x) = 0 \qquad (2.84)$$

The general solution of this equation is of the form

$$\psi(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}, \tag{2.85}$$

where c_1 and c_2 are arbitrary constants, and r_1 and r_2 are the roots of the corresponding characteristic equation:

$$\frac{p}{\lambda} r^2 + \left(\frac{p}{\lambda\mu} - 1 \right) r = 0 \tag{2.86}$$

Thus

$$r_1 = -\frac{\lambda}{p} \left(\frac{p}{\lambda\mu} - 1 \right) = \frac{\lambda}{p} - a = -\frac{1}{\mu} \cdot \frac{\Lambda}{1+\Lambda} =: -R \qquad \text{and} \qquad r_2 = 0. \tag{2.87}$$

Since $\psi(x) \to 0$ for $x \to \infty$ (which will actually be seen in Theorem 2.3.5.), it follows that $c_2 = 0$. $c_1 = \psi(0) = \frac{1}{1+\Lambda}$ by Theorem 2.3.1., and finally

$$\psi(x) = \frac{1}{1+\Lambda} \cdot e^{-\frac{\Lambda}{1+\Lambda} \cdot \frac{x}{\mu}} = \frac{1}{1+\Lambda} \cdot e^{-Rx}, \quad x \geq 0. \tag{2.88}$$

Hence in case of exponential claim amounts, the probability of ruin is an exponential function of the initial surplus measured in mean claim amounts.

Theorem 2.3.5. *Suppose there is a constant $R > 0$ such that the moment generating function of the claim distribution $M_C(R)$ exists, i.e. $\mathbb{E}[e^{RC_n}] < \infty$, and R solves the* ***Cramér-Lundberg condition***

$$\lambda + pR = \lambda M_C(R), \tag{2.89}$$

then R is called the ***Cramér-Lundberg coefficient*** *(or* ***adjustment coefficient****) and the* ***Cramér-Lundberg inequality*** *holds:*

$$\psi(x) \leq e^{-Rx} \quad \forall x \geq 0. \tag{2.90}$$

Proof.
The Cramér-Lundberg inequality for $\psi_n(x)$ can be shown by induction with respect to n:
First, $\psi_0(x) := \mathbb{P}[T_R^c \leq A_0] = \mathbb{P}[T_R^c = 0]$, the probability that ruin occurs immediately, is zero and hence $\psi_0(x) \leq e^{-Rx}$.
Next, use the double integral equation for $\psi_n(x)$ from the proof of Lemma 2.3.2. together with the induction assumption and the Cramér-Lundberg condition to derive:

$$\psi_n(x) = \int_0^\infty \int_0^\infty \psi_{n-1}(x + pt - y) \lambda e^{-\lambda t} dt \, dF(y) \leq \tag{2.91}$$

$$\leq \int_0^\infty \int_0^\infty e^{-R(x+pt-y)} \lambda e^{-\lambda t} dt \, dF(y) = \tag{2.92}$$

$$= \frac{\lambda \, M_C(R)}{\lambda + pR} e^{-Rx} = e^{-Rx} \tag{2.93}$$

Finally, letting $n \to \infty$ and using the fact that $\psi(x) = \lim_{n\to\infty} \psi_n(x)$, one finds the inequality to hold for $\psi(x)$ as well. $\square$

The Cramér-Lundberg inequality shows the probability of ruin to be exponentially decreasing. Therefore, the probability of ruin gets small even with moderate values for the initial surplus.

Remark: According to Willmot and Lin [WL], if the Cramér-Lundberg condition holds, one can even find an improved Lundberg upper bound of the form

$$\psi(x) \leq \beta e^{-Rx} \quad \forall x \geq 0, \tag{2.94}$$

where β is a constant, given by

$$\frac{1}{\beta} = \inf_{0\leq t<\infty} \frac{\int_t^\infty e^{Ry} dF(y)}{e^{Rt}\bar{F}(t)}, \tag{2.95}$$

that satisfies $0 < \beta \leq 1$.

Example 2.3.6. Assume an exponential claim amout distribution $C_n \sim Exp(a)$, with $a > 0$, and $F(x) = 1 - e^{-ax}$ for $x \geq 0$. Then the Cramér-Lundberg condition reduces to

$$\lambda + pR = \frac{a\lambda}{a-R}, \quad \text{with } R < a. \tag{2.96}$$

Solving the arising quadratic equation for R yields the nontrivial solution

$$R = a - \frac{\lambda}{p}, \tag{2.97}$$

which is positive under the usual assumption that $p > \lambda\mu = \frac{\lambda}{a}$. The arising R turns out to be the same as in Example 2.3.4., where the following explicit solution was found:

$$\psi(x) = \frac{1}{1+\Lambda} e^{-Rx} \leq e^{-Rx}, \quad x \geq 0. \tag{2.98}$$

A short calculation shows that, if the claim sizes are exponentially distributed, the improved Lundberg inequality 2.94 is exact with $\beta = \frac{1}{1+\Lambda}$:

$$\frac{1}{\beta} = \inf_{0\leq t<\infty} \frac{\int_t^\infty e^{Ry} dF(y)}{e^{Rt}\bar{F}(t)} = \inf_{0\leq t<\infty} \frac{a\int_t^\infty e^{(R-a)y} dy}{e^{(R-a)t}} = \frac{a}{a-R} = \frac{p}{\lambda\mu} = 1 + \Lambda \tag{2.99}$$

Remark: Obviously, $R = 0$ is a trivial solution of the Cramér-Lundberg condition. The left hand side of equation 2.89 is an affine function, whereas the right hand side is convex. Moreover, the usual assumption that $p > \lambda\mu$ implies that the derivative of the left side exceeds the derivative of the right side at $R = 0$, and thus the adjustment coefficient exists if $M_C(R)$ exists for all $R > 0$.
One can even show that the following two conditions are each sufficient for the existence of the Cramér-Lundberg coefficient (Hubalek [H]):

1. $M_C(R)$ exists for $0 < R < R_{max}$ and $\lim_{R \nearrow R_{max}} M_C(R) = \infty$.
2. There exists an $R' > 0$ such that $M_C(R')$ exists and $\lambda + pR' \leq \lambda M_C(R')$.

Theorem 2.3.7. *The adjustment coefficient fulfills the following two inequalities.*

1. ***Upper bound:***
$$R < 2\Lambda \frac{\mu}{\mu_2} \tag{2.100}$$
2. ***Lower bound:*** *Provided there is a constant $k > 0$ such that $F(k) = 1$ (claims higher than k are not possible),*
$$R > \frac{1}{k}\ln(1+\Lambda). \tag{2.101}$$

Proof.
1. Dividing the Cramér-Lundberg condition 2.89 by λ and using the power series expansion of e^{Ry}, one derives

$$1 + \mu R(1+\Lambda) = \int_0^\infty e^{Ry} dF(y) > \int_0^\infty (1 + Ry + \frac{R^2y^2}{2}) dF(y) = 1 + R\mu + \frac{R^2\mu_2}{2} \tag{2.102}$$
$$\tag{2.103}$$

Therefore
$$R < 2\Lambda \frac{\mu}{\mu_2}. \tag{2.104}$$

2. The lower bound can be derived in the following way:

$$1 + \mu R(1+\Lambda) = \int_0^\infty e^{Ry} dF(y) \leq \int_0^\infty \left(\frac{y}{k}e^{Rk} + (1 - \frac{y}{k})\right) dF(y) = 1 + \frac{\mu}{k}(e^{Rk} - 1) \tag{2.105}$$

Thus
$$1 + \Lambda \leq \frac{e^{Rk} - 1}{Rk} < e^{Rk}, \tag{2.106}$$

where the last inequality is best seen by comparing corresponding terms in the two power series expansions. Finally
$$R > \frac{1}{k}\ln(1+\Lambda). \tag{2.107}$$

□

2.4. Asymptotic formula for the probability of ruin

This section relies on some major results from renewal theory. Asmussens chapter about renewal theory is used at many points [A].

Definition 2.4.1. Let Z, g and h be functions of a non-negative argument, $h(y) \geq 0$ and $\int_0^\infty h(y)dy < \infty$. Then, a **renewal equation** is defined as the following integral equation

$$Z(x) = g(x) + \int_0^x Z(x-y)h(y)dy = g(x) + Z \star h(x),\ x \geq 0, \tag{2.108}$$

where $\star$ means the convolution integral. If g is bounded on finite intervals, equation 2.108 has a unique solution Z that is bounded on finite intervals as well (see Asmussen [A] for a proof). Defining $H(x) := \int_0^x h(y)dy$, the renewal equation is called

$$\begin{cases} \text{proper} & \text{if } H(\infty) = 1, \\ \text{defective} & \text{if } H(\infty) < 1, \\ \text{excessive} & \text{if } H(\infty) > 1. \end{cases} \tag{2.109}$$

Note that in case $H(\infty) = 1$, h is a probability density function and H the corresponding cumulative distribution function.

Example 2.4.2. Consider the classical Cramér-Lundberg model. The ruin probability ψ satisfies a renewal equation (see Lemma 2.3.3.) with

$$g(x) = \frac{\lambda}{p} \int_x^\infty \bar{F}(y)dy \tag{2.110}$$

and

$$h(x) = \frac{\lambda}{p} \bar{F}(x). \tag{2.111}$$

Since by assumption $H(\infty) = \frac{\lambda\mu}{p} < 1$, this renewal equation is defective.

Theorem 2.4.3. *In the proper case, the unique solution of the renewal equation is given by*

$$Z(t) = g(t) + g \star u(t), \quad t \geq 0, \tag{2.112}$$

where $u(t) := \sum_{n=1}^\infty h^{n\star}(t)$.

Proof.
At first, it is shown that $U(t) := \sum_{n=1}^\infty H^{n\star}(t)$ and $u(t)$ satisfy renewal equations of the proper type. For this purpose, consider

$$\int_0^t U(t-y)h(y)dy = \sum_{n=1}^\infty \int_0^t H^{n\star}(t-y)h(y)dy = \sum_{n=2}^\infty H^{n\star}(t) = U(t) - H(t) \tag{2.113}$$

and thus

$$U(t) = H(t) + \int_0^t U(t-y)h(y)dy, \ t \geq 0. \tag{2.114}$$

Upon differentiation,

$$u(t) = h(t) + \int_0^t u(t-y)h(y)dy, \ t \geq 0. \tag{2.115}$$

Using the relation $u = h + u \star h$, one can show that $Z = g + g \star u$ satisfies the renewal equation for Z:

$$Z \star h = (g + g \star u) \star h = g \star (h + u \star h) = g \star u = Z - g, \tag{2.116}$$

which is equivalent to equation 2.108. □

In order to describe the asymptotic behavior of the solution Z to a renewal equation of the proper type one can apply the so-called Key Renewal Theorem (following Asmussen [A] and Resnick [R]). Before presenting the KRT, one needs the following two definitions.

Definition 2.4.4. A random variable X has a **lattice** distribution if there exist $\alpha \in \mathbb{R}$ and $\beta > 0$ such that

$$\mathbb{P}[X \in \{\alpha + z\beta : \ z \in \mathbb{Z}\}] = 1. \tag{2.117}$$

The largest value of β for which the above equality holds is called the **period** (or **span**) of the random variable X. X is called **non-lattice** if no such β exists. Note that in particular all continuous random variables are non-lattice. Furthermore, if $\alpha = 0$ X may also be called arithmetic.

Definition 2.4.5. Suppose for a while that $z : [0, \infty) \to [0, \infty)$. Setting $h > 0$ and $I_n^h := (nh, (n+1)h]$, $n \in \mathbb{N}_0$, let for $x \in I_n^h$ (see Figure 2.2)

$$\bar{z}_h(x) := \sup_{y \in I_n^h} z(y) \qquad \text{and} \qquad \underline{z}_h(x) := \inf_{y \in I_n^h} z(y). \tag{2.118}$$

Then one calls z **directly Riemann integrable** if $\int_0^\infty \bar{z}_h(x)dx < \infty$ for some (and then all) h and as $h \to 0$ $\int_0^\infty (\bar{z}_h(x) - \underline{z}_h(x))dx \to 0$. If z can attain also negative values, one calls z directly Riemann integrable if both $z^+ := \max(z, 0)$ and $z^- := \max(-z, 0)$ are so.

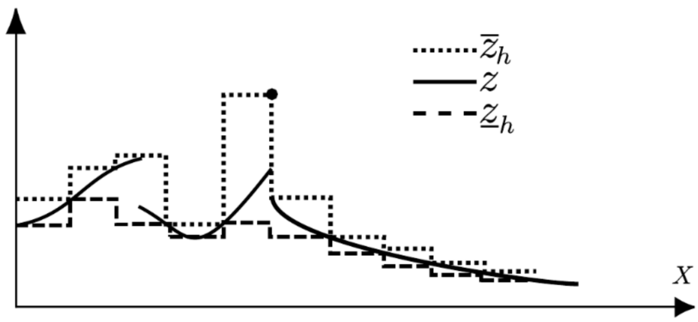


Figure 2.2.: Illustration of functions $\bar{z}_h$ and $\underline{z}_h$

For functions with compact support this concept is the same as Riemann integrability. Since the criterion for direct Riemann integrability is quite technical, there are a few *sufficient conditions* found in the literature, e.g.

(i). z is non-negative, non-increasing and Riemann integrable.

(ii). z is monotone and absolutely integrable.

(iii). z is Riemann integrable and bounded by a directly Riemann integrable function.

Theorem 2.4.6. (Key Renewal Theorem): *Let $Z(t)$ be the solution to a renewal equation of the proper type*

$$Z(t) = g(t) + \int_0^t Z(t-x)dH(x), \quad t \geq 0, \tag{2.119}$$

where H is a non-lattice distribution with mean $\mu := \int_0^\infty x dH(x) > 0$ and $g(t)$ is directly Riemann integrable. Then,

$$\lim_{t\to\infty} Z(t) = \begin{cases} \frac{1}{\mu}\int_0^\infty g(y)dy & \text{if } \mu < \infty, \\ 0 & \text{if } \mu = \infty. \end{cases} \tag{2.120}$$

Proof.
It is referred to Asmussen [A] or Resnick [R] for a detailed proof. □

The following theorem gives an asymptotic formula for the solution to a renewal equation of defective or excessive type. For this purpose, note that two functions $a(x)$ and $b(x)$ are called **asymptotically equivalent** if $\lim_{x\to\infty} \frac{a(x)}{b(x)} = 1$. Asymptotic equivalence is symbolically denoted by $a(x) \sim b(x)$. However, this can be embedded in a more general setting as well: A stronger approximation statement is the so-called little-o notation, where one writes $a(x) = b(x) + o(c(x))$ if $a(x)$ can be approximated by $b(x)$ with an error $e(x)$ getting arbitrarily small compared to a function $c(x)$ as x gets large, i.e. $\lim_{x\to\infty} \frac{e(x)}{c(x)} = 0$. Obviously, asymptotic equivalence can be written in little-o notation as $a(x) = b(x) + o(b(x))$.

Theorem 2.4.7. *If the renewal equation is defective or excessive, g is sufficiently regular (such that $e^{Rx}g(x)$ is bounded on finite intervals as well as directly Riemann integrable or it fulfills one of the sufficient conditions above) and $\int_0^\infty y e^{Ry} h(y)dy < \infty$, then the solution $Z(x)$ is asymptotically exponential:*

$$Z(x) \sim Ce^{-Rx}, \quad \text{for } x \to \infty, \tag{2.121}$$

with

$$C := \frac{\int_0^\infty e^{Ry} g(y)dy}{\int_0^\infty y e^{Ry} h(y)dy} \tag{2.122}$$

and R being the solution of

$$\int_0^\infty e^{Ry} h(y)dy = 1. \tag{2.123}$$

Proof.
Note that R is positive if the renewal equation is defective and negative if it is excessive. Moreover,

$$\tilde{h}(y) := e^{Ry}h(y),\ y \geq 0, \tag{2.124}$$

is a probability density function. Therefore, multiply equation 2.108 by e^{Rx} to obtain the following proper renewal equation:

$$\tilde{Z}(x) = \tilde{g}(x) + \int_0^x \tilde{Z}(x-y)\tilde{h}(y)dy,\ x > 0, \tag{2.125}$$

where

$$\tilde{Z}(x) := e^{Rx}Z(x) \quad \text{and} \quad \tilde{g}(x) := e^{Rx}g(x). \tag{2.126}$$

Hence, one can apply the KRT to obtain

$$\lim_{x\to\infty} \tilde{Z}(x) = \frac{1}{\tilde{\mu}} \int_0^\infty \tilde{g}(y)dy = \frac{\int_0^\infty e^{Ry}g(y)dy}{\int_0^\infty ye^{Ry}h(y)dy} = C. \tag{2.127}$$

□

Corollary 2.4.8. (Cramér-Lundberg approximation):
If the Cramér-Lundberg condition is satisfied with adjustment coefficient $R > 0$ such that $g'(R) = \lambda M_C'(R) < \infty$ (recall that $g(z) = \lambda(M_C(z) - 1)$ is the Lévy exponent of S_t), then the probability of ruin is asymptotically exponential with rate $-R$:

$$\psi(x) \sim Ce^{-Rx}, \quad \text{for } x \to \infty, \tag{2.128}$$

where C is the constant

$$C := \frac{p - g'(0)}{g'(R) - p}. \tag{2.129}$$

Proof.
Recall that $\psi(x)$ satisfies a defective renewal equation (see Lemma 2.3.3.), where $g(x)$ and $h(x)$ were found in Example 2.4.2. Now, substitute the explicit form for $h(x)$ from equation 2.111 into 2.123 to obtain the equation that defines the rate $-R$:

$$\frac{\lambda}{p} \int_0^\infty e^{Ry}\bar{F}(y)dy = 1. \tag{2.130}$$

Upon integration by parts one finds

$$\frac{1}{R}e^{Ry}\bar{F}(y)\Big|_0^\infty + \frac{1}{R}\int_0^\infty e^{Ry}dF(y) = \frac{p}{\lambda}. \tag{2.131}$$

This is equivalent to

$$-\frac{1}{R}+\frac{1}{R}M_C(R)=\frac{p}{\lambda} \quad \Longleftrightarrow \quad \lambda+pR=\lambda M_C(R). \tag{2.132}$$

Hence R satisfies the Cramér-Lundberg condition ($pR = g(R)$) and coincides with the adjustment coefficient. According to Theorem 2.4.7., $\psi(x)$ is asymptotically exponential

$$\psi(x) \sim Ce^{-Rx}, \quad \text{for } x \to \infty, \tag{2.133}$$

which is the famous **asymptotic formula for the probability of ruin**. The constant C is obtained by substituting 2.110 and 2.111 in 2.127:

$$C = \frac{\int_0^\infty e^{Ry}\int_y^\infty \bar{F}(x)dxdy}{\int_0^\infty ye^{Ry}\bar{F}(y)dy} \tag{2.134}$$

The numerator can be simplified by changing the order of integration and using equation 2.130:

$$\int_0^\infty e^{Ry}\int_y^\infty \bar{F}(x)dxdy \;=\; \int_0^\infty \bar{F}(x)\int_0^x e^{Ry}dydx = \tag{2.135}$$

$$= \frac{1}{R}\int_0^\infty \bar{F}(x)(e^{Rx}-1)dx = \frac{1}{R}\left(\frac{p}{\lambda}-\mu\right) \tag{2.136}$$

Since $g'(0)=\lambda\mu$ the constant becomes

$$C = \frac{p-g'(0)}{\lambda R\int_0^\infty ye^{Ry}\bar{F}(y)dy}. \tag{2.137}$$

Applying partial integration and using the Cramér-Lundberg condition yields the stated version for the denominator:

$$\lambda R\int_0^\infty ye^{Ry}\bar{F}(y)dy \;=\; \left(\lambda ye^{Ry}-\frac{\lambda}{R}e^{Ry}\right)\bar{F}(y)\Big|_0^\infty - \int_0^\infty\left(\lambda ye^{Ry}-\frac{\lambda}{R}e^{Ry}\right)d\bar{F}(y) = \tag{2.138}$$

$$= \frac{\lambda}{R}+\lambda\int_0^\infty ye^{Ry}dF(y)-\frac{\lambda}{R}\int_0^\infty e^{Ry}dF(y) = \tag{2.139}$$

$$= \lambda\int_0^\infty ye^{Ry}dF(y)-\frac{\lambda}{R}(M_C(R)-1) = g'(R)-p>0 \tag{2.140}$$

Therefore the final expression for C can be written as

$$C = \frac{p-g'(0)}{g'(R)-p}. \tag{2.141}$$

□

Note that $\psi(x)$ is decreasing in x and its limit for $x \to \infty$ is zero. Hence, the only way to avoid ruin is that the initial surplus increases to infinity.
Moreover, the asymptotic formula for ψ is exact with $C=\beta=\frac{1}{1+\Lambda}$ when the claims are exponentially distributed (see Example 2.3.4.).

3

Queueing theory

In Section 4.3 the proportion of time the Cramér-Lundberg surplus process with a dividend barrier is below some given level will be found. The derivation is based on relations to queueing theory. This motivates the inclusion of a chapter summarizing some important results from queueing theory. They can be found in Asmussen [A].

3.1. Introducing the basic setting

The great diversity of queueing problems gives rise to an enormous variety of models each with their specific features. For the description of a queue the following features are relevant:

(i). The **input or arrival process**, i.e. the way in which the customers arrive to the queue.

(ii). The **service facilities**, i.e. the way in which the system handles a given input stream.

(iii). The **queue discipline**, i.e. the algorithm determining the order in which the customers are served.

In this regard, Kendall's notation system is widely used. It covers some simple and basic queueing systems which have the following characteristics:

(i). Customers arrive one at a time according to a renewal process in discrete or continuous time. That is, the **intervals between successive arrivals** of customers are i.i.d. and governed by a distribution A on $\mathbb{N}$ or $(0, \infty)$. Number the customers $0, 1, 2, ...$ and assume for now, in accordance with Asmussen, that customer 0 arrives at time 0. This assumption can be dropped as well such that the queue starts empty. If T_n denotes the interval between the arrival of customer n and $n+1$, the $\{T_n\}_{n \in \mathbb{N}_0}$ are i.i.d. and the arrival time of customer n is $A_n := \sum_{k=0}^{n-1} T_k$ for $n \in \mathbb{N}$.

(ii). The **service times** of different customers $\{U_n\}_{n \in \mathbb{N}_0}$ are i.i.d. as well and independent of the inter-arrival process $\{T_n\}_{n \in \mathbb{N}_0}$. Denote the governing distribution (concentrated on $(0, \infty)$) by B.

In Kendall's notation, a queueing system of this type is denoted by a string of the type $A/B/s$, where A refers to the form of the **inter-arrival distribution**, B to the form of the **service time distribution** and s is the **number of servers**. The most common values for A and B are as follows:

(i). M **Exponential distribution** (M = Markovian).

(ii). D Degenerate distribution at point $d \in (0, \infty)$, frequently $d = 1$ (D = Deterministic).

(iii). E_k Erlang distribution with k stages.

(iv). H_k Hyperexponential distribution with k parallel channels.

(v). G **General distribution**.

The main types of queue disciplines are listed below. Note that this list is by no means complete and does not cover all aspects.

(i). *FCFS* (**First Come, First Served**) The customers are served in the order of arrival. Throughout the thesis, this is the discipline chosen.

(ii). *LCFS* (Last Come, First Served) After having completed a service the server turns to the latest arrived customer.

(iii). *SIRO* (Service In Random Order) After having served a customer, the server picks the next at random among the remaining ones.

(iv). *PS* (Processor Sharing) The customers share the server, i.e. when n customers are present, the server devotes $1/n$ of his capacity to each.

(v). *RR* (Round Robin) Here the server works on the customers one at a time in a fixed time quantum δ. A customer not having completed service within this time is put back in the queue, and before he can retain service the other customers are each allowed their quantum of δ (or less, if service is completed). As δ becomes infinitely small, PS is obtained as a limiting case of RR.

In connection with a given queueing system, a great variety of stochastic processes and functionals arise. The main ones that one shall study are the following three:

(i). Q_t The **queue length** at time t, i.e. the number of customers currently in the system.

(ii). W_n The **actual waiting time** (or just waiting time) of customer n, i.e. the time from arrival to the system until service starts.

(iii). V_t The **workload** in the system at time t, i.e. the total time the s servers have to work to clear the system provided that no new customers arrive. In the case $s = 1$ of a single server, this is equivalent to the waiting time of a hypothetical customer arriving just after t. For this reason V_t is also denoted the **virtual waiting time** at time t for $s = 1$.

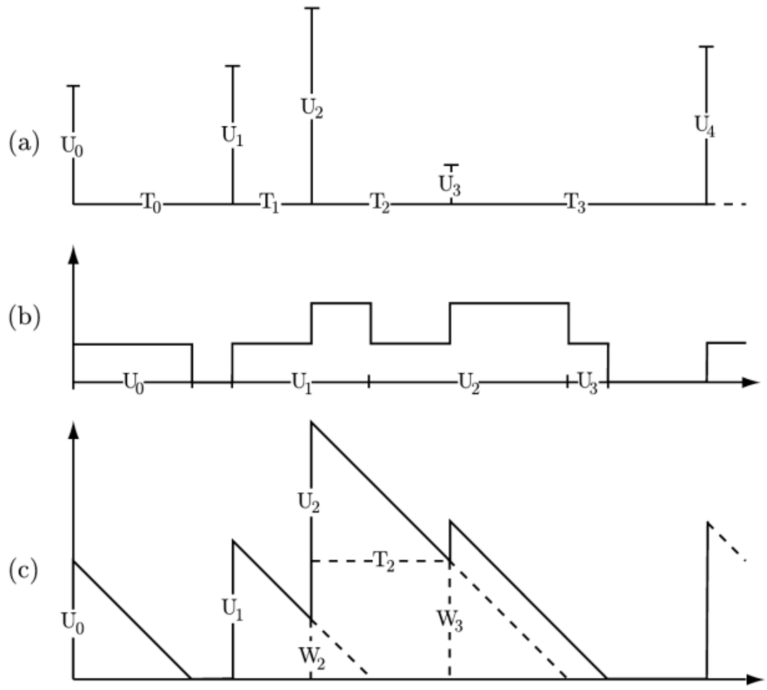


Figure 3.1.: Example of a $G/G/1$ queue: (a) the input of service times $\{U_n\}_{n\in\mathbb{N}_0}$ and inter-arrival times $\{T_n\}_{n\in\mathbb{N}_0}$, (b) the corresponding queue length process $\{Q_t\}_{t\geq 0}$ and (c) the virtual waiting time process $\{V_t\}_{t\geq 0}$.

The following measure of performance of a queueing system is of universal interest.

Definition 3.1.1. The so-called **traffic intensity** ρ of a $G/G/s$ - queue is defined as

$$\rho := \frac{\mathbb{E}[U_k]}{s \cdot \mathbb{E}[T_k]} = \frac{\int_0^\infty x\ B(dx)}{s\int_0^\infty x\ A(dx)} \qquad \text{for } k \in \mathbb{N}_0. \tag{3.1}$$

Suppose that for a very large amount of time t the system is working at full capacity, i.e. that all servers are busy. Then by the law of large numbers there will be about $t/\mathbb{E}[T_k]$ arrivals and a total of $st/\mathbb{E}[U_k]$ services. Thus ρ is about the ratio, i.e. when $\rho > 1$ the number of arrivals exceeds the number of services so that one expects the queue to grow indefinitely. In contrast, when $\rho < 1$ then eventually even a very long initial queue will be cleared (in the sense that not all servers are busy; after that the queue may build up again, but will be cleared up for the same reason, and so on, the system evolving in *cycles*). Thus the behavior should be like *transience* when $\rho > 1$, and like *recurrence* when $\rho < 1$.

The notion of **steady state** is within the setting of Markov processes just what is usually called stationarity: a Markov jump process is in steady state (or in equilibrium) if it is ergodic and stationary.
On intuitive grounds, if the traffic intensity is less than 1, the capacity of the queueing

system is sufficient to deal with the arriving workload. One expects the system to alternate between being *busy* and *idle*, and that the initial conditions will be smoothed away by the stochastic variation in the length of the cycles. Thus, under appropriate conditions there should exist **limiting distributions** of Q_t as well as V_t for $t \to \infty$ and of W_n as $n \to \infty$. In order to study the characteristics of the queueing system one often restricts attention to a steady-state version. This will be represented either by a governing probability distribution $\mathbb{P}_e$ (e for equilibrium) or by denoting the random variable without index, referring to a random variable having the limiting steady-state distribution. The motivation for studying the steady-state versions comes from the following two points. Firstly, a queueing system will frequently be operating for such long periods of time that the steady state is entered rather early in that period. And secondly, the limiting distribution also describes the long-term behavior in terms of time averages.

3.2. Regenerative processes

The classical definition of a stochastic process to be *regenerative* means in intuitive terms that the process can be split into i.i.d. *cycles*. In case of the $G/G/1$ queue length process the cycles are the time intervals separated by the instants S_n with a customer entering an empty system. At each such instant the queue regenerates, i.e. starts completely from scratch independently of the past. Different cycles are independent and all governed by the same probability law. A similar statement holds for the workload.

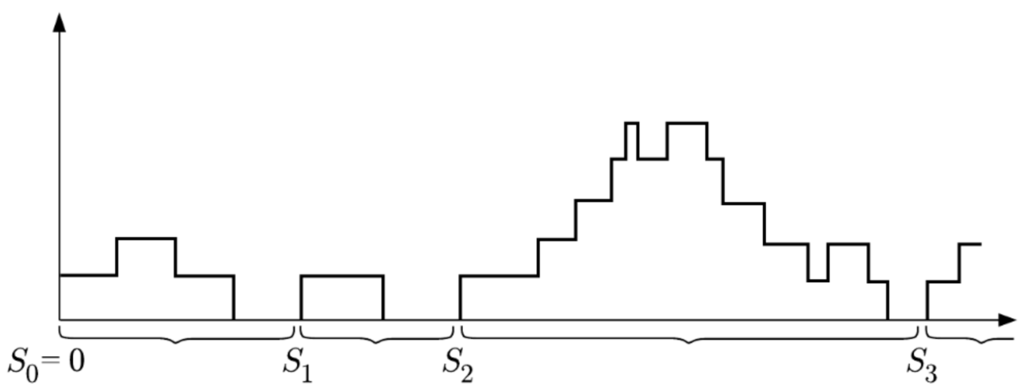


Figure 3.2.: $G/G/1$ queue length process $\{Q_t\}_{t\geq 0}$ as an example for a regenerative process with instants S_n.

However, one should use a slightly wider definition:

Definition 3.2.1. Assume $\{X_t\}_{t\in\mathbb{T}}$ is a stochastic process with $\mathbb{T} = \mathbb{N}_0$ or $\mathbb{T} = [0, \infty)$ and state space E. Then, the process $\{X_t\}_{t\in\mathbb{T}}$ is called (pure or delayed) **regenerative** if there exists a (pure or delayed) renewal process $A_n = \sum_{k=0}^{n} I_k$, $n \in \mathbb{N}_0$, such that the post-A_n process

$$\theta_{A_n} X := (I_{n+1}, I_{n+2}, \ldots, \{X_{A_n+t}\}_{t\in\mathbb{T}}), \qquad \text{for } n \in \mathbb{N}_0, \tag{3.2}$$

is independent of $A_0, \ldots, A_n$ (or, equivalently, of $I_0, \ldots, I_n$) and its distribution does not depend upon n. One calls $\{A_n\}_{n\in\mathbb{N}_0}$ the **embedded renewal process** and refers to the A_n as **regeneration points**. The k-th **cycle** is $\{X_{A_{k-1}+t}\}_{0\le t\le I_k}$ for $k \in \mathbb{N}$.

To a given delayed regenerative process, there clearly exists a zero-delayed one with a unique propability law (e.g. $\{X_{A_0+t}\}_{t\in\mathbb{T}}$). Let $\mathbb{P}_0$ and $\mathbb{E}_0$ correspond to the zero-delayed case and write $I = I_1$ for the length of the first cycle having mean $\mathbb{E}_0[I]$.

A trivial but noteworthy property is that the regenerative property is preserved under mappings (nothing like that is true for say a Markov process):

Proposition 3.2.2. *If $\{X_t\}_{t\in\mathbb{T}}$ is regenerative and $\varphi : E \to F$ any measurable mapping, then $\{\varphi(X_t)\}_{t\in\mathbb{T}}$ is regenerative with the same embedded renewal process.*

The power of the concept of regenerative processes lies in the existence of a limiting distribution under conditions that are very mild and usually easy to verify.

Theorem 3.2.3. *Assume that a (possibly delayed) regenerative process $\{X_t\}_{t\in\mathbb{T}}$ has metric state space, right-continuous paths and non-lattice cycle length distribution F with finite mean. Then the limiting distribution $\mathbb{P}_e$ exists and is given by*

$$\mathbb{E}[f(X)] = \mathbb{E}_e[f(X_t)] = \frac{1}{\mathbb{E}_0[I]}\mathbb{E}_0\left[\int_0^I f(X_s)ds\right]. \tag{3.3}$$

Proof.
It is immediately checked that

$$A \to \frac{1}{\mathbb{E}_0[I]}\mathbb{E}_0\left[\int_0^I \mathbb{1}_{\{X_s\in A\}}ds\right], \quad \text{with } A \in \mathcal{B}(E), \tag{3.4}$$

defines a probability measure. Hence by standard facts on weak convergence it is sufficient to prove that $\mathbb{E}[f(X_t)] \to \mathbb{E}_e[f(X_t)]$ as $t \to \infty$, whenever f is continuous with $0 \le f \le 1$. By defining

$$Z(t) := \mathbb{E}_0[f(X_t)], \quad g(t) := \mathbb{E}_0[f(X_t)\mathbb{1}_{\{t<I\}}] \quad \text{and} \quad F_0^*(x) := \mathbb{P}[I_0 \le x] \tag{3.5}$$

and conditioning on I_0, it follows that

$$\mathbb{E}[f(X_t)] = \mathbb{E}[f(X_t)\mathbb{1}_{\{t<I_0\}}] + \mathbb{E}[f(X_t)\mathbb{1}_{\{t\ge I_0\}}] = \tag{3.6}$$

$$= \mathbb{E}[f(X_t)\mathbb{1}_{\{t<I_0\}}] + \int_0^t \mathbb{E}_0[f(X_{t-x})]F_0^*(dx) \tag{3.7}$$

$$= \mathbb{E}[f(X_t)\mathbb{1}_{\{t<I_0\}}] + \int_0^t Z(t-x)F_0^*(dx). \tag{3.8}$$

Similarly one finds

$$Z(t) = \mathbb{E}_0[f(X_t)] = \mathbb{E}_0[f(X_t)\mathbb{1}_{\{t<I\}}] + \mathbb{E}_0[f(X_t)\mathbb{1}_{\{t\geq I\}}] = \tag{3.9}$$

$$= \mathbb{E}_0[f(X_t)\mathbb{1}_{\{t<I\}}] + \int_0^t \mathbb{E}_0[f(X_{t-x})]F(dx) \tag{3.10}$$

$$= g(t) + \int_0^t Z(t-x)F(dx). \tag{3.11}$$

Hence letting $t \to \infty$ in equation 3.8 one sees that it is even sufficient to show that $Z(t) \to \mathbb{E}_e[f(X_t)]$. Applying the key renewal theorem (Theorem 2.4.6.) to the above renewal equation for $Z(t)$ with $\mathbb{E}_0[I] < \infty$ by assumption yields for $t \to \infty$

$$Z(t) \to \frac{1}{\mathbb{E}_0[I]}\int_0^\infty g(s)ds = \frac{1}{\mathbb{E}_0[I]}\int_0^\infty \mathbb{E}_0[f(X_s)\mathbb{1}_{\{s<I\}}]ds = \frac{1}{\mathbb{E}_0[I]}\mathbb{E}_0\left[\int_0^I f(X_s)ds\right]. \tag{3.12}$$

It remains to show that g is directly Riemann integrable by using the third sufficient condition. g is Riemann integrable since it is bounded and continuous a.e. due to its right-continuity. In particular, $g(t) \leq \mathbb{P}_0[I > t] = \bar{F}(t)$ and $\bar{F}(t)$ is directly Riemann integrable by the first sufficient condition: It is non-negative, non-increasing and Riemann integrable. □

3.3. Lindley processes

It is quite common that in a particular queueing model one or more of the processes of interest may be related to a process that is *Lindley* or at least of a somewhat similar structure.

Definition 3.3.1. By a **Lindley process**, one understands a discrete time process of the form

$$W_0 = w, \qquad W_{n+1} = (W_n + X_n)^+, \quad n \in \mathbb{N}_0, \tag{3.13}$$

where $w \geq 0$ and $\{X_n\}_{n\in\mathbb{N}_0}$ are i.i.d. with common distribution F.

Example 3.3.2. Consider the $G/G/1$ queue with waiting time W_n of customer n. Say that customer n arrives at time t and customer $n+1$ at $t + T_n$. The residual work in the system is W_n just before t, $W_n + U_n$ just after t and W_{n+1} just before $t + T_n$. Since the residual work decreases at a unit linear rate in between arrivals so long as it is positive,

$$W_{n+1} = \max(W_n + U_n - T_n, 0). \tag{3.14}$$

Hence 3.13 holds with $X_n := U_n - T_n$ and clearly the X_n are i.i.d. Therefore, the waiting time process $\{W_n\}_{n\in\mathbb{N}_0}$ in the $G/G/1$ queue is Lindley.

By defining the **partial sums**

$$S_0 := 0, \qquad S_n := \sum_{k=0}^{n-1} X_k, \quad n \in \mathbb{N}, \tag{3.15}$$

the following relation between the paths of $\{S_n\}_{n\in\mathbb{N}_0}$ and $\{W_n\}_{n\in\mathbb{N}_0}$ can be derived.

Proposition 3.3.3. *For $n \in \mathbb{N}_0$ it holds that*

$$W_n = \max(W_0 + S_n, S_n - S_1, \dots, S_n - S_{n-1}, 0). \tag{3.16}$$

Proof.
Both inequalities will be shown.
By comparing the definitions of S_n and W_n one immediately sees that the increments of $\{W_n\}_{n\in\mathbb{N}_0}$ are at least those of $\{S_n\}_{n\in\mathbb{N}_0}$ so that

$$W_n - W_{n-k} \geq S_n - S_{n-k} \quad \text{for } k \in \{0, \dots, n\}. \tag{3.17}$$

Letting $k = n$ yields $W_n \geq W_0 + S_n$ and using $W_{n-k} \geq 0$ one has $W_n \geq S_n - S_{n-k}$ for all $k \in \{0, \dots, n\}$. Thus, one has that

$$W_n \geq \max(W_0 + S_n, S_n - S_1, \dots, S_n - S_{n-1}, 0). \tag{3.18}$$

For the converse, one shows that either $W_n = W_0 + S_n$ or $W_n = S_n - S_{n-k}$ for some k. Obviously, the first case occurs if $W_0 + S_k \geq 0$ for all $k \in \{0, \dots, n\}$. Otherwise, there exists an $l \in \{0, \dots, n\}$ where $W_l = 0$. Choosing k as the largest index l where $W_l = 0$, the definition of W_n in equation 3.13 yields $W_n = S_n - S_{n-k}$. □

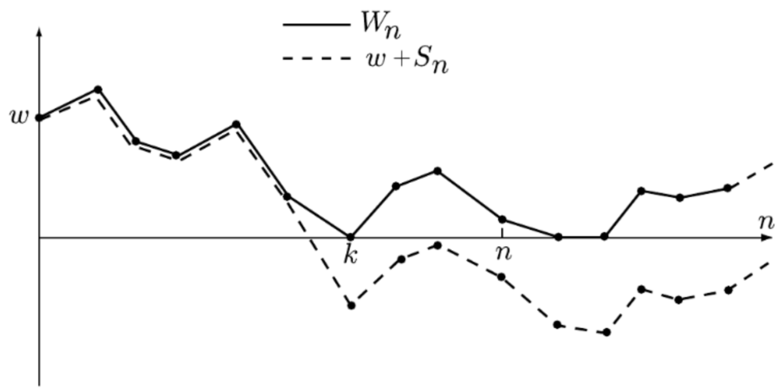


Figure 3.3.: Illustration of the relation between the paths of $\{S_n\}_{n\in\mathbb{N}_0}$ and $\{W_n\}_{n\in\mathbb{N}_0}$.

Now define the **partial maximum** of the first n partial sums as

$$M_n := \max_{0\leq k\leq n} S_k, \quad n \in \mathbb{N}_0, \tag{3.19}$$

with $M := M_\infty = \max_{0 \le k \le \infty} S_k$. Since the distribution of $(S_n, S_n - S_1, \dots, S_n - S_{n-1}, 0)$ is the same as the distribution of $(S_n, S_{n-1}, \dots, S_1, S_0 = 0)$ one gets the following corollary.

Corollary 3.3.4. *It holds that*

$$W_n \overset{d}{=} \max(W_0 + S_n, M_{n-1}) \tag{3.20}$$

and in particular, if $W_0 = 0$, then

$$W_n \overset{d}{=} M_n. \tag{3.21}$$

Suppose now $\mathbb{E}[|X_n|] < \infty$ and define $\mu := \mathbb{E}[X_n]$ for all n.

Corollary 3.3.5. *If $\mu < 0$, then $M < \infty$ a.s. and $W_n \overset{d}{\to} M$ (and in total variation) as $n \to \infty$.*

Proof.
Due to the strong law of large numbers $S_n/n \overset{a.s.}{\to} \mu$ and thus $S_n \overset{a.s.}{\to} -\infty$. This implies in particular $M < \infty$ a.s. Also $W_0 + S_n \overset{a.s.}{\to} -\infty$ and $M_n \nearrow M$ a.s. and in distribution. Thus $W_n \overset{d}{=} \max(W_0 + S_n, M_{n-1}) \overset{d}{=} M_{n-1}$ eventually and $W_n \overset{d}{\to} M$ follows. □

3.4. Reflected Lévy processes

In continuous time, the definition of a **reflected version** $\{V_t\}_{t \ge 0}$ of a Lévy process $\{S_t\}_{t \ge 0}$ is less obvious than in discrete time. The definition used in Asmussen [A] is the continuous-time analogue of Proposition 3.3.3., i.e.

$$V_t := (V_0 + S_t) \vee \max_{0 \le s \le t} (S_t - S_s), \quad t \ge 0. \tag{3.22}$$

Defining $M_T := \sup_{0 \le t \le T} S_t$ for $T \ge 0$ with $M := M_\infty = \sup_{0 \le t \le \infty} S_t$, Asmussen shows

Proposition 3.4.1. *$\{V_t\}_{t \ge 0}$ is a strong Markov process and*

$$V_T \overset{d}{=} (V_0 + S_T) \vee M_T \quad \text{for all } T \ge 0. \tag{3.23}$$

Moreover, if $\mu < 0$, then $M < \infty$ and $V_T \to M$ in total variation as $T \to \infty$.

Example 3.4.2. Consider a compound Poisson process with only positive jumps and a negative drift

$$S_t := \sum_{k=1}^{N_t} U_k - t, \quad t \ge 0, \tag{3.24}$$

where $\{N_t\}_{t\geq 0}$ is a Poisson process with intensity β and $\{U_k\}_{k\in\mathbb{N}_0}$ are i.i.d. with common distribution B concentrated on $(0,\infty)$ and independent of $\{N_t\}_{t\geq 0}$. The reflection then means that the downward drift at unit rate is cut off when $V_t = 0$. Thus, $\{V_t\}_{t\geq 0}$ with $V_0 = U_0$ has the same upward jumps as $\{S_t\}_{t\geq 0}$ and a downward drift at unit rate in positive states so that one recognizes $\{V_t\}_{t\geq 0}$ as the $M/G/1$ virtual waiting time process.

3.5. Steady-state properties of $G/G/1$

From now on consider the $G/G/1$ queue. Let $\mu_A = \mathbb{E}[T_n]$ denote the **inter-arrival mean** and $\mu_B = \mathbb{E}[U_n]$ the **mean service time** (both are assumed finite throughout). In addition, denote the mean of $X_n = U_n - T_n$ by

$$\mu := \mathbb{E}[X_n] = \mu_B - \mu_A. \tag{3.25}$$

Then the cases $\mu < 0$, $\mu = 0$ and $\mu > 0$ correspond to $\rho < 1$, $\rho = 1$ and $\rho > 1$, since $\rho = \mu_B/\mu_A$ with this notation. Putting Example 3.3.2., Proposition 3.3.3. and Corollaries 3.3.4. as well as 3.3.5. together one has that

Proposition 3.5.1. *The (actual) waiting time process $\{W_n\}_{n\in\mathbb{N}_0}$ is a Lindley process generated by $\{S_n\}_{n\in\mathbb{N}_0}$, i.e. $W_{n+1} = (W_n + X_n)^+$ for $n \in \mathbb{N}_0$. In particular,*

$$W_n = \max(S_n, S_n - S_1, \ldots, S_n - S_{n-1}, 0) \overset{d}{=} M_n \tag{3.26}$$

and if $\rho < 1$, then a limiting steady-state distribution exists and is given by

$$\mathbb{P}_e[W_n \leq x] = \mathbb{P}[M \leq x]. \tag{3.27}$$

Now define $\sigma(0) := 0$, $\sigma(1) := \sigma := \inf\{n \geq 1 : \ W_n = 0\}$ and for $k \in \mathbb{N}$ $\sigma(k+1) := \inf\{n > \sigma(k) : \ W_n = 0\}$. Since $W_0 = 0$ one may interpret σ as the number of customers served in the first busy period and $\sigma(k)$ as the **index of the customer initiating the k-th busy cycle**.

Proposition 3.5.2. *The $\sigma(k)$ are regeneration points for the waiting time process $\{W_n\}_{n\in\mathbb{N}_0}$. One has $\sigma < \infty$ a.s. if and only if $\rho \leq 1$, and $\mathbb{E}[\sigma] < \infty$ if and only if $\rho < 1$. Hence for $\rho \leq 1$, $\{W_n\}_{n\in\mathbb{N}_0}$ is regenerative with embedded renewal sequence $\{\sigma(k)\}_{k\in\mathbb{N}_0}$. Furthermore, $\sigma = \sigma(1)$ coincides with the entrance time of the partial sums $\{S_n\}_{n\in\mathbb{N}_0}$ to $(-\infty, 0]$*

$$\sigma = \tau_- := \inf\{n \in \mathbb{N} : \ S_n \leq 0\}. \tag{3.28}$$

One has that

$$W_n = S_n = \sum_{k=0}^{n-1} (U_k - T_k), \quad n = 0, \ldots, \sigma - 1, \tag{3.29}$$

$$-S_\sigma = -S_{\tau_-} = I, \tag{3.30}$$

*where I is the **idle period** corresponding to the first busy cycle.*

Proof.
Since $\{W_n\}_{n\in\mathbb{N}_0}$ is a Lindley process generated by $\{S_n\}_{n\in\mathbb{N}_0}$ according to Proposition 3.5.1., one has

$$\begin{aligned} 0 < W_n &= S_n = \sum_{k=0}^{n-1}(U_k - T_k), \quad n = 0,\dots,\sigma-1, && (3.31)\\ 0 = W_\sigma &= (S_\sigma)^+. && (3.32) \end{aligned}$$

Therefore, $S_\sigma \leq 0$ and $\sigma = \tau_-$.
The idle period I is the amount by which the last inter-arrival time exceeds the residual work at the time of the last arrival in the cycle, yielding

$$I = T_{\sigma-1} - (W_{\sigma-1} + U_{\sigma-1}) = -(S_{\sigma-1} + U_{\sigma-1} - T_{\sigma-1}) = -S_\sigma = -S_{\tau_-}. \tag{3.33}$$

It is clear that the $\sigma(k)$ are regeneration points, and by general random walk results one has finally

$$\sigma = \tau_- < \infty \text{ a.s.} \quad \Longleftrightarrow \quad \mu \leq 0 \quad \Longleftrightarrow \quad \rho \leq 1 \tag{3.34}$$

and

$$\mathbb{E}[\sigma] = \mathbb{E}[\tau_-] = \frac{1}{\mu}\mathbb{E}[S_{\tau_-}] < \infty \quad \Longleftrightarrow \quad \mu < 0 \quad \Longleftrightarrow \quad \rho < 1. \tag{3.35}$$

□

In continuous time, there is a regenerative structure for the workload $\{V_t\}_{t\geq 0}$ similar to the one for $\{W_n\}_{n\in\mathbb{N}_0}$ in Proposition 3.5.2.: the instants with a customer entering an empty queue are regeneration points. Letting C be the first such instant after $t = 0$ and recalling that one starts with customer 0 having just arrived, it is seen that C is just the **length of the first busy cycle**. Furthermore, $C < \infty$ a.s. is equivalent to $\sigma < \infty$ a.s., i.e. to $\rho \leq 1$. In fact, there is a close relation between σ, C and the **first busy period** G: since precisely the customers $0, 1, \dots, \sigma - 1$ are served in the first busy period, one has $G = \sum_{k=0}^{\sigma-1} U_k$ and the first busy cycle ends at the arrival $C = \sum_{k=0}^{\sigma-1} T_k$ of customer σ. One checks immediately that $\{\sigma \leq n\}$ is independent of $T_n, T_{n+1}, \dots, U_n, U_{n+1}, \dots$ and hence Wald's identity yields the first part of

Proposition 3.5.3. *Suppose $\rho \leq 1$. Then the mean busy cycle is $\mathbb{E}[C] = \mu_A\mathbb{E}[\sigma]$, the mean busy period is $\mathbb{E}[G] = \mu_B\mathbb{E}[\sigma]$ and the mean idle period is $\mathbb{E}[I] = \mathbb{E}[C] - \mathbb{E}[G] = -\mu\mathbb{E}[\sigma]$. Moreover the cycle length C is non-lattice if and only if the inter-arrival distribution A is so.*

For a proof of the second part of this proposition see Asmussen [A] for details.

Note that the mean busy cycle can be expressed in terms of the mean idle period as

$$\mathbb{E}[C] = \mu_A \mathbb{E}[\sigma] = \frac{\mu_A}{\mu_A - \mu_B}\mathbb{E}[I] = \frac{1}{1-\rho}\mathbb{E}[I]. \tag{3.36}$$

Corollary 3.5.4. *Suppose $\rho < 1$ and that A is non-lattice. Then a limiting steady-state distribution of the workload $\{V_t\}_{t\geq 0}$ exists and is given by*

$$\mathbb{E}[f(V)] = \frac{1}{\mathbb{E}[C]} \cdot \mathbb{E}\left[\int_0^C f(V_s)ds\right]. \tag{3.37}$$

Proof.
For $\rho < 1$ one has that $\mathbb{E}[\sigma] < \infty$ by Proposition 3.5.2. Hence Proposition 3.5.3. ensures that $\mathbb{E}[C] < \infty$ and that the cycle length C is non-lattice. The basic limit theorem for regenerative processes in Theorem 3.2.3. is applicable. □

As a first application of Corollary 3.5.4., note that the time spent by $\{V_t\}_{t\geq 0}$ in state 0 in the time interval $[0, C)$ is just the idle period. Thus combining with Proposition 3.5.3., one gets

$$\mathbb{P}[V=0] = \frac{1}{\mathbb{E}[C]} \cdot \mathbb{E}\left[\int_0^C \mathbb{1}_{\{V_s=0\}}ds\right] = \frac{\mathbb{E}[I]}{\mathbb{E}[C]} = \frac{(\mu_A - \mu_B)\mathbb{E}[\sigma]}{\mu_A \mathbb{E}[\sigma]} = 1-\rho. \tag{3.38}$$

Theorem 3.5.5. *If $\rho < 1$, the steady-state workload V and the steady-state waiting time W in the $M/G/1$ queue have the same distribution, i.e.*

$$V \stackrel{d}{=} W. \tag{3.39}$$

Proof.
The idea is to compare the maximum representations of the random variables W and V in steady-state. Combining Corollary 3.3.5. with Example 3.3.2. and Proposition 3.4.1. with Example 3.4.2. yields

$$W \stackrel{d}{=} \max\{0, U_0 - T_0, U_0 + U_1 - T_0 - T_1, \ldots\}, \tag{3.40}$$

$$V \stackrel{d}{=} \max_{0\leq t<\infty}\{S_t^{\uparrow} - t\}, \tag{3.41}$$

with $S_t^{\uparrow} := \sum_{k=1}^{N_t} U_k$ for $t \geq 0$. Then $\{S_t^{\uparrow} - t\}_{t\geq 0}$ increases only at the arrival times $A_n = \sum_{k=0}^{n-1} T_k$, $n \in \mathbb{N}$, so that the maximum is attained either at one of these times or at time 0 where $S_0^{\uparrow} = 0$. Now just note that by sample path insection

$$S_{A_n}^{\uparrow} - A_n = \sum_{k=1}^{n} U_k - \sum_{k=0}^{n-1} T_k \stackrel{d}{=} \sum_{k=0}^{n-1}(U_k - T_k), \quad \text{for } n \in \mathbb{N}, \tag{3.42}$$

so that the two maxima above are equal. □

4

Cramér-Lundberg with a dividend barrier

An advanced version of the classical Cramér-Lundberg model is obtained if the risk process in between claims is allowed to vary deterministically but not necessarily linearly. For example, if the surplus is invested at a certain interest rate or **all earnings above a given level are paid out as dividend**.

In this chapter the ideas of J. Irbäck [I] are followed. He studied the Cramér-Lundberg model including a dividend barrier. For this purpose, assume the risk process $\{X_t\}_{t\geq 0}$ to satisfy the storage equation

$$X_t = u + \int_0^t p(X_s)\, ds - S_t, \tag{4.1}$$

where $u > 0$ is the initial surplus and S_t denotes aggregate claims up to time t. As usual $\{S_t\}_{t\geq 0}$ is modeled as a compound Poisson process with intensity $\lambda > 0$ and distribution function F of claim amounts. But now, the premium income rate $p(.)$ depends on the current surplus.

Assuming there is a **dividend barrier** $b > 0$ at which a dividend at rate $p > 0$ is paid out until the next claim, the **premium income rate** has the following form:

$$p(x) := p \cdot \mathbb{1}_{\{x \leq b\}}, \quad x \geq 0. \tag{4.2}$$

Thus the **surplus process** becomes

$$X_t := u + p \int_0^t \mathbb{1}_{\{X_s \leq b\}}\, ds - S_t, \quad t \geq 0. \tag{4.3}$$

In this modified model, ruin occurs with probability 1, but the question of interest is when it does so. Also it is interesting to know the proportion of time the surplus is below some given level. Figure 4.1 shows a typical sample path of the surplus process $\{X_t\}_{t\geq 0}$ in this model.

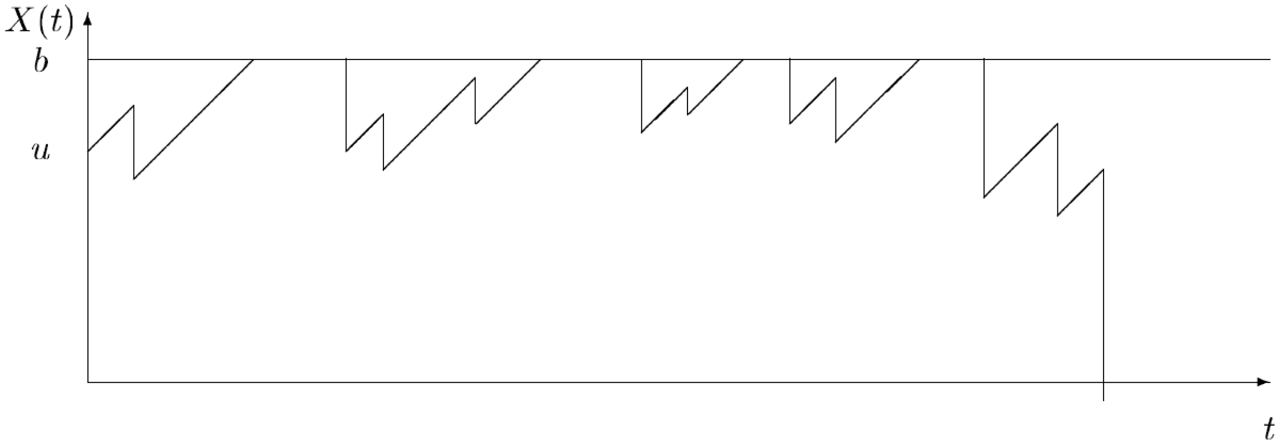


Figure 4.1.: Sample path of the surplus process $\{X_t\}_{t\geq 0}$ in the Cramér-Lundberg model with a dividend barrier

In this chapter a simple approximation formula for the distribution of the time of ruin will be found (in case of a high barrier b). There are two cases that are distinguished subsequently:

1. **An initial surplus just below a high dividend barrier** (Section 4.1)
   The risk process will move towards the dividend barrier and visit it a large geometric number of times before ruin. Ruin will occur after a very long exponentially distributed time.

2. **An initial surplus much lower than a high dividend barrier** (Section 4.2)
   Either ruin will occur quite soon without visit to the dividend barrier or the risk process will reach the dividend barrier and ruin occurs according to the same exponential distribution as in the first case.

4.1. An initial surplus just below a high dividend barrier

Throughout this chapter assume that b is large and x very small in comparison ($x \ll b$). Set the initial surplus to $u = b - x$, which is close to the high dividend barrier. Then the risk process has the form

$$X_t = b - x + p \int_0^t \mathbb{1}_{\{X_s \leq b\}}\, ds - S_t. \tag{4.4}$$

As usual the compound Poisson process $\{S_t\}_{t\geq 0}$ is defined by its moment generating function (recall that $g(z) = \lambda(M_C(z) - 1)$ is the Lévy exponent)

$$M_{S_t}(z) = e^{t\lambda \int_0^\infty (e^{zx}-1)F(dx)} = e^{tg(z)}. \tag{4.5}$$

The **time of ruin** in this model is defined as

$$T_R := \inf\{t \geq 0 | X_t < 0\} \tag{4.6}$$

and one wants to derive an asymptotic formula for its **tail distribution**

$$\mathbb{P}_{b-x}[T_R > t] := \mathbb{P}[T_R > t | X_0 = b - x]. \tag{4.7}$$

Furthermore the following *two assumptions* are useful to be made:

(i). The safety loading Λ is positive, i.e. $p > \lambda\mu = g'(0)$.

(ii). The Cramér-Lundberg condition is satisfied for some $R > 0$, i.e. $\exists\ R > 0$ such that $g(R) = pR$, and in addition $g'(R) < \infty$.

By Markov's inequality, the second assumption implies for $x \geq 0$

$$\begin{aligned} 1 - F(x) &= \mathbb{P}[C_n \geq x] = \mathbb{P}[e^{RC_n} \geq e^{Rx}] \leq & (4.8)\\ &\leq \mathbb{E}[e^{RC_n}]e^{-Rx} = M_C(R)e^{-Rx} = \left(\frac{g(R)}{\lambda} + 1\right)e^{-Rx} = \left(\frac{pR}{\lambda} + 1\right)e^{-Rx}. & (4.9) \end{aligned}$$

This means the probability of a large claim is very small. At the start the risk process will, with a very high probability, move towards the dividend barrier and perform a recurrent motion in the vicinity of it. The path can be partitioned into excursions between successive visits to the dividend barrier. Denote the **time of the k-th visit to the dividend barrier** by T_k. Mathematically, T_k can be defined recursively as ($T_0 = 0$)

$$T_k = \inf\{t > T_{k-1} | X_t = b, \exists\ s \in (T_{k-1}, t) : X_s < b, \nexists\ s \in (T_{k-1}, t) : X_s < 0\},\ k \in \mathbb{N}. \tag{4.10}$$

In addition, denote the **number of completed excursions before ruin** by

$$N := \min((\#T_k | T_k < T_R) - 1, 0). \tag{4.11}$$

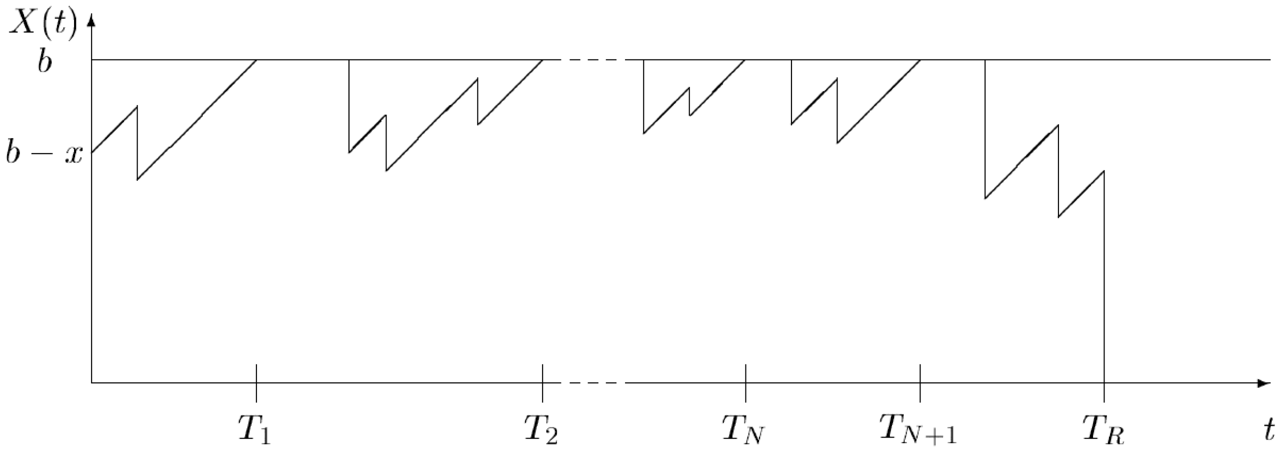


Figure 4.2.: Typical sample path of the surplus process $\{X_t\}_{t\geq 0}$ in the Cramér-Lundberg model with dividend barrier in case of a large initial surplus $b - x$.

Since a compound Poisson process is strong Markov, the surplus process $\{X_t\}_{t\geq 0}$ is **strong Markov** as well, meaning for arbitrary stopping time τ w.r.t. the natural filtration $F_t := \sigma(\{X_s\}_{0\leq s\leq t})$, $t \geq 0$, it holds for any Borel set A a.s. on $\{\tau < \infty\}$ that

$$\mathbb{P}[X_{\tau+t} \in A|F_\tau] = \mathbb{P}[X_t \in A|X_0 = X_\tau], \tag{4.12}$$

where F_τ is the stopping time σ-algebra. Intuitively, given the history F_τ, the process evolves from then on as restarted at time 0 in state X_τ, depending on F_τ through X_τ only. This implies the lengths of completed excursions are independent and identically distributed:

$$T_{k+1} - T_k = \tag{4.13}$$
$$= \inf(t > 0 : \{X_{T_k+t} = b, \exists\, s \in (0,t) : X_{T_k+s} < b, \nexists\, s \in (0,t) : X_{T_k+s} < 0\}|X_{T_k} = b) = \tag{4.14}$$
$$= \inf(t > 0 : \{X_t = b, \exists\, s \in (0,t) : X_s < b, \nexists\, s \in (0,t) : X_s < 0\}|X_0 = b), \tag{4.15}$$

where $k \in \{1, \dots, N\}$. Thus the time of ruin can be written as a random sum of independent excursions

$$T_R = T_1 + \sum_{k=1}^{N}(T_{k+1} - T_k) + (T_R - T_{N+1}). \tag{4.16}$$

There is a small **probability of ruin during each excursion**

$$\rho := \mathbb{P}[T_{k+1} - T_k = \infty|T_k < \infty]. \tag{4.17}$$

Conditional on the surplus process reaching the dividend barrier, the number of completed excursions before ruin N has a **geometric distribution**

$$\begin{aligned}
\mathbb{P}[N = n|T_1 < \infty] &= \mathbb{P}[T_{n+1} < \infty, T_{n+2} - T_{n+1} = \infty|T_1 < \infty] = && (4.18)\\
&= \mathbb{P}[T_{n+1} < \infty|T_1 < \infty] \cdot \mathbb{P}[T_{n+2} - T_{n+1} = \infty|T_{n+1} < \infty] = && (4.19)\\
&= \mathbb{P}[T_2 - T_1 < \infty, \dots, T_{n+1} - T_n < \infty|T_1 < \infty] \cdot \rho = && (4.20)\\
&= \mathbb{P}[T_2 - T_1 < \infty, \dots, T_n - T_{n-1} < \infty|T_1 < \infty] \cdot (1-\rho) \cdot \rho = && (4.21)\\
&= \dots = (1-\rho)^n \cdot \rho, \quad n \in \mathbb{N}_0. && (4.22)
\end{aligned}$$

Since ρ is small, N will be rather large. To be precise, N has expected value $\frac{1-\rho}{\rho}$. Thus the law of large numbers yields that

$$\sum_{k=1}^{N}(T_{k+1} - T_k) \overset{d}{\approx} Nm, \tag{4.23}$$

where

$$m := E[T_{k+1} - T_k|T_{k+1} < \infty] \tag{4.24}$$

is the **expected length of a complete excursion**. If T_1 and $T_R - T_{N+1}$ are small in probability compared with $\sum_{k=1}^{N}(T_{k+1} - T_k)$, one has that

$$T_R \overset{d}{\approx} Nm. \tag{4.25}$$

Now, by the limit representation of the exponential function, the distribution of the time of ruin can be approximated by an exponential distribution with mean $\frac{m}{\rho}$:

$$\mathbb{P}_{b-x}[T_R > t] \approx \mathbb{P}_{b-x}[N > \frac{t}{m}] \approx (1-\rho)^{\lfloor \frac{t}{m} \rfloor} \approx e^{-\frac{\rho}{m}t} \tag{4.26}$$

As opposed to the heuristic considerations above, the following theorem (it is the **main result in this section**) will be proved rigorously. It does not only describe the asymptotic distribution of the time of ruin, it also gives asymptotic formulas for the expected length of a complete excursion and the probability of ruin during an excursion. Its proof is quite extensive and will need a variety of lemmas in advance.

Theorem 4.1.1. (Asymptotic laws): *The time of ruin T_R has asymptotically an exponential distribution with expected value $\frac{m}{\rho}$, i.e.*

$$\mathbb{P}_{b-x}[T_R > t] \sim e^{-\frac{\rho}{m}t} \qquad \text{as } \ b \to \infty. \tag{4.27}$$

The expected length of a complete excursion m satisfies

$$m \sim \frac{1}{\lambda} + \frac{\mu}{p - \lambda\mu} \qquad \text{as } \ b \to \infty, \tag{4.28}$$

and for the probability of ruin during an excursion ρ one finds

$$\rho \sim \frac{CpR}{\lambda} e^{-Rb} \qquad \text{as } \ b \to \infty, \tag{4.29}$$

where R is the adjustment coefficient and $C := \frac{p-g'(0)}{g'(R)-p}$ the constant in the Cramér-Lundberg approximation (see Corollary 2.4.8.).

The proof of this theorem follows from the lemmas below.

First of all, let

$$\psi(b-x, b) := \mathbb{P}[T_R < T_1 | X_0 = b - x] \tag{4.30}$$

be the **probability of ruin without visit to the dividend barrier** (starting at $u = b - x$).

Lemma 4.1.2. *The probability of ruin without visit to the dividend barrier can be computed from the probability of ruin in the classical model (without dividend barrier) as*

$$\psi(b-x, b) = \frac{\psi(b-x) - \psi(b)}{1 - \psi(b)}. \tag{4.31}$$

Moreover, when x is fixed

$$\psi(b-x, b) \sim Ce^{-Rb}(e^{Rx} - 1) \qquad \text{as } b \to \infty. \tag{4.32}$$

Proof.
Let T_1^c be the **time of the first visit to level** b **of the classical risk process** $\{X_t^c\}_{t\geq 0}$ without dividend barrier:

$$T_1^c := \inf\{t \geq 0 | X_t^c \geq b\} \tag{4.33}$$

Then $T_1^c < \infty$ a.s. since the safety loading Λ is positive by assumption. By the strong Markov property of $\{X_t^c\}_{t\geq 0}$, it follows that

$$\begin{aligned}
1-\psi(b-x) &= \mathbb{P}[T_R^c = \infty | X_0^c = b-x] = && (4.34)\\
&= \mathbb{P}[T_R^c = \infty | T_1^c < T_R^c, X_0^c = b-x] \cdot \mathbb{P}[T_1^c < T_R^c | X_0^c = b-x] = && (4.35)\\
&= \mathbb{P}[T_R^c = \infty | T_1^c < T_R^c, X_{T_1^c}^c = b, X_0^c = b-x]\mathbb{P}[T_1^c < T_R^c | X_0^c = b-x] = && (4.36)\\
&= \mathbb{P}[T_R^c = \infty | X_0^c = b] \cdot (1 - \mathbb{P}[T_1^c > T_R^c | X_0^c = b-x]) = && (4.37)\\
&= \mathbb{P}[T_R^c = \infty | X_0^c = b] \cdot (1 - \mathbb{P}[T_1 > T_R | X_0 = b-x]) = && (4.38)\\
&= (1-\psi(b)) \cdot (1-\psi(b-x,b)). && (4.39)
\end{aligned}$$

Note that in the penultimate step it is used that the risk processes $\{X_t^c\}_{t\geq 0}$ and $\{X_t\}_{t\geq 0}$ evolve equally as long as the barrier b is not attained. Hence one finds the stated formula

$$\psi(b-x,b) = \frac{\psi(b-x)-\psi(b)}{1-\psi(b)}. \tag{4.40}$$

In addition, applying the Cramér-Lundberg approximation (see Corollary 2.4.8.) yields

$$\psi(b-x,b) \sim \frac{Ce^{-R(b-x)} - Ce^{-Rb}}{1-Ce^{-Rb}} \sim Ce^{-Rb}(e^{Rx}-1) \qquad \textit{as } b \to \infty. \tag{4.41}$$

□

Since b is chosen large and x rather small in comparison, the probability of hitting the dividend barrier before ruin is almost equal to 1. The next step is to find an asymptotic law for the probability of ruin during an excursion ρ, which is the parameter of the geometric distribution for N.
Each complete excursion can be decomposed in two independent parts, a holding time at the dividend barrier and a time from the risk process leaving the dividend barrier until the next visit. Denote the **holding time at the dividend barrier after the k-th visit** by Y_k. Since the inter-arrival times of a Poisson process $\{I_i\}_{i\in\mathbb{N}}$ are independent and exponentially distributed with parameter λ and due to the memorylessness property of the exponential distribution, the holding times Y_k are exponentially distributed as well: For arbitrary $y > 0$,

$$\mathbb{P}[Y_k > y] = \mathbb{P}[I_{n(k)+1} > T_k - \sum_{i=1}^{n(k)} I_i + y | I_{n(k)+1} > T_k - \sum_{i=1}^{n(k)} I_i] = \mathbb{P}[I_{n(k)+1} > y], \tag{4.42}$$

where $n(k)$ is the **number of claims before the k-th visit** to the dividend barrier.

Lemma 4.1.3. *The probability of ruin during an excursion ρ satisfies*

$$\rho \sim \frac{CpR}{\lambda} e^{-Rb} \qquad as \;\; b \to \infty, \tag{4.43}$$

where R is the adjustment coefficient and $C := \frac{p-g'(0)}{g'(R)-p}$ the constant in the Cramér-Lundberg approximation (see Corollary 2.4.8.).

Proof.
By conditioning on the size of the claim which brings the risk process away from the dividend barrier after the k-th visit (law of total probability) and by the equivalence in distribution of $X_{T_k+Y_k+s}|\{X_{T_k+Y_k} = b-x\}$ and $X_s|\{X_0 = b-x\}$ for $s \geq 0$ due to the strong Markov property of $\{X_t\}_{t\geq 0}$, one derives

$$\begin{aligned} \rho &= \mathbb{P}[T_{k+1} - T_k = \infty | T_k < \infty] = & (4.44)\\ &= \int_0^\infty \mathbb{P}[T_{k+1} - T_k = \infty | T_k < \infty, X_{T_k+Y_k} = b-x] \; F(dx) = & (4.45)\\ &= \int_0^\infty \mathbb{P}[T_R < T_1 | X_0 = b-x] \; F(dx) = \int_0^\infty \psi(b-x,b) \; F(dx). & (4.46) \end{aligned}$$

Now, take a small $\epsilon > 0$ and split the integral in the following way:

$$\rho = \int_0^{b(1-\epsilon)} \psi(b-x,b) \; F(dx) + \int_{b(1-\epsilon)}^\infty \psi(b-x,b) \; F(dx) \tag{4.47}$$

In the first integral, one has for the surplus after the drop from the barrier $b - x \in [\epsilon \cdot b \;, \; b]$. Hence each initial surplus tends to infinity for $b \to \infty$, which is why one can use the asymptotic formula for $\psi(b-x,b)$ from Lemma 4.1.2.:

$$\begin{aligned} \int_0^{b(1-\epsilon)} \psi(b-x,b) \; F(dx) &\sim Ce^{-Rb} \int_0^{b(1-\epsilon)} (e^{Rx} - 1) \; F(dx) \sim & (4.48)\\ &\sim Ce^{-Rb} \underbrace{\int_0^\infty (e^{Rx} - 1) \; F(dx)}_{=M_C(R)-1} = \frac{CpR}{\lambda} e^{-Rb} \quad as \; b \to \infty, & (4.49) \end{aligned}$$

where in the last step the Cramér-Lundberg condition is applied.
Next, one finds the following estimate for the second integral in equation 4.47:

$$\begin{aligned} \int_{b(1-\epsilon)}^\infty \psi(b-x,b) \; F(dx) &\leq \int_{b(1-\epsilon)}^\infty F(dx) = \int_{b(1-\epsilon)}^\infty e^{-Rx(1+2\epsilon)} e^{Rx(1+2\epsilon)} \; F(dx) \leq & (4.50)\\ &\leq e^{-Rb(1-\epsilon)(1+2\epsilon)} \cdot M_C(R(1+2\epsilon)) & (4.51) \end{aligned}$$

Since ϵ is chosen small, $M_C(R(1+2\epsilon)) \approx \frac{pR}{\lambda} + 1 < \infty$ and $R(1-\epsilon)(1+2\epsilon) > R$. Hence the second integral will be very small compared with the first as $b \to \infty$:

$$\frac{\int_{b(1-\epsilon)}^\infty \psi(b-x,b) F(dx)}{\frac{CpR}{\lambda} e^{-Rb}} \leq \frac{\lambda M_C(R(1+2\epsilon))}{CpR} e^{-(R(1-\epsilon)(1+2\epsilon)-R)b} \longrightarrow 0. \tag{4.52}$$

Eventually it follows that

$$\rho \sim \int_0^{b(1-\epsilon)} \psi(b-x,b)\; F(dx) \sim \frac{CpR}{\lambda} e^{-Rb} \qquad as\ b \to \infty. \tag{4.53}$$

□

The goal of the following three lemmas is to show the total length of completed excursions $\sum_{k=1}^{N}(T_{k+1}-T_k)$ to be asymptotically exponential, which is finally achieved by finding its **Laplace transform**. However, this requires some preparation.

Denote the **Esscher transform** of the physical measure $\mathbb{P}$ by $\mathbb{P}^{(z)}$:

$$\frac{d\mathbb{P}^{(z)}}{d\mathbb{P}} = e^{zS_t - tg(z)}, \tag{4.54}$$

where $M_z(t) := e^{zS_t - tg(z)}$ (the likelihood ratio process) can be recognized as **Wald's martingale**. Let us check the martingale property of $\{M_z(t)\}_{t\geq 0}$ w.r.t. the natural filtration $\{F_t\}_{t\geq 0} := \{\sigma((S_u)_{0\leq u\leq t})\}_{t\geq 0}$. Using that $\{S_t\}_{t\geq 0}$ is a Lévy process (independent and stationary increments) with Lévy exponent g, it holds for all $0 \leq s \leq t$

$$\begin{aligned}
\mathbb{E}[M_z(t)|F_s] &= \mathbb{E}[e^{zS_t - tg(z)}|F_s] = e^{zS_s - sg(z)}\mathbb{E}[e^{z(S_t-S_s)-(t-s)g(z)}|F_s] = & (4.55)\\
&= e^{zS_s - sg(z)} \underbrace{\mathbb{E}[e^{z(S_{t-s})-(t-s)g(z)}]}_{=1} = M_z(s). & (4.56)
\end{aligned}$$

Under the new measure $\mathbb{P}^{(z)}$, the moment generating function of S_t is given by

$$\begin{aligned}
M_{S_t}^{(z)}(y) &:= \mathbb{E}^{(z)}[e^{yS_t}] = \mathbb{E}[e^{(z+y)S_t - tg(z)}] = & (4.57)\\
&= e^{t(g(z+y)-g(z))} = e^{t\lambda\int_0^\infty (e^{yx}-1)e^{zx}dF(x)} = & (4.58)\\
&= e^{t\lambda M_C(z)\int_0^\infty (e^{yx}-1)\frac{e^{zx}}{M_C(z)}dF(x)} = e^{t\lambda^{(z)}\int_0^\infty (e^{yx}-1)dF^{(z)}(x)}, & (4.59)
\end{aligned}$$

where $\lambda^{(z)} := \lambda M_C(z) = \lambda + g(z)$ is the intensity and $F^{(z)}(t) := \frac{1}{M_C(z)}\int_0^t e^{zx}dF(x)$, $t \geq 0$, the distribution function of the claim amounts of $\{S_t\}_{t\geq 0}$ under the new measure $\mathbb{P}^{(z)}$. Therefore $\{S_t\}_{t\geq 0}$ is still a compound Poisson process.
The surplus process has the new drift $p - g'(z)$

$$\mathbb{E}^{(z)}[X_t] = b - x + pt - \mathbb{E}^{(z)}[S_t] = b - x + pt - M_{S_t}^{(z)\prime}(0) = b - x + t(p - g'(z)) \tag{4.60}$$

and variance

$$\mathbb{V}^{(z)}[X_t] = M_{S_t}^{(z)\prime\prime}(0) - M_{S_t}^{(z)\prime}(0)^2 = t^2g'(z)^2 + tg''(z) - t^2g'(z)^2 = tg''(z). \tag{4.61}$$

Next, let T_f be the **first passage time out from the interval (0,b)**, i.e. the stopping time

$$T_f := \min(T_1, T_R) = \inf(t \geq 0 | X_t \notin (0,b)), \tag{4.62}$$

which is finite a.s. w.r.t. $\mathbb{P}$ as well as $\mathbb{P}^{(z)}$.
One would like to stop the likelihood ratio martingale $\{M_z(t)\}_{t\geq 0}$ at time T_f. For a general martingale $\{M_t\}_{t\geq 0}$, the criteria for optional stopping at τ (i.e. $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$) usually involve uniform integrability of $\{M_{t\wedge\tau}\}_{t\geq 0}$ (i.e. $\sup_{t\geq 0}\mathbb{E}[|M_{t\wedge\tau}|\cdot \mathbb{1}_{\{M_{t\wedge\tau}>N\}}] \to 0$, as $N\to\infty$). For likelihood ratio martingales, a different sort of criterion is available:
In case of a change of measure to $\tilde{\mathbb{P}}$ with the likelihood ratio process $\{L_t\}_{t\geq 0}$, Asmussen [A] finds for any stopping time τ and non-negative, F_τ-measureable W that

$$\mathbb{E}[W\cdot \mathbb{1}_{\{\tau<\infty\}}] = \tilde{\mathbb{E}}\left[\frac{W}{L_\tau}\cdot \mathbb{1}_{\{\tau<\infty\}}\right]. \tag{4.63}$$

Corollary 4.1.4. *If τ is an almost surely finite stopping time (i.e. $\mathbb{P}[\tau<\infty]=1$), then*

$$\mathbb{E}[L_\tau]=1 \quad\Longleftrightarrow\quad \tilde{\mathbb{P}}[\tau<\infty]=1. \tag{4.64}$$

Proof.
Using Asmussen's result from equation 4.63 with choice $W = L_\tau$ and that τ is a.s. finite w.r.t. $\mathbb{P}$, yields

$$\mathbb{E}[L_\tau] = \tilde{\mathbb{P}}[\tau<\infty]. \tag{4.65}$$

□

Applying the above result to the Esscher transform $\mathbb{P}^{(z)}$, Wald's martingale $\{M_z(t)\}_{t\geq 0}$ and the first passage time T_f gives

$$1 = \mathbb{P}^{(z)}[T_f<\infty] = \mathbb{E}[M_z(T_f)] = \mathbb{E}[e^{zS_{T_f}-T_f g(z)}] = \mathbb{E}_{b-x}[e^{z(b-x-X_{T_f})-T_f(g(z)-pz)}], \tag{4.66}$$

where the index $b-x$ indicates that $X_0 = b-x$.

Now, the function

$$f(z) := g(z) - pz, \tag{4.67}$$

appearing in equation 4.66, is analyzed (see Figure 4.3). It has the following properties:

(i). $f(z)$ is strictly convex.

(ii). $f(0) = f(R) = 0$.

(iii). $f(z)$ takes its minimum at the point z_p satisfying $g'(z_p) = p$.

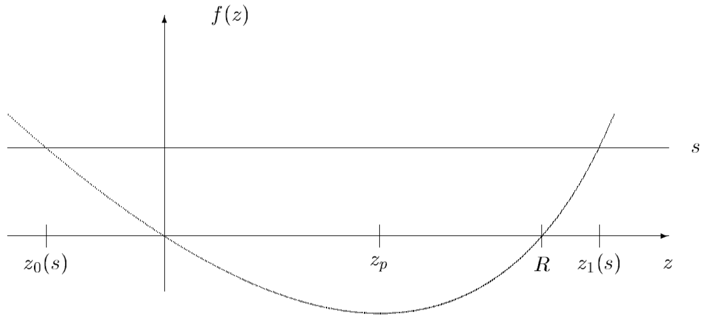


Figure 4.3.: $f(z) = g(z) - pz$

These properties imply that, for any $s > f(z_p)$, the equation

$$g(z) - pz = s \tag{4.68}$$

has **two different roots** $z_0(s)$ and $z_1(s)$, where $z_0(s) < z_1(s)$. Hence relation 4.66 can be written as

$$\mathbb{E}_{b-x}[e^{z_{0,1}(s)(b-x-X_{T_f})-sT_f}] = 1, \quad \text{for } s > f(z_p), \tag{4.69}$$

which is the **key to the remaining proofs** in this section.

The next lemma gives the Laplace transform of the time to the first visit (or next visit) to the dividend barrier from a predetermined level close to b.

Lemma 4.1.5. *Conditional on the risk process reaching the dividend barrier, the Laplace transform of the time to the first visit to the dividend barrier satisfies for $s \geq 0$*

$$\mathbb{E}_{b-x}[e^{-sT_1}|T_1 < T_R] \sim \mathbb{E}_{b-x}[e^{-sT_1}\mathbb{1}_{\{T_1<T_R\}}] \sim e^{xz_0(s)} \quad \textit{as } b \to \infty, \tag{4.70}$$

where $z_0(s)$ is the smallest root of equation 4.68.

Proof.
Equation 4.69 with the root $z_0(s)$ yields

$$\begin{aligned} 1 &= \mathbb{E}_{b-x}[e^{z_0(s)(b-x-X_{T_f})-sT_f}] = & (4.71)\\ &= e^{-xz_0(s)}\mathbb{E}_{b-x}[e^{-sT_1}\mathbb{1}_{\{T_1<T_R\}}] + e^{z_0(s)(b-x)}\mathbb{E}_{b-x}[e^{-z_0(s)X_{T_R}-sT_R}\mathbb{1}_{\{T_R<T_1\}}]. & (4.72)\end{aligned}$$

Take $s \geq 0$, then $z_0(s) \leq 0$ and $-z_0(s)X_{T_R} \leq 0$ on $\{T_R < T_1\}$. Hence

$$\begin{aligned} |\mathbb{E}_{b-x}[e^{-sT_1}\mathbb{1}_{\{T_1<T_R\}}] - e^{xz_0(s)}| &= e^{z_0(s)b}\mathbb{E}_{b-x}[e^{-z_0(s)X_{T_R}-sT_R}\mathbb{1}_{\{T_R<T_1\}}] \leq & (4.73)\\ &\leq e^{z_0(s)b}\mathbb{P}_{b-x}[T_R < T_1] \to 0 \quad \text{as } b \to \infty, & (4.74)\end{aligned}$$

where, by Lemma 4.1.2., $\mathbb{P}_{b-x}[T_R < T_1] \to 0$ as $b \to \infty$. This matches the stated asymptotic behavior

$$\mathbb{E}_{b-x}[e^{-sT_1}|T_1 < T_R] \sim e^{xz_0(s)} \quad \text{as } b \to \infty. \tag{4.75}$$

□

The following lemma finds the Laplace transform of the length of a complete excursion conditional on ruin not occurring during this excursion. Moreover, an asymptotic law for the expected length of a complete excursion is found.

Lemma 4.1.6. *The Laplace transform of the length of a complete excursion satisfies for $s \geq 0$*

$$\mathbb{E}[e^{-s(T_{k+1}-T_k)}|T_{k+1} < \infty] \sim \frac{\lambda + g(z_0(s))}{\lambda + s} \quad \textit{as } b \to \infty, \tag{4.76}$$

and the expected length of a complete excursion satisfies

$$m \sim \frac{1}{\lambda} + \frac{\mu}{p - \lambda\mu} \quad \textit{as } b \to \infty. \tag{4.77}$$

Proof.
By independence of the two parts of an excursion and since the holding time at the barrier Y_k is independent of ruin occurring or not

$$\mathbb{E}[e^{-s(T_{k+1}-T_k)}|T_{k+1}<\infty] = \mathbb{E}[e^{-sY_k}]\cdot\mathbb{E}[e^{-s(T_{k+1}-T_k-Y_k)}|T_{k+1}<\infty] = \tag{4.78}$$

$$= \frac{\lambda}{\lambda+s}\cdot\int_0^b \mathbb{E}[e^{-s(T_{k+1}-T_k-Y_k)}|T_{k+1}<\infty, X_{T_k+Y_k}=b-x]dF(x) \tag{4.79}$$

where the second step follows from the Laplace transform of an exponential distribution as well as the law of iterated expectations conditioning on the size of the claim occurring when the process is at the barrier. Next, using the equivalence in distribution

$$T_{k+1}-T_k-Y_k|\{T_{k+1}<\infty, X_{T_k+Y_k}=b-x\} \stackrel{d}{=} T_1|\{T_1<T_R, X_0=b-x\} \tag{4.80}$$

one finds that

$$\mathbb{E}[e^{-s(T_{k+1}-T_k)}|T_{k+1}<\infty] = \frac{\lambda}{\lambda+s}\int_0^b \mathbb{E}_{b-x}[e^{-sT_1}|T_1<T_R]dF(x). \tag{4.81}$$

Applying the asymptotic law from Lemma 4.1.5., using that the probability of a large jump is small by equations 4.8-4.9 (with $z_0(s)\le 0$) and the definition of g yields

$$\mathbb{E}[e^{-s(T_{k+1}-T_k)}|T_{k+1}<\infty] \sim \frac{\lambda}{\lambda+s}\int_0^\infty e^{xz_0(s)}dF(x) \sim \tag{4.82}$$

$$\sim \frac{\lambda+g(z_0(s))}{\lambda+s} \quad \textit{as } b\to\infty. \tag{4.83}$$

For the second part of the proof, differentiate the Laplace transform and evaluate at 0:

$$m = \mathbb{E}[T_{k+1}-T_k|T_{k+1}<\infty] = -\frac{d}{ds}\mathbb{E}[e^{-s(T_{k+1}-T_k)}|T_{k+1}<\infty]\Big|_{s=0} \sim \tag{4.84}$$

$$\sim \frac{-g'(z_0(s))z_0'(s)(\lambda+s)+\lambda+g(z_0(s))}{(\lambda+s)^2}\Big|_{s=0} \sim \tag{4.85}$$

$$\sim \frac{-g'(0)z_0'(0)\lambda+\lambda+g(0)}{\lambda^2} \sim \frac{1}{\lambda}-\mu z_0'(0) \quad \text{as } b\to\infty. \tag{4.86}$$

The derivative of z_0 at 0 is found by differentiation of $s=g(z_0(s))-pz_0(s)$ w.r.t. s:

$$g'(z_0(s))z_0'(s)-pz_0'(s)=1 \implies z_0'(0)=\frac{1}{g'(0)-p}=\frac{1}{\lambda\mu-p} \tag{4.87}$$

Eventually, the expected length of a complete excursion has the asymptotic law

$$m \sim \frac{1}{\lambda}+\frac{\mu}{p-\lambda\mu} \quad \text{as } b\to\infty. \tag{4.88}$$

□

Recall the lengths of completed excursions $T_{k+1} - T_k,\ k \in \{1, \dots, N\}$, are iid by the strong Markov property. Moreover they are independent of the number of completed excursions before ruin N, which in turn is geometrically distributed with parameter ρ given by Lemma 4.1.3. The next lemma yields the desired asymptotic result for the total length of completed excursions.

Lemma 4.1.7. *The Laplace transform of the total length of completed excursions satisfies for $s \geq 0$*

$$\mathbb{E}[e^{-s\frac{\rho}{m}\sum_{k=1}^{N}(T_{k+1}-T_k)}] \sim \frac{1}{1+s} \quad \text{as } b \to \infty, \tag{4.89}$$

which means the total length of completed excursions is asymptotically exponential with expected value $\frac{m}{\rho}$.

Proof.
The law of iterated expectations together with the independence of the lengths of completed excursions $T_{k+1} - T_k,\ k \in \{1, \dots, N\}$, implies

$$\mathbb{E}[e^{-s\frac{\rho}{m}\sum_{k=1}^{N}(T_{k+1}-T_k)}] = \mathbb{E}[\mathbb{E}[e^{-s\frac{\rho}{m}(T_{k+1}-T_k)}|T_{k+1} < \infty]^N]. \tag{4.90}$$

Now, the Laplace transform of each complete excursion from Lemma 4.1.6. and the **generating function** of $N \sim G(\rho)$

$$g_N(t) := \mathbb{E}[t^N] = \frac{\rho}{1 - t(1-\rho)}, \quad \text{for } t < \left|\frac{1}{1-\rho}\right|, \tag{4.91}$$

yield for $b \to \infty$

$$\mathbb{E}[e^{-s\frac{\rho}{m}\sum_{k=1}^{N}(T_{k+1}-T_k)}] \sim \mathbb{E}\left[\left(\frac{\lambda + g(z_0(\frac{s\rho}{m}))}{\lambda + \frac{s\rho}{m}}\right)^N\right] = \frac{\rho}{1 - \frac{\lambda + g(z_0(\frac{s\rho}{m}))}{\lambda + \frac{s\rho}{m}}(1-\rho)}. \tag{4.92}$$

A first-order Taylor approximation of $g(z_0(.))$ about 0 gives a linear approximation for $\frac{s\rho}{m} \to 0$ (recall that $\rho \to 0$ as $b \to \infty$ by Lemma 4.1.3.):

$$g(z_0(\frac{s\rho}{m})) = g'(0)z_0'(0)\frac{s\rho}{m} + R_1(\frac{s\rho}{m}) \quad \text{as } b \to \infty, \tag{4.93}$$

where the remainder term $R_1(\frac{s\rho}{m}) \in o(\frac{s\rho}{m})$ as $b \to \infty$ and the little-o notation means $\lim_{b\to\infty} \frac{R_1(\frac{s\rho}{m})}{\frac{s\rho}{m}} = 0$. This implies asymptotic equivalence:

$$g(z_0(\frac{s\rho}{m})) \sim g'(0)z_0'(0)\frac{s\rho}{m} = \lambda\mu \cdot \frac{1}{\lambda\mu - p} \cdot \frac{s\rho}{m} = -\frac{\lambda\mu s\rho}{(p-\lambda\mu)m} \quad \text{as } b \to \infty, \tag{4.94}$$

where $z_0'(0)$ was found in equation 4.87. Hence for $b \to \infty$

$$\mathbb{E}[e^{-s\frac{\rho}{m}\sum_{k=1}^{N}(T_{k+1}-T_k)}] \sim \frac{\lambda\rho + \frac{s\rho^2}{m}}{\lambda + \frac{s\rho}{m} + (\frac{\lambda\mu s\rho}{(p-\lambda\mu)m} - \lambda)(1-\rho)} = \tag{4.95}$$

$$= \frac{\lambda + \frac{s\rho}{m}}{\frac{s}{m} + \frac{\lambda\mu s}{(p-\lambda\mu)m} - \frac{\lambda\mu s\rho}{(p-\lambda\mu)m} + \lambda} \sim \frac{\lambda}{\frac{s}{m} + \frac{\lambda\mu s}{(p-\lambda\mu)m} + \lambda} \tag{4.96}$$

where the last step follows since $\rho \to 0$ by Lemma 4.1.3. and $m \sim \frac{1}{\lambda} + \frac{\mu}{p-\lambda\mu} > 0$ by Lemma 4.1.6. Finally, the stated asymptotic law arises:

$$\mathbb{E}[e^{-s\frac{\rho}{m}\sum_{k=1}^{N}(T_{k+1}-T_k)}] \sim \frac{\lambda m}{s + \frac{\lambda\mu s}{p-\lambda\mu} + \lambda m} \sim \frac{\lambda m}{s\lambda m + \lambda m} = \frac{1}{1+s} \quad \text{as } b \to \infty. \tag{4.97}$$

□

The following two lemmas give **exponential bounds** for the time to the first visit T_1 conditional on reaching the dividend barrier, the last holding time at the barrier Y_{N+1} as well as the **time from the risk process leaving the dividend barrier for the last time until ruin** $T_L := T_R - T_{N+1} - Y_{N+1}$. The bounds are then used to show that the times T_1 and $T_R - T_{N+1}$ are small in probability compared with the sum of completed excursions $\sum_{k=1}^{N}(T_{k+1} - T_k)$ as $b \to \infty$. Two of these bounds are given in terms of the **Legendre transform** of $g(z)$, defined by

$$h(x) := \max_z (xz - g(z)). \tag{4.98}$$

Proposition 4.1.8. *The Legendre transform h has the following properties:*

(i). $h(x)$ is strictly convex,

(ii). $h(x) \geq 0$ and $h(g'(0)) = h(\lambda\mu) = 0$,

(iii). $t \cdot h(p + \frac{1}{t}) \geq R$ and strictly convex for $t > 0$, with equality for $\bar{t} := \frac{1}{g'(R)-p}$.

Proof.
See Appendix A. □

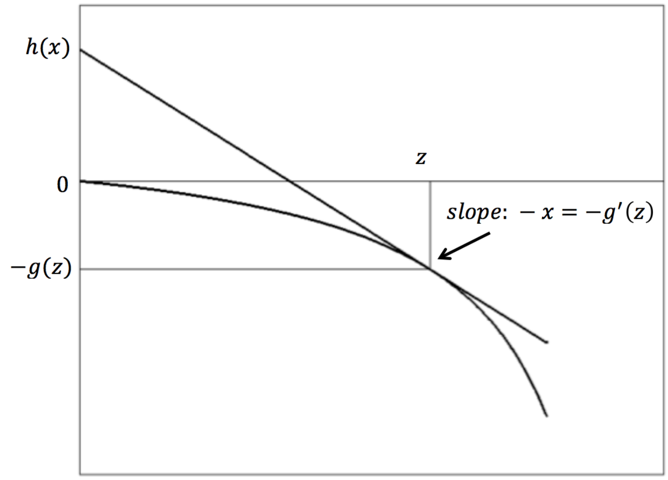


Figure 4.4.: Legendre transform $h(x)$.

Lemma 4.1.9. *The time to the first visit to the dividend barrier T_1 has the following bound. If $t > \frac{x}{p-\lambda\mu}$ then*

$$\mathbb{P}_{b-x}[T_1 > t | T_1 < T_R] \leq \frac{e^{-t\cdot h(p-\frac{x}{t})}}{1 - \psi(b-x, b)} \tag{4.99}$$

and $h(p - \frac{x}{t}) > 0$. Moreover, $h(p - \frac{x}{t}) \nearrow h(p) > 0$ as $t \to \infty$, and thus T_1 is small in probability compared with the sum of completed excursions $\sum_{k=1}^{N}(T_{k+1} - T_k)$ as $b \to \infty$.

Proof.
Recall that equation 4.69 with the root $z_0(s)$, where $s > f(z_p)$, can be written as

$$1 = \underbrace{e^{-xz_0(s)}\mathbb{E}_{b-x}[e^{-sT_1}\mathbb{1}_{\{T_1<T_R\}}]}_{\leq 1} + e^{z_0(s)(b-x)}\mathbb{E}_{b-x}[e^{-z_0(s)X_{T_R}-sT_R}\mathbb{1}_{\{T_R<T_1\}}] \tag{4.100}$$

since $X_{T_1} = b$ on $\{T_1 < T_R\}$. It follows that

$$\mathbb{E}_{b-x}[e^{-sT_1}\mathbb{1}_{\{T_1<T_R\}}] \leq e^{xz_0(s)}. \tag{4.101}$$

If one takes $s < 0$ such that $0 < z_0(s) < z_p$ (see Figure 4.3), then $e^{-st} < e^{-sT_1}$ on $\{T_1 > t\}$ and consequently

$$e^{-st} \cdot \mathbb{P}_{b-x}[\{T_1 > t\} \cap \{T_1 < T_R\}] \leq e^{xz_0(s)}. \tag{4.102}$$

Substituting $s = g(z) - pz$ yields

$$\mathbb{P}_{b-x}[\{T_1 > t\} \cap \{T_1 < T_R\}] \leq e^{-t((p-\frac{x}{t})z-g(z))}. \tag{4.103}$$

The exponent is optimized by the z-value satisfying $g'(z) = p - \frac{x}{t}$. Since $g'(z)$ is strictly increasing, the optimal z-value is positive for $p - \frac{x}{t} > g'(0) = \lambda\mu$, i.e. for $t > \frac{x}{p-\lambda\mu}$. Using that the Legendre transform $h(x)$ has its minimum value 0 at $\lambda\mu$ and is positive for $x \neq \lambda\mu$ (second property) gives that $h(p - \frac{x}{t}) > h(\lambda\mu) = 0$. Thus the optimized exponent is negative:

$$-t\max_z(((p - \frac{x}{t})z - g(z)) = -t \cdot h(p - \frac{x}{t}) < 0, \quad \text{for } t > \frac{x}{p-\lambda\mu}. \tag{4.104}$$

Therefore, optimize inequality 4.103 for z and divide by $\mathbb{P}[T_1 < T_R] = 1 - \psi(b-x, b)$ to derive the stated exponential bound:

$$\mathbb{P}_{b-x}[T_1 > t | T_1 < T_R] \leq \frac{e^{-t \cdot h(p-\frac{x}{t})}}{1-\psi(b-x,b)}, \quad \text{for } t > \frac{x}{p-\lambda\mu}. \tag{4.105}$$

Next, as $t \to \infty$, the optimizing condition becomes $g'(z) = p$. It is solved by z_p and therefore

$$h(p - \frac{x}{t}) = \max_z((p - \frac{x}{t})z - g(z)) \nearrow pz_p - g(z_p) = h(p) > 0, \quad \text{as } t \to \infty, \tag{4.106}$$

which is positive since $p > \lambda\mu$.
Finally, for fixed $\epsilon > 0$, set $t_\epsilon := \frac{m\epsilon}{\rho}$. Since $\rho \to 0$ by Lemma 4.1.3., $t_\epsilon \to \infty$ as $b \to \infty$, and thus

$$\mathbb{P}_{b-x}\left[\frac{\rho T_1}{m} > \epsilon \middle| T_1 < T_R\right] \leq \frac{e^{-t_\epsilon \cdot h(p-\frac{x}{t_\epsilon})}}{1-\psi(b-x,b)} \longrightarrow 0 \qquad \text{as } b \to \infty. \tag{4.107}$$

Consequently T_1 is small in probability compared with the sum of completed excursions $\sum_{k=1}^{N}(T_{k+1} - T_k)$, which has expected length $\frac{m}{\rho}$ as $b \to \infty$. □

Before the main theorem of this section finally can be proved, the following lemma shows the time from the last visit to the dividend barrier until ruin $T_R - T_{N+1}$ to be small as well. The exponential bound is found in a similar way as in the proof above.

Lemma 4.1.10. *The time from the risk process leaving the dividend barrier for the last time until ruin T_L has the following asymptotic bound. If $t > \bar{t}$ then, as $b \to \infty$,*

$$\mathbb{P}[T_L > bt | T_{N+2} - T_{N+1} = \infty, T_{N+1} < \infty] \lesssim \frac{\lambda}{CpR} \left(\frac{g(th(p+\frac{1}{t}))}{\lambda} + 1 \right) e^{-b(th(p+\frac{1}{t})-R)} \quad (4.108)$$

and $t \cdot h(p + \frac{1}{t}) > R$. Hence, T_L is small in probability compared to the sum of completed excursions. Moreover, Y_{N+1} is small as well, and therefore the time from the last visit to the dividend barrier until ruin $T_R - T_{N+1}$ is small compared to the random sum $\sum_{k=1}^{N}(T_{k+1} - T_k)$ as $b \to \infty$.

Proof.
Equation 4.69 with the root $z_1(s)$, where $s > f(z_p)$, can be written as

$$1 = \mathbb{E}_{b-x}[e^{z_1(s)(b-x-X_{T_1})-sT_1}\mathbb{1}_{\{T_1<T_R\}}] + \underbrace{\mathbb{E}_{b-x}[e^{z_1(s)(b-x-X_{T_R})-sT_R}\mathbb{1}_{\{T_R<T_1\}}]}_{\leq 1}. \quad (4.109)$$

Since $z_1(s)X_{T_R} \leq 0$ on $\{T_R < T_1\}$, it follows that

$$\mathbb{E}_{b-x}[e^{-sT_R}\mathbb{1}_{\{T_R<T_1\}}] \leq \mathbb{E}_{b-x}[e^{-z_1(s)X_{T_R}-sT_R}\mathbb{1}_{\{T_R<T_1\}}] \leq e^{-(b-x)z_1(s)}. \quad (4.110)$$

If one takes $s < 0$ such that $z_p < z_1(s) < R$ (see Figure 4.3), then $e^{-s(b-x)t} < e^{-sT_R}$ on $\{T_R > (b-x)t\}$ and consequently

$$e^{-s(b-x)t} \cdot \mathbb{P}_{b-x}[\{T_R > (b-x)t\} \cap \{T_R < T_1\}] \leq e^{-(b-x)z_1(s)}. \quad (4.111)$$

Substituting $s = g(z) - pz$ yields

$$\mathbb{P}_{b-x}[\{T_R > (b-x)t\} \cap \{T_R < T_1\}] \leq e^{-(b-x)t((p+\frac{1}{t})z-g(z))}. \quad (4.112)$$

The exponent is optimized by the z-value satisfying $g'(z) = p + \frac{1}{t}$. Since $g'(z)$ is strictly increasing, the optimal z-value is smaller than R if $g'(R) > p + \frac{1}{t}$, i.e. for $t > \frac{1}{g'(R)-p} = \bar{t}$. By property (iii) of the Legendre transform $h(x)$, one finds that

$$t \max_z((p+\frac{1}{t})z - g(z)) = t \cdot h(p+\frac{1}{t}) > R \quad \text{for } t > \bar{t}, \quad (4.113)$$

and therefore the exponent $-b(th(p+\frac{1}{t}) - R)$ is negative.
Due to the fact that $\{T_R > bt\} \subseteq \{T_R > (b-x)t\}$, it follows that

$$\mathbb{P}_{b-x}[\{T_R > bt\} \cap \{T_R < T_1\}] \leq e^{-(b-x)t \cdot h(p+\frac{1}{t})} \quad \text{for } t > \bar{t}. \quad (4.114)$$

By conditioning on the size of the claim which brings the risk process away from the dividend barrier for the last time (law of total probability) and by the equivalence in distribution of $X_{T_{N+1}+Y_{N+1}+s}|\{X_{T_{N+1}+Y_{N+1}} = b-x\}$ and $X_s|\{X_0 = b-x\}$ for $s \geq 0$ due to the strong Markov property of $\{X_t\}_{t\geq 0}$, one derives for $t > \bar{t}$

$$\mathbb{P}[\{T_L > bt\} \cap \{T_{N+2} - T_{N+1} = \infty\}|T_{N+1} < \infty] = \tag{4.115}$$

$$= \int_0^b \mathbb{P}[\{T_L > bt\} \cap \{T_{N+2} - T_{N+1} = \infty\}|T_{N+1} < \infty, X_{T_{N+1}+Y_{N+1}} = b-x]\, dF(x) = \tag{4.116}$$

$$= \int_0^b \mathbb{P}[\{T_R > bt\} \cap \{T_R < T_1\}|X_0 = b-x]\, dF(x) \leq \int_0^b e^{-(b-x)t\cdot h(p+\frac{1}{t})}\, dF(x) \leq \tag{4.117}$$

$$\leq e^{-bt\cdot h(p+\frac{1}{t})} \int_0^\infty e^{xt\cdot h(p+\frac{1}{t})}\, dF(x) = e^{-bt\cdot h(p+\frac{1}{t})} \left(\frac{g(t\cdot h(p+\frac{1}{t}))}{\lambda} + 1 \right). \tag{4.118}$$

Dividing by the probability of ruin during an excursion, which is asymptotically given as $\rho \sim \frac{CpR}{\lambda} e^{-Rb}$ as $b \to \infty$ (see Lemma 4.1.3.), one ends up with the stated asymptotic bound for $b \to \infty$:

$$\mathbb{P}[T_L > bt|T_{N+2} - T_{N+1} = \infty, T_{N+1} < \infty] \lesssim \frac{\lambda}{CpR} \left(\frac{g(th(p+\frac{1}{t}))}{\lambda} + 1 \right) e^{-b(th(p+\frac{1}{t})-R)} \tag{4.119}$$

For fixed $t > \bar{t}$ and $\epsilon > 0$, as $b \to \infty$ eventually

$$\left\{ \frac{\rho T_L}{m} > \epsilon \right\} \subseteq \{T_L > bt\} \tag{4.120}$$

since one has $\frac{\epsilon m}{\rho} > bt$ for high enough b. Thus T_L can be shown to be small in probability compared to the expected length of the sum of completed excursions:

$$\mathbb{P}\left[\frac{\rho T_L}{m} > \epsilon \Big| T_{N+2} - T_{N+1} = \infty, T_{N+1} < \infty \right] \lesssim \tag{4.121}$$

$$\lesssim \frac{\lambda}{CpR} \left(\frac{g(t\cdot h(p+\frac{1}{t}))}{\lambda} + 1 \right) e^{-b(t\cdot h(p+\frac{1}{t})-R)} \longrightarrow 0 \qquad \text{as } b \to \infty. \tag{4.122}$$

The second part of the last excursion $T_R - T_{N+1}$ (beside T_L) is the holding time at the barrier Y_{N+1}. One finds an exponential bound for Y_{N+1} by applying Markov's inequality:

$$\mathbb{P}\left[\frac{\rho Y_{N+1}}{m} > \epsilon \right] \leq \frac{\rho}{\epsilon m \lambda} \sim \frac{CpR}{\epsilon m \lambda^2} \cdot e^{-Rb} \longrightarrow 0 \quad \text{as } b \to \infty. \tag{4.123}$$

This means Y_{N+1} is small in probability compared to the expected length of the sum of completed excursions as well. Consequently the total last excursion $T_R - T_{N+1}$ is small in this sense. □

Finally, putting all lemmas in this section together, one finds an exponential distribution for the time of ruin T_R as stated in Theorem 4.1.1.

Proof of Theorem 4.1.1.
It is shown that

$$\mathbb{P}_{b-x}[T_R > t] = \mathbb{P}_{b-x}[T_R > t | T_R < T_1] \cdot \underbrace{\mathbb{P}_{b-x}[T_R < T_1]}_{\to 0} + \tag{4.124}$$

$$+\mathbb{P}_{b-x}\left[T_1 + \sum_{k=1}^{N}(T_{k+1} - T_k) + (T_R - T_{N+1}) > t \middle| T_1 < T_R\right] \cdot \underbrace{\mathbb{P}_{b-x}[T_1 < T_R]}_{\to 1} \sim \tag{4.125}$$

$$\sim \mathbb{P}_{b-x}\left[T_1 + \sum_{k=1}^{N}(T_{k+1} - T_k) + (T_R - T_{N+1}) > t \middle| T_1 < T_R\right] \sim \tag{4.126}$$

$$\sim \mathbb{P}_{b-x}\left[\sum_{k=1}^{N}(T_{k+1} - T_k) > t \middle| T_1 < T_R\right] \sim e^{-\frac{t\rho}{m}} \qquad \text{as } b \to \infty. \tag{4.127}$$

Moreover, Lemma 4.1.6. and Lemma 4.1.3. give the asymptotic laws for m and ρ. □

4.2. An initial surplus much lower than a high dividend barrier

Now, set the initial surplus to $u = x > 0$, which is much lower than a high dividend barrier ($x \ll b$). Then the risk process has the form

$$X_t = x + p \int_0^t \mathbb{1}_{\{X_s \le b\}}\, ds - S_t, \quad \text{for } t \ge 0. \tag{4.128}$$

As usual, one wants to derive an asymptotic formula for the distribution of the time of ruin

$$\mathbb{P}_x[T_R > t] := \mathbb{P}[T_R > t | X_0 = x]. \tag{4.129}$$

It will be shown that the probability of ruin without visit to the dividend barrier $\psi(x, b)$ is no longer negligible as $b \to \infty$. Also, if the risk process visits the dividend barrier ($T_1 < T_R$), ruin will occur after the same very long exponentially distributed time as in Section 4.1. These two cases are illustrated in Figure 4.5.

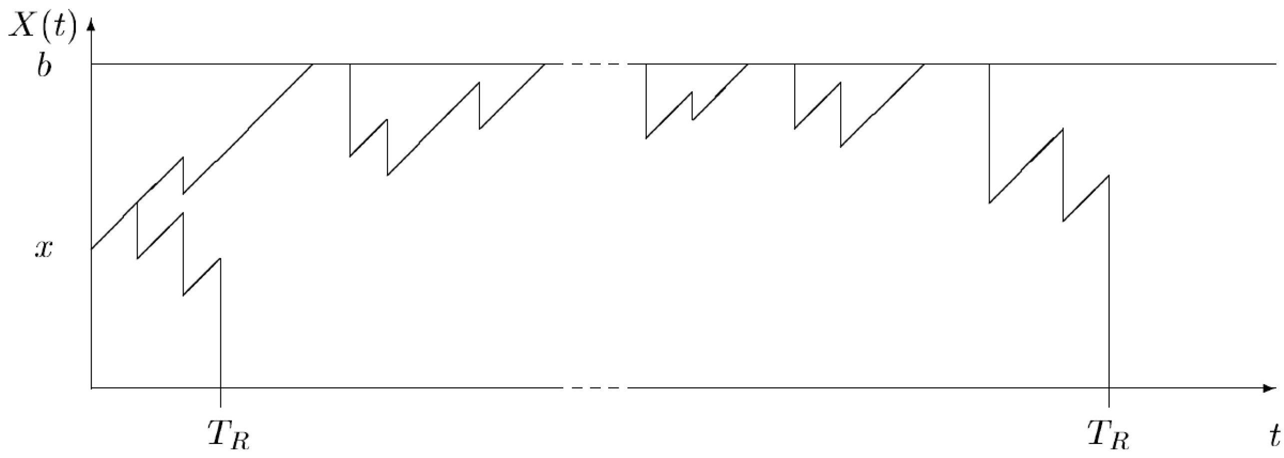


Figure 4.5.: Typical sample paths for the two cases of the surplus process $\{X_t\}_{t\geq 0}$ in the Cramér-Lundberg model with a dividend barrier and a small initial surplus.

The following theorem formulates the **two possible behaviors of the surplus process** in a mathematical way.

Theorem 4.2.1. *As $b \to \infty$, the time of ruin has the following asymptotic distribution*

$$T_R \xrightarrow{d} \begin{cases} Exp(\frac{\rho}{m}) & \text{with probability } 1-\psi(x), \\ T_R^c | \{T_R^c < \infty\} & \text{with probability } \psi(x). \end{cases} \tag{4.130}$$

At this point the following is interesting to mention. It can be shown that $T_R^c|\{T_R^c < \infty\}$ is asymptotically normal as $x \to \infty$ with mean $\frac{x}{g'(R)-p} = x\bar{t}$ and variance $\frac{xg''(R)}{(g'(R)-p)^3}$. For a detailed proof the interested reader is referred to Appendix B.

The proof of the above theorem will follow from the lemmas below.

Lemma 4.2.2. *The probability of ruin without visit to the dividend barrier satisfies*

$$\psi(x,b) \sim \psi(x) \qquad \text{as } b \to \infty. \tag{4.131}$$

Proof.
Lemma 4.1.2. together with the Cramér-Lundberg approximation (see Corollary 2.4.8.) yields

$$\psi(x,b) = \frac{\psi(x)-\psi(b)}{1-\psi(b)} \sim \frac{\psi(x)-Ce^{-Rb}}{1-Ce^{-Rb}} \sim \psi(x) \qquad \text{as } b \to \infty. \tag{4.132}$$

□

The next lemma gives a relation between the time of ruin without visit to the dividend barrier and the time of ruin in the classical Cramér-Lundberg model.

Lemma 4.2.3. *Conditional on ruin occurring before the first visit to the barrier, the time of ruin T_R is asymptotically equal to the time of ruin T_R^c in the classical model given that ruin occurs, i.e.*

$$\mathbb{P}_x[T_R \leq t | T_R < T_1] \sim \mathbb{P}_x[T_R^c \leq t | T_R^c < \infty] \qquad \text{for } t \geq 0 \text{ as } b \to \infty. \tag{4.133}$$

Proof.
Since by Lemma 4.2.2. for $s > 0$

$$\mathbb{E}_x[e^{-sT_R} | T_R < T_1] = \frac{\mathbb{E}_x[e^{-sT_R} \mathbb{1}_{\{T_R < T_1\}}]}{\psi(x,b)} \sim \frac{\mathbb{E}_x[e^{-sT_R} \mathbb{1}_{\{T_R < T_1\}}]}{\psi(x)} \overset{!}{\sim} \tag{4.134}$$

$$\overset{!}{\sim} \frac{\mathbb{E}_x[e^{-sT_R^c}]}{\psi(x)} = \mathbb{E}_x[e^{-sT_R^c} | T_R^c < \infty], \qquad \text{as } b \to \infty, \tag{4.135}$$

it is sufficient to show asymptotic equivalence of the following Laplace transforms

$$\mathbb{E}_x[e^{-sT_R} \mathbb{1}_{\{T_R < T_1\}}] \overset{!}{\sim} \mathbb{E}_x[e^{-sT_R^c}] \qquad \text{as } b \to \infty. \tag{4.136}$$

For this purpose, consider

$$\mathbb{E}_x[e^{-sT_R^c}] = \mathbb{E}_x[e^{-sT_R^c} \mathbb{1}_{\{T_R^c < T_1^c\}}] + \mathbb{E}_x[e^{-sT_R^c} \mathbb{1}_{\{T_1^c < T_R^c\}}] = \tag{4.137}$$

$$= \mathbb{E}_x[e^{-sT_R} \mathbb{1}_{\{T_R < T_1\}}] + \mathbb{E}_x[e^{-s(T_1^c + (T_R^c - T_1^c))} \mathbb{1}_{\{T_1^c < T_R^c\}}]. \tag{4.138}$$

By the strong Markov property of the surplus process $\{X_t^c\}_{t \geq 0}$, the random variables T_1^c and $T_R^c - T_1^c$ are independent on $\{T_1^c < T_R^c\}$ and the distribution of $T_R^c - T_1^c | \{X_{T_1^c}^c = b\}$ is the same as of $T_R^c | \{X_0^c = b\}$. Hence

$$\mathbb{E}_x[e^{-sT_R^c}] = \mathbb{E}_x[e^{-sT_R} \mathbb{1}_{\{T_R < T_1\}}] + \mathbb{E}_x[e^{-sT_1} \mathbb{1}_{\{T_1 < T_R\}}] \cdot \mathbb{E}_b[e^{-sT_R^c}]. \tag{4.139}$$

Since $X_{T_1} = b$ on $\{T_1 < T_R\}$ it follows from equation 4.69 with the root $z_0(s)$ and $X_0 = x$ that

$$\mathbb{E}_x[e^{-sT_1} \mathbb{1}_{\{T_1 < T_R\}}] \leq e^{z_0(s)(b-x)}. \tag{4.140}$$

Take $s > 0$ such that the above exponent is negative because of $z_0(s) < 0$. Therefore,

$$\mathbb{E}_x[e^{-sT_1} \mathbb{1}_{\{T_1 < T_R\}}] \cdot \underbrace{\mathbb{E}_b[e^{-sT_R^c}]}_{\leq 1} \leq e^{z_0(s)(b-x)} \longrightarrow 0 \qquad \text{as } b \to \infty \tag{4.141}$$

and consequently

$$\mathbb{E}_x[e^{-sT_R^c}] \sim \mathbb{E}_x[e^{-sT_R} \mathbb{1}_{\{T_R < T_1\}}] \qquad \text{as } b \to \infty. \tag{4.142}$$

□

Before the proof for Theorem 4.2.1. can finally be given, one needs one more lemma showing the time to the first visit T_1 is also small in probability compared with the sum $\sum_{k=1}^{N}(T_{k+1} - T_k)$ in case of a small initial surplus x.

Lemma 4.2.4. *Conditional on the risk process reaching the dividend barrier, the time to the first visit $T_1|\{X_0 = x\}$ has the following bound. If $t > \frac{1}{p-\lambda\mu}$ then*

$$\mathbb{P}_x[T_1 > (b-x)t | T_1 < T_R] \leq \frac{e^{-(b-x)t \cdot h(p-\frac{1}{t})}}{1-\psi(x,b)} \tag{4.143}$$

and $h(p - \frac{1}{t}) > 0$. This implies that $T_1|\{X_0 = x\}$ is small in probability compared with the sum of completed excursions $\sum_{k=1}^{N}(T_{k+1} - T_k)$ as $b \to \infty$.

Proof.
Changing the initial value from $b - x$ to x in the proof of Lemma 4.1.9. yields that $h(p - \frac{1}{t}) > 0$ and the exponential bound has the form

$$\mathbb{P}_x[T_1 > (b-x)t | T_1 < T_R] \leq \frac{e^{-(b-x)t \cdot h(p-\frac{1}{t})}}{1-\psi(x,b)} \qquad \text{for } \ t > \frac{1}{p-\lambda\mu}. \tag{4.144}$$

Finally, for fixed $\epsilon > 0$, set $t_\epsilon := \frac{m\epsilon}{\rho(b-x)}$ such that $t_\epsilon \to \infty$ as $b \to \infty$ and thus

$$\mathbb{P}_x\left[\frac{\rho T_1}{m} > \epsilon \middle| T_1 < T_R\right] \leq \frac{e^{-\frac{m\epsilon}{\rho} \cdot h(p-\frac{1}{t_\epsilon})}}{1-\psi(x,b)} \longrightarrow 0 \qquad \text{as } b \to \infty. \tag{4.145}$$

Consequently, $T_1|\{X_0 = x\}$ is small in probability compared with the sum of completed excursions $\sum_{k=1}^{N}(T_{k+1} - T_k)$, which has expected length $\frac{m}{\rho}$ as $b \to \infty$. □

Proof of Theorem 4.2.1.
Independent of the initial surplus, the time from the last visit to the dividend barrier until ruin $T_R - T_{N+1}$ is also small compared to the random sum $\sum_{k=1}^{N}(T_{k+1} - T_k)$ as $b \to \infty$ (see Lemma 4.1.10.). Therefore by Lemma 4.1.7.

$$\begin{aligned} \mathbb{P}_x[T_R > t | T_1 < T_R] &= \mathbb{P}_x\left[T_1 + \sum_{k=1}^{N}(T_{k+1} - T_k) + (T_R - T_{N+1}) > t \middle| T_1 < T_R\right] \sim & (4.146) \\ &\sim \mathbb{P}_x\left[\sum_{k=1}^{N}(T_{k+1} - T_k) > t \middle| T_1 < T_R\right] \sim e^{-\frac{t\rho}{m}} \qquad \text{as } b \to \infty. & (4.147) \end{aligned}$$

In summary, it is shown with the last three lemmas that

$$\begin{aligned} \mathbb{P}_x[T_R > t] &= \mathbb{P}_x[T_R > t | T_1 < T_R] \cdot (1 - \psi(x,b)) + \mathbb{P}_x[T_R > t | T_R < T_1] \cdot \psi(x,b) \sim & (4.148) \\ &\sim e^{-\frac{t\rho}{m}} \cdot (1 - \psi(x)) + \mathbb{P}_x[T_R^c > t | T_R^c < \infty] \cdot \psi(x) \qquad \text{as } b \to \infty. & (4.149) \end{aligned}$$

□

The next lemma gives an **exponential bound** for the time of ruin T_R^c in the classical model.

Lemma 4.2.5. *Conditional on ruin occurring, the time of ruin T_R^c in the classical model has the following bound for $t \geq 0$:*

$$\mathbb{P}_x[T_R^c > xt | T_R^c < \infty] \leq \frac{1}{\psi(x)} \cdot \left(e^{-xt \cdot h(p+\frac{1}{t})} \mathbb{1}_{\{t > \bar{t}\}} + e^{-xR} \mathbb{1}_{\{t \leq \bar{t}\}} \right) \tag{4.150}$$

Moreover $t \cdot h(p + \frac{1}{t}) > R$.
Note that for $t = 0$ the Cramér-Lundberg inequality $\psi(x) \leq e^{-xR}$ arises.

Proof.
The proof is similar to the first part of the proof for Lemma 4.1.10. Moreover, A. Martin-Löf [M] uses the same procedure to find a related result.
Since the time of ruin in the classical model T_R^c is not finite a.s. due to the positive drift of the surplus process, one performs a change of measure to the Esscher transform $\mathbb{P}^{(z)}$ such that it has a negative drift ($g'(z) > p$) under the new measure. Then, Asmussen's formula from equation 4.63 becomes

$$\mathbb{E}_x[e^{zS_{T_R^c} - T_R^c g(z)} \mathbb{1}_{\{T_R^c < \infty\}}] = \mathbb{E}^{(z)}[\mathbb{1}_{\{T_R^c < \infty\}}] = \mathbb{P}^{(z)}[T_R^c < \infty] = 1. \tag{4.151}$$

Take $s < 0$ such that $z_p < z_1(s) < R$ implying $g'(z_1(s)) > g'(z_p) = p$ as well as $g(z_1(s)) < pz_1(s)$ due to $f(z_1(s)) < 0$.
Since $S_{T_R^c} \geq x + pT_R^c$ one has

$$\mathbb{E}_x[e^{xz_1(s) - sT_R^c} \mathbb{1}_{\{T_R^c < \infty\}}] \leq 1 \tag{4.152}$$

and further

$$e^{-sxt} \mathbb{P}[\{T_R^c > xt\} \cap \{T_R^c < \infty\}] \leq \mathbb{E}_x[e^{-sT_R^c} \mathbb{1}_{\{xt < T_R^c < \infty\}}] \leq e^{-xz_1(s)}. \tag{4.153}$$

Substituting $s = g(z) - pz$ yields

$$\mathbb{P}[\{T_R^c > xt\} \cap \{T_R^c < \infty\}] \leq e^{-xt((p+\frac{1}{t})z - g(z))}. \tag{4.154}$$

The exponent is optimized for $g'(z) = p + \frac{1}{t}$. Since g' is strictly increasing, the optimal value for z_1 is smaller than R if $g'(R) > p + \frac{1}{t}$, i.e. for $t > \bar{t}$. By property (iii) of the Legendre transform $h(x)$, one finds that

$$t \max_z ((p + \frac{1}{t})z - g(z)) = t \cdot h(p + \frac{1}{t}) > R \quad \text{for } t > \bar{t}. \tag{4.155}$$

Writing the optimized exponent in terms of the Legendre transform and dividing by the probability of ruin $\psi(x)$ gives the desired result in case of $t > \bar{t}$:

$$\mathbb{P}_x[T_R^c > xt | T_R^c < \infty] \leq \frac{e^{-xt \cdot h(p+\frac{1}{t})}}{\psi(x)} \tag{4.156}$$

If $t \leq \bar{t}$ one can choose $z = R$ ($g'(R) > p$ guarantees a negative drift under the new measure $\mathbb{P}^{(R)}$) such that the above procedure simplifies ($s = 0$). One has

$$\mathbb{E}_x[e^{xR}\mathbb{1}_{\{T_R^c<\infty\}}] \leq 1, \tag{4.157}$$

which implies for $\{xt < T_R^c < \infty\} \subseteq \{T_R^c < \infty\}$ the stated exponential bound

$$\mathbb{P}[\{T_R^c > xt\} \cap \{T_R^c < \infty\}] \leq e^{-xR}. \tag{4.158}$$

□

A consequence of the above Lemma is the following. In case of an initial surplus much lower than a high dividend barrier **ruin occurs on different time scales** depending on whether the process reaches the dividend barrier or not. Since the barrier b is chosen high, $\frac{m}{\rho}$ will be high too and therefore

$$\mathbb{P}_x\left[\frac{\rho T_R^c}{m} > t \middle| T_R^c < \infty\right] \approx 0. \tag{4.159}$$

4.3. The proportion of time the surplus is below a given level

If the risk process reaches the high dividend barrier, then the time of ruin will be very long. In this section the proportion of time to ruin where the surplus is below some given level is found. Some results from queuing theory are used to derive the final asymptotic result. For this purpose, one needs the relation of the following theorem.

Consider an $M/G/1$ queue with arrival rate $\lambda > 0$ (equal to the intensity of the compound Poisson process in the surplus process) and service times distributed according to the claim distribution F. Let $\{V_t\}_{t\geq 0}$ be the virtual waiting time at time t starting at $V_0 = 0$ (recall that V_t is the time to clear the system at time t).

Theorem 4.3.1. *Assume for simplicity the premium income rate p is equal to 1. Then, the probability of ruin in the classical model is equivalent to the probability of the virtual waiting time in steady state V exceeding the initial surplus:*

$$\psi(x) = \mathbb{P}[V > x] \tag{4.160}$$

Proof.
The given proof is similar to the procedure of Asmussen and Schock Petersen [ASP]. The key is the following coupling of the virtual waiting time $\{V_t\}_{0\le t\le T}$ with the surplus process $\{X_t^c\}_{0\le t\le T}$ in finite time. Assume that claims occur at times $0 < t_1 < t_2 < \ldots < t_N$ with claim amounts $C_1, C_2, \ldots, C_N$. Let the arrivals in the $M/G/1$ queue occur at times $0 < T - t_N < \ldots < T - t_2 < T - t_1 < T$ with corresponding service times $C_N, \ldots, C_2, C_1$.

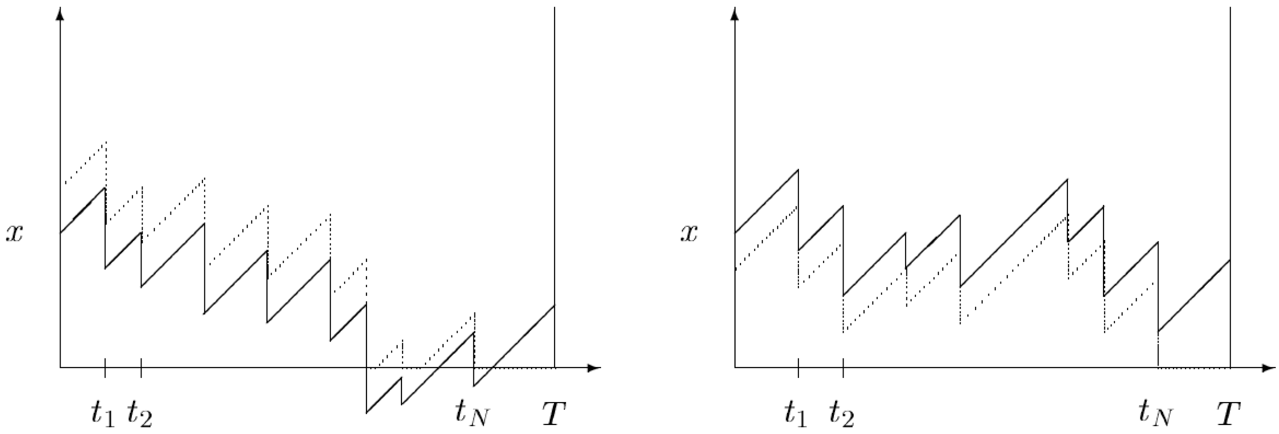


Figure 4.6.: Sample paths of the surplus process $\{X_t^c\}_{0\le t\le T}$ (solid line) and the time-reversed virtual waiting time process $\{V_{T-t}\}_{0\le t\le T}$ of the corresponding $M/G/1$ queue (dotted line), where either ruin does occur (l.h.s.) or not (r.h.s.)

Now one can show the equivalence in finite time:

$$\{T_R^c \le T\} = \{V_{T-} > x\} \text{ a.s.,} \tag{4.161}$$

where V_T is replaced by its limit from the left $V_{T-} := \lim_{s\nearrow T} V_s$. Note that the probability of an arrival at T is 0 a.s.
Suppose $V_{T-} > x$. Then

$$V_{(T-t_1)-} = V_{T-t_1} - C_1 = V_{T-} + t_1 p - C_1 > x + t_1 p - C_1 = X_{t_1}^c \tag{4.162}$$

If $V_{(T-t_1)-} > 0$ repeat the above argument to get $V_{(T-t_2)-} > X_{t_2}^c$, and so on. Hence for t_k where $V_{(T-t_k)-} = 0$ (such a k exists since in any case $V_{(T-t_N)-} = 0$) one finds that

$$0 = V_{(T-t_k)-} > X_{t_k}^c. \tag{4.163}$$

Indeed it holds that $T_R^c \le T$.
Now, suppose that $V_{T-} \le x$. Then

$$V_{(T-t_1)-} = V_{T-t_1} - C_1 = V_{T-} + t_1 p - C_1 \le x + t_1 p - C_1 = X_{t_1}^c \tag{4.164}$$

Repeating the above argument gives $V_{(T-t_2)-} \le X_{t_2}^c$. Thus, proceeding iteratively yields

$$0 \le V_{(T-t_k)-} \le X_{t_k}^c \quad \forall\, k \in \{1, \ldots, N\}. \tag{4.165}$$

Since ruin can only occur at times of claims, one ends up with $T_R^c > T$.
Therefore it holds that

$$\psi(x, T) = \mathbb{P}[V_T > x] \quad \text{for } T > 0. \tag{4.166}$$

The intensity of the queue is

$$\rho = \lambda\mu = \frac{1}{1+\Lambda} < 1 \tag{4.167}$$

since the safety loading Λ is assumed positive. Thus, according to Corollary 3.5.4. a limiting steady-state distribution $V := \lim_{T\to\infty} V_T$ exists and finally one has

$$\psi(x) = \mathbb{P}[V > x]. \tag{4.168}$$

□

Now, some useful results from queueing theory are recalled. For a premium rate not necessarily equal to 1, the positive safety loading in the risk process $\Lambda = p/(\lambda\mu) - 1 > 0$ corresponds to a traffic intensity $\rho = \lambda\mu/p < 1$ in the queueing process. Therefore, the virtual waiting time process is a regenerative process and by Corollary 3.5.4. a limiting steady-state distribution of the virtual waiting time process exists with distribution $(x \geq 0)$

$$\mathbb{P}[V > x] = \frac{1}{\mathbb{E}[C]} \cdot \mathbb{E}\left[\int_0^C \mathbb{1}_{\{V_s > x\}} ds\right], \tag{4.169}$$

where C is the length of a busy cycle. By the forgetfulness property of the exponential distribution, the expected length of the idle period is

$$\mathbb{E}[I] = \frac{1}{\lambda} \tag{4.170}$$

and thus equation 3.36 gives the expected length of a busy cycle:

$$\mathbb{E}[C] = \mathbb{E}[I] \cdot \frac{1}{1-\rho} = \frac{1}{\lambda} \cdot \frac{1}{1 - \frac{\lambda\mu}{p}} = \frac{1}{\lambda} \cdot \frac{p}{p - \lambda\mu} = \frac{1}{\lambda} + \frac{\mu}{p - \lambda\mu} \tag{4.171}$$

Next, consider a **reflected version of the virtual waiting time process about the dividend barrier b**

$$R_t := b - V_t, \quad t \geq 0. \tag{4.172}$$

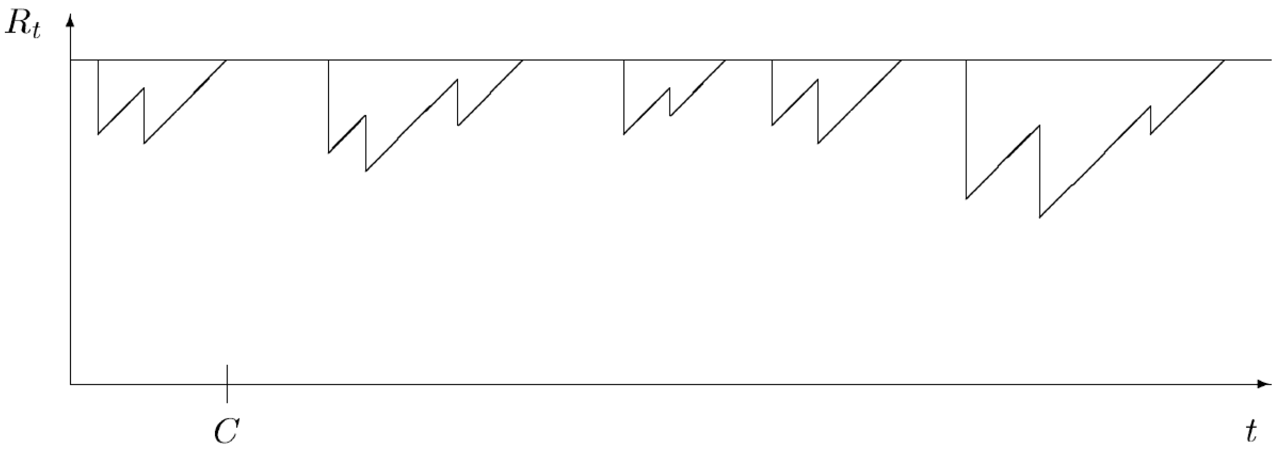


Figure 4.7.: Typical sample path of the reflected virtual waiting time process $\{R_t\}_{t\geq 0}$.

This process behaves in almost the same way as the risk process at the dividend barrier. The main differences are that the risk process has a slope in between jumps not equal to 1 and it is terminated at ruin. The problem with unequal slope in between jumps is easily overcome by a **time transformation**. The process $\{X_{t/p}\}_{t\geq 0}$ has a slope in between jumps equal to 1 and the same safety loading as the original process $\{X_t\}_{t\geq 0}$ (the new premium income rate is 1, however, the new intensity is λ/p). The problem with termination is considered in the proof of the next theorem.

Since one would like to have an asymptotic law for the proportion of time the surplus is below some given level, define the **amount of time the risk process is less than b-x before ruin** as

$$T_{<(b-x)} := \int_0^{T_R} \mathbb{1}_{\{X_s<b-x\}} ds, \quad \text{for } x \in [0,b). \tag{4.173}$$

Theorem 4.3.2. *Conditional on the risk process reaching the dividend barrier, the proportion of time the surplus is less than $b-x$ satisfies for $b \to \infty$*

$$\frac{T_{<(b-x)}}{T_R} \xrightarrow{p} \begin{cases} \psi(x) & \text{if } x \in (0,b), \\ \frac{\lambda\mu}{p} & \text{if } x = 0. \end{cases} \tag{4.174}$$

Proof.
Recall that conditional on the risk process reaching the dividend barrier, one can write

$$T_R = T_1 + \sum_{k=1}^{N} (T_{k+1} - T_k) + (T_R - T_{N+1}). \tag{4.175}$$

Thus the amount of time the risk process is less than $b-x$ can be written as

$$T_{<(b-x)} = \int_0^{T_1} \mathbb{1}_{\{X_s<b-x\}}ds + \sum_{k=1}^{N}\int_{T_k}^{T_{k+1}} \mathbb{1}_{\{X_s<b-x\}}ds + \int_{T_{N+1}}^{T_R} \mathbb{1}_{\{X_s<b-x\}}ds. \tag{4.176}$$

Remember that the times T_1 and $T_R - T_{N+1}$ are small in probability compared with the very long time $\sum_{k=1}^{N}(T_{k+1}-T_k)$ and in the same way, since the number of excursions will be very large, the first and the last integral in equation 4.176 are small in probability compared with the sum of integrals. Using the weak law of large numbers for $N \to \infty$

$$\frac{1}{N}\sum_{k=1}^{N}\int_{T_k}^{T_{k+1}} \mathbb{1}_{\{X_s<b-x\}}ds \xrightarrow{p} \mathbb{E}\left[\int_{T_k}^{T_{k+1}} \mathbb{1}_{\{X_s<b-x\}}ds|T_{k+1}<\infty\right] \tag{4.177}$$

it follows that

$$\frac{T_{<(b-x)}}{T_R} \xrightarrow{p} \frac{\mathbb{E}[\int_{T_k}^{T_{k+1}} \mathbb{1}_{\{X_s<b-x\}}ds|T_{k+1}<\infty]}{\mathbb{E}[T_{k+1}-T_k|T_{k+1}<\infty]} = \tag{4.178}$$

$$= \frac{\frac{1}{p}\cdot\mathbb{E}[\int_{pT_k}^{pT_{k+1}} \mathbb{1}_{\{X_{\frac{t}{p}}<b-x\}}dt|T_{k+1}<\infty]}{\mathbb{E}[T_{k+1}-T_k|T_{k+1}<\infty]} \quad \text{as } b\to\infty. \tag{4.179}$$

Now, consider the virtual waiting time process $\{V_t\}_{t\geq 0}$ of an $M/G/1$ queue with $\mu_A = \frac{p}{\lambda}$ and $\mu_B = \mu$ (such that $\rho = \frac{\lambda\mu}{p}$) with expected cycle length

$$\mathbb{E}[C] = \frac{1}{\frac{\lambda}{p}} + \frac{\mu}{1-\frac{\lambda}{p}\mu} = p\cdot\left(\frac{1}{\lambda}+\frac{\mu}{p-\lambda\mu}\right). \tag{4.180}$$

Due to the asymptotic law for the expected length of a complete excursion from Lemma 4.1.6.

$$\mathbb{E}[T_{k+1}-T_k|T_{k+1}<\infty] \sim \frac{1}{\lambda}+\frac{\mu}{p-\lambda\mu} \quad \textit{as } b\to\infty, \tag{4.181}$$

and since the probability of ruin during each excursion is very small by Lemma 4.1.3., one has for $x\in(0,b)$ as $b\to\infty$

$$\frac{T_{<(b-x)}}{T_R} \xrightarrow{p} \frac{\mathbb{E}[\int_0^C \mathbb{1}_{\{R_s<b-x\}}ds]}{\mathbb{E}[C]} = \frac{\mathbb{E}[\int_0^C \mathbb{1}_{\{V_s>x\}}ds]}{\mathbb{E}[C]} = \mathbb{P}[V>x] = \psi(x). \tag{4.182}$$

Note that the last equation uses formulas 4.172 and 4.169 as well as Theorem 4.3.1.

Finally only the case $x=0$ is missing. Recall that the expected holding time at the barrier is $\frac{1}{\lambda}$ due to the exponentially distributed inter-arrival times. Therefore,

$$\frac{T_{<b}}{T_R} \xrightarrow{p} \frac{\mathbb{E}[\int_{T_k}^{T_{k+1}} \mathbb{1}_{\{X_s<b\}}ds|T_{k+1}<\infty]}{\mathbb{E}[T_{k+1}-T_k|T_{k+1}<\infty]} \sim \frac{\frac{\mu}{p-\lambda\mu}}{\frac{1}{\lambda}+\frac{\mu}{p-\lambda\mu}} = \frac{\lambda\mu}{p} \quad \text{as } b\to\infty. \tag{4.183}$$

□

5 An example: Exponentially distributed claims

In this chapter the approximation formula for the time of ruin T_R is illustrated (compare Theorem 4.2.1.). Recall, as $b \to \infty$, the time of ruin has the following asymptotic distribution

$$T_R \xrightarrow{d} \begin{cases} Exp(\frac{\rho}{m}) & \text{with probability } 1 - \psi(x), \\ T_R^c | \{T_R^c < \infty\} & \text{with probability } \psi(x), \end{cases} \tag{5.1}$$

and ruin occurs on different time scales depending on whether the risk process visits the dividend barrier or not. One would like to **find the density in either of these cases**. For exponentially distributed claims this can be done by **calculating the Laplace transform** of the time of ruin and infer the density by numerical inversion.

To this end, suppose that the claims are **exponentially distributed** with expected size 1, i.e. $C_n \sim Exp(1)$, and that the Poisson parameter $\lambda = 1$. Then the safety loading is $\Lambda = p - 1$, which is assumed to be positive. Recall Example 2.3.4. where the following solution for the probability of ruin in case of exponentially distributed claims in the classical Cramér-Lundberg model was found:

$$\psi(x) = \frac{1}{1+\Lambda} \cdot e^{-Rx} = \frac{1}{p} \cdot e^{-\frac{p-1}{p}x}, \quad x \geq 0. \tag{5.2}$$

Then, by Lemma 4.1.2. , the probability of ruin without visit to the dividend barrier is given by

$$\psi(x,b) = \frac{\psi(x) - \psi(b)}{1 - \psi(b)} = \frac{e^{-\frac{p-1}{p}x} - e^{-\frac{p-1}{p}b}}{p - e^{-\frac{p-1}{p}b}}, \quad x \in [0,b], \tag{5.3}$$

and the probability of ruin during an excursion is obtained by integration with respect to the claim distribution:

$$\rho = \int_0^\infty \psi(b-x,b)\,dF(x) = \int_0^b \frac{e^{-\frac{p-1}{p}(b-x)} - e^{-\frac{p-1}{p}b}}{p - e^{-\frac{p-1}{p}b}} \cdot e^{-x}\,dx + \bar{F}(b) = \tag{5.4}$$

$$= \frac{e^{-\frac{p-1}{p}b}}{p - e^{-\frac{p-1}{p}b}} \cdot \left(\int_0^b e^{-\frac{1}{p}x}\,dx + e^{-b} - 1\right) + e^{-b} = \tag{5.5}$$

$$= \frac{e^{-\frac{p-1}{p}b} \cdot (-pe^{-\frac{1}{p}b} + p + e^{-b} - 1) + e^{-b} \cdot (p - e^{-\frac{p-1}{p}b})}{p - e^{-\frac{p-1}{p}b}} = \tag{5.6}$$

$$= \frac{(p-1) \cdot e^{-\frac{p-1}{p}b}}{p - e^{-\frac{p-1}{p}b}} \sim \frac{p-1}{p} e^{-\frac{p-1}{p}b} \quad \text{as } b \to \infty. \tag{5.7}$$

Now, consider formula 4.69 and take s such that $z_{0,1}(s) < 1$ (such an s exists since $z_{0,1}(s) \leq R = \frac{p-1}{p} < 1$ for $s \leq 0$):

$$1 = \mathbb{E}_x[e^{z_{0,1}(s)(x - X_{T_f}) - sT_f}] = \tag{5.8}$$

$$= e^{z_{0,1}(s)(x-b)} \mathbb{E}_x[e^{-sT_1} \mathbb{1}_{\{T_1 < T_R\}}] + e^{z_{0,1}(s)x} \mathbb{E}_x[e^{-z_{0,1}(s) X_{T_R} - sT_R} \mathbb{1}_{\{T_R < T_1\}}] = \tag{5.9}$$

$$= e^{z_{0,1}(s)(x-b)} \underbrace{\mathbb{E}_x[e^{-sT_1} \mathbb{1}_{\{T_1 < T_R\}}]}_{=:A} + \frac{e^{z_{0,1}(s)x}}{1 - z_{0,1}(s)} \cdot \underbrace{\mathbb{E}_x[e^{-sT_R} \mathbb{1}_{\{T_R < T_1\}}]}_{=:B}, \tag{5.10}$$

where T_R and X_{T_R} are independent on the set $\{T_R < T_1\}$ due to the exponential claim distribution (see Prabhu [P] for a proof) and, by the forgetfulness property, the deficit $-X_{T_R}$ is also exponentially distributed with expected value 1. The derived system of equations

$$\begin{cases} e^{z_0(x-b)} \cdot A + \frac{e^{z_0 x}}{1-z_0} \cdot B = 1 \\ e^{z_1(x-b)} \cdot A + \frac{e^{z_1 x}}{1-z_1} \cdot B = 1 \end{cases} \tag{5.11}$$

can be solved in the following way. Calculate A from the first equation and substitute it in the second:

$$A = \left(1 - \frac{e^{z_0 x}}{1 - z_0} \cdot B\right) \cdot e^{-z_0(x-b)} \tag{5.12}$$

$$\Longrightarrow 1 = e^{(z_1 - z_0)(x-b)} \cdot \left(1 - \frac{e^{z_0 x}}{1 - z_0} \cdot B\right) + \frac{e^{z_1 x}}{1 - z_1} \cdot B \tag{5.13}$$

$$\Longrightarrow 1 - e^{(z_1 - z_0)(x-b)} = B \cdot e^{z_1 x} \left(\frac{1}{1 - z_1} - \frac{1}{1 - z_0} e^{(z_0 - z_1)b}\right) \tag{5.14}$$

$$\Longrightarrow \mathbb{E}_x[e^{-sT_R} \mathbb{1}_{\{T_R < T_1\}}] = \frac{(1 - z_0(s))(1 - z_1(s)) e^{-x z_1(s)} (1 - e^{(b-x)(z_0(s) - z_1(s))})}{1 - z_0(s) - (1 - z_1(s)) e^{b(z_0(s) - z_1(s))}} \tag{5.15}$$

Substituting the solution for B in equation 5.12 gives

$$A = \left(1 - \frac{(1-z_1)e^{x(z_0-z_1)}(1-e^{(b-x)(z_0-z_1)})}{1-z_0-(1-z_1)e^{b(z_0-z_1)}}\right) \cdot e^{(b-x)z_0} = \tag{5.16}$$

$$= \frac{(1-z_0)e^{(b-x)z_0} - (1-z_1)(e^{b(z_0-z_1)} + e^{x(z_0-z_1)} - e^{b(z_0-z_1)}) \cdot e^{(b-x)z_0}}{1-z_0-(1-z_1)e^{b(z_0-z_1)}} \tag{5.17}$$

$$\Longrightarrow \quad \mathbb{E}_x[e^{-sT_1}\mathbb{1}_{\{T_1<T_R\}}] = \frac{(1-z_0(s))e^{(b-x)z_0(s)} - (1-z_1(s))e^{bz_0(s)-xz_1(s)}}{1-z_0(s)-(1-z_1(s))e^{b(z_0(s)-z_1(s))}}, \tag{5.18}$$

where $z_{0,1}(s)$ are the roots of the equation

$$s = g(z) - pz = M_C(z) - 1 - pz = \frac{1}{1-z} - 1 - pz, \quad \text{with } z < 1. \tag{5.19}$$

The arising quadratic equation can be solved using the quadratic formula:

$$z^2 + \frac{1+s-p}{p}z - \frac{s}{p} = 0 \implies z_{0,1}(s) = -\frac{1+s-p}{2p} \pm \frac{\sqrt{(1+s-p)^2+4ps}}{2p} \tag{5.20}$$

Finally, the density of $T_R|\{T_R < T_1\}$ is obtained by numerical inversion of the derived Laplace transform

$$\mathbb{E}_x[e^{-sT_R}|T_R < T_1] = \frac{(1-z_0(s))(1-z_1(s))e^{-xz_1(s)}(1-e^{(b-x)(z_0(s)-z_1(s))})}{(1-z_0(s)-(1-z_1(s))e^{b(z_0(s)-z_1(s))}) \cdot \psi(x,b)} \tag{5.21}$$

and plotted in Figure 5.1 for different premium income rates p and dividend barriers b.

In order to show that this density does indeed converge to the density of the time of ruin in the classical model, the Laplace transform of $T_R^c|\{T_R^c < \infty\}$ is found in addition. The method is similar, however, a lot simpler. Consider formula 4.151 and take s again such that $z_{0,1}(s) < 1$:

$$1 = \mathbb{E}_x[e^{zS_{T_R^c} - T_R^c g(z)}\mathbb{1}_{\{T_R^c<\infty\}}] = e^{xz}\mathbb{E}_x[e^{-zX_{T_R^c} - sT_R^c}\mathbb{1}_{\{T_R^c<\infty\}}] = \frac{e^{xz}}{1-z}\mathbb{E}_x[e^{-sT_R^c}\mathbb{1}_{\{T_R^c<\infty\}}] \tag{5.22}$$

Therefore, the Laplace transform of $T_R^c|\{T_R^c < \infty\}$ is found as

$$\mathbb{E}_x[e^{-sT_R^c}|T_R^c < \infty] = \frac{1-z_1(s)}{\psi(x) \cdot e^{xz_1(s)}}. \tag{5.23}$$

The density of $T_R^c|\{T_R^c < \infty\}$, which is obtained by numerical inversion as well, is also plotted in Figure 5.1. The figure shows nicely that the distribution of $T_R|\{T_R < T_1\}$ converges to the distribution of $T_R^c|\{T_R^c < \infty\}$ as $b \to \infty$.

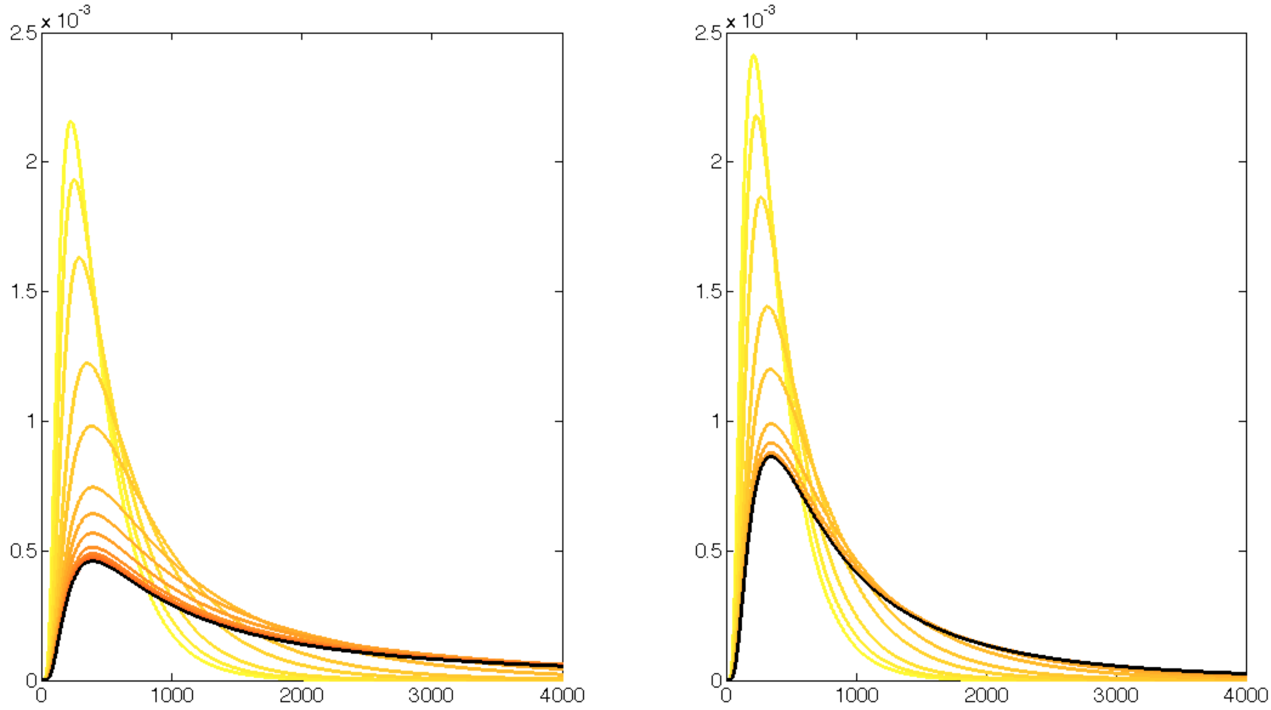


Figure 5.1.: Density of $T_R|\{T_R < T_1\}$ with initial surplus 50, $p = 1.01$ (l.h.s.) and $p = 1.04$ (r.h.s.) and $b \in \{52, 55, 60, 70, 80, 100, 120, 150, 200, 250, 300, 400, 600, 800, 1200\}$. The line color depends on the height of the barrier (yellow for low and red for high values of b respectively). The black curve represents the density of $T_R^c|\{T_R^c < \infty\}$ with same parameters.

Remember that $T_R^c|\{T_R^c < \infty\}$ is asymptotically normal distributed with mean $\mu_c := \frac{x}{g'(R)-p}$ and variance $\sigma_c^2 := \frac{xg''(R)}{(g'(R)-p)^3}$ as $x \to \infty$ (see Appendix B for a detailed proof). In case of exponentially distributed claims of expected size 1 and a Poisson parameter equal to 1, one finds with $R = \frac{p-1}{p}$ that

$$g'(R) = \frac{1}{(1-R)^2} = p^2 \quad \text{as well as} \quad g''(R) = \frac{2}{(1-R)^3} = 2p^3 \tag{5.24}$$

and consequently

$$\mu_c = \frac{x}{p(p-1)} \quad \text{and} \quad \sigma_c^2 = \frac{2x}{(p-1)^3}. \tag{5.25}$$

The Laplace transform of $\frac{T_R^c - \mu_c}{\sigma_c}|\{T_R^c < \infty\}$ is

$$\mathbb{E}_x[e^{-s\frac{T_R^c-\mu_c}{\sigma_c}}|T_R^c < \infty] = e^{s\frac{\mu_c}{\sigma_c}} \frac{1 - z_1(\frac{s}{\sigma_c})}{\psi(x) \cdot e^{xz_1(\frac{s}{\sigma_c})}}. \tag{5.26}$$

The density is plotted in Figure 5.2 for different values of x. The asymptotic normality of $T_R^c|\{T_R^c < \infty\}$ for $x \to \infty$ is clearly visible, however, x needs to get really high.

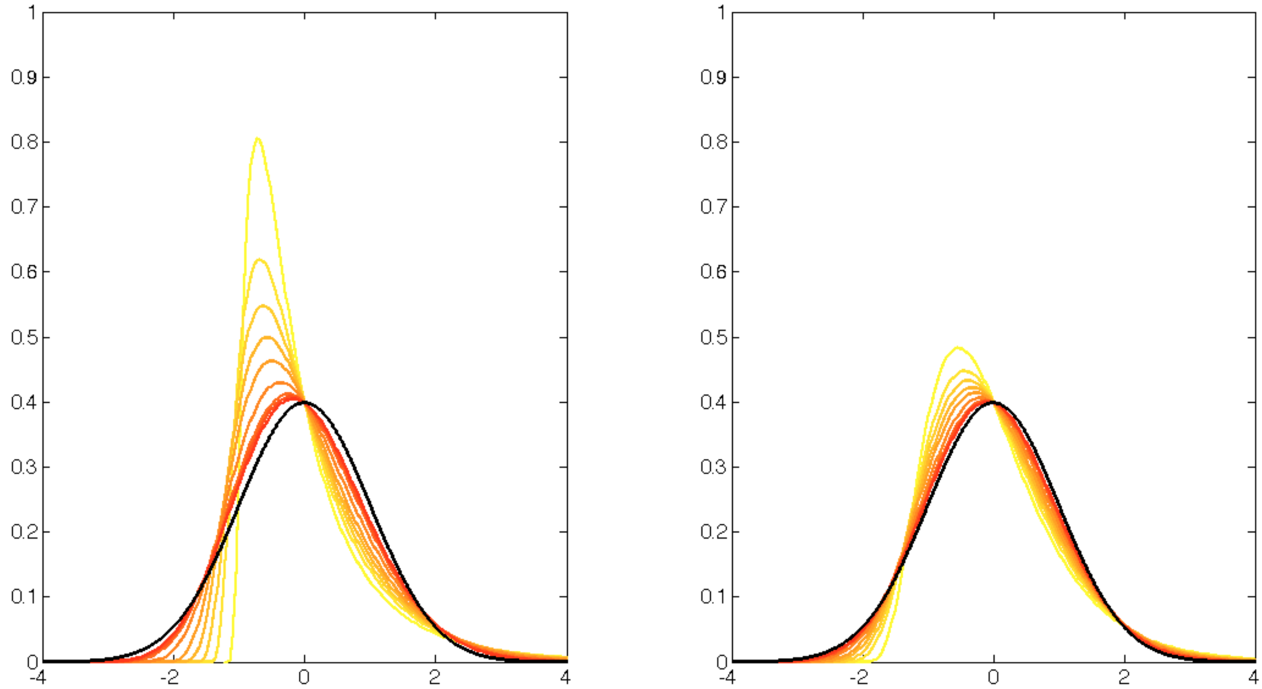


Figure 5.2.: Density of $\frac{T_R^c - \mu_c}{\sigma_c} | \{T_R^c < \infty\}$ for $p = 1.01$ (l.h.s.) and $p = 1.04$ (r.h.s.) with initial surplus $x \in \{300, 500, 700, 1000, 1500, 3000, 7000, 10000, 15000\}$. The line color depends on the height of the initial surplus (yellow for low and red for high values of x respectively). The black curve represents an $N(0,1)$ - distribution.

In addition, by putting the initial surplus on another scale ($x = 5000$), one sees that the limiting distribution of $T_R | \{T_R < T_1\}$ for $b \to \infty$ can be approximated by a normal distribution with mean μ_c and variance σ_c^2 too (Figure 5.3).

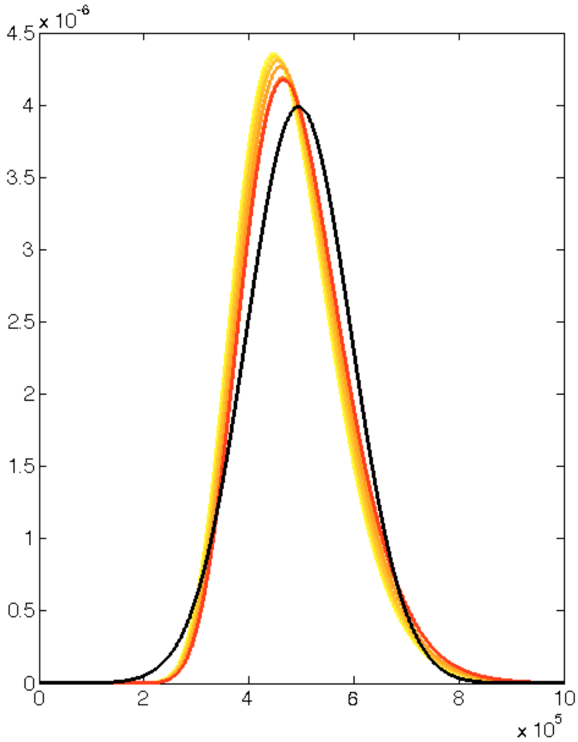
Now, consider the case where the risk process visits the dividend barrier. Then, the Laplace transform of $T_R | \{T_1 < T_R\}$ can be derived by splitting the time of ruin in three independent parts:

$$T_R = T_1 + \sum_{k=1}^{N} (T_{k+1} - T_k) + (T_R - T_{N+1}) \tag{5.27}$$

In equation 5.18 it is already found that

$$\mathbb{E}_x[e^{-sT_1} | T_1 < T_R] = \frac{(1 - z_0(s))e^{(b-x)z_0(s)} - (1 - z_1(s))e^{bz_0(s) - xz_1(s)}}{\beta(1 - \psi(x,b))}, \tag{5.28}$$

where the denominator is abbreviated as $\beta := 1 - z_0(s) - (1 - z_1(s))e^{-b\alpha}$ with $\alpha := z_1(s) - z_0(s)$.



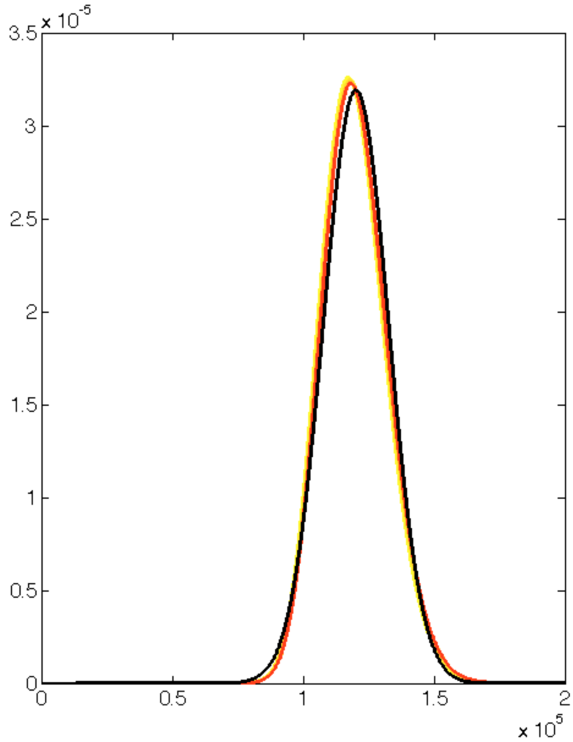


Figure 5.3.: Density of $T_R|\{T_R < T_1\}$ with initial surplus $x = 5000$, $p = 1.01$ (l.h.s.) and $p = 1.04$ (r.h.s.) and $b \in \{5010, 5050, 5100, 5200, 5500, 6000, 7000\}$. The line color depends on the height of the barrier (yellow for low and red for high values of b respectively). The black curve represents a $N(\mu_c, \sigma_c^2)$ - distribution.

Recall the first part of the proof for Lemma 4.1.6. With $\lambda = 1$ one has

$$\mathbb{E}[e^{-s(T_{k+1}-T_k)}|T_{k+1} < \infty] = \mathbb{E}[e^{-sY_k}] \cdot \mathbb{E}[e^{-s(T_{k+1}-T_k-Y_k)}|T_{k+1} < \infty] = \tag{5.29}$$

$$= \frac{\int_0^b \mathbb{E}_{b-x}[e^{-sT_1}\mathbb{1}_{\{T_1<T_R\}}] \cdot e^{-x}dx}{(1-\rho)(1+s)}. \tag{5.30}$$

The numerator can be calculated as

$$\frac{1}{\beta}\int_0^b (1-z_0(s))e^{-x(1-z_0(s))} - (1-z_1(s))e^{-b\alpha}e^{-x(1-z_1(s))}dx = \tag{5.31}$$

$$= \frac{1}{\beta}\left(-\left[e^{-x(1-z_0(s))}\right]_0^b + e^{-b\alpha}\left[e^{-x(1-z_1(s))}\right]_0^b\right) = \frac{1-e^{-b\alpha}}{\beta}. \tag{5.32}$$

Therefore, the length of a complete excursion has the Laplace transform

$$\mathbb{E}[e^{-s(T_{k+1}-T_k)}|T_{k+1} < \infty] = \frac{1-e^{-b\alpha}}{\beta(1-\rho)(1+s)}. \tag{5.33}$$

The random variable N has a geometric distribution with generating function

$$\mathbb{E}[t^N] = \frac{\rho}{1-(1-\rho)t} \qquad \text{for } \; t < \frac{1}{1-\rho} \tag{5.34}$$

and thus the Laplace transform of the total length of completed excursions is by independence

$$\mathbb{E}[e^{-s\sum_{k=1}^{N}(T_{k+1}-T_k)}] = \frac{\rho}{1-(1-\rho)\mathbb{E}[e^{-s(T_{k+1}-T_k)}|T_{k+1}<\infty]} = \tag{5.35}$$

$$= \frac{\beta\rho(1+s)}{\beta(1+s)-1+e^{-b\alpha}} = \frac{\beta\rho(1+s)}{\beta s - z_0(s)+z_1(s)e^{-b\alpha}}. \tag{5.36}$$

The Laplace transform of the time from the last visit to the barrier until ruin is found similarly as for the total length of an excursion:

$$\mathbb{E}[e^{-s(T_R-T_{N+1})}|T_{N+2}=\infty] = \frac{1}{\rho}\cdot\mathbb{E}[e^{-sY_{N+1}}]\cdot\mathbb{E}[e^{-s(T_R-T_{N+1}-Y_{N+1})}\mathbb{1}_{\{T_{N+2}=\infty\}}] = \tag{5.37}$$

$$= \frac{\int_0^b \mathbb{E}_{b-x}[e^{-sT_R}\mathbb{1}_{\{T_R<T_1\}}]\cdot e^{-x}dx + e^{-b}}{\rho(1+s)} \tag{5.38}$$

The integral in the numerator can be solved explicitly:

$$\frac{1}{\beta}\int_0^b (1-z_0(s))(1-z_1(s))e^{-(b-x)z_1(s)}(1-e^{x(z_0(s)-z_1(s))})e^{-x}dx + e^{-b} = \tag{5.39}$$

$$= \frac{e^{-bz_1(s)}}{\beta}\cdot\Big(-(1-z_0(s))e^{-x(1-z_1(s))}\Big|_0^b + (1-z_1(s))e^{-x(1-z_0(s))}\Big|_0^b + \tag{5.40}$$

$$+ (1-z_0(s))e^{-b(1-z_1(s))} - (1-z_1(s))e^{-b(1-z_0(s))}\Big) = \frac{\alpha e^{-bz_1(s)}}{\beta} \tag{5.41}$$

Hence the Laplace transform of the time from the last visit to the barrier until ruin is

$$\mathbb{E}[e^{-s(T_R-T_{N+1})}|T_{N+2}=\infty] = \frac{\alpha e^{-bz_1(s)}}{\beta\rho(1+s)}. \tag{5.42}$$

Since

$$1-\psi(x,b) = 1 - \frac{e^{-\frac{p-1}{p}x} - e^{-\frac{p-1}{p}b}}{p - e^{-\frac{p-1}{p}b}} = \frac{p-e^{-\frac{p-1}{p}x}}{p-e^{-\frac{p-1}{p}b}}, \tag{5.43}$$

multiplying equations 5.28, 5.36 and 5.42 yields

$$\mathbb{E}[e^{-sT_R}|T_1<T_R] = \frac{\alpha((1-z_0(s))e^{-b\alpha-xz_0(s)} - (1-z_1(s))e^{-b\alpha-xz_1(s)})(p-e^{-\frac{p-1}{p}b})}{\beta(\beta s - z_0(s)+z_1(s)e^{-b\alpha})(p-e^{-\frac{p-1}{p}x})}. \tag{5.44}$$

It is interesting to plot the conditional density of $\frac{\rho T_R}{m}|\{T_1<T_R\}$, which is the reason why m needs to be calculated as well. For this purpose, differentiate the Laplace transform of the length of a complete excursion and evaluate at 0.

$$m = \mathbb{E}[T_{k+1} - T_k | T_{k+1} < \infty] = -\frac{d}{ds}\mathbb{E}[e^{-s(T_{k+1}-T_k)} | T_{k+1} < \infty]\Big|_{s=0} = \tag{5.45}$$

$$= -\frac{d}{ds}\left(\frac{1 - e^{b(z_0(s)-z_1(s))}}{(1 - z_0(s) - (1 - z_1(s))e^{b(z_0(s)-z_1(s))})(1-\rho)(1+s)}\right)\Big|_{s=0} \tag{5.46}$$

Using that $z_0(0) = 0$ and $z_1(0) = R = \frac{p-1}{p}$ as well as

$$z_0'(0) = -\frac{1}{p-1} \qquad \text{and} \qquad z_1'(0) = \frac{1}{p(p-1)}, \tag{5.47}$$

one derives

$$m = \frac{-b(\frac{1}{p-1} + \frac{1}{p(p-1)})e^{-b\frac{p-1}{p}}(1 - \frac{1}{p}e^{-b\frac{p-1}{p}}) + (1 - e^{-b\frac{p-1}{p}}) \cdot (1 - \frac{1}{p}e^{-b\frac{p-1}{p}})}{(1-\rho)(1 - \frac{1}{p}e^{-b\frac{p-1}{p}})^2} + \tag{5.48}$$

$$+ \frac{(1 - e^{-b\frac{p-1}{p}}) \cdot \left(\frac{1}{p-1} + \frac{1}{p(p-1)}e^{-b\frac{p-1}{p}} + \frac{1}{p}e^{-b\frac{p-1}{p}} \cdot b(\frac{1}{p-1} + \frac{1}{p(p-1)})\right)}{(1-\rho)(1 - \frac{1}{p}e^{-b\frac{p-1}{p}})^2}. \tag{5.49}$$

Next, use that

$$1 - \rho = 1 - \frac{(p-1) \cdot e^{-\frac{p-1}{p}b}}{p - e^{-\frac{p-1}{p}b}} = \frac{p(1 - e^{-\frac{p-1}{p}b})}{p - e^{-\frac{p-1}{p}b}} \tag{5.50}$$

and simplify further

$$m = 1 + \frac{\frac{b(p+1)}{p(p-1)}(\frac{1}{p} - 1)e^{-b\frac{p-1}{p}} + \frac{1}{p(p-1)}(p + e^{-b\frac{p-1}{p}})(1 - e^{-b\frac{p-1}{p}})}{(1 - e^{-b\frac{p-1}{p}})(1 - \frac{1}{p}e^{-b\frac{p-1}{p}})} = \tag{5.51}$$

$$= 1 + \frac{-b(p+1)(p-1)e^{-b\frac{p-1}{p}} + (p^2 + pe^{-b\frac{p-1}{p}})(1 - e^{-b\frac{p-1}{p}})}{p(p-1)(p - e^{-b\frac{p-1}{p}})(1 - e^{-b\frac{p-1}{p}})} = \tag{5.52}$$

$$= 1 + \frac{p^2 - (p-1)(bp + b + p)e^{-b\frac{p-1}{p}} - pe^{-2b\frac{p-1}{p}}}{p(p-1)(p - e^{-b\frac{p-1}{p}})(1 - e^{-b\frac{p-1}{p}})} \sim \frac{p}{p-1} \quad \text{as } b \to \infty. \tag{5.53}$$

Finally, the density of $\frac{\rho T_R}{m}|\{T_1 < T_R\}$ is obtained by numerical inversion of the derived Laplace transform and plotted in Figure 5.4 for different values of the premium income rate p and the level b of the dividend barrier. One sees clearly that $T_R|\{T_1 < T_R\}$ has asymptotically an $Exp(\frac{\rho}{m})$ - distribution as $b \to \infty$.

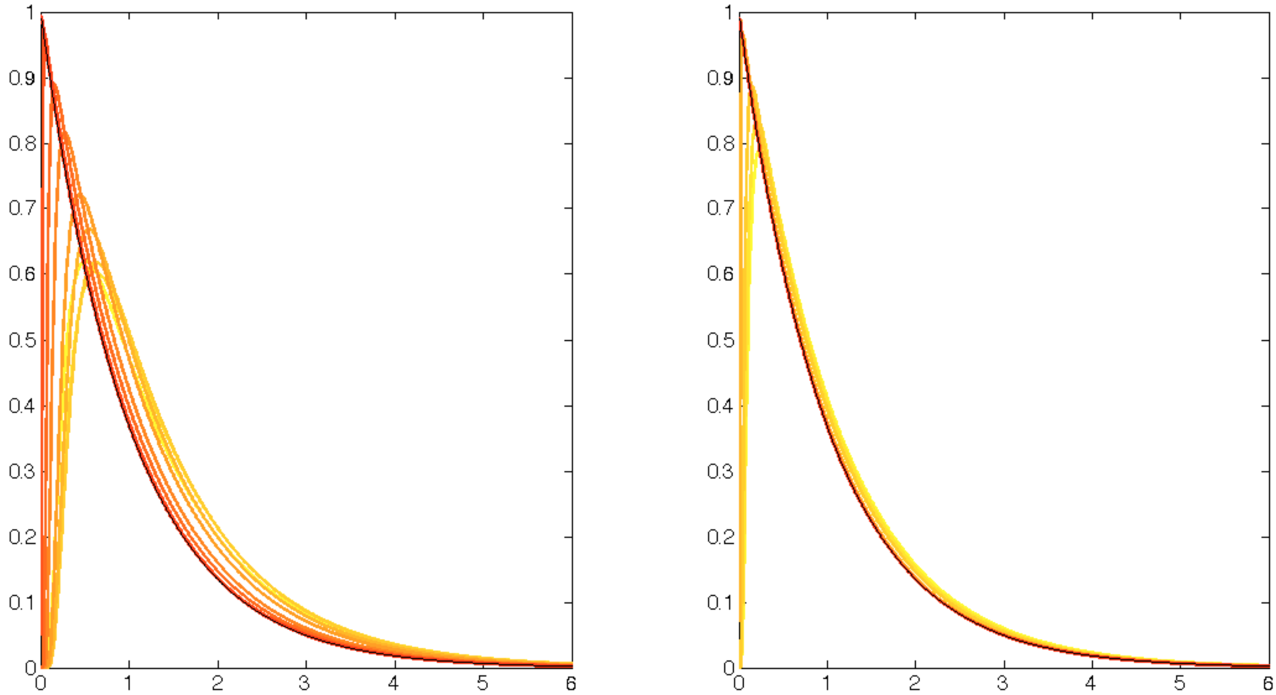


Figure 5.4.: Density of $\frac{\rho T_R}{m}|\{T_1 < T_R\}$ with initial surplus 50, $p = 1.01$ (l.h.s.) and $p = 1.04$ (r.h.s.) and $b \in \{60, 80, 100, 150, 200, 300, 400, 600, 800\}$. The line color depends on the height of the barrier (yellow for low and red for high values of b respectively).

The above example demonstrates the derived asymptotic law for the time of ruin in case of exponentially distributed claims. Nevertheless, calculations are quite extensive even in this basic case. If one would like to provide additional support for the asymptotic formula of T_R for other claim distributions as well, a **Monte Carlo simulation** should be preferable.

For this purpose, consider **Erlang distributed claims** C_n with shape parameter $n \in \mathbb{N}$ and scale parameter $\beta = 1/\alpha = 1/n$ such that the expected size is 1. Recall from Lemma 2.1.5. that an $Erlang(n, \alpha)$ distribution corresponds to a sum of n independent exponentially distributed random variables with mean α. Hence one has exponential claims as special case if $n = 1$. Its density is of the form

$$f(x) = \frac{\alpha^n}{(n-1)!} x^{n-1} e^{-\alpha x}, \qquad x \geq 0, \tag{5.54}$$

and the corresponding moment generating function is

$$M_C(x) = \left(\frac{\alpha}{\alpha - x} \right)^n, \qquad \text{for } x < \alpha. \tag{5.55}$$

Now, let the Poisson parameter be $\lambda = 1$ and $x = b$, meaning that the surplus process starts at the dividend barrier. Recall that by Theorem 4.1.1.

$$m \sim \frac{1}{\lambda} + \frac{\mu}{p - \lambda\mu} \qquad \text{as } b \to \infty \tag{5.56}$$

and

$$\rho \sim \frac{CpR}{\lambda} e^{-Rb} \qquad \text{as } b \to \infty. \tag{5.57}$$

The constant from the Cramér-Lundberg approximation is

$$C = \frac{p - g'(0)}{g'(R) - p} = \frac{p - \frac{n}{\alpha}}{\frac{n\alpha^n}{(\alpha-R)^{n+1}} - p}. \tag{5.58}$$

If one takes the Erlang parameters to be $n = 2$ and $\alpha = 2$ the Cramér-Lundberg condition

$$1 + pR = M_C(R) = \left(\frac{\alpha}{\alpha - R}\right)^n \tag{5.59}$$

has the solution

$$R = \frac{4p - 1 - \sqrt{8p+1}}{2p}. \tag{5.60}$$

One can use $\frac{p}{p-1}$ as approximation for m, and for ρ one has $CpRe^{-Rb}$ in case of an $Erlang(2,2)$ distribution or $\frac{p-1}{p}e^{-\frac{p-1}{p}b}$ in case of an $Exp(1)$ distribution.

Figure 5.6 shows the **histogram of 1000 simulated times of ruin** scaled by the approximation for ρ/m in case of $\lambda = 1$, $\mu = 1$, $p = 1.02$, $Erlang(2,2)$ claims and different levels for the dividend barrier b. Moreover, the surplus process starts at the barrier ($x = b$). It is clearly visible that for increasing dividend barrier the histogram looks more and more like the density of an $Exp(1)$ distribution. The same holds true for $Exp(1)$ claims, see Figure 5.5.

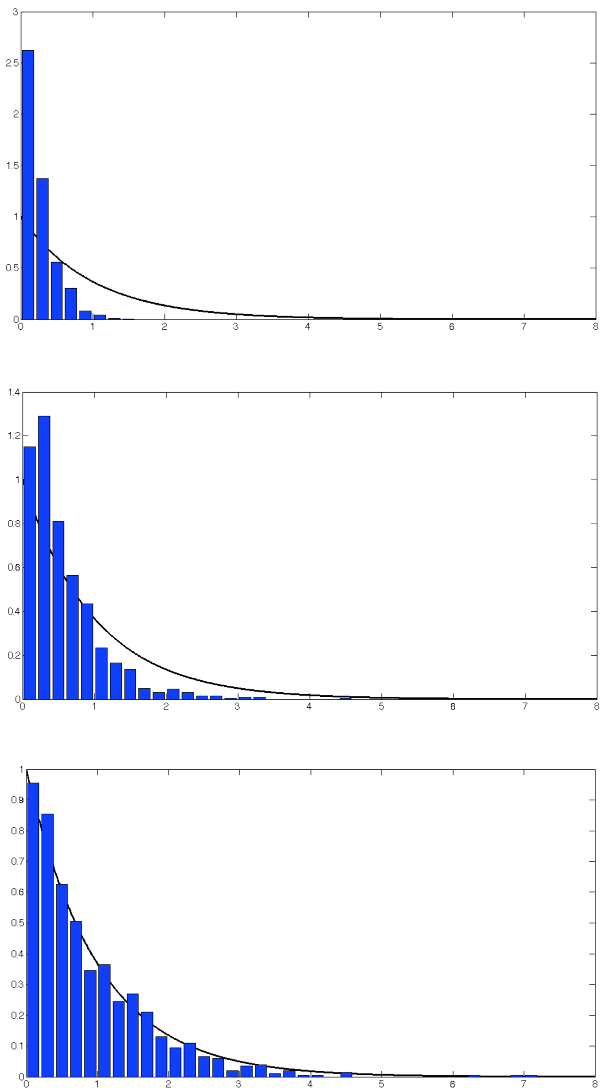


Figure 5.5.: Histograms of $\frac{\rho}{m}T_R$ with $Exp(1)$ claims, $p = 1.02$ and $b \in \{50, 100, 200\}$. The black curves represent the $Exp(1)$ - density.

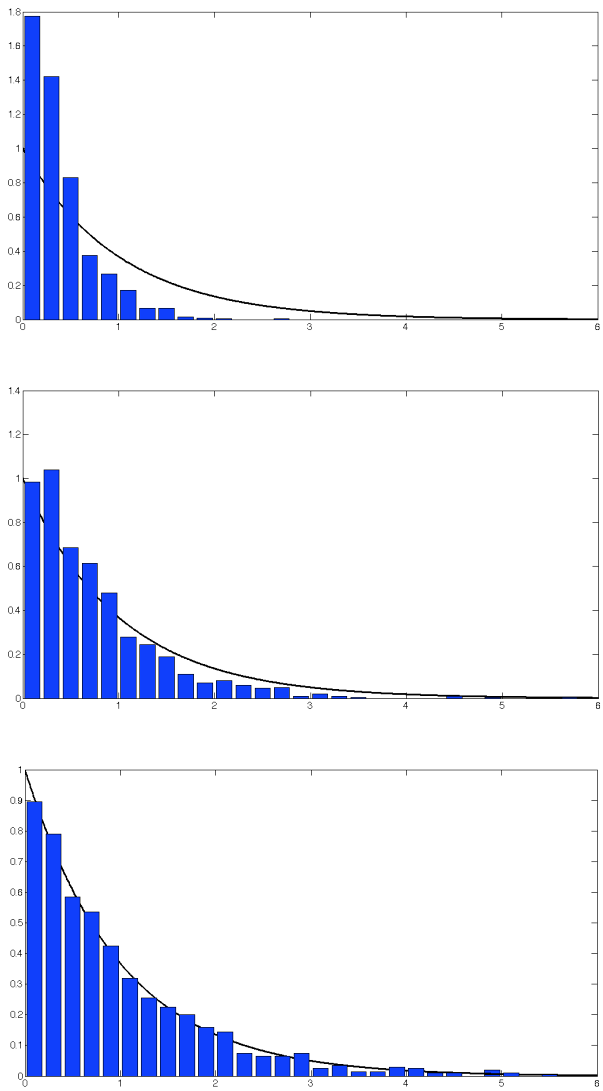


Figure 5.6.: Histograms of $\frac{\rho}{m}T_R$ with $Erlang(2,2)$ claims, $p = 1.02$ and $b \in \{50, 100, 200\}$. The black curves represent the $Exp(1)$ - density.

$\mathcal{A}$

Appendix: Properties of the Legendre transform

Proposition A.1.1. *The Legendre transform of $g(z)$, defined by*

$$h(x) := \max_z (xz - g(z)), \tag{A.1}$$

has the following properties:

(i). $h(x)$ is strictly convex,

(ii). $h(x) \geq 0$ and $h(g'(0)) = h(\lambda\mu) = 0$,

(iii). $H(t) := t \cdot h(p + \frac{1}{t}) \geq R$ and strictly convex for $t > 0$, with equality for $\bar{t} = \frac{1}{g'(R)-p}$.

Proof.
This is a slightly modified version of the proof given by A. Martin-Löf [M].
$g(z)$ is supposed to be finite in some maximal open interval D_g containing positive z-values.
In D_g $g(z)$ is infinitely differentiable and convex with $g(0) = 0$.
The Legendre transform of $g(z)$ is determined parametrically by the equations

$$\begin{cases} h(x) = xz - g(z), \\ x = g'(z) \quad \text{for } z \in D_g. \end{cases} \tag{A.2}$$

(i). Since $g''(z) = \lambda \int_0^\infty y^2 e^{zy} dF(y) > 0$, $g'(z)$ is strictly increasing and thus g' is a bijective function from D_g to some open interval D_h. Because of $h'(x) = z$, it follows that h' is the inverse of g' with $h' : D_h \longrightarrow D_g$ and

$$h''(x) = \frac{dz}{dx} = \frac{1}{g''(z)} > 0. \tag{A.3}$$

This means $h(x)$ is strictly convex and infinitely differentiable in D_h.

(ii). In order to find the minimum of h, look at the optimizing condition $h'(x) = z \stackrel{!}{=} 0$, which is solved for $x = g'(0) = \lambda\mu$. Therefore, the minimum value is $h(\lambda\mu) = 0 - g(0) = 0$

and one has that $h(x) \geq 0$ with $h(\lambda\mu) = 0$.

(iii). Differentiating the function

$$H(t) = t \cdot h\left(p + \frac{1}{t}\right) = t(pz - g(z)) + z, \quad \text{with } g'(z) = p + \frac{1}{t}, \tag{A.4}$$

according to the product rule yields

$$H'(t) = pz - g(z) + t\left(p \cdot \frac{dz}{dt} - g'(z) \cdot \frac{dz}{dt}\right) + \frac{dz}{dt} = \tag{A.5}$$

$$= pz - g(z) + \underbrace{(t \cdot (p - g'(z)) + 1)}_{=0} \cdot \frac{dz}{dt} = pz - g(z). \tag{A.6}$$

Differentiating one more time gives

$$H''(t) = (p - g'(z)) \cdot \frac{dz}{dt} = -\frac{1}{t} \cdot \frac{dz}{dt}. \tag{A.7}$$

The derivative of z w.r.t. t is found from the representation $z = h'(p + \frac{1}{t})$:

$$\frac{dz}{dt} = h''\left(p + \frac{1}{t}\right) \cdot \left(-\frac{1}{t^2}\right) = -\frac{1}{t^2 g''(z)}. \tag{A.8}$$

The final form for the second derivative is

$$H''(t) = \frac{1}{t^3 g''(z)} > 0 \quad \text{for } t > 0, \tag{A.9}$$

which says that $H(t) = t \cdot h(p + \frac{1}{t})$ is strictly convex for $t > 0$.
The minimizer of $H(t)$ is found by the optimizing condition $H'(t) = pz - g(z) \overset{!}{=} 0$. According to the Cramér-Lundberg equality ($g(R) = pR$) that is assumed to hold, this condition is solved by the adjustment coefficient R. Hence, the optimizing value for t is $\bar{t} = \frac{1}{g'(R) - p}$.
It leads to the minimum value

$$H(\bar{t}) = \frac{h(g'(R))}{g'(R) - p} = \frac{Rg'(R) - g(R)}{g'(R) - p} = R. \tag{A.10}$$

In summary, one has $H(t) := t \cdot h(p + \frac{1}{t}) \geq R$ and strictly convex for $t > 0$, with equality for $\bar{t} = \frac{1}{g'(R) - p}$. □

B

Appendix: Asymptotic normality of the time of ruin

It will be shown, conditional on ruin occurring, the time of ruin in the classical Cramér-Lundberg model is asymptotically normal distributed with mean $\mu_c := \frac{x}{g'(R)-p}$ and variance $\sigma_c^2 := \frac{xg''(R)}{(g'(R)-p)^3}$ as the initial surplus tends to infinity.

The concept is borrowed from S. Asmussen [A], and thus the usual notation gets slighly adjusted at this point. Write the surplus process in the following way:

$$X_t^c := x + pt - S_t = x + pt - \sum_{n=1}^{N_t} C_n =: x - R_t, \quad t \geq 0. \tag{B.1}$$

The process $\{R_t\}_{t\geq 0}$ represents the **amount of initial surplus used up to time t**. Ruin occurs if the used amount of surplus exceeds the initial surplus:

$$T_R^c = \inf(t \geq 0 | X_t^c < 0) = \inf(t \geq 0 | R_t > x). \tag{B.2}$$

Moreover, let $B(x) := R_{T_R^c} - x$ be the **overshoot** of used surplus above the initial level at time of ruin.

Theorem B.1.1. *Assume that $g'(R) < \infty$ as well as $g''(R) < \infty$ and that the overshoot $B(x)$ converges in distribution as $x \to \infty$, say to $B(\infty)$ (actually one can even show that $g'(R) < \infty$ is sufficient for this convergence). Then, for every $T > 0$*

$$\mathbb{P}[T_R^c \leq T | \{T_R^c < \infty\}] \sim \Phi\left(\frac{T - \mu_c}{\sigma_c}\right), \quad \text{as } x \to \infty, \tag{B.3}$$

where Φ denotes the cumulative distribution function of the $N(0,1)$ distribution.

The proof rests on two lemmas.

Lemma B.1.2. *As the initial surplus $x \to \infty$, it holds under the Esscher transform $\mathbb{P}^{(R)}$ that:*

(i). $\frac{T_R^c}{x} \xrightarrow{\mathbb{P}^{(R)}} \frac{1}{g'(R)-p}$

(ii). $\mathbb{E}^{(R)}\left[\frac{T_R^c}{x}\right] \longrightarrow \frac{1}{g'(R)-p}$

(iii). T_R^c *is* $\mathbb{P}^{(R)}$*- asymptotically normal with mean* μ_c *and variance* σ_c^2.

Proof.
(i). Equations 4.60 and 4.61 say that

$$\mathbb{E}^{(R)}[R_t] = t(g'(R) - p) \quad \text{and} \quad \mathbb{V}^{(R)}[R_t] = tg''(R). \tag{B.4}$$

By the strong law of large numbers

$$\bar{R}_t := \frac{R_t}{t} \xrightarrow{a.s.} g'(R) - p, \quad \text{as } t \to \infty, \tag{B.5}$$

and by the central limit theorem

$$Z_t := \frac{R_t - t(g'(R) - p)}{(tg''(R))^{\frac{1}{2}}} \xrightarrow{D} N(0,1), \quad \text{as } t \to \infty. \tag{B.6}$$

Now, substitute $t = T_R^c$ and write as above $R_{T_R^c} = x + B(x)$. Since $T_R^c \longrightarrow \infty$ and $B(x) \xrightarrow{D} B(\infty)$, it follows that $\frac{B(x)}{T_R^c} \xrightarrow{\mathbb{P}^{(R)}} 0$, as $x \to \infty$. Hence $\bar{R}_{T_R^c} := \frac{R_{T_R^c}}{T_R^c} \xrightarrow{\mathbb{P}^{(R)}} g'(R) - p$ implies

$$\frac{x}{T_R^c} = \frac{R_{T_R^c} - B(x)}{T_R^c} \xrightarrow{\mathbb{P}^{(R)}} g'(R) - p, \quad \text{as } x \to \infty. \tag{B.7}$$

(ii). The stated asymptotic law follows if one can show

$$\limsup_{x\to\infty} \mathbb{E}^{(R)}\left[\frac{T_R^c}{x}\right] \le \frac{1}{g'(R) - p} \le \liminf_{x\to\infty} \mathbb{E}^{(R)}\left[\frac{T_R^c}{x}\right]. \tag{B.8}$$

Since $\frac{T_R^c}{x}$, $x > 0$, are non-negative random variables that converge in probability to $\frac{1}{g'(R)-p}$ by property (i), the second inequality follows directly by Fatou's lemma.
The proof of the first inequality is more demanding. Recall the definition of $\{R_t\}_{t\ge 0}$:

$$R_t = \sum_{n=1}^{N_t} C_n - pt, \quad t \ge 0. \tag{B.9}$$

The idea in this proof is to define a modified version of the **used amount of surplus with limited claim sizes**

$$\tilde{R}_t := \sum_{n=1}^{N_t} \tilde{C}_n - pt = \sum_{n=1}^{N_t} (C_n \wedge m) - pt, \quad t \ge 0, \tag{B.10}$$

where $m > 0$ is the **maximum amount of a single claim**. Clearly, $\tilde{R}_t \le R_t$ a.s. and

$$\tilde{T}_R^c := \inf(t \ge 0 | \tilde{R}_t > x) \ge T_R^c. \tag{B.11}$$

From the moment generating function of $\{S_t\}_{t\geq 0}$ under the measure $\mathbb{P}^{(R)}$ in equation 4.59 it is found that $\{S_t\}_{t\geq 0}$ is again compound Poisson, where the Poisson process $\{N_t^{(R)}\}_{t\geq 0}$ has intensity $\lambda^{(R)} = \lambda + g(R)$ and the claim distribution has the form $F^{(R)}(t) := \frac{1}{M_C(R)} \int_0^t e^{Rx} dF(x)$, $t \geq 0$. Therefore,

$$\tilde{\mu}^{(R)} := \mathbb{E}^{(R)}[\tilde{C}_n] \nearrow \mathbb{E}^{(R)}[C_n] = \frac{\int_0^\infty x e^{Rx} dF(x)}{M_C(R)} = \frac{g'(R)}{\lambda + g(R)} =: \mu^{(R)}, \quad \text{as } m \to \infty. \tag{B.12}$$

Next, one has that

$$\mathbb{E}^{(R)}[\tilde{R}_{\tilde{T}_R^c}] = \mathbb{E}^{(R)}[N^{(R)}_{\tilde{T}_R^c}] \cdot \mathbb{E}^{(R)}[\tilde{C}_n] - p \cdot \mathbb{E}^{(R)}[\tilde{T}_R^c] = \tag{B.13}$$

$$= ((\lambda + g(R))\tilde{\mu}^{(R)} - p) \cdot \mathbb{E}^{(R)}[\tilde{T}_R^c] \tag{B.14}$$

and

$$\tilde{R}_{\tilde{T}_R^c} = \tilde{R}_{\tilde{T}_R^c -} + \tilde{C}_{N^{(R)}_{\tilde{T}_R^c}} \leq x + m \quad a.s. \tag{B.15}$$

Eventually the first inequality in B.8 is found:

$$\limsup_{x\to\infty} \mathbb{E}^{(R)}\left[\frac{T_R^c}{x}\right] \leq \limsup_{x\to\infty} \mathbb{E}^{(R)}\left[\frac{\tilde{T}_R^c}{x}\right] = \tag{B.16}$$

$$= \limsup_{x\to\infty} \mathbb{E}^{(R)}\left[\frac{\tilde{R}_{\tilde{T}_R^c}}{((\lambda + g(R))\tilde{\mu}^{(R)} - p) \cdot x}\right] \leq \tag{B.17}$$

$$\leq \frac{1}{(\lambda + g(R))\tilde{\mu}^{(R)} - p} \longrightarrow \frac{1}{g'(R) - p}, \quad \text{as } m \to \infty. \tag{B.18}$$

(iii). Since $\frac{T_R^c}{x} \xrightarrow{\mathbb{P}^{(R)}} \frac{1}{g'(R)-p}$ as $x \to \infty$ by (i), one can apply Anscombe's theorem to get a stopped version of the central limit theorem in B.6:

$$Z_{T_R^c} = \frac{R_{T_R^c} - T_R^c(g'(R) - p)}{(T_R^c \cdot g''(R))^{\frac{1}{2}}} \xrightarrow{D} N(0,1), \quad \text{as } x \to \infty. \tag{B.19}$$

Because of $\frac{B(x)}{(T_R^c)^{\frac{1}{2}}} \xrightarrow{\mathbb{P}^{(R)}} 0$, as $x \to \infty$, one ends up with

$$Z_{T_R^c} \overset{D}{\sim} \frac{x - T_R^c(g'(R) - p)}{(T_R^c \cdot g''(R))^{\frac{1}{2}}} \overset{D}{\sim} \frac{T_R^c \left(\frac{x}{T_R^c}\right)^{\frac{3}{2}} - x\left(\frac{T_R^c}{x}\right)^{\frac{1}{2}}(g'(R) - p)}{(x \cdot g''(R))^{\frac{1}{2}}} \overset{D}{\sim} \frac{T_R^c - \frac{x}{g'(R)-p}}{\left(\frac{x \cdot g''(R)}{(g'(R)-p)^3}\right)^{\frac{1}{2}}}. \tag{B.20}$$

□

Lemma B.1.3. *The overshoot $B(x)$ and the time of ruin T_R^c are $\mathbb{P}^{(R)}$- asymptotically independent as $x \to \infty$. That is, for f, g bounded and continuous*

$$\mathbb{E}^{(R)}\left[f(B(x)) \cdot g\left(\frac{T_R^c - \mu_c}{\sigma_c}\right)\right] \longrightarrow \mathbb{E}^{(R)}[f(B(\infty))] \cdot \mathbb{E}[g(Z)], \quad \text{as } x \to \infty, \tag{B.21}$$

where Z is standard normal.

Proof.
In equation B.21 one can replace $T_R^c(x)$ (x indicating the level to be attained) by $T_R^c(x')$, where $x' := x - x^{\frac{1}{4}}$, since

$$\begin{aligned}
\mathbb{E}^{(R)}[T_R^c(x) - T_R^c(x')] &= \mathbb{E}^{(R)}[\underbrace{(T_R^c(x) - T_R^c(x'))}_{=0}\mathbb{1}_{\{B(x')>x^{\frac{1}{4}}\}}]+ &\text{(B.22)}\\
&+ \mathbb{E}^{(R)}[(T_R^c(x) - T_R^c(x'))\mathbb{1}_{\{B(x')\le x^{\frac{1}{4}}\}}] = &\text{(B.23)}\\
&= \mathbb{E}^{(R)}[(T_R^c(x - R_{T_R^c(x')})\mathbb{1}_{\{B(x')\le x^{\frac{1}{4}}\}}] = &\text{(B.24)}\\
&= \mathbb{E}^{(R)}[T_R^c(x^{\frac{1}{4}} - B(x'))\mathbb{1}_{\{B(x')\le x^{\frac{1}{4}}\}}] \le &\text{(B.25)}\\
&\le \mathbb{E}^{(R)}[T_R^c(x^{\frac{1}{4}})] = O(x^{\frac{1}{4}}), &\text{(B.26)}
\end{aligned}$$

where the asymptotic law follows from property (ii) in Lemma B.1.2. Due to the law of total expectation

$$\mathbb{E}^{(R)}\left[f(B(x)) \cdot g\left(\frac{T_R^c(x') - \mu_c}{\sigma_c}\right)\right] = \mathbb{E}^{(R)}\left[\mathbb{E}^{(R)}[f(B(x)) \mid F_{T_R^c(x')}] \cdot g\left(\frac{T_R^c(x') - \mu_c}{\sigma_c}\right)\right] \tag{B.27}$$

and furthermore

$$\begin{aligned}
\mathbb{E}^{(R)}[f(B(x))|F_{T_R^c(x')}] &= \mathbb{E}^{(R)}[f(B(x^{\frac{1}{4}} - B(x'))]\mathbb{1}_{\{B(x')\le x^{\frac{1}{4}}\}} &\text{(B.28)}\\
&+ f(B(x') - x^{\frac{1}{4}})\mathbb{1}_{\{B(x')>x^{\frac{1}{4}}\}} &\text{(B.29)}\\
&\overset{\mathbb{P}^{(R)}}{\longrightarrow} \mathbb{E}^{(R)}[f(B(\infty)], \quad \text{as } x \to \infty, &\text{(B.30)}
\end{aligned}$$

using that $x^{\frac{1}{4}} - B(x') \overset{\mathbb{P}^{(R)}}{\longrightarrow} \infty$ as follows from $B(x') \overset{D}{\longrightarrow} B(\infty)$. Since T_R^c is $\mathbb{P}^{(R)}$- asymptotically normal by property (iii) in Lemma B.1.2.

$$\begin{aligned}
\mathbb{E}^{(R)}\left[f(B(x)) \cdot g\left(\frac{T_R^c(x') - \mu_c}{\sigma_c}\right)\right] &\sim \mathbb{E}^{(R)}[f(B(\infty)] \cdot \mathbb{E}^{(R)}\left[g\left(\frac{T_R^c(x') - \mu_c}{\sigma_c}\right)\right] \sim &\text{(B.31)}\\
&\sim \mathbb{E}^{(R)}[f(B(\infty))] \cdot \mathbb{E}^{(R)}[g(Z)], \quad \text{as } x \to \infty, &\text{(B.32)}
\end{aligned}$$

where Z is an $N(0,1)$ distributed random variable.

□

Finally, the two lemmas above enable to give a proof that, conditional on ruin occurring, the time of ruin in the classical Cramér-Lundberg model is asymptotically normal distributed.

Proof of Theorem B.1.1.
Applying the change of measure to the Esscher transform $\mathbb{P}^{(R)}$ yields

$$\begin{aligned}
\mathbb{P}[T_R^c \le T] &= \mathbb{E}^{(R)}[e^{-R(x-X_{T_R^c})+T_R^c(g(R)-pR)}\mathbb{1}_{\{T_R^c \le T\}}] = && \text{(B.33)}\\
&= \mathbb{E}^{(R)}[e^{-R\cdot R_{T_R^c}}\mathbb{1}_{\{T_R^c \le T\}}] = && \text{(B.34)}\\
&= e^{-Rx}\cdot \mathbb{E}^{(R)}[e^{-R\cdot B(x)}\mathbb{1}_{\{\frac{T_R^c-\mu_c}{\sigma_c}\le\frac{T-\mu_c}{\sigma_c}\}}] \sim && \text{(B.35)}\\
&\sim e^{-Rx}\cdot \mathbb{E}^{(R)}[e^{-R\cdot B(\infty)}]\cdot \Phi\left(\frac{T-\mu_c}{\sigma_c}\right), \quad \text{as } x\to\infty, && \text{(B.36)}
\end{aligned}$$

where the asymptotic law follows from Lemma B.1.3..
The same change of measure yields the Cramér-Lundberg approximation for the probability of ruin in the classical model, where this time the constant is expressed in terms of the overshoot (T_R^c is a.s. finite under the new measure $\mathbb{P}^{(R)}$):

$$\mathbb{P}[T_R^c < \infty] = \mathbb{E}^{(R)}[e^{-R\cdot R_{T_R^c}}] = e^{-Rx}\cdot \mathbb{E}^{(R)}[e^{-R\cdot B(x)}] \sim Ce^{-Rx}, \quad \text{as } x\to\infty, \qquad \text{(B.37)}$$

with $C := \mathbb{E}^{(R)}[e^{-R\cdot B(\infty)}]$. The convergence of $\mathbb{E}^{(R)}[e^{-R\cdot B(x)}]$ relies on the assumption $B(u) \xrightarrow{\mathbb{P}^{(R)}} B(\infty)$ as well as the boundedness and continuity of the function e^{-Rx} for $x > 0$.
Finally

$$\mathbb{P}[T_R^c \le T|\{T_R^c < \infty\}] = \frac{\mathbb{P}[T_R^c \le T]}{\psi(x)} \sim \Phi\left(\frac{T-\mu_c}{\sigma_c}\right), \quad \text{as } x\to\infty. \qquad \text{(B.38)}$$

□

C

Appendix: Matlab code for the Monte Carlo simulation

The below code was run with MATLAB R2012b provided by The MathWorks (2012) on a MacBook Pro Retina (13 inch, early 2013) with OS X Mavericks.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Author: Martin Pleischl
% Master Thesis for the MSc in Financial and Actuarial Mathematics
% Vienna University of Technology, 2014.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%--------------------------------------------------------------------------
% Monte Carlo Simulation of the time to ruin in the Cramer-Lundberg model
% with a high dividend barrier (Exponential or Erlang claims)
%--------------------------------------------------------------------------

clear all, close all;

%--------------------------------------------------------------------------
% Parameter Input
%--------------------------------------------------------------------------

scale=1000; %timescale (discrete time steps of size 1/scale)
Sims=1000; %number of simulated paths

p=1.02; %dividend rate

lambda=1; %Poisson parameter

mu=1; %mean claim size

%shape parameter
%n=1; %Exp(1) claims
n=2; %Erlang(2,2) claims

%scale parameter
beta=mu/n;
```

```

B=[50,100,200]; %dividend barriers


%-------------------------------------------------------------------------
% Calculating m, R and C
%-------------------------------------------------------------------------

%asymptotic law for the expected size of a complete excursion
m=1/lambda+mu/(p-lambda*mu);

%Cramer-Lundberg coefficient
%R=1/mu-lambda/p; %Exp(1) claims
R=(4*p-1-sqrt(8*p+1))/(2*p); %Erlang(2,2) claims

%constant in the Cramer-Lundberg approximation
%C=lambda*mu/p; %Exp(1) claims
C=(p-1)/((2/(2-R))^3-p); %Erlang(2,2) claims


%-------------------------------------------------------------------------
% Allocations
%-------------------------------------------------------------------------

%allocation of the approximate probabilities of ruin during an excursion
rho=zeros(length(B));

%allocation of the simulated time to ruin
TR=zeros(length(B),Sims);


%-------------------------------------------------------------------------
% Loop over different barriers
%-------------------------------------------------------------------------

for i=1:length(B)

    b=B(i); %choose barrier

    %calculate approximate probability of ruin during an excursion
    rho(i)=C*p*R/lambda*exp(-R*b);

    %print size of barrier
    fprintf('\nDividend barrier: %d\n', b);


    %---------------------------------------------------------------------
    % Simulation
    %---------------------------------------------------------------------

    for j=1:Sims

```

```
        %print number of iteration
        fprintf('Iteration: %d/%d\n', j, Sims);

        %start at barrier
        X1=b;

        %simulate time of first claim
        T=exprnd(scale/lambda);

        %simulate size of first claim
        C=gamrnd(n,beta);

        t=1; %number of time step

        %simulate as long as the surplus is positive
        while X1>0

            %surplus is below the barrier
            if b-X1>p/scale

                %next claim does not occur
                if t<T
                    %premium rate is added
                    X2=X1+p/scale;

                %next claim occurs
                else
                    %premium rate is added, claim size subtracted
                    X2=X1+p/scale-C;

                    %simulate size of next claim
                    C=gamrnd(n,beta);

                    %simulate inter-arrival time of next claim
                    I=exprnd(scale/lambda);

                    %add to time of last claim
                    T=T+I;
                end

            %surplus is at the barrier
            else
                %next claim does not occur
                if t<T
                    %surplus stays at the barrier
                    X2=b;

                %next claim occurs
                else
                    %claim size is subtracted
                    X2=b-C;

```

```
                    %simulate size of next claim
                    C=gamrnd(n,beta);

                    %simulate inter-arrival time of next claim
                    I=exprnd(scale/lambda);

                    %add to time of last claim
                    T=T+I;
                end
            end

            %preparing next time step
            t=t+1;
            X1=X2;
        end

        %calculate the simulated time of ruin
        TR(i,j)=t/scale;
    end


    %--------------------------------------------------------------------
    % Post-processing (plotting)
    %--------------------------------------------------------------------

    figure
    binsize = 0.2;
    Tmax=6;

    %plotting an Exp(1) density for comparison
    xvalues=0:0.01:Tmax;
    exp_pdf = exppdf(xvalues,1);
    plot(xvalues,exp_pdf,'LineWidth',1.4,'Color','k')
    hold on;

    %plotting histogram of rho/m*TR
    [counts,TRx] = hist(rho(i)*TR(i,:)/m,binsize/2:binsize:Tmax);
    norm = counts/Sims/binsize;
    bar(TRx,norm,'blue');
    set(gca,'FontSize',16);
    set(gcf,'color','w');

    hold off;
end
```

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