DIPLOMARBEIT

On a uniqueness theorem for the Fokker-Planck equation

Ausgeführt am
Institut für Wirtschaftsmathematik
der Technischen Universität Wien

unter der Anleitung von
Privatdoz. Dipl.-Ing. Dr.techn. Stefan Gerhold

durch
Joel Gotsch, BSc

Gaswerkstraße 2
2000 Stockerau

Wien, im April 2014

TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

Institute of Mathematical Methods in Economics
Faculty of Mathematics and Geoinformation

On a uniqueness theorem for the Fokker-Planck equation

Diploma Thesis

Joel Gotsch

0825654

Vienna, April 2014

Statutory declaration

Joel Gotsch, BSc

Gaswerkstrasse 2
2000 Stockerau

I declare that I have authored this thesis independently, that I have not used other than the declared sources / resources and that I have explicitly marked all material which has been quoted either literally or by content from the used sources.

Vienna, May 2014

Abstract

This thesis aims to prove a uniqueness theorem for the one dimensional driftless Fokker-Planck partial differential equation, i.e.

$$\frac{\partial p(\mathrm{d}x,t)}{\partial t} - \frac{\partial^2}{\partial x^2}(p(\mathrm{d}x,t)a(x,t)) = 0 \quad \textit{in } \mathcal{D}'(U)$$

where $a(x,t)$ is a positive Borel function and for each $t \geq 0$ $p(\mathrm{d}x,t)$ denotes a measure on either $\mathbb{R}$ or $\mathbb{R}_+$, depending on U. We study two cases: $U = \mathbb{R}_+ \times \mathbb{R}_+$ and $U = \mathbb{R} \times \mathbb{R}_+$. The latter case has been examined by M. Pierre, see [3, page 223]. We will replicate the proof which is given there but in more detail in section 5.
However, the main result of this thesis is the case $U = \mathbb{R}_+ \times \mathbb{R}_+$, which is examined in section 4. Unfortunately, the straightforward adaptation of the proof was not successful. This case is of theoretical importance (see e.g. [2]). An additional assumption for the diffusion term $a(x,t)$ should fix this problem, a heuristical proof is shown (see Lemma (4.7) point 5).

Contents

1 Introduction

The Fokker-Planck Equation is a partial differential equation of a family of probability measures $p(\mathrm{d}x,t)_{t\in\mathbb{R}_+}$. It is also known as the Kolmogorov Forward Equation. It is well known, that the probability density $f(x,t)$ of an Ito diffusion X_t satisfies the Fokker-Planck Equation. Hence the importance of the herein discussed equation to financial mathematics. In its general, d-dimensional case the Fokker Planck Equation reads as follows:

$$\frac{\partial}{\partial t}p(\mathbf{x},t) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\Big[b_i(\mathbf{x})p(\mathbf{x},t)\Big] + \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i\,\partial x_j}\Big[a_{ij}(\mathbf{x})p(\mathbf{x},t)\Big]$$

Were $\mathbf{x}$ is a d-dimensional vector, the function vector $\mathbf{b}$ is called the drift and $\mathbf{a}$ is called diffusion.

In this thesis we will only discuss the driftless (i.e. $b = 0$), one dimensional case (one x-dimension and time). But also want to consider distributions which do not admit density functions. Therefor we write $p(\mathrm{d}x,t)$ instead of $p(x,t)$. This leads to

$$\frac{\partial p(\mathrm{d}x,t)}{\partial t} = \frac{\partial^2}{\partial x^2}(p(\mathrm{d}x,t)a(x,t))$$

Here, $p(\mathrm{d}x,t)$ must be seen as a distribution and therefore this equation has to be interpreted in the distributional sence. I.e for some $U \subseteq \mathbb{R}^2$ and for any $\phi \in \mathcal{D}(U)$ the following equation must hold:

$$\iint_U \frac{\partial p(\mathrm{d}x,t)}{\partial t}\mathrm{d}t + \iint_U \frac{\partial^2}{\partial x^2}(p(\mathrm{d}x,t)a(x,t))\mathrm{d}t = 0.$$

We will discuss the cases $U = \mathbb{R}_+ \times \mathbb{R}_+$ and $U = \mathbb{R} \times \mathbb{R}_+$. The latter case has been examined by M. Pierre, see [3, page 223]. We will replicate the proof which is given there but in more detail in section 5.

However, the main result of this thesis is the case $U = \mathbb{R}_+ \times \mathbb{R}_+$, which is examined in section 4. Unfortunately, the straightforward adaptation of the proof was not successful. This case is of theoretical importance (see e.g. [2]). An additional assumption for the diffusion term $a(x,t)$ should fix this problem, a heuristical proof is shown (see Lemma (4.7) point 5).

We haven't discussed the properties of $a(x,t)$ yet. We will require $a(x,t) > 0$ with some boundedness condition, see theorem (3.2). These two conditions are sufficient to proof the case $U = \mathbb{R} \times \mathbb{R}_+$. For the case $U = \mathbb{R}_+ \times \mathbb{R}_+$ however, we need the third condition from theorem (3.2) which states that $a(x,t)$ and $a_x(x,t)$ vanish at $x = 0$ for all $t > 0$, see remark (3.1).

The last distinction we make in this thesis is the boundary condition for p at $t = 0$. In section 3 we will consider the case that $p(x, 0) = f(x)$ for a density function f. In sections 4 and 5 we consider the general case with $p(\mathrm{d}x, 0) = \mu(\mathrm{d}x)$.

2 Notation und definitions

Definition 2.1. Let $I \subseteq \mathbb{R}$. Then we denote by $\chi_I(x) : \mathbb{R} \to \{0, 1\}$ the characteristic function, i.e.

$$\chi_I(x) := \begin{cases} 1, & \text{if } x \in I, \\ 0, & \text{else.} \end{cases}$$

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be an open set (with respect to euclidian topology). By $C^\infty(\Omega)$ we denote the space of smooth functions $f : \Omega \to \mathbb{R}$ (e.g, they have derivatives of all orders). $\mathcal{D}(\Omega) = C_K^\infty(\Omega) = \{f \in C^\infty(\Omega) : \text{supp}(f) \text{ is compact in } \Omega\}$ we denote the class of smooth functions with compact (with respect to euclidian topology) support in Ω.
As usual $\mathbb{R}_+ := (0, \infty), \mathbb{N} := \{1, 2, 3, ...\}, \mathbb{N}_0 := \{0, 1, 2, ...\}$
We will denote the class of continuous and bounded (with respect to euclidian topology) functions $f : \mathbb{R}^d \mapsto \mathbb{R}$ by $C_b(\mathbb{R}^d)$ and the subclass $f : \mathbb{R}^d \mapsto \mathbb{R}_+$ with compact support by $C_K^+(\mathbb{R}^d)$.
We write a function $f : \Omega \to \mathbb{R}$ is in $\mathcal{L}^1(\Omega)$ iff $\int_\Omega |f(x)| \, \mathrm{d}\lambda(x) < \infty$ (λ denotes the Lebesgue measure of $\mathbb{R}^n$). Similarly we denote by $\mathcal{L}^2(\Omega)$ the space of functions which are squareintegrable, i.e $\int_\Omega (f(x))^2 \mathrm{d}\lambda(x) < \infty$.
We denote the space of locally integrable functions f by $\mathcal{L}^1_{loc}(\Omega)$, i.e $\forall K \subseteq \Omega, K$ compact: $\int_K |f(x)| \, \mathrm{d}\lambda(x) < \infty$

Definition 2.3. We now consider the space $\mathcal{M}_d = \mathcal{M}_d(\mathbb{R}^d)$ of locally finite measures on $\mathbb{R}^d$. On $\mathcal{M}_d$ we may introduce the **vague topology**, generated by the mappings $\pi_f : \mu \mapsto \mu f := \int f \mathrm{d}\mu,\ f \in C_K^+(\mathbb{R}^d)$ (the initial topology with respect to these mappings). We see that a family of measures $\mu_n \in \mathcal{M}_d$ **converges vaguely** to $\mu \in \mathcal{M}_d$ iff $\mu_n f \to \mu f \ \forall f \in C_K^+(\mathbb{R}^d)$.
Similarly we introduce the **weak topology** as the initial topology generated by $\pi_f : \mu \mapsto \mu f = \int f \mathrm{d}\mu,\ f \in C_b(\mathbb{R}^d)$. Therefore a family of measures $\mu_n \in \mathcal{M}_d$ **converges weakly** to $\mu \in \mathcal{M}_d$ iff $\mu_n f \to \mu f \ \forall f \in C_b(\mathbb{R}^d)$

Remark 2.4. An equivalent definition for vague convergence is to use the class of continuous functions $f : \mathbb{R}^d \mapsto \mathbb{R}$ with compact support.
Clearly, weak convergence implies vague convergence.
We will later introduce a family of functions $(p(x, t), t \geq 0)$ where for each $t \geq 0$, $x \to p(x, t)$ is a density function on $\mathbb{R}^+$. Weak convergence of $t \to p(x, t)$ means that

$$\forall t \in \mathbb{R}, \forall \phi \in C_b(\mathbb{R}) \text{ and } t_n \to t :$$
$$\int_{\mathbb{R}} \phi(x) p(x, t_n) \mathrm{d}x \to \int_{\mathbb{R}} \phi(x) p(x, t) \mathrm{d}x$$

Similarly, for a family of probability measures $(p(\mathrm{d}x,t), t \geq 0)$ is weakly continuous, iff

$$\forall t \in \mathbb{R}, \forall \phi \in C_b(\mathbb{R}) \text{ and } t_n \to t :$$
$$\int_{\mathbb{R}} \phi(x) p(\mathrm{d}x, t_n) \to \int_{\mathbb{R}} \phi(x) p(\mathrm{d}x, t)$$

Remark 2.5. There are many equivalent statements to weak convergence, which are summarized in the **portmanteau theorem**.

Theorem 2.6 (portmanteu theorem)**.** Let $\mathcal{B}$ be the Borel-σ-Algebra on $\mathbb{R}$. Let $X, X_1, X_2, ...$ be random variables with associated measures $\mu, \mu_1, \mu_2, ...$ and cumulative distribution functions $F, F_1, F_2, ...$ Then, the following statements are equivalent:

- $(X_n)_{n\in\mathbb{N}}/(\mu_n)_{n\in\mathbb{N}}$ converges weakly to X/μ.
- $F_n(x) \to F(x) \quad \forall x : F(x) = F_-(x)$
- $\lim\limits_{n\to\infty} \int f(x)\mathrm{d}\mu_n(x) = \int f(x)\mathrm{d}\mu(x)$ for all bounded f, which are μ-a.s. continuous
- $\lim\limits_{n\to\infty} \int f(x)\mathrm{d}\mu_n(x) = \int f(x)\mathrm{d}\mu(x)$ for all bounded f, which twice differentiable and f' and f'' are uniformly continuous
- $\mu(O) \leq \liminf\limits_{n\in\mathbb{N}} \mu_n(O)$ for all open sets O in the Euclidean topology on $\mathbb{R}$.
- $\mu(C) \geq \limsup\limits_{n\in\mathbb{N}} \mu_n(C)$ for all closed sets C in the Euclidean topology on $\mathbb{R}$.
- $\mu(A) \leq \lim\limits_{n} \mu_n(A)$ for all sets A in the Euclidean topology on $\mathbb{R}$ with $\mu(\partial A) = 0$.

See [6, page 297] for details and proof.

Definition 2.7. We define what convergence for a sequence of functions in $\mathcal{D}(\Omega)$ means.
Let $\Phi_n \in \mathcal{D}(\Omega)$ be a sequence of test functions. Then $\Phi_n \to 0$ iff:

1. $\exists K \subseteq \Omega, K$ compact such that $\forall n \in \mathbb{N} : \mathrm{supp}(\Phi_n) \subseteq K$ and
2. For all $n \in \mathbb{N}$ and multiindices $\alpha \in \mathbb{N}_0^n : \lim_{n\to\infty} \sup_{x\in\Omega} \{|D^\alpha \Phi_n(x)|\} = 0$.

We write $\Phi_n \to \Phi$ iff $|\Phi - \Phi_n| \to 0$.
We may also introduce a norm on $\mathcal{D}(\Omega)$:
For $k \in \mathbb{N}_0, \Phi \in \mathcal{D}(\Omega)$ and $K \subseteq \Omega$ with K compact, let

$$\|\Phi\|_{C^k(K)} = \sum_{|\alpha| \leq k} \sup_{x \in K} \{|D^\alpha(\Phi)|\}.$$

Definition 2.8. We can now define distributions. A distribution is a linear functional $u : \mathcal{D}(\Omega) \to \mathbb{R}$ which is continuous with respect to the convergence of distributions, i.e. $\forall \Phi_n \to 0$ in $\mathcal{D}(\Omega) : u(\Phi_n) \to 0$ in $\mathbb{R}$ (with respect to the Euclidean Topology). We may also wright $\langle u_f, \phi \rangle$ for $u_f(\phi)$.

Lemma 2.9. A linear functional $u : \mathcal{D}(\Omega) \to \mathbb{R}$ is a distribution iff

$$\begin{aligned} &\forall K \subseteq \Omega, K \text{ compact} : \exists C > 0, k \in \mathbb{N}_0 \text{ such that:} \\ &\forall \phi \in \mathcal{D}(\Omega) : |u(\phi)| \leq C\|\phi\|_{C^k(K)} \end{aligned}$$

PROOF : See [4, Lemma 2.4, page 20] □

Definition 2.10. We define now what convergence in $\mathcal{D}'(\Omega)$ means. A sequence of distributions $u_n \in \mathcal{D}'(\Omega)$ converge to $u \in \mathcal{D}'(\Omega)$ iff for all $\phi \in \mathcal{D}(\Omega) : \langle u_n, \phi \rangle \to \langle u, \phi \rangle$ in $\mathbb{R}$.

Lemma 2.11. Let still $\Omega \subseteq \mathbb{R}^n$ be an open set and $\alpha \in \mathbb{N}_0^n$ be a muldiindex. Any distribution u has partial derivatives of any order. It holds for $\phi \in \mathcal{D}(\Omega)$:

$$\langle D^\alpha u, \phi \rangle = (-1)^{|\alpha|} \langle u, D^\alpha \phi \rangle$$

PROOF : See [4, page 23]. □

Lemma 2.12. We remind, that we defined $\mathcal{L}^1_{loc}(\Omega)$ with $\forall K \subseteq \Omega, K$ compact: $\int_K |f(x)| \,\mathrm{d}\lambda(x) < \infty$. An equivalent definition is that $\forall \phi \in \mathcal{D}(\Omega)$: $\int_K \phi(x) f(x) \mathrm{d}\lambda(x) < \infty$.

PROOF : See Appendix. □

Remark 2.13. We can identify every function $f \in \mathcal{L}^1_{loc}(\Omega)$ with a distribution $u_f \in \mathcal{D}(\Omega)$ by $u_f : \mathcal{D}(\Omega) \to \mathbb{R} : \phi \to \int_\Omega \phi(x) f(x) \mathrm{d}\lambda(x)$. u_f is indeed a distribution since $\forall \phi$:supp$(\phi) \subseteq K \subseteq \Omega$ where K is compact. We have by definition $\int_K |f(x)| \,\mathrm{d}\lambda(x) = C_K < \infty$. Therefore,

$$\begin{aligned} u_f(\phi) = &\int_\Omega f(x)\phi(x) \mathrm{d}\lambda(x) \leq \int_\Omega |f(x)| \sup_{x \in \Omega} \{\phi(x)\} \,\mathrm{d}\lambda(x) = \\ =&C_K \|\phi\|_{C^0(K)} \end{aligned}$$

Lemma (2.9) shows now, that u_f is a distribution.
For simplicity we will often identify f with the distribution u_f and write $f\mathrm{d}x$ or „in $\mathcal{D}'(\Omega)$“ to indicate this. Distributions which can be written in this manner are called **regular** distributions.

Lemma 2.14. Let as usual $\Omega \subseteq \mathbb{R}^n$ be an open set. Then $\mathcal{L}^2(\Omega) \subseteq \mathcal{L}^1_{loc}(\Omega)$.

PROOF : Let $f \in \mathcal{L}^2(\Omega)$ and $K \subseteq \Omega$ be compact. We have to show, that $\int_K |f(x)| \,\mathrm{d}\lambda(x) < \infty$. We note, that $\chi_K \in \mathcal{L}^2(\Omega)$ since $\|\chi_K\|_2^2 = \lambda(K)$. Therefore follows from Cauchy Schwarz Inequality (see [6, page 217]) that

$$\int\limits_K |f(x)| \,\mathrm{d}\lambda(x) = \int\limits_\Omega |f(x)| \,\chi_K(x)\mathrm{d}\lambda(x) \leq \|\chi_K\|_2 \|f\|_2 < \infty$$

□

Lemma 2.15. In the following, we will construct functions with the help of $h(x) := e^{-\frac{1}{x}}$ to approximate the characteristic function $\chi_{(0,t)}(x)$ by smooth functions. With their help we can approximate the integration of a function in the distributional sence over an interval $(0, t)$. We will show:

1. $h(0_+) = 0$.
2. $h^{(n)}(x) = \frac{p_n(x)}{x^{2n}} e^{-\frac{1}{x}}$, where $p_n(x)$ is a polynomial with
$$\deg(p_n) \leq 2^n - n - 1.$$
3. $\forall n \in \mathbb{N} : \lim\limits_{x \to 0^+} \frac{1}{x^n} h(x) = 0$.
4. $h^{(n)}(0_+) = 0 \;\forall n \in \mathbb{N}$.

PROOF :

1. Let $0 < \epsilon < 1$. Then
$$e^{-\frac{1}{x}} \leq \epsilon \Leftrightarrow -\frac{1}{x} \leq \ln(\epsilon) \Leftrightarrow x \leq -\frac{1}{\ln(\epsilon)} =: \delta(\epsilon)$$
2. We show this by induction on n. It is true for $n = 1$ since
$$g'(x) = \frac{1}{x^2} e^{-\frac{1}{x}}.$$

Induction step $n \to n+1$:

$$g^{(n+1)}(x) = \left(\frac{p_n(x)}{x^{2^n}} e^{-\frac{1}{x}}\right)' =$$
$$= \left(\frac{p_n(x)' x^{2^n} - p_n(x) 2^n x^{2^n - 1}}{(x^{2^n})^2} + \frac{p_n(x)}{x^{2^n}} \frac{1}{x^2}\right) e^{-\frac{1}{x}} =$$
$$= \left(\frac{p_n(x)' x^{2^n} - p_n(x) 2^n x^{2^n - 1} + p_n(x) x^{2^n - 2}}{(x^{2^n})^2}\right) e^{-\frac{1}{x}} =$$
$$= \left(\frac{p_n(x)' x^{2^n} - p_n(x) 2^n x^{2^n - 1} + p_n(x) x^{2^n - 2}}{(x^{2^{n+1}})}\right) e^{-\frac{1}{x}}$$

We see, that p_{n+1} is recursively defined by

$$p_{n+1}(x) := p_n(x)' x^{2^n} - p_n(x) 2^n x^{2^n - 1} + p_n(x) x^{2^n - 2}$$

Now we show, that $\deg(p_{n+1}) \leq 2^{n+1} - n - 2$. Since

$$\deg(p_{n+1}) \leq \max\{\deg(p_n(x)' x^{2^n}); \deg(p_n(x) 2^n x^{2^n - 1}); \deg(p_n(x) x^{2^n - 2})\},$$

we only have to show, that these 3 polynomials have degree $\leq 2^{n+1} - n - 2$.

$\deg(p_n(x)' x^{2^n}) \leq \deg(p_n(x)') + \deg(x^{2^n}) \leq 2^n - n - 1 - 1 + 2^n =$
$= 2^{n+1} - n - 2$
$\deg(p_n(x) 2^n x^{2^n - 1}) \leq \deg(p_n(x)) + \deg(x^{2^n - 1}) \leq 2^n - n - 1 + 2^n - 1 =$
$= 2^{n+1} - n - 2$
$\deg(p_n(x) x^{2^n - 2}) \leq \deg(p_n(x)) + \deg(x^{2^n - 2}) \leq 2^n - n - 1 + 2^n - 2 =$
$= 2^{n+1} - n - 3 < 2^{n+1} - n - 2$

3. We substitute $y = \frac{1}{x}$.

$$\lim_{x \to 0^+} \frac{1}{x^n} h(x) = \lim_{y \to \infty} \frac{y^n}{e^y} \overset{\text{n times L' Hospital}}{\longrightarrow} \lim_{y \to \infty} \frac{n!}{e^y} = 0$$

4. Follows immediately from 2 and 3.

□

Lemma 2.16. The function

$$g:\mathbb{R}\to\mathbb{R},\quad g(x):=\begin{cases} e^{-\frac{1}{x}}, & \text{if } x>0,\\ 0, & \text{if } x\leq 0\ . \end{cases}$$

is smooth. With its help we define the function $\psi(x):\mathbb{R}\to\mathbb{R}$ by

$$\psi(x):=\frac{g\left(x-1\right)g\left(2-x\right)}{\int_{\mathbb{R}}g\left(x-1\right)g\left(2-x\right)\mathrm{d}x}$$

Then, $\psi(x)$ is smooth, $\int_{\mathbb{R}}\psi(x)\mathrm{d}x=1$ and $\mathrm{supp}(\psi)\subseteq[1,2]$.
Now we can define the desired functions $f_{t,n}(x)$ which approximate the characteristic function $\chi_{(0,t)}(x)$.

$$\text{for } n\in\mathbb{N}:\ \text{we define: } f_{t,n}(x):=\int\limits_{n(x-t)+2}^{nx}\psi(u)\mathrm{d}u \tag{2.1}$$

We will show, that

1. $f_{t,n}\in\mathcal{D}(\mathbb{R}_+)$ with $\mathrm{supp}(f_{t,n}(x))\subseteq[\frac{1}{n},t]$
2. $\lim\limits_{n\to\infty}f_{t,n}(x)=\chi_{(0,t)}(x)$ pointwise.
3. $\forall\ n\in\mathbb{N},t\geq 0: f_{t,n}(x)\leq\chi_{[0,t]}(x)$.
4. $f'_{t,n}(x)=n\left(\psi(nx)-\psi(nx-nt+2)\right)$.

PROOF : The smoothness of g follows immediately from Lemma 2.15.
We begin with showing the properties of ψ.
Since $\mathrm{supp}\left(g\left(x-1\right)\right)\subseteq[1,\infty)$ and $\mathrm{supp}\left(g\left(2-x\right)\right)\subseteq(-\infty,2]$, it follows easily, that $\mathrm{supp}\left(g\left(x-1\right)g\left(2-x\right)\right)\subseteq[1,2]$.
$g\left(x-1\right)$ and $g\left(2-x\right)$ are bounded on $[1,2]$, therefore $g\left(x-1\right)g\left(2-x\right)$ is bounded on $[1,2]$. This ensures the existence of $\int_{\mathbb{R}}g(x-1)g(2-x)\mathrm{d}x$.
As a composition of smooth functions, ψ is smooth. Furthermore,

$$\int\limits_{\mathbb{R}}\psi(u)\mathrm{d}u=\int\limits_{\mathbb{R}}\frac{g\left(u-1\right)g\left(2-u\right)}{\int_{\mathbb{R}}g\left(x-1\right)g\left(2-x\right)\mathrm{d}x}\mathrm{d}u=\frac{\int_{\mathbb{R}}g\left(u-1\right)g\left(2-u\right)\mathrm{d}u}{\int_{\mathbb{R}}g\left(x-1\right)g\left(2-x\right)\mathrm{d}x}=1.$$

Now, we show the properties of $f_{t,n}$.

1. To show: $f_{t,n}\in\mathcal{D}(\mathbb{R}_+)$
   We notice that $\mathrm{supp}(\psi)\subseteq[1,2]$ and $f_{t,n}(x)=0$ if $n(x-t)+2\geq 2$, which is equivalent to $x\geq t$. Similarly for the upper bound $f_{t,n}(x)=0$

if $nx \leq 1$ which is equivalent to $x \leq \frac{1}{n}$. Therefore, $\text{supp}(f_{t,n}(x)) \subseteq [\frac{1}{n}, t]$. Since $\psi(x) \geq 0$ and $\psi \in \mathcal{D}(\mathbb{R}_+)$,

$$f_{t,n}^{(i)}(x) = \left(\int\limits_{n(x-t)+2}^{nx} \psi(u)\mathrm{d}u \right)^{(i)} =$$

$$= |y = u - nx| = \left(\int\limits_{2-nt}^{0} \psi(y + nx)\mathrm{d}y \right)^{(i)} =$$

$$\left(\int\limits_{2-nt}^{0} \frac{\partial^i \psi(y + nx)}{\partial x^i}\mathrm{d}y \right)$$

which shows that $f_{t,n} \in C_K^\infty(\mathbb{R}_+)$.

2. To show: $\lim\limits_{n\to\infty} f_{t,n}(x) = \chi_{(0,t)}(x)$ pointwise.
   After the last point, it is left to show, that $\lim\limits_{n\to\infty} f_{t,n}(x) = 1$ for $x \in (0, t)$. Let $x \in (0, t)$ and $n > \max\left\{\frac{1}{t-x}, \frac{2}{x}\right\}$. Then $n(x-t)+2 \leq 1$ and $nx \geq 2$. Therefor

$$f_{t,n}(x) = \int\limits_{n(x-t)+2}^{nx} \psi(u)\mathrm{d}u = \int\limits_{1}^{2} \psi(u)\mathrm{d}u = 1$$

3. To show: $\forall\ n \in \mathbb{N}, t \geq 0 : f_{t,n}(x) \leq \chi_{[0,t]}(x)$.
   After the first point, it is left to show, that $f_{t,n}(x) \leq 1$. Which is easy to see, since $\psi(x) \geq 0$:

$$f_{t,n}(x) = \int\limits_{n(x-t)+2}^{nx-1} \psi(u)\mathrm{d}u \leq \int\limits_{\mathbb{R}} \psi(u)\mathrm{d}u = 1$$

4. To show: $f'_{t,n}(x) = n\left(\psi(nx) - \psi(nx - nt + 2)\right)$.
   We substiute $y = nx$ and get

$$f'_{t,n}(x) = \frac{\partial \int_{n(x-t)+2}^{nx} \psi(u)\mathrm{d}u}{\partial x} = n\frac{\partial \int_{y-nt+2}^{y} \psi(u)\mathrm{d}u}{\partial y} =$$

$$= n\left(\psi(y) - \psi(y - nt + 2)\right) = n\left(\psi(nx) - \psi(nx - nt + 2)\right)$$

□

Definition 2.17. We define convolution of a distribution u with a test function ψ by

$$\langle u * \psi, \phi \rangle := \langle u, (R\psi) * \phi \rangle$$

Where $R : \psi(x) \to \psi(-x)$.
It holds $u * \psi = \psi * u$ and for $\alpha \in \mathbb{N}_0^n$:

$$D^\alpha(u * \psi) = (D^\alpha u) * \psi = u * (D^\alpha \psi)$$

Therefore, the convolution has partial derivatives of the order of the sum of the convolved functions orders. Especially if one of the functions is smooth, the convolution is smooth. See [4, p. 24]

Lemma 2.18. We can also use the function g to construct for each $n \in \mathbb{N}$ a smooth function k_n with $\text{supp}(k_n) \subseteq [-\frac{1}{n}, \frac{1}{n}]$ and $\int_{\mathbb{R}} k_n(x)\text{d}x = 1$. The sequence $(k_n)_{n\in\mathbb{N}}$ is called a regularizing sequence due to its property that for each $f \in \mathcal{L}^1(\mathbb{R})$ holds

$$\lim_{n\to\infty} \|f - f * k_n\|_1 = 0$$

and as mentioned in the definition above, the convolution $f * k_n$ has derivatives of all orders since k_n has them.

PROOF : See [5, p. 25-28] □

Lemma 2.19. The functions k_n from Lemma (2.18) also approximate in the distributional sense, i.e for $u \in \mathcal{D}'(\mathbb{R})$ holds $\lim_{n\to\infty} u * k_n = u$ in $\mathcal{D}'(\mathbb{R})$.

PROOF : We have to show, that for all $\phi \in \mathcal{D}(\mathbb{R}) : \langle u * k_n, \phi \rangle \to \langle u, \phi \rangle$ in $\mathbb{R}$. It follows from definition (2.17) that $\langle u * k_n, \phi \rangle = \langle u, \phi * (Rk_n) \rangle$. Since k_n are symmetrical $Rk_n = k_n$ pointwise. Since u is per definition continuous with respect to convergence in $\mathcal{D}(\mathbb{R})$ we have to show, $\phi * k_n \to \phi$ in $\mathcal{D}(\mathbb{R})$ for all ϕ in $\mathcal{D}(\mathbb{R})$.
We fix $\phi \in \mathcal{D}(\mathbb{R})$ and $\alpha \in \mathbb{N}_0^n$. Let $\psi = D^\alpha \phi$. Then

$$\psi * k_n(x) = \int_{\mathbb{R}} \psi(y) k_n(x-y)\text{d}y = \int_{\left[x-\frac{1}{n}, x+\frac{1}{n}\right]} \psi(y) k_n(x-y)\text{d}y$$

Let $\epsilon > 0$ and n such that $|\psi(y) - \psi(x)| \leq \epsilon \quad \forall y \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$. Then

$$\int_{\left[x-\frac{1}{n}, x+\frac{1}{n}\right]} \psi(y) k_n(x-y)\text{d}y \leq \int_{\left[x-\frac{1}{n}, x+\frac{1}{n}\right]} (\psi(x) + \epsilon) k_n(x-y)\text{d}y = \psi(x) + \epsilon$$

Analogously we get $\psi * k_n(x) \geq \psi(x) - \epsilon$. Therefore $\lim_{n\to\infty} \|\psi - \psi * k_n\|_\infty = 0$, which concludes the proof. □

3 The case with existing density function

Remark 3.1. Compared with the case $U = R \times R_+$ (see section 5), we need an additional assumption for $a(x,t)$, which is a boundary condition (see point 3 of the following theorem). But still, the proof is not straightforward adaptable and we will only present a heuristic argument to show that this condition is sufficient to ensure $\frac{\partial P}{\partial x}(0,t) = 0$ (see Lemma (3.6) point 3). See also remark 3.7.

Theorem 3.2. Let $U := \mathbb{R}_+ \times \mathbb{R}_+$ and $a : U \to \mathbb{R}_+$ be a Borel function satisfying the following hypothesis:
$\forall 0 < t < T$ and $R > 0 : \exists \epsilon(t,T,R) > 0, m(T,R) > 0$ such that:

- $\forall (x,s) \in (0,R] \times [t,T] : a(x,s) \geq \epsilon(t,T,R)$ and
- $\forall (x,s) \in (0,R] \times (0,T] : a(x,s) \leq m(T,R)$
- $\forall t \in \mathbb{R}_+ : x \to a(x,t)$ is differentiable at $x = 0$ and $a(0,t) = a'(0,t) = 0$

Let μ be a probability measure on $\mathbb{R}_+$ with density function $f(x)$ and $\int_{\mathbb{R}_+} |x| f(x)\mathrm{d}x < \infty$. Then, there exists at most one family of probability measures with density functions $(p(x,t), t \geq 0)$ such that:

(FP 1) $t \geq 0 \to p(x,t)$ is weakly continuous, see Remark 2.4.

(FP 2) $p(0,x) = f(x)$ and

$$\iint\limits_U \frac{\partial \phi(x,t)}{\partial t} p(x,t)\mathrm{d}t\mathrm{d}x + \iint\limits_U \frac{\partial^2 \phi(x,t)}{\partial x^2} a(x,t)p(x,t)\mathrm{d}t\mathrm{d}x = 0 \quad \forall \phi \in \mathcal{D}(U) \tag{3.2}$$

PROOF : We note, that equation (3.2) is the integral representation of the following statement:

$$\frac{\partial p(x,t)}{\partial t} - \frac{\partial^2}{\partial x^2}(p(x,t)a(x,t)) = 0 \quad in\ \mathcal{D}'(U)$$

We will split the proof into several parts. □

Lemma 3.3. Let a probability measure μ with density function f and $p(x,t)$ be as in Theorem 3.2. Then holds $\forall t \geq 0, \phi \in \mathcal{D}(\mathbb{R}_+)$:

$$\int\limits_{\mathbb{R}_+} \phi(x)p(x,t)\mathrm{d}x = \int\limits_{\mathbb{R}_+} \phi(x)f(x)\mathrm{d}x + \int\limits_{\mathbb{R}_+}\int\limits_{(0,t)} \frac{\partial^2 \phi(x)}{\partial x^2} a(x,s)p(x,s)\mathrm{d}s\mathrm{d}x$$

Remark 3.4. We will use the fact, that $\forall \alpha(x), \phi(x) \in \mathcal{D}(\mathbb{R}_+)$ holds, that $\alpha(x)\dot{\phi}(t) \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}_+)$. Note that this lemma reads as $\int_{(0,t)} \frac{\partial^2 a(x,s)p(x,s)}{\partial x^2} \mathrm{d}s = p(x,t) - f(x)$ in $\mathcal{D}'(\mathbb{R}_+)$.

PROOF : For $t \geq 0$ fixed, we define $\alpha_n(x) := f_{t,n}(x)$ for $n \in \mathbb{N} : n \geq \frac{2}{t}$ by equation (2.1). From (FP 2) we know, that $\forall \phi \in \mathcal{D}(\mathbb{R}_+)$:

$$\iint\limits_U \frac{\partial \alpha_n(s)}{\partial s} \phi(x) p(x,s) \mathrm{d}x\mathrm{d}s + \iint\limits_U \frac{\partial^2 \phi(x)}{\partial x^2} \alpha_n(s) a(x,s) p(x,s) \mathrm{d}x\mathrm{d}s = 0$$

Now we take $\lim\limits_{n\to\infty}$ for both integrals individually. First we treat the second integral. In Lemma (2.16) we showed alle necessary requirements for dominated convergence. Therefore,

$$\begin{aligned}
&\lim_{n\to\infty} \iint\limits_U \frac{\partial^2 \phi(x)}{\partial x^2} \alpha_n(s) a(x,s) p(x,s) \mathrm{d}x\mathrm{d}s = \\
&\iint\limits_U \frac{\partial^2 \phi(x)}{\partial x^2} \lim_{n\to\infty} \alpha_n(s) a(x,s) p(x,s) \mathrm{d}x\mathrm{d}s = \\
&\iint\limits_U \frac{\partial^2 \phi(x)}{\partial x^2} \chi_{(0,t)}(s) a(x,s) p(x,s) \mathrm{d}x\mathrm{d}s = \\
&\int\limits_{\mathbb{R}_+} \int\limits_{(0,t)} \frac{\partial^2 \phi(x)}{\partial x^2} a(x,s) p(x,s) \mathrm{d}x\mathrm{d}s
\end{aligned}$$

Now, we examine the first integral.

$$\begin{aligned}
&\iint\limits_U \frac{\partial \alpha_n(s)}{\partial s} \phi(x) p(x,s) \mathrm{d}x\mathrm{d}s = \\
&\iint\limits_U n\left(\psi(ns) - \psi(ns - nt + 2)\right) \phi(x) p(x,s) \mathrm{d}x\mathrm{d}s = \\
&\iint\limits_U n\psi(ns) \phi(x) p(x,s) \mathrm{d}x\mathrm{d}s - \iint\limits_U n\psi(ns - nt + 2) \phi(x) p(x,s) \mathrm{d}x\mathrm{d}s
\end{aligned}$$

And again, we have to treat both integrals individually by substituting:

$$\begin{aligned}
&\iint\limits_U n\psi(ns) \phi(x) p(x,s) \mathrm{d}x\mathrm{d}s = |u = ns| = \\
&\iint\limits_U \psi(u) \phi(x) p\left(x, \frac{u}{n}\right) \mathrm{d}x\mathrm{d}u
\end{aligned}$$

and

$$\iint_U n\psi(ns - nt + 2)\phi(x)p(x,s)\mathrm{d}x\mathrm{d}s = |u = ns - nt + 2| =$$
$$\iint_{\tilde{U}} \psi(u)\phi(x)p\left(x, t + \frac{u-2}{n}\right)\mathrm{d}x\mathrm{d}u$$

Where $\tilde{U} = (2 - nt, \infty) \times \mathbb{R}_+$. Since we required $n \geq \frac{2}{t}$ and $\operatorname{supp}(\psi) \subseteq [1,2]$, we deduce

$$\iint_{\tilde{U}} \psi(u)\phi(x)p\left(x, t + \frac{u-2}{n}\right)\mathrm{d}x\mathrm{d}u =$$
$$\iint_U \psi(u)\phi(x)p\left(x, t + \frac{u-2}{n}\right)\mathrm{d}x\mathrm{d}u$$

By (FP 1), these integrals converge for $n \to \infty$:

$$\lim_{n\to\infty} \iint_U \frac{\partial \alpha_n(s)}{\partial s}\phi(x)p(x,s)\mathrm{d}x\mathrm{d}s =$$
$$\lim_{n\to\infty}\left(\iint_U \psi(u)\phi(x)p\left(x, \frac{u}{n}\right)\mathrm{d}x\mathrm{d}u - \iint_U \psi(u)\phi(x)p\left(x, t + \frac{u-2}{n}\right)\mathrm{d}x\mathrm{d}u\right) =$$
$$\iint_U \psi(u)\phi(x)p(x,0)\mathrm{d}x\mathrm{d}u - \iint_U \psi(u)\phi(x)p(x,t)\mathrm{d}x\mathrm{d}u = \text{ now using Fubini}$$
$$\iint_U \psi(u)\mathrm{d}u\phi(x)p(x,0)\mathrm{d}x - \iint_U \psi(u)\mathrm{d}u\phi(x)p(x,t)\mathrm{d}x =$$
$$\int_{\mathbb{R}_+} \phi(x)p(x,0)\mathrm{d}x - \int_{\mathbb{R}_+} \phi(x)p(x,t)\mathrm{d}x$$

Therefore, by using (FP 2) in the last step, we get

$$
\begin{aligned}
&\lim_{n\to\infty} \iint\limits_{U} \frac{\partial \alpha_n(s)}{\partial s}\phi(x)p(x,s)\mathrm{d}x\mathrm{d}s + \\
&\lim_{n\to\infty} \iint\limits_{U} \frac{\partial^2 \phi(x)}{\partial x^2}\alpha_n(s)a(x,s)p(x,s)\mathrm{d}x\mathrm{d}s = 0 \\
\Leftrightarrow &\int\limits_{\mathbb{R}_+} \phi(x)p(x,0)\mathrm{d}x - \int\limits_{\mathbb{R}_+} \phi(x)p(x,t)\mathrm{d}x + \\
&\int\limits_{\mathbb{R}_+}\int\limits_{(0,t)} \frac{\partial^2 \phi(x)}{\partial x^2}a(x,s)p(x,s)\mathrm{d}x\mathrm{d}s = 0 \\
\Leftrightarrow &\int\limits_{\mathbb{R}_+} \phi(x)p(x,t)\mathrm{d}x = \int\limits_{\mathbb{R}_+} \phi(x)f(x)\mathrm{d}x + \int\limits_{\mathbb{R}_+}\int\limits_{(0,t)} \frac{\partial^2 \phi(x)}{\partial x^2}a(x,s)p(x,s)\mathrm{d}s\mathrm{d}x
\end{aligned}
$$

which concludes the proof. □

Definition 3.5. For a family of probability densities $p(x,t), t \geq 0$ and a Borel function $a(x,t)$ which satisfy the conditions in Theorem 3.2, we define the function $P(x,t) : U \to \mathbb{R}_+$ by

$$P(x,t) := \int\limits_{(0,t)} a(x,s)p(x,s)\mathrm{d}s.$$

Lemma 3.6. Let $P(x,t)$ denote a function as defined in Definition 3.5. Then holds:

1. $\frac{\partial^2 P(x,t)}{\partial x^2} = p(x,t) - f(x)$.
2. $\frac{\partial P}{\partial x}(x,t) - \frac{\partial P}{\partial x}(0,t) = \int_{(x,\infty)} (f(u) - p(u,t))\mathrm{d}u = \int_{(0,x)} (p(u,t) - f(u))\mathrm{d}u$.
3. $\frac{\partial P}{\partial x}(0,t) = 0$.
4. $x \to P(x,t)$ is Lipschitz continuous with Lipschitz constant 1.
5. P is continuous on U and increasing with respect to t.
6. $\forall\, t \in \mathbb{R}_+ : P(0,t) < \infty$.
7. $\forall\, (x,t) \in U : 0 \leq P(x,t) \leq P(0,t) + \int_{\mathbb{R}_+} yf(y)\mathrm{d}y < \infty$.

PROOF :

1. See remark 3.4.

2. Integrating 1, we obtain

$$\frac{\partial P}{\partial x}(v,t) - \frac{\partial P}{\partial x}(0,t) = \int_{(0,v)} (p(u,t) - f(u))\mathrm{d}u$$

3. To show: $\frac{\partial P}{\partial x}(0,t) = 0$.
The technicalities remain to show. Here is an heuristic argument, that the third condition for a, i.e. for all $t \geq 0$ $x \to a(x,t)$ is differentiable and $a'(0,t) = a(0,t) = 0$ is sufficient to conclude this. For that we split $p(x,t)$ in an absolutely continuous density $f_a^t(x)$ and a density as stepfunction $f_T^t(x) = \sum_{i \in I_T} \chi_{[x_i,\infty)}(x) p_i$. Then

$$\frac{\partial P}{\partial x}(x,t) = \frac{\partial}{\partial x}\left(\int_{(0,t)} a(x,s)p(x,s)\mathrm{d}s\right) =$$
$$= \frac{\partial}{\partial x}\int_{(0,t)} a(x,s)(f_a^s(x) + f_T^s(x))\mathrm{d}s =$$
$$= \int_{(0,t)} a'(x,s)(f_a^s(x) + f_T^s(x))\mathrm{d}s +$$
$$\int_{(0,t)} a(x,s)\left(f_a^s(x)' + \sum_{i \in I_T} \chi_{\{x_i\}}(x) p_i\right)\mathrm{d}s$$

Therefore,

$$\frac{\partial P}{\partial x}(0,t) = \frac{\partial}{\partial x}\left(\int_{(0,t)} a(0,s)p(0,s)\mathrm{d}s\right) =$$
$$= \int_{(0,t)} a'(0,s)(f_a^s(0) + f_T^s(0))\mathrm{d}s +$$
$$\int_{(0,t)} a(0,s)\left(f_a^s(0)' + \sum_{i \in I_T} \chi_{\{x_i\}}(0) p_i\right)\mathrm{d}s =$$
$$= 0$$

4. This follows from last two points, since

$$\left|\frac{\partial P}{\partial x}(x,t)\right| =$$

$$\left|\int\limits_{(0,x)} (p(u,t) - f(u))\mathrm{d}u\right| \leq$$

$$\leq 1$$

5. The fact, that P is increasing with respect to t is easy to see, since $a \geq 0$ and $p \geq 0$.
   According to the previous step, $x \to P(x,t)$ is differentiable for all $t \geq 0$ and therefore continuous.
   We show, that $t \to P(x,t)$ is continuous for all $x \geq 0$. In fact, it has a right derivative:

   $$\frac{\partial P}{\partial t}(x,t) = p(x,t) - f(x)$$

   Therefore for $\epsilon > 0$ there exists $\delta > 0$ such that

   $$|P(x,t) - P(x,s)| \leq \frac{\epsilon}{2} \quad \forall\ s \in (t-\delta, t+\delta)$$

   and with point 4 we deduce that for $\delta_1 := \min\{\frac{\epsilon}{2}, \delta\}$ and for all $s \in (t-\delta, t+\delta), y \in (x-\delta_1, x+\delta_1)$ holds

   $$|P(x,s) - P(y,s)| \leq \min\{\frac{\epsilon}{2}, \delta\} \leq \frac{\epsilon}{2}$$

   Therefore $\forall\ (y,s) \in (x-\delta_1, x+\delta_1) \times (t-\delta, t+\delta)$.

   $$|P(y,s) - P(x,t)| \leq |P(y,s) - P(y,t)| + |P(y,t) - P(x,t)| \leq \epsilon$$

6. This follows simply from the fact, that $P(x,t)$ is continuous.

7. $0 \leq P(x,t)$ is trivial. We show the other inequality by using the pre-

vious result.

$$
\begin{aligned}
P(x,t) &= P(0,t) + \int\limits_{(0,x)} \frac{\partial P}{\partial x}(u,t)\mathrm{d}u = \\
&= P(0,t) + \int\limits_{(0,x)} \int\limits_{(u,\infty)} (f(w) - p(w,t))\mathrm{d}w\mathrm{d}u = \\
&= P(0,t) + \int\limits_{(0,\infty)} \int\limits_{(0,\infty)} \chi_{(u,\infty)}(w)\chi_{(0,x)}(u)(f(w) - p(w,t))\mathrm{d}w\ \mathrm{d}u = \\
&= P(0,t) + \int\limits_{(0,\infty)} \int\limits_{(0,\infty)} \chi_{(0,w)}(u)\chi_{(0,x)}(u)(f(w) - p(w,t))\mathrm{d}u\ \mathrm{d}w = \\
&= P(0,t) + \int\limits_{(0,\infty)} \int\limits_{(0,\infty)} \chi_{(0,w\wedge x)}(u)(f(w) - p(w,t))\mathrm{d}u\ \mathrm{d}w = \\
&= P(0,t) + \int\limits_{(0,\infty)} (w \wedge x)(f(w) - p(w,t))\mathrm{d}u\ \mathrm{d}w \leq \\
&\leq P(0,t) + \int\limits_{(0,\infty)} wf(w)\mathrm{d}w
\end{aligned}
$$

From the previous step, we know that $P(0,t) < \infty$. Since $\int_{\mathbb{R}_+} |x|\, f(x)\mathrm{d}x < \infty$ is a requirement for f, we can conclude that the last expression is $< \infty$.

□

Remark 3.7. We will use the knowledge of Lemma (3.6) point (3) only later in the proof to illustrate where exactly the third condition for $a(x,t)$ is needed, i.e. $\forall t \in \mathbb{R}_+ : x \to a(x,t)$ is differentiable at $x = 0$ and $a(0,t) = a'(0,t) = 0$.

PROOF : of Theorem 3.2
Assume $p(x,t)$ and $\hat{p}(x,t)$ are two solutions of formula 3.2. We define

$$
\begin{aligned}
&q(x,t) := p(x,t) - \hat{p}(x,t) \\
&\hat{P}(x,t) := \int\limits_{(0,t)} a(x,s)\hat{p}(x,s)\mathrm{d}s \\
&Q(x,t) := P(x,t) - \hat{P}(x,t) = \int\limits_{(0,t)} a(x,s)q(x,s)\mathrm{d}s
\end{aligned}
$$

From the linearity of the integral and Lemma 3.6 follows, that

1. $\frac{\partial^2 Q}{\partial x^2}(x,t) = q(x,t)$.

2. $\frac{\partial Q}{\partial x}(x,t) - \frac{\partial Q}{\partial x}(0,t) = -\int_{(x,\infty)} q(u,t)\mathrm{d}u = \int_{(0,x)} (q(u,t))\mathrm{d}u.$

3. $\forall\ t \in \mathbb{R}_+ : Q(0,t) < \infty.$

4. $\frac{\partial Q}{\partial t}(x,t) = a(x,t)q(x,t)$

5. $\forall\ (x,t) \in U : Q(x,t) \leq Q(0,t) + 2\int_{\mathbb{R}_+} yf(y)\mathrm{d}y < \infty.$

We want to show now, that $\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} a(x,t)q^2(x,t)\mathrm{d}x\ \mathrm{d}t = 0$ from which we then conclude, that $q(x,t) = 0$ a.s.
We examine the integral $\int_{(t_1,t_2)}\int_{(\frac{1}{R},R)} a(x,t)q^2(x,t)\mathrm{d}x\ \mathrm{d}t$ and then let $t_1 \to 0$, $t_2 \to \infty, R \to \infty$.
We will use the notation $Q_x(y,s) := \frac{\partial Q(x,t)}{\partial x}(y,s)$ and similarly $Q_t(y,s) := \frac{\partial Q(x,t)}{\partial t}(y,s)$, $Q_{xx}(y,s) := \frac{\partial^2 Q(x,t)}{\partial x^2}(y,s)$ as well as $Q_{xt}(y,s) := \frac{\partial^2 Q(x,t)}{\partial t \partial x}(y,s)$. By the just listed facts and integration by parts we see that

$$\int\limits_{(t_1,t_2)}\int\limits_{(\frac{1}{R},R)} a(x,t)q^2(x,t)\mathrm{d}x\ \mathrm{d}t =$$

$$= \int\limits_{(t_1,t_2)}\int\limits_{(\frac{1}{R},R)} Q_{xx}(x,t)Q_t(x,t)\mathrm{d}x\ \mathrm{d}t =$$

$$= \int\limits_{(t_1,t_2)} \Bigg(\left[\left(Q_x(x,t) - Q_x(0,t)\right)Q_t(x,t)\right]_{\frac{1}{R}}^{R} -$$

$$\left. \int\limits_{(\frac{1}{R},R)} \left(Q_x(x,t) - Q_x(0,t)\right)Q_{xt}(x,t)\mathrm{d}x \right)\mathrm{d}t =$$

Now we use Lemma (3.6) point (3). It shows that $Q_x(0,t) = 0$ and therefore $Q_{xt}(0,t) = 0$

$$= \int\limits_{(t_1,t_2)} \Bigg(\left[\left(Q_x(x,t) - Q_x(0,t)\right)Q_t(x,t)\right]_{\frac{1}{R}}^{R} -$$

$$\left. \int\limits_{(\frac{1}{R},R)} \left(Q_x(x,t) - Q_x(0,t)\right)\left(Q_{xt}(x,t) - Q_{xt}(0,t)\right)\mathrm{d}x \right)\mathrm{d}t =$$

$$= \int\limits_{(t_1,t_2)} \Bigg(\left[\left(Q_x(x,t) - Q_x(0,t)\right)Q_t(x,t)\right]_{\frac{1}{R}}^{R} -$$

$$\left. \frac{1}{2}\frac{\partial \int_{(\frac{1}{R},R)} \left(Q_x(x,t) - Q_x(0,t)\right)^2 \mathrm{d}x}{\partial t} \right)\mathrm{d}t =$$

$$= \int\limits_{(t_1,t_2)} \left[\left(Q_x(x,t) - Q_x(0,t)\right) Q_t(x,t)\right]_{\frac{1}{R}}^{R} \mathrm{d}t -$$

$$\frac{1}{2} \int\limits_{(\frac{1}{R},R)} \left(\left(Q_x(x,t_2) - Q_x(0,t_2)\right)^2 - \left(Q_x(x,t_1) - Q_x(0,t_1)\right)^2\right) \mathrm{d}x =$$

$$= \int\limits_{(t_1,t_2)} \left(\left(Q_x(R,t) - Q_x(0,t)\right) Q_t(R,t) - \left(Q_x\left(\frac{1}{R},t\right) - Q_x(0,t)\right) Q_t\left(\frac{1}{R},t\right)\right) \mathrm{d}t -$$

$$\frac{1}{2} \int\limits_{(\frac{1}{R},R)} \left(\left(Q_x(x,t_2) - Q_x(0,t_2)\right)^2 - \left(Q_x(x,t_1) - Q_x(0,t_1)\right)^2\right) \mathrm{d}x =$$

$$= \int\limits_{(t_1,t_2)} \left(\left(- \int\limits_{(R,\infty)} q(u,t)\mathrm{d}u\right) a(R,t)q(R,t) - \left(\int\limits_{(0,\frac{1}{R})} q(u,t)\mathrm{d}u\right) a\left(\frac{1}{R},t\right) q\left(\frac{1}{R},t\right)\right) \mathrm{d}t -$$

$$\frac{1}{2} \int\limits_{(\frac{1}{R},R)} \left(\left(\int\limits_{(0,x)} q(u,t_2)\mathrm{d}u\right)^2 - \left(\int\limits_{(0,x)} q(u,t_1)\mathrm{d}u\right)^2\right) \mathrm{d}x$$

Since $t \geq 0 \to p(x,t)$ is weakly continuous, $t \geq 0 \to q(x,t)$ is also weakly continuous. Therefore,

$$\lim_{t_1 \to 0} \left(\int\limits_{(0,x)} q(u,t_1)\mathrm{d}u\right)^2 = \left(\int\limits_{(0,x)} q(u,0)\mathrm{d}u\right)^2 =$$

$$= \left(\int\limits_{(0,x)} \left(f(u) - f(u)\right) \mathrm{d}u\right)^2 = 0$$

We now estimate the first integral:

$$\int\limits_{(t_1,t_2)} \left(\left(-\int\limits_{(R,\infty)} q(u,t)\mathrm{d}u\right) a(R,t)q(R,t)\right) \mathrm{d}t \leq$$

$$\leq \int\limits_{(t_1,t_2)} \left(\sup\left\{\int\limits_{(R,\infty)} |q(u,t)|\,\mathrm{d}u, t \in (0,t_2)\right\} a(R,t)\,|q(R,t)|\right) \mathrm{d}t \leq$$

$$\leq \sup\left\{\int\limits_{(R,\infty)} |q(u,t)|\,\mathrm{d}u, t \in (0,t_2)\right\} \int\limits_{(0,t_2)} (a(R,t)\,|q(R,t)|)\,\mathrm{d}t \leq$$

$$\leq \sup\left\{\int\limits_{(R,\infty)} (p(u,t)+\hat{p}(u,t))\,\mathrm{d}u, t \in [0,t_2]\right\}$$

$$\left(\int\limits_{(0,t_2)} (a(R,t)p(R,t))\,\mathrm{d}t + \int\limits_{(0,t_2)} (a(R,t)\hat{p}(R,t))\,\mathrm{d}t\right) \leq$$

$$\leq \sup\left\{\int\limits_{(R,\infty)} (p(u,t)+\hat{p}(u,t))\,\mathrm{d}u, t \in [0,t_2]\right\} \left(C+\hat{C}\right)$$

Since $\forall R_2 > R_1$:

$$\int\limits_{(R_2,\infty)} (p(u,t)+\hat{p}(u,t))\,\mathrm{d}u \leq$$

$$\leq \int\limits_{(R_1,\infty)} (p(u,t)+\hat{p}(u,t))\,\mathrm{d}u \text{ and } [0,t_2] \text{ is compact,}$$

We can use Dinis Lemma (A.1) and get

$$\lim_{R\to\infty} \sup\left\{\int\limits_{(R,\infty)} (p(u,t)+\hat{p}(u,t))\,\mathrm{d}u, t \in [0,t_2]\right\} = 0$$

By the same arguments, we obtain

$$\int\limits_{(t_1,t_2)} \left(\left(\int\limits_{(0,\frac{1}{R})} q(u,t)\mathrm{d}u \right) a\left(\frac{1}{R},t\right) q\left(\frac{1}{R},t\right) \right) \mathrm{d}t$$
$$\leq \sup \left\{ \int\limits_{(0,\frac{1}{R})} \left(p(u,t)+\hat{p}(u,t)\right) \mathrm{d}u, t\in[0,t_2] \right\} \left(C+\hat{C}\right)$$

Now $\forall R_2 > R_1$:

$$\int\limits_{\left(0,\frac{1}{R_2}\right)} \left(p(u,t)+\hat{p}(u,t)\right) \mathrm{d}u \leq$$
$$\leq \int\limits_{\left(0,\frac{1}{R_1}\right)} \left(p(u,t)+\hat{p}(u,t)\right) \mathrm{d}u$$

Now using Dinis Lemma one more time:

$$\lim_{R\to\infty} \sup \left\{ \int\limits_{(0,\frac{1}{R})} \left(p(u,t)+\hat{p}(u,t)\right) \mathrm{d}u, t\in[0,t_2] \right\} = 0$$

We have

$$0 \leq \lim_{R\to\infty} \lim_{t_1\to 0} \int\limits_{(t_1,t_2)} \int\limits_{(\frac{1}{R},R)} a(x,t)q^2(x,t)\mathrm{d}x \ \mathrm{d}t \leq$$
$$\leq \lim_{R\to\infty} \lim_{t_1\to 0} -\frac{1}{2} \int\limits_{(\frac{1}{R},R)} \left(\left(\int\limits_{(0,x)} q(u,t_2)\mathrm{d}u \right)^2 \right) \mathrm{d}x \leq 0$$

Therefore,

$$\int\limits_{\mathbb{R}_+} \int\limits_{\mathbb{R}_+} a(x,t)q^2(x,t)\mathrm{d}x \ \mathrm{d}t = 0.$$

□

4 The general version

Remark 4.1. We now generalize the proof for $\mu(\mathrm{d}x)$ which do not necessarily have a density. The steps of the proof are similar, as the reader will observe. As in the case with existing density function, compared with the case $U = R \times R_+$ (see section 5), we need an additional assumption for $a(x,t)$ (the same as in the previous case, see point 3 of the following theorem). Again we will only present a heuristic argument to show that this condition is sufficient to ensure $\frac{\partial P}{\partial x}(0,t) = 0$ (see Lemma (4.7) point 5).

Theorem 4.2. Let $U := \mathbb{R}_+ \times \mathbb{R}_+$ and $a : U \to \mathbb{R}_+$ be a Borel function satisfying the following hypothesis:
$\forall 0 < t < T$ and $R > 0 : \exists \epsilon(t,T,R) > 0, m(T,R) > 0$ such that:

- $\forall (x,s) \in (0,R] \times [t,T] : a(x,s) \geq \epsilon(t,T,R)$ and
- $\forall (x,s) \in (0,R] \times (0,T] : a(x,s) \leq m(T,R)$
- $\forall t \in \mathbb{R}_+ : x \to a(x,t)$ is differentiable at $x = 0$ and $a(0,t) = a'(0,t) = 0$

Let μ be a probability measure on $\mathbb{R}_+$ and $\int_{\mathbb{R}_+} |x| \ \mathrm{d}\mu(x) < \infty$. Then, there exists at most one family of probability measures $(p(\mathrm{d}x,t), t \geq 0)$ such that:

(FP 1) $t \geq 0 \to p(\mathrm{d}x,t)$ is weakly continuous, see Remark 2.4.

(FP 2) $p(0,\mathrm{d}x) = \mu(\mathrm{d}x)$ and

$$\iint\limits_U \frac{\partial \phi(x,t)}{\partial t} p(\mathrm{d}x,t)\mathrm{d}t + \iint\limits_U \frac{\partial^2 \phi(x,t)}{\partial x^2} a(x,t) p(\mathrm{d}x,t)\mathrm{d}t = 0 \quad \forall \phi \in \mathcal{D}(U) \tag{4.3}$$

PROOF : We note, that equation (4.3) is the integral representation of the following statement:

$$\frac{\partial p(\mathrm{d}x,t)}{\partial t} - \frac{\partial^2}{\partial x^2}(p(\mathrm{d}x,t)a(x,t)) = 0 \quad in\ \mathcal{D}'(U)$$

We will split the proof into several parts. □

Lemma 4.3. First, we prove some properties of the function

$$M(x) := -\int\limits_{\mathbb{R}_+} (u \wedge x)\mu(\mathrm{d}u) = -\left(\int\limits_{[0,x]} u\mu(\mathrm{d}u) + \int\limits_{(x,\infty)} x\mu(\mathrm{d}u) \right)$$

which we will need later on. It holds,

1. $M(x)$ is Lipschitz continuous.

2. $M(x)$ is a.s. differentiable and its right derivative is given by

$$M'(x) = -\int_{(x,\infty)} \mu(\mathrm{d}u) = \int_{(0,x]} (u)\mu(\mathrm{d}u) - 1.$$

3. $M'(x)$ is monotonically increasing.

4. $M(x)$ is convex.

5. $\frac{\partial^2 M(x)}{\partial x^2} = \mu(\mathrm{d}x) \quad$ in $\mathcal{D}'(\mathbb{R}_+)$

PROOF :

1. To show: $M(x)$ is Lipschitz continuous
$\forall\ y > x > 0:$

$$|M(y) - M(x)| = \left| -\left(\int_{(x,y]} u\mu(\mathrm{d}u) + \int_{(y,\infty)} y\mu(\mathrm{d}u) - \int_{(x,\infty)} x\mu(\mathrm{d}u) \right) \right| =$$

$$= \left| \int_{(x,y]} u\mu(\mathrm{d}u) - \int_{(x,y]} x\mu(\mathrm{d}u) + (y-x) \int_{(y,\infty)} \mu(\mathrm{d}u) \right| \leq$$

$$\leq \left| \int_{(x,y]} (y-x)\mu(\mathrm{d}u) + (y-x) \int_{(y,\infty)} x\mu(\mathrm{d}u) \right| \leq$$

$$\leq (y-x) \int_{(x,\infty)} \mu(\mathrm{d}u) \leq y - x$$

2. The a.s. differentiability is provided by the Lipschitz continuity, see [1, Theorem 6, page 282]. We calculate the right derivative:

$$\frac{\partial}{\partial_+ x} M(x) = -\frac{\partial}{\partial_+ x} \left(\int_{\mathbb{R}_+} (u \wedge x)\mu(\mathrm{d}u) \right) =$$

$$= -\int_{\mathbb{R}_+} \frac{\partial}{\partial_+ x} (u \wedge x)\mu(\mathrm{d}u) = -\int_{\mathbb{R}_+} \chi_{(0,u)}(x)\mu(\mathrm{d}u) =$$

$$= -\int_{\mathbb{R}_+} \chi_{(x,\infty)}(u)\mu(\mathrm{d}u) = -\int_{(x,\infty)} (u)\mu(\mathrm{d}u) = \int_{(0,x]} (u)\mu(\mathrm{d}u) - 1$$

3. Follows immediately from $\frac{\partial}{\partial_+ x}M(x) = \int_{(0,x]}(u)\mu(\mathrm{d}u) - 1$ a.s. .

4. Follows immediately from the first and last point.

5. Let $f \in \mathcal{D}(\mathbb{R}_+)$. Then

$$\int_{\mathbb{R}_+} f(x)\frac{\partial^2 M(x)}{\partial x^2}\mathrm{d}x =$$

$$= \int_{\mathbb{R}_+} f(x)\frac{\partial}{\partial x}\left(\int_{(0,x]}(u)\mu(\mathrm{d}u) - 1\right)\mathrm{d}x =$$

$$= \int_{\mathbb{R}_+} f(x)\frac{\partial}{\partial x}\mu((0,x])\mathrm{d}x =$$

$$= \int_{\mathbb{R}_+} f(x)\mu(\mathrm{d}x)$$

□

Lemma 4.4. Let a probability measure μ and $p(\mathrm{d}x,t)$ be as in Theorem 4.2. Then holds $\forall t \geq 0, \phi \in \mathcal{D}(\mathbb{R}_+)$:

$$\int_{\mathbb{R}_+} \phi(x)p(\mathrm{d}x,t) = \int_{\mathbb{R}_+} \phi(x)\mu(\mathrm{d}x) + \int_{\mathbb{R}_+}\int_{(0,t)} \frac{\partial^2\phi(x)}{\partial x^2}a(x,s)p(\mathrm{d}x,s)\mathrm{d}s \tag{4.4}$$

Remark 4.5. Note that this lemma reads as $\int_{(0,t)} \frac{\partial^2 a(x,s)p(\mathrm{d}x,s)}{\partial x^2}\mathrm{d}s = p(\mathrm{d}x,t) - \mu(\mathrm{d}x)$ in $\mathcal{D}'(\mathbb{R}_+)$. We will later introduce (analogue to the case with density) the measure $P(\mathrm{d}x,t) := \int_{(0,t)} a(x,s)p(\mathrm{d}x,s)\mathrm{d}s$. Therefore, this Lemma also reads as
$\frac{\partial^2 P(\mathrm{d}x,t)}{\partial x^2}\mathrm{d}s = p(\mathrm{d}x,t) - \mu(\mathrm{d}x)$ in $\mathcal{D}'(\mathbb{R}_+)$.
This proof works analogue to the case with density function.

PROOF : For $t \geq 0$ fixed, we define $\alpha_n(x) := f_{t,n}(x)$ for $n \in \mathbb{N} : n > 2/t$ by equation (2.1). From (FP 2) we know, that $\forall \phi \in \mathcal{D}(\mathbb{R}_+)$:

$$\iint_U \frac{\partial \alpha_n(s)}{\partial s}\phi(x)p(\mathrm{d}x,s)\mathrm{d}s + \iint_U \frac{\partial^2\phi(x)}{\partial x^2}\alpha_n(s)a(x,s)p(\mathrm{d}x,s)\mathrm{d}s = 0$$

Now we take $\lim_{n\to\infty}$ for both integrals individually. First we treat the second integral. In Lemma (2.16) we showed alle necessary requirements for dominated convergence. Therefore,

$$\lim_{n\to\infty} \iint_U \frac{\partial^2 \phi(x)}{\partial x^2} \alpha_n(s) a(x,s) p(\mathrm{d}x, s) \mathrm{d}s =$$
$$\iint_U \frac{\partial^2 \phi(x)}{\partial x^2} \lim_{n\to\infty} \alpha_n(s) a(x,s) p(\mathrm{d}x, s) \mathrm{d}s =$$
$$\iint_U \frac{\partial^2 \phi(x)}{\partial x^2} \chi_{(0,t)}(s) a(x,s) p(\mathrm{d}x, s) \mathrm{d}s =$$
$$\int_{\mathbb{R}_+} \int_{(0,t)} \frac{\partial^2 \phi(x)}{\partial x^2} a(x,s) p(\mathrm{d}x, s) \mathrm{d}s$$

Now, we examine the first integral.

$$\iint_U \frac{\partial \alpha_n(s)}{\partial s} \phi(x) p(\mathrm{d}x, s) \mathrm{d}s =$$
$$\iint_U n \left(\psi(ns) - \psi(ns - nt + 2) \right) \phi(x) p(\mathrm{d}x, s) \mathrm{d}s =$$
$$\iint_U n\psi(ns) \phi(x) p(\mathrm{d}x, s) \mathrm{d}s - \iint_U n\psi(ns - nt + 2) \phi(x) p(\mathrm{d}x, s) \mathrm{d}s$$

And again, we have to treat both integrals individually by substituting:

$$\iint_U n\psi(ns) \phi(x) p(\mathrm{d}x, s) \mathrm{d}s = |u = ns| =$$
$$\iint_U \psi(u) \phi(x) p\left(\mathrm{d}x, \frac{u}{n}\right) \mathrm{d}u$$

and

$$\iint_U n\psi(ns - nt + 2) \phi(x) p(\mathrm{d}x, s) \mathrm{d}s = |u = ns - nt + 2| =$$
$$\iint_{\tilde{U}} \psi(u) \phi(x) p\left(\mathrm{d}x, t + \frac{u-2}{n}\right) \mathrm{d}u$$

Where $\tilde{U} = (2 - nt, \infty) \times \mathbb{R}_+$. Since we required $n \geq \frac{2}{t}$ and $\operatorname{supp}(\psi) \subseteq [1, 2]$, we deduce

$$\iint\limits_{\tilde{U}} \psi(u)\phi(x)p\left(\mathrm{d}x, t + \frac{u-2}{n}\right)\mathrm{d}u =$$
$$\iint\limits_{U} \psi(u)\phi(x)p\left(\mathrm{d}x, t + \frac{u-2}{n}\right)\mathrm{d}u$$

By (FP 1), these integrals converge for $n \to \infty$:

$$\begin{aligned}
&\lim_{n\to\infty} \iint\limits_{U} \frac{\partial \alpha_n(s)}{\partial s}\phi(x)p(\mathrm{d}x, s)\mathrm{d}s = \\
&\lim_{n\to\infty} \left(\iint\limits_{U} \psi(u)\phi(x)p\left(\mathrm{d}x, \frac{u}{n}\right)\mathrm{d}u - \iint\limits_{U} \psi(u)\phi(x)p\left(\mathrm{d}x, t + \frac{u-2}{n}\right)\mathrm{d}u \right) = \\
&\iint\limits_{U} \psi(u)\phi(x)p(\mathrm{d}x, 0)\mathrm{d}u - \iint\limits_{U} \psi(u)\phi(x)p(\mathrm{d}x, t)\mathrm{d}u = \text{ now using Fubini} \\
&\iint\limits_{U} \psi(u)\mathrm{d}u\phi(x)p(\mathrm{d}x, 0) - \iint\limits_{U} \psi(u)\mathrm{d}u\phi(x)p(\mathrm{d}x, t) = \\
&\int\limits_{\mathbb{R}_+} \phi(x)p(\mathrm{d}x, 0) - \int\limits_{\mathbb{R}_+} \phi(x)p(\mathrm{d}x, t)
\end{aligned}$$

Therefore, by using (FP 2) in the last step, we get

$$\begin{aligned}
&\lim_{n\to\infty} \iint\limits_{U} \frac{\partial \alpha_n(s)}{\partial s}\phi(x)p(\mathrm{d}x, s)\mathrm{d}s + \\
&\lim_{n\to\infty} \iint\limits_{U} \frac{\partial^2 \phi(x)}{\partial x^2}\alpha_n(s)a(x, s)p(\mathrm{d}x, s)\mathrm{d}s = 0 \\
\Leftrightarrow &\int\limits_{\mathbb{R}_+} \phi(x)p(\mathrm{d}x, 0) - \int\limits_{\mathbb{R}_+} \phi(x)p(\mathrm{d}x, t) + \\
&\int\limits_{\mathbb{R}_+} \int\limits_{(0,t)} \frac{\partial^2 \phi(x)}{\partial x^2}a(x, s)p(\mathrm{d}x, s)\mathrm{d}s = 0 \\
\Leftrightarrow &\int\limits_{\mathbb{R}_+} \phi(x)p(\mathrm{d}x, t) = \int\limits_{\mathbb{R}_+} \phi(x)\mu(\mathrm{d}x) + \int\limits_{\mathbb{R}_+} \int\limits_{(0,t)} \frac{\partial^2 \phi(x)}{\partial x^2}a(x, s)p(\mathrm{d}x, s)\mathrm{d}s
\end{aligned}$$

which concludes the proof. □

Definition 4.6. For a family of probability measures $p(\mathrm{d}x, t), t \geq 0$ and a Borel function $a(x, t)$ which satisfy the conditions in Theorem 4.2, we define the positive measure $P(\mathrm{d}x, t)$ by

$$P(\mathrm{d}x, t) := \int\limits_{(0,t)} a(x, s)p(\mathrm{d}x, s)\mathrm{d}s.$$

Lemma 4.7. Let $P(\mathrm{d}x, t)$ denote a measure as defined in Definition 4.6. Then holds:

1. $(P(\mathrm{d}x, t), t \geq 0)$ is an increasing family of positive measures.
2. $t \to P(\mathrm{d}x, t)$ is vaguely continuous and $P(\mathrm{d}x, 0) = 0$.
3. $\frac{\partial^2 P(\mathrm{d}x,t)}{\partial x^2} = p(\mathrm{d}x, t) - \mu(\mathrm{d}x) \quad$ in $\mathcal{D}'(U)$.
4. $\forall t \geq 0, P(\mathrm{d}x, t)$ admits a density with respect to the Lebesgue measure, which we will denote by $P(x, t)$.
5. $\frac{\partial P}{\partial x}(0, t) = 0$.
6. The function $x \to P(x, t)$ admits a right derivative denoted by $\frac{\partial P}{\partial x}(x, t)$:
$$\frac{\partial P}{\partial x}(x, t) = \int\limits_{[x,\infty)} (\mu(\mathrm{d}u) - p(\mathrm{d}u, t)) = \int\limits_{(0,x)} (p(\mathrm{d}u, t) - \mu(\mathrm{d}u)). \tag{4.5}$$
7. $\forall\ t \in \mathbb{R}_+ : x \to P(x, t)$ is Lipschitz continuous with Lipschitz constant 1.
8. $\forall\ x \in \mathbb{R}_+ : t \to P(x, t)$ is continuous.
9. $P(x, t)$ is continuous on U.
10. $\forall\ t \in \mathbb{R}_+ : P(0, t) < \infty$.
11. $P(x, t) = -\int_{(0,\infty)}(u \wedge x)p(\mathrm{d}u, t) + \int_{(0,\infty)}(u \wedge x)\mu(\mathrm{d}u) + P(t, 0)$.
12. $\forall\ (x, t) \in U : 0 \leq P(x, t) \leq P(0, t) + \int_{\mathbb{R}_+} y\mu(\mathrm{d}y) < \infty$.
13. $\frac{\partial P}{\partial t}(x, t)\mathrm{d}x = a(x, t)p(\mathrm{d}x, t)$

PROOF :

1. This follows easily since $a(x, t) \geq 0$.

2. $P(\mathrm{d}x, 0) = \int_{(0,0)} a(x,s)p(\mathrm{d}x, s)\mathrm{d}s = 0.$

To show the vague continuity, we fix $f \in C_K^+(\mathbb{R}_+)$. Then there exists $R > 0$ with $\mathrm{supp}(f) \subseteq [0, R]$. We will show, that $\lim_{t \to T} \int_{\mathbb{R}_+} f(x)P(\mathrm{d}x, t) = \int_{\mathbb{R}_+} f(x)P(\mathrm{d}x, T)$. Also, from weak convergence of $p(\mathrm{d}x, t)$, we know, that for all $\epsilon > 0, T \geq 0 : \exists \delta > 0 :$

$$\forall\, t \in [T-\delta, T+\delta] : \left| \int_{\mathbb{R}_+} f(x)p(\mathrm{d}x, t) - \int_{\mathbb{R}_+} f(x)p(\mathrm{d}x, T) \right| \leq \frac{\epsilon}{m(T,R)}$$

$$\Leftrightarrow \sup_{t \in [T-\delta, T+\delta]} \left\{ \left| \int_{\mathbb{R}_+} f(x)p(\mathrm{d}x, t) - \int_{\mathbb{R}_+} f(x)p(\mathrm{d}x, T) \right| \right\} \leq \frac{\epsilon}{m(T,R)} \quad (4.6)$$

Let $\epsilon > 0$ and $t \in [T-\delta, T+\delta]$, then

$$\int_{\mathbb{R}_+} f(x)P(\mathrm{d}x, t) = \int_{\mathbb{R}_+} f(x) \int_{(0,t)} a(x,s)p(\mathrm{d}x, s)\mathrm{d}s =$$
$$= \int_{(0,T)} \int_{\mathbb{R}_+} f(x)a(x,s)p(\mathrm{d}x, s)\mathrm{d}s - \int_{[t,T)} \int_{\mathbb{R}_+} f(x)a(x,s)p(\mathrm{d}x, s)\mathrm{d}s$$

Now, we estimate the second integral:

$$\int_{[t,T)} \int_{\mathbb{R}_+} f(x)a(x,s)p(\mathrm{d}x, s)\mathrm{d}s \leq \text{ using Fubini for positive terms}$$
$$\leq \int_{[t,T)} \sup_{t \in [T-\delta, T+\delta]} \left\{ \int_{\mathbb{R}_+} f(x)m(T,R)p(\mathrm{d}x, s) \right\} \mathrm{d}s \leq \text{using eq. (4.6)}$$
$$\leq \int_{[t,T)} \epsilon \mathrm{d}s = (T-t)\epsilon$$

Putting all together we get:

$$
\begin{aligned}
&\lim_{t\to T}\int\limits_{\mathbb{R}_+} f(x)P(\mathrm{d}x,t) = \\
&\lim_{t\to T}\int\limits_{(0,T)}\int\limits_{\mathbb{R}_+} f(x)a(x,s)p(\mathrm{d}x,s)\mathrm{d}s - \\
&\lim_{t\to T}\int\limits_{[t,T)}\int\limits_{\mathbb{R}_+} f(x)a(x,s)p(\mathrm{d}x,s)\mathrm{d}s = \\
&\quad\int\limits_{(0,T)}\int\limits_{\mathbb{R}_+} f(x)a(x,s)p(\mathrm{d}x,s)\mathrm{d}s - \lim_{t\to T}(T-t)\epsilon = \\
&\quad\int\limits_{\mathbb{R}_+} f(x)P(\mathrm{d}x,T)
\end{aligned}
$$

3. See remark (4.5).

4. To show: $\forall t \geq 0, P(\mathrm{d}x,t)$ admits a continuous density with respect to the Lebesgue measure.
We will show this (for each t) by representing $P(\mathrm{d}x,t)$ as the difference of two convex measures $N(\mathrm{d}x) - M(x)\mathrm{d}x$. We know from Lemma (4.3), that $\frac{\partial^2 M(x)}{\partial x^2} = \mu(\mathrm{d}x) \quad \in \mathcal{D}'(\mathbb{R}_+)$. We use the last point to conclude that

$$
\begin{aligned}
&\frac{\partial^2 P(\mathrm{d}x,t)}{\partial x^2} = p(\mathrm{d}x,t) - \mu(\mathrm{d}x) \quad \text{in } \mathcal{D}'(\mathbb{R}_+) \\
\Rightarrow \quad &\frac{\partial^2 P(\mathrm{d}x,t) + M(x)}{\partial x^2} = p(\mathrm{d}x,t) \quad \text{in } \mathcal{D}'(\mathbb{R}_+)
\end{aligned}
$$

Therefore, the Measure $N(\mathrm{d}x) := P(\mathrm{d}x,t) + M(x)$ is a.s. twice differentiable with $\frac{\partial^2 N(\mathrm{d}x)}{\partial x^2} = p(\mathrm{d}x,t)$ in $\mathcal{D}'(U)$. Furthermore, this shows that $N(x)$ is convex (see [7, p. 54]). Since $N(\mathrm{d}x)$ and $M(x)$ have densities with respect to the Lebesgue Measure, $P(\mathrm{d}x,t) = N(\mathrm{d}x) - M(x)$ also has a density function.

5. To show: $\frac{\partial P}{\partial x}(0,t) = 0$.
As in the case with density function, the technicalities remain to show. Here is a similar heuristic argument, that the third condition for a, i.e. for all $t \geq 0 : \; x \to a(x,t)$ is differentiable and $a'(0,t) = a(0,t) = 0$ is sufficient to conclude this. For that we split $p(\mathrm{d}x,t)$ in an absolutely continuous density $f_a^t(x)$, a density as stepfunction $f_T^t(x) =$

$\sum_{i\in I_T}\chi_{[x_i,\infty)}(x)p_i$ and a point measure $\mu_t((0,x]) = \sum_{i\in I_m} p_{t,i}\chi_{(-\infty,x_i]}(x)$. Then

$$\frac{\partial P}{\partial x}(x,t) = \frac{\partial}{\partial x}\left(\int\limits_{(0,t)} a(x,s)p(\mathrm{d}x,s)\mathrm{d}s\right) =$$
$$=\frac{\partial}{\partial x}\int\limits_{(0,t)} a(x,s)(f_a^s(x) + f_T^s(x) + \mu_s(\mathrm{d}x))\mathrm{d}s =$$
$$=\frac{\partial}{\partial x}\int\limits_{(0,t)} a(x,s)\left(f_a^s(x) + f_T^s(x) + \sum_{i\in I_m}\chi_{\{x_i\}}(x)p_{t,i}\right)\mathrm{d}s =$$
$$= \int\limits_{(0,t)} a'(x,s)\left(f_a^s(x) + f_T^s(x) + \sum_{i\in I_m}\chi_{\{x_i\}}(x)p_{t,i}\right)\mathrm{d}s+$$
$$\int\limits_{(0,t)} a(x,s)\left(f_a^s(x)' + \sum_{i\in I_T}\chi_{\{x_i\}}(x)p_i\right) - \int\limits_{(0,t)}\sum_{i\in I_m}\chi_{\{x_i\}}(x)a'(x,s)p_{t,i}\mathrm{d}s$$

Therefore,

$$\frac{\partial P}{\partial x}(0,t) = \frac{\partial}{\partial x}\left(\int\limits_{(0,t)} a(x,s)p(\mathrm{d}x,s)\mathrm{d}s\right) =$$
$$= \int\limits_{(0,t)} a'(0,s)\left(f_a^s(0) + f_T^s(0) + \sum_{i\in I_m}\chi_{\{x_i\}}(0)a(0,s)p_{t,i}\right)\mathrm{d}s+$$
$$\int\limits_{(0,t)} a(0,s)\left(f_a^s(0)' + \sum_{i\in I_T}\chi_{\{x_i\}}(0)p_i\right) - \int\limits_{(0,t)}\sum_{i\in I_m}\chi_{\{x_i\}}(0)a'(0,s)p_{t,i}\mathrm{d}s =$$
$$=0$$

6. Point 3 also holds for $P(x,t)$ from point 4. By integrating we obtain the right derivative:

$$\frac{\partial P}{\partial x}(x,t) - \frac{\partial P}{\partial x}(0,t) = \int\limits_{(0,x)}(p(\mathrm{d}u,t) - \mu(\mathrm{d}u)) =$$
$$=p((0,x),t) - \mu((0,x))$$

With point 5 we deduce

$$\frac{\partial P}{\partial x}(x,t) = p((0,x),t) - \mu((0,x))$$

7. To show: $\forall\, t \in \mathbb{R}_+ : x \to P(x,t)$ is Lipschitz continuous with Lipschitz constant 1.
   This follows from last point, since the right derivative satisfies
$$\left|\frac{\partial P}{\partial x}(x,t)\right| = |p((0,x),t) - \mu((0,x))| \leq 1$$

8. To show: $\forall\, x \in \mathbb{R}_+ : t \to P(x,t)$ is continuous.
   Let $x \in \mathbb{R}_+$ and suppose there is a discontuinity at t, i.e.
$$\exists (t_n)_{n\in\mathbb{N}} \to t : |P(x,t_n) - P(x,t)| \geq 6\epsilon.$$
   Without loss of generality, let $(t_n)_{n\in\mathbb{N}}$ be monotonically decreasing (since $t \to P(x,t)$ is monotonically increasing $(t_n)_{n\in\mathbb{N}}$ could be chosen either monotonically increasing or decreasing). Then we have
$$P(x,t_n) - P(x,t) \geq 6\epsilon.$$
   We deduce, that for $\epsilon > 0$, there is some $N(\epsilon)$ such that
$$\forall n \geq N(\epsilon) : P(x,t_n) - P(x,t) \geq 5\epsilon.$$
   From point 7 we know that $\forall\, s \in (x-2\epsilon, x+2\epsilon)$:
$$|P(x,t_n) - P(s,t_n)| \leq 2\epsilon \text{ and } |P(x,t) - P(s,t)| \leq 2\epsilon$$
   Therefore, $\forall\, s \in (x-2\epsilon, x+2\epsilon)$:
$$\begin{aligned}
&|P(s,t) - P(s,t_n)| = \\
=& |P(s,t) - P(x,t) + P(x,t) - P(x,t_n) + P(x,t_n) - P(s,t_n)| \geq \\
\geq& |P(x,t) - P(x,t_n)| - |P(s,t) - P(x,t) + P(x,t_n) - P(s,t_n)| \geq \\
\geq& |P(x,t) - P(x,t_n)| - |P(s,t) - P(x,t)| - |P(x,t_n) - P(s,t_n)| \geq \\
\geq& 5\epsilon - 2\epsilon - 2\epsilon = \epsilon
\end{aligned}$$
   We can now define a function $f_x(s) \in C_K^+(\mathbb{R}_+)$ which is 1 in $(x-\epsilon, x+\epsilon)$ and 0 in $\mathbb{R}_+ \backslash (x-2\epsilon, x+2\epsilon)$ by
$$f_x(s) := \begin{cases} 0, & \text{if } |s-x| > 2\epsilon \\ \frac{1}{\epsilon}s - \frac{1}{\epsilon}(x-2\epsilon), & \text{if } s \in [x-2\epsilon, x-\epsilon) \\ -\frac{1}{\epsilon}s + \frac{1}{\epsilon}(x+2\epsilon), & \text{if } s \in [x+\epsilon, x+2\epsilon] \\ 2 & \text{if } s \in (x-\epsilon, x+\epsilon) \end{cases}$$

Then for all $n \geq N(\epsilon)$:

$$
\begin{aligned}
&\int\limits_{\mathbb{R}_+} f_x(s)\left(P(s,t_n) - P(s,t)\right)\mathrm{d}s = \\
&= \int\limits_{(x-2\epsilon,x+2\epsilon)} f_x(s)\left(P(s,t_n) - P(s,t)\right)\mathrm{d}s \geq \\
&\geq \int\limits_{(x-\epsilon,x+\epsilon)} \left(P(s,t_n) - P(s,t)\right)\mathrm{d}s \geq \int\limits_{(x-\epsilon,x+\epsilon)} \epsilon \mathrm{d}s = \\
&= 2\epsilon\epsilon
\end{aligned}
$$

Which contradicts point 2.

9. According to points 7 and 8 it holds, that for $\epsilon > 0$ there exists $\delta > 0$ such that

$$
|P(x,t) - P(x,s)| \leq \frac{\epsilon}{2} \quad \forall\ s \in (t-\delta, t+\delta)
$$

and for $\delta_1 := \min\{\frac{\epsilon}{2}, \delta\}$ holds for all $s \in (t-\delta, t+\delta), y \in (x-\delta_1, x+\delta_1)$

$$
|P(x,s) - P(y,s)| \leq \min\{\frac{\epsilon}{2}, \delta\} \leq \frac{\epsilon}{2}
$$

Therefore $\forall\ (y,s) \in (x-\delta_1, x+\delta_1) \times (t-\delta, t+\delta)$.

$$
|P(y,s) - P(x,t)| \leq |P(y,s) - P(y,t)| + |P(y,t) - P(x,t)| \leq \epsilon
$$

10. This follows simply from the fact, that $P(x,t)$ is continuous.

11. To show: $P(x,t) = -\int_{(0,\infty)}(u \wedge x)p(\mathrm{d}u,t) + \int_{(0,\infty)}(u \wedge x)\mu(\mathrm{d}u) + P(t,0)$

By integrating equation (4.5) we get:

$$\begin{aligned}
P(x,t)-P(0,t) &= \int\limits_{(0,x)} \frac{\partial P}{\partial x}(u,t)\mathrm{d}u = \\
&= \int\limits_{(0,x)} \int\limits_{[u,\infty)} (\mu(\mathrm{d}v)-p(\mathrm{d}v,t))\mathrm{d}u = \\
&= \int\limits_{(0,\infty)} \chi_{(0,x)}(u) \int\limits_{(0,\infty)} \chi_{[u,\infty)}(v)(\mu(\mathrm{d}v)-p(\mathrm{d}v,t))\mathrm{d}u = \\
&= \int\limits_{(0,\infty)} \int\limits_{(0,\infty)} \chi_{(0,x)}(u)\chi_{[u,\infty)}(v)\mathrm{d}u(\mu(\mathrm{d}v)-p(\mathrm{d}v,t)) = \\
&= \int\limits_{(0,\infty)} \int\limits_{(0,\infty)} \chi_{(0,x)}(u)\chi_{(0,v)}(u)\mathrm{d}u(\mu(\mathrm{d}v)-p(\mathrm{d}v,t)) = \\
&= \int\limits_{(0,\infty)} \int\limits_{(0,\infty)} \chi_{(0,x\wedge v)}(u)\mathrm{d}u(\mu(\mathrm{d}v)-p(\mathrm{d}v,t)) = \\
&= \int\limits_{(0,\infty)} (x\wedge v)(\mu(\mathrm{d}v)-p(\mathrm{d}v,t))
\end{aligned}$$

12. $0 \leq P(x,t)$ is trivial. We show the other inequality:

$$\begin{aligned}
P(x,t) =& P(0,t) + \int\limits_{(0,\infty)} (x\wedge v)(\mu(\mathrm{d}v)-p(\mathrm{d}v,t)) \leq \\
\leq& P(0,t) + \int\limits_{(0,\infty)} (x\wedge v)\mu(\mathrm{d}v) \leq \\
\leq& P(0,t) + \int\limits_{(0,\infty)} v\mu(\mathrm{d}v) < \infty
\end{aligned}$$

Were we used the assumption for μ from theorem (4.2) and point 9.

13. To show: $\frac{\partial P}{\partial t}(x,t)\mathrm{d}x = a(x,t)p(\mathrm{d}x,t)$
This follows immediately from the definition of $P(x,t)$.

□

Lemma 4.8. There exists $p \in L^2_{loc}(U)$ such that for almost every $t \geq 0$:

$$p(\mathrm{d}x,t) = p(x,t)\mathrm{d}x \tag{4.7}$$

PROOF : We fix $\alpha, \zeta \in \mathcal{D}(\mathbb{R}_+)$ and assume $\alpha \geq 0$ and $\zeta \geq 0$. There exist $0 < t_1 < t_2$ and $R > 0$ such that $\mathrm{supp}(\alpha) \subseteq [t_1, t_2]$ and $\mathrm{supp}(\zeta) \subseteq [\frac{1}{R}, R]$. We set:

$$\epsilon := \epsilon(t_1, t_2, R) \qquad \text{and} \quad m := m(t_2, R)$$

from the first two assumptions for $a(x,t)$. We define the function

$$\tilde{P}(x,t) := \zeta(x) \left(\alpha(t)P(x,t) - \int\limits_{(0,t)} \alpha'(s)P(x,s)\mathrm{d}s \right) \tag{4.8}$$

Then, by integrating by parts, we obtain

$$\begin{aligned}
&\tilde{P}(x,t)\mathrm{d}x = \\
=&\zeta(x) \left(\alpha(t)P(x,t) - \left(\alpha(t)P(x,t) - \alpha(0)P(x,0) - \right.\right. \\
&\left.\left.\int\limits_{(0,t)} \alpha(s)\frac{\partial P}{\partial s}(x,s)\mathrm{d}s \right)\right) \mathrm{d}x = \\
=&\zeta(x) \left(\alpha(t)P(x,t) - \left(\alpha(t)P(x,t) - \int\limits_{(0,t)} \alpha(s)\frac{\partial P}{\partial s}(x,s)\mathrm{d}s \right)\right) \mathrm{d}x = \\
=&\zeta(x) \left(\int\limits_{(0,t)} \alpha(s)\frac{\partial P}{\partial s}(x,s)\mathrm{d}x\mathrm{d}s \right) =
\end{aligned}$$

Now using Lemma (4.7) point 13 we obtain

$$\begin{aligned}
&=\zeta(x) \left(\int\limits_{(0,t)} \alpha(s)\frac{\partial P}{\partial s}(x,s)\mathrm{d}x\mathrm{d}s \right) = \\
= \tilde{P}(x,t)\mathrm{d}x &=\zeta(x) \int\limits_{(0,t)} \alpha(s)a(x,s)p(\mathrm{d}x,s)
\end{aligned} \tag{4.9}$$

Differentiating equation (4.9) with respect to t, we obtain

$$\frac{\partial \tilde{P}}{\partial t}(x,t) = \zeta(x)\alpha(t)a(x,t)p(\mathrm{d}x,t) \qquad \text{in } \mathcal{D}'(U) \tag{4.10}$$

which implies

$$\frac{\partial \tilde{P}}{\partial t}(x,t) \leq \zeta(x)\alpha(t)mp(\mathrm{d}x,t) \qquad \text{in } \mathcal{D}'(U) \tag{4.11}$$

Differentiating equation (4.8) twice with respect to x, we get in $\mathcal{D}'(U)$

$$\frac{\partial \tilde{P}}{\partial x}(x,t) =$$
$$=\zeta'(x)\left(\alpha(t)P(x,t) - \int\limits_{(0,t)} \alpha'(s)P(x,s)\mathrm{d}s\right) +$$
$$\zeta(x)\left(\alpha(t)\frac{\partial P}{\partial x}(x,t) - \int\limits_{(0,t)} \alpha'(s)\frac{\partial P}{\partial x}(x,s)\mathrm{d}s\right)$$

and therefore in $\mathcal{D}'(U)$

$$\frac{\partial^2 \tilde{P}}{\partial x^2}(x,t) =$$
$$=\zeta''(x)\left(\alpha(t)P(x,t) - \int\limits_{(0,t)} \alpha'(s)P(x,s)\mathrm{d}s\right) +$$
$$2\zeta'(x)\left(\alpha(t)\frac{\partial P}{\partial x}(x,t) - \int\limits_{(0,t)} \alpha'(s)\frac{\partial P}{\partial x}(x,s)\mathrm{d}s\right) +$$
$$\zeta(x)\left(\alpha(t)\frac{\partial^2 P}{\partial x^2}(x,t) - \int\limits_{(0,t)} \alpha'(s)\frac{\partial^2 P}{\partial x^2}(x,s)\mathrm{d}s\right) =$$

using Lemma (4.7) point 3

$$=\zeta''(x)\left(\alpha(t)P(x,t) - \int\limits_{(0,t)} \alpha'(s)P(x,s)\mathrm{d}s\right) +$$
$$2\zeta'(x)\left(\alpha(t)\frac{\partial P}{\partial x}(x,t) - \int\limits_{(0,t)} \alpha'(s)\frac{\partial P}{\partial x}(x,s)\mathrm{d}s\right) +$$
$$\zeta(x)\left(\alpha(t)(p(\mathrm{d}x,t) - \mu(\mathrm{d}x)) - \int\limits_{(0,t)} \alpha'(s)(p(\mathrm{d}x,s) - \mu(\mathrm{d}x))\mathrm{d}s\right) =$$

With expanding the last term we get

$$
\begin{aligned}
=&\zeta''(x)\left(\alpha(t)P(x,t)-\int\limits_{(0,t)}\alpha'(s)P(x,s)\mathrm{d}s\right)+\\
&2\zeta'(x)\left(\alpha(t)\frac{\partial P}{\partial x}(x,t)-\int\limits_{(0,t)}\alpha'(s)\frac{\partial P}{\partial x}(x,s)\mathrm{d}s\right)+\\
&\zeta(x)\left(\alpha(t)(p(\mathrm{d}x,t))-\int\limits_{(0,t)}\alpha'(s)(p(\mathrm{d}x,s))\mathrm{d}s\right)+\\
&\zeta(x)\left(\alpha(t)(-\mu(\mathrm{d}x))+\int\limits_{(0,t)}\alpha'(s)(\mu(\mathrm{d}x))\mathrm{d}s\right)=\\
=&\zeta''(x)\left(\alpha(t)P(x,t)-\int\limits_{(0,t)}\alpha'(s)P(x,s)\mathrm{d}s\right)+\\
&2\zeta'(x)\left(\alpha(t)\frac{\partial P}{\partial x}(x,t)-\int\limits_{(0,t)}\alpha'(s)\frac{\partial P}{\partial x}(x,s)\mathrm{d}s\right)+\\
&\zeta(x)\left(\alpha(t)(p(\mathrm{d}x,t))-\int\limits_{(0,t)}\alpha'(s)(p(\mathrm{d}x,s))\mathrm{d}s\right)+\\
&\zeta(x)\left(\alpha(t)(-\mu(\mathrm{d}x))+\alpha(t)(\mu(\mathrm{d}x))\mathrm{d}s\right)=\\
=&\zeta''(x)\left(\alpha(t)P(x,t)-\int\limits_{(0,t)}\alpha'(s)P(x,s)\mathrm{d}s\right)+\\
&2\zeta'(x)\left(\alpha(t)\frac{\partial P}{\partial x}(x,t)-\int\limits_{(0,t)}\alpha'(s)\frac{\partial P}{\partial x}(x,s)\mathrm{d}s\right)+\\
&\zeta(x)\left(\alpha(t)(p(\mathrm{d}x,t))-\int\limits_{(0,t)}\alpha'(s)(p(\mathrm{d}x,s))\mathrm{d}s\right)=\\
\frac{\partial^2\tilde{P}}{\partial x^2}(x,t)=&\zeta(x)\alpha(t)p(\mathrm{d}x,t)-\zeta(x)\int\limits_{(0,t)}\alpha'(s)p(\mathrm{d}x,s)\mathrm{d}s+\phi(x,t) \qquad (4.12)
\end{aligned}
$$

With

$$\phi(x,t) = \zeta''(x)\left(\alpha(t)P(x,t) - \int\limits_{(0,t)} \alpha'(s)P(x,s)\mathrm{d}s\right) +$$
$$2\zeta'(x)\left(\alpha(t)\frac{\partial P}{\partial x}(x,t) - \int\limits_{(0,t)} \alpha'(s)\frac{\partial P}{\partial x}(x,s)\mathrm{d}s\right)$$

We show now that ϕ is in $\mathcal{L}^\infty(U)$.
Since α and ζ are in $\mathcal{D}(\mathbb{R}_+)$ there exists $C_\alpha, C'_\alpha, C_\zeta, C'_\zeta, C''_\zeta > 0$ such that $\alpha(t) \leq C_\alpha,\ \alpha'(t) \leq C'_\alpha,\ \zeta(x) \leq C_\zeta,\ \zeta'(x) \leq C'_\zeta,\ \zeta''(x) \leq C''_\zeta$. Since $P(x,t)$ is continuous (see lemma (4.7) point 9), There exists some C_P with $P(x,t) \leq C_P$ for all $(x,t) \in [0,R] \times [t_1,t_2]$. From lemma (4.7) point 7 we know, that $\frac{\partial P}{\partial x}(x,t) \leq 1$. Therefore,

$$|\phi(x,t)| \leq C''_\zeta \left(C_\alpha C_P + t_2 C'_\alpha C_P\right) + 2C'_\zeta \left(C_\alpha + t_2 C'_\alpha\right) =: C_\phi < \infty$$

Furthermore we have

$$\zeta(x)\int\limits_{(0,t)} |\alpha'(s)|\, p(\mathrm{d}x,s)\mathrm{d}s \leq \frac{C'_\alpha}{\epsilon}\zeta(x)\int\limits_{(0,t)} a(x,s)p(\mathrm{d}x,s)\mathrm{d}s \leq$$
$$\leq \frac{C'_\alpha}{\epsilon}\zeta(x)P(x,t) \leq \frac{C'_\alpha}{\epsilon}C_\zeta C_P < \infty \tag{4.13}$$

Which shows that equation (4.12) can be written as

$$\frac{\partial^2 \tilde{P}}{\partial x^2}(x,t) = \zeta(x)\alpha(t)p(\mathrm{d}x,t) - \frac{1}{m}\Phi(x,t)$$

with $\Phi \in \mathcal{L}^\infty(U)$. And Therefore,

$$\zeta(x)\alpha(t)p(\mathrm{d}x,t) = \frac{\partial^2 \tilde{P}}{\partial x^2}(x,t) + \frac{1}{m}\Phi(x,t) \tag{4.14}$$

From equations (4.11) and (4.14) we deduce

$$\frac{\partial \tilde{P}}{\partial t}(x,t) \leq \zeta(x)\alpha(t)mp(\mathrm{d}x,t) = m\frac{\partial^2 \tilde{P}}{\partial x^2}(x,t) + \Phi(x,t) \tag{4.15}$$

In the following we want to define the convolution for $\tilde{P}$ with a functions of a regularizing sequence $(\phi_n)_{n\in\mathbb{N}}$ (See Lemma (2.18)). Thatfor we extend the function $\tilde{P}$ from U to $\mathbb{R} \times \mathbb{R}_+$ with $\tilde{P}(x,t) = 0 \forall x \leq 0$. Without loss of

generality let $\mathrm{supp}(\phi) \subseteq [-\frac{1}{n}, \frac{1}{n}]$.
We define

$$\tilde{P}_n(x,t) := \tilde{P}(x,t) * \phi_n(x) = \int_{\mathbb{R}} \tilde{P}(y,t)\phi_n(x-y)\mathrm{d}y$$

As mentioned in Definition (2.17) we know, that

$$\frac{\partial^2 \tilde{P}_n}{\partial x^2}(x,t) = \frac{\partial^2 \tilde{P}}{\partial x^2} * \phi_n(x,t)$$

and similarly with equation (4.10)

$$\frac{\partial \tilde{P}_n}{\partial t}(x,t) = \frac{\partial \tilde{P}}{\partial t} * \phi_n(x,t) = \alpha(t)a(x,t)\zeta(x)p(\mathrm{d}x,t) * \phi_n(x,t)$$

We note, that $\frac{\partial \tilde{P}_n}{\partial t}(x,t)$ is differentiable with respect to x. Now equation (4.15) also holds for the on $\mathbb{R} \times \mathbb{R}_+$ continued and in x convoluted $\tilde{P}_n$ (with $\Phi_n := \Phi * \phi_n$):

$$\frac{\partial \tilde{P}_n}{\partial t}(x,t) \leq m\frac{\partial^2 \tilde{P}_n}{\partial x^2}(x,t) + \Phi_n(x,t) \tag{4.16}$$

From [5, p. 26-27] we know, that

$$\mathrm{supp}\left(\frac{\partial \tilde{P}_n}{\partial t}\right) \subseteq \mathrm{supp}(\phi_n) + \mathrm{supp}\left(\frac{\partial \tilde{P}_n}{\partial t}\right) = \left[\frac{1}{R} - \frac{1}{n}, R + \frac{1}{n}\right] \times [t_1, t_2]. \tag{4.17}$$

and similarly

$$\mathrm{supp}\left(\frac{\partial \tilde{P}_n}{\partial x}\right) \subseteq \mathrm{supp}(\phi_n) + \mathrm{supp}\left(\frac{\partial \tilde{P}_n}{\partial x}\right) = \left[\frac{1}{R} - \frac{1}{n}, R + \frac{1}{n}\right] \times [t_1, \infty). \tag{4.18}$$

We note from [5, p. 23, Fakta 13.3.11-3] that

$$\|\Phi_n\|_\infty \leq \|\Phi\|_\infty \|\phi_n\|_1 = \|\Phi\|_\infty \tag{4.19}$$

and therefore $\Phi_n \in \mathcal{L}^\infty(\mathbb{R} \times \mathbb{R}_+)$. As a little reminder we note the general inequality

$$ab \leq \frac{1}{2}\left(a^2 + b^2\right) \tag{4.20}$$

which is easily proved with the Ansatz $(a-b)^2 \geq 0$.
We want to show, that $\frac{\partial \tilde{P}_n}{\partial t}$ is bounded in $\mathcal{L}^2(\mathbb{R} \times \mathbb{R}_+)$ by a Constant C independent of n. We use equation (4.17) in the first step:

$$\begin{aligned}
&\iint\limits_{\mathbb{R}\times\mathbb{R}_+} \left(\frac{\partial \tilde{P}_n}{\partial t}(x,t)\right)^2 \mathrm{d}t\mathrm{d}x = \\
&= \iint\limits_{[-1,R+1]\times[t_1,t_2]} \frac{\partial \tilde{P}_n}{\partial t}(x,t)\mathrm{d}t\mathrm{d}x \leq \text{ using eq (4.16)} \\
&\leq m \iint\limits_{[-1,R+1]\times[t_1,t_2]} \frac{\partial \tilde{P}_n}{\partial t}(x,t)\frac{\partial^2 \tilde{P}_n}{\partial x^2}(x,t)\mathrm{d}t\mathrm{d}x + \\
&\quad \iint\limits_{[-1,R+1]\times[t_1,t_2]} \frac{\partial \tilde{P}_n}{\partial t}(x,t)\Phi_n(x,t)\mathrm{d}t\mathrm{d}x
\end{aligned} \tag{4.21}$$

We examine the first term. First we use Fubini's Theorem [6, p. 163] to switch integrating order, then partial integration (which is why we convoluted ; to get the necessary smoothness):

$$\begin{aligned}
&m \iint\limits_{[-1,R+1]\times[t_1,t_2]} \frac{\partial \tilde{P}_n}{\partial t}(x,t)\frac{\partial^2 \tilde{P}_n}{\partial x^2}(x,t)\mathrm{d}t\mathrm{d}x = \\
&= m \iint\limits_{[t_1,t_2]\times[-1,R+1]} \frac{\partial \tilde{P}_n}{\partial t}(x,t)\frac{\partial^2 \tilde{P}_n}{\partial x^2}(x,t)\mathrm{d}x\mathrm{d}t = \\
&= m \int\limits_{[t_1,t_2]} \left(\frac{\partial \tilde{P}_n}{\partial t}(x,t)\frac{\partial \tilde{P}_n}{\partial x}(x,t)\bigg|_{x=-1}^{R+1} - \int\limits_{[-1,R+1]} \frac{\partial^2 \tilde{P}_n}{\partial t \partial x}(x,t)\frac{\partial \tilde{P}_n}{\partial x}(x,t)\mathrm{d}x \right) \mathrm{d}t =
\end{aligned}$$

using equation (4.17)

$$= - m \int\limits_{[t_1,t_2]} \int\limits_{[-1,R+1]} \frac{\partial^2 \tilde{P}_n}{\partial t \partial x}(x,t) \frac{\partial \tilde{P}_n}{\partial x}(x,t) \mathrm{d}x \mathrm{d}t =$$

using Fubini's Theorem again

$$= - m \int\limits_{[-1,R+1]} \int\limits_{[t_1,t_2]} \frac{\partial^2 \tilde{P}_n}{\partial t \partial x}(x,t) \frac{\partial \tilde{P}_n}{\partial x}(x,t) \mathrm{d}t \mathrm{d}x =$$

$$= - m \int\limits_{[-1,R+1]} \int\limits_{[t_1,t_2]} \frac{\partial}{\partial t} \left(\frac{\partial \tilde{P}_n}{\partial x}(x,t) \right)^2 \mathrm{d}t \mathrm{d}x =$$

$$= - m \int\limits_{[-1,R+1]} \left(\left(\frac{\partial \tilde{P}_n}{\partial x}(x,t_2) \right)^2 - \left(\frac{\partial \tilde{P}_n}{\partial x}(x,t_1) \right)^2 \right) \mathrm{d}x =$$

using equation (4.18)

$$= - m \int\limits_{[-1,R+1]} \left(\frac{\partial \tilde{P}_n}{\partial x}(x,t_2) \right)^2 \mathrm{d}x \leq 0$$

Therefor

$$4.21 \leq \iint\limits_{[-1,R+1]\times[t_1,t_2]} \frac{\partial \tilde{P}_n}{\partial t}(x,t) \Phi_n(x,t) \mathrm{d}t \mathrm{d}x$$

using equation (4.20)

$$\iint\limits_{\mathbb{R}\times\mathbb{R}_+} \left(\frac{\partial \tilde{P}_n}{\partial t}(x,t) \right)^2 \mathrm{d}t \mathrm{d}x \leq \frac{1}{2} \iint\limits_{[-1,R+1]\times[t_1,t_2]} \left(\left(\frac{\partial \tilde{P}_n}{\partial t}(x,t) \right)^2 + \Phi_n(x,t)^2 \right) \mathrm{d}t \mathrm{d}x$$

Which shows that

$$\iint\limits_{\mathbb{R}\times\mathbb{R}_+} \left(\frac{\partial \tilde{P}_n}{\partial t}(x,t) \right)^2 \mathrm{d}t \mathrm{d}x \leq 2 \iint\limits_{[-1,R+1]\times[t_1,t_2]} \Phi_n(x,t)^2 \mathrm{d}t \mathrm{d}x \leq$$

$$\leq 2 \iint\limits_{[-1,R+1]\times[t_1,t_2]} \left\| \Phi_n(x,t) \right\|_\infty^2 \mathrm{d}t \mathrm{d}x =$$

using equation (4.19)

$$= 2 \iint\limits_{[-1,R+1]\times[t_1,t_2]} \left\| \Phi(x,t) \right\|_\infty^2 \mathrm{d}t \mathrm{d}x =$$

$$= 2(t_2 - t_1)(R-2) \left\| \Phi(x,t) \right\|_\infty^2 =: C < \infty$$

From [1, p. 639, Theorem 3] we know, that since $\frac{\partial \tilde{P}_n}{\partial t}$ is bounded in $\mathcal{L}^2(\mathbb{R}\times\mathbb{R}_+)$ there exists $u \in \mathcal{L}^2(\mathbb{R}\times\mathbb{R}_+)$ and a subsequence $I \subseteq \mathbb{N}$ with

$$\lim_{i\to\infty} \int\limits_{(\mathbb{R}\times\mathbb{R}_+)} f(x,t)\left(\frac{\partial \tilde{P}_i}{\partial t}(x,t) - u(x,t)\right)\mathrm{d}t\mathrm{d}x = 0 \text{ for all } f \in \mathcal{L}^2(\mathbb{R}\times\mathbb{R}_+)$$

On the other hand we know from Lemma (2.19), that $\frac{\partial \tilde{P}_n}{\partial t} \to \frac{\partial \tilde{P}}{\partial t}$ in $\mathcal{D}'(\mathbb{R}\times\mathbb{R}_+)$. Since the limit is unique in distributional sense, $u = \frac{\partial \tilde{P}}{\partial t} \in \mathcal{L}^2(\mathbb{R}\times\mathbb{R}_+)$. □

PROOF : of Theorem 4.2
This proof works analogously to the proof in the case with density function. Suppose $\hat{p}(\mathrm{d}x,t)$ is another solution satisfying (FP 1) and (FP 2). We set

$$\hat{P}(x,t)\mathrm{d}x = \int\limits_{(0,t)} a(x,s)p(\mathrm{d}x,s)\mathrm{d}s$$

and $q = p - \hat{p}$ and $Q = P - \hat{P} = \int_{(0,t)} a(x,s)q(\mathrm{d}x,s)\mathrm{d}s$. The linearity of the differential operator and Lemma 4.7 show that

$$\frac{\partial Q}{\partial t}(x,t)\mathrm{d}x = a(x,t)q(\mathrm{d}x,t) \quad \text{in } \mathcal{D}'(U) \tag{4.22}$$

$$\frac{\partial Q}{\partial x}(x,t)\mathrm{d}x = \int\limits_{(0,x)} q(\mathrm{d}x,t) = q((0,x),t) \quad \text{in } \mathcal{D}'(U) \tag{4.23}$$

and

$$\frac{\partial^2 Q}{\partial x^2}(x,t)\mathrm{d}x = q(\mathrm{d}x,t) \quad \text{in } \mathcal{D}'(U) \tag{4.24}$$

Therefor

$$\frac{\partial Q}{\partial t}(x,t) - a(x,t)\frac{\partial^2 Q}{\partial x^2}(x,t) = 0 \quad \text{in } \mathcal{D}'(U)$$

By Lemma (4.8) $q \in \mathcal{L}^2_{loc}$. Therefore, for $0 < t_1 < t_2$ and $R > 0$ and $V = [t_1,t_2]\times[\frac{1}{R},R]$ holds

$$\iint\limits_V a(x,t)q(x,t)^2\mathrm{d}x\mathrm{d}t = \iint\limits_V \frac{\partial Q}{\partial t}(x,t)\frac{\partial^2 Q}{\partial x^2}(x,t)\mathrm{d}x\mathrm{d}t$$

Let ϕ_n again be a regularizing sequence as in proof of Lemma (4.8). We also extend $Q(x,t)$ on $\mathbb{R} \times \mathbb{R}_+$ with $Q(x,t) = 0$ for $x \leq 0$. Similarly with a and q. We set $Q_n = Q * \phi_n$. By equation (4.24) and Definition (2.17) we have

$$\left(\iint_V aqq \mathrm{d}x\mathrm{d}t \right) * \phi_n * \phi_n = \left(\iint_V \frac{\partial Q}{\partial t}(x,t) \frac{\partial^2 Q}{\partial x^2}(x,t) \mathrm{d}x\mathrm{d}t \right) * \phi_n * \phi_n$$

$$\iint_V (aq * \phi_n(x,t))(q * \phi_n(x,t)) \mathrm{d}x\mathrm{d}t = \iint_V \frac{\partial Q_n}{\partial t}(x,t) \frac{\partial^2 Q_n}{\partial x^2}(x,t) \mathrm{d}x\mathrm{d}t$$

By integration by parts we get

$$\iint_V \frac{\partial Q_n}{\partial t}(x,t) \frac{\partial^2 Q_n}{\partial x^2}(x,t) \mathrm{d}x\mathrm{d}t =$$

$$= \int_{[t_1,t_2]} \left(\frac{\partial Q_n}{\partial t}(x,t) \frac{\partial Q_n}{\partial x}(x,t) \Big|_{x=\frac{1}{R}}^{R} - \int_{\left[\frac{1}{R},R\right]} \frac{\partial^2 Q_n}{\partial t \partial x}(x,t) \frac{\partial Q_n}{\partial x}(x,t) \mathrm{d}x \right) \mathrm{d}t =$$

$$= \int_{[t_1,t_2]} \left(\frac{\partial Q_n}{\partial t}(x,t) \frac{\partial Q_n}{\partial x}(x,t) \Big|_{x=\frac{1}{R}}^{R} - \frac{1}{2} \frac{\partial}{\partial t} \int_{\left[\frac{1}{R},R\right]} \left(\frac{\partial Q_n}{\partial x}(x,t) \right)^2 \mathrm{d}x \right) \mathrm{d}t \leq$$

with equation (4.22)

$$\leq \int_{[t_1,t_2]} (aq) * \phi_n(x,t) \frac{\partial Q_n}{\partial x}(x,t) \Big|_{x=\frac{1}{R}}^{R} \mathrm{d}t + \frac{1}{2} \int_{\left[\frac{1}{R},R\right]} \left(\frac{\partial Q_n}{\partial x}(x,t_1) \right)^2 \mathrm{d}x$$

We estimate $\left| \frac{\partial Q_n}{\partial x}(R,t) \right|$:

$$\left| \frac{\partial Q_n}{\partial x}(R,t) \right| = \left| \int_{\left[R-\frac{1}{n},R+\frac{1}{n}\right]} \frac{\partial Q}{\partial x}(y,t) \phi_n(R-y) \mathrm{d}y \right| \leq$$

$$\leq \sup_{y \in \left[R-\frac{1}{n},R+\frac{1}{n}\right]} \left\{ \left| \frac{\partial Q}{\partial x}(y,t) \right| \right\} \int_{\left[R-\frac{1}{n},R+\frac{1}{n}\right]} \phi_n(R-y) \mathrm{d}y =$$

$$\sup_{y \in \left[R-\frac{1}{n},R+\frac{1}{n}\right]} \left\{ \left| \frac{\partial Q}{\partial x}(y,t) \right| \right\}$$

and analogously we get

$$\left| \frac{\partial Q_n}{\partial x} \left(\frac{1}{R}, t \right) \right| \leq \sup_{y \in \left[\frac{1}{R}-\frac{1}{n},\frac{1}{R}+\frac{1}{n}\right]} \left\{ \left| \frac{\partial Q}{\partial x}(y,t) \right| \right\}$$

and

$$\begin{aligned}
&\int\limits_{[t_1,t_2]} |(aq*\phi_n)(R)|\,\mathrm{d}t = \left(\int\limits_{[t_1,t_2]} |(a(p-\hat{p})|\,\mathrm{d}t\right)*\phi_n)(R) \leq \\
&\leq \left(\int\limits_{[t_1,t_2]} (ap)(R)\mathrm{d}t\right)*\phi_n(R) + \left(\int\limits_{[t_1,t_2]} (a\hat{p})\right)*\phi_n(R)\mathrm{d}t = \\
&=(P(\cdot,t_2)-P(\cdot,t_1))*\phi_n(R) + (\hat{P}(\cdot,t_2)-\hat{P}(\cdot,t_1))*\phi_n(R) \leq \\
&\leq \sup_{x\in\mathbb{R}}\left\{P(x,t_2)+\hat{P}(x,t_2)\right\} \leq C_{t_2}
\end{aligned}$$

In the last step we used Lemma (4.7) point 12. Similarly we get

$$\int\limits_{[t_1,t_2]} \left|(aq*\phi_n)\left(\frac{1}{R}\right)\right| \mathrm{d}t \leq C_{t_2}.$$

Therefore, we have for all $\mathbb{N} \ni n > \frac{1}{R}$:

$$\begin{aligned}
&\iint\limits_V (aq*\phi_n(x,t))(q*\phi_n(x,t))\mathrm{d}x\mathrm{d}t \leq \\
&\leq C_{t_2}\left(\sup_{t\in[t_1,t_2]}\sup_{y\in\left[R-\frac{1}{n},R+\frac{1}{n}\right]}\left\{\left|\frac{\partial Q}{\partial x}(y,t)\right|\right\} + \sup_{t\in[t_1,t_2]}\sup_{y\in\left[\frac{1}{R}-\frac{1}{n},\frac{1}{R}+\frac{1}{n}\right]}\left\{\left|\frac{\partial Q}{\partial x}(y,t)\right|\right\}\right) + \\
&\frac{1}{2}\int\limits_{\left[\frac{1}{R},R\right]}\left(\frac{\partial Q_n}{\partial x}(x,t_1)\right)^2\mathrm{d}x \leq \\
&\leq C_{t_2}\left(\sup_{t\in(0,t_2]}\sup_{y\in[R-1,R+1]}\left\{\left|\frac{\partial Q}{\partial x}(y,t)\right|\right\} + \sup_{t\in(0,t_2]}\sup_{y\in\left[0,\frac{1}{R}+\frac{1}{n}\right]}\left\{\left|\frac{\partial Q}{\partial x}(y,t)\right|\right\}\right) + \\
&\frac{1}{2}\int\limits_{\left[\frac{1}{R},R\right]}\left(\frac{\partial Q_n}{\partial x}(x,t_1)\right)^2\mathrm{d}x
\end{aligned}$$

Now we let $n\to\infty$. Since $q\in\mathcal{L}^2_{loc}(U)$ (see Lemma (4.8)) and with Lemma (2.19) the left hand side converges to $\iint_V (a(x,t)q^2(x,t))\,\mathrm{d}x\mathrm{d}t$. Since there is a $n > 0$ such that $\frac{1}{R} + \frac{1}{n} \leq \frac{1}{R-1}$ we have

$$\iint\limits_V \Big(a(x,t)q^2(x,t)\Big)\,\mathrm{d}x\mathrm{d}t \leq$$

$$\leq C_{t_2}\left(\sup_{t\in(0,t_2]}\sup_{y\in[R-1,R+1]}\left\{\left|\frac{\partial Q}{\partial x}(y,t)\right|\right\} + \sup_{t\in(0,t_2]}\sup_{y\in\left(0,\frac{1}{R-1}\right]}\left\{\left|\frac{\partial Q}{\partial x}(y,t)\right|\right\}\right) +$$

$$\frac{1}{2}\int\limits_{\left[\frac{1}{R},R\right]}\left(\frac{\partial Q}{\partial x}(x,t_1)\right)^2\mathrm{d}x \tag{4.25}$$

By equation (4.23) we have $\left|\frac{\partial Q}{\partial x}(x,t_1)\right| \leq 1$ and $\lim_{t_1\to\infty}\left|\frac{\partial Q}{\partial x}(x,t_1)\right| = 0$. Hence

$$\lim_{t_1\to\infty}\frac{1}{2}\int\limits_{\left[\frac{1}{R},R\right]}\left(\frac{\partial Q}{\partial x}(x,t_1)\right)^2\mathrm{d}x = 0$$

Again by equation (4.23) we have

$$\frac{\partial Q}{\partial x}(x,t)\mathrm{d}x = \int\limits_{(0,x)} q(\mathrm{d}x,t) = \int\limits_{(0,x)} p(\mathrm{d}x,t) - \int\limits_{(0,x)} \hat{p}(\mathrm{d}x,t) =$$

$$= \int\limits_{[x,\infty)} \hat{p}(\mathrm{d}x,t) - \int\limits_{[x,\infty)} p(\mathrm{d}x,t)$$

Now

$$\sup_{t\in(0,t_2]}\sup_{x\in[R-1,R+1]}\left\{\int\limits_{[x,\infty)} p(\mathrm{d}x,t)\right\} = \sup_{t\in(0,t_2]}\left\{\int\limits_{[R-1,\infty)} p(\mathrm{d}x,t)\right\} \leq$$

$$\leq \sup_{t\in[0,t_2]}\left\{\int\limits_{\mathbb{R}_+}\theta(x+2-R)p(\mathrm{d}x,t)\right\}$$

(Note that $p(\mathrm{d}x,t) = \mu(\mathrm{d}x)$) where θ is a continuous function with

$$\theta(x) := \begin{cases} 1, & \text{if } x \geq 1, \\ 0, & \text{if } x \leq 0, \\ x, & \text{else.} \end{cases}$$

For each $t \in [0,t_2]$ holds $\lim_{R\to\infty}\int_{\mathbb{R}_+}\theta(x+2-R)p(\mathrm{d}x,t) = 0$. We also note, that $t \to \int_{\mathbb{R}_+}\theta(x+2-R)p(\mathrm{d}x,t)$ is continuous since $t \to p(\mathrm{d}x,t)$ is weakly continuous and θ is continuous and bounded. It is easy to see, that $[0, t_2]$ is compact and $R \to \int_{\mathbb{R}_+} \theta(x+2-R)p(\mathrm{d}x, t)$ is a decreasing family of continuous functions. Therefore, by Dinis Lemma (A.1)

$$\lim_{R\to\infty} \sup_{t\in[0,t_2]} \left\{ \int_{\mathbb{R}_+} \theta(x+2-R)p(\mathrm{d}x,t) \right\} = 0$$

The same result holds obviously for $\hat{p}$ and therefor

$$\lim_{R\to\infty} C_{t_2} \sup_{t\in(0,t_2]} \sup_{y\in[R-1,R+1]} \left\{ \left| \frac{\partial Q}{\partial x}(y,t) \right| \right\} = 0 \tag{4.26}$$

We treat the second term in equation (4.25) similar:

$$\sup_{t\in(0,t_2]} \sup_{x\in\left(0,\frac{1}{R-1}\right]} \left\{ \int_{(0,x)} p(\mathrm{d}x,t) \right\} = \sup_{t\in(0,t_2]} \left\{ \int_{\left(0,\frac{1}{R-1}\right]} p(\mathrm{d}x,t) \right\} \leq$$

$$\leq \sup_{t\in[0,t_2]} \left\{ \int_{\mathbb{R}_+} \hat{\theta}\left(x(R-1)\right) p(\mathrm{d}x,t) \right\}$$

where $\hat{\theta} : \mathbb{R}_+ \to [0,1]$ is continuous and bounded function.

$$\hat{\theta}(x) := \begin{cases} 0, & \text{if } x \geq 2, \\ 1, & \text{if } 0 < x \leq 1, \\ 2-x, & \text{else.} \end{cases}$$

Therefore $R \to \int_{\mathbb{R}_+} \hat{\theta}\left(x(R-1)\right) p(\mathrm{d}x,t)$ is a decreasing family of continuous functions with $\lim_{R\to\infty} \int_{\mathbb{R}_+} \hat{\theta}\left(x(R-1)\right) p(\mathrm{d}x,t) = 0$. Therefore, again by Dinis Lemma:

$$\lim_{R\to\infty} \sup_{t\in[0,t_2]} \left\{ \int_{\mathbb{R}_+} \hat{\theta}\left(x(R-1)\right) p(\mathrm{d}x,t) \right\} = 0$$

We have the same result for $\hat{p}$ and with equation (4.25) we conclude that

$$\lim_{R\to\infty} \lim_{t_1\to 0} \iint_{[t_1,t_2]\times\left[\frac{1}{R},R\right]} \Big(a(x,t)q^2(x,t)\Big)\,\mathrm{d}x\mathrm{d}t =$$

$$= \iint_{(0,t_2]\times(0,R]} \Big(a(x,t)q^2(x,t)\Big)\,\mathrm{d}x\mathrm{d}t \leq 0$$

for each $t_2 > 0$. This shows, that $q(x,t) = 0$ a.s. □

5 The general version with $U = \mathbb{R} \times \mathbb{R}_+$

Remark 5.1. In this section we reproduce the proof of M. Pierre with details. As we will see, the third condition for $a(x,t)$, i.e. $\forall t \in \mathbb{R}_+ : x \to a(x,t)$ is differentiable at $x = 0$ and $a(0,t) = a'(0,t) = 0$ is not needed here. Remember, it was only needed to proof $\frac{\partial P}{\partial x}(0,t) = 0$, which can here be proven without mentioned condition for $a(x,t)$. The only difference is, that $\mu(\mathrm{d}x)$ and in the following $p(\mathrm{d}x,t)$ are measures on $\mathbb{R}$ instead of $\mathbb{R}_+$. The proof is very similar to the general case.

Theorem 5.2. Let $U := \mathbb{R} \times \mathbb{R}_+$ and $a : U \to \mathbb{R}_+$ be a Borel function satisfying the following hypothesis:
$\forall 0 < t < T$ and $R > 0 : \exists \epsilon(t,T,R) > 0, m(T,R) > 0$ such that:

- $\forall (x,s) \in [-R,R] \times [t,T] : a(x,s) \geq \epsilon(t,T,R)$ and
- $\forall (x,s) \in [-R,R] \times (0,T] : a(x,s) \leq m(T,R)$

Let μ be a probability measure on $\mathbb{R}$ and $\int_{\mathbb{R}} |x| \; \mathrm{d}\mu(x) < \infty$. Then, there exists at most one family of probability measures $(p(\mathrm{d}x,t), t \geq 0)$ such that:

(FP 1) $t \geq 0 \to p(\mathrm{d}x,t)$ is weakly continuous, see Remark 2.4.

(FP 2) $p(0,\mathrm{d}x) = \mu(\mathrm{d}x)$ and

$$\iint\limits_U \frac{\partial \phi(x,t)}{\partial t} p(\mathrm{d}x,t)\mathrm{d}t + \iint\limits_U \frac{\partial^2 \phi(x,t)}{\partial x^2} a(x,t) p(\mathrm{d}x,t)\mathrm{d}t = 0 \quad \forall \phi \in \mathcal{D}(U) \tag{5.27}$$

PROOF : We note, that equation (4.3) is the integral representation of the following statement:

$$\frac{\partial p(\mathrm{d}x,t)}{\partial t} - \frac{\partial^2}{\partial x^2}(p(\mathrm{d}x,t)a(x,t)) = 0 \quad \textit{in } \mathcal{D}'(U)$$

We will split the proof into several parts. □

Lemma 5.3. First, we prove some properties of the function

$$M(x) := \begin{cases} -\int_{\mathbb{R}_+} (u \wedge x)\mu(\mathrm{d}u), & \text{if } x \leq 0, \\ \int_{(-\infty,0)} (u \vee x)\mu(\mathrm{d}u) - x, & \text{else.} \end{cases}$$

Note that $M(x)$ can also be written as

$$M(x) = \begin{cases} -\left(\int_{[0,x]} u\mu(\mathrm{d}u) + \int_{(x,\infty)} x\mu(\mathrm{d}u)\right), & \text{if } x \leq 0, \\ \left(\int_{[x,0]} u\mu(\mathrm{d}u) + \int_{(-\infty,x)} x\mu(\mathrm{d}u)\right) - x, & \text{else.} \end{cases}$$

We will need the function later on. It holds,

1. $M(x)$ is Lipschitz continuous.

2. $M(x)$ is a.s. differentiable and its right derivative is given by

$$M'(x) = -\int\limits_{(x,\infty)} \mu(\mathrm{d}u) = \int\limits_{(-\infty,x]} \mu(\mathrm{d}u) - 1.$$

3. $M'(x)$ is monotonically increasing.

4. $M(x)$ is convex.

5. $\frac{\partial^2 M(x)}{\partial x^2} = \mu(\mathrm{d}x) \quad \text{in } \mathcal{D}'(\mathbb{R})$

PROOF : We note, that we have proven this properties for $x > 0$ in Lemma (4.3).

1. To show: $M(x)$ is Lipschitz continuous
$\forall\ y < x < 0$:

$$\begin{aligned}
&|M(y) - M(x)| = \\
=&\left|\left(\int\limits_{[y,x)} u\mu(\mathrm{d}u) + \int\limits_{(-\infty,y)} y\mu(\mathrm{d}u) - \int\limits_{(-\infty,x)} x\mu(\mathrm{d}u)\right) - y + x\right| = \\
=&\left|\left(\int\limits_{[y,x)} u\mu(\mathrm{d}u) + (y-x)\int\limits_{(-\infty,y)} \mu(\mathrm{d}u) - \int\limits_{[y,x)} x\mu(\mathrm{d}u)\right) - y + x\right| \leq \\
\leq& -\left(\int\limits_{[y,x)} (u-x)\mu(\mathrm{d}u) + (y-x)\int\limits_{(-\infty,y)} \mu(\mathrm{d}u)\right) + x - y \leq \\
\leq& -\left(\int\limits_{[y,x)} (y-x)\mu(\mathrm{d}u) + (y-x)\int\limits_{(-\infty,y)} \mu(\mathrm{d}u)\right) + x - y = \\
=&(x-y)\int\limits_{(-\infty,x)} \mu(\mathrm{d}u) + x - y \leq 2(x-y)
\end{aligned}$$

M is continuous at $x = 0$ with $M(0) = 0$ which is easy to see. Therefore, for $x < 0 < y : M(y) - M(x) \leq y - 0 + 2(0 - x) \leq 2(y - x)$.

2. We only need to calculate the right derivative for $x \leq 0$:

$$\frac{\partial}{\partial_+ x} M(x) = \frac{\partial}{\partial_+ x}\left(\int\limits_{(-\infty,0)} (u \vee x)\mu(\mathrm{d}u) - x \right) =$$

$$= \int\limits_{(-\infty,0)} \frac{\partial}{\partial_+ x}(u \vee x)\mu(\mathrm{d}u) - 1 = \int\limits_{(-\infty,0)} \chi_{[u,0)}(x)\mu(\mathrm{d}u) =$$

$$= \int\limits_{(-\infty,0)} \chi_{(-\infty,x]}(u)\mu(\mathrm{d}u) - 1 = -\int\limits_{(x,\infty)} \mu(\mathrm{d}u)$$

3. Follows immediately from $\frac{\partial}{\partial_+ x} M(x) = \int_{(0,x]}(u)\mu(\mathrm{d}u) - 1$ a.s. .

4. Follows immediately from the first and last point.

5. Let $f \in \mathcal{D}(\mathbb{R})$. Then

$$\int\limits_{\mathbb{R}} f(x)\frac{\partial^2 M(x)}{\partial x^2}\mathrm{d}x =$$

$$= \int\limits_{\mathbb{R}} f(x)\frac{\partial}{\partial x}\left(\int\limits_{(-\infty,x]} \mu(\mathrm{d}u) - 1 \right)\mathrm{d}x =$$

$$= \int\limits_{\mathbb{R}} f(x)\frac{\partial}{\partial x}\mu((-\infty,x])\mathrm{d}x =$$

$$= \int\limits_{\mathbb{R}} f(x)\mu(\mathrm{d}x)$$

□

Lemma 5.4. Let a probability measure μ and $p(\mathrm{d}x,t)$ be as in Theorem (5.2). Then holds $\forall t \geq 0, \phi \in \mathcal{D}(\mathbb{R})$:

$$\int\limits_{\mathbb{R}} \phi(x)p(\mathrm{d}x,t) = \int\limits_{\mathbb{R}} \phi(x)\mu(\mathrm{d}x) + \int\limits_{\mathbb{R}}\int\limits_{(0,t)} \frac{\partial^2\phi(x)}{\partial x^2}a(x,s)p(\mathrm{d}x,s)\mathrm{d}s \qquad (5.28)$$

Remark 5.5. Note that (analogue to the general case) this lemma reads as $\int_{(0,t)} \frac{\partial^2 a(x,s)p(\mathrm{d}x,s)}{\partial x^2}\mathrm{d}s = p(\mathrm{d}x,t) - \mu(\mathrm{d}x)$ in $\mathcal{D}'(\mathbb{R})$ or $\frac{\partial^2 P(\mathrm{d}x,t)}{\partial x^2}\mathrm{d}s = p(\mathrm{d}x,t) - \mu(\mathrm{d}x)$ in $\mathcal{D}'(\mathbb{R})$.

PROOF : This proof works analogously to the general case. See Lemma (5.4).
□

Definition 5.6. Analogously to the general case, we define ror a family of probability measures $p(\mathrm{d}x,t), t \geq 0$ and a Borel function $a(x,t)$ which satisfy the conditions in Theorem (5.2) the positive measure $P(\mathrm{d}x,t)$ by

$$P(\mathrm{d}x,t) := \int\limits_{(0,t)} a(x,s)p(\mathrm{d}x,s)\mathrm{d}s.$$

Lemma 5.7. Let $P(\mathrm{d}x,t)$ denote a measure as defined in Definition 5.6. Then holds:

1. $(P(\mathrm{d}x,t), t \geq 0)$ is an increasing family of positive measures.
2. $t \to P(\mathrm{d}x,t)$ is vaguely continuous and $P(\mathrm{d}x,0) = 0$.
3. $\frac{\partial^2 P(\mathrm{d}x,t)}{\partial x^2} = p(\mathrm{d}x,t) - \mu(\mathrm{d}x) \quad \text{in } \mathcal{D}'(U)$.
4. $\forall t \geq 0, P(\mathrm{d}x,t)$ admits a density with respect to the Lebesgue measure, which we will denote by $P(x,t)$.
5. The function $x \to P(x,t)$ admits a right derivative denoted by $\frac{\partial P}{\partial x}(x,t)$:

$$\frac{\partial P}{\partial x}(x,t) = \int\limits_{[x,\infty)} (\mu(\mathrm{d}u) - p(\mathrm{d}u,t)) = \int\limits_{(-\infty,x)} (p(\mathrm{d}u,t) - \mu(\mathrm{d}u)). \quad (5.29)$$

6. $\forall\ t \in \mathbb{R}_+ : x \to P(x,t)$ is Lipschitz continuous with Lipschitz constant 1.
7. $\forall\ x \in \mathbb{R} : t \to P(x,t)$ is continuous.
8. $P(x,t)$ is continuous on U.
9. $\forall\ t \in \mathbb{R}_+ : P(0,t) < \infty$.
10. $P(x,t) =$

$$\begin{cases} -\int_{(0,\infty)}(u \wedge x)p(\mathrm{d}u,t) + \int_{(0,\infty)}(u \wedge x)\mu(\mathrm{d}u) + P(t,0), & \text{if } x \geq 0, \\ \int_{(-\infty,0)}(u \vee x)p(\mathrm{d}u,t) - \int_{(-\infty,0)}(u \vee x)\mu(\mathrm{d}u) + P(t,0), & \text{else.} \end{cases}$$

11. $\forall\ (x,t) \in U : 0 \leq P(x,t) \leq P(0,t) + \int_{\mathbb{R}} |y|\, \mu(\mathrm{d}y) < \infty$.
12. $\frac{\partial P}{\partial t}(x,t)\mathrm{d}x = a(x,t)p(\mathrm{d}x,t)$

PROOF :

1. This follows easily since $a(x,t) \geq 0$.

2. $P(\mathrm{d}x, 0) = \int_{(0,0)} a(x,s)p(\mathrm{d}x,s)\mathrm{d}s = 0$.
   To show the vague continuity, we fix $f \in C_K^+(\mathbb{R})$. Then there exists $R > 0$ with $\operatorname{supp}(f) \subseteq [-R, R]$. We will show, that $\lim_{t\to T} \int_{\mathbb{R}} f(x)P(\mathrm{d}x,t) = \int_{\mathbb{R}} f(x)P(\mathrm{d}x,T)$. Also, from weak convergence of $p(\mathrm{d}x,t)$, we know, that for all $\epsilon > 0, T \geq 0 : \exists \delta > 0$:

$$\forall\ t \in [T-\delta, T+\delta] : \left| \int_{\mathbb{R}} f(x)p(\mathrm{d}x,t) - \int_{\mathbb{R}} f(x)p(\mathrm{d}x,T) \right| \leq \frac{\epsilon}{m(T,R)}$$

$$\Leftrightarrow \sup_{t\in[T-\delta,T+\delta]} \left\{ \left| \int_{\mathbb{R}} f(x)p(\mathrm{d}x,t) - \int_{\mathbb{R}} f(x)p(\mathrm{d}x,T) \right| \right\} \leq \frac{\epsilon}{m(T,R)} \quad (5.30)$$

   Let $\epsilon > 0$ and $t \in [T-\delta, T+\delta]$, then

$$\int_{\mathbb{R}} f(x)P(\mathrm{d}x,t) = \int_{\mathbb{R}} f(x) \int_{(0,t)} a(x,s)p(\mathrm{d}x,s)\mathrm{d}s =$$
$$= \int_{(0,T)} \int_{\mathbb{R}} f(x)a(x,s)p(\mathrm{d}x,s)\mathrm{d}s - \int_{[t,T)} \int_{\mathbb{R}} f(x)a(x,s)p(\mathrm{d}x,s)\mathrm{d}s$$

   Now, we estimate the second integral:

$$\int_{[t,T)} \int_{\mathbb{R}} f(x)a(x,s)p(\mathrm{d}x,s)\mathrm{d}s \leq \text{ using Fubini for positive terms}$$
$$\leq \int_{[t,T)} \sup_{t\in[T-\delta,T+\delta]} \left\{ \int_{\mathbb{R}} f(x)m(T,R)p(\mathrm{d}x,s) \right\} \mathrm{d}s \leq \text{using eq. (5.30)}$$
$$\leq \int_{[t,T)} \epsilon \mathrm{d}s = (T-t)\epsilon$$

Putting all together we get:

$$
\begin{aligned}
&\lim_{t\to T}\int_{\mathbb{R}} f(x)P(\mathrm{d}x,t) = \\
&\lim_{t\to T}\int_{(0,T)}\int_{\mathbb{R}} f(x)a(x,s)p(\mathrm{d}x,s)\mathrm{d}s - \\
&\lim_{t\to T}\int_{[t,T)}\int_{\mathbb{R}} f(x)a(x,s)p(\mathrm{d}x,s)\mathrm{d}s = \\
&\quad\int_{(0,T)}\int_{\mathbb{R}} f(x)a(x,s)p(\mathrm{d}x,s)\mathrm{d}s - \lim_{t\to T}(T-t)\epsilon = \\
&\quad\int_{\mathbb{R}} f(x)P(\mathrm{d}x,T)
\end{aligned}
$$

3. See remark (5.5).

4. Analogously to the general case.

5. Point 3 also holds for $P(x,t)$ from point 4. By integrating we obtain the right derivative:

$$
\begin{aligned}
\frac{\partial P}{\partial x}(x,t) &= \int_{(-\infty,x)} (p(\mathrm{d}u,t) - \mu(\mathrm{d}u)) + C(t) = \\
&= p((-\infty,x),t) - \mu((-\infty,x)) + C(t)
\end{aligned}
$$

   Suppose $C(t) > 0$. Since $\lim_{|x|\to\infty} \frac{\partial P}{\partial x}(x,t) = C(t)$ there exists for each ϵ some $R > 0$ with $\frac{\partial P}{\partial x}(x,t) > \epsilon$ for all $x \le -R$. As shown in point (4), $x \to P(x,t)$ is Lipschitz continuous and therefore $P(-R,t) < \infty$. Therefore for $y < -R - \frac{P(-R,t)}{\epsilon}$ holds $P(y,t) < 0$ which contradicts the positivity of P. We get a similar contradiction for $C(t) < 0$, we conclude therefore $C(t) = 0$.

6. To show: $\forall\, t \in \mathbb{R}_+ : x \to P(x,t)$ is Lipschitz continuous with Lipschitz constant 1.
   This follows from last point, since the right derivative satisfies

$$
\left|\frac{\partial P}{\partial x}(x,t)\right| = |p((0,x),t) - \mu((0,x))| \le 1
$$

7. Analogously to the general case.

8. Analogously to the general case.

9. This follows simply from the fact, that $P(x,t)$ is continuous.

10. To show: $P(x,t) = \int_{(-\infty,0)} (u \vee x) p(\mathrm{d}u, t) - \int_{(-\infty,0)} (u \vee x) \mu(\mathrm{d}u) + P(t,0)$ for $x \leq 0$
By integrating equation (5.29) we get:

$$\begin{aligned}
&P(0,t) - P(x,t) = \\
&= \int\limits_{(x,0)} \frac{\partial P}{\partial x}(u,t)\mathrm{d}u = \int\limits_{(x,0)} \int\limits_{(-\infty,u)} (p(\mathrm{d}v,t) - \mu(\mathrm{d}v))\mathrm{d}u = \\
&= \int\limits_{(-\infty,0)} \int\limits_{(-\infty,0)} \chi_{(x,0)}(u)\chi_{(-\infty,u)}(v)\mathrm{d}u(p(\mathrm{d}v,t) - \mu(\mathrm{d}v)) = \\
&= \int\limits_{(-\infty,0)} \int\limits_{(-\infty,0)} \chi_{(x,0)}(u)\chi_{(v,\infty)}(u)\mathrm{d}u(p(\mathrm{d}v,t) - \mu(\mathrm{d}v)) = \\
&= \int\limits_{(-\infty,0)} \int\limits_{(-\infty,0)} \chi_{(x\vee v,0)}(u)\mathrm{d}u(p(\mathrm{d}v,t) - \mu(\mathrm{d}v)) = \\
&= \int\limits_{(-\infty,0)} -(x \vee v)(p(\mathrm{d}v,t) - \mu(\mathrm{d}v)) = \int\limits_{(-\infty,0)} (x \vee v)(\mu(\mathrm{d}v) - p(\mathrm{d}v,t))
\end{aligned}$$

11. $0 \leq P(x,t)$ is trivial. We show the other inequality for $x \leq 0$:

$$\begin{aligned}
P(x,t) =& P(0,t) + \int\limits_{(-\infty,0)} (x \vee v)(p(\mathrm{d}v,t) - \mu(\mathrm{d}v)) \leq \\
\leq& P(0,t) - \int\limits_{(-\infty,0)} (x \vee v)\mu(\mathrm{d}v) \leq \\
\leq& P(0,t) + \int\limits_{(-\infty,0)} |v|\,\mu(\mathrm{d}v) < \infty
\end{aligned}$$

Were we used the assumption for μ from theorem (5.2) and point 8.

12. To show: $\frac{\partial P}{\partial t}(x,t)\mathrm{d}x = a(x,t)p(\mathrm{d}x,t)$
This follows immediately from the definition of $P(x,t)$.

□

Lemma 5.8. There exists $p \in L^2_{loc}(U)$ such that for almost every $t \geq 0$:

$$p(\mathrm{d}x,t) = p(x,t)\mathrm{d}x \tag{5.31}$$

PROOF : For the big part, this proof works analogue to the general case. We replace $\frac{1}{R}$ with $-R$ and don't need to extend the functions to $\mathbb{R} \times \mathbb{R}_+$ since that is their original space of definition.
We fix $\alpha, \zeta \in \mathcal{D}(\mathbb{R}_+)$ and assume $\alpha \geq 0$ and $\zeta \geq 0$. There exist $0 < t_1 < t_2$ and $R > 0$ such that $\mathrm{supp}(\alpha) \subseteq [t_1, t_2]$ and $\mathrm{supp}(\zeta) \subseteq [-R, R]$. We set:

$$\epsilon := \epsilon(t_1, t_2, R) \qquad \text{and} \quad m := m(t_2, R)$$

from the first two assumptions for $a(x,t)$. We define the function

$$\tilde{P}(x,t) := \zeta(x) \left(\alpha(t)P(x,t) - \int\limits_{(0,t)} \alpha'(s)P(x,s)\mathrm{d}s \right) \tag{5.32}$$

Then, by integrating by parts, we obtain

$$\begin{aligned}
&\tilde{P}(x,t)\mathrm{d}x = \\
=&\zeta(x) \left(\alpha(t)P(x,t) - (\alpha(t)P(x,t) - \alpha(0)P(x,0) - \right. \\
&\left. \int\limits_{(0,t)} \alpha(s)\frac{\partial P}{\partial s}(x,s)\mathrm{d}s \right) \Bigg) \mathrm{d}x = \\
=&\zeta(x) \left(\alpha(t)P(x,t) - \left(\alpha(t)P(x,t) - \int\limits_{(0,t)} \alpha(s)\frac{\partial P}{\partial s}(x,s)\mathrm{d}s \right) \right) \mathrm{d}x = \\
=&\zeta(x) \left(\int\limits_{(0,t)} \alpha(s)\frac{\partial P}{\partial s}(x,s)\mathrm{d}x\mathrm{d}s \right) =
\end{aligned}$$

Now using Lemma (5.7) point 12 we obtain

$$\begin{aligned}
=&\zeta(x) \left(\int\limits_{(0,t)} \alpha(s)\frac{\partial P}{\partial s}(x,s)\mathrm{d}x\mathrm{d}s \right) = \\
= \tilde{P}(x,t)\mathrm{d}x =&\zeta(x) \int\limits_{(0,t)} \alpha(s)a(x,s)p(\mathrm{d}x,s)
\end{aligned} \tag{5.33}$$

Differentiating equation (5.33) with respect to t, we obtain

$$\frac{\partial \tilde{P}}{\partial t}(x,t) = \zeta(x)\alpha(t)a(x,t)p(\mathrm{d}x,t) \qquad \text{in } \mathcal{D}'(U) \tag{5.34}$$

which implies

$$\frac{\partial \tilde{P}}{\partial t}(x,t) \leq \zeta(x)\alpha(t)mp(\mathrm{d}x,t) \qquad \text{in } \mathcal{D}'(U) \tag{5.35}$$

Differentiating equation (5.32) twice with respect to x, we get in $\mathcal{D}'(U)$

$$\begin{aligned}
&\frac{\partial \tilde{P}}{\partial x}(x,t) = \\
=&\zeta'(x)\left(\alpha(t)P(x,t) - \int\limits_{(0,t)} \alpha'(s)P(x,s)\mathrm{d}s\right) + \\
&\zeta(x)\left(\alpha(t)\frac{\partial P}{\partial x}(x,t) - \int\limits_{(0,t)} \alpha'(s)\frac{\partial P}{\partial x}(x,s)\mathrm{d}s\right)
\end{aligned}$$

and therefore in $\mathcal{D}'(U)$

$$\begin{aligned}
&\frac{\partial^2 \tilde{P}}{\partial x^2}(x,t) = \\
=&\zeta''(x)\left(\alpha(t)P(x,t) - \int\limits_{(0,t)} \alpha'(s)P(x,s)\mathrm{d}s\right) + \\
&2\zeta'(x)\left(\alpha(t)\frac{\partial P}{\partial x}(x,t) - \int\limits_{(0,t)} \alpha'(s)\frac{\partial P}{\partial x}(x,s)\mathrm{d}s\right) + \\
&\zeta(x)\left(\alpha(t)\frac{\partial^2 P}{\partial x^2}(x,t) - \int\limits_{(0,t)} \alpha'(s)\frac{\partial^2 P}{\partial x^2}(x,s)\mathrm{d}s\right) =
\end{aligned}$$

using Lemma (5.7) point 3

$$\begin{aligned}
=&\zeta''(x)\left(\alpha(t)P(x,t) - \int\limits_{(0,t)} \alpha'(s)P(x,s)\mathrm{d}s\right) + \\
&2\zeta'(x)\left(\alpha(t)\frac{\partial P}{\partial x}(x,t) - \int\limits_{(0,t)} \alpha'(s)\frac{\partial P}{\partial x}(x,s)\mathrm{d}s\right) + \\
&\zeta(x)\left(\alpha(t)(p(\mathrm{d}x,t) - \mu(\mathrm{d}x)) - \int\limits_{(0,t)} \alpha'(s)(p(\mathrm{d}x,s) - \mu(\mathrm{d}x))\mathrm{d}s\right) =
\end{aligned}$$

With expanding the last term we get

$$
\begin{aligned}
=&\zeta''(x)\left(\alpha(t)P(x,t)-\int\limits_{(0,t)}\alpha'(s)P(x,s)\mathrm{d}s\right)+\\
&2\zeta'(x)\left(\alpha(t)\frac{\partial P}{\partial x}(x,t)-\int\limits_{(0,t)}\alpha'(s)\frac{\partial P}{\partial x}(x,s)\mathrm{d}s\right)+\\
&\zeta(x)\left(\alpha(t)(p(\mathrm{d}x,t))-\int\limits_{(0,t)}\alpha'(s)(p(\mathrm{d}x,s))\mathrm{d}s\right)+\\
&\zeta(x)\left(\alpha(t)(-\mu(\mathrm{d}x))+\int\limits_{(0,t)}\alpha'(s)(\mu(\mathrm{d}x))\mathrm{d}s\right)=\\
=&\zeta''(x)\left(\alpha(t)P(x,t)-\int\limits_{(0,t)}\alpha'(s)P(x,s)\mathrm{d}s\right)+\\
&2\zeta'(x)\left(\alpha(t)\frac{\partial P}{\partial x}(x,t)-\int\limits_{(0,t)}\alpha'(s)\frac{\partial P}{\partial x}(x,s)\mathrm{d}s\right)+\\
&\zeta(x)\left(\alpha(t)(p(\mathrm{d}x,t))-\int\limits_{(0,t)}\alpha'(s)(p(\mathrm{d}x,s))\mathrm{d}s\right)+\\
&\zeta(x)\left(\alpha(t)(-\mu(\mathrm{d}x))+\alpha(t)(\mu(\mathrm{d}x))\mathrm{d}s\right)=\\
=&\zeta''(x)\left(\alpha(t)P(x,t)-\int\limits_{(0,t)}\alpha'(s)P(x,s)\mathrm{d}s\right)+\\
&2\zeta'(x)\left(\alpha(t)\frac{\partial P}{\partial x}(x,t)-\int\limits_{(0,t)}\alpha'(s)\frac{\partial P}{\partial x}(x,s)\mathrm{d}s\right)+\\
&\zeta(x)\left(\alpha(t)(p(\mathrm{d}x,t))-\int\limits_{(0,t)}\alpha'(s)(p(\mathrm{d}x,s))\mathrm{d}s\right)=\\
\frac{\partial^2\tilde{P}}{\partial x^2}(x,t)=&\zeta(x)\alpha(t)p(\mathrm{d}x,t)-\zeta(x)\int\limits_{(0,t)}\alpha'(s)p(\mathrm{d}x,s)\mathrm{d}s+\phi(x,t) \qquad (5.36)
\end{aligned}
$$

With

$$\phi(x,t) = \zeta''(x)\left(\alpha(t)P(x,t) - \int\limits_{(0,t)} \alpha'(s)P(x,s)\mathrm{d}s\right) +$$
$$2\zeta'(x)\left(\alpha(t)\frac{\partial P}{\partial x}(x,t) - \int\limits_{(0,t)} \alpha'(s)\frac{\partial P}{\partial x}(x,s)\mathrm{d}s\right)$$

We show now that ϕ is in $\mathcal{L}^\infty(U)$.
Since α and ζ are in $\mathcal{D}(\mathbb{R})$ there exists $C_\alpha, C'_\alpha, C_\zeta, C'_\zeta, C''_\zeta > 0$ such that $\alpha(t) \leq C_\alpha,\ \alpha'(t) \leq C'_\alpha,\ \zeta(x) \leq C_\zeta,\ \zeta'(x) \leq C'_\zeta,\ \zeta''(x) \leq C''_\zeta$. Since $P(x,t)$ is continuous (see lemma (5.7) point 8), There exists some C_P with $P(x,t) \leq C_P$ for all $(x,t) \in [0,R] \times [t_1,t_2]$. From lemma (5.7) point 6 we know, that $\frac{\partial P}{\partial x}(x,t) \leq 1$. Therefore,

$$|\phi(x,t)| \leq C''_\zeta \left(C_\alpha C_P + t_2 C'_\alpha C_P\right) + 2C'_\zeta \left(C_\alpha + t_2 C'_\alpha\right) =: C_\phi < \infty$$

Furthermore we have

$$\zeta(x) \int\limits_{(0,t)} |\alpha'(s)|\, p(\mathrm{d}x,s)\mathrm{d}s \leq \frac{C'_\alpha}{\epsilon}\zeta(x) \int\limits_{(0,t)} a(x,s)p(\mathrm{d}x,s)\mathrm{d}s \leq$$
$$\leq \frac{C'_\alpha}{\epsilon}\zeta(x)P(x,t) \leq \frac{C'_\alpha}{\epsilon}C_\zeta C_P < \infty \tag{5.37}$$

Which shows that equation (5.36) can be written as

$$\frac{\partial^2 \tilde{P}}{\partial x^2}(x,t) = \zeta(x)\alpha(t)p(\mathrm{d}x,t) - \frac{1}{m}\Phi(x,t)$$

with $\Phi \in \mathcal{L}^\infty(U)$. And therefore,

$$\zeta(x)\alpha(t)p(\mathrm{d}x,t) = \frac{\partial^2 \tilde{P}}{\partial x^2}(x,t) + \frac{1}{m}\Phi(x,t) \tag{5.38}$$

From equations (5.35) and (5.38) we deduce

$$\frac{\partial \tilde{P}}{\partial t}(x,t) \leq \zeta(x)\alpha(t)mp(\mathrm{d}x,t) = m\frac{\partial^2 \tilde{P}}{\partial x^2}(x,t) + \Phi(x,t) \tag{5.39}$$

In the following we want to define the convolution for $\tilde{P}$ with a functions of a regularizing sequence $(\phi_n)_{n\in\mathbb{N}}$ (See Lemma (2.18)). Without loss of generality

let $\mathrm{supp}(\phi) \subseteq [-\frac{1}{n}, \frac{1}{n}]$.
We define

$$\tilde{P}_n(x,t) := \tilde{P}(x,t) * \phi_n(x) = \int_{\mathbb{R}} \tilde{P}(y,t)\phi_n(x-y)\mathrm{d}y$$

From Definition (2.17) we know, that

$$\frac{\partial^2 \tilde{P}_n}{\partial x^2}(x,t) = \frac{\partial^2 \tilde{P}}{\partial x^2} * \phi_n(x,t)$$

and similarly with equation (5.34)

$$\frac{\partial \tilde{P}_n}{\partial t}(x,t) = \frac{\partial \tilde{P}}{\partial t} * \phi_n(x,t) = \alpha(t)a(x,t)\zeta(x)p(\mathrm{d}x,t) * \phi_n(x,t)$$

We note, that $\frac{\partial \tilde{P}_n}{\partial t}(x,t)$ is differentiable with respect to x. Now equation (5.39) also holds for the in x convoluted $\tilde{P}_n$ (with $\Phi_n := \Phi * \phi_n$):

$$\frac{\partial \tilde{P}_n}{\partial t}(x,t) \leq m\frac{\partial^2 \tilde{P}_n}{\partial x^2}(x,t) + \Phi_n(x,t) \tag{5.40}$$

From [5, p. 26-27] we know, that

$$\mathrm{supp}\left(\frac{\partial \tilde{P}_n}{\partial t}\right) \subseteq \mathrm{supp}(\phi_n) + \mathrm{supp}\left(\frac{\partial \tilde{P}_n}{\partial t}\right) = \left[-R - \frac{1}{n}, R + \frac{1}{n}\right] \times [t_1, t_2]. \tag{5.41}$$

and similarly

$$\mathrm{supp}\left(\frac{\partial \tilde{P}_n}{\partial x}\right) \subseteq \mathrm{supp}(\phi_n) + \mathrm{supp}\left(\frac{\partial \tilde{P}_n}{\partial x}\right) = \left[-R - \frac{1}{n}, R + \frac{1}{n}\right] \times [t_1, \infty). \tag{5.42}$$

We note from [5, p. 23, Fakta 13.3.11-3] that

$$\|\Phi_n\|_\infty \leq \|\Phi\|_\infty \|\phi_n\|_1 = \|\Phi\|_\infty \tag{5.43}$$

and therefore $\Phi_n \in \mathcal{L}^\infty(\mathbb{R} \times \mathbb{R}_+)$.
We want to show, that $\frac{\partial \tilde{P}_n}{\partial t}$ is bounded in $\mathcal{L}^2(\mathbb{R} \times \mathbb{R}_+)$ by a Constant C

independent of n. We use equation (5.41) in the first step:

$$\iint\limits_{\mathbb{R}\times\mathbb{R}_+} \left(\frac{\partial \tilde{P}_n}{\partial t}(x,t)\right)^2 \mathrm{d}t\mathrm{d}x =$$

$$= \iint\limits_{[-1-R,R+1]\times[t_1,t_2]} \frac{\partial \tilde{P}_n}{\partial t}(x,t)\mathrm{d}t\mathrm{d}x \leq \text{ using eq (5.40)}$$

$$\leq m \iint\limits_{[-1-R,R+1]\times[t_1,t_2]} \frac{\partial \tilde{P}_n}{\partial t}(x,t)\frac{\partial^2 \tilde{P}_n}{\partial x^2}(x,t)\mathrm{d}t\mathrm{d}x +$$

$$\iint\limits_{[-1-R,R+1]\times[t_1,t_2]} \frac{\partial \tilde{P}_n}{\partial t}(x,t)\Phi_n(x,t)\mathrm{d}t\mathrm{d}x \tag{5.44}$$

We examine the first term. First we use Fubini's Theorem [6, p. 163] to switch integrating order, then partial integration (which is why we convoluted ; to get the necessary smoothness):

$$m \iint\limits_{[-1-R,R+1]\times[t_1,t_2]} \frac{\partial \tilde{P}_n}{\partial t}(x,t)\frac{\partial^2 \tilde{P}_n}{\partial x^2}(x,t)\mathrm{d}t\mathrm{d}x =$$

$$= m \iint\limits_{[t_1,t_2]\times[-1-R,R+1]} \frac{\partial \tilde{P}_n}{\partial t}(x,t)\frac{\partial^2 \tilde{P}_n}{\partial x^2}(x,t)\mathrm{d}x\mathrm{d}t =$$

$$= m \int\limits_{[t_1,t_2]} \left(\frac{\partial \tilde{P}_n}{\partial t}(x,t)\frac{\partial \tilde{P}_n}{\partial x}(x,t)\Big|_{x=-1-R}^{R+1} - \int\limits_{[-1-R,R+1]} \frac{\partial^2 \tilde{P}_n}{\partial t \partial x}(x,t)\frac{\partial \tilde{P}_n}{\partial x}(x,t)\mathrm{d}x \right) \mathrm{d}t =$$

using equation (5.41)

$$= - m \int\limits_{[t_1,t_2]} \int\limits_{[-1-R,R+1]} \frac{\partial^2 \tilde{P}_n}{\partial t \partial x}(x,t)\frac{\partial \tilde{P}_n}{\partial x}(x,t)\mathrm{d}x\mathrm{d}t =$$

using Fubini's Theorem again

$$= - m \int\limits_{[-1-R,R+1]} \int\limits_{[t_1,t_2]} \frac{\partial^2 \tilde{P}_n}{\partial t \partial x}(x,t)\frac{\partial \tilde{P}_n}{\partial x}(x,t)\mathrm{d}t\mathrm{d}x =$$

$$= - m \int\limits_{[-1-R,R+1]} \int\limits_{[t_1,t_2]} \frac{\partial}{\partial t}\left(\frac{\partial \tilde{P}_n}{\partial x}(x,t)\right)^2 \mathrm{d}t\mathrm{d}x =$$

$$= - m \int\limits_{[-1-R,R+1]} \left(\left(\frac{\partial \tilde{P}_n}{\partial x}(x,t_2)\right)^2 - \left(\frac{\partial \tilde{P}_n}{\partial x}(x,t_1)\right)^2\right) \mathrm{d}x =$$

using equation (5.42)

$$= - m \int\limits_{[-1-R,R+1]} \left(\frac{\partial \tilde{P}_n}{\partial x}(x,t_2)\right)^2 \mathrm{d}x \leq 0$$

Therefor

$$(5.44) \leq \iint\limits_{[-1-R,R+1]\times[t_1,t_2]} \frac{\partial \tilde{P}_n}{\partial t}(x,t)\Phi_n(x,t)\mathrm{d}t\mathrm{d}x$$

using equation (4.20)

$$\iint\limits_{\mathbb{R}\times\mathbb{R}_+} \left(\frac{\partial \tilde{P}_n}{\partial t}(x,t)\right)^2 \mathrm{d}t\mathrm{d}x \leq \frac{1}{2} \iint\limits_{[-1-R,R+1]\times[t_1,t_2]} \left(\left(\frac{\partial \tilde{P}_n}{\partial t}(x,t)\right)^2 + \Phi_n(x,t)^2\right) \mathrm{d}t\mathrm{d}x$$

Which shows that

$$\iint\limits_{\mathbb{R}\times\mathbb{R}_+} \left(\frac{\partial \tilde{P}_n}{\partial t}(x,t)\right)^2 \mathrm{d}t\mathrm{d}x \leq 2 \iint\limits_{[-1-R,R+1]\times[t_1,t_2]} \Phi_n(x,t)^2\mathrm{d}t\mathrm{d}x \leq$$

$$\leq 2 \iint\limits_{[-1-R,R+1]\times[t_1,t_2]} \|\Phi_n(x,t)\|_\infty^2 \mathrm{d}t\mathrm{d}x =$$

using equation (5.43)

$$= 2 \iint\limits_{[-1-R,R+1]\times[t_1,t_2]} \|\Phi(x,t)\|_\infty^2 \mathrm{d}t\mathrm{d}x =$$

$$= 2(t_2 - t_1)(R-2) \|\Phi(x,t)\|_\infty^2 =: C < \infty$$

From [1, p. 639, Theorem 3] we know, that since $\frac{\partial \tilde{P}_n}{\partial t}$ is bounded in $\mathcal{L}^2(\mathbb{R}\times\mathbb{R}_+)$ there exists $u \in \mathcal{L}^2(\mathbb{R} \times \mathbb{R}_+)$ and a subsequence $I \subseteq \mathbb{N}$ with

$$\lim_{i\to\infty} \int\limits_{(\mathbb{R}\times\mathbb{R}_+)} f(x,t) \left(\frac{\partial \tilde{P}_i}{\partial t}(x,t) - u(x,t) \right) \mathrm{d}t\mathrm{d}x = 0 \text{ for all } \in \mathcal{L}^2(\mathbb{R} \times \mathbb{R}_+)$$

On the other hand we know from Lemma (2.19), that $\frac{\partial \tilde{P}_n}{\partial t} \to \frac{\partial \tilde{P}}{\partial t}$ in $\mathcal{D}'(\mathbb{R} \times \mathbb{R}_+)$. Since the limit is unique in distributional sense, $u = \frac{\partial \tilde{P}}{\partial t} \in \mathcal{L}^2(\mathbb{R} \times \mathbb{R}_+)$.
□

PROOF : of Theorem 4.2
This proof works analogously to the proof in the general case. As examined in the above proof, we just have to replace $\frac{1}{R}$ with $-R$ and don't need to extend the function Q nor its derivatives to $\mathbb{R}\times\mathbb{R}_+$ since that is their original space of definition. □

6 Acknowledgement

I want to thank my supervisor, Dr. Stefan Gerhold for his guidance, patience and support. I also want to thank Dr. Harald Woracek for his helpful hints on distributions. Many thanks to Dr. Joachim Schöberl for his immediate answers regarding numerical evaluation.

Literatur

[1] Evans, Lawrence C.: *Partial Differential Equations* -. 2. Aufl. Heidelberg : American Mathematical Soc., 2010. – ISBN 978-0-821-84974-3

[2] Friz, P. ; Gerhold, S. ; Yor, M.: *How to make Dupire's local volatility work with jumps.* 2014 To appear in Quantitative Finance

[3] Hirsch, Francis ; Profeta, Christophe ; Roynette, Bernard ; Yor, Marc: *Peacocks and Associated Martingales, with Explicit Constructions* -. 2011. Aufl. Berlin, Heidelberg : Springer, 2011. – ISBN 978-8-847-01907-2

[4] Jüngel, Ansgar: *Partielle Differentialgleichungen, Vorlesungsmanuskript.* Dezember 2013

[5] Kaltenbäck, Michael: *Vorlesungsskript aus Analysis 3.* Februar 2010

[6] Kusolitsch, Norbert: *Maß- und Wahrscheinlichkeitstheorie - Eine Einführung.* 1st Edition. 2nd Printing. 3rd Printing. Berlin, Heidelberg : Springer, 2011. – ISBN 978-3-709-10684-6

[7] Schwartz, Laurent: *Cours d'analyse* -. Paris : Hermann, 1981. – ISBN 978-2-705-65551-8

A Appendix

PROOF of Lemma 2.12:
We show it for the 1-dimensional case, since we don't need it for higher dimensions. Let $\Omega \subseteq \mathbb{R}$ be an open set. We have to show that $\{f : \forall K \subseteq \Omega, K$ compact : $\int_K |f(x)| \, \mathrm{d}\lambda(x) < \infty\} = \{f : \forall \phi \in \mathcal{D}(\Omega) : \int_K \phi(x) f(x) \mathrm{d}\lambda(x) < \infty\}$.
First, let $\phi \in \mathcal{D}(\Omega)$ and f satisfy the first condition. Since $\operatorname{supp}(\phi)$ is compact,

$$\int\limits_{\operatorname{supp}(\phi)} \phi(x) f(x) \mathrm{d}\lambda(x) \leq \|\phi\|_\infty \int\limits_{\operatorname{supp}(\phi)} f(x) \mathrm{d}\lambda(x) < \infty$$

Therefore, we have the inclusion $\{f : \forall K \subseteq \Omega, K \text{ compact} : \int_K |f(x)| \, \mathrm{d}\lambda(x) < \infty\} \supseteq \{f : \forall \phi \in \mathcal{D}(\Omega) : \int_K \phi(x) f(x) \mathrm{d}\lambda(x) < \infty\}$.
Now let f satisfy the second condition and K a compact set. Since $\Omega \supseteq K$ is open, $\operatorname{dist}(K, \partial\Omega) = \frac{2}{n} > 0$ for some $\mathbb{N} \neq n > 0$. Then $\phi(x) := \chi_K * k_n(x)$ where k_n is the regularizing sequence of Lemma (2.19). Then, according to Definition (2.17), ϕ has derivatives of all orders and since $\operatorname{supp}(k_n) \subseteq \left[-\frac{1}{n}, \frac{1}{n}\right]$ holds $\operatorname{supp}(\phi) \subseteq K + \subseteq \left[-\frac{1}{n}, \frac{1}{n}\right]$ and therefore $\operatorname{dist}(\operatorname{supp}(\phi), \partial\Omega) \geq \frac{1}{n}$. Hence, $\phi \in \mathcal{D}(\Omega)$. Therefor

$$\int\limits_K f(x) \mathrm{d}\lambda(x) \leq \int\limits_{\mathbb{R}} \phi(x) f(x) \mathrm{d}\lambda(x) < \infty$$

Which shows the other inclusion. □

Satz A.1 (von Dini)**.** This Satz can be generalized to topological spaces. For our purposes the case $\mathbb{R}$ with Euclidean Topology is sufficient.
Let K be a compact set with $K \subseteq \mathbb{R}$ and $(f_n)_{n \in \mathbb{N}}$, $f_n : K \to \mathbb{R}$ be a sequence of continuous functions with $f_i(x) \leq f_{i+1}(x)$ for all $x \in K$ (or $f_i(x) \geq f_{i+1}(x)$ for all $x \in K$). Let f be a continuous function with $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in K$, then $\sup_{x \in K} \lim_{n\to\infty} |f_n(x) - f(x)| = 0$.

PROOF : First we prove the case with $f_i(x) \leq f_{i+1}(x)$. Let $\epsilon > 0$. We set $E_n = \{x \in K : |f_n(x) - f(x)| < \epsilon\}$. E_n is open since f_n is continuous. Since $f_n \to f$ pointwise, $(E_n)_{n \in \mathbb{N}}$ is an open cover of K. From $f_i(x) \leq f_{i+1}(x)$ follows $E_i \subseteq E_{i+1}$. Since K is compact, finitely many E_i are sufficient to cover K, i.e

$$\bigcup_{i=1}^{N} E_{n_i} \supseteq K$$

Let N denote the largest index (in the formula above, n_N). From the monotony of the E_n we deduce that $E_N \supseteq K$. Therefore $|f_N(x) - f(x)| < \epsilon$ for all $x \in K$. And again with the monotony we deduce $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and $x \in K$.
To proof the case with $f_i(x) \geq f_{i+1}(x)$ we just consider the sequence $-(f_n)_{n\in\mathbb{N}}$ which now satisfies above conditions for $-f$. □

Remark A.2. As a reminder, we showed that $t \to P(x,t)$ is continuous and needed the vague continuity and monotonicy of $t \to P(x,t)$ together with the Lipschitz continuity of $x \to P(x,t)$. This counterexample shows, that continuity for $x \to P(x,t)$ is not sufficient to show the continuity of $t \to P(x,t)$. We set

$$P(x,t) : \mathbb{R}_+ \times \mathbb{R}_+ \to [0,1]$$

$$P(x,t) := \begin{cases} \frac{1}{1-t}(1-x), & \text{if } t < x < 1, t < 1, \\ \frac{1}{1-t}(x-1), & \text{if } 1 < x < 2-t, t < 1, \\ 0, & \text{if } x = 1, t < 1, \\ 1, & \text{else.} \end{cases}$$

which can also been written as

$$P(x,t) = \begin{cases} \frac{1}{1-t}(1-x), & \text{if } 0 < t < x, x < 1, \\ \frac{1}{1-t}(x-1), & \text{if } 0 < t < 2-x, x > 1, \\ 0, & \text{if } 0 < t < 1, x = 1, \\ 1, & \text{else.} \end{cases}$$

We note, that $P(x,t) \leq 1$. From the second representation one can easily observe, that $t \to P(x,t)$ is monotonically increasing and from the first one, that $x \to P(x,t)$ is continuous for all $t \in \mathbb{R}_+$. It is also obvious, that $t \to P(1,t)$ has a discontuinity at $t = 1$. But $t \to P(x,t)$ is vaguely continuous: Let $f \in C_K^+(\mathbb{R}_+)$. Since the only discontuinity is at $t = 1$ we have to show, that

$$\lim_{t \nearrow 1} \int_{\mathbb{R}_+} f(x)P(x,t)\mathrm{d}x = \int_{\mathbb{R}_+} f(x)P(x,1)\mathrm{d}x = \int_{\mathbb{R}_+} f(x)\mathrm{d}x$$

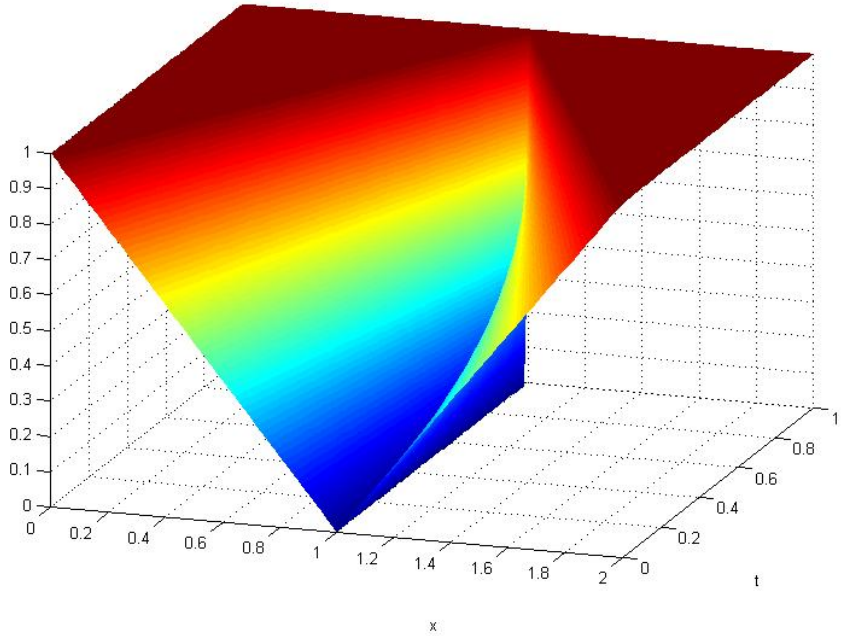

Abbildung 1: illustration of $P(x,t)$

We have

$$\lim_{t\nearrow 1}\int\limits_{\mathbb{R}_+} f(x)P(x,t)\mathrm{d}x = \lim_{t\nearrow 1}\left(\int\limits_{(0,t)} f(x)\mathrm{d}x + \int\limits_{(t,1)} \frac{1}{1-t}(1-x)f(x)\mathrm{d}x + \right.$$

$$\left.\int\limits_{(1,2-t)} \frac{1}{1-t}(x-1)f(x)\mathrm{d}x + \int\limits_{(2-t,\infty)} f(x)\mathrm{d}x\right) =$$

Since $f \in C_K^+(\mathbb{R}_+)$ is bounded and $P(x,t)$ is bounded, we have by dominated convergence

$$= \int\limits_{(0,1)} f(x)\mathrm{d}x + \int\limits_{(1,\infty)} f(x)\mathrm{d}x = \int\limits_{\mathbb{R}_+} f(x)\mathrm{d}x$$