DIPLOMARBEIT

Flat space cosmology and phase transitions in four dimensions

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Wien, 11. Mai 2015

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Abstract

We generalize a phase transition between three-dimensional hot flat space and a certain type of flat space cosmology to four dimensions. To do so, an analogue of this cosmology is constructed in four dimensions and novel flat space boundary conditions are established, that differ from the usual boundary conditions of asymptotically flat space in four dimensions. Also we construct the Lie algebra of asymptotic Killing vectors that preserve these boundary conditions. A generalization of the phase transition can then be found straightforwardly. We will find that there are some differences in possible interpretations as compared to the three-dimensional version, which will also be discussed.

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1. Introduction

Modern theoretical physics has been dominated by two ground-breaking revolutions that stood at the beginning of the 20^{th} century. On the one hand, the emergence of quantum mechanics as a theory to replace classical mechanics in the small-scale regime radically changed the way we understand the dynamics of the fundamental ingredients of our universe, paving the way for modern particle physics as described in terms of the standard model. Einstein's development of general relativity, on the other hand, gave a new, geometrical understanding of the gravitational force, replacing Newton's theory of gravity in describing large-scale dynamics. It is natural, as it lies in the innermost spirit of theoretical physics, to seek a unification of these two concepts in terms of a mathematical theory that includes both, general relativity as well as the standard model of particle physics – hence a theory likewise describing the large- and small-scale regime. Despite many attempts, there is, however, to this day still no full understanding of such a theory of quantum gravity.

One possible – and probably the most promising – theory to unify gravity with the fundamental forces described by the standard model, is string theory. Very generally speaking, string theory aims to describe our universe by proposing one-dimensional objects (strings) living in a ten-dimensional spacetime as the fundamental ingredients of nature, where different observable particles correspond to different oscillation modes of these strings (see e.g. [1]). The additional dimensions, which need to be introduced in order for string theory to be consistent, are typically assumed to be compactified on very small scales, leaving only four dimensions of spacetime observable on the (relatively) large scales we live in. String theory is able to address many of the open questions there are in theoretical physics. In particular, coming back to the above problem, gravity (as a massless spin-2 particle, the graviton) emerges naturally in the framework of string theory. However, though providing answers to a variety of questions, it is not yet a fully understood theory. What makes progress very hard is in particular also the lack of experimental input, due to the fact that corrections from string theory only appear at very high energies, which, at the moment, are above the energies accessible in experiment.

1.1. AdS/CFT and the holographic principle

Despite the lack of having a fully formulated, consistent theory of quantum gravity, there has still been some progress in better understanding the nature of possible such theories. An important step in that respect was marked by the discovery of the Anti-de Sitter/Conformal field theory (AdS/CFT) correspondence [2], in its first version conjectured as a duality (i.e. an equivalence of two different mathematical formulations) between a type IIB string theory (in the supergravity limit) on $\text{AdS}_5 \times S^5$ and four-dimensional $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills theory (in the limit of large N and large 't Hooft coupling). Many generalizations of this duality followed, hinting at an underlying, more fundamental principle. This suggested principle, called the holographic principle [3, 4], states that in a quantum theory of gravity all physics within a region can equivalently be described in terms of a theory on the respective boundary with a sufficiently low number of degrees of freedom per unit area (see e.g. [5]) – i.e. a gravitational theory can always alternatively be described by a theory without gravity in one dimension less. There is yet more that points at the existence of the holographic principle than the AdS/CFT correspondence and its analogues. A prominent example of such is given by the Bekenstein bound, which is an upper bound for the entropy in a certain volume V and given by

$$S \leq \text{Area}(\partial V)/4G = S_{\text{BH}}. \tag{1.1}$$

A black hole of volume V has precisely this maximal entropy S_{BH}. However, since entropy is a measure of information, the Bekenstein bound can be interpreted as maximum amount of information that can be put into a region of volume V. If this maximum amount of information is proportional to the area of the boundary, one could suspect a possibility to describe this content of information (abstractly speaking) via a field theory on the boundary – which precisely matches the statement of the holographic principle. This and many similar arguments, have resulted in a widespread belief among physicists in the correctness of the holographic principle.

1.2. Flat space holography

As briefly mentioned above, there have been many generalizations of the AdS/CFT correspondence. These AdS/CFT-type dualities, as indicated by nomenclature, typically deal with gravitational theories in AdS spacetimes, i.e. spacetimes of constant negative curvature. Though interesting by supporting a general improvement of understanding in the regime of quantum gravity, these types of dualities can not directly be applied to the universe we live in, which, on the contrary, is one of positive curvature described by an extremely small, but non-vanishing cosmological constant. Thus very naturally the question arises, how general AdS/CFT-type dualities can be. This question has in the last years lead to work on possible types of non-AdS holography in different kinds of set-ups.

In particular there is also interest in the possibility of flat space holography. As mentioned above, the cosmological constant in our universe is extraordinary small, making flat space a very good approximation of the spacetime we live in. After success in extracting features of the S-matrix from AdS/CFT correlators, see e.g. [6–9], progress on that subject was specifically made through the development of the BMS/CFT resp. BMS/GCA correspondences [10–12], resulting in a variety of subsequent work (see e.g [13–16] for very recent work, as well as references therein for an overview).

The present work will in particular be concerned with one specific aspect within the development of flat space holographic theories. It was found in [17] that a certain type of a three-dimensional flat space cosmology, developed in [18, 19], can emerge from hot flat space (i.e. Minkowski space with a finite temperature) through a phase transition of the spacetime as a whole. What makes this particularly interesting, is that this flat space cosmology describes an expanding universe, i.e. the phase transition is one from a static solution with little interesting properties (simple, hot Minkowski space) to the time-dependent solution of an expanding universe.

Given the existence of such a phase transition in three spacetime dimensions, the question arises whether or not it is possible to generalize this result to four dimensions. The present work is an attempt to do so. It will be organized as follows. Section 2 gives a brief introduction to the basic concepts needed, which includes in particular the notion of orbifolds (as the flat space cosmology under consideration has orbifold structure) and some background concerning boundary conditions in gravitational theories. Section 3 reviews concrete work on boundary conditions in three-dimensional asymptotically flat space as well as the construction of the flat space cosmology as an orbifold of four-dimensional Minkowski space; section 4 then shows along the lines of [17] how to establish the phase transition. In section 5 we will generalize this to four dimensions. Section 6 summarizes the obtained results and also gives gives a brief outlook on possible further research. Details of longer calculations are given in the appendices.

2. Preliminaries

2.1. Orbifolds

The concept of manifolds – topological spaces that locally look like Euclidean/Lorentzian space – has long been a constantly used concept in various areas of physics. In general relativity and cosmology, in particular, one uses the notion of a Lorentzian manifold for the description of spacetime. However, sometimes one wishes to allow for specifically chosen extensions to usual assumptions for the considered spacetime that go beyond those available in the framework of manifolds, necessitating a more general concept. An often used generalization of the notion of a manifold is that of a so-called orbifold [20], which is defined as the quotient M/G of a manifold M with respect to orbits of a discrete group G [21]. Orbifolds, specifically, allow for the existence of particular singularities: If G has any fixed points – i.e. points where the action of G reduces to the identity – the resulting quotient M/G will have singular points. A prototype of such a construction would be the quotient of $\mathbb{R}^2$ with respect to rotations of angle $2\pi/N$ with some integer N, the resulting space being a cone with a singular point at its tip (the origin) which is left invariant by rotations.

Applications of the concept of orbifolds – in view of the specifics of the present work – have e.g. been found in the construction of orbifold spaces from spaces which are solutions of Einstein's equations, a particular, well-known example being the BTZ black hole [22, 23]. Some more details concerning such orbifold spaces will be given where needed at later points in this work.

2.2. Boundary conditions in gravitational theories

In field theories, the physical content of a theory can be separated into two pieces of information: the field equations and the corresponding boundary conditions. In gravitational theories, where one is interested in the dynamics of the metric itself, boundary conditions play a very special role. This is due to the fact, that what is natural in other field theories, namely fixing boundary conditions simply by demanding the respective field to asymptotically vanish, is not applicable in gravity. On the contrary, it would be highly unnatural to have an asymptotically vanishing, hence singular, metric. Thus when imposing boundary conditions one typically assumes that the metric asymptotes to a certain solution of Einstein's equations. In particular, for the case of flat space considered in this work, one requires g to asymptotically take the simple form of the Minkowski metric. Intuitively this would mean fixing boundary conditions by demanding

$$g \stackrel{r\to\infty}{\longrightarrow} g_{\mathbb{M}} + \text{subleading}. \tag{2.1}$$

It turns out, however, that for flat space it is even possible to allow for leading order fluctuations (with respect to the background Minkowski metric) in some terms of g [10]. This is a rather unusual feature of asymptotically flat spacetimes and has to be handled with care.

2.2.1. Asymptotic symmetries and boundary charges

Given a certain set of boundary conditions, there is in general some freedom left in the sense that one can transform the metric

$$g \to \bar{g} = g + \delta g \tag{2.2}$$

with $\bar{g}$ still satisfying the boundary conditions. The group of all such transformations, that leave the proposed asymptotic form of the metric invariant, is called the asymptotic symmetry group.

This is of particular interest in the context of the holographic principle (which was briefly discussed in section 1), given that the asymptotic symmetries on the boundary of the gravitational theory are precisely the symmetries that govern the dual field theory. It is thus of interest, when studying specific boundary conditions, to construct the corresponding asymptotic symmetry algebra and boundary charges, and, if possible, relate these to a certain type of field theory. For boundary conditions in three-dimensional asymptotically flat space, this was studied in [10], giving rise to the BMS/CFT correspondence. A generalization to four dimensions followed in [11].

2.2.2. Variational principle

It is important from a physical perspective, when having a specific action Γ and an associated set of boundary conditions, to make sure that these give rise to a well defined variational principle, meaning that on-shell variations of the action (under transformations allowed by these boundary conditions) should vanish. This ensures that obtained solutions are actually stable and provide a good semiclassical approximation to the path integral [24]. If the variation $\delta\Gamma$ does not vanish, additional boundary terms have to be introduced in order to fix this problem. A typical example of such a boundary term is the Gibbons-Hawking-York term [25, 26], as introduced for pure Einstein gravity. However, different assumptions concerning the asymptotic behavior of the metric can lead to different forms of $\delta\Gamma$ and therefore possibly need different compensation terms. For the flat space boundary conditions used in this work – in three as well as in four dimensions – this additional compensatory boundary term is exactly one half of the usual Gibbons-Hawking-York term, see [24] and Appendix A for details.

3. Boundary conditions and flat space cosmology in three dimensions

In three-dimensional asymptotically flat space, boundary conditions as well as asymptotic symmetries at null infinity have been studied and the asymptotic symmetry algebra was found to be the three-dimensional BMS [27, 28] algebra [29] with central extensions [10]. Section 3.1 introduces these boundary conditions and summarizes how the BMS algebra and associated charges are obtained. Furthermore in section 3.2 we will review the construction of the flat space cosmology [18, 19] and see that it belongs to the class of solutions described precisely by these boundary conditions.

3.1. Boundary conditions in three dimensions

Working in outgoing Eddington-Finkelstein coordinates (u, r, φ) one imposes boundary conditions [10, 30]

$$\begin{aligned} g_{uu} &= -1 + h_{uu} + O(1/r) & g_{ur} &= -1 + h_{ur}/r + O(1/r^2) \\ g_{u\varphi} &= O(1) & g_{rr} &= O(1/r^2) \\ g_{r\varphi} &= h_{r\varphi} + \tilde{h}_{r\varphi}/r + O(1/r^2) & g_{\varphi\varphi} &= r^2 + r h_{\varphi\varphi} + O(1). \end{aligned} \tag{3.1}$$

The Minkowski metric

$$\mathrm{d}s^2 = -\,\mathrm{d}u^2 - 2\,\mathrm{d}r\,\mathrm{d}u + r^2\,\mathrm{d}\varphi^2 \tag{3.2}$$

is obtained if all functions h and subleading terms are set to zero.

The equations of motion, together with the requirement of a well defined variational principle fix

$$h_{r\varphi} = h_1(\varphi) \tag{3.3}$$
$$h_{\varphi\varphi} = h_2(\varphi) + uh_3(\varphi). \tag{3.4}$$

To determine the asymptotic symmetry algebra, one needs to find gauge transformations that leave the form of (3.1) invariant, i.e. one needs to find the most general asymptotic Killing vector ξ, satisfying

$$\mathcal{L}_\xi g_{ab} = O(\delta g_{ab}). \tag{3.5}$$

This is given by (see Appendix B.1 for details)

$$\xi = [\xi_M(\varphi) + u\xi_L'(\varphi) + O(1/r)]\partial_u + [-r\xi_L'(\varphi) + O(1)]\partial_r \tag{3.6}$$
$$+ [\xi_L(\varphi) - \frac{u}{r}\xi_L''(\varphi) + \frac{1}{r}f_1(\varphi) + O(1/r^2)]\partial_\varphi. \tag{3.7}$$

The part of this which contributes to the asymptotic charges can be separated into two independent Killing vectors,

$$\xi_L = \xi_L(\varphi)\partial_\varphi + \xi_L'(\varphi)(u\partial_u - r\partial_r) - \xi_L''(\varphi)\frac{u}{r}\partial_\varphi + \dots \tag{3.8}$$
$$\xi_M = \xi_M(\varphi)\partial_u + \dots \tag{3.9}$$

where the dots refer to sub-leading terms.

Fourier-expanding $\xi_L(\varphi)$ and $\xi_M(\varphi)$ as

$$\xi_L(\varphi) = \sum_n e^{in\varphi} L_n \tag{3.10}$$
$$\xi_M(\varphi) = \sum_n e^{in\varphi} M_n \tag{3.11}$$

one finds the asymptotic symmetry group to be generated by

$$L_n = ie^{in\varphi}\left(inu\partial_u - inr\partial_r + \left(1 + n^2\frac{u}{r}\right)\partial_\varphi\right) + \dots\ , \tag{3.12}$$
$$M_n = ie^{in\varphi}\partial_u + \dots\ . \tag{3.13}$$

L_n and M_n asymptotically satisfy the three-dimensional BMS algebra without central terms,

$$[L_m, L_n] = (m-n)L_{n+m} \tag{3.14}$$
$$[L_m, M_n] = (m-n)M_{n+m} \tag{3.15}$$
$$[M_m, M_n] = 0. \tag{3.16}$$

Corresponding boundary charges can be calculated with methods developed in [31–33], e.g. by using [34]. This results in

$$Q_{L_n} = \frac{1}{16\pi G} \int \mathrm{d}\varphi e^{in\varphi} \left(-(n^2 + h_3)h_1 + in\,(h_{uu} - h_2 - h_{ur} - h_1') + \partial_u \tilde{h}_{r\varphi} \right) \qquad (3.17)$$

$$Q_{M_n} = \frac{1}{16\pi G} \int \mathrm{d}\varphi e^{in\varphi} \,(h_3 + h_{uu})\,. \qquad (3.18)$$

One can show, see e.g. [10, 30], that these charges are conserved on-shell that their algebra includes a central extension.

3.2. Three-dimensional shifted-boost orbifold

Orbifolds of spaces which are solutions of Einstein's equations can lead to physically interesting cosmologies. Up to certain singularities, they keep their local properties (in particular their property of solving Einstein's equations), however have a different, possibly richer, global structure. A prominent example of such an orbifold in three dimensions is the BTZ black hole, which is an orbifold of three-dimensional AdS. It was in particular the discovery of the BTZ black hole that has motivated following research on orbifolds as possible cosmological scenarios, including generalizations to higher dimensions, see e.g. [35–39], or also to non-AdS orbifolds, see e.g. [18, 19].

We will here consider orbifolds of flat, three-dimensional Minkowski-space, following the lines of [18, 19]. These are constructed by identifying points P along the orbits of a discrete subgroup of the Poincaré group,

$$P \sim e^{\kappa} P \qquad (3.19)$$

where

$$\kappa = 2\pi i(\alpha^a P_a + \beta^{ab} J_{ab}) \qquad (3.20)$$

and P_a, J_{ab} are the Poincaré generators

$$P_a = \frac{1}{i}\partial_a \qquad \text{and} \qquad J_{ab} = \frac{1}{i}\left(X_a \partial_b - X_b \partial_a\right). \qquad (3.21)$$

We will in particular be interested in so-called shifted-boost orbifolds, generated by a Killing vector

$$\kappa = 2\pi i(RP_Y - \Delta J_{TX}) = 2\pi \left[R\partial_Y + \Delta(T\partial_X + X\partial_T)\right], \qquad (3.22)$$

describing a boost with rapidity $\tanh(2\pi\Delta)$ in X-direction and a shift of length $2\pi R$ in Y-direction[1]. The fact that κ contains not only a boost but also a non-vanishing shift is crucial, because it prohibits the existence of any fixed points of κ. If this was not the

[1]For a full classification of possible orbifolds of $\mathbb{M}^3$ see [19].

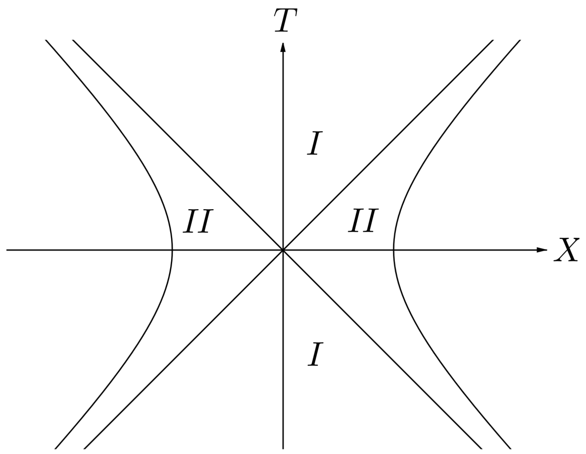

Figure 1: Different regions of the shifted-boost orbifold in the XT-plane

case (i.e. if R was equal to zero) one would encounter severe problems when $\kappa \to 0$ at the origin; in fact the resulting space would not even be Hausdorff.

Having thus fixed $R \neq 0$, in order for the quotient space to constitute a physically meaningful spacetime it is furthermore necessary to exclude regions in which κ would lead to identifications along timelike directions. Such identifications would produce closed timelike curves and thus causal loops. One must therefore excise regions where $\kappa^2 < 0$ from the original Minkowski-space. The boundary of these regions is given by

$$X^2 - T^2 = \frac{1}{E^2}, \tag{3.23}$$

where $E^2 \equiv \Delta^2/R^2$.

The remaining space – the space within the two branches of the hyperbola (3.23) – can furthermore be divided into two regions, depending on their causal properties. The regions to be distinguished (see also Figure 1) are

$$\text{region I:} \qquad X^2 - T^2 < \frac{1}{E^2} \qquad \text{and} \qquad |T| > |X| \tag{3.24}$$

$$\text{region II:} \qquad X^2 - T^2 < \frac{1}{E^2} \qquad \text{and} \qquad |T| < |X|\,. \tag{3.25}$$

In region I, all closed curves resulting from identifications along κ lie completely in region I and are spacelike. In region II the situation is slightly more complicated. If κ points into the formerly excised region, closed timelike curves can form, however they will never close within region II but will end at the boundary (3.23). All closed curves lying completely in region II are again spacelike. The null surface $|T| = |X|$ separating regions I and II acts as a Killing horizon, shielding the singularity at $X^2 - T^2 = \frac{1}{E^2}$. None of the closed timelike curves starting in region II can ever cross this horizon into region I, thus region I is entirely free of singularities.

To actually perform the identifications (3.19), it would be convenient to find coordinates (τ, x, y) for regions I and II, in which the Killing vector κ is simply given by a translation,

$$\kappa = 2\pi R \partial_y. \tag{3.26}$$

These can be found by performing transformations

$$\text{region I:} \qquad T = \tau \cosh\left[E(x+y)\right], \quad X = \tau \sinh\left[E(x+y)\right], \quad Y = y, \tag{3.27}$$

$$\text{region II:} \qquad T = \tau \sinh\left[E(x+y)\right], \quad X = \tau \cosh\left[E(x+y)\right], \quad Y = y, \tag{3.28}$$

leading to line elements of the form

$$\text{region I:} \qquad \mathrm{d}s^2 = -\,\mathrm{d}\tau^2 + \frac{(E\tau)^2}{\Lambda(\tau)}\,\mathrm{d}x^2 + \Lambda(\tau)\left[\mathrm{d}y + \frac{(E\tau)^2}{\Lambda(\tau)}\,\mathrm{d}x\right]^2, \tag{3.29}$$
$$\text{with } \Lambda(\tau) = 1 + (E\tau)^2$$

and

$$\text{region II:} \qquad \mathrm{d}s^2 = \mathrm{d}\tau^2 - \frac{(E\tau)^2}{\Lambda(\tau)}\,\mathrm{d}x^2 + \Lambda(\tau)\left[\mathrm{d}y - \frac{(E\tau)^2}{\Lambda(\tau)}\,\mathrm{d}x\right]^2, \tag{3.30}$$
$$\text{with } \Lambda(\tau) = 1 - (E\tau)^2.$$

Identifying along orbits of the Killing vector (3.26) then reduces to the identification $\varphi \sim \varphi + 2\pi R$.

The conformal diagram corresponding to the constructed orbifold is, in a slightly simplified form, depicted in figure 2: At $r = 0$ one finds the timelike singularity, shielded by the horizon at $r = r_0$ which separates the inner region II from region I. The two different parts of region I can be seen as describing a collapsing respectively expanding universe.

3.3. Shifted-boost orbifold as limit of the BTZ black hole

The BTZ black hole was already mentioned as the probably most prominent example of a physically interesting spacetime with orbifold structure. In the following it will be shown that the shifted-boost orbifold constructed in the previous section can in fact be obtained as flat space limit of the BTZ black hole [18]. We will therefore first briefly recover the BTZ solution [23].

The BTZ black hole can be constructed as an orbifold of three-dimensional Anti-de Sitter space (AdS_3). AdS_3 is defined by the hypersurface

$$-V^2 - U^2 + X^2 + Y^2 = -\ell^2 \tag{3.31}$$

in four-dimensional flat space with two timelike directions,

$$\mathrm{d}s^2 = -\,\mathrm{d}U^2 - \mathrm{d}V^2 + \mathrm{d}X^2 + \mathrm{d}Y^2. \tag{3.32}$$

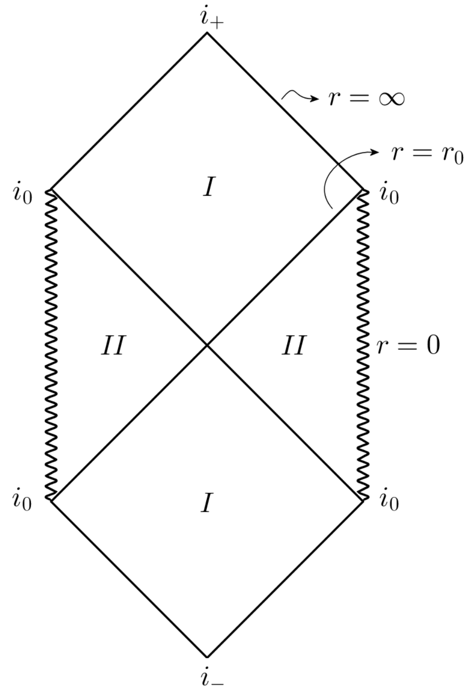

Figure 2: (Simplified) Penrose diagram of the shifted-boost orbifold

It is a space of constant negative curvature $\sim -1/\ell^2$ with ℓ called the AdS radius[2].

AdS_3 is, by construction, invariant under $SO(2,2)$ transformations, generated by Killing vectors

$$J_{ab} = \frac{1}{i}\left(X_a\partial_b - X_b\partial_a\right) \tag{3.33}$$

where $X^a = (X^0, X^1, X^2, X^3) = (V, U, X, Y)$. We will in the following be concerned in particular with identifications

$$P \sim e^{\kappa} P \tag{3.34}$$

along orbits of the discrete subgroup of $SO(2,2)$ generated by the Killing vector

$$\kappa = 2\pi i\left(-\frac{r_+}{\ell}J_{12} + \frac{r_-}{\ell}J_{03}\right) = 2\pi i\left(\frac{r_+}{\ell}(X\partial_U + U\partial_X) - \frac{r_-}{\ell}(Y\partial_V + V\partial_Y)\right). \tag{3.35}$$

This Killing vector satisfies $\kappa^2 > 0$ whenever

$$\frac{r_+^2}{\ell^2}(U^2 - X^2) + \frac{r_-^2}{\ell^2}(V^2 - Y^2) = \frac{r_+^2}{\ell^2}(U^2 - X^2) + \frac{r_-^2}{\ell^2}(X^2 - U^2 + \ell^2) > 0, \tag{3.36}$$

[2]What is usually meant by AdS, however, is actually its universal cover. This is important also from a physical perspective, since AdS without being unwarped to its universal covering would have a timelike periodicity.

i.e. (assuming $r_+ > r_-$),

$$U^2 - X^2 > -\frac{r_-\ell^2}{r_+^2 - r_-^2} \tag{3.37}$$

or equivalently

$$-V^2 + Y^2 > -\frac{r_+\ell^2}{r_+^2 - r_-^2}. \tag{3.38}$$

All regions where $\kappa^2 < 0$ have to be cut out before identifying along κ such that no closed timelike curves can be produced. The remaining part of AdS_3 can be divided into three regions,

$$\begin{aligned} \text{region I:} \qquad & U^2 - X^2 > \ell^2 & (3.39)\\ \text{region II:} \qquad & 0 < U^2 - X^2 < \ell^2 & (3.40)\\ \text{region III:} \qquad & -\frac{r_-\ell^2}{r_+^2 - r_-^2} < U^2 - X^2 < 0. & (3.41) \end{aligned}$$

In each of these regions one can introduce new coordinates (t, r, φ) via

$$\begin{aligned} \text{region I:} \quad U &= \ell\sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}}\cosh f(t,\varphi), & X &= \ell\sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}}\sinh f(t,\varphi)\\ Y &= \ell\sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}}\cosh \tilde{f}(t,\varphi), & V &= \ell\sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}}\sinh \tilde{f}(t,\varphi) \end{aligned} \tag{3.42}$$

$$\begin{aligned} \text{region II:} \quad U &= \ell\sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}}\cosh f(t,\varphi), & X &= \ell\sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}}\sinh f(t,\varphi)\\ Y &= -\ell\sqrt{\frac{r_+^2 - r^2}{r_+^2 - r_-^2}}\sinh \tilde{f}(t,\varphi), & V &= -\ell\sqrt{\frac{r_+^2 - r^2}{r_+^2 - r_-^2}}\cosh \tilde{f}(t,\varphi) \end{aligned} \tag{3.43}$$

$$\begin{aligned} \text{region III:} \quad U &= \ell\sqrt{\frac{r_-^2 - r^2}{r_+^2 - r_-^2}}\sinh f(t,\varphi), & X &= \ell\sqrt{\frac{r_-^2 - r^2}{r_+^2 - r_-^2}}\cosh f(t,\varphi)\\ Y &= -\ell\sqrt{\frac{r_+^2 - r^2}{r_+^2 - r_-^2}}\sinh \tilde{f}(t,\varphi), & V &= -\ell\sqrt{\frac{r_+^2 - r^2}{r_+^2 - r_-^2}}\cosh \tilde{f}(t,\varphi) \end{aligned} \tag{3.44}$$

with functions f, $\tilde{f}$ defined as

$$f(t,\varphi) = \frac{1}{\ell}\left(-\frac{r_- t}{\ell} + r_+\varphi\right), \quad \tilde{f}(t,\varphi) = \frac{1}{\ell}\left(\frac{r_+ t}{\ell} - r_-\varphi\right). \tag{3.45}$$

In these coordinates κ simplifies to

$$\kappa = 2\pi\partial_\varphi \tag{3.46}$$

and the metric becomes

$$\mathrm{d}s^2_{BTZ} = -\frac{\left(r^2-r_-^2\right)\left(r^2-r_+^2\right)}{r^2\ell^2}\,\mathrm{d}t^2+\frac{\ell^2 r^2}{\left(r^2-r_-^2\right)\left(r^2-r_+^2\right)}\,\mathrm{d}r^2+r^2(\mathrm{d}\varphi-\frac{r_-r_+}{\ell r^2}\,\mathrm{d}t)^2, \quad (3.47)$$

where (after identification) $\varphi \sim \varphi + 2\pi$. This is the BTZ solution. It describes a spacetime with a singularity at $r = 0$ (as this describes the boundary to a region including closed timelike curves and with possible non-Hausdorff properties), which has an inner and outer horizon at $r = r_-$ resp. $r = r_+$. The definitions of regions I to III translate to the new coordinates as

$$\text{region I:} \qquad r > r_+ \quad \text{(outer region)} \qquad (3.48)$$

$$\text{region II:} \qquad r_- < r < r_+ \quad \text{(intermediate region)} \qquad (3.49)$$

$$\text{region III:} \qquad 0 < r < r_- \quad \text{(inner region).} \qquad (3.50)$$

In regions I and II no singularities occur, since κ is always spacelike and only connects points within these two regions. Region III of the BTZ black hole is analogous to what was called region II in the case of the flat space cosmology: Closed timelike curves can in principle start there, however will never close within region III, but lead into the singularity $r = 0$.

We want to show that for $r_- < r < r_+$ (3.47) indeed takes the form of (3.29) in the flat space limit, i.e. in the limit of large AdS radius ℓ. To be able to take this limit, one at the same time needs to rescale the outer horizon, $\hat{r}_+ = \ell r_+$. Then sending $\ell \to \infty$ one obtains

$$\mathrm{d}s^2_{BTZ} \to \hat{r}_+^2\,\mathrm{d}t^2 - \frac{r^2}{\hat{r}_+^2\left(r^2-r_-^2\right)}\,\mathrm{d}r^2 - 2\hat{r}_+\,\mathrm{d}t\,\mathrm{d}\varphi r_- + r^2\,\mathrm{d}\varphi^2. \qquad (3.51)$$

Rewriting

$$\hat{r}_+ t = x, \qquad r_-\varphi = y + x, \qquad \left(\frac{r}{r_-}\right)^2 = 1 + (E\tau)^2 \qquad (3.52)$$

with $E = \hat{r}_+/r_-$ this is exactly the flat space cosmology solution[3] (3.29),

$$\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 + \frac{(E\tau)^2}{\Lambda(\tau)}\,\mathrm{d}x^2 + \Lambda(\tau)\left[\mathrm{d}y + \frac{(E\tau)^2}{\Lambda(\tau)}\,\mathrm{d}x\right]^2. \qquad (3.53)$$

4. Cosmic phase transition in three dimensions

It was shown [17] that in three dimensions there exists a cosmological phase transition between hot flat space (i.e. simple, three-dimensional Minkowski space with finite temperature and angular momentum) and a flat space cosmological spacetime, described by a shifted-boost orbifold as introduced in the previous section. This section briefly

[3]By choosing $(r/r_-)^2 = 1 - (E\tau)^2$ one can analogously relate region III of the BTZ black hole to region II of the shifted-boost orbifold.

recovers this result. Although generalizations are possible, we will here focus on Einstein gravity only.

In particular, the action is taken to be the Einstein-Hilbert action plus one half of the usual Gibbons-Hawking-York term. The additional factor of 1/2 is introduced in order to secure a well defined variational principle for the boundary conditions we use, see [24] and appendix A for details. With these conventions, the (Euclidean) action is of the form

$$\Gamma = -\frac{1}{16\pi G}\int \mathrm{d}^3x\sqrt{g}R - \frac{1}{16\pi G}\lim_{r\to\infty}\int \mathrm{d}^2x\sqrt{\gamma}K. \tag{4.1}$$

What is still to be specified is under what conditions two solutions to the equations of motion arising from (4.12) are considered to be in the same ensemble, i.e. under what conditions they form two different realizations of the same physical input. Obviously one wants the two solutions

- to have the same temperature T and angular velocity Ω, and
- not to have any conical singularities.

Furthermore, the two solutions should match asymptotically. However, in the case of asymptotically flat space it is not possible to demand the two metrics to be asymptotically the same, since there are leading order fluctuations in some terms. Thus, the best one can do is to require that

- both solutions obey flat space boundary conditions as given in section 3.1.

All of these conditions are satisfied for the two spacetimes under consideration. To determine which solution is thermodynamically favored, one needs to calculate the free energy for these two solutions. This is done in the following.

4.1. Hot flat space (HFS)

Hot flat space can be obtained from three-dimensional Minkowski space,

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2 \tag{4.2}$$

by introducing a finite, non-zero temperature $T = \beta^{-1}$. To do so, one takes the Euclidean version of $\mathbb{M}^3$,

$$\mathrm{d}s_E^2 = \mathrm{d}\tau_E^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2 \tag{4.3}$$

and defines a periodicity in Euclidean time $\tau_E = -it$, through

$$\tau_E \sim \tau_E + \beta. \tag{4.4}$$

Additionally, it is possible to also introduce an angular momentum $\beta\Omega$ (with Ω denoting angular velocity) by demanding that when moving around a thermal circle in τ_E there is an additional twist in φ such that

$$(\tau_E, \varphi) \sim (\tau_E, \varphi + 2\pi) \sim (\tau_E + \beta, \varphi + \beta\Omega). \tag{4.5}$$

As mentioned above, we work in Einstein gravity with an Euclidean action given by (4.1). On-shell, the first term in (4.1) vanishes, such that only the boundary term is left. In the case of HFS for a $r = const.$ surface one has

$$\sqrt{\gamma} = r, \qquad K = \frac{1}{r}, \tag{4.6}$$

hence the on-shell action is given by

$$\Gamma_{\text{HFS}} = -\frac{1}{16\pi G} \lim_{r\to\infty} \int_0^\beta \mathrm{d}\tau \int_0^{2\pi} \mathrm{d}\varphi = -\frac{\beta}{8G}. \tag{4.7}$$

The corresponding free energy can be determined via the partition function

$$Z(T, \Omega) = \int \mathcal{D}g e^{-\Gamma[g]} \tag{4.8}$$

as

$$F(T, \Omega) = -T \ln Z(T, \Omega). \tag{4.9}$$

For purposes herein it is sufficient to approximate the path integral by only evaluating the exponential for the classical solution g_c. The free energy is then simply given by

$$F(T, \Omega) = T\Gamma[g_c(T, \Omega)], \tag{4.10}$$

which for HFS yields

$$F_{\text{HFS}} = -\frac{1}{8\pi}. \tag{4.11}$$

4.2. Flat space cosmology (FSC)

We will now repeat the same analysis for the expanding region I of the shifted-boost orbifold. In particular we will work with the metric in the form suggested by the limit of the BTZ black hole (3.51),

$$\mathrm{d}s^2 = \hat{r}_+^2 \, \mathrm{d}t^2 - \frac{r^2}{\hat{r}_+^2 \, (r^2 - r_0^2)} \, \mathrm{d}r^2 - 2\hat{r}_+ \, \mathrm{d}t \, \mathrm{d}\varphi r_0 + r^2 \, \mathrm{d}\varphi^2 \tag{4.12}$$

with $\varphi \sim \varphi + 2\pi$. The radius r_0 corresponds to the inner BTZ horizon formerly denoted by r_-; the different notation is motivated by the fact that in the flat space limit r_+ ($\hat{r}_+$) and r_- no longer correspond to equivalent transformations in the orbifold construction.

To look at the thermodynamical properties of FSC, it is best to change to the Euclidean version of (4.12). This can be obtained by choosing

$$t = i\tau_E, \qquad \hat{r}_+ = -ir_+ \tag{4.13}$$

such that the metric becomes

$$\mathrm{d}s_E^2 = r_+^2 \left(1 - \frac{r_0^2}{r^2}\right) \mathrm{d}\tau_E^2 + \frac{\mathrm{d}r^2}{r_+^2 \left(1 - \frac{r_0^2}{r^2}\right)} + r^2(\mathrm{d}\varphi - \frac{r_+ r_0}{r^2}\,\mathrm{d}\tau_E)^2. \tag{4.14}$$

To determine temperature and angular momentum, consider the form of the metric near the horizon, $r^2 = r_0^2 + \epsilon\rho^2$,

$$\mathrm{d}s_E^2 = \frac{\epsilon}{r_+^2}\left(\mathrm{d}\rho^2 + \rho^2 \frac{r_+^4}{r_0^2}\,\mathrm{d}\tau_E^2\right) + r_0^2\left(\mathrm{d}\varphi - \frac{r_+}{r_0}\,\mathrm{d}\tau_E\right)^2 + O(\epsilon^2). \tag{4.15}$$

In order to avoid a conical singularity, the first term in (4.15) requires

$$\tau_E \sim \tau_E + \frac{2\pi r_0}{r_+^2}. \tag{4.16}$$

From the second term in (4.15) one furthermore gets an additional twist in φ, such that in total

$$(\tau_E, \varphi) \sim (\tau_E, \varphi + 2\pi) \sim (\tau_E + \frac{2\pi r_0}{r_+^2}, \varphi + \frac{2\pi}{r_+}). \tag{4.17}$$

Temperature and angular velocity of FSC are thus given by

$$T = \frac{r_+^2}{2\pi r_0} \qquad \text{and} \qquad \Omega = \frac{r_0}{r_+}. \tag{4.18}$$

This agrees with the results obtained (in the Lorentzian version) by looking at the surface gravity of the horizon at $r = r_0$ defined by the normalized Killing vector

$$\chi = \partial_t + \frac{\hat{r}_+}{r_0}\partial_\varphi = \partial_t + \Omega\partial_\varphi \tag{4.19}$$

which is given by

$$\sqrt{-\frac{1}{2}(\nabla_a\chi_b)(\nabla^b\chi^a)} = \frac{\hat{r}_+^2}{r_0} = 2\pi T. \tag{4.20}$$

Having determined temperature and angular momentum, one again needs to evaluate the on-shell action, (4.1). For FSC, one finds asymptotically

$$\sqrt{\gamma} = r_+\sqrt{r^2 - r_0^2} = r_+ r + O\left(\frac{1}{r}\right), \qquad K = \frac{(r^2 + r_0^2)r_+}{r^3} = \frac{r_+}{r} + O\left(\frac{1}{r^3}\right) \tag{4.21}$$

such that

$$\Gamma_{\text{FSC}} = -\frac{1}{16\pi G}\lim_{r\to\infty}\int_0^\beta \mathrm{d}\tau \int_0^{2\pi}\mathrm{d}\varphi(r_+^2) = -\frac{\beta r_+^2}{8G} = -\frac{\pi r_0}{4G}. \tag{4.22}$$

The corresponding free energy,

$$F(T,\Omega) = T\Gamma[g_c(T,\Omega)], \tag{4.23}$$

is given by

$$F_{\text{FSC}} = -\frac{r_+^2}{8G}. \tag{4.24}$$

4.3. Phase transition

Having established expressions for the free energy in both cases, HFS as well as FSC, as

$$F_{\text{HFS}} = -\frac{1}{8G}, \qquad F_{\text{FSC}} = -\frac{r_+^2}{8G}, \tag{4.25}$$

there are two cases to be distinguished. One finds that

- for $r_+ < 1$, i.e. for $T < \frac{1}{2\pi r_0}$ the free energy of HFS is smaller than that of FSC, $F_{\text{HFS}} < F_{\text{FSC}}$, such that HFS is thermodynamically favored, whereas
- for $r_+ > 1$, i.e. for $T > \frac{1}{2\pi r_0}$ one finds that FSC is thermodynamically favored, $F_{\text{HFS}} > F_{\text{FSC}}$.

At the critical value $r_+ = 1$, which corresponds to a critical temperature

$$T_c = \frac{1}{2\pi r_0}, \tag{4.26}$$

the two solutions coexist, and a phase transition between hot flat space and an expanding universe described by FSC takes place. This means, by heating up Minkowski space, when reaching temperature T_c the spacetime will go over to an expanding universe described by the FSC solution. T_c still depends on r_0, i.e. the critical temperature increases with increasing angular velocity $\Omega = r_+/r_0$.

5. Generalization to four dimensions

When trying to generalize the phase transition between three-dimensional hot flat space and the flat space cosmology, as reviewed in detail in the previous chapters, to four-dimensional spacetimes, there are two main questions to be answered. The first of these is whether or not solutions analogous to that of the three-dimensional shifted-boost orbifold also exist in four dimensions, and, if so, what class of boundary conditions they belong to. If successful in constructing such a solution, one can then ask the second question of whether or not a similar kind of phase transition can be established. Whereas the latter of these can be answered rather straightforwardly once knowing what kind of spacetime one is interested in, there are more subtleties to the first of these questions.

Since the three-dimensional shifted-boost orbifold could be obtained as a flat space limit of the BTZ black hole, an obvious way to approach the problem of finding a four-dimensional analogue is by looking at generalizations of the BTZ black hole in four dimensions. Such generalizations exist and have been studied in depth (see e.g. [35–39]). However, despite the existence of a non-rotating ($J = 0$), four-dimensional BTZ black hole, it is not possible to construct a rotating BTZ black hole in four dimensions [36]. This is, because in AdS_4, due to the specifics of this spacetime, it is not possible to construct a horizon shielding the singularity produced by cutting out not allowed regions (with respect to the corresponding Killing vectors).

Despite these difficulties in AdS spacetimes, we will see in the following that it is indeed possible in flat space to generalize the three-dimensional shifted-boost orbifold to four dimensions. However, the four-dimensional generalization lacks some of the features that were found in three dimensions. In particular, in four dimensions one finds that the horizon is no longer compact, but infinitely extended in one direction. Also, the topology at null infinity of the resulting spacetime will not be that of a sphere but rather that of a cylinder. This makes the usual boundary conditions for four-dimensional flat space [11] not applicable, since they assume the spatial part of the boundary metric to be that of a sphere (or, more precisely, something conformally equivalent to a sphere). Section 5.1 aims to establish boundary conditions respecting the specifics needed for the four-dimensional shifted-boost orbifold. We will then, in section 5.2 construct this orbifold, showing that it has indeed properties as just claimed.

5.1. Boundary conditions in four dimensions

Boundary conditions, as well as the asymptotic symmetry algebra for four-dimensional flat space have been studied even earlier than their three-dimensional analogues, giving rise to the original formulation of the BMS algebra [27, 28]. This was generalized and studied in the context of holography in [11], leading to the four-dimensional version of the BMS/CFT correspondence. However, as already mentioned above, boundary conditions for four-dimensional flat space, typically have built-in the assumption of a spherical topology at null infinity. We would here like to establish boundary conditions analogous to those in three dimensions, and with a cylindrical symmetry at null infinity. As a natural ansatz we will therefore use a metric which is of the same form as (3.1) but with an additional z-direction (such that coordinates are given by retarded time u and cylindrical coordinates r, φ, z),

$$\begin{aligned}
g_{uu} &= -1 + h_{uu} + O(1/r) & g_{ur} &= -1 + \frac{h_{ur}}{r} + O(1/r^2) \\
g_{u\varphi} &= h_{u\varphi} + O(1/r) & g_{uz} &= \frac{h_{uz}}{r} + O(1/r^2) \\
g_{rr} &= \frac{h_{rr}}{r^2} + O(1/r^3) & g_{r\varphi} &= h_{r\varphi} + \frac{\tilde{h}_{r\varphi}}{r} + O(1/r^2) \\
g_{rz} &= \frac{h_{rz}}{r} + O(1/r^2) & g_{\varphi\varphi} &= r^2 + r h_{\varphi\varphi} + O(1) \\
g_{\varphi z} &= h_{\varphi z} + O(1/r) & g_{zz} &= 1 + \frac{h_{zz}}{r} + O(1/r^2)
\end{aligned} \tag{5.1}$$

with a priori free functions h. The Minkowski metric

$$\mathrm{d}s^2 = -\,\mathrm{d}u^2 - 2\,\mathrm{d}r\,\mathrm{d}u + r^2\,\mathrm{d}\varphi^2 + \mathrm{d}z^2 \tag{5.2}$$

is obtained if all these functions and subleading terms are set to zero.

Imposing the equations of motion determines

$$h_{r\varphi} = z\tilde{h}_1(\varphi) + h_1(u,\varphi), \tag{5.3}$$

$$h_{\varphi\varphi} = z\tilde{h}_2(u,\varphi) + h_2(u,\varphi), \tag{5.4}$$

$$h_{uu} = z\tilde{h}_3(u,\varphi) + h_3(u,\varphi). \tag{5.5}$$

Furthermore, in order for the boundary charges to be finite and integrable, one needs

$$\partial_u h_{r\varphi} = \partial_z h_{rz} = 0 \tag{5.6}$$

(for finiteness), as well as

$$\partial_z h_{r\varphi} = \partial_z h_{\varphi\varphi} = \partial_z h_{uu} = 0 \tag{5.7}$$

and

$$\partial_z h_{zz} = -2\partial_u h_{rz} \tag{5.8}$$

(for integrability), such that in the end

$$h_{r\varphi} = h_1(\varphi), \qquad h_{\varphi\varphi} = h_2(u,\varphi), \tag{5.9}$$

$$h_{uu} = h_3(u,\varphi), \qquad h_{rz} = h_4(u,\varphi), \tag{5.10}$$

and there is a fixed relation between h_{rz} and h_{zz},

$$h_{zz} = -2z\partial_u h_4(u,\varphi) + h_5(u,\varphi). \tag{5.11}$$

To construct the asymptotic symmetry algebra for boundary conditions (5.1) – (5.11), one needs to solve for the most general asymptotic Killing vector ξ such that

$$\mathcal{L}_\xi g_{ab} = O(\delta g_{ab}). \tag{5.12}$$

This results in (see Appendix B.2)

$$\xi = [\xi_M(\varphi) + u\xi_L'(\varphi) + O(1/r)]\partial_u + [-r\xi_L' + O(1)]\partial_r \tag{5.13}$$

$$+ [\xi_L(\varphi) - \frac{u}{r}\xi_L''(\varphi) + \frac{1}{r}f_1(\varphi) + O(1/r^2)]\partial_\varphi + [\xi_J(\varphi) + O(1/r)]\partial_z, \tag{5.14}$$

which describes the symmetries that also appear for the three-dimensional geometry (ξ_L and ξ_M) plus an additional tower of symmetries under z-translations (ξ_J).

All contributions other than the ones coming from ξ_L, ξ_M and ξ_J turn out to be pure gauge, i.e. they do not appear in the asymptotic charges. We will thus be concerned only with the three asymptotic Killing vectors given by

$$\xi_L = \xi_L(\varphi)\partial_\varphi + \xi_L'(\varphi)(u\partial_u - r\partial_r) - \xi_L''(\varphi)\frac{u}{r}\partial_\varphi + \dots \tag{5.15}$$

$$\xi_M = \xi_M(\varphi)\partial_u + \dots \tag{5.16}$$

$$\xi_J = \xi_J(\varphi)\partial_z + \dots . \tag{5.17}$$

The first two of these are completely equivalent to what was obtained in the three-dimensional case, however there is now an additional asymptotic symmetry along ξ_J. The dots, again, represent sub-leading terms.

The free functions $\xi_L(\varphi)$, $\xi_M(\varphi)$ and $\xi_J(\varphi)$ can be Fourier-expanded as

$$\xi_L(\varphi) = \sum_n e^{in\varphi} L_n \tag{5.18}$$

$$\xi_M(\varphi) = \sum_n e^{in\varphi} M_n \tag{5.19}$$

$$\xi_J(\varphi) = \sum_n e^{in\varphi} J_n \tag{5.20}$$

leading to asymptotic symmetry generators

$$L_n = ie^{in\varphi}\left(inu\partial_u - inr\partial_r + \left(1 + n^2\frac{u}{r}\right)\partial_\varphi\right) + \dots\ , \tag{5.21}$$

$$M_n = ie^{in\varphi}\partial_u + \dots \tag{5.22}$$

$$J_n = ie^{in\varphi}\partial_z + \dots\ . \tag{5.23}$$

They satisfy an extended version of the BMS$_3$ algebra (on the level of generators without central extension),

$$[L_m, L_n] = (m-n)L_{n+m} \tag{5.24}$$

$$[L_m, M_n] = (m-n)M_{n+m} \tag{5.25}$$

$$[M_m, M_n] = 0 \tag{5.26}$$

$$[L_m, J_n] = -nJ_{n+m} \tag{5.27}$$

$$[M_m, J_n] = 0 \tag{5.28}$$

$$[J_m, J_n] = 0. \tag{5.29}$$

The associated charges are given by

$$
\begin{aligned}
Q_{L_n} = \frac{1}{16\pi G}\int \mathrm{d}\varphi e^{in\varphi} \Big(& 2h_{u\varphi} - h_1(\partial_u h_2 + n^2) + h_4(\partial_z h_{u\varphi} - \partial_u h_{\varphi z}) + \partial_u \tilde{h}_{r\varphi} \\
& - in(h_2(1-u) - uh_3 + 2h_5 - h_{ur} - h_4(\partial_u h_4 + \partial_z h_{ur}) - 4z\partial_u h_4 - u\partial_u h_5 \\
& + 2uz\partial_u^2 h_4 + u\partial_z h_{uz}) \Big)
\end{aligned} \tag{5.30}
$$

$$
Q_{M_n} = \frac{1}{16\pi G}\int \mathrm{d}\varphi e^{in\varphi} \left(h_3 + \partial_u h_2 + \partial_u h_5 - 2z\partial_u^2 h_4 - \partial_z h_{uz} \right) \tag{5.31}
$$

$$
Q_{J_n} = \frac{1}{16\pi G}\int \mathrm{d}\varphi e^{in\varphi}\partial_u h_4. \tag{5.32}
$$

These will in general not be conserved. However, as four-dimensional gravity admits gravitational waves arriving at null infinity, this lack of conservation can indeed find a physical interpretation.

5.2. Shifted-boost orbifold in four dimensions

Despite the lack of being able to generalize the rotating BTZ black hole to four dimensions, we will in the following show that for its flat space analogue, the shifted-boost orbifold, such a generalization is indeed possible. However, as already briefly mentioned at the beginning of this section, the spacetime obtained from such a construction has some peculiarities that limit an analogous physical interpretation, one of them being non-compactness of the resulting horizon. We will talk about this in more detail at the end of this section.

To construct the shifted-boost orbifold, we will, in analogy to the procedure of section 3.2, consider identifications in four-dimensional Minkowski space

$$
\mathrm{d}s^2 = -\mathrm{d}T^2 + \mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2 \tag{5.33}
$$

along Killing vectors of the form

$$
\kappa = 2\pi \left[R\partial_Y + \Delta(T\partial_X + X\partial_T) \right]. \tag{5.34}
$$

Since the additional fourth coordinate Z does not enter into the Killing vector at all, the procedures of excising not allowed regions and analyzing causal properties of the remaining spacetime differ from the three-dimensional case solely by the existence of another, completely uninvolved direction. It is worth mentioning that this is different from what one would obtain in four-dimensional AdS, where there exists a connection between coordinates (V, U, X, Y, Z) through the AdS condition $-V^2 - U^2 + X^2 + Y^2 + Z^2 = -\ell^2$.

Knowing this, we can directly transfer the results of section 3.2 from three dimensions to four dimensions: Cutting out regions where $\kappa^2 < 0$, one obtains a spacetime which ends at a singularity at $X^2 - T^2 = 1/E^2$, where E is again defined via $E^2 = \Delta^2/R^2$. The singularity is shielded by a horizon at $|T| = |X|$, which divides the spacetime into two regions,

$$\text{region I:} \qquad X^2 - T^2 < \frac{1}{E^2} \qquad \text{and} \qquad |T| > |X| \tag{5.35}$$

$$\text{region II:} \qquad X^2 - T^2 < \frac{1}{E^2} \qquad \text{and} \qquad |T| < |X| \,. \tag{5.36}$$

with causal properties just as described for the three-dimensional case.

In coordinates (τ, x, y, z) defined via

$$\text{region I}: \quad T = \tau \cosh\left[E(x+y)\right], \quad X = \tau \sinh\left[E(x+y)\right], \quad Y = y, \quad Z = z, \tag{5.37}$$

and

$$\text{region II:} \quad T = \tau \sinh\left[E(x+y)\right], \quad X = \tau \cosh\left[E(x+y)\right], \quad Y = y, \quad Z = z, \tag{5.38}$$

the Killing vector reduces to

$$\kappa = 2\pi R \partial_y, \tag{5.39}$$

leading, in the respective regions, to line elements of the form

$$\begin{aligned} \text{region I:} \qquad & \mathrm{d}s^2 = -\,\mathrm{d}\tau^2 + \frac{(E\tau)^2}{\Lambda(\tau)}\,\mathrm{d}x^2 + \Lambda(\tau)\left[\mathrm{d}y + \frac{(E\tau)^2}{\Lambda(\tau)}\,\mathrm{d}x\right]^2 + \mathrm{d}z^2, \\ & \text{with } \Lambda(\tau) = 1 + (E\tau)^2 \end{aligned} \tag{5.40}$$

and

$$\begin{aligned} \text{region II:} \qquad & \mathrm{d}s^2 = \mathrm{d}\tau^2 - \frac{(E\tau)^2}{\Lambda(\tau)}\,\mathrm{d}x^2 + \Lambda(\tau)\left[\mathrm{d}y - \frac{(E\tau)^2}{\Lambda(\tau)}\,\mathrm{d}x\right]^2 + \mathrm{d}z^2, \\ & \text{with } \Lambda(\tau) = 1 - (E\tau)^2, \end{aligned} \tag{5.41}$$

with $\varphi \sim \varphi + 2\pi R$ after identification.

The two patches of region I again describe a collapsing, respectively expanding universe, however with collapse and expansion only in r- but not in z-direction, like a cylinder shrinking or blowing up only in radial direction. Also the horizon at $r = 0$ is non-compact, due to non-compactness of the z-direction. As the metric (5.40) respectively (5.41) is invariant under z-translations, we can deal with this in two (equivalent) ways, either by considering only densities of physical quantities through dividing by the volume of this non-compact direction, or by compactifying the z-direction, $z \sim z + l$. We choose to do the latter. However, as just stated, there is no genuine difference in the two approaches.

Although we have succeeded in constructing an orbifold that has similar properties to the three-dimensional shifted-boost orbifold (with respect to the causal structure of the different regions), one needs to be careful with a physical interpretation of this spacetime. In particular, the expanding region, which, in three dimensions, had the nice cosmological interpretation of a spatial disk expanding in time, was replaced by something like an expanding cylinder (or torus, when compacitfying the z-direction) instead of something blowing-up like a sphere.

Despite this lack of interpretation as an expanding universe, we will still look at a possible phase transition between such a spacetime and four-dimensional hot Minkowski space. We will, however, only use the terminology flat space cosmology in quotation marks, to indicate this issue. Due to the similarities of the metric the analysis of the two spacetimes is almost identical to the three-dimensinoal case and is given in the following.

5.3. Phase transition in four dimensions

5.3.1. Hot flat space (HFS)

The analysis of HFS in four dimensions is done in full analogy to the three-dimensional case. The metric takes the form

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2 + \mathrm{d}z^2 \tag{5.42}$$

and a finite temperature and angular momentum can be introduced via periodicity conditions in Euclidean time,

$$\tau_E \sim \tau_E + \beta \tag{5.43}$$

and a simultaneous twist in the angle

$$(\tau_E, \varphi) \sim (\tau_E, \varphi + 2\pi) \sim (\tau_E + \beta, \varphi + \beta\Omega). \tag{5.44}$$

Also in four dimensions, consistency of the boundary conditions with a variational principle determines the action to be (see Appendix A)

$$\Gamma = -\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{g}R - \frac{1}{16\pi G}\lim_{r\to\infty}\int \mathrm{d}^3x\sqrt{\gamma}K, \tag{5.45}$$

with the first term vanishing on-shell. Just as in three dimensions, on a $r = const.$ surface $\sqrt{\gamma}$ and K are given by

$$\sqrt{\gamma} = r, \qquad K = \frac{1}{r} \tag{5.46}$$

such that (since the first term in the action vanishes on-shell)

$$\Gamma_{\mathrm{HFS}} = -\frac{1}{16\pi G}\lim_{r\to\infty}\int_0^\beta \mathrm{d}\tau \int_0^{2\pi} \mathrm{d}\varphi \int_0^l \mathrm{d}z = -\frac{\beta l}{8G}. \tag{5.47}$$

and

$$F_{\mathrm{HFS}} = -\frac{l}{8\pi}. \tag{5.48}$$

The three-dimensional result, $F^{(3)}_{\mathrm{HFS}} = -1/8\pi$ could be recovered as a free energy density (with respect to the z-direction), $F^{(3)}_{\mathrm{HFS}} = F^{(4)}_{\mathrm{HFS}}/l$.

5.3.2. “Flat space cosmology” (“FSC”)

The metric (5.40) can, in analogy to the three-dimensional case, be rewritten via

$$\hat{r}_+ t = x, \quad r_0 \varphi = y + x, \quad \left(\frac{r}{r_0}\right)^2 = 1 + (E\tau)^2, \quad z = z \tag{5.49}$$

with $E = \hat{r}_+ / r_0$ as

$$\mathrm{d}s^2 = \hat{r}_+^2 \,\mathrm{d}t^2 - \frac{r^2}{\hat{r}_+^2 \,(r^2 - r_0^2)} \,\mathrm{d}r^2 - 2\hat{r}_+ \,\mathrm{d}t \,\mathrm{d}\varphi r_0 + r^2 \,\mathrm{d}\varphi^2 + \mathrm{d}z^2, \tag{5.50}$$

or, in Euclidean coordinates $t = i\tau_E$, $\hat{r}_+ = -ir_+$, as

$$\mathrm{d}s_E^2 = r_+^2 \left(1 - \frac{r_0^2}{r^2}\right) \mathrm{d}\tau_E^2 + \frac{\mathrm{d}r^2}{r_+^2 \left(1 - \frac{r_0^2}{r^2}\right)} + r^2 (\mathrm{d}\varphi - \frac{r_+ r_0}{r^2} \,\mathrm{d}\tau_E)^2 + \mathrm{d}z^2. \tag{5.51}$$

From the near-horizon approximation, $r^2 = r_0^2 + \epsilon\rho^2$,

$$\mathrm{d}s_E^2 = \frac{\epsilon}{r_+^2} \left(\mathrm{d}\rho^2 + \rho^2 \frac{r_+^4}{r_0^2} \,\mathrm{d}\tau_E^2\right) + r_0^2 \left(\mathrm{d}\varphi - \frac{r_+}{r_0} \,\mathrm{d}\tau_E\right)^2 + \mathrm{d}z^2 + O(\epsilon^2) \tag{5.52}$$

we find the same expressions for temperature and angular momentum as were found in three dimensions,

$$T = \frac{r_+^2}{2\pi r_0} \qquad \text{and} \qquad \Omega = \frac{r_0}{r_+}. \tag{5.53}$$

Also, since there is no contribution coming from the z-directions, $\sqrt{\gamma}$ as well as the extrinsic curvature K at $r \to \infty$ match their three-dimensional analogues,

$$\sqrt{\gamma} = r_+ \sqrt{r^2 - r_0^2} = r_+ r + O\left(\frac{1}{r}\right), \qquad K = \frac{(r^2 + r_0^2) r_+}{r^3} = \frac{r_+}{r} + O\left(\frac{1}{r^3}\right). \tag{5.54}$$

This results in

$$\Gamma_{\text{“FSC”}} = -\frac{1}{16\pi G} \lim_{r\to\infty} \int_0^\beta \mathrm{d}\tau \int_0^{2\pi} \mathrm{d}\varphi \int_0^l \mathrm{d}z r_+^2 = -\frac{\beta r_+^2 l}{8G} = -\frac{\pi r_0 l}{4G}, \tag{5.55}$$

such that the free energy is given by

$$F_{\text{“FSC”}} = -\frac{r_+^2 l}{8\pi}. \tag{5.56}$$

Again, the three dimensional result is recovered as a free energy density, $F^{(3)}_{\text{FSC}} = F^{(4)}_{\text{“FSC”}} / l$.

5.3.3. Phase transition

As the expressions for the free energies are, up to a constant factor l, equivalent to those in three dimensions (i.e. we solely exchanged free energies by free energy densities) we are able to find a phase transition analogous to the phase transition in three dimensions. Namely,

- for $r_+ < 1$, i.e. for $T < 1/2\pi r_0$ HFS is thermodynamically favored, $F_{\text{HFS}} < F_{\text{"FSC"}}$, and
- for $r_+ > 1$, i.e. for $T > 1/2\pi r_0$ one finds $F_{\text{HFS}} > F_{\text{"FSC"}}$ such that "FSC" is favored.

At the critical value $r_+ = 1$, which corresponds to a critical temperature

$$T_c = \frac{1}{2\pi r_0}, \tag{5.57}$$

HFS will tunnel into the "FSC" solution.

6. Conclusions

This work attempted to generalize the results of [17], namely the existence of a phase transition between hot flat space and a certain type of flat space cosmology, to four dimensions. We have succeeded, in the sense that, despite this not being possible for the rotating BTZ black hole, we could construct an analogue of the three-dimensional shifted-boost orbifold in four dimensions with a well defined horizon. This could, however, only be achieved at the price of giving up compactness of the horizon. Also the resulting solution does not fit into the class of boundary conditions given in [11], due to a different topology at null infinity. We thus had to impose boundary conditions more suitable for this solution. These boundary conditions, as well as associated asymptotic symmetries were studied in section 5.1.

Having established appropriate boundary conditions, the free energies of four-dimensional hot Minkowski space, as well as of the four-dimensional shifted-boost orbifold were computed. Due to the strong resemblance between the three- and four-dimensional metrics, the free energies showed the same behavior as in the three-dimensional case; in fact we could recover the exact expressions of the three-dimensional free energies as energy densities along the z-direction. Comparing the free energies of the two different solutions (hot flat space and the four dimensional shifted-boost orbifold) thus results in finding a phase transition analogous to that in three dimensions.

There are, however, certain subtleties about this four-dimensional generalization that need to be handled with care. In particular, as already mentioned at some points, it is not clear in how far the spacetime described by the four-dimensional shifted-boost orbifold can be seen as a cosmological spacetime, as expansion only takes place in the direction of a cylindrical radius.

It would be of interest to study the properties of this orbifold (or also of possibly different flat space orbifolds in four dimensions) in more depth. Furthermore and more generally, in view of the phase transition it would be interesting to further investigate other possible generalizations of the three-dimensional flat space cosmology. In particular in [19] the authors construct a spacetime analogous to the flat spat space cosmology (with an expanding region that blows up like a sphere) from string compactification. Although this spacetime cannot be described as an orbifold of flat space, it would be interesting to further investigate its properties (in particular its asymptotic properties) and check whether or not it represents an alternative (regarding the generalization of the three-dimensional flat space cosmology) to the shifted-boost orbifold considered in this work.

Finally, the constructed boundary conditions in four dimensions deserve further investigation. In particular it would be interesting to study the resulting charge algebra and possible central extensions.

A. Variational principle

In order to have a well defined variational principle, one wants the variation of the action to vanish on-shell. In the case of Einstein gravity, the action will be the Einstein-Hilbert action plus an additional Gibbons-Hawking-York (GHY) term with yet undetermined parameter α,

$$\Gamma_{(\alpha)} = -\frac{1}{16\pi G}\int_M \mathrm{d}^d x\sqrt{g}R - \frac{\alpha}{8\pi G}\int_{\partial M}\mathrm{d}^{d-1}x\sqrt{\gamma}K \tag{A.1}$$

where γ is the $(d-1)$-dimensional metric at the boundary and K as usual the extrinsic curvature

Working with the Euclidean version of the metric and separating $g_{ab} = \gamma_{ab} + n_a n_b$, the variation of (A.1) can be determined in full generality for arbitrary dimension as

$$\begin{aligned}
\delta\Gamma_{(\alpha)} = & \frac{1}{16\pi G}\int_M \mathrm{d}^d x\sqrt{g}G^{ab}\delta g_{ab} \\
& + \frac{1}{16\pi G}\int_{\partial M}\mathrm{d}^{d-1}x\sqrt{\gamma}\left[K^{ab} + (2\alpha-1)n^a n^b K - \alpha g^{ab}K\right]\delta g_{ab} \\
& - \frac{1}{16\pi G}\int_{\partial M}\mathrm{d}^{d-1}x\sqrt{\gamma}(\alpha-1)\left[\gamma^{ab}n^c\nabla_c - \frac{1}{2}n^c\nabla_c(n^a n^b)\right]\delta g_{ab} \\
& + \frac{1}{16\pi G}\int_{\partial^2 M}\mathrm{d}^{d-2}x\sqrt{\gamma'}(2\alpha-1)n^a n'^b\delta g_{ab}
\end{aligned} \tag{A.2}$$

where the last term includes possible corner terms and primed quantities are living on such corners (i.e. on $\partial^2 M$).

It was shown for the case of three dimensions (with boundary conditions as discussed in 3.1) that a well defined variational principle is obtained when choosing $\alpha = 1/2$, in contrast to the usual choice of $\alpha = 1$ [24].

We will here repeat this analysis for the four-dimensional case.

For a Euclidean background metric in coordinates (t, r, φ, z) of the form

$$\mathrm{d}s^2 = h_{tt}(t,\varphi)\,\mathrm{d}t^2 + h_{rr}(t,\varphi)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2 + \mathrm{d}z^2 \tag{A.3}$$

and variations

$$\begin{aligned}
\delta g_{tt} &= \delta h_{tt}(t,\varphi) + O(1/r) & \delta g_{rr} &= \delta h_{rr}(t,\varphi) + O(1/r)\\
\delta g_{\varphi\varphi} &= O(r) & \delta g_{zz} &= O(1/r)\\
\delta g_{tr} &= O(1/r) & \delta g_{t\varphi} &= O(1)\\
\delta g_{tz} &= O(1/r) & \delta g_{r\varphi} &= O(1)\\
\delta g_{rz} &= O(1/r) & \delta g_{\varphi z} &= O(1/r)
\end{aligned} \tag{A.4}$$

one obtains

$$n_a = \delta_a^r \sqrt{h_{rr}} \tag{A.5}$$

$$\sqrt{\gamma} = r\sqrt{\frac{1}{h_{rr}}} \tag{A.6}$$

$$K = \frac{1}{r\sqrt{h_{rr}}} \tag{A.7}$$

and the the various terms in the variation (A.2) behave as

$$K^{ab}\delta g_{ab} = O(1/r^2) \tag{A.8}$$

$$n^a n^b K \delta g_{ab} = \frac{\delta h_{rr}}{r h_{rr}^{3/2}} + O(1/r^2) \tag{A.9}$$

$$g^{ab} K \delta g_{ab} = \frac{h_{rr}\delta h_{tt} + h_{tt}\delta h_{rr}}{r h_{rr}^{3/2} h_{tt}} + O(1/r^2) \tag{A.10}$$

$$\gamma^{ab} n^c \nabla_c \delta g_{ab} = -\frac{\partial_z h_{rr} h_{tt} \delta h_{rz} + \partial_t h_{rr} \delta h_{tr}}{r h_{rr}^{3/2} h_{tt}} + O(1/r^2) \tag{A.11}$$

$$n^c \nabla_c (n^a n^b) \delta g_{ab} = -\frac{\partial_z h_{rr} h_{tt} \delta h_{rz} + \partial_t h_{rr} \delta h_{tr}}{2r h_{rr}^{3/2} h_{tt}} + O(1/r^2). \tag{A.12}$$

Plugging this into equation (A.2), one thus finds the on-shell variation of the action in the $r \to \infty$ limit to be given by

$$\begin{aligned}
\delta\Gamma_{(\alpha)} = &\frac{1}{16\pi G}\int_{\partial M} \mathrm{d}^3x \sqrt{\frac{1}{h_{rr}}}\left[(2\alpha - 1)\frac{\delta h_{rr}}{h_{rr}^{3/2}} - \alpha\frac{h_{rr}\delta h_{tt} + h_{tt}\delta h_{rr}}{h_{rr}^{3/2} h_{tt}}\right]\\
&+ \frac{1}{16\pi G}\int_{\partial M} \mathrm{d}^3x \sqrt{\frac{1}{h_{rr}}}(\alpha - 1)\frac{3}{4}\frac{\partial_z h_{rr} h_{tt}\delta h_{rz} + \partial_t h_{rr}\delta h_{tr}}{h_{rr}^{3/2} h_{tt}}\\
&+ \frac{1}{16\pi G}\int_{\partial^2 M} \mathrm{d}^2x \sqrt{\gamma'}(2\alpha - 1) n^a n'^b \delta g_{ab} + O(1/r).
\end{aligned} \tag{A.13}$$

The terms coming from equations (A.10), (A.11) and (A.12) can be eliminated by demanding

$$\delta(h_{rr}h_{tt}) = \delta h_{rr}h_{tt} + h_{rr}\delta h_{tt} = 0 \tag{A.14}$$

and

$$\partial_t h_{rr} = \partial_z h_{rr} = 0. \tag{A.15}$$

Then (A.13) reduces to

$$\delta\Gamma_{(\alpha)} = \frac{(2\alpha-1)}{16\pi G}\left[\int_{\partial M} \mathrm{d}^3x\sqrt{\frac{1}{h_{rr}}}\frac{\delta h_{rr}}{h_{rr}^{3/2}} + \int_{\partial^2 M} \mathrm{d}^2x\sqrt{\gamma'}n^a n'^b \delta g_{ab}\right] + O(1/r). \tag{A.16}$$

This vanishes (as required for a well defined variational principle) for $\alpha = 1/2$, just as in the three-dimensional case. The action then takes the form

$$\Gamma = -\frac{1}{16\pi G}\int_M \mathrm{d}^4x\sqrt{g}R - \frac{1}{16\pi G}\int_{\partial M}\mathrm{d}^3x\sqrt{\gamma}K. \tag{A.17}$$

B. Flat space asymptotic Killing vectors

B.1. Asymptotic Killing vectors in three dimensions

In order to find the asymptotic Killing vectors corresponding to boundary conditions

$$\begin{aligned}
g_{uu} &= -1 + h_{uu}(\varphi) + O(1/r) & g_{ur} &= -1 + \frac{h_{ur}}{r} + O(1/r^2) \\
g_{u\varphi} &= h_{u\varphi} + O(1/r) & g_{rr} &= \frac{h_{rr}}{r^2} + O(1/r^3) \\
g_{r\varphi} &= h_1(\varphi) + O(1/r) & g_{\varphi\varphi} &= r^2 + \left(h_2(\varphi) + uh_3(\varphi)\right)r + O(1)
\end{aligned} \tag{B.1}$$

one needs to solve for the most general ξ such that

$$\mathcal{L}_\xi g = O(\delta g). \tag{B.2}$$

Using a general ansatz,

$$\xi = \xi_u(u,r,\varphi)\partial_u + \xi_r(u,r,\varphi)\partial_r + \xi_\varphi(u,r,\varphi)\partial_\varphi, \tag{B.3}$$

in order for the ru-, rr- and $r\varphi$-components of the variation $\mathcal{L}_\xi g$ to be of the allowed order,

$$-\partial_r\xi_u - \partial_u\xi_u - \partial_r\xi_r = O(1/r) \tag{B.4}$$
$$-2\partial_r\xi_u = O(1/r^2) \tag{B.5}$$
$$-\partial_\varphi\xi_u + r^2\partial_r\xi_\varphi = O(1) \tag{B.6}$$

one must fix the r-dependencies in ξ as[4]

$$\xi_u(u,r,\varphi) = \tilde{\xi}_u(u,\varphi) + O(1/r) \tag{B.7}$$
$$\xi_r(u,r,\varphi) = -r\partial_u\tilde{\xi}_u(u,\varphi) + O(1) \tag{B.8}$$
$$\xi_\varphi(u,r,\varphi) = \tilde{\xi}_\varphi(u,\varphi) + O(1/r). \tag{B.9}$$

The other components of $\mathcal{L}_\xi g$, together with the boundary conditions (B.1) furthermore determine

$$\xi_u(u,r,\varphi) = \xi_M(\varphi) + u\xi_L'(\varphi) + O(1/r) \tag{B.10}$$
$$\xi_r(u,r,\varphi) = -r\xi_L'(\varphi) + O(1) \tag{B.11}$$
$$\xi_\varphi(u,r,\varphi) = \xi_L(\varphi) - \frac{u}{r}\xi_L''(\varphi) + \frac{1}{r}f_1(\varphi) + O(1/r^2). \tag{B.12}$$

B.2. Asymptotic Killing vectors in four dimensions

We want to solve

$$\mathcal{L}_\xi g = O(\delta g) \tag{B.13}$$

for a metric

$$\begin{aligned}
g_{uu} &= -1 + h_3(u,\varphi) + O(1/r) & g_{ur} &= -1 + \frac{h_{ur}}{r} + O(1/r^2) \\
g_{u\varphi} &= h_{u\varphi} + O(1/r) & g_{uz} &= \frac{h_{uz}}{r} + O(1/r^2) \\
g_{rr} &= \frac{h_{rr}}{r^2} + O(1/r^3) & g_{r\varphi} &= h_1(u,\varphi) + O(1/r) \\
g_{rz} &= \frac{h_4(\varphi)}{r} + O(1/r^2) & g_{\varphi\varphi} &= r^2 + rh_2(u,\varphi) + O(1) \\
g_{\varphi z} &= h_{\varphi z} + O(1/r) & g_{zz} &= 1 + \frac{h_{zz}}{r} + O(1/r^2).
\end{aligned} \tag{B.14}$$

Using the most general ansatz

$$\xi = \xi_u(u,r,\varphi,z)\partial_u + \xi_r(u,r,\varphi,z)\partial_r + \xi_\varphi(u,r,\varphi,z)\partial_\varphi + \xi_z(u,r,\varphi,z)\partial_z, \tag{B.15}$$

we start by looking at variations $\mathcal{L}_\xi\bar{g}$ of a background Minkowski metric $\bar{g}$. The ri components ($i = u, r, \varphi, z$) of this,

$$-\partial_r\xi_u - \partial_u\xi_u - \partial_r\xi_r = O(1/r) \tag{B.16}$$
$$-2\partial_r\xi_u = O(1/r^2) \tag{B.17}$$
$$-\partial_\varphi\xi_u + r^2\partial_r\xi_\varphi = O(1) \tag{B.18}$$
$$-\partial_z\xi_u + \partial_r\xi_z = O(1/r), \tag{B.19}$$

[4]Possible logarithmic terms have been ignored; these could in principle be included but drop out later anyhow.

fix the r-dependencies as

$$\xi_u(u,r,\varphi,z) = \tilde{\xi}_u(u,\varphi,z) + O(1/r) \tag{B.20}$$
$$\xi_r(u,r,\varphi,z) = -r\partial_u\tilde{\xi}_u(u,\varphi,z) + O(1) \tag{B.21}$$
$$\xi_\varphi(u,r,\varphi,z) = \tilde{\xi}_\varphi(u,\varphi,z) + O(1/r) \tag{B.22}$$
$$\xi_z(u,r,\varphi,z) = r\partial_z\tilde{\xi}_u(u,\varphi,z) + O(1). \tag{B.23}$$

Solving also the other constraints furthermore reduces the remaining functions to

$$\xi_u(u,r,\varphi,z) = \xi_M(\varphi) + u\xi_L'(\varphi) + O(1/r) \tag{B.24}$$
$$\xi_r(u,r,\varphi,z) = -r\xi_L'(\varphi) + O(1) \tag{B.25}$$
$$\xi_\varphi(u,r,\varphi,z) = \xi_L(\varphi) - \frac{u}{r}\xi_L''(\varphi) + \frac{1}{r}f_1(\varphi) + O(1/r^2) \tag{B.26}$$
$$\xi_z(u,r,\varphi,z) = \xi_J(\varphi) + O(1/r). \tag{B.27}$$

Plugging this ξ into $\mathcal{L}_\xi g$, one sees that also for the full metric ξ satisfies condition (B.13).

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