Diplomarbeit

Inflation From a Simple Model Based on Weyl Invariance

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Wien, am 8.5.2015 Paul Mezgolits

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Abstract

Despite the many successes of the big bang model of the universe, it lacks the power to explain some key features of the universe including the fact that it is homogeneous, isotropic and spatially flat. A possible explanation of these are given by inflation, a short period before the onset of standard big bang evolution during which the universe expanded exponentially. This thesis starts with an introduction to single field inflation models and explains how they produce primordial fluctuations and then calculates their power spectrum, which can be observed in the cosmic microwave background. Following this discussion, a model based on conformal gravity and conformal coupling of a scalar field is introduced. The tensor and vector perturbations in this model are discussed both classically and quantum mechanically and their power spectra are calculated and are found to share key features with the power spectra of single field inflation.

Contents

1. Introduction

The beginning of the twentieth century saw two developments in physics that would change our understanding of nature. On the one hand quantum mechanics was developed to explain the energy spectrum of black body radiation and was then further developed to describe atoms and the world on small scales in general. The development of quantum mechanics eventually led to the realisation that fields, like the electromagnetic field had to be quantized too. This led to the formulation of quantum field theory which, much like quantum mechanics, has applications in many areas of physics including particle, solid state and statistical physics. Indeed or current understanding of the particles that make up matter is given by the standard model of particle physics, which is formulated in the language of quantum field theory and has so far stood the test of every experiment thrown at it.

On the other hand, special relativity was introduced in 1905 and has challenged our understanding of not only space but time. Time is no longer given by the beat of a universal drum, but a dynamical quantity that passes at different speeds for observers moving relative to each other. Electrodynamics, which can be used to motivate special relativity, is easily incorporated into this new framework and most of Newtonian mechanics can be shown to be the low speed limit of Einstein's 1905 theory. I say most because gravity did not quite fit into the picture and led to the theory of general relativity which is currently our best description of gravity. General relativity quickly led to an explanation of the, until then, anomalous orbit of mercury and introduced the idea of black holes, a region of spacetime from which light cannot escape. The application of general relativity most relevant for this thesis is cosmology.

Applying general relativity to the observable universe as a whole, assuming that the cosmos is homogeneous and isotropic on large scales, then paints the picture of a universe where the distance between free falling observers changes with time. Depending on the details of the energy content of the universe the distance then gets smaller or, what is currently happening in our universe, bigger where the latter is referred to as the expansion of the universe. This expansion has famously been observed by Edwin Hubble, who also discovered the Hubble law, which is the theoretical description of this phenomenon.

It is often said that quantum mechanics and by extension quantum field theory is not compatible with general relativity. Indeed this incompatibility drives a lot of research in an area known as quantum gravity and has brought about such ideas as string theory [1], loop quantum gravity [2] and causal dynamical triangulations [3] among many more ideas of how to tackle this inconsistency. It is however possible to fix a gravitational background and consider how quantum fields behave in the presence of gravity.

The combination of an expanding universe and the standard model of particle physics then led to what we know today as the big bang theory, which describes how the universe and its matter content evolved and still evolves. The big bang explains the abundance of hydrogen and helium and predicted that we should observe radiation coming from all directions describing the spectrum of a black body at roughly $3K$. This radiation which is called the Cosmic Microwave Background (CMB), was indeed discovered by Penzias and Wilson, who were awarded a Nobel prize. The CMB is our main tool to test wether or not the idea of a homogeneous and isotropic background is justified and indeed this holds to a very high precision.

Despite its many successes the big bang theory is not a complete description of our universe. From the observational point of view only 5% of our universes energy content can be described by the standard model of particle physics. 25% seems to be described by matter that only interacts weakly and gravitationally and is therefore called dark matter, while the remaining 70% is called dark energy and while it is described by a constant of the theory of general relativity, the cosmological constant, its very small magnitude remains a mystery. There are two popular ways to address the problem of dark matter. One can either decide that particle physics is missing an ingredient and look for new particles that would account for the missing 25% of energy, or one could argue that general relativity works well on scales the size of our solar systems but needs to be modified for cosmic distances. The second approach is sometimes also invoked to explain the current acceleration of the universe.

From the theoretical side, the initial conditions of the big bang would have had to be very finely tuned to produce our current universe, which is homogeneous, isotropic and spatially flat. This invokes the wish for a causal mechanism that could set these initial conditions for the big bang, without itself having to be finely tuned. Such a mechanism is given by the theory of cosmological inflation first put forward by Guth [4, 5] and later refined by Linde [6].

The first part of this thesis aims to motivate and explain a simple model of inflation. Inflation does not only set the initial fine tuned conditions necessary for the big bang, it also gives rise to inhomogeneities that can be seen as temperature fluctuations in the CMB and seeds the formation of large scale structure in the universe [7]. Here is where the true predictive power of inflation lies. Given an inflationary model properties characterising the resulting deviations from a completely homogeneous and isotropic background radiation can be calculated and compared to observations.

A striking property predicted by many inflationary models is that the perturbations are almost scale invariant. The perturbations can be expanded into modes of different frequency or wavelength, much like the vibrating string of a guitar. Scale invariance then means that all frequencies contribute equally.

Knowing that the temperature fluctuations produced by inflation seem agnostic about scale, one might wonder what perturbations might do in a model that is Weyl invariant and therefore does not know about scales. On the gravity side of things this directly leads to conformal gravity [8, 9]. The second part of this thesis motivates a model built from conformal gravity and the requirement of scale invariance and computes the behaviour of cosmological perturbations and finds, like in simple inflation models, that the perturbations are scale invariant.

2. The Big Bang and its Problems

The big bang model of the universe describes our universe very well. Not only does it describe the expansion of our universe, it also describes how hydrogen and helium were first formed through nucleosynthesis which predicts the abundance of hydrogen and helium in our universe. The big bang also predicts the cosmic microwave background radiation (CMB) which is electromagnetic radiation reaching us from all directions and has the characteristics of a black body at a temperature of roughly $3K$. The details of this model can be found in [10, 11, 12, 13].

While in good agreement with observations, the big bang model has a few short comings. The current homogeneity and flatness of the universe [14] can only be addressed by setting fine tuned initial conditions [5]. This high degree of symmetry also poses a problem for structure formation, as you need some degree of inhomogeneity for things like stars, planets and solar systems to form. This section aims to describe how some of these problems come about and motivate their resolution through the theory of cosmological inflation [5, 6] which will be described in the next section.

2.1. FLRW Spacetime

Our basic model of the observable universe is based on the cosmological principle, the idea that spacetime is spatially homogeneous and isotropic. There is no explanation for the cosmological principle within the big bang theory, but this idea is well supported by observations of the CMB which is homogeneous and isotropic up to fluctuations of the order $\Delta T/T \sim 10^{-5}$. I will first discuss what spacetimes are allowed by these symmetries and some of their properties that will be relevant followed by discussing their dynamics as dictated by the Einstein equations. Let us begin by defining a spatially homogeneous and isotropic spacetime.

Kinematics

A pseudo-riemannian manifold with metric tensor $g_{\mu\nu}$ is spatially homogeneous if there exists a one-parameter family of spacelike hypersurfaces Σ_t that foliate the spacetime such that for each t and $p, q \in \Sigma_t$ there exists an isometry ϕ of $g_{\mu\nu}$ for which $\phi(p) = q$ [11, 15]. Intuitively this translates into the existence of surfaces of a constant timelike parameter t in which the universe still looks the same after walking from p to q. Choosing the parameter t, that labels the spatial hypersurfaces, as a coordinate the line element can be written as $ds^2 = -dt^2 + \gamma_{ij}dx^i dx^j$ where γ_{ij} is the induced metric on Σ_t.

A spacetime is spatially isotropic at each point if there exists a family of timelike curves with tangent vectors u^μ having the property that given a point $p \in \Sigma_t$ and two unit vectors s_1^μ and s_2^μ that are orthogonal to u^μ there exists an isometry ϕ of $g_{\mu\nu}$ that leaves p and u^μ invariant i.e. $\phi(p) = p$ and $(\phi_* u)^\mu = u^\mu$ but transforms s_1^μ into s_2^μ i.e. $(\phi_* s_1)^\mu = s_2^\mu$. Restated less technically this means that for an observer with tangent vector u^μ the universe looks the same in all directions.

For a manifold to be both homogeneous and spatially isotropic at every point the set of all vectors orthogonal to u^μ at a point p must be the tangent space to Σ_t at p. Otherwise an isometry that keeps you at p could take a vector in the tangent space of Σ_t out of it. Isotropy also constrains the curvature. To see this consider the Ricci tensor ${}^3R^i{}_j$ of the induced metric γ_{ij}. This has to be proportional to $\delta^i{}_j$ as otherwise a geometrically preferred direction could be constructed in violation of isotropy. Homogeneity then forces ${}^3R^i{}_j$ to be constant. This puts severe restrictions on the induced metric γ_{ij} and all possible geometries of this kind can be written as

$$\gamma_{ij} dx^i dx^j = a^2(t)\Big(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\Big). \tag{1}$$

The parameter k is a measure of the curvature as the Ricci scalar of such a geometry is ${}^3R = 6k/a^2(t)$. Thus the full line element of a homogeneous and isotropic manifold is given by

$$ds^2 = -dt^2 + a^2(t)\Big(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\Big). \tag{2}$$

All models of this type are collectively referred to as FLRW spacetimes.

The coordinates on the spatial three-slices, r, θ and ϕ are called comoving coordinates. Solving the geodesic equation $\ddot{x}^\mu + \Gamma^\mu{}_{\rho\nu}\dot{x}^\rho \dot{x}^\nu = 0$ gives $x^\mu = (t, c^i)$ where $c^i = const$ i.e. free falling observers remain at their comoving coordinates and they agree on the passage of time between two spatial slices, motivating the name cosmic time for the coordinate t.

Despite free falling observers staying at the same comoving coordinates they move away or towards each other over time depending wether $a(t)$ grows or shrinks. To see this consider two objects of radial coordinates $r = 0$ an $r = \Delta r_{co}$ at time t. Their physical distance is given by the integral over the induced line element in the equal time slice and turns out to give

$$\Delta r_{phys} = a(t)\int_0^{\Delta r_{co}} dr = a(t)\Delta r_{co} \tag{3}$$

making the distance between two free falling observers in an expanding universe increase with time. The speed at which two objects move away from each other is given by

$$v = \frac{d}{dt}\Delta r_{phys} = \dot{a}\Delta r_{co} = H\Delta r_{phys} \tag{4}$$

where $H = \dot{a}/a$ is the Hubble parameter. This is Hubble's law and it tells us that the further objects are away from each other the faster they move away from one another.

Note that so far all differentiation with respect to cosmic time has been denoted with a dot.

It is often useful to transform the time coordinate according to $a(t)dt = d\tau$ where the coordinate τ is called conformal time as in this coordinate the metric can be written as

$$ds^2 = a^2(\tau)\Big(- d\tau^2 + \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin\theta d\phi^2\Big). \tag{5}$$

A flat FLRW metric, where $k = 0$, can for instance be written as the Minkowski metric times a conformal factor $g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$. Along with conformal time come a few conventions. Differentiation with respect to conformal time is denoted with a dash i.e. the derivative of some quantity f with respect to conformal time is written as

$$f' = \frac{df}{d\tau} \tag{6}$$

and it is useful to introduce the conformal Hubble parameter

$$\mathcal{H} = \frac{a'}{a} = \dot{a}. \tag{7}$$

Dynamics

So far I have only described the kinematics of FLRW spacetimes but left the scale factor $a(t)$ undetermined. To find the scale factor one plugs the FLRW ansatz into the Einstein equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{8}$$

where the energy content of the universe sill needs to be specified. On large scales the matter in the universe behaves as a perfect fluid with Energy-Momentum tensor

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 \\ 0 & P\gamma_{ij} \end{pmatrix} \tag{9}$$

where ρ is the energy density measured by the free falling observers and P is the pressure of the fluid. As long as $\rho = \rho(t)$ and $P = P(t)$ this tensor is compatible with the symmetries of the spacetime and the Einstein equations produce the Friedmann equations

$$\frac{\dot{a}^2}{a^2} = -\frac{k}{a^2} + \frac{8\pi}{3}\rho + \frac{\Lambda}{3} \tag{10}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P) + \frac{\Lambda}{3}. \tag{11}$$

Differentiating the first Friedmann equation (10) and then using the second Friedmann equation (11) to replace the terms with second derivatives of the scale factor $\ddot{a}$ produces the continuity equation

$$\dot{\rho} + 3H(\rho + P) = 0 \tag{12}$$

which can also be obtained from the conservation equation $\nabla^\mu T_{\mu\nu} = 0$. The evolution of the scale factor, and therefore of the universe, depends on its matter content and the curvature of the surfaces of constant time. For the rest of this section I will set the curvature of the three-slices to $k = 0$ and solve the Friedmann equations (10) and (11) for matter, radiation and dark energy dominated universes.

Matter and Radiation Dominated Universes

For the time being I set the cosmological constant $\Lambda = 0$. With the equation of state $P = \omega\rho$, the continuity equation (12) can be written as

$$\frac{d}{dt}\ln\rho = -3(1+\omega)\frac{d}{dt}\ln a. \tag{13}$$

Integrating this equation reveals that the energy density scales as

$$\rho = \rho_0 a^{-3(1+\omega)}. \tag{14}$$

Using this expression for the energy density the first Friedmann equation (10) becomes

$$\dot{a}\, a^{1+3\omega} = const \tag{15}$$

and is solved by

$$a(t) = \left(\frac{t}{t_o}\right)^{2/n} \tag{16}$$

where $n = 3(1+\omega)$ and $a(t_o) = 1$ was chosen as an initial condition. The time t_o is to be understood as the time at which some observation was made and could for example be the current time.

So far this is true for perfect fluids with an equation of state $P = \omega\rho$. To make this specific to matter consider a cube with sides of comoving length L. If this cube is filled with matter of energy density ρ_m the total energy contained in the cube is $E = \rho_m a^3(t) L^3$. As the expansion of the universe does not create matter, the energy has to be constant from which follows that the energy density scales as $\rho \propto a^{-3}$. This translates into $\omega = 0$ and $n = 3$ making the scale factor for a matter dominated universe

$$\text{Matter Dominated Universe:} \quad a(t) = \left(\frac{t}{t_o}\right)^{2/3} \tag{17}$$

To describe radiation first note that a single photon of wavelength λ has the energy $E_\lambda = 2\pi/\lambda$. As the wave length will scale as $a(t)$ the energy of a photon will scale as $a^{-1}(t)$. Therefore the total energy of a cube with sides of comoving length L, containing photons $E = \rho_r a^3(t) L^3$ will also scale as $a^{-1}(t)$. In order to produce this behaviour the energy density has to behave as $\rho_r \propto a^{-4}$ making $\omega = 1/3$ and $n = 4$. A universe filled with radiation then evolves according to

$$\text{Radiation Dominated Universe:} \quad a(t) = \left(\frac{t}{t_o}\right)^{1/2}. \tag{18}$$

Dark Energy Dominated Universes

Consider a universe with a cosmological constant but void of radiation and matter. In such a universe the Friedmann equations reduce to

$$\frac{\dot{a}^2}{a^2} = \frac{\Lambda}{3} \tag{19}$$

which is solved by

$$a(t) = \exp\left(\sqrt{\frac{\Lambda}{3}}(t - t_o)\right). \tag{20}$$

It is worth noting that for such a universe the Hubble parameter is constant in time and given by $H = \sqrt{\Lambda/3}$. The line element for this spacetime is

$$ds^2 = -dt^2 + \exp\left(H(t - t_o)\right) d\vec{x}^2. \tag{21}$$

If the cosmological constant is greater than zero this describes de Sitter spacetime, which while spatially flat, is a space for which the curvature of the full four-dimensional geometry is positive and constant. To see this take the trace of the Einstein equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{22}$$

to find $R = 4\Lambda$.

Our Universe

The universe we live in contains radiation, matter and dark energy described by a cosmological constant. To gain some insight into the evolution of our universe it is useful to rewrite the first Friedmann equation (10) by dividing it by the square of the Hubble parameter to obtain the reduced Friedmann equation

$$1 = \Omega_r + \Omega_m + \Omega_\Lambda \tag{23}$$

where

$$\Omega_r = \frac{8\pi}{3H^2}\rho_r, \quad \Omega_m = \frac{8\pi}{3H^2}\rho_m, \quad \Omega_\Lambda = \frac{\Lambda}{3H^2} \tag{24}$$

are called density parameters. Given the density parameters Ω_{ro}, Ω_{mo}, $\Omega_{\Lambda o}$ and the Hubble parameter H_o observed today the equation for the evolution of the scale factor is

$$\frac{H^2}{H_o^2} = \frac{\Omega_{r0}}{a^4} + \frac{\Omega_{m0}}{a^3} + \Omega_{\Lambda 0}. \tag{25}$$

Qualitatively this equations makes it clear that no matter how much radiation is present it will decay in an expanding universe. Matter will also decay but slower than radiation. Eventually the evolution of the scale factor will only be dictated by dark energy.

For our universe the current density and Hubble parameters are roughly [14]

$$H_o \sim 2 \times 10^{18} s^{-1},\ \Omega_{ro} \sim 10^{-4},\ \Omega_{mo} \sim 0.3079,\ \Omega_{\Lambda o} \sim 0.692. \tag{26}$$

This makes the age of the universe $t_o \sim 4 \times 10^{17} s$. After an initial phase in which the universe was dominated by radiation the density parameters for matter and radiation become equal at $t_m \sim 1.5 \times 10^{12} s$ starting the era of matter domination. At $t_\Lambda \sim 3 \times 10^{17} s$ the matter and dark energy density parameters became equal starting the current epoch of the universe which is dominated by dark energy.

2.2. Problems of the Big Bang

2.2.1. Horizon Problem

The horizon problem describes how the homogeneity and isotropy of the universe can not be a product of a causal mechanism in the frame work of the big bang model. A strong indication that our universe has the symmetry properties required by the cosmological principle comes from the CMB which up to fluctuations of order $\Delta T/T \sim 10^{-5}$ is isotropic.

The CMB originates at $t_{rec} \sim 10^{13} s$ where what is known as recombination took place. Before recombination the photons, electrons and protons formed a hot plasma that was opaque as the photons were Thomson scattered of the free electrons causing a very short mean free path. At recombination the plasma cooled down enough to enable the protons and electrons to bind forming hydrogen atoms. With the electrons bound the universe becomes transparent as photons are no longer scattered. The photons that became free to roam the universe at recombination are what we see as the CMB today. Details of recombination can be found in [10, 11, 12].

As light is described by null vectors the equation that describes their path are obtained by setting the line element zero

$$0 = -dt^2 + a^2(t) dr^2. \tag{27}$$

A photon that reaches us today and originated at t_{rec} therefore has travelled a comoving distance of

$$r_{hom} = \int_{t_{rec}}^{t_o} \frac{dt}{a(t)} = \frac{n}{n-2} t_o \Big(1 - \Big(\frac{t_{rec}}{t_o}\Big)^{(n-2)/n}\Big) \tag{28}$$

where the scale factor of a perfect fluid with equation of state $P = \omega\rho$ was assumed. This comoving radial distance is the scale on which we know the universe to have been homogeneous and isotropic at recombination.

This also gives us a tool to probe causality. The distance a photon could have travelled from $t = 0$ to t_{rec}

$$r_{caus} = \int_0^{t_{rec}} \frac{dt}{a(t)} = \frac{n}{n-2} t_o^{2/n}\ t_{rec}^{(n-2)/n} \tag{29}$$

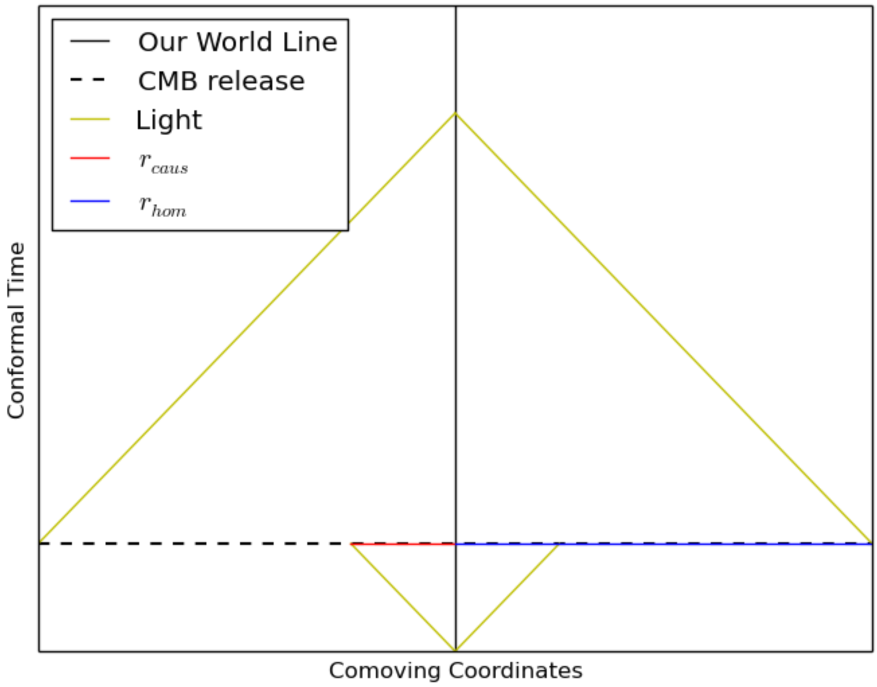

Figure 1: Depiction of the horizon problem. r_{caus} is the comoving scale on which the universe could have been made homogeneous by a causal mechanism, while r_{hom} is the comoving scale on which the universe is homogeneous

is the comoving radial distance on which a causal mechanism could cause the universe to be homogeneous. As $t_o \gg t_{rec}$ in both matter and radiation dominated universes the scale on which the universe was already homogeneous at t_{rec} is far bigger than the scale on which a causal mechanism could have caused this.

This idea can best be visualized in conformal time where the metric becomes $ds^2 = a^2(\tau)(-d\tau^2 + dr^2)$ and light thus obeys $0 = -d\tau^2 + dr^2$, making light straight lines when plotting conformal time against comoving coordinates. Figure 1 depicts the discrepancy between the comoving scales on which the universe is homogeneous versus those on which it could have been made homogeneous by a causal mechanism.

2.2.2. Flatness Problem

To understand the flatness problem turn the spatial curvature back on. Summing the density parameters for radiation, matter and dark energy up into one $\Omega = \Omega_r + \Omega_m + \Omega_\Lambda$, the reduced Friedmann equations (23) becomes

$$1 - \Omega = -\frac{k}{(aH)^2}. \tag{30}$$

In the early universe the Hubble radius $(aH)^{-1}$ is given by

$$\frac{1}{(aH)^2} = \frac{n^2}{4} t_o^{4/n} t^{2(n-2)/n} \tag{31}$$

making it grow as long as $2 < n$. This holds for both the radiation and matter dominated era. From the Planck mission [14] it is known that today the curvature of the universe is at most $1 - \Omega = 0 \pm 5 \times 10^{-3}$. Considering equation (30) in a matter or radiation dominated universe the presence of any initial non-zero spatial curvature would grow with time. To get an idea how finely tuned the universe would have had to have been initially to produce our current universe let's extrapolate back to the Planck time.

To get an estimate on the flatness of the universe at the Planck time t_P it is sufficient to treat the radiation, matter and dark energy dominated epoch as if they evolved according to (17), (18) and (21) respectively. Equation (30) holds for all times. This allows k to be expressed through $1 - \Omega$ at an earlier time. Take $t_1 < t_2$ then

$$1 - \Omega(t_1) = \left(\frac{a(t_2)H(t_2)}{a(t_1)H(t_1)}\right)^2 (1 - \Omega(t_2)). \tag{32}$$

Using this formula to extrapolate from today back to the Planck time through first the dark energy, then matter and then radiation dominated era one finds that the universes curvature had to be very small initially,

$$|1 - \Omega(t_P)| < 10^{-61}. \tag{33}$$

This constitutes a very fine tuned initial condition for the big bang model.

2.2.3. Further Problems of the Big Bang Model

While I will not go into detail it is worth mentioning the monopole problem, one of the original motivations for studying inflation. When introducing a grand unified theory into the picture of the standard big bang model the theory predicts an abundance of particles e.g magnetic monopoles or gravitinos that are not observed today. An inflationary period allows to get rid of these relics [4, 5].

The big bang model describes a homogeneous universe and can therefore not address how structure in the universe is formed making the theory incomplete. The formation of structure can be explained by perturbations of the homogeneous and isotropic spacetime, where the perturbations produced during inflation seed these fluctuations in the matter distribution. Thereby inflation solves this problem too and details of the process of structure formation can for example be found in [7].

The cosmological principle, the idea that the universe is homogeneous and isotropic on large scales, can be understood as an initial condition for the big bang. Given that there is just one universe to observe, setting two of its key features as initial conditions seems somewhat unsatisfying. This also is taken care of by a period of inflation. An era of accelerated expansion of the universe can be understood as setting the initial values for the big bang that create our current universe.

2.3. Solution to the Horizon and Flatness Problems

Both the horizon and flatness problem can be solved by the same mechanism, a shrinking Hubble radius $(aH)^{-1}$. This can be rewritten as

$$\frac{d}{dt}(aH)^{-1} < 0 \Longleftrightarrow \ddot{a} > 0. \tag{34}$$

A period in which the universe under goes accelerated expansion is called inflation and will be discussed in the next chapter. For now I will describe how a shrinking Hubble radius solves the big bang problems.

In case of the flatness problem it is easy to see that this is a solution to the problem as if $(aH)^{-1}$ decreases $|1 - \Omega|$ comes closer to zero. After a sufficiently long period in which the Hubble radius shrinks the universe can go back into the evolution described so far as it will be flat enough to eventually produce the universe we observe today.

To see how a shrinking Hubble radius solves the horizon problem note that the differential dt can be rewritten using the Hubble parameter. As $H = \dot{a}/a$ and $da = \dot{a}dt = aHdt$, the comoving distance traveled by light from time t_1 to time t_2 is given by

$$r = \int_{t_1}^{t_2} \frac{da}{a^2 H} = \int_{t_1}^{t_2} \frac{d\ln a}{aH}. \tag{35}$$

If the universe evolves such that $(aH)^{-1}$ decreases with time then early times will contribute more to the integral (35) increasing the scale on which a causal mechanism can cause the universe to be homoheneous.

While the accelerated expansion of the universe solves the big bang problems in principle, the period of inflation will not get rid of the fine tuning unless it lasts long enough. The length of inflation is given in number of e-folds N. In one e-fold the scale factor $a(t)$ grows by a factor of e, where e is Euler's number.

To obtain the number of e-folds that are needed to solve the flatness problem assume that inflation started when the scale factor had the value a_i and grew by N factors of e to reach $a_f = e^N a_i$ by the end of inflation. The curvature parameter will then change from the beginning to the end of inflation according to

$$1 - \Omega(t_i) = \left(\frac{a_f H_f}{a_i H_i}\right)^2 (1 - \Omega(t_f)). \tag{36}$$

For simplicity assume that inflation is described by de Sitter space. The Hubble parameter is therefore constant meaning that $H_i = H_f$ which combined with $a_f = e^N a_i$ gives the evolution of the curvature parameter as

$$\frac{1 - \Omega_i}{1 - \Omega_f} = e^{2N}. \tag{37}$$

Suppose that the initial curvature parameter is of order one $1-\Omega_i \sim 1$ and that after inflation the universes curvature is given by $1-\Omega_f \sim 10^{-61}$ then the number of e-folds needed is

$$70 \lesssim N. \tag{38}$$

Similar results can be obtained from the Horizon problem though here one needs to assume a time at which inflation ended. The values found in the literature for the number of e-folds required for inflation to solve the big bang problems are usually given by $50-70 \lesssim N$ [11].

2.4. Particle Horizons and Hubble Radius

The Hubble radius, given by $r_H(t) = (a(t)H(t))^{-1}$, is of importance in both formulating and solving the big bang problems and their solutions, and is related to the idea of a particle horizon. A particle horizon has, without being explicitly named, already been used in the formulation of the horizon problem.

Consider an event at time t_1 and an observer at t_2 in an FLRW spacetime. A photon that reaches the observer at time t_2 must have been a comoving radial distance

$$r_{PH}(t_2, t_1) = \int_{t_1}^{t2} \frac{dt}{a(t)} \tag{39}$$

away from the observer at time t_1. The observer at time t_2 can only know about the event time t_1 if it took place at a comoving distance $r < r_{PH}(t_2, t_1)$. This defines the particle horizon $r_{PH}(t_2, t_1)$, the line that separates the events an observer at t_2 can know about from those the observer can not know about if the events took place at time t_1.

In a universe filled with a perfect fluid, where $a(t) = (t/t_2)^{2/n}$, the particle horizon is found at

$$r_{PH}(t_2, t_1) = \frac{n\, t_2}{n-2}\left(1 - \left(\frac{t_1}{t_2}\right)^{(n-2)/n}\right). \tag{40}$$

In such a universe the Hubble parameter is $H(t) = 2(n\, t)^{-1}$ making the Hubble radius $r_H(t) = (a(t)H(t))^{-1}$

$$r_H(t) = \frac{n}{2}\, t_2^{2/n}\, t^{1-2/n}. \tag{41}$$

Therefore if $t_1 \ll t_2$ the particle horizon and Hubble radius are related by

$$r_{PH}(t_2, t_1) \simeq \frac{2}{n-2} r_H(t_2) \tag{42}$$

i.e. the Hubble radius is of the same order of magnitude as the particle horizon.

This is no longer true in a de Sitter universe. For simplicity normalize the scale factor to $a(t) = \exp(Ht)$ and find the particle horizon at

$$r_{PH}(t_2, t_1) = \frac{\exp(-Ht_2)}{H}\Big(\exp(H(t_2 - t_1)) - 1\Big) \tag{43}$$

and the Hubble radius at

$$r_H(t) = \frac{\exp(-H\,t)}{H}. \tag{44}$$

The relation between the two is therefore

$$r_{PH}(t_2, t_1) = r_H(t_2)\Big(\exp\left(H(t_2 - t_1)\right) - 1\Big) \tag{45}$$

making the particle horizon a lot bigger than the Hubble radius for $t_1 \ll t_2$. While in a universe dominated by matter or radiation the Hubble radius is related to the causal structure this is no longer true in an exponentially expanding universe.

3. Inflation in a Homogeneous Universe

An inflationary phase, in which the universe undergoes accelerated expansion, can be constructed in a number of ways [16]. Two popular ways to construct exponentially expanding universes are to add a scalar field to the Einstein-Hilbert action that drives the expansion, or change the gravity theory from Einstein gravity to some alternative theory [17] among which are the popular $f(R)$ theories [18]. In this chapter the two approaches will be explored to both explain inflation and in part motivate the model that will be constructed later using conformal gravity [9].

3.1. Single Field Inflation

3.1.1. Cosmology with a Klein-Gordon Field

A simple model of inflation can be obtained by adding the action of a Klein-Gordon field to the Einstein-Hilbert action

$$S = \int \omega_g \Big(\frac{R}{16\pi} - \frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi - V(\varphi)\Big). \tag{46}$$

Varying this action with respect to the inverse metric $g^{\mu\nu}$ gives the Einstein equations where the energy-momentum tensor of the scalar field is

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}\,\partial_\rho\varphi\,\partial^\rho\varphi - g_{\mu\nu}V(\varphi). \tag{47}$$

As long as the spacetime is an unperturbed FLRW spacetime the scalar field can only depend on time and upon comparing the energy momentum tensor of the scalar field (47) with that of a perfect fluid (9) the energy density and pressure of the Klein-Gordon field are found to be

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V, \quad P = \frac{1}{2}\dot{\varphi}^2 - V. \tag{48}$$

Knowing this, the Friedmann equations become

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}\left(V + \frac{1}{2}\dot{\varphi}^2\right) \tag{49}$$
$$\frac{\ddot{a}}{a} = \frac{8\pi}{3}\left(V - \dot{\varphi}^2\right) \tag{50}$$

and the continuity equation turns out to give the Klein-Gordon equation

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0. \tag{51}$$

Note that in the limit $\varphi \to const.$ the kinematical term in (46) will vanish and the potential term becomes constant and takes the role of a cosmological constant. In the absence of matter and spatial curvature, but with a cosmological constant present, de Sitter space is the only solution an FLRW ansatz will allow. Therefore the limit of constant scalar field in (46) gives the de Sitter limit of this model.

3.1.2. Slow-Roll Parameters

So far, (49)-(51) are the general equations that determine the evolution of the universe in the presence of a Klein-Gordon field. As inflation is given when $\ddot{a} > 0$ the universe undergoes accelerating expansion when the potential is bigger than the derivative of the scalar field squared $V \gg \dot{\varphi}^2$. Wether or not this condition is satisfied can be encoded in a purely geometrical quantity given by

$$\varepsilon = -\frac{\dot{H}}{H^2}. \tag{52}$$

To see this note that with $\dot{H} = -4\pi\dot{\varphi}^2$ the condition $V \gg \dot{\varphi}^2$ can be rewritten as

$$\begin{aligned} V &\gg \dot{\varphi}^2 \\ V + \frac{1}{2}\dot{\varphi}^2 &\gg \frac{3}{2}\dot{\varphi}^2 \\ \frac{1}{8\pi}H^2 &\gg \frac{1}{2}\dot{\varphi}^2 \\ 1 &\gg \frac{4\pi\dot{\varphi}^2}{H^2} = \varepsilon. \end{aligned} \tag{53}$$

As previously discussed too brief a period of inflation will not solve the big bang problems and the condition $V \gg \dot{\varphi}^2$ has to be supplemented. In order for ε to be small sufficiently long the change of $\dot{\varphi}^2$ should be slow. This can be achieved by requiring $\ddot{\varphi} \ll 3H\dot{\varphi}$

making the change in $\dot{\varphi}$ smaller than its value. Again this condition can be expressed through geometrical quantities, namely ε. Differentiate ε with respect to time to find

$$\begin{aligned}\dot{\varepsilon} &= \frac{2}{\dot{\varphi}}\Big(4\pi\frac{\dot{\varphi}^2}{H^2}\Big)\ddot{\varphi} + 2H\Big(4\pi\frac{\dot{\varphi}^2}{H^2}\Big)\Big(-\frac{\dot{H}}{H^2}\Big) \\ &= 2H\,\varepsilon\frac{\ddot{\varphi}}{H\dot{\varphi}} + 2H\varepsilon^2. \end{aligned} \tag{54}$$

Define δ as

$$\delta = \varepsilon - \frac{\dot{\varepsilon}}{2H\varepsilon}. \tag{55}$$

From (54) it is obvious that $\delta = -\ddot{\varphi}/H\dot{\varphi}$ and therefore if $|\delta| \ll 1$ the condition $|\ddot{\varphi}| \ll 3|H|$ required to make inflation last long is satisfied. This describes a broad class of inflationary models summarized under the name slow-roll inflation as in these, the scalar field φ slowly rolls from its original value at the beginning of inflation to its minimum.

The parameters ε and δ are therefore called the slow roll parameters and inflation persists while

$$\varepsilon \ll 1, \quad |\delta| \ll 1 \tag{56}$$

and its end is marked by $\varepsilon \simeq 1$ and $|\delta| \simeq 1$.

The slow-roll parameter ε comes in handy when computing the number of e-folds inflation lasts. In the infinitesimal time from t to $t+dt$ the scale factor grows by $a(t+dt) = a(t)e^{dN}$. The number of e-folds are therefore given by

$$dN = \ln\frac{a(t+dt)}{a(t)} = \frac{\dot{a}(t)}{a(t)}dt + \mathcal{O}(dt^2) \tag{57}$$

making the number of e-folds from some initial time t_i to t_f

$$N = \int_{t_i}^{t_f} dt\,H(t) = \int_{\varphi_i}^{\varphi_f} d\varphi\,\frac{H(t)}{\dot{\varphi}(t)} = -\sqrt{4\,\pi}\int_{\varphi_i}^{\varphi_f} d\varphi\,\frac{1}{\sqrt{\varepsilon}}. \tag{58}$$

Here $\varphi_i = \varphi(t_i)$, $\varphi_f = \varphi(t_f)$ and the last equality follows from $\varepsilon = 4\pi\dot{\varphi}^2/H^2$. Note that as ε increases with time, $\varepsilon^{-1/2}$ will decrease so that when taking the square root of $H^2/\dot{\varphi}^2$ one has to take the negative branch to obtain the physically sensible result of a positive number of e-folds N.

3.1.3. Slow-Roll Approximation and Expansion

During a period of slow-roll inflation an approximation scheme called slow-roll approximation can be used to solve the Friedmann and Klein-Gordon equations (49), (51). The approximation is achieved by simply discarding the $\dot{\varphi}^2$ term in (49) and the $\ddot{\varphi}$ term in (51), as they are by assumption small anyhow, to give the equations

$$H^2 = \frac{8\pi}{3}V, \qquad 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0. \tag{59}$$

These equations can be used to rewrite ε and δ in terms of the potential making them easier to calculate.

Recall from (53) that $\varepsilon = 4\pi\dot{\varphi}^2/H^2$. The slow-roll approximated Klein-Gordon equation makes it possible for $\dot{\varphi}$ to be expressed trough the potential and the Hubble parameter to give $\varepsilon = \frac{4\pi}{9H^2}\left(\frac{dV}{d\varphi}\right)^2$ which, using the approximated Friedmann equation yields

$$\varepsilon = \frac{1}{16\pi}\left(\frac{1}{V}\frac{dV}{d\varphi}\right)^2. \tag{60}$$

To express the other slow-roll parameter $\delta = -\ddot{\varphi}/H\dot{\varphi}$ trough the potential differentiate the approximated Klein-Gordon equation with respect to cosmic time to find an expression for $\ddot{\varphi}$. A short calculation then shows

$$\delta = \frac{1}{8\pi}\left(\frac{1}{V}\frac{d^2V}{d\varphi^2}\right) - \varepsilon = \frac{1}{16\pi}\left(\frac{2}{V}\frac{d^2V}{d\varphi^2} - \left(\frac{1}{V}\frac{dV}{d\varphi}\right)^2\right). \tag{61}$$

The slow-roll parameters prove useful to describe various processes during slow-roll inflation without having to specify a potential in a scheme called slow-roll expansion. The slow roll parameters expressed in conformal time are

$$\varepsilon = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2}, \qquad \delta = 1 - \frac{\varphi''}{\mathcal{H}\varphi'} = \varepsilon - \frac{\varepsilon'}{2\mathcal{H}\varepsilon}. \tag{62}$$

ε can be expressed as

$$\frac{d}{d\tau}\left(\frac{1}{\mathcal{H}}\right) = \varepsilon - 1. \tag{63}$$

The change of ε during inflation is given by $\varepsilon' = 2H\varepsilon(\varepsilon - \delta)/a$ and is therefore expected to be small making it a safe assumption that $\int d\tau\, \varepsilon = \tau\varepsilon$ for times when the slow-roll approximation is valid. Equation (63) can then be integrated to give

$$\mathcal{H} = \frac{1}{\tau\,(\varepsilon - 1)}. \tag{64}$$

Expanding this to first order in ε gives the simple expression

$$\mathcal{H} = -\frac{1}{\tau}(1 + \varepsilon) + \mathcal{O}(\varepsilon^2) \tag{65}$$

which will later be used to express results that hold for all models of slow-roll inflation regardless of the potential.

3.1.4. An Example of Slow-Roll Inflation

As a concrete example consider the potential

$$V(\varphi) = \lambda\,\varphi^n \tag{66}$$

where λ and n are constants. By virtue of (60) and (61) the slow roll parameters for this potential are given by

$$\varepsilon = \frac{n^2}{16\,\pi\,\varphi^2}, \qquad \delta = \frac{n(n-2)}{16\,\pi\,\varphi^2} \tag{67}$$

and inflation ends when the field φ becomes smaller than the value $\varphi_f = \frac{n}{4\sqrt{\pi}}$ making, with the aid of (58), the number of e-folds inflation lasts

$$N = \frac{4\pi}{n}\varphi_i^2 - \frac{n}{4}. \tag{68}$$

Given a minimum number of e-folds N_{MIN} inflation is to last, the initial field value has to satisfy

$$\varphi_i > \sqrt{\frac{n\,N_{MIN}}{4\pi} + \frac{n^2}{16\,\pi}} \tag{69}$$

though one should keep in mind the argument found in [19] that the potential $V(\varphi)$ should always stay low enough to keep the model outside the energy regime, $V(\varphi) < M_P^4$, where quantum gravity is expected to become relevant and the classical description of spacetime used here breaks down.

With the potential (66) the Friedmann and Klein-Gordon equation in the slow-roll approximation read

$$H^2 = \lambda\,\varphi^n, \qquad 3H\dot{\varphi} + \frac{3\lambda}{8\pi}\,n\,\varphi^{n-1} = 0. \tag{70}$$

Plugging the first of these equations into the second gives an equation in just φ, and this in turn can be plugged into the first equation to turn it solvable in terms of φ. The equations now are

$$\frac{\dot{a}}{a} = -\frac{8\pi}{n}\dot{\varphi}\,\varphi, \qquad \dot{\varphi} + \frac{\sqrt{\lambda}}{8\pi}\,n\,\varphi^{n/2-1} = 0. \tag{71}$$

The first is easily integrated to give

$$a(t) = a_i\,\exp\left(\frac{4\pi}{n}(\varphi_i^2 - \varphi^2(t))\right) \tag{72}$$

where a_i and φ_i are the values of $a(t)$ and $\varphi(t)$ at values at the time t_i at the beginning of inflation. To solve the second equation in (71) introduce a field $\widetilde{\varphi}$ according to $\widetilde{\varphi} = \varphi^{2-2/n}$. In terms of this field the equation is given by $\dot{\widetilde{\varphi}} + \frac{\sqrt{\lambda}\,n\,(4-n)}{16\,\pi} = 0$. After a simple integration the result can be transformed back to the original field φ to give

$$\varphi(t) = \varphi_i\Big(1 - \varphi_i^{(n-4)/2}\frac{\sqrt{\lambda}\,n\,(4-n)}{16\,\pi}t\Big)^{2/(4-n)}. \tag{73}$$

To visualize the solutions in a plot without having to specify the constant λ or decide on initial values φ_i and a_i, rescaled functions and time variables $a_R(t_R) = a(t)\exp\left(-4\pi\varphi_i^2/n/a_i\right)$, $\varphi_R(t_R) = \varphi(t)/\varphi_i$ and $t_R = \varphi_i^{(n-4)/2}\sqrt{\lambda}\,n\,(4-n)t/16\,\pi$ are used and the plot for $n = 2$ is

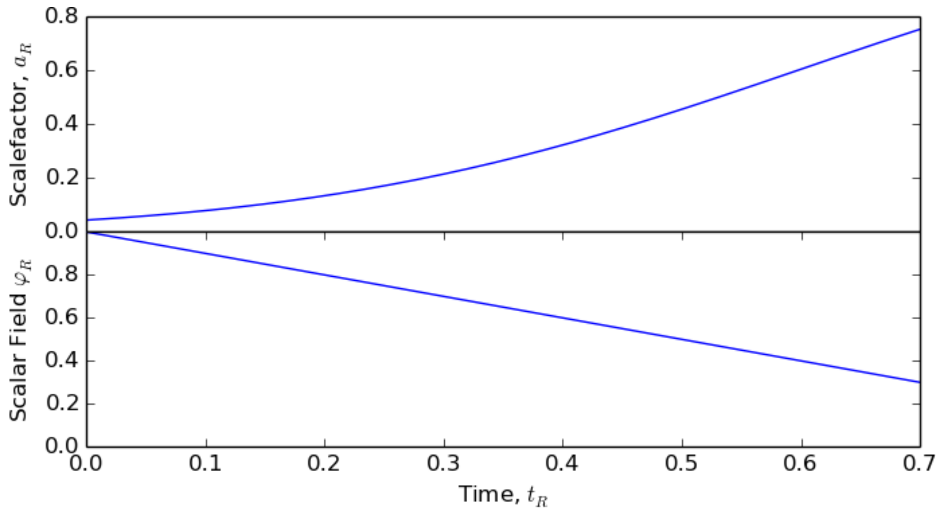

Figure 2: The scale factor and Klein-Gordon field in a potential $V \propto \varphi^2$ during inflation.

given in figure 2. This graph shows the general property of slow-roll inflation where the inflaton decreases while the scale factor grows exponentially. The plot is cut off at the value for which $\ddot{a}_R(t_f) = 0$, the point at which inflation stops. This is also in the general vicinity in which the slow-roll approximation will no longer be valid and a solution of the full Friedmann and Klein-Gordon equations (49) and (51) would be required to investigate the effects the presence of a Klein-Gordon field would have on the expansion of the universe.

3.2. Inflation from Alternative Gravity Theories

The study of theories of gravity different from Einstein gravity has a long history starting with Hermann Weyl who tried to unify GR and electromagnetism in a geometric theory that does not know about length. There are plenty of motivations to study alternatives to Einstein gravity. From a theoretical point of view Einstein gravity is non renormalizable and renormalizable theories of gravity can be constructed by adding terms to the Einstein Hilbert action that introduce derivatives of the metric that are higher than order two [20]. Also, low energy limits of string theory produce effective actions for gravity that are not necessarily pure Einstein Gravity [21].

From an observational point of view, GR agrees with observations made in our solar system. On larger scales however, there are deviations from what can be explained through a combination of GR and the standard model of particle physics. Recall from the discussion that the energy content of our universe is currently made up of around 70% dark energy and 30% matter. However only around 5% of the energy is described by the standard model leaving the remaining 25% of energy attributed to matter unexplained. This is referred to as dark matter. While it is a popular idea that dark matter might be described by the lightest super-symmetric particle [22] or axions [23], it is also a motivation to study alternative gravity theories, as what we see in the framework of Einstein gravity as an unexplained source of energy might be large scale deviations from Einstein gravity. On top of all this, non Einstein gravity is a rich environment for building models of inflation.

A popular example for building models of inflation are $f(R)$ theories, which are theories with actions of the form

$$S_{f(R)} = \int \omega_g \, f(R) \tag{74}$$

where $f(R)$ is some function of the Ricci scalar. This theory is conformally related to Einstein gravity with a Klein-Gordon field. To see this first note that the action can equivalently be written with an auxiliary field χ as

$$S_\chi = \int \omega_g \left(f'(\chi)R - V(\chi)\right) \tag{75}$$

where $V(\chi) = \chi\, f'(\chi) - f(\chi)$. The equations of motion for χ, obtained by setting the variation with respect to the auxiliary field zero, given by

$$0 = \delta_\chi S_\chi = \int \omega_g \, f''(\chi)\big(R - \chi\big)\delta\chi \tag{76}$$

reveal that for $f''(\chi) \neq 0$ the field χ is equal to the ricci scalar, $\chi = R$. Going on shell in the auxiliary field action then reproduces the $f(R)$ action, $S_\chi|_{\chi=R} = S_{f(R)}$.

So far this has all taken place in what is referred to as the Jordan frame, which is given by the metric for which the action takes the form (74). This theory can equivalently be written in the Einstein frame, into which one can change by the conformal transformation $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$. In the Einstein frame (75) is given by

$$S_\varphi = \int \omega_{\widetilde{g}} \Big(\widetilde{R} - \frac{1}{2}\, \widetilde{g}^{\mu\nu} \, \widetilde{\nabla}_\mu \varphi \widetilde{\nabla}_\nu \varphi - U(\varphi)\Big). \tag{77}$$

where $\widetilde{g}_{\mu\nu} = f'(\chi(\varphi))\, g_{\mu\nu}$, $\varphi = \sqrt{3} \ln f'(\chi)$ and $U(\varphi) = V(\chi(\varphi))/\big(f'(\chi(\varphi))\big)^2$. From appendix D of [15] one finds

$$\widetilde{R} = \frac{1}{f'(\chi)}\Big(R - \frac{6\, g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi \sqrt{f'(\chi)}}{\sqrt{f'(\chi)}}\Big). \tag{78}$$

Plugging all of these expressions into (77) recovers (75) up to a boundary term that can, for purposes of the bulk equations of motion, be discarded, thereby showing the equivalence of the actions $S_{f(R)}$ and S_χ.

In the Einstein frame the action (77) is of the form of the action for single field inflation (46). An inflationary model is then obtained by solving the Friedmann and Klein-Gordon equations in the Einstein frame and then transforming back into the Jordan frame in which the theory was originally formulated.

In a later section I will discuss a theory of gravity known as conformal or Weyl gravity and its implications for inflation. Before I do so, let us make the universe more interesting (and complicated) by introducing perturbations.

4. Cosmological Perturbation Theory

So far I have treated inflation and indeed all of cosmology as purely concerned with the evolution of a homogeneous spacetime. However while the universe is very homogeneous on large scales we know from simply observing our solar system that it cannot be completely homogeneous. Indeed the true predictive power of inflation lies not within its homogeneous limit presented above, but as the source of inhomogeneities that are the seeds of structure formation and leave an imprint on the CMB. This chapter is concerned with the formalism used to describe these fluctuations in both matter and metric. A comprehensive review of the topic of cosmological perturbation theory can be found in [24].

4.1. Perturbations and Gauge Freedom

This section deals with the differential geometry behind perturbation theory. To make the presentation clearer indices on tensors and vectors will be dropped.

To formulate perturbation theory introduce two pseudo-riemannian manifolds M and N with metrics g_M and g_N respectively. The pair (M, g_M) is taken to be the physical manifold while (N, g_N) constitutes the background manifold. I will further require a coordinate function $\phi : N \to \mathbb{R}^n$ and two diffeomorphisms D and $\widetilde{D}$ that map N to M. This setup is illustrated in figure 3 for clarity.

Perturbing the background manifold amounts to using one of the diffeomorphisms D, to pull back the physical metric onto the background manifold. This expression can then be expressed as the metric g_N plus a term that is to be thought of as the perturbation

$$D^*(g_M) = g_N + \delta g. \tag{79}$$

The same could have been done with the other diffeomorphism $\widetilde{D}$ to obtain

$$\widetilde{D}^*(g_M) = g_N + \delta\widetilde{g}. \tag{80}$$

As in GR all diffeomorphic manifolds are physically equivalent and there is a priori no reason to choose one diffeomorphism over another to pull back the physical metric onto the background manifold this constitutes a gauge freedom. A less mathematical and for the physicist more intuitive way to think about this gauge freedom is in terms of coordinate transformations.

To see that this can indeed be related to coordinate transformations take the coordinate function ϕ that maps points form an open set in N to $\mathbb{R}^n$ to construct a coordinate system on M. This can be achieved by taking a point $m \in M$ mapping it to N via D^{-1} and then using ϕ to obtain a point $x \in \mathbb{R}$. This whole idea can of course be written down shorter as $x(m) = (\phi \circ D^{-1})(m)$ where $x(p) \in \mathbb{R}$. Using a different diffeomorphism $\widetilde{D}$ one can then construct a different coordinate system $\widetilde{x}(m) = (\phi \circ \widetilde{D}^{-1})(m)$ and a

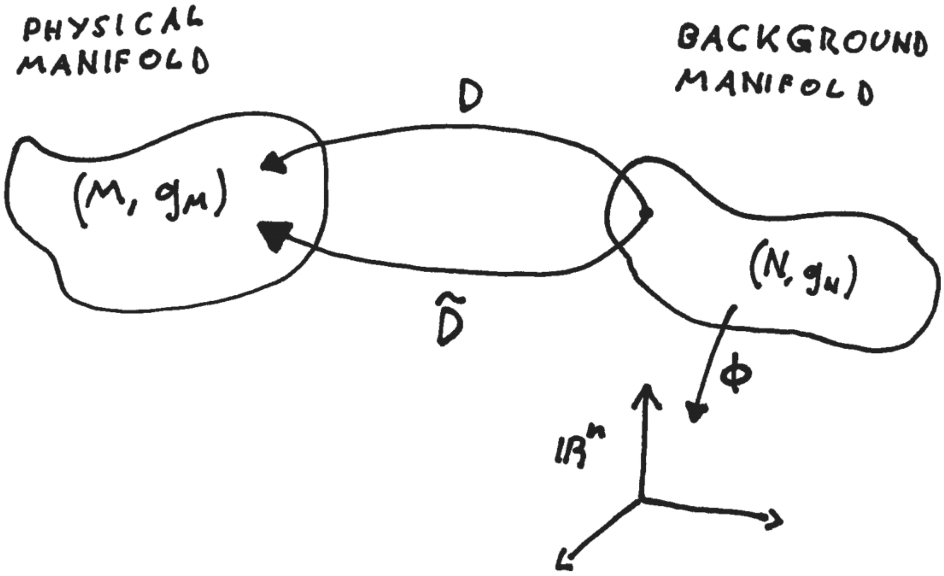

Figure 3: Depiction of the mathematical machinery behind perturbation theory in gravity. The pair (M, g_M) describes the physical spacetime, while (N, g_N) is the background spacetime. D and $\widetilde{D}$ are diffeomorphis that map the physical to the background and choosing between them is a matter of gauge.

change between the coordinate systems can be constructed by $\widetilde{x} = (\phi \circ \widetilde{D}^{-1} \circ D \circ \phi^{-1})(x)$ showing the assertion that gauge transformations i.e. changing the diffeomorphism that pulls back the physical metric to the background manifold is related to a coordinate change.

An infinitesimal coordinate transformation will then cause perturbations to transform via Lie derivative along the vector generating this infinitesimal change. This is not surprising as going from the coordinates x to the coordinates $\widetilde{x}$ is achieved via a diffeomorphism $\widetilde{D}^{-1} \circ D$, yet will explicitly be shown for a scalar field q and a metric g. Tensor and vector indices will be reintroduced.

Given two coordinates x^μ and $\widetilde{x}^\mu$ that are related by the infinitesimal transformation $\widetilde{x}^\mu = x^\mu - \xi^\mu$ the scalar field will take the same value in both coordinate system $q(x^\mu) = q(\widetilde{x}^\mu)$. Splitting of the background value $\bar{q}$ gives the relation

$$\bar{q}(x^\mu) + \delta q = \bar{q}(\widetilde{x}^\mu) + \delta\widetilde{q}. \tag{81}$$

The background value of the right hand side is a function of $\widetilde{x}^\mu = x^\mu - \xi^\mu$ and can therefore be expanded around x^μ to give to first order

$$\bar{q}(x^\mu) + \delta q = \bar{q}(x^\mu) - \xi^\mu \partial_\mu \bar{q}(x^\mu) + \delta\widetilde{q} + \mathcal{O}(\xi^2). \tag{82}$$

As the transformation is infinitesimal the mistake made when ignoring $\mathcal{O}(\xi^2)$ are minimal and it follows that

$$\delta\widetilde{q} = \delta q + \xi^\mu \partial_\mu \, \bar{q}(x^\mu). \tag{83}$$

The same logic can be applied to the metric, though the calculation is slightly more involved due to the metrics tensor nature. Under coordinate transformations the metric changes as

$$\widetilde{g}_{\mu\nu}(\widetilde{x}^\sigma) = \frac{\partial x^\alpha}{\partial \widetilde{x}^\mu}\frac{\partial x^\beta}{\partial \widetilde{x}^\nu} g_{\alpha\beta}(x^\sigma). \tag{84}$$

Taking the derivatives of $x^\alpha = \widetilde{x}^\alpha + \xi^\alpha$ with respect to $\widetilde{x}^\mu$ gives $\delta^\alpha{}_\mu + \widetilde{\partial}_\mu \xi^\alpha$ resulting in

$$\widetilde{g}_{\mu\nu}(\widetilde{x}^\sigma) = g_{\mu\nu}(x^\sigma) + \frac{\partial \xi^\alpha}{\partial \widetilde{x}^\mu} g_{\alpha\nu}(x^\sigma) + \frac{\partial \xi^\beta}{\partial \widetilde{x}^\nu} g_{\mu\beta}(x^\sigma) + \mathcal{O}(\xi^2). \tag{85}$$

The metric in both coordinate systems can now be split into background and perturbation according to

$$\widetilde{g}_{\mu\nu}(\widetilde{x}^\sigma) = \bar{g}_{\mu\nu}(\widetilde{x}^\sigma) + \delta\widetilde{g}_{\mu\nu} \tag{86}$$

$$g_{\mu\nu}(x^\sigma) = \bar{g}_{\mu\nu}(x^\sigma) + \delta g_{\mu\nu}. \tag{87}$$

Now using $x^\alpha = \widetilde{x}^\alpha + \xi^\alpha$, the background metric in coordinates x^σ can be expanded as

$$\bar{g}_{\mu\nu}(x^\sigma) = \bar{g}_{\mu\nu}(\widetilde{x}^\sigma) + \xi^\gamma \frac{\partial}{\partial \widetilde{x}^\gamma} \bar{g}_{\mu\nu}(\widetilde{x}^\sigma). \tag{88}$$

Splitting the remaining metrics in (85) into background and perturbation and expanding them according to (88) gives $\widetilde{g}_{\mu\nu} = \delta g_{\mu\nu} + \mathcal{L}_\xi \, \bar{g}_{\mu\nu}$ where

$$\mathcal{L}_\xi \, \bar{g}_{\mu\nu} = \xi^\gamma \frac{\partial}{\partial \widetilde{x}^\gamma} \bar{g}_{\mu\nu}(\widetilde{x}^\sigma) + \frac{\partial \xi^\alpha}{\partial \widetilde{x}^\mu} \bar{g}_{\alpha\nu}(\widetilde{x}^\sigma) + \frac{\partial \xi^\beta}{\partial \widetilde{x}^\nu} g_{\mu\beta}(x^\sigma) \tag{89}$$

4.2. SVT Decomposition

Having established how perturbation theory and gauge freedom works in the abstract, it is now time to think about the concrete problem at hand, perturbations on a homogeneous and isotropic background manifold. The metric will be split according to $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$. It is simpler to do perturbation theory in conformal time, so $\bar{g}_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$. The symmetries of the background allow one to decompose the metric perturbations into scalars, vectors and tensors.

Under spatial $SO(3)$ transformations the components δg_{00}, δg_{0i} and δg_{ij} of the metric perturbation transform as a scalar, a vector and a tensor respectively. These can then be further decomposed into irreducible representations of $SO(3)$ to give

$$\delta g_{\mu\nu} = a^2(\tau) \begin{pmatrix} -2A & B_i + \partial_i B \\ B_j + \partial_j B & -2\psi\delta_{ij} + \partial_i E_j + \partial_j E_i + 2\partial_i\partial_j E + 2h_{ij} \end{pmatrix} \tag{90}$$

where $\partial^i B_i = \partial^i E_i = 0$, $\partial^i h_{ij} = 0$ and $h^i{}_i = 0$. This is the scalar-vector-tensor (SVT) decomposition of the perturbation. The SVT decomposition can be done for other tensor fields as well.

Note that I have chosen a slightly different convention for the tensor perturbations. The tensor perturbation are often defined as $\delta(g_T)_{ij} = a^2(\tau)h_{ij}$ [25], but I have chosen $\delta(g_T)_{ij} = 2\,a^2(\tau)h_{ij}$ as this is the convention used by *xPand* [26], a Mathematica package designed for cosmological perturbation theory on a number of predefined backgrounds. All perturbations done here were done using *xPand*.

To investigate how the gauge freedom might simplify this decomposition, do an infinitesimal coordinate transformation $x^\mu \to x^\mu - \xi^\mu(x^\nu)$. The vector ξ^μ that generates this change of coordinates, can also be decomposed into irreducible representations of $SO(3)$ according to $\xi^0 = T$ and $\xi^i = L^i + \partial^i L$ where $\partial^i L_i = 0$. This transformation is felt by the metric perturbations as a Lie transport along ξ^μ, $\delta g_{\mu\nu} \to \delta g_{\mu\nu} + \mathcal{L}_\xi(a^2(\tau)\eta_{\mu\nu})$. It is found that the scalar, vector and tensor components of the SVT decomposition transform as

$$A \to A + T' + \mathcal{H}T \tag{91}$$
$$B \to B + L' - T \tag{92}$$
$$C \to C + \mathcal{H}T \tag{93}$$
$$E \to E + L \tag{94}$$
$$B_i \to B_i + L'_j \tag{95}$$
$$E_i \to E_i + L_j \tag{96}$$
$$h_{ij} \to h_{ij} \tag{97}$$

where $\mathcal{H}$ is the conformal Hubble parameter defined in (7). Note that the metric perturbation h_{ij} is gauge invariant. Two further gauge invariant scalar fields Φ and Ψ, and one gauge invariant vector field V_i can be constructed according to

$$\Phi = A + \mathcal{H}(B - E') + \partial_\tau(B - E') \tag{98}$$
$$\Psi = C + \mathcal{H}(B - E') \tag{99}$$
$$V_i = B_i - E'_i \tag{100}$$
$$h_{ij} = h_{ij}. \tag{101}$$

While on the topic of gauge invariant quantities, the Klein-Gordon field introduced to drive inflation will also be perturbed and can be shown to have a gauge invariant perturbation too. The background Klein-Gordon field $\varphi(\tau)$ will be perturbed by the quantity δf as $\varphi(\tau) + \delta f$. Under the usual infinitesimal coordinate transformations δf will transform as $\delta f \to \delta f + \xi^\mu \partial_\mu \varphi = \delta f + T\varphi'$. Therefore a gauge invariant perturbation for the Klein-Gordon field is given by

$$\delta\varphi = \delta f + (B - E')\varphi' \tag{102}$$

The importance of gauge invariant quantities is that they allow one to differentiate between actual physical perturbations and gauge artifacts. A general metric that may look like a perturbed spacetime might actually be an unperturbed spacetime in new coordinates. Gauge invariant quantities will be non zero for physical perturbations.

As is often done for systems with gauge freedom, I will now choose the gauge I will be working with primarily, Newton or Longitudinal gauge as it is known in the literature. To obtain this gauge take the general SVT decomposition introduced above and make a gauge transformation with the transformation vector given by $T = B - E'$, $L = -E$ and $L_i = -E_i$ to obtain

$$\delta g_{\mu\nu} = \begin{pmatrix} -2\Phi & V_i \\ V_j & -2\Psi\delta_{ij} + 2h_{ij} \end{pmatrix} \tag{103}$$

and

$$\delta f = \delta\varphi. \tag{104}$$

All perturbations are expressed trough gauge invariant quantities.

4.3. Equations of Motion for Perturbations

There are two equivalent approaches to finding the equations of motion for the perturbations to first order. First one can simply take the equations of the full theory, in our case Einstein gravity minimally coupled to a Klein-Gordon field and expand them to first order in perturbations. The different types of perturbations evolve separately [25] whereby the equations of motion that follow from the Einstein equations are

$$(\delta G_S)^\mu{}_\nu = 8\pi(\delta T_S)^\mu{}_\nu \tag{105}$$

$$(\delta G_V)^\mu{}_\nu = 0 \tag{106}$$

$$(\delta G_T)^\mu{}_\nu = 0 \tag{107}$$

for scalar, vector and tensor perturbations respectively.

The second method to obtain the equations of motion is to take the action of the theory and expand it to second order in perturbations. The second order term of this expansion then serves as the action from which, after variation with respect to the perturbations, follow their equations of motion. This action also serves as the foundation for the canonical quantization of the perturbations.

As is so often the case, depending on the situation one or the other approach might be more straightforward to finding the equations of motion. Some preliminary work for scalar perturbations is, for example, simpler starting from the Einstein equations.

4.3.1. Vector Perturbations

While tensor and scalar perturbations will be dealt with in a chapter dedicated to them and their quantization, the vector perturbations do not play a role in Einstein gravity inflation and will quickly be dealt with here.

The equation $(\delta G_V)^\mu{}_\nu = 0$ gives two equations for the vectors namely

$$0 = \Delta V_i \tag{108}$$

$$0 = a\,\partial_i V_j' + 2a'\,\partial_i V_j. \tag{109}$$

The second equation (109) can equivalently be written as $(a^2\partial_j V_i)' = 0$ making it clear that V_i is proportional to the squared inverse scale factor $V_i \propto a^{-2}$. The vector perturbations therefore decay fast in an expanding universe and unless they were initially very large they can safely be ignored.

5. Observable Consequences of Inflation

Before I dive into calculating the behaviour of tensor and scalar perturbations during inflation, it seems a good idea to explain why these are of relevance and paint the big picture without going into too much detail. The discussion here follows [25].

The scalar field that drives inflation also dictates when inflation will end. By the uncertainty principle arbitrarily precise measurement of time is not possible and the field will have fluctuations depending on the space coordinates and inflation will be in different stages of evolution at different places. This will then translate into quantum fluctuations of the spacetime metric.

As will be done in the following chapters, both tensor and scalar perturbations can be Fourier expanded into modes of wave vector $\mathbf{k}$. The magnitude $|\mathbf{k}|$ of the wave vector is proportional to the inverse wavelength. Therefore when $k = Ha$ the wave length is of the order of the Hubble radius. The Fourier modes at a time t can be divided into super- and sub-Hubble modes where the former is defined by having a wave number $k \ll aH$ and the latter by $k \gg aH$.

In comoving coordinates the wavelength stays the same, during inflation however, the Hubble radius shrinks and some of the modes become super-Hubble. This is often referred to as inflation stretching the modes to super-Hubble scales. These modes then remain outside the Hubble radius during the period of inflation and once they leave the Hubble sphere the fluctuations are thought to shed their quantum behaviour but keep their expectation values and correlators that are now to be thought of as the statistical properties of the ensemble of values of a classical stochastic field. Once inflation stops the Hubble radius will grow and eventually super-Hubble modes will re-enter the Hubble sphere. For large enough scales this happens close to the release of the CMB and it is

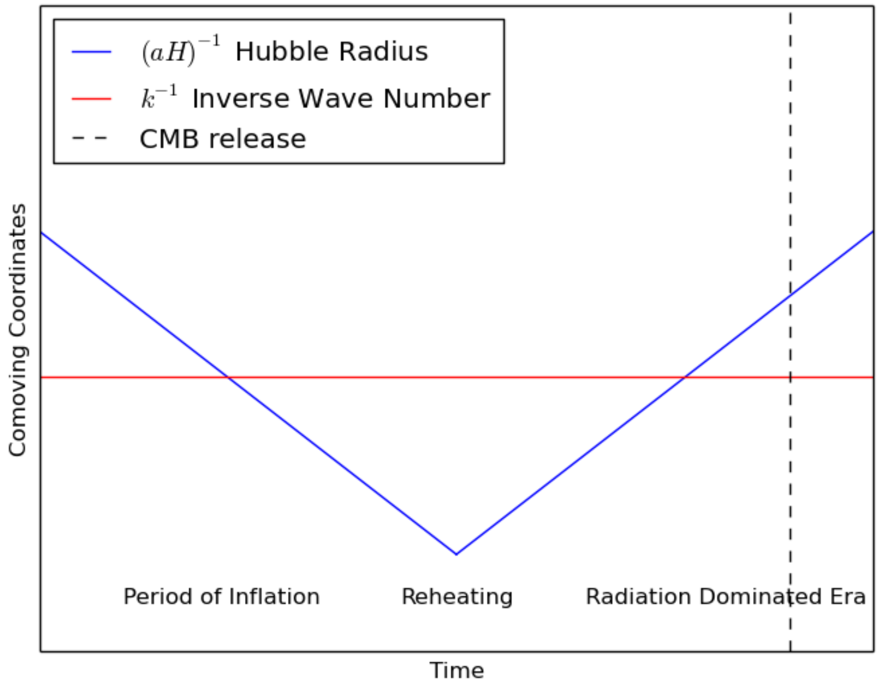

Figure 4: Illustration of a mode of wave number k becoming super-Hubble and re-entering the horizon after inflation. The idea for this depiction was taken from [25]

those modes that are of interest as they cause the CMB anisotropies and are the seeds of large scale structure in the universe. This situation is sketched in figure 4.

Recall from section 2.4 that during the standard big bang evolution of the universe the Hubble radius is roughly the same size as the particle horizon. In this picture, perturbations with wavelengths of the order of the horizon are a bit of a mystery as they could not be caused by a causal mechanism. Inflation remedies this situation and gives an explanation of how these modes could have come about.

There is however one more problem with this picture. During the period following inflation, known as reheating, the physics is not clear and the equations governing the evolution of the perturbations are not well known. This however, is not a problem. A theorem due to Weinberg [12] states that there is a tensor mode with constant amplitude and a constant scalar mode, both of which are super-Hubble. This allows one to extrapolate the behaviour of perturbations that are of order of the Hubble radius during inflation to horizon re-entry and thereby allows one to gain information about the high energy period during which inflation happened by making observations of the low energy period of recombination.

Before I move on to the next section let me introduce some notation. The statistical property of interest for some stochastic field $f(\tau, \mathbf{x})$ is the variance or correlator quantum mechanically given by $\langle 0|\, f(\tau, \mathbf{x}) f(\tau, \mathbf{y})\, |0\rangle$. The fluctuations of metric during inflation will turn out to be statistically isotropic and homogeneous, meaning that the two point function only depends on $r = |\mathbf{x} - \mathbf{y}|$. This then enables the correlator to be expressed as

$$\langle 0| \, f(\tau, \mathbf{x}) f(\tau, \mathbf{y}) \, |0\rangle = \int \frac{dk}{k} \, \mathcal{P}_f^F(k) \, \frac{\sin(kr)}{kr} \tag{110}$$

where $\mathcal{P}_f^F(k)$ is called the power spectrum. This holds for all modes which is why the power spectrum is given the superscript F for full. As the modes of interest are the super-Hubble modes I also introduce the super-Hubble limit of the power spectrum as

$$\mathcal{P}_f = \lim_{k \ll aH} \mathcal{P}_f^F. \tag{111}$$

6. Tensor Fluctuations During Inflation

First I will describe the tensor fluctuations as they are simpler than their scalar counterpart. A general discussion of the equations of motion is followed by their solutions for de Sitter geometry and slow-roll inflation. The tensor perturbations are then quantized and the power spectrum is calculated in general, followed by the specific examples of de Sitter and slow-roll models of inflation.

6.1. Classical Equations of Motion and their Solutions

Expanding (46) to second order in tensor perturbations gives the action

$$S_T = \int d^4x \frac{a^2(\tau)}{16\,\pi} \Big(h'_{ij} h'^{\,ij} - \partial_l \, h_{ij} \, \partial^l h^{ij} \Big) \tag{112}$$

which will later be used as the basis for canonical quantization and gives, after varying with respect to the tensor perturbations h_{ij}, the equations of motion

$$h''_{ij} + 2\mathcal{H} h'_{ij} - \Delta h_{ij} = 0 \tag{113}$$

where $\Delta = \delta^{ij} \partial_i \partial_j$. Note that this equation will depend on the geometry of spacetime, due to the presence of $\mathcal{H}$.

In anticipation of quantization, the tensor perturbations can be expanded into polarisations and fourier modes as

$$h_{ij}(\tau, \vec{x}) = \sum_{p=1}^{2} \int \frac{d^3k}{\sqrt{2}(2\pi)^{3/2}} \Big(a_{\mathbf{k},p} h^*_{\mathbf{k},p}(\tau) \exp(i\mathbf{k} \cdot \mathbf{x}) + a^\dagger_{\mathbf{k},p}(\tau) h_{\mathbf{k},p} \exp(-i\mathbf{k} \cdot \mathbf{x}) \Big) e^p_{ij}. \tag{114}$$

For now $a_{\mathbf{k},p}$ is simply to be thought of as an amplitude, though it will become an operator when quantizing. The 3×3 tensor e^p_{ij} is a polarisation tensor that satisfies

$$e^p{}_{ij} \, e^{mij} = \delta^{pm}. \tag{115}$$

It is symmetric by virtue of h_{ij}, from which it also inherits the properties

$$h^i_{\ i} = 0 \rightarrow e^{p\,i}_{\ \ \ i} = 0, \qquad \partial^i h_{ij} = 0 \rightarrow k^i e^p_{\ ij} = 0. \tag{116}$$

A symmetric 3×3 matrix has at most 6 independent components; supplemented by one equation describing tracelessness and three that describe the transversal nature of the tensor perturbations two independent components remain. Therefore the tensor can be decomposed according to (114) into the sum over two independent polarisation tensors. The functions $h_{\mathbf{k},p}(\tau)$ are called mode functions and are described by the equation of motion

$$h''_{\mathbf{k},p} + 2\mathcal{H}\, h'_{\mathbf{k},p} + \mathbf{k}^2\, h_{\mathbf{k},p} = 0 \tag{117}$$

that follows from plugging the mode expansion (114) into (113). As the polarisations are independent of one another it suffices to treat only one of them and, where neccesary reintroduce the other.

For just one polarisation the action of interest is, after dropping the polarisation index p,

$$S_P = \int d^4x \frac{a^2(\tau)}{16\,\pi}\Big(h'^2 - \partial_i h\, \partial^i h\Big) \tag{118}$$

where $h(\tau, \mathbf{x})$ can be expanded as

$$h(\tau, \mathbf{x}) = \int \frac{d^3k}{\sqrt{2}(2\pi)^{3/2}}\Big(a_{\mathbf{k}}\, h^*_{\mathbf{k}}(\tau) \exp\left(i\mathbf{k}\cdot\mathbf{x}\right) + a^\dagger_{\mathbf{k}}\, h_{\mathbf{k}}(\tau) \exp\left(-i\mathbf{k}\cdot\mathbf{x}\right)\Big) \tag{119}$$

and the mode functions $h_{\mathbf{k}}(\tau)$ satisfy the equation

$$h''_{\mathbf{k}} + 2\mathcal{H}\, h'_{\mathbf{k}} + \mathbf{k}^2 h_{\mathbf{k}} = 0. \tag{120}$$

Note that in this equation the vector $\mathbf{k}$ only shows up squared, whereby the mode functions are not functions of $\mathbf{k}$ but of its magnitude $k = |\mathbf{k}|$, a property that the mode functions inherit from the isotropic FLRW background. This however does not automatically imply that for two unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, $h_{k\mathbf{e}_1} = h_{k\mathbf{e}_2}$ as initial conditions might have been different for $h_{k\mathbf{e}_1}$ and $h_{k\mathbf{e}_2}$. Knowing that the initial conditions will be isotropic I henceforth write $h_{\mathbf{k}} = h_k$.

Next, let us solve these equations for the specific models of de Sitter and slow-roll inflation.

6.1.1. Solutions on de Sitter space

De Sitter space is given by a flat FLRW space with scale factor $a(t) = \exp\left(Ht\right)$ where H is a constant and t is the cosmic time. Integrating the equation $dt = a(t)d\tau$ gives the conformal time $\tau = -\exp\left(-Ht\right)$. Note that conformal time is always negative.

The scale factor expressed through conformal time is $a(\tau) = -1/H\tau$, with which the conformal Hubble parameter becomes $\mathcal{H} = -1/\tau$, making equation (120)

$$h_k'' - \frac{2}{\tau} h_k' + k^2\, h_k = 0. \tag{121}$$

This can be simplified by introducing the variable $z = -k\,\tau$, which often comes in handy, to give

$$\frac{d^2 h_k}{dz^2} - \frac{2}{z}\frac{dh_k}{dz} + h_k = 0. \tag{122}$$

Note that z is always positive. The solutions on de Sitter space are then given by

$$h_k = A\big(z\,\sin(z) + \cos(z)\big) + B\big(\sin(z) - z\,\cos(z)\big) \tag{123}$$

where A and B are integration constants.

Without having to specify initial conditions to fix these integration constants the super-Hubble behaviour of these functions can be investigated. Recall that modes are super-Hubble when their wavelength lies outside the Hubble radius, i.e. when $k \ll \mathcal{H}$. For de Sitter space this can be rewritten as

$$k \ll -\frac{1}{\tau} \quad \rightarrow \quad -k\,\tau = z \ll 1 \tag{124}$$

where the second inequality follows from the fact that the conformal time τ is always negative. The super-Hubble limit is thereby given as the limit $z \rightarrow 0$ in which the functions (123) become

$$h_k = A\cos(z) + B\sin(z). \tag{125}$$

From this it is clear that, as advertised above, the tensor modes have a constant amplitude outside the Hubble radius.

6.1.2. Solutions for Slow-Roll Inflation Models

It might seem surprising that it makes sense to talk about the solutions for slow-roll inflation, instead of talking about solutions for a specific slow-roll model given by a specific potential $V(\varphi)$. At the heart of the possibility to solve equation (120) for slow-roll inflation without using a concrete model lies the slow-roll expansion, essentially given by (65) to first order.

Equation (65) gives the conformal Hubble parameter in terms of the slow-roll parameter ε, a quantity that exists for slow-roll models regardless of the specific potential. Assuming that ε is changing little during inflation (65) can be integrated quite easily to give an expression for the scale factor $a(\tau)$

$$\frac{d}{d\tau}\ln\big(a(\tau)\big) = -(1+\varepsilon)\frac{d}{d\tau}\Big(\ln(\tau) + \ln(a_0)\Big). \tag{126}$$

Here a_0 is some integration constant. To choose a sensible integration constat, observe that in the limit $\varepsilon \to 0$ the slow-roll conformal Hubble parameter becomes that of de Sitter space. One then chooses $a_0 = -1$, making

$$a(\tau) = (-\tau)^{-(1+\varepsilon)} \tag{127}$$

which produces the de Sitter scale factor when ε vanishes. For the scale factor to not become complex conformal time has to be negative, $\tau < 0$, just like in de Sitter space.

The equation for tensor modes in slow-roll inflation is, using (65),

$$\frac{d^2 h_k}{dz^2} - \frac{2(1+\varepsilon)}{z}\frac{dh_k}{dz} + h_k = 0 \tag{128}$$

where again $z = -k\tau$. Therefore, during slow-roll inflation the tensor modes behave as

$$h_k = z^c \Big(A\, J_c(z) + B\, Y_c(z) \Big), \qquad c = \frac{1}{2}(3 + 2\varepsilon). \tag{129}$$

To check the super-Horizon limit of these solutions find in [27] that the Bessel functions behave as

$$J_c(z) \simeq \frac{1}{\Gamma(c+1)} \left(\frac{z}{2}\right)^c, \quad z \to 0 \tag{130}$$

$$Y_c(z) \simeq -\frac{1}{\pi}\Gamma(c) \left(\frac{2}{z}\right)^c, \quad z \to 0 \tag{131}$$

making the super-Hubble tensor fluctuations during slow roll inflation constant

$$h_k = -\frac{2^c B}{\pi}\Gamma(c). \tag{132}$$

6.2. Quantization of Tensor Perturbations

The quantization procedure for the field described by the action

$$S_P = \int d^4x \frac{a^2(\tau)}{16\,\pi} \Big(h'^2 - \partial_i\, h\, \partial^i h \Big) \tag{133}$$

is similar the the usual canonical quantization one does in quantum field theory on Minkowski space. Indeed the only difference working on an FLRW background makes is that the vacuum is no longer uniquely defined and a suitable choice has to be made. The details of why no unique vacuum exists on curved backgrounds are not detailed here, and a discussion can be found in appendix A. Alternatively one could consult [28].

For FLRW spacetimes a suitable choice of vacuum is given by the Bunch-Davies vacuum, which relies on the observation that at early enough times all modes have wavelength much shorter than the Hubble radius and will perceive the spacetime to be Minkowski. The vacuum is then chosen by demanding that the field behaves like it would in Minkowski space for early times.

Again I will first discuss the quantization procedure without specifying a scale function as far as possible and then consider de Sitter space and slow-roll inflation.

6.2.1. Quantization on General FLRW Spacetimes

The first step in quantizing a system is to find its canonical variables. For the action (133) the canonical position variable is simply given by the field $h(\tau, \mathbf{x})$ and its canonical momentum is found to be

$$p(\tau, \mathbf{x}) = \frac{\partial \mathcal{L}}{\partial h'} = \frac{a^2(\tau)}{8\,\pi} h'(\tau, \mathbf{x}). \tag{134}$$

Classically, the canonical variables satisfy the Poisson bracket relation

$$\{h(\tau, \mathbf{x}), p(\tau, \mathbf{y})\} = \delta^{(3)}(\mathbf{x} - \mathbf{y}). \tag{135}$$

To quantize this system the position and momentum variables, $h(\tau, \mathbf{x})$ and $p(\tau, \mathbf{x})$ are promoted to operators and the algebra defined by the Poisson bracket replaced by

$$[h(\tau, \mathbf{x}), p(\tau, \mathbf{y})] = i\,\delta^{(3)}(\mathbf{x} - \mathbf{y}). \tag{136}$$

When promoting $h(\tau, \mathbf{x})$ to an operator what one does is actually promote the amplitudes $a_{\mathbf{k}}$ in the mode expansion (119) to operators. To find the algebra for these operators, plug the mode expansion into the quantization condition (135) to find

$$\int \frac{d^3k\, d^3\widetilde{k}}{(2\pi)^3} \frac{a^2(\tau)}{16\,\pi} [a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}}] \big(h^*_k\, h'_k - h'^*_k\, h_k\big) \exp\left(i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})\right) = i \int \frac{d^3k}{(2\,\pi)^3} \exp\left(i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})\right) \tag{137}$$

where the right hand side is the Fourier expansion of the delta function. This equation is then satisfied if

$$[a_{\mathbf{k}}, a^{\dagger}_{\mathbf{q}}] = \delta^{(3)}(\mathbf{k} - \mathbf{q}), \qquad [a_{\mathbf{k}}, a_{\mathbf{q}}] = 0 \tag{138}$$

and

$$\frac{a^2(\tau)}{16\,\pi} \big(h^*_k\, h'_k - h'^*_k\, h_k\big) = i. \tag{139}$$

where the second equation (139) is called the normalisation condition. A vacuum is then defined by requiring

$$a_{\mathbf{k}}\,|0\rangle = 0. \tag{140}$$

With the quantum machinery in place the 2-point correlator for $h(\tau, \mathbf{x})$ can now be computed as

$$\langle 0|\, h(\tau, \mathbf{x})\, h(\tau, \mathbf{y})\, |0\rangle = \int \frac{d^3k}{2(2\,\pi)^3} |h_k(\tau)|^2 \exp\big(i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})\big). \tag{141}$$

Due to the isotropy of the mode functions the integral can be written in spherical coordinates and the angles can be integrated over to give

$$\langle 0|\, h(\tau, \mathbf{x})\, h(\tau, \mathbf{y})\, |0\rangle = \int \frac{dk\, k^2}{4\,\pi^2}\, |h_k|^2\, \frac{\sin(kr)}{kr} \tag{142}$$

where $r = |\mathbf{x} - \mathbf{y}|$.

The actual aim of this section is to find the 2-point function for the tensor perturbations. To do so we take the full tensor perturbation and calculate

$$\begin{aligned}\langle 0|\, h_{ij}(\tau,\mathbf{x})\, h^{ij}(\tau,\mathbf{y})\, |0\rangle &= \sum_{p,m} \langle 0|\, h_p(\tau,\mathbf{x})\, h_m(\tau,\mathbf{y})\, |0\rangle\, e^p{}_{ij} e^{m\, ij} \\ &= \sum_{p,m} \langle 0|\, h_p(\tau,\mathbf{x})\, h_m(\tau,\mathbf{y})\, |0\rangle\, \delta^{pm} \\ &= \sum_{p=1}^{2} \langle 0|\, h_p(\tau,\mathbf{x})\, h_p(\tau,\mathbf{y})\, |0\rangle\, . \end{aligned} \tag{143}$$

As both polarisations have the same correlator this expression evaluates to

$$\langle 0|\, h_{ij}(\tau,\mathbf{x})\, h^{ij}(\tau,\mathbf{y})\, |0\rangle = 2 \int \frac{dk\, k^2}{4\,\pi^2}\, |h_k|^2\, \frac{\sin{(kr)}}{kr} \tag{144}$$

and the power spectrum for tensor perturbations is given by

$$\mathcal{P}_h^F = \frac{k^3\, |h_k|^2}{2\,\pi^2}. \tag{145}$$

These results hold for any spatially flat FLRW spacetime and I will now make them specific for the usual cases of de Sitter and slow-roll inflation.

6.2.2. Quantization on de Sitter Spacetime

Recall that for de Sitter spacetime the mode functions are given by (123). The essential steps for quantization have been taken above and what remains to be done is to find expressions for the integrations constants A and B. The first step in finding these is to impose the Bunch-Davies vacuum condition on the mode functions.

A quantum field theory on a de Sitter like background is said to have as its ground state the Bunch-Davies vacuum, if the mode functions behave for early enough times like they would in Minkowski space. This makes sense as for early enough times any mode functions would have wavelengths shorter than scales on which the curvature of spacetime would be noticeable.

A good idea of wether or not a mode function would notice the geometry is to check if its wavelength lies far inside the Hubble radius. As the wavelength is inversely proportional the wave number k you are inside the Hubble radius if $k \ll aH = \mathcal{H}$. In terms of the variable z this can be written as $1 \ll z$ i.e. you have to look at the limit $z \to \infty$.

Note that the mode functions for de Sitter are of the form $h_k = z\,\mu(z)$, where $\mu(z)$ is

$$\mu(z) = A\big(\sin{(z)} + \frac{\cos{(z)}}{z}\big) + B\big(\frac{\sin{(z)}}{z} - \cos{(z)}\big) \tag{146}$$

and therefore stays finite in the limit of infinite z as does its derivative. Plugging this decomposition into

$$\frac{d^2 h_k}{dz^2} - \frac{2}{z}\frac{dh_k}{dz} + h_k = 0 \tag{147}$$

gives

$$\frac{d^2\mu}{dz^2} + \mu - \frac{2\mu}{z^2} = 0. \tag{148}$$

For the early time limit $z \to \infty$ the last term vanishes requiring μ to display the early time behaviour

$$\frac{d^2\mu}{dz^2} + \mu = 0 \Longrightarrow \mu \propto \exp(-iz). \tag{149}$$

Therefore the condition to impose on the mode functions to select the Bunch-Davies vacuum is

$$\lim_{z\to\infty} \frac{h_k(\tau)}{z} \propto \exp(-iz). \tag{150}$$

One then finds that this forces $B = -iA$ and the mode functions are

$$h_k = A\big(1 + iz\big)\exp(-iz). \tag{151}$$

The last thing left to do is fix the constant A. This is done by taking (151) and demand that it satisfies the normalisation condition (139). This gives

$$|A|^2 = \frac{8\,\pi H^2}{k^3} \tag{152}$$

thereby fixing $|A|$. The power spectrum for tensor fluctuations on de Sitter space is then

$$\mathcal{P}_h^F = \frac{4H^2}{\pi}(1 + z^2). \tag{153}$$

This holds for all modes. However what we can observe in the CMB are only the modes with a wavelength bigger than the Hubble radius i.e. modes with a wave vector $\mathbf{k}$ that satisfies, $k^{-1} \gg \mathcal{H}^{-1}$, which as shown above can be expressed as, $z \ll 1$. The quantity of interest is therefore

$$\mathcal{P}_h = \lim_{z\to 0} \mathcal{P}_h^F = \frac{4H^2}{\pi}. \tag{154}$$

The super-Hubble power spectrum displays two properties shared with more realistic models of an exponentially expanding early universe. First it tells us about the Hubble parameter, and thereby the energy scale, at which inflation took place. Second it is scale invariant meaning that it is the same for all modes i.e. is independent of $\mathbf{k}$. Similar properties are derived from slow-roll inflation.

6.2.3. Quantization During Slow-Roll Inflation

For slow-roll inflation the solutions for the mode functions are in terms of z given by (129). Again, I start by imposing the Bunch-Davies vacuum condition. In the sub Hubble limit $z \to \infty$ the Bessel functions behave as [27]

$$J_c(z) \simeq \sqrt{\frac{2}{z\,\pi}}\,\cos\left(z - \frac{c\pi}{2} - \frac{\pi}{4}\right), \quad z \to \infty \tag{155}$$

$$Y_c(z) \simeq \sqrt{\frac{2}{z\,\pi}}\,\sin\left(z - \frac{c\pi}{2} - \frac{\pi}{4}\right), \quad z \to \infty. \tag{156}$$

In this limit the mode functions take the form $h_k = z^{c-1/2}\mu(z)$ and equation (128) takes the familiar form

$$\frac{d^2\mu(z)}{dz^2} + \mu(z) = 0. \tag{157}$$

The Bunch-Davies condition is therefore

$$\lim_{z\to\infty} \frac{h_k(\tau)}{z^{c-1/2}} \propto \exp\left(-iz\right) \tag{158}$$

and forces $B = -iA$. The normalisation condition (139) forces A to

$$|A|^2 = \frac{4\pi^2}{k^{3+2\varepsilon}}. \tag{159}$$

The power spectrum of tensor fluctuations during slow-roll inflation is therefore

$$\mathcal{P}_h^F = 2\,k^{-2\,\varepsilon} z^{2\,c}\Big(J_c^2(z) + Y_c^2(z)\Big). \tag{160}$$

Despite what has been advertised, this looks very different from the power spectrum for de Sitter space. To get the characteristic behaviour we would expect to find in the CMB, again take the super Hubble limit, $z \to 0$, which with the help of (155) and (156) is

$$\mathcal{P}_h = \lim_{z\to 0} \mathcal{P}_h^F = \frac{2^{1+2\,c}\,\Gamma^2(c)}{\pi^2} k^{-2\,\varepsilon}. \tag{161}$$

First note that this is not exactly scale invariant, as the power spectrum is a function of the wave number k. The wave number shows up to the power of -2ε, which is a small number whereby the value of the power spectrum for two different values of k will not be far from each other. A spectrum with this behaviour is said to be almost scale invariant and its deviation from scale invariance is measured by the spectral index

$$n_T = \frac{d\ln\mathcal{P}_h}{d\ln k}. \tag{162}$$

For slow-roll models the spectral index is

$$n_T = -2\,\varepsilon. \tag{163}$$

To see how expression (161) relates to the Hubble parameter during inflation first define the time t_k, at which the wavelength crosses into the super Hubble regime by $a(t_k)H(t_k) = k$. As $aH = \mathcal{H} = -(1+\varepsilon)/\tau$, the conformal time at horizon crossing is $\tau_k = \tau(t_k) \simeq -1/k$. The scale dependence of the slow-roll power spectrum can then be rewritten as

$$\mathcal{P}_h = \frac{2^{1+2\,c}\,\Gamma^2(c)}{\pi^2} H^2(t_k) a^2(t_k) k^{-2(1+\varepsilon)} \tag{164}$$

and with $a(t_k) = (-\tau_k)^{-(1+3)} \simeq k^{(1+\varepsilon)}$ we are left with

$$\mathcal{P}_h = \frac{4\,H^2(t_k)}{\pi}\Big(\frac{2^{2\,c-1}\,\Gamma^2(c)}{\pi}\Big). \tag{165}$$

This gives the relation of the power spectrum to the Hubble parameter at the value of horizon exit for the mode of wave vector of magnitude $\mathbf{k}$, where the dependence of the power spectrum on k is hidden in $H(t_k)$.

In the de Sitter limit $\varepsilon \to 0$ the Bessel index c behaves as $c \to 3/2$. With $\Gamma(3/2) = \sqrt{\pi}/2$ the power spectrum then reads

$$\mathcal{P}_h = \frac{4H^2(t_k)}{\pi} \tag{166}$$

which takes the form previously found for de Sitter space, see (154). Note that while it looks like the power spectrum of tensor fluctuations on de Sitter space it differs as the Hubble parameter is the Hubble parameter at the time at which modes of wave number $k = a(t_k)H(t_k)$. Similar expressions are also found for the scalar modes in the next section.

7. Scalar Fluctuations During Inflation

Having calculated the power spectrum for tensor perturbations I now calculate this result for the scalar perturbations. From the perturbation of the metric there are two scalar fields Φ and Ψ, and there is the field $\delta\varphi$ from perturbing the Klein-Gordon field. These evolve according to the Klein-Gordon and Einstein equations, which also constrain them to describe just one degree of freedom. This is the first result I will establish, after which this single degree of freedom will be quantized and its power spectrum calculated.

7.1. The Relevant Field

To see that the three fields Ψ, Φ and $\delta\varphi$ describe one physical degree of freedom consider the Einstein equations perturbed to first order in scalar perturbations

$$(\delta G_S)^{\mu}{}_{\nu} = 8\pi\,(\delta T_S)^{\mu}{}_{\nu}. \tag{167}$$

First note that the energy momentum tensor perturbation for a Klein-Gordon field has the spatial components

$$a^2\,(\delta T_S)^i{}_j = \Big(\varphi'\,\delta\varphi' - \varphi'^2 - a^2\delta\varphi\,\frac{dV(\varphi)}{d\varphi}\Big)\delta^i{}_j \tag{168}$$

while the Einstein tensor has

$$a^2\,(\delta G_S)^i{}_j = \big(2\mathcal{H}^2\Phi + 4\mathcal{H}'\Phi + 2\mathcal{H}\Phi' + 4\mathcal{H}\Psi' + 2\Psi'' + \Delta(\Phi - \Psi)\big)\delta^i{}_j + \partial^i\partial_j(\Psi - \Phi). \tag{169}$$

For $i \neq j$ this gives the equation

$$\partial^i\partial_j(\Psi - \Phi) = 0 \tag{170}$$

which is solved by $\Psi - \Phi = c^i x_i + c$, where c^i and c are constants. If c^i were not zero the perturbations could grow very big far from the origin of the coordinate system, which would be at odds with the idea of small perturbations. Therefore $c^i = 0$. The other integration constant will also be set to zero, $c = 0$, mainly for simplicity. Thus

$$\Phi = \Psi \tag{171}$$

and we are left with only two fields.

The $(i, 0)$ component of the Einstein Equation is then given as

$$-4\pi\,\varphi'\delta\varphi + \mathcal{H}\Phi + \Phi' = 0 \tag{172}$$

which is a constraint equation, allowing $\delta\varphi$ to be expressed through Φ, essentially reducing the two remaining fields to just one independent scalar. In the de Sitter limit, where $\varphi' = 0$ the perturbation of the Klein-Gordon field will still fluctuate but it will no longer couple to the metric. The de Sitter limit will therefore not be of interest for the scalar modes.

Knowing this, one can then define the Mukhanov-Sasaki variable ξ and the useful quantity u as

$$\xi = a\Big(\delta\varphi + \frac{\varphi'}{\mathcal{H}}\Phi\Big), \qquad u = \frac{a\varphi'}{\mathcal{H}} \tag{173}$$

Expanding the action of the single field inflation model (46) to second order in Φ and φ and employing Mukhanov-Sasaki variable (173) gives the action for the scalar degree of freedom [24]

$$S_\xi = \frac{1}{2}\int d^4x\Big(\xi'^2 - \partial_i\xi\,\partial^i\xi + \frac{u''}{u}\xi^2\Big). \tag{174}$$

Following Weinberg [12] I rewrite this in terms of the variable $\mathcal{R}$ via $\xi = u\mathcal{R}$. $\mathcal{R}$ is a gauge invariant variable called the curvature perturbation and shows up in a gauge called comoving gauge, defined by the vanishing of $(\delta T_S)^i{}_j$. For the case of scalar field matter this coincides with the vanishing of the perturbation of the Klein-Gordon field.

$\mathcal{R}$ is of interest as it is the scalar quantity that is conserved on super-Hubble scales and allows one to relate CMB fluctuations to fluctuations during inflation.

The action that describes $\mathcal{R}$ is given by

$$S_{\mathcal{R}} = \frac{1}{2}\int d^4x\, u^2\Big(\mathcal{R}'^{\,2} - \partial_i\mathcal{R}\,\partial^i\mathcal{R}\Big) \tag{175}$$

and is the starting point for all further investigation.

7.2. Classical Equation of Motion and its Solutions

The equation of motion derived from (175) is

$$\mathcal{R}'' + 2\frac{u'}{u}\mathcal{R}' - \Delta\mathcal{R} = 0. \tag{176}$$

To find the expression u'/u during slow-roll inflation, once more evoke the slow-roll expansion. From the definition of u (173) find

$$\frac{u'}{u} = \mathcal{H} + \frac{\varphi''}{\varphi} - \frac{\mathcal{H}'}{\mathcal{H}} \tag{177}$$

which, with the help of the slow roll parameters in conformal time (62) becomes

$$\frac{u'}{u} = \mathcal{H}\big(1+\varepsilon-\delta\big) \simeq -\frac{1}{\tau}\big(1+2\varepsilon-\delta\big) \tag{178}$$

where the last expression comes form using (65) and discarding the terms of second order in slow-roll parameters.

For slow-roll inflation, equation (177) then reads

$$\mathcal{R}'' - 2\frac{(1+2\varepsilon-\delta)}{\tau}\mathcal{R}' - \Delta\mathcal{R} = 0. \tag{179}$$

Again, I introduce the mode expansion of the field $\mathcal{R}$ as

$$\mathcal{R}(\tau,\mathbf{x}) = \int \frac{d^3k}{\sqrt{2}(2\pi)^{3/2}}\Big(b_{\mathbf{k}}\,\mathcal{R}^*_{\mathbf{k}}(\tau)\exp\left(i\,\mathbf{k}\cdot\mathbf{x}\right) + b^\dagger_{\mathbf{k}}\,\mathcal{R}_{\mathbf{k}}(\tau)\exp\left(-i\,\mathbf{k}\cdot\mathbf{x}\right)\Big) \tag{180}$$

where the mode functions are governed by

$$\frac{d^2\mathcal{R}_{\mathbf{k}}}{dz^2} - 2\frac{(1+2\varepsilon-\delta)}{z}\frac{d\mathcal{R}_{\mathbf{k}}}{dz} + \mathcal{R}_{\mathbf{k}} = 0. \tag{181}$$

This equation only depends on the magnitude of the wave vector. Anticipating an isotropic Bunch-Davies condition, I write the mode functions as functions of k.

$$\mathcal{R}_k = z^{c_S}\Big(A\,J_{c_S}(z) + B\,Y_{c_S}(z)\Big), \qquad c_S = \frac{1}{2}\big(3-2\delta+4\varepsilon\big). \tag{182}$$

Comparison with (129) reveals that the only difference between the classical behaviour of the tensor and scalar modes is the Bessel index c_S, which unlike for the tensor perturbations contains the second slow-roll parameter. If the slow-roll parameters are equal, $\varepsilon = \delta$, the Bessel indices for scalar and tensor perturbations are equal too, $c_S|_{\varepsilon=\delta} = c_T$. It is then not surprising that the quantum mechanics of scalar perturbations also gives an almost scale invariant spectrum, as is shown below.

7.3. Quantization of the Scalar Perturbation

From the action (175), for the scalar field $\mathcal{R}$, the canonical momentum

$$p_S(\tau, \mathbf{x}) = \frac{\partial \mathcal{L}}{\partial \mathcal{R}} = u^2\, \mathcal{R}' \tag{183}$$

follows, and after promoting the canonical pair $(\mathcal{R}, p_s)$ to operators, they are to satisfy the usual commutation relation

$$[\mathcal{R}(\tau, \mathbf{x}), p_S(\tau, \mathbf{y})] = i\, \delta^{(3)}(\mathbf{x} - \mathbf{y}). \tag{184}$$

With the mode expansion (180) the operators $b_\mathbf{k}$ and $b_\mathbf{k}^\dagger$ then satisfy the algebra

$$[b_\mathbf{k}, b_\mathbf{q}^\dagger] = \delta^{(3)}(\mathbf{k} - \mathbf{q}), \qquad [b_\mathbf{k}, b_\mathbf{q}] = 0 \tag{185}$$

and the mode functions are normalised by

$$\frac{u^2}{2}\left(\mathcal{R}_k' \mathcal{R}_k^* - \mathcal{R}_k \mathcal{R}_k'^*\right) = i. \tag{186}$$

Defining the vacuum state as the state that gets annihilated according to $b_\mathbf{k}\,|0\rangle = 0$, the power spectrum is given by

$$\mathcal{P}_\mathcal{R}^F = \frac{k^3 |\mathcal{R}_k|^2}{4\pi^2}. \tag{187}$$

Note that this general form of the power spectrum resembles the general form found for tensor perturbations (145) divided by two, the number of polarisations.

Setting the Bunch-Davies vacuum works analogous to the method used above for the tensor modes and restricts the integration constants in (182) to $B = -iA$. The normalisation condition (186) then gives the remaining constant A as

$$|A|^2 = \frac{z^{1-2\,c_S}\, \pi}{2k\, u^2}. \tag{188}$$

Here u^2 has to be expressed in terms of z. This is easily done by integrating equation (178). Note that for $\varepsilon = \delta$, (178) gives $u'/u = \mathcal{H} = a'/a$. This then suggests, that when integrating (178) the initial value is to be chosen such that $u|_{\varepsilon=\delta} = a$, where a has been set, by demanding that it agrees with the scale factor of de Sitter space for $\varepsilon = 0$ in section 6.1.2 and u is therefore

$$u = (-\tau)^{-(1+2\varepsilon-\delta)} = z^{1/2-c_S} k^{-1/2+c_S}. \tag{189}$$

The integration constant A is

$$|A|^2 = \frac{\pi}{2k^{2\,c_S}} \tag{190}$$

and the full power spectrum is given by

$$\mathcal{P}_{\mathcal{R}}^F = \frac{k^{3-2\,c_S}}{8\pi} z^{2\,c_S} |J_{c_S}(z) - iY_{c_S}(z)|^2 \tag{191}$$

and becomes

$$\mathcal{P}_{\mathcal{R}} = \lim_{z\to 0} \mathcal{P}_{\mathcal{R}}^F = 2^{2c_S} \frac{k^{3-2\,c_S}}{8\pi^3} \Gamma^2(c_S) \tag{192}$$

in the super Hubble limit. The first thing to note here is that this spectrum is dependent on k. Keeping in mind $3 - 2c_S = 2\delta - 4\varepsilon$, the spectral index for scalar perturbations, defined as

$$n_S - 1 = \frac{d \ln \mathcal{P}_{\mathcal{R}}}{d \ln k} \tag{193}$$

becomes

$$n_S - 1 = 2\,\delta - 4\,\varepsilon \tag{194}$$

for slow-roll inflation.

As is the case for the tensor modes, this power spectrum can be rewritten in terms of the value of the Hubble parameter at which the mode k leaves the Hubble horizon, $H(t_k)a(t_k) = k$. Recall from the discussion of the slow-roll parameters in 3.1.2 that $-4\pi\varphi = \dot{H}$ and $\varepsilon = -\dot{H}/H^2$. Therefore u can be rewritten as

$$u^2(t_k) = \frac{k^2 \varepsilon}{4\pi H^2(t_k)}. \tag{195}$$

With equation (189) and keeping in mind that at horizon crossing the conformal time $\tau_k \simeq -k^{-1}$, this equation can further be rewritten as

$$1 = \frac{k^{3-2\,c_S}\varepsilon}{4\pi H^2(t_k)} \tag{196}$$

and the power spectrum becomes

$$\mathcal{P}_{\mathcal{R}} = \frac{H^2(t_k)}{\pi\,\varepsilon} \Big(\frac{2^{2c_S-1}\,\Gamma^2(c_S)}{\pi} \Big). \tag{197}$$

Note, that for $\delta = \varepsilon$

$$\mathcal{P}_{\mathcal{R}}|_{\delta=\varepsilon} = \frac{\mathcal{P}_h}{4\,\varepsilon}. \tag{198}$$

The difference between the scalar and tensor power spectra are expected to be small due to the smallness of the slow-roll parameters.

Thus I have calculated the spectral index for tensor and scalar modes in slow-roll inflation and can define the tensor-to-scalar ratio, which are the parameters that are used to compare a model of inflation to observations of the CMB.

7.4. Spectral indices, Tensor-to-Scalar Ratio and Observation

So far I have calculated the power spectrum for scalar and tensor fluctuations, and defined and calculated their spectral indices

$$n_S - 1 = 2\,\delta - 4\,\varepsilon, \qquad n_T = -2\,\varepsilon. \tag{199}$$

These describe the dependence of the power spectra on the wave number k and scalar and tensor spectra are scale invariant for $n_S = 1$ and $n_T = 0$ respectively.

A further quantity that is used to compare models of inflation to CMB measurements is the tensor-to-scalar ratio that is defined as, and given for slow-roll inflation by

$$r = \frac{\mathcal{P}_h}{\mathcal{P}_\mathcal{R}} \simeq 4\,\varepsilon \tag{200}$$

where the last expression follows from (198). Note that this is not the relation usually found in the literature [11, 12]. This is due to the *xPand* convention for the SVT decomposition, where the tensor perturbation is half of what is usually used. Therefore, to translate values found here to values found elsewhere, multiply by two for every factor of h_{ij}.

The power spectrum is proportional to the tensor perturbation squared. Calling the power spectrum in the convention used by most of the world $\widetilde{\mathcal{P}}_h = 4\mathcal{P}_h$, the tensor-to-scalar ratio is then found to be

$$\widetilde{r} = \frac{\widetilde{\mathcal{P}}_h}{\mathcal{P}_\mathcal{R}} \simeq 16\,\varepsilon. \tag{201}$$

The values $\widetilde{r}$ and n_S encode the same information as the slow-roll parameters and therefore constitute a map between inflationary model and CMB observables. The relationship between the tensor-to-scalar ratio and the scalar spectral index is usually given in terms of an $(\widetilde{r}, n_S)$ plot and is best explained through an example.

7.4.1. An Example of Slow-Roll Inflation, continued

In section 3.1.4 I discussed inflation with a potential

$$V(\varphi) = \lambda\,\varphi^n. \tag{202}$$

In this model the slow-roll parameters and number of e-folds for which inflation lasts are

$$\varepsilon = \frac{n^2}{16\,\pi\,\varphi^2}, \qquad \delta = \frac{n(n-2)}{16\,\pi\,\varphi^2}, \qquad N = \frac{4\pi}{n}\varphi_i^2 - \frac{n}{4} \tag{203}$$

where φ_i is the value of the Klein-Gordon field at the beginning of inflation. To calculate spectral indices, these values are only interesting at the time t_k when modes with wave number k grow bigger than the Hubble radius. This can easily be achieved by introducing

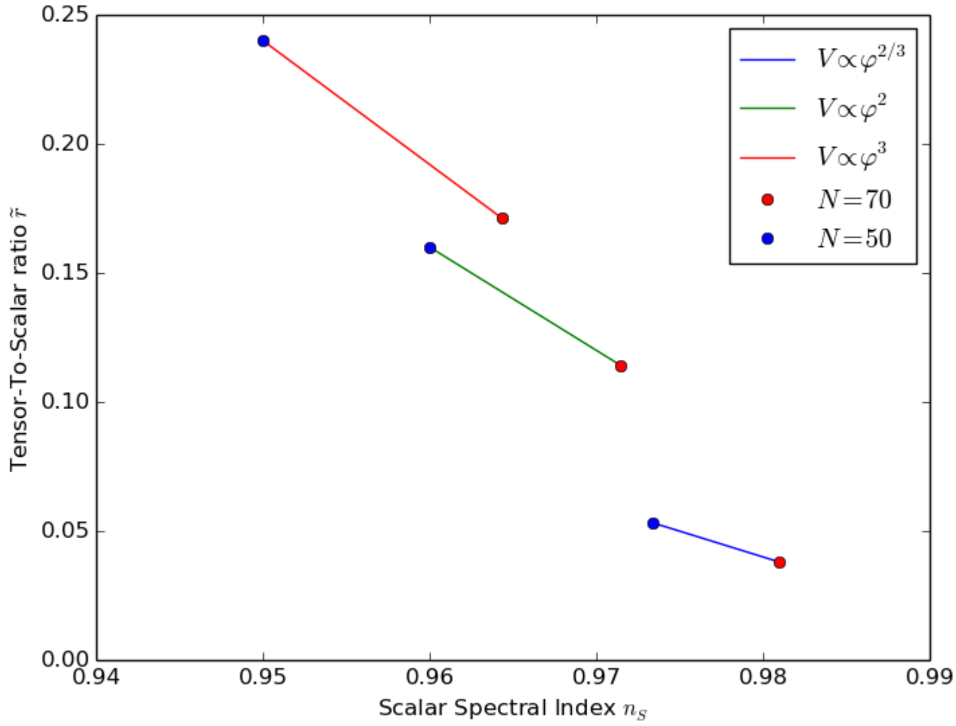

Figure 5: The $(\widetilde{r}, n_S)$ plot.

the number of e-folds from the time t_k to the end of inflation, N_k, and φ_k the value of the Klein-Gordon field at t_k. The relevant quantities are then

$$\varepsilon(t_k) = \frac{n^2}{16\,\pi\,\varphi_k^2} \qquad \delta(t_k) = \frac{n(n-2)}{16\,\pi\,\varphi_k^2}, \qquad N_k = \frac{4\pi}{n}\varphi_k^2 - \frac{n}{4}. \tag{204}$$

Assuming that $n \ll 4\pi$ gives, $\varphi_k^2 \simeq N_k\, n/4\pi$ and the tensor-to-scalar ratio and scalar spectral index are

$$\widetilde{r} = \frac{4\,n}{N_k}, \qquad n_S - 1 = -\frac{(n+2)}{2N_k}. \tag{205}$$

These can be combined to give $\widetilde{r}$ as a function of n_S

$$\widetilde{r}(n_S) = \frac{8n(1-n_S)}{n+2}. \tag{206}$$

and n_S can be understood as a function of N_k. As the modes that are relevant for CMB observations are those that exit the Hubble sphere towards the beginning of inflation, and inflation is thought to have lasted between 50 and 70 e-folds, the domain of interest is $n_S \in [1-(n+2)/100, 1-(n+2)/140]$. $\widetilde{r}(n_S)$ can then be plotted, as has been done in figure 5 for a few different values of n and these values and plots can be compared to observation [29]. This concludes my general discussion of single-field inflationary models.

8. An Inflationary Model Based on Conformal Gravity

Conformal, or Weyl gravity, is a theory of gravity which is invariant under local Weyl rescaling. Historically speaking it was the first alternative proposed to Einstein gravity and has recently become of interest as it shows up in low energy limits of string theory. It is also of theoretical interest as, unlike Einstein gravity it is a renormalisable theory. Its phenomenology has been studied extensively by Mannheim, especially as an attempt to explain galactic rotation curves without dark matter [9, 30, 31]. The holographic properties have recently been studied in [32, 33] and 't Hooft has recently studied conformal gravity in the context of quantum gravity in [34].

As conformal gravity does, by design, not know about scale, it is an interesting question what conformal gravity has to offer in terms of scale invariant spectra. With this aim in mind I will first discuss general properties of conformal gravity and then discuss a model based on conformal gravity for which I will later calculate power spectra of fluctuations.

8.1. Conformal Gravity Action and Equations of Motion

Conformal gravity is based on the Weyl tensor ${C^\mu}_{\nu\sigma\lambda}$. Its most important properties are summed up in appendix B. Given the properties of the Weyl tensor and defining $C^2 := C^{\mu\nu\sigma\lambda}C_{\mu\nu\sigma\lambda}$ the action of conformal garvity is given by

$$S_{CG} = \int \omega_g C^2 = \int \omega_g (2R_{\mu\nu}R^{\mu\nu} - \frac{2}{3}R^2 + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2) \tag{207}$$

and is invariant under conformal transformations

$$g_{\mu\nu} \to \Omega^2(x^\rho)g_{\mu\nu}. \tag{208}$$

Given the Gauß-Bonnet theorem

$$8\pi^2\chi(M) = \int_M \omega_g (R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2) \tag{209}$$

where $\chi(M)$ is a topological invariant of the manifold M, the action can be rewritten as

$$S_{CG} = \int \omega_g (2R_{\mu\nu}R^{\mu\nu} - \frac{2}{3}R^2) + 8\pi^2\chi(M). \tag{210}$$

The Gauß-Bonnet term is then irrelevant for the equations of motion and after a straightforward, but long and tedious calculation, variation of the action gives

$$\delta S_{CG} = 4\int \omega_g B_{\mu\nu}\delta g^{\mu\nu} \tag{211}$$

making the equations of motion the Bach-flatness conditions

$$B_{\mu\nu} = 0 \tag{212}$$

where $B_{\mu\nu} = (\nabla^\rho\nabla^\sigma + \frac{1}{2}R^{\rho\sigma})C_{\rho\mu\sigma\nu}$ is the Bach tensor.

It is obvious that equation (212) is solved by all conformally flat metrics, as for those the Weyl tensor vanishes. The Bach flatness condition is also satisfied by all solutions of the vacuum Einstein equations, with or without cosmological constant. In an empty region of spacetime the Einstein equations reduce to the Ricci flatness condition $R_{\mu\nu} = 0$, which in turn causes $C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}$ and reduces (212) to $\nabla^\rho\nabla^\sigma R_{\rho\mu\sigma\nu} = 0$, which is trivially satisfied as the first Binachi identity, $\nabla^\rho R_{\mu\nu\rho\sigma} = \nabla_\mu R_{\nu\sigma} + \nabla_\nu R_{\mu\sigma}$ reduces to $\nabla^\rho R_{\mu\nu\rho\sigma} = 0$, for a Ricci flat metric. The same holds for metrics of Einstein spaces, for which $R_{\mu\nu} \propto g_{\mu\nu}$. Conformal gravity therefore contains all phenomenology of vacuum Einstein gravity with a cosmological constant. Indeed, choosing specific boundary conditions, these are all solutions that remain for Conformal gravity [35].

The invariance of the action (207) under a Weyl rescaling translates into the Bach tensor transforming as

$$B_{\mu\nu} \to \Omega^{-6}(x^\rho)B_{\mu\nu}. \tag{213}$$

This in turn means that if a metric $g_{\mu\nu}$ solves (212) then so does $\Omega^2(x^\rho)g_{\mu\nu}$. As all FLRW metrics are conformally flat [31] they are all solutions of conformal gravity. However this also implies that conformal gravity will by itself not restrict the scale factor $a(t)$ and an evolution of the universe could randomly be chosen, thereby diminishing the predictive power of pure conformal gravity for cosmology. Ignoring this problem, one could choose a scale factor and study cosmological perturbations on this background. This has been done for de Sitter space in [36]. The path taken here is to introduce scalar field matter that will fix the scale factor.

8.2. Inflation From Conformal Invariance

Keeping in mind that what is of interest are the power spectra of a theory that is conformally invariant, add a conformally coupled scalar field to the conformal gravity action (207) according to

$$S = \int \omega_g\big(C^2 - \frac{1}{2}\nabla^\mu\varphi\nabla_\mu\varphi - \frac{1}{12}R\varphi^2 + \lambda\varphi^4\big) \tag{214}$$

If the field φ transforms as $\varphi \to \Omega^{-1}\varphi$ under conformal transformations (208), i.e. φ is a field of conformal weight minus one the action is indeed conformally invariant.

This model can also be motivated by noting that recently models of inflation based on conformally coupled scalar fields have been under investigation [37, 38, 39] and introducing conformal gravity into the picture amounts to further exploiting what is allowed by the constraint of Weyl invariance.

The scalar field in (214) has a curious property. As the scalar field is of conformal weight minus one and the action is scale invariant, one is free to choose a frame and thereby the field φ. Of course the action and by extension the equations of motion take a simple form in a frame where the scalar field is constant and I choose $\varphi = \sqrt{6}$ making the action

$$S = \int \omega_g \big(C^2 - \frac{1}{2} R + \Lambda \big). \tag{215}$$

where $\Lambda = 36\lambda$. This is a type of Einstein-Weyl gravity with a cosmological constant and theories of this type have been investigated before in [40] and is known to have seven degrees of freedom [40, 41], two massless graviton polarisations and five massive ones.

The equations of motion that follow from (215) are the Einstein-Bach equations given by

$$4B_{\mu\nu} = \frac{1}{2} G_{\mu\nu} + g_{\mu\nu} \frac{\Lambda}{2}. \tag{216}$$

As the Weyl tensor of an FLRW metric vanishes, the ansatz $ds^2 = -dt^2 + a^2(t)\delta_{ij} dx^i dx^j$ produces the equation

$$\frac{\dot{a}^2}{a^2} = \frac{\Lambda}{3} \tag{217}$$

which is solved by $a(t) = \exp{(Ht)}$ where the Hubble parameter is $H^2 = \Lambda/3$. Equation (216) therefore restricts FLRW metrics to de Sitter space. Following [37] the action (214) could be complemented with another scalar field of conformal weight minus one to produce more realistic inflationary scenarios. This will not be done here as de Sitter space gives a simple setting for a first study of fluctuations from (215).

9. Einstein-Weyl Perturbations During Inflation

This section will discuss the classical and quantum behaviour of perturbations in a theory based on the Einstein-Weyl action (216), which is (214) in a frame where the scalar field $\varphi = \sqrt{6}$. This theory has seven degrees of freedom [40, 41] which in terms of the SVT decomposition split up into one scalar, two vector and four tensor degrees of freedom.

Unlike in inflation based on Einstein gravity, like the single field model (46), the vector perturbation in the model under investigation will not decay. This is all together not that surprising as linearised conformal gravity is known to contain a massless spin one degree of freedom [42]. The tensor modes describe the same two degrees of freedom as Einstein gravity plus another two degrees of freedom.These added two degrees of freedom and the vector perturbations describe ghost degrees of freedom and spoil unitarity of the theory.

The scalar degrees of freedom are ignored here. As a more realistic model of inflation would require an extra scalar field that would couple to the scalar degree of freedom already present in (214) the equations of motion would be vastly different, in contrast to the vector and tensor modes where the difference in the equations of motion amounts to changing the scale factor and conformal Hubble parameter a and $\mathcal{H}$ of the resulting slow-roll type model.

9.1. Vector Perturbations

9.1.1. Classical Vector Perturbations

Perturbing the Einstein-Weyl gravity action with a cosmological constant (215) to second order gives the action

$$S_V = \int d^4x\big(-\frac{a^2(\tau)}{4}\partial_i V_j\, \partial^i V^j - \Delta V_j \Delta V^j + \partial_i V_j' \partial^i V'^j\big) \tag{218}$$

for the vector perturbations. The first term in the action comes from the Ricci scalar sector of (215) and the second and third term comes from the conformal gravity sector. Note that only the first term contains the scale factor $a(\tau)$. The scale factor cannot show up in the conformal gravity part as the latter is agnostic regarding the scale factor. The expansion here was done with *xPand* and agrees with the action found in [36].

The equation of motion that follow from (218) are

$$-\frac{a^2}{4}\Delta V_i - \Delta V_i'' + \Delta^2 V_i = 0. \tag{219}$$

In analogy with section 6.1 the vector perturbations can be expanded into mode functions

$$V_i = \sum_{p=1}^{2} \int \frac{d^3k}{\sqrt{2}(2\pi)^{3/2}} \Big(a_{\mathbf{k},p} V_{k,p}^*(\tau) \exp{(i\mathbf{k}\cdot\mathbf{x})} + a_{\mathbf{k},p}^\dagger V_{k,p}(\tau) \exp{(-i\mathbf{k}\cdot\mathbf{x})}\Big) e_i^p \tag{220}$$

where the polarisation tensor satisfies $k^i e_i^p = 0$ and $e_i^p e^{(m)i} = \delta^{pm}$. The action, mode expansion and equations of motion for the mode function, for one polarisation are given by

$$S_{VP} = \int d^4x\Big(-\frac{a^2}{4}\partial_i V\, \partial^i V - \Delta V \Delta V + \partial_i V' \partial^i V'\Big) \tag{221}$$

$$V(\tau, \mathbf{x}) = \int \frac{d^3k}{\sqrt{2}(2\pi)^{3/2}} \Big(a_{\mathbf{k}} V_k^*(\tau) \exp{(i\mathbf{k}\cdot\mathbf{x})} + a_{\mathbf{k}}^\dagger V_k(\tau) \exp{(-i\mathbf{k}\cdot\mathbf{x})}\Big) \tag{222}$$

$$V_k'' + (k^2 + \frac{a^2}{4}) = 0. \tag{223}$$

The scale factor for de Sitter space is $a(\tau) = -1/H\tau$ and employing the usual variable $z = -k\tau$ the equation of motion (223) can be written as

$$\frac{d^2 V_k}{dz^2} + \Big(1 + \frac{1}{4H^2z^2}\Big) V_k = 0 \tag{224}$$

which is solved by

$$V_k(\tau) = \sqrt{z}\Big(A\,J_c(z) + BY_c(z)\Big), \qquad c^2 = \frac{1}{4} - \frac{1}{4H^2}. \tag{225}$$

This will allow us to calculate the power spectrum of vector perturbations on de Sitter space after quantization.

9.1.2. Quantizing Vector Perturbations

From the action (221) the canonical momentum is given by

$$p_V = -2\Delta V'. \tag{226}$$

Following the usual algorithm of canonical quantization the mode functions and the amplitudes turned operators $a_{\mathbf{k}}$, satisfy the normalisation condition and algebra

$$V_k^* V_k' - V_k V_k'^* = \frac{i}{k^2}, \qquad [a_{\mathbf{k}}, a_{\mathbf{q}}^\dagger] = \delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{227}$$

and the vacuum is given by $a_{\mathbf{k}}\left|0\right\rangle = 0$. The two point function for one polarisation is then

$$\left\langle 0\right| V(\tau, \mathbf{x}) V(\tau, \mathbf{y}) \left|0\right\rangle = \int \frac{dk}{k}\frac{k^3}{4\pi^2}|V_k|^2 \frac{\sin{(kr)}}{kr}. \tag{228}$$

The 2-point correlator for the vector perturbation is then simply the sum over the two point functions of the polarisations and the power spectrum reads

$$\mathcal{P}_V^F = \frac{k^3|V_k|^2}{2\pi^2}. \tag{229}$$

Given the equation (224), its solution (225) with the large z behaviour

$$V_k \to \sqrt{\frac{2}{\pi}}\Big(A\cos{(z - \frac{c\pi}{2} - \frac{\pi}{4})} + B\sin{(z - \frac{c\pi}{2} - \frac{\pi}{4})}\Big) \tag{230}$$

the Bunch-Davies vacuum condition, ignoring the phase in (230), is given by

$$\lim_{z\to\infty} V_k(\tau) \propto \exp{(-iz)}. \tag{231}$$

This fixes $B = -iA$ giving

$$V_k = \sqrt{z}A\Big(\,J_c(z) - iY_c(z)\Big). \tag{232}$$

Fixing A then amounts to solving the normalisation condition (227). This equation does not only constrain A but also c and thereby H^2 and Λ. The normalisation condition will result in an equation of the type $i|A|^2 f(k) = i/k^2$. As the square of the magnitude of the possibly complex number A is a real number, for this equation to make sense $f(k)$ has to be a real number. For complex c this expression is however not guaranteed to be real. To guarantee the realness of $|A|$, c has to be real, which forces $H^2 \geq 1$. This then translates into the cosmological constant to be constrained from below, $\Lambda \geq 3$.

The constant A is then found to be

$$|A|^2 = \frac{\pi}{4k^3} \tag{233}$$

making the power spectrum

$$\mathcal{P}_V^F = \frac{z}{8\pi}|J_c(z) - iY_c(z)|^2. \tag{234}$$

Given the behaviour of the Bessel functions in the $z \to 0$ limit, the mode functions behave as

$$V_k = -\frac{i\,2^c A}{\pi}\Gamma(c) z^{1/2-c}. \tag{235}$$

Their exact behaviour in the super-Hubble regime now depends on the value of the constant c. From (225) and the fact that c is a real number it follows that $c \in [0, \frac{1}{2}]$. For the value $c = \frac{1}{2}$, which corresponds to $\Lambda = 3$, the dependence on z vanishes making the vector modes constant. For any other possible value of c the mode functions will decay approaching $z = 0$. Recall from section 4.3.1 that similar behaviour is observed for vector perturbations in Einstein gravity. The difference is that on de Sitter space the Einstein vector modes decay as $V_i \propto (-\tau)^2$ approaching $\tau = 0$, while for Einstein-Weyl gravity they decay at most with $V_i \propto (-\tau)^{1/2}$ and can become constant.

The power spectrum in the $z = 0$ limit is therefore

$$\mathcal{P}_V = \lim_{z\to 0} \mathcal{P}_V^F = \frac{1}{4\pi^2}\delta_{\Lambda 3}. \tag{236}$$

9.2. Tensor Perturbations

9.2.1. Classical Solutions

Perturbing the action (215) to second order in tensor perturbations gives the action

$$S_T = \int d^4x \Big(-\frac{a^2(\tau)}{2}\big(h'_{ij}h'^{ij} - \partial_l h_{ij}\,\partial^l h^{ij}\big) + 2h''_{ij}h''^{ij} - 4\,\partial_l h'_{ij}\,\partial^l h'^{ij} + 2\Delta h_{ij}\Delta h^{ij}\Big) \tag{237}$$

which again is made up of a sector that comes from the Ricci scalar part of (215), marked by the appearance of the scale factor $a(\tau)$, and by the rest which stems from the C^2 part. The equations of motion that then follow for de Sitter space are

$$h''''_{ij} - 2\Delta h''_{ij} + \Delta^2 h_{ij} = \frac{1}{4H^2\tau^2}\big(- h''_{ij} + \frac{2}{\tau}h'_{ij} + \Delta h_{ij}\big). \tag{238}$$

This equation can be written in a number of equivalent forms.

First note that with the d'Alambertian of de Sitter space $\Box_{dS}$ [31] this equation can be rewritten as

$$\big(\Box_{dS} - 2H^2\big)\big(\Box_{dS} - 4H^2 - \frac{1}{4}\big)\big(a^2 h_{ij}\big) = 0 \tag{239}$$

which is the equation found in [40] and shows that the approach taken there and here are compatible. This also suggests that equation (238) can be written as a product of two linear second order differential operators. For the purposes here there is however a better decomposition.

From [43] it is known that (238) can be decomposed on de Sitter space as

$$\Big(\frac{d^2}{d\tau^2} + \frac{2}{\tau}\frac{d}{d\tau} - \Delta + \frac{1}{4H^2\tau^2}\Big)\Big(\frac{d^2}{d\tau^2} - \frac{2}{\tau}\frac{d}{d\tau} - \Delta\Big)h_{ij} = 0. \tag{240}$$

To discuss how to solve this, think of this equation as $L_1 L_2 h_{ij} = 0$, where L_1 and L_2 are the two linear second order differential operators. Any solution of the equation $L_2 f_2 = 0$ will automatically be a solution of (240). Consider a solution f_1 of the equation $L_1 f_1 = 0$. f_1 will not solve (240), but a function f_3 that is a solution of $L_2 f_3 = f_1$ will, as $L_1 L_2 f_3 = L_1(L_2 f_3) = L_1 f_1 = 0$. This solution strategy then suggests that the tensor modes will be of the form

$$h_{ij} = h_{ij}^{(1)} + h_{ij}^{(2)} \tag{241}$$

prompting the mode expansion

$$h_{ij}(\tau,\mathbf{x}) = \sum_{p=1}^{2}\int \frac{d^3k}{\sqrt{2}(2\pi)^{3/2}}\Big(a_{\mathbf{k},p}\, h_k^{(1)*}(\tau)\exp\left(i\mathbf{k}\cdot\mathbf{x}\right) + b_{\mathbf{k},p}\, h_k^{(2)*}(\tau)\exp\left(i\mathbf{k}\cdot\mathbf{x}\right) +$$
$$+ a^\dagger_{\mathbf{k},p}\, h_k^{(1)}(\tau)\exp\left(i\mathbf{k}\cdot\mathbf{x}\right) + b^\dagger_{\mathbf{k},p}\, h_k^{(2)}(\tau)\exp\left(i\mathbf{k}\cdot\mathbf{x}\right)\Big)e^p_{ij} \tag{242}$$

where the polarisation tensor satisfies the usual traceless and transversal condition. Just one polarisation has to be considered and the relevant action reads

$$S_{TP} = \int d^4x\Big(-\frac{1}{2H^2\tau^2}\big(h'^2 - \partial_l h\,\partial^l h\big) + 2h''^2 - 4\,\partial_l h'\,\partial^l h' + 2\Delta h\Delta h\Big). \tag{243}$$

The relevant mode expansion for one polarisation is given by (242) without the sum and the equation of motion for the mode function is given by

$$\Big(\frac{d^2}{dz^2} + \frac{2}{z} + 1 + \frac{1}{4H^2\tau^2}\Big)\Big(\frac{d^2}{dz^2} - \frac{2}{z}\frac{d}{dz} + 1\Big)h_k^{(i)} = 0 \tag{244}$$

where $z = -k\tau$ of course. This is solved by

$$h_k^{(1)} = A\big(z\sin\left(z\right) + \cos\left(z\right)\big) + B\big(\sin\left(z\right) - z\cos\left(z\right)\big) \tag{245}$$

$$h_k^{(2)} = z^{3/2}\big(D\,J_c(z) + F\,Y_c(z)\big), \qquad c^2 = \frac{1}{4} - \frac{1}{4H^2}. \tag{246}$$

The first solution has been encountered before in section 6.1.1 for tensor perturbations on de Sitter space in single field inflation. The second solution, along with the vector solution (225) and the scalar solution not discussed here, are the degrees of freedom added to the usual Einstein degrees of freedom (245) through Weyl gravity in (215)

9.2.2. Quantizing Tensor Perturbations

Much like solving the equations of motion, quantizing the tensor perturbations described by (237) is more complicated than what has been encountered so far, due to the second order derivative with respect to conformal time showing up in the action (243). In classical mechanics the Legendre transformation replaces the first derivative of the generalized coordinate with a generalized momentum, taking you from configuration to phase space. In a Lagrangian with derivatives of order two or higher this transformation will not take you into what is normally thought of as phase space and a different route has to be taken.

The Lagrangian can be reduced to second order by introducing extra fields. Following [43, 44] introduce the auxiliary field β and fix the condition $\beta = h'$ with a Lagrange multiplier λ. The action

$$S_\beta = \int d^4x(-\frac{1}{2H^2\tau^2}(h'^2 - \partial_k h \partial^k h) + 2(\beta'^2 - 2\partial_k h' \partial^k h' + (\Delta h)^2 - \frac{\lambda}{2}(\beta - h')) \tag{247}$$

is equivalent to (243). The equations of motion for the auxiliary field and Lagrange multiplier are

$$8\beta'' + \lambda = 0 \tag{248}$$
$$\beta - h' = 0. \tag{249}$$

Going on-shell with β gives back the action (243), $S_\beta|_{\beta=h'} = S_{TP}$. As the Lagrange multiplier will not show up in the total Hamiltonian constructed from (247) it is not of interest for quantization and the canonical momenta of interest are

$$p_h = \frac{\partial \mathcal{L}}{\partial h'} = -\frac{h'}{H^2\tau^2} + 8\Delta h' + \frac{\lambda}{2} = -\frac{h'}{H^2\tau^2} + 8\Delta h' - 4h''' \tag{250}$$
$$p_\beta = \frac{\partial \mathcal{L}}{\partial \beta'} = 4\beta' = 4h'' \tag{251}$$

where the last equality employed the auxiliary and Lagrange multiplier equations of motion. This system can now be quantized by promoting the canonical coordinates and momenta to operators and demand that they satisfy

$$[h(\tau, \mathbf{x}), p_h(\tau, \mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \qquad [\beta(\tau, \mathbf{x}), p_\beta(\tau, \mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}). \tag{252}$$

Expressing these commutators in terms of the mode expansion (242) then gives restrictions on the algebra of the operators $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$, and gives two normalisation conditions for the mode functions $h_k^{(1)}$ and $h_k^{(2)}$. Here the algebra is not uniquely fixed and I make the ansatz

$$[a_{\mathbf{k}}, a_{\mathbf{q}}^\dagger] = \delta^{(3)}(\mathbf{k} - \mathbf{q}), \qquad [b_{\mathbf{k}}, b_{\mathbf{q}}^\dagger] = \sigma\, \delta^{(3)}(\mathbf{k} - \mathbf{q}) \tag{253}$$

where σ is a constant to be determined. With this ansatz the commutation relations (253) produce the normalisation conditions

$$\Big(\frac{1}{H^2\tau^2}+8k^2\Big)\Big(h_k^{(1)*}h_k^{(1)'}-h_k^{(1)'*}h_k^{(1)}+\sigma(h_k^{(2)*}h_k^{\prime(2)}-h_k^{\prime(2)*}h_k^{(2)})\Big)+$$
$$+4\Big(h_k^{(1)*}h_k^{(1)'''}-h_k^{(1)}h_k^{(1)'''*}+\sigma(h_k^{(2)*}h_k^{(2)'''}-h_k^{(2)}h_k^{(2)'''*})\Big)=-2i \tag{254}$$

$$h^{(1)'*}h^{(1)''}-h^{(1)'}h^{(1)''*}+\sigma\big(h^{(2)'*}h^{(2)''}-h^{(2)'}h^{(2)''*}\big)=\frac{i}{2} \tag{255}$$

and the 2-point function for one polarisation gives

$$\langle 0|\,h(\tau,\mathbf{x})h(\tau,\mathbf{y})\,|0\rangle=\int\frac{dk}{k}\frac{k^3}{4\pi^2}(|h_k^{(1)}|^2+\sigma|h_k^{(2)}|^2)\frac{\sin{(kr)}}{kr} \tag{256}$$

where the vacuum has been defined as $a_\mathbf{k}\,|0\rangle=b_\mathbf{k}\,|0\rangle=0$. This makes the power function for both polarisations of the tensor perturbations

$$\mathcal{P}_h^F=\frac{k^3}{2\pi^2}(|h_k^{(1)}|^2+\sigma|h_k^{(2)}|^2). \tag{257}$$

The Bunch-Davies vacuum has to be imposed on the mode functions $h_k^{(1)}$ and $h_k^{(2)}$ separately and the conditions for the solutions (245) and (246) are given by

$$\lim_{z\to\infty}\frac{h_k^{(1)}}{z}=\lim_{z\to\infty}\frac{h_k^{(2)}}{z}\propto\exp{(-iz)} \tag{258}$$

which constrains the integrations constants in (245) and (246) to $B=-iA$ and $F=-iD$. The two normalisation conditions of the mode functions therefore link the three constants A, D and σ. The normalisation condition (255) then gives

$$-\frac{k^3}{\pi}\Big((9-4c^2)\sigma|D|^2+2k^2\tau^2(\pi|A|^2+2\sigma|D|^2)\Big)=\frac{1}{2}. \tag{259}$$

The second term in the big parentheses poses a problem as it depends on conformal time. As A and D are constants they can not depend on time. This problem would be solved if $\pi|A|^2+2\sigma|D|^2=0$. This can be achieved for different ratios between σ and $|D|^2$ and I choose $A=D$ and $\sigma=-\pi/2$.

The other normalisation condition (254) then essentially gives the same equation for A as (255)

$$|A|^2=\frac{1}{k^3(9-4c^2)}=\frac{H^2}{k^3(1+8H^2)}. \tag{260}$$

The power spectrum of tensor perturbations in the theory (215) is then given by

$$\mathcal{P}_h^F=\frac{H^2}{2\pi(1+8H^2)}\Big(|1+iz|^2-\frac{z^3\pi}{2}|J_c(z)-iY_c(z)|^2\Big). \tag{261}$$

The first term comes from the usual Einstein gravitons and the second is a consequence of the presence of the ghost gravitons added by conformal gravity. As usual it comes with a negative sign, signifying the loss of unitarity.

In the super Hubble limit, where $z \to 0$, the second term vanishes due to the constraint $c > 1$ and the power spectrum becomes

$$\mathcal{P}_h = \lim_{z \to 0} \mathcal{P}_h^F = \frac{H^2}{2\pi(1+8H^2)}. \tag{262}$$

This resembles the power spectrum of tensor perturbations on de Sitter space in Einstein gravity (154). Much like (154), (262) is scale invariant and is dependent on the Hubble parameter. While the direct contribution of the ghost gravitons vanish on super Hubble scales they make their presence known via the normalisation constant.

10. Conclusion

I have investigated what the properties of power spectra of cosmological perturbations during inflation are in a Weyl invariant theory (214), where conformal gravity is conformally coupled to a real scalar field of conformal weight minus one. Due to the invariance of the action under local Weyl rescaling, the scalar field can be chosen to take any desired form. In a frame where it is constant the theory is simply given as Einstein-Weyl gravity with a cosmological constant (215). Reminiscent of the Higgs mechanism, this can be interpreted as the partially massless gravitons of conformal gravity eating up the scalar field, becoming massive in the process.

For an FLRW ansatz Einstein-Weyl gravity with a cosmological constant (215) only allows de Sitter space, which can be interpreted as a model of eternal inflation. While this is not a very realistic scenario as inflation would have to end in order to allow the universe to evolve according to the Big Bang theory, it is nevertheless a simple setting allowing a first investigation of vector and tensor perturbations in Einstein-Weyl gravity.

There are two main results, one from the vector and one from the tensor perturbation sectors of the theory. On the vector side, the presence of a C^2 term introduces vector modes that decay with time (235), but slower than their Einstein gravity counter part, see section 4.3.1.

The power spectrum of vector perturbations in the super-Hubble limit (236) is then either zero or constant, not only in time but also in scale making the vector perturbations scale invariant. This sector of the theory however, is only relevant for observables for a specific value of the cosmological constant $\Lambda = 3$, which corresponds to $\lambda = 1/12$ in the original action (214). This behaviour is somewhat surprising as in the classical theory the value $\lambda = 1/12$ has no significance and, apart from stemming from calculation, currently lacks a deeper explanation. The appearance of a non-vanishing vector perturbation in the super-Hubble regime however is not surprising. Setting the cosmological constant in (215) zero and requiring the background to be maximally symmetric produces Minkowski space, on which vector perturbations from Einstein-Weyl gravity show the same behaviour as the massive tensor perturbations [45]. The possibility of a vanishing power spectrum could have been expected from the discussion found in [46].

The tensor sector, unlike the vectors, produces a power spectrum on super Hubble scales (262), for all allowed values of Λ, with qualities similar to the power spectrum of pure Einstein gravity on de Sitter space (154). They are both scale invariant and contain information about the energy scale at which inflation took place, as the square of the Hubble parameter shows up in both power spectra. The same result has independently been found in [47].

As mentioned in section 8.2, a next step to investigate the possibilities of models based on Weyl invariance and conformal gravity would be to conformally couple another scalar field as has been done in [37]. This would then allow to build a more realistic single field type inflation model and vector perturbations could be studied in a more realistic setting. The work done here would serve as a consistency check as it gives the expressions to be expected of the de Sitter limit of the single field type inflation model. It is worth mentioning here that, due to the vanishing of the Noether charge of Weyl symmetry, this approach to inflation model building has come under criticism recently [48]. It is argued that what is presented in [37] is merely a field redefinition.

The results found here have to be taken with a grain of salt. It is established that conformal gravity and Einstein Weyl gravity contain ghost degrees of freedom [42, 8]. On general grounds this spoils unitarity of the theory and it is not at all obvious if such a theory can or will give sensible results. A possible solution of this problem is PT-symmetric quantum mechanics due to Bender [49]. It is argued that the problem of ghosts is not a principle problem of the theory but a problem of choosing the correct conjugated state $\langle\psi|$, which is argued to work for the Pais-Uhlenbeck oscillator [50], though it is not yet clear under what circumstances this conjecture holds [51]. Further investigation into this direction would be in order.

With this in mind, my main result are the method of how to quantize the vector and tensor perturbations of Einstein-Weyl gravity, and that vector perturbations during inflation might lead to observables in the CMB spectrum as their mode functions do not necessarily decay, but might approach a constant value (235).

A. The Non-Uniqueness of the Vacuum

Throughout the main text I have talked about "choosing the Bunch-Davies vacuum" and have then demanded that the mode functions satisfy some asymptotic condition. This prompts two questions. First, how does a condition for the mode functions relate to a vacuum, and second, why is the vacuum not unique? This appendix aims to answer these questions and closely follows [28] and sometimes borrows from [25].

Consider a massive Klein-Gordon field in a flat FLRW spacetime described by the action

$$S = \int d^4x \, a^4 \Big(\frac{1}{2a^2} \varphi'^2 - \frac{1}{2a^2} \partial_i \varphi \, \partial^i \varphi - m^2 \varphi^2 \Big). \tag{263}$$

Redefining the field as $\chi = a\varphi$ this action can equivalently be rewritten as

$$S = \frac{1}{2} \int d^4x \Big(\chi'^2 - \partial_i \varphi \, \partial^i \varphi - m_{eff}^2(\tau) \chi^2 \Big) \tag{264}$$

where $m_{eff}^2(\tau) = a^2 m^2 - a''/a$. This is the action of a Klein-Gordon field with time dependent mass in Minkowski space, where the geometric information about the curved background the field lives on is encoded in the effective time dependent mass m_{eff}. The equation of motion for χ is given by

$$\chi'' - \Delta \chi + m_{eff}^2(\tau) \chi = 0. \tag{265}$$

Fourier transforming the field χ according to

$$\chi(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \, \chi_{\mathbf{k}}(\tau) \exp(i \mathbf{k} \cdot \mathbf{x}) \tag{266}$$

gives the equation

$$\chi_{\mathbf{k}}'' + \Big(|\mathbf{k}|^2 + m^2 a^2 - \frac{a''}{a} \Big) \chi_{\mathbf{k}} = 0. \tag{267}$$

which is of the type $\chi'' + \omega^2(\tau)\chi = 0$. It is known from the theory of ordinary differential equations that such equations have a two dimensional space of solutions.

Consider two linearly independent solutions $\chi_{1\mathbf{k}}(\tau)$ and $\chi_{2\mathbf{k}}(\tau)$ of (267). These can be used as a basis $\{\chi_{1\mathbf{k}}(\tau), \chi_{2\mathbf{k}}(\tau)\}$, but more importantly for the application here, these can be used to motivate a complex basis made up of the functions $v_{\mathbf{k}}(\tau) = \chi_{1\mathbf{k}}(\tau) + i\chi_{2\mathbf{k}}(\tau)$ and $v_{\mathbf{k}}^*(\tau) = \chi_{1\mathbf{k}}(\tau) - i\chi_{2\mathbf{k}}(\tau)$. In terms of the basis $\{v_{\mathbf{k}}, v_{\mathbf{k}}^*\}$ the function $\chi_{\mathbf{k}}$ can be decomposed as

$$\chi_{\mathbf{k}} = \frac{1}{\sqrt{2}} \big(a_{\mathbf{k}} v_{\mathbf{k}}^* + a_{-\mathbf{k}}^\dagger v_{-\mathbf{k}} \big). \tag{268}$$

Plugging the decomposition (268) into (266) then gives the mode expansion

$$\chi(\tau, \mathbf{x}) = \int \frac{d^3k}{\sqrt{2}(2\pi)^{3/2}} \Big(a_{\mathbf{k}} v_{\mathbf{k}}^*(\tau) \exp(i\mathbf{k} \cdot \mathbf{x}) + a_{\mathbf{k}}^\dagger v_{\mathbf{k}}(\tau) \exp(-i\mathbf{k} \cdot \mathbf{x}) \Big). \tag{269}$$

Having set this up, I am now ready to quantize the theory.

The canonical momentum that follows from (264) is $p = \chi'$. Promoting the fields $\chi(\tau, \mathbf{x})$ and $p(\tau, \mathbf{x})$, and thereby $a_\mathbf{k}$ and $a^\dagger_\mathbf{k}$, to operators and demanding that they satisfy $[\chi(\tau, \mathbf{x}), p(\tau, \mathbf{y})] = \delta^{(3)}(\mathbf{x} - \mathbf{y})$ gives the normalisation of the mode functions and the algebra for $a_\mathbf{k}$ as

$$v'_\mathbf{k} v^*_\mathbf{k} - v_\mathbf{k} v'^*_\mathbf{k} = 2i, \qquad [a_\mathbf{k}, a^\dagger_\mathbf{q}] = \delta^{(3)}(\mathbf{k} - \mathbf{q}). \tag{270}$$

The vacuum for the operator a is then defined as $a_\mathbf{k} |0_{(a)}\rangle = 0$, which gives the last piece of information required to address the question: how does a condition on the mode functions relate to a vacuum?

The mode functions $v_\mathbf{k}$ in (269) can be any complex solutions spanning the solutions space of (267), as long as the satisfy the normalisation condition (270). The mode expansion could also have been expressed in a different basis $\{u_\mathbf{k}, u^*_\mathbf{k}\}$ that is related to the basis $\{v_\mathbf{k}, v^*_\mathbf{k}\}$ via

$$v^*_\mathbf{k} = \alpha_\mathbf{k} u^*_\mathbf{k} + \beta_\mathbf{k} u_\mathbf{k}, \qquad |\alpha_\mathbf{k}|^2 - |\beta_\mathbf{k}|^2 = 1 \tag{271}$$

where the second equation that restricts $\alpha_\mathbf{k}$ and $\beta_\mathbf{k}$ is a consequence of the normalisation condition (270) and ensures that it is also satisfied by the basis $\{u_\mathbf{k}, u^*_\mathbf{k}\}$. Use this to express the mode expansion in terms of the new basis and find

$$\chi(\tau, \mathbf{x}) = \int \frac{d^3k}{\sqrt{2}(2\pi)^{3/2}} \Big(u^*_\mathbf{k} \big(\alpha_\mathbf{k} a_\mathbf{k} + \beta^*_\mathbf{k} a^\dagger_{-\mathbf{k}}\big) \exp(i\mathbf{k}\cdot\mathbf{x}) + u_\mathbf{k} \big(\alpha^*_\mathbf{k} a_{-\mathbf{k}} + \beta_\mathbf{k} a^\dagger_\mathbf{k}\big) \exp(-i\mathbf{k}\cdot\mathbf{x}) \Big). \tag{272}$$

Therefore, when expanding the field χ in terms of $\{u_\mathbf{k}, u^*_\mathbf{k}\}$ the creation and annihilation operators in the decomposition are

$$b_\mathbf{k} = \alpha_\mathbf{k} a_\mathbf{k} + \beta^*_\mathbf{k} a^\dagger_{-\mathbf{k}} \tag{273}$$

$$b^\dagger_\mathbf{k} = \beta_\mathbf{k} a_{-\mathbf{k}} + \alpha^*_\mathbf{k} a^\dagger_\mathbf{k}. \tag{274}$$

I can then define a b vacuum by $b_\mathbf{k} |0_{(b)}\rangle = 0$. Transformations (271), (273) and (274) between different mode functions and creation and annihilation operators are called Bogolyubov transformations.

This shows that a given set of mode functions correspond to a certain pair of creation and annihilation operators that define different vacua and choosing a mode function by requiring a certain asymptotic behaviour therefore selects a vacuum for the theory. The b vacuum $|0_{(b)}\rangle$ in general will be an excited state for the $a_\mathbf{k}$ operators and vice versa. To explicitly see this define the operator $N^{(b)} = b^\dagger_\mathbf{k} b_\mathbf{k}$ that gives the particle number of b particles in a given state. For the b vacuum this is, as is to be expected,

$$\langle 0_{(b)}| N^{(b)} |0_{(b)}\rangle = \langle 0_{(b)}| b^\dagger_\mathbf{k} b_\mathbf{k} |0_{(b)}\rangle = 0 \tag{275}$$

meaning that there are no b particles in the b vacuum. For the a vacuum however

$$\begin{aligned}\langle 0_{(a)}| N^{(b)} |0_{(a)}\rangle &= \langle 0_{(a)}| b^\dagger_{\mathbf{k}} b_{\mathbf{k}} |0_{(a)}\rangle &(276)\\ &= |\beta_{\mathbf{k}}|^2 \langle 0_{(a)}| a_{\mathbf{k}} a^\dagger_{-\mathbf{k}} |0_{(a)}\rangle &(277)\\ &= |\beta_{\mathbf{k}}|^2 \delta^{(3)}(0) &(278)\end{aligned}$$

i.e. there is an infinite number of a particles in the b vacuum if $\beta_{\mathbf{k}} \neq 0$. This infinity might seem alarming; it is however to be understood as number of particles in the entire universe and the relevant number is $|\beta_{\mathbf{k}}|$ which is the density of a particles. This then answers the first of the two questions posed at the beginning of this appendix.

Next construct the Hamiltonian of the theory. With the canonical momentum given above the Hamilton function is found to be

$$H(\tau) = \int d^3x \Big(p^2 + \partial_i \chi \, \partial^i \chi + m_{eff}^2 \chi^2 \Big). \tag{279}$$

Upon quantization and after using the mode expansion this is expressed as

$$\begin{aligned}H(\tau) = \frac{1}{4} \int d^3k \Big(& a_{\mathbf{k}} a_{-\mathbf{k}} \big(v'^{*\,2}_{\mathbf{k}} + \omega_k^{*\,2}(\tau) v^{*\,2}_{\mathbf{k}} \big) + a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} \big(v'^2_{\mathbf{k}} + \omega_k^2(\tau) v^2_{\mathbf{k}} \big) + \\ & + \big(2 a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + \delta^{(3)}(0) \big) \big(|v'_{\mathbf{k}}|^2 + \omega_k^2(\tau) |v_{\mathbf{k}}|^2 \big) \Big) \end{aligned} \tag{280}$$

where $\omega_k^2(\tau) = k^2 + m^2 a^2 - a''/a$.

The expectation value of the energy of the ground state is then

$$\langle 0_{(a)}| H(\tau) |0_{(a)}\rangle = \frac{1}{4} \delta^{(3)}(0) \int d^3k \big(|v'_{\mathbf{k}}|^2 + \omega_k^2(\tau) |v_{\mathbf{k}}|^2 \big) + \frac{1}{4} \delta^{(3)}(0). \tag{281}$$

The two infinities that show up in this expression are harmless. The last term, is simply the sum over the zero point energy of infinitely many harmonic oscillators and can simply be dropped. Once more this expression is to be understood as accounting for the energy in the whole universe whereby the physically relevant information becomes the energy density which is given by

$$\lim_{V \to \infty} \frac{E}{V} = \frac{1}{4} \int d^3k \big(|v'_{\mathbf{k}}|^2 + \omega_k^2(\tau) |v_{\mathbf{k}}|^2 \big). \tag{282}$$

For a quantum field theory in flat spacetime the vacuum is now defined as being given by the mode functions that minimize the energy density (282). In curved spacetime this does not work.

In Minkowski space, the frequency ω is time independent, which is no longer the case for curved spacetime. One could then suggest that a vacuum could be chosen by minimizing the energy density for a fixed time τ_0. This type of vacuum is called instantaneous vacuum. Minimizing (282) at a fixed time τ_0 then gives a specific mode function $v_k(\tau)$

and therefore gives the annihilation operator $a_{\mathbf{k}}$ that defines the vacuum at τ_0. If one were to then go and do the same procedure at a later time τ_1 one would find a vacuum that does not agree with the vacuum found at τ_0. Note that (282) can only have a global minimum for times when $\omega_k^2(\tau) > 0$ i.e. there could be times when an instantaneous vacuum can not be defined at all.

This then answers the question why there is no unique vacuum. It is worth noting that despite what most introductory quantum field theory texts would make you believe, there also is no unique vacuum in Minkowski space. The correct statement would be that in Minkowski space all inertial observers define the same vacuum. Indeed an accelerated observer in empty Minkowski space would measure a temperature proportional to the acceleration. This is called the Unruh effect and is a consequence of the inertial and accelerated observer defining different ground states. It would go to far to discuss this here, but a quick discussion of the Unruh effect can be found in [28] and a more thorough review is given in [52].

In de Sitter and quasi de Sitter space a preferred vacuum, called the Bunch-Davies vacuum, can be defined. Here the time dependent frequency is given by

$$\omega^2(\tau) = k^2 - \frac{2}{\tau}. \tag{283}$$

As τ is negative big magnitudes of τ correspond to early times. This then means that for early enough times for all values of k there will be a period of time when the time dependence of ω_k can be ignored and the field will behave as if it were in Minkowski space. This then allows to define a unique vacuum for sufficiently early times.

B. Weyl Tensor

Contracting the first and third index of the Riemann tensor gives the Ricci tensor $R^{\mu}{}_{\nu\mu\sigma} = R_{\nu\sigma}$, which can be understood as the trace of the Riemann tensor. The trace free part of the Riemann tensor is the Weyl tensor $C_{\mu\nu\sigma\lambda}$ and is in n dimensions given by [15]

$$R_{\mu\nu\sigma\lambda} = C_{\mu\nu\sigma\lambda} + \frac{2}{(n-2)}\left(g_{\mu[\sigma}R_{\lambda]\nu} - g_{\nu[\sigma}R_{\lambda]\mu}\right) - \frac{2}{(n-1)(n-2)}Rg_{\mu[\sigma}g_{\lambda]\nu}. \tag{284}$$

The Weyl tensor then inherits the symmetries of the Riemann tensor

$$C_{\mu\nu\sigma\lambda} = -C_{\mu\nu\lambda\sigma} = -C_{\nu\mu\sigma\lambda} \tag{285}$$

$$C_{\mu\nu\sigma\lambda} = C_{\sigma\lambda\mu\nu} \tag{286}$$

$$C_{\mu\nu\sigma\lambda} + C_{\mu\sigma\lambda\nu} + C_{\mu\lambda\nu\sigma} = 0 \tag{287}$$

and is additionally trace free

$$C^{\mu}{}_{\nu\mu\lambda} = 0. \tag{288}$$

The property that is of most interest here is that under Weyl transformations $g_{\mu\nu} \to \Omega^2(x^\rho)g_{\mu\nu}$ the Weyl tensor is invariant

$$C^\mu{}_{\nu\sigma\lambda} \to C^\mu{}_{\nu\sigma\lambda}. \tag{289}$$

From this then follows that $C^2 := C_{\mu\nu\sigma\lambda}C^{\mu\nu\sigma\lambda}$ transforms as

$$C^2 \to \Omega^{-4}(x^\rho)C^2 \tag{290}$$

under local rescaling. As the volume form transforms according to $\omega_g \to \Omega^4(x^\rho)\,\omega_g$ the product $\omega_g C^2$ is invariant

$$\omega_g\, C^2 \to \omega_g\, C^2 \tag{291}$$

making conformal gravity (207) invariant under Weyl transformations.

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