Diplomarbeit
Master thesis

Boundaries in $\mathcal{N} = (2, 2)$ supersymmetric field theories

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Contents

1 Introduction

In this thesis we study boundaries in $\mathcal{N} = (2,2)$ supersymmetric theories in two dimensions. These models are of great importance to string compactifications from 10 to 4 dimensions, because they can be used to describe the dynamics of the 6 dimensional internal space of the target spacetime. For the open string one has to impose boundary conditions at the worldsheet boundary. There a two possibilities for boundary conditions: *Neumann* or *Dirichlet*. The boundary conditions for the string map to extended objects in the target space, called D-branes. In this thesis we are discussing D-branes in the internal dimensions. The presence of D-branes breaks supersymmetry. At most half of the supersymmetry can be preserved. There are two inequivalent ways to do so, called *A-type* and *B-type*. In this thesis we study D-branes preserving *B-type* supersymmetry, called B-branes. We focus on gauged linear sigma models [1] that describe *Calabi-Yau* compactifications. *Calabi-Yau*s have moduli spaces describing deformations of the *Kähler* class and the *complex* structure. In the gauged linear sigma model the parameters for the *Kähler* deformation are identified with certain coupling constants (the *Fayet-Iliopoulos-*θ parameters). We consider a class of *Calabi-Yau*s with one *Kähler* parameter. The *Kähler* moduli space has three distinctive points called large radius, *Landau-Ginzburg* and conifold point. The effective theory describing the *Calabi-Yau* compactification at the large radius point is a non-linear sigma model, at the *Landau-Ginzburg* point it is described by a *Landau-Ginzburg* theory. We are particularly interested in the behaviour of D-branes under transport through the moduli space. To study the behaviour we use the hemisphere partition function of the gauged linear sigma model [2, 3, 4], to compute D-brane charges and monodromy matrices in examples of one-parameter *Calabi-Yau* compactifications. Thereby we were able to reproduce the results in [5], obtained via mirror symmetry.

This thesis is divided into two parts. In part I we outline the necessary prerequisites in order to do the calculations given in part II. The fast-track reader familiar with the subject can directly start in partII.

In chapter 2 we recall the mathematical tools required for this thesis.

In chapter 3 we state the basics of the gauged linear sigma model [1].

In chapter 4 we study boundary conditions in $\mathcal{N} = (2,2)$ supersymmetric theories. We discuss the non-linear sigma model, the *Landau-Ginzburg* model and the gauged linear sigma model, focusing on B-type boundary conditions. Also we introduce further mathematical structures to describe D-branes in these models.

Chapter 5 deals with the low energy behaviour of D-branes and transport of branes in the *Kähler* moduli space.

The hemisphere partition function is introduced in chapter 6.

The field content of the models we focus on and the corresponding hemisphere partition function are given in chapter 7.

In chapter 8 we give a hands-on approach on how to use the methods described in chapter 4 in order to describe D-branes in the models under consideration.

Chapter 9 gives an overview of examples of branes in the gauged linear sigma model, realized as matrix factorisations.

In chapter 10 we calculate the monodromy matrices.

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Part I

Theoretical Background

2
Mathematical Preliminaries

In this chapter we will give an overview of the required mathematical background in this thesis. Thereby we will only scratch the surface of these materials. The presented material is taken out of [6] and [7], where further details can be found. Also we will mostly use the notation of [7].

2.1 Manifolds

We will gradually develop the definition of a complex manifold, following the discussion of [7]. By doing so we will develop all necessary tools to understand *Calabi-Yau* manifolds, which play an important role in this thesis.

Topolocial Manifold

We begin with a topological space. Let X be a set of points endowed with a topology $\mathcal{T}$. $\mathcal{T}$ consists of open subsets U_i of points in X, called open sets. For $\mathcal{T}$ to be a topology we have to require

- X and the empty set $\emptyset \in \mathcal{T}$.
- $\bigcup\limits_i U_i \in \mathcal{T}$ for arbitrary $U_i \in \mathcal{T}$.
- $\bigcap\limits_i U_i \in \mathcal{T}$ for finitely many $U_i \in \mathcal{T}$.

X together with $\mathcal{T}$ is called a topological space. Additionally we want some notion of continuity. Therefore we consider functions ϕ from one topological space to another $\phi : X \to Y$. A function is said to be continuous if $\phi^{-1}(V_j)$ is an open set in X and V_j is open in Y. Of course Y itself has to be a topological space for the definition to make sense.

If we now can cover our topological space X with open subsets $U_i \in \mathcal{T}$ and can find a continuous map $\phi_i : U_i \to \mathbb{R}^n$ for each U_i, with a continuous inverse map ϕ^{-1}, then X is a topological manifold. (U_i, ϕ_i) are usually referred to as charts on X, because the ϕ_i allow us to introduce a local coordinate system for points lying in U_i. Given a point p in U_i we can deduce its coordinates by its image under ϕ_i. The advantage of these local coordinates is, that we now can give a coordinate representation of an abstract function $f : X \to \mathbb{R}$, by considering $f \circ \phi_i^{-1} : \phi_i(U_i) \to \mathbb{R}$.

Compact Manifold A manifold X is said to be compact if every collection $V_j \in \mathcal{T}$, with $X = \bigcup_i V_j$, has a finite subcover. If the collection $\{V_j\}$ consists of finitely many elements, we already have a finite subcover of X. If not we have to find $\{W_k\} \subset \{V_j\}$ such that $X = \bigcup_k W_k$, with k running over finitely many values. We see that compactness depends on the choice of topology $\mathcal{T}$ on X.

Differentiable Manifold

In order to establish a notion of differentiation we have to introduce additional structure on our topological manifold X and thereby obtain a differentiable manifold.

Again we start with a function $f : X \to \mathbb{R}$ on X. Consider the coordinate representation of f in the patch U_i, $f \circ \phi_i^{-1} : \phi_i(U_i) \to \mathbb{R}$. This map is now a map from $\mathbb{R}^n \to \mathbb{R}$ and we can apply ordinary differentiation from multi-variable calculus. If we now consider an overlap of two patches $U_i \cap U_j$, one would reasonably demand that the result of differentiation on the overlap is independent of the chosen coordinate representation. This is only fulfilled if the transition functions $\phi_j \circ \phi_i^{-1}$ are infinitely differentiable.

This restriction gives the additional structure necessary to define a differentiable manifold.

Definiton 2.1.1. *A differentiable manifold X is a topological manifold, with infinitely differentiable transition functions.*

Complex Manifold

Consider a manifold with n real dimensions with n even. Locally we can always introduce local complex coordinates by $z_j = x_j + ix_{\frac{n}{2}+j}$, where the x_j are local real coordinates. But this does not guarantee, that the transition functions are holomorphic, if we express them in local complex coordinates. If the transition functions fulfil the *Cauchy-Riemann* equations the manifold is a complex manifold of dimension $d = \frac{n}{2}$. The local complex coordinates and transition functions endow the manifold with a complex structure.

Definiton 2.1.2. *A complex manifold is a differentiable manifold where the transition functions fulfil the Cauchy-Riemann equations.*

2.2 Tangent Spaces

In order to have a notion of vectors on a manifold one introduces the tangent space. The tangent space can be viewed as a local flat approximation of X at the point p [7]. One can view the tangent space as an $\mathbb{R}^n$ if X has dimension n, or as an equivalence class of curves through the point p, which have the same tangent vector at p

[6]. The tangent space at a point p carries the structure of a vector space. A local basis of the tangent space at the point p is given by the n linearly independent partial derivative operators

$$T_pX : \left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}, \tag{2.1}$$

which allows one to represent a vector v in this basis by $v = v^\alpha \left.\frac{\partial}{\partial x^\alpha}\right|_p$.

As every vector space, also the tangent space has a dual space of linear maps. This dual space is denoted by T_p^*X and usually referred to as cotangent space. A basis is given by

$$T_p^*X : \left\{ \mathrm{d}x^1\big|_p, \dots, \mathrm{d}x^n\big|_p \right\}. \tag{2.2}$$

By definition, $dx^i : T_pX \to \mathbb{R}$ is a linear map with $\mathrm{d}x^i_p \left(\left.\frac{\partial}{\partial x^j}\right|_p \right) = \delta^i_j$. A general element α of T_p^*X can be written as $\alpha = \alpha_i \mathrm{d}x^i$ and is called a one-form.

In the case of a complex manifold X of complex dimension $d = n/2$, we can construct the complexified tangent space of X, $T_pX^{\mathbb{C}} = T_pX \otimes \mathbb{C}$. The only difference to T_pX is, that now the coefficients can be complex-valued. To see the underlying complex structure one rearranges the basis elements to obtain

$$T_pX^{\mathbb{C}} : \left\{ \left.\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^{d+1}} \right)\right|_p, \dots, \left.\left(\frac{\partial}{\partial x^d} + i\frac{\partial}{\partial x^{2d}} \right)\right|_p, \left.\left(\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^{d+1}} \right)\right|_p, \dots, \left.\left(\frac{\partial}{\partial x^d} - i\frac{\partial}{\partial x^{2d}} \right)\right|_p \right\}. \tag{2.3}$$

Re-expressing this basis in terms of complex coordinates gives

$$T_pX^{\mathbb{C}} : \left\{ \left.\frac{\partial}{\partial z^1}\right|_p, \dots, \left.\frac{\partial}{\partial z^d}\right|_p, \left.\frac{\partial}{\partial \bar{z}^1}\right|_p, \dots, \left.\frac{\partial}{\partial \bar{z}^d}\right|_p \right\}. \tag{2.4}$$

A similar construction can be done to get the complexified cotangent space $T_p^*X^{\mathbb{C}} = T_p^*X \otimes \mathbb{C}$, with basis

$$T_p^*X^{\mathbb{C}} : \left\{ \mathrm{d}z^1\big|_p, \dots, \mathrm{d}z^d\Big|_p, \mathrm{d}\bar{z}^1\big|_p, \dots, \mathrm{d}\bar{z}^d\Big|_p \right\}. \tag{2.5}$$

It is possible to separate the holomorphic and anti-holomorphic directions in the tangent and cotangent space, such that

$$T_pX^{\mathbb{C}} = T_pX^{(1,0)} \oplus T_pX^{(0,1)} \tag{2.6}$$
$$T_p^*X^{\mathbb{C}} = T_p^*X^{(1,0)} \oplus T_p^*X^{(0,1)}. \tag{2.7}$$

$T_pX^{(1,0)}$ is called holomorphic tangent space and $T_p^*X^{(1,0)}$ holomorphic cotangent space. The other subspaces as refered as anti-holomorphic tangent space and anti-holomorphic cotangent-space.

2.3 Differential Forms

One can generalize one-forms to forms of higher degree. For example a $q-$form α_p is an antisymmetric multi-linear map

$$\alpha_p : \underbrace{T_pX \times T_pX \times \cdots \times T_pX}_{q-\text{times}} \to \mathbb{R}. \tag{2.8}$$

We know that the $\mathrm{d}x^i$ are a basis for one-forms, but we can also use them to construct a basis for $q-$forms by introducing the wedge-product, denoted by $\wedge$ and defined by

$$\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_q} = \frac{1}{q!}\sum_P \operatorname{sgn} P \mathrm{d}x^{i_{P(1)}} \otimes \mathrm{d}x^{i_{P(2)}} \otimes \cdots \otimes \mathrm{d}x^{i_{P(q)}}. \tag{2.9}$$

P denotes a permutation of $\{1 \dots q\}$ and $\operatorname{sgn} P = 1$ for an even permutation and -1 otherwise. Having a basis we can write a $q-$form ω as

$$\omega = \omega_{i_1 \dots i_q} \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_q}. \tag{2.10}$$

The space of $q-$forms is usually denoted by $\bigwedge^q T^*X$. Now let us consider a complex manifold X of complex dimension $d = n/2$. Similar to the construction we did for the real case, we can use basis elements of $T^*X^{\mathbb{C}}$ to construct a basis of $\bigwedge^q T^*X^{\mathbb{C}}$. Using dt^i, where dt^i is an arbitrary basis element of $\in T_p^*X^{\mathbb{C}}$ (see eq. (2.5)), we can write a $q-$form ω as

$$\omega = \omega_{i_1 \dots i_q} \mathrm{d}t^{i_1} \wedge \mathrm{d}t^{i_2} \wedge \cdots \wedge \mathrm{d}t^{i_q}. \tag{2.11}$$

Also we can label each term in eq. (2.11) by the number r of holomorphic one forms and $s = q - r$ the number of anti-holomorphic one forms. We can rearrange the indices and get

$$\omega = \sum_r \omega_{i_1 \dots i_r \bar{\jmath}_1 \dots \bar{\jmath}_{q-r}} \mathrm{d}z^{i_1} \wedge \mathrm{d}z^{i_2} \wedge \cdots \wedge \mathrm{d}z^{i_r} \wedge \mathrm{d}\bar{z}^{\bar{\jmath}_1} \wedge \mathrm{d}\bar{z}^{\bar{\jmath}_2} \wedge \cdots \wedge \mathrm{d}\bar{z}^{\bar{\jmath}_{q-r}}. \tag{2.12}$$

The terms in the sum live in $\Omega^{r,s}(X)$, which is the space of anti-symmetric tensors with r holomorphic and s anti-holomorphic indices. So we can write $\Omega^{r,s}(X)$ as $\Omega^{r,s}(X) = \bigwedge^r T^{*(1,0)} \otimes \bigwedge^s T^{*(0,1)}$.

2.4 Exterior Differentiation and Cohomology

Differential forms have a natural differentiation operation associated to them, the exterior differentiation d. d is a map

$$\mathrm{d} : \bigwedge^q T^*X \to \bigwedge^{q+1} T^*X. \tag{2.13}$$

Expressing the map d in local coordinates gives

$$\mathrm{d}:\omega\to\mathrm{d}\omega=\frac{\partial\omega_{i_1\dots i_p}}{\partial x^{i_{q+1}}}\mathrm{d}x^{i_{q+1}}\wedge\mathrm{d}x^{i_2}\wedge\cdots\wedge\mathrm{d}x^{i_q}, \tag{2.14}$$

where ω is a $q-$form and $\mathrm{d}\omega$ a $q+1$-form. On a complex manifold X we can refine the exterior differentiation. We consider

$$\omega^{r,s}=\omega_{i_1\dots i_r\bar{\jmath}_1\dots\bar{\jmath}_s}\mathrm{d}z^{i_1}\wedge\cdots\wedge\mathrm{d}z^{i_r}\wedge\mathrm{d}\bar{z}^{\bar{\jmath}_1}\wedge\cdots\wedge\mathrm{d}\bar{z}^{\bar{\jmath}_s}. \tag{2.15}$$

We can view $\omega^{r,s}$ as a real $r+s$-form on X and conversely $\mathrm{d}\omega^{r,s}$ is an $(r+s+1)$-form. Again we can decompose this $(r+s+1)$-form, using the complex structure on X, into $\Omega^{r+1,s}(X)\oplus\Omega^{r,s+1}(X)$.

As before we can express this fact in local coordinates by

$$\begin{aligned}\mathrm{d}\omega^{r,s}=&\frac{\partial\omega_{i_1\dots i_r\bar{\jmath}_1\dots\bar{\jmath}_s}}{\partial z^{i_{r+1}}}\mathrm{d}z^{i_{r+1}}\wedge\mathrm{d}z^{i_1}\wedge\cdots\wedge\mathrm{d}z^{i_r}\wedge\mathrm{d}\bar{z}^{\bar{\jmath}_1}\wedge\cdots\wedge\mathrm{d}\bar{z}^{\bar{\jmath}_s}\\&+\frac{\partial\omega_{i_1\dots i_r\bar{\jmath}_1\dots\bar{\jmath}_s}}{\partial\bar{z}^{\bar{\jmath}_{s+1}}}\mathrm{d}z^{i_1}\wedge\cdots\wedge\mathrm{d}z^{i_r}\wedge\mathrm{d}\bar{z}^{\bar{\jmath}_{s+1}}\wedge\mathrm{d}\bar{z}^{\bar{\jmath}_1}\wedge\cdots\wedge\mathrm{d}\bar{z}^{\bar{\jmath}_s}.\end{aligned} \tag{2.16}$$

This can be summerized by the following shorthand notation

$$\mathrm{d}\omega^{r,s}=\partial\omega^{r,s}+\bar{\partial}\omega^{r,s}. \tag{2.17}$$

Thereby we decomposed the real exterior differential operator d into exterior differentials along holomorphic (∂) and anti-holomorphic $(\bar{\partial})$ directions.

Because of the anti-symmetry of the wedge-product, applying d twice results in

$$\mathrm{d}\left(\mathrm{d}\alpha\right)=0, \tag{2.18}$$

where α can be any form, so $\mathrm{d}^2=0$.

DeRham cohomology group

If a $q-$form ω fulfils $\mathrm{d}\omega=0$ we call it closed. Having found a q-from ω with $\mathrm{d}\omega=0$, there are now the following possibilities. ω can be written locally as $\omega=\mathrm{d}\alpha$, where α is a $q-1$-form and by $\mathrm{d}^2=0$, $\mathrm{d}\omega=0$ is trivially fulfilled. If this is the case we call ω exact. In the other case ω is a non-trivial solution of $\mathrm{d}\omega=0$. We use this property to define the $q-$th *DeRahm*-cohomology group $H_d^q(X)$, where X is a real manifold, by

$$H_d^q(X,\mathbb{R})=\frac{\{\omega|\mathrm{d}\omega=0\}}{\{\alpha|\alpha=\mathrm{d}\beta\}}, \tag{2.19}$$

both ω and α are $q-$forms.

Dolbeault cohomology

By considering a complex manifold X one can use the operator $\bar{\partial}$ to define the *Dolbeault*-cohomology by

$$H_{\bar{\partial}}^{r,s}(X,\mathbb{C})=\frac{\left\{\omega^{r,s}|\bar{\partial}\omega^{r,s}=0\right\}}{\left\{\alpha^{r,s}|\alpha^{r,s}=\bar{\partial}\beta^{r,s-1}\right\}}. \tag{2.20}$$

Equivalently one can make a similar construction using the ∂ operator.

2.5 Hermitian and Kähler Manifolds

In order to define *Hermitian* and *Kähler*-manifolds, we have to introduce an additional structure, namely a metric. A metric g is a map which takes tangent vectors to a number

$$g : T_pX \times T_pX \to \mathbb{R}. \tag{2.21}$$

g is a symmetric and positive map. Establishing a local coordinate system we can write g in these coordinates as $g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$. Because of the symmetry of g the coefficients fulfil $g_{ij} = g_{ji}$. The metric gives a notion of measuring the length of a tangent vector v_p by $g(v_p, v_p)$. This allows us to measure distances on X. On a complex manifold X we extend the metric g to a map from the complexified tangent space

$$g : T_pX^{\mathbb{C}} \times T_pX^{\mathbb{C}} \to \mathbb{C}. \tag{2.22}$$

In the following we describe the extension of g to complex spaces. For this purpose we take four vectors r, s, u, v which lie in T_pX. Now we use these vectors to construct vectors in $T_pX^{\mathbb{C}}$ by

$$\begin{aligned} \omega_{(1)} &= r + is \\ \omega_{(2)} &= u + iv. \end{aligned}$$

Both are elements of $T_pX^{\mathbb{C}}$. Now we act with the metric g on $\omega_{(1)}$ and $\omega_{(2)}$. By linearity of g we get

$$\begin{aligned} g\left(\omega_{(1)}, \omega_{(2)}\right) &= g(r + is, u + iv) \\ &= g(r, u) - g(s, v) + i\{g(r, v) + g(s, u)\}. \end{aligned} \tag{2.23}$$

The components of the metric g, extended to the complexified tangent space, can be obtained by its action on the basis vectors of $T_pX^{\mathbb{C}}$

$$g_{ij} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) \tag{2.24}$$

$$g_{i\bar{j}} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^{\bar{j}}}\right). \tag{2.25}$$

The symmetry and the reality of the original metric imply further constraints on the components:

$$g_{ij} = g_{ji} \tag{2.26}$$

$$g_{i\bar{j}} = g_{\bar{j}i} \tag{2.27}$$

$$\overline{g_{ij}} = g_{\bar{i}\bar{j}} \tag{2.28}$$

$$\overline{g_{i\bar{j}}} = g_{\bar{i}j}. \tag{2.29}$$

Hermitian metric and hermitian manifold

If we express a metric g in local coordinates and its components fulfil $g_{ij} = g_{\bar{i}\bar{j}} = 0$, we call it hermitian and g can be written as

$$g = g_{i\bar{j}}\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^{\bar{j}} + g_{\bar{i}j}\mathrm{d}\bar{z}^{\bar{i}} \otimes \mathrm{d}z^j. \tag{2.30}$$

Definiton 2.5.1. *A complex manifold equipped with a hermitian metric is called hermitian manifold.*

Kähler-form and Kähler manifold

If we have a hermitian metric g on X, we can construct a form in $\Omega^{1,1}(X)$ by

$$J = ig_{i\bar{\jmath}}\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^{\bar{\jmath}} - ig_{\bar{\jmath}i}\mathrm{d}\bar{z}^{\bar{\jmath}} \otimes \mathrm{d}z^i. \tag{2.31}$$

Using the symmetry of g the above formula can be rewritten into

$$J = ig_{i\bar{\jmath}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}}. \tag{2.32}$$

We call this form *Kähler*-form. Having a *Kähler*-from we can define:

Definiton 2.5.2. *A hermitian manifold with a closed Kähler-form,* $\mathrm{d}J = 0$*, is called Kähler manifold.*

2.6 Differential Geometry

Before we concentrate our discussion to *Kähler*-manifolds, we recall that if we have a metric g on a manifold X we can calculate the associated *Levi-Civitá* connection

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right). \tag{2.33}$$

This equation gets simplified on a *Kähler* manifold. Remember that on a *Kähler* manifold the *Kähler*-form J is closed. In local coordinates this condition reads

$$\mathrm{d}J = \left(\partial + \bar{\partial}\right) ig_{i\bar{\jmath}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}} = 0. \tag{2.34}$$

A bit of manipulation gives

$$\begin{aligned}\mathrm{d}J =& \frac{i}{2}\left(\frac{\partial g_{i\bar{\jmath}}}{\partial z^k} - \frac{\partial g_{k\bar{\jmath}}}{\partial z^i}\right)\mathrm{d}z^k \wedge \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}}\\ &+ \frac{i}{2}\left(\frac{\partial g_{i\bar{\jmath}}}{\partial \bar{z}^{\bar{k}}} - \frac{\partial g_{i\bar{k}}}{\partial \bar{z}^{\bar{\jmath}}}\right)\mathrm{d}\bar{z}^{\bar{k}} \wedge \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}} = 0.\end{aligned} \tag{2.35}$$

The above equations are locally solved by expressing the components of the metric g by

$$g_{i\bar{\jmath}} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^{\bar{\jmath}}} \tag{2.36}$$

and therefore $J = i\partial\bar{\partial}K$. K is the *Kähler*-potential, which is a locally defined function in the chosen coordinate patch.

These conditions on the metric components result in the following non-vanishing components of the *Christoffel*-symbols in complex coordinates

$$\Gamma^l_{jk} = g^{l\bar{s}}\frac{\partial g_{k\bar{s}}}{\partial z^j} \tag{2.37}$$

$$\Gamma^{\bar{l}}_{\bar{\jmath}\bar{k}} = g^{\bar{l}s}\frac{\partial g_{\bar{k}s}}{\partial \bar{z}^{\bar{\jmath}}}. \tag{2.38}$$

All other components of the connection are zero. With the *Christoffel*-symbols at hand one can calculate the components of the curvature tensor $R_{i\bar{j}k\bar{l}}$ in complex coordinates

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{s}} \frac{\partial \Gamma^{\bar{s}}_{\bar{j}\bar{l}}}{\partial z^k}. \tag{2.39}$$

The other components are given by the usual symmetries of the curvature tensor.

From the curvature tensor on calculates the *Ricci* tensor

$$R_{\bar{i}j} = R^{\bar{k}}_{\bar{i}\bar{k}j} = -\frac{\partial \Gamma^{\bar{k}}_{\bar{i}\bar{k}}}{\partial z^j}. \tag{2.40}$$

Parallel Transport and Holonomy

If we parallel transport a vector in an flat euclidean space around a closed loop we obtain the same vector. On an arbitrary curved manifold this does not have to be the case. Let us first consider a real manifold X of dimension n. X has to be equipped with a metric g, from which we can obtain the associated *Levi-Civita* connection Γ. Given a curve C in X, starting and ending at p, we can now parallel transport a vector $v \in T_pM$ along the given curve and obtain v'. Now we can compare v and v', and if X is orientable, they are related by a $SO(n)$ transformation A_C

$$v' = A_C v. \tag{2.41}$$

Applying this procedure to all possible curves starting and ending at p we get a set of $SO(n)$ matrices $A_{C_1}, A_{C_2}, A_{C_3}, \ldots$. The subscript labels the curve. On can now consider a parallel transport around C_i followed by a transport around C_j. The associated matrix is given by $A_{C_j} A_{C_i}$. Traversing around C_j in the opposite direction is described by $A^{-1}_{C_j}$. By considering all possible combinations the set of generated matrices in this procedure carries a group structure. The obtained group is a subgroup of $SO(n)$. Similar to the above reasoning, we obtain a group structure if we consider all points p in X. We call the group, describing the effect of parallel transport on X, holonomy of X.

Whether the holonomy group is the whole $SO(n)$ or only a subgroup depends on the properties of X. For example if X is flat the holonomy group consist solely of the identity element of $SO(n)$. An important point is, that if we consider a complex *Kähler* manifold X of complex dimension $d = n/2$ we see by looking at eqs. (2.37) and (2.38) that parallel transport does not mix holomorphic and anti-holomorphic components. This means that the decomposition of $T_pX^{\mathbb{C}} = T^{(1,0)}_p X \oplus T^{(0,1)}_p X$ is untouched by parallel transport away from p. Therefore the holonomy matrices consist of parts acting only on the holomorphic and a part acting only on anti-holomorphic basis elements. Consequently these matrices lie in a $U(d)$ subgroup of $SO(n)$. If the holonomy group of a complex *Kähler* manifold is further restricted to lie in $SU(d)$, we call the manifold *Calabi-Yau*[1].

[1] Details are given in section 2.8.

2.7 Harmonic Analysis

Let us first introduce the *Hodge*-star operator $\star$, which maps p-forms to $(n-p)-$forms on a manifold X with dimension n. If X is equipped with a metric g the *Hodge*-star operator $\star$ is defined by

$$\omega \to \star\omega = \frac{1}{(n-p)!p!}\varepsilon_{i_1\dots i_n}\sqrt{|\det g|}g^{i_1 j_1}\dots g^{i_p j_p}\omega_{j_1\dots j_p}\mathrm{d}x^{i_{p+1}}\wedge\dots\wedge\mathrm{d}x^{i_n}. \tag{2.42}$$

$\varepsilon_{i_1\dots i_n}$ are the components of the totally antisymmetric tensor ε, defined by

$$\varepsilon_{i_1\dots i_n}\begin{cases}+1 & \text{if } (i_1\dots i_n) \text{ is an even permutation of } (12\dots n)\\ -1 & \text{if } (i_1\dots i_n) \text{ is an odd permutation of } (12\dots n)\\ 0 & \text{otherwise.}\end{cases} \tag{2.43}$$

The $\star$-operator is bijective and coordinate independent. With the $\star$ operator at hand we can construct a further map $\star\mathrm{d}\star$. This maps p-forms to $(p-1)$ forms. Further we can construct the adjoint operator $\mathrm{d}^\dagger$ of the exterior derivative d, which maps p-forms to $(p-1)$-forms, by

$$\mathrm{d}^\dagger = (-1)^{np+n+1}\star\mathrm{d}\star. \tag{2.44}$$

In local coordinates this map reads

$$\mathrm{d}^\dagger : \omega \to \mathrm{d}^\dagger\omega = -\frac{1}{(p-1)!}\omega^{\mu}_{\mu_1\dots\mu_{p-1};\mu}\mathrm{d}x^{\mu_1}\wedge\dots\wedge\mathrm{d}x^{\mu_{p-1}}, \tag{2.45}$$

$\omega^{\mu}_{\nu_1\dots\nu_q;\rho}$ is defined by acting with the covariant derivative[2] on $\omega^{\mu}_{\nu_1\dots\nu_q}$:

$$\begin{aligned}\omega^{\mu}_{\nu_1\dots\nu_q;\rho} \equiv{}& \partial_\rho\omega^{\mu}_{\nu_1\dots\nu_q} + \Gamma^{\mu}_{\rho\kappa}\omega^{\kappa}_{\nu_1\dots\nu_q}\\ &- \Gamma^{\kappa}_{\rho\nu_1}\omega^{\mu}_{\kappa\dots\nu_q} - \dots - \Gamma^{\kappa}_{\rho\nu_q}\omega^{\mu}_{\kappa\dots\nu_{q-1}\kappa}.\end{aligned} \tag{2.46}$$

[2] Details regarding the covariant derivative can be found in [6, Chapter 7].

Additionally, the *Hodge* star operator enables one to introduce an inner product on $p-$forms by

$$\langle\omega|\omega'\rangle = \int_X \omega\wedge\star\omega'. \tag{2.47}$$

An inner product gives the possibility to define the adjoint of d in a more familiar way. Let β be a $(p-1)$-form and ω a $p-$form. Then the adjoint of d is defined by the condition

$$\langle\omega|\mathrm{d}\beta\rangle = \left\langle\mathrm{d}^\dagger\omega\middle|\beta\right\rangle. \tag{2.48}$$

An important application of the d and $\mathrm{d}^\dagger$ operators is the *Hodge* decomposition theorem. The theorem says that any p-form on X can be decomposed uniquely into

$$\omega = \mathrm{d}\beta + \mathrm{d}^\dagger\gamma + \omega', \tag{2.49}$$

where β is a $(p-1)$-form, γ a $(p+1)$-form and ω' is a *harmonic* $p-$form. A *harmonic* $p-$form fulfils

$$\Delta\omega' = \mathrm{d}^\dagger \mathrm{d}\omega' + \mathrm{d}\mathrm{d}^\dagger\omega' = 0. \tag{2.50}$$

Δ is the *Laplacian*, which is the generalization of the *Laplacian* on $\mathbb{R}^n$. A closed form can be written as

$$\omega = \mathrm{d}\beta + \omega', \tag{2.51}$$

because γ has to vanish. Also $\omega - \mathrm{d}\beta$ is an element of $H^p(X, \mathbb{R})$ and as consequence there is a unique harmonic p-form representative in each cohomology class of $H^p(X, \mathbb{R})$. On a complex manifold X the *Hodge* decomposition of an (r,s)-form $\omega^{r,s}$ is given by

$$\omega^{r,s} = \overline{\partial}\alpha^{r,s-1} + \overline{\partial}^\dagger\beta^{r,s+1} + \omega'^{r,s}. \tag{2.52}$$

$\omega'^{r,s}$ gets annihilated by the *Laplacian* $\Delta_{\overline{\partial}} = \overline{\partial}^\dagger\overline{\partial} + \overline{\partial}\overline{\partial}^\dagger$ and is called harmonic. As in the real case, if one considers a closed form $\omega^{r,s}$ with respect to $\overline{\partial}$, there is a unique $\Delta_{\overline{\partial}}$ harmonic representative in the class $H^{r,s}_{\overline{\partial}}(X, \mathbb{C})$. On a *Kähler* manifold all *Laplacians* Δ, $\Delta_{\overline{\partial}}$ and Δ_∂ are related by

$$\Delta = 2\Delta_{\overline{\partial}} = 2\Delta_\partial. \tag{2.53}$$

Now we define $h^{r,s}_X$ to be the complex dimension of $H^{r,s}_{\overline{\partial}}(X, \mathbb{C})$, which is equal to the dimension of the vector space of harmonic (r,s)-forms on X. Extending the *Hodge*-star operator into the complex numbers, gives the following relation, valid on a manifold X with complex dimension d,

$$h^{r,s}_X = h^{d-r,d-s}_X \tag{2.54}$$

and from complex conjugation and *Kählerity* we get

$$h^{r,s}_X = h^{s,r}_X. \tag{2.55}$$

Kählerity relates d and $\overline{\partial}$ cohomology by

$$H^p_d(X) = \bigoplus_{r+s=p} H^{r,s}_{\overline{\partial}}(X). \tag{2.56}$$

2.8 Calabi-Yau Manifolds

With the previous definitions at hand, we can now define a *Calabi-Yau*-manifold. As we will see, there a some equivalent definitions. Let us consider a complex manifold X, with complex dimension d,

Definiton 2.8.1. *A Calabi-Yau manifold is a compact, complex, Kähler manifold which has SU(d) holonomy.*

Vanishing of the *Ricci*-tensor is equivalent to the above statement. Additionally, the theorem of *Yau* states that a complex *Kähler* manifold of vanishing first *Chern*-class admits a *Ricci*-flat metric. This theorem proved a conjecture by *Calabi*. The k-th *Chern* class $c_k(X)$ is an element of $H_d^k(X)$ given by the expansion

$$\begin{aligned} c(X) &= 1 + \sum_j c_j(X) \\ &= \det(1+\mathcal{R}) \\ &= 1 + \operatorname{tr}\mathcal{R} + \operatorname{tr}\left(\mathcal{R}\wedge\mathcal{R} - 2(\operatorname{tr}\mathcal{R})^2\right) + \ldots . \end{aligned} \tag{2.57}$$

$\mathcal{R}$ is the curvature tensor of the tangent bundle T_X of X:

$$\mathcal{R} = R^k_{li\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{j}}. \tag{2.58}$$

The matrix indices are in the fiber direction. The power of *Yau*'s theorem is, that if we have a *Kähler* manifold with vanishing first *Chern* class, we know that there has to be a *Ricci*-flat metric, but we do not need to construct it explicitly. Constructing a *Ricci* flat metric explicitly has not been possible on non trivial *Calabi-Yau* manifolds up to now. On a *Calabi-Yau* manifold there are simplifications regarding the *Hodge* numbers. The $SU(d)$ holonomy ensures that $h^{0,s} = h^{s,0} = 0$ for $1 \le s \le d$ and $h^{0,d} = h^{d,0} = 1$. The element of $h^{d,0}$ is a nowhere vanishing holomorphic from of type $(d,0)$ on the *Calabi-Yau* manifold and usually denoted by Ω and referred to as holomorphic d-form. The connectedness of the space X gives $h^{0,0} = 1$. In table 2.1 the structure of the *Hodge* diamond for $d = 3$, called *Calabi-Yau* threefold, is indicated. In table 2.1 all previously stated simplifications and symmetries where used.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Table 2.1: *Hodge*-diamond for a *Calabi-Yau*-threefold.

2.9 Projective Spaces

We define the complex projective space $\mathbb{P}^n$ by introducing $n+1$ homogeneous complex coordinates $z_1, \ldots, z_{n+1}$. These coordinates are related by $(z_1, \ldots, z_{n+1}) \sim (\lambda z_1, \ldots, \lambda z_{n+1})$, $\lambda \in \mathbb{C}$. One can show that $\mathbb{P}^n$ is *Kähler* [7]. We can generalize the concept of $\mathbb{P}^n$ to the weighted projective space $\mathbb{P}^n[w_1, w_2, \ldots w_{n+1}]$, where the w_i are called weights. From now on we will write $\mathbb{P}^n$ for $\mathbb{P}^n[w_1, w_2, \ldots w_{n+1}]$. On $\mathbb{P}^n$ the coordinate identification is extended to

$$(z_1, \ldots, z_{n+1}) \sim (\lambda^{w_1} z_1, \ldots, \lambda^{w_{n+1}} z_{n+1}). \tag{2.59}$$

Calabi-Yaus as Subspaces

A *Calabi-Yau* manifold can be constructed as vanishing locus of a quasi-homogeneous polynomial P in $\mathbb{P}^n$. Given weights w_i a polynomial P is quasi homogeneous if and only if

$$P(\lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n) = \lambda^w P(x_1, \ldots, x_n), \tag{2.60}$$

where w is called the degree of P.

The *Kählerity* of the subspace defined by $P = 0$ is inherited from $\mathbb{P}^n$. For the subspace to be *Calabi-Yau* the first *Chern* class has to vanish. This is guaranteed if we choose the degree w of P to be given by

$$w = \sum_i w_i, \tag{2.61}$$

where w_i are the individual weights of the x_i.

2.10 Moduli Space of Calabi-Yaus

An interesting question is, given a *Calabi-Yau*-manifold X with metric g can we deform the metric in a continuous was such that, the vanishing of the *Ricci*-tensor is still given? This problem was discussed, for example in [8] and the result is that there are two sorts of perturbations δg

$$\delta g = \delta g_{ij} \mathrm{d}z^i \mathrm{d}z^j + \delta g_{i\overline{\jmath}} \mathrm{d}z^i \mathrm{d}\overline{z}^{\overline{\jmath}} + c.c. \tag{2.62}$$

To preserve the vanishing of the *Ricci*-tensor $\delta g_{i\overline{\jmath}} \mathrm{d}z^i \mathrm{d}\overline{z}^{\overline{\jmath}}$ must be harmonic and consequently related to an element of $H^{1,1}_{\overline{\partial}}(X)$. Also with the help of the holomorphic three-form Ω, one can show that $\Omega_{ijk} g^{k\overline{k}} \partial g_{\overline{kl}} \mathrm{d}z^i \wedge \mathrm{d}z^j \wedge \mathrm{d}\overline{z}^{\overline{l}}$ is an element of $H^{2,1}_{\overline{\partial}}(X)$. We can now identify these two cohomology groups with the space of deformations of the initial *Ricci*-flat metric on X to a nearby *Ricci*-flat metric. This space is called *moduli*-space. Elements of $H^{2,1}_{\overline{\partial}}(X)$ are related to deformations of the complex structure on X and elements of $H^{1,1}_{\overline{\partial}}$ are deformations of the *Kähler*-class J of X. In [9] it was shown that deformations of the complex structure can be encoded into homogeneous re-parametrizations of the defining polynomials P.

2.11 Grassmann Numbers

Later on we will consider anti-commuting coordinates and therefore we will outline some useful techniques for using anti-commuting, or *Grassmann* numbers. Let θ_1 and θ_2 be two anti-commuting variables so consequently

$$\theta_1^2 = \theta_2^2 = 0 \tag{2.63}$$

$$\theta_1\theta_2 = -\theta_2\theta_1. \tag{2.64}$$

Considering a function f of a single anti-commuting variable θ. We can expand the function in a power series

$$f(\theta) = a + b\theta, \tag{2.65}$$

where a is a commuting number and b is *Grassmann* number. Because of the nilpotency of the variables the power series terminates. It is now natural to define differentiation with respect to anti-commuting variables by

$$\frac{\partial f}{\partial \theta} = -b \tag{2.66}$$

and as consequence

$$\frac{\partial\theta}{\partial\theta} = 1. \tag{2.67}$$

Integration of anti-commuting variables can be defined if we demand invariance of the integral under translations, $\theta \to \theta + \eta$,

$$\int \mathrm{d}\theta \left(a + b\theta\right) = \int \mathrm{d}\theta \left(a + b\eta + b\theta\right). \tag{2.68}$$

This is only valid if

$$\int \mathrm{d}\theta = 0 \tag{2.69}$$

and

$$\int \mathrm{d}\theta\theta = 1. \tag{2.70}$$

Let us now consider two *Grassmann*-variables θ_1,θ_2 and a function f of both variables. Again we can expand f in a power series

$$f\left(\theta_1, \theta_2\right) = a + b\theta 1 + c\theta_2 + d\theta_1\theta_2. \tag{2.71}$$

If we now integrate the above expansion, using rules above, we get [3]

$$\int \mathrm{d}\theta_1 \int \mathrm{d}\theta_2 f = -d. \tag{2.72}$$

[3] The minus in eq. (2.72) comes from exchanging $\mathrm{d}\theta_2$ and θ_1.

We see that we can use the integration to pick a specific term in the expansion. This will be very useful in the later chapter where we use the superspace formalism to obtain *Lagrangians* in supersymmetric theories[4].

[4] See section 3.1.

3
Gauged Linear Sigma Model

In 1993 *Witten* [1] constructed a theory, which gives a relation between nonlinear sigma models on *Calabi-Yau*-manifolds and *Landau-Ginzburg*-models. This theory is called gauged linear sigma model. The gauged linear sigma model is a 2-dimensional theory with $\mathcal{N} = 2$ supersymmetry and gauge group $\mathcal{G}$. In the following sections we will closely follow the discussion of [1].

3.1 Notation and Conventions

The gauged linear sigma model can be obtained by dimensional reduction of 4 dimensional gauge theories with $\mathcal{N} = 1$ supersymmetry. In our exposition we will follow [10] and [1].

Supersymmetric theories can be written down most conveniently by using the superspace formalism. Therefore we extend the familiar coordinates x^m by fermionic coordinates θ^α and $\overline{\theta}^{\dot{\alpha}}$. These fermionic coordinates are two component *Weyl*-spinors[1] and α, $\dot{\alpha}$ distinguish between the two chiralities. Superspace furnishes a representation of the supercharges[2] Q_α and $\overline{Q}_{\dot{\alpha}}$, given in terms of derivatives with respect to the coordinates

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} - i\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^m} \tag{3.1}$$

$$\overline{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial \overline{\theta}^{\dot{\alpha}}} + i\sigma^m_{\alpha\dot{\alpha}}\theta^\alpha\frac{\partial}{\partial x^m}. \tag{3.2}$$

$$\tag{3.3}$$

Also, one uses the *Levi-Civita*-symbol ϵ, with $\epsilon^{12} = -\epsilon_{12} = 1$ to write

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \tag{3.4}$$

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta. \tag{3.5}$$

Relations for the other chirality can simply be obtained, by dotting α. For an explicit representation of $\sigma^m_{\alpha\dot{\alpha}}$ the reader is refered to [10, 1]. In order to write down a Lagrangian one further introduces operators which commute with the supersymmetry generators

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^m} \tag{3.6}$$

$$\overline{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \overline{\theta}^{\dot{\alpha}}} - i\sigma^m_{\alpha\dot{\alpha}}\theta^\alpha\frac{\partial}{\partial x^m}. \tag{3.7}$$

[1] A *Weyl*-spinor is a spinor which is an eigenstate of the chirality operator. In 4 dimensions the chirality operator is given by γ_5.

[2] These are the generators of the supersymmetry algebra.

We also introduce the notion of a superfield. The components of a superfield are various fermionic and bosonic fields. Similar to ordinary functions we can perform a power series expansions in terms of the fermionic coordinates. These are finite series, as a consequence of the nilpotency of the fermionic coordinates. In order to obtain a superfield with field content representing an certain supermultiplet, one has to impose further constraints on a superfield[3]. Therefore one introduces the chiral superfield Φ with the property $\overline{D}_{\dot{\alpha}}\Phi = 0$. The expansion of a chiral superfield is given by

[3] A nice account of supersymmetric extensions of the standard model and superspace formalism can be found in [11].

$$\Phi\left(x,\theta\right) = \phi\left(y\right) + \sqrt{2}\theta^{\alpha}\psi_{\alpha}\left(y\right) + \theta^{\alpha}\theta_{\alpha}F\left(y\right), \tag{3.8}$$

with $y^m = x^m + i\theta^{\alpha}\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}$. Expanding the chiral superfield Φ in the x coordinates results in

$$\begin{aligned}\Phi(x) =& \phi(x) + i\theta^{\alpha}\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_m A(x) + \frac{1}{4}\theta^{\alpha}\theta_{\alpha}\overline{\theta}_{\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\Box A(x) \\ &+ \sqrt{2}\theta^{\alpha}\psi_{\alpha}(x) - \frac{i}{\sqrt{2}}\theta^{\alpha}\theta_{\alpha}\partial_m\psi^{\alpha}(x)\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}} + \theta^{\alpha}\theta_{\alpha}F(x).\end{aligned} \tag{3.9}$$

An antichiral superfield is obtained by complex conjugation of Φ and obeys $D_{\alpha}\overline{\Phi} = 0$.

In order the write down gauge invariant Lagrangians one introduces a gauge field in superspace and gauge covariant derivatives defined by

$$\mathcal{D}_{\alpha} = e^{-V}D_{\alpha}e^{V} \tag{3.10}$$
$$\overline{\mathcal{D}}_{\dot{\alpha}} = e^{V}\overline{D}_{\dot{\alpha}}e^{-V}. \tag{3.11}$$

V is a vector superfield and takes values in the *Lie*-algebra of the gauge group. A vector superfield fulfils $V^{\dagger} = V$ and reads in components

$$\begin{aligned}V\left(x,\theta,\bar{\theta}\right) =& C\left(x\right) + i\theta^{\alpha}\chi_{\alpha}\left(x\right) - i\overline{\theta}_{\dot{\alpha}}\overline{\chi}^{\dot{\alpha}}(x) \\ &+ \frac{i}{2}\theta^{\alpha}\theta_{\alpha}\left[M(x) + iN(x)\right] - \frac{i}{2}\overline{\theta}_{\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\left[M(x) - iN(x)\right] \\ &- \theta^{\alpha}\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}v_m(x) + i\theta^{\alpha}\theta_{\alpha}\overline{\theta}_{\dot{\alpha}}\left[\overline{\lambda}^{\dot{\alpha}}(x) + \frac{i}{2}\overline{\sigma}^{m\dot{\alpha}\alpha}\partial_m\chi_{\alpha}(x)\right] \\ &- i\overline{\theta}_{\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\theta^{\alpha}\left[\lambda_{\alpha}(x) + \frac{i}{2}\sigma^m_{\alpha\dot{\alpha}}\partial_m\overline{\chi}^{\dot{\alpha}}(x)\right] + \frac{1}{2}\theta^{\alpha}\theta_{\alpha}\overline{\theta}_{\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\left[D(x) + \frac{1}{2}\Box C(x)\right].\end{aligned} \tag{3.12}$$

The components C,D, M, N and v_m have to be real. The name vector superfield is related to the vector field v_m. One can now generalize the notion of a gauge transformation to superfields, which for an abelian gauge group, is given by

$$V \to V + i\left(\Lambda - \overline{\Lambda}\right), \tag{3.13}$$

where Λ is a chiral superfield. This transformation reads for the component fields

$$\begin{aligned} C &\to C + i\left(\phi - \overline{\phi}\right) \\ M + iN &\to M + iN + 2F \\ v_m &\to v_m + \partial_m\left(\phi + \overline{\phi}\right) \\ \chi &\to \chi + \sqrt{2}\psi \\ \lambda &\to \lambda \\ D &\to D, \end{aligned} \tag{3.14}$$

where we did not explicitly denote the x dependence of the fields. ϕ, F and ψ are the component fields of Λ. The transformation of the vector field v_m is similar to the known gauge transformations. Now one can use this gauge transformation to choose the gauge in such a way that C, χ, M and N vanish. This gauge is called *Wess-Zumino* gauge. In this gauge V reads in components[4]

$$V = -\theta^\alpha \sigma^m_{\alpha\dot{\alpha}} \overline{\theta}^{\dot{\alpha}} v_m + i\theta^\alpha\theta_\alpha\overline{\theta}_{\dot{\alpha}}\overline{\lambda}^{\dot{\alpha}} - i\overline{\theta}_{\dot{\alpha}}\theta^\alpha\lambda_\alpha + \frac{1}{2}\theta^\alpha\theta_\alpha\overline{\theta}_{\dot{\alpha}}\overline{\theta}^{\dot{\alpha}} D. \tag{3.15}$$

[4] The *Wess-Zumino* gauge breaks supersymmtry, but ordinary gauge transformations $v_m \to v_m + \partial_m a$ are still possible.

The transformation of a chrial superfield, with charge Q , for a $U(1)$ gauge theory is given by

$$\Phi \to e^{-iQ\Lambda}\Phi. \tag{3.16}$$

The next task is to dimensionally reduce to 2 dimensions. Following the convention of [1] we will chose our fields to be independent of x^1 and x^2. So our fields are functions of x^0 and x^3. We introduce $\sigma = \frac{(v_1 - v_2)}{\sqrt{2}}$, $\overline{\sigma} = \frac{(v_1 + v_2)}{\sqrt{2}}$ and write $x^0 = y^0$ and $x^3 = y^1$. In 2 dimensions we will label fermionic components by

$$\left(\psi^1, \psi^2\right) = \left(\psi^-, \psi^+\right) \tag{3.17}$$

$$\left(\psi_1, \psi_2\right) = \left(\psi_-, \psi_+\right). \tag{3.18}$$

Dotted components are written in a similar way. Also the components of the fermionic coordinates are denoted in that way:

$$\theta = \begin{pmatrix} \theta^- \\ \theta^+ \end{pmatrix}, \tag{3.19}$$

and likewise for $\overline{\theta}$. The corresponding supersymmetry transformations of the fields can be found in [1]. They are obtained by dimensional reduction of the ones given in [10]. In two dimensions there exists an additional superfield, the twisted chiral superfield, with the property $\overline{D}_+\Sigma = D_-\Sigma = 0$. The gauge invariant field strength in 2 dimensions is given by such a twisted chiral field

$$\Sigma = \frac{1}{\sqrt{2}}\left\{\overline{\mathcal{D}}_+, \mathcal{D}_-\right\}. \tag{3.20}$$

In the case of an abelian gauge group the components of Σ read

$$\begin{aligned} \Sigma = \frac{1}{\sqrt{2}}\overline{D}_+ D_- V &= \sigma - i\sqrt{2}\theta^+\overline{\lambda}_+ - i\sqrt{2}\overline{\theta}\lambda_- + \sqrt{2}\theta^+\overline{\theta}^-\left(D - iv_{01}\right) \\ &- i\overline{\theta}^-\theta^-\left(\partial_0 - \partial_1\right)\sigma - i\theta^+\overline{\theta}^+\left(\partial_0 + \partial_1\right)\sigma + \sqrt{2}\overline{\theta}^-\theta^+\theta^-\left(\partial_0 - \partial_1\right)\overline{\partial}_+ \\ &+ \sqrt{2}\theta^+\overline{\theta}^-\overline{\theta}^+\left(\partial_0 + \partial_+\right)\lambda_- - \theta^+\overline{\theta}^-\theta^-\overline{\theta}^+\left(\partial_0^2 - \partial_1^2\right)\sigma \end{aligned} \tag{3.21}$$

with $v_{01} = \partial_0 v_1 - \partial_1 v_0$.

With the above definitions at hand we can write down the Lagrangian of the gauged linear sigma model. We will always denote a superfield with an upper case letter and the scalar component with the corresponding lower case letter.

3.2 Lagrangian

Following [1] we will consider the case of an abelian gauge group $U(1)^s$, with vector superfields V_a, $a = 1 \ldots s$. The matter content of the theory is given by k chiral superfields Φ_i of charge $Q_{i,a}$. One can view the bosonic components of Φ_i as coordinates on $\mathbb{C}^k$ and their kinetic term determined by a *Kähler* metric on $\mathbb{C}^k$. The Lagrangian of the gauged linear sigma model has the form

$$L = L_{kin} + L_W + L_{gauge} + L_{\mathrm{D},\theta}. \tag{3.22}$$

L_{kin} is given by

$$L_{kin} = \int \mathrm{d}^2 y \mathrm{d}^4\theta \sum_i \overline{\Phi}_i e^{2\sum_a Q_{i,a} V_a} \Phi_i. \tag{3.23}$$

We can also write out this Lagrangian into components and get

$$\begin{aligned} L_{kin} = \sum_i \int \mathrm{d}^2 y \Big(& -D_\rho \bar{\phi}_i D^\rho \phi_i + i\bar{\psi}_{-,i} \left(D_0 + D_1\right) \psi_{-,i} + i\bar{\psi}_{+,i} \left(D_0 - D_1\right) \psi_{+,i} + |F_i|^2 \\ & - 2\sum_a \bar{\sigma}_a \sigma_a Q_{i,a}^2 \bar{\phi}_i \phi_i - \sqrt{2} \sum_a Q_{i,a} \left(\bar{\sigma}_a \bar{\psi}_{+,i} \psi_{-,i} + \sigma_a \bar{\psi}_{-,i} \psi_{+,i}\right) + \sum_a D_a Q_{i,a} \bar{\psi}\psi \\ & - \sum_a i\sqrt{2} Q_{i,a} \bar{\phi}_i \left(\psi_{-,i} \lambda_{+,a} - \psi_{+,i} \lambda_{-,a}\right) - \sum_a i\sqrt{2} Q_{i,a} \phi_i \left(\bar{\lambda}_{-,a} \bar{\psi}_{+,i} - \bar{\lambda}_{+,a} \bar{\psi}_{-,i}\right) \Big). \end{aligned} \tag{3.24}$$

The next part contains a gauge invariant holomorphic function W on $\mathbb{C}^k$. For our cases we will always assume that W is a polynomial function of the chiral superfields. W is usually referred to as the superpotential. The Lagrangian is given by

$$L_W = -\int \mathrm{d}^2 y \mathrm{d}\theta^+ \mathrm{d}\theta^- W\left(\Phi_i\right)\big|_{\bar{\theta}^+ = \bar{\theta}^- = 0} - h.c. \tag{3.25}$$

where *h.c.* stands for hermitian conjugation. Writing L_W in components results in

$$L_W = -\int \mathrm{d}^2 y \left(F_i \frac{\partial W}{\partial \phi_i} + \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \psi_{-,i} \psi_{+,j} \right) - h.c. \tag{3.26}$$

By using the twisted chiral superfield Σ_a we can set up the gauge kinetic term

$$L_{gauge} = -\sum_a \frac{1}{4e_a^2} \int \mathrm{d}^2 y \mathrm{d}^4\theta \overline{\Sigma}_a \Sigma_a, \tag{3.27}$$

with coupling constants e_a, $a = 1 \dots s$. Decomposing Σ into components results in

$$L_{gauge} = \sum_a \frac{1}{e_a^2} \int \mathrm{d}^2 y \left(\frac{1}{2} v_{01,a}^2 + \frac{1}{2} \mathrm{D}_a^2 + i\bar{\lambda}_{+,a} \left(\partial_0 - \partial_1\right) \lambda_{+,a} \right.$$
$$\left. + i\bar{\lambda}_{-,a} \left(\partial_0 + \partial_1\right) \lambda_{-,a} - \left|\partial_\rho \sigma_a\right|^2 \right). \tag{3.28}$$

Supersymmetry allows the addition of a further term. We focus on theories with gauge group $U(1)$, which will also be the case of interest in the following chapters. We can add the so called *Fayet-Iliopoulos*-term

$$-r \int \mathrm{d}^2 y \mathrm{d}^4 \theta V. \tag{3.29}$$

This term is indeed supersymmetric, but not gauge invariant. To cure that deficit we add the θ coupling term

$$\frac{\theta}{2\pi} \int \mathrm{d}v = \frac{\theta}{2\pi} \int \mathrm{d}^2 y v_{01}. \tag{3.30}$$

We can combine both eqs. (3.29) and (3.30) with the help of the twisted chiral superfield Σ, as one can see by using

$$\int \mathrm{d}^2 y \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^- \Sigma\big|_{\theta^- = \bar{\theta}^+ = 0} = \sqrt{2} \int \mathrm{d}^2 y \left(\mathrm{D} - i v_{01}\right), \tag{3.31}$$
$$\int \mathrm{d}^2 y \mathrm{d}\theta^- \mathrm{d}\bar{\theta}^+ \bar{\Sigma}\big|_{\theta^+ = \bar{\theta}^- = 0} = \sqrt{2} \int \mathrm{d}^2 y \left(\mathrm{D} + i v_{01}\right). \tag{3.32}$$
$$\tag{3.33}$$

Both equations can be used to write the last missing term in the Lagrangian,with $t = ir + \frac{\theta}{2\pi}$,

$$L_{\mathrm{D},\theta} = \int \mathrm{d}^2 y \left(-r\mathrm{D} + \frac{\theta}{2\pi} v_{01} \right)$$
$$= \frac{it}{2\sqrt{2}} \int \mathrm{d}^2 y \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^- \Sigma\big|_{\theta^- = \bar{\theta}^+ = 0} - \frac{i\bar{t}}{2\sqrt{2}} \int \mathrm{d}^2 y \mathrm{d}\theta^- \mathrm{d}\bar{\theta}^+ \bar{\Sigma}\big|_{\theta^+ = \bar{\theta}^- = 0}. \tag{3.34}$$

Equation (3.34) can be generalized by introducing a holomorphic "twisted superpotential" $\widetilde{W}(\Sigma)$

$$\Delta L = \int \mathrm{d}^2 y \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^- \widetilde{W}(\Sigma)\big|_{\theta^- = \bar{\theta}^+ = 0} + h.c. \tag{3.35}$$

Writing ΔL in components gives

$$\Delta L = \int \mathrm{d}^2 y \left(\sqrt{2} \widetilde{W}'(\sigma) \left(\mathrm{D} - i v_{01}\right) + 2\widetilde{W}''(\sigma) \bar{\lambda}_+ \lambda_- \right) + h.c. \tag{3.36}$$

Equation (3.34) is recovered by setting $\widetilde{W}(\sigma) = \frac{it\sigma}{2\sqrt{2}}$. Combining all the previous terms results in the Lagrangian of the gauged linear sigma model

$$
\begin{aligned}
L = &\int \mathrm{d}^2 y \mathrm{d}^4\theta \sum_i \overline{\Phi}_i e^{2Q_i V}\Phi_i \\
&- \int \mathrm{d}^2 y \mathrm{d}\theta^+\theta^- W\left(\Phi_i\right)\big|_{\overline{\theta}^+=\overline{\theta}^-=0} \\
&- \frac{1}{4e_a^2}\int \mathrm{d}^2 y \mathrm{d}^4\theta \overline{\Sigma}\Sigma \\
&+ \int \mathrm{d}^2 y \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^- \widetilde{W}\left(\Sigma\right)\big|_{\theta^-=\overline{\theta}^+=0} + h.c. \qquad (3.37)
\end{aligned}
$$

Symmetries

The gauged linear sigma model has $\mathcal{N}=(2,2)$ supersymmetry. To be more precise, it has 2 left moving and 2 right moving supersymmetries. The left-moving supersymmetries act on θ^- and $\overline{\theta}^-$, whereas the right-moving ones act on θ^+ and $\overline{\theta}^+$. An additional symmetry present in all supersymmetric models with $\mathcal{N}>1$ is the so called R-symmetry. A right-moving R symmetry acts in the following way

$$
\begin{aligned}
\theta^+ &\to e^{i\alpha}\theta^+, \\
\overline{\theta}^+ &\to e^{-i\alpha}\overline{\theta}^+, \qquad (3.38)
\end{aligned}
$$

whereas θ^- and $\overline{\theta}^-$ remain invariant. Left-moving R-symmetry acts in a similar way, except for $+$ and $-$ exchanged.

The GLSM with superpotential W has a right- and left-moving $R-$symmetries only if the following conditions are fulfilled

1.
$$\sum_i Q_{i,a} = 0 \qquad (3.39)$$
for $a = 1\ldots s$,
2. W is a quasi-homogeneous.

Also eq. (3.39) ensures that the charges related to the R-symmetries are anomaly free. More details regarding the symmetries can again be found in [1].

3.3 Calabi-Yau/Landau-Ginzburg Correspondence

In the following we discuss the vacuum behaviour of the GLSM. Depending on the value of the r parameter we will encounter different vacuum configurations of the GLSM called phases. To simplify our task we consider gauge group $U(1)$. At first we take a look a the equations of motion of D and F_i in eq. (3.37), which are purely algebraic. By solving them one gets

$$\mathrm{D} = -e^2\left(\sum_i Q_i|\phi_i|^2 - r\right), \qquad (3.40)$$

$$F_i = \frac{\partial W}{\partial \phi_i}. \qquad (3.41)$$

Next we consider the potential energy of the scalar fields ϕ_i, σ

$$U(\phi_i,\sigma) = \frac{1}{2e^2}\mathrm{D}^2 + \sum_i |F_i|^2 + 2\overline{\sigma}\sigma \sum_i Q_i^2|\phi_i|^2. \tag{3.42}$$

Of particular interest in this thesis are models with field content given in table 3.1. The X_is are chiral superfields of gauge charge w_i. Here we changed the notation of the chiral superfields to make the relation between the bosonic components and coordinates on the target spacetime more apparent. P is also a chiral superfield of charge $-N = -\sum\limits_i w_i$. The choice of the gauge charges satisfies eq. (3.39).

	P	X_1	X_2	X_3	X_4	X_5
$U(1)$	$-N=-\sum_i w_i$	w_1	w_2	w_3	w_4	w_5
R	2	0	0	0	0	0

Table 3.1: Field content with gauge and R-charges.

For the superpotential we choose

$$W = PG\left(X_1,\ldots,X_5\right), \tag{3.43}$$

where G is a weighted homogenous polynomial of degree $N = \sum\limits_i w_i$. This satisfies both conditions for having a model with R-symmetry. Also W is gauge invariant under

$$\begin{aligned} X_i &\to g^{w_i}X_i \\ P &\to g^{-N}P, \end{aligned} \tag{3.44}$$

and carries R-charge 2. Another restriction on G is, that the only solution of

$$0 = \frac{\partial G}{\partial X_1} = \cdots = \frac{\partial G}{\partial X_5} \tag{3.45}$$

is $X_i = 0$ for $\forall i$. This condition is necessary for the smoothness of the hypersurface X in $\mathbb{P}[w_1,w_2,w_3,w_4,w_5][N]$[5], described by $G=0$, that we are going to find as solution to eqs. (3.40) and (3.41). By choosing G as quasi-homogenous polynomial, this requirement is automatically fulfilled.

[5] $\mathbb{P}[w_1,w_2,w_3,w_4,w_5][N]$, w_i denotes the weight of the homogeneous coordinates on the projective space and N denotes the degree of G.

Writing the potential eq. (3.42) for this model gives

$$U = |G(x_i)|^2 + |p|^2 \sum_i \left|\frac{\partial G}{\partial x_i}\right|^2 + \frac{1}{2e^2}\mathrm{D}^2 + 2|\sigma|^2\left(\sum_i w_i^2|x_i|^2 + N^2|p|^2\right) \tag{3.46}$$

and

$$\mathrm{D} = -e^2\left(\sum_i w_i\overline{x}_i x_i - N\overline{p}p - r\right), \tag{3.47}$$

where x_i and p_i are the scalar components of X_i and P_i Of particular interest is now the vacuum configuration, i.e. the ground state. Therefore we look at the zeros of the potential U. We will find that there are different possibilities, characterised by the value of r.

Calabi-Yau Phase, $r \gg 0$

By looking at eq. (3.47) we see that $D = 0$ implies that there have to have at least one $x_i \neq 0$ and consequently the vanishing of $|p|^2 \sum_i |\partial_i G|^2$ in eq. (3.46) requires $p = 0$. With these restrictions we obtain from eq. (3.47)

$$\sum_i w_i \overline{x}_i x_i = r, \tag{3.48}$$

to ensure the vanishing of the D-term. Since the x_i carry weights w_i, we see that the space of solutions is a copy of weighted complex projective space $\mathbb{P}[w_1, w_2, w_3, w_4, w_5]$, with *Kähler* class proportional to r [1]. Further we have to require $G = 0$ and $\sigma = 0$ for the vanishing of eq. (3.46).

As one can see, the space of classical vacua is the hypersurface $X \subset \mathbb{P}[w_1, w_2, w_3, w_4, w_5]$ defined by $G = 0$. We denote this by $\mathbb{P}[w_1, w_2, w_3, w_4, w_5][N]$. Having chosen $N = \sum_i w_i$ ensures the vanishing of the first *Chern*-class and hence the obtained hypersurface is *Calabi-Yau*. The effective low energy theory is a non-linear sigma model with target space X [6].

[6] Details are given in [12] and chapter 4.

Landau-Ginzburg Phase, $r \ll 0$

To set eq. (3.47) equal to zero we have to demand $p \neq 0$ and therefore $|p|^2 \sum_i |\partial_i G|^2 = 0$ requires the vanishing of all x_i. So we obtain from eq. (3.47)

$$|p| = \sqrt{\frac{-r}{N}}. \tag{3.49}$$

Via a gauge transformation we can cancel the complex part of p. Because p acquires a vacuum expectation value the gauge group gets broken down to $\mathbb{Z}_N$. Expanding around the vacuum leads to the observation, that the x_i remain massless (as long as $n \geq 3$). The effective superpotenital $\widetilde{W}$ for the x_i can be obtained by integrating out p, which is easily accomplished by setting p equal to its expectation value. So that $\widetilde{W} = G(x_i)$. For $n \geq 3$ the effective superpotential has a degenerate critical point at the origin, where it vanishes up to the n^{th}- order. Such a theory is called a *Landau-Ginzburg*-orbifold[7].

[7] Further details on the *Landau-Ginzburg* theory can be found in chapter 4.

Reinterpretation of the Parameters

One can identify the previously defined parameter t as the parameter of the stringy *Kähler*-moduli-space $\mathcal{M}_K$. For later use we will slightly redefine t

$$t = r - i\theta. \tag{3.50}$$

In subsequent chapters we will use $D-$branes to probe different regions in $\mathcal{M}_K$. A subtlety arises when we move from one phase to the other. There is a singularity in $\mathcal{M}_K$. Classically the singularity is located at $r = 0$, where σ becomes unconstrained. This is called the *Coulomb*-branch. By calculating quantum corrections one can show[8], that the locus of the singularity is determined by

[8] Details can be found in [13].

$$\partial_\sigma \widetilde{W}_{eff}(\sigma) = 0. \tag{3.51}$$

$\widetilde{W}_{eff}(\sigma)$ is the effective twisted potential given by

$$\widetilde{W}_{eff}(\sigma) = -t\sigma - \sum_{i=0}^{n} Q_i \sigma \log(Q_i\sigma) \qquad \mod 2\pi i. \tag{3.52}$$

This exact form of the effective twisted potential is obtained through a 1-loop-calculation [13].

Next, we will calculate the position of the singularity for our model, with 5 chiral superfields of charge w_i and one with charge $-N = -\sum_i w_i$.

$$\begin{aligned}\partial_\sigma \widetilde{W}_{eff}(\sigma) &= -t - \sum_{i=1}^{5} w_i \log(w_i) + N\log(-N) \\ &= -t - \sum_{i=1}^{5} w_i \log(w_i) + N\log\left(e^{-i\pi}N\right) \\ &= -t - \sum_{i=1}^{5} w_i \log(w_i) + N\log(N) - iN\pi \quad \stackrel{!}{=} 0.\end{aligned} \tag{3.53}$$

Now comparing real and imaginary parts gives for the position of the singularity

$$(r,\theta) = \left(-\sum_i Q_i \log|Q_i|, N\pi + 2\pi\mathbb{Z}\right). \tag{3.54}$$

In the next chapter we will formulate the GLSM on a worldsheet with boundaries. For this purpose we will introduce boundary interactions to keep as much supersymmetry as possible. This will result in the identification of D-branes as matrix factorisation of the superpotential.

4
Boundary Interactions and D-Branes

In this chapter we will gently develop various aspects of *D*-branes in supersymmetric theories. We will follow the discussion given in [14, 15, 16] and [17], where also further details can be found. The models of interest in this work have $\mathcal{N} = (2,2)$ supersymmetry. Generically imposing a boundary breaks all of the supersymmetry, but there are special types of boundaries which only break half of the supersymmerty. This boundaries are called *A*-branes and *B*-branes. The main interest of this thesis are *B*-branes and therefore we will not discuss $A-$branes in detail. Starting with the non-linear σ-model we will develop the interpretation of D-branes as objects in derived categories of coherent sheaves. Then boundary conditions in *Landau-Ginzburg*-models are discussed following [15]. The D-branes are matrix factorizations of the *Landau-Ginzburg* superpotential. A good overview can be found in [18]. Also matrix factorisations can be described in terms of categories. As seen in chapter 3 the gauged linear sigma model provides a connection between *Landau-Ginzburg*-models and non-linear-sigma-models. The connection between the category of matrix factorisations and the category of coherent sheaves was given by *Orlov* [19], a review is given in [20]. At the end of this chapter we will describe *B*-branes in the gauged linear sigma model.

4.1 Boundaries in Non-linear Sigma Models

In the following we will give an introduction to D-branes on *Calabi-Yau* manifolds. The presented material is mostly taken from [14, 12]. We consider the non linear sigma model with *Calabi-Yau* target manifold.

Action

The starting point for the non linear sigma model are maps ϕ from a worldsheet Σ into a target spacetime X. In order to obtain an $\mathcal{N} = (2,2)$ supersymmetric theory X is restricted to be *Kähler*. Introducing worldsheet fermions we get the following action

$$S = \frac{i}{4\pi i} \int_\Sigma \mathrm{d}^2 z \left\{ g_{i\bar{\jmath}} \left(\frac{\partial \phi^i}{\partial z} \frac{\partial \phi^{\bar{\jmath}}}{\partial \bar{z}} + \frac{\partial \phi^i}{\partial \bar{z}} \frac{\partial \phi^{\bar{\jmath}}}{\partial z} \right) + i B_{i\bar{\jmath}} \left(\frac{\partial \phi^i}{\partial z} \frac{\partial \phi^{\bar{\jmath}}}{\partial \bar{z}} - \frac{\partial \phi^i}{\partial \bar{z}} \frac{\partial \phi^{\bar{\jmath}}}{\partial z} \right) \right.$$
$$\left. + i g_{i\bar{\jmath}} \psi_-^{\bar{\jmath}} D \psi_-^i + i g_{i\bar{\jmath}} \psi_+^{\bar{\jmath}} \overline{D} \psi_+^i + R_{i\bar{\imath}j\bar{\jmath}} \psi_+^i \psi_+^{\bar{\jmath}} \psi_-^j \psi_-^{\bar{\jmath}} \right\}, \tag{4.1}$$

where z, $\bar{z}$ are complex coordinates on Σ. The $\phi^i(z,\bar{z})$ and $\phi^{\bar{\imath}}(z,\bar{z})$ can be interpreted as local complex coordinates on the target space X. $g_{i\bar{\jmath}}$ is the *Kähler* metric on X and $R_{i\bar{\imath}j\bar{\jmath}}$ is the curvature tensor. The B-field degrees of freedom are given by the real $(1,1)$-form $B_{i\bar{\jmath}}$. D is the covariant derivative, $D\psi_-^i = \partial \psi_-^i + \partial^j \Gamma^i_{jk} \psi_-^j$. The fermions are interpreted as sections of bundels over Σ

$$\begin{aligned} \psi_+^i &\in \Gamma\left(K^{1/2} \otimes \phi^* T_X^{(1,0)}\right) \\ \psi_+^{\bar{\jmath}} &\in \Gamma\left(K^{1/2} \otimes \phi^* T_X^{(0,1)}\right) \\ \psi_-^i &\in \Gamma\left(\overline{K}^{1/2} \otimes \phi^* T_X^{(1,0)}\right) \\ \psi_-^{\bar{\jmath}} &\in \Gamma\left(\overline{K}^{1/2} \otimes \phi^* T_X^{(0,1)}\right) \end{aligned}. \tag{4.2}$$

K is the holomorphic cotangent bundle on Σ, $T_X^{(1,0)}$ the holomorphic tangent bundle and $T_X^{(0,1)}$ the anti-holomorphic tangent bundle on X. The action is invariant under $\mathcal{N} = (2,2)$ supersymmetry, with following transformations of the fields

$$\begin{aligned} \delta\phi^i &= i\alpha_- \psi_+^i + i\alpha_+ \psi_-^i \\ \delta\phi^{\bar{\imath}} &= i\bar{\alpha}_- \psi_+^{\bar{\imath}} + i\bar{\alpha}_+ \psi_-^{\bar{\imath}} \\ \delta\psi_+^i &= -\bar{\alpha}_- \partial\phi^i - i\alpha_+ \psi_-^j \Gamma^i_{jk} \psi_+^k \\ \delta\psi_+^{\bar{\imath}} &= -\alpha_- \partial\phi^{\bar{\imath}} - i\bar{\alpha}_+ \psi_-^{\bar{\jmath}} \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} \psi_+^{\bar{k}} \\ \delta\psi_-^i &= -\bar{\alpha}_+ \overline{\partial}\phi^i - i\alpha_- \psi_+^j \Gamma^i_{jk} \psi_-^k \\ \delta\psi_-^{\bar{\imath}} &= -\alpha_+ \overline{\partial}\phi^{\bar{\imath}} - i\bar{\alpha}_- \psi_+^{\bar{\jmath}} \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} \psi_-^{\bar{k}}, \end{aligned} \tag{4.3}$$

where α is the fermionic parameter of the transformation. In the following we will restrict to the case where X is *Calabi-Yau*. This extends the supersymmetry to a superconformal symmetry. Closed string states form a representation of the superconformal algebra.

Superconformal Algebra

We give only a small excerpt of the $\mathcal{N} = (2,2)$ algebra, details are given in [14, 7].

The generators of the left-moving algebra are given by:

$$\begin{aligned} T(z) &= -g_{i\bar{\jmath}} \frac{\partial \phi^i}{\partial z} \frac{\partial \phi^{\bar{\jmath}}}{\partial z} + \frac{1}{2} g_{i\bar{\jmath}} \psi_+^i \frac{\partial \psi_+^{\bar{\jmath}}}{\partial z} + \frac{1}{2} g_{i\bar{\jmath}} \psi_+^{\bar{\jmath}} \frac{\partial \psi_+^i}{\partial z} \\ G(z) &= \frac{1}{2} g_{i\bar{\jmath}} \psi^i{}_+ \frac{\partial \phi^{\bar{\jmath}}}{\partial z} \\ \widetilde{G}(z) &= \frac{1}{2} g_{i\bar{\jmath}} \psi^{\bar{\jmath}}{}_+ \frac{\partial \phi^i}{\partial z} \\ J(z) &= \frac{1}{4} g_{i\bar{\jmath}} \psi_+^i \psi_+^{\bar{\jmath}}. \end{aligned} \tag{4.4}$$

The generators of the right-moving part, denoted by $\overline{T}(\overline{z})$,$\overline{G}(\overline{z})$,$\widetilde{\overline{G}}(\overline{z})$ and $\overline{J}(\overline{z})$ are given by replacing un-bared quantities in eq. (4.4) by bared ones and vice versa . The eigenvalue of the operator $T(z)$ is denoted by h and called conformal weight of a state. For the operator $J(z)$ we denote the eigenvalue of a given state by q, which is the charge of the $U(1)$ R-symmetry. The eigenvalues for the right moving part are denoted by $\overline{h}$ and $\overline{q}$. q takes values in $\mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$, depending on the boundary conditions chosen for the fermions. The first case corresponds to *Neveu-Schwarz* sector, NS for short and the second case to the *Ramond* sector, abbreviated by R. Many aspects of the superconformal algebra are given by states fulfilling

$$h = |q/2|, \qquad\qquad q = -3, -2, \ldots, 3 \tag{4.5}$$

in the NS sector. These states are a finite subset of the infinitely many closed string states. Operators creating states fulfilling the stated equality are called chiral primary operators for $q > 0$ and antichiral primary operators for $q < 0$. These operators are closed under the operator product and form the chiral algebra, referred to as chiral ring [21]. The analysis of the chiral ring is best done by using methods from topological field theory. There are two natural ways to obtain a topological field theory from the described $\mathcal{N} = (2,2)$ superconformal theory. The procedure is called topological twist [12, 22]. There are two independent ways of performing the twist that lead to the so called A-model and B-model respectively.

A-Model

The A-model is obtained by a redefinition of the fermions such that

$$\begin{aligned}
\chi^i = \psi^i_+ &\in \Gamma\left(\phi^*\left(T_X^{(1,0)}\right)\right)\\
\chi^{\bar{\imath}} = \psi^{\bar{\imath}}_- &\in \Gamma\left(\phi^*\left(T_X^{(0,1)}\right)\right)\\
\psi^{\bar{\imath}}_z = \psi^{\bar{\imath}}_+ &\in \Gamma\left(K \otimes \phi^*\left(T_X^{(0,1)}\right)\right)\\
\psi^{i}_{\bar{z}} = \psi^{i}_- &\in \Gamma\left(\overline{K} \otimes \phi^*\left(T_X^{(1,0)}\right)\right)
\end{aligned}. \tag{4.6}$$

The resulting theory is still invariant under eq. (4.3), with a redefinition of the α parameters. The parameters are given by $\alpha = \alpha_- = \tilde{\alpha}_+$ and $\alpha_+ = \tilde{\alpha}_- = 0$. Thus the transformation now depends on a single scalar parameter. By Q we denote the operator generating this symmetry, by

$$\delta\mathcal{O} = -i\alpha\{Q, \mathcal{O}\}, \tag{4.7}$$

where $\mathcal{O}$ is an arbitrary operator.

Q generates a $BRST$-symmetry, because

$$Q^2 = 0, \tag{4.8}$$

which is satisfied up to equations of motion. The action can now be rewritten as

$$S = \int_\Sigma i\{Q,V\} - 2\pi i \int_\Sigma \phi^* \left(B + iJ\right), \tag{4.9}$$

$$V = 2\pi g_{i\bar{\jmath}} \left(\psi_z^{\bar{\jmath}} \bar{\partial}\phi^i + \partial\phi^{\bar{\jmath}} \psi_{\bar{z}}^i\right), \tag{4.10}$$

where $B + iJ \in H^2(X, \mathbb{C})$ is the complexified *Kähler* form. Now one considers $Q-$closed operators $\mathcal{O}$, which fulfil $\{Q, \mathcal{O}\} = 0$. As a consequence also the states are restricted to $Q-$closed states. These are states with $h = q/2$, $\bar{h} = -\bar{q}/2$.

Correlation functions of Q-exact operators $\mathcal{O}$, given by $\mathcal{O} = \{Q, \mathcal{O}'\}$, where $\mathcal{O}'$ is some operator, vanish and as a consequence Q-exact operators can be viewed as zeros in the chiral algebra. Changes of the complex structure only affect V and thus are trivial. Thus correlation functions only depend on the comlpexified *Kähler*- form. By the form of the action only the cohomology class of $B + iJ$ is important. One can also show that in the $A-$model $Q-$cohomology is *de Rham* cohomolgy. Further details on the $A-$model can be found in [14, 12] and in the references given therein.

B-Model

The second possibility of twisting the non linear σ-model is given by

$$\begin{aligned} \psi_\pm^{\bar{\jmath}} &\in \Gamma\left(\phi^*\left(T_X^{(0,1)}\right)\right) \\ \psi_+^j &\in \Gamma\left(K \otimes \pi^*\left(T_X^{(1,0)}\right)\right) \\ \psi_-^j &\in \Gamma\left(\overline{K} \otimes \pi^*\left(T_X^{(1,0)}\right)\right) \end{aligned} \tag{4.11}$$

we define

$$\begin{aligned} \eta^{\bar{\jmath}} &= \psi_+^{\bar{\jmath}} + \psi_-^{\bar{\jmath}} \\ \theta_j &= g_{j\bar{k}}\left(\psi_+^{\bar{k}} - \psi_-^{\bar{k}}\right) \\ \rho^j &= \psi_+^j + \psi_-^j. \end{aligned} \tag{4.12}$$

The parameters of the transformation given in eq. (4.3) fulfil $\alpha_\pm = 0$ and $\tilde{\alpha}_\pm = \alpha$. Similar to the case of the A-model we denote the operator corresponding to the supersymmetry transformation with Q. As in A-model Q is nilpotent $Q^2 = 0$ and therefore again a generator of a *BRST* variation.

The action of the B-model is given by

$$S = i\int \{Q,V\} + U, \tag{4.13}$$

with

$$\begin{aligned} V &= g_{j\bar{k}}\left(\rho_z^j \bar{\partial}\phi^{\bar{k}} + \rho_{\bar{z}}^j \partial\phi^{\bar{k}}\right), \\ U &= \int_\Sigma \left(\theta_j D\rho^j - \frac{i}{2} R_{j\bar{\jmath}k\bar{k}} \rho^j \wedge \rho^k \eta^{\bar{\jmath}} \theta_l g^{l\bar{k}}\right). \end{aligned} \tag{4.14}$$

On a general target space X the B-model has a chiral anomaly [12], which is cured if we demand that the target space is *Calabi-Yau*. In contrast to the A- model the B-model only depends on the complex structure on X. This dependence is of course also present in the correlation functions. In the B-model the Q-cohomology is *Dolbeault*-cohomology on forms valued in exterior powers of the holomorphic tangent bundle. A further interesting aspect of the $B-$model is, that is has no instanton corrections. Through the independence of the B-model of the metric and the *Kähler* form on X, the model has only "algebraic" knowledge of X [14]. This is to be understood as follows. Suppose X is given as subspace of $\mathbb{P}^N$ defined by intersection of homogeneous polynomials $f_1 = f_2 = \cdots = 0$[1]. These polynomials define the B-model completely. There is a relation between the A-model and B-model given by mirror symmetry, which we will sketch in the next subsection.

[1] Such a subspace is called algebraic variety

Mirror Symmetry

Mirror symmetry can be defined in various ways[2]. For our purpose we require only a weak definition, which says that two *Calabi-Yau* threefolds X and Y are mirror if the operator algebra on the A-model with target space X is isomorphic to the operator algebra of the B-model with target space Y. Further analysis of the dimensions of the vector spaces of the operator algebra gives the following relation between the *Hodge* numbers

[2] See for instance [14] for a review.

$$h^{p,q}(X) = h^{3-p,q}(Y). \tag{4.15}$$

As stated in the previous sections the operator algebra of the A-model depends on the choice of the complexifed *Kähler* -form and on the B-model it depends on the complex structure on Y. Therefore mirror symmetry relates the moduli space of $B + iJ$ on X to the moduli space of complex structures on Y. One can construct a map between the two moduli spaces called mirror map. Since we will need these concepts later, we will follow [14] and construct the mirror map for the quintic. The quintic is the subspace in $\mathbb{P}^4$ given by the vanishing of a homogeneous polynomial of degree 5. The moduli space of the complexified *Kähler* class is one dimensional, $h^{1,1}(X) = 1$. The dimension of the complex structure moduli is given by $h^{2,1}(X) = 101$. In order to obtain the mirror Y of X, X is divided by a $(\mathbb{Z}_5)^3$ orbifold action and take a crepant resolution thereof[3]. Y is specified by giving the quintic polynomial

[3] The explanation why this orbifolding yields the mirror is given in [7] following [23].

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0x_1x_2x_3x_4, \tag{4.16}$$

where ψ is a complex parameter which parametrises the complex structure. The mirror map will give a mapping between $B + iJ$ on the A-model side and ψ on the B-model side. The mirror map is not globally well defined and therefore one constructs the map at a specific basepoint in the moduli space. In the A-model we choose the large radius point. The maps at the other points can be obtained by analytic continuation. The moduli space has the structure of a special *Kähler* manifold [24]. This special structure leads to preferred coordinates, which on the A-model side are given by the components of $B+iJ$ and on the B-model side are given by periods. These periods are defined in the following way

$$\varpi_m = \int\limits_{\alpha_m} \Omega. \tag{4.17}$$

Ω is the holomorphic 3-form and the α_m, $m = 0 \dots h^{2,1}(Y)$, form a symplectic basis of $H_3(Y, \mathbb{Z})$. We will now sketch the procedure for the quintic. The periods ϖ_m are solutions of a so called *Picard-Fuchs* differential equation. To write down the *Picard-Fuchs* equation we first introduce a new coordinate z on the B-model moduli space. z is given by $z = (5\psi)^{-5}$. Using this coordinate the equation reads

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^4 \varpi_m - 5^5 z \left(z\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1}{5}\right)\left(z\frac{\mathrm{d}}{\mathrm{d}z} + \frac{2}{5}\right)\left(z\frac{\mathrm{d}}{\mathrm{d}z} + \frac{3}{5}\right)\left(z\frac{\mathrm{d}}{\mathrm{d}z} + \frac{4}{5}\right)\varpi_m = 0. \tag{4.18}$$

In the vicinity of $z = 0$ a basis of solutions is given by

$$\begin{aligned} \varpi_0 &= \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n \\ \varpi_k &= \frac{1}{(2\pi i)^k} \log(z)^k \varpi_0 + \dots, \quad k = 1,2,3 \end{aligned} \tag{4.19}$$

These solutions can by associated with the large radius limit $J = \infty$ on the A-model side through monodromy considerations. This results in the following mirror map

$$B + iJ = \frac{\varpi_1}{\varpi_0} = \frac{1}{2\pi i}\left(\log(z) + 770z + 717825z^2 + \dots\right). \tag{4.20}$$

There are three distinct points in the moduli space namely $z = 0$, $z = \infty$ (or $\psi = 0$) and $z = 5^{-5}$(or $\psi = \exp(2\pi i n/5)$). At these points the *Picard-Fuchs* equation is singular. These points are usually called large complex structure point, *Landau-Ginzburg* orbifold point and conifold point, respectively. In the next subsections we will study the different possible boundary conditions in the non linear sigma model following [14].

A-branes

One can view D-branes as subspaces L_a in the target space X. If we now introduce a boundary on our worldsheet Σ denoted by $\partial\Sigma$ we demand that the maps ϕ fulfil

$$\phi(\partial\Sigma) \subset \bigcup_a L_a. \tag{4.21}$$

Variation of the action on a worldsheet with boundaries gives now two terms, the bulk term and the boundary term. As familiar from variation of the action without boundary, setting the bulk variation to zero gives the equations of motion for the bulk fields. Requiring the vanishing of the boundary variation gives further constraints on the fields. For the boundary at $z = \overline{z}$, these conditions on the fields can be summarized as

$$\frac{\partial \phi^I}{\partial z} = R^I_J(\phi) \frac{\partial \phi^J}{\partial \overline{z}} + \text{fermions}. \tag{4.22}$$

R is a matrix with $g_{IJ} R^I_K R^J_L = g_{KL}$[4]. Directions normal to L are called *Dirichlet* directions and are eigenvectors of R with eigenvalue -1. Tangent directions of L are referred to as *Neumann* directions and have eigenvalue $+1$ with respect to R. The boundary conditions on the fermions are also described by the matrix R

[4] In flat space this is equal to $RR^T = \mathbb{1}$, so R is an orthogonal matrix.

$$\psi^I_+ = R^I_J(\phi)\, \psi^J_-. \tag{4.23}$$

In order for the reflection of the fermionic modes at the boundary to be compatible with the A-twist R must obey, in holomorphic coordinates,

$$R^i_{\overline{j}} = R^{\overline{\imath}}_{j} = 0. \tag{4.24}$$

An important fact is now that the almost complex structure on X exchanges the tangent and normal directions of L. As consequence L has to be of middle dimension. In the case of *Calabi-Yau* threefolds L has real dimension 3. By looking at the definition of the *Kähler*-form (see section 2.5) one sees, that the *Kähler*-form restricted to L vanishes. Sub-manifolds of middle dimension with vanishing *Kähler*-form are called *Lagrangian*-submanifolds. Before we noted that the $A-$model correlation functions only depend on the cohomology class of the $B-$field. In the case of a worldsheet with boundaries additional degrees of freedom appear. The description of these additional degrees of freedom is done by introducing a 1-form A on X and with an additional boundary term in the action

$$S_{\partial\Sigma} = -2\pi i \oint_{\partial\Sigma} \phi^*(A)\,. \tag{4.25}$$

To preserve supersymmetry or BRST invariance one has the introduce additional fermions on the boundary $\partial\Sigma$, but details are omitted here. The A field can be interpreted as a connection on a $U(1)$ bundle[5] associated to the gauge theory on the on the worldvolume of the D-brane. These further degrees of freedom obtained by introducing a boundary are encoded in a so-called *Chan Paton* space. Also the preservation of BRST symmetry puts further constraints on the connection A. At quantization a further constraint on A-branes emerges from an anomaly. Skipping the details, which are given in [14], the anomaly cancellation is given by the vanishing of the *Maslov*-class.

[5] One can also consider larger gauge groups related to the D-brane.

B-branes

The difference to the case of A-branes is, that R has now the property $R^{\overline{\imath}}_j = R^i_{\overline{j}} = 0$. In that case the almost complex structure preserves the direction tangent and normal to L. As consequence B-type $D-$branes are holomorphically embedded submanifolds of X. This restricts the dimension of possible branes to be even: $0, 2, 4$ or 6.

In the case of $B-$branes it is sufficient to take a look at the 6-brane first, because from this brane we can deduce the properties of the other branes. The 6-brane fills the entire target manifold and is given by purely *Neumann*-boundary conditions on the open string. Similar to $A-$branes, adding a boundary results in additional degrees of freedom, such that one can consider a bundle $E \to X$ over the $B-$brane. Restricting to the case $B = 0$ gives a restriction on the curvature F of the bundle. Then F has to be a $(1,1)-$form, which is equivalent to the statement that $E \to X$ is a holomorphic bundle. A holomorphic bundle can be described using methods from algebraic geometry [6]. Therefore one uses *sheaves*, which are the algebraic geometry counterparts of holomorphic vector bundles.

[6] An introduction to holomorphic vector bundels can be found in [25, chap.15]

Mathematical Description of Branes

In the following we describe mathematical aspects of D-branes following [14]. The natural way to describe D-branes is in terms of *categories*. The definition of a category reads

Definiton 4.1.1. *A category $\mathcal{C}$ consists of a class of objects* $\mathrm{obj}(\mathcal{C})$ *and a set* $\mathrm{Hom}_{\mathcal{C}}(A, B)$ *of morphisms for every ordered pair* (A, B) *of objects. Further for every* $A \in \mathrm{obj}(\mathcal{C})$ *there exists an identity morphism* $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$. *A composition function is given by*

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C) \tag{4.26}$$

for every ordered triple (A, B, C). *For* $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ *and* $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ *the composition is denoted by* fg. *In addition the following axioms hold*

1. *Associativity axiom:* $(hg)f = h(gf)$ *for* $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ *and* $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$
2. *Unit axiom:* $\mathrm{id}_B f = f = f\,\mathrm{id}_A$ *for* $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

In terms of category-language D-branes are the objects of the category and the morphisms between different objects are given by open-string states. Depending on our model, either $A-$ or $B-$type, the possible D-branes are objects of specific categories. In the geometric phases the A-branes are objects in a so called *Fukaya*-category and B-branes are objects in the *derived category of coherent sheaves*. Whereas in *Landau-Ginzburg* phases branes are described by matrix factorisations (see section 4.2). Because A-branes are not further considered in this thesis, we will not give additional details on their description. Subsequently we will only consider B-branes.

Starting with the definition of a sheaf, we will give a rough overview of the mathematical picture describing B-branes. In order to define the notion of a *sheaf*, we first define a *presheaf*.

Definiton 4.1.2. *Let X be a topological space. A presheaf $\mathscr{F}$ on X is given by the following data* *(a) To every open set $U \subset X$ we assign an abelian group $\mathscr{F}(U)$*

(b) If $V \subset U$ we have a 'restriction' homomorphism $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$.

Additionally this data has to fulfil

1) $\mathscr{F}(\emptyset) = 0$

2) ρ_{UU} is the identity map.

3) If $W \subset V \subset U$ then $\rho_{UW} = \rho_{VW}\rho_{UV}$.

By $\sigma|_V$ we denote the restriction $\rho_{UV}(\sigma)$ of $\sigma \in \mathscr{F}$.

An example of a sheaf, which we will again encounter in the calculations, is the sheaf of holomorphic functions on X or structure sheaf $\mathscr{O}_X$. The set of holomorphic functions forms a group with addition as group law. Therefore the sheaf $\mathscr{O}_X$ is constructed by choosing $\mathscr{F}(U)$ to be the group of holomorphic functions. Starting from $\mathscr{O}_X$ one can construct more complicated sheaves, but in order to do so we have to introduce additional algebraic structures [7]. First we introduce the notion of a *ring*.

[7] A good reference for the basic algebraic structures is [26]. The given definitions were taken from there.

Definiton 4.1.3. *A ring[8] is a set $\mathcal{R}$ with two operations $(x, y) \to xy$ and $(x, y) \to x + y$. These operations are called multiplication and addition respectively. The operations are subject to the following axioms*

[8] Physicists are more familiar with the notion of a *field*. A field is a ring with an identity element for the multiplication and an inverse element for every element, except for 0, with respect to the multiplication.

1. *$\mathcal{R}$ is an abelian group under addition.*

2. *Multiplication is associative and distributive with respect to addition.*

$$
\begin{aligned}
(xy)z &= x(yz) \\
x(y+z) &= xy + xz \\
(y+z)x &= yx + zx \qquad\qquad \forall x, y, z \in \mathcal{R}
\end{aligned}
$$

Further we need the concept of a *module*.

Definiton 4.1.4. *A module[9] $\mathcal{M}$ over the ring $\mathcal{R}$ is an abelian group $\mathcal{M}$ together with an operation called scalar multiplication, $\mathcal{R} \times \mathcal{M} \to \mathcal{M}$ by $(\alpha, x) \to \alpha x$ such that*

[9] A familiar module is the module over a field usually called *vector* space.

$$
\begin{aligned}
\alpha(x+y) &= \alpha x + \alpha y \\
(\alpha+\beta)x &= \alpha x + \beta x \\
(\alpha\beta)x &= \alpha(\beta x)
\end{aligned}
$$

$\forall \alpha, \beta \in \mathcal{R}$ and $\forall x, y \in \mathcal{M}$. If the ring has an identity element 1 then

$$
1x = x \qquad\qquad \forall x \in \mathcal{M}
$$

Since the set of holomorphic functions is an abelian group under addition and multiplication, it gets the structure of a ring. We can construct a sheaf of $\mathscr{O}_X$-modules. Namely let $\mathscr{E}$ be a sheaf such that $\mathscr{E}(U)$ is an $\mathscr{O}_X(U)$-module for any open $U \in X$. Further we can take

$$\mathscr{O}_X^{\oplus n} = \underbrace{\mathscr{O}_X \oplus \mathscr{O}_X \oplus \cdots \oplus \mathscr{O}_X}_{n}, \tag{4.27}$$

and construct another sheaf of $\mathscr{O}_X$-modules called *free* sheaf of $\mathscr{O}_X$-modules of rank n. If a sheaf $\mathscr{E}$ fulfils $\mathscr{E}(U_\alpha) \cong \mathscr{O}_X(U_\alpha)^{\oplus n}$ for all α, where $\{U_\alpha\}$ is an open covering of X. Then $\mathscr{E}$ is called *locally free* of rank n. We want to note that holomorphic vector bundles of rank n on X and locally free sheaves of rank n on X are in a one-to-one correspondence. Locally free sheaves are not sufficient to describe all possible B-branes and therefore one needs the concept of coherent sheaves. The mathematical aspects of coherent sheaves can be found in [14]. Coherent sheaves form a category. By introducing sheaves we have replaced vector bundles with algebraic objects. The next step is to find an algebraic notion of *Dolbeault* cohomology. This is done by *sheaf* cohomology. By doing so we get a notion of open strings in the category setup. Skipping mathematical details we give the result from [14]

> An open string from the B-brane associated to the locally-free sheaf $\mathscr{E}$ to another B-brane given by the locally-free sheaf $\mathscr{F}$ is given by an element of the group $\mathrm{Ext}^q(\mathscr{E}, \mathscr{F})$.

Here q is related to the fact that, local operators in the B-model are written in terms of $(0, q)$ forms valued in $\bigwedge^q T_X$. Usually q is referred to as ghost number. In order to find a relation between $\mathrm{Ext}^q(\mathscr{E}, \mathscr{F})$ and the cohomology of sheaves H^q, we suppose to have two vector bundles E and F. We can construct the vector bundle $\mathrm{Hom}(E, F)$ from the given bundles. Additionally we can associate locally free sheaves $\mathscr{E}$ and $\mathscr{F}$ respectively to the given vector bundles. The locally free sheaf associated to $\mathrm{Hom}(E, F)$ is denoted by $\mathscr{H}(E, F)$. Details of this procedure are given in [14]. Now we can relate *Dolbeault* cohomology and sheaf cohomology by

$$H^{0,q}\left(X, \mathrm{Hom}(E, F)\right) = H^q\left(X, \mathscr{H}(\mathscr{E}, \mathscr{F})\right) \tag{4.28}$$

and the relation between sheaf cohomology and the group $\mathrm{Ext}^q(\mathscr{E}, \mathscr{F})$ is given by

$$H^q\left(X, \mathscr{H}(\mathscr{E}, \mathscr{F})\right) = \mathrm{Ext}^q(\mathscr{E}, \mathscr{F}). \tag{4.29}$$

The problem of the previous discussion is, that not all possible B-branes are covered by it. To get a solution of this deficit, we need to notion of a *complex*.

In general a *complex*[10] is a sequence of abeliean groups or modules B_i, with homomorphisms between them $\mathrm{d}_{n-1} : B_{n-1} \to B_n$. The homomorphisms fulfil $d_n \circ \mathrm{d}_{n-1} = 0 \quad \forall n$, consequently $\mathrm{Im}\left(\mathrm{d}_{n-1}\right) \subseteq \mathrm{Ker}\left(\mathrm{d}_n\right)$. Often a complex is represented by a diagram of the form

$$\cdots \xrightarrow{\mathrm{d}_{n-2}} B_{n-1} \xrightarrow{\mathrm{d}_{n-1}} B_n \xrightarrow{\mathrm{d}_n} B_{n+1} \xrightarrow{\mathrm{d}_{n+1}} \cdots \tag{4.30}$$

[10] The given definition is a so-called *cochain complex*. There are different sorts of complexes, but in this thesis it is sufficient to focus on *cochain complexes*.

To apply the techniques of complexes to B-branes, we construct a general collection of D-branes as a locally-free sheaf $\mathscr{E}$, with a decomposition

$$\mathscr{E} = \bigoplus_{n\in\mathbb{Z}} \mathscr{E}^n. \tag{4.31}$$

$\mathscr{E}^n$ is a B-brane with ghost number n[11].

Additionally we introduce morphisms d_n between the $\mathscr{E}^n$

$$\mathrm{d}_n \in \mathrm{Ext}^0\left(\mathscr{E}^n, \mathscr{E}^{n+1}\right) = \mathrm{Hom}\left(\mathscr{E}^n, \mathscr{E}^{n+1}\right). \tag{4.32}$$

[11] In the B-model one can assign a ghost number to a brane in order to match the ambiguity of the open string ghost number on the A-model side [14].

The collection of $\mathscr{E}^n$ and d_n gives a complex, subsequently denoted by $\mathscr{E}^\bullet$.

The open string spectrum between two complexes of B-branes, denoted by $\mathscr{E}^\bullet$ and $\mathscr{F}^\bullet$, is given by $\mathrm{Ext}^n\left(\mathscr{E}^\bullet, \mathscr{F}^\bullet\right)$ as shown in [14]. The structures given above form a category, namely the *dervied category of locally-free sheaves.* Further analysis shows that B-branes are related to *coherent sheaves.* Although we performed the previous analysis in terms of locally-free sheaves, the obtained results are also valid in the case of coherent sheaves, because on a smooth space every complex of locally-free sheaves $\mathscr{F}^\bullet$ is quasi-isomorphic[12] to a coherent sheaf. Therefore we close this section with a final statement [14]

[12] See the definition in section 5.1.

The category of B-branes is the derived category of coherent sheaves.

4.2 Boundaries in Landau-Ginzburg models

This section follows the discussion given in [15] and also the notation is chosen in accordance with this reference. These considerations were first done by [27].

We will consider a 2-dimensional *Landau-Ginzburg* theory with $\mathcal{N} = (2,2)$ supersymmetry and place this theory on a worldsheet Σ with a boundary. By doing so we can at most preserve half of the supersymmetries. Similar to the non-linear sigma model, there are two choices of supersymmetry to preserve, namely $A-$type and $B-$type supersymmetry [28]. Depending on our choice we also get so called $A-$ and $B-$type D-branes, respectively. In this thesis we are mainly interested in $B-$type branes and so we will focus on the $B-$type supersymmetry. Preserving $B-$type supersymmetry is done by introducing boundary interactions. Then one does not need to impose boundary conditions on the fields.

Bulk-Lagrangian

As in chapter 3 we are using the $(2,2)$-superspace, spanned by two bosonic coordinates x^0, x^1 and four fermionic ones $\theta^\pm, \overline{\theta}^\pm$. The supercharges, covariant derivatives and the supersymmetry algebra in the conventions of [15] are

$$
\begin{aligned}
Q_\pm &= \frac{\partial}{\partial\theta^\pm} + i\overline{\theta}^\pm \frac{\partial}{\partial\pm}, & \overline{Q}_\pm &= -\frac{\partial}{\partial\overline{\theta}^\pm} - i\theta^\pm \frac{\partial}{\partial x^\pm} \\
D_\pm &= \frac{\partial}{\partial\theta^\pm} - i\overline{\theta}^\pm \frac{\partial}{\partial\pm}, & \overline{D}_\pm &= -\frac{\partial}{\partial\overline{\theta}^\pm} + i\theta^\pm \frac{\partial}{\partial x^\pm} \\
\left\{Q_\pm, \overline{Q}_\pm\right\} &= -2i\partial_\pm, & \left\{D_\pm, \overline{D}_\pm\right\} &= 2i\partial_\pm.
\end{aligned}
\tag{4.33}
$$

The chiral Φ and anti-chiral $\overline{\Phi}$ superfields fulfil the common conditions $\overline{D}_\pm\Phi = 0$ and $D_\pm\overline{\Phi} = 0$. Writing down a chiral superfield in components results in

$$
\begin{aligned}
\Phi\left(y^\pm, \theta^\pm\right) = \phi\left(y^\pm\right) &+ \theta^+\psi_+\left(y^\pm\right) \\
&+ \theta^-\psi_-\left(y^\pm\right) + \theta^+\theta^- F\left(y^\pm\right),
\end{aligned}
\tag{4.34}
$$

with $y^\pm = x^\pm - i\theta^\pm\overline{\theta}^\pm$. Writing the variation operator δ as $\delta = \epsilon_+ Q_- - \epsilon_- Q_+ - \overline{\epsilon}_+\overline{Q}_- + \overline{\epsilon}_-\overline{Q}_+$ results in the following transformations of the components

$$
\begin{aligned}
\delta\phi &= \epsilon_+\psi_- - \epsilon_-\psi_+, & \delta\overline{\phi} &= -\overline{\epsilon}_+\overline{\psi}_- + \overline{\epsilon}_-\overline{\psi}_+, \\
\delta\psi_+ &= +2i\overline{\epsilon}_-\partial_+\phi + \epsilon_+ F, & \delta\overline{\psi}_+ &= -2i\epsilon_-\partial_+\overline{\phi} + \overline{\epsilon}_+\overline{F}, \\
\delta\psi_- &= -2i\overline{\epsilon}_+\partial_-\phi + \epsilon_- F, & \delta\overline{\psi}_- &= +2i\epsilon_+\partial_-\overline{\phi} + \overline{\epsilon}_-\overline{F}.
\end{aligned}
\tag{4.35}
$$

The Lagrangian of the *Landau-Ginzburg* model is given by

$$
\begin{aligned}
S_\Sigma = \int_\Sigma \mathrm{d}^2x \Bigg(&-\partial^\mu\overline{\phi}\partial_\mu\phi + \frac{i}{2}\overline{\psi}_-\left(\overleftrightarrow{\partial_0} + \overleftrightarrow{\partial_1}\right)\psi_- + \frac{i}{2}\overline{\psi}_+\left(\overleftrightarrow{\partial_0} - \overleftrightarrow{\partial_1}\right)\psi_+ \\
&-\frac{1}{4}\left|W'\right|^2 - \frac{1}{2}W''\psi_+\psi_- - \frac{1}{2}\overline{W}''\overline{\psi}_-\overline{\psi}_+\Bigg),
\end{aligned}
\tag{4.36}
$$

with $\phi\overleftrightarrow{\partial_i}\overline{\phi} = \phi\partial_i\overline{\phi} - \partial_i\phi\overline{\phi}$. To obtain eq. (4.36) we used the algebraic equation of motion of $F = -\frac{1}{2}\overline{W}'\left(\overline{\phi}\right)$ and W is the superpotential. S_Σ is invariant under eq. (4.35) up to total derivatives.

Adding a Boundary

The next step is to place the *Landau-Ginzburg* model on a worldsheet with boundary. Following [15] we place the model on the strip $\left(x^0, x^1\right) \in (\mathbb{R}, [0, \pi])$. The conserved supercharge for B-type supersymmetry is given by $Q = \overline{Q}_+ + \overline{Q}_-$. B-type supersymmetry can also be obtained by setting the parameters of the previous supersymmetry transformations to

$$
\epsilon = \epsilon_+ = -\epsilon_-. \tag{4.37}
$$

If we now consider the B-type supersymmetry transformations, given by $\delta = \epsilon\overline{Q} - \overline{\epsilon}Q$, we get

$$
\begin{aligned}
\delta\phi &= \epsilon\eta, & \delta\overline{\phi} &= -\overline{\epsilon\eta}, \\
\delta\eta &= -2i\overline{\epsilon}\partial_0\phi, & \delta\overline{\eta} &= 2i\epsilon\partial_0\overline{\phi}, \\
\delta\zeta &= 2i\overline{\epsilon}\partial_1\phi + \epsilon\overline{W}'\left(\overline{\phi}\right), & \delta\overline{\zeta} &= -2i\epsilon\partial_1\overline{\phi} + \overline{\epsilon}W'\left(\phi\right),
\end{aligned}
\tag{4.38}
$$

where we introduced $\eta = \psi_- + \psi_+$ and $\zeta = \psi_- - \psi_+$. The fermionic coordinates on the boundary superspace are given by $\theta^0 = \frac{1}{2}(\theta^- + \theta^+)$ and $\overline{\theta}^0 = \frac{1}{2}\left(\overline{\theta}^- + \overline{\theta}^+\right)$. Now we can write the supercharges as

$$\overline{Q} = \frac{\partial}{\partial\theta^0} + i\overline{\theta}^0\frac{\partial}{\partial x^0}, \qquad Q = -\frac{\partial}{\partial\overline{\theta}^0} - i\theta^0\frac{\partial}{\partial x^0}. \tag{4.39}$$

If we now vary the action eq. (4.36) with respect to transformations eq. (4.38) we get surface terms. To cancel these terms we have to introduce a boundary action. In the case of $W = 0$ we add the following term to the Lagrangian

$$S_{\partial\Sigma,\psi} = \frac{i}{4}\int \mathrm{d}x^0 \left(\overline{\zeta}\eta - \overline{\eta}\zeta\right)\Big|_0^\pi, \tag{4.40}$$

to obtain an invariant action. If $W \neq 0$ the following surface terms appear after variation

$$\delta\left(S_\Sigma + S_{\partial\Sigma,\psi}\right) = \frac{i}{2}\int \mathrm{d}x^0 \left(\epsilon\overline{\eta}\overline{W}' + \overline{\epsilon}\eta W'\right)\Big|_0^\pi. \tag{4.41}$$

In contrast to the previous case this leftover term can not be cancelled by adding additional terms containing only bulk fields. Therefore we have to introduce a boundary fermionic superfield Π[13]. Π satisfies $D\Pi = E(\Phi')$, where $\Phi'\left(y^0,\theta^0\right) = \phi\left(y^0\right) + \theta^0\eta\left(y^0\right)$. The component expression of Π is given by

[13] We will later call Π boundary fermion.

$$\Pi\left(y^0,\theta^0,\overline{\theta}^0\right) = \pi\left(y^0\right) + \theta^0 l\left(y^0\right) - \overline{\theta}^0\left(E(\phi) + \theta^0\eta\left(y^0\right)E'(\phi)\right), \tag{4.42}$$

with $y^0 = x^0 - i\theta^0\overline{\theta}^0$. The components of the field transform under the B-type supersymmetry as

$$\begin{aligned} \delta\pi &= \epsilon l - \overline{\epsilon}E(\phi), & \delta\overline{\pi} &= \overline{\epsilon}\overline{l} - \epsilon\overline{E}\left(\overline{\phi}\right), \\ \delta l &= -2i\overline{\epsilon}\partial_0\pi + \overline{\epsilon}\eta E'(\phi), & \delta\overline{l} &= -2i\epsilon\partial_0\overline{\pi} + \epsilon\overline{\eta}\overline{E}'\left(\overline{\phi}\right). \end{aligned} \tag{4.43}$$

The action of the boundary fields is given by

$$S_{\partial\Sigma} = -\frac{1}{2}\int\limits_{\partial\Sigma} \mathrm{d}x^0\mathrm{d}^2\overline{\Pi}\Pi\Big|_0^\pi - \frac{i}{2}\int\limits_{\partial\Sigma} \mathrm{d}x^0\mathrm{d}\theta\Pi J(\Phi)_{\overline{\theta}=0}\Big|_0^\pi + c.c. \tag{4.44}$$

Rewriting the action in components and using the equation of motion $l = -i\overline{J}\left(\overline{\phi}\right)$ results in

$$S_{\partial\Sigma} = \int \mathrm{d}x^0\left(i\overline{\pi}\partial_0\pi - \frac{1}{2}\overline{J}J - \frac{1}{2}\overline{E}E + \frac{i}{2}\pi\eta J' + \frac{i}{2}\overline{\pi}\overline{\eta}\overline{J}' - \frac{1}{2}\overline{\pi}\eta E' + \frac{1}{2}\pi\overline{\eta}\overline{E}'\right)\Big|_0^\pi. \tag{4.45}$$

By employing the equation of motion of l, the variations of π simplify to

$$\begin{aligned} \delta\pi &= -i\epsilon\overline{J}\left(\overline{\phi}\right) - \overline{\epsilon}E(\phi) \\ \delta\overline{\pi} &= i\overline{\epsilon}J(\phi) - \epsilon\overline{E}\left(\overline{\phi}\right). \end{aligned} \tag{4.46}$$

Under the B-type variation the action eq. (4.45) is invariant except for the following term

$$\delta S_{\partial\Sigma} = -\frac{i}{2}\int \mathrm{d}x^0 \left(\epsilon\overline{\eta}\left(\overline{EJ}\right)' + \overline{\epsilon}\eta\left(EJ\right)'\right). \tag{4.47}$$

If we now compare eq. (4.41) with eq. (4.47) we see that we get a supersymmetric action if we demand

$$W = EJ + \text{const.} \tag{4.48}$$

$J(\phi)$ and $E(\phi)$ are usually referred to as boundary potentials.

Generalization

As explained for instance in [18] quantization leads to a *Hilbert* space of the boundary fermions which is a two-dimensional vector space $\mathbb{C}^2$, and is graded by fermion number. In string theory this space is interpreted as the *Chan-Paton*-space $\mathcal{V}$ of a D-brane anti-D-brane system and E, J describe the tachyon configuration stretched between the D-branes. The supercharge also gets an additional boundary contribution. Now let us generalize the above considerations. In general, boundary interactions preserving $B-$type supersymmetry are described by two $N \times N$ matrices $J(x_1, \ldots, x_n)$, $E(x_1, \ldots, x_n)$. These matrices have polynomial entries and fulfil

$$J \cdot E = E \cdot J = W \cdot \mathbb{1}_{N\times N}. \tag{4.49}$$

Again one can think of the matrices J and E as describing the tachyon configuration between a stack of N branes and N anti-branes. Considering the contribution to the supercharge arising from the boundary, denoted by Q, we get

$$Q = \begin{pmatrix} 0 & J \\ E & 0 \end{pmatrix}. \tag{4.50}$$

The condition for preserving the $B-$type supersymmetry can now be written as

$$Q^2 = W \cdot \mathbb{1}_{2N\times 2N}. \tag{4.51}$$

The action of Q on open string states is given by a supercommutator. The open string ground states are given as cohomology classes of Q. These states are given by matrices with polynomial entries and found by action with Q on them [27, 29, 15, 18]

$$\{Q, \mathrm{Y}\} = 0 \qquad\qquad \mathrm{Y} \equiv \mathrm{Y} + \{Q, \mathrm{Y}'\}. \tag{4.52}$$

Considering a quasi-homogeneous superpotential W the bulk theory has an additional $U(1)$ $R-$symmetry under which W has charge 2:

$$W\left(\lambda_i^r x_i\right) = \lambda^2 W\left(r_i\right). \tag{4.53}$$

This symmetry gives a further constraint on Q [30]:

$$R(\lambda)Q\left(\lambda^{r_i}x_i\right)R(\lambda)^{-1} = \lambda Q\left(x_i\right). \tag{4.54}$$

$R(\lambda)$ is a representation of the $U(1)$ symmetry group on the *Chan-Paton*-space $\mathcal{V}$, with $\lambda = e^{i\alpha}$.

Mathematical Description of Branes in LG-models

Equation (4.51) states that all possible boundary conditions, preserving $B-$type supersymmetry, can be obtained trough matrix factorisations of the superpotential W. Similar to the case in the non-linear sigma model, the possible matrix factorisations form a category. The objects in this category are given by the possible matrix factorisations. These factorisations[14] can be represented in terms of the *Chan-Paton* spaces $\mathcal{V}$, corresponding to the D-branes. A matrix factorisation Q, representing the brane $\mathcal{B}$, acts as an odd operator on the *Chan-Paton* space of $\mathcal{B}$ by: Let $V = \mathcal{V}^{even} \oplus \mathcal{V}^{odd}$ by the *Chan-Paton* space corresponding to a matrix factorisation, with

[14] We only give here a fairly short introduction to these concepts and also the given discussion lacks mathematical rigour. Details can be found in [16], especially in chapter 2.

$$Q = \begin{pmatrix} 0 & J \\ E & 0 \end{pmatrix}. \tag{4.55}$$

Then the objects M of the category of matrix factorisations $\mathrm{MF}\left(W\right)$ are given by

$$\mathcal{B} : M \cong \left(\mathcal{V}^{even} \underset{J}{\overset{E}{\rightleftharpoons}} \mathcal{V}^{odd} \right). \tag{4.56}$$

The morphisms $\mathrm{Hom}\left(\mathcal{V}, \tilde{\mathcal{V}}\right)$ in this category are given by the open string states Y, stretching between the D-branes with *Chan-Paton* spaces V and $\tilde{\mathcal{V}}$ respectively.

4.3 Boundaries in Gauged Linear Sigma models

In this section we consider the main model of interest in this thesis. We follow [16]. The previous sections concerning the *Landau-Ginzburg* and non-linear sigma model will help us to develop the necessary tools to describe $D-$branes in the gauged linear sigma model. We will solely concentrate on the implications arising by introducing a boundary, because aspects regarding the phases have been already discussed in chapter 3. We will also adopt the conventions of [16] for the gauged linear sigma model. In these conventions the *Lagrangian* density for an *Abelian* gauge group $T \cong U(1)_1 \times U(1)_2 \times \cdots \times U(1)_k$ and N matter chiral superfields Φ_i is given by

$$\begin{aligned} \mathcal{L} = & \int \mathrm{d}^4\theta \left(-\frac{1}{2} \sum_{a,b=1}^{k} \left(e^{-2}\right)^{ab} \overline{\Sigma}_a \Sigma_b + \sum_{i=1}^{N} \overline{\Phi}_i e^{Q_i \cdot V} \Phi_i \right) \\ & + \mathrm{Re} \int \mathrm{d}^2\tilde{\theta} \left(-\sum_{a=1}^{k} t^a \Sigma_a \right) + \mathrm{Re} \int \mathrm{d}^2\theta W\left(\Phi\right). \end{aligned} \tag{4.57}$$

Luckily the conventions of [16] mostly agree with the conventions used in [1] and chapter 3.

Boundary Counter Terms

Applying an $\mathcal{N}=2$ B-type supersymmetry variation[15] on eq. (4.57) results in a boundary term. Instead of imposing boundary conditions on the bulk fields, we will add additional terms to the action, whose variation cancels the boundary terms of the variation of the bulk action. Let us denote by $\mathbf{Q}=\overline{\mathbf{Q}}_+ + \overline{\mathbf{Q}}_-$ and $\mathbf{Q}^\dagger = \mathbf{Q}_+ + \mathbf{Q}_-$ the generators of the B-type supersymmetry variation. If we can now write down the action eq. (4.57) as $\int \mathrm{d}^2 s \mathbf{Q}\mathbf{Q}^\dagger(\dots)$, then we have found the desired $\mathcal{N}=2_B$-type supersymmetry invariant action. This procedure automatically gives the boundary counter terms. As stated in [16] the $\mathcal{N}=2_B$ invariant parts in the action for the gauge kinetic, matter kinetic and the FI-θ terms are given by[16]

[15] In the following we will denote this variation by $\mathcal{N}=2_B$ as in [16].

[16] Here we denote the worldsheet by S and coordinates on S by s in agreement with [16], but differently as in chapter 3.

$$\begin{aligned} S_g + S_m + S_{FI\theta} = \frac{1}{4}\int\limits_{\mathsf{S}} \mathrm{d}^2 s \mathbf{Q}\mathbf{Q}^\dagger \left[\overline{Q}_+, Q_+\right] \left(-\frac{1}{2e^2}\sum_{a=1}^{k}|\sigma_a|^2 + \sum_{i=1}^{N}|\phi_i|^2\right) \\ + \frac{1}{2\pi}\mathrm{Re}\int\limits_{\mathsf{S}} \mathrm{d}^2 s \mathbf{Q}\mathbf{Q}^\dagger \left(-\sum_{a=1}^{k} t^a\sigma_a\right). \end{aligned} \tag{4.58}$$

By comparison of the above result with the *Lagrangian* density given in eq. (4.57) the boundary counter terms can be read off. The total counter term action reads

$$\begin{aligned} S^{c.t.}_{tot} = \frac{1}{2\pi}\int\limits_{\partial\mathsf{S}} \mathrm{d}t \Bigg\{ \frac{1}{2e^2}\sum_{a=1}^{k}\left(\frac{1}{2}\partial_1|\sigma_a|^2 + \mathrm{Im}(\sigma_a)D_a + \mathrm{Re}(\sigma_a)(v_a)_{01}\right) \\ + \frac{i}{2}\sum_{i=1}^{N}\left(\overline{\psi}_{i-}\psi_{i+} - \overline{\psi}_{i+}\psi_{i-}\right) + \mathrm{Im}\sum_{a=1}^{k}\left\{\left(\sum_{i=1}^{N} Q_i^a|\phi_i|^2 - t^a\right)\sigma_a\right\}\Bigg\}. \end{aligned} \tag{4.59}$$

Next we take a look at boundary terms which are $\mathcal{N}=2_B$ supersymmetry invariant, but cannot be written with the help of the supersymmetry generators.

Wilson Line Branes

The term[17]

[17] v_0 is the 0-component of the gauge potential, see chapter 3.

$$\frac{1}{2}\int \mathrm{d}\theta \mathrm{d}\overline{\theta} V = -\left(v_0 - \mathrm{Re}(\sigma)\right) \tag{4.60}$$

is invariant under the $B-$type supersymmetry variation, but lacks $U(1)$ gauge invariance under which $iv_0 \to iv_0 + g\partial_0 g^{-1}$. This can be cured, by exponentiation of the term [16]

$$W_q\left(t_f, t_i\right) = \exp\left(-i\int\limits_{t_i}^{t_f} q\left[v_0 - \mathrm{Re}(\sigma)\right]\mathrm{d}t\right). \tag{4.61}$$

This term is a *Wilson* line and transforms under the gauge group as

$$W_q\left(t_f, t_i\right) \to g\left(t_f\right)^q \cdot W_q\left(t_f, t_i\right) \cdot g\left(t_i\right)^{-q}, \tag{4.62}$$

so the *Wilson*-line is gauge covariant whenever q is an integer. In accordance with [16] we will denote the brane supporting this *Wilson* line by

$$\mathcal{W}(q). \tag{4.63}$$

This allows the interpretation of a charged *Chan-Paton* space $\mathcal{V}$ with charge q under the gauge group. The full boundary *Lagrangian* is

$$\begin{aligned} S_{bdry} = S_g^{c.t.} + \frac{1}{2\pi} \int\limits_{\partial\mathbb{S}} \mathrm{d}t \Bigg\{ & \frac{i}{2} \sum_{i=1}^{N} \left(\overline{\psi}_{i-}\psi_{i+} - \overline{\psi}_{i+}\psi_{i-} \right) \\ & + \sum_{a=1}^{k} \left(\sum_{i=1}^{N} Q_i^a |\phi_i|^2 - r^a \right) \operatorname{Im}\sigma_a - \sum_{a=1}^{k} \left(\theta^a + 2\pi q^a \right) \left[(v_a)_0 - \operatorname{Re}(\sigma_a) \right] \Bigg\}. \end{aligned} \tag{4.64}$$

$S_g^{c.t.}$ denotes the counter term for the gauge kinetic term, which is the first line in eq. (4.59). In the above expression the bulk theta-angle term was converted into a boundary term.

With the *Wilson*-line at hand one can build further supersymmetric boundary interactions. A possible generalization is to consider direct sums of *Wilson*-line branes

$$\mathcal{W} = \bigoplus_{i=1}^{n} \mathcal{W}(q_i). \tag{4.65}$$

This results in a matrix valued boundary interaction

$$\mathcal{A}_t = \sum_{a=1}^{k} \begin{pmatrix} q_1^a & & \\ & \ddots & \\ & & q_n^a \end{pmatrix} \left\{ (v_a)_0 - \operatorname{Re}(\sigma_a) \right\}. \tag{4.66}$$

$\mathcal{A}_t$ transforms under the gauge group $U(1)^k$ as

$$i\mathcal{A}_t \to \rho(g) i\mathcal{A}_t \rho(g)^{-1} + \rho(g)\partial_t \rho(g)^{-1}, \tag{4.67}$$

where $\rho(g)$ is given by

$$\rho(g) = \begin{pmatrix} g^{q_1} & & \\ & \ddots & \\ & & g^{q_n} \end{pmatrix}, \tag{4.68}$$

with $g^q := g_1^{q^1} \cdots g_k^{q^k}$. The *Chan-Paton* space $\mathcal{V}$ of the brane is then $\mathcal{V} = \oplus_{i=1}^{k} \mathcal{W}(q_i)$. $\mathcal{V}$ carries the representation ρ of the gauge group $T \cong U(1)^k$.

Also the introduction of a $\mathbb{Z}_2$ graded sum of *Wilson*-line branes $\mathcal{V} = \mathcal{W}^{ev} \oplus \mathcal{W}^{od}$ and a tachyon profile Q is possible. As in section 4.2, Q describes an interaction between *Wilson*-line branes. This is done by considering a $\mathbb{Z}_2$ graded *Chan-Paton* space

$$\mathcal{V} = \mathcal{W}^{ev} \oplus \mathcal{W}^{odd}. \tag{4.69}$$

$\mathcal{V}$ carries a representation ρ of the gauge group and Q is an odd operator on $\mathcal{V}$. Q is a holomorphic function of the fields $\phi_1, \ldots, \phi_N$. The corresponding boundary interaction is given by

$$\begin{aligned}\mathcal{A}_t = \rho_* \left(v_0 - \mathrm{Re}(\sigma)\right) + \frac{1}{2}\left\{Q, Q^\dagger\right\} \\ - \frac{1}{2}\sum_{i=1}^{N} \psi^i \frac{\partial}{\partial \phi_i} Q + \frac{1}{2}\sum_{i=1}^{N} \overline{\psi_i} \frac{\partial}{\partial \overline{\phi}_i} Q^\dagger,\end{aligned} \tag{4.70}$$

where ρ_* is given by $\rho_*(X) = -i\frac{\mathrm{d}}{\mathrm{d}t}\rho\left(e^{itX}\right)\big|_{t=0}$, with iX an element of the *Lie* algebra of the gauge group. Demanding that the transformation law for $\mathcal{A}_t$ is given by eq. (4.67), restricts Q to fulfil

$$\rho\left(g\right)^{-1} Q\left(g\cdot\phi\right)\rho\left(g\right) = Q\left(\phi\right), \tag{4.71}$$

where $g\cdot\phi$ is the action of the gauge group on $\phi = (\phi_1, \ldots, \phi_N)$ given by $\left(g^{Q_1}, \ldots, g^{Q_N}\phi_N\right)$. The $\mathcal{N} = 2_B$ variation of eq. (4.70) results in

$$\begin{aligned}\delta\mathcal{A}_t = -\,\mathrm{Re}\left\{\sum_{i=1}^{N}\left(\overline{\epsilon}\psi_i\frac{\partial}{\partial\phi_i}Q^2\right) - \left[\overline{\epsilon}Q^\dagger, Q^2\right]\right\} \\ + i\mathcal{D}_t\left(\overline{\epsilon}Q + \epsilon Q^\dagger\right) - i\left(\dot{\overline{\epsilon}}Q + \dot{\epsilon}Q^\dagger\right).\end{aligned} \tag{4.72}$$

In the case of a gauged linear sigma model without a superpotential the condition for $\mathcal{A}_t$ to be $\mathcal{N} = 2_B$ supersymmetric is

$$Q^2 = c\cdot \mathrm{id}_{\mathcal{V}}\,. \tag{4.73}$$

We will not consider this case in detail, because in this thesis we only discuss models with non-vanishing superpotential, which we focus on next.

Matrix Factorisations

To obtain insight into the case of a non-vanishing superpotential W, we first consider the $\mathcal{N} = 2_B$ variation of the part in the action containing the superpotential

$$\delta\int_{\mathbb{S}} \mathrm{d}^2 s\mathcal{L}_W = -\,\mathrm{Re}\int_{\partial\mathbb{S}}\mathrm{d}t\sum_{i=1}^{N}\overline{\epsilon}\psi_i\frac{\partial W}{\partial\phi_i}. \tag{4.74}$$

The resulting term is called *Warner* term [28]. As in the *Landau-Ginzburg* model, we consider a $\mathbb{Z}_2$ graded sum of *Wilson*-line branes, given by a $\mathbb{Z}_2$ graded *Chan-Paton* space $\mathcal{V} = \mathcal{V}^{ev}\oplus\mathcal{V}^{od}$. The tachyon profile Q acts again as an odd operator on $\mathcal{V}$ and is a polynomial in $\phi = (\phi_1, \ldots, \phi_N)$. Obviously Q has to fulfil eq. (4.71) and the boundary interaction $\mathcal{A}_t$ is given by eq. (4.70).

By comparison of the variation of $\mathcal{A}_t$ eq. (4.72) with eq. (4.74) one sees, that Q has to fulfil

$$Q^2 = W\cdot\mathrm{id}_{\mathcal{V}} \tag{4.75}$$

in order to cancel the variation of the superpotential term.

The gauged linear sigma model also has a bulk vector $U(1)$ R-symmetry. The action of the $R-$symmetry on ϕ is denoted by $R_\lambda\phi$. In addition, the symmetry commutes with the gauge group

$$R_\lambda\left(g\cdot\phi\right)=g\cdot R_\lambda\left(\phi\right). \tag{4.76}$$

In the presence of a superpotential, $U(1)$ R-symmetry is only possible if W is quasi-homogeneous and we set, by convention,

$$W\left(R_\lambda\phi\right)=\lambda^2 W\left(\phi\right). \tag{4.77}$$

Consistency with eq. (4.75) requires Q to have $R-$charge 1 [16]. This is equivalent to demand that there are linear operators $R(\lambda)$ on the *Chan-Paton* space $\mathcal{V}$ commuting with the gauge symmetry

$$R\left(\lambda\right)\rho(g)R\left(\lambda\right)^{-1}=\rho(g), \tag{4.78}$$

such that

$$R\left(\lambda\right)Q\left(R_\lambda(\phi)\right)R\left(\lambda\right)^{-1}=\lambda Q\left(\phi\right). \tag{4.79}$$

The $R-$symmetry induces a additional grading on $\mathcal{V}$. It is possible to choose the R-grading in such a way that it respects the $\mathbb{Z}_2$ grading on $\mathcal{V}$. For example let $\mathcal{V}^j$ be the $R(\lambda)=\lambda^j$ subspace of $\mathcal{V}$ such that

$$\mathcal{V}=\bigoplus_{j_{min}}^{j_{max}}\mathcal{V}^j, \tag{4.80}$$

then

$$\mathcal{V}^{ev}=\bigoplus_{j:even}\mathcal{V}^j \qquad\qquad \mathcal{V}^{od}=\bigoplus_{j:odd}\mathcal{V}^j. \tag{4.81}$$

Each $\mathcal{V}^j$ corresponds to a direct sum of *Wilson*-line branes. A brane with R-grading is called R-graded D-brane $\mathfrak{B}$, in the notation of [16], and is determined by the quadruple $(\mathcal{V},Q,\rho,R)$.

Mathematical Description

D-branes of the GLSM form a category. The objects in this category are the various D-branes $\mathfrak{B}_i$ and open string states are described by morphisms between two branes $\mathfrak{B}_i$ and $\mathfrak{B}_j$. The data $(\mathcal{V},Q,\rho,R)$ of a brane $\mathfrak{B}$ is nicely represented as a complex of *Chan-Paton* spaces $\mathcal{V}$:

$$\cdots\xrightarrow{J}\mathcal{V}^{even}\xrightarrow{E}\mathcal{V}^{odd}\xrightarrow{J}\mathcal{V}^{even}\xrightarrow{E}\mathcal{V}^{odd}\xrightarrow{J}\cdots, \tag{4.82}$$

with

$$Q=\begin{pmatrix}0 & J\\ E & 0\end{pmatrix}. \tag{4.83}$$

Each $\mathcal{V}$ in the above complex corresponds to a direct sum of *Wilson*-line branes $\mathcal{W}$, where *even* and *odd* stands for the R-grading.

5
D-Brane Low Energy Behaviour and D-Brane Transport

The first task of this chapter is to consider the low energy behaviour of D-branes in the GLSM. Therefore we study so called D-term deformations and the process of brane anti-brane annihilation. Both are in the following collected in the term *D-ismorphisms* as in [16].

We will use this in the context of D-brane transport in the *Kähler* moduli space of a *Calabi Yau* described in terms of a GLSM. The following discussion consists of material taken from [16].

5.1 D-Brane Deformations and Tachyon condensation

In the following we will consider different possibilities of brane deformations and a process of combing two branes, called *cone* construction, whereby our main interest lies in the low energy behaviour of the various constructions.

D-Brane Deformations

An $\mathcal{N} = 2_B$ supersymmetric D-brane on a *Kähler* manifold X is described by the data (E, A, Q). Where E is the vector bundle corresponding to the brane, A a connection on the bundle and Q is the holomorphic part of the tachyon. $\mathcal{N} = 2$ invariant *Lagrangians* consist of two possible terms of the form

$$\int V \mathrm{d}\theta \mathrm{d}\bar{\theta} \qquad \text{D-terms} \tag{5.1}$$

$$\int W \mathrm{d}\theta \qquad \text{F-terms.} \tag{5.2}$$

V is any superfield and W is any chiral superfield. Given a D-brane the choice of a fibre metric on E fixes the D-term. The F-term is determined by the choice of a complex structure on E and of Q. The important part is now, that under the renormalization group flow from the ultra violet theory (GLSM) to the infra red theory (phase) the F-term is unchanged and every D-term flows to a unique expression in the infra-red. Therefore the low energy behaviour is determined by the F-term. As consequence a D-term deformations do not influence the low energy behaviour.

The upshot is, that D-branes which differ by a D-term have the same low energy behaviour.

Brane-Anti-Brane Annihilation

For the next aspect of infra-red behaviour we take a look at the tachyon potential U

$$U(x) = \frac{1}{2}\mathbf{T}(x)^2, \tag{5.3}$$

which appears in the boundary interaction eq. (4.70) where

$$\mathbf{T}^2 = \left\{Q, Q^\dagger\right\}. \tag{5.4}$$

$\mathbf{T} = iQ - iQ^\dagger$ can be interpreted as tachyon profile. As stated in [16], if $\det(\mathbf{T}(x))$ is nowhere vanishing, $U(x)$ is everywhere positive on the target space and has therefore no influence at the low energy behaviour. If now $\mathbf{T}$ is of block diagonal form, then a block, which is invertible everywhere, can be ignored in the infra red. In other words, in the infra-red the full potential is equivalent to a potential without the everywhere positive block. In $\mathcal{N} = 2$ theories the condition of everywhere positive $\mathbf{T}^2$ is equivalent to the statement, that the complex $\mathcal{C}(\mathcal{E}, Q)$ is *exact*[1]. Consider a complex of the form eq. (4.30). This complex is said to be exact if

$$\mathrm{Im}\left(\mathrm{d}_{n-1}\right) = \mathrm{Ker}\left(\mathrm{d}_n\right). \tag{5.5}$$

[1] This result can be readily applied to the case of a gauged linear sigma model, because the boundary conditions are also given in terms of complexes, as seen in eqs. (4.80) to (4.82).

To summarize, we quote [16]

> A D-brane corresponding to an exact complex can be ignored in the infra-red limit.

The described process originates from brane-antibrane annihilation first discussed in [31]. It was argued by *Sen* in [31], that in a system, consisting of an equal number of coincident branes and antibranes, the classical minimum of the tachyon potential has zero energy and therefore can be identified with the supersymmetric vacuum state.

Brane-antibrane annihilation may introduce additional interactions between infra-red non-trivial parts of $\mathbf{T}$. As an example consider a tachyon of the form

$$\mathbf{T} = \left(\begin{array}{c|c} \mathbf{T}_0 & * \\ \hline * & \mathbf{T}' \end{array}\right), \tag{5.6}$$

with $\mathbf{T}_0$ everywhere invertible. $\mathbf{T}_0$ is interacting with the remaining part by the off-diagonal terms denoted by $*$. If one can now remove the off-diagonal parts by row/column addition and subtraction, $\mathbf{T}_0$ can be ignored in the low energy regime. As consequence of this procedure further terms may appear in $\mathbf{T}'$.

The next task is to combine two branes and study their infra-red behaviour.

Cone Construction

In order to discuss the binding of two branes, we need to clarify the meaning of *cochain maps* and *quasi-isomorphisms.*

Definiton 5.1.1. *Let $\mathcal{C}(\mathcal{E})$ and $\mathcal{C}(\mathcal{F})$ be two complexes. A cochain map is a sequence of maps $\mathcal{E}^j \to \mathcal{F}^j$, with the property that the diagram*

$$\begin{tikzcd} \cdots \arrow[r] & \mathcal{E}^{j-1} \arrow[r] \arrow[d] & \mathcal{E}^{j} \arrow[r] \arrow[d] & \mathcal{E}^{j+1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & \mathcal{F}^{j-1} \arrow[r] & \mathcal{F}^{j} \arrow[r] & \mathcal{F}^{j+1} \arrow[r] & \cdots \end{tikzcd} \tag{5.7}$$

commutes.

Definiton 5.1.2. *A quasi-isomorphism is a cochain map, which induces an isomorphism at the level of the cohomology groups of the complexes.*

The notion of a cochain complex allows us to discuss the binding of two branes $(\mathcal{E}, Q_E)$ and $(\mathcal{F}, Q_F)$. Let φ be a cochain map $\varphi : \mathcal{C}(\mathcal{E}) \to \mathcal{C}(\mathcal{F})$. At the level of the associated bundles a cochain map φ is equivalent to a degree zero bundle map $\varphi : \mathcal{E} \to \mathcal{F}$, which obeys $Q_F\varphi - \varphi Q_E = 0$. By using the cochain map one can build a new brane called cone of φ. Following the notation of [16] we denote the cone by $C(\varphi) = C(\varphi : \mathcal{C}(\mathcal{E}) \to \mathcal{C}(\mathcal{F}))$. The new brane $C(\varphi)$ corresponds to the graded vector bundle $\mathcal{E}_{C(\varphi)} = \mathcal{E}[1] \oplus \mathcal{F}$. The holomorphic part of the tachyon can be written as

$$Q_{C(\varphi)} = \begin{pmatrix} -Q_E & 0 \\ \varphi & Q_F \end{pmatrix}. \tag{5.8}$$

The corresponding diagram is called cone complex and reads

$$\begin{tikzcd} \cdots \arrow[r, "-Q_E"] \arrow[dr, "\varphi"] & \mathcal{E}^{j} \arrow[r, "-Q_E"] \arrow[dr, "\varphi"] \arrow[d, phantom, "\oplus"] & \mathcal{E}^{j+1} \arrow[r, "-Q_E"] \arrow[dr, "\varphi"] \arrow[d, phantom, "\oplus"] & \mathcal{E}^{j+2} \arrow[r, "-Q_E"] \arrow[dr, "\varphi"] \arrow[d, phantom, "\oplus"] & \cdots . \\ \cdots \arrow[r, "Q_F"] & \mathcal{F}^{j-1} \arrow[r, "Q_F"] & \mathcal{F}^{j} \arrow[r, "Q_F"] & \mathcal{F}^{j+1} \arrow[r, "Q_F"] & \cdots \end{tikzcd} \tag{5.9}$$

Additionally in [16] a proof was given that, if φ is a quasi-ismorphism, then the corresponding cone complex is exact. Also the converse is true. From our previous discussion it follows, that the cone of quasi-isomorphic branes has no influence on the low energy behaviour.

Are D-Isomorphic Branes Quasi-Isomorphic?

The above question is equivalent to the following: If we have a quasi-isomorphism φ between the complexes $\mathcal{C}_A$ and $\mathcal{C}_B$ corresponding to the two branes A and B. Are A and B equivalent in the infra-red?

To show the infra-red equivalence the following line of arguments, valid at the low energy regime, was used in [16]

$$A \cong A + \left(\overline{B} + B\right)_{\mathrm{id}_B} \cong \left(A + \overline{B}\right)_{\varphi} + B \cong B. \tag{5.10}$$

$\left(\overline{B}+B\right)_{\mathrm{id}_B}$ is the brane-antibrane system with the tachyon id_B and $\left(A+\overline{B}\right)_{\varphi}$ is the previously described cone C between A and B. In [16] the above line of arguments was proofed, by showing that quasi-isomorphic D-branes are related by D-isomorphisms. As a consequence quasi-isomorphic branes are infra-red equivalent. By using the previously given results one sees the validity of the first and last equivalence relations. The equivalence relations in the middle requires more work. The steps necessary to prove the middle one can be found in [16].

The relation between D-isomorphic branes and quasi-isomorphic branes will be useful later on.

5.2 D-Brane Transport in the Kähler Moduli Space

As we have seen in chapter 3 the gauged linear sigma model allows to interpolate between different low energy effective theories describing a *Calabi-Yau* compactification, depending on the value of the complexified *Kähler* parameter. We called the different parameter regions *phases* of the gauged linear sigma model. An interesting question is now, how boundary interactions are influenced under transport between phases. The main problem is how to transport D-branes across phase boundaries, which involves the so called *grade restriction rule*. Also in the following we will focus on $B-$branes in the linear sigma model, expressed in terms of matrix factorisations of the gauged linear sigma model superpotential. We will not describe the map between the GLSM branes and the corresponding branes in the low energy theory. For details on that issue the reader is encouraged to consult [16].

Grade Restriction Rule

Here we will consider the gauged linear sigma model with a single $U(1)-$gauge group. The $U(1)$ charges Q_i of the matter fields fulfil the *Calabi-Yau* condition

$$\sum_i Q_i = 0. \tag{5.11}$$

The existence of the *Coulomb* branch makes the transport of a *D*-brane in the *Kähler* moduli space from one phase to another a non-trivial process. By performing a general discussion of A-type boundary conditions in *Landau-Ginzburg* theories[2], as done in [16], one sees the emergence of a boundary potential V_{bdry} on the *Coulomb* branch. For a *Lagrangian* A-brane L, which encodes the boundary conditions for the σ-fields, we require that V_{bdry} is bounded from below on a brane L. Furthermore by using $D-$term boundary deformations, positivity of V_{bdry} on L can be imposed. Now consider the gauged linear sigma model on the *Coulomb* branch, where σ is unconstrained. By integrating out the charged matter fields on finds the effective

[2] This is done because, A-type boundary conditions on chiral superfields are equivalent to B-type boundary conditions on twisted chiral superfields.

boundary potential, given a *Wilson*-line brane of charge q

$$V_{bdry}^{eff} = \frac{1}{2\pi}\underbrace{\left(r+\sum_{i=1}^{n} Q_i \log|Q_i|\right)}_{r_{eff}} \mathrm{Im}(\sigma) - \left(\frac{\theta}{2\pi}+q\right)\mathrm{Re}(\sigma) + \frac{1}{4}\sum_{i=1}^{n}|Q_i\,\mathrm{Re}(\sigma)|. \tag{5.12}$$

The potential V_{bdry}^{eff} must be positive on L. Deep in a phase ($r_{eff} \gg 0$ or $r_{eff} \ll 0$) one can always find an L such that V_{bdry}^{eff} stays positive. The situation is more subtle when $r_{eff} = 0$. For $r_{eff} = 0$ the potential takes the form

$$V_{bdry}^{eff}\big|_{r_{eff}=0} = -\left[\operatorname{sgn}(\sigma)\left(\frac{\theta}{2\pi}+q\right) - \frac{1}{4}\sum_{i=1}^{n}|Q_i|\right]|\mathrm{Re}(\sigma)|. \tag{5.13}$$

If now one wants to transport a brane from the phase $r_{eff} \gg 0$ to $r_{eff} \ll 0$ the first step is to choose a path in $\mathcal{M}_K$. This path is chosen such that one avoids the singularity lying at[3] $r_{eff} = 0$ and $\theta \in N\pi + 2\pi\mathbb{Z}$. Thereby one gets possible windows for the D-brane transport, which are situated between the singularities (see fig. 5.1).

[3] This is in accordance with eq. (3.54).

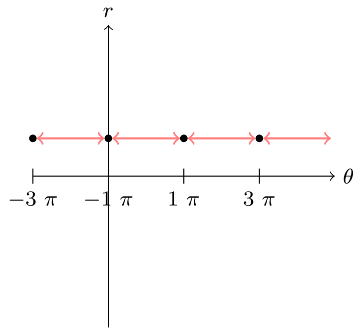

Figure 5.1: Windows in the FI-θ parameter space.

Now we fix a specific window, i.e. an interval of size 2π for the θ-angle. We can further analyse eq. (5.13). Suppose now $\operatorname{sgn}(\sigma) = -1$. Consequently for eq. (5.13) to be positive we have to fulfil

$$\left(\frac{\theta}{2\pi}+q\right) + \frac{1}{4}\sum_{i=1}^{n}|Q_i| > 0 \tag{5.14}$$

and in the case $\operatorname{sgn}(\sigma) = 1$

$$\left(\frac{\theta}{2\pi}+q\right) - \frac{1}{4}\sum_{i=1}^{n}|Q_i| < 0. \tag{5.15}$$

Combing both gives a restriction on the charges of a *Wilson*-line brane

$$-\frac{1}{4}\sum_{i=1}^{n}|Q_i| < \frac{\theta}{2\pi}+q < \frac{1}{4}\sum_{i=1}^{n}|Q_i|. \tag{5.16}$$

Thus we can conclude that only a *Wilson*-line brane fulfilling eq. (5.16) can safely cross a phase boundary. Writing a matrix factorisation as a complex of *Wilson*-line branes, the rule also holds. In this case, each charge q_i of the complex must fulfil the given inequality. Such a complex of *Wilson*-line branes is called *grade restricted*. As argued in [16] and in section 6.2, choosing $\mathrm{Im}(\sigma) = 0$ on the boundary for a grade restricted brane, keeps the V_{bdry}^{eff} positive for arbitrary values of r_{eff}.

D-Brane Transport

In the following the large gauge coupling limit $e \gg 0$ is considered and therefore the gauge multiplet can be integrated out. Given a matrix factorization $(\mathcal{V}, Q, \rho, R)$ in the linear sigma model it becomes a matrix factorization of W over a *Calabi-Yau* space X_r in this limit. As in [16] we denote the category of all possible matrix factorization in the linear sigma model by $\mathfrak{MF}_W\left(\mathbb{C}^N, T\right)$ and the category of matrix factorizations of W over X by $MF_W\left(X_r\right)$. The set $MF_W\left(X_r\right)$ consists of matrix factorizations of the linear sigma model up to $D-$isomorphisms, because two branes in $\mathfrak{MF}_W\left(\mathbb{C}^N, T\right)$ which differ by a $D-$isomorphism flow in the infra-red limit to the same brane in $MF_W\left(X_r\right)$. This is represented by the following projection

$$\pi_r : \mathfrak{MF}_W\left(\mathbb{C}^N, T\right) \to MF_W\left(X_r\right). \tag{5.17}$$

The different phases of the linear sigma model lead to a "pyramid of maps" [16]:

$$\begin{tikzcd}
 & & \mathfrak{MF}_W\left(\mathbb{C}^N, T\right) \arrow[dll, "\pi_{\mathrm{I}}"'] \arrow[ddl, "\pi_{\mathrm{III}}"'] \arrow[ddr, "\pi_{\mathrm{IV}}"] \arrow[drr, "\pi_{\mathrm{II}}"] & & \\
MF_W\left(X_{\mathrm{I}}\right) & & & & MF_W\left(X_{\mathrm{IV}}\right) \\
 & MF_W\left(X_{\mathrm{II}}\right) & & MF_W\left(X_{\mathrm{III}}\right) &
\end{tikzcd} \tag{5.18}$$

The crucial part is now, that the low energy behaviour of a brane depends on the phase in the *Kähler* moduli space. A brane which is infra-red empty[4] in the phase $r \gg 0$, does not necessarily descend to an empty brane in the $r \ll 0$ phase. This is best understood by looking at an example. Let us consider the gauged linear sigma model with a single $U(1)$ gauge group and focus on the case of the quintic[5]. The considered model has the following superpotential

[4] Empty means that in the low energy regime the brane is absent in a phase.

[5] The notation is in accordance with section 3.3.

$$W = PG\left(X\right), \tag{5.19}$$

where G is a generic polynomial of degree 5. We are interested in the vacuum configuration and look therefore at zero locus of the bosonic potential eq. (3.46). Also eq. (3.47) has to be zero. The vanishing of eq. (3.47) gives some additional requirements on the fields. For example in the case $r \gg 0$ not all x_i can be zero simultaneously and therefore the point $x_1 = x_2 = x_3 = x_4 = x_5 = 0$ is excluded. As in [16] we call these points the deleted set and denote it by $\triangle_{\pm}$. The subscript $+$ or $-$ stands for $r \gg 0$ or $r \ll 0$, respectively. Similar considerations lead to the deleted set $\triangle_{-}$ in the phase $r \ll 0$, which is given by $\triangle_{-} = \{p = 0\}$. We are now interested at the low energy behaviour of the brane $\mathcal{B}_{-}$, represented by the matrix factorization

$$Q = \begin{pmatrix} 0 & G\left(x\right) \\ p & 0 \end{pmatrix}, \rho\left(g\right) = \begin{pmatrix} g^4 & 0 \\ 0 & g^{-1} \end{pmatrix}, R\left(\lambda\right) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}. \tag{5.20}$$

This data can also be encoded into the complex

$$\mathcal{B}_{-} : \left(\mathcal{W}\left(-1\right)_{-1} \underset{p}{\overset{G}{\rightleftarrows}} \mathcal{W}\left(4\right)_0 \right). \tag{5.21}$$

If we denote the target space of the vacuum configuration in the phase $r \ll 0$ by X_-, we see, considering the positivity of the tachyon potential, that $\{Q, Q^\dagger\} = |p|^2 + |G|^2$ is nowhere vanishing on X_-, because $p \neq 0$. According to the result in 5.1, $\pi_-(\mathcal{B}_-)$ corresponds to the empty brane. A further example is the brane $\mathcal{B}_+$ given by the complex

$$\mathcal{W}(0)_0 \underset{pX^4}{\overset{X}{\rightleftarrows}} \mathcal{W}(1)_1^{\oplus 5} \underset{pX^4}{\overset{X}{\rightleftarrows}} \mathcal{W}(2)_2^{\oplus 10} \underset{pX^4}{\overset{X}{\rightleftarrows}} \mathcal{W}(3)_3^{\oplus 10} \underset{pX^4}{\overset{X}{\rightleftarrows}} \mathcal{W}(4)_4^{\oplus 5} \underset{pX^4}{\overset{X}{\rightleftarrows}} \mathcal{W}(5)_5 \,. \tag{5.22}$$

The related matrix factorization is

$$Q = \underbrace{\sum_{i=1}^{5} x_i \eta_i}_{X} + \underbrace{\sum_{i=1}^{5} p x_i^4 \overline{\eta}_i}_{pX^4} \,. \tag{5.23}$$

Here we have written down Q in terms of anti-commuting matrices[6] η, $\overline{\eta}$. In the $r \gg 0$ phase, the tachoyn potential, $\{Q, Q^\dagger\} = \sum_i \left(|x_i|^2 + |p|^2 |x^4|^2\right)$, is everywhere positive on the target space X_+. Therefore the brane $\mathcal{B}_+$ flows to the empty brane in the infra-red $\pi_+(\mathcal{B}_+) \to 0$. We want to remark, that none of the previously described branes was grade restricted with respect to any window. With the brane $\mathcal{B}_+$ and $\mathcal{B}_-$ at hand, we can build the cone $\mathcal{B}$ consisting of these two branes . Now depending on the considered phase the cone $\mathcal{B}$ is infra-red equivalent to $\mathcal{B}_-$ or $\mathcal{B}_+$. In the $r \gg 0$ phase $\mathcal{B}_+$ is infra-red empty and therefore we have

[6] Details are given in chapter 8.

$$\pi_+(\mathcal{B}) \cong \pi_+(\mathcal{B}_-) \,. \tag{5.24}$$

Consequently in the $r \ll 0$ phase the equivalence reads

$$\pi_-(\mathcal{B}) \cong \pi_-(\mathcal{B}_+) \,. \tag{5.25}$$

The process of binding empty branes will be useful in the description of monodromies around special points in the *Kähler* moduli space.

Monodromies

As mentioned in chapter 3, in the *Kähler* moduli space $\mathfrak{M}_K$ there are three distinct points. The large volume limit $r \to \infty$, the *Landau-Ginzburg* orbifold point $r \to \infty$ and the singular or conifold point located at eq. (3.54).

Monodromies around these points can be described by transporting branes around them. In 5.2 we saw that $\mathcal{M}_K$ is parametrized by the complexified *Kähler* parameter $t = r - i\theta$. The monodromies around the large volumen limit and the *Landau-Ginzburg* point are simply achieved by shifting $\theta \to \theta \pm 2\pi$. By looking at the boundary interaction of the *Wilson*-line brane $\mathcal{W}(q)$ eq. (4.64), we see that a shift of θ by $\pm 2\pi$ is equivalent to a shift in the gauge charge q by 1. The monodromy around the conifold point is more subtle.

The subtlety arises from the fact that only *Wilson*-line branes in certain charge windows can be transported in the *Kähler* moduli space in a sensible way. Given a brane outside a window, one can use empty branes to restrict the brane to the window by the cone construction. This does not change the brane in the infra-red. In chapter 10 examples of monodromies can be found, along with further details on the process.

6
Hemisphere Partition Function

In this chapter we will give an introduction to the *hemisphere partition function*. We will not go into details on the derivation and only describe aspects of the hemisphere partition function, which were crucial for this thesis. Thereby we will follow [2] and [32], where further details can be found.

6.1 General Aspects of the Hemisphere Partition Function

The hemisphere partition function Z_{D^2} is obtained by placing the gauged linear sigma model on a hemisphere and calculating the partition function. Because of the boundary at most half of the supersymmetry can be preserved. These are the familiar $\mathcal{N} = 2_A$ and $\mathcal{N} = 2_B$ symmetries. The hemisphere partition function is calculated through supersymmetric localization. This was done for a generic gauged linear sigma model in [2, 3, 4]. Here we focus on the *Calabi-Yau* cause in which the hemisphere partition function is given by:

$$Z_{D^2}\left(\mathcal{B}\right) = \mathcal{C}\int\limits_{\gamma} \mathrm{d}^{l_G}\sigma \prod_{\alpha>0} \alpha\left(\sigma\right)\sinh\left(\pi\alpha\left(\sigma\right)\right)\prod_i \Gamma\left(iQ_i\left(\sigma\right) + \frac{R_i}{2}\right) \exp\left(it\left(\sigma\right)\right)\mathrm{tr}_M\left(e^{\pi i \mathbf{r}_*}e^{2\pi\rho(\sigma)}\right) \tag{6.1}$$

where α denotes the roots of the matter representation of the gauge group G. Q_i are the charges of the chiral fields and R_i are the corresponding $R-$charges. l_G is the rank of the gauge group. tr_M is the trace on the boundary *Chan-Paton* space M and $\mathbf{r}_*$ is the representation of the $R-$symmetry on the boundary. $\rho\left(\sigma\right)$ is a representation of the gauge group on the boundary *Chan-Paton* space. For this thesis we can focus on the case of a gauged linear sigma model with $U(1)$ gauge symmetry and N chiral fields X_i of gauge charges $Q_i = w_i$ and $R-$charges $R_i = 0$. Additionally we introduce a chiral field P of gauge charge $-N$, with $N = \sum_i w_i$ and $R-$charge 2. In this specific model the hemisphere partition function simplifies to

$$Z_{D^2} = \mathcal{C}\int\limits_{\gamma} \mathrm{d}\sigma \prod_i \Gamma\left(iQ_i\sigma + \frac{R_i}{2}\right) e^{it\sigma} f_{\mathcal{B}}. \tag{6.2}$$

Where γ is the integration contour and $\mathcal{C}$ is a constant. Let us comment on the so-called brane factor $f_{\mathcal{B}}$. For a boundary condition given by a complex of *Wilson* line branes

$$\cdots \rightleftharpoons \bigoplus_{i=1}^{L_j} \mathcal{W}(q_j^{(i)})_{r_j^{(i)}}^{\oplus n_j^{(i)}} \rightleftharpoons \bigoplus_{i=1}^{L_{j+1}} \mathcal{W}(q_{j+1}^{(i)})_{r_{j+1}^{(i)}}^{\oplus n_{j+1}^{(i)}} \rightleftharpoons \cdots, \tag{6.3}$$

the brane factor reads

$$f_{\mathcal{B}} = \sum_j \sum_{i=1}^{L_j} n_j^{(i)} e^{i\pi r_j^{(i)}} e^{2\pi q_i^{(i)}\sigma}, \tag{6.4}$$

where we chose the notation as in [32].

Comparing the above complex with eq. (6.4), we see that the sum j runs over the different positions in the complex. The sum i goes over the different *Wilson* line components in the direct sum at complex position j. $q_j^{(i)}$ and $r_j^{(i)}$ are the gauge charge and R-charge of the *Wilson* line brane at position i at complex position j. The factor $n_j^{(i)}$ is the multiplicity of the considered *Wilson* line brane.

Next we have to choose an integration contour γ. Thereby we will reencounter the grade restriction rule described in section 5.2.

6.2 Convergence of Integrand and Grade Restriction Rule

In order to study possible integration contours γ, we consider the asymptotic behaviour of the integrand in eq. (6.2), because convergence considerations should constrain the possible contours. We approximate the gamma functions by the *Stirling* formula

$$\Gamma(z) \approx \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z}. \tag{6.5}$$

Because we are only interested in the asymptotic behaviour we will drop oscillatory and sub-leading terms in the following calculations whenever possible. At first we look at the asymptotic behaviour of the Γ-functions

$$\begin{aligned}
\prod_i \Gamma\left(iQ_i\sigma + \frac{R_i}{2}\right) &\approx \prod_i e^{-iQ_i\sigma} e^{-\frac{R_i}{2}} (iQ_i\sigma)^{iQ_i\sigma + \frac{R}{2} - \frac{1}{2}} \\
&\approx \underbrace{e^{-i\sigma\sum_i Q_i}}_{=0 \text{ in CY- case}} e^{i\sigma\sum_i Q_i \log(iQ_i\sigma)} = e^{i\sigma\sum_i Q_i(\log(|iQ_i\sigma|) + i\operatorname{Arg}(iQ_i\sigma))} \\
&\approx \underbrace{e^{i\sigma\sum_i Q_i\log(|i|)}}_{=0 \text{ in CY- case}} e^{i\sigma\sum_i Q_i\log(|Q_i|)} \underbrace{e^{i\sigma\sum_i Q_i\log(|\sigma|)}}_{=0 \text{ in CY- case}} e^{-\sigma\operatorname{Arg}(iQ_i\sigma)} \\
&\approx e^{i\operatorname{Re}(\sigma)\sum_i Q_i\log|Q_i|} e^{-\operatorname{Im}(\sigma)\sum_i Q_i\log|Q_i|} e^{\operatorname{Re}(\sigma)\sum_i\operatorname{Arg}(iQ_i\sigma)} e^{i\operatorname{Im}(\sigma)\operatorname{Arg}(iQ_i\sigma)} \\
&\approx e^{-\operatorname{Im}(\sigma)\sum_i Q_i\log|Q_i|} e^{-\sum_i\operatorname{Re}(\sigma)Q_i\operatorname{sgn}(\operatorname{Re}(Q_i\sigma))\left(\frac{\pi}{2} + \arctan\left(\frac{Q_i\operatorname{Im}(\sigma)}{|Q_i\operatorname{Re}(\sigma)|}\right)\right)} \\
&\approx e^{-\operatorname{Im}(\sigma)\sum_i Q_i\log|Q_i|} e^{-\sum_i|Q_i\operatorname{Re}(\sigma)|\frac{\pi}{2}} \\
&\qquad\qquad e^{-|\operatorname{Re}(\sigma)|\sum_i^5 w_i\arctan\left(\frac{\operatorname{Im}(\sigma)}{|\operatorname{Re}(\sigma)|}\right)} e^{-|\operatorname{Re}(\sigma)|N\arctan\left(-\frac{\operatorname{Im}(\sigma)}{|\operatorname{Re}(\sigma)|}\right)} \\
&\approx e^{-\operatorname{Im}(\sigma)\sum_i Q_i\log|Q_i|} e^{-|\operatorname{Re}(\sigma)|N\pi}.
\end{aligned} \tag{6.6}$$

In the second line we inserted the expression of the logarithm of a complex number. To calculate $\operatorname{Arg}(iz)$, where z is a complex number, we used the following formula given in [2]:

$$\operatorname{Arg}(iz) = \operatorname{sgn}(\operatorname{Re}(z)) \left(\frac{\pi}{2} + \arctan \frac{\operatorname{Im}(z)}{|\operatorname{Re}(z)|} \right). \tag{6.7}$$

In the sixth line we used that all w_i are positive, which results in the same arguments in the arctan functions. The last line is a consequence of the *Calabi-Yau* condition and that arctan is an odd function. The second term we consider is

$$\begin{aligned} e^{it\sigma} &= e^{(ir+\theta)(\operatorname{Re}(\sigma)+i\operatorname{Im}(\sigma))} \\ &= e^{ir\operatorname{Re}(\sigma)} e^{-r\operatorname{Im}(\sigma)} e^{\theta\operatorname{Re}(\sigma)} e^{i\operatorname{Im}(\sigma)\theta} \\ &\approx e^{-r\operatorname{Im}(\sigma)} e^{\theta\operatorname{Re}(\sigma)}. \end{aligned} \tag{6.8}$$

To study convergence it is sufficient to consider the case of a single *Wilson* line brane with charge q. Therefore we get for $f_{\mathcal{B}}$:

$$\begin{aligned} f_{\mathcal{B}} &\approx e^{2\pi q\sigma} \\ &\approx e^{2\pi q\operatorname{Re}(\sigma)} e^{2\pi iq\operatorname{Im}(\sigma)} \\ &\approx e^{2\pi q\operatorname{Re}(\sigma)}. \end{aligned} \tag{6.9}$$

Let us write the remaining exponent as

$$e^{-\mathcal{A}_q(\sigma)}, \tag{6.10}$$

with

$$\mathcal{A}_q(\sigma) = \left(r + \sum_i Q_i \log(|Q_i|) \right) \operatorname{Im}(\sigma) + \{N\pi - \operatorname{sgn}(\operatorname{Re}(\sigma))(\theta + 2\pi q)\} |\operatorname{Re}(\sigma)|. \tag{6.11}$$

Now we can conclude that the hemisphere partition function only converges if $\mathcal{A}_q(\sigma)$ is positive for all values of σ. This positivity condition restricts the possible integration contours γ. An integration path which respects the convergence condition is called admissible [2]. Furthermore we have to analyse the singularities of the integrand. The possible singular contributions are the Γ-functions. These have poles whenever their argument hits a negative integer. Therefore we conclude that the γ-functions get singular whenever

$$iQ_i\sigma + \frac{R_i}{2} = n \qquad \forall n \in \mathbb{Z}_-. \tag{6.12}$$

From the above equation we conclude that the singularities reside along the imaginary axis. Now we consider different values of r for fixed θ and q and ask if we could find a path γ such that A_q is positive.

To simplify the notation we define

$$r_{eff} = r + \sum_i Q_i \log(|Q_i|). \tag{6.13}$$

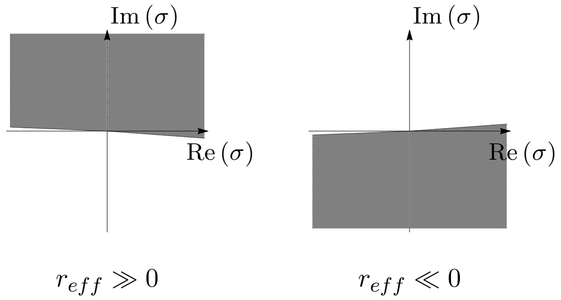

Figure 6.1: σ-plane.

For $r_{eff} \gg 0$ or $r_{eff} \ll 0$ we can always find a path in the σ-plane such that $\mathcal{A}_q(\sigma)$ is everywhere positive for arbitrary q and θ values. See fig. 6.1 where the region of positive $\mathcal{A}_q(\sigma)$ is shaded.

A subtlety arises when $r_{eff} = 0$. This happens when r hits the singularity found in eq. (3.54) if $\theta = N\pi + 2\pi n$ with $n \in \mathbb{Z}$. In the case of $r_{eff} = 0$ and $\theta + 2\pi q \geq N\pi$, σ has to avoid the complete right half plane in order for $\mathcal{A}_q(\sigma)$ to stay positive. When $\theta + 2\pi q \leq -N\pi$, σ is not allowed to take values in the entire left half plane. It follows if $-\frac{N}{2} < \frac{\theta}{2\pi} + q < \frac{N}{2}$, nearly the entire σ plane is admissible. Of course we have to avoid the imaginary axis where the poles sit. Therefore the real line $\mathbb{R}$ is a possible contour γ and all possible deformations of it which avoid the imaginary axis. The above analysis restricts the possible *Chan-Paton* charges for a brane $\mathcal{B} = (V, Q, \rho, R)$ as follows:

If $r_{eff} = 0$ the *Chan-Paton* charges q_i of $\mathcal{B}$ have to fulfil

$$-\frac{N}{2} < \frac{\theta}{2\pi} + q_i < \frac{N}{2}. \tag{6.14}$$

By fixing θ to a value inside $(N\pi + 2\pi\mathbb{Z}, N\pi + 2\pi(\mathbb{Z} + 1))$ eq. (6.14) gives allowed values of q. We call these allowed values charge window $\mathcal{W}$ and a brane $\mathcal{B}$ with charges only lying inside $\mathcal{W}$ *grade restricted* [16]. This is the same rule as the *grade restriction rule* defined in section 5.2, which we discovered by considering the transport of a brane trough the moduli space. In fig. 6.2 we shaded

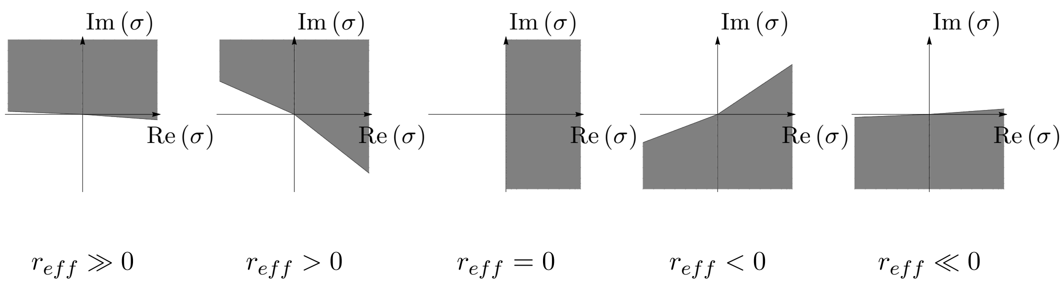

Figure 6.2: Positive $\mathcal{A}_q(\sigma)$-region for a brane with charges outside a window.

the positive region of $\mathcal{A}_q(\sigma)$ for a non-grade-restricted brane as one goes from $r_{eff} \gg 0$ to $r_{eff} \ll 0$. As one can see there is no possible way to deform the integration contour γ in a continuous way, without crossing the imaginary axis, such that $\mathcal{A}_q(\sigma)$ is always positive. The situation is different if we consider a brane with charges inside a window $\mathcal{W}$. In fig. 6.3 we see the behaviour of the positive region. We see that at any r-value there is a possible deformation of the contour γ such that $\mathcal{A}_q(\sigma)$ stays positive.

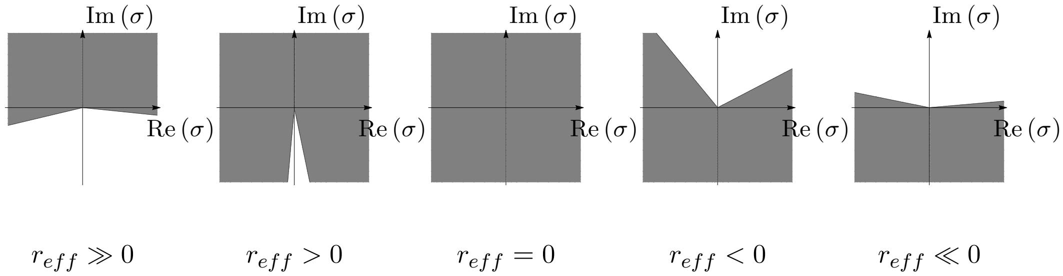

Figure 6.3: Positive $\mathcal{A}_q(\sigma)$-region for a brane with charges inside a window.

We can now conclude that only branes with charges inside a window $\mathcal{W}$ can be transported through the *Kähler* moduli space. This is exactly the grade restriction rule discussed in section 5.2.

Let as compare $\mathcal{A}_q$ with the the boundary potential (eq. (5.12)) in the *Calabi-Yau* case

$$\begin{aligned} 2\pi V_{bdry}^{eff} &= \left(r + \sum_{i=1}^{n} Q_i \log|Q_i|\right) \mathrm{Im}(\sigma) - (\theta + 2\pi q)\,\mathrm{Re}(\sigma) + \frac{\pi}{2}\sum_{i=1}^{n} |Q_i\,\mathrm{Re}(\sigma)| \\ &= \left(r + \sum_{i=1}^{n} Q_i \log|Q_i|\right) \mathrm{Im}(\sigma) + \{N\pi - \mathrm{sgn}\,(\mathrm{Re}(\sigma))\,(\theta + 2\pi q)\}\,|\mathrm{Re}(\sigma)| \end{aligned} \tag{6.15}$$

This is the same as $\mathcal{A}_q(x)$ (eq. (6.11)) and therefore we have shown that both rules have the same origin. Having found a possible contour γ we can now evaluate the hemisphere partition function.

6.3 The Hemisphere Partition Function And Picard-Fuchs Equation

As analysed in [32] for a particular basis of branes on a degree N *Calabi-Yau* hypersurface in $\mathbb{P}^{N-1}$ the hemisphere partition function solves a linear homogeneous differential equation. This differential equation is a generalized hypergeometric differential equation and the same as the *Picard-Fuchs* equation solved by the periods of the mirror hypersurface. Although [32] obtained their results for a hypersurface in $\mathbb{P}^{N-1}$, the result can be extended to the case of a hypersurface in

$\mathbb{P}[w_1, w_2, w_3, w_4, w_5]$, where we get a fourth order differential equation. Evaluation of the hemisphere partition function in a phase corresponds to calculation a contour integral for $r \gg 0$ or $r \ll 0$. The choice of the contour depends on the phase and we will comment on this issue in section 7.2. Evaluation of the hemisphere partition function near the singular point is a more difficult issue and was analysed in [32].

6.4 Interpretation Of The Hemisphere Partition Function

In [2] the authors showed that the hemisphere partition function corresponds to the fully quantum corrected central charge of the branes in the geometric phases. An interpretation in the *Landau-Ginzburg* orbifold phases is not known until now. By evaluating the hemisphere partition function for an arbitrary gauged linear sigma model brane for $r \gg 0$ we obtain the central charge of the brane in the non-linear sigma models with *Calabi-Yau* target space. This allows us to find a correspondence between branes in the high energy theory and branes in the low energy effective theories, at least at the level of charges. Having settled this last issue we now have all the necessary ingredients to compute central charges and monodromies of D-branes on one-parameter *Calabi-Yau* hypersurfaces.

Part II

Calculations

7
GLSMs of 1-parameter Calabi-Yau hypersurfaces

The main focus of this thesis are one parameter *Calabi-Yau* threefolds, realized as a hypersurface in a weighted projective space. The most prominent example is the quintic in $\mathbb{P}^4$. In the context of the gauged linear sigma model the quintic was studied in [1, 16]. We will focus on GLSMs whose large radius phases are *Calabi-Yau* hypersurfaces in $\mathbb{P}[1,1,1,1,2]$ [6],$\mathbb{P}[1,1,1,1,4]$ [8] and $\mathbb{P}[1,1,1,2,5]$ [10]. These hypersurfaces are described in terms of a gauged linear sigma model with $U(1)$ gauge group and a superpotential of the form $W = PG(X)$, where $G(X)$ is a quasi-homogeneous polynomial of degree 6, 8 or 10.[1] Below we state $G(X)$ and the field content of the corresponding gauged linear sigma models. Afterwards we will focus on the evaluation of the hemisphere partition function in the large radius phase.

[1] Details can be found in chapter 3.

7.1 Field Content

In order to perform calculations we have to set up the gauged linear sigma model corresponding to the *Calabi-Yau*s of interest. In the following we simply give the field content and G of the gauged linear sigma models, with superpotential $W = PG$. In general G is a generic quasi-homogeneous polynomial of degree 6, 8 or 10, but for this thesis it is enough to just consider the *Fermat*-polynomials. This is sufficient because Z_{D^2} is insensitive to deformations of the complex structure. We will denote the $U(1)$ charges of the fields by Q_i and the R-charges by R_i. The R-charges of the fields have to be in the range[2]

[2] See [2] for a recent discussion.

$$0 \leq R_i \leq 2. \tag{7.1}$$

We will write down the R-charges with a parameter κ, which can be chosen such that eq. (7.1) is fulfilled.

$\mathbb{P}[1,1,1,1,2]$ [6]

The *Fermat*-polynomial reads

$$G_6 = X_1^6 + X_2^6 + X_3^6 + X_4^6 + X_5^3. \tag{7.2}$$

This gives the following gauge and R charges

$$\begin{array}{c|c|c|c} & P & X_{1\ldots4} & X_5 \\ \hline Q_i & -6 & 1 & 2 \\ R_i & 2-6\kappa & \kappa & 2\kappa \end{array} . \tag{7.3}$$

$\mathbb{P}\left[1,1,1,1,4\right]\left[8\right]$

The *Fermat*-polynomial is

$$G_8 = X_1^8 + X_2^8 + X_3^8 + X_4^8 + X_5^2. \tag{7.4}$$

As before we can deduce the gauge and R charges

$$\begin{array}{c|c|c|c} & P & X_{1\ldots4} & X_5 \\ \hline Q_i & -8 & 1 & 4 \\ R_i & 2-8\kappa & \kappa & 4\kappa \end{array} . \tag{7.5}$$

$\mathbb{P}\left[1,1,1,2,5\right]\left[10\right]$

The quasi-homogeneous degree and the weights restrict the possible form of the *Fermat*-polynomial to

$$G_{10} = X_1^{10} + X_2^{10} + X_3^{10} + X_4^5 + X_5^2. \tag{7.6}$$

The required gauge invariance and R charge of W lead to the charges

$$\begin{array}{c|c|c|c|c} & P & X_{1\ldots3} & X_4 & X_5 \\ \hline Q_i & -10 & 1 & 2 & 5 \\ R_i & 2-10\kappa & \kappa & 2\kappa & 5\kappa \end{array} . \tag{7.7}$$

For further calculations we choose $\kappa = 0$ in the $R-$charge of the considered models. Also we note that the *Calabi-Yau* condition, $\sum_i Q_i = 0$, is fulfilled in all models[3]. With the bulk information at hand we can further focus on evaluating the hemisphere partition function.

[3] In section 3.3 details on the *Calabi-Yau* conditions are given.

7.2 Hemisphere Partition Function

As described in chapter 6 the hemisphere partition function for a gauged linear sigma model with gauge group $U(1)$ reads

$$Z_{D^2}\left(\mathcal{B}\right) = \mathcal{C}\int_\gamma \mathrm{d}\sigma \prod_i \Gamma\left(iQ_i\sigma + \frac{R_i}{2}\right) e^{it\sigma} f_{\mathcal{B}}\left(\sigma\right). \tag{7.8}$$

Because we are interested in the evaluation of $Z_{D^2}\left(\mathcal{B}\right)$ in a phase, $|r| \gg 0$, the real line $\mathbb{R}$ is always a admissible contour as argued in chapter 6 and [2]. To simplify the calculation we will use the following Γ-function identity

$$\Gamma\left(z\right)\Gamma\left(1-z\right) = \frac{\pi}{\sin \pi z}, \tag{7.9}$$

which is known as the reflection formula. By looking at the above section we see, that all of our models consist of 5 fields of gauge charge Q_i and R-charge 0 and 1 field of gauge charge $-N = -\sum_i Q_i$ and R-charge 2. Writing $Z_{D^2}(\mathcal{B})$ for these fields gives

$$Z_{D^2}(\mathcal{B}) = \mathcal{C} \int_{-\infty}^{\infty} \mathrm{d}\sigma \prod_{i=1}^{5} \Gamma(iQ_i\sigma)\, \Gamma(iN\sigma + 1)\, e^{it\sigma} f_{\mathcal{B}}(\sigma)\,. \tag{7.10}$$

Now we perform the following transformation $s = i\sigma$ and get

$$Z_{D^2}(\mathcal{B}) = -i\mathcal{C} \int_{-i\infty}^{i\infty} \mathrm{d}s \prod_{i=1}^{5} \Gamma(Q_i s)\, \Gamma(1 - Ns)\, e^{ts} f_{\mathcal{B}}(-is)\,. \tag{7.11}$$

We want to evaluate the above integral with the help of the residue theorem, and therefore we have to close the contour. The contour can be closed in two possible ways. The possible contours and poles are given in fig. 7.1.

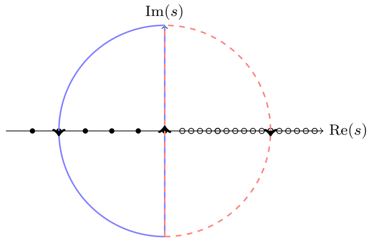

Figure 7.1: Possible integration contours.

Which contour one chooses depends on the convergence behaviour of the integrand. This behaviour is determined by the e^{ts} factor. Therefore we take a closer look at the asymptotic behaviour of this factor and decompose s as $s = \mathrm{Re}(s) + i\,\mathrm{Im}(s)$ and get

$$\begin{aligned} e^{ts} &= e^{(r-i\theta)(\mathrm{Re}(s)+i\,\mathrm{Im}(s))} \\ &\approx e^{r\,\mathrm{Re}(s)}. \end{aligned} \tag{7.12}$$

In the last line we kept only the leading order terms. We see that in the case $r \gg 0$ we have to close the contour to the left and in the case $r \ll 0$ the contour is closed to the right. Consequently for $r \gg 0$ we encounter the singularities of $\prod_i^5 \Gamma(Q_i s)$ and for $r \ll 0$ only $\Gamma(1 - Ns)$ gives poles. Subsequently we fill focus on the geometric phase.

Geometric Phase $r \gg 0$

For $r \gg 0$ we have to close the contour to the left and so only the poles of $\prod_i^5 \Gamma(Q_i s)$ contribute. In the following we write[4]

$$z = e^{-t}. \tag{7.13}$$

[4] Details of this choice are given in section 7.3.

To further evaluate the integral we deform the contour and write the integral as a sum over integrals around the various poles. For this purpose we transform $s \to s_P + \epsilon$, where s_P is the position of a pole, and get

$$Z_{D^2}^{r\gg 0}(\mathcal{B}) = \lim_{\epsilon\to 0} \sum_{s_P} -i\mathcal{C} \oint \frac{\mathrm{d}\epsilon}{2\pi i} \prod_i^5 \Gamma(Q_i(s_P+\epsilon))\, \Gamma(1 - N(s_P+\epsilon))\, z^{-(s_P+\epsilon)} e^{iN\pi(s_P+\epsilon)} f_{\mathcal{B}}(-i(s_P+\epsilon))\,. \tag{7.14}$$

The factor $e^{iN\pi(s_P+\epsilon)}$ comes from the fact that the θ angle gets shifted in the geometric phase [16].

To simplify the calculation of the residues we apply the reflection formula (eq. (7.9))

$$Z_{D^2}^{r\gg 0}(\mathcal{B}) = \lim_{\epsilon\to 0}\sum_{s_P} -i\mathcal{C}\oint \frac{\mathrm{d}\epsilon}{2\pi i}\pi^5 \prod_i^5 \frac{1}{\sin\left(\pi Q_i\left(s_P+\epsilon\right)\right)}\frac{1}{\Gamma\left(1-Q_i\left(s_P+\epsilon\right)\right)} \\ \Gamma\left(1-N\left(s_P+\epsilon\right)\right) z^{-(s_P+\epsilon)} e^{iN\pi(s_P+\epsilon)} f_{\mathcal{B}}\left(-i\left(s_P+\epsilon\right)\right). \tag{7.15}$$

The singularities are at

$$Q_i\left(s_P+\epsilon\right) = -n \qquad \forall n\in\mathbb{N}. \tag{7.16}$$

Let us rewrite the sum in eq. (7.15) into a sum over the poles at $s_P \in \mathbb{Z}_-$ and the poles at $s_P \in \mathbb{Q}_-$, with $s_P \notin \mathbb{Z}_-$. To do so we transform for the first contributions $s_P \to -n$ and for the second $s_P \to -\frac{k}{Q_l}$ and get

$$Z_{D^2}^{r\gg 0}(\mathcal{B}) = -i\mathcal{C}\lim_{\epsilon\to 0}\Bigg[\sum_{n\in\mathbb{N}}\oint \frac{\mathrm{d}\epsilon}{2\pi i}\pi^5\prod_i^5\frac{1}{\sin\left(\pi Q_i\left(-n+\epsilon\right)\right)}\frac{1}{\Gamma\left(1+Q_i n-Q_i\epsilon\right)} \\ \Gamma\left(1+Nn-N\epsilon\right) z^{-(-n+\epsilon)}\left(-1\right)^{Nn} e^{iN\pi\epsilon} f_{\mathcal{B}}\left(-i\left(-n+\epsilon\right)\right) \\ +\sum_{Q_l}\sum_{\frac{k}{Q_l}\notin\mathbb{N}}\oint\frac{\mathrm{d}\epsilon}{2\pi i}\pi^5\prod_i^5\frac{1}{\sin\left(\pi Q_i\left(-\frac{k}{Q_l}+\epsilon\right)\right)}\frac{1}{\Gamma\left(1+Q_i\frac{k}{Q_l}-Q_i\epsilon\right)} \\ \Gamma\left(1+N\frac{k}{Q_l}-N\epsilon\right) z^{-\left(-\frac{k}{Q_l}+\epsilon\right)}\left(-1\right)^{N\frac{k}{Q_l}} e^{iN\pi\epsilon} f_{\mathcal{B}}\left(-i\left(-\frac{k}{Q_l}+\epsilon\right)\right)\Bigg]. \tag{7.17}$$

Of course these last steps may not be necessary to compute $Z_{D^2}^{r\gg 0}$, but we found it useful in order to evaluate $Z_{D^2}^{r\gg 0}(\mathcal{B})$ with the help of a computer algebra system. In general the solution of $Z_{D^2}^{r\gg 0}(\mathcal{B})$ is an infinite sum, but as explained in the following section we can express the result of $Z_{D^2}^{r\gg 0}(\mathcal{B})$ in terms of the periods of the mirror *Calabi-Yau*

7.3 Hemisphere Partition Function and Periods

As noted in section 6.3 the heimsphere partition function solves the *Picard-Fuchs* equation associated to the *Calabi-Yau* spaces of our models.

The *Picard-Fuchs* operators of one-parameter *Calabi-Yau*s have the general structure

$$\mathcal{L} = \theta^4 - \alpha_0 z\prod_{i=1}^4\left(\alpha_i\theta+\beta_i\right), \tag{7.18}$$

with $\theta = z\frac{\mathrm{d}}{\mathrm{d}z}$. The coefficients for our models can be found in table 7.1. These operators were obtained from [33].

The periods ϖ_i on the complex structure moduli space of the mirror *Calabi-Yau* solve the respective *Picard-Fuchs* equation[5]

$$\mathcal{L}\varpi_i = 0. \tag{7.19}$$

		$\mathbb{WP}[6]$		$\mathbb{WP}[8]$		$\mathbb{WP}[10]$	
α_0		36		16		80	
α_1	β_1	6	1	8	1	10	1
α_2	β_2	3	1	8	3	10	3
α_3	β_3	3	2	8	5	10	7
α_3	β_3	6	5	8	7	10	9

Table 7.1: Coefficients of the *Picard-Fuchs* operator.

[5] Further background information is stated in section 4.1.

Before writing down an ansatz for the periods, we note that z and the coordinate $t = r - i\theta$ are related by

$$z = e^{-t}. \tag{7.20}$$

Since we are interested in expressing the solution of $Z_{D^2}^{r\gg 0}$ in terms of the periods, we have to find a solution to the *Picard-Fuchs* equation near $z = 0$. For our models possible periods for $z \approx 0$ are given by

$$\varpi_k\left(z\right) = \frac{1}{\left(2\pi i\right)^k} \sum_{m=0}^{\infty} \frac{\partial^k}{\partial \epsilon^k} \left[\frac{\Gamma\left(N(m+\epsilon)+1\right)}{\Gamma\left(N\epsilon+1\right)} \prod_{i=1}^{5} \frac{\Gamma\left(Q_i\epsilon+1\right)}{\Gamma\left(Q_i(m+\epsilon)+1\right)} z^{m+\epsilon} \right]_{\epsilon=0} \tag{7.21}$$

Having obtained the periods we can now expand the solution of $Z_{D^2}^{r\gg 0}\left(\mathcal{B}\right)$ in terms of the periods

$$Z_{D^2}^{r\gg 0}\left(\mathcal{B}\right) = \alpha\varpi_0 + \beta\varpi_1 + \gamma\varpi_2 + \delta\varpi_3. \tag{7.22}$$

With the above result we can now write down results for $Z_{D^2}^{r\gg 0}\left(\mathcal{B}\right)$ for various branes. This will be done in chapter 9.

8 Matrix Factorisations

By placing the GLSM on a world sheet with boundaries we have to add a boundary term to preserve some of the supersymmetry. We specify the modified GLSM Lagrangian by defining a boundary datum $\mathcal{B}$ [1]. For the subsequent discussion a matrix factorisation Q is the quantity of interest. One can view Q as an odd graded map on the *Chan-Paton* space $\mathcal{V}$ which satisfies

[1] Further information can be found in section 4.3.

$$Q^2 = W\,\mathrm{id}_{\mathcal{V}}\,. \tag{8.1}$$

The *Chan-Paton* space $\mathcal{V} = \mathcal{V}^{even} \oplus \mathcal{V}^{odd}$ decomposes into an even and odd part. Additionally Q has to be invariant under the action of the gauge group and must carry $R-$charge one. As discussed in section 4.3 the conditions on Q read

$$\rho(g)^{-1} Q\,(g\phi)\,\rho(g) = Q\,(\phi) \tag{8.2}$$

$$\lambda^{\mathbf{r}_*} Q\left(\lambda^R \phi\right) \lambda^{-\mathbf{r}_*} = \lambda Q\,(\phi)\,. \tag{8.3}$$

Obtaining an appropriate matrix factorisation is a tedious task, which can be simplified by working with complexes of *Wilson* line branes[2]. Let $\mathbb{C}_{x_i} = \mathbb{C}\left[x_1, \ldots, x_5\right]$ be the polynomial ring in the variables x_i. The *Chan-Paton* space $\mathcal{V}$ can be built from a vacuum state $|0\rangle \in \mathbb{C}_{x_i}$ by acting with anti-commuting creation operators $\bar{\eta}_\alpha$ and annihilation operators η_β

[2] See [16] and section 4.3 for background information on *Wilson* line branes.

$$\mathcal{V} = \bigoplus_{k=0}^{j} c_{\beta_1 \ldots \beta_k} \bar{\eta}_{\beta_1} \ldots \bar{\eta}_{\beta_k} \,|0\rangle \tag{8.4}$$

with $c_{\beta_1 \ldots \beta_k} \in \mathbb{C}_{x_i}$ and

$$\eta_\beta\,|0\rangle = 0 \qquad\qquad \forall\beta. \tag{8.5}$$

The creation and annihilation operator satisfy the following algebra

$$\left\{\eta_\alpha, \bar{\eta}_\beta\right\} = \delta_{\alpha\beta} \qquad\qquad \left\{\eta_\alpha, \eta_\beta\right\} = \left\{\bar{\eta}_\alpha, \bar{\eta}_\beta\right\} = 0. \tag{8.6}$$

The superpotential can be written as

$$W = \sum_{\alpha=1}^{j} a_\alpha \cdot b_\alpha, \tag{8.7}$$

where a_α, b_α are homogeneous polynomials $\in \mathbb{C}_{x_i}$. A matrix factorisation has then the form

$$Q = \sum_{\alpha=1}^{j} a_\alpha \eta_\alpha + b_\alpha \bar{\eta}_\alpha. \tag{8.8}$$

In the following we will give a short guide on how to use eqs. (8.2) and (8.3) to obtain the gauge charges and R-charges of the creation and annihilation operators and of an arbitrary element in $\mathcal{V}$.

8.1 *Charges of the Clifford basis*

To use eq. (8.2) we write down $Q(g\phi)$ and$Q(\lambda^R\phi)$ using eq. (8.8)

$$Q(g\phi) = \sum_{\alpha=1}^{j} g^{q_{a_\alpha}} a_\alpha \eta_\alpha + g^{q_{b_\alpha}} b_\alpha \bar{\eta}_\alpha \tag{8.9}$$

$$Q(\lambda^R\phi) = \sum_{\alpha=1}^{j} \lambda^{r_{a_\alpha}} a_\alpha \eta_\alpha + \lambda^{r_{b_\alpha}} b_\alpha \bar{\eta}_\alpha, \tag{8.10}$$

which allows us to recast eqs. (8.2) and (8.3) into conditions for the creation and annihilation operators

$$\begin{aligned} \rho(g)^{-1} g^{q_{a_\alpha}} \eta_\alpha \rho(g) &\stackrel{!}{=} \eta_\alpha \\ \rho(g)^{-1} g^{q_{b_\alpha}} \bar{\eta}_\alpha \rho(g) &\stackrel{!}{=} \bar{\eta}_\alpha \end{aligned}, \tag{8.11}$$

and

$$\begin{aligned} \lambda^{\mathbf{r}_*} \lambda^{r_{a_\alpha}} \eta_\alpha \lambda^{-\mathbf{r}_*} &\stackrel{!}{=} \lambda \eta_\alpha \\ \lambda^{\mathbf{r}_*} \lambda^{r_{b_\alpha}} \bar{\eta}_\alpha \lambda^{-\mathbf{r}_*} &\stackrel{!}{=} \lambda \bar{\eta}_\alpha. \end{aligned} \tag{8.12}$$

From these conditions we can easily read off the respective charges of η_α and $\bar{\eta}_\alpha$ and get,

$$(q_{\eta_\alpha}, r_{\eta_\alpha}) = (-q_{a_\alpha}, -r_{a_\alpha} + 1) \qquad (q_{\bar{\eta}_\alpha}, r_{\bar{\eta}_\alpha}) = (-q_{b_\alpha}, -r_{b_\alpha} + 1)\,. \tag{8.13}$$

This can also be seen directly from eq. (8.8) and eq. (8.2). We also have to deduce how an operator $a_\alpha \eta_\alpha$ alters the charges of a state $|S\rangle$ with charges $\left(q_{|S\rangle}, r_{|S\rangle}\right) = (q, r)$:

$$\rho\left(a_\alpha \eta_\alpha |S\rangle\right) = \underbrace{\rho(g) a_\alpha \eta_\alpha \rho(g)^{-1}}_{g^{q_{a_\alpha}} a_\alpha \eta_\alpha} \underbrace{\rho(g)|S\rangle}_{g^q |S\rangle} = g^{q+q_{a_\alpha}} \left(a_\alpha \eta_\alpha |S\rangle\right) \tag{8.14}$$

where eq. (8.2) was used. The calculation for $b_\alpha \bar{\eta}_\alpha$ is similar. The change of the R-charge is obtained through

$$\lambda^{\mathbf{r}_*}\left(a_\alpha \eta_\alpha |S\rangle\right) = \underbrace{\lambda^{\mathbf{r}_*} a_\alpha \eta_\alpha \lambda^{-\mathbf{r}_*}}_{\lambda^{1-r_{a_\alpha}} a_\alpha \eta_\alpha} \underbrace{\lambda^{\mathbf{r}_*}|S\rangle}_{\lambda^r |S\rangle} = \lambda^{1+r-r_{a_\alpha}} \left(a_\alpha \eta_\alpha |S\rangle\right). \tag{8.15}$$

We used the following statement[3]

[3] See eq. (8.3).

$$\begin{aligned}\lambda^{\mathbf{r}_*}\lambda^{r_{a_\alpha}}a_\alpha\eta_\alpha\lambda^{-\mathbf{r}_*} &= \lambda a_\alpha\eta_\alpha \quad \Rightarrow \\ \lambda^{\mathbf{r}_*}a_\alpha\eta_\alpha\lambda^{-\mathbf{r}_*} &= \lambda^{1-r_{a_\alpha}}a_\alpha\eta_\alpha.\end{aligned} \tag{8.16}$$

Again the result for $b_\alpha\bar{\eta}_\alpha$ is given by an analogous calculation. Now we are going to deduce the gauge and R charges of the state $\bar{\eta}_{\alpha_1}\bar{\eta}_{\alpha_2}\dots\bar{\eta}_{\alpha_n}|0\rangle$, where $|0\rangle$ is the vacuum with charges $\left(q_{|0\rangle}, r_{|0\rangle}\right) = (q,r)$.

$$\begin{aligned}\rho\left(\bar{\eta}_{\alpha_1}\bar{\eta}_{\alpha_2}\dots\bar{\eta}_{\alpha_n}|0\rangle\right) &= \rho(g)\bar{\eta}_{\alpha_1}\rho(g)^{-1}\rho(g)\bar{\eta}_{\alpha_2}\rho(g)^{-1}\rho(g)\dots\bar{\eta}_{\alpha_n}\rho(g)^{-1}\rho(g)|0\rangle \\ &= g^{-q_{\bar{\eta}_{\alpha_1}}-q_{\bar{\eta}_{\alpha_2}}\cdots-q_{\bar{\eta}_{\alpha_n}}+q}\left(\bar{\eta}_{\alpha_1}\bar{\eta}_{\alpha_2}\dots\bar{\eta}_{\alpha_n}|0\rangle\right),\end{aligned} \tag{8.17}$$

where eq. (8.11) and eq. (8.13) were used. The $R-$ charge is calculated by using eq. (8.12):

$$\begin{aligned}\lambda^{\mathbf{r}_*}\left(\bar{\eta}_{\alpha_1}\bar{\eta}_{\alpha_2}\dots\bar{\eta}_{\alpha_n}|0\rangle\right) &= \lambda^{\mathbf{r}_*}\bar{\eta}_{\alpha_1}\lambda^{-\mathbf{r}_*}\lambda^{\mathbf{r}_*}\bar{\eta}_{\alpha_2}\lambda^{-\mathbf{r}_*}\lambda^{\mathbf{r}_*}\dots\bar{\eta}_{\alpha_n}\lambda^{-\mathbf{r}_*}\lambda^{\mathbf{r}_*}|0\rangle \\ &= \lambda^{(1-r_{b_{\alpha_1}})+(1-r_{b_{\alpha_2}})\cdots+(1-r_{b_{\alpha_n}})+r}\left(\bar{\eta}_{\alpha_1}\bar{\eta}_{\alpha_2}\dots\bar{\eta}_{\alpha_n}|0\rangle\right) \\ &= \lambda^{r_{\bar{\eta}_{\alpha_1}}+r_{\bar{\eta}_{\alpha_2}}\cdots+r_{\bar{\eta}_{\alpha_n}}+r}\left(\bar{\eta}_{\alpha_1}\bar{\eta}_{\alpha_2}\dots\bar{\eta}_{\alpha_n}|0\rangle\right).\end{aligned} \tag{8.18}$$

In the last line eq. (8.13) was employed.

8.2 Concrete Examples

Let us consider a special matrix factorisation,

$$Q = \sum_i^N\left(x_i\eta_i + \frac{1}{w_i}p\frac{\partial W}{\partial x_i}\bar{\eta}_i\right). \tag{8.19}$$

This factorisation will be later referred to as large-radius-empty-brane [4].

[4] This name is a consequence of the fact, that eq. (8.19) has zero central charge when we evaluate its hemisphere-partition-function in the large radius phase.

$\mathbb{P}[1,1,1,1,2][6]$

By using eq. (8.19) we obtain the following factorisation:

$$Q = x_1\eta_1 + x_1^5p\bar{\eta}_1 + x_2\eta_2 + x_2^5p\bar{\eta}_2 + x_3\eta_3 + x_3^5p\bar{\eta}_3 + x_4\eta_4 + x_4^5p\bar{\eta}_4 + x_5\eta_5 + px_5^2\bar{\eta}_5 \tag{8.20}$$

From eq. (8.11) and eq. (8.12) we can read off the respective charges and obtain, by denoting the gauge charge by q and $R-$charge by r

$$\begin{aligned}(q_{\eta_\alpha}, r_{\eta_\alpha}) &= (-1, -\kappa+1) \quad \alpha = 1,\dots,4 \qquad & (q_{\bar{\eta}_\alpha}, r_{\bar{\eta}_\alpha}) &= (1, \kappa-1) \quad \alpha = 1,\dots,4 \\ (q_{\eta_5}, r_{\eta_5}) &= (-2, -2\kappa+1) & (q_{\bar{\eta}_5}, r_{\bar{\eta}_5}) &= (2, 2\kappa-1)\end{aligned} \tag{8.21}$$

With the charges at hand we can write down the complex of Wilson line branes. We set κ zero and choose for the vacuum

$$\left(q_{|0\rangle}, r_{|0\rangle}\right) = (6,5). \tag{8.22}$$

We get

$$\mathcal{W}(0)_0^{\oplus 1} \rightleftharpoons \begin{matrix} \mathcal{W}(2)_1^{\oplus 1} \\ \bigoplus \\ \mathcal{W}(1)_1^{\oplus 4} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(3)_2^{\oplus 4} \\ \bigoplus \\ \mathcal{W}(2)_2^{\oplus 6} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(4)_3^{\oplus 6} \\ \bigoplus \\ \mathcal{W}(3)_3^{\oplus 4} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(5)_4^{\oplus 4} \\ \bigoplus \\ \mathcal{W}(4)_4^{\oplus 1} \end{matrix} \rightleftharpoons \mathcal{W}(6)_5^{\oplus 1} \tag{8.23}$$

From the complex of Wilson line branes we can read off the brane factor $f_{\mathcal{B}}$

$$f_{\mathcal{B}} = 1 - 4e^{2\pi\sigma} + 5e^{4\pi\sigma} - 5e^{8\pi\sigma} + 4e^{10\pi\sigma} - e^{12\pi\sigma}. \tag{8.24}$$

$\mathbb{P}\,[1,1,1,1,4]\,[8]$

For this model we get the following charges for the *Clifford* basis

$$\begin{matrix} (q_{\eta_\alpha}, r_{\eta_\alpha}) = (-1, -\kappa+1) & \alpha = 1,\dots,4 & (q_{\bar\eta_\alpha}, r_{\bar\eta_\alpha}) = (1, \kappa-1) & \alpha = 1,\dots,4 \\ (q_{\eta_5}, r_{\eta_5}) = (-4, -4\kappa+1) & & (q_{\bar\eta_5}, r_{\bar\eta_5}) = (4, 4\kappa-1) & \end{matrix} \tag{8.25}$$

To construct the complex of Wilson line branes of eq. (8.20) we set $\kappa = 0$ and $\left(q_{|0\rangle}, r_{|0\rangle}\right) = (8,5)$

$$\mathcal{W}(0)_0^{\oplus 1} \rightleftharpoons \begin{matrix} \mathcal{W}(1)_1^{\oplus 4} \\ \bigoplus \\ \mathcal{W}(4)_1^{\oplus 1} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(2)_2^{\oplus 6} \\ \bigoplus \\ \mathcal{W}(5)_2^{\oplus 4} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(3)_3^{\oplus 4} \\ \bigoplus \\ \mathcal{W}(6)_3^{\oplus 6} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(4)_4^{\oplus 1} \\ \bigoplus \\ \mathcal{W}(7)_4^{\oplus 4} \end{matrix} \rightleftharpoons \mathcal{W}(8)_5^{\oplus 1} \tag{8.26}$$

The brane factor is given by

$$f_{\mathcal{B}} = 1 - 4e^{2\pi\sigma} + 6e^{4\pi\sigma} + 4e^{10\pi\sigma} - 4e^{6\pi\sigma} - 6e^{12\pi\sigma} + 4e^{14\pi\sigma} - e^{16\pi\sigma}. \tag{8.27}$$

$\mathbb{P}\,[1,1,1,2,5]\,[10]$

The relevant charges are given by

$$\begin{matrix} (q_{\eta_\alpha}, r_{\eta_\alpha}) = (-1, -\kappa+1) & \alpha = 1,\dots,3 & (q_{\bar\eta_\alpha}, r_{\bar\eta_\alpha}) = (1, \kappa-1) & \alpha = 1,\dots,3 \\ (q_{\eta_4}, r_{\eta_4}) = (-2, -2\kappa+1) & & (q_{\bar\eta_4}, r_{\bar\eta_4}) = (2, 2\kappa-1) & \\ (q_{\eta_5}, r_{\eta_5}) = (-5, -5\kappa+1) & & (q_{\bar\eta_5}, r_{\bar\eta_5}) = (5, 5\kappa-1) & \end{matrix}. \tag{8.28}$$

The charges of the vacuum are set to $\left(q_{|0\rangle}, r_{|0\rangle}\right) = (10,5)$ and $\kappa = 0$.
We construct the following complex associated to eq. (8.20):

$$\mathcal{W}(0)_0^{\oplus 1} \rightleftharpoons \begin{matrix} \mathcal{W}(1)_1^{\oplus 3} \\ \bigoplus \\ \mathcal{W}(2)_1^{\oplus 1} \\ \bigoplus \\ \mathcal{W}(5)_1^{\oplus 1} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(2)_2^{\oplus 3} \\ \bigoplus \\ \mathcal{W}(3)_2^{\oplus 3} \\ \bigoplus \\ \mathcal{W}(6)_2^{\oplus 3} \\ \bigoplus \\ \mathcal{W}(7)_2^{\oplus 1} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(3)_3^{\oplus 1} \\ \bigoplus \\ \mathcal{W}(4)_3^{\oplus 3} \\ \bigoplus \\ \mathcal{W}(7)_3^{\oplus 3} \\ \bigoplus \\ \mathcal{W}(8)_3^{\oplus 3} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(5)_4^{\oplus 1} \\ \bigoplus \\ \mathcal{W}(8)_4^{\oplus 1} \\ \bigoplus \\ \mathcal{W}(9)_4^{\oplus 3} \end{matrix} \rightleftharpoons \mathcal{W}(10)_5^{\oplus 1} \tag{8.29}$$

The brane factor is

$$f_{\mathcal{B}} = 1 - 3e^{2\pi\sigma} + 2e^{4\pi\sigma} + 2e^{6\pi\sigma} + 3e^{12\pi\sigma} - 2e^{14\pi\sigma} - 3e^{8\pi\sigma} - 2e^{16\pi\sigma} + 3e^{18\pi\sigma} - e^{20\pi\sigma}. \tag{8.30}$$

9
Catalogue of large radius branes

This section is an overview of possible matrix factorisations in one-parameter *Calabi-Yau* spaces. Our aim is to identify the GLSM matrix factorisations corresponding to a basis of $(D0, D2, D4, D6)$ branes in the large radius phase. We will simply state the factorisation Q, the corresponding complex of *Wilson* line branes and the central charge obtained by evaluation the hemisphere partition function in the large radius phase. We write the central charges in the form

$$Z_{D^2}^{\zeta \gg 0}(\mathcal{B}) = \alpha\varpi_0 + \beta\varpi_1 + \gamma\varpi_2 + \delta\varpi_3, \tag{9.1}$$

where the ϖ_i are a basis of solutions of the *Picard-Fuchs* equation of the mirror *Calabi Yau* [1]. For later use we state the topological numbers of the consider spaces in table 9.1 [33].

[1] For details the reader is referred to section 4.1 and chapter 7.

	H^3	c_2H	c_3
$\mathbb{P}[6]$	3	42	-204
$\mathbb{P}[8]$	2	44	-296
$\mathbb{P}[10]$	1	34	-288

Table 9.1: Topological numbers of the consider spaces

9.1 $\mathbb{P}[1,1,1,1,2][6]$

In this case we choose the vacuum to have charges

$$\left(q_{|0\rangle}, R_{|0\rangle}\right) = (6, 5). \tag{9.2}$$

In the following we used various constants in the stated matrix factorisations. These constants are solutions of the equations

$$\begin{aligned} a^6 &= -1 & b^3 &= -1 \\ c^3 &= 1 & d^2 &= -1 \\ f^6 &= -1 & l^3 &= 1 \\ h^3 &= -1 & o^2 &= -1. \end{aligned} \tag{9.3}$$

D0-branes

We will simply denote different D0-branes by subscript numbers. To describe D0-branes we consider matrix factorisations of the *Fermat* polynomial that have the following structure:

$$Q = f_1\eta_1 + x_\alpha\eta_2 + x_\beta\eta_3 + x_\gamma\eta_4 + pg_1\bar{\eta}_1 + px_\alpha^{q_\alpha}\bar{\eta}_2 + px_\beta^{q_\beta}\bar{\eta}_3 + px_\gamma^{q_\gamma}\bar{\eta}_4 \tag{9.4}$$

with different f_1 and g_1. Also x_α, x_β, x_γ and q_α, q_β, q_γ have to be appropriately chosen such that

$$Q^2 = W \cdot \mathbb{1} \tag{9.5}$$

is fulfilled.

$D0_1$ or $D0_4$

$$\begin{aligned} f_1^{D0_1} &= x_1 + a x_2 && (9.6)\\ g_1^{D0_1} &= x_1^5 - a x_1^4 x_2 + a^2 x_1^3 x_2^2 - a^3 x_1^2 x_2^3 + a^4 x_1 x_2^4 - a^5 x_2^5 && (9.7)\\ f_1^{D0_4} &= x_1^2 - b x_5 && (9.8)\\ g_1^{D0_4} &= x_1^4 + b x_1^2 x_5 + b^2 x_5^2. && (9.9) \end{aligned}$$

In both cases the complex reads

$$\mathcal{W}(1)_1 \rightleftharpoons \begin{matrix} \mathcal{W}(2)_2^{\oplus 3} \\ \bigoplus \\ \mathcal{W}(3)_2 \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(3)_3^{\oplus 3} \\ \bigoplus \\ \mathcal{W}(4)_3^{\oplus 3} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(5)_4^{\oplus 3} \\ \bigoplus \\ \mathcal{W}(4)_4 \end{matrix} \rightleftharpoons \mathcal{W}(6)_5\,. \tag{9.10}$$

The correct brane factor is:

$$f_{D0_{1,4}} = -e^{2\pi\sigma} + 3e^{4\pi\sigma} - 2e^{6\pi\sigma} - 2e^{8\pi\sigma} + 3e^{10\pi\sigma} - e^{12\pi\sigma} \tag{9.11}$$

The central charge was calculated to

$$Z_{D^2}^{\zeta \gg 0}(D0_{1,4}) = \varpi_0. \tag{9.12}$$

$D0_2$

$$\begin{aligned} f_1^{D0_2} &= x_1^2 + c x_2^2 && (9.13)\\ g_1^{D0_2} &= x_1^4 - c x_1^2 x_2^2 + c^2 x_2^4 && (9.14) \end{aligned}$$

With complex[2]:

$$\mathcal{W}(0)_1 \rightleftharpoons \begin{matrix} \mathcal{W}(2)_2^{\oplus 2} \\ \mathcal{W}(1)_2^{\oplus 2} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(3)_3^{\oplus 4} \\ \mathcal{W}(4)_3 \\ \mathcal{W}(2)_3 \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(4)_4^{\oplus 2} \\ \mathcal{W}(5)_4^{\oplus 2} \end{matrix} \rightleftharpoons \mathcal{W}(6)_5\,. \tag{9.15}$$

[2] Henceforth we will drop the direct sum between to complexes at the same R-charge till the end of this thesis.

and brane factor:

$$f_{D0_2} = 2e^{2\pi\sigma} + e^{4\pi\sigma} - 4e^{6\pi\sigma} + e^{8\pi\sigma} + 2e^{10\pi\sigma} - e^{12\pi\sigma} - 1. \tag{9.16}$$

This brane carries central charge

$$Z_{D^2}^{r \gg 0}(D0_2) = 2\varpi_0 \tag{9.17}$$

$D0_3$

$$\begin{aligned} f_1^{D0_3} &= x_1^3 - d x_2^3 && (9.18)\\ g_1^{D0_3} &= x_1^3 + d x_2^3 && (9.19) \end{aligned}$$

The complex is given by

$$\mathcal{W}(-1)_1 \rightleftarrows \begin{matrix} \mathcal{W}(0)_2^{\oplus 2} \\ \mathcal{W}(2)_2 \\ \mathcal{W}(1)_2 \end{matrix} \rightleftarrows \begin{matrix} \mathcal{W}(3)_3^{\oplus 2} \\ \mathcal{W}(2)_3^{\oplus 2} \\ \mathcal{W}(1)_3 \\ \mathcal{W}(4)_3 \end{matrix} \rightleftarrows \begin{matrix} \mathcal{W}(5)_4^{\oplus 2} \\ \mathcal{W}(4)_4 \\ \mathcal{W}(3)_4 \end{matrix} \rightleftarrows \mathcal{W}(6)_5 \,. \tag{9.20}$$

The corresponding brane factor reads:

$$f_{D0_3} = -e^{-2\pi\sigma} - e^{4\pi\sigma} - e^{6\pi\sigma} + 2e^{10\pi\sigma} - e^{12\pi\sigma} + 2. \tag{9.21}$$

The central charge is

$$Z_{D^2}^{r \gg 0}(D0_3) = 3\varpi_0. \tag{9.22}$$

D2-branes

We state a general D2-brane matrix factorisation:

$$Q = f_1\eta_1 + f_2\eta_2 + x_\alpha\eta_4 + pg_1\bar{\eta}_1 + pg_2\bar{\eta}_2 + px_\alpha^{q_\alpha}\bar{\eta}_4, \tag{9.23}$$

where $f_1(f_2)$, $g_1(g_2)$ and x_α, q_α are chosen appropriately to accomplish eq. (9.5).

$D2_1$ or $D2_4$ are given by the assignment

$$f_1^{D2_1} = x_1 + ax_2 \tag{9.24}$$
$$g_1^{D2_1} = x_1^5 - ax_1^4x_2 + a^2x_1^3x_2^2 - a^3x_1^2x_2^3 + a^4x_1x_2^4 - a^5x_2^5 \tag{9.25}$$
$$f_2^{D2_1} = x_3 + fx_4 \tag{9.26}$$
$$g_2^{D2_1} = x_3^5 - fx_3^4x_4 + f^2x_3^3x_4^2 - f^3x_3^2x_4^3 + f^4x_3x_4^4 - f^5x_4^5 \tag{9.27}$$

or equivalently by

$$f_1^{D2_4} = x_1 + ax_2 \tag{9.28}$$
$$g_1^{D2_4} = x_1^5 - ax_1^4x_2 + a^2x_1^3x_2^2 - a^3x_1^2x_2^3 + a^4x_1x_2^4 - a^5x_2^5 \tag{9.29}$$
$$f_1^{D2_4} = x_3^2 - bx_5 \tag{9.30}$$
$$g_1^{D2_4} = x_3^4 + bx_3^2x_5 + b^2x_5^2. \tag{9.31}$$

These factorisations yield the following complex

$$\mathcal{W}(2)_2 \rightleftarrows \begin{matrix} \mathcal{W}(3)_3^{\oplus 2} \\ \mathcal{W}(4)_3 \end{matrix} \rightleftarrows \begin{matrix} \mathcal{W}(4)_4 \\ \mathcal{W}(5)_4^{\oplus 2} \end{matrix} \rightleftarrows \mathcal{W}(6)_5 \,. \tag{9.32}$$

We read off the brane factor:

$$f_{D2_{1,4}} = e^{4\pi\sigma} - 2e^{6\pi\sigma} + 2e^{10\pi\sigma} - e^{12\pi\sigma}. \tag{9.33}$$

Inserting the brane factor into the hemisphere partition function gives

$$Z_{D^2}^{r \gg 0}(D2_{1,4}) = \varpi_0 + \varpi_1. \tag{9.34}$$

$D2_2$ or $D2_7$ We get the following polynomials for the factorisation:

$$f_1^{D2_2} = x_1 + ax_2 \tag{9.35}$$
$$g_1^{D2_2} = x_1^5 - ax_1^4x_2 + a^2x_1^3x_2^2 - a^3x_1^2x_2^3 + a^4x_1x_2^4 - a^5x_2^5 \tag{9.36}$$
$$f_2^{D2_2} = x_3^2 + cx_4^2 \tag{9.37}$$
$$g_2^{D2_2} = x_3^4 - cx_3^2x_4^2 + c^2x_4^4 \tag{9.38}$$

Also the following polynomials gives the same central charge:

$$f_1^{D2_7} = x_1^2 + cx_2^2 \tag{9.39}$$
$$g_1^{D2_7} = x_1^4 - cx_1^2x_2^2 + c^2x_2^4 \tag{9.40}$$
$$f_2^{D2_7} = x_3^2 - bx_5 \tag{9.41}$$
$$g_2^{D2_7} = x_3^4 + bx_3^2x_5 + b^2x_5^2, \tag{9.42}$$

The complex and brane factor are given by:

$$\mathcal{W}(1)_2 \rightleftarrows \begin{matrix} \mathcal{W}(3)_3^{\oplus 2} \\ \mathcal{W}(2)_3 \end{matrix} \rightleftarrows \begin{matrix} \mathcal{W}(5)_4 \\ \mathcal{W}(4)_4^{\oplus 2} \end{matrix} \rightleftarrows \mathcal{W}(6)_5\,, \tag{9.43}$$

$$f_{D2_{2,7}} = e^{2\pi\sigma} - e^{4\pi\sigma} - 2e^{6\pi\sigma} + 2e^{8\pi\sigma} + e^{10\pi\sigma} - e^{12\pi\sigma}. \tag{9.44}$$

The result for the hemisphere partition function is :

$$Z_{D^2}^{r\gg 0}(D2_{2,7}) = \varpi_0 + 2\varpi_1. \tag{9.45}$$

$D2_3$ or $D2_9$ The functions are given by

$$f_1^{D2_3} = x_1 + ax_2 \tag{9.46}$$
$$g_1^{D2_3} = x_1^5 - ax_1^4x_2 + a^2x_1^3x_2^2 - a^3x_1^2x_2^3 + a^4x_1x_2^4 - a^5x_2^5 \tag{9.47}$$
$$f_2^{D2_3} = x_3^3 - dx_4^3 \tag{9.48}$$
$$g_2^{D2_3} = x_3^3 + dx_4^3 \tag{9.49}$$

and

$$f_1^{D2_9} = x_1^3 - bx_2^3 \tag{9.50}$$
$$g_1^{D2_9} = x_1^4 - bx_1^2x_2^2 + b^2x_2^4 \tag{9.51}$$
$$f_2^{D2_9} = x_3^2 - hx_5 \tag{9.52}$$
$$g_2^{D2_9} = x_3^4 + hx_3^2x_5 + h^2x_5^2. \tag{9.53}$$

Writing down the complex gives

$$\mathcal{W}(0)_2 \rightleftarrows \begin{matrix} \mathcal{W}(2)_3 \\ \mathcal{W}(3)_3 \\ \mathcal{W}(1)_3 \end{matrix} \rightleftarrows \begin{matrix} \mathcal{W}(5)_4 \\ \mathcal{W}(3)_4 \\ \mathcal{W}(4)_4 \end{matrix} \rightleftarrows \mathcal{W}(6)_5\,. \tag{9.54}$$

The corresponding brane factor is given by

$$f_{D2_{3,9}} = -e^{2\pi\sigma} - e^{4\pi\sigma} + e^{8\pi\sigma} + e^{10\pi\sigma} - e^{12\pi\sigma} + 1. \tag{9.55}$$

The hemisphere partition function gives:

$$Z_{D^2}^{r\gg 0}(D2_{3,9}) = 3\varpi_1. \tag{9.56}$$

$D2_5$ is obtained by

$$f_1^{D2_5} = x_1^2 + cx_2^2 \tag{9.57}$$
$$g_1^{D2_5} = x_1^4 - cx_1^2x_2^2 + c^2x_2^4 \tag{9.58}$$
$$f_2^{D2_5} = x_3^3 - dx_4^3 \tag{9.59}$$
$$g_2^{D2_5} = x_3^3 + dx_4^3 \tag{9.60}$$

The complex is given by

$$\mathcal{W}(-1)_2 \rightleftarrows \begin{matrix} \mathcal{W}(1)_3^{\oplus 2} \\ \mathcal{W}(2)_3 \end{matrix} \rightleftarrows \begin{matrix} \mathcal{W}(4)_4^{\oplus 2} \\ \mathcal{W}(3)_4 \end{matrix} \rightleftarrows \mathcal{W}(6)_5\,. \tag{9.61}$$

Reading off the brane factor results in

$$f_{D2_5} = e^{-2\pi\sigma} - 2e^{2\pi\sigma} - e^{4\pi\sigma} + e^{6\pi\sigma} + 2e^{8\pi\sigma} - e^{12\pi\sigma}, \tag{9.62}$$

and the hemisphere partition function is evaluated to give

$$Z_{D^2}^{r\gg 0}(D2_5) = -3\varpi_0 + 6\varpi_1. \tag{9.63}$$

$D2_6$ is given by

$$f_1^{D2_6} = x_1^2 + cx_2^2 \tag{9.64}$$
$$g_1^{D2_6} = x_1^4 - cx_1^2x_2^2 + c^2x_2^4 \tag{9.65}$$
$$f_2^{D2_6} = x_3^3 + lx_4^2 \tag{9.66}$$
$$g_2^{D2_6} = x_3^4 - lx_3^2x_4^2 + l^2x_4^4 \tag{9.67}$$

Setting up the complex gives

$$\mathcal{W}(0)_2 \rightleftarrows \mathcal{W}(2)_3^{\oplus 3} \rightleftarrows \mathcal{W}(4)_4^{\oplus 3} \rightleftarrows \mathcal{W}(6)_5\,, \tag{9.68}$$

and the corresponding brane factor is given by

$$f_{D2_6} = -3e^{4\pi\sigma} + 3e^{8\pi\sigma} - e^{12\pi\sigma} + 1. \tag{9.69}$$

Plugging the brane factor into the hemisphere partition function results in

$$Z_{D^2}^{r\gg 0}(D2_6) = 4\varpi_1. \tag{9.70}$$

$D2_8$ is defined by the following functions

$$f_1^{D2_8} = x_1^3 - dx_2^3 \tag{9.71}$$
$$g_1^{D2_8} = x_1^3 + dx_2^3 \tag{9.72}$$
$$f_2^{D2_8} = x_3^3 - ox_4^3 \tag{9.73}$$
$$g_2^{D2_8} = x_3^3 + ox_4^3 \tag{9.74}$$
$$\tag{9.75}$$

The corresponding complex is of the form

$$\mathcal{W}(-2)_2 \rightleftarrows \begin{matrix} \mathcal{W}(1)_3^{\oplus 2} \\ \mathcal{W}(0)_3 \end{matrix} \rightleftarrows \begin{matrix} \mathcal{W}(3)_4^{\oplus 2} \\ \mathcal{W}(4)_4 \end{matrix} \rightleftarrows \mathcal{W}(6)_5\,, \tag{9.76}$$

with brane factor

$$f_{D2_8} = e^{-4\pi\sigma} - 2e^{2\pi\sigma} + 2e^{6\pi\sigma} + e^{8\pi\sigma} - e^{12\pi\sigma} - 1. \tag{9.77}$$

Evaluation of the hemisphere partition function results in

$$Z_{D^2}^{r\gg 0}(D2_8) = -9\varpi_0 + 9\varpi_1. \tag{9.78}$$

D4-branes

D4-branes are slightly different to the matrix factorisations for D0- and D2-branes. A D4-brane is obtained by intersecting a linear devisor $h = 0$ with G resulting in

$$Q = h\eta_1 + G\eta_2 + p\bar{\eta}_2, \tag{9.79}$$

with complex

$$\mathcal{W}(-1)_3 \rightleftarrows \begin{matrix} \mathcal{W}(0)_4 \\ \mathcal{W}(5)_4 \end{matrix} \rightleftarrows \mathcal{W}(6)_5\,. \tag{9.80}$$

The brane factor reads

$$f_{D4} = -e^{-2\pi\sigma} + e^{10\pi\sigma} - e^{12\pi\sigma} + 1, \tag{9.81}$$

and gives central charge

$$Z_{D^2}^{r\gg 0}(D4) = \left(\frac{c_2 H}{24} + \frac{H^3}{6}\right)\varpi_0 - \frac{H^3}{2}\varpi_1 + \frac{H^3}{2}\varpi_2. \tag{9.82}$$

D6-branes

The factorisation giving the D6-brane that describes the structure sheaf $\mathcal{O}_X$ in the large radius phase is given by:

$$Q = G\eta_1 + p\bar{\eta}_1. \tag{9.83}$$

The corresponding complex reads:

$$\mathcal{W}(0)_4 \rightleftarrows \mathcal{W}(6)_5 \tag{9.84}$$

Evaluating the hemisphere partition function, with brane factor

$$f_{D6} = 1 - e^{12\pi\sigma}, \tag{9.85}$$

gives

$$Z_{D^2}^{r\gg 0}(D6) = \frac{c_3\zeta(3)}{(2\pi i)^3}\varpi_0 + \frac{c_2 H}{24}\varpi_1 + \frac{H^3}{6}\varpi_3. \tag{9.86}$$

9.2 $\mathbb{P}[1,1,1,1,4][8]$

Similar to the previous section we state our results for various D-branes. Because the form of the matrix factorisation are similar to those considered in section 9.1, we will only write down the different terms. We set the vacuum charges to

$$\left(q_{|0\rangle}, R_{|0\rangle}\right) = (8,5). \tag{9.87}$$

The constant coefficients in the subsequent given matrix factorisations fulfil

$$\begin{aligned} b^8 &= -1 \quad a^8 = -1 \\ e^2 &= -1 \quad f^2 = -1 \\ c^4 &= -1 \quad h^4 = -1. \end{aligned} \tag{9.88}$$

D0-branes

$D0_1$ or $D0_4$ The corresponding terms in the factorisations for these branes are

$$f_1^{D0_1} = x_1 + bx_2 \tag{9.89}$$
$$g_1^{D0_1} = x_1^7 - bx_1^6x_2 + b^2x_1^5x_2^2 - b^3x_1^4x_2^3 + b^4x_1^3x_2^4 - b^5x_1^2x_2^5 + b^6x_1x_2^6 - b^7x_2^7 \tag{9.90}$$

and

$$f_1^{D0_4} = x_1^4 - ex_5 \tag{9.91}$$
$$g_1^{D0_4} = x_1^4 + ex_5. \tag{9.92}$$

The complex is given by

$$\mathcal{W}(1)_1 \rightleftarrows \begin{array}{c} \mathcal{W}(2)_2^{\oplus 3} \\ \mathcal{W}(5)_2 \end{array} \rightleftarrows \begin{array}{c} \mathcal{W}(6)_3^{\oplus 3} \\ \mathcal{W}(3)_3^{\oplus 3} \end{array} \rightleftarrows \begin{array}{c} \mathcal{W}(7)_4^{\oplus 3} \\ \mathcal{W}(4)_4 \end{array} \rightleftarrows \mathcal{W}(8)_5\,, \tag{9.93}$$

which leads to the brane factor

$$f_{D0_{1,4}} = -e^{2\pi\sigma} + 3e^{4\pi\sigma} - 3e^{6\pi\sigma} + e^{8\pi\sigma} + e^{10\pi\sigma} - 3e^{12\pi\sigma} + 3e^{14\pi\sigma} - e^{16\pi\sigma}. \tag{9.94}$$

The hemisphere partition function gives

$$Z_{D^2}^{r\gg 0}(D0_{1,4}) = \varpi_0. \tag{9.95}$$

$D0_2$ is obtained by setting

$$f_1^{D0_2} = x_1^2 + cx_2^2 \tag{9.96}$$
$$g_1^{D0_2} = x_1^6 - cx_1^4x_2^2 + c^2x_1^2x_2^4 - c^3x_2^6. \tag{9.97}$$

The complex of *Wilson* line branes reads

$$\mathcal{W}(0)_1 \rightleftarrows \begin{matrix} \mathcal{W}(1)_2^{\oplus 2} \\ \mathcal{W}(2)_2 \\ \mathcal{W}(4)_2 \end{matrix} \rightleftarrows \begin{matrix} \mathcal{W}(3)_3^{\oplus 2} \\ \mathcal{W}(6)_3 \\ \mathcal{W}(5)_3^{\oplus 2} \\ \mathcal{W}(2)_3 \end{matrix} \rightleftarrows \begin{matrix} \mathcal{W}(7)_4^{\oplus 2} \\ \mathcal{W}(4)_4 \\ \mathcal{W}(6)_4 \end{matrix} \rightleftarrows \mathcal{W}(8)_5 . \tag{9.98}$$

From the above complex we can read off the corresponding brane factor

$$f_{D0_2} = 2e^{2\pi\sigma} - 2e^{6\pi\sigma} + 2e^{8\pi\sigma} - 2e^{10\pi\sigma} + 2e^{14\pi\sigma} - e^{16\pi\sigma} - 1. \tag{9.99}$$

Plugging the brane factor into the hemisphere partition function results in

$$Z_{D^2}^{r\gg 0}(D0_2) = 2\varpi_0. \tag{9.100}$$

$D0_3$ has the following factorisation

$$f_1^{D0_3} = x_1^4 - e x_2^4 \tag{9.101}$$
$$g_1^{D0_3} = x_1^4 + e x_2^4, \tag{9.102}$$

and corresponding complex

$$\mathcal{W}(-2)_1 \rightleftarrows \begin{matrix} \mathcal{W}(2)_2^{\oplus 2} \\ \mathcal{W}(-1)_2^{\oplus 2} \end{matrix} \rightleftarrows \begin{matrix} \mathcal{W}(3)_3^{\oplus 4} \\ \mathcal{W}(6)_3 \\ \mathcal{W}(0)_3 \end{matrix} \rightleftarrows \begin{matrix} \mathcal{W}(7)_4^{\oplus 2} \\ \mathcal{W}(4)_4^{\oplus 2} \end{matrix} \rightleftarrows \mathcal{W}(8)_5 . \tag{9.103}$$

Reading off the brane factor gives

$$\begin{aligned} f_{D0_3} =& - e^{-4\pi\sigma} + 2e^{-2\pi\sigma} + 2e^{4\pi\sigma} - 4e^{6\pi\sigma} + 2e^{8\pi\sigma} \\ & - e^{12\pi\sigma} + 2e^{14\pi\sigma} - e^{16\pi\sigma} - 1. \end{aligned} \tag{9.104}$$

Evaluating the hemisphere partition function results in

$$Z_{D^2}^{r\gg 0}(D0_3) = 4\varpi_0. \tag{9.105}$$

D2-branes

$D2_1$ and $D2_4$ The relevant terms read

$$f_1^{D2_1} = x_1 + b x_2 \tag{9.106}$$
$$g_1^{D2_1} = x_1^7 - b x_1^6 x_2 + b^2 x_1^5 x_2^2 - b^3 x_1^4 x_2^3 + b^4 x_1^3 x_2^4 - b^5 x_1^2 x_2^5 + b^6 x_1 x_2^6 - b^7 x_2^7 \tag{9.107}$$
$$f_2^{D2_1} = x_3 + a x_4 \tag{9.108}$$
$$g_2^{D2_1} = x_3^7 - a x_3^6 x_4 + a^2 x_3^5 x_4^2 - a^3 x_3^4 x_4^3 + a^4 x_3^3 x_4^4 - a^5 x_3^2 x_4^5 + a^6 x_3 x_4^6 - a^7 x_4^7. \tag{9.109}$$

For $D2_4$ we have

$$f_1^{D2_4} = x_1 + bx_2 \tag{9.110}$$
$$g_1^{D2_4} = x_1^7 - bx_1^6x_2 + b^2x_1^5x_2^2 - b^3x_1^4x_2^3 + b^4x_1^3x_2^4 - b^5x_1^2x_2^5 + b^6x_1x_2^6 - b^7x_2^7 \tag{9.111}$$
$$f_2^{D2_4} = x_3^4 - ex_5 \tag{9.112}$$
$$g_2^{D2_4} = x_3^4 + ex_5. \tag{9.113}$$

Setting up the complex results in

$$\mathcal{W}(2)_2 \rightleftharpoons \begin{matrix} \mathcal{W}(3)_3^{\oplus 2} \\ \mathcal{W}(6)_3 \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(4)_4 \\ \mathcal{W}(7)_4^{\oplus 2} \end{matrix} \rightleftharpoons \mathcal{W}(8)_5 \,. \tag{9.114}$$

The brane factor can simply be read off and is given by

$$f_{D2_4} = e^{4\pi\sigma} - 2e^{6\pi\sigma} + e^{8\pi\sigma} - e^{12\pi\sigma} + 2e^{14\pi\sigma} - e^{16\pi\sigma}, \tag{9.115}$$

resulting in

$$Z_{D^2}^{r\gg 0}(D2_{1,4}) = \varpi_0 + \varpi_1. \tag{9.116}$$

$D2_2$ and $D2_7$ Corresponding to $D2_2$ we have

$$f_1^{D2_2} = x_1 + bx_2 \tag{9.117}$$
$$g_1^{D2_2} = x_1^7 - bx_1^6x_2 + b^2x_1^5x_2^2 - b^3x_1^4x_2^3 + b^4x_1^3x_2^4 - b^5x_1^2x_2^5 + b^6x_1x_2^6 - b^7x_2^7 \tag{9.118}$$
$$f_2^{D2_2} = x_3^2 + cx_4^2 \tag{9.119}$$
$$g_2^{D2_2} = x_3^6 - cx_3^4x_4^2 + c^2x_3^2x_4^4 - c^3x_4^6 \tag{9.120}$$

and for $D2_7$ we set

$$f_1^{D2_7} = x_1^2 + cx_2^2 \tag{9.121}$$
$$g_1^{D2_7} = x_1^6 - cx_1^4x_2^2 + c^2x_1^2x_2^4 - c^3x_2^6 \tag{9.122}$$
$$f_2^{D2_7} = x_3^4 - ex_5 \tag{9.123}$$
$$g_2^{D2_7} = x_3^4 + ex_5. \tag{9.124}$$

The associated complex reads

$$\mathcal{W}(1)_2 \rightleftharpoons \begin{matrix} \mathcal{W}(2)_3 \\ \mathcal{W}(3)_3 \\ \mathcal{W}(5)_3 \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(6)_4 \\ \mathcal{W}(4)_4 \\ \mathcal{W}(7)_4 \end{matrix} \rightleftharpoons \mathcal{W}(8)_5 \,, \tag{9.125}$$

with brane factor

$$f_{D2_{2,7}} = e^{2\pi\sigma} - e^{4\pi\sigma} - e^{6\pi\sigma} + e^{8\pi\sigma} - e^{10\pi\sigma} + e^{12\pi\sigma} + e^{14\pi\sigma} - e^{16\pi\sigma}. \tag{9.126}$$

The hemisphere partition function is given by

$$Z_{D^2}^{r\gg 0}(D2_{2,7}) = \varpi_0 + 2\varpi_1. \tag{9.127}$$

$D2_3$ and $D2_9$ The corresponding factorisation terms are

$$f_1^{D2_3} = x_1 + bx_2 \tag{9.128}$$
$$g_1^{D2_3} = x_1^7 - bx_1^6x_2 + b^2x_1^5x_2^2 - b^3x_1^4x_2^3 + b^4x_1^3x_2^4 - b^5x_1^2x_2^5 + b^6x_1x_2^6 - b^7x_2^7 \tag{9.129}$$
$$f_2^{D2_3} = x_3^4 - ex_4^4 \tag{9.130}$$
$$g_2^{D2_3} = x_3^4 + ex_4^4, \tag{9.131}$$

and

$$f_1^{D2_9} = x_1^4 - ex_2^4 \tag{9.132}$$
$$g_1^{D2_9} = x_1^4 + ex_2^4 \tag{9.133}$$
$$f_2^{D2_9} = x_3^4 - fx_5 \tag{9.134}$$
$$g_2^{D2_9} = x_3^4 + fx_5. \tag{9.135}$$

These branes are represented by the following complex

$$\mathcal{W}(-1)_2 \rightleftarrows \begin{array}{c} \mathcal{W}(3)_3^{\oplus 2} \\ \mathcal{W}(0)_3 \end{array} \rightleftarrows \begin{array}{c} \mathcal{W}(4)_4^{\oplus 2} \\ \mathcal{W}(7)_4 \end{array} \rightleftarrows \mathcal{W}(8)_5 \,. \tag{9.136}$$

This complex results in the brane factor

$$f_{D2_{3,9}} = e^{-2\pi\sigma} - 2e^{6\pi\sigma} + 2e^{8\pi\sigma} + e^{14\pi\sigma} - e^{16\pi\sigma} - 1. \tag{9.137}$$

The charges are given by

$$Z_{D^2}^{r\gg 0}(D2_{3,9}) = -2\varpi_0 + 4\varpi_1. \tag{9.138}$$

$D2_5$ is given by the following terms

$$f_1^{D2_5} = x_1^2 + cx_2^2 \tag{9.139}$$
$$g_1^{D2_5} = x_1^6 - cx_1^4x_2^2 + c^2x_1^2x_2^4 - c^3x_2^6 \tag{9.140}$$
$$f_2^{D2_5} = x_3^2 + hx_4^2 \tag{9.141}$$
$$g_2^{D2_5} = x_3^6 - hx_3^4x_4^2 + h^2x_3^2x_4^4 - h^3x_4^6. \tag{9.142}$$

The corresponding complex is given by

$$\mathcal{W}(0)_2 \rightleftarrows \begin{array}{c} \mathcal{W}(2)_3^{\oplus 2} \\ \mathcal{W}(4)_3 \end{array} \rightleftarrows \begin{array}{c} \mathcal{W}(6)_4^{\oplus 2} \\ \mathcal{W}(4)_4 \end{array} \rightleftarrows \mathcal{W}(8)_5 \,. \tag{9.143}$$

resulting in the brane factor

$$f_{D2_5} = -2e^{4\pi\sigma} + 2e^{12\pi\sigma} - e^{16\pi\sigma} + 1. \tag{9.144}$$

Evaluating the hemisphere partition function results in

$$Z_{D^2}^{r\gg 0}(D2_5) = 4\varpi_1. \tag{9.145}$$

$D2_6$ The corresponding functions read

$$f_1^{D2_6} = x_1^2 + cx_2^2 \tag{9.146}$$
$$g_1^{D2_6} = x_1^6 - cx_1^4x_2^2 + c^2x_1^2x_2^4 - c^3x_2^6 \tag{9.147}$$
$$f_2^{D2_6} = x_3^4 - ex_4^4 \tag{9.148}$$
$$g_2^{D2_6} = x_3^4 + ex_4^4. \tag{9.149}$$

Setting up the Wilson line complex gives

$$\mathcal{W}(-2)_2 \rightleftarrows \begin{matrix} \mathcal{W}(2)_3^{\oplus 2} \\ \mathcal{W}(0)_3 \end{matrix} \rightleftarrows \begin{matrix} \mathcal{W}(4)_4^{\oplus 2} \\ \mathcal{W}(6)_4 \end{matrix} \rightleftarrows \mathcal{W}(8)_5\,, \tag{9.150}$$

with brane factor

$$f_{D2_6} = e^{-4\pi\sigma} - 2e^{4\pi\sigma} + 2e^{8\pi\sigma} + e^{12\pi\sigma} - e^{16\pi\sigma} - 1. \tag{9.151}$$

This brane has charges given by

$$Z_{D^2}^{r\gg 0}(D2_6) = -8\varpi_0 + 8\varpi_1. \tag{9.152}$$

$D2_8$ is given by

$$f_1^{D2_8} = x_1^4 - ex_2^4 \tag{9.153}$$
$$g_1^{D2_8} = x_1^4 + ex_2^4 \tag{9.154}$$
$$f_2^{D2_8} = x_3^4 - fx_4^4 \tag{9.155}$$
$$g_2^{D2_8} = x_3^4 + fx_4^4. \tag{9.156}$$

The corresponding complex is

$$\mathcal{W}(-4)_2 \rightleftarrows \mathcal{W}(0)_3^{\oplus 3} \rightleftarrows \mathcal{W}(4)_4^{\oplus 3} \rightleftarrows \mathcal{W}(8)_5\,. \tag{9.157}$$

From this complex we can read off the following brane factor

$$f_{D2_8} = e^{-8\pi\sigma} + 3e^{8\pi\sigma} - e^{16\pi\sigma} - 3. \tag{9.158}$$

Evaluating the hemisphere partition function gives

$$Z_{D^2}^{r\gg 0}(D2_8) = -32\varpi_0 + 16\varpi_1. \tag{9.159}$$

D4-brane

Again we construct a D4-brane by intersecting the G with a linear divisor h. The complex is given by

$$\mathcal{W}(-1)_3 \rightleftarrows \begin{matrix} \mathcal{W}(7)_4 \\ \mathcal{W}(0)_4 \end{matrix} \rightleftarrows \mathcal{W}(8)_5\,. \tag{9.160}$$

Plugging the brane factor,

$$f_{D4} = -e^{-2\pi\sigma} + e^{14\pi\sigma} - e^{16\pi\sigma} + 1, \tag{9.161}$$

into the hemisphere partition function results in

$$Z_{D^2}^{r\gg 0}(D4) = \left(\frac{c_2 H}{24} + \frac{H^3}{6}\right)\varpi_0 - \frac{H^3}{2}\varpi_1 + \frac{H^3}{2}\varpi_2. \tag{9.162}$$

D6-brane

A D6-brane is given by the same factorisation as stated in the previous section. Only the complex has to be adjusted and gives

$$\mathcal{W}(0)_4 \rightleftarrows \mathcal{W}(8)_5\,, \tag{9.163}$$

with brane factor

$$f_{D6} = 1 - e^{16\pi\sigma} \tag{9.164}$$

and central charge

$$Z_{D^2}^{r\gg 0}(D6) = \frac{c_3\zeta(3)}{(2\pi i)^3}\varpi_0 + \frac{c_2 H}{24}\varpi_1 + \frac{H^3}{6}\varpi_3. \tag{9.165}$$

9.3 $\mathbb{P}[1,1,1,2,5][10]$

The vacuum charges are set to

$$\left(q_{|0\rangle}, R_{|0\rangle}\right) = (10,5). \tag{9.166}$$

The coefficients in the matrix factorisations are solutions of

$$\begin{aligned} b^{10} &= -1 \\ c^5 &= 1 \\ d^2 &= -1 \\ a^2 &= 1 \\ f^2 &= -1. \end{aligned} \tag{9.167}$$

D0-branes

$D0_1$, $D0_4$ and $D0_5$ The factorisations of these branes are given by

$$f_1^{D0_1} = x_1 + bx_2 \tag{9.168}$$

$$\begin{aligned} g_1^{D0_1} &= x_1^9 - bx_1^8x_2 + b^2x_1^7x_2^2 - b^3x_1^6x_2^3 + b^4x_1^5x_2^4 - b^5x_1^4x_2^5 \\ &\quad + b^6x_1^3x_2^6 - b^7x_1^2x_2^7 + b^8x_1x_2^8 - b^9x_2^9, \end{aligned} \tag{9.169}$$

$$f_1^{D0_4} = x_1^2 + cx_4 \tag{9.170}$$

$$g_1^{D0_4} = x_1^8 - cx_1^6x_4 + c^2x_1^4x_4^2 - c^3x_1^2x_4^3 + c^4x_4^4 \tag{9.171}$$

and

$$f_1^{D0_5} = x_1^5 - dx_5 \tag{9.172}$$

$$g_1^{D0_5} = x_1^5 + dx_5 \tag{9.173}$$

The complex reads

$$\mathcal{W}(1)_1 \rightleftarrows \begin{matrix} \mathcal{W}(2)_2^{\oplus 2} \\ \mathcal{W}(3)_2 \\ \mathcal{W}(6)_2 \end{matrix} \rightleftarrows \begin{matrix} \mathcal{W}(3)_3 \\ \mathcal{W}(4)_3^{\oplus 2} \\ \mathcal{W}(7)_3^{\oplus 2} \\ \mathcal{W}(8)_3 \end{matrix} \rightleftarrows \begin{matrix} \mathcal{W}(5)_4 \\ \mathcal{W}(8)_4 \\ \mathcal{W}(9)_4^{\oplus 2} \end{matrix} \rightleftarrows \mathcal{W}(10)_5\,. \tag{9.174}$$

The brane factor is given by

$$f_{D0_{1,4,5}} = -e^{2\pi\sigma} + 2e^{4\pi\sigma} - 2e^{8\pi\sigma} + e^{10\pi\sigma} + e^{12\pi\sigma} - 2e^{14\pi\sigma} + 2e^{18\pi\sigma} - e^{20\pi\sigma}. \tag{9.175}$$

The hemisphere partition function gives

$$Z_{D^2}^{r\gg 0}(D0_{1,4,5}) = \varpi_0. \tag{9.176}$$

$D0_2$ is given by setting

$$f_1^{D0_2} = x_1^2 + cx_2^2 \tag{9.177}$$
$$g_1^{D0_2} = x_1^8 - cx_1^6x_2^2 + c^2x_1^4x_2^4 - c^3x_1^2x_2^6 + c^4x_2^8, \tag{9.178}$$

which leads to the complex

$$\mathcal{W}(0)_1 \rightleftarrows \begin{matrix}\mathcal{W}(2)_2^{\oplus 2}\\ \mathcal{W}(1)_2\\ \mathcal{W}(5)_2\end{matrix} \rightleftarrows \begin{matrix}\mathcal{W}(4)_3\\ \mathcal{W}(3)_3^{\oplus 2}\\ \mathcal{W}(7)_3^{\oplus 2}\\ \mathcal{W}(6)_3\end{matrix} \rightleftarrows \begin{matrix}\mathcal{W}(5)_4\\ \mathcal{W}(9)_4\\ \mathcal{W}(8)_4^{\oplus 2}\end{matrix} \rightleftarrows \mathcal{W}(10)_5\,. \tag{9.179}$$

The corresponding brane factor is

$$\begin{aligned} f_{D0_2} = &-e^{2\pi\sigma} + 2e^{4\pi\sigma} - 2e^{6\pi\sigma} - e^{8\pi\sigma} + 2e^{10\pi\sigma} - e^{12\pi\sigma} - 2e^{14\pi\sigma} \\ &+ 2e^{16\pi\sigma} + e^{18\pi\sigma} - e^{20\pi\sigma} - 1. \end{aligned} \tag{9.180}$$

Calculating the hemisphere partition function results in

$$Z_{D^2}^{r\gg 0}(D0_2) = 2\varpi_0. \tag{9.181}$$

$D0_3$ is given by

$$f_1^{D0_3} = x_1^5 + dx_2^5 \tag{9.182}$$
$$g_1^{D0_3} = x_1^5 - dx_2^5, \tag{9.183}$$

with corresponding complex:

$$\mathcal{W}(-3)_1 \rightleftarrows \begin{matrix}\mathcal{W}(2)_2^{\oplus 2}\\ \mathcal{W}(-2)_2\\ \mathcal{W}(-1)_2\end{matrix} \rightleftarrows \begin{matrix}\mathcal{W}(7)_3\\ \mathcal{W}(4)_3^{\oplus 2}\\ \mathcal{W}(3)_3^{\oplus 2}\\ \mathcal{W}(0)_3\end{matrix} \rightleftarrows \begin{matrix}\mathcal{W}(9)_4\\ \mathcal{W}(8)_4\\ \mathcal{W}(5)_4^{\oplus 2}\end{matrix} \rightleftarrows \mathcal{W}(10)_5\,. \tag{9.184}$$

The brane factor results in

$$\begin{aligned} f_{D0_3} = &-e^{-6\pi\sigma} + e^{-4\pi\sigma} + e^{-2\pi\sigma} + 2e^{4\pi\sigma} - 2e^{6\pi\sigma} - 2e^{8\pi\sigma} \\ &+ 2e^{10\pi\sigma} - e^{14\pi\sigma} + e^{16\pi\sigma} + e^{18\pi\sigma} - e^{20\pi\sigma} - 1, \end{aligned} \tag{9.185}$$

and leads to the following central charge

$$Z_{D^2}^{r\gg 0}(D0_3) = 5\varpi_0. \tag{9.186}$$

D2 branes

$D2_1$,$D2_2$ *or* $D2_3$ are a result of the following factorisations

$$
\begin{aligned}
f_1^{D2_1} &= x_1 + bx_2 && (9.187)\\
g_1^{D2_1} &= x_1^9 - bx_1^8x_2 + b^2x_1^7x_2^2 - b^3x_1^6x_2^3 + b^4x_1^5x_2^4\\
&\quad - b^5x_1^4x_2^5 + b^6x_1^3x_2^6 - b^7x_1^2x_2^7 + b^8x_1x_2^8 - b^9x_2^9 && (9.188)\\
f_2^{D2_1} &= x_3^2 + cx_4 && (9.189)\\
g_2^{D2_1} &= x_3^8 - cx_3^6x_4 + c^2x_3^4x_4^2 - c^3x_3^2x_4^3 + c^4x_4^4, && (9.190)
\end{aligned}
$$

$$
\begin{aligned}
f_1^{D2_2} &= x_1 + bx_2 && (9.191)\\
g_1^{D2_1} &= x_1^9 - bx_1^8x_2 + b^2x_1^7x_2^2 - b^3x_1^6x_2^3 + b^4x_1^5x_2^4\\
&\quad - b^5x_1^4x_2^5 + b^6x_1^3x_2^6 - b^7x_1^2x_2^7 + b^8x_1x_2^8 - b^9x_2^9 && (9.192)\\
f_2^{D2_2} &= x_3^5 + dx_5 && (9.193)\\
g_2^{D2_2} &= x_3^5 - dx_5 && (9.194)
\end{aligned}
$$

and

$$
\begin{aligned}
f_1^{D2_3} &= x_1^2 + cx_4 && (9.195)\\
g_1^{D2_3} &= x_1^8 - cx_1^6x_4 + c^2x_1^4x_4^2 - c^3x_1^2x_4^3 + c^4x_4^4 && (9.196)\\
f_2^{D2_3} &= x_2^5 + dx_5 && (9.197)\\
g_2^{D2_3} &= x_2^5 - dx_5. && (9.198)
\end{aligned}
$$

The Wilson line brane complex is given by

$$
\mathcal{W}(2)_2 \rightleftarrows \begin{matrix} \mathcal{W}(3)_3 \\ \mathcal{W}(4)_3 \\ \mathcal{W}(7)_3 \end{matrix} \rightleftarrows \begin{matrix} \mathcal{W}(5)_4 \\ \mathcal{W}(8)_4 \\ \mathcal{W}(9)_4 \end{matrix} \rightleftarrows \mathcal{W}(10)_5\,. \tag{9.199}
$$

The brane factor reads

$$
f_{D2_{1,2,3}} = e^{4\pi\sigma} - e^{6\pi\sigma} - e^{8\pi\sigma} + e^{10\pi\sigma} - e^{14\pi\sigma} + e^{16\pi\sigma} + e^{18\pi\sigma} - e^{20\pi\sigma}. \tag{9.200}
$$

The evaluation of the hemisphere partition function gives

$$
Z_{D^2}^{r\gg 0}(D0_3) = \varpi_0 + \varpi_1. \tag{9.201}
$$

$D2_4$ *or* $D2_5$ are given by

$$
\begin{aligned}
f_1^{D2_4} &= x_1^2 + cx_2^2 && (9.202)\\
g_1^{D2_4} &= x_1^8 - cx_1^6x_2^2 + c^2x_1^4x_2^4 - c^3x_1^2x_2^6 + c^4x_2^8 && (9.203)\\
f_2^{D2_4} &= x_3^2 + ax_4 && (9.204)\\
g_2^{D2_4} &= x_3^8 - ax_3^6x_4 + a^2x_3^4x_4^2 - a^3x_3^2x_4^3 + a^4x_4^4 && (9.205)
\end{aligned}
$$

and

$$f_1^{D2_5} = x_1^2 + cx_2^2 \tag{9.206}$$
$$g_1^{D2_5} = x_1^8 - cx_1^6x_2^2 + c^2x_1^4x_2^4 - c^3x_1^2x_2^6 + c^4x_2^8 \tag{9.207}$$
$$f_2^{D2_5} = x_3^5 + dx_5 \tag{9.208}$$
$$g_2^{D2_5} = x_3^5 - dx_5. \tag{9.209}$$

The corresponding complex is

$$\mathcal{W}(1)_2 \rightleftharpoons \begin{matrix} \mathcal{W}(3)_3^{\oplus 2} \\ \mathcal{W}(6)_3 \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(5)_4 \\ \mathcal{W}(8)_4^{\oplus 2} \end{matrix} \rightleftharpoons \mathcal{W}(10)_5\,, \tag{9.210}$$

which leads to the brane factor

$$f_{D2_{4,5}} = e^{2\pi\sigma} - 2e^{6\pi\sigma} + e^{10\pi\sigma} - e^{12\pi\sigma} + 2e^{16\pi\sigma} - e^{20\pi\sigma}. \tag{9.211}$$

The hemisphere partition function is given by

$$Z_{D^2}^{r\gg 0}(D2_{4,5}) = \varpi_0 + 2\varpi_1. \tag{9.212}$$

$D2_6$ or $D2_7$ can be obtained by setting

$$f_1^{D2_6} = x_1^5 + dx_2^5 \tag{9.213}$$
$$g_1^{D2_6} = x_1^5 - dx_2^5 \tag{9.214}$$
$$f_2^{D2_6} = x_3^2 + cx_4 \tag{9.215}$$
$$g_2^{D2_6} = x_3^8 - cx_3^6x_4 + c^2x_3^4x_4^2 - c^3x_3^2x_4^3 + c^4x_4^4 \tag{9.216}$$

and

$$f_1^{D2_7} = x_1^5 + dx_2^5 \tag{9.217}$$
$$g_1^{D2_7} = x_1^5 - dx_2^5 \tag{9.218}$$
$$f_2^{D2_7} = x_3^5 + fx_5 \tag{9.219}$$
$$g_2^{D2_7} = x_3^5 - fx_5. \tag{9.220}$$

We deduce the complex

$$\mathcal{W}(-2)_2 \rightleftharpoons \begin{matrix} \mathcal{W}(3)_3^{\oplus 2} \\ \mathcal{W}(0)_3 \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(8)_4 \\ \mathcal{W}(5)_4^{\oplus 2} \end{matrix} \rightleftharpoons \mathcal{W}(10)_5\,. \tag{9.221}$$

The brane factor,

$$f_{D2_{6,7}} = e^{-4\pi\sigma} - 2e^{6\pi\sigma} + 2e^{10\pi\sigma} + e^{16\pi\sigma} - e^{20\pi\sigma} - 1, \tag{9.222}$$

leads to the following result

$$Z_{D^2}^{r\gg 0}(D2_{6,7}) = -5\varpi_0 + 5\varpi_1. \tag{9.223}$$

D4-brane

A D4- brane is obtained as before and is given by the complex

$$\mathcal{W}(-1)_3 \rightleftarrows \begin{matrix} \mathcal{W}(0)_4 \\ \mathcal{W}(9)_4 \end{matrix} \rightleftarrows \mathcal{W}(10)_5 \,. \tag{9.224}$$

The brane factor corresponding to this complex is

$$f_{D4} = -e^{-2\pi\sigma} + e^{18\pi\sigma} - e^{20\pi\sigma} + 1. \tag{9.225}$$

This brane carries the expected central charge:

$$Z_{D^2}^{r\gg 0}(D4) = \left(\frac{c_2 H}{24} + \frac{H^3}{6}\right)\varpi_0 - \frac{H^3}{2}\varpi_1 + \frac{H^3}{2}\varpi_2. \tag{9.226}$$

D6-brane

Also the D6-brane given by

$$\mathcal{W}(0)_4 \rightleftarrows \mathcal{W}(10)_5 \,, \tag{9.227}$$

with brane factor

$$f_{D6} = 1 - e^{20\pi\sigma}, \tag{9.228}$$

gives the expected central charge

$$Z_{D^2}^{r\gg 0}(D6) = \frac{c_3\zeta(3)}{(2\pi i)^3}\varpi_0 + \frac{c_2 H}{24}\varpi_1 + \frac{H^3}{6}\varpi_3. \tag{9.229}$$

10 Monodromy

In this chapter we study the monodromy around various distinctive points in the *Kähler* moduli space of the previously discussed *Calabi-Yau* spaces[1]. Therefore we will set up a basis $\mathcal{W}$ consisting of $(\mathcal{W}_{pt}, \mathcal{W}_l, \mathcal{W}_H, \mathcal{W}_X)$ [2]. The central charge of the corresponding D-branes used as basis in this chapter are[3]

$$\begin{aligned}
Z_{D^2}^{r\gg 0}(D0) &= \varpi_0 \\
Z_{D^2}^{r\gg 0}(D2) &= \varpi_1 \\
Z_{D^2}^{r\gg 0}(D4) &= \left(\frac{c_2 H}{24} + \frac{H^3}{6}\right)\varpi_0 + -\frac{H^3}{2}\varpi_1 + \frac{H^3}{2}\varpi_2 \\
Z_{D^2}^{r\gg 0}(D6) &= \frac{c_3\zeta(3)}{(2\pi i)^3}\varpi_0 + -\frac{c_2 \cdot H}{24}\varpi_1 + \frac{H^3}{+}\varpi_3.
\end{aligned} \tag{10.1}$$

The monodromy acts as a transformation on our basis $\mathcal{W} \to \mathcal{W}' = \left(\mathcal{W}'_{pt}, \mathcal{W}'_l, \mathcal{W}'_H, \mathcal{W}'_X\right)$. Expressing $\mathcal{W}'$ in terms of $\mathcal{W}$ leads to the relation

$$Z_{D^2}\left(\mathcal{W}'\right) = M Z_{D^2}\left(\mathcal{W}\right), \tag{10.2}$$

where M is the monodromy matrix.

Also we want to emphasise that a direct evaluation of the hemisphere partition function is not necessary. Because of the properties of a integral we can read off the monodromy matrix directly by comparing brane factors. This feature decouples our result from a particular phase of the GLSM. In the following we will discuss mondromies around the large-radius-, the *Landau-Ginzburg*- and the conifold-point. For setting up a basis we will rely on the the results stated in sections 9.1, 9.2 and 9.3. To perform a monodromy we have to specify which charge windows we are looking at. This is necessary for the grade restriction process, because only grade restricted branes can be transported in a sensible manner[4].

Subsequently we will use a condensed notation to write down the process of grade restriction in order to discuss all considered cases at once.

Therefore we will use the following parameter

$$\alpha_{nmk} = q - (N - nw_1 - mw_2 - kw_3), \tag{10.3}$$

[1] In section 5.2 further theoretical background on the process of D-brane transport is given.

[2] By $\mathcal{W}_X$ we denote the GLSM lift of the structure sheaf $\mathcal{O}_X$. Consequently we denote with the subscripts H, l and pt the lifts of sheaves of a hyperplane, a line and a point respectively.

[3] The numerical values of the used topological constants can be found in table 9.1 of chapter 9.

[4] The process of grade restriction is described in section 5.2 and to some extent also in 6.2.

where N is the quasi-homogeneous degree of the embedding polynomial, and the symbol

$$C_n = \binom{\#w_1}{n}.$$

The charges $Q_i w_i$ of the fields/coordinates of our models are summarized in tables 10.1 to 10.3.

	weight	#
w_1	1	4
w_2	2	1
w_3	-	-

Table 10.1: Different weights in $\mathbb{P}[6]$

	weight	#
w_1	1	3
w_2	4	1
w_3	-	-

Table 10.2: Weights in $\mathbb{P}[8]$

	weight	#
w_1	1	3
w_2	2	1
w_3	5	1

Table 10.3: Example of weights in $\mathbb{P}[10]$

With these definitions we write a generic *Wilson* line brane of a complex representing a GLSM brane as

$$\mathcal{W}\left(\alpha^{ijk}_{lmn}\right)^{\oplus C_n}_{r}, \tag{10.4}$$

where we use the superscript indices to denote the appropriate values for $\mathbb{P}[6]$ and $\mathbb{P}[8]$. Subscript indices are used for the values in $\mathbb{P}[10]$. In all cases q refers to the chosen vacuum charge and N to the total degree of the polynomial. Writing down α with subscript indices and superscript indices denotes that the corresponding term in the complex is appearing in $\mathbb{P}[10]$ and $\mathbb{P}[8]/\mathbb{P}[6]$. In order to get used to this notation we will write down the complexes corresponding to empty branes in the large radius(LR)- and *Landau-Ginzburg*(LG)-phase. The notation α^{ijk}_{lmn} means α^{ijk} for $\mathbb{P}[6]$ and $\mathbb{P}[8]$ and α_{lmn} for $\mathbb{P}[10]$. The LR-empty brane is represented by

$$\mathcal{W}(q-N)^{\oplus 1}_{r-5} \rightleftharpoons \begin{matrix} \mathcal{W}(\alpha^{010})^{\oplus C_4}_{r-4} \\ \mathcal{W}(\alpha^{100}_{100})^{\oplus C_3}_{r-4} \\ \mathcal{W}(\alpha_{001})^{\oplus C_3}_{r-4} \\ \mathcal{W}(\alpha_{010})^{\oplus C_3}_{r-4} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(\alpha^{110}_{011})^{\oplus C_3}_{r-3} \\ \mathcal{W}(\alpha^{200}_{101})^{\oplus C_2}_{r-3} \\ \mathcal{W}(\alpha_{200})^{\oplus C_1}_{r-3} \\ \mathcal{W}(\alpha_{110})^{\oplus C_2}_{r-3} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(\alpha^{210}_{111})^{\oplus C_2}_{r-2} \\ \mathcal{W}(\alpha^{300}_{201})^{\oplus C_1}_{r-2} \\ \mathcal{W}(\alpha_{210})^{\oplus C_1}_{r-2} \\ \mathcal{W}(\alpha_{300})^{\oplus 1}_{r-2} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(\alpha^{310}_{211})^{\oplus C_1}_{r-1} \\ \mathcal{W}(\alpha^{400}_{301})^{\oplus 1}_{r-1} \\ \mathcal{W}(\alpha_{310})^{\oplus 1}_{r-1} \end{matrix} \rightleftharpoons \mathcal{W}(q)^{\oplus 1}_{r} \tag{10.5}$$

In the above complex we assumed, that the weights w_2 and w_3 only appear once, which is fulfilled in the examined spaces. Writing down the LG-empty brane results in

$$\mathcal{W}(q-N)^{\oplus 1}_{r-1} \rightleftharpoons \mathcal{W}(q)^{\oplus 1}_{r}. \tag{10.6}$$

Now let Q be the maximum charge in a chosen window. The minimum charge in a window is given by $Q-(N-1)$. Using the minimum charge we get a lower bound for the charges of the *Wilson*-line branes in a complex, which is given by [5]

$$\begin{aligned} \alpha_{nmk} &\geq Q-(N-1) \\ nw_1 + mw_2 + kw_3 &\geq 1 + (Q-q). \end{aligned}$$

As a first step we will set up a basis of branes and then perform the various monodromies.

10.1 *Grade Restriction process*

In order to transport the basis of branes, we have to grade restrict them to the considered window. Therefore we will outline the general grade restriction procedure. A window consists of $N-1$ consecutive

[5] In later calculations we also assumed $w_1 \geq 1$ and $w_2 \geq 2$, which is true for the considered spaces.

integers. To write down a procedure valid for all considered examples, we will consider a window of the form

$$\{q-N+1, q-N+2, \dots, q\}\,. \tag{10.7}$$

Also we assume that we are in the large radius phase and use *LR*-empty branes to restrict the studied branes. The procedure in the *Landau-Ginzburg* phase is equivalent, but now one uses *LG*-empty branes for the restriction.

*D*6-*brane*

The complex representing a $D6$-brane is the same as the structure of the LG-empty brane and is given by

$$\mathcal{W}(q-N+1)^{\oplus 1}_{r-1} \rightleftharpoons \mathcal{W}(q+1)^{\oplus 1}_{r}\,. \tag{10.8}$$

In all considered cases we cannot choose q such that the $D6$ lies entirely in a window. To describe the restriction process we define

$$\hat{\alpha}_{nmk} = \alpha_{nmk} + 1 \tag{10.9}$$

Now we bind a appropriate LR-empty brane to remove the spurious $\mathcal{W}(q+1)$, in the following way

$$\begin{tikzcd}
 & & & & & \mathcal{W}(q-N+1)^{\oplus 1}_{r-1} \arrow[r, shift left] & \mathcal{W}(q+1)^{\oplus 1}_{r} \arrow[l, shift left] \\
\mathcal{W}(q-N+1)^{\oplus 1}_{r-6} \arrow[r, shift left] & \begin{matrix} \mathcal{W}(\hat{\alpha}^{010})^{\oplus \mathrm{C}_4}_{r-5} \\ \mathcal{W}(\hat{\alpha}^{100}_{100})^{\oplus \mathrm{C}_3}_{r-5} \\ \mathcal{W}(\hat{\alpha}_{001})^{\oplus \mathrm{C}_3}_{r-5} \\ \mathcal{W}(\hat{\alpha}_{010})^{\oplus \mathrm{C}_3}_{r-5} \end{matrix} \arrow[l, shift left] \arrow[r, shift left] & \begin{matrix} \mathcal{W}(\hat{\alpha}^{110}_{011})^{\oplus \mathrm{C}_3}_{r-4} \\ \mathcal{W}(\hat{\alpha}^{200}_{101})^{\oplus \mathrm{C}_2}_{r-4} \\ \mathcal{W}(\hat{\alpha}_{200})^{\oplus \mathrm{C}_1}_{r-4} \\ \mathcal{W}(\hat{\alpha}_{110})^{\oplus \mathrm{C}_2}_{r-4} \end{matrix} \arrow[l, shift left] \arrow[r, shift left] & \begin{matrix} \mathcal{W}(\hat{\alpha}^{210}_{111})^{\oplus \mathrm{C}_2}_{r-3} \\ \mathcal{W}(\hat{\alpha}^{300}_{201})^{\oplus \mathrm{C}_1}_{r-3} \\ \mathcal{W}(\hat{\alpha}_{210})^{\oplus \mathrm{C}_1}_{r-3} \\ \mathcal{W}(\hat{\alpha}_{300})^{\oplus 1}_{r-3} \end{matrix} \arrow[l, shift left] \arrow[r, shift left] & \begin{matrix} \mathcal{W}(\hat{\alpha}^{310}_{211})^{\oplus \mathrm{C}_1}_{r-2} \\ \mathcal{W}(\hat{\alpha}^{400}_{301})^{\oplus 1}_{r-2} \\ \mathcal{W}(\hat{\alpha}_{310})^{\oplus 1}_{r-2} \end{matrix} \arrow[l, shift left] \arrow[r, shift left] \arrow[ur, "\phi"] & \mathcal{W}(q+1)^{\oplus 1}_{r-1} \arrow[l, shift left] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd} \tag{10.10}$$

The identity map can be removed and one obtains

$$\mathcal{W}(q-N+1)^{\oplus 1}_{r-6} \rightleftharpoons \begin{matrix} \mathcal{W}(\hat{\alpha}^{010})^{\oplus \mathrm{C}_4}_{r-5} \\ \mathcal{W}(\hat{\alpha}^{100}_{100})^{\oplus \mathrm{C}_3}_{r-5} \\ \mathcal{W}(\hat{\alpha}_{001})^{\oplus \mathrm{C}_3}_{r-5} \\ \mathcal{W}(\hat{\alpha}_{010})^{\oplus \mathrm{C}_3}_{r-5} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(\hat{\alpha}^{110}_{011})^{\oplus \mathrm{C}_3}_{r-4} \\ \mathcal{W}(\hat{\alpha}^{200}_{101})^{\oplus \mathrm{C}_2}_{r-4} \\ \mathcal{W}(\hat{\alpha}_{200})^{\oplus \mathrm{C}_1}_{r-4} \\ \mathcal{W}(\hat{\alpha}_{110})^{\oplus \mathrm{C}_2}_{r-4} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(\hat{\alpha}^{210}_{111})^{\oplus \mathrm{C}_2}_{r-3} \\ \mathcal{W}(\hat{\alpha}^{300}_{201})^{\oplus \mathrm{C}_1}_{r-3} \\ \mathcal{W}(\hat{\alpha}_{210})^{\oplus \mathrm{C}_1}_{r-3} \\ \mathcal{W}(\hat{\alpha}_{300})^{\oplus 1}_{r-3} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(\hat{\alpha}^{310}_{211})^{\oplus \mathrm{C}_1}_{r-2} \\ \mathcal{W}(\hat{\alpha}^{400}_{301})^{\oplus 1}_{r-2} \\ \mathcal{W}(\hat{\alpha}_{310})^{\oplus 1}_{r-2} \end{matrix} \rightleftharpoons \mathcal{W}(q-N+1)^{\oplus 1}_{r-1} \tag{10.11}$$

*D*4-*brane*

A $D4$-brane, with a geometric interpretation as sheaf of a hyperplane, is of the form

$$\mathcal{W}(q-N)^{\oplus 1}_{r-2} \rightleftharpoons \begin{matrix} \mathcal{W}(q)^{\oplus 1}_{r-1} \\ \mathcal{W}(q-N+1)^{\oplus 1}_{r-1} \end{matrix} \rightleftharpoons \mathcal{W}(q+1)^{\oplus 1}_{r}\,. \tag{10.12}$$

As on can see, $\mathcal{W}(q-N)$ and $\mathcal{W}(q+1)$ are not in the window. Therefore we have to bind various LR-empty branes and use an algorithm developed by [32].

In table 10.4 we used $\hat{\alpha}_{nmk}$ defined in eq. (10.9).

									#
				$\mathcal{W}(q-N)^{\oplus 1}_{r-2}$	$\begin{matrix}\mathcal{W}(q)^{\oplus 1}_{r-1} \\ \mathcal{W}(q-N+1)^{\oplus 1}_{r-1}\end{matrix}$	$\mathcal{W}(q+1)^{\oplus 1}_{r}$			
			$\mathcal{W}(q-N)^{\oplus 1}_{r-3}$	$\begin{matrix}\mathcal{W}(\alpha^{010})^{\oplus C_4}_{r-2} \\ \mathcal{W}(\alpha^{100}_{100})^{\oplus C_3}_{r-2} \\ \mathcal{W}(\alpha_{001})^{\oplus C_3}_{r-2} \\ \mathcal{W}(\alpha_{010})^{\oplus C_3}_{r-2}\end{matrix}$	$\begin{matrix}\mathcal{W}(\alpha^{110}_{011})^{\oplus C_3}_{r-1} \\ \mathcal{W}(\alpha^{200}_{101})^{\oplus C_2}_{r-1} \\ \mathcal{W}(\alpha_{200})^{\oplus C_1}_{r-1} \\ \mathcal{W}(\alpha_{110})^{\oplus C_2}_{r-1}\end{matrix}$	$\begin{matrix}\mathcal{W}(\alpha^{110}_{111})^{\oplus C_2}_{r} \\ \mathcal{W}(\alpha^{300}_{201})^{\oplus C_1}_{r} \\ \mathcal{W}(\alpha_{210})^{\oplus C_1}_{r} \\ \mathcal{W}(\alpha_{300})^{\oplus 1}_{r}\end{matrix}$	$\begin{matrix}\mathcal{W}(\alpha^{310}_{211})^{\oplus C_1}_{r+1} \\ \mathcal{W}(\alpha^{400}_{301})^{\oplus 1}_{r+1} \\ \mathcal{W}(\alpha_{310})^{\oplus 1}_{r+1}\end{matrix}$	$\mathcal{W}(\tilde{q})^{\oplus 1}_{r+2}$	1
$\mathcal{W}(q-N+1)^{\oplus 1}_{r-6}$	$\begin{matrix}\mathcal{W}(\hat{\alpha}^{010})^{\oplus C_4}_{r-5} \\ \mathcal{W}(\hat{\alpha}^{100}_{100})^{\oplus C_3}_{r-5} \\ \mathcal{W}(\hat{\alpha}_{001})^{\oplus C_3}_{r-5} \\ \mathcal{W}(\hat{\alpha}_{010})^{\oplus C_3}_{r-5}\end{matrix}$	$\begin{matrix}\mathcal{W}(\hat{\alpha}^{110}_{011})^{\oplus C_3}_{r-4} \\ \mathcal{W}(\hat{\alpha}^{200}_{101})^{\oplus C_2}_{r-4} \\ \mathcal{W}(\hat{\alpha}_{200})^{\oplus C_1}_{r-4} \\ \mathcal{W}(\hat{\alpha}_{110})^{\oplus C_2}_{r-4}\end{matrix}$	$\begin{matrix}\mathcal{W}(\hat{\alpha}^{110}_{111})^{\oplus C_2}_{r-3} \\ \mathcal{W}(\hat{\alpha}^{300}_{201})^{\oplus C_1}_{r-3} \\ \mathcal{W}(\hat{\alpha}_{210})^{\oplus C_1}_{r-3} \\ \mathcal{W}(\hat{\alpha}_{300})^{\oplus 1}_{r-3}\end{matrix}$	$\begin{matrix}\mathcal{W}(\hat{\alpha}^{310}_{211})^{\oplus C_1}_{r-2} \\ \mathcal{W}(\hat{\alpha}^{400}_{301})^{\oplus 1}_{r-2} \\ \mathcal{W}(\hat{\alpha}_{310})^{\oplus 1}_{r-2}\end{matrix}$	$\mathcal{W}(q+1)^{\oplus 1}_{r-1}$				1

Table 10.4: General grade restriction process with a $D4$-brane

D2-brane

To write down a $D2$-brane in the most general manner we define

$$\tilde{C}_n = \binom{\#w_1 - 2}{n}$$
$$\beta^{nmk}_{nmk} = \alpha^{nmk}_{nmk} + 2w_1.$$

$$\mathcal{W}(q-N+2w_1)^{\oplus 1}_{r-3} \rightleftharpoons \begin{matrix}\mathcal{W}(\beta^{010})^{\oplus \tilde{C}_2}_{r-2} \\ \mathcal{W}(\beta^{100}_{001})^{\oplus \tilde{C}_1}_{r-2} \\ \mathcal{W}(\beta_{010})^{\oplus \tilde{C}_1}_{r-2} \\ \mathcal{W}(\beta_{100})^{\oplus 1}_{r-2}\end{matrix} \rightleftharpoons \begin{matrix}\mathcal{W}(\beta^{110}_{011})^{\oplus \tilde{C}_1}_{r-1} \\ \mathcal{W}(\beta^{200}_{101})^{\oplus 1}_{r-1} \\ \mathcal{W}(\beta_{110})^{\oplus 1}_{r-1}\end{matrix} \rightleftharpoons \mathcal{W}(q)^{\oplus 1}_{r}. \tag{10.13}$$

For these $D2$-brane grade restriction is not required because its charges lie entirely in a given window.

D0-brane

Before writing down a $D0$-brane we define

$$\tilde{\tilde{C}}_n = \binom{\#w_1 - 1}{n}$$
$$\gamma^{nmk}_{nmk} = \alpha^{nmk}_{nmk} + w_1.$$

By using the above definitions we can express a $D0$-brane by the following complex

$$\mathcal{W}(q-N+w_1)^{\oplus 1}_{r-4} \rightleftharpoons \begin{matrix}\mathcal{W}(\gamma^{010})^{\oplus \tilde{\tilde{C}}_3}_{r-3} \\ \mathcal{W}(\gamma^{100}_{001})^{\oplus \tilde{\tilde{C}}_2}_{r-3} \\ \mathcal{W}(\gamma_{010})^{\oplus \tilde{\tilde{C}}_2}_{r-2} \\ \mathcal{W}(\gamma_{100})^{\oplus \tilde{\tilde{C}}_1}_{r-2}\end{matrix} \rightleftharpoons \begin{matrix}\mathcal{W}(\gamma^{110}_{011})^{\oplus \tilde{\tilde{C}}_2}_{r-2} \\ \mathcal{W}(\gamma^{200}_{101})^{\oplus \tilde{\tilde{C}}_1}_{r-2} \\ \mathcal{W}(\gamma_{110})^{\oplus \tilde{\tilde{C}}_1}_{r-2} \\ \mathcal{W}(\gamma_{200})^{\oplus 1}_{r-2}\end{matrix} \rightleftharpoons \begin{matrix}\mathcal{W}(\gamma^{210}_{111})^{\oplus \tilde{\tilde{C}}_1}_{r-1} \\ \mathcal{W}(\gamma^{300}_{201})^{\oplus 1}_{r-1} \\ \mathcal{W}(\gamma_{210})^{\oplus 1}_{r-1}\end{matrix} \rightleftharpoons \mathcal{W}(q)^{\oplus 1}_{r}. \tag{10.14}$$

Also in the case of the $D0$-brane no grade restriction is needed as long as we choose q to be equal to the maximum charge of the considered window.

10.2 $\mathbb{P}[1,1,1,1,2][6]$

To perform a monodromy we have to fix a window. We choose

$$w1 : \theta \in (-6\pi, -4\pi) \qquad q \in (0,1,2,3,4,5) \tag{10.15}$$
$$w2 : \theta \in (-8\pi, -6\pi) \qquad q \in (1,2,3,4,5,6)\,. \tag{10.16}$$

In the following we give the basis of branes for this model explicitly.

*D*0*-brane*

$$\mathcal{W}(0)_1^{\oplus 1} \rightleftharpoons \begin{matrix} \mathcal{W}(2)_2^{\oplus 1} \\ \mathcal{W}(1)_2^{\oplus 3} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(3)_3^{\oplus 3} \\ \mathcal{W}(2)_3^{\oplus 3} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(4)_4^{\oplus 3} \\ \mathcal{W}(3)_4^{\oplus 1} \end{matrix} \rightleftharpoons \mathcal{W}(5)_5^{\oplus 1}\,. \tag{10.17}$$

The corresponding brane-factor is given by

$$f_{\mathcal{B}}^{D0} = 3e^{2\pi\sigma} - 2e^{4\pi\sigma} - 2e^{6\pi\sigma} + 3e^{8\pi\sigma} - e^{10\pi\sigma} - 1. \tag{10.18}$$

*D*2*-brane* We get the following complex

$$\mathcal{W}(1)_2^{\oplus 1} \rightleftharpoons \begin{matrix} \mathcal{W}(3)_3^{\oplus 1} \\ \mathcal{W}(2)_3^{\oplus 2} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(4)_4^{\oplus 2} \\ \mathcal{W}(3)_4^{\oplus 1} \end{matrix} \rightleftharpoons \mathcal{W}(5)_5^{\oplus 1}\,, \tag{10.19}$$

with brane-factor

$$f_{\mathcal{B}}^{D2} = e^{2\pi\sigma} - 2e^{4\pi\sigma} + 2e^{8\pi\sigma} - e^{10\pi\sigma}. \tag{10.20}$$

*D*4*-brane* We will not write down the grade restricted complex of a $D4$-brane, which can be obtained by using table 10.4. The brane-factor is

$$f_{\mathcal{B}}^{D4} = 9e^{2\pi\sigma} - 5e^{4\pi\sigma} - 5e^{6\pi\sigma} + 9e^{8\pi\sigma} - 4e^{10\pi\sigma} - 4. \tag{10.21}$$

*D*6*-brane* A complex presenting a $D6$-brane is given by

$$\mathcal{W}(0)_{-1}^{\oplus 1} \rightleftharpoons \begin{matrix} \mathcal{W}(2)_0^{\oplus 1} \\ \mathcal{W}(1)_0^{\oplus 4} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(3)_1^{\oplus 4} \\ \mathcal{W}(2)_1^{\oplus 6} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(4)_2^{\oplus 6} \\ \mathcal{W}(3)_2^{\oplus 4} \end{matrix} \rightleftharpoons \begin{matrix} \mathcal{W}(5)_3^{\oplus 4} \\ \mathcal{W}(4)_3^{\oplus 1} \end{matrix} \rightleftharpoons \mathcal{W}(0)_4^{\oplus 1}\,. \tag{10.22}$$

Reading off the brane-factor gives

$$f_{\mathcal{B}}^{D6} = 4e^{2\pi\sigma} - 5e^{4\pi\sigma} + 5e^{8\pi\sigma} - 4e^{10\pi\sigma}. \tag{10.23}$$

With the basis at hand we can begin calculating monodromies.

Large-Radius-Monodromy

A *LR*-monodromy can simply be done by shifting the charges of a basis brane complex by 1 and grade restricting back to the old window[6]. For the grade restriction one uses a *LR*-empty brane. After the grade restriction one can simply compare the brane factors and write down the desired monodromy-matrix. We will simply state the brane-factors of our transported basis, because the procedure for grade restriction should be familiar from previous sections. We get the following transported and restricted brane-factors:

[6] An explanation why this is a *LR*-monodromy is given in section 5.2.

$$f_{\mathcal{B}'}^{D0} = 3e^{2\pi\sigma} - 2e^{4\pi\sigma} - 2e^{6\pi\sigma} + 3e^{8\pi\sigma} - e^{10\pi\sigma} - 1 \tag{10.24}$$
$$f_{\mathcal{B}'}^{D2} = 4e^{2\pi\sigma} - 4e^{4\pi\sigma} - 2e^{6\pi\sigma} + 5e^{8\pi\sigma} - 2e^{10\pi\sigma} - 1 \tag{10.25}$$
$$f_{\mathcal{B}'}^{D4} = 12e^{2\pi\sigma} - 11e^{4\pi\sigma} - 5e^{6\pi\sigma} + 15e^{8\pi\sigma} - 7e^{10\pi\sigma} - 4 \tag{10.26}$$
$$f_{\mathcal{B}'}^{D6} = 16e^{2\pi\sigma} - 16e^{4\pi\sigma} - 5e^{6\pi\sigma} + 20e^{8\pi\sigma} - 11e^{10\pi\sigma} - 4. \tag{10.27}$$

Comparison with the previously chosen basis gives the following monodromy matrix

$$M_{LR} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 1 \end{pmatrix}. \tag{10.28}$$

Landau-Ginzburg-Monodromy

A LG-monodromy can be performed similar to a LR-monodromy, except that we shift the gauge charge by -1 and use the LG-empty-brane for grade restriction. Applying this procedure to our basis results in

$$f_{\mathcal{B}'}^{D0} = -2e^{2\pi\sigma} - 2e^{4\pi\sigma} + 3e^{6\pi\sigma} - e^{8\pi\sigma} - e^{10\pi\sigma} + 3 \tag{10.29}$$
$$f_{\mathcal{B}'}^{D2} = -2e^{2\pi\sigma} + 2e^{6\pi\sigma} - e^{8\pi\sigma} + 1 \tag{10.30}$$
$$f_{\mathcal{B}'}^{D4} = -5e^{2\pi\sigma} - 5e^{4\pi\sigma} + 9e^{6\pi\sigma} - 4e^{8\pi\sigma} - 4e^{10\pi\sigma} + 9 \tag{10.31}$$
$$f_{\mathcal{B}'}^{D6} = -5e^{2\pi\sigma} + 5e^{6\pi\sigma} - 4e^{8\pi\sigma} + 4. \tag{10.32}$$

By matching with the basis-brane-factors we get the following monodromy matrix

$$M_{LG} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 3 & -3 & -3 & 4 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \tag{10.33}$$

Conifold-Monodromy

Performing a conifold-monodromy is a more subtle task. At first one has to choose a path in the moduli space to transport the brane around the singularities. We choose the path sketched in fig. 10.1.

Let us describe the process of transporting a brane in an algorithmic way:

1. Restrict the brane to window $w2$ by using large-radius-empty-branes and transport the brane to the *Landau-Ginzburg*-phase

2. Use LG-empty-branes to restrict to window $w1$. The restricted brane can now be transported in a sensible way to the large-radius-phase

3. In the large-radius-phase use large-radius-empty-branes to restrict the brane back to $w2$

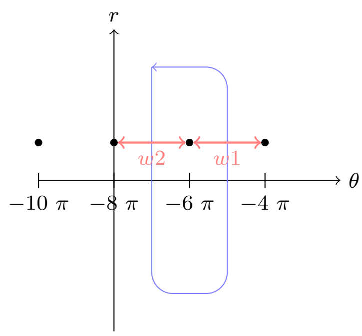

Figure 10.1: Path in the moduli space.

4. Compare brane-factors of the transported and untransported brane and read off the monodromy.

As before, we simply state the brane-factors of the transported branes

$$f_{\mathcal{B}'}^{D0} = -5e^{2\pi\sigma} + 8e^{4\pi\sigma} - 2e^{6\pi\sigma} - 7e^{8\pi\sigma} + 7e^{10\pi\sigma} - e^{12\pi\sigma} \tag{10.34}$$
$$f_{\mathcal{B}'}^{D2} = e^{2\pi\sigma} - 2e^{4\pi\sigma} + 2e^{8\pi\sigma} - e^{10\pi\sigma} \tag{10.35}$$
$$f_{\mathcal{B}'}^{D4} = -23e^{2\pi\sigma} + 35e^{4\pi\sigma} - 5e^{6\pi\sigma} - 31e^{8\pi\sigma} + 28e^{10\pi\sigma} - 4e^{12\pi\sigma} \tag{10.36}$$
$$f_{\mathcal{B}'}^{D6} = 4e^{2\pi\sigma} - 5e^{4\pi\sigma} + 5e^{8\pi\sigma} - 4e^{10\pi\sigma}. \tag{10.37}$$

The conifold-monodromy matrix is given by

$$M_C = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10.38}$$

Consistency Check

Monodromy matrices furnish a group. They should satisfy the following relation

$$M_{LG} M_{LR} M_C = \mathbb{1}. \tag{10.39}$$

By inserting eqs. (10.28), (10.33) and (10.38) we see that they obey this condition. Also M_{LG} has to have the following property

$$M_{LG}^6 = \mathbb{1}, \tag{10.40}$$

which is indeed fulfilled by eq. (10.33).

10.3 $\mathbb{P}[1,1,1,1,4][8]$

As before we have to fix windows. We choose

$$w1 : \theta \in (-8\pi, -6\pi) \qquad q \in (0,1,2,3,4,5,6,7) \tag{10.41}$$
$$w2 : \theta \in (-10\pi, -8\pi) \qquad q \in (1,2,3,4,5,6,7,8)\,. \tag{10.42}$$

As important as choosing a window is to set up a basis of branes consistent with the chosen window. As in section 10.2, we can set up our basis by using the branes given in eqs. (10.11) to (10.14).

Because the complex representing the branes can be obtained as in section 10.2 we will only state the brane-factors of the basis branes.

D0-brane

$$f_{\mathcal{B}}^{D0} = 3e^{2\pi\sigma} - 3e^{4\pi\sigma} + e^{6\pi\sigma} + e^{8\pi\sigma} - 3e^{10\pi\sigma} + 3e^{12\pi\sigma} - e^{14\pi\sigma} - 1 \tag{10.43}$$

D2-brane

$$f_{\mathcal{B}}^{D2} = e^{2\pi\sigma} - 2e^{4\pi\sigma} + 2e^{8\pi\sigma} - e^{10\pi\sigma} \tag{10.44}$$

D4-brane

$$f_{\mathcal{B}}^{D4} = 10e^{2\pi\sigma} - 10e^{4\pi\sigma} + 4e^{6\pi\sigma} + 4e^{8\pi\sigma} - 10e^{10\pi\sigma} + 10e^{12\pi\sigma} - 4e^{14\pi\sigma} - 4 \tag{10.45}$$

D6-brane

$$f_{\mathcal{B}}^{D6} = 4e^{2\pi\sigma} - 6e^{4\pi\sigma} + 4e^{6\pi\sigma} - 4e^{10\pi\sigma} + 6e^{12\pi\sigma} - 4e^{14\pi\sigma} \tag{10.46}$$

Having set up a basis we can now transport it.

Large-Radius-Monodromy

We give only the results for the transported branes. Details can be found in section 10.2.

$$f_{\mathcal{B}'}^{D0} = 3e^{2\pi\sigma} - 3e^{4\pi\sigma} + e^{6\pi\sigma} + e^{8\pi\sigma} - 3e^{10\pi\sigma} + 3e^{12\pi\sigma} - e^{14\pi\sigma} - 1 \tag{10.47}$$
$$f_{\mathcal{B}'}^{D2} = 4e^{2\pi\sigma} - 5e^{4\pi\sigma} + 2e^{6\pi\sigma} + e^{8\pi\sigma} - 4e^{10\pi\sigma} + 5e^{12\pi\sigma} - 2e^{14\pi\sigma} - 1 \tag{10.48}$$
$$f_{\mathcal{B}'}^{D4} = 12e^{2\pi\sigma} - 14e^{4\pi\sigma} + 6e^{6\pi\sigma} + 4e^{8\pi\sigma} - 12e^{10\pi\sigma} + 14e^{12\pi\sigma} - 6e^{14\pi\sigma} - 4 \tag{10.49}$$
$$f_{\mathcal{B}'}^{D6} = 16e^{2\pi\sigma} - 20e^{4\pi\sigma} + 10e^{6\pi\sigma} + 4e^{8\pi\sigma} - 16e^{10\pi\sigma} + 20e^{12\pi\sigma} - 10e^{14\pi\sigma} - 4. \tag{10.50}$$

Comparing with the basis brane-factors gives the following monodromy matrix

$$M_{LR} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{pmatrix}. \tag{10.51}$$

Landau-Ginzburg-Monodromy

Again, we will only state the results, given by

$$f_{\mathcal{B}'}^{D0} = -3e^{2\pi\sigma} + e^{4\pi\sigma} + e^{6\pi\sigma} - 3e^{8\pi\sigma} + 3e^{10\pi\sigma} - e^{12\pi\sigma} - e^{14\pi\sigma} + 3 \tag{10.52}$$
$$f_{\mathcal{B}'}^{D2} = -2e^{2\pi\sigma} + e^{4\pi\sigma} - e^{8\pi\sigma} + 2e^{10\pi\sigma} - e^{12\pi\sigma} + 1 \tag{10.53}$$
$$f_{\mathcal{B}'}^{D4} = -10e^{2\pi\sigma} + 4e^{4\pi\sigma} + 4e^{6\pi\sigma} - 10e^{8\pi\sigma} + 10e^{10\pi\sigma} - 4e^{12\pi\sigma} - 4e^{14\pi\sigma} + 10 \tag{10.54}$$
$$f_{\mathcal{B}'}^{D6} = -6e^{2\pi\sigma} + 4e^{4\pi\sigma} - 4e^{8\pi\sigma} + 6e^{10\pi\sigma} - 4e^{12\pi\sigma} + 4. \tag{10.55}$$

The monodromy matrix is given by

$$M_{LG} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & -3 & 4 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \tag{10.56}$$

Conifold-Monodromy

By performing the steps stated in section 10.2 we obtain the following brane-factors

$$f_{\mathcal{B}'}^{D0} = -5e^{2\pi\sigma} + 9e^{4\pi\sigma} - 7e^{6\pi\sigma} + e^{8\pi\sigma} + 5e^{10\pi\sigma} - 9e^{12\pi\sigma} + 7e^{14\pi\sigma} - e^{16\pi\sigma} \quad (10.57)$$

$$f_{\mathcal{B}'}^{D2} = e^{2\pi\sigma} - 2e^{4\pi\sigma} + e^{6\pi\sigma} - e^{10\pi\sigma} + 2e^{12\pi\sigma} - e^{14\pi\sigma} \quad (10.58)$$

$$f_{\mathcal{B}'}^{D4} = -22e^{2\pi\sigma} + 38e^{4\pi\sigma} - 28e^{6\pi\sigma} + 4e^{8\pi\sigma} + 22e^{10\pi\sigma} - 38e^{12\pi\sigma} + 28e^{14\pi\sigma} - 4e^{16\pi\sigma} \quad (10.59)$$

$$f_{\mathcal{B}'}^{D6} = 4e^{2\pi\sigma} - 6e^{4\pi\sigma} + 4e^{6\pi\sigma} - 4e^{10\pi\sigma} + 6e^{12\pi\sigma} - 4e^{14\pi\sigma}. \quad (10.60)$$

The result for the monodromy-matrix reads

$$M_C = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (10.61)$$

Consistency

Again our matrices (eqs. (10.51), (10.56) and (10.61) fulfil eq. (10.39). Also eq. (10.56) has the property

$$M_{LG}^8 = \mathbb{1}. \quad (10.62)$$

10.4 $\mathbb{P}[1,1,1,2,5][10]$

We fix the windows to

$$w1 : \theta \in (-10\pi, -8\pi) \qquad q \in (0,1,2,3,4,5,6,7,8,9) \quad (10.63)$$

$$w2 : \theta \in (-12\pi, -10\pi) \qquad q \in (1,2,3,4,5,6,7,8,9,10)\,. \quad (10.64)$$

By employing eqs. (10.11) to (10.14) we get the following results for our basis brane-factors.

*D*0*-brane*

$$f_{\mathcal{B}}^{D0} = 2e^{2\pi\sigma} - 2e^{6\pi\sigma} + e^{8\pi\sigma} + e^{10\pi\sigma} - 2e^{12\pi\sigma} + 2e^{16\pi\sigma} - e^{18\pi\sigma} - 1 \quad (10.65)$$

*D*2*-brane*

$$f_{\mathcal{B}}^{D2} = e^{2\pi\sigma} - e^{4\pi\sigma} - e^{6\pi\sigma} + e^{8\pi\sigma} - e^{12\pi\sigma} + e^{14\pi\sigma} + e^{16\pi\sigma} - e^{18\pi\sigma} \quad (10.66)$$

*D*4*-brane*

$$f_{\mathcal{B}}^{D4} = 5e^{2\pi\sigma} - 5e^{6\pi\sigma} + 3e^{8\pi\sigma} + 3e^{10\pi\sigma} - 5e^{12\pi\sigma} + 5e^{16\pi\sigma} - 3e^{18\pi\sigma} - 3 \quad (10.67)$$

D6-brane

$$f_{\mathcal{B}}^{D6} = 3e^{2\pi\sigma} - 2e^{4\pi\sigma} - 2e^{6\pi\sigma} + 3e^{8\pi\sigma} - 3e^{12\pi\sigma} + 2e^{14\pi\sigma} + 2e^{16\pi\sigma} - 3e^{18\pi\sigma} \tag{10.68}$$

Subsequently we will only give the results of the calculations, because they are analogous to section 10.2.

Large-Radius-Monodromy

Our transported brane-factors read

$$f_{\mathcal{B}'}^{D0} = 2e^{2\pi\sigma} - 2e^{6\pi\sigma} + e^{8\pi\sigma} + e^{10\pi\sigma} - 2e^{12\pi\sigma} + 2e^{16\pi\sigma} - e^{18\pi\sigma} - 1 \tag{10.69}$$
$$f_{\mathcal{B}'}^{D2} = 3e^{2\pi\sigma} - e^{4\pi\sigma} - 3e^{6\pi\sigma} + 2e^{8\pi\sigma} + e^{10\pi\sigma} - 3e^{12\pi\sigma} + e^{14\pi\sigma} + 3e^{16\pi\sigma} - 2e^{18\pi\sigma} - 1 \tag{10.70}$$
$$f_{\mathcal{B}'}^{D4} = 6e^{2\pi\sigma} - e^{4\pi\sigma} - 6e^{6\pi\sigma} + 4e^{8\pi\sigma} + 3e^{10\pi\sigma} - 6e^{12\pi\sigma} + e^{14\pi\sigma} + 6e^{16\pi\sigma} - 4e^{18\pi\sigma} - 3 \tag{10.71}$$
$$f_{\mathcal{B}'}^{D6} = 9e^{2\pi\sigma} - 3e^{4\pi\sigma} - 8e^{6\pi\sigma} + 7e^{8\pi\sigma} + 3e^{10\pi\sigma} - 9e^{12\pi\sigma} + 3e^{14\pi\sigma} + 8e^{16\pi\sigma} - 7e^{18\pi\sigma} - 3. \tag{10.72}$$

Comparison with the basis gives the following monodromy-matrix

$$M_{LR} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \tag{10.73}$$

Landau-Ginzburg-Monodromy

Doing a LG-monodromy results in

$$f_{\mathcal{B}'}^{D0} = -2e^{4\pi\sigma} + e^{6\pi\sigma} + e^{8\pi\sigma} - 2e^{10\pi\sigma} + 2e^{14\pi\sigma} - e^{16\pi\sigma} - e^{18\pi\sigma} + 2 \tag{10.74}$$
$$f_{\mathcal{B}'}^{D2} = -e^{2\pi\sigma} - e^{4\pi\sigma} + e^{6\pi\sigma} - e^{10\pi\sigma} + e^{12\pi\sigma} + e^{14\pi\sigma} - e^{16\pi\sigma} + 1 \tag{10.75}$$
$$f_{\mathcal{B}'}^{D4} = -5e^{4\pi\sigma} + 3e^{6\pi\sigma} + 3e^{8\pi\sigma} - 5e^{10\pi\sigma} + 5e^{14\pi\sigma} - 3e^{16\pi\sigma} - 3e^{18\pi\sigma} + 5 \tag{10.76}$$
$$f_{\mathcal{B}'}^{D6} = -2e^{2\pi\sigma} - 2e^{4\pi\sigma} + 3e^{6\pi\sigma} - 3e^{10\pi\sigma} + 2e^{12\pi\sigma} + 2e^{14\pi\sigma} - 3e^{16\pi\sigma} + 3, \tag{10.77}$$

and a monodromy-matrix given by

$$M_{LG} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & -2 & 3 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \tag{10.78}$$

Conifold-Monodromy

Under a conifold-monodromy the basis transforms to

$$f_{\mathcal{B}'}^{D0} = -4e^{2\pi\sigma} + 4e^{4\pi\sigma} + 2e^{6\pi\sigma} - 5e^{8\pi\sigma} + e^{10\pi\sigma} + 4e^{12\pi\sigma} - 4e^{14\pi\sigma} - 2e^{16\pi\sigma} + 5e^{18\pi\sigma} - e^{20\pi\sigma} \tag{10.79}$$
$$f_{\mathcal{B}'}^{D2} = e^{2\pi\sigma} - e^{4\pi\sigma} - e^{6\pi\sigma} + e^{8\pi\sigma} - e^{12\pi\sigma} + e^{14\pi\sigma} + e^{16\pi\sigma} - e^{18\pi\sigma} \tag{10.80}$$
$$f_{\mathcal{B}'}^{D4} = -13e^{2\pi\sigma} + 12e^{4\pi\sigma} + 7e^{6\pi\sigma} - 15e^{8\pi\sigma} + 3e^{10\pi\sigma} + 13e^{12\pi\sigma} - 12e^{14\pi\sigma} - 7e^{16\pi\sigma} + 15e^{18\pi\sigma} - 3e^{20\pi\sigma} \tag{10.81}$$
$$f_{\mathcal{B}'}^{D6} = 3e^{2\pi\sigma} - 2e^{4\pi\sigma} - 2e^{6\pi\sigma} + 3e^{8\pi\sigma} - 3e^{12\pi\sigma} + 2e^{14\pi\sigma} + 2e^{16\pi\sigma} - 3e^{18\pi\sigma}. \tag{10.82}$$

Comparing the old basis elements with the transported one gives

$$M_C = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10.83}$$

Consistency

As expected, the obtained matrices again fulfil eq. (10.39). Also M_{LG} (eq. (10.78)) fulfils

$$M_{LG}^{10} = \mathbb{1}. \tag{10.84}$$

10.5 Comparison of our Result with known Results

The monodromy matrices of the studied space have been calculated before by [5]. We want to remark that the authors of [5] used a rather different method, namely mirror symmetry, to obtain the monodromy matrices. By comparison with their result we noticed that the matrices calculated by us, are the inverse matrices of the results given in [5]. For completeness we give the transformation matrices from our basis to the basis used for calculating the matrices given in [5, pp.10-11] explicitly. The relations between the matrices are of the form

$$\begin{aligned} S &= U \left(M_{LR}^{-1}\right)^N U^{-1} \\ A &= U M_{LG}^{-1} U^{-1} \\ T &= U M_C^{-1} U^{-1}, \end{aligned} \tag{10.85}$$

where U is the transformation matrix. N is the homogeneous degree of the polynomial, whose vanishing locus gives the *Calabi-Yau* manifold.

$\mathbb{P}[1,1,1,1,2][6]$

The transformation matrix reads

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 4 & 3 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{10.86}$$

$\mathbb{P}[1,1,1,1,4][8]$

For this configuration we obtained

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 4 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{10.87}$$

$\mathbb{P}[1,1,1,2,5][10]$

To get the results of [5] one applies the following transformation matrix

$$U = \begin{pmatrix} 3 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{10.88}$$

11
Conclusion

In this thesis we discussed boundaries in $\mathcal{N} = (2,2)$ supersymmetric theories. After developing the necessary theoretical prerequisites, we used the hemisphere partition function to calculate central charges and monodromy matrices for D-branes in the gauged linear sigma model. Thereby we were able to reproduce the results for the monodromy matrices in [5], obtained by mirror symmetry.

A possible extension would be to consider monodromies in models with more than one *Kähler* parameter. This can be achieved by considering higher rank gauge groups. Higher rank abelian gauge groups where studied in [16] and in the context of the hemisphere partition function in [2]. In [2] also non abelian gauge groups where discussed. Of particular interest would be to find a interpretation of the hemisphere partition function in the *Landau-Ginzburg* phase.

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