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DIPLOMARBEIT

Twisted Warped Entanglement Entropy

Ausgeführt am Institut für

Theoretische Physik

der Technischen Universität Wien

unter der Anleitung von **Ass. Prof. Dr. Daniel Grumiller**

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12.12.2016

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Abstract

The aim of the thesis is to calculate the entanglement entropy of an interval in different two-dimensional warped conformal field theories. The result by Castro et al [5] is generalized to a second WCFT with a different symmetry algebra. This is done in two ways: First using the Rindler method and second using the replica trick. The new WCFT is particularly interesting because it appears as holographic dual of a boosted Rindler-spacetime. On the gravitational side, entanglement entropy is much easier to compute and I show that the results agree if one locates the field theory on the horizon at $r = 0$ rather then at $r \to \infty$. This statement is shown to be also true for the slightly more involved case of boosted Rindler-AdS.

Contents

1 Introduction

The holographic principle has been one the most exciting developments in theoretical physics in the last few decades. The statement is that a theory of gravity in a d-dimensional spacetime is equivalent to a $d-1$-dimensional conformal field theory which lives on the boundary of this spacetime. This connection between gravitational theories and quantum field theories could provide a better understanding of quantum gravity, which is maybe the most important unsolved problem in theoretical physics. One of the most famous realizations of the holographic principle is the *AdS(3)/CFT(2) correspondence*, which states that gravity in a 3-dimensional Anti-de-Sitter-space is equivalent to a relativistic conformal field theory in two dimensions. This choice of dimension is particularly interesting for two reasons: First of all, gravity in three dimensions has no local degrees of freedom and all information about the state of the system is contained in topological properties. Furthermore, it can be formulated as a *Chern-Simons-gauge theory*, so all the well-known formalisms of gauge theories can be used. This means that gravity in three dimensions is simple enough that many problems can be treated analytically. On the other hand it is complicated enough to admit interesting features like black holes and boundary gravitons. Secondly, conformal field theory is also special in two dimensions since the symmetry algebra then becomes infinite-dimensional. This large amount of symmetries makes the theory more accessible and enables us to do a lot of analytical computations.

There have been many attempts to generalize the holographic principle to different spacetimes with dual field theories which are not relativistic CFTs. In this thesis I will investigate so-called *warped conformal field theories* which do not have a symmetry group of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ but rather $SL(2,\mathbb{R}) \times U(1)$. The associated algebra admits two qualitatively different central extensions which I will introduce in detail in section 3.6. The quantity that I will calculate is the *entanglement entropy* of an interval, a measure for how "strongly entangled" the interval is with its complement. Entanglement entropy is not only interesting to its own right but also provides a popular check of the holographic theorem since it can be calculated on both sides. In my thesis, the main focus lies on the field theory side where I will derive a Cardy-type formula for the entanglement entropy following the paper by A. Castro, D. Hofman and N. Iqbal [5]:

$$S_A = -4\ln\left[\frac{L}{\pi\epsilon}\sin\frac{\pi l}{L}\right] L_0^{vac} + il\left(\frac{\overline{L}}{L} - \frac{\bar{l}}{l}\right) P_0^{vac} \tag{1}$$

While in [5] this formula was only derived for one of the two possible symmetry algebras, I will show that it holds for both.

Despite the fact that warped conformal field theories are interesting to study from a purely mathematical point of view, they appear as holographic duals of near-horizon approximations of non-extremal black holes. As argued in [10] such a near-horizon limit is given by the *boosted Rindler spacetime*

$$ds^2 = -2a(u)r\,du^2 - 2dr\,du + 2\eta(u)du\,dx + dx^2 \tag{2}$$

whose asymptotic symmetry algebra is exactly one of the warped CFT algebras. On the gravity side, entanglement entropy is usually much easier to compute since it only involves the length of a geodesic. However, in this case it is a priori not clear where to locate the field theory because $r \to \infty$ as well as $r = 0$ are possible options. I will argue that after choosing $r = 0$ the result for the entanglement entropy in the ground state indeed agrees with the result on the field theory side. Furthermore, equality also holds for the slightly more complicated case of boosted Rindler-AdS.

The work is organized as follows: In the second chapter I give some basic information on entanglement entropy in general. In the third chapter I introduce conformal field theories in two dimensions, first of all the ordinary relativistic CFTs and then I specialize to warped CFTs. In the fourth chapter I calculate the entanglement entropy of an interval using the Rindler method for both algebras and in the fifth chapter I repeat this calculation using the twist field method. The sixth chapter is devoted to the dual bulk theory where the basics of boosted Rindler space are explained and the gravitational calculations are performed.

I will use natural units with $c = \hbar = k_B = 1$ and the Einstein sum convention throughout the thesis.

2 Entanglement Entropy

Entanglement entropy is a very important quantity to measure "how strongly entangled" a quantum mechanical system is. If we have a system described by a density operator ρ, then the expectation value of the von-Neumann-entropy can be calculated to

$$S = -Tr(\rho \ln \rho). \tag{3}$$

Now consider the case if the system consists of two subsystems A and B and the Hilbert space can be written as a direct product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Then the *reduced density operator* ρ_A is defined by tracing out the subsystem B: $\rho_A = \sum\limits_{\psi \in \mathcal{H}_B} \langle\psi| \rho |\psi\rangle$ and the *entanglement entropy* is equal to the von-Neumann-entropy of the reduced density operator,

$$S_A = -Tr(\rho_A \ln \rho_A). \tag{4}$$

S_A has a few important features which make it a good candidate for measuring the entanglement between A and B: First of all, it is invariant under unitary transformations $\rho_A \to U \rho_A U^{-1}$ which typically means independent of the observer, as we will see later. Secondly, if ρ_A describes a pure state i. e. $\rho_A^2 = \rho_A$, we expect S_A to vanish because a pure state cannot be entangled. This is indeed the case:

$$S_A = -Tr(\rho_A \ln \rho_A) = -Tr(\rho_A \ln \rho_A^2) = -2Tr(\rho_A \ln \rho_A) = 0 \tag{5}$$

Thirdly, at zero temperature also the equality $S_A = S_B$ holds.

In field theories, entanglement entropy is usually quite difficult to calculate and there exist only a few analytical results. Because of the infinitely many degrees of freedom, S diverges and one has to introduce an ultraviolet cutoff e. g. in form of a lattice spacing a. It has been proven that in $2+1$ or higher dimensional systems the continuum limit is given by

$$S_A \sim \frac{Area(\partial A)}{a^{d+1}} + ... \tag{6}$$

where ∂A denotes the boundary of the subsytem A. This looks very similar to the familiar Bekenstein-Hawking law which gives the entropy of a black hole in terms of its horizon area

$$S_{BH} = \frac{A}{4G_N} \tag{7}$$

and therefore motivates a holographic description of entanglement entropy.

There is another notion of entanglement entropy which will become useful in later calculations which is called the *Renyi entropy*

$$S_n^{Ren} = \frac{1}{1-n} \ln(Tr\, \rho_A^n) \tag{8}$$

with index n. Since this expression is defined for all natural numbers n, it can be analytically continued to any real n. The limit $n \to 1^+$ can be obtained by using de l´Hospital´s rule and yields

$$\begin{aligned} S_1^{Ren} &= -\lim_{n\to 1} \partial_n \ln(Tr\, \rho_A^n) = -\lim_{n\to 1} \partial_n \ln(Tr\, e^{n \ln \rho_A}) \\ &= -\lim_{n\to 1} \frac{Tr(\ln(\rho_A)\rho_A^n)}{Tr\, \rho_A^n} = -Tr(\ln(\rho_A)\rho_A) = S_A \end{aligned} \tag{9}$$

The relation also holds after a small modification:

$$-\lim_{n\to 1} \partial_n (Tr\, \rho_A^n) = -\lim_{n\to 1} \partial_n (Tr\, e^{n \ln \rho_A}) = -\lim_{n\to 1} Tr(\ln(\rho_A)\rho_A^n) = S_A \tag{10}$$

This way to compute S_A is often advantageous because it does not involve a logarithm anymore but just powers of ρ_A. It will be used as part of the *replica trick* in the section about twist fields.

3 Warped Conformal Field Theories

3.1 Basics of Conformal Field Theories

This chapter serves as a short introduction into Conformal Field Theory (CFT) in general to recapitulate the most important concepts and to specify the notation (see [1]). The defining property of a CFT is that it is invariant under conformal transformations. A conformal transformation is a diffeomorphism which preserves the metric up to a scale factor, i. e. a coordinate transformation which fulfills

$$g_{\mu\nu}(x') \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} = \Lambda(x) g_{\rho\sigma}(x) \tag{11}$$

with $\Lambda(x) > 0$. If one considers infinitesimal transformations $x'^{\mu} = x^{\mu} + \epsilon^{\mu}$ this reduces to the *Conformal Killing Equation*

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = (\Lambda - 1)g_{\mu\nu} \tag{12}$$

We will be mainly interested in the two-dimensional Euclidean (or, after Wick rotation, Lorentzian) plane where $g_{\mu\nu} = \delta_{\mu\nu} = const.$ After taking the trace and plugging in again the conformal Killing equation then can be written as

$$\partial_0\epsilon_0 = \partial_1\epsilon_1 \qquad \partial_0\epsilon_1 = -\partial_1\epsilon_0. \tag{13}$$

These are just the *Cauchy-Riemann-Equations,* which is important for two reasons: Firstly, it means that we can use the powerful framework of complex analysis and secondly it implies that the algebra of conformal symmetries is infinite dimensional in 2d. One can now introduce complex coordinates $z = x^0 + ix^1$, $\overline{z} = x^0 - ix^1$ and $\partial_z = \frac{1}{2}(\partial_0 - i\partial_1)$, $\partial_{\overline{z}} = \frac{1}{2}(\partial_0 + i\partial_1)$ and treat $\epsilon(z) = \epsilon^0 + i\epsilon^1$ as a complex function. Therefore, $\epsilon(z)$ induces an infinitesimal conformal transformation iff it is holomorphic. It is now also quite easy to determine the scaling factor: If z goes to $f(z) = z + \epsilon(z)$ and $\overline{z}$ likewise, then

$$ds^2 = dz\, d\overline{z} \to \frac{\partial f}{\partial z}\frac{\partial \overline{f}}{\partial \overline{z}} dz\, d\overline{z} \tag{14}$$

and $\Lambda = \mid \frac{\partial f}{\partial z} \mid^2$.

3.2 Virasoro algebra

To determine the conformal symmetry algebra we first of all have to define an appropriate basis. Every holomorphic function can be expanded in a Taylor series, however, we shall also include functions which are only locally holomorphic and have poles elsewhere. We therefore write $\epsilon(z)$ as a Laurent series

$$\epsilon(z) = \sum_{n=-\infty}^{\infty} \epsilon_n(-z^{n+1}). \tag{15}$$

The generators of conformal transformations can now be read off as $L_n = (-z^{n+1})\partial_z$ and have the following commutator algebra called the *Witt algebra*:

$$[L_n, L_m] = (n - m)L_{n+m} \tag{16}$$

We shall also allow for a *central extension* of this algebra, which is necessary to have non-trivial representations. This central extension is unique up to trivial redefinitions of the generators and reads

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{m+n} \tag{17}$$

with *central charge c*. The form of the central extension can be calculated by writing out the various Jacobi identities, see [1]. This central extended algebra is called the *Virasoro algebra* and will be of great importance throughout the thesis. For the antiholomorphic sector the above considerations are the same and one additionally obtains

$$[\overline{L}_n, \overline{L}_m] = (n-m)\overline{L}_{n+m} + \frac{c}{12}(n^3-n)\delta_{m+n}$$
$$[L_n, \overline{L}_m] = 0 \tag{18}$$

The fact that the two copies of the Virasoro algebra commute suggests that the holomorphic and antiholomorphic coordinates z and $\overline{z}$ can be somehow treated as independent variables. Of course, one should still have in mind that in truth they are complex conjugated to each other.

Since we have included meromorphic functions, not all of the generators are defined globally: In fact, only L_{-1}, L_0, L_1(and $\overline{L}_{-1}, \overline{L}_0, \overline{L}_1$) are well defined on the whole Riemann sphere $\mathbb{C} \cup \{\infty\}$ and therefore generate global conformal transformations. As can be seen from the definition, L_{-1} generates translations, L_0 generates rotations and dilatations and L_1 generates special conformal transformations, which are a mix of inversions and translations. The whole global conformal group is isomorphic to the *Moebius group* which consists of transformations

$$z \to \frac{az+b}{cz+b} \tag{19}$$

with $a, b, c, d \in \mathbb{C}$ and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$, i. e. $SL(2, \mathbb{C})$. One should note that the central extension vanishes for those generators; this is related to global conformal invariance of the vacuum state.

3.3 Conformal fields and conserved currents

Let us now investigate the behaviour of field operators under conformal transformations: A conformal field Φ is called *primary* if it transforms as

$$\Phi'(z, \overline{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \overline{f}}{\partial \overline{z}}\right)^{\overline{h}} \Phi(f(z), \overline{f}(\overline{z})) \tag{20}$$

where h and $\overline{h}$ are called the *conformal weights* of Φ. If this only holds for global conformal transformations it is called *quasi-primary.* When the transformation is infinitesimal, one gets after Taylor-expanding $\Phi(z+\epsilon, \overline{z}+\overline{\epsilon})$:

$$\Phi'(z, \overline{z}) = (1 + h\partial_z\epsilon(z) + \epsilon(z)\partial_z + \overline{h}\partial_{\overline{z}}\overline{\epsilon}(\overline{z}) + \overline{\epsilon}(\overline{z})\partial_{\overline{z}})\Phi(z, \overline{z}) \tag{21}$$

In many cases we will have to do with purely holomorphic fields $\Phi(z)$, so in the upcoming sections I often will suppress the $\overline{z}$-dependence since it usually does not contain any new information. A conformal field with $h = 1$ is called a *current.* Due to Noether´s theorem, there exist conserved currents j^μ with

$\partial_\mu j^\mu = 0$ associated to conformal symmetry. Since we have written the conformal symmetries as infinitesimal translations $x^\mu \to x^\mu + \epsilon^\mu$ they can be expressed as

$$j_\mu = T_{\mu\nu}\epsilon^\nu \tag{22}$$

with $T_{\mu\nu}$ the *energy-momentum-tensor.* $T_{\mu\nu}$ is symmetric, conserved, and, in addition, traceless, which can be seen as follows: For constant ϵ^μ we have

$$\partial_\mu(T^{\mu\nu}\epsilon_\nu) = \partial_\mu T^{\mu\nu}\epsilon_\nu = 0 \Rightarrow \partial_\mu T^{\mu\nu} = 0 \tag{23}$$

and for x-dependent ϵ^μ

$$\begin{aligned}\partial_\mu(T^{\mu\nu}\epsilon_\nu) = T^{\mu\nu}\partial_\mu\epsilon_\nu &= \frac{1}{2}T^{\mu\nu}(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) \\ &= \frac{1}{2}T^{\mu\nu}(\Lambda - 1)g_{\mu\nu} = 0 \Rightarrow T^\mu{}_\mu = 0\end{aligned} \tag{24}$$

where the conformal Killing equation has been used. If one goes over to the complex coordinates z and $\overline{z}$ there are two non-vanishing components T_{zz} and $T_{\overline{zz}}$ which are because of $\partial_\mu T^{\mu\nu} = 0$ purely (anti-)holomorphic and denoted by $T(z)$ and $\overline{T}(\overline{z})$.

For many applications it is necessary to put a CFT on a cylinder where the spatial axis is compactified: $x^1 \sim x^1 + 2\pi$. [1] This can be achieved by the conformal mapping $z \to e^z$, which maps the spatial axis to concentrical circles and time evolution now goes radially outwards. If we now want to define conserved charges, the integral has to run over such a circle, i. e.

$$Q(\epsilon) = \frac{1}{2\pi i}\oint dz\, T(z)\epsilon(z) + anti - hol. \tag{25}$$

However, this equation should be regarded with care because the fields involved transform non-trivially under $z \to e^z$. As it will be shown later, $T(z)$ transforms in a way that the powers of $\left(\frac{dz'}{dz}\right)$ cancel away and what remains is an anomaly proportional to the central charge which can be absorbed in $T(z)$.

Usually in quantum theories, the conserved charge is the generator of the associated symmetry, so by taking $\epsilon(z) = z^{n+1}$ there should exist a correspondence $L_n = Q(\epsilon = z^{n+1}) \equiv Q_n$. This is interesting because the Q_n are just the Laurent modes of $T_{\mu\nu}$, hence we can establish the following useful relations:

$$L_n = \frac{1}{2\pi i}\oint dz\, T(z) z^{n+1} \quad \text{and} \quad T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2} \tag{26}$$

[1] One should notice that this is not a compactification in the topological sense where a manifold is a subset of its compactification; here we have the set of eqivalence classes $x^1 \sim x^1 + 2\pi$

3.4 Operator product expansion (OPE)

The *operator product expansion* is a kind of algebraic structure on the space of conformal fields and can be motivated as follows: After the mapping discussed above the origin represents the inifinite past, that means that incoming states can be constructed by conformal primaries acting on the vacuum state, i. e.

$$|\phi\rangle = \phi(0)\,|0\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})\,|0\rangle\,. \tag{27}$$

This relation is called the *operator-state-correspondence.* If we now perform a Laurent expansion of the field

$$\phi(z) = \sum_n z^{-n-h}\phi_n \tag{28}$$

(for simplicity I assume ϕ to be chiral) one sees that in the limit $z \to 0$ the modes with $n > -h$ create singularities. This is not what we expect of well-behaved asymptotic states, so we require that those modes *annihilate* the vacuum. The relation above can therefore be written as

$$|\phi\rangle = \lim_{z\to 0}\phi(z)\,|0\rangle = \phi_{-h}\,|0\rangle \tag{29}$$

If we now have a product of two conformal fields, it should automatically be time-ordered which means *radially ordered* in our theory: $R\,O_i(z)O_j(w) = O_i(z)O_j(w)$ if $\mid z \mid > \mid w \mid$ and vice versa. Their action on the vacuum now gives a state of the Hilbert space, which could also have been created at the earlier time ($\mid w \mid$ in the previous case) by a (in general highly non-trivial) linear combination of operators. So it should be possible to write

$$R\,O_i(z)O_j(w) = \sum_n c_{ij}^n(z-w)O_n(w) \tag{30}$$

where the set $O_n(w)$ has to be complete (more details can be found in [2]). Of particular interest is the OPE of a conformal primary field $\Phi(z)$ with $T(z)$: From the above considerations we deduce that the variation of a field under an infinitesimal conformal transformation is given by

$$\delta_\epsilon\Phi(w) = [Q(\epsilon),\Phi(w)] = \frac{1}{2\pi i}\oint dz\,\epsilon(z)[T(z),\Phi(w)]. \tag{31}$$

The commutator can be rewritten as a radial ordered product:

$$\delta_\epsilon\Phi(w) = \frac{1}{2\pi i}\left(\oint_{|z|>|w|} dz - \oint_{|z|<|w|} dz\right)\epsilon(z)R\,T(z)\Phi(w) \tag{32}$$

The difference of the two contour integrals is a contour integral around w [2, 1] so we have

$$\delta_\epsilon\Phi(w) = \frac{1}{2\pi i}\oint_{C(w)} dz\,\epsilon(z)R\,T(z)\Phi(w) \tag{33}$$

This result has to be equal to that from chapter 3.3, $\delta_\epsilon \Phi(w) = (h\partial_w \epsilon(w) + \epsilon(w)\partial_w)\Phi(w)$, which is only possible if

$$RT(z)\Phi(w) = \frac{h}{(z-w)^2}\Phi(w) + \frac{1}{z-w}\partial_w \Phi(w) + \text{regular terms} \qquad (34)$$

To show this one has to expand $\epsilon(z)$ in a Taylor series and use Cauchy´s residue theorem [2]. One should notice that in an OPE the singular terms are most interesting because they determine the short-distance behaviour and the regular terms vanish inside a contour integral. The last equation can also serve as an equivalent definition of a conformal primary field. With a similar calculation we can also establish the relation

$$[L_m, \Phi_n] = ((h-1)m - n)\,\Phi_{m+n} \qquad (35)$$

for the corresponding Laurent modes.

To end this section we state that the OPE of $T(z)$ with itself is given by

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}, \qquad (36)$$

a proof can be found in [1]. Consequently, $T(z)$ is conformal primary with $h = 2$ only for $c = 0$, otherwise it transforms anomalously. The infinitesimal transformation law can be computed to

$$\begin{aligned} \delta_\epsilon T(w) &= \frac{1}{2\pi i} \oint_{C(w)} dz\, \epsilon(z) R T(z) T(w) \\ &= \frac{c}{12}\partial_w^3 \epsilon(w) + 2\partial_w \epsilon(w) T(w) + \epsilon(w)\partial_w T(w) \end{aligned} \qquad (37)$$

To integrate this one has to perform some non-trivial steps described in [3] to obtain

$$T´(w) = \left(\frac{\partial z}{\partial w}\right)^2 T(z) + \frac{c}{12}\{z, w\} = \left(\frac{\partial z}{\partial w}\right)^2 T(z) + \frac{c}{12}\left(\frac{\frac{\partial^3 z}{\partial w^3}}{\frac{\partial z}{\partial w}} - \frac{3}{2}\left(\frac{\frac{\partial^2 z}{\partial w^2}}{\frac{\partial z}{\partial w}}\right)^2\right). \qquad (38)$$

for finite transformations. The anomalous term $\{z, w\}$ is called the *Schwarzian derivative* and will also appear later in some modifications.

3.5 Conformal Ward identity and n-point functions

The fundamental objects in every quantum field theory are n-point correlation functions and, remarkably, in CFTs the 2- and 3-point function are fixed up to a constant only by employing the symmetries. For example consider a general 2-point function of two holomorphic quasi-primary fields $\langle \Phi_i(z)\Phi_j(w)\rangle = f(z, w)$. Translational invariance implies that $f(z, w) = f(z - w)$. Invariance under dilatations gives the condition

$$f(z-w) = \lambda^{h_i}\lambda^{h_j} f(\lambda(z-w)) \Rightarrow f = \frac{c_{ij}}{(z-w)^{h_i+h_j}} \qquad (39)$$

with some constant c_{ij}. At last, invariance under special conformal transformations fixes $h_i = h_j$, see [1], so that the final result becomes

$$\langle \Phi_i(z)\Phi_j(w)\rangle = \frac{c_{ij}\,\delta_{h_i,h_j}}{(z-w)^{2h_i}} \tag{40}$$

The 3-point function can be fixed similarly to

$$\langle \Phi_i(z)\Phi_j(w)\Phi_k(u)\rangle = \frac{c_{ijk}}{(z-w)^{h_i+h_j-h_k}(w-u)^{h_j+h_k-h_i}(z-u)^{h_k+h_i-h_j}}. \tag{41}$$

For higher n-point function one can derive a *Ward identity* as follows: Consider the expression

$$\begin{aligned}
\frac{1}{2\pi i}\oint dz\,\epsilon(z)\,\langle T(z)\Phi_1(w_1)\Phi_2(w_2)...\Phi_n(w_n)\rangle &= \\
\left\langle \frac{1}{2\pi i}\oint dz\,\epsilon(z)T(z)\Phi_1(w_1)\Phi_2(w_2)...\Phi_n(w_n)\right\rangle &= \\
\sum_{i=1}^{n}\left\langle \Phi_1(w_1)...\frac{1}{2\pi i}\oint\limits_{C(w_i)} dz\,\epsilon(z)T(z)\Phi_i(w_i)...\Phi_n(w_n)\right\rangle &= \\
\sum_{i=1}^{n}\left\langle \Phi_1(w_1)...\frac{1}{2\pi i}\oint\limits_{C(w_i)} dz\,\epsilon(z)\left(\frac{h_i}{(z-w_i)^2}+\frac{1}{z-w_i}\partial_{w_i}\right)\Phi_i(w_i)...\Phi_n(w_n)\right\rangle &= \\
\frac{1}{2\pi i}\oint dz\,\epsilon(z)\sum_{i=1}^{n}\left(\frac{h_i}{(z-w_i)^2}+\frac{1}{z-w_i}\partial_{w_i}\right)\langle\Phi_1(w_1)...\Phi_n(w_n)\rangle &
\end{aligned} \tag{42}$$

for primary fields $\Phi_i(w_i)$ where in the third line the contour integral has been deformed, see [1] and then the OPE has been used. Since $\epsilon(z)$ is arbitrary, the integrands have to be identical, so

$$\langle T(z)\Phi_1(w_1)\Phi_2(w_2)...\Phi_n(w_n)\rangle = \sum_{i=1}^{n}\left(\frac{h_i}{(z-w_i)^2}+\frac{1}{z-w_i}\partial_{w_i}\right)\langle\Phi_1(w_1)...\Phi_n(w_n)\rangle\,. \tag{43}$$

This conformal Ward identity contains all the information about conformal symmetry and will be used later in the section about *twist fields.*

3.6 The Warped Conformal Field Theory algebra

A *Warped Conformal Field Theory* (WCFT) differs from a conventional CFT in the point that the symmetry algebra is not two copies of the Virasoro algebra but a *Virasoro-Kac-Moody-algebra.* That means that the group of global transformations is not $SL(2,\mathbb{C})$ but $SL(2,\mathbb{R})\times U(1)$. The theory is formulated firstly on a plane described by two coordinates z and w which will be treated as complex coordinates to employ the advantages of complex analysis (analogously to z and $\overline{z}$ before). A WCFT is now per definition invariant under global coordinate transformations

$$z = f(z') \text{ and } w = w' + g(z'). \tag{44}$$

So, as before, z can be transformed by a holomorphic function but for w only *z-dependent translations* are allowed. The symmetries are generated by an energy-momentum-operator $T(z)$ (of course only the holomorphic component) and a *current operator* $P(z)$ responsible for the translations. They can be decomposed in Laurent modes according to

$$L_n = \frac{1}{2\pi i}\oint dz\, T(z) z^{n+1} \tag{45}$$

$$P_n = -\frac{1}{2\pi}\oint dz\, P(z) z^{n} \tag{46}$$

and generate an algebra which reads in its simplest form

$$\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m}\\
&[L_n, P_m] = -mP_{n+m}\\
&[P_n, P_m] = 0.
\end{aligned} \tag{47}$$

This special symmetry algebra admits at most three central extensions which are compatible with the Jacobi-identities:

$$\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{m+n}\\
&[L_n, P_m] = -mP_{m+n} + ik(n^2+n)\delta_{m+n}\\
&[P_n, P_m] = K\frac{n}{2}\delta_{m+n}
\end{aligned} \tag{48}$$

The first one is the already familiar Virasoro extension but the other two are interesting and provide the main motivation for this thesis. An important point is that for $K \neq 0$ one can eliminate the second extension by quite a simple redefinition of the generators: First consider the shift

$$\overline{P}_n = P_n + ik\delta_n \tag{49}$$

k is a central element so this change only contributes to the right-hand side of the second equation and removes the linear term in n:

$$[L_n, \overline{P}_m] = -m\overline{P}_{m+n} + ikn^2\delta_{m+n} \tag{50}$$

Now define

$$\overline{L}_n = L_n - i\frac{2k}{K}nP_n \tag{51}$$

so that the second commutator becomes

$$[\overline{L}_n, \overline{P}_m] = -m\overline{P}_{m+n} + ikn^2\delta_{m+n} - i\frac{2k}{K}n[P_n, \overline{P}_m] = -m\overline{P}_{m+n}. \tag{52}$$

After a short algebraic calculation one can also show that the $\overline{L}_n$s also satisfy a Virasoro algebra but with a different central charge, $[\overline{L}_n, \overline{L}_m] = (n-m)\overline{L}_{n+m} + \frac{\bar{c}}{12}(n^3 - n)\delta_{m+n}$. That means, that it should suffice to work on the case $k = 0$ as it has been done in [4, 5]. However, if $K = 0$, the redefinition above is not possible anymore and the algebra

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{m+n} \\
[L_n, P_m] &= -mP_{m+n} + ik(n^2 + n)\delta_{m+n} \\
[P_n, P_m] &= 0
\end{aligned}
\tag{53}
$$

is non-trivial.[2] It will be the main task of the thesis to determine entanglement entropy in a theory with this special algebra generalizing the calculations of [5] to this case.

3.7 Geometrical aspects of WCFTs

WCFTs exhibit quite a strange geometric structure, which shall be analyzed in this chapter [5]. First of all, for most of the applications we want to put the theory on a cylinder described by the two coordinates x and t. It can be mapped to the plane by the warped conformal transformation

$$
z = e^{ix} \text{ and } w = t + 2\alpha x. \tag{54}
$$

with the constant tilt parameter α. The symmetry algebra remains the same under this transformation up to a linear shift of the zero modes L_0 and P_0, which can be absorbed in the definition and does not influence the results of this chapter (see 4.5 for more information). The invariance under warped conformal transformations has important consequences for the geometry, e. g. our theory should be invariant under the linear transformation $t \to t + vx$ or in matrix notation

$$
\begin{pmatrix} x \\ t \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \quad \text{or} \quad x^a \to \Lambda^a{}_b x^b. \tag{55}
$$

There obviously exist a vector $\bar{q}^a = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and a one-form $q_a = \begin{pmatrix} 1 & 0 \end{pmatrix}$ which are preserved under this transformation. The fixed points are exactly those which fulfill $x = 0$ so we conclude that the t-axis is *preferred* because of the symmetry structure. We can also find two invariant tensors namely a degenerate metric $g_{ab} = q_a q_b \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and a symplectic structure $h_{ab} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The norm defined with respect to g_{ab} only measures the length in x-direction. To investigate the whole symmetry algebra it is useful to introduce the one-form $J_{a,n} = \begin{pmatrix} L_n & P_n \end{pmatrix}$. The non-extended algebra can then be written as

$$
[J_{a,n}, J_{b,m}] = \frac{n-m}{2}(q_a J_{b,n+m} + q_b J_{a,n+m}) - \frac{n+m}{2}(q_a J_{b,n+m} - q_b J_{a,n+m}) \tag{56}
$$

[2] In the following, we shall always assume that if $K \neq 0$, then $k = 0$ and vice-versa.

By using the tensors defined above it is possible to include the first two central extensions:

$$[J_{a,n}, J_{b,m}] = \frac{n-m}{2}(q_a J_{b,n+m} + q_b J_{a,n+m}) - \frac{n+m}{2}(q_a J_{b,n+m} - q_b J_{a,n+m}) + \frac{c}{12} g_{ab}(n^3 - n)\delta_{n+m} + ikh_{ab}(n^2 + n)\delta_{n+m} \tag{57}$$

However, for the third one it is necessary to introduce a second one-form $\overline{q}_a = \begin{pmatrix} 0 & 1 \end{pmatrix}$ which gives rise to a second degenerate metric $\overline{g}_{ab} = \overline{q}_a \overline{q}_b \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ to write

$$[J_{a,n}, J_{b,m}] = \frac{n-m}{2}(q_a J_{b,n+m} + q_b J_{a,n+m}) - \frac{n+m}{2}(q_a J_{b,n+m} - q_b J_{a,n+m}) + \frac{c}{12} g_{ab}(n^3 - n)\delta_{n+m} + ikh_{ab}(n^2 + n)\delta_{n+m} + \frac{K}{2}\overline{g}_{ab} n \delta_{n+m}. \tag{58}$$

This observation is crucial because it means that for $K \neq 0$ there exists a *second preferred axis* namely the x-axis, which consists of null-vectors with respect to $\overline{g}_{ab}$.

4 The Rindler method

4.1 Basic ideas

We turn now to the actual aim of the thesis: Calculating the entanglement entropy of an interval A in a $1+1$ dimensional WCFT. The basic idea of the Rindler method seems quite natural since it consists of performing a warped conformal transformation from the whole cylinder to the "interior" of the interval A and then evaluate a thermal entropy via the partition function. Following [5], the background should be a spacetime cylinder described by two coordinates T and X which correspond to the two preferred axes of the last chapter. In the most general case, the cylinder identification is oriented arbitrarily so we write

$$(T, X) \sim (T + \overline{L}, X - L). \tag{59}$$

The interval A also does not necessarily have to be aligned with the coordinates or the cylinder and can be denoted as

$$(T, X) \in \left[\left(\frac{\overline{l}}{2}, -\frac{l}{2} \right), \left(-\frac{\overline{l}}{2}, \frac{l}{2} \right) \right]. \tag{60}$$

So, what do we mean by the "interior" of the interval? In a relativistic theory it would be given by the Rindler wedge (see [5]), bounded by the future and past light-cone, which is of course independent of the observer. In a WCFT, we shall also use the preferred axis of the theory, which is in this case the T-axis, as the boundary of the interior region. So we look for a transformation which maps the whole cylinder to the strip between the straight lines $X = -\frac{l}{2}$ and

$X = \frac{l}{2}$. One candidate which has the necessary structure of a warped conformal transformation is given by

$$\frac{\tan\frac{\pi X}{L}}{\tan\frac{\pi l}{2L}} = \tanh\frac{\pi x}{\kappa} \quad \text{and} \quad T + \frac{\overline{L}}{L}X = t + \frac{\overline{\kappa}}{\kappa}x \tag{61}$$

with two arbitrary scaling factors κ and $\overline{\kappa}$. It maps, as desired, the interval $[-\frac{l}{2}, \frac{l}{2}]$ to the whole real axis and respects the cylinder identification which is of course not visible anymore in the x, t-coordinates. However, a different identification appears naturally in the new coordinates, namely

$$(t, x) \sim (t - i\overline{\kappa}, x + i\kappa). \tag{62}$$

It can be seen as a thermal identification with potentials $\overline{\kappa}$ and κ, which allows us to calculate a thermal entropy S_{th}. Since the entanglement entropy is independent of the observer as stated above, we expect that it equals this thermal entropy,

$$S_A = S_{th}. \tag{63}$$

4.2 Handling of divergencies

There is one issue which has to be discussed concerning the last equation and is related to the divergence structure of both sides: S_A is ultraviolet divergent as already stated while S_{th} is infrared divergent because of the infinite extension in x. To relate these divergencies, we introduce a cutoff parameter ϵ and change the interval A to

$$(T, X) \in \left[\left(\frac{\bar{l}}{2} - \frac{\bar{l}}{l}\epsilon, -\frac{l}{2} + \epsilon\right), \left(-\frac{\bar{l}}{2} + \frac{\bar{l}}{l}\epsilon, \frac{l}{2} - \epsilon\right)\right]. \tag{64}$$

If we plug this in into the transformation above we see, using

$$\tan\frac{\pi\left(-\frac{l}{2} + \epsilon\right)}{L} \approx \tan\left(-\frac{\pi l}{2L}\right) + \epsilon\frac{\pi}{L\cos^2(-\frac{\pi l}{2L})}, \tag{65}$$

that the lower bound of X gets mapped to

$$\begin{aligned}
x &\approx \frac{\kappa}{\pi}\text{artanh}\left(\frac{\tan(-\frac{\pi l}{2L})}{\tan\frac{\pi l}{2L}} + \epsilon\frac{\pi}{L\cos^2(-\frac{\pi l}{2L})\tan\frac{\pi l}{2L}}\right) \\
&= \frac{\kappa}{\pi}\text{artanh}\left(-1 + \epsilon\frac{\pi}{L\cos\frac{\pi l}{2L}\sin\frac{\pi l}{2L}}\right) = \frac{\kappa}{\pi}\text{artanh}\left(-1 + \epsilon\frac{2\pi}{L\sin\frac{\pi l}{L}}\right) \\
&= \frac{\kappa}{2\pi}\ln\left(\frac{\epsilon\frac{2\pi}{L\sin\frac{\pi l}{L}}}{2 - \epsilon\frac{2\pi}{L\sin\frac{\pi l}{L}}}\right) = \frac{\kappa}{2\pi}\ln\left(-1 + \frac{1}{1 - \epsilon\frac{\pi}{L\sin\frac{\pi l}{L}}}\right) \\
&\approx \frac{\kappa}{2\pi}\ln\left(\epsilon\frac{\pi}{L\sin\frac{\pi l}{L}}\right) = -\frac{\kappa}{2\pi}\ln\left(\frac{L\sin\frac{\pi l}{L}}{\pi\epsilon}\right) \equiv -\frac{\kappa}{2\pi}\zeta.
\end{aligned} \tag{66}$$

The quantity ζ diverges as ϵ goes to zero, as expected. For the lower bound of T one gets

$$t = T + \frac{\overline{L}}{L}X - \frac{\overline{\kappa}}{\kappa}x = \frac{\overline{l}}{2} - \frac{\overline{l}}{l}\epsilon - \frac{l\overline{L}}{2L} + \frac{\overline{L}}{L}\epsilon + \frac{\overline{\kappa}}{2\pi}\zeta \approx \frac{\overline{l}}{2} - \frac{l\overline{L}}{2L} + \frac{\overline{\kappa}}{2\pi}\zeta \tag{67}$$

if one ignores contributions of order ϵ. Doing the same calculations with the upper bounds yields the regularized interval

$$(t,x) \in \left[\left(\frac{\overline{\kappa}}{2\pi}\zeta + \frac{\overline{l}}{2} - \frac{l\overline{L}}{2L}, -\frac{\kappa}{2\pi}\zeta\right), \left(-\frac{\overline{\kappa}}{2\pi}\zeta - \frac{\overline{l}}{2} + \frac{l\overline{L}}{2L}, \frac{\kappa}{2\pi}\zeta\right)\right]. \tag{68}$$

4.3 The partition function in WCFTs

In ordinary quantum statistics, the *partition function* in the canonical ensemble is defined as

$$Z = \mathrm{Tr}\ e^{-\beta H}. \tag{69}$$

The trace can be viewed as a result of an identification in imaginary time with period β. We will be mainly interested to calculate Z on a torus with an additional spatial identification. To keep the torus completely arbitrary we define

$$(t,x) \sim (t + 2\pi\overline{a}, x - 2\pi a) \sim (t + 2\pi\overline{\tau}, x - 2\pi\tau) \tag{70}$$

where $\overline{\tau}$ plays the role of β (up to prefactors).

The next question we have to ask is what the Hamiltonian in our WCFT is? It should be the operator which generates translations in the direction of the thermal identification above. We know that L_0 generates dilatations on the plane which correspond to x-translations on the cylinder. P_0 generates translations in the plane which can be seen from the definition and, since the transformation to the cylinder is linear in P, also generates time translations on the cylinder. The whole expression for Z should therefore be given by

$$Z_{\overline{a},a}(\overline{\tau},\tau) = \mathrm{Tr}_{\overline{a},a} e^{-2\pi i\tau L_0 + 2\pi i\overline{\tau}P_0} \tag{71}$$

where I write the spatial identification parameters $\overline{a}$ and a as subscripts to avoid confusion. For later purposes it is useful to transform this expression to the *canonical circle* with $\overline{a}, a = 0, 1$ which can be achieved by the transformation

$$u = \frac{x}{a} \quad \text{and} \quad v = t + \frac{\overline{a}}{a}x. \tag{72}$$

The identification now reads

$$(u,v) \sim (u - 2\pi, v) \sim \left(u - \frac{2\pi\tau}{a}, v - \frac{2\pi\tau\overline{a}}{a} + 2\pi\overline{\tau}\right). \tag{73}$$

Let us first consider the case $k = 0$ and $K \neq 0$: The transformation law of T and P has been calculated in [4, 5] and reads

$$\begin{aligned} P'(z') &= \frac{\partial z}{\partial z'}\left(P(z) + \frac{K}{2}\frac{\partial w'}{\partial z}\right) \\ T'(z') &= \left(\frac{\partial z}{\partial z'}\right)^2 (T(z) - \frac{c}{12}\{z', z\}) + \frac{\partial z}{\partial z'}\frac{\partial w}{\partial z'}P(z) - \frac{K}{4}\left(\frac{\partial w}{\partial z'}\right)^2 \end{aligned} \tag{74}$$

The first equation could have been guessed: The z-dependent translations in w can be seen as gauge transformations in a $U(1)$-bundle, so $P(z)$ transforms as a connection. The second equation is more surprising, because despite of the first term which is familiar from ordinary CFTs there appear two anomalous terms, which become necessary if one has two consecutive transformations. Applying these results to the transformation above one finds

$$\begin{aligned} P'(u) &= aP(x) + \frac{K\overline{a}}{2} \\ T'(u) &= a^2T(x) - a\overline{a}P(x) - \frac{K}{4}\overline{a}^2. \end{aligned} \tag{75}$$

The correct expression for the Fourier modes on the cylinder is given by

$$\begin{aligned} P_n &= -\frac{1}{2\pi}\int dx\, P(x)e^{\frac{inx}{a}} \\ L_n &= -\frac{1}{2\pi}\int dx\, T(x)e^{\frac{inx}{a}}, \end{aligned} \tag{76}$$

and respectively

$$\begin{aligned} P'_n &= -\frac{1}{2\pi}\int du\, P'(u)e^{inu} \\ L'_n &= -\frac{1}{2\pi}\int du\, T'(u)e^{inu} \end{aligned} \tag{77}$$

in the new coordinates. It is now easy to relate the modes to get

$$\begin{aligned} P'_n &= P_n - \frac{K\overline{a}}{2}\delta_n \\ L'_n &= aL_n - \overline{a}P_n + \frac{K}{4}\overline{a}^2\delta_n. \end{aligned} \tag{78}$$

Plugging this into the expression for Z yields

$$\begin{aligned} Z_{0,1}\left(\overline{\tau} - \frac{\tau\overline{a}}{a}, \frac{\tau}{a}\right) &= \mathrm{Tr}_{\overline{a},a} e^{-2\pi i\frac{\tau}{a}(aL_0 - \overline{a}P_0 + \frac{K}{4}\overline{a}^2) + 2\pi i(\overline{\tau} - \frac{\tau\overline{a}}{a})(P_0 - \frac{K\overline{a}}{2})} \\ &= \mathrm{Tr}_{\overline{a},a} e^{-2\pi i\tau L_0 + 2\pi i\overline{\tau}P_0} e^{\pi iK\overline{a}(-\overline{\tau} + \frac{\tau\overline{a}}{2a})} \\ &= Z_{\overline{a},a}(\overline{\tau}, \tau)e^{\pi iK\overline{a}(\frac{\tau\overline{a}}{2a} - \overline{\tau})} \end{aligned} \tag{79}$$

so Z transforms anomalously due to the central element K.

For the entropy calculation it will be necessary to introduce an operation called *S-transformation* which exchanges the spatial and the thermal circle, $\overline{a}, a; \overline{\tau}, \tau \to \overline{\tau}, \tau; -\overline{a}, -a$. Since S belongs to the modular group and does not change the torus, we expect the partition function to be invariant:

$$Z_{0,1}\left(\overline{\tau} - \frac{\tau\overline{a}}{a}, \frac{\tau}{a}\right) = Z_{\overline{\tau}-\frac{\tau\overline{a}}{a}, \frac{\tau}{a}}(0, -1) \tag{80}$$

Using the above result gives the relation

$$Z_{0,1}\left(\overline{\tau} - \frac{\tau\overline{a}}{a}, \frac{\tau}{a}\right) = Z_{0,1}\left(\frac{\overline{\tau}a}{\tau} - \overline{a}, -\frac{a}{\tau}\right) e^{\frac{\pi i K a}{2\tau}(\overline{\tau}-\frac{\tau\overline{a}}{a})^2}. \tag{81}$$

One should note that this simple form of the S-transformation only holds in the canonical circle, for other choices of $\overline{a}, a$ it would be more complicated. [3]

Let us now investigate the other non-trivial algebra with $K = 0$ and $k \neq 0$: The transformation laws of T and P are novel and are derived in detail in Appendix A. The result is

$$\begin{aligned}
P'(z') &= \frac{\partial z}{\partial z'} P(z) - k \frac{\partial^2 z}{\partial z'^2} \frac{\partial z'}{\partial z} \\
T'(z') &= \left(\frac{\partial z}{\partial z'}\right)^2 (T(z) - \frac{c}{12}\{z', z\}) + \frac{\partial z}{\partial z'} \frac{\partial w}{\partial z'} P(z) + k \left(\frac{\partial^2 w}{\partial z'^2} - \frac{\partial w}{\partial z} \frac{\partial^2 z}{\partial z'^2}\right)
\end{aligned} \tag{82}$$

If one now performs the transformation to the canonical circle one finds

$$\begin{aligned}
P'(u) &= aP(x) \\
T'(u) &= a^2 T(x) - \overline{a} a P(x)
\end{aligned} \tag{83}$$

This is quite remarkable because it is just the above result with $K = 0$, the new anomaly k does nowhere appear! What looks surprising at first sight, could yet have been expected from a geometrical point of view: Since there is no preferred spatial axis for $K = 0$, there is also no preferred canonical circle and the transformation should not contain any anomalies. This simplifies the entropy calculation in the next chapter because we do not have to distinguish between the two possible algebras.

4.4 Entropy calculation

What remains to do is to evaluate a thermal entropy on the cylinder after the coordinate transformation to x and t. The interval covered by x is diverging with $\epsilon \to 0$ so it should make no difference to consider the identification of the

[3] For more information about the torus partition function see [1].

interval and calculate the entropy on a torus. The identification parameters can be read off to

$$
\begin{aligned}
2\pi a &= \frac{\kappa}{\pi}\zeta & \qquad 2\pi\overline{a} &= \frac{\overline{\kappa}}{\pi}\zeta - \frac{\overline{L}}{L}l + \bar{l} \\
2\pi\tau &= -i\kappa & \qquad 2\pi\overline{\tau} &= -i\overline{\kappa}.
\end{aligned} \tag{84}
$$

Starting from the partition function one can obtain the entropy

$$
S_{\overline{a},a}(\overline{\tau},\tau) = (1 - \tau\partial_\tau - \overline{\tau}\partial_{\overline{\tau}}) \ln\, Z_{\overline{a},a}(\overline{\tau},\tau). \tag{85}
$$

This formula is the analog of

$$
S = \frac{E-F}{T} = \beta(-\partial_\beta \ln\, Z_k + \frac{1}{\beta}\ln\, Z_k) = (1-\beta\partial_\beta)\ln\, Z_k \tag{86}
$$

in classical statistical physics with the only difference that in our case the thermal circle also has a spatial contribution which is included by the τ-term. This expression is now quite hard to evaluate directly because the trace is taken over various states of the theory. However, if we perform an S-transformation and exchange the two circles, the new inverse temperature gets very large and we can assume that in the limit $\epsilon \to 0$ only the vacuum contributes. Moreover, entropy is a classical observable which takes the same value for any observer, so it can be as well calculated in the canonical circle. To prove this statement use the anomalous transformation rule of Z to obtain (with the abbreviations $z = \overline{\tau} - \frac{\tau\overline{a}}{a}$ and $y = \frac{\tau}{a}$)

$$
\begin{aligned}
S_{\overline{a},a}(\overline{\tau},\tau) &= (1 - \tau\partial_\tau - \overline{\tau}\partial_{\overline{\tau}})(\ln\, Z_{0,1}(z,y) + iK\overline{a}\left(\overline{\tau} - \frac{\tau\overline{a}}{2a}\right)) \\
&= \ln\, Z_{0,1}(z,y) - \tau(\partial_z \ln\, Z_{0,1}(z,y)\left(-\frac{\overline{a}}{a}\right) - \partial_y \ln\, Z_{0,1}(z,y)\left(\frac{1}{a}\right)) \\
&- \overline{\tau}\partial_z \ln\, Z_{0,1}(z,y) \\
&= \left(1 - \left(\overline{\tau} - \frac{\tau\overline{a}}{a}\right)\partial_z - \frac{\tau}{a}\partial_y\right)\ln\, Z_{0,1}(z,y) \\
&= (1 - z\partial_z - y\partial_y)\ln\, Z_{0,1}(z,y) \\
&= S_{0,1}(z,y)
\end{aligned} \tag{87}
$$

Now, the S-transformation gives

$$
Z_{0,1}(z,y) = e^{\pi i K\frac{z^2}{2y}} Z_{0,1}\left(\frac{z}{y}, -\frac{1}{y}\right) = e^{\pi i K\frac{z^2}{2y}} e^{2\pi i \frac{z}{y}P_0^{vac}} e^{2\pi i \frac{1}{y}L_0^{vac}} \tag{88}
$$

where I have already inserted the vacuum expectation values (VEVs) of P_0 and L_0. For the entropy we get

$$\begin{aligned}
S_{0,1}(z,y) &= (1 - z\partial_z - y\partial_y)\ln Z_{0,1}(z,y) \\
&= \pi i K\frac{z^2}{2y} + 2\pi i\frac{z}{y}P_0^{vac} + 2\pi i\frac{1}{y}L_0^{vac} - \pi i K\frac{z^2}{y} \\
&- 2\pi i\frac{z}{y}P_0^{vac} + \pi i K\frac{z^2}{2y} + 2\pi i\frac{z}{y}P_0^{vac} + 2\pi i\frac{1}{y}L_0^{vac} \\
&= 2\pi i\frac{1}{y}(2L_0^{vac} + zP_0^{vac})
\end{aligned} \tag{89}$$

One can finally insert

$$\begin{aligned}
z &= \overline{\tau} - \frac{\tau\overline{a}}{a} = -\frac{i\overline{\kappa}}{2\pi} - \frac{-(\frac{\overline{\kappa}}{\pi}\zeta - \frac{\overline{L}}{L}l + \overline{l})i\kappa}{2\kappa\zeta} = \frac{il}{2\zeta}\left(\frac{\overline{l}}{l} - \frac{\overline{L}}{L}\right) \\
y &= \frac{\tau}{a} = -\frac{i\pi}{\zeta}
\end{aligned} \tag{90}$$

to obtain

$$S_A = -4\zeta L_0^{vac} + il\left(\frac{\overline{L}}{L} - \frac{\overline{l}}{l}\right)P_0^{vac} \tag{91}$$

This Cardy-type formula gives the entanglement entropy in terms of the VEVs and is the main result of this section. To recap, $\zeta = \ln[\frac{L}{\pi\epsilon}\sin\frac{\pi l}{L}]$ measures the length of the interval in the thermal coordinates, L and $\overline{L}$ are the cylinder identification parameters and l and $\overline{l}$ give the spacelike and timelike size of the arbitrarily oriented interval. There are a few things to notice: First of all it does not depend on the scales $\overline{\kappa}, \kappa$ which is good because they were chosen completely arbitrary. Secondly, one can see that despite of the diverging term there is a finite term proportional to the misalignment $\frac{\overline{L}}{L} - \frac{\overline{l}}{l}$ of the interval and the cylinder identification. And, most importantly, this formula is universal in the sense it holds for both non-trivial cases of symmetry algebras, $K \neq 0$ and $k \neq 0$. The central elements only contribute over the VEVs.

4.5 Calculation of the vacuum expectation values

The vacuum state is defined, not only in field theories but also in the dual gravity theories, as the state of *maximal symmetry.* Since L_0 and P_0 generate global symmetries, there action on the vacuum state should give zero which would imply $L_0^{vac} = P_0^{vac} = 0$. However, this statement only holds on the plane, where the symmetry algebra in its initial form was defined. After the transformation to the cylinder, the zero modes get changed by a linear shift, which shall be calculated now. First, for the algebra with $k = 0$ the transformation law 74 applied to the transformation $z = e^{ix}$ and $w = t + 2\alpha x$ yields

$$\begin{aligned}
T'(x) &= -z^2T(z) + \frac{c}{24} + 2\alpha i z P(z) - K\alpha^2 \\
P'(x) &= izP(z) - k\alpha.
\end{aligned} \tag{92}$$

This implies for the modes

$$L'_n = L_n + 2\alpha P_n + (K\alpha^2 - \frac{c}{24})\delta_{n+m}$$
$$P'_n = P_n + K\alpha\delta_n \tag{93}$$

and in particular for the zero modes

$$L'_0 = L_0 + 2\alpha P_0 + K\alpha^2 - \frac{c}{24}$$
$$P'_0 = P_0 + K\alpha. \tag{94}$$

Plugging this into the entropy formula 91 yields

$$S_A = -4\zeta(K\alpha^2 - \frac{c}{24}) + il\left(\frac{\overline{L}}{L} - \frac{\bar{l}}{l}\right)K\alpha \tag{95}$$

The algebra of the new modes looks almost the same, only the linear term of the Virasoro extension has vanished:

$$[L'_n, L'_m] = (n-m)L_{n+m} + \frac{c}{12}n^3\delta_{m+n}$$
$$[L'_n, P'_m] = -mP_{m+n}$$
$$[P'_n, P'{}_m] = K\frac{n}{2}\delta_{m+n} \tag{96}$$

For the algebra 53 we have to use the transformation law 82 to get

$$T'(x) = -z^2T(z) + 2\alpha izP(z) + \frac{c}{24} - 2k\alpha i$$
$$P'(x) = izP(z) - ik \tag{97}$$

and furthermore

$$L'_0 = L_0 + 2\alpha P_0 + 2k\alpha i - \frac{c}{24}$$
$$P'_0 = P_0 + ik \tag{98}$$

The entanglement entropy now becomes

$$S_A = -4\zeta(2k\alpha i - \frac{c}{24}) - l\left(\frac{\overline{L}}{L} - \frac{\bar{l}}{l}\right)k \tag{99}$$

and the algebra reads

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n^3\delta_{m+n} \tag{100}$$
$$[L_n, P_m] = -mP_{m+n} + ikn^2\delta_{m+n}$$
$$[P_n, P_m] = 0, \tag{101}$$

again without the linear terms in the central extensions.

5 The twist field method

5.1 The replica trick

There is another different method for evaluating entanglement entropy in two-dimensional field theories [6, 7] which is motivated by the Renyi entropy. In the first chapter we already encountered the relation

$$S_A = -\lim_{n\to 1}\partial_n \ln(Tr\,\rho_A^n) = -\lim_{n\to 1}\partial_n Tr\,\rho_A^n. \tag{102}$$

Since it contains merely powers of ρ_A and no logarithm anymore, it can be used to calculate S_A via a path integral. At zero temperature we usually have $\rho = \psi\psi^\dagger$ where ψ is the ground state of the theory. If ψ and $\psi^\dagger$ are represented by fields $\phi'(x,0)$ and $\phi''(x,0)$ respectively, this can be written as a path integral

$$\rho = Z^{-1}\int\limits_{t=-\infty}^{0}\mathcal{D}\phi\, e^{-S_E[\phi]}\delta(\phi(x,0)-\phi'(x,0))\int\limits_{t=0}^{\infty}\mathcal{D}\phi\, e^{-S_E[\phi]}\delta(\phi(x,0)-\phi''(x,0)). \tag{103}$$

Here S_E denotes the Euclidean action of the theory and $Z = \int_{t=-\infty}^{\infty}\mathcal{D}\phi\, e^{-S_E[\phi]}$ is a normalization factor which ensures that $Tr\,\rho = 1$. The reduced density matrix ρ_A is now obtained by setting $\phi'(x,0) = \phi''(x,0)$ and integrating over it for all x which are *not* in A, i. e. integrating over the whole plane and merely letting an *open cut* at the interval A. To construct the powers of ρ_A we just take n copies of that and glue them together cyclically such that the every lower edge of the cut is glued to the upper edge of the next copy, see [6] for a picture. If one additionally glues the n^{th} lower edge to the first upper edge one obtains an *n-sheeted Riemann surface* $\mathcal{R}^n$ with the property

$$Tr\,\rho_A^n = Z^{-n}\int\limits_{\mathcal{R}^n}\mathcal{D}\phi\, e^{-S_E} \equiv \frac{Z_n}{Z^n} \tag{104}$$

One should notice that we now need the n^{th} power of the normalization factor because of the n sheets involved. This special Riemann surface consists of flat planes, except for the two endpoints a and b of A which become branch points. We therefore conclude that it should be possible to determine the above path integral in the x,t-plane and implement the non-trivial topology with local fields inserted at a and b. [8, 9] More precisely, those so-called *twist fields* are always present if there exists an internal symmetry σ of the Lagrangian such that

$$\int dx\, dt\, \mathcal{L}[\sigma\phi](x,t) = \int dx\, dt\, \mathcal{L}[\phi](x,t) \tag{105}$$

The above path integral can also be calculated on n copies of the x,t-plane

$$Z_n = \int\limits_{C(a,b)}\mathcal{D}\phi_1\mathcal{D}\phi_2 ... \mathcal{D}\phi_n e^{-\int dx\,dt\,\mathcal{L}[\phi_1]+\mathcal{L}[\phi_2]+...+\mathcal{L}[\phi_n]} \tag{106}$$

where $C(a,b)$ restricts the path integral according to $\phi_i(x,0^+) = \phi_{i+1}(x,0^-)$ for $a < x < b$. The total Lagrangian density is simply written as a sum over all the sheets. Now, the symmetry σ corresponds to cyclic permutations of the sheets, which is possible in two directions:

$$\sigma_+ : \ \phi_i \to \phi_{i+1} \ \text{ and } \ \sigma_- : \ \phi_i \to \phi_{i-1} \ \ (i \bmod n) \tag{107}$$

In fact, the twist fields Φ can now be *defined* through the path integral

$$\langle \Phi_+(a,0)...\rangle \propto \int\limits_{C'(a,0)} \mathcal{D}\phi \, e^{-\int dx\, dt\, \mathcal{L}[\phi]}... \tag{108}$$

where the condition $C'(a,0)$ now implies $\phi(x,0^+) = \sigma_+\phi(x,0^-)$ for $x > a$ and likewise for Φ_-. If we combine two twist fields, we get something which is proportional to Z_n

$$Z_n \propto \langle \Phi_+(a,0)\Phi_-(b,0)\rangle \propto \int\limits_{C''(a,b)} \mathcal{D}\phi \, e^{-\int dx\, dt\, \mathcal{L}[\phi]} \tag{109}$$

because in the interval (a,b) every sheet is connected to the next one though for $x > b$ the operations σ_+ and σ_- cancel and the sheets are connected to themselves. The correlation function on the left hand side of course depends on n, or more explicitly, the conformal dimension of $\Phi_\pm$ depends linearly on n. For an arbitrary operator O we can now write down the equation

$$Z^{-n} \int\limits_{\mathcal{R}^n} \mathcal{D}\phi \, O(x,t) \, e^{-S_E} = \frac{\langle O(x,t)\Phi_+(a,0)\Phi_-(b,0)\rangle}{\langle \Phi_+(a,0)\Phi_-(b,0)\rangle} \tag{110}$$

where the constant of proportionality cancels away. The fields $\Phi_\pm$ are *local* fields in the sense that they do not depend on the energy density at space-like distances. Moreover, $\sigma_\pm$ commutes with all the symmetries present in (W)CFTs. We therefore conclude that also the twist fields respect those symmetries, so that we can use them to determine the correlation functions. This statement is quite powerful because correlation functions are highly constrained by the symmetries in (W)CFTs.

5.2 Entanglement entropy in ordinary CFT

At first we want to use the above results to calculate the entanglement entropy S_A in a relativistic CFT on the complex plane with $z = x + it$. We already know that Z_n is proportional to a two-point function but we do not know yet the conformal dimension of the twist fields. It is possible to determine by computing the expectation value of $T(z)$ on the n-sheeted Riemann surface in two different ways: Firstly we can map $\mathcal{R}_n$ to the plane by the conformal map

$$z = \left(\frac{w-a}{w-b}\right)^{\frac{1}{n}} \tag{111}$$

where $z = \frac{w-a}{w-b}$ maps the branch cut to $(-\infty, 0)$ and the n^{th} root removes it. In the plane we have $\langle T(z)\rangle = 0$ because of translational and rotational invariance, so $\langle T(z)\rangle$ on $\mathcal{R}_n$ is given solely by the Schwarzian derivative $\frac{c}{12}\{z, w\}$. The evaluation has been done with Mathematica and yields

$$\langle T_i(w)\rangle = \frac{c}{24}\left(1 - \frac{1}{n^2}\right)\frac{(a-b)^2}{(w-a)^2(w-b)^2} \tag{112}$$

for the i^{th} sheet. To get the full $\langle T(w)\rangle$ we just have to multiply by n :

$$\langle T(w)\rangle = \frac{c}{24}\left(n - \frac{1}{n}\right)\frac{(a-b)^2}{(w-a)^2(w-b)^2} \tag{113}$$

On the other hand from the formula above we have

$$\langle T(w)\rangle = \frac{\langle T(w)\Phi_+(a)\Phi_-(b)\rangle}{\langle \Phi_+(a)\Phi_-(b)\rangle} \tag{114}$$

and, additionally, the conformal Ward identity

$$\langle T(w)\Phi_+(a)\Phi_-(b)\rangle = \left(\frac{h_+}{(w-a)^2} + \frac{h_-}{(w-b)^2} + \frac{1}{w-a}\partial_a + \frac{1}{w-b}\partial_b\right)\langle \Phi_+(a)\Phi_-(b)\rangle\,. \tag{115}$$

It is reasonable to assume that the conformal dimensions h_+ and h_- of the twist fields are the same, otherwise their correlation function would vanish. Moreover, in this equation we can normalize the two-point function without loss of generality to

$$\langle \Phi_+(a)\Phi_-(b)\rangle = (b-a)^{-2h} \tag{116}$$

Plugging in yields

$$\begin{aligned}
&\langle T(w)\Phi_+(a)\Phi_-(b)\rangle \\
&= \left(\frac{h}{(w-a)^2} + \frac{h}{(w-b)^2}\right)(b-a)^{-2h} - 2h(b-a)^{-2h-1}\left(\frac{1}{w-b} - \frac{1}{w-a}\right) \\
&= (b-a)^{-2h}\frac{(h((w-b)^2 + (w-a)^2) - 2h(w-a)(w-b)}{(w-a)^2(w-b)^2} \\
&= (b-a)^{-2h}\frac{h(b-a)^2}{(w-a)^2(w-b)^2}
\end{aligned} \tag{117}$$

Dividing by $\langle \Phi_+(a)\Phi_-(b)\rangle$ immediately gives

$$\langle T(w)\rangle = \frac{h(b-a)^2}{(w-a)^2(w-b)^2} \tag{118}$$

and after comparison with the result before

$$h = \frac{c}{24}\left(n - \frac{1}{n}\right). \tag{119}$$

So far we only considered the holomorphic dependence in the two-point function and the Ward identity, however, we expect the antiholomorphic sector to be the same because the symmetry σ does not distinguish between holomorphic and anti-holomorphic fields. Therefore after inserting an extra factor of two we get the proportionality relation

$$Tr\,\rho_A^n = C_n \left(\frac{b-a}{\epsilon}\right)^{-\frac{c}{6}(n-\frac{1}{n})} \tag{120}$$

where the extra factor ϵ is introduced for dimensional reasons and the constant C_n remains undetermined. This can be plugged into the formula for the entanglement entropy to yield the famous result first derived by Holzhey, Wilzcek and Larsen

$$S_A = \frac{c}{3} \ln\left(\frac{b-a}{\epsilon}\right) + C^* \tag{121}$$

C^* is minus the derivative of C_n at $n = 1$ and can be calculated for special systems, see the references in [7].

5.3 Renyi entropy in WCFT

We will generalize the discussion a bit and calculate the Renyi entropy

$$S_n^{Ren} = \frac{1}{1-n} \ln(Tr\,\rho_A^n) \tag{122}$$

instead of the entanglement entropy using first the Rindler and then the twist field method. From chapter 4.1 we know that $Tr\,\rho_A^n = \frac{Z_n}{Z^n}$ where Z_n denotes the partition function of the replica manifold. Now, since $Tr\,\rho_A^n$ is not affected by a unitary coordinate transformation, we may calculate S_n^{Ren} equivalently in the x, t-coordinates. This makes it quite easy because in those "thermal" coordinates $\mathcal{R}_n$ is simply a torus without branch points so all we have to do is multiply the thermodynamic potentials by n to get

$$S_n^{Ren} = \frac{1}{1-n} \ln\left(\frac{Z_{\overline{a},a}(n\overline{\tau}, n\tau)}{Z_{\overline{a},a}(\overline{\tau},\tau)^n}\right) \tag{123}$$

One can now perform the same modular manipulations as above; the new potentials z and y depend linearly on τ and $\overline{\tau}$ so one obtains

$$\begin{aligned} Z_{0,1}(nz, ny) &= e^{\pi i K \frac{nz^2}{2y}} e^{2\pi i \frac{z}{y} P_0^{vac}} e^{2\pi i \frac{1}{ny} L_0^{vac}} \\ Z_{0,1}(z,y)^n &= e^{\pi i K \frac{nz^2}{2y}} e^{2\pi i \frac{nz}{y} P_0^{vac}} e^{2\pi i \frac{n}{y} L_0^{vac}} \end{aligned} \tag{124}$$

and plugging in yields

$$\begin{aligned} S_n^{Ren} &= \frac{1}{1-n}(2\pi i \frac{z}{y} P_0^{vac}(1-n) + 2\pi i \frac{1}{y} L_0^{vac}\left(\frac{1}{n} - n\right)) \\ &= 2\pi i \frac{z}{y} P_0^{vac} + 2\pi i \frac{1}{y} L_0^{vac} \left(\frac{1}{n} + 1\right) \\ &= il \left(\frac{\overline{L}}{L} - \frac{\overline{l}}{l}\right) P_0^{vac} - 2\zeta L_0^{vac} \left(\frac{1}{n} + 1\right) \end{aligned} \tag{125}$$

In the limit $n \to 1$ we get back the result for the entanglement entropy.

5.4 Calculation of $\langle T \rangle$ and $\langle P \rangle$ in WCFT

To use the twist field method we have to calculate the expectation values of T and P on the replica manifold. However, this is not as straightforward as before in ordinary CFTs because we do not have a warped conformal mapping which maps $\mathcal{R}_n$ to the plane. Instead we will again use the thermal coordinates x and t where $\mathcal{R}_n$ is topologically trivial. To keep the calculations a bit easier we will send $\overline{L}$ and L to infinity and only keep track of the angle of the cylinder identification $\frac{\overline{L}}{L}$. The transformation to x and t then reduces to

$$\frac{2X}{l} = \tanh\frac{\pi x}{\kappa} \quad \text{and} \quad T + \frac{\overline{L}}{L}X = t + \frac{\overline{\kappa}}{\kappa}x. \tag{126}$$

What we have to do now is to compute $\langle T \rangle$ and $\langle P \rangle$ on the n-fold copy of the x, t-torus, i. e. with potentials $n\overline{\tau}$ and $n\tau$. If we assume translational invariance then $T(x)$ and $P(x)$ are proportional to their zero modes, or more precisely, from the definition it follows that

$$L_0 = -aT(x) \quad \text{and} \quad P_0 = -aP(x). \tag{127}$$

$\langle L_0 \rangle$ and $\langle P_0 \rangle$ can be calculated from the partition function:

$$\begin{aligned}
\langle L_0 \rangle &= -\frac{1}{2\pi i}\frac{\frac{\partial}{\partial\tau}Z_{\overline{a},a}(\overline{\tau},\tau)}{Z_{\overline{a},a}(\overline{\tau},\tau)} = -\frac{1}{2\pi i}\frac{\partial}{\partial\tau}\ln Z_{\overline{a},a}(\overline{\tau},\tau) \\
\langle P_0 \rangle &= \frac{1}{2\pi i}\frac{\partial}{\partial\overline{\tau}}\ln Z_{\overline{a},a}(\overline{\tau},\tau)
\end{aligned} \tag{128}$$

We now want to express $Z_{\overline{a},a}(\overline{\tau},\tau)$ in terms of the vacuum expectation values of L_0 and P_0, so we use our result in the canonical frame and transform it back to an arbitrary frame:

$$\begin{aligned}
Z_{\overline{a},a}(\overline{\tau},\tau) &= e^{\pi i K\overline{a}(\overline{\tau}-\frac{\tau\overline{a}}{2a})} Z_{0,1}\left(\overline{\tau} - \frac{\overline{a}\tau}{a}, \frac{\tau}{a}\right) \\
&= e^{\pi i K\overline{a}(\overline{\tau}-\frac{\tau\overline{a}}{2a})} e^{i\pi K\frac{(\overline{\tau}-\frac{\overline{a}\tau}{a})^2 a}{2\tau}} e^{2\pi i(\frac{\overline{\tau}a}{\tau}-\overline{a})P_0^{vac}} e^{2\pi i\frac{a}{\tau}L_0^{vac}} \\
&= e^{\pi i K\frac{\overline{\tau}^2 a}{2\tau}} e^{2\pi i(\frac{\overline{\tau}a}{\tau}-\overline{a})P_0^{vac}} e^{2\pi i\frac{a}{\tau}L_0^{vac}}
\end{aligned} \tag{129}$$

The anomaly k does not appear here, so in the algebra with $k \neq 0$ the K-term simply vanishes. It follows immediately that

$$\begin{aligned}
\langle L_0 \rangle &= \frac{K}{4}\frac{\overline{\tau}^2 a}{\tau^2} + \frac{\overline{\tau}a}{\tau^2}P_0^{vac} + \frac{a}{\tau^2}L_0^{vac} \\
\langle P_0 \rangle &= \frac{K}{2}\frac{\overline{\tau}a}{\tau} + \frac{a}{\tau}P_0^{vac}
\end{aligned} \tag{130}$$

and if one plugs in the potentials $\overline{\tau} = -\frac{ni\overline{\kappa}}{2\pi}$ and $\tau = -\frac{in\kappa}{2\pi}$ one can write

$$\begin{aligned}
\langle T(x)\rangle &= -\frac{K}{4}\frac{\overline{\kappa}^2}{\kappa^2} - \frac{2\pi\overline{\kappa}i}{n\kappa^2}P_0^{vac} + \frac{4\pi^2}{\kappa^2 n^2}L_0^{vac} \\
\langle P(x)\rangle &= -\frac{K}{2}\frac{\overline{\kappa}}{\kappa} - \frac{2\pi i}{n\kappa}P_0^{vac}.
\end{aligned} \tag{131}$$

These results can now be transformed to the original replica manifold using the transformation laws of $T(x)$ and $P(x)$. For $K \neq 0$ they read

$$\begin{aligned}
P'(X) &= \frac{\partial x}{\partial X}P(x) + \frac{K}{2}\frac{\partial T}{\partial X} \\
T'(X) &= \left(\frac{\partial x}{\partial X}\right)^2 T(x) + \frac{c}{12}\{x, X\} + \frac{\partial x}{\partial X}\frac{\partial t}{\partial X}P(x) - \frac{K}{4}\left(\frac{\partial t}{\partial X}\right)^2
\end{aligned} \tag{132}$$

The derivatives can be evaluated straightforwardly to

$$\begin{aligned}
\frac{\partial x}{\partial X} &= -\frac{\kappa l}{2\pi}\frac{1}{(X-\frac{l}{2})(X+\frac{l}{2})} \qquad & \frac{\partial t}{\partial X} &= \frac{\overline{L}}{L} + \frac{\overline{\kappa}l}{2\pi}\frac{1}{(X-\frac{l}{2})(X+\frac{l}{2})} \\
\{x, X\} &= \frac{l^2}{2(X-\frac{l}{2})^2(X+\frac{l}{2})^2} & \frac{\partial T}{\partial X} &= -\frac{\overline{L}}{L} - \frac{\overline{\kappa}l}{2\pi}\frac{1}{(X-\frac{l}{2})(X+\frac{l}{2})}.
\end{aligned} \tag{133}$$

Plugging in everything is tedious but we arrive at

$$\begin{aligned}
P'(X) &= -\frac{\kappa l}{2\pi}\frac{1}{(X-\frac{l}{2})(X+\frac{l}{2})}\left(-\frac{K}{2}\frac{\overline{\kappa}}{\kappa} - \frac{2\pi i}{n\kappa}P_0^{vac}\right) + \frac{K}{2}\left(-\frac{\overline{L}}{L} - \frac{\overline{\kappa}l}{2\pi}\frac{1}{(X-\frac{l}{2})(X+\frac{l}{2})}\right) \\
&= \frac{il}{n}\frac{1}{(X-\frac{l}{2})(X+\frac{l}{2})}P_0^{vac} - \frac{K}{2}\frac{\overline{L}}{L}
\end{aligned} \tag{134}$$

$$\begin{aligned}
T'(X) &= \frac{\kappa^2 l^2}{4\pi^2}\frac{1}{(X-\frac{l}{2})^2(X+\frac{l}{2})^2}\left(-\frac{K}{4}\frac{\overline{\kappa}^2}{\kappa^2} - \frac{2\pi\overline{\kappa}i}{n\kappa^2}P_0^{vac} + \frac{4\pi^2}{\kappa^2 n^2}L_0^{vac}\right) + \frac{c}{24}\frac{l^2}{(X-\frac{l}{2})^2(X+\frac{l}{2})^2} \\
&\quad - \frac{\kappa l}{2\pi}\frac{1}{(X-\frac{l}{2})(X+\frac{l}{2})}\left(\frac{\overline{L}}{L} + \frac{\overline{\kappa}l}{2\pi}\frac{1}{(X-\frac{l}{2})(X+\frac{l}{2})}\right)\left(-\frac{K}{2}\frac{\overline{\kappa}}{\kappa} - \frac{2\pi i}{n\kappa}P_0^{vac}\right) \\
&\quad - \frac{K}{4}\left(\frac{\overline{L}}{L} + \frac{\overline{\kappa}l}{2\pi}\frac{1}{(X-\frac{l}{2})(X+\frac{l}{2})}\right)^2 \\
&= \frac{l^2}{(X-\frac{l}{2})^2(X+\frac{l}{2})^2}\left(\frac{c}{24} + \frac{L_0^{vac}}{n^2}\right) - \frac{K}{4}\left(\frac{\overline{L}}{L}\right)^2 + \frac{il}{n}\frac{\overline{L}}{L}\frac{1}{(X-\frac{l}{2})(X+\frac{l}{2})}P_0^{vac}
\end{aligned} \tag{135}$$

For the algebra with $k \neq 0$ the transformation law derived in Appendix A reads

$$\begin{aligned}
P'(X) &= \frac{\partial x}{\partial X}P(x) - k\frac{\partial^2 x}{\partial X^2}\frac{\partial X}{\partial x} \\
T\text{'}(X) &= \left(\frac{\partial x}{\partial X}\right)^2 T(x) + \frac{c}{12}\{x, X\} + \frac{\partial x}{\partial X}\frac{\partial t}{\partial X}P(x) + k\left(\frac{\partial^2 t}{\partial X^2} - \frac{\partial t}{\partial x}\frac{\partial^2 x}{\partial X^2}\right)
\end{aligned} \tag{136}$$

and plugging in the transformation 126 gives

$$
\begin{aligned}
P'(X) &= -\frac{\kappa l}{2\pi}\frac{1}{(X-\frac{l}{2})(X+\frac{l}{2})}\left(-\frac{2\pi i}{n\kappa}P_0^{vac}\right)- \\
&\quad k\frac{\kappa l}{2\pi}\left(\frac{1}{(X-\frac{l}{2})^2(X+\frac{l}{2})}+\frac{1}{(X-\frac{l}{2})(X+\frac{l}{2})^2}\right)\left(-\frac{2\pi}{\kappa l}\left(X-\frac{l}{2}\right)\left(X+\frac{l}{2}\right)\right) \\
&= \frac{1}{(X-\frac{l}{2})(X+\frac{l}{2})}\frac{il}{n}P_0^{vac}+k\left(\frac{1}{(X-\frac{l}{2})}+\frac{1}{(X+\frac{l}{2})}\right) \\
&= \frac{\frac{i}{n}P_0^{vac}+k}{X-\frac{l}{2}}+\frac{-\frac{i}{n}P_0^{vac}+k}{X+\frac{l}{2}}
\end{aligned}
\tag{137}
$$

$$
\begin{aligned}
T'(X) &= \frac{\kappa^2 l^2}{4\pi^2}\frac{1}{(X-\frac{l}{2})^2(X+\frac{l}{2})^2}\left(-\frac{2\pi\overline{\kappa}i}{n\kappa^2}P_0^{vac}+\frac{4\pi^2}{\kappa^2n^2}L_0^{vac}\right)+\frac{c}{24}\frac{l^2}{(X-\frac{l}{2})^2(X+\frac{l}{2})^2}- \\
&\quad \frac{\kappa l}{2\pi}\frac{1}{(X-\frac{l}{2})(X+\frac{l}{2})}\left(\frac{\overline{L}}{L}+\frac{\kappa l}{2\pi}\frac{1}{(X-\frac{l}{2})(X+\frac{l}{2})}\right)\left(-\frac{2\pi i}{n\kappa}P_0^{vac}\right)- \\
&\quad \frac{k\overline{\kappa}l}{2\pi}\left(\frac{1}{(X-\frac{l}{2})^2(X+\frac{l}{2})}+\frac{1}{(X-\frac{l}{2})(X+\frac{l}{2})^2}\right)- \\
&\quad k\left(-\frac{\overline{\kappa}}{\kappa}+\frac{\overline{L}}{L}\left(-\frac{2\pi}{\kappa l}\left(X-\frac{l}{2}\right)\left(X+\frac{l}{2}\right)\right)\right)\left(\frac{\kappa l}{2\pi}\left(\frac{1}{(X-\frac{l}{2})^2(X+\frac{l}{2})}+\frac{1}{(X-\frac{l}{2})(X+\frac{l}{2})^2}\right)\right) \\
&= \frac{l^2}{(X-\frac{l}{2})^2(X+\frac{l}{2})^2}\left(\frac{c}{24}+\frac{L_0^{vac}}{n^2}\right)+\frac{il}{n}\frac{\overline{L}}{L}\frac{1}{(X-\frac{l}{2})(X+\frac{l}{2})}P_0^{vac}+k\frac{\overline{L}}{L}\left(\frac{1}{(X-\frac{l}{2})}+\frac{1}{(X+\frac{l}{2})}\right) \\
&= \frac{l^2}{(X-\frac{l}{2})^2(X+\frac{l}{2})^2}\left(\frac{c}{24}+\frac{L_0^{vac}}{n^2}\right)+\frac{i}{n}\frac{\overline{L}}{L}\left(\frac{P_0^{vac}-ink}{(X-\frac{l}{2})}+\frac{-P_0^{vac}-ink}{(X+\frac{l}{2})}\right)
\end{aligned}
\tag{138}
$$

5.5 Two-point function and Ward identity in WCFT

We now want to relate these results to those we can compute via the twist fields

$$
\langle T(X)\rangle = \frac{\langle T(X)\Phi_+\Phi_-\rangle}{\langle\Phi_+\Phi_-\rangle} \quad \text{and} \quad \langle P(X)\rangle = \frac{\langle P(X)\Phi_+\Phi_-\rangle}{\langle\Phi_+\Phi_-\rangle} \tag{139}
$$

so we have to know how the two-point function is constrained by the symmetries in WCFT. Translational invariance in X implies that it should only depend on $\Delta X = l$. With the help of the invariant degenerate metric we express this as

$$
l = \sqrt{\Delta X^a \Delta X^b g_{ab}} \tag{140}
$$

where X^a stands for $\begin{pmatrix} X \\ T \end{pmatrix}$. The next point we have to consider is boost invariance $T \to T + vX$ from which we immediately see that $\bar{l}$ is not a good measure because it is not boost invariant. However, we can construct a boost invariant quantity which contains $\bar{l}$ as follows: Denote by $V^a = \begin{pmatrix} L \\ \overline{L} \end{pmatrix}$ the spatial identification vector of the cylinder and by n^a its normalization

$$n^a = \frac{V^a}{\sqrt{V^b V^c g_{bc}}} = \begin{pmatrix} 1 \\ \frac{\overline{L}}{L} \end{pmatrix}. \tag{141}$$

Now s defined by

$$s = n^a \Delta X^b h_{ab} = \bar{l} - l\frac{\overline{L}}{L} \tag{142}$$

has the desired property:

$$\begin{aligned} \Delta X^a \quad &\to \quad \begin{pmatrix} l \\ \bar{l} + vl \end{pmatrix} \quad V^a \to \begin{pmatrix} L \\ \overline{L} + vL \end{pmatrix} \quad n^a \to \begin{pmatrix} 1 \\ \frac{\overline{L}}{L} + v \end{pmatrix} \\ &\Rightarrow s \to \bar{l} + vl - l\frac{\overline{L}}{L} - vl = s \end{aligned} \tag{143}$$

From the scaling symmetry on the X-axis we conclude that the two-point function should contain a factor l^{-2h_n} where h_n is the conformal dimension of the twist fields. Since s measures the X-dependent translations in T, s should act in form of a “translation operator” $e^{iQ_n s}$ so that we can write

$$\langle \Phi_+ \Phi_- \rangle \propto l^{-2h_n} e^{iQ_n s}. \tag{144}$$

The next thing we need is the Ward identity in WCFT: While the one for $T(X)$ is similar to that for the holomorphic part in ordinary CFT

$$\langle T(X)\Phi_+(a)\Phi_-(b)\rangle = \left(\frac{h_n}{(X-a)^2} + \frac{h_n}{(X-b)^2} + \frac{1}{X-a}\partial_a + \frac{1}{X-b}\partial_b\right)\langle \Phi_+(a)\Phi_-(b)\rangle . \tag{145}$$

the one for $P(X)$ gets modified. A Kac-Moody primary field is defined through the OPE

$$P(z)\phi(w) = \frac{iQ}{z-w} + reg. \tag{146}$$

where the Q is the same as in the correlation function. If one uses this expression in the derivation of the conformal Ward identity of chapter 2.5 one gets

$$\langle P(X)\Phi_+(a)\Phi_-(b)\rangle = \left(\frac{iQ_{n+}}{X-a} + \frac{iQ_{n-}}{X-b}\right)\langle \Phi_+(a)\Phi_-(b)\rangle . \tag{147}$$

5.6 Entropy calculation

Now we can combine all the results and calculate the Renyi entropy for both symmetry algebras $K \neq 0$ and $k \neq 0$. In both cases we have

$$\langle P(X)\rangle = \frac{\langle P(X)\Phi_+(a)\Phi_-(b)\rangle}{\langle \Phi_+(a)\Phi_-(b)\rangle} = \frac{iQ_{n+}}{X+\frac{l}{2}} + \frac{iQ_{n-}}{X-\frac{l}{2}} \tag{148}$$

If we additionally assume $Q_{n-} = -Q_{n+}$ we get the expression

$$\langle P(X)\rangle = \frac{iQ_n}{X+\frac{l}{2}} - \frac{iQ_n}{X-\frac{l}{2}} = -\frac{ilQ_n}{(X+\frac{l}{2})(X-\frac{l}{2})} \tag{149}$$

which can be compared to n times the above result which was valid on a single sheet. For $K \neq 0$ we find an equivalence by taking $Q_n = -P_0^{vac}$ if we ignore the constant, anomalous term $\frac{K}{2}\frac{\overline{L}}{L}$. However, we cannot expect to reproduce this term because we neglected all the regular terms in the OPE so all we can compare are the singular terms. For $k \neq 0$ there is no regular term and the two results match perfectly for $Q_n = -P_0^{vac} - ikn$. To evaluate h we have to do the same calculation for $T(X)$ which involves a bit more algebra:

$$\begin{aligned}
\langle T(X)\rangle &= \frac{\left(\frac{h_n}{(X-a)^2} + \frac{h_n}{(X-b)^2} + \frac{1}{X-a}\partial_a + \frac{1}{X-b}\partial_b\right)\langle\Phi_+(a)\Phi_-(b)\rangle}{\langle\Phi_+(a)\Phi_-(b)\rangle} \\
&= \frac{h_n}{(X-a)^2} + \frac{h_n}{(X-b)^2} + (b-a)^{2h_n}e^{-iQ_n(\bar{l}-(b-a)\frac{\overline{L}}{L})} \\
&\left(\frac{1}{X-a}\partial_a + \frac{1}{X-b}\partial_b\right)(b-a)^{-2h_n}e^{iQ_n(\bar{l}-(b-a)\frac{\overline{L}}{L})} \\
&= \frac{h_n}{(X+\frac{l}{2})^2} + \frac{h_n}{(X-\frac{l}{2})^2} + \frac{2h_n}{(X-a)(b-a)} - \frac{2h_n}{(X-b)(b-a)} + \frac{iQ_n\frac{\overline{L}}{L}}{X-a} - \frac{iQ_n\frac{\overline{L}}{L}}{X-b} \\
&= \frac{h_n l^2}{(X-\frac{l}{2})^2(X+\frac{l}{2})^2} - iQ_n\frac{\overline{L}}{L}\frac{l}{(X-\frac{l}{2})(X+\frac{l}{2})}
\end{aligned} \tag{150}$$

If we plug in for Q_n and compare to the previous result we get for both algebras

$$h_n = n\left(\frac{c}{24} + \frac{L_0^{vac}}{n^2}\right). \tag{151}$$

Again, in the $K \neq 0$ case, the results differ by the anomaly $\frac{K}{4}\left(\frac{\overline{L}}{L}\right)^2$ which was neglected in the OPE. Now when we have the expression

$$\langle\Phi_{n+}(a)\Phi_{n-}(b)\rangle \propto l^{-2n(\frac{c}{24}+\frac{L_0^{vac}}{n^2})}e^{-i(\bar{l}-l\frac{\overline{L}}{L})(P_0^{vac}+ikn)} \tag{152}$$

it is straightforward to compute the Renyi entropy:

$$\begin{aligned}
S_n^{Ren} &= \frac{1}{1-n} \ln \frac{\langle \Phi_{n+}(a)\Phi_{n-}(b)\rangle}{\langle \Phi_{1+}(a)\Phi_{1-}(b)\rangle^n} + c_n \\
&= \frac{1}{1-n} \ln(l) \left(-2n \left(\frac{c}{24} + \frac{L_0^{vac}}{n^2} \right) + 2n \left(\frac{c}{24} + L_0^{vac} \right) \right) + \\
&\frac{1}{1-n} \left[-i \left(\bar{l} - l \frac{\overline{L}}{L} \right) (P_0^{vac} + ikn) + in \left(\bar{l} - l \frac{\overline{L}}{L} \right) (P_0^{vac} + ik) \right] + c_n \\
&= \frac{1}{1-n} \left(\ln(l) L_0^{vac} \left(2n - \frac{2}{n} \right) + i \left(\bar{l} - l \frac{\overline{L}}{L} \right) P_0^{vac}(n-1) \right) + c_n \\
&= -il P_0^{vac} \left(\frac{\bar{l}}{l} - \frac{\overline{L}}{L} \right) - 2L_0^{vac} \ln(l) \left(1 + \frac{1}{n} \right) + c_n
\end{aligned} \tag{153}$$

Since we took the limit $\overline{L}, L \to \infty$, ζ got replaced by $\ln l$, so this is the desired result we also had before. Although the last calculation was made for the $k \neq 0$ case, we see immediately that the result also holds for $K \neq 0$ because the k-anomaly simply dropped out. Therefore, this last formula is universal since it holds for both non-trivial algebras. However, the VEVs depend crucially on the central extensions.

6 Holographic entanglement entropy

6.1 Rindler spacetime

The main motivation behind the calculation of entanglement entropy was a check of the holographic theorem, so this section is dedicated to the dual bulk theory. As described in [10], we consider the near-horizon approximation of non-extremal black holes. For extremal black holes there exists the generic near-horizon metric [11]

$$ds^2 = -F(x^m)r^2 du^2 - 2dr\, du + 2rh_a(x^m) du\, dx^a + \gamma_{ab}(x^m) dx^a\, dx^b, \tag{154}$$

where r is a light-like coordinate which measures the distance to the horizon, v is a coordinate on the horizon which becomes light-like at $r = 0$ and the x^a are transverse coordinates. For non-extremal black holes there does not exist a general form of the metric but one example is the three-dimensional *boosted Rindler-spacetime*

$$ds^2 = -2a(u) r\, du^2 - 2dr\, du + 2\eta(u) du\, dx + dx^2 \tag{155}$$

which differs from 154 by the powers of r and the dependencies of the functions. This metric is locally flat for arbitrary functions a and η, i. e. it solves the vacuum Einstein equations for vanishing cosmological constant. At this point we can also assume that all the coordinates run from $-\infty$ to $+\infty$. Another example is the *boosted Rindler-AdS-spacetime*

$$ds^2 = -2a(u) r\, du^2 - 2dr\, du + 2 \left(\eta(u) + \frac{2r}{R} \right) du\, dx + dx^2 \tag{156}$$

which also solves the Einstein equations but for $\Lambda = -\frac{1}{R^2}$.

To study these spacetimes holographically, first of all one has to find suitable boundary conditions and then determine the asymptotic symmetry algebra. However, in this case it is not clear where the boundary should be located: The first choice would probably be at $r \to \infty$, as it was done in [10]; the fall-off-conditions for the metric then become

$$g_{\mu\nu} = \begin{pmatrix} -2a(u)r + \mathcal{O}(1) & -1 + \mathcal{O}\left(\frac{1}{r}\right) & \eta(u) + \mathcal{O}\left(\frac{1}{r}\right) \\ -1 + \mathcal{O}\left(\frac{1}{r}\right) & \mathcal{O}\left(\frac{1}{r^2}\right) & \mathcal{O}\left(\frac{1}{r}\right) \\ \eta(u) + \mathcal{O}\left(\frac{1}{r}\right) & \mathcal{O}\left(\frac{1}{r}\right) & 1 + \mathcal{O}\left(\frac{1}{r}\right) \end{pmatrix} \tag{157}$$

But since we deal with an approximation for a near-horizon region, also $r = 0$ would be a possible boundary. I shall give evidence that $r = 0$ is indeed the right choice in the next chapters. The behaviour of the metric for $r \to 0$ is then determined by

$$g_{\mu\nu} = \begin{pmatrix} -2a(u)r + \mathcal{O}(1) & -1 + \mathcal{O}(r) & \eta(u) + \mathcal{O}(r) \\ -1 + \mathcal{O}(r) & \mathcal{O}\left(r^2\right) & \mathcal{O}(r) \\ \eta(u) + \mathcal{O}(r) & \mathcal{O}(r) & 1 + \mathcal{O}(r) \end{pmatrix} \tag{158}$$

Fortunately, the asymptotic symmetry group turns out to be independent of r, so for the moment it does not matter which option we choose. Most of the calculations in [10] are done in the Chern-Simons-formulation which I will not introduce here, hence I just summarize the results. The conventional approach would now be to calculate the canonical boundary charges as an integral over the space-like coordinate x. However, this results in a trivial theory because all of these boundary charges will be identically zero which means that all asymptotic symmetries are gauge symmetries. The quite unusual procedure which was done in [10] consists of making the retarded time coordinate u periodic with period L and define the surface charges as integrals over u. Of course it is now difficult to asign a physical meaning to this spacetime which has closed causal curves and is not Poincare-invariant anymore. But it is still interesting to study it from a more pragmatical point of view because now one can construct a non-trivial dual field theory. The infinitesimal transformations which preserve the boundary conditions are

$$u \to u + \epsilon t(u) \qquad x \to x + \epsilon p(u) \tag{159}$$

with two arbitrary functions $t(u)$ and $p(u)$. The integrated canonical charge is then given by

$$Q(t,p) = \frac{1}{8\pi G_N} \int_0^L du\, t(u)T(u) + p(u)P(u) \tag{160}$$

with $T(u) = \frac{d\eta}{du} + a(u)\eta(u)$ and $P(u) = a(u)$. If one now defines modes according to [4]

$$L_n = \frac{L}{16\pi^2 G_N} \int_0^L du\, e^{\frac{2\pi i n u}{L}} T(u)$$
$$P_n = \frac{1}{8\pi G_N} \int_0^L du\, e^{\frac{2\pi i n u}{L}} P(u) \tag{161}$$

their Poisson brackets read

$$i\{L_n, L_m\} = (n-m)L_{n+m}$$
$$i\{L_n, P_m\} = -mP_{m+n} + ikn^2\delta_{m+n}$$
$$i\{P_n, P_m\} = 0 \tag{162}$$

with $k = \frac{1}{4G_N}$ which is after canonical quantization exactly the cylinder algebra 100 with vanishing central charge c. Since we have already extensively studied the WCFT with that algebra and derived a formula for the entanglement entropy on the field theory side, we now want to compare the results with a gravitational calculation in the bulk.

6.2 Entanglement entropy of flat Rindler space

Due to the famous proposal of Ryu and Takanayagi [12] one can determine the entanglement entropy on the gravity side in the following way: If one has a $d+1$-dimensional field theory then the boundary of the entangling region is $d-1$-dimensional. The entanglement entropy is now given in terms of the *area of a minimal surface* γ in the bulk which shares the same boundary as the entangling region as

$$S_A = \frac{Area(\gamma)}{4G_N}. \tag{163}$$

This formula is of course motivated by the Bekenstein-Hawking-law 7 where γ now plays the role of the horizon. In our case of a $1+1$-dimensional field theory γ becomes a geodesic attached to the endpoints of the interval A on the boundary. So for calculating S_A we just have to compute the length s of a geodesic which is comparatively easy to do in general. The first thing to check now is which *state* we have on the gravity side and determine the functions $a(u)$ and $\eta(u)$ in the metric. The formula 91 was derived for the ground state of the system, i. e. for $T=0$, so we have to know what the ground state of the bulk theory is. The common procedure is to define the ground state as the *maximal symmetric* spacetime which possesses the highest number of linear independent Killing vector fields. The metric 155 in general has six local Killing vector fields

[4] This definition of the modes is not the same as 76, hence the charges $T(u)$ and $P(u)$ differ by constants from their analogues on the field theory side.

given by

$$
\begin{aligned}
\xi_1 &= \partial_u & \xi_4 &= e^{au}(\partial_u - \eta\partial_x - (ar + \frac{1}{2}\eta^2)\partial_r \\
\xi_2 &= \partial_x & \xi_5 &= a(\eta u + x)\xi_3 + e^{-au}\partial_x \\
\xi_3 &= e^{-au}\partial_r & \xi_6 &= a(\eta u + x)\xi_4 + e^{au}(ar + \frac{1}{2}\eta^2)\partial_x
\end{aligned} \tag{164}
$$

Due to the identification $u \sim u + L$ all six of them are only defined globally if $a = \frac{2\pi i n}{L}$ and $\eta = 0$, hence our vacuum state is given by

$$
ds^2 = -\frac{4\pi i n r}{L} du^2 - 2du\, dr + dx^2 \tag{165}
$$

where $n \neq 0$ is an arbitrary integer. By examining the Euclidean theory one can see that n should be fixed to ± 1, see [10]. For simplicity we choose $n = 1$. The factor of i in the first term may seem disturbing because an imaginary distance between two points is quite unphysical. On the other hand, the spacetime was constructed in an abstract way without a clear physical interpretation from the beginning, so we shall accept this factor i.

At this point we can already determine the final result on the field theory side: Since u is the periodic $SL(2)$-axis and x the preferred $U(1)$-axis, we state that (u, x) correspond to (X, T) on the cylinder. The periodicity condition which was given by $(X, T) \sim (X - L, T + \overline{L})$ can be achieved by just setting $\overline{L}$ to zero, i. e. the identification of u has no contribution in x-direction. Moreover, the map from the cylinder to the plane is defined in the simplest way with tilt parameter $\alpha = 0$. Noticing also that c is equal to zero in the algebra one can plug into the formula 99 and get

$$
S_A = \bar{l}k = \frac{\bar{l}}{4G_N}. \tag{166}
$$

Remarkably, this result is finite, while, as it can be seen from 121, in an ordinary CFT the entanglement entropy is divergent. The corresponding geodesic in AdS also has infinite length because it is attached to the cutoff surface $r = r_0$ with $r_0 \to \infty$. Now in our case we need a finite geodesic, which suggests choosing $r_0 = 0$ instead of $r_0 \to \infty$. This is the first hint that the dual field theory of the near horizon metric lives on the horizon rather than at infinity.

6.3 Calculation of the geodesic

It is now straight forward to solve the geodesic equation

$$
\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0. \tag{167}
$$

with an arbitrary affine parameter t. The Christoffel symbols can be computed from the metric according to

$$
\Gamma^\rho_{\mu\nu} = \frac{g^{\rho\sigma}}{2}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}); \tag{168}
$$

the result is

$$\Gamma^r_{ru} = \Gamma^r_{ur} = \frac{2\pi i}{L}, \quad \Gamma^u_{uu} = -\frac{2\pi i}{L}, \quad \Gamma^r_{uu} = -\frac{8\pi^2 r}{L^2}, \tag{169}$$

all others vanish. The geodesic equation in components now reads

$$\begin{aligned}
&\frac{d^2u}{dt^2} - \frac{2\pi i}{L}\left(\frac{du}{dt}\right)^2 = 0 \\
&\frac{d^2r}{dt^2} + \frac{4\pi i}{L}\frac{dr}{dt}\frac{du}{dt} - \frac{8\pi^2 r}{L^2}\left(\frac{du}{dt}\right)^2 = 0 \\
&\frac{d^2x}{dt^2} = 0
\end{aligned} \tag{170}$$

This coupled system of diffential equations has to be solved with the right boundary conditions. The interval A was given by

$$(u, x) \in \left[\left(-\frac{l}{2}, \frac{\bar{l}}{2}\right), \left(\frac{l}{2}, -\frac{\bar{l}}{2}\right)\right], \tag{171}$$

so the equation for $x(t)$ can be solved easily:

$$x(t) = \frac{\bar{l}}{2} - \frac{\bar{l}}{t_f}t \tag{172}$$

t_f denotes the final value of the affine parameter; of course, the end result has to be independent of t_f which serves as a consistency check. The equation for $u(t)$ is separable and the general solution is

$$u(t) = -\frac{\ln\left(\frac{2\pi i}{L}t + c_1\right)}{\frac{2\pi i}{L}} + c_2 \tag{173}$$

Moreover, the equation for $r(t)$ is a Sturm-Liouville equation and can be solved to

$$r(t) = c_3\left(\frac{2\pi i}{L}t + c_1\right) + c_4\left(\frac{2\pi i}{L}t + c_1\right)^2. \tag{174}$$

Now, determining the constants to obtain $r(0) = r(t_f) = r_0$ for an arbitrary r_0 is tedious and not very promising; instead we will try the other approach with $r_0 = 0$. The solution then becomes trivial:

$$r(t) \equiv 0 \tag{175}$$

so the geodesic never leaves the horizon. If we now take a first glance at the final result, we see that with 175 things simplify a lot:

$$s = \int_0^{t_f} \sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}dt = \int_0^{t_f} \sqrt{\frac{\bar{l}^2}{t_f^2}}dt = \bar{l} \tag{176}$$

Plugging this into the Ryu-Takayanagi-formula gives indeed the right result

$$S_A = \frac{\bar{l}}{4G_N}. \tag{177}$$

In the next chapter we shall analyze the slightly more complicated case of Rindler-AdS.

6.4 Entanglement entropy of Rindler-AdS

Instead of a locally flat near-horizon geometry one can consider spacetimes with constant negative curvature in the near-horizon limit like

$$ds^2 = -2a(u)r\,du^2 - 2dr\,du + 2\left(\eta(u) + \frac{2r}{R}\right) du\,dx + dx^2, \tag{178}$$

the state of maximal symmetry then becomes

$$ds^2 = -\frac{4\pi i r}{L}\,du^2 - 2dr\,du + 4\frac{r}{R}du\,dx + dx^2. \tag{179}$$

The asymptotic symmetry algebra also changes a bit, namely it gains a second central extension:

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} \\
[L_n, P_m] &= -mP_{m+n} + ikn^2\delta_{m+n} \\
[P_n, P_m] &= K\frac{n}{2}\delta_{n+m}
\end{aligned} \tag{180}$$

with $K = -\frac{1}{G_N R}$. It would be possible now to remove the twist term by a redefinition of the generators, however, we shall proceed in another way and compute the vacuum expectation values directly. The transformation rule for this algebra is just the combination of 74 and 82 and reads

$$\begin{aligned}
P'(x) &= \frac{\partial z}{\partial x}\left(P(z) + \frac{K}{2}\frac{\partial t}{\partial z}\right) - k\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial z} \\
T'(x) &= \left(\frac{\partial z}{\partial x}\right)^2 T(z) + \frac{\partial z}{\partial x}\frac{\partial w}{\partial x}P(z) + k\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial z}\frac{\partial^2 z}{\partial x^2}\right) - \frac{K}{4}\left(\frac{\partial w}{\partial x}\right)^2.
\end{aligned} \tag{181}$$

Applying this to the transformation $z = e^{ix}$, $w = t$ gives the relation for the zero modes

$$\begin{aligned}
L_0' &= L_0 \\
P_0' &= P_0 + ik
\end{aligned} \tag{182}$$

so the vacuum expectation values are exactly the same as before and therefore the result for the entanglement entropy is again

$$S_A = \frac{\bar{l}}{4G_N}. \tag{183}$$

The calculation of the geodesic is now a bit more tedious but still straightforward; the non-vanishing Christoffel-symbols are

$$\begin{aligned}
&\Gamma^x_{ru} = \Gamma^x_{ur} = \Gamma^u_{ux} = \Gamma^u_{xu} = \frac{1}{R}, \quad \Gamma^r_{ru} = \Gamma^r_{ur} = \frac{2r}{R^2} + \frac{2\pi i}{L} \\
&\Gamma^u_{uu} = -\frac{2\pi i}{L}, \quad \Gamma^x_{uu} = \frac{4\pi i r}{RL}, \quad \Gamma^r_{uu} = \frac{8\pi i r^2}{R^2 L} - \frac{8\pi^2 r}{L^2} \\
&\Gamma^r_{rx} = \Gamma^r_{xr} = -\frac{1}{R}, \quad \Gamma^x_{ux} = \Gamma^x_{xu} = -\frac{2r}{R^2}, \quad \Gamma^r_{ux} = \Gamma^r_{xu} = -\frac{4r^2}{R^3} - \frac{4\pi i r}{RL}
\end{aligned} \tag{184}$$

and the geodesic equation reads

$$\begin{aligned}
\frac{d^2u}{dt^2} - \frac{2\pi i}{L}\left(\frac{du}{dt}\right)^2 + \frac{2}{R}\frac{du}{dt}\frac{dx}{dt} &= 0 \\
\frac{d^2r}{dt^2} + 2\left(\frac{2r}{R^2} + \frac{2\pi i}{L}\right)\frac{du}{dt}\frac{dr}{dt} + \left(\frac{8\pi i r^2}{R^2 L} - \frac{8\pi^2 r}{L^2}\right)\left(\frac{du}{dt}\right)^2 & \\
-\frac{2}{R}\frac{dr}{dt}\frac{dx}{dt} - \left(\frac{8r^2}{R^3} + \frac{8\pi i r}{RL}\right)\frac{du}{dt}\frac{dx}{dt} &= 0 \\
\frac{d^2x}{dt^2} + \frac{2}{R}\frac{du}{dt}\frac{dr}{dt} + \frac{4\pi i r}{RL}\left(\frac{du}{dt}\right)^2 - \frac{4r}{R^2}\frac{du}{dt}\frac{dx}{dt} &= 0.
\end{aligned} \tag{185}$$

The general solution of this system is much more difficult to find, however, the choice $r_0 = 0$ simplifies it a lot again. Since every term in the equation for $r(t)$ contains a factor of r, $r(t) \equiv 0$ is again a solution. Afterwards, the equation for $x(t)$ has simplified to $\frac{d^2x}{dt^2} = 0$ which has the same solution as before,

$$x(t) = \frac{\bar{l}}{2} - \frac{\bar{l}}{t_f}t \tag{186}$$

It is not even necessary to calculate $u(t)$ because in the expression for the length again only one term survives after setting $r(t)$ equal to zero:

$$s = \int_0^{t_f} \sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}dt = \int_0^{t_f} \sqrt{\frac{\bar{l}^2}{t_f^2}}dt = \bar{l} \tag{187}$$

We see that the result is the same as in the flat case and also agrees with the field theory result:

$$S_A = \frac{\bar{l}}{4G_N} \tag{188}$$

7 Conclusion

In this last section I want to summarize all the results and give an interpretation as far as possible. At first, in section 3.6 we saw that there exist two qualitatively different WCFT algebras namely $K \neq 0$ and $k \neq 0$. For $K \neq 0$ the entanglement entropy of an interval for zero temperature has already been calculated in [5], the result was the Cardy-type formula

$$S_{EE} = -4\ln\left[\frac{L}{\pi\epsilon}\sin\frac{\pi l}{L}\right] L_0^{vac} + il\left(\frac{\bar{L}}{L} - \frac{\bar{l}}{l}\right) P_0^{vac}. \tag{189}$$

I have shown now that this formula is also valid for the algebra 53 and that the central elements only contribute over the vacuum expectation values. This second algebra is particularly interesting because it appears as the holographic dual of the boosted Rindler spacetime 155. After calculating the entanglement entropy on the gravity side I found that the results agree if one chooses $r = 0$ as the location of the boundary. Hence, it can be conjectured that for a near-horizon metric the dual field theory should live on the horizon instead of at $r \to \infty$. This statement was also true for the slightly more complicated case of boosted Rindler-AdS.

However, the check of this conjecture is not very strong because in both cases most of the terms in 91 vanish. It we would be interesting now to examine more complicated cases like excited states at $T > 0$ and compare the results to get a more profound check of the $r = 0$ - conjecture.

A Appendix

A.1 Transformation properties of $P(z)$ and $T(z)$

First of all one has to derive the *infinitesimal* transformation laws: The global transformations $z = f(z')$ and $w = w' + g(z')$ are generated by $T(z)$ and $P(z)$ respectively so the infinitesimal version $z = z' - \epsilon(z')$ and $w = w' - \gamma(w')$ can be written as a commutator [5]

$$\begin{aligned}
\delta_\epsilon\phi(z) &= -i[T_\epsilon, \phi(z)] = \frac{i}{2\pi}\oint_{C(z)} dw\, \epsilon(w)T(w)\phi(z) \\
\delta_\gamma\phi(z) &= -i[P_\gamma, \phi(z)] = \frac{i}{2\pi}\oint_{C(z)} dw\, \gamma(w)P(w)\phi(z)
\end{aligned} \tag{190}$$

All in all there are four commutators to calculate (the four combinations of T and P); the first one is already known from ordinary CFTs and reads

$$\delta_\epsilon T(z) = -\frac{c}{12}\partial_z^3\epsilon(z) - 2\partial_z\epsilon(z)T(z) - \epsilon(z)\partial_z T(z) \tag{191}$$

The commutator of P with itself is zero since the P_n commute with $K = 0$:

$$\delta_\gamma P(z) = 0 \tag{192}$$

[5]The sign convention is now different from the introduction.

The other two are a bit harder to calculate; starting from the algebra one can write

$$
\begin{aligned}
[P_n, L_m] &= -\oint \frac{dz}{2\pi} z^n \oint \frac{dw}{2\pi i} w^{m+1} [P(z), T(w)] \\
&= -\oint_{C(0)} \frac{dw}{2\pi i} w^{m+1} \oint_{C(w)} \frac{dz}{2\pi} z^n P(z) T(w) \\
&=! -\oint \frac{dz}{2\pi} n z^{m+n} P(z) - ikm(m+1)\delta_{m+n} \qquad (193)
\end{aligned}
$$

Now one can apply Cauchy´s residue theorem from the reversed way and make a guess for the OPE of P and T:

$$
P(z)T(w) = \frac{P(w)}{(z-w)^2} + \frac{2k}{(z-w)^3} + reg \qquad (194)
$$

Plugging in yields

$$
\begin{aligned}
[P_n, L_m] &= -\oint_{C(0)} \frac{dw}{2\pi i} w^{m+1} \oint_{C(w)} \frac{dz}{2\pi} z^n \left(\frac{P(w)}{(z-w)^2} + \frac{2k}{(z-w)^3} \right) \\
&= -\oint_{C(0)} \frac{dw}{2\pi i} w^{m+1} \oint_{C(w)} \frac{dz}{2\pi} (w + (z-w))^n \left(\frac{P(w)}{(z-w)^2} + \frac{2k}{(z-w)^3} \right) \\
&= -\oint_{C(0)} \frac{dw}{2\pi i} w^{m+1} (n w^{n-1} i P(w) + n(n-1) w^{n-2} ik) \\
&= nP_{n+m} - ikn(n-1)\delta_{n+m} = nP_{n+m} - ikm(m+1)\delta_{n+m} \qquad (195)
\end{aligned}
$$

where the binomial theorem has been used to extract the right powers of $z - w$. The reversed OPE is obtained by a Taylor expansion around z:

$$
T(w)P(z) = \frac{P(z)}{(w-z)^2} + \frac{\partial_z P(z)}{w-z} - \frac{2k}{(w-z)^3} \qquad (196)
$$

Now it is straightforward to compute the infinitesimal transformations

$$
\begin{aligned}
\delta_\gamma T(z) &= \frac{i}{2\pi} \oint_{C(z)} dw\, \gamma(w) P(w) T(z) \\
&= \frac{i}{2\pi} \oint_{C(z)} dw\, (\gamma(z) + \partial_z \gamma(z)(w-z) + \\
&\quad \frac{1}{2}\partial_z^2 \gamma(z)(w-z)^2) \left(\frac{P(z)}{(w-z)^2} + \frac{2k}{(w-z)^3} \right) \\
&= -\partial_z \gamma(z) P(z) - k \partial_z^2 \gamma(z) \qquad (197)
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_\epsilon P(z) &= \frac{i}{2\pi}\oint\limits_{C(z)} dw\, \epsilon(w)T(w)P(z) \\
&= \frac{i}{2\pi}\oint\limits_{C(z)} dw\, (\epsilon(z)+\partial_z\epsilon(z)(w-z)+ \\
&\frac{1}{2}\partial_z^2\epsilon(z)(w-z)^2\left(\frac{P(z)}{(w-z)^2}+\frac{\partial_z P(z)}{w-z}-\frac{2k}{(w-z)^3}\right) \\
&= -\partial_z\epsilon(z)P(z)-\epsilon(z)\partial_z P(z)+k\partial_z^2\epsilon(z) \qquad (198)
\end{aligned}
$$

The next task is to find finite transformation laws which reduce to those infinitesimal ones and can be composed properly so that it makes no difference if I go from (z,w) to (z',w') and from (z',w') to (z'',w'') or from (z,w) to (z'',w'') in one step. Since $\delta_\gamma P(z)$ is zero the total variation of $P(z)$ is given by the last equation. The first two terms can be generated by a simple tensorial transformation

$$P'(z') = \frac{\partial z}{\partial z'}P(z). \qquad (199)$$

which also composes properly; one gets

$$P'(z') = P(z) - \partial_{z'}\epsilon(z')P(z) = P(z') - \epsilon(z')\partial_{z'}P(z') - \partial_{z'}\epsilon(z')P(z). \qquad (200)$$

The last term looks similar to the anomalous term in the transformation law of $T(z)$ with the only difference that we have here only a second and no third derivative. So we have to search for the right modification of the Schwarzian derivative denoted by $\{z', z\}$ to write

$$P'(z') = \frac{\partial z}{\partial z'}(P(z) + k\,\{z', z\}) \qquad (201)$$

It should fulfill

$$\{z+\epsilon, z\} = \partial_z^2\epsilon(z) + O(\epsilon^2) \qquad (202)$$

and

$$
\begin{aligned}
P''(z'') = \frac{\partial z'}{\partial z''}\left(\frac{\partial z}{\partial z'}(P(z)+k\,\{z',z\})+k\,\{z'',z'\}\right) &= \frac{\partial z}{\partial z''}(P(z)+k\,\{z'',z\}) \\
\rightarrow \{z'',z\} = \frac{\partial z'}{\partial z}\,\{z'',z'\}+\{z',z\}\,. & \qquad (203)
\end{aligned}
$$

A simple second derivative $\{z', z\} = \frac{\partial^2 z'}{\partial z^2}$ does not solve this equation, so the next guess will be, guided by the form of the Schwarzian derivative,

$$\{z', z\} = \frac{\partial^2 z'}{\partial z^2}\frac{\partial z}{\partial z'} \qquad (204)$$

By expanding the second derivative one can write

$$\begin{aligned}\frac{\partial^2 z''}{\partial z^2}\frac{\partial z}{\partial z''} &= \frac{\partial}{\partial z}\left(\frac{\partial z''}{\partial z'}\frac{\partial z'}{\partial z}\right)\frac{\partial z}{\partial z'}\frac{\partial z'}{\partial z''} = \left(\frac{\partial^2 z''}{\partial z'^2}\left(\frac{\partial z'}{\partial z}\right)^2 + \frac{\partial z''}{\partial z'}\frac{\partial^2 z'}{\partial z^2}\right)\frac{\partial z}{\partial z'}\frac{\partial z'}{\partial z''} \\ &= \frac{\partial z'}{\partial z}\frac{\partial^2 z''}{\partial z'^2}\frac{\partial z'}{\partial z''} + \frac{\partial^2 z'}{\partial z^2}\frac{\partial z}{\partial z'} \end{aligned} \tag{205}$$

to see that the guess was right. So the whole transformation rule for $P(z)$ becomes

$$P'(z') = \frac{\partial z}{\partial z'}\left(P(z) + k\frac{\partial^2 z'}{\partial z^2}\frac{\partial z}{\partial z'}\right) = \frac{\partial z}{\partial z'}P(z) - k\frac{\partial^2 z}{\partial z'^2}\frac{\partial z'}{\partial z}. \tag{206}$$

For $T(z)$ the algebraic effort is a bit more but the steps to go through are nearly the same: The total variation is given by

$$\delta T(z) = -\frac{c}{12}\partial_z^3\epsilon(z) - 2\partial_z\epsilon(z)T(z) - \epsilon(z)\partial_z T(z) - \partial_z\gamma(z)P(z) - k\partial_z^2\gamma(z). \tag{207}$$

For the first three terms the solution is already known: $T\,'(z') = (\frac{\partial z}{\partial z'})^2 T(z) + \frac{c}{12}\{z, z'\}$; the fourth term also appears in the better known version of the algebra with $K \neq 0$ and can be generated by an additional $\frac{\partial z}{\partial z'}\frac{\partial w}{\partial z'}P(z)$. To include the last term one could try as a first naive guess a second derivative $k\frac{\partial^2 w}{\partial z'^2}$. However, the resulting expression

$$T\,'(z') = \left(\frac{\partial z}{\partial z'}\right)^2 T(z) + \frac{c}{12}\{z, z'\} + \frac{\partial z}{\partial z'}\frac{\partial w}{\partial z'}P(z) + k\frac{\partial^2 w}{\partial z'^2} \tag{208}$$

does not compose properly. If one defines $w = w' + g(z') = w'' + h(z'') + g(z')$ one can see this by writing (I supress the first two terms in the transformation)

$$\begin{aligned} T''(z'') = \left(\frac{\partial z'}{\partial z''}\right)^2 &\left(... + \frac{\partial z}{\partial z'}\frac{\partial w}{\partial z'}P(z) + k\frac{\partial^2 w}{\partial z'^2}\right) + \\ &\frac{\partial z'}{\partial z''}\frac{\partial w'}{\partial z''}\left(\frac{\partial z}{\partial z'}P(z) - k\frac{\partial^2 z}{\partial z'^2}\frac{\partial z'}{\partial z}\right) + k\frac{\partial^2 w'}{\partial z''^2} \\ &=!... + \frac{\partial z}{\partial z''}\frac{\partial w}{\partial z''}P(z) + k\frac{\partial^2 w}{\partial z''^2} \end{aligned} \tag{209}$$

$$\begin{aligned} &\rightarrow \frac{\partial z}{\partial z''}\frac{\partial z'}{\partial z''}g'P(z) + k\left(\frac{\partial z'}{\partial z''}\right)^2 g'' + \frac{\partial z}{\partial z''}h'P(z) - \frac{\partial z'}{\partial z''}h'k\frac{\partial^2 z}{\partial z'^2}\frac{\partial z'}{\partial z} + kh'' \\ &=!\frac{\partial z}{\partial z''}\left(g'\frac{\partial z'}{\partial z''} + h'\right)P(z) + k\left(g''\left(\frac{\partial z'}{\partial z''}\right)^2 + g'\frac{\partial^2 z'}{\partial z''^2} + h''\right) \\ &\rightarrow -\frac{\partial z'}{\partial z''}h'k\frac{\partial^2 z}{\partial z'^2}\frac{\partial z'}{\partial z} = kg'\frac{\partial^2 z'}{\partial z''^2} \end{aligned} \tag{210}$$

which is in general not fulfilled (the prime of a function means differentiation with respect to its argument). So the $k\frac{\partial^2 w}{\partial z'^2}$-term has to be modified and an educated guess for this modification would be to include a contribution proportional to the modified Schwarzian derivative $k\frac{\partial^2 z}{\partial z'^2}\frac{\partial z'}{\partial z}$. If we replace

$$k\frac{\partial^2 w}{\partial z'^2} \to k\frac{\partial^2 w}{\partial z'^2} - k\frac{\partial w}{\partial z'}\frac{\partial^2 z}{\partial z'^2}\frac{\partial z'}{\partial z} = k\frac{\partial^2 w}{\partial z'^2} - k\frac{\partial w}{\partial z}\frac{\partial^2 z}{\partial z'^2} \tag{211}$$

this correction vanishes to linear order in ϵ and γ and has the right composition properties:

$$\begin{aligned}
T''(z'') = &\left(\frac{\partial z'}{\partial z''}\right)^2 \left(... + \frac{\partial z}{\partial z'} g' P(z) + k\left(g'' - g'\frac{\partial^2 z}{\partial z'^2}\frac{\partial z'}{\partial z}\right)\right) + \\
&\frac{\partial z}{\partial z''} h' P(z) - k\frac{\partial z'}{\partial z''} h' \frac{\partial^2 z}{\partial z'^2}\frac{\partial z'}{\partial z} + k\left(h'' - h'\frac{\partial^2 z'}{\partial z''^2}\frac{\partial z''}{\partial z'}\right) \\
&=! ... + \frac{\partial z}{\partial z''}\left(g'\frac{\partial z'}{\partial z''} + h'\right) P(z) + \\
&k\left(g''\left(\frac{\partial z'}{\partial z''}\right)^2 + g'\frac{\partial^2 z'}{\partial z''^2} + h'' - \left(h' + g'\frac{\partial z'}{\partial z''}\right)\frac{\partial^2 z}{\partial z''^2}\frac{\partial z''}{\partial z}\right) \\
\\
&\to g'\left(\frac{\partial z'}{\partial z''}\right)^2 \frac{\partial^2 z}{\partial z'^2}\frac{\partial z'}{\partial z} + \frac{\partial z'}{\partial z''} h' \frac{\partial^2 z}{\partial z'^2}\frac{\partial z'}{\partial z} + h'\frac{\partial^2 z'}{\partial z''^2}\frac{\partial z''}{\partial z'} \\
&=! - g'\frac{\partial^2 z'}{\partial z''^2} + h'\frac{\partial^2 z}{\partial z''^2}\frac{\partial z''}{\partial z} + g'\frac{\partial z'}{\partial z}\frac{\partial^2 z}{\partial z''^2}
\end{aligned} \tag{212}$$

This equation is trivially fulfilled after applying the relation

$$\frac{\partial^2 z}{\partial z''^2} = \frac{\partial}{\partial z''}\left(\frac{\partial z}{\partial z'}\frac{\partial z'}{\partial z''}\right) = \frac{\partial z}{\partial z'}\frac{\partial^2 z'}{\partial z''^2} + \left(\frac{\partial z'}{\partial z''}\right)^2 \frac{\partial^2 z}{\partial z'^2}. \tag{213}$$

So, the full transformation law for $T(z)$ reads

$$T'(z') = \left(\frac{\partial z}{\partial z'}\right)^2 T(z) + \frac{c}{12}\{z, z'\} + \frac{\partial z}{\partial z'}\frac{\partial w}{\partial z'} P(z) + k\left(\frac{\partial^2 w}{\partial z'^2} - \frac{\partial w}{\partial z}\frac{\partial^2 z}{\partial z'^2}\right). \tag{214}$$

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