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DISSERTATION

Analysis of Sobolev Spaces in the Context of Convex Geometry and Investigations of the Busemann-Petty Problem

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Kurzfassung

Diese Arbeit besteht aus drei Teilen.

Im ersten Teil wird die Stabilität des unterdimensionalen Busemann-Petty Problems für beliebige Maße gezeigt. Daraus resultiert eine Verallgemeinerung der sogenannten Hyperebenen Ungleichung für Schnittkörper, wobei das Volumen durch ein beliebiges Maß mit einer stetigen Dichte ersetzt wird und Schnitte von beliebigen Dimensionen $n-k$, $1 \leq k < n$ betrachtet werden. Dieser Teil beruht auf der gemeinsamen Arbeit mit Koldobsky [40].

Bourgain, Brezis & Mironescu [7] haben gezeigt, dass unter geeigneter Skalierung, die gebrochene Sobolev s-Halbnorm einer Funktion $f \in W^{1,p}(\mathbb{R}^n)$ zur Sobolev Halbnorm von f konvergiert, wenn $s \to 1^-$. Ludwig [55] benutzte ein Minkowski-Funktional mit der Einheitskugel K um eine anisotrope gebrochene Sobolev s-Halbnorm von f einzuführen. Diese konvergieren, wenn $s \to 1^-$ zu der anisotropen Sobolev Halbnorm von f, die durch ein Minkowski-Funktional, das als Einheitskugel den polaren L_p Momentenkörper von K besitzt, definiert ist. In diesem Teil wird gezeigt, dass asymmetrische anisotrope s-Halbnormen zu der anisotropen Sobolev Halbnorm von f konvergieren wenn $s \to 1^-$, welche durch ein Minkowski-Funktional mit dem polaren asymmetrischen L_p Momentenkörper von K als Einheitskugel definiert werden. Dieser Abschnitt basiert auf [65].

Im letzten, dritten Teil werden reellwertige, stetige, $\mathrm{SL}(n)$ und translationsinvariante Bewertungen auf Sobolevräumen klassifiziert.

Abstract

This thesis consists of three parts.

In the first part, the stability of the lower dimensional Busemann-Petty problem for arbitrary measures is shown. This further yields a generalization of the hyperplane inequality for intersection bodies, where volume is replaced by an arbitrary measure with even continuous density and sections are of arbitrary dimension $n-k$, $1 \leq k < n$. This is based on joint work with Koldobsky [40].

Bourgain, Brezis & Mironescu [7] showed that (with suitable scaling) the fractional Sobolev s-seminorm of a function $f \in W^{1,p}(\mathbb{R}^n)$ converges to the Sobolev seminorm of f as $s \to 1^-$. Ludwig [55] introduced the anisotropic fractional Sobolev s-seminorms of f defined by the norm on $\mathbb{R}^n$ whose unit ball is K. She showed that they converge to the anisotropic Sobolev seminorm of f defined by the norm whose unit ball is the polar L_p moment body of K as $s \to 1^-$. In the second part, the asymmetric anisotropic s-seminorms are shown to converge to the anisotropic Sobolev seminorm of f defined by the Minkowski functional of the polar asymmetric L_p moment body of K as $s \to 1^-$. This is based on [65].

In the third part, continuous, $\mathrm{SL}(n)$ and translation invariant real-valued valuations on Sobolev spaces are classified.

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Chapter 1

Introduction

The concept of convex bodies (i.e. compact convex sets) is the main theme throughout this thesis. In the first part, we study sections of convex bodies. In the second part, the anisotropic Sobolev seminorm defined by the Minkowski functional of the polar asymmetric L_p moment body of a convex body is shown to be the limit of the asymmetric anisotropic Sobolev norm. In the third part, the classification of real-valued valuations on Sobolev spaces makes use of the centro-affine Hadwiger characterization theorem for convex polytopes.

The first part of this thesis starts with Minkowski's uniqueness theorem for sections. It states that an origin-symmetric star body (i.e. compact sets such that every line through the origin meets them in a line segment) in $\mathbb{R}^n$ is uniquely determined by the volumes of its central hyperplane sections in all directions (see, for example, [36, Corollary 3.9]). Does there exist a volume comparison via central hyperplane sections? Busemann and Petty [10] asked if given two origin-symmetric convex bodies K and L in $\mathbb{R}^n$ such that

$$\mathrm{Vol}_{n-1}(K \cap H) \leq \mathrm{Vol}_{n-1}(L \cap H)$$

for every central hyperplane H in $\mathbb{R}^n$, does it follow that

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L).$$

After more than forty years, the answer turned out to be affirmative if $n \leq 4$ and negative if $n \geq 5$ (see, for example, [18, 36]).

For $1 \leq k \leq n$, an origin-symmetric star body is also uniquely determined by the volumes of all of its $(n-k)$-dimensional subspaces sections (see, for example, [36, Corollary 3.10]). It is natural to ask for a volume comparison via sections of lower dimensions. Suppose that for every $(n-k)$-dimensional subspace $H \subset \mathbb{R}^n$,

$$\mathrm{Vol}_{n-k}(K \cap H) \leq \mathrm{Vol}_{n-k}(L \cap H).$$

Does it follow that

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L)?$$

Bourgain and Zhang [9] provided a negative answer to this lower dimensional Busemann-Petty problem when $n - k > 3$. However, it still remains open in the cases of two- and three-dimensional sections.

Instead of Lebesgue measure, one can also ask for volume comparisons via arbitrary measures. Let f be an even continuous non-negative function on $\mathbb{R}^n$, and denote by μ the measure on $\mathbb{R}^n$ with density f, that is, for every compact set $B \subset \mathbb{R}^n$, we define

$$\mu(B) = \int_B f.$$

Zvavitch [87] gave an affirmative answer to the Busemann-Petty problem for arbitrary measures for $n \leq 4$, while the general answer is negative for $n \geq 5$.

Stability is a step further than the volume comparison. Here we ask how the perturbations of the volumes (or measures) of sections affect the volume comparison. Suppose that $\varepsilon > 0$ and for every $\xi \in S^{n-1}$,

$$g_K(\xi) \leq g_L(\xi) + \varepsilon,$$

where g_K is a volume (or measure) of sections of K. Does there exist a constant c not dependent on ε and such that for every ε

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \leq \mathrm{Vol}_n(L)^{\frac{n-1}{n}} + c\varepsilon?$$

Koldobsky [37,38] obtained the stabilities for the Busemann-Petty problem (for arbitrary measures). Stabilities for other geometric inequalities can be found in [21] and references therein.

In this part, in joint work with Koldobsky [40], we establish the stability in the affirmative case of the lower dimensional Busemann-Petty problem.

Theorem 1.1. *Let K and L be origin-symmetric star bodies in $\mathbb{R}^n$, and $1 \le k < n$. Suppose K is a generalized k-intersection body (see Chapter 2 for definitions and properties) and $\varepsilon > 0$. If for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$*

$$\mathrm{Vol}_{n-k}(K \cap H) \le \mathrm{Vol}_{n-k}(L \cap H) + \varepsilon, \tag{1.1}$$

then

$$\mathrm{Vol}_n(K)^{\frac{n-k}{n}} \le \mathrm{Vol}_n(L)^{\frac{n-k}{n}} + c_{n,k}\ \varepsilon, \tag{1.2}$$

where $c_{n,k} = |B_2^n|^{(n-k)/n}/|B_2^{n-k}|$ and $|B_2^n|$ is the volume of the unit Euclidean ball.

The stability of the lower dimensional Busemann-Petty problem for arbitrary measures is as follows.

Theorem 1.2. *Let K and L be origin-symmetric star bodies in $\mathbb{R}^n$, and $1 < k < n$. Suppose K is a generalized k-intersection body and $\varepsilon > 0$. If for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$*

$$\mu(K \cap H) \le \mu(L \cap H) + \varepsilon, \tag{1.3}$$

then

$$\mu(K) \le \mu(L) + \frac{n}{n-k} c_{n,k} \mathrm{Vol}_n(K)^{k/n} \varepsilon.$$

Furthermore, we apply these stability results to the slicing problem (or hyperplane conjecture), which is one of the main open problems in the asymptotic theory of convex bodies. It asks whether every origin-symmetric convex body of volume 1 has a hyperplane section through the origin whose volume is greater than an absolute constant $1/C$. This problem was posed by Bourgain [5]. The best-to-date estimate $C \sim n^{1/4}$ is due to Klartag [34], who removed the logarithmic term from the previous estimate of Bourgain [6]. As shown in Milman and Pajor's famous paper [70], it is equivalent to ask whether there exists an absolute constant C such that for every $n \in \mathbb{N}$ and every origin-symmetric convex body $K \subset \mathbb{R}^n$

$$(\mathrm{Vol}_n(K))^{\frac{n-k}{n}} \le C^k \max_{H \in G(n,n-k)} \mathrm{Vol}_{n-k}(K \cap H), \tag{1.4}$$

where $G(n, n-k)$ is the Grassmannian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$. We prove a generalization of this inequality to arbitrary measures for generalized k-intersection bodies.

Corollary 1.3. *Let $1 \leq k < n$, and suppose that K is a generalized k-intersection body in $\mathbb{R}^n$. Then*

$$\mu(K) \leq \frac{n}{n-k} c_{n,k} \max_{H \in G(n,n-k)} \mu(K \cap H) \ \mathrm{Vol}_n(K)^{k/n}. \tag{1.5}$$

In the second part of this thesis, we study the limit behavior of asymmetric anisotropic fractional Sobolev norms. Let Ω be an open set in $\mathbb{R}^n$. For $p \geq 1$ and $0 < s < 1$, Gagliardo introduced the *fractional Sobolev spaces*

$$W^{s,p}(\Omega) = \left\{ f \in L^p(\Omega) : \frac{|f(x) - f(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\},$$

and the *fractional Sobolev s-seminorm* of a function $f \in L^p(\Omega)$

$$\|f\|^p_{W^{s,p}(\Omega)} = \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+ps}} dxdy$$

(see [16]). They have found many applications in pure and applied mathematics (see [8, 13, 66]).

Although $\|f\|_{W^{s,p}(\Omega)} \to \infty$ as $s \to 1^-$, Bourgain, Brezis and Mironescu showed in [7] that

$$\lim_{s \to 1^-} (1-s) \|f\|^p_{W^{s,p}(\Omega)} = \frac{K_{n,p}}{p} \|f\|^p_{W^{1,p}(\Omega)}, \tag{1.6}$$

for $f \in W^{1,p}(\Omega)$ and $\Omega \subset \mathbb{R}^n$ a smooth bounded domain, where

$$K_{n,p} = \frac{2\Gamma((p+1)/2)\pi^{(n-1)/2}}{\Gamma((n+p)/2)}$$

is a constant depending on n and p,

$$\|f\|^p_{W^{1,p}} = \int_\Omega |\nabla f(x)|^p \, dx$$

is the *Sobolev seminorm* of f, and $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ denotes the *L^p weak derivative* of f. Throughout this thesis, the Sobolev space on $\mathbb{R}^n$ with indices k and p is denoted by $W^{k,p}(\mathbb{R}^n)$ (see Chapter 4 for precise definitions).

If instead of the Euclidean norm $|\cdot|$, we consider an arbitrary norm $\|\cdot\|_K$ with unit ball K, we obtain the *anisotropic Sobolev seminorm*,

$$\|f\|_{W^{1,p},K}^p = \int\limits_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}^p \, dx,$$

where $K^* = \{v \in \mathbb{R}^n : v \cdot x \leq 1 \text{ for all } x \in K\}$ is the *polar body* of K, and $v \cdot x$ denotes the inner product between v and x. Anisotropic Sobolev seminorms and the corresponding Sobolev inequalities attracted a lot of attentions in recent years (see [3, 12, 15, 22]).

Anisotropic s-seminorms, introduced very recently by Ludwig [55], reflect a fine structure of the anisotropic fractional Sobolev spaces. She established that

$$\lim_{s\to 1^-} (1-s) \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+ps}} dxdy = \frac{2}{p} \int\limits_{\mathbb{R}^n} \|\nabla f(x)\|_{Z_p^* K}^p \, dx,$$

for $f \in W^{1,p}(\mathbb{R}^n)$ with compact support, where the norm associated with $Z_p^* K$, the polar L_p moment body of K, is defined as

$$\|v\|_{Z_p^* K}^p = \frac{n+p}{2} \int\limits_K |v \cdot x|^p \, dx,$$

for $v \in \mathbb{R}^n$, and a convex body $K \subset \mathbb{R}^n$. Several different other cases were considered in [54, 55, 76].

In this part, by replacing the absolute value $|\cdot|$ by the positive part $(\cdot)_+$, for $x \in \mathbb{R}$, where $(x)_+ = \max\{0, x\}$, we obtain the following generalization [65]. Note that here it is no longer required that K is origin-symmetric. As a consequence, for $K \subset \mathbb{R}^n$ a convex body containing the origin in its interior and $x \in \mathbb{R}^n$,

$$\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}$$

just defines the *Minkowski functional* of K and no longer a norm.

Theorem 1.4. *If $f \in W^{1,p}(\mathbb{R}^n)$ has compact support, then*

$$\lim_{s\to 1^-} (1-s) \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{(f(x)-f(y))_+^p}{\|x-y\|_K^{n+sp}} dxdy = \frac{1}{p} \int\limits_{\mathbb{R}^n} \|\nabla f(x)\|_{Z_p^{+,*} K}^p \, dx,$$

where $Z_p^{+,*} K$ is the polar asymmetric L_p moment body of K.

For a convex body $K \subset \mathbb{R}^n$, the polar *asymmetric L_p moment body* is the unit ball of the Minkowski functional defined by

$$\|v\|^p_{Z_p^{+,*}K} = (n+p) \int\limits_K (v \cdot x)_+^p \, dx,$$

for $v \in \mathbb{R}^n$, $Z_p^- K = Z_p^+(-K)$. For $p > 1$, in [48], Ludwig introduced and characterized the two-parameter family

$$c_1 \cdot Z_p^+ K +_p c_2 \cdot Z_p^- K$$

as all possible L_p analogs of moment bodies, including the symmetric case

$$Z_p K = \frac{1}{2} \cdot Z_p^+ K +_p \frac{1}{2} \cdot Z_p^- K,$$

where $\|\cdot\|^p_{(\alpha \cdot K +_p \beta \cdot L)^*} = \alpha \|\cdot\|^p_{K^*} + \beta \|\cdot\|^p_{L^*}$, for $\alpha, \beta \geq 0$, defines the *L_p Minkowski combination*. In recent years, this family of convex bodies have found important applications within convex geometry, probability theory, and the local theory of Banach spaces (see [18, 25, 29, 47, 48, 50, 59–64, 72–75, 84]).

The proof given in this part makes use of an asymmetric version of the one-dimensional case of result (1.6) by Bourgain, Brezis and Mironescu and an asymmetric decomposition of Blaschke-Petkantschin type.

In the third part of this thesis, continuous, $\mathrm{SL}(n)$ and translation invariant real-valued valuations on Sobolev spaces are classified. A function z defined on a lattice $(\mathcal{L}, \vee, \wedge)$ and taking values in an abelian semigroup is called a *valuation* if

$$z(f \vee g) + z(f \wedge g) = z(f) + z(g) \tag{1.7}$$

for all $f, g \in \mathcal{L}$. A function z defined on some subset $\mathcal{M}$ of $\mathcal{L}$ is called a valuation on $\mathcal{M}$ if (1.7) holds whenever $f, g, f \vee g, f \wedge g \in \mathcal{M}$. Valuations were a key part of Dehn's solution of Hilbert's Third Problem in 1901; they are closely related to dissections and lie at the very heart of geometry. Here, valuations were considered on the space of convex bodies in $\mathbb{R}^n$, denoted by $\mathcal{K}^n$. Perhaps the most famous result is the Hadwiger characterization theorem on this space which classifies all continuous and rigid motion invariant real-valued valuations. Important later contributions can be found in [30, 33, 67, 68]. As for recent results, we refer to [1, 2, 23–

26, 28, 31, 32, 44, 46, 47, 49, 56, 57, 74, 75, 77–79, 84]. For later reference, we state here a centro-affine version of the Hadwiger characterization theorem on the space of convex polytopes containing the origin in their interiors, which is denoted by $\mathcal{P}_0^n$.

Theorem 1.5 ([27]). *A map $Z : \mathcal{P}_0^n \to \mathbb{R}$ is an upper semicontinuous and* $\mathrm{SL}(n)$ *invariant valuation if and only if there exist constants $c_0, c_1, c_2 \in \mathbb{R}$ such that*

$$Z(P) = c_0 + c_1 \mathrm{Vol}_n(P) + c_2 \mathrm{Vol}_n(P^*)$$

for all $P \in \mathcal{P}_0^n$.

Valuations are also considered on spaces of real-valued functions. Here, we take pointwise maximum and minimum as join and meet, respectively. Two important functions associated with every convex body K in $\mathbb{R}^n$ are the *indicator function* $\mathbb{1}_K$ and the *support function* $h(K, \cdot)$, where $h(K, u) = \max \{u \cdot x : x \in K\}$ for every $u \in \mathbb{R}^n$. As each of them is in one-to-one correspondence with K, valuations on these function spaces are often considered to be valuations on convex bodies.

Since 2010, valuations on other classical function spaces started to be characterized. Tsang [81] characterized real-valued valuations on L^p-spaces.

Theorem A ([81]). *A functional $z : L^p(\mathbb{R}^n) \to \mathbb{R}$ is a continuous translation invariant valuation if and only if there exists a continuous function on $\mathbb{R}$ with the property that there exists $c \geq 0$ such that $|h(x)| \leq c\,|x|^p$ for all $x \in \mathbb{R}$ and*

$$z(f) = \int_{\mathbb{R}^n} h \circ f$$

for every $f \in L^p(\mathbb{R}^n)$.

Kone [42] generalized this characterization to Orlicz spaces. As for valuations on Sobolev spaces, Ludwig characterized the Fisher information matrix and the optimal Sobolev body. The additive group of real symmetric $n \times n$ matrices is denoted by $\langle \mathbb{M}^n, + \rangle$. An operator $z : W^{1,2}(\mathbb{R}^n) \to \langle \mathbb{M}^n, + \rangle$ is called $\mathrm{GL}(n)$ *contravariant* if for some $p \in \mathbb{R}$,

$$z\left(f \circ \phi^{-1}\right) = |\det \phi|^p \, \phi^{-t} z(f) \phi^{-1}$$

for all $f \in W^{1,2}(\mathbb{R}^n)$ and $\phi \in \mathrm{GL}(n)$, where $\det \phi$ is the determinant of ϕ and ϕ^{-t} denotes the inverse of the transpose of ϕ. An operator

$z : W^{1,2}(\mathbb{R}^n) \to \langle \mathbb{M}^n, + \rangle$ is called *affinely contravariant* if it is $\mathrm{GL}(n)$ contravariant, translation invariant and homogeneous (see Chapter 4 for precise definitions).

Theorem B ([51]). *An operator $z : W^{1,2}(\mathbb{R}^n) \to \langle \mathbb{M}^n, + \rangle$, where $n \geq 3$, is a continuous and affinely contravariant valuation if and only if there is a constant $c \in \mathbb{R}$ such that*

$$z(f) = c \int_{\mathbb{R}^n} \nabla f \otimes \nabla f$$

for every $f \in W^{1,2}(\mathbb{R}^n)$.

Other recent and interesting characterizations can be found in [4, 11, 53, 71, 82, 83].

In this part, we classify real-valued valuations on $W^{1,p}(\mathbb{R}^n)$. The result regarding homogeneous valuations is stated first. Let $1 \leq p < n$ within this part. We say that a valuation is *trivial* if it is identically zero.

Theorem 1.6. *A functional $z : W^{1,p}(\mathbb{R}^n) \to \mathbb{R}$ is a non-trivial continuous, $\mathrm{SL}(n)$ and translation invariant valuation that is homogeneous of degree q if and only if $p \leq q \leq \frac{np}{n-p}$ and there exists a constant $c \in \mathbb{R}$ such that*

$$z(f) = c \, \|f\|_q^q \tag{1.8}$$

for every $f \in W^{1,p}(\mathbb{R}^n)$.

It is natural to consider the same characterization without the assumption of homogeneity. It turns out to be more complicated and costs additional assumptions. We first fix the following notation. Let $C^k(\mathbb{R}^n)$ denote the space of functions on $\mathbb{R}^n$ that have k times continuous partial derivatives for a positive integer k; let $BV_{loc}(\mathbb{R})$ denote the space of functions on $\mathbb{R}$ that are of locally bounded variation. We denote by $\mathcal{G}_p$ the class of functions g that belong to $BV_{loc}(\mathbb{R})$ and satisfy

$$g(x) \sim \begin{cases} O(x^p), & \text{as } x \to 0; \\ O(x^{\frac{np}{n-p}}), & \text{as } x \to \infty. \end{cases} \tag{1.9}$$

and by $\mathcal{B}_p$ the class of functions g that belong to $C^n(\mathbb{R})$ with $g^{(n)} \in BV_{loc}(\mathbb{R})$ and $x^k g^{(k)}(x)$ satisfying (1.9) for each integer $1 \leq k \leq n$. Let $P^{1,p}(\mathbb{R}^n)$ be the set of functions ℓ_P with $P \in \mathcal{P}_0^n$ that enclose pyramids of height 1 on P (see Chapter 4 for the precise definition).

Theorem 1.7. *A functional* $z : W^{1,p}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation with* $z(0) = 0$ *and* $s \mapsto z(sf)$ *in* $\mathcal{B}_p$ *for* $s \in \mathbb{R}$ *and* $f \in P^{1,p}(\mathbb{R}^n)$ *if and only if there exists a continuous function* $h \in \mathcal{G}_p$ *such that*

$$z(f) = \int_{\mathbb{R}^n} h \circ f,$$

for every $f \in W^{1,p}(\mathbb{R}^n)$.

Chapter 2

Stability and slicing inequalities for intersection bodies

The *radial function* of a star body K is defined by

$$\rho_K(x) = \|x\|_K^{-1}, \qquad x \in \mathbb{R}^n.$$

If $x \in S^{n-1}$ then $\rho_K(x)$ is the radius of K in the direction of x.

Writing the volume of K in polar coordinates, one gets

$$\mathrm{Vol}_n(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^n(\theta)d\theta = \frac{1}{n}\int_{S^{n-1}} \|\theta\|_K^{-n}d\theta. \tag{2.1}$$

The *spherical Radon transform* $R : C(S^{n-1}) \mapsto C(S^{n-1})$ is a linear operator defined by

$$Rf(\xi) = \int_{S^{n-1}\cap\xi^\perp} f(x)\ dx, \quad \xi \in S^{n-1}$$

for every function $f \in C(S^{n-1})$.

The polar formula (2.1) for the volume of a hyperplane section expresses this volume in terms of the spherical Radon transform (see for example [36, p.15]):

$$S_K(\xi) = \mathrm{Vol}_{n-1}(K \cap \xi^\perp) = \frac{1}{n-1}R(\|\cdot\|_K^{-n+1})(\xi). \tag{2.2}$$

The spherical Radon transform is self-dual (see [21, Lemma 1.3.3]): for any functions $f, g \in C(S^{n-1})$

$$\int_{S^{n-1}} Rf(\xi)\ g(\xi)\ d\xi = \int_{S^{n-1}} f(\xi)\ Rg(\xi)\ d\xi. \tag{2.3}$$

Using self-duality, one can extend the spherical Radon transform to measures. Let μ be a finite Borel measure on S^{n-1}. We define the spherical Radon transform of μ as a functional $R\mu$ on the space $C(S^{n-1})$ acting by

$$(R\mu, f) = (\mu, Rf) = \int_{S^{n-1}} Rf(x)d\mu(x).$$

By Riesz's characterization of continuous linear functionals on the space $C(S^{n-1})$, $R\mu$ is also a finite Borel measure on S^{n-1}. If μ has continuous density g, then by (2.3) the Radon transform of μ has density Rg.

The class of intersection bodies was introduced by Lutwak [58]. Let K, L be origin-symmetric star bodies in $\mathbb{R}^n$. We say that K is the intersection body of L if the radius of K in every direction is equal to the $(n-1)$-dimensional volume of the section of L by the central hyperplane orthogonal to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\rho_K(\xi) = \|\xi\|_K^{-1} = \mathrm{Vol}_{n-1}(L \cap \xi^\perp). \tag{2.4}$$

All the bodies K that appear as intersection bodies of different star bodies form *the class of intersection bodies of star bodies.*

Note that the right-hand side of (2.4) can be written in terms of the spherical Radon transform using (2.2):

$$\|\xi\|_K^{-1} = \frac{1}{n-1}\int_{S^{n-1}\cap\xi^\perp} \|\theta\|_L^{-n+1} d\theta = \frac{1}{n-1} R(\|\cdot\|_L^{-n+1})(\xi).$$

It means that a star body K is the intersection body of a star body if and only if the function $\|\cdot\|_K^{-1}$ is the spherical Radon transform of a continuous positive function on S^{n-1}. This allows to introduce a more general class of bodies. A star body K in $\mathbb{R}^n$ is called an *intersection body* if there exists a finite Borel measure μ on the sphere S^{n-1} so that $\|\cdot\|_K^{-1} = R\mu$ as functionals on $C(S^{n-1})$, i.e. for every continuous function f on S^{n-1},

$$\int_{S^{n-1}} \|x\|_K^{-1} f(x)\ dx = \int_{S^{n-1}} Rf(x)\ d\mu(x). \tag{2.5}$$

Intersection bodies played the crucial role in the solution of the original Busemann-Petty problem due to the following connection found by Lutwak [58]. If K in an origin-symmetric intersection body in $\mathbb{R}^n$ and L is any origin-symmetric star body in $\mathbb{R}^n$, then the inequalities $S_K(\xi) \leq S_L(\xi)$ for all $\xi \in S^{n-1}$ imply that $\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L)$, i.e. the answer to the Busemann-Petty problem in this situation is affirmative. For more information about intersection bodies, see [36, Chapter 4], [41], [18, Chapter 8] and references there. In particular, every origin-symmetric convex body in $\mathbb{R}^n$, $n \leq 4$ is an intersection body; see [17, 19, 86]. Also the unit ball of any finite dimensional subspace of L_p, $0 < p \leq 2$ is an intersection body; see [35].

Zhang in [85] introduced a generalization of intersection bodies. For $1 \leq k \leq n-1$, the *(n − k)-dimensional spherical Radon transform* is an operator $\mathcal{R}_{n-k} : C(S^{n-1}) \mapsto C(G(n, n-k))$ defined by

$$\mathcal{R}_{n-k}\,(f)\,(H) = \int_{S^{n-1}\cap H} f(x)dx, \quad H \in G(n, n-k).$$

Denote the image of the operator $\mathcal{R}_{n-k}$ by X:

$$\mathcal{R}_{n-k}\left(C(S^{n-1})\right) = X \subset C(G(n, n-k)).$$

Let $M^+(X)$ be the space of linear positive continuous functionals on X, i.e. for every $\nu \in M^+(X)$ and non-negative function $f \in X$, we have $\nu(f) \geq 0$.

An origin-symmetric star body K in $\mathbb{R}^n$ is called a *generalized k-intersection body* if there exists a functional $\nu \in M^+(X)$, so that for every $f \in C(S^{n-1})$,

$$\int_{S^{n-1}} \|x\|_K^{-k} f(x)dx = \nu(\mathcal{R}_{n-k}\,(f)).$$

When $k = 1$ we get the class of intersection bodies. It was proved by Grinberg and Zhang [20, Lemma 6.1] that every intersection body in $\mathbb{R}^n$ is a generalized k-intersection body for every $k < n$. More generally, as proved later by Milman [69], if m divides k, then every generalized m-intersection body is a generalized k-intersection body. Zhang [85] showed that the answer to the lower dimensional Busemann-Petty problem is affirmative if and only if every origin-symmetric convex body in $\mathbb{R}^n$ is a generalized k-intersection body.

Denote by $1_S \equiv 1$ and $1_G \equiv 1$ the functions which are equal to 1 everywhere on the unit sphere S^{n-1} and the Grassmannian $G(n, n-k)$, correspondingly. Then, $\mathcal{R}_{n-k}\,(1_S) = |S^{n-k-1}|\; 1_G$.

We are now ready to prove the stability in the lower dimensional Busemann-Petty problem.

Proof of Theorem 1.1. By the polar formula for volume (2.1), for each $H \in G(n, n-k)$ we have

$$\mathrm{Vol}_{n-k}(K \cap H) = \frac{1}{n-k} \mathcal{R}_{n-k} \left(\|\cdot\|_K^{-n+k} \right)(H), \tag{2.6}$$

Then the inequality (1.1) can be written as

$$\mathcal{R}_{n-k} \left(\|\cdot\|_K^{-n+k} \right)(H) \leq \mathcal{R}_{n-k} \left(\|\cdot\|_L^{-n+k} \right)(H) + (n-k)\varepsilon. \tag{2.7}$$

Since K is a generalized k-intersection body, there exists $\mu_0 \in M^+$, such that for each $\psi \in C\left(S^{n-1}\right)$,

$$\int_{S^{n-1}} \|x\|_K^{-k} \psi(x) dx = \mu_0(\mathcal{R}_{n-k}(\psi)). \tag{2.8}$$

Since μ_0 is a positive functional, by (2.7) and (2.8), we have

$$\begin{aligned} n\mathrm{Vol}_n(K) &= \int_{S^{n-1}} \|x\|_K^{-k} \|x\|_K^{-n+k} dx \\ &= \mu_0 \left(\mathcal{R}_{n-k} \left(\|\cdot\|_K^{-n+k} \right) \right) \\ &\leq \mu_0 \left(\mathcal{R}_{n-k} \left(\|\cdot\|_L^{-n+k} \right) \right) + (n-k)\varepsilon\mu_0(1_G) \\ &:= \mathrm{I} + \mathrm{II}. \end{aligned} \tag{2.9}$$

Using (2.8), Hölder's inequality and polar formula for the volume, we get

$$\begin{aligned} \mathrm{I} &= \int_{S^{n-1}} \|x\|_K^{-k} \|x\|_L^{-n+k} dx \\ &\leq \left(\int_{S^{n-1}} \|x\|_K^{-n} dx \right)^{k/n} \left(\int_{S^{n-1}} \|x\|_L^{-n} dx \right)^{(n-k)/n} \\ &= n\mathrm{Vol}_n(K)^{k/n} \mathrm{Vol}_n(L)^{(n-k)/n}. \end{aligned} \tag{2.10}$$

Now, by (2.8), the well-known formula $|S^{n-1}| = n|B_2^n|$ (see [36, p. 33]) and

Hölder's inequality,

$$\begin{aligned}
\mathrm{II} &= (n-k)\varepsilon\mu_0(1_G) = \frac{(n-k)\varepsilon}{|S^{n-k-1}|}\int_{S^{n-1}} \|x\|_K^{-k}\ 1_S(x)\ dx \\
&\leq \frac{(n-k)\varepsilon}{|S^{n-k-1}|}\left(\int_{S^{n-1}} \|x\|_K^{-n} dx\right)^{k/n} |S^{n-1}|^{\frac{n-k}{n}} \\
&= \frac{n^{k/n}(n-k)|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|}\mathrm{Vol}_n(K)^{k/n}\varepsilon \\
&= \frac{n|B_2^n|^{\frac{n-k}{n}}}{|B_2^{n-k-1}|}\mathrm{Vol}_n(K)^{k/n}\varepsilon.
\end{aligned}$$

Combining this with (2.9) and (2.10), we get the result. $\square$

Remark 2.1. Note that in (1.2) $c_{n,k} < 1$, which immediately follows from the log-convexity of the gamma-function (see, for example, [39, Lemma 2.1]). Also, in the formulation of Theorem 1 in [38] the constant $c_{n,1}$ was replaced by 1, though the proof there gives the result with $c_{n,1}$.

We now pass to stability for arbitrary measures. Let μ be a measure on $\mathbb{R}^n$ with even continuous density f. The measure μ of a star body K can be expressed in polar coordinates as follows:

$$\begin{aligned}
\mu(K) &= \int_K f(x)\ dx = \int_{\mathbb{R}^n} \mathbb{1}_{[0,1]}(\|x\|_K) f(x)\ dx \\
&= \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta)\ dr\right) d\theta. \qquad (2.11)
\end{aligned}$$

Similarly, we can express the volume of a section of K by an $(n-k)$ -dimensional subspace H of $\mathbb{R}^n$ as

$$\begin{aligned}
\mu(K\cap H) &= \int_H \mathbb{1}_{[0,1]}\left(\|x\|_K\right) f(x) dx \\
&= \int_{S^{n-1}\cap H} \left(\int_0^{\|\theta\|_K^{-1}} t^{n-k-1} f\left(t\theta\right) dt\right) d\theta \\
&= \mathcal{R}_{n-k}\left(\int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\ dr\right)(H), \qquad (2.12)
\end{aligned}$$

where the Radon transform is applied to a function of the variable $\theta \in S^{n-1}$.

We need the following lemma, which was also used by Zvavitch in his proof.

Lemma 2.2. *Let $a, b, k \in \mathbb{R}^+$, and α be a non-negative function on $(0, \max\{a, b\})$, such that the integral below converges. Then*

$$\int_0^a r^{n-1}\alpha(r)\ dt - a^k \int_0^a r^{n-k-1}\alpha(r)\ dr$$

$$\leq \int_0^b r^{n-1}\alpha(r)\ dr - a^k \int_0^b r^{n-k-1}\alpha(r)\ dr$$

Proof. The result follows from

$$a^k \int_a^b r^{n-k-1}\alpha(r)\ dr \leq \int_a^b r^{n-1}\alpha(r)\ dr.$$

$\square$

Proof of Theorem 1.2. Using (2.12), inequality (1.3) can be written as

$$\mathcal{R}_{n-k}\left(\int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\ dr\right)(H) \tag{2.13}$$

$$\leq \mathcal{R}_{n-k}\left(\int_0^{\|\theta\|_L^{-1}} r^{n-k-1} f(r\theta)\ dr\right)(H) + \varepsilon, \qquad \forall H \in G(n, n-k).$$

As in the proof of Theorem 1.1, let μ_0 be the positive functional associated with the generalized k-intersection body K. Applying μ_0 to both sides of (2.13) and then using (2.8), we get

$$\int_{S^{n-1}} \|\theta\|_K^{-k}\left(\int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\ dr\right) d\theta \tag{2.14}$$

$$\leq \int_{S^{n-1}} \|\theta\|_K^{-k}\left(\int_0^{\|\theta\|_L^{-1}} r^{n-k-1} f(r\theta)\ dr\right) d\theta + \varepsilon\mu_0(1_G).$$

Applying Lemma 2.2 with $a = \|\theta\|_K^{-1}$, $b = \|\theta\|_L^{-1}$ and $\alpha(r) = f(r\theta)$ and then integrating over the sphere, we get

$$\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta)\ dr - \|\theta\|_K^{-k} \int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\ dr$$

$$\leq \int_0^{\|\theta\|_L^{-1}} r^{n-1} f(r\theta)\, dr - \|\theta\|_K^{-k} \int_0^{\|\theta\|_L^{-1}} r^{n-k-1} f(r\theta)\, dr,$$

and

$$\begin{aligned} &\int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta)\, dr \right) d\theta \qquad (2.15)\\ - &\int_{S^{n-1}} \|\theta\|_K^{-k} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\, dr \right) d\theta \\ \leq &\int_{S^{n-1}} \left(\int_0^{\|\theta\|_L^{-1}} r^{n-1} f(r\theta)\, dr \right) d\theta \\ - &\int_{S^{n-1}} \|\theta\|_K^{-k} \left(\int_0^{\|\theta\|_L^{-1}} r^{n-k-1} f(r\theta)\, dr \right) d\theta. \end{aligned}$$

Adding (2.14) and (2.15) and using (2.11) we get

$$\mu(K) \leq \mu(L) + \varepsilon \mu_0(1_G).$$

As shown in the proof of Theorem 1.1,

$$\mu_0(1_G) \leq \frac{n}{n-k} \mathrm{Vol}_n(K)^{k/n},$$

which completes the proof. □

Remark 2.3. In Theorem 1.2, in the case $f \equiv 1$, we get another stability result for volume which is weaker than what is provided by Theorem 1.1. This is the reason why we state Theorem 1.1 separately. However, for arbitrary measures the constant in Theorem 1.2 is the best possible, as follows from the example after Corollary 1.3.

In the case where $k = 1$ and K is an intersection body, the inequality (1.4) is known for sections of arbitrary dimension with the best possible constant. In particular, if the dimension $n \leq 4$, then (1.4) is true for any origin-symmetric convex body K. The proof is an immediate consequence of Zhang's connection between generalized intersection bodies and the lower dimensional Busemann-Petty problem; apply this connection to any generalized k-intersection body K and $L = B_2^n$. Then use the fact that every intersection body is a generalized k-intersection body for every k (see [20] or [69]). For every fixed k, the inequality (1.4) holds for any generalized k-intersection body.

We prove several generalizations of (1.4) using the stability results formulated above. First, interchanging K and L in Theorem 1.1, we get the following "difference" inequality, previously established in [38, Corollary 1] in the hyperplane case.

Corollary 2.4. *Let K and L be origin-symmetric star bodies in $\mathbb{R}^n$, and $1 \le k < n$. Suppose K and L are generalized k-intersection bodies, then*

$$\left|\mathrm{Vol}_n(K)^{\frac{n-k}{n}} - \mathrm{Vol}_n(L)^{\frac{n-k}{n}}\right|$$

$$\le c_{n,k} \max_{H \in G(n,n-k)} \left|\mathrm{Vol}_{n-k}(K \cap H) - \mathrm{Vol}_{n-k}(L \cap H)\right|.$$

Putting $L = \varnothing$ in the latter inequality, we get (1.4) for any generalized k-intersection body K.

Interchanging K and L in Theorem 1.2, we get the following inequality, which was earlier proved for $k = 1$ in [37, Corollary 1].

Corollary 2.5. *Let K and L be origin-symmetric star bodies in $\mathbb{R}^n$, and $1 \le k < n$. Suppose that K and L are generalized k-intersection bodies. Then*

$$|\mu(K) - \mu(L)| \le$$

$$\frac{n}{n-k} c_{n,k} \max_H |\mu(K \cap H) - \mu(L \cap H)| \max\left\{\mathrm{Vol}_n(K)^{k/n}, \mathrm{Vol}_n(L)^{k/n}\right\},$$

where maximum is taken over all $(n-k)$-dimensional subspaces H of $\mathbb{R}^n$.

Putting $L = \varnothing$, we have Corollary 1.3.

The constant in the right-hand side of (1.5) is the best possible. In fact, let $K = B_2^n$ and, for every $j \in N$, let f_j be a non-negative continuous function on $[0, 1]$ supported in $(1 - \frac{1}{j}, 1)$ and such that $\int_0^1 f_j(t)dt = 1$. Let μ_j be the measure on $\mathbb{R}^n$ with density $f_j(|x|_2)$, where $|x|_2$ is the Euclidean norm. We have

$$\mu_j(B_2^n) = |S^{n-1}| \int_0^1 r^{n-1} f_j(r)dr,$$

where $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit sphere in $\mathbb{R}^n$. For every $H \in G(n, n-k)$,

$$\mu_j(B_2^n \cap H) = |S^{n-k-1}| \int_0^1 r^{n-k-1} f_j(r) dr.$$

Clearly,

$$\lim_{j \to \infty} \frac{\int_0^1 r^{n-1} f_j(r) dr}{\int_0^1 r^{n-k-1} f_j(r) dr} = 1.$$

Using $|S^{n-1}| = n|B_2^n|$, we get

$$\lim_{j \to \infty} \frac{\mu_j(B_2^n)}{\max_H \mu_j(B_2^n \cap H) \ \mathrm{Vol}_n(B_2^n)^{k/n}} = \frac{|S^{n-1}|}{|S^{n-k-1}||B_2^n|^{k/n}} = \frac{n}{n-k} c_{n,k},$$

which shows that the constant is asymptotically optimal.

Chapter 3

Asymmetric anisotropic fractional Sobolev norms

This chapter is devoted to the proof of Theorem 1.4.

First, we need the asymmetric one-dimensional analogue of (1.6). For its proof we require the following result from [7].

Lemma 3.1. *Let $\rho \in L^1(\mathbb{R}^n)$ and $\rho \geq 0$. If $f \in W^{1,p}(\mathbb{R}^n)$ is compactly supported and $1 \leq p < \infty$, then*

$$\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho(x-y)dxdy \leq C\,\|f\|^p_{W^{1,p}}\,\|\rho\|_{L^1}$$

where C depends only on p and the support of f.

Let $\Omega \subset \mathbb{R}$ be a bounded domain.

Proposition 3.2. *If $f \in W^{1,p}(\Omega)$, then*

$$\lim_{s\to 1^-}(1-s)\int\limits_{\Omega}\int\limits_{\Omega\cap\{x>y\}} \frac{(f(x)-f(y))^p_+}{|x-y|^{1+ps}}dxdy = \frac{1}{p}\int\limits_{\Omega}(f'(x))^p_+\,dx. \tag{3.1}$$

Proof. Take a sequence (ρ_ε) of radial mollifiers, i.e. $\rho_\varepsilon(x) = \rho_\varepsilon(|x|)$; $\rho_\varepsilon \geq 0$; $\int_0^\infty \rho_\varepsilon(x)dx = 1$; $\lim\limits_{\varepsilon\to 0}\int_\delta^\infty \rho_\varepsilon(r)dr = 0$ for every $\delta > 0$. Let $F_\varepsilon(x,y) = \frac{(f(x)-f(y))_+}{|x-y|}\rho_\varepsilon^{1/p}(x-y)$, for $x > y$. It suffices to prove that

$$\lim_{\varepsilon\to 0}\int\limits_{\Omega}\int\limits_{\Omega\cap\{x>y\}} {F_\varepsilon}^p(x,y)dxdy = \int\limits_{\Omega}(f'(x))^p_+\,dx. \tag{3.2}$$

Indeed, as in [80], let $R > \max\{|x-y| : x, y \in \Omega\}$, $\varepsilon = 1 - s$ and

$$\rho_\varepsilon(x) = \frac{\mathbb{1}_{[0,R]}(|x|)}{R^{\varepsilon p}} \frac{p\varepsilon}{|x|^{1-p\varepsilon}}.$$

Then one obtains (3.1) from (3.2) as desired.

By Lemma 3.1, we have, for any $\varepsilon > 0$ and $f, g \in W^{1,p}(\Omega)$

$$\left| \|F_\varepsilon\|_{L^p(\Omega\times\Omega)} - \|G_\varepsilon\|_{L^p(\Omega\times\Omega)} \right| \leq \|F_\varepsilon - G_\varepsilon\|_{L^p(\Omega\times\Omega)} \leq C \|f-g\|_{W^{1,p}},$$

for some constant C dependent on ε, f and g. Therefore, it suffices to establish (3.2) for f in some dense subset of $W^{1,p}(\Omega)$, e.g., for $f \in C^2(\bar{\Omega})$, where $\bar{\Omega}$ is the closure of Ω.

Fix $f \in C^2(\bar{\Omega})$. Since for $t \in \mathbb{R}$ and $\lambda > 0$, $(\lambda t)_+ = \lambda (t)_+$, there exists $\delta > 0$, such that for $y < x < y + \delta$ and a constant c,

$$\left| \frac{(f(x)-f(y))_+^p}{|x-y|^p} - (f'(y))_+^p \right| \leq c(x-y).$$

We have

$$\begin{aligned} &\int\limits_{\Omega\cap\{x>y\}} \frac{(f(x)-f(y))_+^p}{|x-y|^p} \rho_\varepsilon(x-y)dx \\ &= \int\limits_{\Omega\cap\{y<x<y+\delta\}} \frac{(f(x)-f(y))_+^p}{|x-y|^p} \rho_\varepsilon(x-y)dx \\ &\quad + \int\limits_{\Omega\cap\{x\geq y+\delta\}} \frac{(f(x)-f(y))_+^p}{|x-y|^p} \rho_\varepsilon(x-y)dx, \end{aligned}$$

yet, only the former integral on the right hand side need be considered, as the latter vanishes. In fact, for each fixed $y \in \Omega$, since

$$\begin{aligned}
&\left| \int_{y}^{y+\delta} \left(\frac{(f(x)-f(y))_+^p}{|x-y|^p} - (f'(y))_+^p \right) \rho_\varepsilon(x-y)dx \right| \\
\leq &\int_{y}^{y+\delta} \left| \frac{(f(x)-f(y))_+^p}{|x-y|^p} - (f'(y))_+^p \right| \rho_\varepsilon(x-y)dx \\
\leq &c \int_{y}^{y+\delta} (x-y)\rho_\varepsilon(x-y)dx \\
= &c \int_{0}^{\delta} r\rho_\varepsilon(r)dr \to 0, \text{ as } \varepsilon \to 0,
\end{aligned}$$

we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{y}^{y+\delta} \frac{(f(x)-f(y))_+^p}{|x-y|^p} \rho_\varepsilon(x-y)dx \\
&= (f'(y))_+^p \lim_{\varepsilon\to 0} \int_{y}^{y+\delta} \rho_\varepsilon(x-y)dx \\
&= (f'(y))_+^p \lim_{\varepsilon\to 0} \int_{0}^{\delta} \rho_\varepsilon(r)dr \\
&= (f'(y))_+^p .
\end{aligned}$$

Therefore,

$$\lim_{\varepsilon\to 0} \int_{\Omega\cap\{x>y\}} \frac{(f(x)-f(y))_+^p}{|x-y|^p} \rho_\varepsilon(x-y)dx = (f'(y))_+^p . \tag{3.3}$$

Since $f \in C^2(\bar{\Omega})$, there exists $L > 0$ is such that $|f(x)-f(y)| < L\,|x-y|$, for every $x, y \in \Omega$, then

$$\int_{\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho_\varepsilon(x-y)dx \leq L^p, \quad \text{for each } y \in \Omega. \tag{3.4}$$

Hence, for $f \in C^2(\Omega)$, (3.2) follows by dominated convergence theorem from (3.3) and (3.4). $\square$

Now, for $u \in S^{n-1}$, the Euclidean unit sphere, let $[u] = \{\lambda u : \lambda \in \mathbb{R}\}$ and $[u]^+ = \{\lambda u : \lambda > 0\}$. Denote the k-dimensional Hausdorff measure on $\mathbb{R}^n$ by H^k. For $f \in W^{1,p}(\mathbb{R}^n)$, we denote by $\bar{f}$ its precise representative (see [14, Section 1.7.1]). We require the following result. For every $u \in S^{n-1}$, the precise representative $\bar{f}$ is absolutely continuous on the lines $L = \{x + \lambda u : \lambda \in \mathbb{R}\}$ for H^{n-1}-a.e. $x \in u^\perp$ and its first-order (classical) partial derivatives belong to $L^p(\mathbb{R}^n)$ (see [14, Section 4.9.2]). Hence, we have for the restriction of $\bar{f}$ to L,

$$\bar{f}\big|_L \in W^{1,p}(L) \tag{3.5}$$

for a.e. line L parallel to u.

Proof of Theorem 1.4. By the polar coordinate formula and Fubini's theorem, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))_+^p}{\|x-y\|_K^{n+sp}} dH^n(x)dH^n(y)\\
&= \int_{\mathbb{R}^n}\int_{S^{n-1}} \|u\|_K^{-(n+ps)} \int_0^\infty \frac{(f(y+ru)-f(y))_+^p}{r^{1+sp}} dH^1(r)d\sigma(u)dH^n(y)\\
&= \int_{S^{n-1}} \|u\|_K^{-(n+ps)} \int_0^\infty \int_{u^\perp} \int_{[u]+z} \frac{(f(w+ru)-f(w))_+^p}{r^{1+sp}} dH^1(w)dH^{n-1}(z)dH^1(r)d\sigma(u)\\
&= \int_{S^{n-1}} \|u\|_K^{-(n+ps)} \int_{u^\perp} \int_{[u]+z} \int_0^\infty \frac{(f(w+ru)-f(w))_+^p}{r^{1+sp}} dH^1(r)dH^1(w)dH^{n-1}(z)d\sigma(u)\\
&= \int_{S^{n-1}} \|u\|_K^{-(n+ps)} \int_{u^\perp} \int_{[u]+z} \int_{[u]^++w} \frac{(f(t)-f(w))_+^p}{|t-w|^{1+sp}} dH^1(t)dH^1(w)dH^{n-1}(z)d\sigma(u),
\end{aligned} \tag{3.6}$$

where σ denotes the standard surface area measure on S^{n-1}. By Proposition

3.2 and (3.5), we obtain

$$\lim_{s\to 1^-}(1-s)\int\limits_{[u]+z}\int\limits_{[u]^++w}\frac{(f(t)-f(w))_+^p}{|t-w|^{1+sp}}dH^1(t)dH^1(w)$$
$$=\frac{1}{p}\int\limits_{[u]+z}(\nabla f(t)\cdot u)_+^p\,dH^1(t). \tag{3.7}$$

By Fubini's theorem and the polar coordinate formula, we get

$$\frac{1}{p}\int\limits_{S^{n-1}}\|u\|_K^{-(n+p)}\int\limits_{u^\perp}\int\limits_{[u]+z}(\nabla f(t)\cdot u)_+^p\,dH^1(t)dH^{n-1}(z)d\sigma(u)$$
$$=\frac{1}{p}\int\limits_{S^{n-1}}\int\limits_{\mathbb{R}^n}\|u\|_K^{-(n+p)}(\nabla f(x)\cdot u)_+^p\,dH^n(x)d\sigma(u)$$
$$=\frac{n+p}{p}\int\limits_{K}\int\limits_{\mathbb{R}^n}(\nabla f(x)\cdot y)_+^p\,dH^n(x)dH^n(y)$$

Using Fubini's theorem and the definition of the asymmetric L_p moment body of K, we obtain

$$\int\limits_{S^{n-1}}\|u\|_K^{-(n+p)}\int\limits_{u^\perp}\int\limits_{[u]+z}(\nabla f(t)\cdot u)_+^p\,dH^1(t)dH^{n-1}(z)d\sigma(u)$$
$$=\int\limits_{\mathbb{R}^n}\|\nabla f(x)\|_{Z_p^{+,*}K}^p\,dH^n(x). \tag{3.8}$$

So, in particular, we have

$$\int\limits_{S^{n-1}}\int\limits_{u^\perp}\int\limits_{[u]+z}(\nabla f(t)\cdot u)_+^p\,dH^1(t)dH^{n-1}(z)d\sigma(u)$$
$$=\frac{n+p}{4}K_{n,p}\int\limits_{\mathbb{R}^n}|\nabla f(x)|^p\,dH^n(x)<+\infty. \tag{3.9}$$

Using the dominated convergence theorem with Lemma 3.1 and (3.9), we obtain from (3.6), (3.7) and (3.8) that

$$\lim_{s\to 1^-}(1-s)\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\frac{(f(x)-f(y))_+^p}{\|x-y\|_K^{n+sp}}dxdy=\frac{1}{p}\int\limits_{\mathbb{R}^n}\|\nabla f(x)\|_{Z_p^{+,*}K}^p\,dx.$$

$\square$

Remark 3.3. In Theorem 1.4, let $g = -f$ and $(x)_- = -\min\{0, x\} = (-x)_+$, for $x \in \mathbb{R}$. Then, we get

$$\lim_{s \to 1^-} (1-s) \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{(f(x) - f(y))_-^p}{\|x-y\|_K^{n+sp}} dx dy = \frac{1}{p} \int\limits_{\mathbb{R}^n} \|\nabla f(x)\|_{Z_p^{-,*}K}^p \, dx.$$

Chapter 4

Real-valued valuations on Sobolev spaces

For $p \geq 1$ and a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, let

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}.$$

Define $L^p(\mathbb{R}^n)$ to be the class of measurable functions with $\|f\|_p < \infty$ and $L^p_{loc}(\mathbb{R}^n)$ to be the class of measurable functions with $\|f \mathbb{1}_K\|_p < \infty$ for every compact $K \subset \mathbb{R}^n$.

A measurable function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be the *weak gradient* of $f \in L^p(\mathbb{R}^n)$ if

$$\int_{\mathbb{R}^n} \nu(x) \cdot \nabla f(x) dx = - \int_{\mathbb{R}^n} f(x) \nabla \cdot \nu(x) dx \tag{4.1}$$

for every compactly supported smooth vector field $\nu : \mathbb{R}^n \to \mathbb{R}^n$, where $\nabla \cdot \nu = \frac{\partial \nu}{\partial x_1} + \cdots + \frac{\partial \nu}{\partial x_n}$. A function $f \in L^1(\mathbb{R}^n)$ is said to be of *bounded variation* on $\mathbb{R}^n$ if there exists a finite signed vector-valued Radon measure λ on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \nu(x) \cdot \nabla f(x) dx = \int_{\mathbb{R}^n} \nu(x) \cdot d\lambda(x)$$

for every ν as mentioned before. A function $f \in L^1_{loc}(\mathbb{R}^n)$ is said to be of *locally bounded variation* on $\mathbb{R}^n$ if f is of bounded variation on all open subset of $\mathbb{R}^n$.

The *Sobolev space* $W^{1,p}(\mathbb{R}^n)$ consists of all functions $f \in L^p(\mathbb{R}^n)$ whose weak gradient belongs to $L^p(\mathbb{R}^n)$ as well. For each $f \in W^{1,p}(\mathbb{R}^n)$, we define the *Sobolev norm* to be

$$\|f\|_{W^{1,p}(\mathbb{R}^n)} = \left(\|f\|_p^p + \|\nabla f\|_p^p\right)^{1/p},$$

where $\|\nabla f\|_p$ denotes the L^p norm of $|\nabla f|$. Equipped with the Sobolev norm, the Sobolev space $W^{1,p}(\mathbb{R}^n)$ is a Banach space.

Theorem 4.1 ([45]). *Let $\{f_i\}$ be a sequence in $W^{1,p}(\mathbb{R}^n)$ that converges to $f \in W^{1,p}(\mathbb{R}^n)$. Then, there exists a subsequence $\{f_{i_j}\}$ that converges to f a.e. as $j \to \infty$.*

Furthermore, for $1 \le p < n$, $W^{1,p}(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for all $p \le q \le p^*$, where $p^* = \frac{np}{n-p}$ is the *Sobolev conjugate* of p, due to the Sobolev-Gagliardo-Nirenberg inequality stated as the following theorem.

Theorem 4.2 ([43]). *Let $1 \le p < n$. There exists a positive constant C, depending only on p and n, such that*

$$\|f\|_{p^*} \le C\,\|\nabla f\|_p$$

for all $f \in W^{1,p}(\mathbb{R}^n)$.

Remark 4.3. By Theorem 4.2, the expression in (1.8) is well defined.

For $f, g \in W^{1,p}(\mathbb{R}^n)$, we have $f \vee g$, $f \wedge g \in W^{1,p}(\mathbb{R}^n)$ and for almost every $x \in \mathbb{R}^n$,

$$\nabla(f \vee g)(x) = \begin{cases} \nabla f(x), & \text{when } f(x) > g(x) \\ \nabla g(x), & \text{when } f(x) < g(x) \\ \nabla f(x) = \nabla g(x), & \text{when } f(x) = g(x) \end{cases}$$

and

$$\nabla(f \wedge g)(x) = \begin{cases} \nabla f(x), & \text{when } f(x) < g(x) \\ \nabla g(x), & \text{when } f(x) > g(x) \\ \nabla f(x) = \nabla g(x), & \text{when } f(x) = g(x) \end{cases}$$

(see [45]). Hence $(W^{1,p}(\mathbb{R}^n), \vee, \wedge)$ is a lattice.

Let $L^{1,p}(\mathbb{R}^n) \subset W^{1,p}(\mathbb{R}^n)$ be the space of piecewise affine functions on $\mathbb{R}^n$. Here, a function $\ell : \mathbb{R}^n \to \mathbb{R}$ is called *piecewise affine*, if it is continuous and there exists a finite number of n-dimensional simplices $\triangle_1, \ldots, \triangle_m \subset \mathbb{R}^n$ with pairwise disjoint interiors such that the restriction of ℓ to each $\triangle_i$ is affine and $\ell = 0$ outside $\triangle_1 \cup \cdots \cup \triangle_m$. The simplices $\triangle_1, \ldots, \triangle_m$ are called a triangulation of the support of ℓ. Let V denote the set of vertices of this triangulation. We further have that V and the values $\ell(v)$ for $v \in V$ completely determine ℓ. Piecewise affine functions lie dense in $W^{1,p}(\mathbb{R}^n)$ (see [43]).

For $P \in \mathcal{P}_0^n$, define the piecewise affine function ℓ_P by requiring that $\ell_P(0) = 1$, that $\ell_P(x) = 0$ for $x \notin P$, and that ℓ_P is affine on each simplex with apex at the origin and base among facets of P. Define $P^{1,p}(\mathbb{R}^n) \subset L^{1,p}(\mathbb{R}^n)$ as the set of all ℓ_P for $P \in \mathcal{P}_0^n$. For $\phi \in \mathrm{GL}(n)$, $\ell_{\phi P} = \ell_P \circ \phi^{-1}$. We remark that multiples and translates of $\ell_P \in P^{1,p}(\mathbb{R}^n)$ correspond to linear elements within the theory of finite elements.

For $P \in \mathcal{P}_0^n$, let $F_1, \ldots, F_m$ be the facets of P. For the facet F_i, let u_i be its unit outer normal vector and T_i the convex hull of F_i and the origin. Since for $x \in T_i$,

$$\ell_P(x) = -\frac{u_i}{h(P, u_i)} \cdot x + 1$$

and

$$\nabla \ell_P(x) = -\frac{u_i}{h(P, u_i)},$$

we have

$$\begin{aligned}
\|\ell_P\|_p^p &= \int_{\mathbb{R}^n} |\ell_P|^p \, dx \\
&= p \int_0^1 t^{p-1} \mathrm{Vol}_n\left(\{\ell_P > t\}\right) dt \\
&= p \mathrm{Vol}_n(P) \int_0^1 t^{p-1} (1-t)^n dt \\
&= c_{p,n} \mathrm{Vol}_n(P),
\end{aligned}$$

where $c_{p,n} = \frac{\Gamma(p+1)\Gamma(n+1)}{\Gamma(n+p+1)} = \binom{n+p}{n}^{-1}$, and

$$\begin{aligned}
\|\nabla \ell_P\|_p^p &= \int_{\mathbb{R}^n} |\nabla \ell_P(x)|^p \, dx \\
&= \sum_{i=1}^m \int_{T_i} \left| \frac{u_i}{h(P,u_i)} \right|^p dx \\
&= \sum_{i=1}^m \frac{\mathrm{Vol}_n(T_i)}{h^p(P,u_i)} \\
&= \frac{1}{n} \sum_{i=1}^m \mathrm{Vol}_{n-1}(F_i) h^{1-p}(P,u_i) \\
&= \frac{1}{n} S_p(P),
\end{aligned}$$

where $S_p(P)$ is the L_p surface area of P.

Let $z : W^{1,p}(\mathbb{R}^n) \to \mathbb{R}$ be a functional. It is called *continuous* if for every sequence $f_k \in W^{1,p}(\mathbb{R}^n)$ with $f_k \to f$ as $k \to \infty$ with respect to the Sobolev norm, we have $|z(f_k) - z(f)| \to 0$ as $k \to \infty$. It is called *translation invariant* if $z(f \circ \tau^{-1}) = z(f)$ for all $f \in W^{1,p}(\mathbb{R}^n)$ and translations τ. It is called *homogeneous* if for some $q \in \mathbb{R}$, we have $z(sf) = |s|^q z(f)$ for all $f \in W^{1,p}(\mathbb{R}^n)$ and $s \in \mathbb{R}$. It is called $\mathrm{SL}(n)$ *invariant* if $z(f \circ \phi^{-1}) = z(f)$ for all $f \in W^{1,p}(\mathbb{R}^n)$ and $\phi \in \mathrm{SL}(n)$. Denote the derivative of the map $s \mapsto z(sf)$ by

$$Dz_f(s) = \lim_{\varepsilon \to 0} \frac{z\left((s+\varepsilon)f\right) - z(sf)}{\varepsilon},$$

whenever it exists.

We have the following examples of valuations on $W^{1,p}(\mathbb{R}^n)$.

Theorem 4.4. *Let $h \in \mathcal{G}_p$ be a continuous function. Then, for every $f \in W^{1,p}(\mathbb{R}^n)$, the functional*

$$z(f) = \int_{\mathbb{R}^n} h \circ f$$

is a continuous, $\mathrm{SL}(n)$ and translation invariant valuation. Furthermore, $z(0) = 0$ and the map $s \mapsto z(sf)$ belongs to $\mathcal{B}_p$ for every $s \in \mathbb{R}$ and $f \in P^{1,p}(\mathbb{R}^n)$.

Proof. **1. Valuation.** Let $f, g \in W^{1,p}(\mathbb{R}^n)$ and $E = \{x \in \mathbb{R}^n : f(x) \geq g(x)\}$. Then

$$\begin{aligned} z(f \vee g) + z(f \wedge g) &= \int_{\mathbb{R}^n} h \circ (f \vee g) + \int_{\mathbb{R}^n} h \circ (f \wedge g) \\ &= \int_E h \circ (f \vee g) + \int_{\mathbb{R}^n \setminus E} h \circ (f \vee g) \\ &\quad + \int_E h \circ (f \wedge g) + \int_{\mathbb{R}^n \setminus E} h \circ (f \wedge g) \\ &= \int_E h \circ f + \int_{\mathbb{R}^n \setminus E} h \circ g + \int_E h \circ g + \int_{\mathbb{R}^n \setminus E} h \circ f \\ &= \int_{\mathbb{R}^n} h \circ f + \int_{\mathbb{R}^n} h \circ g \\ &= z(f) + z(g). \end{aligned}$$

2. Continuity. Let $f \in W^{1,p}(\mathbb{R}^n)$ and $\{f_i\}$ be a sequence in $W^{1,p}(\mathbb{R}^n)$ that converges to f. For every subsequence $\left\{z(f_{i_j})\right\} \subset \{z(f_i)\}$, we are going to show that there exists a subsequence $\{z(f_{i_{j_k}})\}$ that converges to $z(f)$. Let $\{f_{i_j}\}$ be a subsequence of $\{f_i\}$. Then, $\{f_{i_j}\}$ converges to f in $W^{1,p}(\mathbb{R}^n)$. Thus, there exists a subsequence $\{f_{i_{j_k}}\} \subset \{f_{i_j}\}$ with $f_{i_{j_k}} \to f$ a.e. as $k \to \infty$. Furthermore, since h is continuous, we obtain $h \circ f_{i_{j_k}} \to h \circ f$ a.e. as $k \to \infty$. Since h satisfies (1.9), there exist $\delta > 0$ and $M_1 > 0$, such that whenever $|x| < \delta$, we have $|h(x)| \leq M_1 |x|^p$. Let $E_1 = \{|f| < 3\delta/4\}$. Since $f_{i_{j_k}} \to f$ a.e. as $k \to \infty$, for such $\delta > 0$, there exists $N_1 > 0$, such that whenever $k > N_1$, we obtain $|f_{i_{j_k}} - f| < \delta/4$ a.e. Thus, $|f_{i_{j_k}}| < \delta$ a.e. on E_1. Hence, for such k, $|h \circ f_{i_{j_k}}| \leq M_1 |f_{i_{j_k}}|^p$ a.e. on E_1. Since $M_1 \int_{E_1} |f_{i_{j_k}}|^p \leq M_1 \|f_{i_{j_k}}\|_p^p < \infty$, by the dominated convergence theorem, we have

$$\lim_{k \to \infty} \int_{E_1} h \circ f_{i_{j_k}} = \int_{E_1} h \circ f.$$

On the other hand, there exist $M_0 > 0$ and $M_2 > 0$, such that whenever $|x| > M_0$, we obtain $|h(x)| \leq M_2 |x|^{p^*}$. Let $E_2 = \{|f| > 3M_0/2\}$. Since $f_{i_{j_k}} \to f$ a.e. as $k \to \infty$, for such $M_0 > 0$, there exists $N_2 > 0$, such that whenever $k > N_2$, we have $|f_{i_{j_k}} - f| < M_0/2$ a.e. Thus, $|f_{i_{j_k}}| > M_0$ a.e. on E_2. Hence, for such k, $|h \circ f_{i_{j_k}}| \leq M_2 |f_{i_{j_k}}|^{p^*}$ a.e. on E_2. Since

$$M_2 \int_{E_2} \left| f_{i_{j_k}} \right|^{p^*} \leq M_2 \left\| f_{i_{j_k}} \right\|_{p^*}^{p^*} \leq C M_2 \left\| \nabla f_{i_{j_k}} \right\|_p^{p^*} < \infty,$$

by the dominated convergence theorem, we obtain

$$\lim_{k\to\infty}\int_{E_2} h\circ f_{i_{j_k}} = \int_{E_2} h\circ f.$$

Now, let $E_3=\mathbb{R}^n\setminus(E_1\cup E_2)$ and $N=\max\{N_1,N_2\}$. Then, for $k>N$, we have $\delta/2\le |f_{i_{j_k}}|\le 2M_0$ a.e. on E_3. Thus, for such k, since h is continuous, there exists $\gamma>0$, such that $|h\circ f_{i_{j_k}}|\le\gamma|f_{i_{j_k}}|$ a.e. on E_3. Since

$$\gamma\int_{E_3}\left|f_{i_{j_k}}\right|\le\gamma\left\|f_{i_{j_k}}\right\|_1\le\gamma\left\|f_{i_{j_k}}\right\|_p<\infty,$$

again by the dominated convergence theorem, we obtain

$$\lim_{k\to\infty}\int_{E_3} h\circ f_{i_{j_k}} = \int_{E_3} h\circ f.$$

3. SL(n) **invariance.** Let $f\in W^{1,p}(\mathbb{R}^n)$ and $\phi\in\mathrm{SL}(n)$. Then

$$z(f\circ\phi^{-1})=\int_{\mathbb{R}^n} h\circ f\circ\phi^{-1}=\int_{\mathbb{R}^n} h\left(f\left(\phi^{-1}x\right)\right)dx.$$

By setting $y=\phi^{-1}x$, we obtain

$$\begin{aligned} z(f\circ\phi^{-1}) &= \int_{\mathbb{R}^n} h\left(f(y)\right)dy \\ &= \int_{\mathbb{R}^n} h\circ f = z(f). \end{aligned}$$

4. Translation invariance. Let $f\in W^{1,p}(\mathbb{R}^n)$ and τ be a translation. Then

$$z(f\circ\tau^{-1})=\int_{\mathbb{R}^n} h\circ f\circ\tau^{-1}=\int_{\mathbb{R}^n} h\left(f\left(\tau^{-1}x\right)\right)dx.$$

By setting $y=\tau^{-1}x$, we obtain

$$\begin{aligned} z(f\circ\tau^{-1}) &= \int_{\mathbb{R}^n} h\left(f(y)\right)dy \\ &= \int_{\mathbb{R}^n} h\circ f = z(f). \end{aligned}$$

5. $z(0)=0$ follows from the continuity of z and (1.9).

6. Differentiability. Let $\ell_P \in P^{1,p}(\mathbb{R}^n)$, where $P \in \mathcal{P}_0^n$. Without loss of generality, we assume $s > 0$. Indeed, set

$$h^e(x) = \frac{h(x) + h(-x)}{2} \text{ and } h^o(x) = \frac{h(x) - h(-x)}{2}$$

for every $x \in \mathbb{R}$, and the case $s < 0$ follows from

$$\begin{aligned} z(-s\ell_P) &= \int_{\mathbb{R}^n} (h^e + h^o) \circ (-s\ell_P) \\ &= \int_{\mathbb{R}^n} (h^e \circ (-s\ell_P) + h^o \circ (-s\ell_P)) \\ &= \int_{\mathbb{R}^n} (h^e \circ (s\ell_P) - h^o \circ (s\ell_P)) \,. \end{aligned}$$

Since $h \in BV_{loc}(\mathbb{R})$, there exists a signed measure ν on $\mathbb{R}$, such that $h(s) = \nu([0, s))$ for every $s > 0$ (this can be done by setting $\nu = \mathbb{1}_{[0,s)}$ in (4.1)). By the layer cake representation, we have

$$\begin{aligned} z(s\ell_P) &= \int_{\mathbb{R}^n} h \circ (s\ell_P) \\ &= \int_0^s \mathrm{Vol}_n\left(\{s\ell_P > t\}\right) d\nu(t) = \mathrm{Vol}_n(P) \int_0^s \left(\frac{s-t}{s}\right)^n d\nu(t). \end{aligned}$$

In other words,

$$s^n z(s\ell_P) = \mathrm{Vol}_n(P) \int_0^s (s-t)^n d\nu(t). \tag{4.2}$$

We are going to show the differentiability by induction. Let $k \geq 2$ and $\psi_k(s)$ be the kth derivative of $\int_0^s (s-t)^n d\nu(t)$ with respect to s. We have

$$\psi_k(s) = \frac{n!}{(n-k)!} \int_0^s (s-t)^{n-k} d\nu(t). \tag{4.3}$$

In particular, we obtain $\psi_n(s) = n! h(s)$. On the other hand, differentiating the left hand side of (4.2), we have

$$\psi_1(s)\mathrm{Vol}_n(P) = ns^{n-1} z(s\ell_P) + s^n D z_{\ell_P}(s) \left\|\ell_P\right\|_{W^{1,p}(\mathbb{R}^n)} .$$

By induction, we get

$$\psi_k(s)\mathrm{Vol}_n(P) = \sum_{j=0}^{k} \binom{k}{j} \frac{n!}{(n-k+j)!} s^{n-k+j} D^j z_{\ell_P}(s) \left\|\ell_P\right\|^j_{W^{1,p}(\mathbb{R}^n)} . \tag{4.4}$$

In particular, we obtain

$$\psi_n(s)\mathrm{Vol}_n(P) = n! \sum_{j=0}^{n} \binom{n}{j} s^j D^j z_{\ell_P}(s) \, \|\ell_P\|^j_{W^{1,p}(\mathbb{R}^n)} \, ,$$

which coincides with $n!\mathrm{Vol}_n(P)h(s)$. Since h is a continuous locally BV function, we have the desired differentiability of $s \mapsto z(s\ell_P)$.

7. Growth condition. First of all, by (4.2),

$$\begin{aligned} z(s\ell_P) &= \mathrm{Vol}_n(P) \int_0^s \left(\frac{s-t}{s} \right)^n d\nu(t) \\ &\leq \mathrm{Vol}_n(P) \int_0^s d\nu(t) \\ &= \mathrm{Vol}_n(P) h(s) \end{aligned}$$

satisfies (1.9). As shown in the previous step ((4.3) and (4.4)), for every integer $1 \leq k \leq n$,

$$\begin{aligned} &\sum_{j=0}^{k} \binom{k}{j} \frac{n!}{(n-k+j)!} s^{n-k+j} D^j z_{\ell_P}(s) \, \|\ell_P\|^j_{W^{1,p}(\mathbb{R}^n)} \\ = \; & \frac{n!}{(n-k)!} \mathrm{Vol}_n(P) \int_0^s (s-t)^{n-k} d\nu(t) \end{aligned}$$

i.e.

$$\begin{aligned} &\sum_{j=0}^{k} \binom{k}{j} \frac{n!}{(n-k+j)!} s^{j} D^j z_{\ell_P}(s) \, \|\ell_P\|^j_{W^{1,p}(\mathbb{R}^n)} \\ = \; & \frac{n!}{(n-k)!} \mathrm{Vol}_n(P) \int_0^s \left(\frac{s-t}{s} \right)^{n-k} d\nu(t) \\ \leq \; & \frac{n!}{(n-k)!} \mathrm{Vol}_n(P) h(s) \end{aligned}$$

also satisfies (1.9). $\square$

4.1 The characterization of homogeneous valuations

First, we need the following reduction similar to [52, Lemma 8]. We include the proof for the sake of completeness.

Lemma 4.5. *Let $z_1, z_2 : L^{1,p}(\mathbb{R}^n) \to \mathbb{R}$ be continuous and translation invariant valuations satisfying $z_1(0) = z_2(0) = 0$. If $z_1(sf) = z_2(sf)$ for all $s \in \mathbb{R}$ and $f \in P^{1,p}(\mathbb{R}^n)$, then*

$$z_1(f) = z_2(f) \tag{4.5}$$

for all $f \in L^{1,p}(\mathbb{R}^n)$.

Proof. We first make the following reduction steps for (4.5).

1. Non-negative $f \in L^{1,p}(\mathbb{R}^n)$. Since z_1 and z_2 are valuations satisfying $z_1(0) = z_2(0) = 0$, we have for $i = 1, 2$,

$$z_i(f \vee 0) + z_i(f \wedge 0) = z_i(f) + z_i(0) = z_i(f).$$

For $i = 1, 2$, let

$$z_i^e(f) = \frac{z_i(f) + z_i(-f)}{2}, z_i^o(f) = \frac{z_i(f) - z_i(-f)}{2}$$

and hence $z_i(f) = z_i^e(f) + z_i^o(f)$ for all $f \in L^{1,p}(\mathbb{R}^n)$. Therefore, we have

$$z_i^e(f \wedge 0) = z_i^e\left(-((-f) \wedge 0)\right) = z_i^e\left((-f) \wedge 0\right)$$

and

$$z_i^o(f \wedge 0) = z_i^o\left(-((-f) \wedge 0)\right) = -z_i^o\left((-f) \wedge 0\right).$$

Thus, it suffices to show that (4.5) holds for all non-negative $f \in L^{1,p}(\mathbb{R}^n)$.

2. $f \in L^{1,p}(\mathbb{R}^n)$ where the values $f(v)$ are distinct for $v \in V$ with $f(v) > 0$. Let f not vanish identically and $\mathcal{S}$ be the triangulation of the support of f in n-dimensional simplices, such that $f\big|_{\triangle}$ is affine for each simplex $\triangle \in \mathcal{S}$. Denote by V the (finite) set of vertices of $\mathcal{S}$. Note that f is determined by its value on V. Since there always exists an approximation of f by $g \in L^{1,p}(\mathbb{R}^n)$ where the values $g(v)$ are distinct for $v \in V$ with $g(v) > 0$, by continuity of z_1 and z_2, we have the reduction.

3. $f \in L^{1,p}(\mathbb{R}^n)$ that are concave on their supports. Let $f_1, \ldots, f_m \in L^{1,p}(\mathbb{R}^n)$ non-negative and concave on their supports such that

$$f = f_1 \vee \cdots \vee f_m. \tag{4.6}$$

For $i = 1, 2$, by the inclusion-exclusion principle, we obtain

$$z_i(f) = z_i(f_1 \vee \cdots \vee f_m) = \sum_J (-1)^{|J|-1} z_i(f_J),$$

where J is a non-empty subset of $\{1, \ldots, m\}$ and

$$f_J = f_{j_1} \wedge \cdots \wedge f_{j_k}$$

for $J = \{j_1, \ldots, j_k\}$. Indeed, such representation in (4.6) exists. We determine f_i's by their value on V. Set $f_i(v) = f(v)$ on the vertices v of the simplex $\triangle_i$ of $\mathcal{S}$. Choose a polytope P_i containing $\triangle_i$ and set $f_i(v) = 0$ on the vertices v of P_i. Then (4.6) holds, if P_i's are chosen suitably small. The reduction follows since the meet of concave functions is still concave.

4. $f \in L^{1,p}(\mathbb{R}^n)$ such that F is not singular. Given a function $f \in L^{1,p}(\mathbb{R}^n)$. Let $F \subset \mathbb{R}^{n+1}$ be the compact polytope bounded by the graph of f and the hyperplane $\{x_{n+1} = 0\}$. We say F is *singular* if F has n facet hyperplanes that intersect in a line L parallel to $\{x_{n+1} = 0\}$ but not contained in $\{x_{n+1} = 0\}$. Similar to the second step, by continuity of z_1 and z_2, it suffices to show (4.5) for $f \in L^{1,p}(\mathbb{R}^n)$ such that F is not singular.

Let a function f satisfying reduction steps 1-4 be given. Denote by $\bar{p}$ the vertex of F with the largest x_{n+1}-coordinate. We are now going to show (4.5) by induction on the number m of facet hyperplanes of F that are not passing through $\bar{p}$. In the case $m = 1$, a scaled translate of f is in $P^{1,p}(\mathbb{R}^n)$. Since z_1 and z_2 are translation invariant, (4.5) holds. Let $m \geq 2$. Let $p_0 = (x_0, f(x_0))$ be a vertex of F with minimal x_{n+1}-coordinate and $H_1, \ldots, H_j$ be the facet hyperplanes of F through p_0 which do not contain $\bar{p}$. Notice that there exists at least one such hyperplane. Write $\bar{F}$ as the polytope bounded by the intersection of all facet hyperplanes of F other than $H_1, \ldots, H_j$. Since F is not singular, $\bar{F}$ is bounded. Thus, there exists an $f \in L^{1,p}(\mathbb{R}^n)$ that corresponds to F. Note that $\bar{F}$ has at most $(m-1)$ facet hyperplanes not containing $\bar{p}$. Let $\bar{H}_1, \ldots, \bar{H}_i$ be the facet hyperplanes of $\bar{F}$ that contain p_0. Choose hyperplanes $\bar{H}_{i+1}, \ldots, \bar{H}_k$ also containing p_0 such that the hyperplanes $\bar{H}_1, \ldots, \bar{H}_k$ and $\{x_{n+1} = 0\}$ enclose a pyramid with apex at p_0 that is contained in $\bar{F}$ and has x_0 in its base with $\bar{H}_1, \ldots, \bar{H}_i$ among its facet hyperplanes. Therefore, there exists a piecewise affine function ℓ corresponding to this pyramid. Moreover, a scaled translate of ℓ is in $P^{1,p}(\mathbb{R}^n)$. We also obtain that a scaled translate of $\bar{\ell} = f \wedge \ell$ is in $P^{1,p}(\mathbb{R}^n)$. To summarize, scaled translates of $\bar{\ell}$ and ℓ are in $P^{1,p}(\mathbb{R}^n)$, the polytope $\bar{F}$ has at most $(m-1)$ facet hyperplanes not containing $\bar{p}$, and

$$f \vee \ell = \bar{f} \text{ and } f \wedge \ell = \bar{\ell}.$$

Applying valuations z_1 and z_2, we have for $i = 1, 2$,

$$z_i(f) + z_i(\ell) = z_i(\bar{f}) + z_i(\bar{\ell}).$$

Thus, the induction hypotheses yields the desired result. $\square$

The classification will also make use of the following elementary fact.

Remark 4.6. Let f and g be functions on $\mathbb{R}$. If $f(x) \sim o\,(g(x)h(x))$ as $x \to 0$, for each function h on $\mathbb{R}$ with $\lim_{x\to 0} h(x) = \infty$, then

$$f(x) \sim O(g(x)) \text{ as } x \to 0.$$

This can be seen by the following simple argument. Suppose $\left|f(x)/g(x)\right| \to \infty$ as $x \to 0$. Let $h = \sqrt{|f/g|}$. It is clear that $h(x) \to \infty$ as $x \to 0$. But now

$$\left|f(x)/\left(g(x)h(x)\right)\right| = \sqrt{\left|f(x)/g(x)\right|} = h(x) \to \infty \text{ as } x \to 0,$$

which yields a contradiction. The similar argument also works for the limit $x \to \infty$.

Lemma 4.7. *Let $z : L^{1,p}(\mathbb{R}^n) \to \mathbb{R}$ be a continuous,* SL(n) *and translation invariant valuation with $z(0) = 0$. Then there exists a continuous function $c : \mathbb{R} \to \mathbb{R}$ satisfying* (1.9) *such that*

$$z(s\ell_P) = c(s)\mathrm{Vol}_n(P),$$

for every $s \in \mathbb{R}$ and $\ell_P \in P^{1,p}(\mathbb{R}^n)$.

Proof. Similar to the proof of Lemma 5 in [52]. Define the functional $Z : \mathcal{P}_0^n \to \mathbb{R}$ by setting

$$Z(P) = z(s\ell_P),$$

for every $s \in \mathbb{R}$ and $\ell_P \in P^{1,p}(\mathbb{R}^n)$. If $\ell_P, \ell_Q \in P^{1,p}(\mathbb{R}^n)$ are such that $\ell_P \vee \ell_Q \in P^{1,p}(\mathbb{R}^n)$, then $\ell_P \vee \ell_Q = \ell_{P\cup Q}$ and $\ell_P \wedge \ell_Q = \ell_{P\cap Q}$. Since z is a valuation on $L^{1,p}(\mathbb{R}^n)$, it follows for $P, Q, P \cup Q \in \mathcal{P}_0^n$ that

$$\begin{aligned} Z(P) + Z(Q) &= z(s\ell_P) + z(s\ell_Q) \\ &= z(s(\ell_P \vee \ell_Q)) + z(s(\ell_P \wedge \ell_Q)) \\ &= Z(P \cup Q) + Z(P \cap Q). \end{aligned}$$

Thus, $Z : \mathcal{P}_0^n \to \mathbb{R}$ is a valuation.

By Theorem 1.5, there exist $c_0, c_1, c_2 \in \mathbb{R}$ now depending on s such that

$$z(s\ell_P) = c_0(s) + c_1(s)\mathrm{Vol}_n(P) + c_2(s)\mathrm{Vol}_n(P^*), \tag{4.7}$$

for all $s \in \mathbb{R}$ and $\ell_P \in P^{1,p}(\mathbb{R}^n)$. We now investigate the behavior of these constants by studying valuations on different $s\ell_P$'s and their translations, for $s \in \mathbb{R}$.

We start with c_0 and c_2.

Example 1. Let $P \in \mathcal{P}_0^n$. Take translations $\tau_1, \dots, \tau_k$, such that the $\phi_i P$'s are pairwise disjoint, where $\phi_i P = \tau_i\left(P/k^i\right)$. Consider the function $f_k = s\left(\ell_{\phi_1 P} \vee \cdots \vee \ell_{\phi_k P}\right)$, $s \in \mathbb{R}$. Then, we have

$$\begin{aligned}
\|f_k\|_p^p &= |s|^p \sum_{i=1}^k \int_{\phi_i P} \ell_{\phi_i P}^p = |s|^p \sum_{i=1}^k \int_{\phi_i P} \left(\ell_P\left(\phi_i^{-1}x\right)\right)^p dx \\
&= |s|^p \sum_{i=1}^k k^{-in} \int_P \ell_P^p = |s|^p \|\ell_P\|_p^p \sum_{i=1}^k k^{-in} \to 0 \text{ as } k \to \infty.
\end{aligned}$$

Further,

$$\begin{aligned}
\|\nabla f_k\|_p^p &= |s|^p \sum_{i=1}^k \int_{\phi_i P} |\nabla \ell_{\phi_i P}|^p = |s|^p \sum_{i=1}^k \int_{\phi_i P} \left|\nabla\left(\ell_P \circ \phi_i^{-1}\right)\right|^p \\
&= |s|^p \sum_{i=1}^k \int_{\phi_i P} \left|\phi_i^{-t} \nabla \ell_P\left(\phi_i^{-1}x\right)\right|^p dx \\
&= |s|^p \sum_{i=1}^k k^{ip} \int_{\phi_i P} \left|\nabla \ell_P\left(\phi_i^{-1}x\right)\right|^p dx \\
&= |s|^p \sum_{i=1}^k k^{-i(n-p)} \|\nabla \ell_P\|_p^p \to 0 \text{ as } k \to \infty.
\end{aligned}$$

Thus, $f_k \to 0$ in $W^{1,p}(\mathbb{R}^n)$ as $k \to \infty$.

By translation invariance of z and (4.7), we have

$$\begin{aligned} z(f_k) &= \sum_{i=1}^{k} z(s\ell_{P/k^i}) = \sum_{i=1}^{k}\left(c_0(s) + \frac{c_1(s)}{k^{in}}\mathrm{Vol}_n(P) + c_2(s)k^{in}\mathrm{Vol}_n(P^*)\right) \\ &= kc_0(s) + c_1(s)\mathrm{Vol}_n(P)\sum_{i=1}^{k} k^{-in} \\ &\quad + c_2(s)\mathrm{Vol}_n(P^*)\sum_{i=1}^{k} k^{in} \to 0 \text{ as } k \to \infty. \end{aligned}$$

Therefore, $c_2(s)$ has to vanish, as the geometric series diverges, as well as $c_0(s)$, for every $s \in \mathbb{R}$.

Now, let's further determine c_1 by two different examples.

Example 2. For each function f with $\lim_{x\to 0} f(x) = \infty$, let $P \in \mathcal{P}_0^n$ and $P_k = P\left(k^p/f(1/k)\right)^{\frac{1}{n}}$, for $k = 1, 2, \ldots$. Then, we have

$$\|\ell_{P_k}/k\|_p^p = c_{n,p}k^{-p}\mathrm{Vol}_n(P_k) = c_{n,p}\mathrm{Vol}_n(P)/f(1/k) \to 0 \text{ as } k \to \infty,$$

$$\|\nabla\ell_{P_k}/k\|_p^p = \frac{1}{n}k^{-p}S_p(P_k) = \frac{1}{n}S_p(P)k^{-\frac{p^2}{n}}\left(f(1/k)\right)^{\frac{p-n}{n}} \to 0 \text{ as } k \to \infty.$$

Thus, $\ell_{P_k}/k \to 0$ in $W^{1,p}(\mathbb{R}^n)$ as $k \to \infty$.

By (4.7), we obtain

$$z\left(\ell_{P_k}/k\right) = c_1(1/k)k^p\mathrm{Vol}_n(P)/f(1/k) \to 0 \text{ as } k \to \infty.$$

Therefore, $c_1(1/k) \sim o\left(f(1/k)/k^p\right)$ as $k \to \infty$. Similarly, considering $-\ell_{P_k}/k$, we obtain the same estimate as $x \to 0^-$. Hence, $c_1(x) \sim o(x^p f(x))$ as $x \to 0$. It follows that $c_1(x) \sim O(x^p)$ as $x \to 0$ via Remark 4.6.

Example 3. For each function f with $\lim_{x\to\infty} f(x) = \infty$, let $P \in \mathcal{P}_0^n$ and $P_k = P/\left(k^{p^*}f(k)\right)^{\frac{1}{n}}$, for $k = 1, 2, \ldots$. Then, we have

$$\|k\ell_{P_k}\|_p^p = c_{n,p}k^p\mathrm{Vol}_n(P_k) = c_{n,p}k^{p-p^*}\left(f(k)\right)^{-1}\mathrm{Vol}_n(P) \to 0 \text{ as } k \to \infty,$$

$$\|\nabla k\ell_{P_k}\|_p^p = \frac{1}{n}k^p S_p(P_k) = \frac{1}{n}S_p(P)\left(f(k)\right)^{\frac{p-n}{n}} \to 0 \text{ as } k \to \infty.$$

Thus, $k\ell_{P_k} \to 0$ in $W^{1,p}(\mathbb{R}^n)$ as $k \to \infty$.

By (4.7), we obtain

$$z\left(k\ell_{P_k}\right) = c_1\left(k\right)k^{-p^*}\left(f(k)\right)^{-1}\mathrm{Vol}_n(P) \to 0 \text{ as } k \to \infty.$$

Therefore, $c_1(k) \sim o(k^{p^*} f(k))$ as $k \to \infty$. Similarly, considering $-k\ell_{P_k}$, we obtain the same estimate as $x \to -\infty$. Hence, $c_1(x) \sim o\left(x^{p^*} f(x)\right)$ as $x \to \infty$. It follows that $c_1(x) \sim O(x^{p^*})$ as $x \to \infty$ via Remark 4.6. $\square$

Now, we are ready to proof the result on homogeneous valuations.

Proof of Theorem 1.6. In the light of Lemma 4.5, it suffices to consider the case $f = s\ell_P$ for every $s \in \mathbb{R}$ and $\ell_P \in P^{1,p}(\mathbb{R}^n)$. In this case, due to Lemma 4.7, there exists a continuous function $c : \mathbb{R} \to \mathbb{R}$ satisfying (1.9) such that

$$z(s\ell_P) = c(s)\mathrm{Vol}_n(P) \tag{4.8}$$

for every $s \in \mathbb{R}$ and $\ell_P \in P^{1,p}(\mathbb{R}^n)$. On the other hand, by homogeneity, there exists a constant $c \in \mathbb{R}$ such that

$$z(s\ell_P) = c\,|s|^q\,\mathrm{Vol}_n(P) \tag{4.9}$$

for every $s \in \mathbb{R}$ and $\ell_P \in P^{1,p}(\mathbb{R}^n)$. Formulas (4.8) and (4.9) yield

$$c(s) = c\,|s|^q \tag{4.10}$$

for every $s \in \mathbb{R}$.

For $q < p$ or $q > p^*$, since $c(s)$ satisfies (1.9), which is impossible with the expression (4.10), we have $c = 0$. It follows that $z(s\ell_P) = 0$ for every $s \in \mathbb{R}$ and $\ell_P \in P^{1,p}(\mathbb{R}^n)$.

For $p \le q \le p^*$, set $\tilde{c} = \binom{n+q}{q}c$. By properties of the beta and the gamma function and the layer cake representation, we have

$$\begin{aligned}
c(s) &= \tilde{c}\,|s|^q \binom{n+q}{q}^{-1} \\
&= \tilde{c}q\,|s|^q \frac{\Gamma(q)\Gamma(n+1)}{\Gamma(n+q+1)} \\
&= \tilde{c}q\,|s|^q \int_0^1 t^{q-1}(1-t)^n dt \\
&= \tilde{c}q \int_0^1 (|s|\,t)^{q-1}(1-t)^n d\,|s|\,t \\
&= \tilde{c}q \int_0^{|s|} t^{q-1}\left(\frac{|s|-t}{|s|}\right)^n dt.
\end{aligned}$$

Thus,

$$\begin{aligned} c(s)\mathrm{Vol}_n(P) &= \tilde{c}q\int_0^{|s|} t^{q-1}\mathrm{Vol}_n\left(\{|s|\,\ell_P > t\}\right)dt \\ &= \tilde{c}\int_{\mathbb{R}^n}(|s|\,\ell_P(x))^q dx \\ &= \tilde{c}\,\|s\ell_P\|_q^q\,. \end{aligned}$$

□

4.2 A more general characterization

We finish the proof of Theorem 1.7 by the following crucial representation.

Lemma 4.8. *Let the functional $z : L^{1,p}(\mathbb{R}^n) \to \mathbb{R}$ satisfy $z(0) = 0$ and let $s \mapsto z(sf)$ belong to $\mathcal{B}_p$ for $s \in \mathbb{R}$ and $\ell_P \in P^{1,p}(\mathbb{R}^n)$. If there exists a continuous function $c : \mathbb{R} \to \mathbb{R}$ satisfying* (1.9) *such that*

$$z(s\ell_P) = c(s)\mathrm{Vol}_n(P),$$

for every $s \in \mathbb{R}$ and $\ell_P \in P^{1,p}(\mathbb{R}^n)$, then there exists a continuous function $h \in \mathcal{G}_p$ such that

$$z(s\ell_P) = \int_{\mathbb{R}^n} h\circ(s\ell_P).$$

Proof. It suffices to consider the case $s > 0$. Since there exists a continuous function $c : \mathbb{R} \to \mathbb{R}$ satisfying (1.9), such that

$$z(s\ell_P) = c(s)\mathrm{Vol}_n(P),$$

we have

$$Dz_{\ell_P}(s) = c'(s)\mathrm{Vol}_n(P).$$

It follows that $c(s)$ is continuously differentiable in the usual sense. Hence $c(s) \in C^n(\mathbb{R})$, due to $s \mapsto z(sf)$ belongs to $C^n(\mathbb{R})$ for every $s \in \mathbb{R}$ and $f \in P^{1,p}(\mathbb{R}^n)$. Moreover,

$$D^\alpha z_{\ell_P}(s) = c^{(\alpha)}(s)\mathrm{Vol}_n(P), \tag{4.11}$$

for every non-negative integer $\alpha \leq n$, and $c^{(n)} \in BV_{loc}(\mathbb{R})$.

Now, let

$$h(s) = \sum_{j=0}^{n} \frac{1}{j!} \binom{n}{j} s^j c^{(j)}(s). \tag{4.12}$$

We show by induction that there exists a signed measure ν on $\mathbb{R}$ such that

$$c(s) = \int_0^s \left(\frac{s-t}{s}\right)^n d\nu(t).$$

Since $c \in C^n(\mathbb{R})$ and $c^{(n)} \in BV_{loc}(\mathbb{R})$, there exists a signed measure ν on $\mathbb{R}$ such that $h(s) = \nu([0,s))$ for every $s \geq 0$. Let $h_1(s) = \int_0^s h(x)dx$. Then, by Fubini theorem, we obtain

$$\begin{aligned} h_1(s) &= \int_0^s \int_0^x d\nu(t)dx \\ &= \int_0^s \int_t^s dxd\nu(t) \\ &= \int_0^s (s-t)d\nu(t). \end{aligned}$$

Let $k \geq 2$ and $h_k(s) = \int_0^s h_{k-1}(x)dx$. Assume $h_k(x) = \frac{1}{k!}\int_0^x (x-t)^k d\nu(t)$. Again, by Fubini theorem, we have

$$\begin{aligned} h_{k+1}(s) &= \frac{1}{k!}\int_0^s \int_0^x (x-t)^k d\nu(t)dx \\ &= \frac{1}{k!}\int_0^s \int_t^s (x-t)^k dxd\nu(t) \\ &= \frac{1}{(k+1)!}\int_0^s (s-t)^{k+1} d\nu(t). \end{aligned}$$

Thus, in particular, we have

$$h_n(s) = \frac{1}{n!}\int_0^s (s-t)^n d\nu(t).$$

On the other hand, by (4.12), we have

$$\begin{aligned}
h(x) &= c(x) + \frac{1}{n!}x^n c^{(n)}(x) + \sum_{j=1}^{n-1} \frac{1}{j!}\left(\binom{n-1}{j} + \binom{n-1}{j-1}\right) x^j c^{(j)}(x) \\
&= \sum_{j=0}^{n-1} \frac{1}{j!}\binom{n-1}{j} x^j c^{(j)}(x) + \sum_{j=0}^{n-1} \frac{1}{(j+1)!}\binom{n-1}{j} x^{j+1} c^{(j+1)}(x) \\
&= \sum_{j=0}^{n-1} \frac{1}{(j+1)!}\binom{n-1}{j} \left(x^{j+1} c^{(j)}(x)\right)'.
\end{aligned}$$

Hence,

$$h_1(s) = \int_0^s h(x)dx = \sum_{j=0}^{n-1} \frac{1}{(j+1)!}\binom{n-1}{j} s^{j+1} c^{(j)}(s).$$

Assume that $h_k(x) = \sum_{j=0}^{n-k} \frac{1}{(j+k)!}\binom{n-k}{j} x^{j+k} c^{(j)}(x)$. Similarly, we obtain

$$\begin{aligned}
h_k(x) &= \frac{1}{k!}x^k c(x) + \frac{1}{n!}x^n c^{(n-k)}(x) \\
&\quad + \sum_{j=1}^{n-k-1} \frac{1}{(j+k)!}\left(\binom{n-k-1}{j} + \binom{n-k-1}{j-1}\right) x^{j+k} c^{(j)}(x) \\
&= \sum_{j=0}^{n-k-1} \frac{1}{(j+k)!}\binom{n-k-1}{j} x^{j+k} c^{(j)}(x) \\
&\quad + \sum_{j=0}^{n-k-1} \frac{1}{(j+k+1)!}\binom{n-k-1}{j} x^{j+k+1} c^{(j+1)}(x) \\
&= \sum_{j=0}^{n-k-1} \frac{1}{(j+k+1)!}\binom{n-k-1}{j} \left(x^{j+k+1} c^{(j)}(x)\right)'.
\end{aligned}$$

It follows that

$$h_{k+1}(s) = \int_0^s h_k(x)dx = \sum_{j=0}^{n-(k+1)} \frac{1}{(j+k+1)!}\binom{n-(k+1)}{j} s^{j+k+1} c^{(j)}(s).$$

Thus, in particular, we have $h_n(s) = \frac{1}{n!}s^n c(s)$. Therefore, by the layer cake representation, we have

$$\begin{aligned} z(s\ell_P) = c(s)\mathrm{Vol}_n(P) &= \int_0^s \left(\frac{s-t}{s}\right)^n \mathrm{Vol}_n(P) d\nu(t) \\ &= \int_0^s \mathrm{Vol}_n\left(\{s\ell_P > t\}\right) d\nu(t) \\ &= \int_{\mathbb{R}^n} h \circ (s\ell_P). \end{aligned}$$

Furthermore, for fixed $P \in \mathcal{P}_0^n$,

$$s^k D^k z_{\ell_P}(s) \left\|\ell_P\right\|_{W^{1,p}(\mathbb{R}^n)}^k = s^k c^{(k)}(s)\mathrm{Vol}_n(P)$$

satisfies (1.9) for every integer $0 \leq k \leq n$. Therefore, as defined in (4.12), h also satisfies (1.9). $\square$

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