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Standard Completeness: Proof-theoretic and algebraic methods

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Abstract

The thesis is a contribution to Mathematical Fuzzy Logic. This is a prominent research area within the broader field of nonclassical logics, with significant mathematical interest and also computer science and engineering applications, e.g. in expert systems, control theory and knowledge representation. Mathematical Fuzzy Logic aims to deal formally with statements involving vague predicates, such as "X is tall","X is young", "X is small", which in many cases seem to be neither completely true nor completely false. Such statements pose a serious challenge for classical logic, whose semantics admits only two truth values, 0 for false and 1 for true. Fuzzy Logic addresses this issue by admitting various *degrees of truth*. Its intended or *standard semantics* is based on the continuum of values in the real interval $[0, 1]$, ranging from absolute falsity 0 to absolute truth 1.

Showing that a logic is *standard complete*, i.e. it is complete with respect to the standard semantics, is a task of crucial importance in the field of Mathematical Fuzzy Logic. It is typically achieved using purely *algebraic* methods. However, an alternative *proof-theoretic* approach has been recently introduced based on the study of formal proofs in the considered logic. The key idea is to show the admissibility of a particular rule, called *density*, for the logic under consideration. The techniques used for proving the admissibility of the density rule are closely related to those that have been developed for showing the admissibility of the *cut* rule, one of the central topics of investigation in proof theory.

Thus far, both the algebraic and proof-theoretic approaches to standard completeness have been tailored to specific logics. In our work we prove general results on standard completeness that apply to a large class of logics in a uniform way. Our results subsume many known results on standard completeness and also yield standard completeness for infinitely many new logics. We begin from the basic systems for uninorm logic UL and *monoidal t-norm* logic MTL, the logics of left-continuous uninorms and left-continuous t-norms, respectively. We then obtain standard completeness for large classes of axiomatic extensions of these two logics via suitable modifications of the proof-theoretic method. In particular, we obtain sufficient conditions which guarantee standard completeness. These conditions, formulated on the proof systems for the logics under consideration, can be verified in an automated fashion using the Prolog program AxiomCalc[1].

Furthermore, we introduce a new algebraic method for proving standard completeness by translating the proof-theoretic method into an algebraic framework. The new method is not only more accessible to the algebraic community, it also simplifies some technical combinatorial

[1]Available online at `http://www.logic.at/tinc/webaxiomcalc`.

arguments and thus leads to standard completeness proofs also for the technically involved case of non-commutative logics.

Our results contribute to the theoretical tools of algebraic proof-theory, a new field of research, which aims to combine proof-theoretic and algebraic techniques in the investigation of non-classical logics.

Kurzfassung

Die vorliegende Thesis leistet einen Beitrag zur *mathematischen Fuzzylogik*, einem prominenten Forschungsgebiet innerhalb der nicht-klassischen Logik, welches Anwendungen in der Mathematik, der Informatik und den Ingenieurwissenschaften findet. Beispielhaft hierfür seien die Forschungsgebiete der Expertensysteme, der Kontrolltheorie und der Wissensrepräsentation genannt. Ziel der mathematischen Fuzzylogik ist es, vage Aussagen, wie "X ist gross", "X ist jung" oder "X ist klein", formal zu behandeln, also Aussagen, die in vielen Fällen weder vollständig wahr noch vollständig falsch genannt werden können. Derartige Aussagen sind in der klassischen Logik schwerlich zu erfassen, da deren Semantik lediglich zwei Wertigkeiten besitzt, nämlich wahr und falsch. Der Ansatz der Fuzzylogik besteht darin, mehr als diese beiden Wahrheitswerte zuzulassen. Insbesondere werden als Wahrheitswerte in der *Standardsemantik* für Fuzzylogik alle Zahlen des reellen Einheitsintervalls $[0, 1]$ zugelassen, also ein Kontinuum von Wahrheitswerten zwischen absoluter Falschheit 0 und absoluter Wahrheit 1.

Ein wesentlicher Aspekt der mathematischen Fuzzylogik sind Vollständigkeitsresultate bezüglich der Standardsemantik. Diese Eigenschaft, im Folgenden *Standardvollständigkeit* (standard completeness) genannt, wird üblicherweise mit rein *algebraischen* Methoden hergeleitet. Basierend auf einer Analyse der formalen Beweise der verschiedenen Fuzzylogiken wurde jedoch inzwischen eine *beweistheoretische* Alternative zu diesen Methoden entwickelt. Bei dieser wird im Wesentlichen gezeigt, dass die sogenannte *Dichtheitsregel* (density rule) in einem Beweiskalkül für die betrachtete Logik redundant ist. Die herbei verwendeten Techniken ähneln stark denen in Gentzens Beweis seines Hauptsatzes, einem zentralen Resultat der Beweistheorie.

Standardvollständigkeitsbeweise folgend beider Ansätze, des algebraischen sowie des beweisetheoretischen, waren bisher auf die jeweils gegebene Logik zugeschnitten. In der vorliegenden Arbeit werden hingegen allgemeine Standardvollständigkeitsaussagen für große Klassen von Logiken bewiesen. Hierdurch lassen sich, neben bereits bekannten, unendlich viele neue Logiken abdecken. Als Ausgangspunkt dienen uns die Systeme der *Uninorm-Logik* (uninorm logic, UL) und der *Logik monoidealer T-Normen* (monoidal t-norm logic, MTL), gegeben durch linksseitig stetige Uninormen beziehungsweise linksseitig stetige T-Normen. Darauf aufbauend erhalten wir durch entsprechende Anpassungen der beweistheoretischen Methoden Standardvollständigkeitsresultate für große Klassen axiomatischer Erweiterungen dieser beiden Logiken, einschließlich hinreichender Bedingungen. Diese Bedingungen können, in der beweistheoretischen Formulierung der jeweiligen Logik, mithilfe des Prolog Programms "AxiomCalc" [2] über-

[2]Online abrufbar unter `http://www.logic.at/tinc/webaxiomcalc`.

prüft werden.

Weiterhin entwickeln wir eine neue algebraische Methode für Standardvollständigkeitsbeweise, indem wir die beweistheoretische Methode in algebraische Begriffe übertragen. Einerseits erleichtert dies den algebraischen Zugang zu unseren allgemeinen Resultaten, andererseits werden hierbei auch einige technische kombinatorische Argumente vereinfacht. Letzteres führt insbesondere auch zu Beweisen der Standardvollständigkeit für nicht-kommutative Logiken.

Mit dieser Arbeit wird ein Beitrag zum noch jungen Forschungsgebiet der *algebraischen Beweistheorie* (algebraic proof-theory) geleistet, welches beweistheoretische und algebraische Techniken zur Untersuchung nicht-klassischer Logiken vereint.

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Contents

CHAPTER 1

Introduction

1.1 Algebraic and proof-theoretic methods in nonclassical logics

Classical logic in its modern form was mainly conceived and developed for the formalization of mathematics, at the beginning of the 20th century. It therefore cannot be expected to be a particularly adequate and efficient instrument for reasoning about non mathematical notions, e.g. those involving dynamic data structures, vague propositions, beliefs, time, etc.
The increasing demands, coming from computer scientists, linguists and philosophers, for a rigorous study of aspects of reasoning which classical logic seems to neglect, has led in the last decades to an explosion of research on *nonclassical logics* and to the definition of many new logics. Just as classical logic, nonclassical logics present themselves typically in two facets: their *syntax*, with the related notions of proofs and formal derivability, and *semantics*, which gives rise to a corresponding notion of validity.

Proof-theoretic methods

Pure syntactical, i.e. *proof-theoretical* methods, play a fundamental role in the investigation of logics. The origins of modern proof theory trace back to the work of David Hilbert, in the context of a broader investigation of the foundations of mathematics. Hilbert introduced a formalization of classical logic, using what is known today as *Hilbert calculus*. Hilbert-style calculi usually consist of a set of axioms and few inference rules (e.g. modus ponens) for obtaining new derivable formulas. While these calculi are flexible for presenting logics and showing their connection with classes of algebras, they are not so helpful when it comes to searching for, analyzing, and reasoning about proofs. Proofs in a Hilbert calculus are indeed heavily based on guessing the right axiom or instance of the rule to be used and therefore they lack a clear, discernible structure. These drawbacks are overcome by a different type of formalism, the *sequent calculus*, which was introduced by Gentzen [47] for classical and intuitionistic logic and is now, in its numerous variants, among the most popular frameworks for proof-theoretic investigations. The advantage of Gentzen-style calculi over Hilbert-style ones lies in the important property of *analyticity* enjoyed by the former: in a good (i.e. analytic) sequent calculus, proofs use only subformulas of the formula to be proved and therefore can be constructed in a more mechanical way. Analyticity for a given sequent calculus is usually shown by proving the admissibility (or in its algorithmic form, the elimination) of a particular rule, the *cut* rule. Once a calculus is shown to be analytic, it can be used as a basis for automated deduction and for investigating properties of the corresponding logic. Proofs of logical properties obtained via proof-theoretic methods usually have a constructive nature: for instance a decidability result proved using a Gentzen-style calculus would give a concrete decision procedure, and similarly an interpolation result would provide a procedure to construct a concrete interpolating formula. These proof-theoretic approaches suffer however from some limitations. First, it is not always easy to introduce Gentzen-style analytic calculi for logics presented Hilbert-style. Results about specific calculi are usually difficult to generalize to related logics, e.g. obtained by adding or removing axioms. Moreover, negative and limitative results (e.g. non provability of a formula or non-existence of an analytic calculus for a certain logic) are particularly hard to achieve by proof-theoretic methods alone.

Algebraic methods

One of the simplest and most natural semantics for nonclassical logics are provided by suitable *algebraic* structures. The study of the interplay between proof systems and classes of algebraic structures is the main focus of the field of Abstract Algebraic Logic [15]. Results in this area provide, in very general settings, completeness theorems (*algebraization*) connecting nonclassical logics presented axiomatically and corresponding general classes of algebras. This is only the starting point towards an extensive use of algebraic methods for the investigation of nonclassical logics: one can show decidability, via the finite embeddability property [16, 52], interpolation, via amalgamation [67], and the disjunction property, via well-connectedness [54], to name a few (see also [46]).

Algebraic proof theory

The algebraic and proof-theoretic approaches to logics have traditionally developed in parallel, non-intersecting ways. In recent years, however, an increasing number of investigations proceeded towards an integration of the two approaches and of their methods, to overcome their respective limitations. In particular, [24–27] inaugurated a new research area, dubbed *algebraic proof theory*, which joins algebra and proof theory in a novel way, going beyond the mere combination of results of the two fields, but rather integrating their techniques. Algebraic proof theory is based on two fundamental ideas: (1) a proof-theoretic treatment of algebraic equations over residuated lattices and (2) the algebraization of proof-theoretic methods. On the one hand, (1) led to the investigation of the transformation of equations into equivalent quasiequations (corresponding to rules in proof theory) and allowed for uniform proofs of preservation under completions, for large classes of subvarieties of residuated lattices [26]. On the other hand, for (2), a strong form of cut-admissibility was proved algebraically and in a uniform way for classes of substructural logics [24, 25, 27] (see [14, 74] for earlier algebraic proofs of cut-admissibility). Beyond the achievement of these general results on completions and cut-admissibility, the interplay of algebraic and proof-theoretic techniques also shed light on the expressive power of some Gentzen-style calculi: what can (and more importantly, what cannot) be formalized in the frameworks of of sequent calculi and in their generalization, hypersequent calculi [3].

1.2 Fuzzy Logic and standard completeness

In the present work, we show how the interaction of algebraic and proof-theoretic methods can be profitable to address an important problem in the field of *Fuzzy Logic*. The field is a prominent member of the broader area of nonclassical logics: it is the object of a growing research literature, as witnessed e.g. by [35, 36] and has applications in many areas of computer science, in field such as expert systems [80], knowledge representation and the semantic web [62, 77]. Fuzzy Logic is generally motivated by the intuition that the usual two truth values of classical logic, 0 for false and 1 for true, do not suffice to model reasoning about vaguely defined predicates, such as “tall”, “young”, “warm”, etc. In other words, its crucial underlying assumption is that *truth comes in degrees*. The current research on Fuzzy Logic arises historically from the following traditions:

- The philosophical and linguistic investigations concerning vagueness and the paradoxes involved therein, such as the well known Sorites paradox, see e.g. [56].
- The tradition of many-valued logics, which were investigated among others by Post, Kleene, Łukasiewicz, Gödel and Belnap. There, truth values different from 0 and 1 (e.g. a third truth-value standing for “unknown”) were investigated out of various philosophical or purely algebraic motivations.
- The theory of Fuzzy Sets, first developed by Zadeh [85], which attempted to model vague concepts as functions $v\colon V \to [0,1]$. The intended meaning of the real value $v(x)$ is the degree of membership of the element x to V, ranging from 1 representing absolute, crisp membership and 0, absolute non membership.

The theory of Fuzzy Sets, which was mainly oriented towards engineering applications, is usually referred to under the name of “Fuzzy Logic”, or sometimes “Fuzzy Logic in broad sense” [86], but despite its name, the objects of investigation are often unrelated to those typical of mathematical logic, see e.g. [19]. In the present work, by Fuzzy Logic we mean instead the “Mathematical Fuzzy Logic” or “Fuzzy Logic in narrow sense”. This emerged as a discipline of its own by the end of the ’90s, in particular with Hájek’s book “Metamathematics of Fuzzy Logic” [49], which represented one of the first systematizations of the field. One of the aims of this book was the translation of part of the issues of “Fuzzy Logic in broad sense” in the traditional framework of mathematical logic. In Hájek’s approach, the real interval $[0,1]$ was assumed as the set of truth values and the so-called *continuous t-norms* and related *residua* were proposed as the most adequate operations over $[0,1]$ to interpret the logical connectives conjunction and implication. To capture this intended semantics, an axiomatic system was introduced: the *Basic logic* BL. This represented a turning point for the discipline: on the one hand, well-known older many-valued logics in the literature, such as Gödel [48] or Łukasiewicz [61] logic, could then be seen as axiomatic extensions of BL. On the other hand, it paved the way for the introduction of many new logics, usually obtained as axiomatic systems which either extended or restricted the axiomatic system for BL. Among them worth mentioning are the *monoidal t-norm logic* MTL [42], *uninorm logic* UL [66] and *pseudo monoidal t-norm logic* $\mathrm{psMTL}^{\mathrm{r}}$. Fuzzy Logic inherited then the typical agenda of mathematical logic (showing completeness, decidability, complexity results etc), and researchers started to develop and apply a full range of proof-theoretical [63] and algebraic [52] methods for them.

Standard completeness

As for the vast majority of nonclassical logics, the axiomatic systems introduced in the area of Mathematical Fuzzy Logic can be shown to be complete with respect to classes of corresponding algebraic structures, using methods from Abstract Algebraic Logic. These results are however in some respect unsatisfactory: general algebraic semantics may appear just as a reformulation of the syntax in algebraic terms.

For instance, the completeness of Basic logic BL (or for that matters MTL, UL) with respect to the corresponding class of BL-algebras (MTL, UL-algebras, respectively) can be easily established. But, coherently with the initial motivations to model vague predicates and the roots in the theory of fuzzy sets, it is the algebras over the real interval $[0, 1]$ which represent the intended semantics for these logics. For instance, for BL the intended or standard semantics is the algebraic structure over $[0, 1]$, where conjunction is interpreted as a continuous t-norm and implication as its residuum. *Standard completeness*, i.e. the completeness of a logic with respect to the algebras over the real interval $[0, 1]$, is thus a fundamental issue for Mathematical Fuzzy Logic. This can be reformulated, in purely algebraic term, as the problem of establishing whether the algebras over $[0, 1]$ in certain varieties (or quasivarieties) generate the whole variety (resp. quasivariety).

Proofs of standard completeness were (and still are) usually tailored to specific logics, and developed ad hoc using purely algebraic techniques, see e.g. [23, 31, 34, 41, 42, 49, 51, 58, 71]. While the early algebraic proofs of standard completeness for BL [31], and for the three main continuous t-norm based logics, i.e. Łukasiewicz [22], Gödel [40] and Product [57] logics each have their own peculiar structure, since Jenei and Montagna's [58] algebraic proof of standard completeness of MTL most proofs of standard completeness attacked the problem along the following lines.

Given a logic L, described as a Hilbert-style system, the first step is usually to identify a general class of algebraic structures (L-algebras), for which the logic is complete.

A further, and more interesting step, consists of verifying whether the completeness result can be sharpened to the class of linearly ordered L-algebras, so called L-chains, that is algebras where every two elements can be compared. Completeness of a logic L with respect to the corresponding class of L-chains has been advocated as a defining feature of fuzzy logic [20] and is now understood in very general terms, see e.g. [33, 52]. These investigations have led to the identification of the class of so-called *semilinear* (or representable) algebraic varieties, i.e. the varieties whose subdirectly irreducible members are chains, and for which therefore the step from the general algebraic completeness result to the completeness with respect to chains is guaranteed to hold. Once completeness with respect to chains is achieved, to show standard completeness two more steps remain to be proved:

(i) Show that any countable L-chain can be embedded into a countable *dense* L-chain by adding countably many new elements to the algebra and extending the operations appropriately. This establishes *rational completeness*: a formula is derivable in L iff it is valid in all dense L-chains.

(ii) To move from rational to standard completeness, a countable dense L-chain has to be embedded into a standard L-algebra (an L-algebra with lattice reduct $[0, 1]$, see Definition 2.2.1) using a Dedekind-MacNeille style completion, that is a generalization of Dedekind's embedding of the rational numbers into the extended real field (i.e. $\mathbb{R}$ with $\pm\infty$).

While some general results exist for (ii) (see [26]), no systematic approach seems to exist for (i) (rational completeness), which relies on finding the right embedding, if any.

A different approach to (i) was introduced in [66] and is based on *proof-theoretic* techniques. The main idea is that the admissibility of a particular syntactic rule (called *density*) in a logic L leads to a proof of rational completeness for L. Formalized Hilbert-style, the density rule has the following form

$$\frac{(\varphi \to p) \vee (p \to \chi) \vee \psi}{(\varphi \to \chi) \vee \psi} \ (density)$$

where p is a propositional variable not occurring in φ, χ, or ψ. Ignoring ψ and reading $\to$ as the ordering $\leq$, this can be read contrapositively as saying (very roughly) "if $\varphi > \chi$, then $\varphi > p$ and $p > \chi$ for some p". Hence the name "density" to correspond to the density of the usual order of the rational numbers, and the intuitive connection with rational completeness.

The proof-theoretic approach was used to establish standard completeness for various logics, where, in some cases, the algebraic techniques did not appear to work, e.g. for UL [66]. Following this method, to establish rational completeness for a logic L expressed Hilbert-style we need to check that the density rule is eliminable (or admissible), i.e. that density does not enlarge the set of provable formulas. Proving this is not easy in a Hilbert-style formulation of a logic, but requires instead analytic calculi. *Hypersequent* calculi have proved to be suitable calculi for this purpose.

1.3 Aims and outline of the thesis

Many papers in the literature are devoted to ad-hoc proofs of standard completeness, which use either algebraic or proof-theoretic techniques.

The aim of our work is to address the problem of standard completeness in a more systematic way. Using tools from algebraic proof theory, we aim at developing methods which apply to large classes of logics in a uniform way. More precisely we introduce:

- General methods for density elimination for axiomatic extensions of $\mathrm{UL}\forall$ and $\mathrm{MTL}\forall$ (the first-order version of UL and MTL). Our methods extend the proof-theoretic approach introduced in [66] and refined in [28].

- A new algebraic method for standard completeness, which is inspired by the proof-theoretic technique of density elimination. Using this method, we obtain general results on extensions of the logic $\mathrm{psMTL^r}$, the noncommutative version of MTL.

The thesis is organized as follows. Chapter 2 and 3 are devoted to preliminaries and to a survey of already known proofs and results on standard completeness.

In Chapter 2 we give the necessary algebraic preliminaries, introduce the notion of Dedekind–MacNeille completion, and recall results on preservation of equations under this construction.

In Chapter 3 we recall basic preliminaries on propositional and first-order logics, and explore the issues involved in the algebraic and proof-theoretic methods which have been developed to address standard completeness. Furthermore, we recall the concept of the substructural hierarchy [25] and the systematic introduction of analytic calculi which is based on it.

In Chapters 4, 5 and 6 we present our original results. More precisely, in Chapter 4 we address standard completeness for axiomatic extensions of the logic $\mathrm{MTL}\forall$, using the proof-theoretic approach based on density elimination. We extend the method of density elimination introduced in [28] to a large class of of logics, in a uniform and systematic way. The main result is a sufficient condition for standard completeness which works for large classes of axiomatic extensions of $\mathrm{MTL}\forall$. A check of this condition, which applies to hypersequent calculus rules, is implemented in a program which is available online.

In Chapter 5 we address standard completeness for axiomatic extensions of the logic $\mathrm{UL}\forall$. The proof of density elimination is complicated here by the absence of the weakening rules in the corresponding hypersequent calculi (in algebraic terms, integrality of the corresponding residuated lattice). The main result here is a sufficient condition for standard completeness of classes of axiomatic extensions of $\mathrm{UL}\forall$.

Chapter 6 contains a new algebraic method to prove standard completeness. The method is inspired by the proof-theoretic approach and uses residuated frames [45]. This chapter has a slightly different structure with respect to the others and requires the introduction of some additional definitions and concepts from the literature. Hence it starts with preliminaries on residuated frames and noncommutative logics. Then it proceeds with the reformulation of the results in Chapter 4 and part of Chapter 5 in this new algebraic framework and the extension of the results in Chapter 4 to the noncommutative case.

In Chapter 7 we summarize the results of the thesis and discuss open problems and future research directions.

1.4 Publications

This thesis is based on the following publications:

- P. Baldi, A. Ciabattoni and L. Spendier. Standard Completeness for Extensions of MTL: an Automated Approach. Proceedings of Workshop on Logic, Language, Information and Computation (WoLLIC 2012), L. Ong and R. de Queiroz (Eds.), LNCS 7456, pp. 154–167. Springer, Heidelberg (2012).

- P. Baldi. A note on standard completeness for some axiomatic extensions of uninorm logic. Soft Computing. 18(8): 1463-1470 (2014).
- P. Baldi and A. Ciabattoni. Uniform proofs of standard completeness for extensions of first-order MTL. Theoretical Computer Science. Accepted for publication.
- P. Baldi and K. Terui. Densification of FL chains via residuated frames. Algebra Universalis. Accepted for publication.
- P. Baldi and A. Ciabattoni. Standard completeness for uninorm-based logics. Proceedings of IEEE International Symposium on Multiple-Valued Logic (ISMVL 2015), pp. 78–83, Waterloo (Canada).

and on some unpublished results [9].

CHAPTER 2

Algebraic preliminaries

In this chapter we briefly recall basic notions on residuated lattices and completions.

2.1 Residuated lattices

Residuated lattices are fundamental algebraic structures which provide general semantics for the logics we will investigate (see [46] for an extensive treatment). We start by recalling some basic notions on lattice and order theory, see e.g. [18] for more details.

Definition 2.1.1. *A* lattice *is an algebraic structure* $(A, \wedge, \vee)$ *with* A *a nonempty set, and* $\wedge$ *and* $\vee$ *commutative, associative, idempotent operations, satisfying in addition the following* absorption laws*:*

- $x = x \vee (x \wedge y)$
- $x = x \wedge (x \vee y)$

The following is a well known fact, connecting the algebraic presentation of lattices with partial order relations. Recall that a partial order is a transitive, asymmetric, reflexive binary relation on a set. Let $(A, \leq)$ be a partially ordered set and $X \subseteq A$. We denote by $\sup X, \inf X$ the supremum and infimum element of the set X.

Lemma 2.1.2. *Let* $\mathbf{A} = (A, \wedge, \vee)$ *be a lattice and* $\leq$ *be the binary relation on* A *defined by*

$$(*) \quad x \leq y \Leftrightarrow x = x \wedge y.$$

The relation $\leq$ *is a* partial order *such that, for any* $x, y \in A$*, there is a supremum element* $\sup\{x, y\}$ *and an infimum* $\inf\{x, y\}$ *in* A*, with respect to* $\leq$*. Conversely, let* $(A, \leq)$ *be a partially ordered set such that for any* $x, y \in A$ *there are* $\sup\{x, y\}$ *and* $\inf\{x, y\}$ *in* A*. We obtain a lattice, letting* $x \vee y = \sup\{x, y\}$ *and* $x \wedge y = \inf\{x, y\}$*.*

In what follows we will freely move between the algebraic and the order-theoretic presentation of lattices. In virtue of the equivalence $(*)$, we call *equation* not only an expression of the form $t = s$, but also $t \leq s$. The following are important properties of partial order relations. As usual, by $<$ we denote the strict order associated with a partial order $\leq$, i.e. we let $x < y$ iff $x \leq y$ and $x \neq y$.

Definition 2.1.3. *Let $\boldsymbol{A} = (A, \leq)$ be a partially ordered set. We say that*

- $\boldsymbol{A}$ *is a* chain *if the order $\leq$ is total, i.e. for any $x, y \in A$, either $x \leq y$ or $y \leq x$.*
- $\boldsymbol{A}$ *contains a* gap (g, h) *if $g < h$ ($g, h \in A$) and there is no element $p \in A$ such that $g < p < h$.*
- $\boldsymbol{A}$ *is* dense *if it does not contain any gap, i.e. if $g < h$ ($g, h \in A$) implies $g < p < h$ for some $p \in A$.*
- $\boldsymbol{A}$ *is* bounded *if it has a least element $\bot$ and a greatest element $\top$.*
- $\boldsymbol{A}$ *is* complete *if, for every $X \subseteq A$, $\sup X \in A$ and $\inf X \in A$ (where X might be an infinite set as well).*

In what follows, we say that an algebra $\boldsymbol{A}$ satisfies one of the properties above, if it has a lattice reduct, whose corresponding order does.

We recall now basic notions of universal algebra mainly taken from [18].

Definition 2.1.4. *A* language $\mathcal{L}$ *or* type *of algebras is a function $ar\colon C_{\mathcal{L}} \to \mathbb{N}$, where $C_{\mathcal{L}}$ is a countable set of function symbols, giving for each one its* arity. *The 0-ary functions are also called* constants. *Let Var be a fixed countable set of symbols called* variables. *The set $Fm_{\mathcal{L}}$ is the least set containing Var and closed under the functions in $C_{\mathcal{L}}$, i.e. for each n-ary function symbol $f \in C_{\mathcal{L}}$ and every $x_1, \ldots, x_n \in Fm_{\mathcal{L}}$, $f(x_1, \ldots, x_n)$ is in $Fm_{\mathcal{L}}$. Elements of $Fm_{\mathcal{L}}$ will be denoted in this chapter by s, t, u, v. We denote by $\boldsymbol{Fm_{\mathcal{L}}}$ the* formula algebra*, or* absolutely free algebra *(see e.g. [18, 52]) for the language $\mathcal{L}$, whose support is the set $Fm_{\mathcal{L}}$.*[1]

We can now define the notion of evaluation and a consequence relation based on an algebra.

Definition 2.1.5. *Let $\boldsymbol{A}$ be an algebra of type $\mathcal{L}$. An* evaluation *into $\boldsymbol{A}$ is a homomorphism v from the formula algebra $\boldsymbol{Fm_{\mathcal{L}}}$ into $\boldsymbol{A}$, determined uniquely by the images of variables. Let $E = \{t_i = s_i \mid i \in I\}$ and $t = s$ be equations in $\boldsymbol{Fm_{\mathcal{L}}}$. The* equational consequence relation *$\models_{\boldsymbol{A}}$ is defined as follows:*

$$E \models_{\boldsymbol{A}} t = s \quad \text{iff} \quad \begin{array}{l}\text{for all evaluations } v \text{ we have } v(t) = v(s)\\ \text{whenever } v(t_i) = v(s_i) \text{ for all } i \in I.\end{array}$$

[1]The formula algebra defined here is usually known as *term algebra*, see [18] and its element are called terms. We preferred formula algebra, as in the next chapters, under the same notion of language, functions will stand for *connectives* and elements of the algebra will stand for propositional formulas and be denoted by $\alpha, \beta, \ldots$

The definition of $\models_{\boldsymbol{A}}$ can be extended to an arbitrary class of algebras V, by setting $E \models_{\mathrm{V}} t = s$ iff $E \models_{\boldsymbol{A}} t = s$ holds for every $\boldsymbol{A} \in \mathrm{V}$. A class of algebras V is an *equational class* if there is a set of equations E such that $\boldsymbol{A} \in \mathrm{V}$ if and only if $\models_{\boldsymbol{A}} t = s$ for every equation $t = s$ in E. In what follows, we say that two equations $t_1 = s_1$ and $t_2 = s_2$ are *equivalent* in a class of algebras V, if $\models_{\mathrm{V}} t_1 = s_1$ if and only if $\models_{\mathrm{V}} t_2 = s_2$.

Definition 2.1.6. *A class of algebras is said to be a* variety *if it is closed under subalgebras, homomorphic images and direct products.*

A famous result by Birkhoff shows that varieties coincide with equational classes.

Theorem 2.1.7. *(Birkhoff) [18]* V *is an equational class if and only if* V *is a variety.*

Let V be a variety, defined by the equations E. By a *subvariety* V' we mean a subclass of V that is a variety as well. Clearly, in view of Theorem 2.1.7, V' is an equational class defined by a set $E' \supseteq E$ of equations.

Other important concepts that we need from universal algebra are that of subdirect product and subdirectly irreducible algebras. In what follows, we denote by $\prod_{i \in I} \boldsymbol{A}_i$ the usual direct product of a family $\langle \boldsymbol{A}_i \mid i \in I \rangle$ of algebras.

Definition 2.1.8. *An algebra* $\boldsymbol{A}$ *is a* subdirect product *of a family* $\langle \boldsymbol{A}_i \mid i \in I \rangle$ *of algebras if the following hold:*

1. $\boldsymbol{A}$ *is a subalgebra of* $\prod_{i \in I} \boldsymbol{A}_i$*,*
2. $\pi_i(\boldsymbol{A}) = \boldsymbol{A}_i$ *for all* $i \in I$*, where* π_i *denotes the projection to the* i*-th component.*

Given an algebra $\boldsymbol{A}$, a family $\langle \boldsymbol{A}_i \mid i \in I \rangle$ of algebras and an embedding $f \colon A \to \prod_{i \in I} A_i$, we say the f is *subdirect* if $f[\boldsymbol{A}]$ is a subdirect product of $\langle \boldsymbol{A}_i \mid i \in I \rangle$.

Definition 2.1.9. *An algebra* $\boldsymbol{A}$ *is said to be* subdirectly irreducible *if it is nontrivial and for every subdirect embedding* $f \colon A \to \prod_{i \in I} A_i$ *there is* $i \in I$ *such that* $\pi_i \circ f \colon A \to A_i$ *is an isomorphism.*

We are now ready to move to residuated lattices.

Definition 2.1.10. *A* residuated lattice *is a structure* $\boldsymbol{A} = (A, \wedge, \vee, \cdot, \backslash, /, e)$ *where*

- $(A, \wedge, \vee)$ *is a lattice,*
- $(A, \cdot, e)$ *is a monoid, i.e.* $\cdot$ *is an associative operation and* e *is a neutral element for* $\cdot$*,*
- *for each* $x, y, z \in A$ *the* residuation *property holds, i.e.*

$$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z/.y$$

Definition 2.1.11. *An* FL-algebra *is a* pointed *residuated lattice, i.e. a residuated lattice* $\boldsymbol{A} = (A, \wedge, \vee, \cdot, \backslash, /, e, f)$ *with a distinguished element* $f \in A$*. An* FL*-algebra is* *(e)* commutative *if $x \cdot y = y \cdot x$ for all $x, y \in A$,*

(c) contractive *if $x \leq x \cdot x$ for every $x \in A$,*

(i) integral *if* e *is the greatest element,*

(o) f-bounded *if f is the least element,*

($\bot$) bounded *if there is a least element, denoted by $\bot$.*

Remark 2.1.12. *The element f allows us to define the two unary operations $\sim a = a \backslash f$ and $-a = f/a$, which are called respectively* left *and* right negation. *Note that, if an* FL*-algebra has a least element $\bot$, then it also has a greatest element $\bot/\bot = \bot\backslash\bot$, which we denote by $\top$.*

Recall that all the defining properties of FL-algebras, including residuation, can be expressed equationally [17], hence these structures form algebraic varieties. We use the subscripts $e, c, i, o, \bot$ to indicate the properties (e), (c), (i), (o), $(\bot)$ above[2]. If a variety satisfies both (i) and (o), we use the subscript w. For instance, FL_{ew} denotes the variety of commutative, f-bounded, integral FL-algebras. We will mostly consider the variety FL_e as our basic algebraic semantics. It is easy to show that for FL_e-algebras we have $\backslash = /$. Hence we use the operation symbol $\to$ to denote both of them. Negation in FL_e-algebras can be thus simply defined as $\neg x = x \to f$. Moreover, we can simplify the residuation property as follows:

$$x \cdot y \leq z \iff x \leq y \to z.$$

The variety FL_w is by definition bounded (i.e. FL_w coincides with $\mathrm{FL}_{\bot w}$) and the constants e and f coincide with $\top$ and $\bot$. Hence we can dispense with the latter constants. More precisely, in correspondence with each variety, we will use the following languages:

- $\mathcal{L}_{\mathrm{FL}}$ denotes the language of FL-algebras, i.e. the language consisting of binary functions $\{\cdot, \wedge, \vee, /, \backslash\}$ and constants $\{e, f\}$
- $\mathcal{L}_{\mathrm{FL}_e}$ denotes the language of FL_e-algebras, i.e. with binary functions $\{\cdot, \wedge, \vee, \to\}$ and constants $\{e, f\}$
- $\mathcal{L}_{\mathrm{FL}_\bot}$ denotes the language of $\mathrm{FL}_\bot$-algebras, i.e. with binary functions $\{\cdot, \wedge, \vee, /, \backslash\}$ and constants $\{e, f, \top, \bot\}$
- $\mathcal{L}_{\mathrm{FL}_w}$ denotes the language of FL_w-algebras, i.e. with binary functions $\{\cdot, \wedge, \vee, /, \backslash\}$ and constants $\{e, f\}$.

The languages are combined in obvious ways, for instance the language $\mathcal{L}_{\mathrm{FL}_{ew}}$ for FL_{ew}-algebras contains the binary functions $\{\cdot, \wedge, \vee, \to\}$ and constants $\{e, f\}$. We assume in the following that any variety of FL_x-algebras is defined over the language $\mathcal{L}_{\mathrm{FL}_x}$, for any $x \subseteq$

[2] (e), (c), (i), (o) correspond to structural rules in the sequent calculus: (e) correspond to the exchange rule, (i) to the weakening left rule (wl), (o) to the weakening right rule (wr), and (c) to the contraction left rule, see Table 3.2.

$\{e, c, i, o, \bot\}$. In what follows we adopt the usual convention of writing xy for $x \cdot y$. We also write $x\backslash y/z$ for $x\backslash(y/z)$ and $(x\backslash y)/z$, since the latter two are equal in every FL algebra. In the absence of parentheses, we assume that $\cdot$ is performed first, followed by $/, \backslash$, and finally the lattice operations. If $\to$ is present in the language, it comes after the lattice operations.

A subvariety V of FL is said to be *semilinear* if $\models_{\mathrm{V}} = \models_{\mathrm{V}_C}$, where V_C consists of all the chains in V. This is equivalent to requiring that every subdirectly irreducible algebra in V is a chain, see [52]. Let V be a a subvariety of FL. We denote by V^ℓ the smallest semilinear variety containing V.

Theorem 2.1.13. *[52] Let* V *be a subvariety of* FL.

- V^ℓ *is axiomatized over* V *by the equation* $\lambda_a((x \vee y)\backslash y) \vee \rho_b((x \vee y)\backslash x) = e$,
- *If* V *is a subvariety of* $\mathrm{FL_e}$, V^ℓ *is axiomatized over* V *by* $((x \to y) \wedge e) \vee ((y \to x) \wedge e) = e$.
- *If* V *is a subvariety of* $\mathrm{FL_{ei}}$, V^ℓ *is axiomatized over* V *by* $(x \to y) \vee (y \to x) = e$.

λ_a *and* ρ_b *are conjugate operators defined by:*

$$\lambda_a(x) := (a\backslash xa) \wedge e, \qquad \rho_b(x) := (bx/b) \wedge e.$$

The varieties of $\mathrm{FL}^\ell_{e\bot}$-algebras, FL^ℓ_w, FL^ℓ_{ew}-algebras are also known in the literature as UL-*algebras*, psMTL^r-*algebras*, MTL-*algebras*, respectively, see e.g. [21]. These structures provide an algebraic semantics for the logics UL, psMTL^r, MTL, which we will introduce in the next chapters.

2.2 Examples: standard FL-algebras

FL-algebras constitute a very broad abstraction of many different algebraic structures introduced in the literature, ranging from the ideals of a ring [60] to Boolean algebras. Well-known varieties, such as the MV-algebras [32], Heyting algebras and the above mentioned Boolean algebras [18, 46], though originally defined in different signatures, can be easily shown to be term-equivalent to subvarieties of FL-algebras. We give now some concrete examples of FL-algebras, from the class of the so-called *standard* FL-algebras. The "standard" here comes from Mathematical Fuzzy Logic, as these algebras form the intended semantics for Fuzzy Logic, see Chapter 1 and [21, 49]. Proving completeness of logics expressed Hilbert-style with respect to standard algebras will be the main focus of this thesis.

Definition 2.2.1. *A* standard FL*-algebra is an* FL*-algebra whose lattice reduct is the real interval* $[0, 1]$*, with the usual order.*

To illustrate some examples of standard FL-algebras, we start by recalling the notion of (pseudo) t-norms and uninorms.

Definition 2.2.2. *Let* $([0, 1], \leq)$ *be the real interval* $[0, 1]$ *equipped with the usual real ordering* $\leq$.

- *A* pseudo-uninorm *is an associative, monotone operation* $*\colon [0,1]^2 \to [0,1]$ *compatible with the order* $\leq$*, with neutral element* $e \in [0,1]$.
- *A* uninorm *[84] is a commutative pseudo-uninorm*
- *A (pseudo)* t-norm *[49] is a (pseudo) uninorm with neutral element* $e = 1$.

Definition 2.2.3.

- *Let* $*$ *be a* t*-norm or a uninorm. The* residuum *of* $*$ *is defined as*

$$x \to_* y = \sup\{z \mid x * z \leq y\}$$

for any x, y *in* $[0,1]$.

- *Let* $*$ *be a pseudo t-norm or a pseudo-uninorm. We need to distinguish two residua of* $*$ *:*
  - *The* left residuum
  $$x \backslash_* y = \sup\{z \mid x * z \leq y\}$$
  - *The* right residuum
  $$y /_* x = \sup\{z \mid z * x \leq y\}$$

In the following examples, we say that a (pseudo) t-norm or uninorm $*\colon [0,1]^2 \to [0,1]$ is *continuous* if both functions $f_x(y) = x * y$ and $f_y(x) = x * y$ are continuous with respect to the standard topology over $[0,1]$, i.e. they commute with suprema

$$\sup_{x<z} x * y = z * y \qquad \sup_{y<z} x * y = x * z$$

and with infima

$$\inf_{x>z} x * y = z * y \qquad \inf_{y>z} x * y = x * z$$

A (pseudo) t-norm or uninorm is said to be *left-continuous* (resp. *right continuous*) if the functions f_x and f_y only commute with suprema (resp. infima). All uninorms are either *conjunctive* or *disjunctive* (see [84]), where by conjunctive we mean that $0 * 1 = 0$ and by disjunctive that $0 * 1 = 1$. It can be easily shown that continuous conjunctive uninorms are already continuous t-norms, see e.g. [66]. In the following we recall the connection between (pseudo) uninorms, t-norms and FL-algebras.

Lemma 2.2.4.

- *[42] A t-norm* $*$ *and its residuum* $\to_*$ *satisfy the residuation property in Definition 2.1.10 if and only if* $*$ *is left-continuous. Hence* $([0,1], max, min, *, \to_*, 0, 1)$ *is an* FL_{ew}*-algebra if and only if* $*$ *is left-continuous.*
- *[49] Let* $*$ *be a* t*-norm and* $\to_*$ *its residuum. Then* $*$ *is continuous if and only if the algebra* $([0,1], max, min, *, \to_*, 0, 1)$ *is an* FL_{ew}*-algebra, which satisfies in addition the* divisibility *equation*

$$(div) \quad 1 \leq (x \wedge y) \to_* (x * (x \to_* y))$$

- *[59] Let $*$ be a pseudo t-norm and $\backslash_*, /_*$ its residua. Then $*$ is left-continuous if and only if the algebra $([0,1], max, min, *, \backslash_*, /_*, 0, 1)$ is an FL_w-algebra.*
- *[66] Let $*$ be a uninorm and $\to_*$ its residuum. The algebra $([0,1], max, min, *, \to_*, e, f, 0, 1)$ is an FL_e-algebra iff $*$ is a left-continuous conjunctive uninorm.*

Note that FL^{ℓ}_{ew}-algebras satisfying (div) are known in the literature as BL-*algebras*, as they form the algebraic semantics of *basic logic* BL, see Example 3.1.4 and [49].

Example 2.2.5. *The following are the most important examples of continuous t-norms:*

- $x * y = min(x, y)$ *(Gödel t-norm)*
- $x * y = max(x + y - 1, 0)$ *(Łukasiewicz t-norm)*
- $x * y = x \cdot y$ *(Product t-norm)*

(all the operations to the right of the equality symbol denote the usual operations over the reals). The three operations above are in a sense paradigmatic examples of continuous t-norms. Indeed, it was proved in [72] that any *continuous t-norm can be represented as an ordinal sum of Gödel, Łukasiewicz and Product t-norms. For any of these three continuous t-norms, the value of the residuum $x \to_* y$ is 1 when $x \leq y$. In case $x > y$, we have*

- $x \to_* y = y$ *if $*$ is Gödel t-norm*
- $x \to_* y = min(1, 1 - x + y)$ *if $*$ is Łukasiewicz t-norm*
- $x \to_* y = y/x$ *if $*$ is Product t-norm*

(all the operations to the right of the equality symbol denote the usual operations over the reals, and in particular / stands here for the usual division). We call the above implications Gödel, Łukasiewicz, and Product implication, respectively. The corresponding negations are defined as $\neg_* x = x \to_* 0$. It can be easily checked that

- $\neg_* x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$ *if $\to_*$ is Gödel or Product implication*
- $\neg_* x = 1 - x$ *if $\to_*$ is Łukasiewicz implication.*

Gödel and Product negation are called strict *or* pseudocomplement *[21], while Łukasiewicz negation is said to be* involutive, *as it satisfies $\neg_*\neg_* x = x$ for any $x \in [0,1]$.*

Example 2.2.6. *An example of a left-continuous but not continuous t-norm is the following [42]. Let an order-reversing function $n\colon [0,1] \to [0,1]$ with $n(n(x)) \geq x$ for all $x \in [0,1]$ and $n(1) = 0$ be called a* weak negation. *Given a weak negation n, the* weak nilpotent minimum *t-norm $*_{\mathrm{WNM}(n)}$ is defined as:*

$$x * y = \begin{cases} 0 & \text{if } x \leq n(y) \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

Each $*_{\mathrm{WNM}(n)}$ is left-continuous, but not (right)-continuous.

Example 2.2.7. *An example of a left-continuous pseudo t-norm is as follows [21], letting* $0 < a < b < 1$ *and, for any* $x, y \in [0,1]$

$$x * y = \begin{cases} 0 & \text{if } x \leq a \text{ and } y \leq b \\ \min\{x,y\} & \text{otherwise.} \end{cases}$$

Example 2.2.8. *Prominent examples of left-continuous conjunctive uninorms that are not t-norms are idempotent left-continuous conjunctive uninorms [6]. Consider for instance the following, where* n *is a weak negation (see Example 2.2.6)* $n\colon [0,1] \to [0,1]$ *such that* $n(e) = e$

$$x * y = \begin{cases} \min\{x,y\} & \text{if } y \leq n(x) \\ \max\{x,y\} & \text{otherwise.} \end{cases}$$

2.3 The Dedekind-MacNeille completion

Completions are a deeply investigated topic in the field of lattices and ordered structures, see e.g. [46,50]. The abstract notion of completion emerged as a generalization of the construction of the real numbers from the rationals via Dedekind cuts. Intuitively, a completion can be thought of as a uniform way of "filling" the gaps in an ordered structure. More formally, given an algebra $\boldsymbol{A}$ with a lattice reduct, a *completion* of $\boldsymbol{A}$ consists of a complete (see Definition 2.1.3) algebra $\mathbf{A}^+$ together with an embedding $v : A \longrightarrow A^+$. A completion (A^+, v) is *join-dense* if $x = \bigvee\{a \in v[A] : a \leq_{A^+} x\}$, and *meet-dense* if $x = \bigwedge\{a \in v[A] : x \leq_{A^+} a\}$ for every $x \in A^+$. A join-dense and meet-dense completion is called a *Dedekind-MacNeille completion* (DM completions in the following). This is a generalization of Dedekind's embedding of the rational numbers into the extended real field (i.e. $\mathbb{R}$ with $\pm\infty$). It is known that the lattice reduct of a DM completion is uniquely determined (up to isomorphism fixing A) by join and meet density [13, 76]. For instance, the DM completion of the rational unit interval $([0,1]_{\mathbb{Q}}, \leq)$ is just $([0,1]_{\mathbb{R}}, \leq)$. In what follows, we give an explicit definition of the DM completion of an FL-algebra.

Definition 2.3.1 (Dedekind-MacNeille completion)**.** *[46, 66]*
Let $\boldsymbol{A} = (A, \wedge, \vee, \cdot, \backslash, /, f, e)$ *be an* FL*-algebra and* $X \subseteq A$*. The sets of upper and lower bounds of* X *are defined as follows:*
$X^u = \{y \in A \mid x \leq y \text{ for all } x \in X\}$ $\quad X^l = \{y \in A \mid y \leq x \text{ for all } x \in X\}$
Let $DM(A) = \{X \subseteq A \mid (X^u)^l = X\}$*. For any,* $X, Y \in DM(A)$ *we define*

- $X \vee_{DM} Y = ((X \cup Y)^u)^l$
- $X \wedge_{DM} Y = X \cap Y$
- $X \cdot_{DM} Y = (\{x \cdot y \mid x \in X, y \in Y\}^u)^l$
- $Y \backslash_{DM} X = \{z \in A \mid y \cdot z \in X \text{ for all } y \in Y\}$
- $X /_{DM} Y = \{z \in A \mid z \cdot y \in X \text{ for all } y \in Y\}$

- $e_{DM} = \{e\}^l \quad f_{DM} = \{f\}^l$

The structure $\boldsymbol{A}^+ = (DM(A), \wedge_{DM}, \vee_{DM}, /_{DM}, \backslash_{DM}, \cdot_{DM}, f_{DM}, e_{DM})$ *is a complete* FL-*algebra, the* Dedekind-MacNeille completion *of* $\boldsymbol{A}$.

The following is a well known fact about DM completions, see e.g. [46, 66].

Lemma 2.3.2. *Let* $\boldsymbol{A}$ *be an* FL*-algebra and* $\boldsymbol{A}^+$ *its DM completion. The map* $e\colon A \to DM(A)$*, associating to any* $x \in A$ *the set* $\{x\}^l$ *in* $DM(A)$*, is a* regular embedding*, i.e. an injective map, preserving all operations and all existing (also infinite) meets and joins in* $\boldsymbol{A}$.

DM completions can be easily shown to preserve all the important ordering properties in Definition 2.1.3 and 2.1.11.

Lemma 2.3.3. *[52, 66] Let* $\boldsymbol{A}$ *be an* FL_x*-algebra, with* $x \subseteq \{e, o, i, c\}$*. Then its DM completion* $\boldsymbol{A}^+$ *is an* FL_x*-algebra as well. Moreover if* $\boldsymbol{A}$ *is a dense algebra and/or a chain,* $\boldsymbol{A}^+$ *is a dense algebra and/or a chain, respectively.*

Note that we did not consider the properties of *boundedness* and *completeness* from Definition 2.1.3, as DM completions are by definition always complete, hence bounded.

The equations $(e), (o), (i), (c)$ in Definition 2.1.11 are just a few examples of equations that are preserved under DM completion. We recall in the following the more general results proved in [26, 27], concerning the preservation of equations under DM completion. A key concept in [26, 27] is the classification of equations in the so-called *substructural hierarchy*, whose classes we define below. This classification was originally introduced in [25], from a proof-theoretical perspective, as we will see in the next chapter.

Definition 2.3.4. *For each* $n \geq 0$*, we define the sets* $\mathcal{P}_n, \mathcal{N}_n$ *of elements of the formula algebra* $Fm_{\mathcal{L}}$ *in the language* $\mathcal{L} = \mathcal{L}_{\mathrm{FL}}$ *as follows:*

(0) $\mathcal{P}_0 = \mathcal{N}_0 =$ *is the set of variables.*

(P1) e *and all* $t \in \mathcal{N}_n$ *belong to* $\mathcal{P}_{n+1}$.

(P2) If $t, u \in \mathcal{P}_{n+1}$*, then* $t \vee u, t \cdot u \in \mathcal{P}_{n+1}$.

(N1) f *and all* $t \in \mathcal{P}_n$ *belong to* $\mathcal{N}_{n+1}$.

(N2) If $t, u \in \mathcal{N}_{n+1}$*, then* $t \wedge u \in \mathcal{N}_{n+1}$.

(N3) If $t \in \mathcal{P}_{n+1}$ *and* $u \in \mathcal{N}_{n+1}$*, then* $t\backslash u, u/t \in \mathcal{N}_{n+1}$.

In other words, $\mathcal{P}_n$ *and* $\mathcal{N}_n$ *(*$n \geq 1$*) are generated by the following BNF grammar:*

$$\begin{array}{lcllllllll} \mathcal{P}_n & ::= & \mathcal{N}_{n-1} & \mid & e & \mid & \mathcal{P}_n \vee \mathcal{P}_n & \mid & \mathcal{P}_n \cdot \mathcal{P}_n, & \\ \mathcal{N}_n & ::= & \mathcal{P}_{n-1} & \mid & f & \mid & \mathcal{N}_n \wedge \mathcal{N}_n & \mid & \mathcal{P}_n \backslash \mathcal{N}_n & \mid \ \mathcal{N}_n / \mathcal{P}_n. \end{array}$$

By residuation, any equation $u = v$ *can be written as* $e \leq t$*. We say that* $u = v$ *belongs to* $\mathcal{P}_n$ *(*$\mathcal{N}_n$*, resp.) if* t *does.*

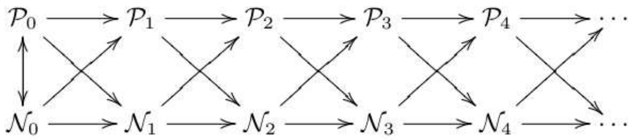

Figure 2.1: Substructural hierarchy $(\mathcal{N}_n, \mathcal{P}_n)$ [25]

The classes $(\mathcal{P}_n, \mathcal{N}_n)$ constitute the *substructural hierarchy*. Figure 2.1 depicts the hierarchy, with the arrows representing the inclusion relation between the classes. Among those classes, relevant to subsequent arguments are $\mathcal{N}_2$ and $\mathcal{P}_3$. The former includes:

$xy \leq yx$	(e)
$x \leq xx$	(c)
$x \leq e$	(i)
$f \leq x$	(o)
$x^m \leq x^n$	(knotted axioms, $m, n \geq 0$)
$e \leq \sim(x \wedge \sim x)$	(no-contradiction)

$\mathcal{P}_3$ includes:

$e \leq x \vee \sim x$	(excluded middle)
$e \leq \sim x \vee \sim\sim x$	(weak excluded middle)
$e \leq \sim(x \cdot y) \vee ((x \wedge y) \backslash (x \cdot y))$	(weak nilpotent minimum)
$e \leq \sim(x \cdot y)^n \vee ((x \wedge y)^{n-1} \backslash (x \cdot y)^n)$	(wnm^n)
$e \leq p_0 \vee (p_0 \backslash p_1) \vee \cdots \vee ((p_0 \wedge \cdots \wedge p_{k-1}) \backslash p_k)$	(bounded size k)

We give now a normal form for equations within the classes $\mathcal{N}_2$ and $\mathcal{P}_3$, and a definition of the subclass $\mathcal{P}_3'$ of $\mathcal{P}_3$, which we need to consider in the absence of integrality.

Lemma 2.3.5. *[27]*

- *Any equation in $\mathcal{N}_2$ is equivalent to a finite set of equations of the form $t_1 \cdots t_m \leq u$ where*
  - *$u = f$ or $u = u_1 \vee \cdots \vee u_k$, where each u_i is a product of variables.*
  - *Each t_i is of the form $\bigwedge_{1 \leq j \leq n_i} l_j \backslash v_j / r_j$ where $v_j = f$ or a variable, l_j and r_j are products of variables.*
- *Any equation in $\mathcal{P}_3$ is equivalent to $e \leq t$ where $t = \bigvee_{1 \leq i \leq n} \bigodot_{1 \leq j \leq n_i} s_{ij}$, each s_{ij} is in $\mathcal{N}_2$ and $\bigodot$ stands for the product $\cdot$ of finitely many elements.*

Definition 2.3.6. *An equation $e \leq t$ is in the class $\mathcal{P}_3'$ if and only if $t = \bigvee_{1 \leq i \leq n} \bigodot_{1 \leq j \leq n_i} (s_{ij})_{\wedge e}$ where each s_{ij} is in $\mathcal{N}_2$ and $(s_{ij})_{\wedge e}$ is a compact notation for $s_{ij} \wedge e$.*

The algorithm : from equations to analytic clauses

The preservation of equations in $\mathcal{N}_2$ and $\mathcal{P}_3$ (or subclasses thereof, such as $\mathcal{P}_3'$) under DM completion is shown in [26, 27], adapting an algorithm which was first introduced in [25] in a proof-theoretic context. The algorithm works as follows

(a) Convert equations into equivalent *structural clauses*, see Definition 2.3.7.

(b) Transform structural clauses into equivalent "good" clauses, so-called *analytic* clauses, see Definition 2.3.12.

(c) Show that analytic clauses are preserved under DM completion.

Below we present a proof of the steps of the algorithm, which is tailored to *chains* and hence is simpler than that in [26]. We start from the definition of structural clauses and quasiequations.

Definition 2.3.7. *By a* clause*, we mean a classical first-order formula of the form:*

$$t_1 \leq u_1 \ \textit{and} \ \cdots \ \textit{and} \ t_m \leq u_m \implies t_{m+1} \leq u_{m+1} \ \textit{or} \ \cdots \ \textit{or} \ t_n \leq u_n, \qquad (q)$$

where and, or, $\implies$ *stand for the classical connectives conjunction, disjunction and implication respectively,* $0 \leq m < n$*, the* t_i, u_i *are in* $\mathrm{Fm}_{\mathcal{L}}$ *for* $\mathcal{L} = \mathcal{L}_{\mathrm{FL}}$*, and all variables are assumed to be universally quantified. Each* $t_i \leq u_i$ *(*$1 \leq i \leq m$*) is called a* premise*, while each* $t_j \leq u_j$ *(*$m+1 \leq j \leq n$*) is a* conclusion*.* (q) *is a* quasiequation *if* $n = m+1$*. If the set of premises is empty (*$m = 0$*), i.e.* (q) *is of the form*

$$\implies t_1 \leq u_1 \ \textit{or} \ \cdots \ \textit{or} \ t_n \leq u_n, \qquad (q)$$

the clause is said to be positive*. A clause* (q) *is* structural *if* $t_1, \ldots, t_n$ *are products of variables (including the empty product* e*) and any of* $u_1, \ldots, u_n$ *is either a variable or* f*.*

Given a structural clause (q), we let $L(q)$ be the set of variables occurring in $t_{m+1}, \ldots, t_n$, and $R(q)$ the set of variables occurring in $u_{m+1}, \ldots, u_n$. We say that an algebra $\boldsymbol{A}$ *satisfies* a clause (q) iff, for any evaluation v into $\boldsymbol{A}$ (see Definition 2.1.5), if $v(t_i) \leq v(u_i)$ for all premises $t_i \leq u_i$, then there is at least one conclusion $t_j \leq u_j$ such that $v(t_j) \leq v(u_j)$. Clearly $\boldsymbol{A}$ satisfies a positive clause iff for any evaluation v into $\boldsymbol{A}$ there is at least one conclusion $t_j \leq u_j$ such that $v(t_j) \leq v(u_j)$.

We are now ready to recall the step (a) of the algorithm in [26], i.e. the transformation of equations into equivalent structural clauses or quasiequations. Here, by saying that a clause (q) is equivalent in a class of algebras V to an equation $t \leq s$ (or a clause $(q)'$), we mean that for any algebra $\boldsymbol{A}$ in V, $\boldsymbol{A}$ satisfies (q) if and only if $\boldsymbol{A}$ satisfies $t \leq s$ (resp. $(q)'$).

(a) From equations to structural clauses

A crucial tool for the proof of step (a) is the following algebraic observation, which is also known as Ackermann Lemma [25, 38]. Below, $\vec{\varepsilon}_1$ (resp., $\vec{\varepsilon}_2$) stands for a Boolean conjunction (resp., disjunction) of equations.

Lemma 2.3.8. *[26, 27] The following are equivalent in* FL*, where x is a fresh variable, and $l, t, r, u \in \mathrm{Fm}_{\mathcal{L}}$ for $\mathcal{L} = \mathcal{L}_{\mathrm{FL}}$.*

1. $\vec{\varepsilon}_1 \implies \vec{\varepsilon}_2$ *or* $ltr \leq u$.
2. $\vec{\varepsilon}_1$ *and* $u \leq x \implies \vec{\varepsilon}_2$ *or* $ltr \leq x$.
3. $\vec{\varepsilon}_1$ *and* $x \leq t \implies \vec{\varepsilon}_2$ *or* $lxr \leq u$.

Theorem 2.3.9. *[26]*

1. *Every equation in $\mathcal{N}_2$ is equivalent in* FL *to a set of structural quasiequations.*
2. *Every equation in $\mathcal{P}_3'$ is equivalent in the chains in FL_e to a set of structural clauses.*
3. *Every equation in $\mathcal{P}_3$ is equivalent in the chains in FL_i to a set of structural clauses.*

Proof. 1. By Lemma 2.3.5, any equation in $\mathcal{N}_2$ is equivalent to a finite set of equations of the form $t_1 \cdots t_n \leq u$, where $u = f$ or $u = u_1 \vee \cdots \vee u_m$, and each u_i is a product of variables. Moreover, each t_i is of the form $\bigwedge_{1 \leq j \leq n_i} l_j \backslash v_j / r_j$ where $v_j = f$ or a variable, and l_j and r_j are products of variables. By repeated applications of Lemma 2.3.8, the equation becomes

$$(1) \quad x_1 \leq t_1 \text{ and } \ldots \text{ and } x_n \leq t_n \implies x_1 \ldots x_n \leq u$$

Recall that each t_i is of the form $\bigwedge_{1 \leq j \leq n_i} l_{ij} \backslash v_{ij} / r_{ij}$, Hence, the equations $x_i \leq t_i$ can be read as

$$x_i \leq \bigwedge_{1 \leq j \leq n_i} l_{ij} \backslash v_{ij} / r_{ij},$$

Using residuation and basic properties of $\wedge$, the quasiequation (1) then becomes

$$(2) \qquad \underset{\substack{1 \leq i \leq m \\ 1 \leq j \leq n_i}}{\mathrm{AND}} \quad l_{ij} x_i r_{ij} \leq v_{ij} \implies x_1 \cdots x_n \leq u.$$

If $u = f$, the resulting quasiequation is already structural. Otherwise, $u = u_1 \vee \cdots \vee u_m$, where each u_i is a product of variables. By applying again Lemma 2.3.8 to the quasiequation (2), we get

$$(3) \quad u \leq z \text{ and } x_1 \leq t_1 \text{ and } \ldots \text{ and } x_n \leq t_n \implies x_1 \ldots x_n \leq z$$

where $u \leq z$ can be read as $\bigvee_{1 \leq i \leq m} u_i \leq z$. Using basic properties of $\wedge, \vee$, the quasiequation (3) finally becomes

$$\underset{1 \leq i \leq m}{\mathrm{AND}} \quad u_i \leq z \text{ and } \underset{\substack{1 \leq i \leq m \\ 1 \leq j \leq n_i}}{\mathrm{AND}} \quad l_{ij} x_i r_{ij} \leq v_{ij} \implies x_1 \cdots x_n \leq z,$$

which is structural.

2. Recall from Definition 2.3.6 that any equation in $\mathcal{P}_3'$ is of the form $e \leq \bigvee \bigodot (s_{ij})_{\wedge_e}$ with $s_{ij} \in \mathcal{N}_2$. The following equivalences

$$\begin{array}{lcl} \models_{\boldsymbol{A}} e \leq (t)_{\wedge e} \vee (u)_{\wedge e} & \Longleftrightarrow & \models_{\boldsymbol{A}} e \leq t \ \text{ or } \ \models_{\boldsymbol{A}} e \leq u \\ \models_{\boldsymbol{A}} e \leq (t)_{\wedge e} \cdot (u)_{\wedge e} & \Longleftrightarrow & \models_{\boldsymbol{A}} e \leq t \ \text{ and } \ \models_{\boldsymbol{A}} e \leq u \end{array}$$

can be easily shown to hold for every chain $\boldsymbol{A}$ in FL_e. Thus we obtain that every $\mathcal{P}_3'$ equation is equivalent to a conjunction of disjunctions of equations of the form $e \leq s_{ij}$, namely to a finite set of clauses of the form

$$e \leq s_1 \quad \text{or} \quad e \leq s_2 \quad \text{or} \quad \ldots \quad \text{or} \quad e \leq s_m,$$

where each $e \leq s_i$ is in $\mathcal{N}_2$. We can then proceed for each $e \leq s_i$ as in the proof of 1, until we obtain a structural clause.

3. The proof works as that in 2., noticing that in presence of integrality we simply have

$$\begin{array}{lcl} \models_{\boldsymbol{A}} e \leq t \vee u & \Longleftrightarrow & \models_{\boldsymbol{A}} e \leq t \ \text{ or } \ \models_{\boldsymbol{A}} e \leq u \\ \models_{\boldsymbol{A}} e \leq t \cdot u & \Longleftrightarrow & \models_{\boldsymbol{A}} e \leq t \ \text{ and } \ \models_{\boldsymbol{A}} e \leq u \end{array}$$

□

(b) From structural to analytic clauses

In the case without integrality (see Definition 2.1.11), the step from structural to analytic clauses only works for a subclass of structural clauses, the *acyclic* ones.

Definition 2.3.10. *Given a structural clause (q), we build its* dependency graph $D(q)$ *in the following way:*

- *The vertices of $D(q)$ are the variables occurring in the premises (we do not distinguish occurrences).*
- *There is a directed edge $x \longrightarrow y$ in $D(q)$ if and only if there is a premise of the form $t \leq y$ with x occurring in t.*

(q) is acyclic *if the graph $D(q)$ is acyclic (i.e., it has no directed cycles or loops).*

In what follows, we call *acyclic* an equation which is transformed into an equivalent acyclic structural clause by Theorem 2.3.9.

Example 2.3.11. *Consider the $\mathcal{N}_2$ equation $x\backslash x \leq x/x$. By applying the procedure in Theorem 2.3.9 we obtain the equivalent (in* FL*) quasiequation*

$$xy \leq x \implies yx \leq x. \qquad (we)$$

The quasiequation (we) is not acyclic, since we have a loop at the vertex x in $D(we)$.

We recall now the notion of analytic clause.

Definition 2.3.12. *A structural clause* (q)

$$t_1 \leq u_1 \text{ and } \cdots \text{ and } t_m \leq u_m \implies t_{m+1} \leq u_{m+1} \text{ or } \cdots \text{ or } t_n \leq u_n, \qquad (q)$$

is said to be analytic *if the following conditions are satisfied:*

Separation *$L(q)$ and $R(q)$ are disjoint.*

Linearity *Each variable in $L(q) \cup R(q)$ occurs exactly once in the conclusions.*

Inclusion *Each of $t_1, \ldots, t_m$ is a product of variables in $L(q)$, while each of $u_1, \ldots, u_m$ is either a variable in $R(q)$ or f.*

We are now ready to show the transformation of structural clauses into analytic ones. We restrict the result to the case of chains over FL, thus obtaining a simpler form for the resulting analytic clauses with respect to those in [26].

Theorem 2.3.13. *[26]*

- *Every acyclic structural clause is equivalent in* FL*-chains to a set of analytic clauses.*
- *Every structural clause is equivalent in* FL_i*-chains to a set of analytic clauses.*

Proof. We sketch the procedure from [26]. Let (q) be a structural clause

$$t_1 \leq u_1 \text{ and } \cdots \text{ and } t_m \leq u_m \implies t_{m+1} \leq u_{m+1} \text{ or } \cdots \text{ or } t_n \leq u_n. \qquad (q)$$

The transformation is performed in two steps.

1. Restructuring. For each $i \in \{m+1, \ldots, n\}$, assume that t_i is $y_1 \cdots y_p$. Let $x_0, x_1, \ldots, x_p$ be distinct fresh variables. Depending on whether u_i is f or a variable, we transform (q) into either

$$S \text{ and } x_1 \leq y_1 \text{ and } \ldots \text{ and } x_p \leq y_p \implies S' \text{ or } x_1 \ldots x_p \leq f \qquad (q_1)$$

or

$$S \text{ and } x_1 \leq y_1 \text{ and } \ldots \text{ and } x_p \leq y_p \text{ and } u_i \leq x_0 \implies S' \text{ or } x_1 \ldots x_p \leq x_0 \qquad (q_2)$$

where S denotes the set of premises of (q) and S' denotes the conclusion of (q) without $t_i \leq u_i$ (i.e. $t_{m+1} \leq u_{m+1}$ or $\cdots$ or $t_{i-1} \leq u_{i-1}$ or $t_{i+1} \leq u_{i+1}$ or $\cdots$ or $t_n \leq u_n$). We apply this procedure iteratively, for all $i \in \{m+1, \ldots, n\}$.

2. Cutting. Let (q') be the clause obtained after step 1 (restructuring). (q') satisfies the properties of separation and linearity of Definition 2.3.12. An analytic clause equivalent to (q') is obtained by suitably removing all the *redundant variables* from its premises, i.e. variables other than $L(q') \cup R(q')$. This is done as follows. Given a redundant variable z, we distinguish the following cases:

- If z appears in the premises of (q') only on right-hand sides (RHS), we simply remove all such premises, say $s_1 \leq z, \ldots, s_k \leq z$ from (q'). It is easy to see that the resulting clause is equivalent to (q') in FL. Indeed, observe that all premises $s_i \leq z$ in (q') hold, by instantiating z with $\bigvee s_i$, and this instantiation does not affect the other premises and the conclusion. Hence (q') implies the new clause. The other direction is trivial.

- If z appears only on left-hand sides (LHS) of premises of (q'), we again remove all such premises, say $l_1 \cdot z \cdot r_1 \leq v_1, \ldots, l_k \cdot z \cdot r_k \leq v_k$ from (q'). We argue similarly as in the previous case, instantiating z with $\bigwedge_{1\leq i\leq k} l_i \backslash v_i / r_i$.

- Assume that z appears both on RHS and LHS of premises of (q'). Let S_r and S_l be the sets of premises of (q') which contain z on RHS and LHS, respectively. Namely, S_r consists of the premises $s_1 \leq z, \ldots, s_k \leq z$, with $k \geq 1$ and S_l of the premises $t(z, \ldots, z) \leq u$, where the occurrences of z in t are all indicated. Because of acyclicity, S_r and S_l are disjoint. We replace $S_r \cup S_l$ with a new set S_{cut} of premises, which consists of all the equations of the form
$$t(s_i, \ldots, s_i) \leq u$$
for any $t(z, \ldots, z) \leq u \in S_l$ and $s_i \leq z \in S_r$. Let us call the resulting clause (q''). (q'') clearly implies (q'), from the transitivity of $\leq$. To show the converse, let us take an arbitrary FL-chain $\boldsymbol{A}$ and an evaluation v on $\boldsymbol{A}$ such that the premises of (q'') hold in $\boldsymbol{A}$. Let us evaluate the variable z in (q') as $v(z) = \bigvee\{v(s_i) : s_i \leq z \in S_r\}$. All premises in S_r clearly hold in $\boldsymbol{A}$. Moreover, $\boldsymbol{A}$ being an FL-*chain*, there is a maximum in the finite set $\{v(s_i) : s_i \leq z \in S_r\}$, say $v(s_m)$ with $m \in \{1, \ldots, k\}$. Hence we have $v(z) = \bigvee\{v(s_i) : s_i \leq z \in S_r\} = v(s_m)$ and consequently
$$v(t(z, \ldots, z)) = v(t(s_m, \ldots, s_m)) \leq v(u)$$
for any $t(z, \ldots, z) \leq u \in S_l$. Hence we can apply (q') to obtain the conclusion.

Note that the hypothesis that the equations are acyclic is used only in step 2. (Cutting), for excluding the case where S_r and S_l are not disjoint. For FL_i-algebras, an equation that belongs to both S_r and S_l, i.e. of the form $t(z, \ldots, z) \leq z$ can be safely removed, as it follows from integrality. □

Remark 2.3.14. *The main difference between the proof above and that in [26], which is not restricted to chains, is in the cutting step. In [26], whenever premises $s_i \leq z$, with $i \in \{1, \ldots, k\}$ and $t(z, \ldots, z) \leq v$ are present, we need to replace them with all possible combinations $t(s_{i_1}, \ldots, s_{i_m})$ where, for any $i \in \{1, \ldots, m\}$ the $i_1, \ldots, i_m$ are indices, not necessarily distinct, in $\{1, \ldots, k\}$.*

Example 2.3.15. *The equation $x^m \leq x^n$ is in the class $\mathcal{N}_2$. Using our version of the algorithm, the equation is equivalent in FL_e-chains to*
$$x_1^n \leq z \text{ and } \ldots \text{ and } x_m^n \leq z \implies x_1 \cdots x_m \leq z. \qquad (knot_m^n)$$
In the general case of FL_e-algebras the equation is instead equivalent to the following weaker quasiequation (see [27]):
$$\underset{i=1,\ldots,m}{\text{AND}} \{x_{i_1} \cdots x_{i_n} \leq z\} \implies x_1 \cdots x_m \leq z \qquad (*)$$
where the $i_1, \ldots, i_n$ are indices, not necessarily distinct, in $\{1, \ldots, n\}$. It is the clear that $(*)$ is weaker than $(knot_m^n)$, as its premises contain those of $(knot_m^n)$.

Example 2.3.16. *Consider the weak nilpotent minimum equation (see page 18):*

$$e \leq \sim(x \cdot y) \vee ((x \wedge y)\backslash xy)$$

that belongs to $\mathcal{P}_3$*. In* FL_i*-chains the equation is equivalent to the analytic clause:*

$$xy \leq z \textit{ and } xv \leq z \textit{ and } vy \leq z \textit{ and } vv \leq z \implies xy \leq f \textit{ or } v \leq z. \qquad (wnm)$$

(c) Preservation under DM completion

The importance of analytic clauses lies in the fact that it is easy to prove their preservation under DM completion. Preservation of analytic clauses under DM completion is proved in [26], although the equivalent theorem we give below is not explicitly stated there.

Theorem 2.3.17. *[26] Let* V *be a subvariety of* FL*-algebras. If* V *is defined by equations equivalent in* FL *to analytic clauses and* $\boldsymbol{A}$ *belongs to* V*, then its DM completion belongs to* V *as well.*

We can finally summarize the results in this section as follows.

Theorem 2.3.18.

- *Let* V *be a subvariety of* FL_e *defined by acyclic equations in* $\mathcal{N}_2$ *and* $\mathcal{P}'_3$*. The chains in* V *are preserved under DM completion*
- *Let* V *be a subvariety of* FL_i*-algebras defined by equations in* $\mathcal{N}_2$ *and* $\mathcal{P}_3$*. The chains in* V *are preserved under DM completion.*

Proof. Follows from Theorems 2.3.9, 2.3.13 and 2.3.17 □

CHAPTER 3

Standard completeness

3.1 Preliminaries on logics

We revise state-of-the-art results concerning standard completeness and set the general framework for the original results contained in the rest of the thesis.

Following standard practice, for propositional logics we use the same notion of language as the one for algebras introduced in Definition 2.1.4. The only difference will be that here we call the function symbols *connectives*, the constants *truth constants*, and the terms *formulas*. We fix the language $\mathcal{L}$ to be $\mathcal{L}_{\mathrm{FL}_{e\perp}}$, unless stated otherwise. We make use of the derived connectives:

$$\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi) \qquad \neg\varphi = \varphi \to f.$$

A (finitary) *Hilbert system* is a pair $\langle Ax, R\rangle$ where Ax is a set of *axiom* schemas and R is a set of *rule* schemas. Axiom schemas can be instantiated with any concrete formulas: with a slight abuse of notation, the same symbols $\varphi, \chi, \psi, \ldots$ will be used for both schemas and concrete formulas instantiating them. Rule schemas are pairs $\langle \Gamma, \varphi\rangle$ where Γ can be instantiated with a *finite* non-empty set of formulas and φ with a formula. We often use the notation

$$\frac{\varphi_1 \cdots \varphi_n}{\varphi}$$

for a rule $\langle \Gamma, \varphi\rangle$ with $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$. Note that axioms could be seen as nullary rules. Unless stated otherwise, in what follows, by *axioms* and *rules* we will mean axiom and rule schemas.

Definition 3.1.1. *Let* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*.*

- *A* proof *of* φ *from* Γ *in* $\langle Ax, R\rangle$ *is a finite sequence of formulas* $\langle \varphi_0, \ldots, \varphi_n\rangle$ *such that* $\varphi_n = \varphi$ *and, for every* $i \leq n$*, one of the following holds:*
  - $\varphi_i \in \Gamma$ *or is an instance of* Ax
  - *There is a pair* $\langle \Delta, \varphi_i\rangle$*, with* $\Delta \subseteq \{\varphi_0, \ldots, \varphi_{i-1}\}$ *that is an instance of a rule in* R*.*

- *We define the* provability relation generated by $\langle Ax, R\rangle$ *as the relation* $\Gamma \vdash_{\langle Ax,R\rangle} \varphi$ *iff there is a proof of* φ *from* Γ *in* $\langle Ax, R\rangle$.

Observe that the provability relation is finitary, i.e., if $\Gamma \vdash_{\langle Ax,R\rangle} \varphi$*, then there is a finite* $\Gamma' \subseteq \Gamma$ *such that* $\Gamma' \vdash_{\langle Ax,R\rangle} \varphi$*. In what follows, by a* logic *we mean the provability relation generated by some Hilbert system* $\langle Ax, R\rangle$.

The weakest logic UL that we consider in this chapter is generated by the following Hilbert system [66], with axioms

(UL1)	$\varphi \to \varphi$
(UL2)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(UL3)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(UL4)	$(\varphi \cdot \psi \to \chi) \leftrightarrow (\varphi \to (\psi \to \chi))$
(UL5a, b)	$\varphi \wedge \psi \to \varphi, \quad \varphi \wedge \psi \to \psi$
(UL6)	$(\varphi \to \psi) \wedge (\varphi \to \chi) \to (\varphi \to \psi \wedge \chi)$
(UL7a, b)	$\varphi \to (\varphi \vee \psi), \quad \psi \to (\varphi \vee \psi)$
(UL8)	$(\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$
(UL9)	$\varphi \leftrightarrow (e \to \varphi)$
(UL10a, b)	$\bot \to \varphi, \quad \varphi \to \top$
(UL11)	$((\varphi \to \psi) \wedge e) \vee ((\psi \to \varphi) \wedge e)$

and the derivation rules of modus ponens and $\wedge$-adjunction:

$$\text{(MP)} \qquad \frac{\varphi \quad \varphi \to \psi}{\psi}$$

$$(\wedge - \text{Adj}) \qquad \frac{\varphi \quad \psi}{\varphi \wedge \psi}$$

We denote by $\vdash_{\text{UL}}$ the provability relation generated by the Hilbert system above.

Remark 3.1.2. *Axiom* (UL3) *accounts for the exchange property* (e) *(see Definition 2.1.11), and the axiom* (UL11) *corresponds to the prelinearity equation (see Theorem 2.1.13), which forces the corresponding algebraic semantics of* UL *to be the semilinear variety* $\text{FL}^{\ell}_{e\bot}$*, hence, guaranteeing the completeness with respect to chains.*

An axiomatic extension L of UL is a provability relation $\vdash_{\text{L}}$ generated by extending the Hilbert system for UL above with axioms and rule schemas. For any logic L and axiom schema α, we denote by $\text{L} + (\alpha)$ the axiomatic extension of L obtained by adding the axiom schema α to the Hilbert system for L. In what follows, unless stated otherwise, by L we mean any axiomatic extension of UL.

Example 3.1.3. *By adding to* UL *the axioms*

$$(o) \quad f \to \varphi \qquad (i) \quad \varphi \to e$$

we get a Hilbert system for the Monoidal t-norm logic MTL *[42, 66]. These axioms correspond to the equations* (o) *and* (i) *in Definition 2.1.11 and to the rules weakening right* (wr) *and left* (wl) *in Gentzen-style calculi (see Table 3.2 in Section 3.3). It is immediate to see that in* MTL *we have* $f \leftrightarrow \bot$ *and* $e \leftrightarrow \top$*, hence, we can identify the constants* $\bot$ *and* $\top$ *with* f *and* e *respectively. Moreover, the rule* (adj) *becomes derivable, hence unnecessary. The axiom (*$\mathrm{UL9}$*) can be equivalently replaced with* $\varphi \to (\psi \to \varphi)$ *and* $(\mathrm{UL11})$ *can be simplified as* $(\varphi \to \psi) \vee (\psi \to \varphi)$ *(see the corresponding equation in Theorem 2.1.13). Hence, a Hilbert system for* MTL *has the following axioms:*

$(\mathrm{MTL1})$	$\varphi \to \varphi$
$(\mathrm{MTL2})$	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
$(\mathrm{MTL3})$	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
$(\mathrm{MTL4})$	$(\varphi \cdot \psi \to \chi) \leftrightarrow (\varphi \to (\psi \to \chi))$
$(\mathrm{MTL5}a, b)$	$\varphi \wedge \psi \to \varphi, \quad \varphi \wedge \psi \to \psi$
$(\mathrm{MTL6})$	$(\varphi \to \psi) \wedge (\varphi \to \chi) \to (\varphi \to \psi \wedge \chi)$
$(\mathrm{MTL7}a, b)$	$\varphi \to (\varphi \vee \psi), \quad \psi \to (\varphi \vee \psi)$
$(\mathrm{MTL8})$	$(\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$
$(\mathrm{MTL9})$	$\varphi \to (\psi \to \varphi)$
$(\mathrm{MTL10}a, b)$	$f \to \varphi, \quad \varphi \to e$
$(\mathrm{MTL11})$	$(\varphi \to \psi) \vee (\psi \to \varphi)$

and the derivation rule of modus ponens :

(MP) $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$

The algebraic semantics of this logic is the class of FL_{ew}^{ℓ}-algebras, also known as MTL-algebras, see Definition 2.1.11. Most of the important fuzzy logics in the literature are axiomatic extensions of MTL, as we will see in the following.

Example 3.1.4. *Hájek's Basic Logic* BL *[49] is obtained adding to* MTL *the axiom*

$$(div) \quad (\varphi \wedge \psi) \to (\varphi \cdot (\varphi \to \psi)).$$

The algebraic semantics for this logic is given by the BL*-algebras, considered in Lemma 2.2.4. Recall that in the* BL*-algebras over* $[0, 1]$*, the operation* $\cdot$ *is a continuous t-norm and* $\to$ *is its residuum. Axiomatic systems corresponding to the three main continuous t-norms (Gödel, Łukasiewicz, Product) and their residua can be obtained from* BL *by the addition of the following axioms:*

(c)	$\varphi \to \varphi \cdot \varphi$		
(inv)	$\neg\neg\varphi \to \varphi$		
(S)	$\neg(\varphi \wedge \neg\varphi)$	(Π)	$\neg\neg\chi \to ((\varphi \cdot \chi \to \psi \cdot \chi) \to (\varphi \to \psi))$

In particular, we have:

- *the logic* $\mathrm{BL} + (c)$*, known as* Gödel logic *(or Gödel–Dummett logic) corresponds to the Gödel* t*-norm and its residuum;*

- *the logic* $\mathrm{BL} + (inv)$*, known as* Łukasiewicz logic *corresponds to the Łukasiewicz* t*-norm and its residuum;*

- *the logic* $\mathrm{BL} + (S) + (\Pi)$*, known as* Product logic *corresponds to the Product* t*-norm and its residuum.*

An alternative axiomatization of Gödel logic is given by $\mathrm{MTL} + (c)$*, see e.g. [21]. Indeed, the axiom* (div) *of* BL *is already derivable in the Hilbert system for* $\mathrm{MTL} + (c)$*.*

As shown in the following, the algebraic semantics for axiomatic extensions of UL are given by subvarieties of $\mathrm{FL}_{e\bot}^{\ell}$ (or equivalently UL)-algebras.

Definition 3.1.5. *Let* $\mathrm{L} = \mathrm{UL} + C$*, with* C *being any finite set of axiom schemas. An* L*-algebra* $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, f, e, \bot, \top \rangle$ *is an* $\mathrm{FL}_{e\bot}^{\ell}$*-algebra that satisfies the equations* $e \leq \varphi$ *for every* $\varphi \in C$*.* L*-algebras that are chains, are called shortly* L*-chains.*

As the class of L-algebras is semilinear, we obtain a general completeness theorem for propositional axiomatic extension of UL with respect to the corresponding classes of L-chains.

Theorem 3.1.6. *[33,52] Let* L *be any axiomatic extension of* UL*. Then for every set of formulas* T *and every formula* φ *the following are equivalent:*

- $T \vdash_{\mathrm{L}} \varphi$*,*

- $\{e \leq \psi\}_{\psi \in T} \models_{\boldsymbol{A}} e \leq \varphi$ *for every* L*-chain* $\boldsymbol{A}$*,*

where $\models_{\boldsymbol{A}}$ *is as in Definition 2.1.5.*

Remark 3.1.7. *The completeness result above holds for any theory* T*, including infinite ones. This is usually known as* strong completeness *in the literature, see e.g. [34, 52]. Slightly departing from this convention, in what follows, by* completeness *we always mean the strong version, which applies to any set* T*. We use* finitely strong completeness *for a completeness result which holds only for finite* T*.*

We now consider first-order logics. The language that we use is the same as for classical first-order logic. In order to fix the notation and terminology we give an explicit definition:

Definition 3.1.8. *A* predicate language $\mathcal{P}$ *is a triple* $\langle Pred_{\mathcal{P}}, Func_{\mathcal{P}}, Ar_{\mathcal{P}} \rangle$ *where* $Pred_{\mathcal{P}}$ *is a non-empty set of* predicate symbols, $Func_{\mathcal{P}}$ *is a set (disjoint with* $Pred_{\mathcal{P}}$*) of* function symbols, *and* $Ar_{\mathcal{P}}$ *is the* arity function *assigning to each predicate or function symbol a natural number called the* arity *of the symbol. The function symbols* F *with* $Ar_{\mathcal{P}}(F) = 0$ *are called* object *or* individual constants. *The predicate symbols* P *for which* $Ar_{\mathcal{P}}(P) = 0$ *are called* propositional variables.

$\mathcal{P}$-terms and (atomic) $\mathcal{P}$-formulas of a given predicate language are defined as in classical logic. A $\mathcal{P}$-*theory* is a set of $\mathcal{P}$-formulas. The notions of free occurrence of a variable, substitution, open formula, and closed formula (or, synonymously, *sentence*) are defined in the same way as in classical logic.

For any propositional logic L extending UL, we define its first-order extension L$\forall$ as follows:

Definition 3.1.9. *Let $\mathcal{P}$ be a predicate language, and* L *be an axiomatic extension of* UL. *The logic* L$\forall$ *in the language $\mathcal{P}$ is generated by the following Hilbert system, consisting of the axiom schemas*[1]

(P)	*The axioms of* L
$(\forall 1)$	$(\forall x)\varphi(x) \to \varphi(t)$, *where the $\mathcal{P}$-term t is substitutable for x in φ*
$(\exists 1)$	$\varphi(t) \to (\exists x)\varphi(x)$, *where the $\mathcal{P}$-term t is substitutable for x in φ*
$(\forall 2)$	$(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, *where x is not free in χ*
$(\exists 2)$	$(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$, *where x is not free in χ*
$(\forall 3)$	$(\forall x)(\chi \vee \varphi) \to \chi \vee (\forall x)\varphi$, *where x is not free in χ.*

and the same deduction rules of L *plus the rule of* generalization:

$$(\text{Gen}) \qquad \frac{\varphi}{(\forall x)\varphi}.$$

The notions of proof and provability are defined in the same way as for classical logic. The fact that the formula φ is provable in L$\forall$ from a theory T will be denoted by $T \vdash_{\mathrm{L}\forall} \varphi$. An important syntactic property of our logics is the so-called local deduction theorem.

Lemma 3.1.10. *Let* L *be an axiomatic extension of* UL. *For any $\mathcal{P}$-theory T and sentence φ $T \cup \varphi \vdash_{\mathrm{L}\forall} \chi$ iff $T \vdash_{\mathrm{L}\forall} \varphi^* \to \chi$ for some $\varphi^* = (\varphi_1 \wedge e) \cdot \ldots \cdot (\varphi_m \wedge e)$ where $\varphi_i \in T$, for $i = 1, \ldots, m$ are not necessarily distinct formulas.*

We now introduce the semantics counterpart of our first-order logics. In what follows, we fix $\mathcal{P} = \langle \mathit{Pred}, \mathit{Func}, \mathit{Ar} \rangle$ to be a predicate language and $\boldsymbol{A}$ to be an L-chain.

Definition 3.1.11. *An* $\boldsymbol{A}$-structure $\mathfrak{M}$ *for the predicate language $\mathcal{P}$ has the form* $\mathfrak{M} = \langle M, (P_{\mathfrak{M}})_{P \in \mathit{Pred}}, (F_{\mathfrak{M}})_{F \in \mathit{Func}} \rangle$ *where*

- *M is a non-empty domain;*
- *for each n-ary predicate symbol $P \in \mathit{Pred}$, $P_{\mathfrak{M}}$ is an n-ary fuzzy relation on M, i.e., a function $M^n \to A$ (identified with an element of A if $n = 0$);*
- *for each n-ary function symbol, $F \in \mathit{Func}$ $F_{\mathfrak{M}}$ is a function $M^n \to M$ (identified with an element of M if $n = 0$).*

[1]When we speak about axioms or deduction rules of propositional logic, we actually consider them with $\mathcal{P}$-formulas substituted for propositional formulas.

Let $\mathfrak{M}$ be an $\boldsymbol{A}$-structure for $\mathcal{P}$. An $\mathfrak{M}$-evaluation *of the object variables is a mapping* v *that assigns an element from M to each object variable. Let* v *be an $\mathfrak{M}$-evaluation, x be a variable, and $a \in M$. Then by* $\mathrm{v}[x{\mapsto}a]$ *we denote the $\mathfrak{M}$-evaluation such that* $\mathrm{v}[x{\mapsto}a](x) = a$ *and* $\mathrm{v}[x{\mapsto}a](y) = \mathrm{v}(y)$ *for each object variable y different from x. Let $\mathfrak{M}$ be an $\boldsymbol{A}$-structure for $\mathcal{P}$ and* v *be an $\mathfrak{M}$-evaluation. We define the* values *of terms and the* truth values *of formulas in $\mathfrak{M}$ for an evaluation* v *recursively as follows:*

$$\begin{aligned}
\|x\|^{\boldsymbol{A}}_{\mathfrak{M},\mathrm{v}} &= \mathrm{v}(x), \\
\|F(t_1,\dots,t_n)\|^{\boldsymbol{A}}_{\mathfrak{M},\mathrm{v}} &= F_{\mathfrak{M}}(\|t_1\|^{\boldsymbol{A}}_{\mathfrak{M},\mathrm{v}},\dots,\|t_n\|^{\boldsymbol{A}}_{\mathfrak{M},\mathrm{v}}) && \textit{for } F \in \textit{Func}, \\
\|P(t_1,\dots,t_n)\|^{\boldsymbol{A}}_{\mathfrak{M},\mathrm{v}} &= P_{\mathfrak{M}}(\|t_1\|^{\boldsymbol{A}}_{\mathfrak{M},\mathrm{v}},\dots,\|t_n\|^{\boldsymbol{A}}_{\mathfrak{M},\mathrm{v}}) && \textit{for } P \in \textit{Pred}, \\
\|c(\varphi_1,\dots,\varphi_n)\|^{\boldsymbol{A}}_{\mathfrak{M},\mathrm{v}} &= c_{\boldsymbol{A}}(\|\varphi_1\|^{\boldsymbol{A}}_{\mathfrak{M},\mathrm{v}},\dots,\|\varphi_n\|^{\boldsymbol{A}}_{\mathfrak{M},\mathrm{v}}) && \textit{for } c \in \mathcal{L}_{\mathrm{FL}_{e\perp}}, \\
\|\forall x\varphi\|^{\boldsymbol{A}}_{\mathfrak{M},\mathrm{v}} &= \inf\{\|\varphi\|^{\boldsymbol{A}}_{\mathfrak{M},\mathrm{v}[x\mapsto a]} \mid a \in M\}, \\
\|\exists x\varphi\|^{\boldsymbol{A}}_{\mathfrak{M},\mathrm{v}} &= \sup\{\|\varphi\|^{\boldsymbol{A}}_{\mathfrak{M},\mathrm{v}[x\mapsto a]} \mid a \in M\}.
\end{aligned}$$

Note that in the last two clauses, if the infimum or supremum does not exist, then the corresponding value is taken to be undefined, and in all clauses, if one of the arguments is undefined, then the result is undefined. We say that the $\boldsymbol{A}$-structure $\mathfrak{M}$ is safe *if $\|\varphi\|^{\boldsymbol{A}}_{\mathfrak{M},\mathrm{v}}$ is defined for each $\mathcal{P}$-formula φ and each $\mathfrak{M}$-evaluation* v.

In what follows, we write $\langle \boldsymbol{A}, \mathfrak{M}\rangle \models \varphi$ if $e \leq_{\boldsymbol{A}} \|\varphi\|^{\boldsymbol{A}}_{\mathfrak{M},\mathrm{v}}$ for each $\mathfrak{M}$-evaluation v.

Definition 3.1.12. *Let $\mathfrak{M}$ be an $\boldsymbol{A}$-structure for $\mathcal{P}$ and T be a $\mathcal{P}$-theory. Then $\mathfrak{M}$ is called an* $\boldsymbol{A}$-model of T *if it is safe and $\langle \boldsymbol{A}, \mathfrak{M}\rangle \models \varphi$ for each $\varphi \in T$.*

Observe that models are safe structures by definition. As obviously each safe $\boldsymbol{A}$-structure is an $\boldsymbol{A}$-model of the empty theory, we shall use the term *model* for both models and safe structures in the rest of the text. By a slight abuse of language we use the term model also for the pair $\langle \boldsymbol{A}, \mathfrak{M}\rangle$.

The following completeness theorem shows that the syntactic presentations introduced above succeed in capturing the intended general chain semantics for first-order fuzzy logics.

Theorem 3.1.13. *[33, 66] Let* L *be an axiomatic extension of* UL, *$\mathcal{P}$ be a predicate language, T be a $\mathcal{P}$-theory, and φ be a $\mathcal{P}$-formula. Then the following are equivalent:*

- $T \vdash_{\mathrm{L}\forall} \varphi$.
- $\langle \boldsymbol{A}, \mathfrak{M}\rangle \models \varphi$ *for each* L*-chain $\boldsymbol{A}$ and each model $\langle \boldsymbol{A}, \mathfrak{M}\rangle$ of the theory T.*

3.2 The density rule and rational completeness

We recall from [63, 70] that the addition of the density rule to a first-order logic L∀ extending UL∀ makes the logic complete with respect to the class of countable dense L-chains. Introduced by Takeuti and Titani in [79] for axiomatizing Gödel logic, the density rule formalized Hilbert-style has the following form

$$\frac{(\varphi \to p) \vee (p \to \chi) \vee \psi}{(\varphi \to \chi) \vee \psi} \ (density)$$

where p is a propositional variable not occurring in φ, χ, or ψ.

Definition 3.2.1. *Let* L *be a an axiomatic extension of* UL, $\mathcal{P}$ *be a predicate language,* T *be a* $\mathcal{P}$*-theory, and* φ *be a* $\mathcal{P}$*-formula.* L$\forall$ *is said to be* rational complete *if the following are equivalent:*

- $T \vdash_{\mathrm{L}\forall} \varphi$.
- $\langle \boldsymbol{A}, \mathfrak{M} \rangle \models \varphi$ *for each* L*-algebra* $\boldsymbol{A}$ *with lattice reduct* $\mathsf{Q} \cap [0,1]$ *and each model* $\langle \boldsymbol{A}, \mathfrak{M} \rangle$ *of the theory* T.

It is a well known fact that a bounded dense countable chain is order isomorphic to $\mathsf{Q} \cap [0,1]$. *Hence, we can reformulate the second item as*

- $\langle \boldsymbol{A}, \mathfrak{M} \rangle \models \varphi$ *for each dense countable* L*-chain* $\boldsymbol{A}$ *and each model* $\langle \boldsymbol{A}, \mathfrak{M} \rangle$ *of the theory* T.

Let L be an axiomatic extension of UL and L$\forall$ be the corresponding first-order logic, generated by a Hilbert system $\langle Ax, R \rangle$, like the one in Definition 3.1.9. In the following, we extend the notion of L$\forall$-proof in such a way as to include applications of $(density)$.

Definition 3.2.2.
- *Let* $\Gamma' \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$. *An* L$\forall^{\mathrm{D}}$-proof *of* φ *from* Γ' *is a finite sequence of formulas* $\langle \varphi_0, \ldots, \varphi_n \rangle$ *with* $\varphi_n = \varphi$ *such that, for every* $i \leq n$*, one of the following holds:*
  - $\varphi_i \in \Gamma'$ *or* φ_i *is an instance of an axiom in* Ax.
  - *There is a pair* $\langle \Delta, \varphi_i \rangle$ *with* $\Delta \subseteq \{\varphi_0, \ldots, \varphi_{i-1}\}$ *that is an instance of a rule in* R.
  - $\varphi_i = (\alpha \to \beta) \vee \gamma$ *and there is* $\varphi \in \{\varphi_0, \ldots, \varphi_{i-1}\}$ *such that* $\varphi = (\alpha \to p) \vee (p \to \beta) \vee \gamma$ *where* p *is a propositional variable not occurring in* Γ', α, β*, or* γ *(*φ_i *is obtained from a previous member of the sequence by* $(density)$*).*

  The notion of L$\forall^{\mathrm{D}}$*-proof naturally defines a consequence relation* $\vdash_{\mathrm{L}\forall^{\mathrm{D}}}$ *as follows:*
- $\Gamma \vdash_{\mathrm{L}\forall^{\mathrm{D}}} \varphi$ *iff there is an* L$\forall^{\mathrm{D}}$*-proof of* φ *from a finite subset* Γ' *of* Γ.

The following theorem shows that L$\forall^{\mathrm{D}}$ succeeds in capturing the intended semantics of dense L-chains for first-order fuzzy logics.

Theorem 3.2.3. *[70] Let* L *be any axiomatic extension of* UL, $\mathcal{P}$ *be a predicate language,* T *be a* $\mathcal{P}$*-theory, and* φ *be a* $\mathcal{P}$*-formula.* L$\forall^{\mathrm{D}}$ *is rational complete.*

Proving that a logic L∀ is rational complete can thus be reduced to showing that L∀ and $L\forall^D$ define the same consequence relation. This means that any $L\forall^D$-proof (i.e. a proof containing applications of the density rule) can be converted into a proof in L∀ (a proof without density). Arguing on the properties of proofs from Hilbert-style calculi is extremely difficult because proofs have no discernible structure. The solution is to consider proofs for the logic in question in an *analytic calculus*, i.e. a calculus where a proof of a given formula uses only subformulas of that formula.

3.3 Hypersequent calculi and the substructural hierarchy

We show how to define analytic calculi for axiomatic extensions of UL∀, which are well suited for proving the admissibility of the density rule. The results we recall here were first shown in [25] for axiomatic extensions of the (bounded) Full Lambek Calculus with exchange $FL_{e\bot}$[2]. The calculi we define are based on hypersequents, a natural extension of Gentzen sequents. We assume that the reader has basic knowledge of sequent calculi for classical and substructural logics, see e.g. [46, 78] for an overview.

Definition 3.3.1. *[3] A* hypersequent *is a non-empty finite multiset $S_1 \mid \ldots \mid S_n$ where each $S_i, i = 1 \ldots n$ is a sequent, called a* component *of the hypersequent. A* (single-conclusioned) sequent *is in turn an object of the form $\Gamma \Rightarrow \Pi$, where Γ is a multiset* [3] *of formulas and Π is either empty or a single formula.*

The symbol "$\mid$" is intended to denote a disjunction at the meta-level. A *meta(hyper)sequent* is a (hyper)sequent formed by metavariables for multisets and formulas, instead of concrete multisets and formulas. A concrete hypersequent is then obtained by instantiating the corresponding metavariables with concrete multisets and formulas. In what follows, we will distinguish between meta(hyper)sequent and (hyper)sequent only when required from the context. In other cases, we will simply speak of (hyper)sequents.

Notation 3.3.2. *(Meta)hypersequents will be denoted by G, H and (meta)sequents by S, C. Within a (meta)sequent $S := \Gamma_1, \ldots, \Gamma_n \Rightarrow \Pi$ we will denote by $L(S)$ the multiset $\langle \Gamma_1, \ldots, \Gamma_n \rangle$ of (metavariables for) multisets of formulas occurring in its left hand side, and by $R(S)$ (the metavariable in) its right hand side Π. We will use $\Gamma, \Delta, \Sigma, \Theta$ and Λ for (metavariables for) multisets, and Π, Ψ for (metavariable for)* stoups*, i.e. either a formula or the empty set. The notation Γ^k will stand for $\Gamma, \ldots, \Gamma$, i.e. k comma-separated occurrence of (the metavariable for) a multiset Γ. By α^k we will denote both the multiplicative conjunction $\alpha \cdots \alpha$ of k occurrences of the (metavariable for the) formula α, and k comma-separated occurrences $\alpha, \ldots, \alpha$. The meaning will be clear from the context.*

Definition 3.3.3. *A* hypersequent rule *is as an ordered pair $(\{G_1, \ldots, G_n\}, H)$ consisting of a finite set of metahypersequents $G_1, \ldots, G_n$ (the premises) and a metahypersequent H for the*

[2]The same calculus is denoted by FL_e in [25], but it includes the constants $\bot, \top$. Hence, we prefer here to use $FL_{e\bot}$

[3]The use of multisets avoids to consider the exchange rule explicitly.

conclusion, written as

$$\frac{G_1 \quad \dots \quad G_n}{H}$$

A rule application *is obtained by instantiating the rule with concrete hypersequents. A* structural rule *is a rule not containing metavariables for formulas. We distinguish* internal *and* external *structural rules: internal rules operate only on one component of the conclusion, while external structural rules operate on multiple components of the conclusion*[4]*. The components of the premises and the conclusion on which a rule operates are said to be* active *components of the rule.*

Following standard practice, we use the same notation for hypersequent rules and concrete rule applications.

As in sequent calculus, a hypersequent calculus consists of initial axioms, cut, and logical and structural rules. Axioms, cut, and logical and internal structural rules are the same as in sequent calculi; the only difference being the presence of a context G representing a (possibly empty) hypersequent. External structural rules, by permitting the interaction between components, increase the expressive power of the hypersequent calculus with respect to the sequent calculus.

Definition 3.3.4. *A* derivation d *of a hypersequent* G *from* $G_1, \dots, G_n$ *is a tree, whose nodes are hypersequents, edges correspond to rule applications,* G *is the root, and leaves are* $G_1, \dots, G_n$*, or axioms of the calculus. The* length $|d|$ of a derivation d *is the maximal number of inference rules occurring on any branch, + 1.*

Henceforth, by

$$G_1, \dots, G_n \vdash_{\mathrm{HL}} G$$

we denote the fact that there is a derivation of G from $G_1, \dots, G_n$ in a hypersequent calculus HL.

A hypersequent calculus HUL∀ for UL∀ is shown in Table 3.1. Note that the *eigenvariable* condition for the rules $(\forall r)$ and $(\exists l)$ refer to the whole hypersequent: in other words, the rules can be applied only when the variable a does not appear in the whole conclusion, including the hypersequent context G. Otherwise, using (com) we could easily derive $\exists x\alpha(x) \Rightarrow \forall x\alpha(x)$ for any formula α.

Remark 3.3.5. *By removing the external structural rule* (com) *from Table 3.1 we get a (hypersequent version of a) calculus for* $\mathrm{FL}_{\mathrm{e}\bot}\forall$ *(bounded first-order Full Lambek calculus with exchange). The rule* (com) *allows us indeed to derive the axioms* (UL11) *and* (∀3)*, see e.g. [25, 63] and Theorem 3.3.8.*

The calculus HUL∀ admits *cut elimination*, i.e. any derivation in HUL∀ containing applications of the rule (cut) can be transformed into a derivation which does not contain any application of (cut) (a *cut-free* derivation). In a cut-free derivation, all the rule applications except for

[4]Internal and external structural rules are sometimes referred to in the literature as sequent and hypersequent structural rules, respectively.

the quantifiers rule enjoy the *subformula property*, i.e. the formulas occurring in the premises are subformulas of the formulas in the conclusion. This is an essential property, which will be needed for our proof of density elimination.

Notation 3.3.6. *We denote by* $(\overline{r})$ *multiple applications of a rule* (r). *Let* S *be a (meta)sequent.*

$$S[^{\Lambda \Rightarrow \Pi}/_{\Gamma \Rightarrow \Psi}]$$

denotes the (meta)sequent obtained by replacing one *occurrence of* Γ *in* $L(S)$ *with* Λ *and* $R(S)$ *with* Π, *provided that* $R(S) = \Psi$. *If there is no occurrence of* Γ *in* $L(S)$, *we let* $L(S[^{\Lambda \Rightarrow \Pi}/_{\Gamma \Rightarrow \Psi}]) = L(S)$. *Similarly, if* $R(S) \neq \Psi$, *we let* $R(S[^{\Lambda \Rightarrow \Pi}/_{\Gamma \Rightarrow \Psi}]) = R(S)$. *We denote by*

$$S[^{\Lambda \Rightarrow \Pi}/_{\overline{\Gamma} \Rightarrow \Psi}]$$

the sequent obtained by replacing each *occurrence of* Γ *in* $L(S)$ *with an occurrence of* Λ *and* $R(S)$ *with* Π, *provided that* $R(S) = \Psi$. *If there is no occurrence of* Γ *in* $L(S)$, *we let* $L(S[^{\Lambda \Rightarrow \Pi}/_{\Gamma \Rightarrow \Psi}]) = L(S)$. *Similarly if* $R(S) \neq \Psi$, *we let* $R(S[^{\Lambda \Rightarrow \Pi}/_{\Gamma \Rightarrow \Psi}]) = R(S)$.

Example 3.3.7. *Let* S *be the sequent* $\Theta, \Gamma, \Gamma \Rightarrow \Psi$. *We have:*

$$S[^{\Lambda \Rightarrow \Pi}/_{\Gamma \Rightarrow \Psi}] = \Theta, \Lambda, \Gamma \Rightarrow \Pi \qquad S[^{\Lambda \Rightarrow \Pi}/_{\overline{\Gamma} \Rightarrow \Psi}] = \Theta, \Lambda, \Lambda \Rightarrow \Pi.$$

For calculi with exchange, the formula-interpretation I of a hypersequent $H = \Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$ (see, e.g., [4, 25, 37, 70]), is as follows:

- $I(\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n) = (I(\Gamma_1 \Rightarrow \Pi_1) \wedge e) \vee \cdots \vee (I(\Gamma_n \Rightarrow \Pi_n) \wedge e)$

where the interpretation of a sequent $\Gamma \Rightarrow \Pi$ is

- $I(\Gamma \Rightarrow \Pi) = \odot\Gamma \rightarrow \beta$ if Π is a formula β,
- $I(\Gamma \Rightarrow) = \odot\Gamma \rightarrow f$ otherwise.

$\odot\Gamma$ stands for the multiplicative conjunction $\cdot$ of all the formulas in Γ and is e when Γ is empty.

Theorem 3.3.8. *The hypersequent calculus* HUL$\forall$ *admits cut elimination, and* $\vdash_{\mathrm{HUL}\forall} H$ *if and only if* $\vdash_{\mathrm{UL}\forall} I(H)$, *for any hypersequent* H.

Proof. Cut elimination is proved e.g. in [28, 70]. For proving that $\vdash_{\mathrm{HUL}\forall} H$ if and only if $\vdash_{\mathrm{UL}\forall} I(H)$, we deal only with some cases, referring the reader to [70] for the others. For the left to right direction, we proceed by induction on the length of the derivation of H in HUL$\forall$. Among the quantifier rules, the only non-trivial case is $(\forall r)$. This case is handled by using axiom $(\forall 3)$, see Definition 3.1.9. Indeed, assume that $I(G) \vee I(\Gamma \Rightarrow \alpha(a))$ is derivable in UL$\forall$. By the generalization rule (Gen), $\forall x(I(G) \vee I(\Gamma \Rightarrow \alpha(x)))$ is also derivable in UL$\forall$. Recall that, for the eigenvariable condition, a must not occur in $I(G)$. Then we may assume that x does not occur there either. Hence, using axiom $(\forall 3)$ we obtain that $I(G) \vee \forall x I(\Gamma \Rightarrow \alpha(x))$ is derivable. The result follows using the fact that $\forall x I(\Gamma \Rightarrow \alpha(x)) \rightarrow I(\Gamma \Rightarrow \forall x \alpha(x))$ is derivable in UL$\forall$ (from axiom $(\forall 2)$). The case of $(\exists l)$ can be proved in a similar way, using in a suitable way

$$\dfrac{G\,|\,\Gamma\Rightarrow\alpha \quad G\,|\,\alpha,\Delta\Rightarrow\Pi}{G\,|\,\Gamma,\Delta\Rightarrow\Pi}\;(cut) \qquad \dfrac{}{G\,|\,\alpha\Rightarrow\alpha}\;(id) \qquad \dfrac{}{G\,|\,f\Rightarrow}\;(fl)$$

$$\dfrac{G\,|\,\Gamma\Rightarrow\alpha \quad G\,|\,\Delta\Rightarrow\beta}{G\,|\,\Gamma,\Delta\Rightarrow\alpha\cdot\beta}\;(\cdot\, r) \qquad \dfrac{G\,|\,\alpha,\beta,\Gamma\Rightarrow\Pi}{G\,|\,\alpha\cdot\beta,\Gamma\Rightarrow\Pi}\;(\cdot\, l) \qquad \dfrac{G\,|\,\Gamma\Rightarrow\Pi}{G\,|\,e,\Gamma\Rightarrow\Pi}\;(el)$$

$$\dfrac{G\,|\,\Gamma\Rightarrow\alpha \quad G\,|\,\beta,\Delta\Rightarrow\Pi}{G\,|\,\Gamma,\alpha\to\beta,\Delta\Rightarrow\Pi}\;(\to l) \qquad \dfrac{G\,|\,\alpha,\Gamma\Rightarrow\beta}{G\,|\,\Gamma\Rightarrow\alpha\to\beta}\;(\to r) \qquad \dfrac{G\,|\,\Gamma\Rightarrow}{G\,|\,\Gamma\Rightarrow f}\;(fr)$$

$$\dfrac{G\,|\,\Gamma\Rightarrow\alpha \quad G\,|\,\Gamma\Rightarrow\beta}{G\,|\,\Gamma\Rightarrow\alpha\wedge\beta}\;(\wedge r) \qquad \dfrac{G\,|\,\alpha_i,\Gamma\Rightarrow\Pi}{G\,|\,\alpha_1\wedge\alpha_2,\Gamma\Rightarrow\Pi}\;(\wedge l)_{i=\{1,2\}} \qquad \dfrac{}{G\,|\;\Rightarrow e}\;(er)$$

$$\dfrac{G\,|\,\alpha,\Gamma\Rightarrow\Pi \quad G\,|\,\beta,\Gamma\Rightarrow\Pi}{G\,|\,\alpha\vee\beta,\Gamma\Rightarrow\Pi}\;(\vee l) \qquad \dfrac{G\,|\,\Gamma\Rightarrow\alpha_i}{G\,|\,\Gamma\Rightarrow\alpha_1\vee\alpha_2}\;(\vee r)_{i=\{1,2\}} \qquad \dfrac{}{\Gamma,\bot\Rightarrow\Pi}\;(\bot l)$$

$$\dfrac{G\,|\,\Gamma,\alpha(t)\Rightarrow\Pi}{G\,|\,\Gamma,\forall x\alpha(x)\Rightarrow\Pi}\;(\forall l) \qquad \dfrac{G\,|\,\Gamma\Rightarrow\alpha(a)}{G\,|\,\Gamma\Rightarrow\forall x\alpha(x)}\;(\forall r)\quad (a \textit{ eigenvariable}) \qquad \dfrac{}{\Gamma\Rightarrow\top}\;(\top r)$$

$$\dfrac{G\,|\,\Gamma\Rightarrow\alpha(t)}{G\,|\,\Gamma\Rightarrow\exists x\alpha(x)}\;(\exists r) \qquad \dfrac{G\,|\,\Gamma,\alpha(a)\Rightarrow\Pi}{G\,|\,\Gamma,\exists x\alpha(x)\Rightarrow\Pi}\;(\exists l)\quad (a \textit{ eigenvariable})$$

$$\dfrac{G\,|\,\Gamma\Rightarrow\Pi\,|\,\Gamma\Rightarrow\Pi}{G\,|\,\Gamma\Rightarrow\Pi}\;(ec) \qquad \dfrac{G}{G\,|\,\Gamma\Rightarrow\Pi}\;(ew) \qquad \dfrac{G\,|\,\Gamma\Rightarrow}{G\,|\,\Gamma\Rightarrow\Pi}\;(wr)$$

$$\dfrac{G\,|\,\Gamma_2,\Sigma_1\Rightarrow\Pi_1 \quad G\,|\,\Gamma_1,\Sigma_2\Rightarrow\Pi_2}{G\,|\,\Gamma_1,\Sigma_1\Rightarrow\Pi_1\,|\,\Gamma_2,\Sigma_2\Rightarrow\Pi_2}\;(com)$$

Table 3.1: Hypersequent calculus HUL∀ for UL∀

(Gen) and axioms (∀3), (∃2).
For the right to left direction, it is enough to show that all axioms and rules in UL∀ are derivable in HUL∀. This is immediate for (∀1), (∀2), (∃1), (∃2) and (Gen). Modus ponens is simulated by (cut) and (∧-Adj) by $(\wedge r)$. We show how to derive the prelinearity axiom (UL11) and (∀3) in HUL∀ using (com). (UL11) can be derived as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi\Rightarrow\varphi \quad \psi\Rightarrow\psi}{\varphi\Rightarrow\psi\,|\,\psi\Rightarrow\varphi}\;(com)}{\Rightarrow\varphi\to\psi\,|\,\psi\Rightarrow\varphi}\;(\to r)}{\Rightarrow\varphi\to\psi\,|\,\Rightarrow\psi\to\varphi}\;(\to r) \qquad \Rightarrow e}{\Rightarrow\varphi\to\psi\,|\,\Rightarrow(\psi\to\varphi)\wedge e}\;(\wedge r) \qquad \Rightarrow e}{\Rightarrow(\varphi\to\psi)\wedge e\,|\,\Rightarrow(\psi\to\varphi)\wedge e}\;(\wedge r)}{\Rightarrow((\varphi\to\psi)\wedge e)\vee((\psi\to\varphi)\wedge e)\,|\,\Rightarrow(\varphi\to\psi)\wedge e}\;(\vee r)}{\Rightarrow((\varphi\to\psi)\wedge e)\vee((\psi\to\varphi)\wedge e)\,|\,\Rightarrow((\varphi\to\psi)\wedge e)\vee((\psi\to\varphi)\wedge e)}\;(\vee r)}{\Rightarrow((\varphi\to\psi)\wedge e)\vee((\psi\to\varphi)\wedge e)}\;(ec)$$

A derivation of (∀3) proceeds as as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi(a) \Rightarrow \varphi(a) \qquad \dfrac{\varphi(a) \Rightarrow \varphi(a) \quad \psi \Rightarrow \psi}{\psi \Rightarrow \varphi(a) \mid \varphi(a) \Rightarrow \psi}\,(com)}{\varphi(a) \vee \psi \Rightarrow \varphi(a) \mid \varphi(a) \Rightarrow \psi}\,(\vee l) \qquad \psi \Rightarrow \psi}{\varphi(a) \vee \psi \Rightarrow \varphi(a) \mid \varphi(a) \vee \psi \Rightarrow \psi}\,(\vee l)}{\forall x(\varphi(x) \vee \psi) \Rightarrow \varphi(a) \mid \forall x(\varphi(x) \vee \psi) \Rightarrow \psi}\,(\overline{\forall l})}{\forall x(\varphi(x) \vee \psi) \Rightarrow \forall x\varphi(x) \mid \forall x(\varphi(x) \vee \psi) \Rightarrow \psi}\,(\forall r)}{\forall x(\varphi(x) \vee \psi) \Rightarrow \forall x\varphi(x) \vee \psi \mid \forall x(\varphi(x) \vee \psi) \Rightarrow \forall x\varphi(x) \vee \psi}\,(\overline{\vee r})}{\forall x(\varphi(x) \vee \psi) \Rightarrow \forall x\varphi(x) \vee \psi}\,(ec)}{\Rightarrow \forall x(\varphi(x) \vee \psi) \to (\forall x\varphi(x) \vee \psi)}\,(\to r)
$$

□

We can now consider the issue of finding analytic calculi for axiomatic extension of UL∀. A systematic way to extract (hyper)sequent rules from some classes of axioms extending the full (bounded) Lambek calculus with exchange $\mathrm{FL}_{e\perp}$ was introduced in [25]. We recall this result below, adapting it to first-order logics extending UL∀. The substructural hierarchy, which we have already recalled in Chapter 2 for proving preservation under DM completion, plays a key role also here. We apply the classification to axioms instead of equations and adapt it to the commutative case, i.e. considering the language $\mathcal{L}_{\mathrm{FL}_e}$. The main idea behind the classes $(\mathcal{N}_n, \mathcal{P}_n)$, from a proof-theoretic point of view, is that an axiom belongs to a class $\mathcal{N}$ (resp. $\mathcal{P}$) if its most external connective has negative (resp. positive) polarity (see [2]), i.e. its right (resp. left) introduction rule is invertible. Here by invertible rules we mean that, whenever the conclusion of the rule is derivable, the premises are derivable as well. In particular, the right rules for the connectives $\to$ and $\wedge$ of UL∀ are invertible, hence, these connectives are negative, while $\vee$ and $\cdot$ have left invertible rules, hence they are positive. The general grammar for determining the classes $\mathcal{N}_n$ and $\mathcal{P}_n$, has the following structure:

$\mathcal{P}_0 ::= \mathcal{N}_0 ::=$ the set of atomic formulas. For $n \geq 1$ we have

$\mathcal{P}_n ::= \mathcal{N}_{n-1} \mid \mathcal{P}_n \cdot \mathcal{P}_n \mid \mathcal{P}_n \vee \mathcal{P}_n \mid e$

$\mathcal{N}_n ::= \mathcal{P}_{n-1} \mid \mathcal{P}_n \to \mathcal{N}_n \mid \mathcal{N}_n \wedge \mathcal{N}_n \mid f$

The relation between the classes is the same as in Figure 2.1 in page 18.

In Theorems 2.3.9, 2.3.13, and 2.3.17 of Chapter 2 we have seen that the acyclic equations in the classes $\mathcal{N}_2$ and $\mathcal{P}_3'$ over FL_e-chains can be transformed into equivalent analytic clauses, which are preserved under DM completions. As a proof-theoretical reformulation of this, [25] presents an algorithm which transforms axioms from these classes into equivalent analytic rules, preserving cut elimination. In other words, cut elimination for analytic rules mirrors preservation under DM completion for analytic clauses (see [27] for details of the interplay between the algebraic and proof-theoretic results). The notion of analytic internal and external structural rules matches that of analytic quasiequations and clauses, see Definition 2.3.12.

Definition 3.3.9. *A structural rule:*

$$\frac{G \,|\, S_1 \quad \dots \quad G \,|\, S_n}{G \,|\, C_1 \,|\, \dots \,|\, C_q}\ (r)$$

is analytic *if* (r) *satisfies the following conditions:*

- Strong subformula property *: Each metavariable occurring in* $L(S_i)$ *(respectively in* $R(S_i)$*) with* $i = \{1, \dots, n\}$ *occurs also in* $L(C_j)$ *(respectively in* $R(C_j)$*), for some* j *in* $\{1, \dots, q\}$.
- Linear conclusion *: Each metavariable occurs at most once in* $G \,|\, C_1 \,|\, \dots \,|\, C_q$.
- Coupling*: Let* $R(C_j) = \Pi$ *with* Π *being non empty,* $j \in \{1, \dots, q\}$. *There is a metavariable* Σ *in* $L(C_j)$ *such that* Π *belongs to* $R(S_i)$*, for a given* i *in* $\{1, \dots, n\}$*, if and only if* Σ *belongs to* $L(S_i)$ *as well.*

In the following, we say that an axiom is *acyclic* if the corresponding equation is, see Definition 2.3.10. We can give now a proof-theoretic analogue of Theorems 2.3.9 and 2.3.13.

Theorem 3.3.10. *The algorithm in [25] transforms*

- *any acyclic* $\mathcal{N}_2$ *axiom* α *into an analytic internal structural rule* (r) *such that* $\vdash_{\mathrm{HUL\forall}+(r)} S$ *if and only if* $\vdash_{\mathrm{UL\forall}+\alpha} I(S)$*, for any sequent* S*;*
- *any acyclic* $\mathcal{P}'_3$ *axiom* α *into an analytic external structural rule* (r) *such that* $\vdash_{\mathrm{HUL\forall}+(r)} S$ *if and only if* $\vdash_{\mathrm{UL\forall}+\alpha} I(S)$*, for any sequent* S.

Proof. The theorem is shown in [25] for axiomatic extensions of the propositional logic $\mathrm{FL}_{e\bot}$. In virtue of Theorem 3.3.8, we can lift this result to axiomatic extensions of UL∀. The proof is analogous to that of Theorems 2.3.9 and 2.3.13 (we will sketch the algorithm in its proof-theoretic form in Lemma 5.1.3). □

The equivalence between hypersequent calculi and Hilbert-style systems given in Theorem 3.3.10 can be lifted to hypersequents, due to the presence of (com) in UL∀. We show the following.

Lemma 3.3.11. *Let* (r) *be an analytic rule and* α *an axiom in the language of* UL∀*. The following are equivalent:*

1. $\vdash_{\mathrm{HUL\forall}+(r)} S$ *if and only if* $\vdash_{\mathrm{UL\forall}+\alpha} I(S)$*, for any sequent* S.
2. $\vdash_{\mathrm{HUL\forall}+(r)} H$ *if and only if* $\vdash_{\mathrm{UL\forall}+\alpha} I(H)$*, for any hypersequent* H

Proof. That 2. implies 1. is obvious. Let us assume 1. and $\vdash_{\mathrm{HUL\forall}+(r)} H$. From this, using (cut) and easy derivations in HUL∀, we can prove that $\vdash_{\mathrm{HUL\forall}+(r)} \Rightarrow I(H)$. By 1. we then obtain $\vdash_{\mathrm{UL\forall}+\alpha} I(H)$.

Let us now assume 1. and $\vdash_{\mathrm{UL\forall}+(\alpha)} I(H)$. By 1. we get $\vdash_{\mathrm{HUL\forall}+(r)} \Rightarrow I(H)$. Let us assume, for simplicity, that $H = \Gamma_1 \Rightarrow \psi_1 \,|\, \Gamma_2 \Rightarrow \psi_2$, the generalization being easy. Thus, we have $\vdash_{\mathrm{HUL}\forall+(r)} \Rightarrow \varphi_1 \vee \varphi_2$, where the formula φ_1 stands for $(\odot\Gamma_1 \to \psi_1) \wedge e$, the formula φ_2 stands for $(\odot\Gamma_2 \to \psi_2) \wedge e$, and clearly $I(H) = \varphi_1 \vee \varphi_2$. By the following derivation:

$$\dfrac{\Rightarrow \varphi_1 \vee \varphi_2 \qquad \dfrac{\dfrac{\dfrac{\varphi_2 \Rightarrow \varphi_2 \quad \dfrac{\varphi_1 \Rightarrow \varphi_1 \quad \varphi_2 \Rightarrow \varphi_2}{\varphi_1 \Rightarrow \varphi_2 \mid \varphi_2 \Rightarrow \varphi_1}(com)}{\varphi_1 \vee \varphi_2 \Rightarrow \varphi_2 \mid \varphi_2 \Rightarrow \varphi_1}(\vee l) \qquad \varphi_1 \Rightarrow \varphi_1}{\varphi_1 \vee \varphi_2 \Rightarrow \varphi_2 \mid \varphi_1 \vee \varphi_2 \Rightarrow \varphi_1} \qquad \Rightarrow \varphi_1 \vee \varphi_2}{\Rightarrow \varphi_2 \mid \varphi_1 \vee \varphi_2 \Rightarrow \varphi_1}(cut)}{\Rightarrow \varphi_2 \mid \;\Rightarrow \varphi_1}(cut)$$

we obtain $\vdash_{\mathrm{HUL}\forall+(r)} \Rightarrow (\odot\Gamma_1 \to \psi_1) \wedge e \mid \;\Rightarrow (\odot\Gamma_2 \to \psi_2) \wedge e$, under the assumption $\vdash_{\mathrm{HUL}\forall+(r)} \Rightarrow (\odot\Gamma_1 \to \psi_1) \wedge e \vee (\odot\Gamma_2 \to \psi_2) \wedge e$. Finally, we obtain $\vdash_{\mathrm{HUL}\forall+(r)} H$ as follows:

$$\dfrac{\begin{matrix}\vdots\\ \Gamma_2, (\odot\Gamma_2 \to \psi_2) \wedge e \Rightarrow \psi_2\end{matrix} \qquad \dfrac{\Rightarrow (\odot\Gamma_1 \to \psi_1) \wedge e \mid \;\Rightarrow (\odot\Gamma_2 \to \psi_2) \wedge e \qquad \begin{matrix}\vdots\\ \Gamma_1, (\odot\Gamma_1 \to \psi_1) \wedge e \Rightarrow \psi_1\end{matrix}}{\Gamma_1 \Rightarrow \psi_1 \mid \;\Rightarrow (\odot\Gamma_2 \to \psi_2) \wedge e}(cut)}{\Gamma_1 \Rightarrow \psi_1 \mid \Gamma_2 \Rightarrow \psi_2}(cut)$$

where the sequents $\Gamma_1, (\odot\Gamma_1 \to \psi_1) \wedge e \Rightarrow \psi_1$ and $\Gamma_2, (\odot\Gamma_2 \to \psi_2) \wedge e \Rightarrow \psi_2$ can be easily derived in HUL∀. □

Theorem 3.3.12. *The algorithm in [25] transforms*

- *any acyclic $\mathcal{N}_2$ axiom α into an analytic internal structural rule (r) such that $\vdash_{\mathrm{HUL}\forall+(r)} H$ if and only if $\vdash_{\mathrm{UL}\forall+\alpha} I(H)$, for any hypersequent H;*
- *any acyclic $\mathcal{P}_3'$ axiom α into an analytic external structural rule (r) such that $\vdash_{\mathrm{HUL}\forall+(r)} S$ if and only if $\vdash_{\mathrm{UL}\forall+\alpha} I(H)$, for any hypersequent H.*

Proof. Follows from Theorem 3.3.10 and Lemma 3.3.11. □

The result in Theorem 3.3.12 can be extended and simplified if we consider the logic MTL∀ as the basic system. Recall that this logic is obtained by adding to UL∀ the axioms $f \to \varphi$ and $\varphi \to e$. The rules corresponding to these axioms are the well known weakening rules:

$$\frac{G \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma, \alpha \Rightarrow \Pi}\,(wl) \qquad \frac{G \mid \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow \Pi}\,(wr)$$

which can be obtained just applying Theorem 3.3.12. Letting $\mathrm{HMTL}\forall = \mathrm{HUL}\forall + (wl) + (wr)$, we would, thus, have that $\vdash_{\mathrm{HMTL}\forall} H$ if and only if $\vdash_{\mathrm{MTL}\forall} I(H)$, for any hypersequent H, as a consequence of Theorem 3.3.12. Taking a hypersequent calculus with weakening, such as HMTL∀ as a starting point, we obtain the following result, which translates the Theorems 2.3.9 and 2.3.13 for FL_i-algebras into the proof-theoretical context and restricts them to the commutative case.

Axiom	Rule
$f \to \alpha$	$\dfrac{G_1 \mid \Gamma_1 \Rightarrow}{G_1\Gamma_1 \Rightarrow \Pi}\ (wr)$
$\alpha \to e$	$\dfrac{G_1 \mid \Sigma \Rightarrow \Pi}{G_1 \mid \Gamma_1, \Sigma \Rightarrow \Pi}\ (wl)$
$\alpha \to \alpha^2$	$\dfrac{G_1 \mid \Gamma_1, \Gamma_1, \Sigma \Rightarrow \Pi}{G_1 \mid \Gamma_1, \Sigma \Rightarrow \Pi}\ (c)$
$\alpha^2 \to \alpha$	$\dfrac{G_1 \mid \Gamma_1, \Sigma \Rightarrow \Pi \quad G_1 \mid \Gamma_2, \Sigma \Rightarrow \Pi}{G_1 \mid \Gamma_1, \Gamma_2, \Sigma \Rightarrow \Pi}\ (mgl)$
$\alpha^k \to \alpha^n$	$\dfrac{G_1 \mid \Gamma_1^n, \Sigma \Rightarrow \Pi \ldots G_1 \mid \Gamma_k^n, \Sigma \Rightarrow \Pi}{G_1 \mid \Gamma_1, \ldots, \Gamma_k, \Sigma \Rightarrow \Pi}\ (knot_k^n)$
$f \cdot \alpha^k \to \alpha^n$	$\dfrac{\{G_1 \mid \Gamma_i^n, \Sigma \Rightarrow \Pi\}_{i=1,\ldots,k} \quad G_1 \mid \Gamma_{k+1} \Rightarrow}{G_1 \mid \Gamma_1, \ldots, \Gamma_k, \Gamma_{k+1}, \Sigma \Rightarrow \Pi}\ (fknot_k^n)$

Table 3.2: Some $\mathcal{N}_2$ axioms and the corresponding internal structural rules

Theorem 3.3.13. *The algorithm in [25] transforms*

- *any $\mathcal{N}_2$ axiom α into an analytic internal structural rule (r) such that $\vdash_{\mathrm{HMTL}\forall+(r)} H$ if and only if $\vdash_{\mathrm{MTL}\forall+\alpha} I(H)$ for any hypersequent H;*
- *any $\mathcal{P}_3$ axiom α into an analytic external structural rule (r) such that $\vdash_{\mathrm{HMTL}\forall+(r)} H$ if and only if $\vdash_{\mathrm{MTL}\forall+\alpha} I(H)$ for any hypersequent H.*

Proof. Follows from the results in [25], Theorem 3.3.12 and Lemma 3.3.11. ☐

The program *AxiomCalc* [11], which is available online at `http://www.logic.at/people/lara/axiomcalc.html`, implements the algorithm, automating the transformation of axioms into rules for axiomatic extensions of FL_{ew}. Tables 3.2 and 3.3 show some examples of axioms within $\mathcal{N}_2$ and $\mathcal{P}_3$ and the corresponding analytic rules obtained using AxiomCalc.

The analytic rules produced by the algorithm in [25] have the important property of preserving cut elimination when added to HUL∀. We give a sketch of the proof below, starting with the following technical lemma.

Lemma 3.3.14. *[8]*

1. *Let $d(a)$ be a derivation $G_1(a), \ldots, G_n(a) \vdash_{\mathrm{HL}\forall} G(a)$ and t be a term whose variables are all free and do not occur in $d(a)$. Then $d(t)$ is a derivation of $G_1(t), \ldots, G_n(t) \vdash_{\mathrm{HL}\forall} G(t)$, where $d(t)$ and $G(t)$ denote the results of substituting the term t for all free occurrences of a in the derivation $d(a)$ and hypersequent $G(a)$ respectively.*

Axiom	Rule
$\alpha \vee \neg\alpha$	$\dfrac{G \,\vert\, \Gamma, \Sigma \Rightarrow \Pi}{G \,\vert\, \Gamma \Rightarrow \;\vert\, \Sigma \Rightarrow \Pi}$ *(em)*
$(\alpha \to \beta) \vee (\beta \to \alpha)$	$\dfrac{G \,\vert\, \Gamma_2, \Sigma_1 \Rightarrow \Pi_1 \quad G \,\vert\, \Gamma_1, \Sigma_2 \Rightarrow \Pi_2}{G \,\vert\, \Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \,\vert\, \Gamma_2, \Sigma_2 \Rightarrow \Pi_2}$ (com)
$\alpha \vee \neg\alpha^n$ [23]	$\dfrac{G \,\vert\, \Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \ldots G \,\vert\, \Gamma_n, \Sigma_1 \Rightarrow \Pi_1}{G \,\vert\, \Gamma_1, \ldots, \Gamma_n \Rightarrow \;\vert\, \Sigma_1 \Rightarrow \Pi_1}$ *(em$_n$)*
$\neg\alpha \vee \neg\neg\alpha$	$\dfrac{G \,\vert\, \Gamma_1, \Gamma_2 \Rightarrow}{G \,\vert\, \Gamma_1 \Rightarrow \;\vert\, \Gamma_2 \Rightarrow}$ *(lq)*
$\alpha_0 \vee (\alpha_0 \to \alpha_1) \vee \cdots \vee (\alpha \wedge \cdots \wedge \alpha_{n-1} \to \alpha_n)$	$\dfrac{\{G \,\vert\, \Sigma_j, \Sigma_i \Rightarrow \Pi_i\}_{0 \le i \le n-1; i+1 \le j \le n}}{G \,\vert\, \Sigma_1 \Rightarrow \Pi_1 \,\vert\, \ldots \,\vert\, \Sigma_{n-1} \Rightarrow \Pi_{n-1} \,\vert\, \Gamma_n \Rightarrow}$ *(bcn)*
$\neg(\alpha \cdot \beta) \vee (\alpha \wedge \beta \to \alpha \cdot \beta)$ [42]	$\dfrac{\begin{array}{c} G \,\vert\, \Gamma_2, \Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \quad G \,\vert\, \Gamma_1, \Gamma_3, \Sigma_1 \Rightarrow \Pi_1 \\ G \,\vert\, \Gamma_1, \Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \quad G \,\vert\, \Gamma_2, \Gamma_3, \Sigma_1 \Rightarrow \Pi_1 \end{array}}{G \,\vert\, \Gamma_2, \Gamma_3 \Rightarrow \;\vert\, \Gamma_1, \Sigma_1 \Rightarrow \Pi_1}$ *(wnm)*
$\neg(\alpha \cdot \beta)^n \vee ((\alpha \wedge \beta)^{n-1} \to (\alpha \cdot \beta)^n)$ [11]	$\dfrac{\begin{array}{c} \{G \,\vert\, \Gamma_i^n, \Gamma_{i+(2p-1)}^n, \Sigma \Rightarrow \Pi\}_{1 \le p \le n, n \le i \le (3n-2p)} \\ \{G \,\vert\, \Gamma_i^n, \Gamma_j^n, \Sigma \Rightarrow \Pi\}_{1 \le i \le (n-1), 1 \le j \le (3n-1)} \end{array}}{G \,\vert\, \Gamma_n, \cdots, \Gamma_{3n-1} \Rightarrow \;\vert\, \Gamma_1, \cdots, \Gamma_{n-1}, \Sigma \Rightarrow \Pi}$ (wnm^n)
$\neg(\alpha^n) \vee (\alpha^{n-1} \to \alpha^n)$ [1]	$\dfrac{\{\Gamma_i^n, \Sigma \Rightarrow \Pi\}_{2n-1}}{\Gamma_1, \ldots, \Gamma_n \Rightarrow \;\vert\, \Gamma_{n+1}, \ldots, \Gamma_{2n-1}, \Sigma \Rightarrow \Pi}$ $(wnm1^n)$
$\bigvee_{i<k}(\neg\alpha_i \to \neg\alpha_{i+1})$ [73]	$\dfrac{\{G \,\vert\, \Gamma_{2i}, \Gamma_{2i+1} \Rightarrow \;\}_{1 \le i \le (k-2)}}{G \,\vert\, \Gamma_1, \Gamma_2 \Rightarrow \;\vert\, \ldots \,\vert\, \Gamma_{2k-3}, \Gamma_{2k-2} \Rightarrow}$ (inv_k)
$(\alpha^{n-1} \to \alpha \cdot \beta) \vee (\beta \to \alpha \cdot \beta)$ [53]	$\dfrac{\begin{array}{c} \{\mathrm{G} \,\vert\, \Gamma_i, \Gamma_1, \Sigma_1 \Rightarrow \Pi_1\}_{2 \le i \le n} \\ \{G \,\vert\, \Gamma_i, \Gamma_1, \Sigma_2 \Rightarrow \Pi_2\}_{2 \le i \le n} \end{array}}{G \,\vert\, \Gamma_2, \ldots, \Gamma_n, \Sigma_2 \Rightarrow \Pi_2 \,\vert\, \Gamma_1, \Sigma_1 \Rightarrow \Pi_1}$ (Ω_n)

Table 3.3: Some $\mathcal{P}_2$ and $\mathcal{P}_3$ axioms and the corresponding external structural rules

2. *Any derivation* d *of a hypersequent* H *can be transformed into a derivation of* $H[^{\alpha \Rightarrow \alpha}/_{\overline{p} \Rightarrow p}]$, *for any formula* α *and propositional variable* p.

In the following, by the *complexity* of a formula φ, we mean the number of occurrences of its connectives and quantifiers.

Theorem 3.3.15. *Any hypersequent calculus extending* HUL$\forall$ *with analytic structural rules admits cut elimination.*

Proof. The proof is a reformulation of that in [30].

We consider two generalized cut rules $(mcut_1)$ and $(mcut_2)$, which allow us to deal with applications of (ec) and any form of internal contraction:

$$\frac{G \,|\, \Gamma \Rightarrow \alpha \quad G \,|\, \Sigma_1, \alpha^{i_1} \Rightarrow \Pi_1 \,|\, \ldots \,|\, \Sigma_n, \alpha^{i_n} \Rightarrow \Pi_n}{G \,|\, \Sigma_1, \Gamma^{i_1} \Rightarrow \Pi_1 \,|\, \ldots \,|\, \Sigma_n, \Gamma^{i_n} \Rightarrow \Pi_n} \, (mcut_1)$$

and

$$\frac{G \,|\, \Gamma_1 \Rightarrow \alpha \,|\, \ldots \,|\, \Gamma_n \Rightarrow \alpha \quad G \,|\, \Sigma, \alpha \Rightarrow \Pi}{G \,|\, \Gamma_1, \Sigma \Rightarrow \Pi \,|\, \ldots \,|\, \Gamma_n, \Sigma \Rightarrow \Pi} \, (mcut_2)$$

Let d be a derivation containing applications of $(mcut_1)$ and $(mcut_2)$. Let us denote by $\rho(d)$ the *cut-rank* of d, i.e. the maximal complexity of the cut-formulas in the applications of $(mcut_1)$ and $(mcut_2)$ in d, plus 1. Let d_- and d_+ be the subderivations of d ending in the topmost cut (either $(mcut_1)$ or $(mcut_2)$) with cut-formula of complexity $\rho(d)$. We denote by $h(d)$ the sum $|d_-| + |d_+|$ of the lengths of d_- and d_+. Finally, we let $n(d)$ be the number of cuts ($(mcut_1)$ or $(mcut_2)$) with cut-formula of complexity $\rho(d)$. We reason by induction on the triple $(\rho(d), n(d), h(d))$ with the lexicographic order. Let us consider the topmost application of either $(mcut_1)$ or $(mcut_2)$ that has complexity $\rho(d)$. First, assume that this is an application of $(mcut_1)$. We show how to shift or reduce this application of $(mcut_1)$, in such a way as to reduce $(\rho(d), n(d), h(d))$. Let the application of $(mcut_1)$ be of the form

$$\frac{\begin{array}{c} \vdots\, d_- \\ G \,|\, \Gamma \Rightarrow \alpha \end{array} \qquad \begin{array}{c} \vdots\, d_+ \\ G \,|\, \Sigma_1, \alpha^{i_1} \Rightarrow \Pi_1 \,|\, \ldots \,|\, \Sigma_n, \alpha^{i_n} \Rightarrow \Pi_n \end{array}}{G \,|\, \Sigma_1, \Gamma^{i_1} \Rightarrow \Pi_1 \,|\, \ldots \,|\, \Sigma_n, \Gamma^{i_n} \Rightarrow \Pi_n} \, (mcut_1)$$

Let us assume that d_- ends in a logical rule introducing α. We distinguish cases, according to the last rule (r) applied in d_+. If d_+ ends in an axiom, we can just remove the application of $(mcut_1)$. The case for the rules $(el), (fr), (ew)$ and (ec) are easy and follow by reducing $h(d)$. If (r) is an application of an analytic rule, we can reduce $h(d)$ by shifting upwards $(mcut_1)$ over d_+ and then applying (r). The latter is a correct rule application, due to linearity and the strong subformula property of analytic rules. If (r) is a logical rule not introducing α, we can perform a similar shift of the $(mcut_1)$ application over the premises of (r), taking care of eigenvariables by using Lemma 3.3.14, if needed. If (r) is any logical rule introducing α, we apply $(mcut_1)$ between the premises of (r) and of the last applied rule in d_-, which by assumption is a logical rule introducing α. We examine the cases for quantifiers in more details, as they are not present in [30], which only deals with propositional logics. Assume that (r) is $(\forall l)$ and that the cut-formula α is of the form $\forall x \beta(x)$, i.e. the derivations d_- and d_+ end as follows (a being an eigenvariable):

$$\frac{\dfrac{\begin{array}{c} \vdots\, d'_- \\ G \,|\, \Gamma \Rightarrow \beta(a) \end{array}}{G \,|\, \Gamma \Rightarrow \forall x \beta(x)} \, (\forall r) \qquad \dfrac{\begin{array}{c} \vdots\, d'_+ \\ G \,|\, \Sigma_1, \forall x \beta(x)^{i_1} \Rightarrow \Pi_1 \,|\, \ldots \,|\, \Sigma_n, \forall x \beta(x)^{i_n - 1}, \beta(t) \Rightarrow \Pi_n \end{array}}{G \,|\, \Sigma_1, \forall x \beta(x)^{i_1} \Rightarrow \Pi_1 \,|\, \ldots \,|\, \Sigma_n, \forall x \beta(x)^{i_n} \Rightarrow \Pi_n} \, (\forall l)}{G \,|\, \Sigma_1, \Gamma^{i_1} \Rightarrow \Pi_1 \,|\, \ldots \,|\, \Sigma_n, \Gamma^{i_n} \Rightarrow \Pi_n} \, (mcut_1)$$

We shift upwards the application of $(mcut_1)$, applying it between the end-hypersequent of d_- and the end-hypersequent of d'_+. Thus, we obtain a derivation d_1 of

$$G \,|\, \Sigma_1, \Gamma^{i_1} \Rightarrow \Pi_1 \,|\, \dots \,|\, \Sigma_n, \Gamma^{i_n-1}, \beta(t) \Rightarrow \Pi_n.$$

We then apply Lemma 3.3.14(1) to d'_-, to obtain a derivation d_2 of $G \,|\, \Gamma \Rightarrow \beta(t)$. The required derivation is finally obtained by

$$\dfrac{\begin{matrix} \vdots\, d_2 \\ G \,|\, \Gamma \Rightarrow \beta(t) \end{matrix} \qquad \begin{matrix} \vdots\, d_1 \\ G \,|\, \Sigma_1, \Gamma^{i_1} \Rightarrow \Pi_1 \,|\, \dots \,|\, \Sigma_n, \Gamma^{i_n-1}, \beta(t) \Rightarrow \Pi_n \end{matrix}}{G \,|\, \Sigma_1, \Gamma^{i_1} \Rightarrow \Pi_1 \,|\, \dots \,|\, \Sigma_n, \Gamma^{i_n} \Rightarrow \Pi_n}\,(mcut_1)$$

where the cut-formula of $(mcut_1)$ has complexity less than $\rho(d)$. Similarly, for the rule $(\exists l)$, assume that we have:

$$\dfrac{\dfrac{\begin{matrix} \vdots\, d'_- \\ G \,|\, \Gamma \Rightarrow \beta(t) \end{matrix}}{G \,|\, \Gamma \Rightarrow \exists x \beta(x)}\,(\exists r) \qquad \dfrac{\begin{matrix} \vdots\, d'_+ \\ G \,|\, \Sigma_1, \exists x\beta(x)^{i_1} \Rightarrow \Pi_1 \,|\, \dots \,|\, \Sigma_n, \exists x\beta(x)^{i_n-1}, \beta(a) \Rightarrow \Pi_n \end{matrix}}{G \,|\, \Sigma_1, \exists x\beta(x)^{i_1} \Rightarrow \Pi_1 \,|\, \dots \,|\, \Sigma_n, \exists x\beta(x)^{i_n} \Rightarrow \Pi_n}\,(\exists l)}{G \,|\, \Sigma_1, \Gamma^{i_1} \Rightarrow \Pi_1 \,|\, \dots \,|\, \Sigma_n, \Gamma^{i_n} \Rightarrow \Pi_n}\,(cut)$$

where a is an eigenvariable. We shift upwards the application of $(mcut_1)$, applying it between the end-hypersequent of d_- and the end-hypersequent of d'_+. Thus, we obtain a derivation d_1 of

$$G \,|\, \Sigma_1, \Gamma^{i_1} \Rightarrow \Pi_1 \,|\, \dots \,|\, \Sigma_n, \Gamma^{i_n-1}, \beta(a) \Rightarrow \Pi_n.$$

We then apply Lemma 3.3.14(1) to d_1, to obtain a derivation d_2 of

$$G \,|\, \Sigma_1, \Gamma^{i_1} \Rightarrow \Pi_1 \,|\, \dots \,|\, \Sigma_n, \Gamma^{i_n-1}, \beta(t) \Rightarrow \Pi_n.$$

The required derivation is finally obtained by

$$\dfrac{\begin{matrix} \vdots\, d'_- \\ G \,|\, \Gamma \Rightarrow \beta(t) \end{matrix} \qquad \begin{matrix} \vdots\, d_2 \\ G \,|\, \Sigma_1, \Gamma^{i_1} \Rightarrow \Pi_1 \,|\, \dots \,|\, \Sigma_n, \Gamma^{i_n-1}, \beta(t) \Rightarrow \Pi_n \end{matrix}}{G \,|\, \Sigma_1, \Gamma^{i_1} \Rightarrow \Pi_1 \,|\, \dots \,|\, \Sigma_n, \Gamma^{i_n} \Rightarrow \Pi_n}\,(mcut_1)$$

where the $(mcut_1)$ has complexity less than $\rho(d)$. In case d_- ends with a rule different from a logical rule, we shift first the application of $(mcut_1)$ upwards on d_-, using Lemma 3.3.14 (1) to rename variables, when needed, and then reason in the same way as above. Let us consider now the case where the topmost application of complexity $\rho(d)$ is an application of $(mcut_2)$ of the form

$$\dfrac{\begin{matrix} \vdots\, d_- \\ G \,|\, \Sigma, \alpha \Rightarrow \Pi \end{matrix} \qquad \begin{matrix} \vdots\, d_+ \\ G \,|\, \Gamma_1 \Rightarrow \alpha \,|\, \dots \,|\, \Gamma_n \Rightarrow \alpha \end{matrix}}{G \,|\, \Gamma_1, \Sigma \Rightarrow \Pi \,|\, \dots \,|\, \Gamma_n, \Sigma \Rightarrow \Pi}\,(mcut_2)$$

We need to show how to shift or reduce this application of $(mcut_2)$, in such a way as to reduce $(\rho(d), n(d), h(d))$. First, we assume that d_- ends with a logical rule and proceed in a similar way as for the case of $(mcut_1)$. Note that, in case the last applied rule in d_+ is an analytic rule, we can shift the application of $(mcut_2)$ upwards, due to linearity, strong subformula and coupling properties of analytic rules. In case d_- ends with a rule different from a logical rule, we shift first the application of $(mcut_2)$ upwards on d_-, using Lemma 3.3.14 (1) to rename variables, when needed, and then proceed as above.

□

Henceforth, by *an* $\mathcal{N}_2$ (respectively $\mathcal{P}_3$,$\mathcal{P}_3'$)*-extension of* L we mean any axiomatic extension of a logic L with axioms within $\mathcal{N}_2$ (respectively, $\mathcal{P}_3, \mathcal{P}_3'$). By *acyclic-*$\mathcal{N}_2$ (respectively $\mathcal{P}_3, \mathcal{P}_3'$)-extension, we mean an $\mathcal{N}_2$ (respectively $\mathcal{P}_3$,$\mathcal{P}_3'$)-extension where axioms are acyclic. We can thus summarize the main result of this section with the next Theorem, which is a proof-theoretic counterpart of Theorem 2.3.18.

Theorem 3.3.16. *[25] Let* L *be any acyclic* $\mathcal{P}_3'$*-extension of* UL *or a* $\mathcal{P}_3$*-extension of* MTL*. We can construct a hypersequent calculus* HL∀ *extending* UL∀ *(*MTL∀ *respectively) such that*

- HL∀ *admits cut elimination,*
- $\vdash_{\mathrm{HL}\forall} H$ *if and only if* $\vdash_{\mathrm{L}\forall} I(H)$*, for any hypersequent* H*.*

Proof. Follows from Theorems 3.3.12, 3.3.13 and 3.3.15. □

In what follows we will always denote by HL∀ the first-order hypersequent calculus corresponding to the logic L∀ via Theorem 3.3.16.

Remark 3.3.17. *A calculus for first-order Gödel logic (see Example 3.1.4) is obtained by adding the rule* (c) *(see Table 3.2) to* HMTL∀*. Analytic calculi for Basic, Product and Łukasiewicz logics cannot be obtained by adding structural rules to* $\mathrm{HFL_e}$*. Indeed the defining axiom of Basic Logic* (div) *(see Example 3.1.4) belongs to the class* $\mathcal{N}_3$*, which cannot be treated by the algorithm in [25]. The same applies to the axiom* (Π) *of Product Logic. Hypersequent calculi have been introduced in the literature for Product [68] and Łukasiewicz [69] logics. However, these calculi use logical rules different from that of* $\mathrm{FL_e}$ *and require a different interpretation of hypersequents than the one used here, see [70] for more details.*

3.4 From density elimination to standard completeness

We have seen how to introduce a hypersequent calculus HL∀ admitting cut elimination for many axiomatic extensions L∀ of UL∀. We can easily extend them to cover $\mathrm{L}\forall^{\mathrm{D}}$, i.e. L∀ with the density rule, see Definition 3.2.2. Indeed, the density rule in hypersequent calculi is just an *external structural rule* of the form

$$\frac{G \,|\, p \Rightarrow \Pi \,|\, \Lambda \Rightarrow p}{G \,|\, \Lambda \Rightarrow \Pi}\ (D)$$

where p is a propositional variable not occurring in (any instance of) Λ, Π or G (p is an *eigenvariable*).

The density rule is not always admissible: its addition might enable us indeed to derive new theorems, even resulting in inconsistency.

Example 3.4.1. *Consider the calculus* $\mathrm{HMTL}\forall + (em_2)$*, cf. Table 3.3. Adding the density rule would allow us to derive an inconsistency as follows:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p \quad p \Rightarrow p}{p, p \Rightarrow \,|\, \Rightarrow p}\,(em_2) \qquad \dfrac{q \Rightarrow q \quad q \Rightarrow q}{q, q \Rightarrow \,|\, \Rightarrow q}\,(em_2)}{p, q \Rightarrow \,|\, p, q \Rightarrow \,|\, \Rightarrow p \,|\, \Rightarrow q}\,(com)}{p, q \Rightarrow \,|\, \Rightarrow p \,|\, \Rightarrow q}\,(ec)}{q \Rightarrow \,|\, \Rightarrow q}\,(D)}{\Rightarrow}\,(D)$$

This coheres with the fact that the logic $\mathrm{MTL} + (em_n)$ *is not standard complete, see [53]. A similar situation arises for* $\mathrm{HMTL}\forall + (bc2)$ *(cf. Table 3.3), where the empty sequent can be derived as follows:*

$$\dfrac{\dfrac{\dfrac{\dfrac{q \Rightarrow q \qquad \dfrac{p \Rightarrow p \quad q \Rightarrow q}{q \Rightarrow p \,|\, p \Rightarrow q}\,(com) \qquad p \Rightarrow p}{\Rightarrow p \,|\, p \Rightarrow q \,|\, p \Rightarrow q \,|\, q \Rightarrow}\,(bc2)}{\Rightarrow p \,|\, p \Rightarrow q \,|\, q \Rightarrow}\,(ec)}{\Rightarrow p \,|\, p \Rightarrow}\,(D)}{\Rightarrow}\,(D)$$

For some hypersequent calculi we are able instead to show the admissibility of (D), proving in fact its *elimination*: we find concrete procedures to rewrite any derivation containing an application of (D) into a derivation not containing this rule. Note the similarity between (D) and the cut rule:

$$\dfrac{G \,|\, \Gamma \Rightarrow \alpha \quad G \,|\, \alpha, \Delta \Rightarrow \Pi}{G \,|\, \Gamma, \Delta \Rightarrow \Pi}\,(cut)$$

Hence, in developing techniques for density elimination, we can benefit from insights coming from cut elimination procedures, one of the most developed topics in proof theory.

The first proofs of density elimination in the literature [5, 66] were calculi-specific. They were developed by analogy with Gentzen-style cut elimination proofs, making use of heavy combinatorial arguments to shift the density rule upwards. For instance, in order to show density elimination for HUL in [66], the following generalized density rule was considered

$$\dfrac{G \,|\, \{\Gamma_i, p^{\lambda_i} \Rightarrow \Pi_i\}_{i=1,\ldots,m} \,|\, \{\Sigma_k, p^{\mu_k+1} \Rightarrow p\}_{k=1,\ldots,o} \,|\, \{\Delta_j \Rightarrow p\}_{j=1,\ldots,m}}{G \,|\, \{\Gamma_i, \Delta_j^{\lambda_i} \Rightarrow \Pi_i\}_{i=1,\ldots,n}^{j=1,\ldots,m} \,|\, \{\Sigma_k, \Delta_j^{\mu_k} \Rightarrow e\}_{k=1,\ldots,o}^{j=1,\ldots,m}}\,(D)$$

where p is an eigenvariable, $n, m \geq 1$, and $\lambda_i \geq 1$ for some i, $1 \leq i \leq n$. This rule plays the same role as Gentzen's mix rule for the proof of cut elimination of the sequent calculi LK and LJ, see e.g. [78]. Showing that the generalized density rule can be "pushed" above other rules of the calculus requires checking many cases, especially for external structural rules such as (com). A different method to eliminate applications of the density rule from derivations was introduced in [28] and called density elimination *by substitution*. In this approach, similar to normalization for natural deduction systems, applications of the density rule are removed by making suitable substitutions for the newly introduced propositional variables. In Chapters 4 and 5 we will explain the method of density elimination by substitution in detail and extend it to classes of axiomatic extensions of MTL$\forall$ and UL$\forall$, respectively.

Let HL$\forall$ be a first-order hypersequent calculus. We denote by HL$\forall^{\mathrm{D}}$ the extension of the calculus with the rule (D). The addition of (D) matches the density rule in Hilbert systems i.e. $\vdash_{\mathrm{HL}\forall^{\mathrm{D}}} H$ if and only if $\vdash_{\mathrm{L}\forall^{\mathrm{D}}} I(H)$, for any hypersequent H. We do not prove this fact here, but we directly show, adapting from [63, 70] that the elimination of the density rule in a hypersequent calculus implies the admissibility of the density rule in the corresponding Hilbert system. First, we prove a simple technical lemma, which will be needed in the following.

Theorem 3.4.2. *Let* L *be any acyclic* $\mathcal{P}_3'$*-extension of* UL *or* $\mathcal{P}_3$*-extension of* MTL*. If the calculus* HL$\forall^{\mathrm{D}}$ *admits density elimination, then* $\vdash_{\mathrm{L}\forall^{\mathrm{D}}} = \vdash_{\mathrm{L}\forall}$.

Proof. We need to show that any L$\forall^{\mathrm{D}}$-proof can be converted in an L$\forall$-proof. We reason by induction on the length of an L$\forall^{\mathrm{D}}$-proof. Note that L$\forall^{\mathrm{D}}$-proofs differ from L$\forall$-proofs only for applications of $(density)$ (item 3 in Definition 3.2.2), hence it suffices to consider only these cases. Assume that $T \vdash_{\mathrm{L}\forall^{\mathrm{D}}} (\varphi \to p) \vee (p \to \psi) \vee \chi$, for some set of formulas $T \cup \{\varphi, \psi\}$, where p is a propositional variable. By induction hypothesis we have $T \vdash_{\mathrm{L}\forall} (\varphi \to p) \vee (p \to \psi) \vee \chi$. By Lemma 3.1.10, for some $\varphi^* = (\varphi_1 \wedge e) \cdot \ldots \cdot (\varphi_m \wedge e)$ where $\varphi_i \in T$ for $i = 1, \ldots, m$, we have

$$\vdash_{\mathrm{L}\forall} \varphi^* \to ((\varphi \to p) \vee (p \to \psi) \vee \chi)$$

and, using some derivabilities in UL$\forall$

$$\vdash_{\mathrm{L}\forall} ((\varphi^* \cdot \varphi) \to p) \vee (p \to (\varphi^* \to \psi)) \vee (\varphi^* \to \chi).$$

But then, by Theorem 3.3.16, we obtain that the hypersequent

$$\varphi^*, \varphi \Rightarrow p \mid p \Rightarrow \varphi^* \to \psi \mid \varphi^* \Rightarrow \chi$$

is cut-free derivable in HL$\forall$. We apply then (D) to the latter hypersequent and, by density elimination, we obtain that

$$\varphi^*, \varphi \Rightarrow \varphi^* \to \psi \mid \varphi^* \Rightarrow \chi$$

is derivable in HL$\forall$. Consider now the following derivation in HL$\forall$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots \\ \varphi^*, \varphi \Rightarrow \varphi^* \to \psi \mid \varphi^* \Rightarrow \chi\end{matrix} \qquad \dfrac{\varphi^* \Rightarrow \varphi^* \quad \psi \Rightarrow \psi}{\varphi^*, \varphi^* \to \psi \Rightarrow \psi}\,(\to l)}{\varphi^*, \varphi, \varphi^* \Rightarrow \psi \mid \varphi^* \Rightarrow \chi}\,(cut)}{\varphi^*, \varphi, \varphi^* \Rightarrow \psi \mid \varphi^*, e, \ldots, e \Rightarrow \chi}\,(\overline{el})}{\varphi^*, \varphi, \varphi^* \Rightarrow \psi \mid \varphi^*, \varphi_1 \wedge e, \ldots, \varphi_m \wedge e \Rightarrow \chi}\,(\overline{\wedge} l)}{\varphi^*, \varphi, \varphi^* \Rightarrow \psi \mid \varphi^*, \varphi^* \Rightarrow \chi}\,(\cdot l)}{\varphi^*, \varphi^* \Rightarrow \varphi \to \psi \mid \varphi^*, \varphi^* \Rightarrow \chi}\,(\to r)}{\dfrac{\varphi^*, \varphi^* \Rightarrow (\varphi \to \psi) \vee \chi \mid \varphi^*, \varphi^* \Rightarrow (\varphi \to \psi) \vee \chi}{\varphi^*, \varphi^* \Rightarrow (\varphi \to \psi) \vee \chi}\,(ec)}\,(\vee r)
$$

By Theorem 3.3.16 we have

$$\vdash_{\mathrm{L}\forall} (\varphi^* \cdot \varphi^*) \to ((\varphi \to \psi) \vee \chi).$$

Finally, by the other direction of Lemma 3.1.10, we get

$$T \vdash_{\mathrm{L}\forall} (\varphi \to \psi) \vee \chi.$$

□

Remark 3.4.3. *The density rule considered here is slightly different from the one originally introduced in [66], which has the form*

$$\frac{G \mid \Lambda \Rightarrow p \mid p, \Sigma \Rightarrow \Pi}{G \mid \Lambda, \Sigma \Rightarrow \Pi}$$

As shown in the proof above, our weaker version actually suffices to obtain the desired equivalence between the elimination of the rule in the hypersequent framework and the admissibility of the Hilbert-style density rule. This simpler version of the density rule simplifies the forthcoming proofs, but it comes at the price of using an application of (cut) *(see the derivation above), that in our calculi can anyway be eliminated. Modifying the proof above in a suitable way, we could also show that even a weaker form of density, where* Λ *is restricted to be a single formula rather than a multiset, suffices. This latter rule however would not cause any relevant simplifications of the proofs, hence, we do not consider it in what follows.*

Combining the results obtained so far, we see how density elimination entails rational completeness.

Corollary 3.4.4. *Let* L *be any acyclic* $\mathcal{P}_3'$*-extension of* UL *or* $\mathcal{P}_3$*-extension of* MTL*. If the calculus* $\mathrm{HL}\forall^{\mathrm{D}}$ *admits density elimination, the logic* $\mathrm{L}\forall$ *is rational complete*

Proof. Follows from Theorems 3.3.16,3.4.2 and 3.2.3. □

Let us take stock of what we have achieved so far. We have shown that axiomatic extensions of UL$\forall$ are rational complete, provided that we have corresponding hypersequent calculi admitting density elimination. Our last step is now to show standard completeness, which is defined as follows (recall Definition 2.2.1 of standard FL-algebras).

Definition 3.4.5. *Let* L *be a an axiomatic extension of* UL*,* $\mathcal{P}$ *be a predicate language,* T *be a* $\mathcal{P}$*-theory, and* φ *be a* $\mathcal{P}$*-formula.* L$\forall$ *is said to be* standard complete *if the following are equivalent:*

- $T \vdash_{\mathrm{L}\forall} \varphi$.
- $\langle \boldsymbol{A}, \mathfrak{M} \rangle \models \varphi$ *for each standard* L*-algebra* $\boldsymbol{A}$ *and each model* $\langle \boldsymbol{A}, \mathfrak{M} \rangle$ *of the theory* T.

As mentioned in Chapter 1, the traditional way to prove standard completeness uses the following lemma (see e.g. [34, 52]).

Lemma 3.4.6. *(Embedding Lemma) Let* L *be any axiomatic extension of* UL*. If countable* L*-chains are regularly embeddable into complete dense* L*-chains, then* L$\forall$ *is standard complete.*

Usually Lemma 3.4.6 is proved in two steps, see e.g. [41, 58] and Chapter 1.

(i) Prove rational completeness, showing that countable L-chains are regularly embeddable into dense countable L-chains.

(ii) Prove that dense countable L-chains are regularly embeddable into complete L-chains.

Step (i) is the algebraic counterpart of density elimination, while for step (ii) a general result has been presented in Theorem 2.3.18.

We can then summarize what we have revised on standard completeness in this chapter as follows.

Theorem 3.4.7 (Standard Completeness)**.** *Let* L *be any acyclic* $\mathcal{P}_3'$*-extension of* UL *or* $\mathcal{P}_3$*-extension of* MTL*.*

- *If the corresponding hypersequent calculus* $\mathrm{HL}\forall^{\mathrm{D}}$ *admits density elimination, then* L$\forall$ *is standard complete.*
- *If countable* L*-chains are regularly embeddable into dense countable* L*-chains, then* L$\forall$ *is standard complete.*

Proof. For the first item, L$\forall$ is rational complete by Corollary 3.4.4. Standard completeness is obtained by Lemma 3.4.6 and Theorem 2.3.18. The second item follows from Lemma 3.4.6 and Theorem 2.3.18. □

We are then left with two possible ways for obtaining the conclusion of Theorem 3.4.7, i.e. standard completeness: the proof-theoretic method already discussed, which relies on density elimination, and an algebraic one, consisting in the construction of an embedding from L-chains into dense L-chains. Such embeddings were found for Gödel logic in [49] and for the logics *nilpotent minimum* NM and *weak nilpotent minimum* WNM in [42]. The results were later generalized to MTL in [58] and refined and extended to MTL∀ in [71]. We briefly sketch the construction of the dense embedding for MTL, following the survey paper [34]. Let $\boldsymbol{A} = (A, \wedge, \vee, \cdot, \rightarrow, f, e)$ be a countable MTL-chain. The embedding into a dense countable MTL-chain is constructed as follows:

- For every $a \in A$, define

$$succ(a) = \begin{cases} \text{the successor of } a \text{ in } (A, \leq) & \text{if it exists,} \\ a & \text{otherwise.} \end{cases}$$

- Let B be the union of the sets $\{(a, 1) \mid a \in A\}$ (an isomorphic copy of A) and $\{(a, q) \mid \exists a' \in A$ such that $a \neq a'$ and $succ(a') = a, q \in \mathbb{Q} \cap (0, 1)\}$.

- Consider the lexicographical order $\preceq$ on B.

- Define the following monoidal operation on B:

$$\langle a, q \rangle \circ \langle b, r \rangle = \begin{cases} \min_{\preceq}\{\langle a, q \rangle, \langle b, r \rangle\} & \text{if } a \cdot b = \min\{a, b\} \\ \langle a \cdot b, 1 \rangle & \text{otherwise.} \end{cases}$$

It suffices then to check the following statements:

- The ordered monoid $(A, \cdot, e, \leq)$ is embeddable into $(B, \circ, (e, 1), \preceq)$ by mapping every $a \in A$ to $(a, 1)$.

- $\boldsymbol{B} = (B, \circ, (e, 1), \preceq)$ is a densely ordered countable monoid with maximum and minimum, so it is isomorphic to a monoid $\boldsymbol{B'} = ([0, 1] \cap \mathbb{Q}, \circ', 1, \preceq')$. Obviously, $(A, \cdot, e, \leq)$ is also embeddable into $\boldsymbol{B'}$. Let h be such an embedding. Moreover, restricted to $h[A]$, the residuum of $\circ'$ exists, call it $\Rightarrow$, and $h(a) \Rightarrow h(b) = h(a \rightarrow b)$.

The algebraic approach sketched above is hard to extend to other logics. In particular, it is not clear how to extend it to UL and to its axiomatic extensions that are not extensions of MTL, see [52]. In Chapter 6 we will show how the general proof-theoretic methods described in Chapters 4 and 5 can be translated back into the algebraic setting, thus obtaining a general method for constructing our required embeddings into dense chains.

CHAPTER 4

Axiomatic extensions of $\mathrm{MTL}\forall$

We provide general sufficient conditions for standard completeness for a large class of $\mathcal{P}_3$-extensions of $\mathrm{MTL}\forall$. This leads to a *uniform* proof of standard completeness, which applies to all $\mathcal{P}_3$-extensions of $\mathrm{MTL}\forall$ already known to be standard complete and also to infinitely many new ones. This chapter is based on [8, 11] and on the unpublished work [9].

4.1 Density elimination and semianchored rules

Recall that for any $\mathcal{P}_3$-extension L$\forall$ of $\mathrm{MTL}\forall$ we have a hypersequent calculus HL$\forall$ admitting cut elimination (see Theorem 3.3.16 in Chapter 3). Moreover, by Theorem 3.4.7, showing density elimination for the extension HL$\forall^{\mathrm{D}}$ of HL$\forall$ with (D) suffices to obtain standard completeness for L$\forall$. Density elimination will be proved here for any extension of HMTL$\forall^{\mathrm{D}}$ with a class of external structural rules – called *semianchored* – which correspond to a subclass of axioms in the class $\mathcal{P}_3$ of the substructural hierarchy.

Definition 4.1.1. *Let H be a hypersequent. We call* pp-component *any component of H of the form $\Theta, p \Rightarrow p$ where Θ is any multiset of formulas.*

Note that, for axiomatic extensions of $\mathrm{MTL}\forall$, any hypersequent containing pp-components is derivable from the axiom $p \Rightarrow p$ using (external and internal) weakenings.

Definition 4.1.2. *Let (r) be any analytic structural rule:*

$$\frac{G \,|\, S_1 \quad \dots \quad G \,|\, S_m}{G \,|\, C_1 \,|\, \dots \,|\, C_q}\ (r)$$

Let $L(C) = L(C_1) \cup \cdots \cup L(C_q)$ and $R(C) = R(C_1) \cup \cdots \cup R(C_q)$ (cf. Notation 3.3.2) and let (Γ, Π) be a pair in the cartesian product $L(C) \times R(C)$, with $\Pi \neq \emptyset$. We say that (Γ, Π) is an anchored *pair for (r) if $(\Gamma, \Pi) \in L(C_s) \times R(C_s)$, for some conclusion component C_s. We call (Γ, Π) a* unanchored *pair otherwise.*

Note that, by linearity (see Definition 3.3.9), if $(\Gamma_1, \Pi), \ldots, (\Gamma_n, \Pi)$ are anchored pairs for a rule (r), then all the $\Gamma_1, \ldots, \Gamma_n$ belong to the *same* component of the conclusion. Let B be a set of (un)anchored pairs $B = \{(\Gamma_1, \Pi), \ldots, (\Gamma_n, \Pi)\}$ for an analytic rule (r). In what follows, we say that B *is contained* in a premise $G \mid S_i$ of (r), if all the metavariables $\Gamma_1, \ldots, \Gamma_n$ belong to $L(S_i)$ and $R(S_i) = \Pi$.

Note that, by coupling (see Definition 3.3.9), any premise of an analytic structural rule with nonempty right hand side is of the form $G \mid \Theta, \Sigma_i \Rightarrow \Pi_i$ where Θ is a multiset of metavariables and Σ_i is the metavariable witnessing the coupling property for Π_i.

Definition 4.1.3. *Let (r) be an analytic structural rule as in the previous definition. We say that (r) is* anchored *iff for each premise $G \mid S_i$, either $R(S_i) = \emptyset$ or S_i contains only anchored pairs. (r) is* semianchored *iff for any set of unanchored pairs $\{(\Gamma_1, \Pi), \ldots, (\Gamma_n, \Pi)\}$ which is contained in a premise $G \mid S_i = G \mid \Theta, \Gamma_1^{i_1}, \ldots, \Gamma_n^{i_n}, \Sigma_i \Rightarrow \Pi_i$, there is a premise $G \mid S_j$ such that one of the following holds:*

1. *$S_j = \Theta, \Delta_1^{i_1}, \ldots, \Delta_n^{i_n}, \Sigma_i \Rightarrow \Pi_i$ and $(\Delta_1, \Pi_i), \ldots, (\Delta_n, \Pi_i)$ are anchored pairs (with $\Delta_1, \ldots, \Delta_n$ not necessarily distinct metavariables).*

2. *$S_j = \Theta, \Gamma_1^{i_1}, \ldots, \Gamma_n^{i_n}, \Sigma_j \Rightarrow \Pi_j$ and $(\Gamma_1, \Pi_j), \ldots, (\Gamma_n, \Pi_j)$ are anchored pairs.*

3. *$S_j = \Theta, \Delta_1^{i_1}, \ldots, \Delta_n^{i_n}, \Sigma_j \Rightarrow \Pi_j$ and $(\Gamma_1, \Pi_j), \ldots, (\Gamma_n, \Pi_j), (\Delta_1, \Pi_i), \ldots, (\Delta_n, \Pi_i)$ are anchored pairs (with $\Delta_1, \ldots, \Delta_n$ not necessarily distinct metavariables).*

It is clear that any anchored rule is semianchored. Indeed, by definition, anchored rules do not contain any set of unanchored pairs, hence they vacuously verify the properties in Definition 4.1.3.

Example 4.1.4. *All analytic internal structural rules are anchored, by the strong subformula property, see Definition 3.3.9. Consider now the external structural rules in Table 3.3:*

- *(lq) and (inv_k) are anchored, as all their premises have empty right hand side.*
- *The rules (wnm) (see Example 4.1.6 below), (wnm^n), (Ω_n) (see Example 4.1.7 below) and (com) (see Example 4.1.8 below) are semianchored.*
- *(em), (em_n) (see Example 4.1.5 below) and (bc_n) are not semianchored.*

Example 4.1.5. *For any $n > 0$, the rule*

$$\frac{G \mid \Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \ldots G \mid \Gamma_n, \Sigma_1 \Rightarrow \Pi_1}{G \mid \Gamma_1, \ldots, \Gamma_n \Rightarrow \ \mid \Sigma_1 \Rightarrow \Pi_1}\ (em_n)$$

is not semianchored. Indeed, any premise of the form $G \mid \Gamma_i, \Sigma_1 \Rightarrow \Pi_1$ for $i \in \{1, \ldots, n\}$ contains the set of unanchored pairs $\{(\Gamma_i, \Pi_1)\}$, for which none of the conditions $(1) - (3)$ in Definition 4.1.3 is satisfied.

Example 4.1.6. *We show that the rule* (wnm)

$$\frac{G\,|\,\Gamma_1,\Gamma_1,\Sigma_1 \Rightarrow \Pi_1 \quad G\,|\,\Gamma_2,\Gamma_1,\Sigma_1 \Rightarrow \Pi_1 \quad G\,|\,\Gamma_1,\Gamma_3,\Sigma_1 \Rightarrow \Pi_1 \quad G\,|\,\Gamma_2,\Gamma_3,\Sigma_1 \Rightarrow \Pi_1}{G\,|\,\Gamma_2,\Gamma_3 \Rightarrow\ |\,\Gamma_1,\Sigma_1 \Rightarrow \Pi_1}$$

is semianchored. Indeed:

- *The premise* $G\,|\,\Gamma_1,\Gamma_1,\Sigma_1 \Rightarrow \Pi_1$ *does not contain any set of unanchored pairs.*
- *The premises* $G\,|\,\Gamma_2,\Gamma_1,\Sigma_1 \Rightarrow \Pi_1$ *and* $G\,|\,\Gamma_3,\Gamma_1,\Sigma_1 \Rightarrow \Pi_1$ *contain the sets of unanchored pairs* $\{(\Gamma_2,\Pi_1)\}$ *and* $\{(\Gamma_3,\Pi_1)\}$*, respectively. For both, the premise* $G\,|\,\Gamma_1,\Gamma_1,\Sigma_1 \Rightarrow \Pi_1$ *satisfies Condition 1 in Definition 4.1.3.*
- *The premise* $G\,|\,\Gamma_2,\Gamma_3,\Sigma_1 \Rightarrow \Pi_1$ *contains the set of unanchored pairs* $B_1 = \{(\Gamma_2,\Pi_1), (\Gamma_3,\Pi_1)\}$*,* $B_2 = \{(\Gamma_2,\Pi_1)\}$ *and* $B_3 = \{(\Gamma_3,\Pi_1)\}$*. For* B_1*, the premise* $G\,|\,\Gamma_1,\Gamma_1,\Sigma_1 \Rightarrow \Pi_1$ *satisfies Condition 1 in Definition 4.1.3, while for* B_2 *and* B_3*, the same condition is satisfied by* $G\,|\,\Gamma_3,\Gamma_1,\Sigma_1 \Rightarrow \Pi_1$ *and* $G\,|\,\Gamma_2,\Gamma_1,\Sigma_1 \Rightarrow \Pi_1$*, respectively.*

Example 4.1.7. *The rule* (Ω_3) *(see Table 3.3):*

$$\frac{\begin{array}{c} G\,|\,\Gamma_3,\Gamma_1,\Sigma_2 \Rightarrow \Pi_2 \;\; G\,|\,\Gamma_3,\Gamma_1,\Sigma_1 \Rightarrow \Pi_1 \\ G\,|\,\Gamma_2,\Gamma_1,\Sigma_2 \Rightarrow \Pi_2 \;\; G\,|\,\Gamma_2,\Gamma_1,\Sigma_1 \Rightarrow \Pi_1 \end{array}}{G\,|\,\Gamma_3,\Gamma_2,\Sigma_2 \Rightarrow \Pi_2\,|\,\Gamma_1,\Sigma_1 \Rightarrow \Pi_1}\ (\Omega_3)$$

is semianchored. Indeed, the premise $G\,|\,\Gamma_3,\Gamma_1,\Sigma_2 \Rightarrow \Pi_2$ *contains the set of unanchored pairs* $\{(\Gamma_1,\Pi_2)\}$*. Condition 2 in Definition 4.1.3 is satisfied by the premise* $G\,|\,\Gamma_3,\Gamma_1,\Sigma_1 \Rightarrow \Pi_1$*. The premise* $G\,|\,\Gamma_3,\Gamma_1,\Sigma_1 \Rightarrow \Pi_1$ *contains the set of unanchored pairs* $\{(\Gamma_3,\Pi_1)\}$*. Condition 2 of Definition 4.1.3 is satisfied by* $G\,|\,\Gamma_3,\Gamma_1,\Sigma_2 \Rightarrow \Pi_2$*. The check is similar for the remaining premises.*

The check that any (Ω_n) *is semianchored is similar.*

Example 4.1.8. *The rule*

$$\frac{G\,|\,\Gamma_2,\Sigma_1 \Rightarrow \Pi_1 \quad G\,|\,\Gamma_1,\Sigma_2 \Rightarrow \Pi_2}{G\,|\,\Gamma_1,\Sigma_1 \Rightarrow \Pi_1\,|\,\Gamma_2,\Sigma_2 \Rightarrow \Pi_2}\ (com)$$

is semianchored. Indeed, the premise $G\,|\,\Gamma_2,\Sigma_1 \Rightarrow \Pi_1$ *contains the set of unanchored pairs* $\{(\Gamma_2,\Pi_1)\}$*. The premise* $G\,|\,\Gamma_1,\Sigma_2 \Rightarrow \Pi_2$ *satisfies Condition 3 of Definition 4.1.3, as both* (Γ_1,Π_1) *and* (Γ_2,Π_2) *are anchored pairs. The case of* $G\,|\,\Gamma_1,\Sigma_2 \Rightarrow \Pi_2$ *is symmetric.*

We describe below, first in an informal way, our proof of density elimination. This uses and refines the method of *density elimination by substitution* introduced in [28].

Let d be a cut-free derivation ending in a topmost application of the density rule

$$\frac{\begin{array}{c}\vdots\, d \\ G\,|\,\Lambda \Rightarrow p\,|\,p \Rightarrow \Pi\end{array}}{G\,|\,\Lambda \Rightarrow \Pi}\ {\scriptstyle (D)}$$

The application of (D) is removed by substituting the occurrences of p in d in an "asymmetric" way, according to whether p occurs in the left or in the right hand side of a sequent. More precisely, each component S of every hypersequent in d is replaced with $S[^{\Lambda \Rightarrow \Pi}/_{\overline{p} \Rightarrow p}]$ (cf. Notation 3.3.6). Following this substitution, the application of (D) above can be replaced with an application of (ec).

A problem

The labeled tree that results by applying the "asymmetric" substitution to d (we denote it by d^*) is in general not a correct derivation anymore. This is due to the possible presence in d of hypersequents containing *pp-components*, such as $G \mid \Theta, p^k \Rightarrow p$. Such hypersequents are transformed in d^* into hypersequents of the form $G \mid \Theta, \Lambda^k \Rightarrow \Pi$, which are no longer derivable. To solve the problem and obtain a correct density-free derivation, we need to restructure all subderivations of d^* containing the old pp-components. Looking at the original (cut-free) derivation d bottom-up it is clear that pp-components can originate only from applications of external structural rules that "mix" the content of various components in the conclusion. We discuss below some cases, starting with the case of (com), addressed and solved in [28].

The communication rule [28]:

Density elimination was proved in [28] for calculi containing only (com) as external structural rule "mixing" the content of components. The *only* problematic case to handle there was when one of the premises of (com) in d led to a pp-component as, e.g., in the following case:

$$\dfrac{\begin{array}{cc} \vdots & \vdots \\ G \mid \Gamma_1, \Gamma_2 \Rightarrow \Psi & \Sigma, p \Rightarrow p \end{array}}{G \mid \Gamma_1, \Sigma \Rightarrow p \mid \Gamma_2, p \Rightarrow \Psi} \, (com)$$

The restructuring of d^* was handled in [28] by *removing* this application of (com) and replacing it with a (sub)derivation starting from the premise $G \mid \Gamma_1, \Gamma_2 \Rightarrow \Psi$ and containing suitable applications of (cut) (see Lemma 4.1.10 below) and (wl).

Semianchored rules

In this chapter we prove density elimination for $\mathrm{HMTL}\forall^{\mathrm{D}}$ extended with any *semianchored* rule. Though mixing components, semianchored rules allow for a suitable restructuring of d^*. Indeed these rules have the property that, whenever the active component S of a premise is a pp-component, we can always find another premise to be used to derive its substituted version $S[^{\Lambda \Rightarrow \Pi}/_{\overline{p} \Rightarrow p}]$. Note that the case of semianchored rules subsumes the case above, since (com) itself is a semianchored rule, see Example 4.1.8. We illustrate the idea behind semianchored rules and the way we restructure the derivation d^* in Theorem 4.1.11 first with an example.

Example 4.1.9. *Consider the rule* (wnm) *of Table 3.3*

$$\dfrac{G \mid \Gamma_1, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 \quad G \mid \Gamma_2, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 \quad G \mid \Gamma_2, \Gamma_3, \Delta_1 \Rightarrow \Pi_1 \quad G \mid \Gamma_1, \Gamma_3, \Delta_1 \Rightarrow \Pi_1}{G \mid \Gamma_2, \Gamma_3 \Rightarrow \ \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1}$$

Assume that the derivation d of page 51 contains the following application of (wnm):

$$\dfrac{\begin{array}{c}\vdots\, d_1\\ \Gamma_1,\Gamma_1\Rightarrow p\end{array} \quad \dfrac{p\Rightarrow p}{p,\Gamma_1\Rightarrow p}\,(\mathrm{wl}) \quad \dfrac{p\Rightarrow p}{p,p\Rightarrow p}\,(\mathrm{wl}) \quad \dfrac{p\Rightarrow p}{p,\Gamma_1\Rightarrow p}\,(\mathrm{wl})}{p,p\Rightarrow\ \mid \Gamma_1\Rightarrow p}\,(\mathrm{wnm})$$

The substitution $[{}^{\Lambda\Rightarrow\Pi}/_{\overline{p}\Rightarrow p}]$ applied to all the hypersequents above yields the following incorrect subderivation in d^* (d_1^* is obtained by applying the asymmetric substitution $[{}^{\Lambda\Rightarrow\Pi}/_{\overline{p}\Rightarrow p}]$ to all the sequents in d_1):

$$\dfrac{\begin{array}{c}\vdots\, d_1^*\\ \Gamma_1,\Gamma_1\Rightarrow \Pi\end{array} \quad \dfrac{\begin{array}{c}\vdots\, ??\\ \Lambda\Rightarrow\Pi\end{array}}{\Lambda,\Gamma_1\Rightarrow \Pi}\,(\mathrm{wl}) \quad \dfrac{\begin{array}{c}\vdots\, ??\\ \Lambda\Rightarrow\Pi\end{array}}{\Lambda,\Lambda\Rightarrow \Pi}\,(\mathrm{wl}) \quad \dfrac{\begin{array}{c}\vdots\, ??\\ \Lambda\Rightarrow\Pi\end{array}}{\Lambda,\Gamma_1\Rightarrow \Pi}\,(\mathrm{wl})}{\Lambda,\Lambda\Rightarrow\ \mid \Gamma_1\Rightarrow \Pi}\,(\mathrm{wnm})$$

Now the idea is to use the derivation d_1^* of $\Gamma_1,\Gamma_1\Rightarrow\Pi$ (the substituted version of the non pp-component $\Gamma_1,\Gamma_1\Rightarrow p$) to derive the substituted version of the other premises, i.e. $(p,\Gamma_1\Rightarrow p)\ [{}^{\Lambda\Rightarrow\Pi}/_{\overline{p}\Rightarrow p}]$ and $(p,p\Rightarrow p)\ [{}^{\Lambda\Rightarrow\Pi}/_{\overline{p}\Rightarrow p}]$. More precisely, the incorrect (sub)derivation in d^* above is replaced with:

$$\dfrac{\begin{array}{c}\vdots\, d_1^*\\ \Gamma_1,\Gamma_1\Rightarrow \Pi\end{array} \quad \begin{array}{c}\vdots\, d_1^*\\ \Gamma_1,\Gamma_1\Rightarrow \Pi\\ \vdots\,(*)\\ \Lambda,\Gamma_1\Rightarrow\Pi\end{array} \quad \begin{array}{c}\vdots\, d_1^*\\ \Gamma_1,\Gamma_1\Rightarrow \Pi\\ \vdots\,(**)\\ \Lambda,\Lambda\Rightarrow\Pi\end{array} \quad \begin{array}{c}\vdots\, d_1^*\\ \Gamma_1,\Gamma_1\Rightarrow \Pi\\ \vdots\,(*)\\ \Lambda,\Gamma_1\Rightarrow\Pi\end{array}}{\Lambda,\Lambda\Rightarrow\ \mid \Gamma_1\Rightarrow \Pi}\,(\mathrm{wnm})$$

where $(*)$ and $(**)$ are obtained by suitable cuts (i.e. applications of Lemma 4.1.10 below).

The technical lemma below, which enables us to suitably "move" multisets of formulas between components of the same hypersequents, is the key for our proof of density elimination.

Lemma 4.1.10. *Let* HL∀ *be a hypersequent calculus extending* HUL∀ *with analytic rules. Let d be a cut-free, density-free derivation of $G \mid \Lambda\Rightarrow p \mid p\Rightarrow\Pi$ ($p\notin G,\Pi,\Lambda$). Then the rule*

$$\dfrac{G \mid \Theta,\Delta\Rightarrow\Psi}{G\mid \Theta,\Lambda\Rightarrow\Psi\mid\Delta\Rightarrow\Pi}\,(split_d)$$

is derivable in HL∀.

Proof. Applying Lemma 3.3.14(2) to d, we have a derivation d' of $G \mid \Lambda\Rightarrow\odot\Delta\mid\odot\Delta\Rightarrow\Pi$. The conclusion of the rule $(split_d)$ is obtained as follows (the sequent $\Delta\Rightarrow\odot\Delta$ is easily derivable):

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, d'\\ G\mid\Lambda\Rightarrow\odot\Delta\mid\odot\Delta\Rightarrow\Pi\end{array}\quad \dfrac{G\mid\Theta,\Delta\Rightarrow\Psi}{G\mid\Theta,\odot\Delta\Rightarrow\Psi}\,(\overline{\cdot l})}{G\mid\Theta,\Lambda\Rightarrow\Psi\mid\odot\Delta\Rightarrow\Pi}\,(cut)\quad \Delta\Rightarrow\odot\Delta}{G\mid\Theta,\Lambda\Rightarrow\Psi\mid\Delta\Rightarrow\Pi}\,(cut)$$

□

We are now ready to present the main theorem of this chapter. In what follows, a (D)-free derivation is a derivation not containing the rule (D).

Theorem 4.1.11. *Let* HL∀ *be any hypersequent calculus extending* MTL∀ *with any set of semi-anchored rules.* $\mathrm{HL}\forall^{\mathrm{D}}$ *admits density elimination.*

Proof. It is enough to consider topmost applications of (D). Assume that this is

$$\dfrac{\begin{array}{c}\vdots\, d\\ G\,|\,\Lambda \Rightarrow p\,|\,p \Rightarrow \Pi\end{array}}{G\,|\,\Lambda \Rightarrow \Pi}\,(D)$$

By Theorem 3.3.16, we can assume that d is cut-free. We show that we can obtain a (D)-free derivation of $G\,|\,\Lambda \Rightarrow \Pi$. Let H be $S_1 \,|\, \dots \,|\, S_n$. We define H^* as $S_1^* \,|\, ... \,|\, S_n^*$ where for each $i = 1, ..., n$

$$S_i^* = S_i[{}^{\Lambda\Rightarrow\Pi}/_{\overline{p}\Rightarrow p}]$$

We prove the following:

Claim: *For each hypersequent* H *in* d *that does not contain a pp-component, one can find a* (D)*-free derivation of* $G\,|\,H^*$.

The result on density elimination follows by this claim. Indeed let H be $G\,|\,\Lambda \Rightarrow p\,|\,p \Rightarrow \Pi$. We get that $G\,|\,H^* = G\,|\,(G\,|\,\Lambda \Rightarrow p\,|\,p \Rightarrow \Pi)^* = G\,|\,G\,|\,\Lambda \Rightarrow \Pi\,|\,\Lambda \Rightarrow \Pi$ is derivable (note that $G^* = G$ by the eigenvariable condition on p). The desired (D)-free proof of $G\,|\,\Lambda \Rightarrow \Pi$ follows by (ec).

The proof of the claim proceeds by induction on the length of the cut-free subderivation d_H of H in HL∀. If (r) is an axiom, the claim easily follows by applying $(\overline{ew})$. For the inductive step, we distinguish cases according to the last rule (r) applied in d_H. If (r) is (ec),(ew),(wl),(wr), the claim holds by the induction hypothesis followed by a suitable application of the corresponding rule.

Logical rules. Let (r) be any logical rule, for instance of the form

$$\dfrac{G_1\,|\,S_1 \quad G_1\,|\,S_2}{G_1\,|\,S}\,(r)$$

Recall that, by assumption, the conclusion of (r) does not contain *pp*-components. Hence, by the form of our logical rules (see Table 3.1), none of the premises of (r) can contain *pp*-components as well. We then apply the induction hypothesis to the premises of (r), obtaining a derivation of $G\,|\,G_1^*\,|\,S_1^*$, and $G\,|\,G_1^*\,|\,S_2^*$. Note that, since p is a propositional variable, it cannot be the principal formula of the original rule application. Hence

$$\dfrac{G\,|\,G_1^*\,|\,S_1^* \quad G\,|\,G_1^*\,|\,S_2^*}{G\,|\,G_1^*\,|\,S^*}\,(r)$$

is a correct rule application. This gives the desired hypersequent. In case (r) is a quantifier rule, we use Lemma 3.3.14 to rename variables, if needed.

Semianchored rules. Assume now that the last applied rule (r) is a semianchored rule of the form

$$\frac{G_1 \mid S_1 \quad \dots \quad G_1 \mid S_m}{G_1 \mid C_1 \mid \dots \mid C_q} \; (r)$$

and that its conclusion does not contain any *pp*-component. We show how to find a (D)-free derivation of

$$G \mid G_1^* \mid C_1^* \mid \dots \mid C_q^*.$$

Take a premise $G_1 \mid S_i$ of (r). All the possible cases that can arise are listed in Figure 4.1.

(a) S_i is not a *pp*-component.
(b) S_i is a *pp*-component and does not contain unanchored tuples.
(c) S_i is a *pp*-component and contains unanchored tuples.

Figure 4.1: Cases for the premises of a semianchored rule

For case (a), we can just apply the induction hypothesis to get a derivation of $G \mid G_1^* \mid S_i^*$. (b) cannot occur, as otherwise we would have a *pp*-component in the conclusion, contradicting our assumption. Therefore, the only nontrivial case to handle is (c). Assume, to fix the ideas, that S_i is obtained as the instantiation of the metasequent $\Theta, \Gamma_1^{i_1}, \dots, \Gamma_n^{i_n}, \Sigma_i \Rightarrow \Pi_i$, where $\Gamma_1, \dots, \Gamma_n$ are the metavariables whose instantiation contain some *p*s, and Π_i is instantiated with p. Clearly every $(\Gamma_1, \Pi_i), \dots, (\Gamma_n, \Pi_i)$ has to be an unanchored pair, as otherwise the conclusion would contain a *pp*-component. Following Definition 4.1.3 of semianchored rules, we consider three cases.

1. There is a premise $G_1 \mid S_j$, where S_j instantiates a metasequent of the form

   $$\Theta, \Delta_1^{i_1}, \dots, \Delta_n^{i_n}, \Sigma_i \Rightarrow \Pi_i$$

   and any $(\Delta_1, \Pi_i), \dots, (\Delta_n, \Pi_i)$ is anchored. From this and the coupling property, it follows that $L(S_j)$ does not contain any p, hence S_j is not a *pp*-component. By the induction hypothesis we have a derivation of $G \mid G_1^* \mid S_j^*$, i.e. $G \mid G_1^* \mid \Theta, \Delta_1^{i_1}, \dots, \Delta_n^{i_n}, \Sigma_i \Rightarrow \Pi$. By multiple applications of the rule $(split_d)$ (see Lemma 4.1.10) we get a derivation of

   $$G \mid G_1^* \mid \Theta, \Lambda_1^{i_1}, \Delta_2^{i_2}, \dots, \Delta_n^{i_n}, \Sigma_i \Rightarrow \Pi \mid \Delta_1 \Rightarrow \Pi \mid \dots \mid \Delta_1 \Rightarrow \Pi$$

   and by applying $(\overline{ec})$ we get

   $$G \mid G_1^* \mid \Theta, \Lambda_1^{i_1}, \Delta_2^{i_2}, \dots, \Delta_n^{i_n}, \Sigma_i \Rightarrow \Pi \mid \Delta_1 \Rightarrow \Pi.$$

   Proceeding similarly for every $\Delta_1, \dots, \Delta_n$, we eventually obtain a derivation of

   $$(*) \; G \mid G_1^* \mid \Theta, \Lambda^{i_1 + \dots + i_n}, \Sigma_i \Rightarrow \Pi \mid \Delta_1 \Rightarrow \Pi \mid \dots \mid \Delta_n \Rightarrow \Pi.$$

Note that all the $(\Delta_1, \Pi_i), \ldots, (\Delta_n, \Pi_i)$ are anchored pairs. By Definition 4.1.2 there is a component of the conclusion, say C_s, such that $\Delta_1, \ldots, \Delta_n \in L(C_s)$ and $\Pi_i \in R(C_s)$. Hence, by applying (wl) to all the sequents in $(*)$ of the form $\Delta_i \Rightarrow \Pi$, we obtain

$$G \,|\, G_1^* \,|\, \Theta, \Lambda^{i_1+\cdots+i_n}, \Sigma_i \Rightarrow \Pi \,|\, C_s^* \,|\, \ldots \,|\, C_s^*.$$

Applying $(\overline{ec})$, we get

$$(**)\ G \,|\, G_1^* \,|\, \Theta, \Lambda^{i_1+\cdots+i_n}, \Sigma_i \Rightarrow \Pi \,|\, C_s^*.$$

Note that S_i contains at least $i_1 + \cdots + i_n$ times p, as we assumed that the instantiations of all the metavariables $\Gamma_1, \ldots, \Gamma_n$ contained at least one p. Hence S_i^* should contain at least $i_1 + \cdots + i_n$ times Λ, and it can be derived by applying (wl) to the component $\Theta, \Lambda^{i_1+\cdots+i_n}, \Sigma_i \Rightarrow \Pi$ of $(**)$. We have thus obtained a derivation of $G \,|\, G_1^* \,|\, S_i^* \,|\, C_s^*$.

2. There is a premise $G \,|\, S_j$, where S_j instantiates a metasequent of the form

$$\Theta, \Gamma_1^{i_1}, \ldots, \Gamma_n^{i_n}, \Sigma_j \Rightarrow \Pi_j$$

and $(\Gamma_1, \Pi_j), \ldots, (\Gamma_n, \Pi_j)$ are anchored pairs. Clearly S_j is not a *pp*-component, hence by the induction hypothesis we have a derivation of $G \,|\, G_1^* \,|\, S_j^*$. Note that S_j^* is a sequent of the form

$$\Theta, (\Gamma_1^*)^{i_1}, \ldots, (\Gamma_n^*)^{i_n}, \Sigma_j^* \Rightarrow \Pi_j$$

where Γ_i^*, Σ_j^* denote the multiset Γ_i, Σ_j where any occurrence of p has been replaced with Λ.

We apply the rule $(split_d)$ to $G \,|\, G_1^* \,|\, S_j^*$, obtaining

$$(*)\ G \,|\, G_1^* \,|\, \Lambda, \Sigma_j^* \Rightarrow \Pi_j \,|\, \Theta, (\Gamma_1^*)^{i_1}, \ldots, (\Gamma_n^*)^{i_n} \Rightarrow \Pi.$$

We obtain then S_i^*, by applying (wl) on the component $\Theta, (\Gamma_1^*)^{i_1}, \ldots, (\Gamma_n^*)^{i_n} \Rightarrow \Pi$ of $(*)$. Moreover, recall that all the $(\Gamma_1, \Pi_j), \ldots, (\Gamma_n, \Pi_j)$ are anchored and all the $\Gamma_1, \ldots, \Gamma_n$ contain at least one p. Hence we can obtain a component C_s^* of the conclusion of (r), by applying (wl) on the component $\Lambda, \Sigma_j^* \Rightarrow \Pi_j$ of $(*)$. Thus, we have obtained a derivation of

$$G \,|\, G_1^* \,|\, S_i^* \,|\, C_s^*$$

3. There is a premise $G \,|\, S_j$, where S_j instantiates a metasequent of the form

$$\Theta, \Delta_1^{i_1}, \ldots, \Delta_n^{i_n}, \Sigma_j \Rightarrow \Pi_j$$

and all the pairs $(\Gamma_1, \Pi_j), \ldots, (\Gamma_n, \Pi_j)$, $(\Delta_1, \Pi_i), \ldots, (\Delta_n, \Pi_i)$ are anchored. Recall that the instantiation of all the $\Gamma_1, \ldots, \Gamma_n$ and Π_i contain a p. Hence, as $(\Gamma_1, \Pi_j), \ldots,$ $(\Gamma_n, \Pi_j), (\Delta_1, \Pi_i), \ldots, (\Delta_n, \Pi_i)$ are anchored, we can assume that neither the instantiation of Π_j nor of any of the $\Delta_1, \ldots, \Delta_n$ can contain a p (otherwise we would have a *pp*-component in the conclusion). Thus, by induction hypothesis we have a derivation of $G \mid G_1^* \mid S_j^*$, i.e.

$$G \mid G_1^* \mid \Theta, \Delta_1^{i_1}, \ldots, \Delta_n^{i_n}, \Sigma_j^* \Rightarrow \Pi_j$$

where Σ_j^* denotes the multiset Σ_j where any occurrence of p has been replaced with Λ, if any. We apply Lemma 4.1.10 $(i_1 + \cdots + i_n)$ times to the latter hypersequent, obtaining

$$G \mid G_1^* \mid \Theta, \Lambda^{i_1+\cdots+i_n}, \Sigma_j^* \Rightarrow \Pi_j \mid \Delta_1 \Rightarrow \Pi \mid \ldots \mid \Delta_n \Rightarrow \Pi$$

Applying once again Lemma 4.1.10 we get

$$G \mid G_1^* \mid \Lambda, \Sigma_j^* \Rightarrow \Pi_j \mid \Theta, \Lambda^{i_1+\cdots+i_n} \Rightarrow \Pi \mid \Delta_1 \Rightarrow \Pi \mid \ldots \mid \Delta_n \Rightarrow \Pi$$

Now, by applying (wl) on the component $\Theta, \Lambda^{i_1+\cdots+i_n} \Rightarrow \Pi$ we get S_i^*. Moreover, as $(\Gamma_1, \Pi_j), \ldots, (\Gamma_n, \Pi_j)$ are anchored and all the instantiations of $\Gamma_1, \ldots, \Gamma_n$ contain a p, applying (wl) to $\Lambda, \Sigma_j^* \Rightarrow \Pi_j$ we get a conclusion component, say C_s^*. Similarly, being $(\Delta_1, \Pi_i), \ldots, (\Delta_n, \Pi_i)$ anchored and Π_i instantiated with p, we get another conclusion component, say C_t^*, by applying (wl) on the components $\Delta_1 \Rightarrow \Pi, \ldots, \Delta_n \Rightarrow \Pi$. Thus, we have a derivation of the hypersequent $G \mid G_1^* \mid S_i^* \mid C_s^* \mid C_t^* \mid \ldots \mid C_t^*$. By $(\overline{ec})$, we obtain $G \mid G_1^* \mid S_i^* \mid C_s^* \mid C_t^*$.

By summarizing, when the last rule (r) in d_H is semianchored, for each premise $G_1 \mid S_i$:

- If S_i is not a *pp*-component, then $G \mid G_1^* \mid S_i^*$ is (D)-free derivable by the induction hypothesis.
- If S_i is a *pp*-component, then either $G \mid G_1^* \mid S_i^* \mid C_s^*$ or $G \mid G_1^* \mid S_i^* \mid C_s^* \mid C_t^*$ is (D)-free derivable, for some conclusion components C_s, C_t.

Applying $(\overline{ew})$ to every hypersequent obtained above, we get a derivation of $G_1 \mid S_i^* \mid C_1^* \mid \ldots \mid C_q^*$, for each premise $G_1 \mid S_i$. Let G_c be the hypersequent $C_1^* \mid \ldots \mid C_q^*$. We obtain our required derivation of $G \mid G_1^* \mid G_c$ as follows:

$$\dfrac{\dfrac{G \mid G_1^* \mid S_1^* \mid G_c \quad \ldots \quad G \mid G_1^* \mid S_m^* \mid G_c}{G \mid G_1^* \mid G_c \mid G_c}\,(r)}{G \mid G_1^* \mid G_c}\,(\overline{ec})$$

The application of the semi-anchored rule above is correct, being just the original rule application in d, where the substitution $[^{\Lambda \Rightarrow \Pi}/_{\overline{p} \Rightarrow p}]$ has been applied to every component in the premises and in the conclusion. This completes the proof of the main claim. □

For simplicity, by a semianchored $\mathcal{P}_3$-extension of $\mathrm{MTL}\forall$, we mean an extension of $\mathrm{MTL}\forall$ with axioms which are transformed into semianchored rules by the algorithm in [25].

Theorem 4.1.12. *Any semianchored $\mathcal{P}_3$-extension of* $\mathrm{MTL}\forall$ *is standard complete*

Proof. Follows from Theorem 3.4.7(i) and 4.1.11. □

Figure 4.2: Screenshot of AxiomCalc

4.2 The program AxiomCalc

The results that we have obtained in the previous section are based on purely syntactic considerations. We have shown standard completeness for axiomatic extensions of $\mathrm{MTL}\forall$, provided that the additional axioms are in the class $\mathcal{P}_3$ and the corresponding hypersequent rules are semianchored. We also have automated the (otherwise tedious) check whether an analytic rule is semianchored within the program AxiomCalc.

AxiomCalc was originally developed to automate the conversion of axioms into corresponding analytic rules (see page 39), in the context of the larger project TINC (Tools for the Investigation of Non-Classical logics, see [29]). By our additional feature, AxiomCalc can also check whether the generated analytic rules are semianchored. The program offers to the user an online interface, see the screenshot in Figure 4.2. It receives as input an axiom for a logic extending $\mathrm{MTL}\forall$ and provides as output a paper written in LaTeXthat contains the generated analytic rules and the result of checking the semianchored rules condition. The program performs also an additional check: it verifies whether the generated analytic rules are *convergent*[1]. In what follows we present the definition of convergent rules given in [8]. This definition is equivalent to the one first presented in [11], but has a simpler form.

Definition 4.2.1. *Let (r) be any analytic structural rule:*

$$\frac{G \mid S_1 \quad \dots \quad G \mid S_n}{G \mid C_1 \mid \dots \mid C_q} \; (r)$$

and V be the set of different metavariables appearing in $L(S_1) \cup \dots \cup L(S_n)$.

- *A premise $G \mid S_i$ of (r) is said to be a* pivot*-premise if there is a component C_j of the conclusion such that $R(S_i) = R(C_j)$ and the (set of) metavariables in $L(S_i)$ are all contained in $L(C_j)$.*

- *The rule (r) is said to be* convergent *if for each premise $G \mid S_i$ of (r), either $R(S_i) = \emptyset$ or there is a map $\sigma\colon V \to V$ such that:*

  (i) $G \mid S_i[\{^{\sigma(\Gamma)\Rightarrow}/_{\Gamma\Rightarrow}\}_{\Gamma \in V}]$ is a premise of (r) which is a pivot.

  (ii) For any $W \subset V$, the hypersequent $G \mid S_i[\{^{\sigma(\Gamma)\Rightarrow}/_{\Gamma\Rightarrow}\}_{\Gamma \in W}]$ is a premise of (r).

[1]Convergent rules were introduced in [11] before semianchored rules

Note that both conditions (i) and (ii) are trivially satisfied if $G \mid S_i$ is a pivot premise itself, by letting σ be the identity function. In the following we show that convergent rules are a proper subclass of semianchored rules, roughly corresponding to the Condition 1 in Definition 4.1.3

Lemma 4.2.2. *Every convergent rule is semianchored.*

Proof. Let (r) be a convergent rule and $\{(\Gamma_1, \Pi_i), \ldots, (\Gamma_n, \Pi_i)\}$ be any set of unanchored pairs contained in a premise of (r). For simplicity, assume $S_i = \Theta, \Gamma_1, \ldots \Gamma_n, \Sigma_i \Rightarrow \Pi_i$. We show that Condition 1 in Definition 4.1.3 is satisfied. As (r) is convergent, we have a map $\sigma \colon V \to V$ such that:

(i) $G \mid S_i[\{^{\sigma(\Gamma)\Rightarrow}/_{\overline{\Gamma}\Rightarrow}\}_{\Gamma \in V}] = G \mid \sigma(\Theta), \sigma(\Gamma_1), \cdots, \sigma(\Gamma_n), \sigma(\Sigma_i) \Rightarrow \Pi_i$ is a pivot premise.

(ii) $G \mid \Theta, \sigma(\Gamma_1), \ldots, \sigma(\Gamma_n), \Sigma_i \Rightarrow \Pi_i$ is a premise of (r).

By (i) and the definition of pivot premise, all the metavariables $\sigma(\Gamma_1), \cdots, \sigma(\Gamma_n)$ and Π_i belong to a component of the conclusion. Hence $(\sigma(\Gamma_1), \Pi_i), \ldots, (\sigma(\Gamma_n), \Pi_i)$ are all anchored pairs and the premise in (ii) satisfies Condition 1 of Definition 4.1.3.

□

The converse of Lemma 4.2.2 does not hold, as we see in what follows.

Example 4.2.3. *The rules (Ω_n) and (com) (see Table 3.3) are semianchored but not convergent, as they do not contain any pivot premise.*

CHAPTER 5

Axiomatic extensions of UL∀

We provide general sufficient conditions for standard completeness for a large class of acyclic $\mathcal{N}_2$-extensions of UL∀. This leads to a *uniform* proof of standard completeness, which applies to all $\mathcal{N}_2$-extensions of UL∀ already known to be standard complete and also to infinitely many new ones. This chapter is based on [7, 10].

5.1 Density elimination and nonlinear rules

Recall that any acyclic $\mathcal{N}_2$-extension L∀ of UL∀ has an analytic calculus HL∀. Moreover, by Theorem 3.4.7, showing density elimination for the extension $\mathrm{HL}\forall^{\mathrm{D}}$ of HL∀ with (D) suffices to obtain standard completeness for L∀. Density elimination will be proven for any extension of $\mathrm{HUL}\forall^{\mathrm{D}}$ with a class of internal structural rules – called *nonlinear* – which correspond to a subclass of axioms in $\mathcal{N}_2$ (nonlinear axioms). Recall that the calculus HUL∀ does not contain the following rules, which were crucial for the results in Chapter 4:

$$\frac{G\,|\,\Gamma \Rightarrow \Pi}{G\,|\,\Gamma, \alpha \Rightarrow \Pi}\ (wl) \qquad \frac{G\,|\,\Gamma \Rightarrow}{G\,|\,\Gamma \Rightarrow \Pi}\ (wr)$$

Their absence makes the proofs here more difficult. Indeed, standard completeness has been shown so far only for few axiomatic extensions of UL∀, see e.g. [43, 66, 81], most of which fall within the class $\mathcal{N}_2$. We start by giving the normal form of the axioms in the class $\mathcal{N}_2$, adapting Lemma 2.3.5 in the commutative case.

Lemma 5.1.1. *Any formula in $\mathcal{N}_2$ is equivalent over FL_e to a formula $\alpha = \bigwedge_{1\leq i\leq n}\delta_i$, where every δ_i is of the form $\alpha_1\cdots\alpha_m \to \beta$ and*

- *$\beta = f$ or $\beta_1 \vee \cdots \vee \beta_k$ and each β_l is a multiplicative conjunction of propositional variables or e.*
- *$\alpha_i = \bigwedge_{1\leq j\leq p}\gamma_i^j \to \beta_i^j$ where $\beta_i^j = f$ or propositional variable, and γ_i^j is a multiplicative conjunction or disjunction of propositional variables (or e).*

We can give now the definition of acyclic axioms and rules.

Definition 5.1.2.

- *Let α be any acyclic $\mathcal{N}_2$ axiom, which has the normal form in Lemma 5.1.1. α is said to be* nonlinear *if no propositional variable appears only once in any β_l, γ_i^j.*

- *Let (r) be any internal analytic rule*

$$\frac{G \mid S_1 \quad \ldots G \mid S_m}{G \mid \Sigma, \Gamma_1, \ldots, \Gamma_n \Rightarrow \Psi} \ (r)$$

  (r) is said to be nonlinear *if, for each premise $G \mid S_i$ such that $R(S_i) \neq \emptyset$, none of the multisets $\Gamma_1, \ldots, \Gamma_n$ appears only once in $L(S_i)$.*

In the lemma below, we show that nonlinear axioms correspond to nonlinear rules.

Lemma 5.1.3. *Let α be any acyclic nonlinear $\mathcal{N}_2$ axiom. The algorithm in [25] transforms α into a nonlinear analytic rule (r) such that $\vdash_{\mathrm{HUL}\forall+(r)} H$ iff $\vdash_{\mathrm{UL}\forall+\alpha} I(H)$, for any hypersequent H.*

Proof. By Lemma 3.3.12, we only need to show that (r) is nonlinear. We provide a sketch of the algorithm below, which is analogous to that in Theorem 2.3.9 and 2.3.13. First, as in Theorem 2.3.9 we transform the axiom into structural rules. We apply backwards the invertible propositional logical rules of HUL$\forall$ (i.e. (el), (fr), $(\cdot l)$, $(\wedge, r)$, $(\vee l)$ and $(\rightarrow r)$) as much as possible and we then use Ackermann's lemma (see [26] or Lemma 2.3.8), which proof-theoretically asserts (using (id) and (cut)) the interderivability of the rule

$$\frac{S_1 \quad \cdots \quad S_m}{\psi_1, \ldots, \psi_n \Rightarrow \xi} \ (r')$$

and of each of the rules

$$\frac{\vec{S} \quad \Lambda_1 \Rightarrow \psi_1 \quad \cdots \quad \Lambda_n \Rightarrow \psi_n}{\Lambda_1, \ldots, \Lambda_n \Rightarrow \xi} \ (r_1) \qquad \frac{\vec{S} \quad \xi, \Sigma \Rightarrow \Pi}{\psi_1, \ldots, \psi_n, \Sigma \Rightarrow \Pi} \ (r_2)$$

where $\vec{S} = S_1 \ \cdots \ S_m$ and $\Lambda_1, \ldots, \Lambda_n, \Sigma$ are fresh metavariables for multisets of formulas and Π is either a formula or the empty set. Let $\alpha = \bigwedge_{1 \leq i \leq n} \delta_i$ be any nonlinear axiom. We start with $\Rightarrow \alpha$ and, applying backwards $(\wedge r)$, we get n sequents $\Rightarrow \delta_i$ that, as shown below, all give rise to structural rules satisfying nonlinearity. Let δ_i be $\alpha_1 \cdots \alpha_m \rightarrow \beta$ as in Definition 5.1.2. By applying backwards $(\rightarrow r)$ and $(\cdot l)$ we get $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$. Assume that $\beta \neq f$ (as otherwise we can simply remove f by (fr)), Ackermann's lemma and subsequent applications of the invertible rules give

$$\frac{\{\beta_j, \Sigma \Rightarrow \Pi\}_{j=1,\ldots,k} \quad \{\Gamma_i, \gamma_i^j \Rightarrow \beta_i^j\}_{j=1,\ldots,p}}{\Gamma_1 \cdots \Gamma_m, \Sigma \Rightarrow \Pi}$$

By applying backwards the invertible rules for e and f, together with $(\cdot l)$, to all β_j, and $(\vee l)$ and $(\cdot l)$ to all γ_i^j, we remove all connectives and constants from the premises. Now we conclude the transformation procedure as in the proof of Theorem 2.3.13. If there are propositional variables that appear in the premises only on the same side, then the premises containing these variables are simply removed. We apply (cut) to the remaining premises. Since the propositional variables on the left hand side of the premises appear all with multiplicities (being α a nonlinear axiom), the resulting rule satisfies nonlinearity. □

Note that the rules obtained by the algorithm sketched in Lemma 5.1.3 are sequent rules. In the following we will consider their hypersequent version, which is simply obtained by adding a hypersequent context G to the premises and the conclusion.

Example 5.1.4. *The rules*

$$\frac{G_1 \mid \Gamma_1^n, \Sigma \Rightarrow \Pi \ldots G_1 \mid \Gamma_k^n, \Sigma \Rightarrow \Pi}{G_1 \mid \Gamma_1, \ldots, \Gamma_k, \Sigma \Rightarrow \Pi} \; (knot_k^n)$$

for $n > 1$ (and corresponding axioms) are nonlinear. (c) and $(fknot_k^n)$, for $n > 1$, in Table 3.2 are nonlinear as well. This is not the case of

$$\frac{G_1 \mid \Gamma_1, \Sigma \Rightarrow \Pi \quad G_1 \mid \Gamma_2, \Sigma \Rightarrow \Pi}{G_1 \mid \Gamma_1, \Gamma_2, \Sigma \Rightarrow \Pi} \; (mgl)$$

and of any rule $(knot_k^1)$.

We now prove density elimination for any hypersequent calculus extending $\mathrm{HUL}\forall^{\mathrm{D}}$ with nonlinear rules and possibly (mgl). Recall that in Chapter 4 we applied the method of density elimination by substitution for extensions of MTL∀, i.e. for calculi containing weakening. The absence of weakening here makes things more complicated. Assume indeed that we have a derivation d ending in an application of (D)

$$\frac{\begin{array}{c}\vdots\, d \\ G \mid \Lambda \Rightarrow p \mid p \Rightarrow \Pi\end{array}}{G \mid \Lambda \Rightarrow \Pi} \; (D)$$

By Lemma 5.1.3 we can safely assume d to be cut-free. As in Chapter 4, we may think of removing the application of (D) simply by substituting all occurrences of p in d in an "asymmetric" way, i.e. applying the substitution $S[^{\Lambda \Rightarrow \Pi}/_{\overline{p} \Rightarrow p}]$ (see Notation 3.3.6) to every sequent S in d. The above application of (D) would then be replaced with (ec).

The communication rule [28]:

A problematic case may arise however in the following application of (com):

$$\frac{\begin{array}{c}\vdots\, d_1 \\ G \mid \Gamma_1, \Gamma_2 \Rightarrow \Psi\end{array} \quad \begin{array}{c}\vdots\, d_2 \\ G \mid \Sigma, p \Rightarrow p\end{array}}{G \mid \Gamma_1, \Sigma \Rightarrow p \mid \Gamma_2, p \Rightarrow \Psi} \; (com)$$

For calculi with weakening, we can just discard the premise $G \mid \Sigma, p \Rightarrow p$ and then restructure the derivation above as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, d_1\\ G \mid \Gamma_1, \Gamma_2 \Rightarrow \Psi\end{array}}{G \mid \Gamma_1 \Rightarrow \Pi \mid \Gamma_2, \Lambda \Rightarrow \Psi}\,(split_d)}{G \mid \Gamma_1, \Sigma \Rightarrow \Pi \mid \Gamma_2, \Lambda \Rightarrow \Psi}\,(wl)$$

where $(split_d)$ is an application of Lemma 4.1.10. However, this is no longer possible for calculi without weakening, as we do not have any way to recover the multiset Σ, if we discard the hypersequent $G \mid \Sigma, p \Rightarrow p$. The solution of [28] is to replace each component below left by the component below right:

$$\Sigma, p^k \Rightarrow p \quad \rightsquigarrow \quad \Sigma, \Lambda^{k-1} \Rightarrow e$$

i.e., using Notation 3.3.6, we replace $\Sigma, p^k \Rightarrow p$ with $(\Sigma, p^k \Rightarrow p[{}^{\Rightarrow e}/_{p \Rightarrow p}])[{}^{\Lambda \Rightarrow}/_{\overline{p} \Rightarrow}]$. Note that the axiom $p \Rightarrow p$ is thus replaced by the axiom $\Rightarrow e$. We then perform the usual asymmetric substitution $S[{}^{\Lambda \Rightarrow \Pi}/_{\overline{p} \Rightarrow p}]$ to each component S of the derivation that is not a *pp*-component. We add suitable cuts and applications of the Lemma 4.1.10 to handle the problematic applications of (com). For example, the application of (com) in page 63 would be replaced by the following (by d_2^* we mean the derivation d_2 where the substitution sketched above has been applied to every sequent)

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\, d_1\\ G \mid \Gamma_1, \Gamma_2 \Rightarrow \Psi\end{array}}{G \mid \Gamma_1, \Gamma_2, e \Rightarrow \Psi}\,(el) \qquad \begin{array}{c}\vdots\, d_2^*\\ G \mid \Sigma \Rightarrow e\end{array}}{G \mid \Gamma_1, \Gamma_2, \Sigma \Rightarrow \Psi}\,(cut)}{G \mid \Gamma_1, \Sigma \Rightarrow \Pi \mid \Gamma_2, \Lambda \Rightarrow \Psi}\,(split_d)$$

This method was introduced in [28] and applied to extensions of HUL∀ with *balanced* rules, i.e. internal structural rules for which the number of occurrences of metavariables in the premises and in the conclusion is the same. Note that important rules, such as contraction (c) and mingle (mgl)

$$\dfrac{G \mid \Sigma_1, \Gamma_1, \Gamma_1 \Rightarrow \Pi_1}{G \mid \Sigma_1, \Gamma_1 \Rightarrow \Pi_1}\,(c) \qquad \dfrac{G \mid \Sigma_1, \Gamma_1 \Rightarrow \Pi_1 \quad G \mid \Sigma_1, \Gamma_2 \Rightarrow \Pi_1}{G \mid \Sigma_1, \Gamma_1, \Gamma_2 \Rightarrow \Pi_1}\,(mgl)$$

are not balanced, hence the method of density elimination by substitution cannot be directly applied.

A new idea

To illustrate the idea for handling rules more complicated than the balanced rules in [28], we consider the case of contraction (c). Assume that the derivation d (below left) contains an application of (c). After the substitution, we get the derivation below right, where the "incorrect" application of (c) is marked with (?).

$$
\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\vdots}}{\Sigma, p, p \Rightarrow p}}{\dfrac{\Sigma, p \Rightarrow p}{\vdots}}\,(c)
\qquad
\dfrac{\dfrac{\Rightarrow e}{\vdots}}{\Sigma, \Lambda \Rightarrow e}\!\!\!\dfrac{}{\Sigma \Rightarrow e}\,(?)
$$

$$
\frac{G \,|\, \Lambda \Rightarrow p \,|\, p \Rightarrow \Pi}{G \,|\, \Lambda \Rightarrow \Pi}\,(D)
\qquad
\frac{G \,|\, \Lambda \Rightarrow \Pi \,|\, \Lambda \Rightarrow \Pi}{G \,|\, \Lambda \Rightarrow \Pi}\,(ec)
$$

To solve the problem, we look back at the whole original derivation d, with the additional knowledge that $\Sigma, \Lambda \Rightarrow e$ is derivable (this is always the case if we start the proof transformation from the uppermost application of (c) in d). The idea is to apply a new substitution to d, in such a way that an axiom $p \Rightarrow p$ is still replaced with something derivable. This time, instead of $\Rightarrow e$, we let the sequent $\Sigma, \Lambda \Rightarrow e$ do the job. More precisely, we apply the following substitution to the whole derivation d: we replace each component below left by the component below right:

$$\Theta, p^k \Rightarrow p \quad \rightsquigarrow \quad \Theta, \Lambda^{k-1}, \Sigma, \Lambda \Rightarrow e.$$

In the other sequents we replace any p occurring on the left with Λ and any p occurring on the right with Π. As we will show below in Lemma 5.1.6, this new substitution eventually leads to a derivation d_1 of $G \,|\, \Lambda \Rightarrow \Pi \,|\, \Sigma, \Lambda, \Lambda \Rightarrow \Pi$. The $(?)$ above is then replaced by the subderivation

$$
\dfrac{\dfrac{\dfrac{\Sigma, \Lambda \Rightarrow e}{\vdots\; d_1}}{G \,|\, \Lambda \Rightarrow \Pi \,|\, \Sigma, \Lambda, \Lambda \Rightarrow \Pi}\!\!\!\dfrac{}{G \,|\, \Lambda \Rightarrow \Pi \,|\, \Sigma, \Lambda \Rightarrow \Pi}\,(c) \qquad \Rightarrow e}{G \,|\, \Lambda \Rightarrow \Pi \,|\, \Lambda \Rightarrow \Pi \,|\, \Sigma \Rightarrow e}\,(com)
$$

Note that the additional components $G \,|\, \Lambda \Rightarrow \Pi$ can be removed by applications of (ec) at the end of our restructured derivation. A similar procedure can be applied in a uniform way to any nonlinear rule, and hence to almost all $\mathcal{N}_2$-extensions of UL∀ considered in the literature (see [7, 66, 70]). A simple variant of the method used for the nonlinear rules applies also to the rule (mgl), which is not nonlinear. Consider, for instance, the following:

$$
\dfrac{\dfrac{p \Rightarrow p}{\vdots}}{\Sigma, p \Rightarrow p}\!\!\!\dfrac{}{\dfrac{\Sigma, p, p \Rightarrow p}{\vdots}}\,(mgl)
\qquad
\dfrac{\dfrac{\Rightarrow e}{\vdots}}{\Sigma \Rightarrow e}\!\!\!\dfrac{}{\dfrac{\Sigma, \Lambda \Rightarrow e}{\vdots}}\,(?)
$$

$$
\frac{G \,|\, \Lambda \Rightarrow p \,|\, p \Rightarrow \Pi}{G \,|\, \Lambda \Rightarrow \Pi}\,(D)
\qquad
\frac{G \,|\, \Lambda \Rightarrow \Pi \,|\, \Lambda \Rightarrow \Pi}{G \,|\, \Lambda \Rightarrow \Pi}\,(ec)
$$

where, for space reasons, in the application of (mgl) the single premise actually stands for two identical premises. We perform a new substitution in d: this time we replace each component below left with the component below right:

$$\Theta, p^k \Rightarrow p \quad \rightsquigarrow \quad \Theta, \Sigma \Rightarrow \Pi.$$

In the other sequents we replace any p occurring on the left with Λ and any p occurring on the right with Π. As we will see in Lemma 5.1.7, this new substitution eventually leads to a derivation d_1 of $G \,|\, \Lambda \Rightarrow \Pi \,|\, \Sigma, \Lambda \Rightarrow \Pi$, which can be used to replace the $(?)$ above. The idea that we have sketched for nonlinear rules and (mgl) is formalized in the following. Henceforth, whenever we say that a premise or the conclusion of a rule contains (resp. do not contain) *pp*-components, we refer only to their *active* components.

Theorem 5.1.5. *Let* $\mathrm{HL}\forall$ *be any hypersequent calculus extending* $\mathrm{HUL}\forall$ *with any nonlinear rule and possibly* (mgl). *The calculus* $\mathrm{HL}\forall^{\mathrm{D}}$ *admits density elimination.*

Proof. Let d be a cut-free derivation in $\mathrm{HL}\forall^{\mathrm{D}}$ ending in a uppermost application of (D) as follows:

$$\dfrac{\begin{array}{c}\vdots\\ G \,|\, \Lambda \Rightarrow p \,|\, p \Rightarrow \Pi\end{array}}{G \,|\, \Lambda \Rightarrow \Pi}\ (D)$$

We show that we can obtain a (D)-free derivation of $G \,|\, \Lambda \Rightarrow \Pi$. Let H be a hypersequent $H = S_1 \,|\, \ldots \,|\, S_n$. We apply a different substitution to any component S_i, depending on whether S_i is a *pp*-component or not. More precisely, we define H^* as $S_1^* \,|\, \ldots \,|\, S_n^*$, where for each $i = 1, \ldots, n$

- $S_i^* = (S_i[{}^{\Rightarrow e}/_{p \Rightarrow p}])[{}^{\Lambda \Rightarrow}/_{\overline{p} \Rightarrow}]$, if S_i is a *pp*-component.
- $S_i^* = S_i[{}^{\Lambda \Rightarrow \Pi}/_{\overline{p} \Rightarrow p}]$ otherwise.

We prove the following:

Claim : *For each hypersequent H in d we can find a (D)-free derivation of* $G \,|\, \Lambda \Rightarrow \Pi \,|\, H^*$.

Density elimination follows by applying the claim to the hypersequent $G \,|\, \Lambda \Rightarrow p \,|\, p \Rightarrow \Pi$. We have indeed $G \,|\, \Lambda \Rightarrow \Pi \,|\, (G \,|\, \Lambda \Rightarrow p \,|\, p \Rightarrow \Pi)^* = G \,|\, \Lambda \Rightarrow \Pi \,|\, G \,|\, \Lambda \Rightarrow \Pi \,|\, \Lambda \Rightarrow \Pi$ (observe that $G^* = G$ by the eigenvariable condition on p). The desired hypersequent is obtained by $(\overline{ec})$.

We now prove the claim by induction on the length of the derivation of H in d. If H is the axiom $p \Rightarrow p$, we derive $G \,|\, \Lambda \Rightarrow \Pi \,|\, H^* = G \,|\, \Lambda \Rightarrow \Pi \,|\, \Rightarrow e$ by applying $(\overline{ew})$ to the axiom $\Rightarrow e$. Other axioms, (ec) and (ew) are easy to handle.

Logical rules. Logical rules (but $(\rightarrow l)$) are easy to handle (see [28]), as we can have a *pp*-component in the conclusion if and only if all the premises contain *pp*-components. For $(\rightarrow l)$ we can have instead an application with a *pp*-component in the conclusion and no *pp*-components in the premises, e.g. as the following:

$$\dfrac{G_1 \,|\, \Gamma_1, p \Rightarrow \varphi \quad G_1 \,|\, \Gamma_2, \chi \Rightarrow p}{G_1 \,|\, \Gamma_1, \Gamma_2, p, \varphi \rightarrow \chi \Rightarrow p}$$

By induction hypothesis, we have derivations of $G \,|\, \Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Gamma_1, \Lambda \Rightarrow \varphi$ and $G \,|\, \Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Gamma_2, \chi \Rightarrow \Pi$. The desired derivation is obtained as follows:

$$\dfrac{\dfrac{G \,|\, \Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Gamma_1, \Lambda \Rightarrow \varphi \quad G \,|\, \Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Gamma_2, \chi \Rightarrow \Pi}{G \,|\, \Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Gamma_1, \Gamma_2, \Lambda, \varphi \to \chi \Rightarrow \Pi \qquad\qquad \Rightarrow e}}{\dfrac{G \,|\, \Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Gamma_1, \Gamma_2, \varphi \to \chi \Rightarrow e \,|\, \Lambda \Rightarrow \Pi}{G \,|\, \Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Gamma_1, \Gamma_2, \varphi \to \chi \Rightarrow e} \, (ec)}$$

The communication rule. We recall the case of (com) (taken from [28]). The different possibilities that we need to check are displayed in Figure 5.1.

1. The number of pp-components in the premises and in the conclusion is the same.
2. The premises contain more pp-components than the conclusion. We distinguish the subcases:
   (a) There is one pp-component in the premises and no pp-component in the conclusion.
   (b) There are two pp-components in the premises and one pp-component in the conclusion.
3. The premises contain less pp-components than the conclusion. We distinguish the subcases:
   (a) There is no pp-component in the premises and one pp-component in the conclusion.
   (b) There is one pp-component in the premises and two pp-components in the conclusion.

Figure 5.1: Cases of (com)

Case (1) is handled by simply applying the induction hypothesis and then (com). We recall cases $(2a)$ and $(3a)$ from [28]. The cases $(2b)$ and $(3b)$ are similar to $(2a)$ and $(3a)$, respectively.

$(2a)$ Assume that we have an application of (com) of the form

$$\dfrac{G_1 \,|\, \Gamma_1, \Gamma_2, p \Rightarrow p \quad G_1 \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Delta_1}{G_1 \,|\, \Gamma_1, \Sigma_1, p \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2 \Rightarrow p} \, (com)$$

Our aim is to get a (D)-free derivation of $G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Gamma_1, \Sigma_1, \Lambda \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi$. By induction hypothesis we have derivations of $G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Delta_1$ and $G|\Lambda \Rightarrow$

$\Pi \,|\, G_1^* \,|\, \Gamma_1, \Gamma_2 \Rightarrow e$. We proceed as follows:

$$\dfrac{\dfrac{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Delta_1}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, e, \Sigma_1, \Sigma_2 \Rightarrow \Delta_1}\,(el) \qquad G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Gamma_1, \Gamma_2 \Rightarrow e}{\dfrac{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Gamma_1, \Gamma_2, \Sigma_1, \Sigma_2 \Rightarrow \Delta_1}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Gamma_1, \Sigma_1, \Lambda \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi}\,(split_d)}\,(cut)$$

$(3a)$ Assume that we have an application of (com) of the form

$$\dfrac{G_1 \,|\, \Gamma_1, \Gamma_2, p \Rightarrow \Delta_1 \quad G_1 \,|\, \Sigma_1, \Sigma_2 \Rightarrow p}{G_1 \,|\, \Gamma_1, \Sigma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2, p \Rightarrow p}\,(com)$$

Our aim is to get a (D)-free derivation of $G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Gamma_1, \Sigma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2 \Rightarrow e$. By induction hypothesis we have derivations of $G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Gamma_1, \Gamma_2, \Lambda \Rightarrow \Delta_1$ and $G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Pi$. We obtain the desired hypersequent as follows:

$$\dfrac{\dfrac{\dfrac{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Gamma_1, \Gamma_2, \Lambda \Rightarrow \Delta_1 \quad G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Pi}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Gamma_1, \Sigma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2, \Lambda \Rightarrow \Pi}\,(com) \qquad \Rightarrow e}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Gamma_1, \Sigma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2 \Rightarrow e \,|\, \Lambda \Rightarrow \Pi}\,(com)}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Gamma_1, \Sigma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2 \Rightarrow e}\,(ec)$$

Nonlinear rules. Assume now that the last rule (r) applied in d to derive H is a nonlinear rule. In case the conclusion does not contain a *pp*-component, by the strong subformula property and coupling (see Definition 3.3.9), none of the premises contains a *pp*-component as well. The claim then simply follows by the induction hypothesis and an application of (r). Let us now consider the case where the conclusion contains a *pp*-component, i.e. the application of (r) has the form

$$\dfrac{G_1 \,|\, S_1 \quad \ldots \quad G_1 \,|\, S_m}{G_1 \,|\, \Sigma, \Gamma_1, \ldots, \Gamma_n, p^k \Rightarrow p}\,(r)$$

where Σ is the instantiation of the metavariable which witnesses the coupling property for (r) (see Definition 3.3.9). We distinguish the following cases:

1. None of the premises contains a *pp*-component.
2. Some premises contain a *pp*-component and Σ is instantiated with at least one p.
3. Some premises contain a *pp*-component and Σ is not instantiated with any p.

Figure 5.2: Cases for a nonlinear rule (r)

In case (1), we simply apply the induction hypothesis and consider the following derivation:

$$\dfrac{\dfrac{\dfrac{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, S_1^* \quad \dots \quad G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, S_m^*}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1, \dots, \Gamma_n, \Lambda^k \Rightarrow \Pi}\,(r) \qquad \Rightarrow e}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1, \dots, \Gamma_n, \Lambda^{k-1} \Rightarrow e \,|\, \Lambda \Rightarrow \Pi}\,(com)}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1, \dots, \Gamma_n, \Lambda^{k-1} \Rightarrow e}\,(ec)$$

In case (2), by the strong subformula property and coupling, all the premises $G_1 \,|\, S_i$ of (r) are either of the form $G_1 \,|\, \Theta_i, p^{n_i} \Rightarrow p$, for $n_i \geq 1$ or $G_1 \,|\, \Theta_i, p^{n_i} \Rightarrow$, for $n_i \geq 0$. Consequently any S_i^* will be either of the form $\Theta_i, \Lambda^{n_i - 1} \Rightarrow e$ or $\Theta_i, \Lambda^{n_i-1} \Rightarrow$. The claim just follows by applying the rule (r) to all the $G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, S_i^*$ that we obtain by the induction hypothesis.

In case (3) we can assume that the S_i are as follows:

- $\Sigma, \Theta_i, p^{n_i} \Rightarrow p$, say the S_i's for $i \in \{1, \dots, q\}$ with $q \leq m$, $n_i \geq 2$.
- $\Sigma, \Theta_i \Rightarrow p$, say the S_i's for $i \in \{q+1, \dots, r\}$ with $r \leq m$.
- $\Theta_i, p^{n_i} \Rightarrow$, say for the S_i 's, for $i \in \{r+1, \dots, m\}$, $n_i \geq 0$.

Note that, by nonlinearity (see Definition 5.1.3), whenever S_i is a pp-component, the number n_i of ps appearing on the left hand side is greater or equal then 2. By the induction hypothesis, we have derivations of $G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, S_1^*, \dots, G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, S_m^*$, which cannot be used as premises of (r). Note that, for any premise of the form

$$G_1 \,|\, \Sigma, \Theta_i, p^{n_i} \Rightarrow p$$

we have by induction hypothesis a derivation of

$$G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Theta_i, \Lambda^{n_i - 1} \Rightarrow e.$$

We apply Lemma 5.1.6 below to the latter, to obtain a derivation of

$$G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Theta_i, \Lambda^{n_i} \Rightarrow \Pi.$$

Hence, we have:

- for any premise $G_1 \,|\, \Sigma, \Theta_i, p^{n_i} \Rightarrow p$, a derivation of $G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Theta_i, \Lambda^{n_i} \Rightarrow \Pi$;
- for any premise $G_1 \,|\, \Sigma, \Theta_i \Rightarrow p$, a derivation of $G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Theta_i \Rightarrow \Pi$;
- for any premise $G_1 \,|\, \Theta_i, p^{n_i} \Rightarrow$, a derivation of $G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Theta_i, \Lambda^{n_i} \Rightarrow$.

We then apply the rule (r) as follows:

$$\dfrac{\dfrac{\dfrac{\{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Theta_i, \Lambda^{n_i} \Rightarrow \Pi\}_{i=1,\dots,q} \quad \{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, S_i^*\}_{i=q+1,\dots,m}}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1, \dots, \Gamma_n, \Lambda^k \Rightarrow \Pi}\,(r) \qquad \Rightarrow e}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1, \dots, \Gamma_n, \Lambda^{k-1} \Rightarrow e \,|\, \Lambda \Rightarrow \Pi}\,(com)}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1, \dots, \Gamma_n, \Lambda^{k-1} \Rightarrow e}\,(ec)$$

The above application of (r) is correct, being nothing more (apart from the hypersequent context) than the original rule application in which every premise S_i is replaced by $S_i[^{\Lambda\Rightarrow\Pi}/_{\overline{p}\Rightarrow p}]$.

The mingle rule. Assume now that the last rule applied in d to derive H is (mgl). If the conclusion of (mgl) does not contain a *pp*-component, by the strong subformula property and coupling (see Definition 3.3.9) none of the premises contains a *pp*-component. In case the conclusion of (mgl) contains a *pp*-component, two subcases can occur: both premises of (mgl) contain a *pp*-component or only one does. The latter case can be reduced to the former. Indeed, assume that we have:

$$\frac{G_1 \,|\, \Sigma, \Gamma_1 \Rightarrow p \quad G_1 \,|\, \Sigma, \Gamma_2, p \Rightarrow p}{G_1 \,|\, \Sigma, \Gamma_1, \Gamma_2, p \Rightarrow p}\,(mgl)$$

We can replace this application by the following derivation:

$$\dfrac{\dfrac{G_1 \,|\, \Sigma, \Gamma_1 \Rightarrow p \quad G_1 \,|\, \Sigma, \Gamma_1 \Rightarrow p}{G_1 \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow p}\,(mgl) \qquad \dfrac{G_1 \,|\, \Sigma, \Gamma_2, p \Rightarrow p \quad G_1 \,|\, \Sigma, \Gamma_2, p \Rightarrow p}{G_1 \,|\, \Sigma, \Gamma_2, \Gamma_2, p, p \Rightarrow p}\,(mgl)}{\dfrac{G_1 \,|\, \Sigma, \Gamma_1, \Gamma_2, p \Rightarrow p \,|\, \Sigma, \Gamma_1, \Gamma_2, p \Rightarrow p}{G_1 \,|\, \Sigma, \Gamma_1, \Gamma_2, p \Rightarrow p}\,(ec)}\,(com)$$

where, in the only application of (mgl) with a *pp*-component in the conclusion, both premises contain *pp*-components. We can now assume that both premises of an application of (mgl) contain a *pp*-component, e.g. as in

$$\frac{G_1 \,|\, \Sigma, \Gamma_1, p \Rightarrow p \quad G_1 \,|\, \Sigma, \Gamma_2, p \Rightarrow p}{G_1 \,|\, \Sigma, \Gamma_1, \Gamma_2, p, p \Rightarrow p}\,(mgl)$$

The induction hypothesis provides us with the derivations $G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1 \Rightarrow e$ and $G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_2 \Rightarrow e$. Consider the following derivation d_1:

$$\dfrac{\dfrac{\dfrac{\dfrac{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1 \Rightarrow e \quad G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1 \Rightarrow e}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma^2, \Gamma_1^2 \Rightarrow e}\,(mgl)}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1, \Lambda \Rightarrow e \,|\, \Sigma, \Gamma_1 \Rightarrow \Pi}\,(split_d)}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1, \Lambda \Rightarrow e \,|\, \Sigma, \Gamma_1, \Lambda \Rightarrow \Pi}\,(*) \qquad G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_2 \Rightarrow e}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1, \Gamma_2, \Lambda \Rightarrow e \,|\, \Sigma, \Gamma_1, \Lambda \Rightarrow \Pi}\,(mgl)$$

where $(*)$ stands for an application of Lemma 5.1.7 below. Similarly, we obtain the following derivation d_2:

$$\dfrac{\dfrac{\dfrac{\dfrac{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_2 \Rightarrow e \quad G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_2 \Rightarrow e}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma^2, \Gamma_2^2 \Rightarrow e}\,(mgl)}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_2, \Lambda \Rightarrow e \,|\, \Sigma, \Gamma_2 \Rightarrow \Pi}\,(split_d)}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_2, \Lambda \Rightarrow e \,|\, \Sigma, \Gamma_2, \Lambda \Rightarrow \Pi}\,(*) \qquad G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1 \Rightarrow e}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1, \Gamma_2, \Lambda \Rightarrow e \,|\, \Sigma, \Gamma_2, \Lambda \Rightarrow \Pi}\,(mgl)$$

Hence we have:

$$\dfrac{\dfrac{\vdots\, d_1}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1, \Gamma_2, \Lambda \Rightarrow e \,|\, \Sigma, \Gamma_1, \Lambda \Rightarrow \Pi} \qquad \dfrac{\vdots\, d_2}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1, \Gamma_2, \Lambda \Rightarrow e \,|\, \Sigma, \Gamma_2, \Lambda \Rightarrow \Pi}}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1, \Gamma_2, \Lambda \Rightarrow e \,|\, \Sigma, \Gamma_1, \Gamma_2, \Lambda^2 \Rightarrow \Pi}\,(mgl)$$

The desired hypersequent is finally obtained as follows:

$$\dfrac{\dfrac{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1, \Gamma_2, \Lambda \Rightarrow e \,|\, \Sigma, \Gamma_1, \Gamma_2, \Lambda^2 \Rightarrow \Pi \quad \Rightarrow e}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1, \Gamma_2, \Lambda \Rightarrow e \,|\, \Sigma, \Gamma_1, \Gamma_2, \Lambda \Rightarrow e \,|\, \Lambda \Rightarrow \Pi}\,(com)}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1, \Gamma_2, \Lambda \Rightarrow e}\,(\overline{ec})$$

□

We now prove the two technical lemmas that we have used in the proof of Theorem 5.1.5.

Lemma 5.1.6 (Nonlinear rules). *Let* HL∀ *and* d *be as in Theorem 5.1.5 and assume to have a derivation of a hypersequent* $G_1 \,|\, \Sigma, \Lambda^{k-1} \Rightarrow e$*, where* $k \geq 2$*, and no* p *appears. We can find a* (D)*-free derivation of the hypersequent* $H_1 = G|\Lambda \Rightarrow \Pi \,|\, G_1 \,|\, \Sigma, \Lambda^k \Rightarrow \Pi$.

Proof. Let H be a hypersequent $H = S_1 \,|\, \ldots \,|\, S_n$. We define H^{**} as $S_1^{**} \,|\, \ldots \,|\, S_n^{**}$, where for each $i = 1, \ldots, n$

- $S_i^{**} = (S_i[^{\Sigma,\Lambda^{k-1}\Rightarrow e}/_{p\Rightarrow p}])[^{\Lambda\Rightarrow}/_{\overline{p}\Rightarrow}]$ if S_i is a *pp*-component.
- $S_i^{**} = S_i[^{\Lambda\Rightarrow\Pi}/_{\overline{p}\Rightarrow p}]$ otherwise.

The statement of the lemma is a consequence of the following:

Claim: *For each hypersequent* H *in* d *we can find a* (D)*-free derivation of* $H_1 \,|\, H^{**}$

Indeed, applying the claim to the end-hypersequent of d we obtain a (D)-free derivation of $(G \,|\, \Lambda \Rightarrow p \,|\, p \Rightarrow \Pi)^{**} = H_1 \,|\, G^{**} \,|\, \Lambda \Rightarrow \Pi \,|\, \Lambda \Rightarrow \Pi$. Note that $G^{**} = G$ by the eigenvariable condition on (D) and $\Lambda \Rightarrow \Pi$ and G are components of H_1. Hence, we obtain the desired derivation of H_1 by applying $(\overline{ec})$.

We now prove the claim by induction on the length of the derivation of H in d. Let H be the axiom $p \Rightarrow p$. Recall that the hypersequent $G_1 \,|\, \Sigma, \Lambda^{k-1} \Rightarrow e$ is derivable by assumption. Hence, applying $(\overline{ew})$ to the latter, we obtain $H_1 \,|\, (p \Rightarrow p)^{**} = G|\Lambda \Rightarrow \Pi \,|\, G_1 \,|\, \Sigma, \Lambda^k \Rightarrow \Pi \,|\, \Sigma, \Lambda^{k-1} \Rightarrow e$. Other axioms, (ec) and (ew) are easy to handle.

Logical rules. Logical rules (but $(\to l)$) are easy to handle, as the conclusion contains a *pp*-component if and only if all the premises do. For $(\to l)$ we can have instead an application where the conclusion contains a *pp*-component and none of the premises does, e.g. as in the following:

$$\dfrac{G_2 \,|\, \Gamma_1, p \Rightarrow \varphi \quad G_2 \,|\, \Gamma_2, \chi \Rightarrow p}{G_2 \,|\, \Gamma_1, \Gamma_2, p, \varphi \to \chi \Rightarrow p}\,(\to l)$$

By induction hypothesis we have derivations of $H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Lambda \Rightarrow \varphi$ and $H_1 \,|\, G_2^{**} \,|\, \Gamma_2, \chi \Rightarrow \Pi$. The desired derivation is obtained as follows:

$$\dfrac{\dfrac{\dfrac{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Lambda \Rightarrow \varphi \quad H_1 \,|\, G_2^{**} \,|\, \Gamma_2, \chi \Rightarrow \Pi}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Lambda, \varphi \to \chi \Rightarrow \Pi}\,(\to l) \qquad G_1 \,|\, \Sigma, \Lambda^{k-1} \Rightarrow e}{H_1 \,|\, G_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \varphi \to \chi, \Sigma, \Lambda^{k-1} \Rightarrow e \,|\, \Lambda \Rightarrow \Pi}\,(com)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \varphi \to \chi, \Sigma, \Lambda^{k-1} \Rightarrow e}\,(\overline{ec})$$

where $G_1 \,|\, \Sigma, \Lambda^{k-1} \Rightarrow e$ is derivable by assumption and the application of $(\overline{ec})$ is justified by the fact that $\Lambda \Rightarrow \Pi$ and G_1 are components of H_1.

The communication rule. Assume now that the last applied rule in a derivation of H is (com). We follow the case distinction in Figure 5.1 as for the proof of Theorem 5.1.5. For case (1) we just apply the induction hypothesis and then the rule (com). Let us consider now the case $(2a)$ and assume that the pp-component in the premises contains only one p in its left-hand side (the case with more ps being easy to generalize, but confusing). We assume to have an application of (com) the form

$$\dfrac{G_2 \,|\, \Gamma_1, \Gamma_2, p \Rightarrow p \quad G_2 \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Delta_1}{G_2 \,|\, \Gamma_1, \Sigma_1, p \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2 \Rightarrow p}\,(com)$$

Our aim is to get a (D)-free derivation of $H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Lambda \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi$. By induction hypothesis we have derivations of $H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Delta_1$ and $H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Sigma, \Lambda^{k-1} \Rightarrow e$. The desired hypersequent is obtained as follows (the application of (ec) is justified as $\Sigma, \Lambda^k \Rightarrow \Pi$ is a component of H_1):

$$\dfrac{\dfrac{\dfrac{\dfrac{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Sigma, \Lambda^{k-1} \Rightarrow e \qquad \dfrac{H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Delta_1}{H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2, e \Rightarrow \Delta_1}\,(el)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Sigma_1, \Sigma_2, \Sigma, \Lambda^{k-1} \Rightarrow \Delta_1}\,(cut)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Sigma, \Lambda^k \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi}\,(split_d)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Lambda \Rightarrow \Delta_1 \,|\, \Sigma, \Lambda^k \Rightarrow \Pi \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi}\,(split_d)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Lambda \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi}\,(ec)$$

We consider now the similar case $(2b)$, i.e. an application of (com) of the form

$$\dfrac{G_2 \,|\, \Gamma_1, \Gamma_2, p \Rightarrow p \quad G_2 \,|\, \Sigma_1, \Sigma_2, p \Rightarrow p}{G_2 \,|\, \Gamma_1, \Sigma_1, p, p \Rightarrow p \,|\, \Gamma_2, \Sigma_2 \Rightarrow p}\,(com)$$

Our aim is to obtain a derivation of

$$H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Lambda, \Sigma, \Lambda^{k-1} \Rightarrow e \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi.$$

By induction hypothesis we have derivations of the hypersequents $H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2, \Sigma, \Lambda^{k-1} \Rightarrow e$ and $H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Sigma, \Lambda^{k-1} \Rightarrow e$. The desired hypersequent is obtained as follows (the application of (ec) is justified as $\Sigma, \Lambda^k \Rightarrow \Pi$ is a component of H_1):

$$\dfrac{\dfrac{\dfrac{\dfrac{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Sigma, \Lambda^{k-1} \Rightarrow e \qquad \dfrac{H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2, \Sigma, \Lambda^{k-1} \Rightarrow e}{H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2, e, \Sigma, \Lambda^{k-1} \Rightarrow e}\,(el)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Sigma_1, \Sigma_2, \Sigma, \Lambda^{k-1}, \Sigma, \Lambda^{k-1} \Rightarrow e}\,(cut)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Sigma, \Lambda^{k}, \Sigma, \Lambda^{k-1} \Rightarrow e \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi}\,(split_d)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Lambda, \Sigma, \Lambda^{k-1} \Rightarrow e \,|\, \Sigma, \Lambda^{k} \Rightarrow \Pi \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi}\,(split_d)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Lambda, \Sigma, \Lambda^{k-1} \Rightarrow e \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi}\,(ec)$$

Let us now consider the case $(3a)$. We assume that the pp-component in the conclusion contains only one p on its left hand side (the case with more ps being easy to generalize, but confusing). We assume to have an application of (com) of the form

$$\dfrac{G_2 \,|\, \Gamma_1, \Gamma_2, p \Rightarrow \Delta_1 \quad G_2 \,|\, \Sigma_1, \Sigma_2 \Rightarrow p}{G_2 \,|\, \Gamma_1, \Sigma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2, p \Rightarrow p}\,(com)$$

Our aim is to get a (D)-free derivation of $H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2, \Sigma, \Lambda^{k-1} \Rightarrow e$. By induction hypothesis we have derivations of $H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Lambda \Rightarrow \Delta_1$ and of $H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Pi$. We obtain the desired hypersequent as follows (the application of (ec) is justified as $\Lambda \Rightarrow \Pi$ is a component of H_1):

$$\dfrac{\dfrac{\dfrac{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Lambda \Rightarrow \Delta_1 \quad H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Pi}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2, \Lambda \Rightarrow \Pi}\,(com) \qquad H_1 \,|\, \Sigma, \Lambda^{k-1} \Rightarrow e}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2, \Sigma, \Lambda^{k-1} \Rightarrow e \,|\, \Lambda \Rightarrow \Pi}\,(com)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2, \Sigma, \Lambda^{k-1} \Rightarrow e}\,(ec)$$

Let us now consider the case $(3b)$, i.e. an application of (com) of the form

$$\dfrac{G_2 \,|\, \Gamma_1, \Gamma_2, p, p \Rightarrow p \quad G_2 \,|\, \Sigma_1, \Sigma_2 \Rightarrow p}{G_2 \,|\, \Gamma_1, \Sigma_1, p \Rightarrow p \,|\, \Gamma_2, \Sigma_2, p \Rightarrow p}\,(com)$$

Our aim is to get a (D)-free derivation of $H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Sigma, \Lambda^{k-1} \Rightarrow e \,|\, \Gamma_2, \Sigma_2, \Sigma, \Lambda^{k-1} \Rightarrow e$. By induction hypothesis we have derivations of $H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Lambda, \Sigma, \Lambda^{k-1} \Rightarrow e$ and $H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Pi$. We obtain the desired hypersequent as follows (the application of (ec) is justified as $\Lambda \Rightarrow \Pi$ is a component of H_1)

$$\dfrac{\dfrac{\dfrac{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Lambda, \Sigma, \Lambda^{k-1} \Rightarrow e \quad H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Pi}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Sigma, \Lambda^{k-1} \Rightarrow e \,|\, \Gamma_2, \Sigma_2, \Lambda \Rightarrow \Pi}\,(com) \qquad H_1 \,|\, \Sigma, \Lambda^{k-1} \Rightarrow e}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Sigma, \Lambda^{k-1} \Rightarrow e \,|\, \Gamma_2, \Sigma_2, \Sigma, \Lambda^{k-1} \Rightarrow e \,|\, \Lambda \Rightarrow \Pi}\,(com)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Sigma, \Lambda^{k-1} \Rightarrow e \,|\, \Gamma_2, \Sigma_2, \Sigma, \Lambda^{k-1} \Rightarrow e}\,(ec)$$

Nonlinear rules and mingle. Assume now that the last rule (r) applied in a derivation of H is a nonlinear rule or (mgl). If the conclusion does not contain a pp-component, by the strong subformula property (see Definition 3.3.9), we have that none of the premises contains a *pp*-component as well. Hence we can just apply the induction hypothesis and then the rule. Assume that the conclusion contains a *pp*-component, i.e. we have an application of the form

$$\frac{G_2 \mid P_1 \quad \dots \quad G_2 \mid P_m}{G_2 \mid \Sigma_1, \Theta_1, \dots, \Theta_n, p^s \Rightarrow p} \, (r)$$

where Σ_1 is the multiset witnessing the coupling property. By the strong subformula property (see Definition 3.3.9), each P_j can only have one of the following forms:

- $\Sigma_1, \Xi_j, p^{n_j} \Rightarrow p$, say the P_j 's for $j \in \{1, \dots, l\}$ with $l \leq m$, $n_j \geq 1$.
- $\Sigma_1, \Xi_j \Rightarrow p$, say the P_j's for $j \in \{l+1, \dots, r\}$ with $r \leq m$.
- $\Xi_j, p^{n_j} \Rightarrow$, say for the P_j 's, for $j \in \{r+1, \dots, m\}$, $n_j \geq 0$.

The induction hypothesis gives us derivations of $H_1 \mid G_2^{**} \mid P_1^{**}, \dots, H_1 \mid G_2^{**} \mid P_m^{**}$. In particular, for the first case, i.e. the $G_2 \mid P_j$'s with $j \in \{1, \dots, l\}$, we have derivations of

$$H_1 \mid G_2^{**} \mid \Sigma_1, \Xi_j, \Lambda^{m_j - 1}, \Sigma, \Lambda^{k-1} \Rightarrow e.$$

For the second case, the induction hypothesis applied to the premises of the form $G_2 \mid \Sigma_1, \Xi_j \Rightarrow p$ leads to derivations of

$$H_1 \mid G_2^{**} \mid \Sigma_1, \Xi_j \Rightarrow \Pi$$

From the latter, recalling that $H_1 \mid \Sigma, \Lambda^{k-1} \Rightarrow e$ is derivable and that $k \geq 2$ by assumption, we get:

$$\dfrac{\dfrac{H_1 \mid G_2^{**} \mid \Sigma_1, \Xi_j \Rightarrow \Pi \qquad H_1 \mid \Sigma, \Lambda^{k-1} \Rightarrow e}{H_1 \mid G_2^{**} \mid \Sigma_1, \Xi_j, \Sigma, \Lambda^{k-2} \Rightarrow e \mid \Lambda \Rightarrow \Pi} \, (com)}{H_1 \mid G_2^{**} \mid \Sigma_1, \Xi_j, \Sigma, \Lambda^{k-2} \Rightarrow e} \, (ec)$$

Summing up, we have:

- for any premise $G_2 \mid \Sigma_1, \Xi_j, p^{m_j} \Rightarrow p$, a derivation of $H_1 \mid G_2^{**} \mid \Sigma_1, \Xi_j, \Lambda^{m_j}, \Sigma, \Lambda^{k-2} \Rightarrow e$;
- for any premise $G_2 \mid \Sigma_1, \Xi_j \Rightarrow p$, a derivation of $H_1 \mid G_2^{**} \mid \Sigma_1, \Xi_j, \Sigma, \Lambda^{k-2} \Rightarrow e$;
- for any premise $G_2 \mid P_j$ with $R(S_j) = \emptyset$, a derivation of $H_1 \mid G_2^{**} \mid P_j^{**}$.

We can then apply the rule (r) as follows:

$$\frac{\begin{array}{c}\{H_1 \mid G_2^{**} \mid \Sigma_1, \Xi_j, \Sigma, \Lambda^{k-2} \Rightarrow e\}_{j=l+1,\dots,r} \quad \{H_1 \mid G_2^{**} \mid P_j^{**}\}_{j=r+1,\dots,m} \\ \{H_1 \mid G_2^{**} \mid \Sigma_1, \Xi_j, \Lambda^{m_j}, \Sigma, \Lambda^{k-2} \Rightarrow e\}_{j=1,\dots,l}\end{array}}{H_1 \mid G_2^{**} \mid \Sigma_1, \Theta_1, \dots, \Theta_n, \Lambda^s, \Sigma, \Lambda^{k-2} \Rightarrow e} \, (r)$$

Note that, apart from the hypersequent context, the rule application above is the original rule application, where Σ_1 is replaced by $\Sigma_1, \Sigma, \Lambda^{k-2}$, each p on the left is replaced by Λ and each p on the right by e. Note that what we derived above can also be written as

$$H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Theta_1, \ldots, \Theta_n, \Lambda^{s-1}, \Sigma, \Lambda^{k-1} \Rightarrow e,$$

which is the desired hypersequent. □

Lemma 5.1.7 (Mingle). *Let* HL∀ *and* d *be as in Theorem 5.1.5. Assume that* HL∀ *includes the rule* (mgl) *and that we have a derivation of a hypersequent* $G_1 \,|\, \Sigma \Rightarrow \Pi$ *where no* p *appears. We can find a* (D)*-free derivation of the hypersequent* $H_1 = G|\Lambda \Rightarrow \Pi \,|\, G_1 \,|\, \Sigma, \Lambda \Rightarrow \Pi$.

Proof. Let H be $H = S_1 \,|\, \ldots \,|\, S_n$. We define H^{**} as $S_1^{**} \,|\, \ldots \,|\, S_n^{**}$, where for each $i = 1, \ldots, n$

- $S_i^{**} = (S_i[^{\Sigma \Rightarrow \Pi}/_{p \Rightarrow p}])[^{\Rightarrow}/_{\overline{p} \Rightarrow}]$, if S_i is a *pp*-component.
- $S_i^{**} = S_i[^{\Lambda \Rightarrow \Pi}/_{\overline{p} \Rightarrow p}]$ otherwise.

The statement of the lemma is a consequence of the following:

Claim: *For each hypersequent* H *in* d *we can find a* (D)*-free derivation of* $H_1 \,|\, H^{**}$.

Indeed, applying the claim to the end-hypersequent of d, we obtain a (D)-free derivation of $(G \,|\, \Lambda \Rightarrow p \,|\, p \Rightarrow \Pi)^{**} = H_1 \,|\, G^{**} \,|\, \Lambda \Rightarrow \Pi \,|\, \Lambda \Rightarrow \Pi$. Note that $G^{**} = G$ by the eigenvariable condition on (D) and $\Lambda \Rightarrow \Pi$ and G are components of H_1. Hence the desired derivation of H_1 is obtained by applying $(\overline{ec})$. We now prove the claim by induction on the length of the derivation of H in d. If $H = p \Rightarrow p$ then $H_1 \,|\, H^{**} = H_1 \,|\, \Sigma \Rightarrow \Pi$ is derivable by applying $(\overline{ew})$ to the hypersequent $G_1 \,|\, \Sigma \Rightarrow \Pi$, which is derivable by assumption. Other axioms, (ec) and (ew) are easy to handle.

Logical rules. Logical rules (but $(\rightarrow l)$) are easy to handle, as the conclusion contains a *pp*-component if and only if all the premises do. For $(\rightarrow l)$ where the conclusion contains a *pp*-component and none of the premises does, e.g. as in the following:

$$\frac{G_2 \,|\, \Gamma_1, p \Rightarrow \varphi \quad G_2 \,|\, \Gamma_2, \chi \Rightarrow p}{G_2 \,|\, \Gamma_1, \Gamma_2, p, \varphi \rightarrow \chi \Rightarrow p}$$

By induction hypothesis we have derivations of $H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Lambda \Rightarrow \varphi$ and $H_1 \,|\, G_2^{**} \,|\, \Gamma_2, \chi \Rightarrow \Pi$. The desired derivation is obtained as follows:

$$\dfrac{\dfrac{\dfrac{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Lambda \Rightarrow \varphi \quad H_1 \,|\, G_2^{**} \,|\, \Gamma_2, \chi \Rightarrow \Pi}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Lambda, \varphi \rightarrow \chi \Rightarrow \Pi}(\rightarrow l) \qquad G_1 \,|\, \Sigma \Rightarrow \Pi}{H_1 \,|\, G_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \varphi \rightarrow \chi, \Sigma \Rightarrow \Pi \,|\, \Lambda \Rightarrow \Pi}(com)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \varphi \rightarrow \chi, \Sigma \Rightarrow \Pi}(\overline{ec})$$

where the application of $(\overline{ec})$ is justified by the fact that $\Lambda \Rightarrow \Pi$ and G_1 are components of H_1.

The communication rule. Assume now that the last applied rule in a derivation of H is (com). We follow the same case distinction as in Figure 5.1. For case (1) we just apply the induction hypothesis and then (com). Let us consider now the case $(2a)$ and assume that we have an application of (com) as:

$$\frac{G_2 \,|\, \Gamma_1, \Gamma_2, p^k \Rightarrow p \quad G_2 \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Delta_1}{G_2 \,|\, \Gamma_1, \Sigma_1, p^k \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2 \Rightarrow p}\;(com)$$

By induction hypothesis we have derivations of $H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Delta_1$ and $H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Sigma \Rightarrow \Pi$. We restructure the derivation as follows (the last application of (ec) is justified by the fact that $\Sigma, \Lambda \Rightarrow \Pi$ is in H_1):

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Sigma \Rightarrow \Pi}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Gamma_2, \Sigma \Rightarrow \Pi}\,(mgl) \quad \cfrac{H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Delta_1}{H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2, \Sigma_2 \Rightarrow \Delta_1}\,(mgl)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Sigma_1, \Sigma_2, \Sigma \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi}\,(com)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Sigma, \Lambda \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi}\,(split_d)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Sigma, \Lambda \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi}\,(ec)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Sigma, \Lambda^k \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi}\,(mgl)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Lambda^k \Rightarrow \Delta_1 \,|\, \Sigma, \Lambda \Rightarrow \Pi \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi}\,(split_d)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Lambda^k \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi}\,(ec)$$

For space reasons, in all the above applications of (mgl) the single premise actually stands for two equal premises. We consider now the case $(2b)$, i.e. we assume to have an application of (com) as:

$$\frac{G_2 \,|\, \Gamma_1, \Gamma_2, p^k \Rightarrow p \quad G_2 \,|\, \Sigma_1, \Sigma_2, p^l \Rightarrow p}{G_2 \,|\, \Gamma_1, \Sigma_1, p^{k+l} \Rightarrow p \,|\, \Gamma_2, \Sigma_2 \Rightarrow p}\;(com)$$

Our aim is to obtain a derivation of

$$H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Sigma \Rightarrow \Pi \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi$$

By induction hypothesis we have derivations of the hypersequents $H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2, \Sigma \Rightarrow \Pi$ and $H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Sigma \Rightarrow \Pi$. We restructure the derivation as follows (the last application of (ec) is justified by the fact that $\Lambda, \Sigma \Rightarrow \Pi$ is in H_1):

$$\cfrac{\cfrac{\cfrac{\cfrac{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Sigma \Rightarrow \Pi \quad H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2, \Sigma \Rightarrow \Pi}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Sigma_1, \Sigma_2, \Sigma \Rightarrow \Pi}\,(mgl)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Lambda, \Sigma \Rightarrow \Pi \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi}\,(split_d) \quad H_1 \,|\, \Sigma \Rightarrow \Pi}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Sigma \Rightarrow \Pi \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi \,|\, \Lambda, \Sigma \Rightarrow \Pi}\,(com)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Sigma \Rightarrow \Pi \,|\, \Gamma_2, \Sigma_2 \Rightarrow \Pi}\,(ec)$$

We consider now the case $(3a)$, i.e. an application of (com) of the form

$$\frac{G_2 \,|\, \Gamma_1, \Gamma_2, p^k \Rightarrow \Delta_1 \quad G_2 \,|\, \Sigma_1, \Sigma_2 \Rightarrow p}{G_2 \,|\, \Gamma_1, \Sigma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2, p^k \Rightarrow p} \,(com)$$

By induction hypothesis we have derivations of $H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Lambda^k \Rightarrow \Delta_1$ and $H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Pi$. Applying (com), we obtain:

$$\frac{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Lambda^k \Rightarrow \Delta_1 \quad H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Pi}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2, \Lambda^k \Rightarrow \Pi} \,(com)$$

Then, recalling that $H_1 \,|\, \Sigma \Rightarrow \Pi$ is derivable, the desired hypersequent is obtained as follows (the (ec)s are justified as $\Lambda \Rightarrow \Pi$ and $\Sigma, \Lambda \Rightarrow \Pi$ are in H_1):

$$\dfrac{\dfrac{\dfrac{\dfrac{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2, \Lambda^k \Rightarrow \Pi \quad H_1 \,|\, \Sigma \Rightarrow \Pi}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2, \Sigma, \Lambda^{k-1} \Rightarrow \Pi \,|\, \Lambda \Rightarrow \Pi} \,(com)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2, \Sigma, \Lambda^{k-1} \Rightarrow \Pi} \,(ec) \qquad H_1 \,|\, \Sigma \Rightarrow \Pi}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2, \Sigma, \Lambda^{k-2} \Rightarrow \Pi \,|\, \Sigma, \Lambda \Rightarrow \Pi} \,(com)}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2, \Sigma, \Lambda^{k-2} \Rightarrow \Pi} \,(ec)$$

$$\vdots$$

$$H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2, \Sigma_2, \Sigma \Rightarrow \Pi$$

Let us now consider the case $(3b)$, i.e. an application of (com) of the form

$$\frac{G_2 \,|\, \Gamma_1, \Gamma_2, p^{k+l} \Rightarrow p \quad G_2 \,|\, \Sigma_1, \Sigma_2 \Rightarrow p}{G_2 \,|\, \Gamma_1, \Sigma_1, p^k \Rightarrow p \,|\, \Gamma_2, \Sigma_2, p^l \Rightarrow p} \,(com)$$

Our aim is to get a (D)-free derivation of $H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Sigma \Rightarrow \Pi \,|\, \Gamma_2, \Sigma_2, \Sigma \Rightarrow \Pi$. By induction hypothesis we have derivations of $H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Sigma \Rightarrow \Pi$ and $H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Pi$. We obtain the desired hypersequent as follows:

$$\dfrac{\dfrac{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Sigma \Rightarrow \Pi \quad H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Sigma \Rightarrow \Pi}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Gamma_2, \Sigma, \Sigma \Rightarrow \Pi} \,(mgl) \qquad H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Sigma_2 \Rightarrow \Pi}{H_1 \,|\, G_2^{**} \,|\, \Gamma_1, \Sigma_1, \Sigma \Rightarrow \Pi \,|\, \Gamma_2, \Sigma_2, \Sigma \Rightarrow \Pi} \,(com)$$

The mingle rule. Assume now the last rule applied to derive H is (mgl). If the conclusion does not contain a *pp*-component, then the premises do not contain *pp*-components as well. Hence we just apply the induction hypothesis and the rule. The only problematic case can arise when both premises contain *pp*-components and the conclusion contains a *pp*-component as well (recall from the proof of Theorem 5.1.5 that the case where the conclusion and only one premise contain *pp*-components can be reduced to the one with *pp*-components in both premises). Consider e.g. the following:

$$\frac{G_2 \,|\, \Sigma_1, \Gamma_1, p \Rightarrow p \quad G_2 \,|\, \Sigma_1, \Gamma_2, p \Rightarrow p}{G_2 \,|\, \Sigma_1, \Gamma_1, \Gamma_2, p, p \Rightarrow p}\,(mgl)$$

By induction hypothesis we have derivations of the hypersequents $H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Gamma_1, \Sigma \Rightarrow \Pi$ and $H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Gamma_2, \Sigma \Rightarrow \Pi$. We simply replace the original application with:

$$\frac{H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Gamma_1, \Sigma \Rightarrow \Pi \quad H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Gamma_2, \Sigma \Rightarrow \Pi}{H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Gamma_1, \Gamma_2, \Sigma \Rightarrow \Pi}\,(mgl)$$

Nonlinear rules. Assume now that the last rule applied to derive H is a non linear rule (r). In case the conclusion is not a pp component, none of the premises is a pp-component as well, by the strong subformula property (see Definition 3.3.9). Hence we can just apply the induction hypothesis and then (r). In case the conclusion contains a pp-component, i.e. as in

$$\frac{G_2 \,|\, P_1 \quad \ldots \quad G_2 \,|\, P_m}{G_2 \,|\, \Sigma_1, \Theta_1, \ldots, \Theta_n, p^s \Rightarrow p}\,(r)$$

we distinguish two cases:

1. None of the premises contains a pp-component.
2. Some premises contain a pp-component.

Figure 5.3: Cases for a nonlinear rule (r)

In case (1) we can simply apply the induction hypothesis and restructure the derivation as follows (the applications of (ec) are justified by the fact that $\Sigma, \Lambda \Rightarrow \Pi$ and $\Lambda \Rightarrow \Pi$ belongs to H_1) :

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{H_1 \,|\, G_2^{**} \,|\, P_1^{**} \quad \ldots \quad H_1 \,|\, G_2^{**} \,|\, P_m^{**}}{H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Theta_1, \ldots, \Theta_n, \Lambda^s \Rightarrow \Pi}\,(r) \quad G_1 \,|\, \Sigma \Rightarrow \Pi}{H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Theta_1, \ldots, \Theta_n, \Sigma, \Lambda^{s-1} \Rightarrow \Pi \,|\, \Lambda \Rightarrow \Pi}\,(com)}{H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Theta_1, \ldots, \Theta_n, \Sigma, \Lambda^{s-1} \Rightarrow \Pi}\,(ec) \quad G_1 \,|\, \Sigma \Rightarrow \Pi}{H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Theta_1, \ldots, \Theta_n, \Sigma, \Lambda^{s-2} \Rightarrow \Pi \,|\, \Sigma, \Lambda \Rightarrow \Pi}\,(com)}{H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Theta_1, \ldots, \Theta_n, \Sigma, \Lambda^{s-2} \Rightarrow \Pi}\,(ec)$$

$$\vdots$$

$$H_1 \,|\, G_2^{**} \,|\, \Sigma_1, \Theta_1, \ldots, \Theta_n, \Sigma \Rightarrow \Pi$$

Consider now the case (2). By the strong subformula property (Definition 3.3.9), each P_j can only have one of the following forms:

- $\Sigma_1, \Xi_j, p^{n_j} \Rightarrow p$, say the P_j 's for $j \in \{1, \ldots, l\}$, with $l \leq m$, $n_j \geq 1$.

- $\Sigma_1, \Xi_j \Rightarrow p$, say the P_j 's for $j \in \{l+1, \ldots, r\}$, with $r \leq m$.
- $\Xi_j, p^{n_j} \Rightarrow$, say the P_j 's for $j \in \{r+1, \ldots, m\}$, with $n_j \geq 0$.

The induction hypothesis gives us derivations of $H_1 \mid G_2^{**} \mid P_1^{**}, \ldots, H_1 \mid G_2^{**} \mid P_m^{**}$. In particular, for $G_2 \mid P_j$, for $j \in \{1, \ldots, l\}$, we have derivations of $H_1 \mid G_2^{**} \mid \Sigma_1, \Xi_j, \Sigma \Rightarrow \Pi$. For the premises of (r) of the form $G_2 \mid \Sigma_1, \Xi_j \Rightarrow p$, we have derivations of $H_1 \mid G_2^{**} \mid \Sigma_1, \Xi_j \Rightarrow \Pi$. From the latter, recalling that $H_1 \mid \Sigma \Rightarrow \Pi$ is derivable, we get:

$$\frac{H_1 \mid G_2^{**} \mid \Sigma_1, \Xi_j \Rightarrow \Pi \qquad H_1 \mid \Sigma \Rightarrow \Pi}{H_1 \mid G_2^{**} \mid \Sigma_1, \Xi_j, \Sigma \Rightarrow \Pi} \, (mgl)$$

Consider now the premises of the form $G_2 \mid \Xi_j, p^{n_j} \Rightarrow$. By induction hypothesis, we have derivations of $H_1 \mid G_2^{**} \mid \Xi_j, \Lambda_j^n \Rightarrow$. Consider the following, recalling that $H_1 \mid \Sigma \Rightarrow \Pi$ is derivable and that $\Sigma, \Lambda \Rightarrow \Pi$ is in H_1 (this justifies the $(\overline{ec})$ below):

$$\dfrac{\dfrac{H_1 \mid G_2^{**} \mid \Xi_j, \Lambda_j^n \Rightarrow \qquad H_1 \mid \Sigma \Rightarrow \Pi}{H_1 \mid G_2^{**} \mid \Xi_j, \Lambda^{n_j - 1} \Rightarrow \;\mid \Sigma, \Lambda \Rightarrow \Pi} \, (com) \qquad H_1 \mid \Sigma \Rightarrow \Pi}{H_1 \mid G_2^{**} \mid \Xi_j, \Lambda^{n_j - 2} \Rightarrow \;\mid \Sigma, \Lambda \Rightarrow \Pi \mid \Sigma, \Lambda \Rightarrow \Pi} \, (com)$$
$$\vdots$$
$$\frac{H_1 \mid G_2^{**} \mid \Xi_j \Rightarrow \;\mid \Sigma, \Lambda \Rightarrow \Pi \mid \ldots \mid \Sigma, \Lambda \Rightarrow \Pi}{H_1 \mid G_2^{**} \mid \Xi_j \Rightarrow} \, (\overline{ec})$$

Summing up, we have:

- For any premise $G_2 \mid \Sigma_1, \Xi_j, p^{m_j} \Rightarrow p$, a derivation of $H_1 \mid G_2^{**} \mid \Sigma_1, \Xi_j, \Sigma \Rightarrow \Pi$.
- For any premise $G_2 \mid \Sigma_1, \Xi_j \Rightarrow p$, a derivation of $H_1 \mid G_2^{**} \mid \Sigma_1, \Xi_j, \Sigma \Rightarrow \Pi$.
- For any premise $G_2 \mid \Xi_j, p_j^n \Rightarrow$, a derivation of $H_1 \mid G_2^{**} \mid \Xi_j \Rightarrow$.

We can then apply the rule (r) as follows:

$$\frac{\{H_1 \mid G_2^{**} \mid \Sigma_1, \Xi_j, \Sigma \Rightarrow \Pi\}_{j=l+1,\ldots,r} \quad \{H_1 \mid G_2^{**} \mid \Xi_j \Rightarrow\}_{j=r+1,\ldots,m} \quad \{H_1 \mid G_2^{**} \mid \Sigma_1, \Xi_j, \Sigma \Rightarrow \Pi\}_{j=1,\ldots,l}}{H_1 \mid G_2^{**} \mid \Sigma_1, \Theta_1, \ldots, \Theta_n, \Sigma \Rightarrow \Pi} \, (r)$$

which is a correct application of (r). Indeed, apart from the hypersequent context, it is the original rule application, where Σ_1 has been replaced by Σ_1, Σ, each p on the left is removed and each p on the right is replaced by Π. □

Thus, we have the following.

Theorem 5.1.8. *Let* L∀ *be an extension of* UL∀ *with any nonlinear axiom and/or mingle. The logic* L∀ *is standard complete.*

Proof. Follows from Theorems 3.4.7(i) and 5.1.5. □

5.2 A particular case: knotted axioms

A class of axioms in $\mathcal{N}_2$, which is strictly contained in the class of nonlinear axioms, allows for an easier proof of density elimination, based on a simple variant of the original idea of the proof in [28]. These are the knotted axioms (see Table 3.2 and [55]), i.e. $\alpha^k \to \alpha^n$, with $k, n > 1$, whose corresponding internal structural rules have the form

$$\frac{G_1 \,|\, \Gamma_1^n, \Sigma \Rightarrow \Pi \ldots G_1 \,|\, \Gamma_k^n, \Sigma \Rightarrow \Pi}{G_1 \,|\, \Gamma_1, \ldots, \Gamma_k, \Sigma \Rightarrow \Pi} \; (knot_k^n)$$

We show below density elimination for extensions of HUL∀ with any rule $(knot_k^n)$, for $n, k > 1$. This excludes all the rules $(knot_1^k)$ and $(knot_n^1)$. However, it can be easily shown in HUL∀ (due to the presence of (com)) that any rule $(knot_1^k)$ is equivalent to $(knot_1^2)$, and any rule $(knot_n^1)$ is equivalent to $(knot_2^1)$. Hence, the knotted rules that we are not able to handle by the method presented below are actually only $(knot_1^2)$ and $(knot_2^1)$, i.e. (c) and (mgl), respectively. In the following we show how to adapt the method in [28] for all the remaining $(knot_k^n)$. Recall the derivation d and the substitution sketched on page 63: we replaced each sequent of the form $\Sigma, p^k \Rightarrow p$ with $(\Sigma, p^k \Rightarrow p[{}^{\Rightarrow e}/_{p \Rightarrow p}])[{}^{\Lambda \Rightarrow}/_{\overline{p} \Rightarrow}]$ and any other sequent S with $S[{}^{\Lambda \Rightarrow \Pi}/_{\overline{p} \Rightarrow p}]$. The key observation for the proof to go through is that the knotted rules allow us to use a restricted form of contraction and weakening on the left, when the right hand side of a sequent is equal to e. This is expressed by the following lemma.

Lemma 5.2.1. *Let $n, k > 1$. The following rules are derivable in the calculus* HUL∀+$(knot_k^n)$*:*

$$\frac{G \,|\, \Sigma, \Gamma_1 \Rightarrow e}{G \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e} \; (w_e) \qquad\qquad \frac{G \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e}{G \,|\, \Sigma, \Gamma_1 \Rightarrow e} \; (c_e)$$

Proof. Note that in HUL∀, for any $m > 1$, the rule

$$\frac{G \,|\, \Sigma \Rightarrow e}{G \,|\, \Sigma^m \Rightarrow e} \; (*_m)$$

is derivable. We reason by induction on m. First, we show that $(*_2)$ is derivable as follows:

$$\dfrac{G \,|\, \Sigma \Rightarrow e \qquad \dfrac{G \,|\, \Sigma \Rightarrow e}{G \,|\, \Sigma, e \Rightarrow e} \; (el)}{G \,|\, \Sigma, \Sigma \Rightarrow e} \; (cut)$$

Assuming that $(*_{m-1})$ is derivable, we derive $(*_m)$ as follows:

$$\dfrac{G \,|\, \Sigma \Rightarrow e \qquad \dfrac{\dfrac{G \,|\, \Sigma \Rightarrow e}{G \,|\, \Sigma^{m-1} \Rightarrow e} \; (*_{m-1})}{G \,|\, \Sigma^{m-1}, e \Rightarrow e} \; (el)}{G \,|\, \Sigma^m \Rightarrow e} \; (cut)$$

Similarly, we can prove that for any $m > 1$ the rule

$$\frac{G \,|\, \Sigma^m \Rightarrow e}{G \,|\, \Sigma \Rightarrow e}\ (*^m)$$

is derivable. The base case $(*^2)$ can be derived as follows:

$$\dfrac{\dfrac{G \,|\, \Sigma, \Sigma \Rightarrow e \qquad \dfrac{\Rightarrow e}{G \,|\, \Rightarrow e}\,(ew)}{G \,|\, \Sigma \Rightarrow e \,|\, \Sigma \Rightarrow e}\,(com)}{G \,|\, \Sigma \Rightarrow e}\,(ec)$$

Assuming that $(*^{m-1})$ is derivable, we get:

$$\dfrac{\dfrac{\dfrac{G \,|\, \Sigma^m \Rightarrow e \qquad \dfrac{\Rightarrow e}{G \,|\, \Rightarrow e}\,(ew)}{G \,|\, \Sigma^{m-1} \Rightarrow e \,|\, \Sigma \Rightarrow e}\,(com)}{G \,|\, \Sigma \Rightarrow e \,|\, \Sigma \Rightarrow e}\,(*^{m-1})}{G \,|\, \Sigma \Rightarrow e}\,(ec)$$

Using $(*_2)$ and $(*^2)$, we can easily show that the two rules (c_e) and (w_e) are interderivable. Indeed, if we have (w_e), we can derive (c_e) as follows:

$$\dfrac{\dfrac{G \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e}{G \,|\, \Sigma, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e}\,(w_e)}{G \,|\, \Sigma, \Gamma_1 \Rightarrow e}\,(*^2)$$

And analogously:

$$\dfrac{\dfrac{G \,|\, \Sigma, \Gamma_1 \Rightarrow e}{G \,|\, \Sigma, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e}\,(*_2)}{G \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e}\,(c_e)$$

In what follows it is therefore enough to prove that either (c_e) or (w_e) is derivable. In particular, we show that (c_e) is derivable in case the knotted rule $(knot^n_k)$ has $n > k$, and that (w_e) is derivable otherwise.

1. Assume $n > k$. Suppose that we are given a derivation of $G \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e$. Consider the following application of $(knot^n_k)$:

$$\dfrac{\dfrac{G \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e}{G \,|\, \Sigma^n, \Gamma_1^n, \Gamma_1^n \Rightarrow e}\,(*_n) \quad \ldots \quad \dfrac{G \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e}{G \,|\, \Sigma^n, \Gamma_1^n, \Gamma_1^n \Rightarrow e}\,(*_n)}{G \,|\, \Sigma^n, \Gamma_1^k, \Gamma_1^k \Rightarrow e}\,(knot^n_k)$$

If $2k > n$ we apply $(knot_k^n)$ with k identical premises $G_1 \,|\, \Sigma^n, \Gamma_1^{2k-n}, \Gamma_1^n \Rightarrow e$, to obtain $G_1 \,|\, \Sigma^n, \Gamma_1^{2k-n+k} \Rightarrow e$. We apply the rule $(knot_k^n)$ once more using this sequent as the premises. We repeat in this way until we get $G_1 \,|\, \Sigma^n, \Gamma_1^l \Rightarrow e$, for some $l \leq n$. The proof that (c_e) is derivable is then completed as follows:

$$\dfrac{\dfrac{\dfrac{\vdots}{G \,|\, \Sigma^n, \Gamma_1^l \Rightarrow e} \quad G \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e}{G \,|\, \Sigma^n, \Gamma_1^{l+1} \Rightarrow e \,|\, \Sigma, \Gamma_1 \Rightarrow e}\,(com) \qquad \dfrac{G \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e}{G \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e \,|\, \Sigma, \Gamma_1 \Rightarrow e}\,(ew)}{G \,|\, \Sigma^n, \Gamma_1^{l+2} \Rightarrow e \,|\, \Sigma, \Gamma_1 \Rightarrow e \,|\, \Sigma, \Gamma_1 \Rightarrow e}\,(com)$$

$$\vdots$$

$$\dfrac{\dfrac{G \,|\, \Sigma^n, \Gamma_1^n \Rightarrow e \,|\, \Sigma, \Gamma_1 \Rightarrow e \,|\, \ldots \,|\, \Sigma, \Gamma_1 \Rightarrow e}{G \,|\, \Sigma, \Gamma_1 \Rightarrow e \,|\, \ldots \,|\, \Sigma, \Gamma_1 \Rightarrow e}\,(*^n)}{G \,|\, \Sigma, \Gamma_1 \Rightarrow e}\,(\overline{ec})$$

2. Consider now the case where $n < k$. Suppose that we are given a derivation of $G \,|\, \Sigma, \Gamma_1 \Rightarrow e$. We prove that (w_e) is derivable in our calculus. First, consider the following application of $(knot_k^n)$:

$$\dfrac{\dfrac{G \,|\, \Sigma, \Gamma_1 \Rightarrow e}{G \,|\, \Sigma^n, \Gamma_1^n \Rightarrow e}\,(*_n) \quad \ldots \quad \dfrac{G \,|\, \Sigma, \Gamma_1 \Rightarrow e}{G \,|\, \Sigma^n, \Gamma_1^n \Rightarrow e}\,(*_n)}{G \,|\, \Sigma^n, \Gamma_1^k \Rightarrow e}\,(knot_k^n)$$

We iterate similar application of $(knot_k^n)$, increasing the occurrences of Γ_1 by $(k-n)$, until we get $G \,|\, \Sigma^n, \Gamma_1^l \Rightarrow e$, for some $l \geq 2n$. The proof that (w_e) is derivable is then completed as follows:

$$\dfrac{\dfrac{\dfrac{\vdots}{G \,|\, \Sigma^n, \Gamma_1^l \Rightarrow e} \quad G \,|\, \Sigma, \Gamma_1 \Rightarrow e}{G \,|\, \Sigma^n, \Gamma_1^{l-1} \Rightarrow e \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e}\,(com) \qquad \dfrac{G \,|\, \Sigma, \Gamma_1 \Rightarrow e}{G \,|\, \Sigma, \Gamma_1 \Rightarrow e \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e}\,(ew)}{G \,|\, \Sigma^n, \Gamma_1^{l-2} \Rightarrow e \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e}\,(com)$$

$$\vdots$$

$$\dfrac{\dfrac{G \,|\, \Sigma^n, \Gamma_1^n, \Gamma_1^n \Rightarrow e \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e \,|\, \ldots \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e}{G \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e \,|\, \ldots \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e}\,(*^n)}{G \,|\, \Sigma, \Gamma_1, \Gamma_1 \Rightarrow e}\,(\overline{ec})$$

□

We are now ready to prove density elimination. The proof follows closely that in [28].

Theorem 5.2.2. *Let* HL∀ *be a hypersequent calculus extending* HUL∀ *with any set of rules of the form* $(knot_k^n)$*, with* $n, k > 1$*. The calculus* $\mathrm{HL}\forall^{\mathrm{D}}$ *admits density elimination.*

Proof. It proceeds by induction on the length of the derivations. Consider a derivation d ending in a topmost application of the density rule

$$\dfrac{\begin{array}{c}\vdots\\ G \mid \Lambda \Rightarrow p \mid p \Rightarrow \Pi\end{array}}{G \mid \Lambda \Rightarrow \Pi}\ (D)$$

As usual, we can safely assume d to be cut-free. Let H be a hypersequent $S_1 \mid \ldots \mid S_n$. We let $H^* = S_1^* \mid \ldots \mid S_n^*$ where, for each component S_i, the sequent S_i^* is defined as follows:

- $S_i^* = (S_i[^{\Rightarrow e}/_{p \Rightarrow p}])[^{\Lambda \Rightarrow}/_{\overline{p} \Rightarrow}]$, if S_i is a pp-component.
- $S_i^* = S_i[^{\Lambda \Rightarrow \Pi}/_{\overline{p} \Rightarrow p}]$ otherwise.

As for Theorem 5.1.5, it is enough to prove the following:
Claim: *For each hypersequent H in d one can find a (D)-free derivation of $G \mid \Lambda \Rightarrow \Pi \mid H^*$*

For proving the claim, we reason by induction on the length of the derivation of a hypersequent H in d. We only show the case when the last applied rule is $(knot_k^n)$, the other cases being already contained in the proof of Theorem 5.1.5 (and originally considered in [25]). We distinguish three cases, according to the presence of pp-components in the premises:

1. None of the premises contains a pp-component.
2. All the premises contain a pp-component.
3. Only some of the premises contain pp-components.

Figure 5.4: Cases for knotted rules

For case (1), also the conclusion does not contain any pp-component: the claim hence holds, by using the induction hypothesis and applying the knotted rule again. For case (2), assume that we have an application of $(knot_k^n)$ as the following:

$$\dfrac{G_1 \mid \Sigma, \Gamma_1^n, p^{n_1} \Rightarrow p \quad \ldots \quad G_1 \mid \Sigma, \Gamma_k^n, p^{n_k} \Rightarrow p}{G_1 \mid \Sigma, \Gamma_1, \ldots, \Gamma_k, p^l \Rightarrow p}\ (knot_k^n)$$

By the induction hypothesis we have density-free derivations of $G|\Lambda \Rightarrow \Pi \mid G_1^* \mid \Sigma, \Gamma_1^n, \Lambda^{n_1-1} \Rightarrow e, \ldots, G|\Lambda \Rightarrow \Pi \mid G_1^* \mid \Sigma, \Gamma_k^n, \Lambda^{n_k-1} \Rightarrow e$. Consider the following derivation:

$$\dfrac{G|\Lambda \Rightarrow \Pi \mid G_1^* \mid \Sigma, \Gamma_1^n, \Lambda^{n_1-1} \Rightarrow e \qquad \dfrac{G|\Lambda \Rightarrow \Pi \mid G_1^* \mid \Sigma, \Gamma_2^n, \Lambda^{n_2-1} \Rightarrow e}{G|\Lambda \Rightarrow \Pi \mid G_1^* \mid \Sigma, \Gamma_2^n, e, \Lambda^{n_2-1} \Rightarrow e}\ (el)}{G|\Lambda \Rightarrow \Pi \mid G_1^* \mid \Sigma^2, \Gamma_1^n, \Gamma_2^n, \Lambda^{n_1-1}, \Lambda^{n_2-1} \Rightarrow e}\ (cut)$$

Starting from the end-hypersequent above, we can iterate similar applications of (cut) with each of the $G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_3^n, \Lambda^{n_3-1} \Rightarrow e, \ldots, G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_k^n, \Lambda^{n_k-1} \Rightarrow e$ until we get:

$$G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma^k, \Gamma_1^n, \Gamma_2^n, \ldots, \Gamma_k^n, \Lambda^{n_1+\cdots+n_k-k} \Rightarrow e.$$

The desired hypersequent $G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1, \ldots, \Gamma_k, \Lambda^{l-1} \Rightarrow e$ is then obtained by applications of $(\overline{c_e})$ and $(\overline{w_e})$ to the hypersequent above.

We consider now the case (3), where only some premises (say m, with $m < k$) contain active pp-components. Assume, without loss of generality, that the application of $(knot_k^n)$ has the following form:

$$\frac{\begin{array}{c} G_1 \,|\, \Sigma, \Gamma_1^n, p^{n_1} \Rightarrow p \ \ldots \ G_1 \,|\, \Sigma, \Gamma_m^n, p^{n_m} \Rightarrow p \\ G_1 \,|\, \Sigma, \Gamma_{m+1}^n \Rightarrow p \ \ldots G_1 \,|\, \Sigma, \Gamma_k^n \Rightarrow p \end{array}}{G_1 \,|\, \Sigma, \Gamma_1, \ldots, \Gamma_k, p^l \Rightarrow p} \ (knot_k^n)$$

By induction hypothesis we have (D)-free derivations of $G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1^n, \Lambda^{n_1-1} \Rightarrow e, \ldots, G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_m^n, \Lambda^{n_m-1} \Rightarrow e$ and $G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_{m+1}^n \Rightarrow \Pi, \ldots, G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_k^n \Rightarrow \Pi$. As in the previous case, we do repeated cuts, but only on the m premises containing pp-components, thus obtaining:

$$G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma^m, \Gamma_1^n, \ldots, \Gamma_m^n, \Lambda^{n_1+\cdots+n_m-m} \Rightarrow e.$$

We repeatedly apply (c_e) or (w_e) to the previous hypersequent, to get:

$$G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma^m, \Gamma_1^n, \ldots, \Gamma_m^n, \Lambda^{(l-1)+(k-m)} \Rightarrow e.$$

We "remove" then the extra-occurrences of Λ from this hypersequent, using applications of (com) as the following :

$$\frac{\dfrac{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma^m, \Gamma_1^n, \ldots, \Gamma_m^n, \Lambda^{(l-1)+(k-m)} \Rightarrow e \quad G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_{m+1}^n \Rightarrow \Pi}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma^{m+1}, \Gamma_1^n, \ldots, \Gamma_{m+1}^n, \Lambda^{(l-1)+(k-m-1)} \Rightarrow e \,|\, \Lambda \Rightarrow \Pi} \ (com)}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma^{m+1}, \Gamma_1^n, \ldots, \Gamma_{m+1}^n, \Lambda^{(l-1)+(k-m-1)} \Rightarrow e} \ (ec)$$

Similarly, by an application of (com) to the conclusion of the derivation above and the premise $G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_{m+2}^n \Rightarrow \Pi$, we get

$$G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma^{m+2}, \Gamma_1^n, \ldots, \Gamma_{m+2}^n, \Lambda^{(l-1)+(k-m-2)} \Rightarrow e.$$

We can iterate applications of (com) of this kind for all the $(k-m)$ premises of $(knot_k^n)$ which do not contain pp-components, until we finally get:

$$\frac{\begin{array}{c} \vdots \\ G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma^k, \Gamma_1^n, \ldots, \Gamma_k^n, \Lambda^{l-1} \Rightarrow e \end{array}}{G|\Lambda \Rightarrow \Pi \,|\, G_1^* \,|\, \Sigma, \Gamma_1, \ldots, \Gamma_k, \Lambda^{l-1} \Rightarrow e} \ (\overline{c_e})$$

This concludes the proof of the main claim, thus establishing density elimination for our calculus. □

CHAPTER 6

A new algebraic approach

The proofs of standard completeness that we have presented in Chapter 4 and Chapter 5 were based on the purely syntactic method of density elimination. In some sense we bypassed there the original algebraic problem: showing that a chain from a subvariety V of FL-algebras can be embedded into a *dense* chain belonging to the same subvariety V, see Theorem 3.4.7. The construction of this embedding will be referred to as *densification* in what follows. Recall that proofs of density elimination show the completeness of a logic L with respect to dense L-chains. Hence, using e.g. Theorem 3.4.6 in [33] we obtain indirectly that any countable L-chain is embeddable into a dense one. However, showing this offers no information neither on the structure of this dense algebra nor on the embedding. Moreover, in the previous chapters we have used specific methods of proof theory, i.e. induction on the length of derivations and complicated substitutions, etc., which are perhaps not the everyday tools of the "working algebraist".

In this chapter we try to overcome these drawbacks, by presenting a proof of densification which, though inspired by density elimination, can be understood in purely algebraic terms. The crucial idea is to divide the proof of densification in two parts. First, we show that for some subvarieties of FL-algebras we can find an embedding v from any non-dense chain containing a "gap" (two elements $g < h$ such that there is no p satisfying $g < p < h$) into another chain of the same subvariety which "fills the gap" (there is an element p satisfying $v(g) < p < v(h)$). We call such varieties *densifiable*. We then apply a general result stating that, whenever a subvariety of FL is densifiable, one can embedd any chain in the subvariety into a dense one of the same subvariety.

This method contrasts with the usual approach in the literature, where one directly looks for the embedding of an FL-chain into a dense one, usually by adding countably many new elements (a copy of the rationals $\mathbb{Q}$) to fill the gap (see the sketch of proof on page 48).

Most of this chapter is devoted to prove densifiability, by reformulating in this new setting the proof-theoretic techniques developed in Chapter 4 and (part of) Chapter 5. Towards this aim, we use *residuated frames* [45], which are objects that can be used to construct (complete) FL-algebras with various properties. Residuated frames have been already used to connect proof theory with algebras, and played a crucial role in the development of "algebraic proof theory", see Chapter 1. More precisely, one can naturally define a residuated frame $\boldsymbol{W}$ from the full Lambek calculus FL, so that validity in the dual algebra $\boldsymbol{W}^{+}$ directly implies cut-free derivability in FL [27].

Note that the results in this chapter are restricted to propositional logic, for which Theorem 3.4.7 still applies, dropping the requirement that the embedding should be regular. However, the use of an algebraic framework avoids some syntactic bureaucracy, permitting us to extend, in an easier way, the results from Chapter 4 to the noncommutative case.

Plan of the chapter

In Section 6.1 we introduce the notion of densifiability and recall some preliminaries on noncommutative logics. In Section 6.2 we recall some basic notions of residuated frames, mainly from [27,45]. In Section 6.3 we present a uniform proof of densification for FL_i-algebras, more precisely for FL_x^ℓ with $\{i\} \subseteq x \subseteq \{e, c, i, o\}$. As a corollary follows the (known) result that the logics MTL, psMTL^r and Gödel logic G are standard complete. We prefer to give first a proof of these simpler cases, to show more clearly the connection with the proof-theoretic method for $\mathrm{HMTL}\forall$ presented in Chapter 4. In Section 6.4, we provide a similar proof only for the variety FL_x^ℓ with $\{e\} \subseteq x \subseteq \{e, i, o\}$, this time in connection with the proof theoretic method for $\mathrm{HUL}\forall$ in Chapter 5. Here, as a corollary, we obtain that the logic UL is standard complete. In Section 6.5 we then translate in our new algebraic setting the general results on semi-anchored rules of Chapter 4 and in Section 6.6 the results on knotted axioms from Chapter 5.

6.1 Densifiability and noncommutative logics

We begin with general considerations on densification and densifiability.

Definition 6.1.1. *Let* $\mathbf{A}$ *be a* chain *of cardinality* $\kappa > 1$ *which is not dense. By Definition 2.1.3 it contains a* gap (g, h)*. We say that a chain* $\mathbf{B}$ fills *a gap* (g, h) *of* $\mathbf{A}$ *if there is an embedding* $v : \mathbf{A} \longrightarrow \mathbf{B}$ *and an element* $p \in B$ *such that* $v(g) < p < v(h)$*, see Figure 6.1.*

A nontrivial variety V *is said to be* densifiable *if every gap of a chain in* V *can be filled by another chain in* V.

Note that, although by filling a gap one may introduce some undesirable elements that have nothing to do with the gap, densifiability is a sufficient condition for densification.

Lemma 6.1.2. *Let* $\mathcal{L}$ *be a countable language as in Definition 2.1.4 and* V *a densifiable variety over* $\mathcal{L}$*. Then every countable chain* $\mathbf{A}$ *in* V *is embeddable into a dense countable chain.*

Proof. Let Var be a countable set of variables and $Fm_{\mathcal{L}}$ be defined as in Definition 2.1.4. Let (t_0, u_0), (t_1, u_1), … be a countable sequence of elements of $Fm_{\mathcal{L}} \times Fm_{\mathcal{L}}$ such that each $(t, u) \in Fm_{\mathcal{L}} \times Fm_{\mathcal{L}}$ occurs infinitely many times in it.

For each $n \in \mathbb{N}$, we define a chain $\mathbf{B}_n$ in V as well as a partial valuation $f_n : Var \rightharpoonup B_n$. Let $\mathbf{B}_0 := \mathbf{A}$ and f_0 be any surjective partial function onto A such that $Var \backslash dom(f_0)$ is infinite.

For $n \geq 0$, if one of $f_n(t_n)$, $f_n(u_n)$ is undefined or $f_n(t_n) \not< f_n(u_n)$, then let $\mathbf{B}_{n+1} := \mathbf{B}_n$ and $f_{n+1} := f_n$.

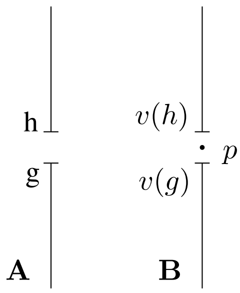

Figure 6.1: The chain $\mathbf{B}$ ‘fills” the gap (g, h) of $\mathbf{A}$

Otherwise, let x be a variable taken from $Var\backslash dom(f_n)$. If there is $p \in B_n$ such that $f_n(t_n) < p < f_n(u_n)$, then let $\mathbf{B}_{n+1} := \mathbf{B}_n$. If not, let $\mathbf{B}_{n+1}$ be a chain in V that fills the gap $(f_n(t_n), f_n(u_n))$ by $p \in B_{n+1}$. We assume $B_n \subseteq B_{n+1}$ and define $f_{n+1} : Var \rightharpoonup B_{n+1}$ by extending f_n with $f_{n+1}(x) := p$.

Let $\mathbf{B} := \bigcup \mathbf{B}_n$, $f := \bigcup f_n$ and $\mathbf{C}$ be the subalgebra of $\mathbf{B}$ generated by $f[Var]$ (so that $C = f[Fm_{\mathcal{L}}]$). Clearly $\mathbf{C}$ is a countable chain in V that has $\mathbf{A}$ as subalgebra since $A \subseteq f[Var]$. Moreover $\mathbf{C}$ is dense, since for every pair $(g, h) \in C^2$ with $g < h$, there is $n \in \mathbb{N}$ such that $g = f_n(t_n)$ and $h = f_n(u_n)$ so that we have $g < f_{n+1}(x) < h$. □

Remark 6.1.3. *For our purposes it was enough to restrict the lemma above to countable chains. This is however not necessary. Using the axiom of choice the lemma can be easily extended to chains of arbitrary cardinality (see [12]).*

The results described in this chapter apply to the noncommutative variant of MTL, known as psMTL^r, see [21]. In contrast to the logics introduced in Chapter 3, psMTL^r is not an axiomatic extension of UL, hence we recall an Hilbert system for it from [21], in a slightly different notation.

$(\mathrm{psMTL^r}1a, b)$	$(\varphi\backslash\psi)\backslash((\chi\backslash\varphi)\backslash(\chi\backslash\psi)),\ \ ((\psi/\chi)/(\varphi/\chi))/(\psi/\varphi)$
$(\mathrm{psMTL^r}2a, b)$	$\varphi\cdot\psi\backslash\varphi,\quad \varphi\cdot\psi\backslash\psi$
$(\mathrm{psMTL^r}3)$	$\varphi\vee\psi\backslash\psi\vee\varphi$
$(\mathrm{psMTL^r}4a)$	$\varphi\wedge\psi\backslash\varphi$
$(\mathrm{psMTL^r}4b)$	$\varphi\wedge\psi\backslash\psi\wedge\varphi$
$(\mathrm{psMTL^r}4c, d)$	$\varphi\cdot(\varphi\backslash\psi)\backslash\varphi\wedge\psi,\ \ \varphi\wedge\psi/(\psi/\varphi)\cdot\varphi$
$(\mathrm{psMTL^r}5a, b)$	$(\varphi\backslash(\psi\backslash\chi)\backslash\!\!-\!\!\backslash(\psi\cdot\varphi\backslash\chi)),\ \ ((\chi/\psi)/\varphi)/\!\!-\!\!/(\chi/\varphi\cdot\psi)$
$(\mathrm{psMTL^r}6a)$	$((\varphi\backslash\psi)\backslash\chi)\backslash(((\psi\backslash\varphi)\backslash\chi)\backslash\chi)$
$(\mathrm{psMTL^r}6b)$	$(\chi/(\chi/(\varphi/\psi)))/(\chi/(\psi/\varphi))$
$(\mathrm{psMTL^r}7a, b)$	$f\backslash\varphi,\ \ \varphi\backslash e$
$(\mathrm{psMTL^r}8a, b)$	$(\varphi\backslash\psi)\vee(\chi\cdot(\psi\backslash\varphi)/\chi),\ \ (\psi/\varphi)\vee(\chi\backslash(\varphi/\psi)\cdot\chi)$

with the definitions:

$$\varphi /\!\!-\!\!/ \psi \equiv (\varphi/\psi) \wedge (\psi/\varphi)$$
$$\varphi \backslash\!\!-\!\!\backslash \psi \equiv (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$$

and the derivation rules:

(MP) $\dfrac{\varphi \quad \varphi\backslash\psi}{\psi}$

(Imp$_{a.b}$) $\dfrac{\varphi\backslash\psi}{\psi/\varphi} \quad \dfrac{\psi/\varphi}{\varphi\backslash\psi}$

Note that in the logic psMTL^r the connective $\cdot$ is noncommutative and we have two implications $\backslash$ and $/$. The algebraic semantics of psMTL^r is given by the class of FL_w^ℓ-algebras (see Definition 2.1.13), also known as psMTL^r-algebras (*representable pseudo* MTL-*algebras*), see [21, 52]. The logic is already known to be standard complete, see e.g. [39, 59]. In a standard psMTL^r-algebra, the connectives $\cdot$ and $\backslash$ and $/$ are interpreted by any left-continuous pseudo t-norms and their residua, respectively, see Definition 2.2.1 and Example 2.2.7.

Let L be any axiomatic extension of psMTL^r with an additional set of axiom schemas C (in the same language). In what follows we extend Definition 3.1.5, calling L-algebra any FL_w^ℓ-algebra satisfying the equations $e \leq \varphi$, for every $\varphi \in C$. In analogy with Theorem 3.1.6, we have a general completeness theorem for any propositional axiomatic extension L of psMTL^r with respect to the corresponding classes of L-chains,.

Theorem 6.1.4. *[33, 52] Let* L *be any axiomatic extension of* psMTL^r. *For every set of formulas T and every formula φ the following are equivalent:*

- $T \vdash_{\mathrm{L}} \varphi$,
- $\{e \leq \psi\}_{\psi \in T} \models_{\boldsymbol{A}} e \leq \varphi$ *for every* L*-chain* $\boldsymbol{A}$

Where $\models_{\boldsymbol{A}}$ is as in Definition 2.1.5.

Finally, we present a modified version of Theorem 3.4.7, which will be used in what follows.

Theorem 6.1.5 (Standard Completeness)**.** *Let* L *be any acyclic $\mathcal{P}_3'$-extension of* UL *or a $\mathcal{P}_3$-extension of* psMTL^r. *If countable* L*-chains are embeddable into dense countable* L*-chains, then the logic* L *is standard complete.*

Proof. Lemma 3.4.6 can be applied to axiomatic extensions of psMTL^r, see e.g. [21, 52]. The statement then follows by Theorem 2.3.18. ☐

Before introducing our general method for densifiability, we show an example of a simple proof of the densifiability of FL_i^ℓ.

Theorem 6.1.6. FL_i^ℓ *is a densifiable variety.*

Proof. Let $\mathbf{A}$ be an FL_i-chain with a gap (g, h). We insert a new element p between g and h:

$$A^p := A \cup \{p\}, \qquad g < p < h.$$

(see Figure 6.1). The meet and join operations are naturally extended to A^p. To extend multiplication $\cdot$ and divisions $\backslash, /$, note that for every $a \in A$, either $ah = h$ or $ah \leq g$ holds. For every $a \in A$, we define:

$$\begin{array}{rclcrcllc} p \cdot p & := & p & (h^2 = h) & p\backslash p & := & e & \\ & := & h^2 & (h^2 \leq g) & p\backslash a & := & h\backslash a & \\ a \cdot p & := & p & (ah = h) & a\backslash p & := & p & (ah = h) \\ & = & ah & (ah \leq g) & & = & a\backslash g & (ah \leq g) \end{array}$$

The remaining cases $p \cdot a$, p/p, a/p and p/a are defined analogously. This gives rise to a new algebra $\boldsymbol{A}^p$ in FL_i^ℓ that fills the gap (g, h) of $\boldsymbol{A}$. □

Remark 6.1.7. *While it is possible to check manually that the algebra $\boldsymbol{A}^p$ defined above is in FL_i^ℓ, the idea is rather to derive $\boldsymbol{A}^p$ by a general construction (Section 6.3). Our approach will explain the rationale behind $\boldsymbol{A}^p$, and provide a general recipe for proving further densifiability results.*

6.2 Residuated frames and Dedekind-MacNeille completions

Just as Kripke frames are useful devices to build various Heyting and modal algebras, residuated frames are useful devices to build various FL algebras. In this section, we introduce residuated frames and recall some relevant facts from [27, 45].

Definition 6.2.1. *A* frame $\boldsymbol{W}$ *(for* FL*-algebras) is a tuple* $(W, W', N, \circ, \varepsilon, \epsilon)$ *where* $(W, \circ, \varepsilon)$ *is a monoid,* $N \subseteq W \times W'$ *and* $\epsilon \in W'$*. It is* residuated *if there are functions* $\backslash\!\backslash : W \times W' \longrightarrow W'$ *and* $/\!/ : W' \times W \longrightarrow W'$ *such that*

$$x \circ y \; N \; z \iff y \; N \; x \backslash\!\backslash z \iff x \; N \; z /\!/ y.$$

We often omit $\circ$ and write xy for $x \circ y$.

Given a frame $\boldsymbol{W} = (W, W', N, \circ, \varepsilon, \epsilon)$, there is a canonical way to make it residuated: let $\tilde{W}' := W \times W' \times W$ and define $\tilde{N} \subseteq W \times \tilde{W}'$ by

$$x \; \tilde{N} \; (v_1, z, v_2) \iff v_1 x v_2 \; N \; z.$$

Then $\tilde{\boldsymbol{W}} := (W, \tilde{W}', \tilde{N}, \circ, \varepsilon, (\varepsilon, \epsilon, \varepsilon))$ is a residuated frame, since

$$\begin{array}{rcl} x \circ y \; \tilde{N} \; (v_1, z, v_2) & \iff & v_1 x y v_2 \; N \; z \\ & \iff & y \; \tilde{N} \; (v_1 x, z, v_2) \\ & \iff & x \; \tilde{N} \; (v_1, z, y v_2). \end{array}$$

As said at the beginning, the primary purpose of residuated frames is to build residuated lattices. Let us now describe the construction.

Let $\boldsymbol{W} = (W, W', N, \circ, \varepsilon, \epsilon)$ be a residuated frame. Given $X, Y \subseteq W$ and $Z \subseteq W'$, let:

$$\begin{array}{rcl} X \circ Y & := & \{x \circ y : x \in X, y \in Y\}, \\ X^{\rhd} & := & \{z \in W' : X \; N \; z\}, \\ Z^{\lhd} & := & \{x \in W : x \; N \; Z\}, \end{array}$$

where $X \; N \; z$ holds iff $x \; N \; z$ for every $x \in X$, and $x \; N \; Z$ iff $x \; N \; z$ for every $z \in Z$.

The pair $(\rhd, \lhd)$ forms a *Galois connection*, i.e. :

$$X \subseteq Z^{\lhd} \quad \Longleftrightarrow \quad X^{\rhd} \supseteq Z,$$

so that $\gamma(X) := X^{\rhd\lhd}$ defines a closure operator on $\mathcal{P}(W)$ (the powerset of W):

1. $X \subseteq \gamma(X)$,
2. $X \subseteq Y \Longrightarrow \gamma(X) \subseteq \gamma(Y)$,
3. $\gamma(\gamma(X)) = \gamma(X)$.

Furthermore, γ is a *nucleus*, namely it satisfies

4. $\gamma(X) \circ \gamma(Y) \subseteq \gamma(X \circ Y)$.

It is for this property that a frame is required to be residuated.

Let $\mathcal{P}(W)$ be the powerset of W and $\gamma[\mathcal{P}(W)] \subseteq \mathcal{P}(W)$ be its image under γ. Then a set X belongs to $\gamma[\mathcal{P}(W)]$ iff it is *Galois-closed*, namely $X = \gamma(X)$, iff $X = Z^{\lhd}$ for some $Z \subseteq W'$. For $X, Y \in \mathcal{P}(W)$, let

$$\begin{array}{rcl} X \circ_\gamma Y & := & \gamma(X \circ Y), \\ X \cup_\gamma Y & := & \gamma(X \cup Y), \\ X \backslash Y & := & \{y : X \circ \{y\} \subseteq Y\}, \\ Y / X & := & \{y : \{y\} \circ X \subseteq Y\}. \end{array}$$

Proposition 6.2.2. *[45] Let $\boldsymbol{W} = (W, W', N, \circ, \varepsilon, \epsilon)$ be a residuated frame. The* dual algebra *defined by*

$$\boldsymbol{W}^+ := (\gamma[\mathcal{P}(W)], \cap, \cup_\gamma, \circ_\gamma, \backslash, /, \gamma(\{\varepsilon\}), \{\epsilon\}^{\lhd})$$

is a complete FL*-algebra.*

As an example, let $\boldsymbol{A} = (A, \wedge, \vee, \cdot, \backslash, /, e, f)$ be an FL-algebra. Then we may define a frame by $\boldsymbol{W_A} := (A, A, N, \cdot, e, f)$, where N is the lattice ordering $\leq$ of $\boldsymbol{A}$. $\boldsymbol{W_A}$ is residuated precisely because $\boldsymbol{A}$ is residuated:

$$a \cdot b \; N \; c \iff b \; N \; a \backslash c \iff a \; N \; c/b.$$

Hence by the previous proposition, $\boldsymbol{W}^+_{\boldsymbol{A}}$ is a complete FL-algebra. We want $\boldsymbol{W}^+_{\boldsymbol{A}}$ to be commutative (resp. contractive, integral, f-bounded, totally ordered) whenever $\boldsymbol{A}$ is. To this purpose, it is useful to recall the notion of structural clauses and quasiequations in Definition 2.3.7. These can be easily expressed as rules for residuated frames. Indeed, let

$$t_1 \leq u_1 \text{ and } \cdots \text{ and } t_m \leq u_m \implies t_{m+1} \leq u_{m+1} \text{ or } \cdots \text{ or } t_n \leq u_n. \qquad (q)$$

be a structural clause and $\boldsymbol{W} = (W, W', N, \circ, \varepsilon, \epsilon)$ a residuated frame. We can naturally translate each t_i into a term over $(\circ, \varepsilon)$, and each u_i into either a variable or ϵ. The resulting terms are still denoted by t_i, u_i. Corresponding to the clause (q), we have:

$$t_1 \; N \; u_1 \text{ and } \cdots \text{ and } t_m \; N \; u_m \implies t_{m+1} \; N \; u_{m+1} \text{ or } \cdots \text{ or } t_n \; N \; u_n. \qquad (q^N)$$

To stress the connection with sequent and hypersequent rules, in what follows we write a structural clause (or quasiequation) such as (q^N) above also in its compact form :

$$\frac{t_1 \; N \; u_1 \text{ and } \cdots \text{ and } t_m \; N \; u_m}{t_{m+1} \; N \; u_{m+1} \text{ or } \cdots \text{ or } t_n \; N \; u_n} \; (q^N)$$

Example 6.2.3. *The clause (wnm) in Example 2.3.16 corresponds to the following rule for residuated frames:*

$$\frac{xy \; N \; z \textit{ and } xv \; N \; z \textit{ and } vy \; N \; z \textit{ and } vv \; N \; z}{xy \; N \; \epsilon \textit{ or } v \; N \; z} \; (wnm^N)$$

Let us come back now to our example, the residuated frame $\boldsymbol{W_A} := (A, A, N, \cdot, e, f)$. The following clauses ensure that $\boldsymbol{W_A^+}$ is commutative (resp. contractive, integral, f-bounded, totally ordered) whenever $\boldsymbol{A}$ is.

$$\frac{xy \; N \; z}{yx \; N \; z} \; (e^N) \qquad \frac{xx \; N \; z}{x \; N \; z} \; (c^N) \qquad \frac{\varepsilon \; N \; z}{x \; N \; z} \; (i^N)$$

$$\frac{x \; N \; \epsilon}{x \; N \; z} \; (o^N) \qquad \frac{x \; N \; z \text{ and } y \; N \; w}{x \; N \; w \text{ or } y \; N \; z} \; (com^N)$$

It is clear that $\boldsymbol{W_A}$ satisfies (e^N) (resp. (c^N), (i^N), (o^N), (com^N)), whenever $\boldsymbol{A}$ is commutative (resp. contractive, integral, f-bounded, totally ordered). These properties are in turn propagated to the dual algebra $\boldsymbol{W_A^+}$. This holds for any residuated frame, as the next lemma shows, in analogy with Theorem 2.3.3.

Lemma 6.2.4. *Let $\boldsymbol{W}$ be a residuated frame. If $\boldsymbol{W}$ satisfies (e^N) (resp. (c^N), (i^N), (o^N), (com^N)), then $\boldsymbol{W}^+$ is commutative (resp. contractive, integral, f-bounded, linearly ordered).*

Proof. We only prove that (com^N) implies that $\boldsymbol{W}^+$ is totally ordered, as for the other rules the proof is straightforward. Suppose that there are $X, Y \in \gamma[\mathcal{P}(W)]$ for which $X \not\subseteq Y$ and $Y \not\subseteq X$. The former means that there are $x \in X$ and $w \in Y^{\triangleright}$ such that $x \; N \; w$ does not hold (since $Y = Y^{\triangleright\triangleleft}$). Similarly, the latter means that there are $y \in Y$ and $z \in X^{\triangleright}$ such that $y \; N \; z$ does not hold. On the other hand, we have $x \; N \; z$ and $y \; N \; w$ by definition of $X^{\triangleright}, Y^{\triangleright}$. Hence the rule (com^N) implies that at least one of $x \; N \; w$ and $y \; N \; z$ should hold, a contradiction. □

Finally, we would like to have an embedding of $\boldsymbol{A}$ into $\boldsymbol{W_A^+}$. The notion of *Gentzen frame* and *Gentzen rules* serve to this purpose.

Definition 6.2.5. *Let $\boldsymbol{A}$ be an* FL-*algebra, $\boldsymbol{W} = (W, W', N, \circ, \varepsilon, \epsilon)$ a residuated frame and $i : A \longrightarrow W$ and $i' : A \longrightarrow W'$ injections by means of which we identify A with a subset of W and of W'. $(\boldsymbol{W}, \boldsymbol{A})$ is said to be a* Gentzen frame *if it satisfies all the* Gentzen rules *in Figure 6.1, for every $x \in W$, $z \in W'$ and $a, b \in A$.*

$$\frac{x\ N\ a \ \text{ and } \ a\ N\ z}{x\ N\ z}\ (\text{cut}) \qquad \frac{}{a\ N\ a}\ (\text{id})$$

$$\frac{x\ N\ a \ \text{ and } \ b\ N\ z}{a\backslash b\ N\ x\backslash\!\!\backslash z}\ (\backslash\text{L}) \qquad \frac{x\ N\ a\backslash\!\!\backslash b}{x\ N\ a\backslash b}\ (\backslash\text{R})$$

$$\frac{x\ N\ a \ \text{ and } \ b\ N\ z}{b/a\ N\ z/\!\!/x}\ (/\text{L}) \qquad \frac{x\ N\ b/\!\!/a}{x\ N\ b/a}\ (/\text{R})$$

$$\frac{a \circ b\ N\ z}{a \cdot b\ N\ z}\ (\cdot\text{L}) \qquad \frac{x\ N\ a \ \text{ and } \ y\ N\ b}{x \circ y\ N\ a \cdot b}\ (\cdot\text{R})$$

$$\frac{a\ N\ z}{a \wedge b\ N\ z}\ (\wedge\text{L}\ell) \qquad \frac{b\ N\ z}{a \wedge b\ N\ z}\ (\wedge\text{L}r) \qquad \frac{x\ N\ a \ \text{ and } \ x\ N\ b}{x\ N\ a \wedge b}\ (\wedge\text{R})$$

$$\frac{a\ N\ z \ \text{ and } \ b\ N\ z}{a \vee b\ N\ z}\ (\vee\text{L}) \qquad \frac{x\ N\ a}{x\ N\ a \vee b}\ (\vee\text{R}\ell) \qquad \frac{x\ N\ b}{x\ N\ a \vee b}\ (\vee\text{R}r)$$

$$\frac{\varepsilon\ N\ z}{e\ N\ z}\ (e\text{L}) \qquad \frac{}{\varepsilon\ N\ e}\ (e\text{R}) \qquad \frac{}{f\ N\ \epsilon}\ (\text{fL}) \qquad \frac{x\ N\ \epsilon}{x\ N\ f}\ (\text{fR})$$

Table 6.1: Gentzen rules

Note that the Gentzen rules in Table 6.1 basically correspond to the logical rules, (cut), and (id) in Table 3.1.

Lemma 6.2.6. *[45]*

1. *If $(\boldsymbol{W}, \boldsymbol{A})$ is a Gentzen frame, then for every $a \in A$, $v(a) := \gamma(\{a\})$ defines a homomorphism $v : \boldsymbol{A} \longrightarrow \boldsymbol{W}^+$.*

2. *If $a\ N\ b$ implies $a \leq_{\boldsymbol{A}} b$ for every $a, b \in A$, then v is an embedding.*

Remark 6.2.7. *Actually Lemma 6.2.6 holds in a more general setting. For instance, $\boldsymbol{A}$ can be an arbitrary, even partial, algebra in the language of* FL*, and i and i' need not be injections as far as the Gentzen rules are satisfied.*

Since $(\boldsymbol{W}_{\boldsymbol{A}}, \boldsymbol{A})$ trivially satisfies the Gentzen rules, we see that $(\boldsymbol{W}^+_{\boldsymbol{A}}, v)$ is a completion of $\boldsymbol{A}$. Moreover, it is join-dense and meet-dense since

$$\begin{array}{rcl} X &=& \bigcup_\gamma\{\gamma(\{a\}) : a \in X\} &=& \bigcup_\gamma\{v(a) : v(a) \subseteq X\} \\ &=& \bigcap\{\{a\}^{\lhd} : X\ N\ a\} &=& \bigcap\{v(a) : X \subseteq v(a)\} \end{array} \qquad (*)$$

holds for every Galois-closed set X. The last equality holds because $v(a) = \{a\}^{\rhd\lhd} = \{a\}^{\lhd}$ and $X\ N\ a$ iff $X \subseteq \{a\}^{\lhd}$.

We have thus found an indirect proof of Lemma 2.3.3.

Corollary 6.2.8. *$(\boldsymbol{W}^{+}_{\boldsymbol{A}}, v)$ is a DM completion of $\boldsymbol{A}$. Hence for every $x \subseteq \{e, c, i, o\}$, every chain $\boldsymbol{A} \in \mathrm{FL}^{\ell}_{x}$ has a DM completion in FL^{ℓ}_{x}.*

In Theorem 2.3.9 and 2.3.13 we have seen that many equations in the classes $\mathcal{N}_2$ and $\mathcal{P}_3$ can be converted into equivalent analytic clauses which are preserved under DM completion (Theorem 2.3.17). Recall that any structural clause (q) can also be seen as a clause (q^N) over residuated frames. Hence, we can reasonably expect that, whenever a residuated frame satisfies an analytic clause, the dual algebra would satisfy the corresponding equation. To make things more precise, we adapt Theorem 2.3.17 as follows.

Theorem 6.2.9. *[24, 26, 27] Let (q) be an analytic clause. If a residuated frame $\boldsymbol{W} = (W, W', N, \circ, \varepsilon, \epsilon)$ satisfies (q^N), then the dual algebra $\boldsymbol{W}^{+} = (\gamma[\mathcal{P}(W)], \cap, \cup_{\gamma}, \circ_{\gamma}, \backslash, /, \gamma(\{\varepsilon\}), \{\epsilon\}^{\lhd})$ satisfies (q).*

Proof. Assume that a residuated frame $\boldsymbol{W}$ satisfies the analytic clause

$$t_1 \; N \; u_1 \;\text{ and }\; \cdots \;\text{ and }\; t_m \; N \; u_m \implies t_{m+1} \; N \; u_{m+1} \;\text{ or }\; \cdots \;\text{ or }\; t_n \; N \; u_n. \qquad (q^N)$$

By the properties of analytic clauses, we can assume that any premise $t_i \; N \; u_i$ has the form $x_{i1} \circ \cdots \circ x_{ik_i} \; N \; u_i$ where the $x_{i_1}, \ldots, x_{ik_i}$ are variables, not necessarily distinct, in $L(q)$ and $u_i \in R(q)$, or $u_i = \epsilon$. Any conclusion $t_i \; N \; u_i$ has the form $x_{i1} \circ \cdots \circ x_{ik_i} \; N \; u_i$, where all the x_{ij}, for $i = m+1, \ldots, n$ and $j = 1, \ldots, k_i$ are *distinct* variables in $L(q)$, and $u_i \in R(q)$, or $u_i = \epsilon$.

To show that the dual algebra $\boldsymbol{W}^{+}$ satisfies the analytic clause (q), we take an arbitrary evaluation that makes the premises of (q) true in $\boldsymbol{W}^{+}$. Hence, for any premise $t_i \; N \; u_i$ with $i \in \{1, \ldots, m\}$, we have $T_i = X_{i1} \circ_{\gamma} \cdots \circ_{\gamma} X_{ik_i} \subseteq U_i$, for some closed sets $X_{i1}, \ldots, X_{ik_i}, U_i$ in $\boldsymbol{W}^{+}$ (where U_i equals $\{\epsilon\}^{\lhd}$, in case the corresponding u_i is ϵ). This can be easily shown to be equivalent to

$$(!) \quad X_{i1} \circ \cdots \circ X_{ik_i} \subseteq U_i$$

for any $i = 1, \ldots, m$. Let us assume now, for contradiction, that $\boldsymbol{W}^{+}$ does not satisfy any of the conclusions of (q) i.e. that for any $i = m+1, \ldots, n$, we have $T_i = X_{i1} \circ_{\gamma} \cdots \circ_{\gamma} X_{ik_i} \not\subseteq U_i$, for the closed sets $X_{i1}, \ldots, X_{ik_i}, U_i$. As for the premises, this is equivalent to say

$$(*) \quad X_{i1} \circ \cdots \circ X_{ik_i} \not\subseteq U_i$$

for any $i = m+1, \ldots, n$. Recall that all the X_{ij}, for $i = m+1, \ldots, n$ and $j = 1, \ldots, k_i$ are distinct. Hence, by $(*)$, for any distinct X_{ij}, we can take an element $x^{\bullet}_{ij} \in X_{ij}$, such that the following holds:

$$(**) \quad x^{\bullet}_{i1} \circ \cdots \circ x^{\bullet}_{ik_i} \in X_{i1} \circ \cdots \circ X_{ik_i} \qquad x^{\bullet}_{i1} \circ \cdots \circ x^{\bullet}_{ik_i} \notin U_i$$

for any $i \in \{m+1, \ldots, n\}, j \in \{1, \ldots, k_i\}$. By the inclusion property of analytic clauses all the $x^{\bullet}_{ij}$ can be used as an evaluation of the variables appearing on $t_1, \ldots t_m$ in the premises of (q^N). By (!) we will have that $x^{\bullet}_{i_1} \circ \cdots \circ x^{\bullet}_{ik_i} \in X_{i1} \circ \cdots \circ X_{ik_i} = T_i \subseteq U_i$, for any $i \in \{1, \ldots, m\}$. As each U_i is Galois-closed, we have $x^{\bullet}_{i1} \circ \cdots \circ x^{\bullet}_{ik_i} \in U_i^{\rhd\lhd}$, which is equivalent to say that

$x_{i1}^{\bullet} \circ \cdots \circ x_{ik_i}^{\bullet} N u_i^{\bullet}$ for a certain $u_i^{\bullet} \in U_i^{\triangleright} \subseteq W'$. We have then $x_{i1}^{\bullet} \circ \cdots \circ x_{ik_i}^{\bullet} N u_i^{\bullet}$ for any $i \in \{1, \ldots, m\}$. Thus, we apply the clause (q^N), obtaining at least one of the conclusions, say without loss of generality, $t_s N u_s$ with $s \in \{m+1, \ldots, k\}$. The conclusion $t_s N u_s$ is evaluated in $\boldsymbol{W}$ as $x_{s1}^{\bullet} \circ \cdots \circ x_{sk_s}^{\bullet} N u_s^{\bullet}$, with $u_s^{\bullet} \in U_s^{\triangleright} \subseteq W'$. This entails $x_{s1}^{\bullet} \circ \ldots \circ x_{sk_s}^{\bullet} \in U_s^{\triangleright\triangleleft} = U_s$, which contradicts $(**)$. □

Example 6.2.10. *Suppose that $\boldsymbol{W}$ satisfies (wnm^N), see Example 6.2.3. Then $\boldsymbol{W}^+$ satisfies (wnm), namely*

$$XY \subseteq Z \text{ and } XV \subseteq Z \text{ and } VY \subseteq Z \text{ and } VV \subseteq Z \implies XY \subseteq \epsilon^{\triangleleft} \text{ or } V \subseteq Z$$

holds for every Galois-closed sets X, Y, V, Z.

6.3 Densification of FL_i-chains

Residuated frames are useful not only for completions, but also for densification. In this section, we use them to prove the densifiability of FL_x^{ℓ} with $\{i\} \subseteq x \subseteq \{e, c, i, o\}$. Our proof gives a rationale behind the concrete definition of $\boldsymbol{A}^p$ in Section 6.1, and moreover serves as a warm-up before the more involved case of (nonintegral) FL_e-chains in the next section.

Let us fix an FL_i-chain $\boldsymbol{A}$, a gap (g, h) in it and a new element p. Our purpose is to define a residuated frame whose dual algebra is an FL_i-chain filling the gap (g, h) by p, as summarized in Figure 6.2.

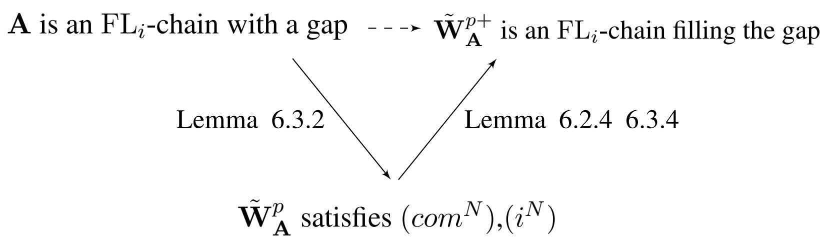

Figure 6.2: The structure of our proof.

We define a frame $\mathbf{W}_{\mathbf{A}}^{p} = (W, W', N, \circ, \varepsilon, \epsilon)$ such that

- $(W, \circ, \varepsilon)$ is the free monoid generated by $A \cup \{p\}$.
- $W' := A \cup \{p\}$, $\epsilon := f \in A$.

Thus each element $x \in W$ is a finite sequence of elements from $A \cup \{p\}$. We denote by A^* the subset of W that consists of finite sequences of elements from A (without any occurrence of p). Also, given $x \in W$ we denote by $\overline{x}$ the product (in $\boldsymbol{A}$) of all elements of x *where p is replaced by h*. For instance, if $x = papb \in W$ with $a, b \in A$, then $\overline{x} = hahb \in A$.

Let us now define the relation N between the two sets of the frame. With the intuition that $g < p < h$ should hold and N should be an extension of $\leq_{\boldsymbol{A}}$, it is natural to require that $a\ N\ p$

iff $a \leq g$, and $p\ N\ a$ iff $h \leq a$ for every $a \in A$. We also require that $p\ N\ p$. The definition below embodies these requirements. For every $x \in W$ and $a \in A$:

$$\begin{array}{llll} x\ N\ a & \Longleftrightarrow & \overline{x} \leq_{\boldsymbol{A}} a & \\ x\ N\ p & \Longleftrightarrow & \overline{x} \leq_{\boldsymbol{A}} g & (\text{if } x \in A^*) \\ x\ N\ p & \text{always holds} & & (\text{otherwise}) \end{array}$$

Compare this definition with the proof of Theorem 4.1.11 in Chapter 4: the substitution of p when appearing on the left with h and when appearing on the right with g perfectly mirrors the substitution $S[^{\Lambda\Rightarrow\Pi}/_{\overline{p}\Rightarrow p}]$ there. Similarly, the stipulation that $x\ N\ p$ always holds if p is in x and its role in the following proof of densification, perfectly matches the role of pp-components in Theorem 4.1.11. As explained in Section 6.2, the frame $\mathbf{W}^p_{\mathbf{A}}$ induces a residuated frame $\tilde{\mathbf{W}}^p_{\mathbf{A}}$. To have a closer look at the residuated frame $\tilde{\mathbf{W}}^p_{\mathbf{A}}$, it is convenient to partition the set $\tilde{W}' = W \times W' \times W$ into three, in accordance with the case distinctions in the definition of N:

$$\begin{array}{lll} \tilde{W}'_1 & := & \{(u,a,v) \in \tilde{W}' : a \in A\}, \\ \tilde{W}'_2 & := & \{(u,p,v) \in \tilde{W}' : u, v \in A^*\}, \\ \tilde{W}'_3 & := & \{(u,p,v) \in \tilde{W}' : u \notin A^* \text{ or } v \notin A^*\}. \end{array}$$

Just as we associated an element $\overline{x} \in A$ to each $x \in W$, we associate an element $\overline{z} \in A$ to each $z \in \tilde{W}'$ as follows:

$$\begin{array}{llll} \overline{z} & := & \overline{u}\backslash a/\overline{v} & (z = (u,a,v) \in \tilde{W}'_1) \\ & := & \overline{u}\backslash g/\overline{v} & (z = (u,p,v) \in \tilde{W}'_2) \\ & := & e & (z = (u,p,v) \in \tilde{W}'_3) \end{array}$$

We finally define $A^\circ := \tilde{W}'_1 \cup \tilde{W}'_3$. A pair $(x,z) \in W \times \tilde{W}'$ is said to be *stable* if either $x \in A^*$ or $z \in A^\circ$. We also say that a statement $x\ \tilde{N}\ z$ is *stable* if (x,z) is. The following lemma explains why we have defined the sets A^*, A° and the concept of stability.

Lemma 6.3.1.

1. *If (x,z) is stable, then $x\ \tilde{N}\ z$ iff $\overline{x} \leq_{\boldsymbol{A}} \overline{z}$. If not, $x\ \tilde{N}\ z$ always holds.*

2. *If $x \notin A^*$, then $\overline{x} \leq_{\boldsymbol{A}} h$.*

3. *If $z \notin A^\circ$, then $g \leq_{\boldsymbol{A}} \overline{z}$.*

Proof. (1) When $z = (u,a,v) \in \tilde{W}'_1$, we have $x\ \tilde{N}\ z$ iff $uxv\ N\ a$ iff $\overline{uxv} \leq a$ iff $\overline{x} \leq \overline{z}$. When $z = (u,p,v) \in \tilde{W}'_3$, both $uxv\ N\ p$ and $\overline{x} \leq e = \overline{z}$ hold. When $z \in \tilde{W}'_2$ and $x \in A^*$, $x\ \tilde{N}\ z$ iff $uxv\ N\ p$ iff $\overline{uxv} \leq g$ iff $\overline{x} \leq \overline{z}$. When $z \in \tilde{W}'_2$ and $x \notin A^*$, $x\ \tilde{N}\ z$ always holds.

(2) $x \notin A^*$ means that the sequence x contains an occurrence of p, that is interpreted by h. Hence the claim holds by integrality. (3) is proved in a similar way. ☐

Lemma 6.3.2. *Let $\boldsymbol{A}$ be an FL_i-chain with a gap (g,h). The residuated frame $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ satisfies the rules (com^N),(i^N).*

Proof. Being $\boldsymbol{A}$ an integral chain, $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ clearly satisfies

$$\frac{uv\ N\ z}{uxv\ N\ z}\ (i^N)$$

Hence the dual algebra $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ is integral by Lemma 6.2.4. We verify:

$$\frac{x\ \tilde{N}\ z \text{ and } y\ \tilde{N}\ w}{x\ \tilde{N}\ w \quad \text{or} \quad y\ \tilde{N}\ z}\ (com^N)$$

In case at least one of the conclusions is not stable, (com^N) holds by Lemma 6.3.1(1). Note that this is always the case when both premises $x\ \tilde{N}\ z$ and $y\ \tilde{N}\ w$ are not stable. Hence we only need to consider the cases when both conclusions are stable and either both or only one of the premises is stable.

(i) If both premises $x\ \tilde{N}\ z$ and $y\ \tilde{N}\ w$ are stable, (com^N) boils down to

$$\overline{x} \leq \overline{z} \text{ and } \overline{y} \leq \overline{w} \implies \overline{x} \leq \overline{w} \quad \text{or} \quad \overline{y} \leq \overline{z},$$

that holds by the communication property in A.

(ii) Assume only one premise is stable. For instance, let $y\ \tilde{N}\ w$ be stable and $x\ \tilde{N}\ z$ not stable. We have $x \notin A^*$, $z \notin A^\circ$. Moreover, as both conclusions $x\ \tilde{N}\ w$ and $y\ \tilde{N}\ z$ are assumed to be stable, we have $w \in A^\circ$, $y \in A^*$. We have either $\overline{y} \leq g$ or $h \leq \overline{y}$ since (g, h) is a gap. If $\overline{y} \leq g$, then $\overline{y} \leq g \leq \overline{z}$ by Lemma 6.3.1(3), so the right conclusion holds. If $h \leq \overline{y}$, Lemma 6.3.1(2) and the right premise imply $\overline{x} \leq h \leq \overline{y} \leq \overline{w}$, so the left conclusion holds. The case where $y\ \tilde{N}\ w$ is not stable and $x\ \tilde{N}\ z$ is stable is symmetrical. □

Lemma 6.3.3. *Let $\boldsymbol{A}$ be an FL_i-chain with a gap (g, h). $(\tilde{\mathbf{W}}^p_{\mathbf{A}}, \boldsymbol{A})$, with the injections i, i' from A to W and W' given by $i(a) := a \in W$ and $i'(a) := (\varepsilon, a, \varepsilon) \in \tilde{W}'$, is a Gentzen frame. Moreover $a\ \tilde{N}\ b$ implies $a \leq_{\mathbf{A}} b$ for every $a, b \in A$. Hence $v(a) := \gamma(\{a\})$ is an embedding of $\boldsymbol{A}$ into $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$.*

Proof. Observe that all Gentzen rules (Figure 6.1, where N is replaced by $\tilde{N}$) have stable premises. If the conclusion is also stable, then it is obtained from the premises by Lemma 6.3.1(1). Otherwise (as it may happen for the rule (cut)), the conclusion holds automatically. □

Lemma 6.3.4. *Let v be the embedding of $\boldsymbol{A}$ into $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ in Lemma 6.3.3. The following hold:*

1. *For every $z \in A \cup \{p\}$, $v(z) = \{z\}^{\rhd\lhd} = \{z\}^{\lhd}$.*
2. *$v(g) \subsetneq \{p\}^{\rhd\lhd} \subsetneq v(h)$.*

Proof. (1) Suppose that $z = a \in A$. We have $a \in \{a\}^{\lhd}$ by (id). Hence $\{a\}^{\rhd\lhd} \subseteq \{a\}^{\lhd}$. To show the other inclusion, let $x \in \{a\}^{\lhd}$ and $z \in \{a\}^{\rhd}$. Then $x\ \tilde{N}\ a$ and $a\ \tilde{N}\ z$, so $x\ \tilde{N}\ z$ by (cut). This shows that $\{a\}^{\lhd} \subseteq \{a\}^{\rhd\lhd}$.

For $z = p$, the above reasoning suggests that it is enough to show :

$$\frac{}{p\ \tilde{N}\ p}\ (\mathrm{Id}) \qquad \frac{x\ \tilde{N}\ p \text{ and } p\ \tilde{N}\ z}{x\ \tilde{N}\ z}\ (\mathrm{cut})$$

(Here we identify p on the right hand side with $(\varepsilon, p, \varepsilon) \in \tilde{W}'$). (id) is obvious. For (cut), if the conclusion is unstable, it holds automatically. Otherwise, we distinguish three cases. If $x \in A^*$ and $z \notin A^\circ$, Lemma 6.3.1(3) and the left premise (which is stable) imply $\overline{x} \leq g \leq \overline{z}$. If $x \notin A^*$ and $z \in A^\circ$, Lemma 6.3.1(2) and the right premise (which is stable) imply $\overline{x} \leq h \leq \overline{z}$. If $x \in A^*$ and $z \in A^\circ$, we have $\overline{x} \leq g < h \leq \overline{z}$.
(2) We have $g \ \tilde{N} \ p$ and $p \ \tilde{N} \ h$, so $g \in \{p\}^{\lhd}$ and $p \in \{h\}^{\lhd}$, that imply $v(g) = \gamma(\{g\}) \subseteq \gamma(\{p\}) \subseteq v(h) = \gamma(\{h\})$ by (1). On the other hand, we have neither $p \ \tilde{N} \ g$ nor $h \ \tilde{N} \ p$ (that would mean $h \leq g$). Hence the two inclusions are strict. □

We have proved that the chain $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ fills the gap (g, h) of $\boldsymbol{A}$. Hence we conclude:

Theorem 6.3.5. FL^{ℓ}_{x} *with* $\{i\} \subseteq x \subseteq \{e, c, i, o\}$ *is densifiable.*

Proof. Let $\boldsymbol{A}$ be an in FL^{ℓ}_{x} with $\{i\} \subseteq x \subseteq \{e, c, i, o\}$ and (g, h) a gap in $\boldsymbol{A}$. By Lemma 6.3.2 $\tilde{\mathbf{W}}^{p}_{\mathbf{A}}$ satisfies $(com), (i)$. Moreover it is easy to see that $\tilde{\mathbf{W}}^{p}_{\mathbf{A}}$ satisfies (e^N), (c^N), (o^N) whenever $\boldsymbol{A}$ satisfies (e), (c), (o). Hence by Lemma 6.2.4, $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ is in FL^{ℓ}_{x} whenever $\boldsymbol{A}$ is. Lemma 6.3.4 shows that $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ fills the gap (g, h) of $\boldsymbol{A}$ and Lemma 6.3.3 shows the existence of an embedding from $\boldsymbol{A}$ to $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$. □

Corollary 6.3.6. *The logics* MTL, psMTL^r *and Gödel logic* G *are standard complete.*

Proof. Recall that the algebraic semantics for the three logics are $\mathrm{FL}^{\ell}_{ew}, \mathrm{FL}^{\ell}_{w}, \mathrm{FL}^{\ell}_{ecw}$, respectively. Hence the claim follows from Lemma 6.1.2 and Theorems 6.3.5 and 6.1.5 □

Structure of $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$

We have obtained a chain $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ filling a gap of $\boldsymbol{A}$, but we have not yet examined what kind of chain it is. By looking into its structure, it turns out that it is just a DM completion of the chain $\boldsymbol{A}^p$ presented in Section 6.1. We will show that the restriction of $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ to $v[A] \cup \{\gamma(\{p\})\}$ forms a subalgebra by giving a concrete description. To simplify the notation, we write

$$\hat{x} := v(x) = \{x\}^{\rhd\lhd} = \{x\}^{\lhd}$$

for every $x \in A \cup \{p\}$ (see Lemma 6.3.4 (1)), and $\hat{x} \cdot \hat{y} := \hat{x} \circ_\gamma \hat{y}$. The lattice structure of $v[A] \cup \{\hat{p}\}$ is already clear (see Lemma 6.3.4 (2)). Moreover, since $v(a) = \hat{a}$ is an embedding, we have $\hat{a} \star \hat{b} = \widehat{a \star b}$ for every $a, b \in A$ and $\star \in \{\cdot, \backslash, /\}$. Hence it is sufficient to determine the operations $\cdot, \backslash, /$ applied to $\hat{a}$ and $\hat{p}$.

Proposition 6.3.7. *For every* $a \in A$, *we have:*

$$\begin{array}{rclcrcll}
\hat{p} \cdot \hat{p} & = & \hat{p} & (h^2 = h) & \hat{p} \backslash \hat{p} & = & \hat{e} & \\
& = & \hat{h}^2 & (h^2 \leq g) & \hat{p} \backslash \hat{a} & = & \widehat{h \backslash a} & \\
\hat{a} \cdot \hat{p} & = & \hat{p} & (ah = h) & \hat{a} \backslash \hat{p} & = & \hat{p} & (ah = h) \\
& = & \widehat{ah} & (ah \leq g) & & = & \widehat{a \backslash g} & (ah \leq g)
\end{array}$$

Similar equalities hold for $\hat{p} \cdot \hat{a}$, $\hat{p}/\hat{p}$, $\hat{a}/\hat{p}$ *and* $\hat{p}/\hat{a}$. *Hence the restriction of* $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ *to* $v[A] \cup \{\hat{p}\}$ *forms a subalgebra that is isomorphic to* $\boldsymbol{A}^p$, *and* $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ *is its DM completion.*

Proof. Note that $\hat{x} \cdot \hat{y} = \gamma(\gamma(\{x\}) \circ \gamma(\{y\})) = \{xy\}^{\rhd\lhd}$. Hence to see the equivalence between $\hat{x} \cdot \hat{y}$ and $\hat{u}$, it is sufficient to check $\{xy\}^{\rhd} = \{u\}^{\rhd}$, which holds exactly when $xy\ \tilde{N}\ z$ iff $u\ \tilde{N}\ z$ for every $z \in \tilde{W}'$.

If $z \in A^{\circ}$, stability implies:

- $pp\ \tilde{N}\ z$ iff $h^2 \leq \overline{z}$ iff $h \leq \overline{z}$ iff $p\ \tilde{N}\ z$ (when $h^2 = h$).
- $pp\ \tilde{N}\ z$ iff $h^2 \leq \overline{z}$ iff $h^2\ \tilde{N}\ z$ (when $h^2 \leq g$).
- $ap\ \tilde{N}\ z$ iff $ah \leq \overline{z}$ iff $h \leq \overline{z}$ iff $p\ \tilde{N}\ z$ (when $ah = h$).
- $ap\ \tilde{N}\ z$ iff $ah \leq \overline{z}$ iff $ah\ \tilde{N}\ z$ (when $ah \leq g$).

If $z \notin A^{\circ}$, both sides of the above four hold by Lemma 6.3.1(1) and (3).

To prove the equalities for $\backslash$, note that $\hat{w} = \{w\}^{\lhd}$ and $\hat{x}\backslash\hat{z} = \{x\}^{\rhd\lhd}\backslash\{z\}^{\lhd} = \{x\}\backslash\{z\}^{\lhd}$ for every $w, x, z \in A \cup \{p\}$. Hence to see $\hat{x}\backslash\hat{z} = \hat{w}$, it is sufficient to check that $xy\ \tilde{N}\ z$ iff $y\ \tilde{N}\ w$ for every $y \in W$.

We have $\hat{p}\backslash\hat{p} = \hat{e}$ and $\hat{p}\backslash\hat{a} = \widehat{h\backslash a}$ since:

- both $py\ \tilde{N}\ p$ and $y\ \tilde{N}\ e$ hold,
- $py\ \tilde{N}\ a$ iff $h\overline{y} \leq a$ iff $\overline{y} \leq h\backslash a$ iff $y\ \tilde{N}\ h\backslash a$.

For the equality for $\hat{a}\backslash\hat{p}$, we distinguish two cases. If $y \in A^{*}$, stability implies:

- $ay\ \tilde{N}\ p$ iff $a\overline{y} \leq g$ iff $\overline{y} \leq g$ iff $y\ \tilde{N}\ p$ (when $ah = h$). For the second equivalence, note that $\overline{y} \leq g$ obviously implies $a\overline{y} \leq g$. Conversely, suppose that $\overline{y} \leq g$ does not hold. Then $h \leq \overline{y}$, so $h = ah \leq a\overline{y}$. Hence $a\overline{y} \leq g$ does not hold.
- $ay\ \tilde{N}\ p$ iff $a\overline{y} \leq g$ iff $\overline{y} \leq a\backslash g$ iff $y\ \tilde{N}\ a\backslash g$ (when $ah \leq g$).

If $y \notin A^{*}$, both sides of the above two hold. In particular, $y\ \tilde{N}\ a\backslash g$ holds since $a\overline{y} \leq ah \leq g$ by Lemma 6.3.1(2), so $\overline{y} \leq a\backslash g$.

Finally $\tilde{\mathbf{W}}_{\mathbf{A}}^{p+}$ is a DM completion of $\boldsymbol{A}^{p}$, since $v[A] \cup \{\hat{p}\} = \{\hat{x} : x \in W\} = \{\{z\}^{\lhd} : z \in W'\}$, and any Galois-closed set X is both a join of elements from the second set and a meet of elements from the third set (see (*) in page 92). ☐

6.4 Densification of FL_e-chains

We now turn to another class of algebras: FL_e-chains. Recall that bounded algebras of this class form the algebraic semantics for UL, see Definition 3.1.5 and Theorem 3.1.6. In this section, we translate the proof-theoretic argument in Chapter 5 into an algebraic one, based on residuated frames. This gives rise to an algebraic proof of standard completeness for uninorm logic.

Let $\boldsymbol{A}$ be an FL_e-chain with a gap (g, h) and p a new element. We again build a residuated frame whose dual algebra fills the gap (g, h). Although we could define W as before, we can exploit commutativity to simplify the construction. We define a frame $\mathbf{W}_{\mathbf{A}}^{p} := (W, W', N, \circ, \varepsilon, \epsilon)$ as follows:

- $W := A \times \mathbb{N}$. Each element $(a, m) \in W$ is denoted by ap^m as if it were a polynomial in the variable p. We identify A with the subset $\{ap^0 : a \in A\}$ of W.
- $ap^m \circ bp^n := (ab)p^{m+n}$, $\varepsilon := e = ep^0$.
- $W' := A \cup \{p\}$, $\epsilon := f \in A$.
- There are three types of elements in $W \times W'$: (ap^n, b), (a, p) and (ap^{n+1}, p) with $a, b \in A$ and $n \in \mathbb{N}$. N is defined accordingly:

$$\begin{array}{rcl} ap^n \ N \ b & \iff & ah^n \leq_{\boldsymbol{A}} b, \\ a \ N \ p & \iff & a \leq_{\boldsymbol{A}} g, \\ ap^{n+1} \ N \ p & \iff & ah^n \leq_{\boldsymbol{A}} e. \end{array}$$

Note that this is compatible with the previous definition. In particular, $ap^{n+1} \ N \ p$ always holds if $\boldsymbol{A}$ is integral. As for the integral case, there is a strong connection between this definition and the substitution used in the proof of density elimination for extensions of HUL∀ in Theorem 5.1.5. Indeed, the definition of N in the first and the second case exactly matches the substitution $S[{}^{\Lambda \Rightarrow \Pi}/_{\overline{p} \Rightarrow p}]$ while the third case matches the substitution for pp-components $S[{}^{\Rightarrow e}/_{p \Rightarrow \underline{p}}])[{}^{\Lambda \Rightarrow}/_{\overline{p} \Rightarrow}]$. As before, from the frame $\mathbf{W}^p_{\mathbf{A}}$ we obtain a residuated frame $\tilde{\mathbf{W}}^p_{\mathbf{A}} := (W, \tilde{W}', \tilde{N}, \circ, \varepsilon, (\varepsilon, \epsilon))$. Because of commutativity, the definitions of $\tilde{W}'$ and $\tilde{N}$ are slightly simplified:

$$\tilde{W}' := W \times W', \qquad x \ \tilde{N} \ (y, z) \text{ iff } x \circ y \ N \ z.$$

As in the integral case the set $\tilde{W}'$ can be partitioned into three:

$$\begin{array}{rl} \tilde{W}'_1 := & \{(ap^n, b) : a, b \in A, n \geq 0\}, \\ \tilde{W}'_2 := & \{(a, p) : a \in A\}, \\ \tilde{W}'_3 := & \{(ap^{n+1}, p) : a \in A, n \geq 0\}. \end{array}$$

Elements of $W, \tilde{W}'$ are again interpreted by elements of A. For $x = ap^n \in W$, let $\overline{x} := ah^n \in A$. For $z \in \tilde{W}'$, we define:

$$\begin{array}{rcll} \overline{z} & := & ah^n \to b & (z = (ap^n, b) \in \tilde{W}'_1), \\ & := & a \to g & (z = (a, p) \in \tilde{W}'_2), \\ & := & ah^n \to e & (z = (ap^{n+1}, p) \in \tilde{W}'_3). \end{array}$$

As before, $A^\circ := \tilde{W}'_1 \cup \tilde{W}'_3$. A pair $(x, z) \in W \times \tilde{W}'$ is *stable* if either $x \in A$ or $z \in A^\circ$. Similarly to Lemma 6.3.1(1), we have:

Lemma 6.4.1. *If (x, z) is stable, then $x \ \tilde{N} \ z$ iff $\overline{x} \leq_{\boldsymbol{A}} \overline{z}$.*

Proof. When $x \in A$ and $z = (a, p) \in \tilde{W}'_2$, we have $x \ \tilde{N} \ z$ iff $xa \ N \ p$ iff $xa \leq_{\boldsymbol{A}} g$ iff $x \leq a \to g$ iff $\overline{x} \leq_{\boldsymbol{A}} \overline{z}$.

When $x = x'p^m$ and $z = (ap^n, b) \in \tilde{W}'_1$, we have $x \ \tilde{N} \ z$ iff $x'ap^{m+n} \ N \ b$ iff $x'ah^{m+n} \leq_{\boldsymbol{A}} b$ iff $x'h^m \leq_{\boldsymbol{A}} ah^n \to b$ iff $\overline{x} \leq_{\boldsymbol{A}} \overline{z}$.

When $x = x'p^m$ and $z = (ap^{n+1}, p) \in \tilde{W}'_3$, we have $x \ \tilde{N} \ z$ iff $x'ap^{m+n+1} \ N \ p$ iff $x'ah^{m+n} \leq_{\boldsymbol{A}} e$ iff $x'h^m \leq_{\boldsymbol{A}} ah^n \to e$ iff $\overline{x} \leq_{\boldsymbol{A}} \overline{z}$. □

Lemma 6.4.2. $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ *satisfies the rule* (com^N).

Proof. We verify:

$$\frac{x\ \tilde{N}\ z \ \text{ and } \ y\ \tilde{N}\ w}{x\ \tilde{N}\ w \ \text{ or } \ y\ \tilde{N}\ z}\ (com^N)$$

(i) If $x, y \in A$ or $w, z \in A^\circ$, all the pairs $\{x, y\} \times \{z, w\}$ are stable. By Lemma 6.4.1, the rule boils down to

$$\overline{x} \leq \overline{z} \text{ and } \overline{y} \leq \overline{w} \implies \overline{x} \leq \overline{w} \text{ or } \overline{y} \leq \overline{z},$$

that holds by the communication property in A.
(ii) Suppose that $w \notin A^\circ$ and $z \notin A^\circ$. Then $w = (a, p)$ and $z = (b, p)$ so that (com^N) becomes:

$$\frac{xb\ N\ p \ \text{ and } \ ya\ N\ p}{xa\ N\ p \ \text{ or } \ yb\ N\ p} \quad \iff \quad \frac{b\ \tilde{N}\ (x,p) \ \text{ and } \ a\ \tilde{N}\ (y,p)}{a\ \tilde{N}\ (x,p) \ \text{ or } \ b\ \tilde{N}\ (y,p)}$$

Since $a, b \in A$, it reduces to the case (i).
(iii) Suppose that $w \in A^\circ$ and $z \notin A^\circ$. We write $w = (w_1, w_2)$ and $z = (a, p)$. There are three subcases.

First, suppose that $x, y \notin A$. Then we may write $x = x'p$ and $y = y'p$ so that (com^N) becomes:

$$\frac{x'\ \tilde{N}\ (pa,p) \ \text{ and } \ y'\ \tilde{N}\ (pw_1,w_2)}{x'\ \tilde{N}\ (pw_1,w_2) \ \text{ or } \ y'\ \tilde{N}\ (pa,p)}$$

Since $(pa, p), (pw_1, w_2) \in A^\circ$, it reduces to the case (i).

Second, suppose that $x \in A$ and $y \notin A$, so that we may write $y = y'p$. Note that $x\ \tilde{N}\ z$ iff $xa\ N\ p$ iff $xa \leq g$. Also, $y\ \tilde{N}\ z$ iff $y'pa\ N\ p$ iff $\overline{y}'a \leq e$. Thus what we have to check is

$$xa \leq g \text{ and } \overline{y} \leq \overline{w} \implies x \leq \overline{w} \text{ or } \overline{y}'a \leq e.$$

By the communication property, the premises imply either $x \leq \overline{w}$ or $\overline{y}a \leq g$. If $x \leq \overline{w}$, we are done. Otherwise, we have $\overline{y}'ha = \overline{y}a \leq g$, so $\overline{y}'a \leq h \to g < e$ (since $g < h$). So we are done.

Finally, suppose that $x \notin A$ and $y \in A$, so that we may write $x = x'p$. Note that $x\ \tilde{N}\ z$ iff $x'pa\ N\ p$ iff $\overline{x}'a \leq e$. Also, $y\ \tilde{N}\ z$ iff $ya\ N\ p$ iff $ya \leq g$. Thus what we have to check is

$$\overline{x}'a \leq e \text{ and } y \leq \overline{w} \implies \overline{x} \leq \overline{w} \text{ or } ya \leq g.$$

If $ya \leq g$, we are done. Otherwise $h \leq ya$. Hence together with the premises we obtain $\overline{x} = \overline{x}'h \leq \overline{x}'ya \leq y \leq \overline{w}$. □

Lemma 6.4.3. *Let* $\boldsymbol{A}$ *be an* FL_e*-chain with a gap* (g, h). $(\tilde{\mathbf{W}}^p_{\mathbf{A}}, \boldsymbol{A})$*, with the injections* i, i' *from* A *to* W *and* W' *given by* $i(a) := a \in W$ *and* $i'(a) := (\varepsilon, a, \varepsilon) \in \tilde{W}'$*, is a Gentzen frame. Moreover* $a\ \tilde{N}\ b$ *implies* $a \leq_{\boldsymbol{A}} b$ *for every* $a, b \in A$*. Hence* $v(a) := \gamma(\{a\})$ *is an embedding of* $\boldsymbol{A}$ *into* $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$.

Proof. As in the proof of Lemma 6.3.3, note that all Gentzen rules except (cut) have stable premises and conclusion. Hence we only have to check the (cut) rule

$$\frac{x \ \tilde{N} \ a \ \text{ and } \ a \ \tilde{N} \ z}{x \ \tilde{N} \ z} \ (\text{cut})$$

where (x, z) is unstable. We may write $x = x'p$ and $z = (b, p)$. By noting that $x \ \tilde{N} \ z$ iff $x'pb \ N \ p$ iff $\overline{x}'b \leq e$, it amounts to

$$\overline{x}'h \leq a \ \text{ and } \ ab \leq g \implies \overline{x}'b \leq e.$$

Now the premises imply $\overline{x}'hb \leq g$, so $\overline{x}'b \leq h \to g < e$.

□

Lemma 6.4.4. *Let v be the embedding of $\boldsymbol{A}$ into $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ in Lemma 6.4.3. The following hold.*

1. *For every $z \in W'$, $v(z) = \{z\}^{\rhd\lhd} = \{z\}^{\lhd}$.*
2. *$v(g) \subsetneq \{p\}^{\rhd\lhd} \subsetneq v(h)$.*

Proof. In view of the proof of Lemma 6.3.4, it is enough to show

$$\frac{x \ \tilde{N} \ p \ \text{ and } \ p \ \tilde{N} \ z}{x \ \tilde{N} \ z} \ (\text{cut}) \qquad \frac{}{pNp} \ (id)$$

(id) is clear. For (cut), If $x \in A$ and $z \in A^\circ$, then all of (x, z), (x, p) and (p, z) are stable. Hence the premises imply $\overline{x} \leq g < h \leq \overline{z}$.

If $x \notin A$ and $z \in A^\circ$, we may write $x = x'p$. The premises amount to $\overline{x}' \leq e$ and $h \leq \overline{z}$, so we obtain $\overline{x} = \overline{x}'h \leq \overline{z}$.

If $x \notin A$ and $z \notin A^\circ$, we may write $x = x'p$ and $z = (a, p)$. The premises amount to $\overline{x}' \leq e$ and $a \leq e$, so we obtain $\overline{x}'a \leq e$.

Finally if $x \in A$ and $z \notin A^\circ$, we may write $z = (a, p)$. The premises amount to $x \leq g$ and $a \leq e$, so we obtain $xa \leq g$.

In any case we obtain the conclusion $x \ \tilde{N} \ z$. □

We have proved that the chain $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ fills the gap (g, h) of $\boldsymbol{A}$. We have the following.

Theorem 6.4.5. *Every variety* FL^{ℓ}_{x} *with* $\{e\} \subseteq x \subseteq \{e, i, o, \bot\}$ *is densifiable.*

Proof. Let $\boldsymbol{A}$ be an in FL^{ℓ}_{x} with $\{e\} \subseteq x \subseteq \{e, i, o, \bot\}$ and (g, h) a gap in $\boldsymbol{A}$. By Lemma 6.4.2 $\tilde{\mathbf{W}}^{p}_{\mathbf{A}}$ satisfies (com). It is easy to see that $\tilde{\mathbf{W}}^{p}_{\mathbf{A}}$ satisfies (e^N) and (i^N), (o^N) whenever $\boldsymbol{A}$ does. Note that $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ is a complete algebra, hence it always satisfies $(\bot)$. By Lemma 6.2.4, $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ is thus in FL^{ℓ}_{x}. Lemma 6.4.4 shows that $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ fills the gap (g, h) of $\boldsymbol{A}$ and Lemma 6.4.3 shows the existence of an embedding from $\boldsymbol{A}$ to $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$. □

Hence we have provided a purely algebraic proof of the following

Corollary 6.4.6. *The logic* UL *is standard complete*

Proof. Follows from Lemma 6.1.2 and Theorem 6.1.5 and 6.4.5. □

Remark 6.4.7. *[44] contains a proof of densification for* FL_e*-chains, which makes use of a simplified version of the residuated frame presented here. The crucial observation there is that, with our definition of the relation* N*, the nucleus* γ_N *identifies the elements* $p \circ p$ *and* $p \circ h$*, i.e.* $\gamma_N(\{p \circ p\}) = \gamma_N(\{p \circ h\})$*. Hence, instead of letting* W *be the full free monoid* $A \cup \{p\}^*$*, it suffices to consider* $(A \cup Ap, \circ)$ *where* $Ap = \{ap \mid a \in A\}$ *and the operation* $\circ$ *is defined in such a way as to extend the operation* $\cdot$ *in* $\boldsymbol{A}$ *and satisfy* $p \circ p = ph$*. Similarly, the set* $\tilde{W}'$ *is restricted in [44] only to* $A \cup A \times \{p\}$*. Under these restrictions, the resulting dual algebra is the same as the one in Lemma 6.4.3, but showing that it is a chain becomes easier. This semplification of the residuated frame in [44] hides however the original connection with the proof-theoretic method of density elimination, which was our main concern here. Note that in [44], also an alternative proof of densification for* FL_e*-chains is provided, based on linear polynomials. This proof, though inspired by the structure of the dual algebra of residuated frames, can be understood in principle without any reference to residuated frames.*

6.5 Densification of subvarieties of FL_i^ℓ

We now focus on subvarieties of FL_i^ℓ defined by $\mathcal{P}_3$ equations. By Theorem 2.3.18 such varieties are always closed under DM completions (applied to chains). However, not all such varieties admit densification. A typical example is the variety BA of Boolean algebras, whose only nontrivial chain is the two element chain. Note that BA is defined by the excluded middle equation $x \vee \neg x = e \in \mathcal{P}_2$, which is equivalent to

$$xy \leq z \quad \Longrightarrow \quad x \leq f \quad \text{or} \quad y \leq z. \tag{em}$$

Inspired by the proof-theoretical approach in Chapter 4, we will reformulate the definition of semi-anchored rules in our setting, so to obtain some criteria for densifiability. Before we proceed further, let us make it precise what it means that the specific residuated frame $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ defined in Section 6.3 satisfies (q^N). Recall that an analytic clause (q) is of the form

$$t_1 \leq z_1 \text{ and } \cdots \text{ and } t_m \leq z_m \implies t_{m+1} \leq z_{m+1} \text{ or } \cdots \text{ or } t_n \leq z_n.$$

For the purpose of this section, it is convenient to write (q) as $P \Longrightarrow C$, where

$$\begin{array}{rcl} P & := & \{t_1 \leq z_1, \ldots, t_m \leq z_m\}, \\ C & := & \{t_{m+1} \leq z_{m+1}, \ldots, t_n \leq z_n\}. \end{array}$$

Recall that each equation in P and C consists of variables $L(q)$ and $R(q)$. To each $x \in L(q)$ we associate an element $x^\bullet \in W = (A \cup \{p\})^*$, so that each term t is interpreted by $t^\bullet \in W$. Likewise, to each $z \in R(q)$ we associate a triple $z^\bullet \in \tilde{W}' = W \times W' \times W$, where $W' = A \cup \{p\}$. The interpretations of constants e, f are already fixed: $e^\bullet := \varepsilon \in W$ and $f^\bullet := (\varepsilon, \epsilon, \varepsilon) = (\varepsilon, f, \varepsilon) \in \tilde{W}'$. It is now clear when $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ satisfies (q^N). It is true just in case the following holds for each such interpretation $\bullet$:

$$\{t^\bullet \ \tilde{N}\ z^\bullet : t \leq z \in P\} \implies \{t^\bullet \ \tilde{N}\ z^\bullet : t \leq z \in C\}. \tag{*}$$

Let us now come back to criteria for densifiability. In the following we adapt Definition 4.1.2 to our context.

Definition 6.5.1. *Let $(q) : P \Longrightarrow C$ be an analytic clause and $(x, z) \in L(q) \times R(q)$. We say that (x, z) is an* anchored pair *for (q) if there is a conclusion of (q) of the form $t \leq z$ such that x is in t. We call $(x, z) \in L(q) \times R(q)$* unanchored pair *if $z \neq f$ and (x, z) is not anchored. We say that a set of (un)anchored pairs $\{(x_1, z), \ldots, (x_n, z)\}$ is* contained *in a premise $t \leq z$ if all the variables $x_1, \ldots, x_n$ appear in t.*

Definition 6.5.2. *A clause (q) is said to be* anchored *if, for each premise $t \leq z$, either $z = f$ or $t \leq z$ contains only anchored pairs.*

Clearly the clause (em) is not anchored, as (x, z) is an unanchored pair, while any analytic quasiequation is anchored, due to the inclusion condition, see Definition 2.3.12.

Lemma 6.5.3. *Let $\boldsymbol{A}$ be an FL_i-chain with a gap (g, h) and (q) an anchored analytic clause. If $\boldsymbol{A}$ satisfies (q), then the residuated frame $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ in Section 6.3 satisfies (q^N). In particular, if $\boldsymbol{A}$ satisfies an analytic quasiequation (q), $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ satisfies (q^N).*

Proof. Assume that $\boldsymbol{A}$ satisfies an anchored clause (q). Our goal is to verify $(*)$ above. If there is a conclusion $t \leq z \in C$ such that $(t^\bullet, z^\bullet)$ is not stable, then we have $t^\bullet\ \tilde{N}\ z^\bullet$ by Lemma 6.3.1(1), so $(*)$ holds.

Otherwise, $(t^\bullet, z^\bullet)$ is stable for every $t \leq z \in C$, so that, by Lemma 6.3.1(1), $t^\bullet\ \tilde{N}\ z^\bullet$ iff $\overline{t}^\bullet \leq \overline{z}^\bullet$.

We claim that the same holds for each premise $t \leq z \in P$. Suppose for a contradiction that $t^\bullet \notin A^*$ and $z^\bullet \notin A^\circ$. The former means that t contains a variable x such that $x^\bullet \notin A^*$, i.e., the sequence $x^\bullet$ contains p. Since x is anchored, there must be a conclusion $u \leq z \in C$ (with x occurring in u), so that $(u^\bullet, z^\bullet)$ is not stable. But that has been already ruled out.

As a consequence, $(*)$ amounts to

$$\{\overline{t}^\bullet \leq_{\boldsymbol{A}} \overline{z}^\bullet : t \leq z \in P\} \implies \{\overline{t}^\bullet \leq_{\boldsymbol{A}} \overline{z}^\bullet : t \leq z \in C\},$$

that holds since $\boldsymbol{A}$ satisfies (q). □

The previous lemma does not apply to many clauses. For instance, it does not apply to (wnm):

$$xy \leq z \text{ and } xv \leq z \text{ and } vy \leq z \text{ and } vv \leq z \implies xy \leq 0 \text{ or } v \leq z, \qquad (wnm)$$

since (x, z) and (y, z) are unanchored. To deal with this and more involved clauses, we need to extend the definition of anchoredness. In the sequel, we write $t = t(x_1, \ldots, x_n)$ to indicate variables $x_1, \ldots, x_n$ occurring in the term t. $t(y_1, \ldots, y_n)$ then denotes the result of substituting y_i for x_i. In analogy with Definition 4.1.3, we can now give the definition of semi-anchored clauses.

Definition 6.5.4. *Let $(q) : P \Longrightarrow C$ be an analytic clause. We say that (q) is* semi-anchored *iff for every set of unanchored pairs $\{(x_1, z), \ldots, (x_j, z)\}$ which is contained in some premise $t(x_1, \ldots, x_j) \leq z$ of (q), one of the following holds:*

1. *There is a premise $t(y_1, \dots, y_j) \leq z$ and $(y_1, z), \dots, (y_j, z)$ are anchored pairs (with $y_1, \dots, y_j$ not necessarily distinct variables).*

2. *There is a premise $t(x_1, \dots, x_j) \leq w$ and $(x_1, w), \dots, (x_j, w)$ are anchored pairs.*

3. *There is a premise $t(y_1, \dots, y_j) \leq w$ and $(x_1, w), \dots, (x_j, w), (y_1, z), \dots, (y_j, z)$ are anchored pairs (with $y_1, \dots, y_j$ not necessarily distinct variables).*

Example 6.5.5. *The analytic clauses*

$$xy \leq z \text{ and } xv \leq z \text{ and } vy \leq z \text{ and } vv \leq z \implies xy \leq 0 \text{ or } v \leq z, \qquad (wnm)$$

and

$$yx \leq z_1 \text{ and } wx \leq z_1 \text{ and } yx \leq z_2 \text{ and } wx \leq z_2 \implies wy \leq z_2 \text{ or } x \leq z_1 \qquad (\Omega_3)$$

and

$$x \leq z \text{ and } y \leq w \implies x \leq w \text{ or } y \leq z \qquad (com)$$

are semi-anchored (see Examples 4.1.6, 4.1.7, 4.1.8).

Lemma 6.5.6. *Let $\boldsymbol{A}$ be an FL_i-chain with a gap (g, h) and $(q) : P \implies C$ a semi-anchored analytic clause. If $\boldsymbol{A}$ satisfies (q), then $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ satisfies (q^N).*

Proof. Our purpose is again to show that

$$\{t^\bullet \ \tilde{N} \ z^\bullet : t \leq z \in P\} \implies \{t^\bullet \ \tilde{N} \ z^\bullet : t \leq z \in C\} \qquad (*)$$

holds in $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ for every interpretation $\bullet$. As in the previous proof, we may assume that $(t^\bullet, z^\bullet)$ is stable for every conclusion $t \leq z$ in C. But this time we cannot assume that this holds for all premises.

So let $t_i \leq z_i$ be a premise that violate stability, namely $z_i^\bullet \notin A^\circ$ and $t_i^\bullet \notin A^*$. We can write our premise as $t_i(x_1, \dots, x_j) \leq z_i$ where $x_1, \dots, x_j$ are all the variables appearing in t_i, such that $x_1^\bullet, \dots, x_j^\bullet \notin A^*$.

By Lemma 6.3.1 (2) and (3), we have:

$$g \leq \overline{z_i^\bullet} \qquad \overline{x_j^\bullet} \leq h \ (1 \leq j \leq n). \qquad (!)$$

Note that, if any of the pairs $(x_1, z_i), \cdots, (x_j, z_i)$ would be anchored, a conclusion component, $t \leq z_i$ would contain it, and hence $(t^\bullet, z_i^\bullet)$ would not be stable. But we have already ruled out this possibility, so we can assume that $(x_1, z_i), \dots, (x_j, z_i)$ are all unanchored pairs. By the definition of semi-anchored clause three cases can occur.

1. There is a premise $t_s \leq z_i$ with $t_s = t_i(y_1, \dots, y_j)$ and $(y_1, z_i), \dots, (y_j, z_i)$ anchored pairs. We can safely assume that $(t_s^\bullet, z_i^\bullet)$ is stable, as otherwise we would have a non stable conclusion. Hence, by Lemma 6.3.1(1) we have $\overline{t_s^\bullet} \leq \overline{z_i^\bullet}$. Since (g, h) is a gap, we have that either $\overline{y_k^\bullet} \leq g$ or $h \leq \overline{y_k^\bullet}$ for each $1 \leq k \leq j$. We distinguish two cases.

   (a) There is some $y_k^\bullet$ such that $\overline{y_k^\bullet} \leq g$. As (y_k, z_i) is anchored, this means that there is a conclusion $t \leq z_i$ in C such that t contains y_k. We have hence $\overline{t^\bullet} \leq \overline{y_k^\bullet} \leq g \leq \overline{z_i^\bullet}$ by integrality and (!). So we have obtained a true conclusion $t^\bullet \ \tilde{N}\ z_i^\bullet$.

   (b) For every $y_k^\bullet$ we have $h \leq \overline{y_k^\bullet}$. Using this fact and (!) we obtain

   $\overline{t_i^\bullet} = \overline{t_i(x_1, \dots, x_j)}^\bullet \leq \overline{t_i(h, \dots, h)}^\bullet \leq \overline{t_i(y_1, \dots, y_j)}^\bullet = \overline{t_s^\bullet} \leq z_i^\bullet$. In other words, we have that for the nonstable premise $t_i^\bullet \ \tilde{N}\ z_i^\bullet$, the inequation $\overline{t_i}^\bullet \leq \overline{z_i^\bullet}$ holds in $\boldsymbol{A}$.

2. There is a premise $t_i \leq w$ with $(x_1, w), \dots, (x_j, w)$ anchored pairs. We can safely assume that $(t_i^\bullet, w^\bullet)$ is stable, hence by Lemma 6.3.1 (1) we have $\overline{t_i^\bullet} \leq \overline{w}^\bullet$. As (g, h) is a gap, we have that either $h \leq \overline{w}^\bullet$ or $\overline{w}^\bullet \leq g$.

   (a) In case $h \leq \overline{w}^\bullet$, recall that $(x_1, w), \dots, (x_j, w)$ are anchored, hence we have a conclusion of the kind $t \leq w$ such that t contains $x_1, \dots, x_j$. From this follows that $t^\bullet \notin A^*$. Hence by Lemma 6.3.1 (2) we have $\overline{t}^\bullet \leq h \leq \overline{w}^\bullet$ which means that we have a true conclusion $t^\bullet \ \tilde{N}\ w^\bullet$.

   (b) In case $\overline{w}^\bullet \leq g$, by (!) we have $\overline{t_i^\bullet} \leq \overline{w}^\bullet \leq g \leq \overline{z_i^\bullet}$. In other words, we have that for the nonstable premise $t_i \ \tilde{N}\ z_i$, the inequation $\overline{t_i}^\bullet \leq \overline{z_i^\bullet}$ holds in $\boldsymbol{A}$.

3. There is a premise $t_s \leq w$ where $t_s = t_i(y_1, \dots, y_j)$ and all the pairs $(x_1, w), \dots, (x_j, w)$, $(y_1, z_i), \dots, (y_j, z_i)$ are anchored tuples. We can safely assume that $(t_s^\bullet, w^\bullet)$ is stable, hence by Lemma 6.3.1 (1) we have $\overline{t_s^\bullet} \leq \overline{w}^\bullet$. As (g, h) is a gap, we have that either $h \leq \overline{w}^\bullet$ or $\overline{w}^\bullet \leq g$. If $h \leq \overline{w}^\bullet$, we can proceed as in case $(2a)$. Assume now that $\overline{w}^\bullet \leq g$. By $\overline{t_s^\bullet} \leq \overline{w}^\bullet$ and (!) we get that $\overline{t_s^\bullet} \leq \overline{z_i}^\bullet$. From here we can proceed exactly as in case 1.

Summing up, for any nonstable premise $t_i \ \tilde{N}\ z_i$, we have shown that either a conclusion is true (cases (1a) and (2a)), in which case we are done, or that (cases (1b) and (2b)) $\overline{t_i^\bullet} \leq \overline{z_i^\bullet}$ holds in $\boldsymbol{A}$. Recall that the latter inequality is also true for all the stable premises by Lemma 6.3.1 (1). Hence in the worst case our claim follows, just applying

$$\{\overline{t}^\bullet \leq_{\boldsymbol{A}} \overline{z}^\bullet : t \leq z \in P\} \implies \{\overline{t}^\bullet \leq_{\boldsymbol{A}} \overline{z}^\bullet : t \leq z \in C\},$$

which holds since $\boldsymbol{A}$ satisfies (q).

□

To state our main theorem, let us call an equation *semi-anchored* if it is equivalent to a set of semi-anchored analytic clauses in the FL_i-chains.

Theorem 6.5.7. *Let* V *be a nontrivial subvariety of* FL_i^ℓ *defined over* FL_i^ℓ *by semi-anchored equations. Then* V *is densifiable.*

Proof. By Theorems 2.3.9 and 2.3.13 we have a set Q of semi-anchored analytic clauses equivalent to the defining equations of V. Let $\boldsymbol{A} \in$ V be a chain with a gap (g, h). $\boldsymbol{A}$ satisfies all the clauses in Q, hence, by Lemma 6.5.6, the residuated frame $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ defined there satisfies (q^N) for all $(q) \in Q$. By Lemma 6.3.3 $\boldsymbol{A}$ is embeddable in the dual algebra $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ and by Lemma 6.3.4 $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ fills the gap (g, h) of $\boldsymbol{A}$. Finally, by Theorem 6.2.9 $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ satisfies Q, hence it belongs to V. ☐

We thus obtained the following

Theorem 6.5.8. *Let* L *be any axiomatic extension of* psMTL^r *with semi-anchored axioms.* L *is standard complete*

Proof. Follows by Lemma 6.1.2 and Theorems 6.5.7 and 6.1.5. ☐

6.6 Densification of subvarieties of FL^ℓ_e

Now we turn our attention to subvarieties of FL^ℓ_e-algebras. The situation here is considerably more complicated than for FL^ℓ_i, and for instance it is not clear how to translate the general result on nonlinear axioms in Chapter 4. We thus limit ourselves to the subvarieties of FL^ℓ_e defined by *knotted axioms* $x^m \leq x^n$, with distinct $m, n > 1$, reformulating the proof-theoretic result in Section 5.2 in algebraic terms. To begin with, we translate the Lemma 5.2.1 in the algebraic context.

Lemma 6.6.1. *Let $\boldsymbol{A}$ be an* FL_e*-chain satisfying $x^m \leq x^n$ for some distinct $m, n > 0$. Then $\boldsymbol{A}$ satisfies the following quasiequations.*

$$xxy \leq e \implies xy \leq e, \tag{c_e}$$

$$xy \leq e \implies xxy \leq e. \tag{w_e}$$

Proof. Note that (c_e) and (w_e) are mutually derivable in FL_e-chains. Therefore we will only show that (c_e) holds in case $m < n$. For $n < m$, we can prove in a symmetric way that (w_e) holds. Given $a, b \in A$, assume that $aab \leq e$ holds in $\boldsymbol{A}$. It implies (1) $a^{2n}b^n \leq e$. We have either $e \leq a$ or $a \leq e$. In the former case, we immediately obtain $ab \leq aab \leq e$. In the latter case, we have (2) $a^n \leq a^l$ for every $l \leq n$. Now choose $k, l \in \mathbb{N}$ such that $2n = k(n-m) + l$ and $m \leq l < n$. Note that, given that $m \leq l$, we have by the knotted axiom $a^l = a^{(l-m)}a^m \leq a^{(l-m)}a^n = a^l a^{n-m}$. Hence we get (3) $a^l \leq a^l a^{k(n-m)} = a^{2n}$. By combining (1) - (3), we obtain $a^n b^n \leq a^l b^n \leq a^{2n} b^n \leq e$. Since $\boldsymbol{A}$ is a chain, it can be easily shown that $ab \leq e$ follows from the latter. ☐

Recall that any knotted equation $x^m \leq x^n$ is an acyclic $\mathcal{N}_2$ equation, and hence it is equivalent in FL_e-chains to the simple analytic quasiequation (see Example 2.3.15):

$$x_1^n \leq z \text{ and } \ldots \text{ and } x_m^n \leq z \implies x_1 \cdots x_m \leq z. \tag{$knot^n_m$}$$

Lemma 6.6.2. *Let $\boldsymbol{A}$ be an* FL_e*-chain with a gap* (g,h)*, satisfying* $(knot^n_m)$ *for some* $m,n>1$*. The residuated frame* $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ *defined in Section 6.4 satisfies* $(knot^{n\,N}_m)$*.*

Proof. We need to show that $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ satisfies

$$x_1^n \ \tilde{N}\ z \ \text{ and } \ \ldots \ \text{ and } \ x_m^n \ \tilde{N}\ z \implies x_1\cdots x_m \ \tilde{N}\ z, \qquad (knot^{n\,N}_m)$$

for every $x_1,\ldots,x_m \in W = A\times\mathbb{N}$ and $z\in\tilde{W}' = W\times W' = W\times(A\cup\{p\})$.

The conclusion is stable if and only if all the premises are. If this is the case, the claim easily follows from Lemma 6.4.1 and from the fact that $\boldsymbol{A}$ satisfies $(knot^n_m)$.

So assume that some of the premises violate stability, for instance, without loss of generality, $x_1^n\ \tilde{N}\ z,\ldots,x_k^n\ \tilde{N}\ z$ with $1\le k\le m$. This means that there are $a_1,\ldots,a_m,b\in A$ and natural numbers $e_1,\ldots,e_k\ge 1$ such that

$$z=(b,p),\quad x_i=a_ip^{e_i}\ (1\le i\le k),\quad x_j=a_j\ (k+1\le j\le m).$$

Then $(knot^{n\,N}_m)$ amounts to:

$$\left.\begin{array}{ccc} a_1^n b h^{ne_1-1}\le e, & \ldots, & a_k^n b h^{ne_k-1}\le e,\\ a_{k+1}^n b\le g, & \ldots, & a_m^n b\le g \end{array}\right\} \implies a_1\cdots a_m b h^{e_1+\cdots+e_k-1}\le e.$$

By combining all the premises on the first line and by applying Lemma 6.6.1 (noting that $n>1$), we obtain

$$a_1^n\cdots a_k^n b^k h^l\le e \qquad (*)$$

for any $l\ge 1$. By combining all those on the second line, we obtain

$$a_{k+1}^n\cdots a_m^n b^{m-k}\le g^{m-k}\le h^{m-k}. \qquad (**)$$

If $e_1+\cdots+e_k-1\ge 1$, the two inequalities (*) and (**) with $l:=m-k+1$ imply $a_1^n\cdots a_m^n b^m h\le e$, which leads to the conclusion by Lemma 6.6.1.

Otherwise $k=1$ and $e_1=1$. Since $m>1$, we have $m-k\ge 1$. Hence (*) and (**) with $l:=m-k$ implies $a_1^n\cdots a_m^n b^m\le e$, which leads to the conclusion. □

Finally, we obtain the main theorem of this section.

Theorem 6.6.3. *Let* V *be a subvariety of* FL_e^ℓ *defined by* $x^m\le x^n$ *with* $m,n>1$*. Then* V *is densifiable.*

Proof. Let $\boldsymbol{A}\in$ V be a chain with a gap (g,h). By Lemma 6.6.2, $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ satisfies $(knot^{n\,N}_m)$. Hence $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ satisfies $(knot^{n\,N}_m)$ by Theorem 6.2.9, i.e. $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}\in$ V. The rest of the proof follows that of Theorem 6.4.5. □

Corollary 6.6.4. *Let* L *be any axiomatic extension of* UL *with knotted axioms* $x^m\le x^n$ *with distinct* $m,n>1$*.* L *is standard complete*

Proof. Follows by Lemma 6.1.2 and Theorems 6.6.3 and 6.1.5. □

Note that Corollary 6.6.4 does not apply to $x^m \leq x^1$ and $x^1 \leq x^n$ with $m, n > 1$, which are respectively equivalent to $x^2 \leq x$ and $x \leq x^2$ in FL^ℓ. These cases have been covered in the purely proof-theoretical result for nonlinear axioms in Chapter 5. Translating that result into our algebraic framework is viable, but it would require the construction of a residuated frame different from the one given in Section 6.4.

CHAPTER 7

Conclusions and open problems

We have presented general sufficient conditions for standard completeness for both integral and non-integral logics, via proof theoretic methods. Moreover, we have introduced a new algebraic approach to address standard completeness, inspired by the proof-theoretic techniques. Using the proof-theoretic approach, we have obtained the following results:

- Any semianchored $\mathcal{P}_3$-extension of $\mathrm{MTL}\forall$ is standard complete (Theorem 4.1.12).
- Any extension of $\mathrm{UL}\forall$ with nonlinear axioms and/or mingle is standard complete (Theorem 5.1.8).

Using the new algebraic method, we have reformulated the proof-theoretical results above, and in addition we obtained the following:

- Any semianchored $\mathcal{P}_3$-extension of $\mathrm{psMTL}^{\mathrm{r}}$ is standard complete (Theorem 6.5.8).

Theorems 4.1.12, 5.1.8 and 6.5.8 show in a uniform way standard completeness for logics which have thus far been treated individually. Our results also extend to infinitely many logics not known before to be standard complete. In Table 7.1 we summarize all standard completeness results already known in the literature, for extensions of $\mathrm{UL}\forall$, $\mathrm{MTL}\forall$ and $\mathrm{psMTL}^{\mathrm{r}}$ with axioms within the class $\mathcal{P}_3$ of the substructural hierarchy. This table also includes some logics, not previously known in the literature and for which our methods apply. We refer the reader to Tables 3.2 and 3.3 for the axioms (and corresponding rules) mentioned in Table 7.1.

Further research directions and open problems

Our results naturally raise many further research questions, some of which are discussed in this section.

	Known results	Selected new results
Theorem 4.1.12	• $\mathrm{MTL}\forall$ [58,71] • $\mathrm{SMTL}\forall = \mathrm{MTL}\forall + (lq)$ [42] • $\mathrm{MTL}\forall + (c_n)$ [23] [1] • $\mathrm{WNM}\forall = \mathrm{MTL}\forall + (wnm)$ [42] • $\mathrm{GHP}\forall = \mathrm{MTL}\forall + (wnm1)$ [1] • $\mathrm{WNM}\forall + (inv_k)$ [73] [1] • $\Omega_{\mathrm{n}}\mathrm{MTL}\forall = \mathrm{MTL}\forall + (\Omega_n)$ [53][1] • $\mathrm{G}\forall$ [49]	• $\mathrm{MTL}\forall + (wnm^n)$ • $\mathrm{MTL}\forall + (wnm1^n)$ • ... use *AxiomCalc (see Section 4.2) to find more.*
Theorem 5.1.8	• $\mathrm{UL}\forall$ [66] • $\mathrm{UML}\forall = \mathrm{UL}\forall + (c) + (mgl)$ [66][1] • $\mathrm{UL}\forall + (\alpha^{n-1} \leftrightarrow \alpha^n)$ for $n > 1$ [81][1]	• $\mathrm{UL}\forall + (c)$ • $\mathrm{UL}\forall + (mgl)$ • $\mathrm{UL}\forall + (knot_k^n)$ for $k, n > 1$ • $\mathrm{UL}\forall + (fknot_k^n)$ for $n > 1$. • ...
Theorem 6.5.8	• $\mathrm{psMTL^r}$ [59]	• $\mathrm{psMTL^r} + (lq)$ • $\mathrm{psMTL^r} + (c_n)$ • $\mathrm{psMTL^r} + (wnm^n)$ • $\mathrm{psMTL^r} + (wnm1^n)$ • $\mathrm{psMTL^r} + (\Omega_n)$ • $\mathrm{psMTL^r} + (inv_k)$ • ...

Table 7.1: An overview: examples of standard complete logics.

Involutive logics

Our results do not apply to logics with an involutive negation, i.e. logics where the axiom

$$(inv) \quad \neg\neg\varphi \to \varphi$$

is valid. The axiom (inv) is in the class $\mathcal{N}_3$ in the substructural hierarchy and there is no equivalent external or internal structural rule for it. Calculi for logics including the axiom (inv) usually employ *multiple conclusioned hypersequents*, i.e. hypersequents whose components can contain a multiset of formulas on the right hand side, see e.g. [30,66]. Standard completeness was proved for few involutive logics only, using ad hoc proofs. For the involutive monoidal t-norm logic IMTL, i.e. $\mathrm{MTL} + (inv)$, standard completeness has been proved proof-theoretically in [66]

[1] The result was actually proven in the literature only for the propositional case.

and algebraically in [41]. Algebraic proofs of standard completeness have also been given for IMTL with the n-contraction axioms [23]. The situation is more complicated in the case without weakening. For the involutive uninorm logic IUL, i.e. $\mathrm{UL}+(inv)$, standard completeness is still an open problem. However, the problem was settled positively, proof-theoretically for IUML, i.e. IUL with the addition of mingle and contraction [66], and algebraically for IUL with the addition of both n-contraction and n-mingle [83]. For IUL, neither algebraic nor proof-theoretic methods have been successful so far. Note that a hypersequent calculus HIUL for IUL is just the multiple conclusion version of the hypersequent calculus HUL. Hence, standard completeness for IUL (or, in case it is not standard complete, of some minimal extension of it) could be shown by proving density elimination in the corresponding hypersequent calculus. However, there is no immediate way to show this, by adapting the substitution method used in Chapter 5. Indeed a derivation ending in an application of density could contain hypersequents of the form $H \mid \Theta_1, p, p \Rightarrow \Pi_1 \mid \Theta_2 \Rightarrow p, p, \Pi_2$, with multiple occurrences of p both on the left and on the right hand side of the components (p being, as usual, the propositional eigenvariable appearing in the premises of the application of density). Consider, for instance, the following:

$$\frac{p \Rightarrow p \quad p \Rightarrow p}{p, p \Rightarrow \ \mid \ \Rightarrow p, p}\,(com)$$

It is unclear which substitution procedure is needed to deal with a derivation containing a rule application like the one above. No simple variant of the trick used in Chapter 5, based on substituting $p \Rightarrow p$ with $\Rightarrow e$, seems possible. This problem calls for new approaches for proving density elimination, which go perhaps beyond the idea of performing a global substitution method. Tracking all the occurrences of axioms of the form $p \Rightarrow p$ in a derivation, and a careful preliminary analysis of the structure of proofs seems to be needed. Insights facilitating this effort might also come from the algebraic method based on residuated frames, which, though closely related to the proof-theoretic method, has the advantage of avoiding some of the technical details appearing in the proofs of density elimination.

Noncommutative logics

Another open problem concerns the noncommutative variant of UL, i.e. the logic psUL^r (see [21]). It was shown in [82] that this logic is not complete with respect to dense chains. It is still not clear which axioms should be added to psUL^r to obtain the smallest logic complete with respect to dense chains, and how this reflects in the proofs of density elimination.

Logics axiomatized by $\mathcal{N}_3$ axioms

An important open question is related to the substructural hierarchy: it is not yet known how to deal proof-theoretically with the axioms in the class $\mathcal{N}_3$. For these axioms, no equivalent structural internal or external rules admitting cut-elimination can be defined along the lines of [25]. On the algebraic side indeed, the corresponding equations are in general not preserved under completions. The class is nevertheless very important for mathematical fuzzy logic, as it contains the defining axioms of logics such as BL, Łukasiewicz and Product logic. These logics have been proved to be standard complete at the propositional level, though only in the finite strong form (see the Remark 3.1.7), but density elimination has not been yet addressed for any of the calculi developed for them, see [68–70].

Necessary conditions for $\mathcal{P}_3$-extensions of $\mathrm{MTL}\forall$

In Chapter 4 we have identified a broad class of hypersequent rules that preserve density elimination when added to the calculus $\mathrm{HMTL}\forall$. Note however that our condition is sufficient but not necessary for density elimination. In particular, since the condition depends only on the syntactic shape of single rules, it cannot take into account the possible effects of the interaction of different rules in the density elimination procedure. Consider for instance the axiom $(\alpha^{n-1} \to \beta) \vee (\beta \to \alpha \cdot \beta)$. It is proved in [53] that the logic obtained by adding to MTL this axiom and the n-contraction axiom $\alpha^{n-1} \to \alpha^n$ is standard complete. However, the rule corresponding to $(\alpha^{n-1} \to \beta) \vee (\beta \to \alpha \cdot \beta)$ i.e.

$$\frac{\Gamma_1, \Sigma_2 \Rightarrow \Pi_2 \quad \{\Gamma_1, \Gamma_i, \Sigma_1 \Rightarrow \Pi_1\}_{2 \leq i \leq n}}{\Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \mid \Gamma_2, \ldots, \Gamma_n, \Sigma_2 \Rightarrow \Pi_2} \; (*)$$

is not semi-anchored. In Example 4.1.7 we considered instead the rules (Ω_n), which are semi-anchored and equivalent to the conjunction of $(\alpha^{n-1} \to \beta) \vee (\beta \to \alpha \cdot \beta)$ and $\alpha^{n-1} \to \alpha^n$. Hence, even though $\mathrm{HMTL}\forall + (\Omega_n)$ and $\mathrm{HMTL}\forall + (*) + (c_n)$ are calculi for the same logic, our density elimination method works only for the former.

$\mathcal{N}_2$ and $\mathcal{P}'_3$-extensions of $\mathrm{UL}\forall$

In Chapter 5 we have shown standard completeness for extensions of $\mathrm{UL}\forall$ with nonlinear axioms and (mgl). This covers all the proofs of standard completeness known in the literature for $\mathcal{N}_2$-extensions of $\mathrm{UL}\forall$, see [66,70,81]. Nonlinear axioms are however a proper subset of acyclic $\mathcal{N}_2$ axioms; in particular our proof does not extend to *every* analytic internal rule (i.e. any acyclic $\mathcal{N}_2$-extension of $\mathrm{UL}\forall$). We conjecture that density elimination holds for the extension of $\mathrm{HUL}\forall$ with any internal structural rule, and hence that every acyclic $\mathcal{N}_2$-extension of $\mathrm{UL}\forall$ is standard complete.

A further step would be to investigate standard completeness for $\mathcal{P}'_3$-extensions of $\mathrm{UL}\forall$. This means proving density elimination for $\mathrm{HUL}\forall$ extended with external structural rules. Note that the results on semi-anchored rules for extensions of $\mathrm{HMTL}\forall$ in Chapter 4 make heavy use of the weakening rules and hence there is no clear way to transfer them to $\mathrm{HUL}\forall$.

Density and the admissibility of rules

The density rule is a striking example of a rule whose admissibility (elimination) enables us to establish an important logical property, i.e. rational completeness. Showing the admissibility of rules for proving other important algebraic and logical results (see e.g. [64, 65]) would be an interesting direction for future research. In the framework of sequent and hypersequent calculi, the investigation of the admissibility of rules can benefit from the examination of the methods and syntactic criteria established so far for cut and density elimination. An interesting research direction would be to understand, in more general terms, the relation between the syntactic form of a given rule (r) (not necessarily cut or density) and the criteria to be imposed on (hyper)sequent calculi, to obtain the admissibility of (r).

Residuated frames

Residuated frames have been a fundamental tool for the development of algebraic proof-theory. In [24, 27] these structures were used for obtaining algebraic proofs of cut-admissibility. In our work, we employed residuated frames to translate proofs of density elimination in an algebraic setting. A future research direction might be to extend the use of residuated frames (or variants thereof) to find algebraic counterparts for other proof-theoretic arguments, e.g. for the admissibility of other rules.

Keeping the Density rule

In our work, we used the density rule mainly as an instrument towards proving completeness. A natural question is whether keeping this rule might bring any added value. For instance, a restricted use of the rule might help in proof search, see e.g. [70] or, just as for the cut rule, bring to a significant reduction of proof lengths.

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