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To my parents

To my supervisor

I declare that I carried out this dissertation thesis independently, and only with the cited sources, literature and other professional sources.

Wien, 2017 Lenka Kovářová

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Introduction

We often encounter objects that can be described only with some uncertainty. We have only vague information about them. Imagine everyday terms like little, less, more, much, small, big, cold, hot and so on. One can definitely see that these are meant to give us information - however uncertain.

The Greek philosopher Zenon [9] had been already dealing with problems of uncertainty and vague terms. Let us imagine a small sandhill in front of us on a beach. The following question can be asked: "What if we take a small grain of sand away from that sandhill? Will it still be a sandhill?" Should we take one grain only, there will be still a sandhill in front of us, most likely.

Yet, when we are taking grains of sand away a longer time the sandhill will diminish to a fistful of sand. When will the turning point happen? How many grains of sand should a fistful contain at utmost to turn it into a small sandhill with one grain of sand added? This is the question which can help us to determine the exact borderline between a fistfull and a small hill. Alternatively, having used a certain measure of uncertainty, we can lower gradually and steadily the weight of classification of the sandformation in front of us as a small sandhill with each grain of sand taken away.

The idea of generalized sets was originally presented by Karl Menger [7] and the term fuzzy sets was introduced by Lofti A. Zadeh [12]. This theory provides a scheme for handling a variety of problems in which a fundamental role is played by an indefiniteness arising more from a sort of intrinsic ambiguity than from statistical variation.

Measurements are basic for all quantitative science and for many human activities. Usually real numbers are the foundation for the description of measurement results of one-dimensional quantities. By the limited accuracy of every measurement equipment the result of one measurement of a continuous quantity is not a precise number but more or less non-precise. This unavoidable imprecision - which is different from measurement errors - has to be analyzed in order to obtain realistic results. Using the concept of so-called *fuzzy numbers*, which are special fuzzy subsets of the set of real numbers, a more realistic description of measurement results is possible.

In Chapter 1 there are the elementary fuzzy sets introduced. Chapter 2 is dealing with fuzzy numbers and Chapter 3 with fuzzy vectors. In Chapter 4 there are analyzed the basics of vectors of/with fuzzy numbers. In Chapter 5 the characterizing functions are constructed. Some of the computing with fuzzy numbers is presented in Chapter 6. Chapter 7 demonstrates certain fuzzy valued functions. Chapter 8 outlines practical application to measurement data.

All proofs in this dissertation thesis are made by the author of this thesis (except the appendix part).

1. Fuzzy Sets

Definition 1.1. Let M denote a classical universal set. *A crisp set* A in M is is a set, where for each element x of M exactly one of the following holds true: $x \in A$ or $x \notin A$.

Definition 1.2. Let M denote a classical universal set. *A fuzzy set* A in M is characterized by a so-called *membership function* $\mu_A : M \to [0,1]$, which associates each object in M with a real number in the interval $[0,1]$.

The value $\mu_A(x)$ is representing the *grade of membership* of x in A for each $x \in M$. A value of $\mu_A(x)$ close to unity means high grade of membership of x in A.

Remark. When A is a crisp set, i.e. a classical set, its membership function can take only two values 0 or 1, where

$$\mu_A(x) = 1 \ \ \forall x \in A \quad \textit{and} \quad \mu_A(x) = 0 \ \ \forall x \notin A.$$

Note that the set of all crisp subsets of M is a subset of the set of all fuzzy sets in M. Examples of fuzzy sets can be a blurry outline of an object, the set of old or young men, a number close to zero and many others. We cannot determine an exact border to decide if an object is or isn't a part of our set.

Fuzzy sets are sometimes incorrectly assumed to indicate some form of probability. Despite the fact that they can take on similar values, it is important to realize that membership grades are *not* probabilities. One immediately apparent difference is that the summation of probabilities on a finite universal set must equal 1, while there is no such requirement for membership grades.

Definition 1.3. The *support* of a fuzzy set A in the universal set M is the crisp set that contains all the elements of M that have a non-zero membership grade in A. That is, the support of a fuzzy set A in M is defined by

$$\operatorname{supp}(A) := \{x \in M \colon \mu_A(x) > 0\}.$$

Definition 1.4. The *kernel* of a fuzzy set A in the universal set M is the crisp set that contains all the elements of M that have a membership grade in A equal to 1. That is, the kernel of a fuzzy set A in M is defined by

$$\operatorname{kern}(A) := \{x \in M \colon \mu_A(x) = 1\}.$$

Definition 1.5. An *empty* fuzzy set A, $A = \emptyset$, is a fuzzy set with empty support. Its membership function is identically zero on the universal set M,

$$\mu_A(x) = 0 \quad \forall x \in M.$$

Definition 1.6. A fuzzy set A is called *normalized* when at least one of its elements reaches the maximum possible membership grade 1,

$$\exists x \in M \quad \mu_A(x) = 1.$$

Definition 1.7. For $\delta \in (0,1]$ the *δ-cut* of a fuzzy set A is a crisp set $\mathcal{C}_\delta(A)$ that contains all the elements of the universal set M that have a membership grade in A greater than or equal to the specified value of δ,

$$\mathcal{C}_\delta(A) := \{x \in M \colon \mu_A(x) \geq \delta\}.$$

Remark. Observe that the δ-cuts of any fuzzy set on M are nested crisp subsets of M, i.e.

$$\delta_1 < \delta_2 \Rightarrow \mathcal{C}_{\delta_1}(A) \supseteq \mathcal{C}_{\delta_2}(A).$$

1.1 Operations on Fuzzy Sets

In this section let M denote a universal set, let A and B be fuzzy sets in the universal set M with membership functions μ_A and μ_B respectively.

Definition 1.8. Two fuzzy sets A and B are *equal*, $A = B$, if and only if their membership functions are identical.

Definition 1.9. The *complement* of a fuzzy set A is a fuzzy set denoted by A^c and defined by its membership function

$$\mu_{A^c}(x) := 1 - \mu_A(x) \quad \forall x \in M.$$

Definition 1.10. A fuzzy set A is a *subset of* a fuzzy set B (equivalently, A is contained in B, A is smaller than or equal to B), if and only if the following is fulfilled:

$$\mu_A(x) \leq \mu_B(x) \ \forall x \in M$$

More precisely:

$$A \subseteq B \quad \Leftrightarrow \quad \forall x \in M \colon \ \mu_A(x) \leq \mu_B(x)$$

Definition 1.11. The *union* of fuzzy sets A_i, $i \in I$, is a fuzzy set C,

$$C = \bigcup_{i \in I} A_i,$$

whose membership function is defined by:

$$\mu_C(x) := \sup\{\mu_{A_i}(x) \colon i \in I\} \quad \forall x \in M$$

Definition 1.12. The *intersection* of fuzzy sets A_i, $i \in I$, is a fuzzy set D,

$$D = \bigcap_{i \in I} A_i,$$

whose membership function is defined by:

$$\mu_D(x) := \inf\{\mu_{A_i}(x) \colon i \in I\} \quad \forall x \in M$$

Definition 1.13. The *difference* of fuzzy sets A and B is a fuzzy set defined by:

$$A \setminus B = A \cap B^c$$

Definition 1.14. Fuzzy sets A_i, $i \in I$, are *disjoint* if their intersection is empty:

$$\bigcap_{i \in I} A_i = \emptyset$$

Lemma 1.1. *Let A and B be two fuzzy sets. The union of A and B is the smallest fuzzy set containing both A and B.*

Proof. Let A, B be fuzzy sets with membership functions μ_A and μ_B respectively.

Let $C = A \cup B$ with membership function μ_C according to the definition of union,

$$\mu_C(x) = \sup\{\mu_A(x), \mu_B(x)\} \quad \forall x \in M, \quad \text{equivalently}$$

$$\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\} \quad \forall x \in M.$$

We note first, that C is containing the fuzzy sets A and B, $A \subseteq C$ and $B \subseteq C$. It is sufficient to prove, according to definition 1.10, that

$$\mu_A \leq \mu_C \quad \text{and} \quad \mu_B \leq \mu_C.$$

To do this, we just realize that

$$\forall x \in M \qquad \begin{array}{l} \mu_A(x) \leq \max\{\mu_A(x), \mu_B(x)\} = \mu_C(x) \\ \mu_B(x) \leq \max\{\mu_A(x), \mu_B(x)\} = \mu_C(x). \end{array}$$

Now we note that C is the smallest fuzzy set containing both A and B. Let D be a fuzzy set containing both A and B, then

$$\mu_D(x) \geq \mu_A(x) \quad \text{and} \quad \mu_D(x) \geq \mu_B(x) \quad \forall x \in M,$$

$$\mu_D(x) \geq \max\{\mu_A(x), \mu_B(x)\} = \mu_C(x) \quad \forall x \in M,$$

which implies that C is a subset of D, in symbols $C \subseteq D$. □

Lemma 1.2. *Let A and B be fuzzy sets. Then the intersection of A and B is the largest fuzzy set which is contained in both A and B.*

Proof. The proof is an analogy to the previous one. □

1.2 Fuzzy Set and Generating Family

Theorem 1.3. *Let be a family of nested sets $(A_\delta; \delta \in (0,1])$ where $A_\delta \subseteq M$ $\forall \delta \in (0,1]$ and $A_\delta \subseteq A_\alpha$ $\forall \alpha < \delta$, where $\alpha, \delta \in (0,1]$. We define a fuzzy set A^* with membership function*

$$\xi(x) := \sup\{\delta \cdot \mathbb{1}_{A_\delta}(x) : \delta \in (0,1]\} \quad \forall x \in M.$$

For $\delta \in (0,1]$ the following conditions are equivalent:

○ *$\mathcal{C}_\delta(A^*) = A_\delta$, where $\mathcal{C}_\delta(A^*)$ is the δ-cut of the fuzzy set A^**

○ $\bigcap\limits_{\alpha\in(0,\delta)} A_\alpha = A_\delta$

Proof. Firstly we recall the definition of supremum.

Let be $N \subseteq \mathbb{R}$ a set of real numbers, then we call $s \in \mathbb{R}$ supremum of N if and only if

1. $\forall x \in N \quad x \leq s$,
2. $\forall s' \in \mathbb{R}, s' < s \quad \exists x \in N \quad s' < x$.

Choose arbitrary $\tilde{\delta} \in (0,1]$ and arbitrary $\tilde{x} \in M$. We will investigate if $\tilde{x} \in A_{\tilde{\delta}}$ or $\tilde{x} \notin A_{\tilde{\delta}}$ according to the value $\xi(\tilde{x})$. We define

$$s := \xi(\tilde{x}) = \sup\left\{\delta \cdot \mathbb{1}_{A_\delta}(\tilde{x}) : \delta \in (0,1]\right\}.$$

From the first condition of the definition of supremum we have $\forall \delta \in (0,1]$

$$\delta \cdot \mathbb{1}_{A_\delta}(\tilde{x}) = \left\{\begin{array}{ll} 0 & \text{for } \tilde{x} \notin A_\delta \\ \delta & \text{for } \tilde{x} \in A_\delta \end{array}\right\} \leq s.$$

Therefore $\tilde{x} \notin A_\delta$ is valid for $\delta > s$ (in opposite case $\tilde{x} \in A_\delta$ the first condition of supremum will not be fulfilled).

From the second condition of the definition of supremum we have

$$\forall s' < s \quad \exists \alpha \in (0,1]: \ s' < \alpha \cdot \mathbb{1}_{A_\alpha}(\tilde{x}).$$

We will investigate this condition only for $s' > 0$ (for $s' \leq 0$ we can choose arbitrary $\alpha \in (0,1]$ where $\tilde{x} \in A_\alpha$).

We already know that $\alpha \leq s$ from the first condition of the definition of supremum (for $\alpha > s$ is $\tilde{x} \notin A_\alpha$ and $\alpha \cdot \mathbb{1}_{A_\alpha}(\tilde{x}) = 0$).

We can deduce that $s' < \alpha$, because there is to be fulfilled $s' < \alpha \cdot \mathbb{1}_{A_\alpha}(\tilde{x})$ and we have $\alpha \cdot \mathbb{1}_{A_\alpha}(\tilde{x}) = \left\{\begin{array}{ll} 0 & \text{for } \tilde{x} \notin A_\alpha \\ \alpha & \text{for } \tilde{x} \in A_\alpha \end{array}\right\}$.

We can now reformulate the second condition of supremum in the following way

$$\forall s' < s \quad \exists \alpha \in (s', s]: \ \tilde{x} \in A_\alpha.$$

For $\delta < s$ we can choose arbitrary $s' \in (\delta, s)$ and according to the second condition of supremum there exists $\alpha \in (s', s]$ where $\tilde{x} \in A_\alpha$. From the nested structure of the generating family $(A_\delta; \delta \in (0,1])$ and the fact $\delta < s' < \alpha$ we know that $A_\alpha \subseteq A_\delta$ and hence $\tilde{x} \in A_\delta$.

Let us investigate the δ-cut $\mathcal{C}_{\tilde{\delta}}(A^*)$.

$$\begin{array}{lll} \xi(\tilde{x}) \geq \tilde{\delta} & \Rightarrow & \tilde{x} \in \{x \in M : \xi(x) \geq \tilde{\delta}\} = \mathcal{C}_{\tilde{\delta}}(A^*) \\ \xi(\tilde{x}) < \tilde{\delta} & \Rightarrow & \tilde{x} \notin \{x \in M : \xi(x) \geq \tilde{\delta}\} = \mathcal{C}_{\tilde{\delta}}(A^*) \end{array}$$

We can summarize our findings in the following way:

$$\begin{array}{ll} \xi(\tilde{x}) > \tilde{\delta} & \text{for } \tilde{x} \in \mathcal{C}_{\tilde{\delta}}(A^*) \text{ and } \tilde{x} \in A_{\tilde{\delta}} \\ \xi(\tilde{x}) = \tilde{\delta} & \text{for } \tilde{x} \in \mathcal{C}_{\tilde{\delta}}(A^*) \\ \xi(\tilde{x}) < \tilde{\delta} & \text{for } \tilde{x} \notin \mathcal{C}_{\tilde{\delta}}(A^*) \text{ and } \tilde{x} \notin A_{\tilde{\delta}} \end{array}$$

Because we have chosen $\tilde{x} \in M$ and $\tilde{\delta} \in (0,1]$ arbitrary we have $\forall \delta \in (0,1]$

$$A_\delta \subseteq \mathcal{C}_\delta(A^*) \quad \text{and} \quad \mathcal{C}_\delta(A^*) \subseteq A_\delta \cup \{x : \xi(x) = \delta\}.$$

We want to recognize such $\delta \in (0,1]$ for which $\{x : \xi(x) = \delta\} \subseteq A_\delta$, then we will have $\mathcal{C}_\delta(A^*) = A_\delta$ for recognized δ.

- Firstly we will prove that from the fact that $\bigcap\limits_{\alpha \in (0,\delta)} A_\alpha = A_\delta$ follows that $\mathcal{C}_\delta(A^*) = A_\delta$, where $\mathcal{C}_\delta(A^*)$ is the δ-cut of the fuzzy set A^*.

  Let be $\bigcap\limits_{\alpha \in (0,\hat{\delta})} A_\alpha = A_{\hat{\delta}}$, where $\hat{\delta} \in (0,1]$. Choose arbitrary $\tilde{x} \in M$.

  $\tilde{x} \in A_{\hat{\delta}}$ **:** $\xi(\tilde{x}) = \sup\{\delta \cdot \mathbb{1}_{A_\delta}(\tilde{x}) : \delta \in (0,1]\} \geq \hat{\delta}$.

  $\tilde{x} \notin A_{\hat{\delta}}$ **:** We can find $\alpha < \hat{\delta}$, where $\tilde{x} \notin A_\alpha$, because $\tilde{x} \notin A_{\hat{\delta}} = \bigcap\limits_{\alpha \in (0,\hat{\delta})} A_\alpha$.

  From the nested structure of the generating family $(A_\delta; \delta \in (0,1])$ we can deduce

  $$\forall \beta > \alpha \quad A_\beta \subseteq A_\alpha, \text{ especially } \tilde{x} \notin A_\beta,$$

  and hence

  $$\xi(\tilde{x}) = \sup\left\{\delta \cdot \mathbb{1}_{A_\delta}(\tilde{x}) : \delta \in (0,1]\right\} \leq \alpha < \hat{\delta}.$$

  We have $\{x : \xi(x) = \hat{\delta}\} \subseteq A_{\hat{\delta}}$ and hence, using the previous part of this proof, we have $\mathcal{C}_{\hat{\delta}}(A^*) = A_{\hat{\delta}}$.

- Secondly we will prove the opposite implication, from $\mathcal{C}_\delta(A^*) = A_\delta$, where $\mathcal{C}_\delta(A^*)$ is the δ-cut of the fuzzy set A^*, follows that $\bigcap\limits_{\alpha \in (0,\delta)} A_\alpha = A_\delta$.

  Let be $\mathcal{C}_{\hat{\delta}}(A^*) = A_{\hat{\delta}}$, where $\hat{\delta} \in (0,1]$. Choose arbitrary $\tilde{x} \in M$.

  $$A_{\hat{\delta}} = \mathcal{C}_{\hat{\delta}}(A^*) = \{x \in M : \xi(x) \geq \hat{\delta}\}$$

  $$M \setminus A_{\hat{\delta}} = \{x \in M : \xi(x) < \hat{\delta}\}$$

  $\tilde{x} \in A_{\hat{\delta}}$ **:** From the nested structure of the generating family $(A_\delta; \delta \in (0,1])$ we have $\forall \alpha \in (0,\hat{\delta})$ $\tilde{x} \in A_\alpha$ and hence $\tilde{x} \in \bigcap\limits_{\alpha \in (0,\hat{\delta})} A_\alpha$.

  $\tilde{x} \notin A_{\hat{\delta}}$ **:** We have $\xi(\tilde{x}) < \hat{\delta}$. We choose arbitrary $\beta \in (\xi(\tilde{x}), \hat{\delta})$, now we have $\xi(\tilde{x}) < \beta$ and hence $\tilde{x} \notin A_\beta$ it follows to $\tilde{x} \notin \bigcap\limits_{\alpha \in (0,\hat{\delta})} A_\alpha$.

Because we have chosen $\tilde{x} \in M$ arbitrary, we have $\bigcap\limits_{\alpha \in (0,\hat{\delta})} A_\alpha = A_{\hat{\delta}}$. □

2. Fuzzy Numbers

Definition 2.1. A *fuzzy number* a^* is a fuzzy set in $\mathbb{R}$ determined by its membership function ξ called *characterizing function*, which is a real function of one real variable, fulfilling

1. $\xi : \mathbb{R} \to [0,1]$,
2. $\forall \delta \in (0,1]$ the δ-cut $\mathcal{C}_\delta(a^*)$ is a non-empty, finite union of compact intervals,

$$\exists\, k_\delta \in \mathbb{N} \quad \exists\, ([a_{\delta,i}, b_{\delta,i}])_{i=1}^{k_\delta} : \; \mathcal{C}_\delta(a^*) = \bigcup_{i=1}^{k_\delta} [a_{\delta,i}, b_{\delta,i}]\,,$$

3. the support of a^*, $\operatorname{supp}(a^*) = \{x \in \mathbb{R} \colon \xi(x) > 0\}$, is bounded.

The set of all fuzzy numbers is denoted by $\mathcal{F}(\mathbb{R})$.

Remark. A precise number $x_0 \in \mathbb{R}$ is represented by its characterizing function:

$$\xi(x) = \left\{ \begin{array}{ll} 1 & \text{for } x = x_0 \\ 0 & \text{for } x \neq x_0 \end{array} \right\} \quad \forall x \in \mathbb{R}$$

This characterizing function is the one-point indicator function $\mathbb{1}_{\{x_0\}}(\cdot)$ of the crisp set $\{x_0\}$.

Remark. Each fuzzy number is a special case of a fuzzy set defined in $\mathbb{R}$.

Definition 2.2. A *fuzzy interval* a^* is a fuzzy number with characterizing function ξ, where each δ-cut $\mathcal{C}_\delta(a^*)$ is a compact interval,

$$\forall \delta \in (0,1] \quad \exists a_\delta, b_\delta \in \mathbb{R} \quad \mathcal{C}_\delta(a^*) = [a_\delta, b_\delta]\,.$$

The set of all fuzzy intervals is denoted by $\mathcal{F}_\mathcal{I}(\mathbb{R})$.

Example 2.1. Examples of characterizing functions of fuzzy numbers are given in Figure 2.1. In Figure 2.1(a) is a characterizing function of a precise number x_0. The characterizing functions in Figures 2.1(a)-(e) are characterizing functions of fuzzy intervals.

Lemma 2.1 (Representation)**.** *For the characterizing function $\xi(\cdot)$ of a fuzzy number a^* the following holds true:*

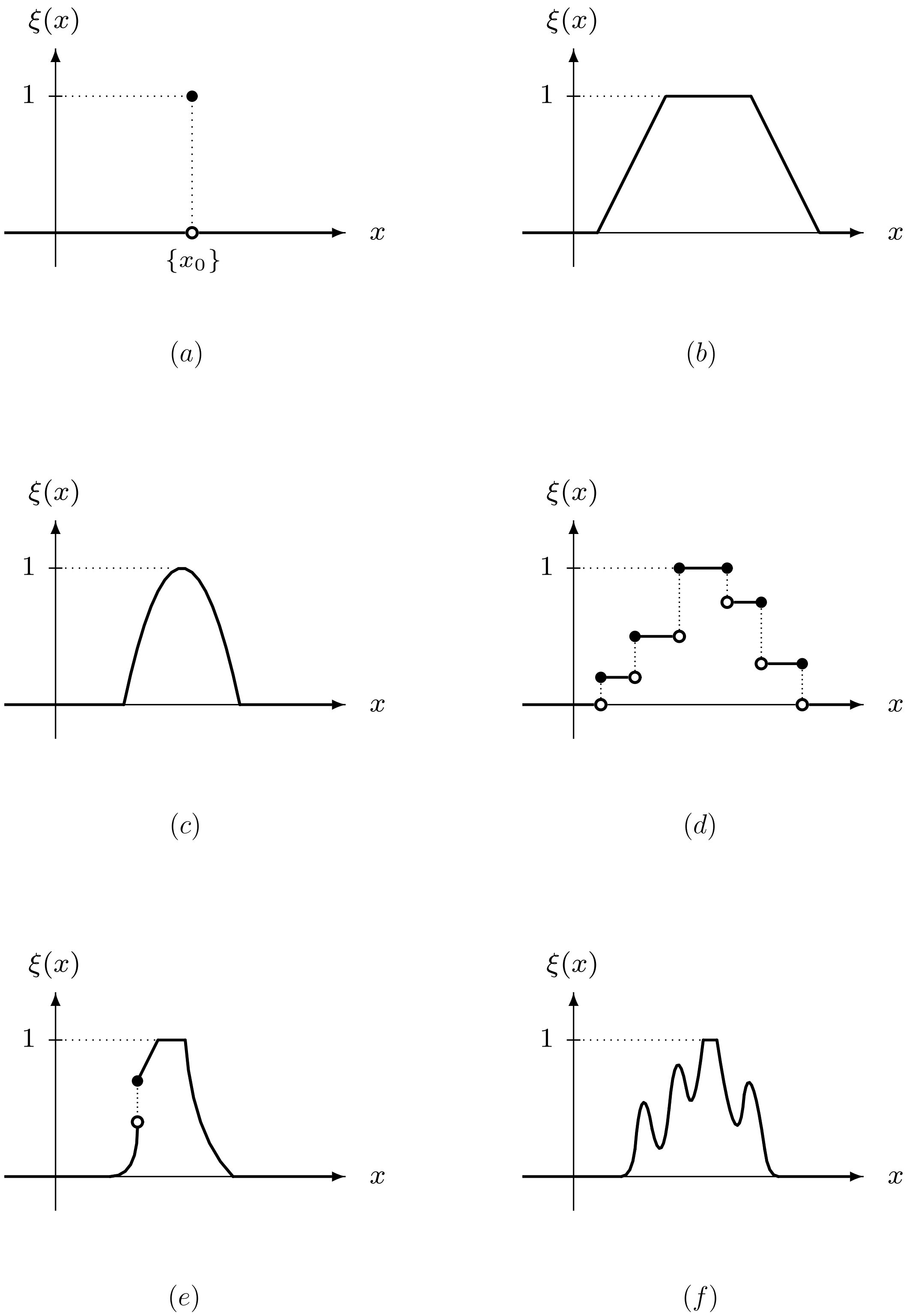
$$\xi(x) = \max\left\{\delta \cdot \mathbb{1}_{\mathcal{C}_\delta(a^*)}(x) : \delta \in [0,1]\right\} \quad \forall x \in \mathbb{R}$$

Proof. This is a well known proof, taken from (Viertl, 2011).

For fixed $x_0 \in \mathbb{R}$ we have:

$$\delta \cdot \mathbb{1}_{\mathcal{C}_\delta(a^*)}(x_0) = \delta \cdot \mathbb{1}_{\{x \colon \xi(x) \geq \delta\}}(x_0) = \left\{ \begin{array}{ll} \delta & \text{for } \xi(x_0) \geq \delta \\ 0 & \text{for } \xi(x_0) < \delta \end{array} \right\}$$

Figure 2.1: Examples of characterizing functions of fuzzy numbers



Therefore we have for every $\delta \in [0,1]$:

$$\delta \cdot \mathbb{1}_{\mathcal{C}_\delta(a^*)}(x_0) \leq \xi(x_0)$$
$$\sup\{\mathcal{C}_\delta(a^*) \colon \delta \in [0,1]\} \leq \xi(x_0)$$

On the other hand we have for $\delta_0 = \xi(x_0)$:

$$\delta_0 \cdot \mathbb{1}_{\mathcal{C}_\delta(a^*)}(x_0) = \delta_0$$
$$\sup\{\mathcal{C}_\delta(a^*) \colon \delta \in [0,1]\} \geq \delta_0$$

From that follows:

$$\sup\{\mathcal{C}_\delta(a^*) \colon \delta \in [0,1]\} = \max\{\mathcal{C}_\delta(a^*) \colon \delta \in [0,1]\} = \delta_0 = \xi(x_0)$$

□

2.1 Fuzzy Number and Generating Family

Theorem 2.2. *Let $(A_\delta; \delta \in (0,1])$ be a generating family of uniformly bounded nested closed intervals, where $A_\delta = [a_\delta, b_\delta]$ $\forall \delta \in (0,1]$ with $a_\delta, b_\delta \in \mathbb{R}$ and $A_\delta \subseteq A_\alpha$ $\forall \alpha < \delta$, $\alpha, \delta \in (0,1]$ and $\bigcup_{\alpha \in (0,1]} A_\alpha$ is bounded. We define a fuzzy set a^* with membership function*

$$\xi(x) := \sup\left\{\delta \cdot \mathbb{1}_{A_\delta}(x) : \delta \in (0,1]\right\} \quad \forall x \in \mathbb{R}.$$

The fuzzy set a^* is a fuzzy interval and for $\delta \in (0,1]$ the following conditions are equivalent:

- $\mathcal{C}_\delta(a^*) = A_\delta$, *where $\mathcal{C}_\delta(a^*)$ is the δ-cut of the fuzzy interval a^**
- $\bigcap\limits_{\alpha \in (0,\delta)} A_\alpha = A_\delta$
- *The functions $f(\alpha) := a_\alpha$ and $g(\alpha) := b_\alpha$, defined $\forall \alpha \in (0,1]$, are continuous from the left at the point δ*

Proof. We have to prove that the fuzzy set a^* in $\mathbb{R}$ described by the membership function $\xi(\cdot)$ is a fuzzy interval. Firstly we need to verify, that a^* is a fuzzy number:

1. $\xi : \mathbb{R} \to (0,1]$.
2. $\forall \delta \in (0,1]$ the δ-cut $\mathcal{C}_\delta(a^*)$ is a compact interval:

$$\begin{aligned}
\mathcal{C}_\delta(a^*) &= \{x \in \mathbb{R} : \xi(x) \geq \delta\} \\
&= \{x \in \mathbb{R} : \sup\left\{\alpha \cdot \mathbb{1}_{A_\alpha}(x) : \alpha \in (0,1]\right\} \geq \delta\} \\
&= \{x \in \mathbb{R} : x \in \bigcap_{\alpha \in (0,\delta)} A_\alpha\} = \bigcap_{\alpha \in (0,\delta)} A_\alpha \\
&= \left[\lim_{\alpha \uparrow \delta} a_\alpha,\ \lim_{\alpha \uparrow \delta} b_\alpha\right]
\end{aligned}$$

3. The support of ξ is bounded:

$$\operatorname{supp}(\xi(\cdot)) = \{x \in \mathbb{R} : \xi(x) > 0\} = \{x \in \mathbb{R} : \exists \delta \in (0,1] \text{ with } x \in A_\delta\}$$

From the nested structure of the generating family $(A_\delta; \delta \in (0,1])$, where $A_\delta = [a_\delta, b_\delta] \quad \forall \delta \in (0,1]$, we have

$$\operatorname{supp}(\xi(\cdot)) = \left[\lim_{\delta \downarrow 0} a_\delta,\ \lim_{\delta \downarrow 0} b_\delta\right].$$

We have verified that $\xi(\cdot)$ fulfils all conditions required for a characterizing function and hence a^* is a fuzzy number, moreover a^* is a fuzzy interval.

We have already proved the equivalence of the first two points from this theorem in the more general theorem 1.3. It is enough to prove the implication from the third to the second point and the implication from the first to the third point.

We define the functions $f(\alpha) := a_\alpha$ and $g(\alpha) := b_\alpha$ for all $\alpha \in (0,1]$, where $A_\delta = [a_\delta, b_\delta]$. The function f is non-decreasing and the function g is non-increasing as follows from the nested structure of the generating family $(A_\delta; \delta \in (0,1])$. For all $\alpha, \delta \in (0,1]$ with $\alpha < \delta$ we have:

$$A_\delta \subseteq A_\alpha$$

$$[a_\delta, b_\delta] \subseteq [a_\alpha, b_\alpha]$$

$$a_\alpha \le a_\delta \text{ and } b_\delta \le b_\alpha$$

$\Rightarrow$ Let the functions f and g be continuous from the left at point δ, where $\delta \in (0,1]$.

$$\forall \varepsilon > 0 \quad \exists \delta' < \delta \quad \forall \alpha \in (\delta', \delta] \quad \begin{array}{l} f(\alpha) \in (f(\delta) - \varepsilon,\ f(\delta) + \varepsilon) \\ g(\alpha) \in (g(\delta) - \varepsilon,\ g(\delta) + \varepsilon) \end{array}$$

Because f is non-decreasing, $\alpha < \delta$ and $f(\delta) = a_\delta$ we have $f(\alpha) \in (a_\delta - \varepsilon, a_\delta]$. Because g is non-increasing, $\alpha < \delta$ and $g(\delta) = b_\delta$ we have $g(\alpha) \in [b_\delta, b_\delta + \varepsilon)$. We have

$$A_\alpha = [a_\alpha, b_\alpha] \subset (a_\delta - \varepsilon, b_\delta + \varepsilon).$$

For arbitrary small $\varepsilon > 0$ we can find $\alpha < \delta$, where

$$A_\delta = [a_\delta, b_\delta] \subseteq A_\alpha \subset (a_\delta - \varepsilon, b_\delta + \varepsilon),$$

due to continuity of the functions f and g from the left at point δ.

We choose arbitrary $x \in \mathbb{R}$.

$x \in A_\delta$ **:** From the nested structure of $(A_\delta, \delta \in (0,1])$ we have

$$\forall \alpha < \delta \quad x \in A_\alpha \quad \text{and hence} \quad x \in \bigcap_{\alpha \in (0,\delta)} A_\alpha.$$

$x \notin A_\delta$ **:** For $x < a_\delta$, we choose $\varepsilon = \frac{a_\delta - x}{2}$, for $x > b_\delta$, we choose $\varepsilon = \frac{x - b_\delta}{2}$. There exists $\alpha < \delta$ where $A_\alpha \subset (a_\delta - \varepsilon, b_\delta + \varepsilon)$, and hence $x \notin A_\alpha$ and $x \notin \bigcap\limits_{\alpha \in (0,\delta)} A_\alpha$.

Because we have chosen arbitrary $x \in \mathbb{R}$ we have proved that $A_\delta = \bigcap\limits_{\alpha \in (0,\delta)} A_\alpha$.

$\Leftarrow$ Let be $A_\delta = \mathcal{C}_\delta(a^*)$, where $\delta \in (0,1]$.

$$\begin{array}{ccl} A_\delta & = & [a_\delta, b_\delta] \\ \shortparallel & & \\ \mathcal{C}_\delta(a^*) & = & \{x \in \mathbb{R} \mid \xi(x) \geq \delta\}. \end{array}$$

So we have

$$\forall x \in A_\delta \quad \xi(x) \geq \delta \quad \text{and} \quad \forall x \notin A_\delta \quad \xi(x) < \delta.$$

We want to prove that the function $f(\alpha) = a_\alpha$ defined for all $\alpha \in (0,1]$ is continuous from the left at point δ. Let be $\varepsilon > 0$ arbitrary, we want to find $\delta' < \delta$ with

$$\begin{array}{ccc} \forall \alpha \in (\delta', \delta) \quad f(\alpha) & \in & (f(\delta) - \varepsilon, f(\delta) + \varepsilon) \\ \shortparallel & & \shortparallel \\ a_\alpha & & (a_\delta - \varepsilon, a_\delta + \varepsilon). \end{array}$$

We know that $a_\alpha = f(\alpha) \leq f(\delta) = a_\delta \quad \forall \alpha < \delta$ from non-decreasing characteristic of the function f.

Possibility 1: $\quad \forall \alpha < \delta \quad f(\alpha) = a_\alpha \leq a_\delta - \varepsilon = f(\delta) - \varepsilon$

We will show by proof ad absurdum that this isn't possible. We choose $x' \in (a_\delta - \varepsilon, a_\delta)$, for example $x' = a_\delta - \frac{\varepsilon}{2}$.

From the nested structure of the generating family $(A_\delta, \delta \in (0,1])$ we know that

$$\forall \alpha \in (0,\delta) \quad a_\alpha \leq a_\delta - \varepsilon < x' < a_\delta < b_\delta \leq b_\alpha,$$

and hence

$$\forall \alpha \in (0,\delta) \quad x' \in A_\alpha.$$

We also know that $x' \notin A_\delta = [a_\delta, b_\delta]$, and from the nested structure of the generating family $(A_\delta, \delta \in (0,1])$ we have

$$\forall \alpha' \in [\delta, 1] \quad x' \notin A_{\alpha'}.$$

Now we can calculate

$$\xi(x') = \sup\left\{\alpha \cdot \mathbb{1}_{A_\alpha}(x') : \alpha \in (0,1]\right\} = \sup\{\alpha : \alpha \in (0,\delta)\} = \delta.$$

But we have proved that $\forall x \notin A_\delta \quad \xi(x) < \delta$ and $x' \notin A_\delta$. $\lightning$

Possibility 2: $\quad \exists \delta' < \delta \quad f(\delta') = a_{\delta'} > a_\delta - \varepsilon = f(\delta) - \varepsilon$

From the nested structure of the generating family $(A_\delta, \delta \in (0,1])$ we have

$$\forall \alpha \in (\delta', \delta) \qquad \begin{array}{ccccc} A_\delta & \subseteq & A_\alpha & \subseteq & A_{\delta'} \\ [a_\delta, b_\delta] & \subseteq & [a_\alpha, b_\alpha] & \subseteq & [a_{\delta'}, b_{\delta'}] \end{array}$$

$$\text{and hence} \quad a_\delta \ \geq \ a_\alpha \ \geq \ a_{\delta'} \ > \ a_\delta - \varepsilon.$$

So we have for arbitrary selected $\varepsilon > 0$ found $\delta' < \delta$, where

$$\forall \alpha \in (\delta', \delta) \quad f(\alpha) = a_\alpha \in (a_\delta - \varepsilon, a_\delta] = (f(\delta) - \varepsilon, f(\delta)],$$

and the function $f(\cdot)$ is continuous from the left at point δ.

Analogically we can prove that the function $g(\cdot)$ continuous from the left at point δ.

$\square$

Example 2.2. We define a generating family of nested sets $(A_\delta; \delta \in (0,1])$ in the following way:

$$A_\delta := \left\{ \begin{array}{cc} [-1,\ 1] & \text{for } \delta \in (0, \frac{1}{2}) \\ [-1+\delta,\ 1-\delta] & \text{for } \delta \in [\frac{1}{2}, 1] \end{array} \right\}$$

This structure fulfils the conditions required in Theorem 2.2. We define functions $f(\cdot)$ and $g(\cdot)$ on the interval $(0,1]$ according to Theorem 2.2 as

$$f(\alpha) := \left\{ \begin{array}{cc} -1 & \text{for } \alpha \in (0, \frac{1}{2}) \\ -1+\alpha & \text{for } \alpha \in [\frac{1}{2}, 1] \end{array} \right\} \quad \text{and} \quad g(\alpha) := \left\{ \begin{array}{cc} 1 & \text{for } \alpha \in (0, \frac{1}{2}) \\ 1-\alpha & \text{for } \alpha \in [\frac{1}{2}, 1] \end{array} \right\}.$$

An outline of the structure of the generating family of nested sets $(A_\delta; \delta \in (0,1])$ is given in Figure 2.2 and the graph of the functions f and g is given in Figure 2.3.

Similarly to Theorem 2.2 we define a fuzzy number a^* with characterizing function $\xi(x) := \sup \{\delta \cdot \mathbb{1}_{A_\delta}(x) | \delta \in (0,1]\} \quad \forall x \in \mathbb{R}$. In this particular case we can calculate the function $\xi(\cdot)$ as

$$\xi(x) = \left\{ \begin{array}{cl} 0 & \text{for } x < -1 \\ \frac{1}{2} & \text{for } x \in [-1, -\frac{1}{2}] \\ 1-|x| & \text{for } x \in [-\frac{1}{2}, \frac{1}{2}] \\ \frac{1}{2} & \text{for } x \in [\frac{1}{2}, 1] \\ 0 & \text{for } x > 1 \end{array} \right\}.$$

We can see that for $\delta = \frac{1}{2}$, where the set $A_{\frac{1}{2}}$ has the value $\left[-\frac{1}{2}, \frac{1}{2}\right]$, no one of the three equivalent conditions from Theorem 2.2 is fulfilled:

- $\mathcal{C}_{\frac{1}{2}}(a^*) = \{x \in \mathbb{R} \ : \xi(x) \geq \frac{1}{2}\} = [-1, 1]$ and hence $\mathcal{C}_{\frac{1}{2}}(a^*) \neq A_{\frac{1}{2}}$.
- $\bigcap\limits_{\alpha \in (0, \frac{1}{2})} A_\alpha = [-1, 1]$ and hence $\bigcap\limits_{\alpha \in (0, \frac{1}{2})} A_\alpha \neq A_{\frac{1}{2}}$.
- The functions $f(\cdot)$ and $g(\cdot)$ are non-continuous at the point $\frac{1}{2}$ from the left.

Figure 2.2: Borderlines for sets A_δ in the nested structure $(A_\delta; \delta \in (0,1])$ from Example 2.2.

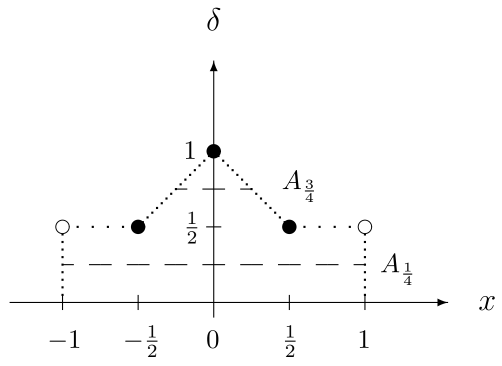
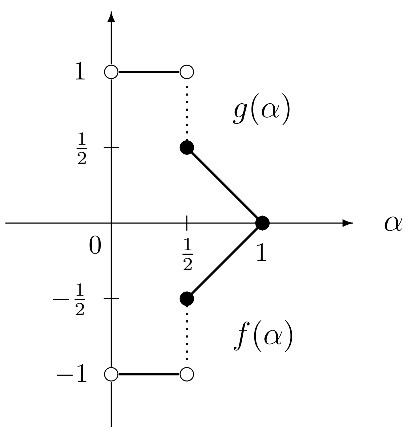


Figure 2.3: Functions $f(\cdot)$ and $g(\cdot)$ defined in Example 2.2.



If we want to fulfil these conditions, we need to redefine the set $A_{\frac{1}{2}}$ to be the biggest set keeping the nested structure of the generating family as $A_{\frac{1}{2}} = [-1, 1]$. In this redefined structure $\forall \delta \in (0, 1]$ the δ-cuts are the same as the A_δ sets, i.e. $\forall \delta \in (0, 1]$ is fulfilled $\mathcal{C}_\delta(a^*) = A_\delta$.

3. Fuzzy Vectors

Definition 3.1. A *n-dimensional fuzzy vector* $\underline{x}^*$ is a fuzzy set in $\mathbb{R}^n$ determined by its membership function $\mu_{\underline{x}^*}(\cdot, \ldots, \cdot)$ called *vector-characterizing function*, which is a real function of n real variables, fulfilling the following:

1. $\mu_{\underline{x}^*}: \mathbb{R}^n \to [0,1]$
2. $\forall \delta \in (0,1]$ the δ-cut $\mathcal{C}_\delta(\underline{x}^*) := \{\underline{x} \in \mathbb{R}^n : \mu_{\underline{x}^*}(\underline{x}) \geq \delta\}$ is non-empty and a finite union of simply connected, closed and bounded sets
3. The support of $\underline{x}^*$, $\operatorname{supp}(\underline{x}^*) = \{\underline{x} \in \mathbb{R}^n : \mu_{\underline{x}^*}(\underline{x}) > 0\}$, is a bounded set

Definition 3.2. A *n-dimensional fuzzy interval* is a n-dimensional fuzzy vector where each δ-cut $\mathcal{C}_\delta(\underline{x}^*)$ is a simply connected, closed and a bounded set.

Definition 3.3. A *n-dimensional convex fuzzy vector* $\underline{x}^*$ is a n-dimensional fuzzy vector in $\mathbb{R}^n$ where each δ-cut $\mathcal{C}_\delta(\underline{x}^*)$ is a finite union of convex, closed and bounded sets.

Definition 3.4. A *n-dimensional convex fuzzy interval* is a n-dimensional fuzzy vector where each δ-cut $\mathcal{C}_\delta(\underline{x}^*)$ is a convex, closed and a bounded set.

Remark. A precise vector $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ can be represented by its vector-characterizing function:

$$\mu_{\underline{a}^*}(\underline{x}) = \left\{ \begin{array}{ll} 1 & \text{for } \underline{x} = \underline{a} \\ 0 & \text{for } \underline{x} \neq \underline{a} \end{array} \right\} \quad \forall \underline{x} \in \mathbb{R}^n,$$

This vector-characterizing function is the one-point indicator function $\mathbb{1}_{\{\underline{a}\}}(\cdot, \ldots, \cdot)$ of the crisp set $\{\underline{a}\}$.

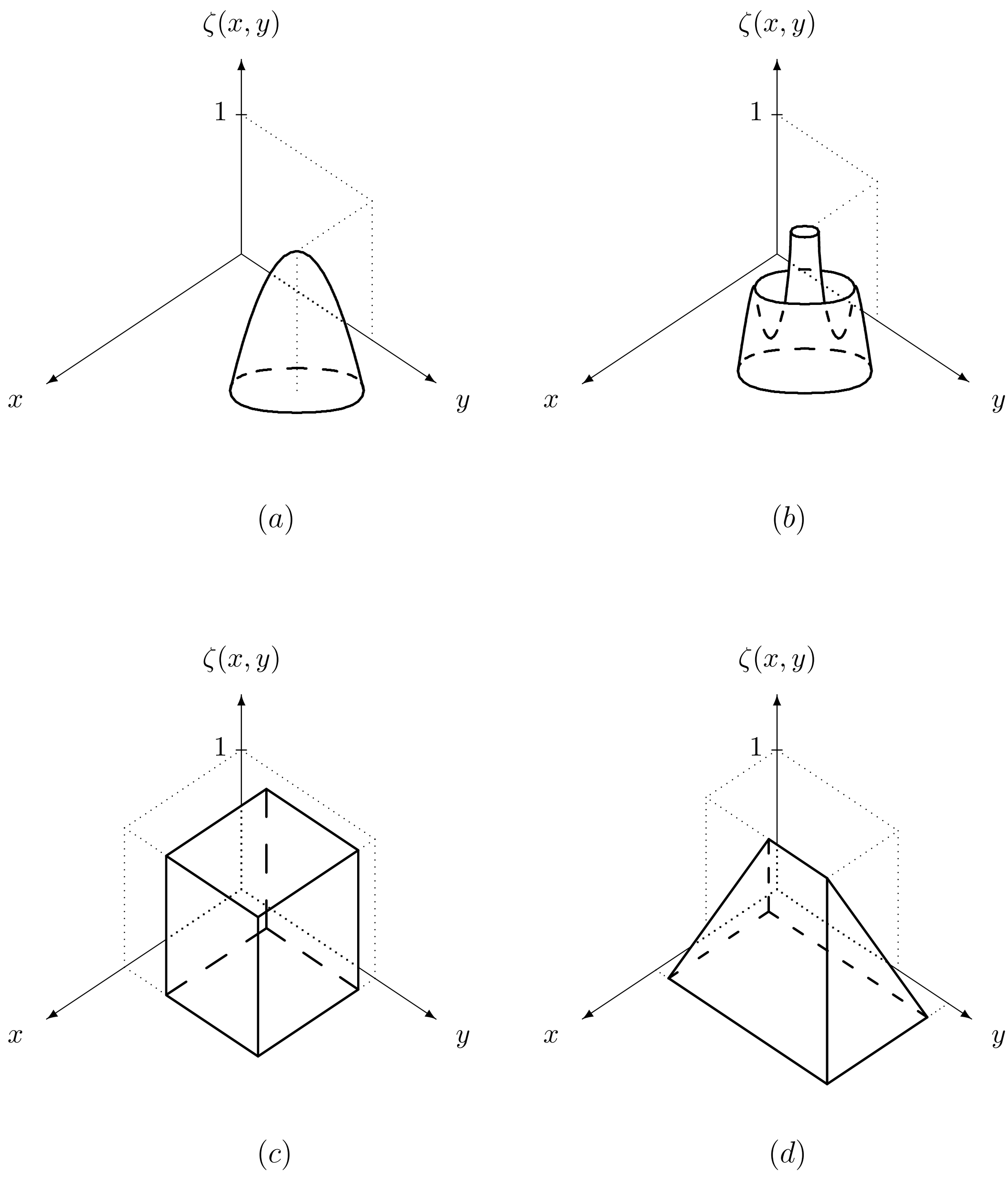
Remark. Each one-dimensional fuzzy vector is a fuzzy number and each fuzzy number is a one-dimensional fuzzy vector.

Remark. Each n-dimensional convex fuzzy vector is a n-dimensional fuzzy vector and each n-dimensional convex fuzzy interval is a n-dimensional fuzzy interval.

Remark. Each n-dimensional fuzzy vector and n-dimensional fuzzy interval is a special case of a fuzzy set defined in $\mathbb{R}^n$.

Example 3.1. Examples of vector-characterizing functions of fuzzy vectors are given in Figure 3.1. The vector-characterizing functions in Figures 3.1(a), 3.1(c), and 3.1(d) are characterizing functions of 2-dimensional fuzzy intervals. The function in Figure 3.1(b) is a vector-characterizing function a of 2-dimensional fuzzy vector. In Figure 3.1(c) is displayed a vector-characterizing function of a crisp set.

Figure 3.1: Examples of vector-characterizing functions of fuzzy vectors



3.1 Simply Connected Set

In this section we will define and describe the concept of simply connected sets. Also some basic properties will be proved.

Definition 3.5. Let $M \subseteq \mathbb{R}^n$ be a set. We call the set M *path-connected* if and only if we can find a continuous curve between arbitrary two points in M such that this curve is a part of the set M, more precisely:

$$\forall x_1, x_2 \in M \quad \exists\, h\colon [0,1] \to M\colon \quad h \text{ is continuous}, \quad h(0) = x_1, \quad h(1) = x_2$$

Definition 3.6. Let $M \subseteq \mathbb{R}^n$ be a set. We call the set M *simply connected* if and only if M is path-connected and $\forall i \in \{1, \ldots, n\}$ any continuous mapping

$$f\colon S_i \to M$$

can be contracted to a point in the set M, where $S_i = \{\underline{x} \in \mathbb{R}^i : \|\underline{x}\| = 1\}$ is a sphere.

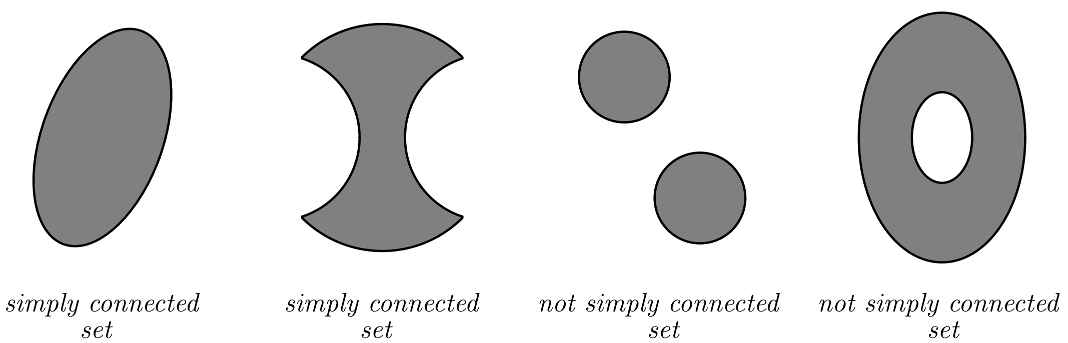
We can imagine the contraction to the point as a continuous sequence of images of a sphere with gradually decreasing diameter. More precisely that there exists a continuous mapping

$$F\colon D_i \to M$$

such that F restricted to S_i if equal to f, where $D_i = \{\underline{x} \in \mathbb{R}^i : \|\underline{x}\| \leq 1\}$ is a ball.

Example 3.2. Some graphical examples of simply connected sets in $\mathbb{R}^2$ and sets which are not simply connected in $\mathbb{R}^2$ are given in Figure 3.2.

Figure 3.2: Examples of simply connected and not simply connected sets in $\mathbb{R}^2$



Definition 3.7. Let M be a set in a vector space over $\mathbb{R}$. Then M is called *convex set* if the whole line segment joining any pair of points of M lies entirely in M:

$$\forall x, y \in M \quad \forall \lambda \in [0,1] \quad \lambda x + (1-\lambda) y \in M$$

Lemma 3.1. *Let $M \subseteq \mathbb{R}^n$ be a convex set. Then M is a simply connected set.*

Proof. Firstly we have to prove that M is path-connected. Let $\underline{x}, \underline{y} \in M$ be two arbitrary points. We have to construct a continuous curve between points $\underline{x}$ and $\underline{y}$ contained in M. From the fact that M is a convex set, the continuous curve $h : [0,1] \to M$ fulfilling $h(0) = \underline{x}$, $h(1) = \underline{y}$ can be constructed as a line segment in the following way:

$$h(\lambda) = \lambda \underline{y} + (1 - \lambda)\underline{x} \qquad \forall \lambda \in [0,1]$$

Now we have to prove that any continuous mapping on a sphere can be contracted to a point in the set M. Let $i \in \{1, \dots, n\}$ be a dimension of a sphere $S_i = \{\underline{x} \in \mathbb{R}^i : \|\underline{x}\| = 1\}$ and let $f : S_i \to M$ be a continuous mapping of the sphere S_i to the set M. We have to prove that the mapping can be contracted to a point in the set M.

We choose an arbitrary point $\underline{x} \in M$. From the fact that M is a convex set we know that there exists a line segment between each point of the image of f and the point $\underline{x}$ in the set M. We define a continuous mapping $F : D_i \to M$, where $D_i = \{\underline{x} \in \mathbb{R}^i : \|\underline{x}\| \leq 1\}$ is a ball, such that F restricted to S_i is equal to f in the following way:

$$F(\alpha) = \|\alpha\| f\left(\frac{\alpha}{\|\alpha\|}\right) + (1 - \|\alpha\|)\,\underline{x} \qquad \forall \alpha \in D_i$$

□

Lemma 3.2. *Let $M \subseteq \mathbb{R}$ be a non-empty and bounded set. Then the following statements are is equivalent:*

- *M is a simply connected set*
- *M is an interval such that* $(\inf(M), \sup(M)) \subseteq M \subseteq [\inf(M), \sup(M)]$

Proof. We will prove both implications separately. Firstly we will prove that from the fact that M is a simply connected set follows that M is an interval such that $(\inf(M), \sup(M)) \subseteq M \subseteq [\inf(M), \sup(M)]$.

Let M be a simply connected set. For the proof ad absurdum assume that there exists a point $x \in (\inf(M), \sup(M))$ such that $x \notin M$. Then the set M is not pathwise-connected (in this one-dimensional case) and hence M is not a simply connected set.

Secondly we will prove the opposite implication. Let M be an interval such that $(\inf(M), \sup(M)) \subseteq M \subseteq [\inf(M), \sup(M)]$. The interval M is a convex set and thus this implication is a special case of Lemma 3.1. □

3.2 Extension principle

Definition 3.8. The extension principle generalizes classical functions from an arbitrary set M to a second set N for fuzzy elements of M to fuzzy elements in N. Fuzzy numbers and fuzzy vectors are special cases of fuzzy elements.

Let g: $M \to N$ be a classical function. For a fuzzy element A^* in M with membership function $\xi(\cdot)$ the generalized value $g(A^*)$ has to be defined in reasonable way such that it is a fuzzy element in N. In order to obtain the membership function $\eta(\cdot)$ which characterizes the fuzzy element $B^* = g(A^*)$ in N, the values $\eta(\cdot)$ are defined in the following way:

$$\eta(b) = \left\{ \begin{array}{ll} \sup\left\{\xi(a)\colon a \in g^{-1}(\{b\})\right\} & \text{if } g^{-1}(\{b\}) \neq \emptyset \\ 0 & \text{if } g^{-1}(\{b\}) = \emptyset \end{array} \right\} \qquad \forall b \in N$$

By this definition a membership function $\eta(\cdot)$ of a fuzzy element in N is obtained.

Theorem 3.3. *Let $\underline{x}^*$ be a n-dimensional convex fuzzy interval with vector-characterizing function $\zeta(\cdot,\ldots,\cdot)$. Let $M \subseteq \mathbb{R}^n$ be a set containing the support of $\underline{x}^*$. Let $g : M \to \mathbb{R}^m$ be a continuous and bounded function on M such that the image of an arbitrary convex set is a convex set.*

We define a fuzzy set $\underline{y}^* = g(\underline{x}^*)$ in $\mathbb{R}^m$ described by its membership function $\psi(\cdot,\ldots,\cdot)$ based on the extension principle by

$$\psi(\underline{y}) = \sup(\{\zeta(\underline{x}) : \underline{x} \in \mathbb{R}^n, g(\underline{x}) = \underline{y}\} \cup \{0\}) \quad \forall \underline{y} \in \mathbb{R}^m \ .$$

Then $\underline{y}^*$ is a m-dimensional convex fuzzy interval.

Proof. We define a function $f : \mathbb{R}^n \times \mathbb{R}^m \to [0,1]$ by the following formula:

$$f(\underline{x},\underline{y}) := \left\{ \begin{array}{ll} \zeta(\underline{x}) & \text{if } g(\underline{x}) = \underline{y} \\ 0 & \text{elswere} \end{array} \right\} \quad \forall (\underline{x},\underline{y}) \in \mathbb{R}^n \times \mathbb{R}^m$$

An example of this type of function is given in Example 3.3 on Figure 3.5.

We can define functions $f_1 : \mathbb{R}^n \to [0,1]$ and $f_2 : \mathbb{R}^m \to [0,1]$ as a kind of restriction of the function $f(\underline{x},\underline{y})$ in the following way:

$$f_1(\underline{x}) := \sup\{f(\underline{x},\underline{y}) : \underline{y} \in \mathbb{R}^m\} \quad \forall \underline{x} \in \mathbb{R}^n$$

$$f_2(\underline{y}) := \sup\{f(\underline{x},\underline{y}) : \underline{x} \in \mathbb{R}^n\} \quad \forall \underline{y} \in \mathbb{R}^m$$

We can see that $f_1(\cdot,\ldots,\cdot)$ and $f_2(\cdot,\ldots,\cdot)$ are related to the vector-characterizing function $\zeta(\cdot,\ldots,\cdot)$ and the membership function $\psi(\cdot,\ldots,\cdot)$:

$$f_1(\underline{x}) = \sup\{f(\underline{x},\underline{y}) : \underline{y} \in \mathbb{R}^m\} = \sup\{0, f(\underline{x}, g(\underline{x}))\} = \max\{0, \zeta(\underline{x})\} = \zeta(\underline{x}) \qquad \forall \underline{x} \in \mathbb{R}^n$$

$$\begin{aligned} f_2(\underline{y}) &= \sup\{f(\underline{x},\underline{y}) : \underline{x} \in \mathbb{R}^n\} = \sup(\{f(\underline{x}, g(\underline{x})) : \underline{x} \in \mathbb{R}^n, g(\underline{x}) = \underline{y}\} \cup \{0\}) = \\ &= \sup(\{\zeta(\underline{x}) : \underline{x} \in \mathbb{R}^n, g(\underline{x}) = \underline{y}\} \cup \{0\}) = \psi(\underline{y}) \qquad \forall \underline{y} \in \mathbb{R}^m \end{aligned}$$

We have shown that the function f is containing the vector-characterizing function of the n-dimensional fuzzy interval $\underline{x}^*$ in the first coordinate and all information about the membership function of the fuzzy set y^* in the second coordinate. For better vision of this idea see Example 3.3.

We want to prove, that δ-cuts $\mathcal{C}_\delta(\underline{y}^*)$ are convex, closed and bounded sets. We will construct a family of sets $B_\delta \subseteq \mathbb{R}^n \times \mathbb{R}^m$, which will have δ-cut-like structure in $(n+m)$ dimensions containing information about both $\mathcal{C}_\delta(\underline{x}^*)$ and $\mathcal{C}_\delta(\underline{y}^*)$. We define B_δ for $\delta \in (0,1]$ as:

$$B_\delta := \{(\underline{x},\underline{y}) : (\underline{x},\underline{y}) \in \mathbb{R}^n \times \mathbb{R}^m,\ f(\underline{x},\underline{y}) \geq \delta\} =$$

$$= \{(\underline{x},\underline{y}) : (\underline{x},\underline{y}) \in \mathbb{R}^n \times \mathbb{R}^m,\ \zeta(\underline{x}) \geq \delta,\ g(\underline{x}) = \underline{y}\} \quad \forall \delta \in (0,1]$$

We will investigate sets $\{\underline{x} : (\underline{x},\underline{y}) \in B_\delta\}$ and $\{\underline{y} : (\underline{x},\underline{y}) \in B_\delta\}$ and find that they correspond with the δ-cuts $\mathcal{C}_\delta(\underline{x}^*)$ and $\mathcal{C}_\delta(\underline{y}^*)$ respectively.

$$\{\underline{x} : (\underline{x},\underline{y}) \in B_\delta\} = \{\underline{x} : (\underline{x},\underline{y}) \in \mathbb{R}^n \times \mathbb{R}^m,\ \zeta(\underline{x}) \geq \delta,\ g(\underline{x}) = \underline{y}\} =$$

$$= \{\underline{x} : \underline{x} \in \mathbb{R}^n,\ \zeta(\underline{x}) \geq \delta\} = \mathcal{C}_\delta(\underline{x}^*)$$

Remind, that $\mathcal{C}_\delta(\underline{x}^*)$ is non-empty, simply connected, closed and a bounded set in $\mathbb{R}^n$ by definition.

$$\{\underline{y} : (\underline{x},\underline{y}) \in B_\delta\} = \{\underline{y} : (\underline{x},\underline{y}) \in \mathbb{R}^n \times \mathbb{R}^m,\ \zeta(\underline{x}) \geq \delta,\ g(\underline{x}) = \underline{y}\} =$$

$$= \{\underline{y} : \sup\{\zeta(\underline{x}) : \underline{x} \in \mathbb{R}^n, g(\underline{x}) = \underline{y}\} \geq \delta\} =$$
$$= \{\underline{y} : \psi(\underline{y}) \geq \delta\} = \mathcal{C}_\delta(\underline{y}^*)$$

We will prove that $\underline{y}^*$ is a m-dimensional convex fuzzy interval with characterizing function $\psi(\cdot,\ldots,\cdot)$ by verifying the conditions from Definition 3.4 of a m-dimensional fuzzy vector and m-dimensional convex fuzzy interval:

- $\psi : \mathbb{R}^m \to [0,1]$.

  This is fulfilled, because of the construction of the function $\psi(\cdot,\ldots,\cdot)$.

- $\forall \delta \in (0,1]$ the δ-cut $\mathcal{C}_\delta(\underline{y}^*)$ is a non-empty convex, closed, and a bounded set.

  We have proved, that $\mathcal{C}_\delta(\underline{y}^*) = \{\underline{y} : (\underline{x},\underline{y}) \in B_\delta\}$. We can express this set also in the following way:

  $$\{\underline{y} : (\underline{x},\underline{y}) \in B_\delta\} = \{\underline{y} : (\underline{x},\underline{y}) \in \mathbb{R}^n \times \mathbb{R}^m,\ \zeta(\underline{x}) \geq \delta,\ g(\underline{x}) = \underline{y}\} =$$

  $$= \{\underline{y} : \underline{x} \in \mathcal{C}_\delta(\underline{x}^*),\ g(\underline{x}) = \underline{y}\} = g(\mathcal{C}_\delta(\underline{x}^*))$$

  So we have $\mathcal{C}_\delta(\underline{y}^*) = g(\mathcal{C}_\delta(\underline{x}^*))$, where $\underline{x}^*$ is a n-dimensional convex fuzzy interval.

  We know that $\mathcal{C}_\delta(\underline{x}^*)$ is non-empty, convex, closed, and a bounded set of M from the definition for all $\delta \in (0,1]$. Now we will prove that $\mathcal{C}_\delta(\underline{y}^*)$ also fulfils all these attributes.

– $\mathcal{C}_\delta(\underline{y}^*)$ is non-empty.

$\mathcal{C}_\delta(\underline{x}^*)$ is non-empty, therefore exists $\underline{x} \in \mathcal{C}_\delta(\underline{x}^*)$. The function $g(\cdot, \ldots, \cdot)$ is defined on $M \supseteq \operatorname{supp}(\zeta(\cdot, \ldots, \cdot))$ so we can take $\underline{y} := g(\underline{x})$. From $\mathcal{C}_\delta(\underline{y}^*) = g(\mathcal{C}_\delta(\underline{x}^*))$ we have $\underline{y} \in \mathcal{C}_\delta(\underline{y}^*)$ and therefore $\mathcal{C}_\delta(\underline{y}^*)$ is non-empty.

– $\mathcal{C}_\delta(\underline{y}^*)$ is convex.

$\mathcal{C}_\delta(\underline{x}^*)$ is convex and the function $g(\cdot, \ldots, \cdot)$ by assumption displays a convex set to a convex set, so $\mathcal{C}_\delta(\underline{y}^*) = g(\mathcal{C}_\delta(\underline{x}^*))$ is a convex set.

– $\mathcal{C}_\delta(\underline{y}^*)$ is closed.

The function $g(\cdot, \ldots, \cdot)$ is continuous on M and the set $\mathcal{C}_\delta(\underline{x}^*) \subset M$ is closed, so the set $\mathcal{C}_\delta(\underline{y}^*) = g(\mathcal{C}_\delta(\underline{x}^*))$ is closed as an continuous image of a closed set.

– $\mathcal{C}_\delta(\underline{y}^*)$ is bounded.

The function $g(\cdot, \ldots, \cdot)$ is bounded on M, so there exists $r \in \mathbb{R}$ such that $g(M) \subseteq B(0, r) := \{\underline{y} \in \mathbb{R}^m : |\underline{y}| < r\}$. We have $\mathcal{C}_\delta(\underline{x}^*) \subseteq \operatorname{supp}(\zeta(\cdot, \ldots, \cdot)) \subseteq M$ and hence $\mathcal{C}_\delta(\underline{y}^*) = g(\mathcal{C}_\delta(\underline{x}^*)) \subseteq B(0, r)$ is bounded.

○ Support of $\psi(\cdot, \ldots, \cdot)$ is bounded.

The support of the function $\psi(\cdot, \ldots, \cdot)$ can be expressed as

$$\operatorname{supp}(\psi(\cdot, \ldots, \cdot)) = \{g(\underline{x}) : \underline{x} \in \operatorname{supp}(\zeta(\cdot, \ldots, \cdot))\},$$

where the function $g(\cdot, \ldots, \cdot)$ is bounded, therefore the support of the function $\psi(\cdot, \ldots, \cdot)$ is bounded.

We have proved that the fuzzy set $\underline{y}^*$ is an m-dimensional convex fuzzy interval.

□

Theorem 3.4. *Let $\underline{x}^*$ be a n-dimensional convex fuzzy interval with vector-characterizing function $\zeta(\cdot, \ldots, \cdot)$. Let $M \subseteq \mathbb{R}^n$ be a set containing the support of $\underline{x}^*$. Let $g : M \to \mathbb{R}$ be a continuous and bounded function on M such that the image of an arbitrary convex set is an interval.*

We define a fuzzy set $y^* = g(\underline{x}^*)$ in $\mathbb{R}$ described by its membership function $\psi(\cdot)$ based on the extension principle by

$$\psi(y) = \sup(\{\zeta(\underline{x}) : \underline{x} \in \mathbb{R}^n, g(\underline{x}) = y\} \cup \{0\}) \quad \forall y \in \mathbb{R} \ .$$

Then y^* is a one-dimensional convex fuzzy interval and each δ-cut $\mathcal{C}_\delta(y^*)$ fulfils the following:

$$\mathcal{C}_\delta(y^*) = \left[\min_{\underline{x} \in \mathcal{C}_\delta(\underline{x}^*)} g(\underline{x}), \max_{\underline{x} \in \mathcal{C}_\delta(\underline{x}^*)} g(\underline{x}) \right] \quad \forall \delta \in (0, 1]$$

Proof. This Theorem is a special one-dimensional case of Theorem 3.3. The only part we have to prove is the part describing the shape of $\mathcal{C}_\delta(y^*)$.

We have proved, that $g(\mathcal{C}_\delta(\underline{x}^*)) = \mathcal{C}_\delta(y^*)$. We also know, that both $\mathcal{C}_\delta(\underline{x}^*)$ and $\mathcal{C}_\delta(y^*)$ are non-empty, closed, bounded, and convex sets.

Then the following must be fulfilled:

$$\mathcal{C}_\delta(y^*) = \left[\min_{y\in\mathcal{C}_\delta(y^*)} y, \max_{y\in\mathcal{C}_\delta(y^*)} y\right] = \left[\min_{\underline{x}\in\mathcal{C}_\delta(\underline{x}^*)} g\left(\underline{x}\right), \max_{\underline{x}\in\mathcal{C}_\delta(\underline{x}^*)} g\left(\underline{x}\right)\right]$$

□

Theorem 3.5. *Let $\underline{x}^*$ be a n-dimensional convex fuzzy vector with vector-characterizing function $\zeta(\cdot,\dots,\cdot)$. Let $M \subseteq \mathbb{R}^n$ be a set containing the support of $\underline{x}^*$. Let $g : M \to \mathbb{R}^m$ be a continuous and bounded function on M such that the image of an arbitrary convex set is a convex set.*

We define a fuzzy set $\underline{y}^* = g(\underline{x}^*)$ in $\mathbb{R}^m$ described by its membership function $\psi(\cdot,\dots,\cdot)$ based on the extension principle by

$$\psi(\underline{y}) = \sup(\{\zeta(\underline{x}) : \underline{x} \in \mathbb{R}^n, g(\underline{x}) = \underline{y}\} \cup \{0\}) \quad \forall \underline{y} \in \mathbb{R}^m$$

Then $\underline{y}^*$ is a m-dimensional convex fuzzy vector.

Proof. We will prove this Theorem 3.5 by using Theorem 3.3.

We know, that the δ-cut $\mathcal{C}_\delta(\underline{x}^*)$ is non-empty and a finite union of convex, closed, and bounded sets, for all $\delta \in (0,1]$. We can split each δ-cut into a finite number k_δ of convex, closed, and bounded sets and denote them by $X_\delta^1, \dots, X_\delta^{k_\delta}$.

We also know, that $\mathcal{C}_{\tilde{\delta}}(\underline{x}^*) \subseteq \mathcal{C}_\delta(\underline{x}^*)$ for all $\delta, \tilde{\delta} \in (0,1]$, where $\delta \leq \tilde{\delta}$. Especially we know that $\mathcal{C}_1(\underline{x}^*) \subseteq \mathcal{C}_\delta(\underline{x}^*)$ for all $\delta \in (0,1]$, and hence $k_1 \geq k_\delta$ for all $\delta \in (0,1]$.

We will construct k_1 n-dimensional convex fuzzy intervals with the δ-cuts:

$$\mathcal{C}_\delta(\underline{x_i}^*) := X_\delta^j, \quad \text{where} \quad X_1^i \subseteq X_\delta^j \quad \text{for} \quad i \in \{1,\dots,k_1\}$$

According to Theorem 3.3 there exist m-dimensional convex fuzzy intervals $\underline{y_i}^* = g(\underline{x_i}^*)$ for all $i \in \{1,\dots,k_1\}$.

We can construct a m-dimensional convex fuzzy vector $\underline{y}^*$ as (some sort of) union of $\underline{y_1},\dots,\underline{y_{k_1}}$ with δ-cuts fulfilling the following formula:

$$\mathcal{C}_\delta(\underline{y}^*) := \bigcup_{i=1}^{k_1} \mathcal{C}_\delta(\underline{y_i}^*) \quad \forall\delta \in (0,1]$$

□

Example 3.3. We define the (one-dimensional) fuzzy interval x^* as a fuzzy interval x^* with the characterizing function $\xi(\cdot)$ (see Figure 3.3):

$$\xi(x) := \left\{ \begin{array}{cl} (x-1)/3 & \text{for } x \in [1,4] \\ 1 & \text{for } x \in [4,7] \\ 8-x & \text{for } x \in [7,8] \\ 0 & \text{elswere} \end{array} \right\} \quad \forall x \in \mathbb{R}$$

We define a function $g(\cdot)$ using the following formula (see Figure 3.4):

$$g(x) := 5 - 2\sin(x) \qquad \forall x \in \mathbb{R}$$

Then we can construct a function $f(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ like in the proof of Theorem 3.3 in the following way:

$$f(x,y) := \left\{ \begin{array}{ll} \xi(x) & \text{where } g(x) = y \\ 0 & \text{otherwise} \end{array} \right\} \quad \forall (x,y) \in \mathbb{R} \times \mathbb{R}$$

The function $f(\cdot,\cdot)$ has a 3-dimensional structure, as shown in Figure 3.5.

If we look to this structure in the direction of the y-axis, we will see the characterizing function $\xi(\cdot)$ (see Figure 3.6), and if we look in the direction of the z-axis, we will see a part of the function $g(\cdot)$ (see Figure 3.7).

We define a fuzzy set $y^* = g(x^*)$ described by its membership function:

$$\psi(y) := \sup(\{\xi(x) : x \in \mathbb{R}, g(x) = y\} \cup \{0\}) \quad \forall y \in \mathbb{R}$$

According to Theorem 3.3 the function $\psi(\cdot)$ is the characterizing function of a fuzzy interval.

We can look at the function $f(\cdot,\cdot)$ in the direction of x-axis and we will see the function $\psi(\cdot)$ (see Figures 3.8 and 3.9). This discovery of the function $\psi(\cdot)$ is the reason for the construction of the function $f(\cdot,\cdot)$.

Figure 3.3: Characterizing function $\xi(\cdot)$ of a fuzzy interval x^*

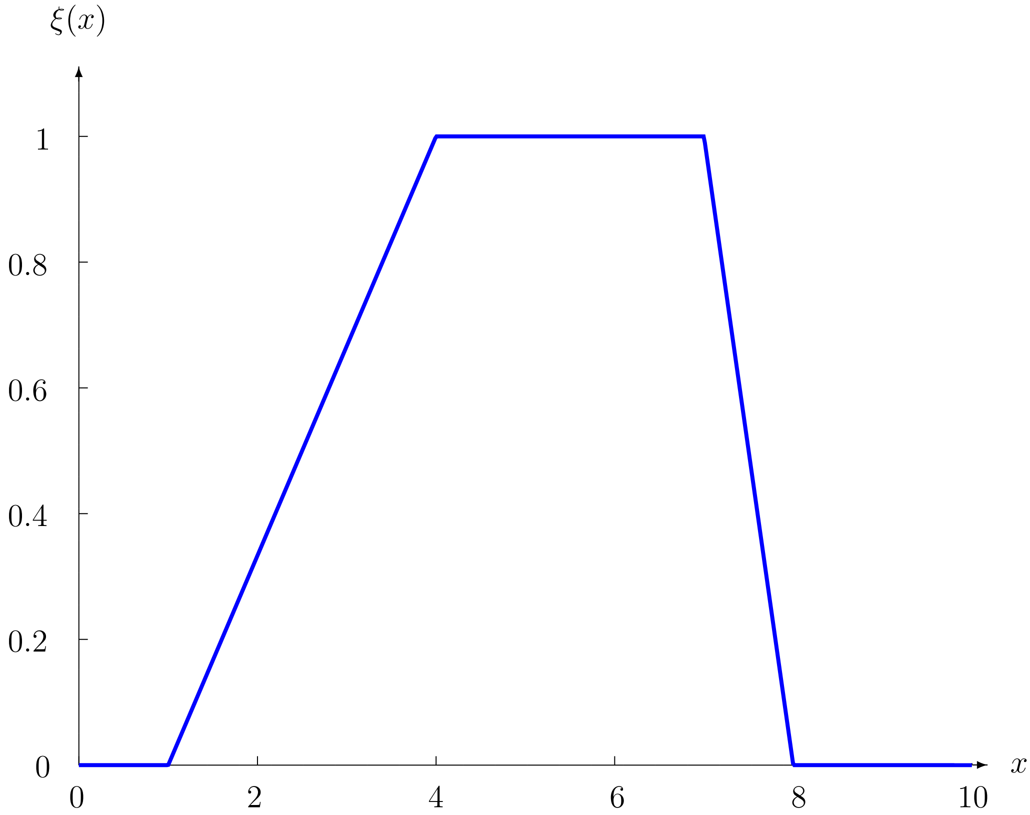


Figure 3.4: Function $g(\cdot)$.

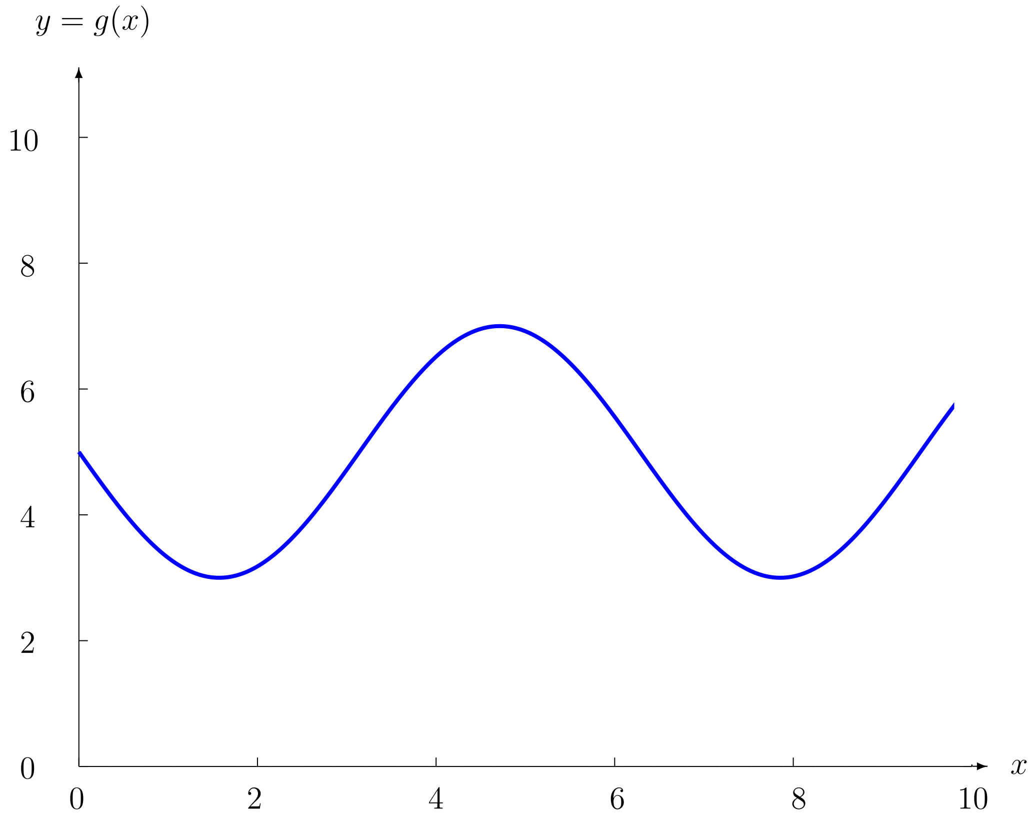


Figure 3.5: Function $f(\cdot,\cdot)$ from two different viewpoints

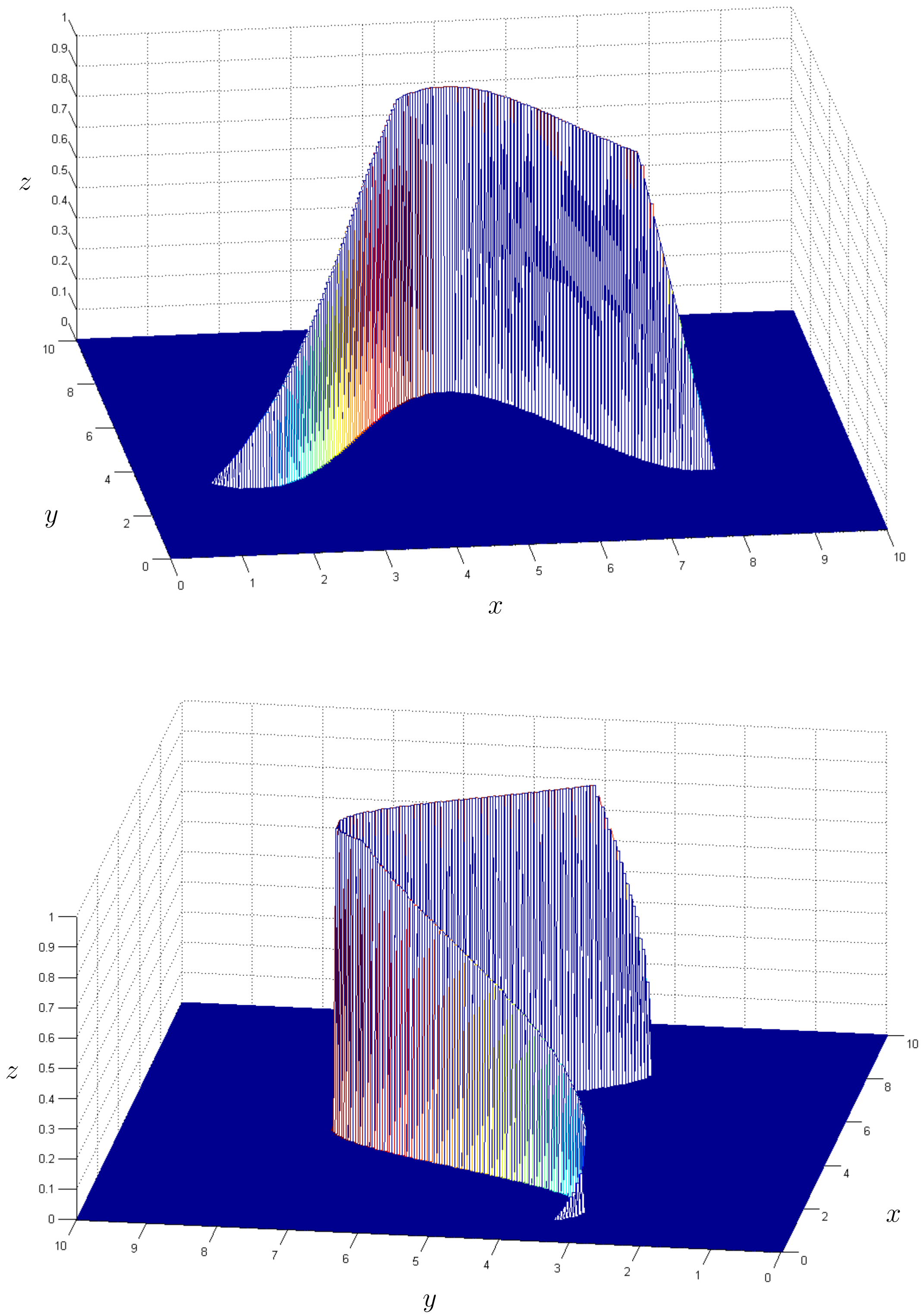


Figure 3.6: Function $f(\cdot,\cdot)$ viewed in direction of the y-axis

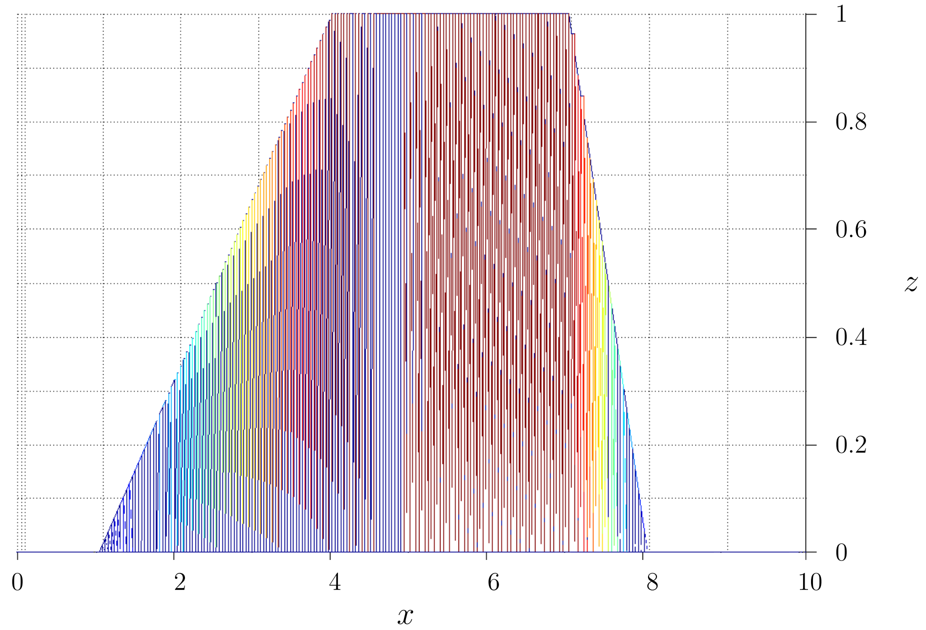


Figure 3.7: Function $f(\cdot,\cdot)$ viewed in direction of the z-axis

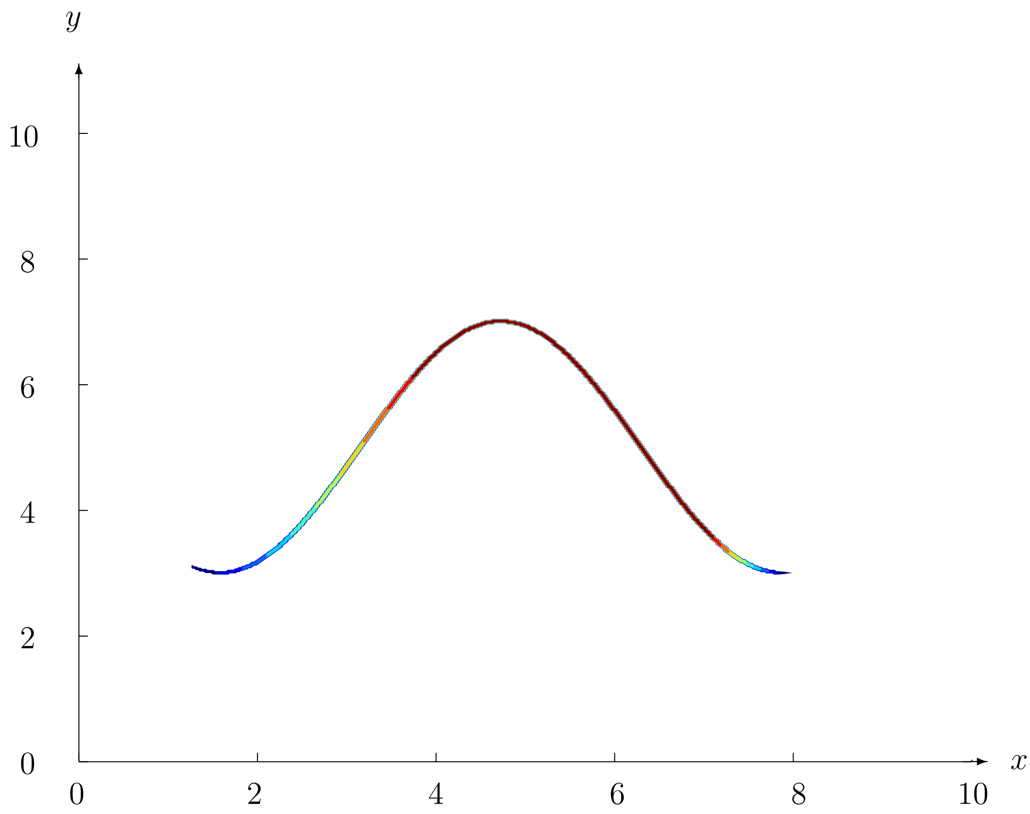


Figure 3.8: Function $f(\cdot,\cdot)$ viewed in direction of the x-axis

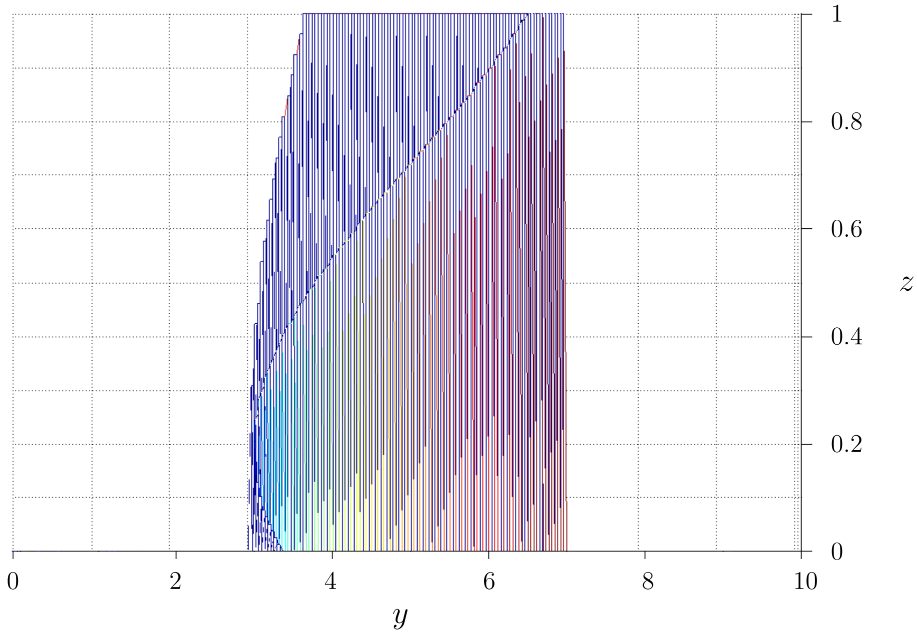
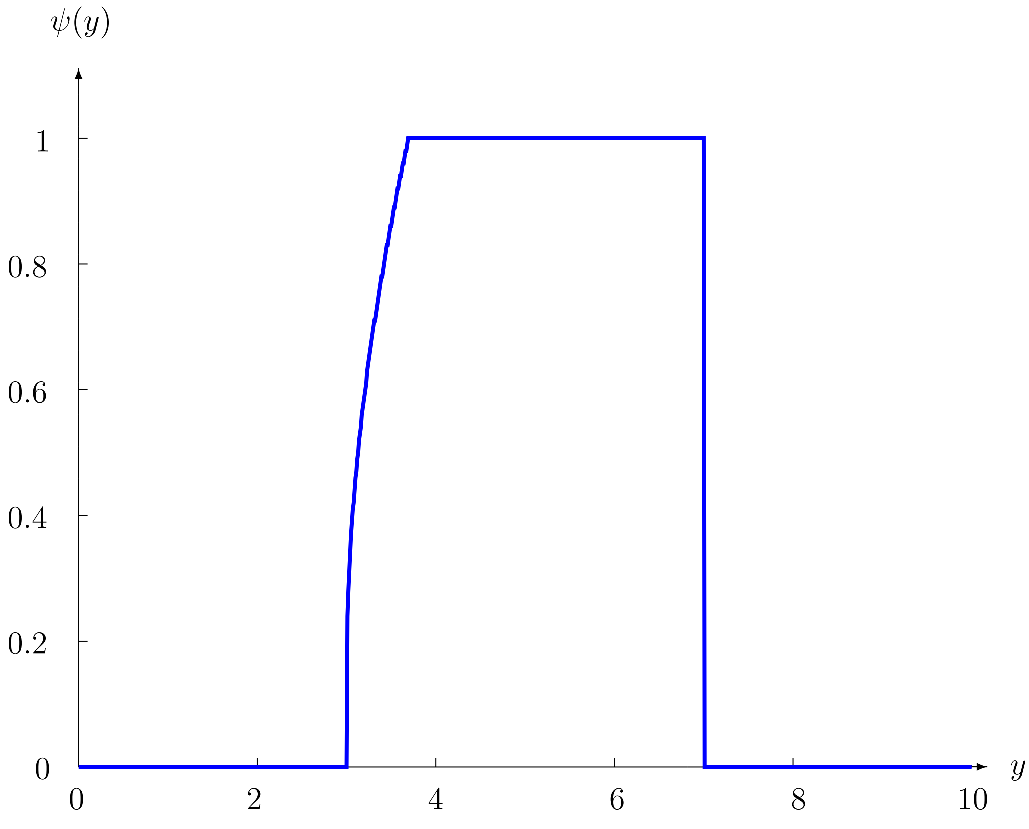


Figure 3.9: Characterizing function $\psi(\cdot)$ of a fuzzy interval y^*



4. Vector of Fuzzy Numbers

Definition 4.1. A n-dimensional *vector of fuzzy numbers* $(x_1^*, \ldots, x_n^*)$ is a vector containing n fuzzy numbers $x_1^*, \ldots, x_n^*$. It is determined by n characterizing functions $\mu_{x_1^*}(\cdot), \ldots, \mu_{x_n^*}(\cdot)$ belonging to fuzzy numbers $x_1^*, \ldots, x_n^*$.

Definition 4.2. A function $T : [0,1] \times [0,1] \to [0,1]$ is called *triangular norm* or *t-norm*, if $\forall x, y, z \in [0,1]$ the following conditions are fulfilled:

1. $T(x,y) = T(y,x)$, T is commutative
2. $T(T(x,y),z) = T(x,T(y,z))$, T is associative
3. $T(x,1) = x$, value 1 is neutral to T
4. $x \le y \Rightarrow T(x,z) \le T(y,z)$, T is transitive in one coordinate

Definition 4.3. In this definition are given some examples of t-norms.

Minimum t-norm:

$$T_{\min}(x,y) = \min\{x,y\} \qquad \forall (x,y) \in [0,1]^2$$

Product t-norm:

$$T_{\text{prod}}(x,y) = x \cdot y \qquad \forall (x,y) \in [0,1]^2$$

Limited sum t-norm:

$$T_{\text{lsum}}(x,y) = \max\{x+y-1, 0\} \qquad \forall (x,y) \in [0,1]^2$$

Drastic product t-norm:

$$T_{\text{dp}}(x,y) = \min\{x,y\} \cdot \mathbb{1}_{\{\max\{x,y\}\}}(1) \qquad \forall\, (x,y) \in [0,1]^2$$

Example 4.1. In Figure 4.1 are given fuzzy vectors obtained by application of the above defined t-norms on a vector of two fuzzy numbers.

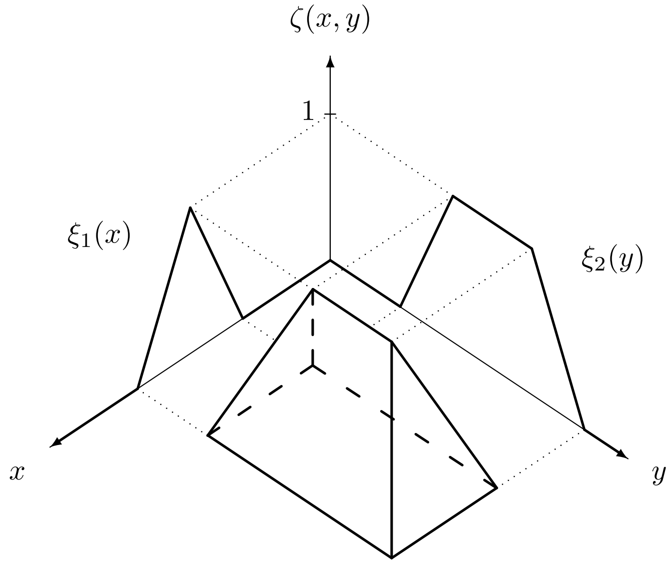
Remark. Combination of fuzzy numbers into a fuzzy vector is possible based on t-norms. For two fuzzy numbers a^* and b^* with corresponding characterizing functions $\mu_{a^*}(\cdot)$, $\mu_{b^*}(\cdot)$ a fuzzy vector $\underline{x}^* = (a,b)^*$ is given by its vector-characterizing function $\mu_{\underline{x}^*}(\cdot,\cdot)$ whose values $\mu_{\underline{x}^*}(x,y)$ are defined based on a t-norm T by:

$$\mu_{\underline{x}^*}(x,y) := T(\mu_{a^*}(x), \mu_{b^*}(y)) \quad \forall (x,y) \in \mathbb{R}^2$$

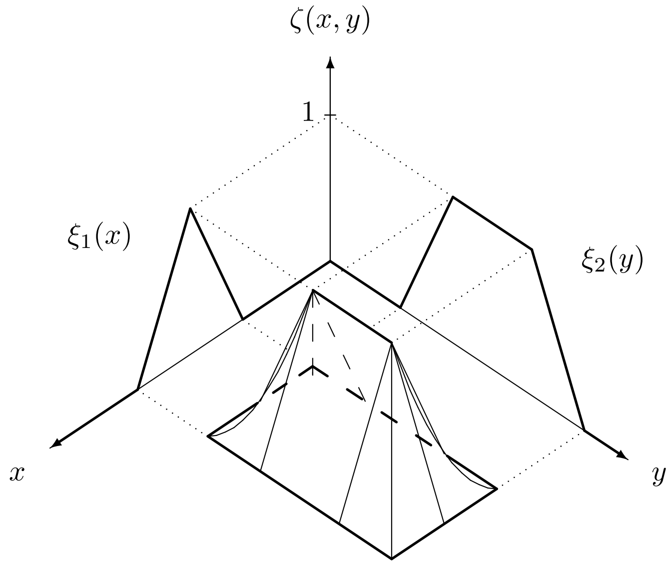
By the associativity of t-norms this can be extended to n fuzzy numbers, $n \in \mathbb{N}$. Let $a_1^*, \ldots, a_n^*$ be fuzzy numbers with corresponding characterizing functions $\mu_{a_1^*}(\cdot), \ldots, \mu_{a_n^*}(\cdot)$. Then we can define a fuzzy vector $\underline{x}^* = (x_1, \ldots, x_n)^*$ by its vector-characterizing function $\mu_{\underline{x}^*}(\cdots)$ fulfilling the following formula:

$$\mu_{\underline{x}^*}(x_1, \ldots, x_n) := T(\mu_{a_1^*}(x_1), T(\mu_{a_2^*}(x_2), \ldots T(\mu_{a_{n-1}^*}(x_{n-1}), \mu_{a_n^*}(x_n)) \ldots)) \qquad \forall (x_1, \ldots, x_n) \in \mathbb{R}^n$$

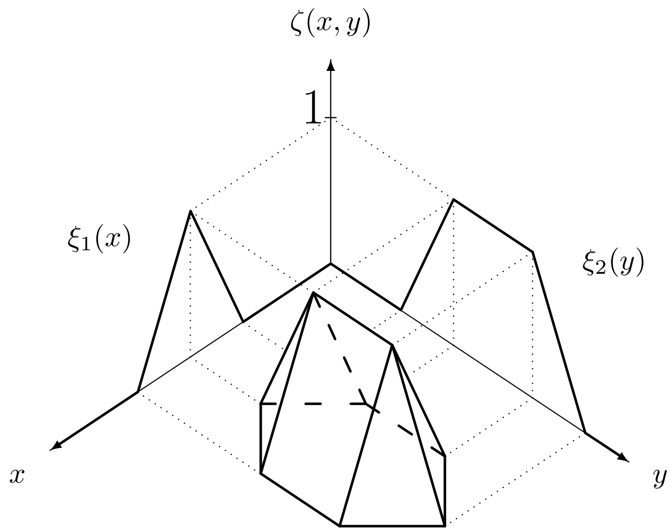
Figure 4.1: Combination of fuzzy numbers



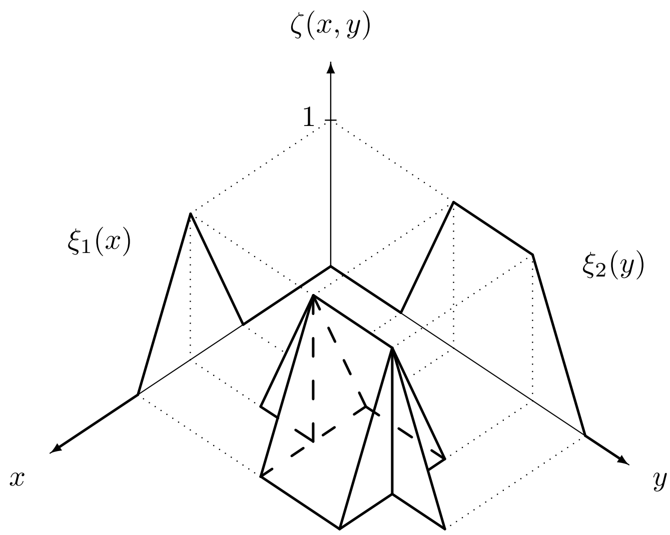
(a) Combination of two fuzzy numbers using the minimum t-norm



(b) Combination of two fuzzy numbers using the product t-norm



(c) Combination of two fuzzy numbers using the limited sum t-norm



(d) Combination of two fuzzy numbers using the drastic product t-norm

5. Construction of Characterizing Functions

How to construct a membership function, a characterizing function or a vector-characterizing function is an important topic of fuzzy set theory. There are many approaches, determined by the structure of the data. Some ways how to create a characterizing function or a vector-characterizing function are mentioned in this chapter.

Let us have some metric space M with a metric $\rho : M \to [0, \infty)$. We want to construct a fuzzy set on M. More specifically we want to construct a fuzzy number where $M = \mathbb{R}$ and ρ is the Euclidean metric or a fuzzy vector with $M = \mathbb{R}^n$ and ρ is the Euclidean metric in $\mathbb{R}^n$. For $I \subseteq M$ let $h\colon I \to \mathbb{R}$ be some function defined on a subset of M. In order to describe a fuzzy number or a fuzzy vector, the function h has to be extended to the whole set M.

Usually we will construct firstly a function $g\colon M \to \mathbb{R}$ from the function $h\colon I \to \mathbb{R}$. Then we will be able to construct the required characterizing function or vector-characterizing function $\xi : M \to [0,1]$ by normalizing the function $g\colon M \to \mathbb{R}$.

5.1 Function h defined on whole $\mathbb{R}^n$

Let the function $h(\cdot)$ be defined on the set $M = \mathbb{R}^n$, let $h(\cdot)$ be partly continuous, and let there exist a bounded subset $B \subseteq \mathbb{R}^n$, where $h(x) = 0 \quad \forall x \in \mathbb{R}^n \setminus B$. Then we define the function $g\colon \mathbb{R}^n \to \mathbb{R}$ in the following way:

$$g(x) := \left\{ \begin{array}{ll} |h(x)| & \text{for } h \text{ continuous in } x \\ \max\left\{ |h(a)|, \limsup\limits_{x \to a} |h(x)| \right\} & \text{for } h \text{ not continuous in } x \end{array} \right\} \qquad \forall x \in \mathbb{R}^n$$

To construct the characterizing function or the vector-characterizing function $\xi : \mathbb{R}^n \to [0,1]$ from the function $g : \mathbb{R}^n \to \mathbb{R}$, we need to normalize the function g. One possible way to do so is using the following equation:

$$\xi(x) := \frac{|g(x)|}{\max\{|g(z)| \colon z \in \mathbb{R}^n\}} \qquad \forall x \in \mathbb{R}^n$$

We have transformed the problem of constructing a characterizing function or a vector-characterizing function $\xi\colon \mathbb{R}^n \to [0,1]$ to the problem of constructing some suitable function $g\colon \mathbb{R}^n \to \mathbb{R}$, which will be normalized afterwards.

Example 5.1. Let $M=\mathbb{R}$ and let the function $h(\cdot)$ be defined as follows:

$$h(x)=\left\{\begin{array}{cl} \frac{3}{2}(x-1) & \text{for } x\in[1,3) \\ 3 & \text{for } x\in[3,5) \\ x^2-14x+48 & \text{for } x\in[5,8] \\ 1 & \text{for } x\in(8,10) \\ 0 & \text{for } x\in(-\infty,1)\cup[10,\infty) \end{array}\right\} \qquad \forall x\in\mathbb{R}$$

Then we set $g(x):=|h(x)| \quad \forall x\in\mathbb{R}\backslash\{8,10\}$ and $g(x):=1$ for $x\in\{8,10\}$. Then we compute the characterizing function $\xi(\cdot)$ by normalization and get:

$$\xi(x):=\frac{g(x)}{3} \qquad \forall x\in\mathbb{R}$$

The function $h(\cdot)$ is shown in Figure 5.1 and the resulting characterizing function $\xi(\cdot)$ is shown in Figure 5.2.

Example 5.2. Let $M=\mathbb{R}^2$ and let the function $h(\cdot,\cdot)$ be defined as follows:

$$h(x,y)=\max\left\{\begin{array}{r} 0,\ 3\,(1-x)^2\,e^{\left(-x^2-(y+1)^2\right)}-10\left(\frac{1}{5}x-x^3-y^5\right)e^{\left(-x^2-y^2\right)} \\ -\frac{1}{3}e^{\left(-(x+1)^2-y^2\right)}-\frac{1}{100} \end{array}\right\}$$

$$\forall (x,y)\in\mathbb{R}^2$$

Then we can set $g(x,y):=h(x,y) \quad \forall (x,y)\in\mathbb{R}^2$ and compute the vector-characterizing function $\zeta(\cdot,\cdot)$ by normalization and get:

$$\zeta(x,y):=\frac{h(x,y)}{\max\{h(u,v)\colon (u,v)\in\mathbb{R}^2\}} \qquad \forall (x,y)\in\mathbb{R}^2$$

The resulting vector-characterizing function $\zeta(\cdot,\cdot)$ is shown in Figure 5.3.

Example 5.3. Let $M=\mathbb{R}^2$ and let the function $h(\cdot,\cdot)$ be defined as follows:

$$h(x,y)=\left\{\begin{array}{ll} 1 & \text{for } x=y=0 \\ \frac{\sin\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}+\frac{1}{12}\left(4\pi-\sqrt{x^2+y^2}\right) & \text{for } \ 0<\sqrt{x^2+y^2}\leq 4\pi \\ 0 & \text{for } 4\pi<\sqrt{x^2+y^2} \end{array}\right\}$$

$$\forall (x,y)\in\mathbb{R}^2$$

Then we set $g(x,y):=h(x,y) \quad \forall (x,y)\in\mathbb{R}^2\backslash\{(0,0)\}$ and $g(0,0):=1+\frac{1}{3}\pi$. Then we compute the vector-characterizing function $\zeta(\cdot,\cdot)$ by normalization as:

$$\zeta(x,y):=\frac{g(x,y)}{1+\frac{1}{3}\pi} \qquad \forall (x,y)\in\mathbb{R}^2$$

The resulting vector-characterizing function $\zeta(\cdot,\cdot)$ is shown in Figrure 5.4

Figure 5.1: Function $h(\cdot)$ according to Example 5.1

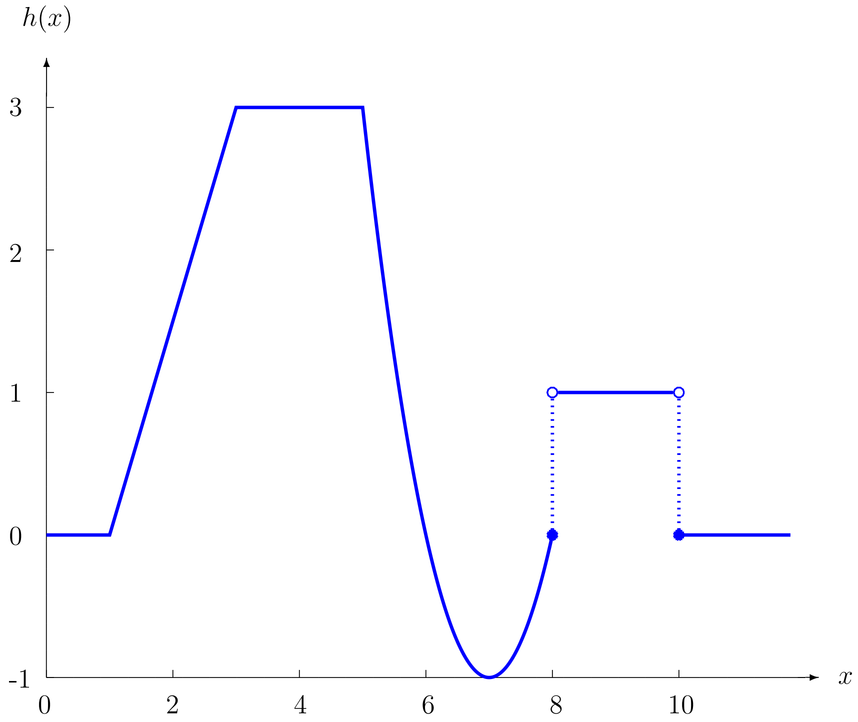


Figure 5.2: Characterizing function $\xi(\cdot)$ constructed from the function $h(\cdot)$ according to Example 5.1

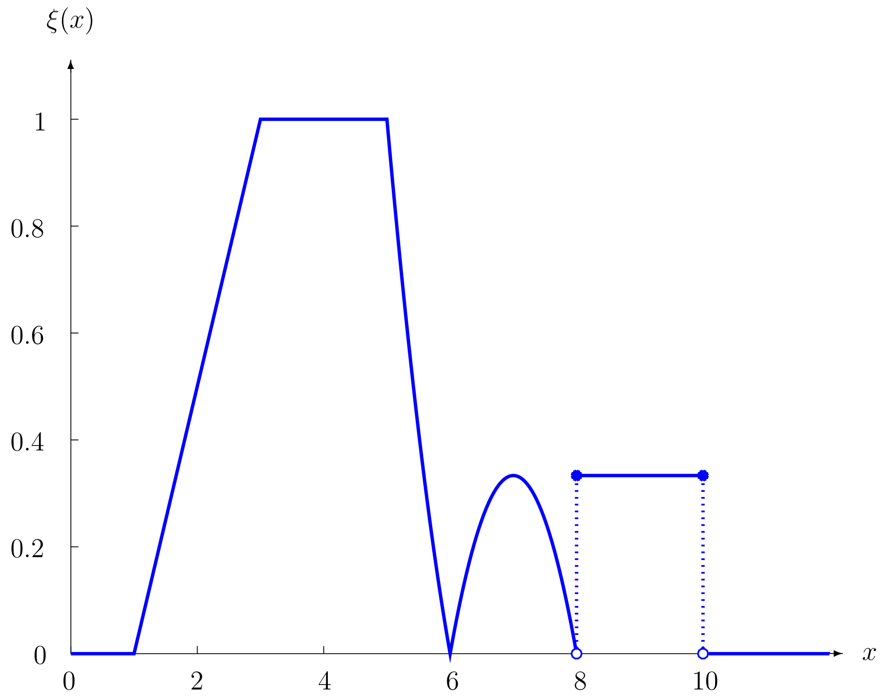


Figure 5.3: Vector-characterizing function $\zeta(\cdot,\cdot)$ constructed from the function $h(\cdot,\cdot)$ according to Example 5.2

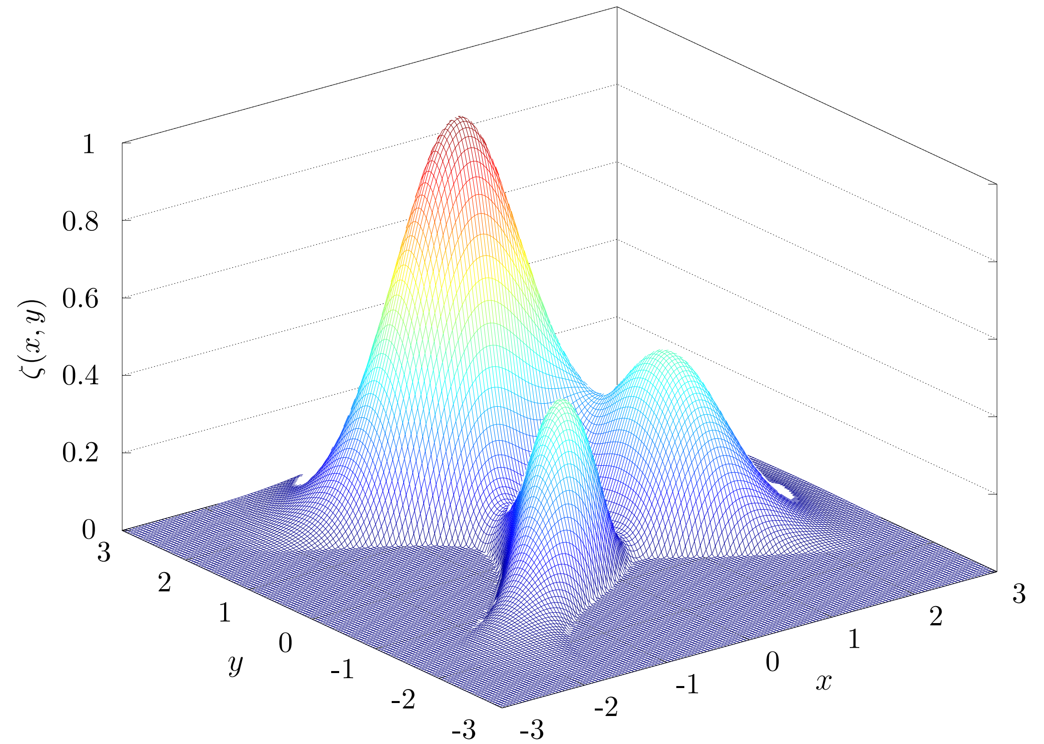
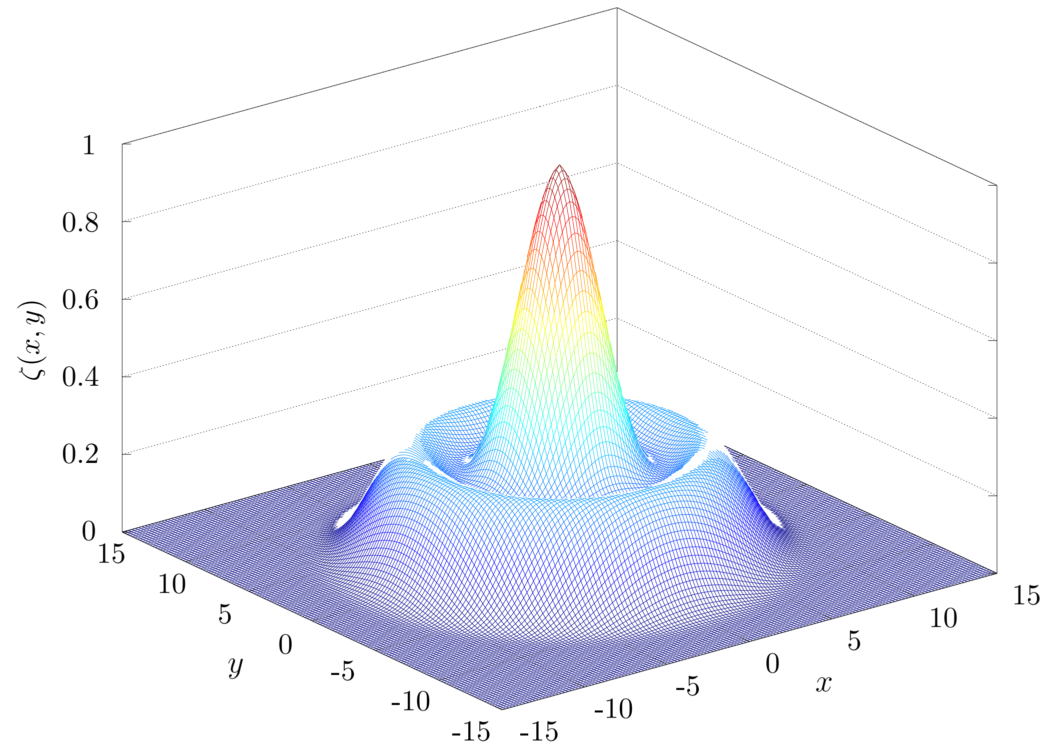


Figure 5.4: Vector-characterizing function $\zeta(\cdot,\cdot)$ constructed from the function $h(\cdot,\cdot)$ according to Example 5.3



5.2 Function h defined for one value $a \in \mathbb{R}^n$

Let the function $h(\cdot)$ be defined on the set $M = \{a\} \subset \mathbb{R}^n$, in other words, we have only one $a \in \mathbb{R}$, for which the function $h(\cdot)$ is defined. We can interpret the value a as the measurement result. Then we have different approaches, how to construct the characterizing function, in dependence with the measurement uncertainity of the measurement machine.

5.2.1 Crisp approach

In the case, where there is no uncertainty and we know it, the crisp approach can be used. The characterizing function is representing a crisp number. Results of calculation with this type of characterizing functions are analogue to calculations with standard real numbers.

Let $a \in \mathbb{R}^n$ and let there be no uncertainty in the measurement. Then we define the characterizing $\xi\colon \mathbb{R}^n \to [0,1]$ in the following way:

$$\xi(x) := \left\{ \begin{array}{ll} 1 & \text{for } x = a \\ 0 & \text{for } x \neq a \end{array} \right\} \qquad \forall x \in \mathbb{R}^n$$

An equivalent way of describing this characterizing function is using the indicator function of the set $\{a\}$ as follows:

$$\xi(x) := \mathbb{1}_{\{a\}}(x) \qquad \forall x \in \mathbb{R}^n$$

The function $\xi(\cdot)$ defined above fulfils the conditions for characterizing functions (for $n = 1$) and vector-characterizing functions (for $n > 1$).

Example 5.4. Let us have a one-dimensional measurement, $n = 1$, let $a = 2$ be the measurement result and let there be no uncertainty in the measurement. Then the characterizing function $\xi\colon \mathbb{R} \to [0,1]$ is:

$$\xi(x) := \mathbb{1}_{\{2\}}(x) \qquad \forall x \in \mathbb{R}$$

The resulting characterizing function is displayed in Figure 5.5.

Example 5.5. Let us have a two-dimensional measurement, $n = 2$, let $a = (3,4)$ be the measurement result and let there be no uncertainty in the measurement. Then the vector-characterizing function $\xi\colon \mathbb{R}^2 \to [0,1]$ is defined as:

$$\xi(x,y) := \left\{ \begin{array}{ll} 1 & \text{for } x = 3 \text{ and } y = 4 \\ 0 & \text{elsewhere} \end{array} \right\} \qquad \forall (x,y) \in \mathbb{R}^2$$

The resulting vector-characterizing function is displayed in Figure 5.6.

Figure 5.5: Characterizing function $\xi(\cdot)$ according to Example 5.4

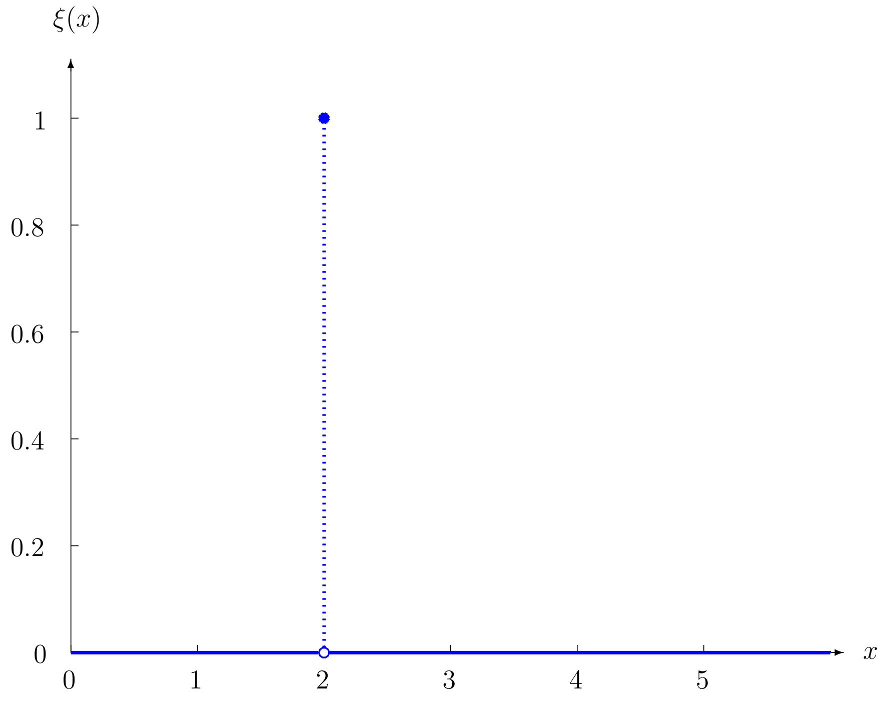
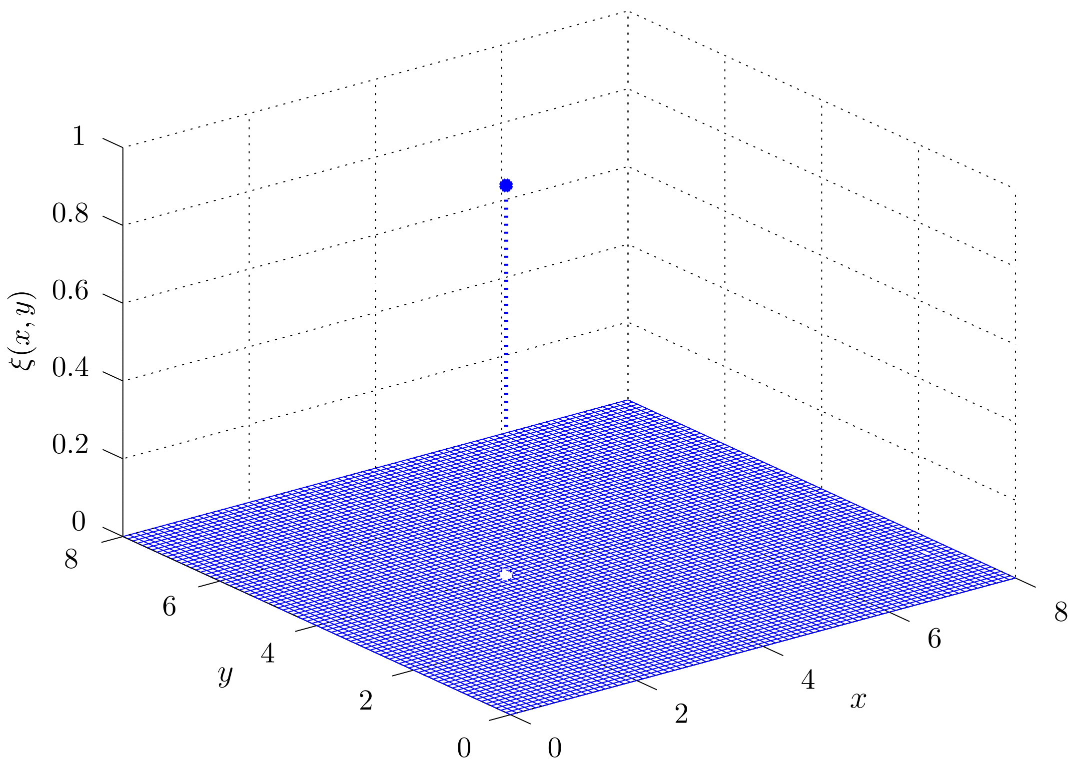


Figure 5.6: Characterizing function $\xi(\cdot,\cdot)$ according to Example 5.5



5.2.2 Interval approach

In the case, where there is some uncertainty and we know the value of the uncertainty, the interval approach can be used. The uncertainty can be represented by a real number or a vector of real numbers. The characterizing function is then representing an interval $[a - u, a + u]$. Results of calculation with this type of characterizing functions are analogue to interval calculus.

Firstly we will investigate the construction of the characterizing function in one-dimensional case ($n = 1$). Let $a \in \mathbb{R}$ be a measurement result and let $u \in [0, \infty)$ be the measurement uncertainty. Then we define the characterizing function $\xi : \mathbb{R} \to [0, 1]$ as indicator function of the interval $[a-u,\ a+u]$ in the following way:

$$\xi(x) := \mathbb{1}_{[a-u,\ a+u]}(x) = \left\{ \begin{array}{ll} 1 & \text{for } x \in [a-u,\ a+u] \\ 0 & \text{for } x \in (-\infty, a-u) \cup (a+u, \infty) \end{array} \right\} \qquad \forall x \in \mathbb{R}$$

The function $\xi(\cdot)$ defined above fulfils the conditions for characterizing functions.

Similarly we construct the vector-characterizing function for the more-dimensional case ($n > 1$). Let $\underline{a} \in \mathbb{R}^n$ be a measurement result and let $\underline{u} \in [0, \infty)^n$ be a measurement uncertainty. The measurement $\underline{a} = (a_1, \ldots, a_n)$ and measurement uncertainty $\underline{u} = (u_1, \ldots, u_n)$ are n-dimensional vectors of real numbers. We define multi-dimensional interval in the following way:

$$[\underline{a} - \underline{u}, \underline{a} + \underline{u}] = [a_1 - u_1, a_1 + u_1] \times \ldots \times [a_n - u_n, a_n + u_n]$$

Then we define the vector-characterizing function $\zeta : \mathbb{R}^n \to [0, 1]$ as indicator function of the multi-dimensional interval $[\underline{a} - \underline{u}, \underline{a} + \underline{u}]$ in the following way:

$$\zeta(\underline{x}) := \mathbb{1}_{[\underline{a}-\underline{u}, \underline{a}+\underline{u}]}(\underline{x}) = \left\{ \begin{array}{ll} 1 & \text{for } \underline{x} \in [\underline{a}-\underline{u}, \underline{a}+\underline{u}] \\ 0 & \text{for } \underline{x} \in \mathbb{R}^n \setminus [\underline{a}-\underline{u}, \underline{a}+\underline{u}] \end{array} \right\} \qquad \forall \underline{x} \in \mathbb{R}^n$$

The function $\zeta(\cdot, \ldots, \cdot)$ defined above fulfils the conditions for vector-characterizing functions.

Example 5.6. Let us have a one-dimensional measurement, $n = 1$, let $a = 2.5$ be the measurement result and let the uncertainty in the measurement be $u = 0.5$. Then the characterizing function $\xi : \mathbb{R} \to [0, 1]$ is defined as:

$$\xi(x) := \mathbb{1}_{[2,3]}(x) = \left\{ \begin{array}{ll} 1 & \text{for } x \in [2, 3] \\ 0 & \text{elsewhere} \end{array} \right\} \qquad \forall x \in \mathbb{R}$$

The resulting characterizing function is displayed in Figure 5.7

Example 5.7. Let us have a two-dimensional measurement, $n = 2$, let $a = (3, 4)$ be the measurement result, let the uncertainty in the measurement be $u = (1, 1)$. Then the vector-characterizing function $\zeta : \mathbb{R}^2 \to [0, 1]$ is defined as:

$$\zeta(x, y) := \mathbb{1}_{[2,4]\times[3,5]}(x, y) = \left\{ \begin{array}{ll} 1 & \text{for } x \in [2, 4], y \in [3, 5] \\ 0 & \text{elsewhere} \end{array} \right\} \qquad \forall (x, y) \in \mathbb{R}^2$$

The resulting vector-characterizing function is displayed in Figure 5.8.

Figure 5.7: Characterizing function $\xi(\cdot)$ according to Example 5.6

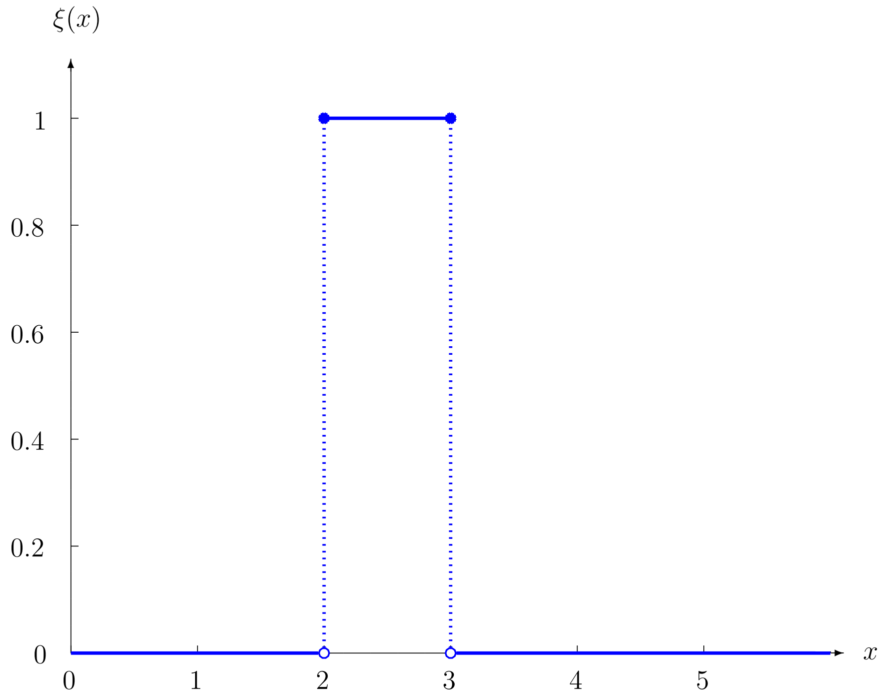
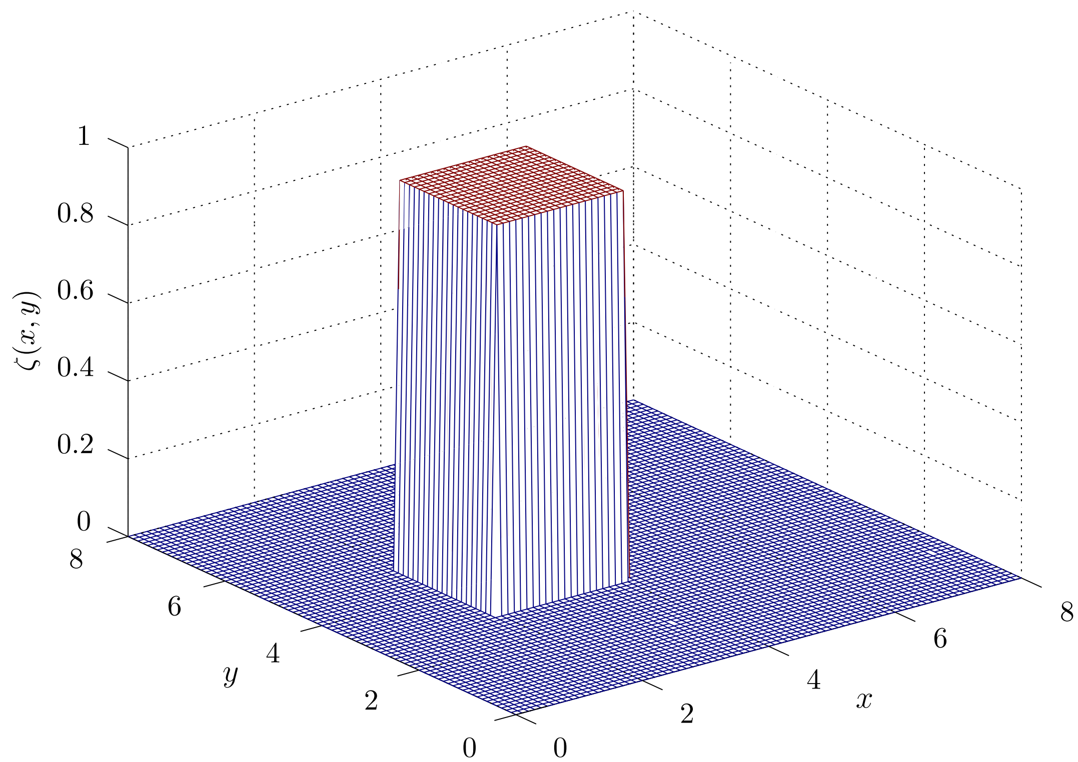


Figure 5.8: Characterizing function $\zeta(\cdot,\cdot)$ according to Example 5.7



5.2.3 Trapezoidal approach

In the case, where there is some uncertainty and we know, that the uncertainty has a trapezoidal shape, the trapezoidal approach can be used. The uncertainty can be represented by two real numbers (in one-dimensional case) or two vectors of real numbers (in more-dimensional case). The characterizing function is represented by a trapezoid.

Firstly we will investigate the construction of the characterizing function in one-dimensional case ($n = 1$). Let $a \in \mathbb{R}$ be a measurement result and let $u, U \in [0, \infty)$ be measurement uncertainties, such that $0 < u < U$. Then we define the characterizing function $\xi \colon \mathbb{R} \to [0, 1]$ in the following way:

$$\xi(x) := \left\{ \begin{array}{cl} \frac{x-(a-U)}{U-u} & \text{for } x \in [a-U, a-u) \\ 1 & \text{for } x \in [a-u, a+u] \\ \frac{(a+U)-x}{U-u} & \text{for } x \in (a+u, a+U] \\ 0 & \text{for } x \in \mathbb{R} \setminus [a-U, a+U] \end{array} \right\} \qquad \forall x \in \mathbb{R}$$

The function $\xi(\cdot)$ defined above fulfils the conditions for characterizing functions.

For more-dimensional case ($n > 1$) we will firstly construct the vector of fuzzy numbers $(x_1^*, \dots, x_n^*)$. Then we will construct the fuzzy vector $\underline{x}^*$ by applying one of the t-norms to the vector of fuzzy numbers. The vector-characterizing function of the fuzzy vector $\underline{x}^*$ can vary depending on the selected t-norm.

Let $\underline{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ be a measurement result and let $\underline{u} = (u_1, \dots, u_n)$, $\underline{U} = (U_1, \dots, U_n) \in [0, \infty)^n$ be measurement uncertainties, such that $0 < u_i < U_i$ for all $i \in \{1, \dots, n\}$. We construct the characterizing functions $\xi_1, \dots, \xi_n \colon \mathbb{R} \to [0, 1]$ in the same way as in one-dimensional case:

$$\xi_i(x) := \left\{ \begin{array}{cl} \frac{x-(a_i-U_i)}{U_i-u_i} & \text{for } x \in [a_i-U_i, a_i-u_i) \\ 1 & \text{for } x \in [a_i-u_i, a_i+u_i] \\ \frac{(a_i+U_i)-x}{U_i-u_i} & \text{for } x \in (a_i+u_i, a_i+U_i] \\ 0 & \text{for } x \in \mathbb{R} \setminus [a_i-U_i, a_i+U_i] \end{array} \right\} \qquad \forall x \in \mathbb{R} \quad \forall i \in \{1, \dots, n\}$$

Then we define the vector-characterizing function $\zeta : \mathbb{R}^n \to [0, 1]$ using a t-norm. Some t-norms were defined in Chapter 4. The vector-characterizing function can be constructed using the minimum t-norm:

$$\zeta(x_1, \dots, x_n) = \min\{\xi_1(x_1), \dots, \xi_n(x_n)\} \qquad \forall (x_1, \dots, x_n) \in \mathbb{R}^n$$

Other choices of t-norms are also possible. The function $\zeta(\cdot, \dots, \cdot)$ fulfils the conditions for vector-characterizing functions.

Example 5.8. Let us have a one-dimensional measurement, let $a = 2.5$ be the measurement result and let the uncertainties in the measurement be $u = 0.5$ and $U = 1.5$. Then the characterizing function $\xi \colon \mathbb{R} \to [0, 1]$ is defined as:

$$\xi(x) := \left\{ \begin{array}{cl} x-1 & \text{for } x \in [1, 2) \\ 1 & \text{for } x \in [2, 3] \\ 4-x & \text{for } x \in (3, 4] \\ 0 & \text{for } x \in \mathbb{R} \setminus [1, 4] \end{array} \right\} \qquad \forall x \in \mathbb{R}$$

The resulting characterizing function is displayed in Figure 5.9

Figure 5.9: Characterizing function $\xi(\cdot)$ according to Example 5.8

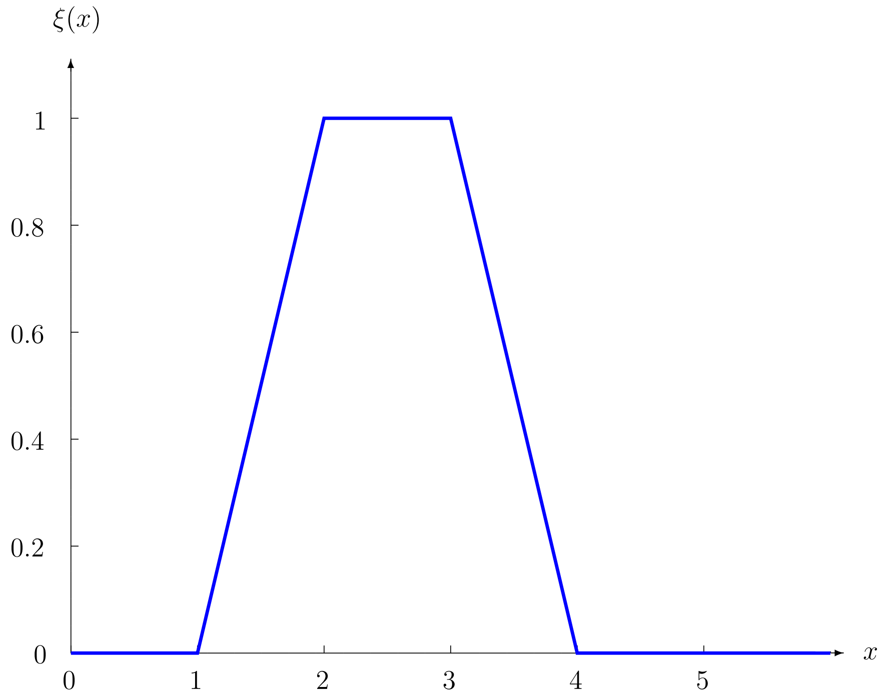
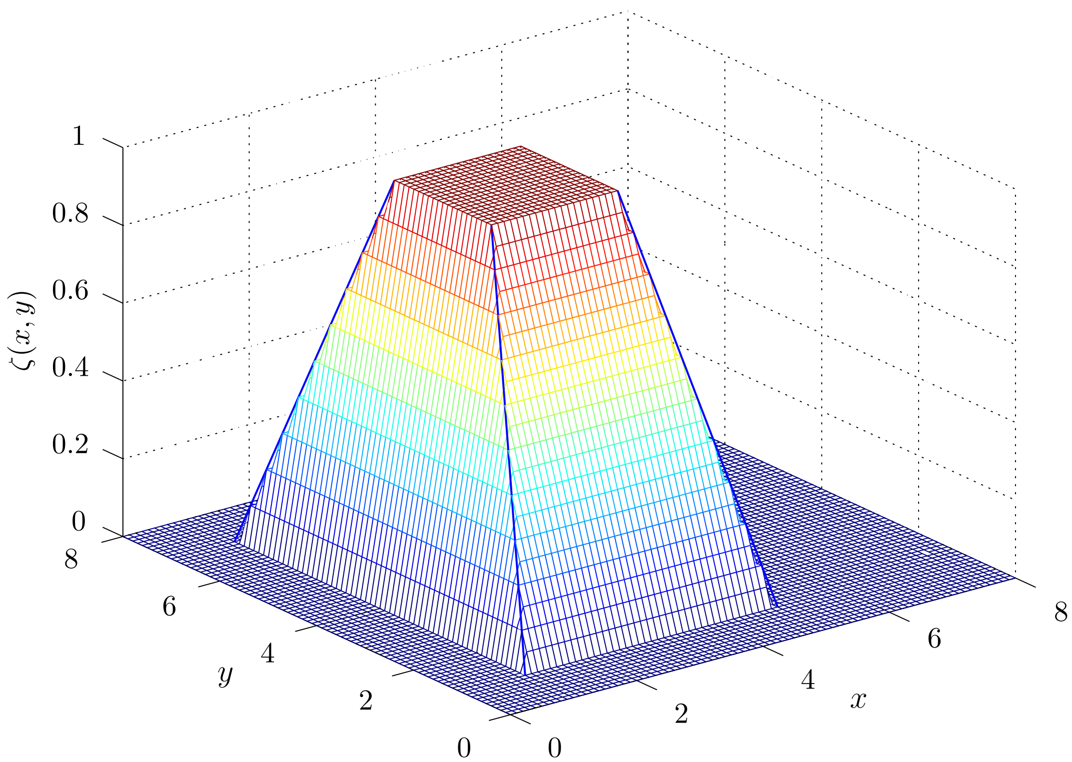


Figure 5.10: Characterizing function $\zeta(\cdot,\cdot)$ according to Example 5.9



Example 5.9. Let us have a two-dimensional measurement, let $a = (3, 4)$ be the measurement result and let the uncertainties in the measurement be $u = (1, 1)$, $U = (2, 3)$. Then the characterizing functions $\xi_1, \xi_2 \colon \mathbb{R} \to [0, 1]$ are defined as:

$$\xi_1(x) := \left\{ \begin{array}{cl} x - 1 & \text{for } x \in [1, 2) \\ 1 & \text{for } x \in [2, 4] \\ 5 - x & \text{for } x \in (4, 5] \\ 0 & \text{for } x \in \mathbb{R} \backslash [1, 5] \end{array} \right\}, \quad \xi_2(y) := \left\{ \begin{array}{cl} \frac{1}{2}(x - 1) & \text{for } y \in [1, 3) \\ 1 & \text{for } y \in [3, 5] \\ \frac{1}{2}(7 - x) & \text{for } y \in (5, 7] \\ 0 & \text{for } y \in \mathbb{R} \backslash [1, 7] \end{array} \right\}$$

$$\forall x \in \mathbb{R} \qquad\qquad \forall y \in \mathbb{R}$$

The resulting vector-characterizing function $\zeta \colon \mathbb{R}^2 \to [0, 1]$ constructed by the minimum t-norm, i.e.

$$\zeta(x, y) = \min\{\xi_1(x), \xi_2(y)\} \qquad \forall (x, y) \in \mathbb{R}^2$$

is displayed in Figure 5.10.

5.2.4 Rotation approach

This approach should be used for more-dimensional measurements where there is some uncertainty which is not independent in individual dimensions, but the uncertainty of individual dimensions is dependent on the distance from the measurement result. The uncertainty can be represented by two real numbers, similarly to the trapezoidal approach, or by one real number, similarly to the interval approach. The vector-characterizing function is then represented by a rotated trapezoid or a rotated interval.

Let $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ be a measurement result and let $u, U \in [0, \infty)$ be measurement uncertainties, such that $0 < u < U$. We construct a helping function $f \colon [0, \infty] \to [0, 1]$ in a similar way as in the one-dimensional case of the trapezoidal approach:

$$f(x) := \left\{ \begin{array}{cl} 1 & \text{for } x \in [0, u] \\ \frac{U - x}{U - u} & \text{for } x \in (u, U] \\ 0 & \text{for } x \in (U, \infty) \end{array} \right\} \quad \forall x \in [0, \infty)$$

Then we construct the vector-characterizing function $\zeta \colon \mathbb{R}^n \to [0, 1]$ using some metric $d \colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ on $\mathbb{R}^n$ and the above defined function $f(\cdot)$ in the following way:

$$\zeta(\underline{x}) = f\left(d(\underline{x}, \underline{a})\right) \qquad \forall \underline{x} \in \mathbb{R}^n$$

In the case when $d(\cdot, \cdot)$ is the Euclidean metric, the vector-characterizing function $\zeta \colon \mathbb{R}^n \to [0, 1]$ is then defined in the following way:

$$\zeta(x_1, \ldots, x_n) = f\left(\sqrt{\sum_{i=1}^{n} (x_i - a_i)^2}\right) \qquad \forall (x_1, \ldots, x_n) \in \mathbb{R}^n$$

The function $\zeta(\cdot, \ldots, \cdot)$ fulfils the conditions for vector-characterizing functions.

For the rotation interval approach, the only difference would be using one value for uncertainty and in the definition of the function $f(\cdot)$.

Let $\underline{a} \in \mathbb{R}^n$ be a measurement result and let $u \in [0, \infty)$ be the measurement uncertainty. We construct a helping function $f: [0, \infty] \to [0, 1]$ in a similar way as in the one-dimensional case of the trapezoidal approach:

$$f(x) := \left\{ \begin{array}{ll} 1 & \text{for } x \in [0, u] \\ 0 & \text{for } x \in (u, \infty) \end{array} \right\} \quad \forall x \in [0, \infty)$$

The construction of the vector-characterizing function $\zeta: \mathbb{R}^n \to [0, 1]$ using a metric $d: \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ and the above defined function $f(\cdot)$ is similar to the rotation trapezoid case:

$$\zeta(\underline{x}) = f\left(d(\underline{x}, \underline{a})\right) \qquad \forall \underline{x} \in \mathbb{R}^n$$

The function $\zeta(\cdot, \ldots, \cdot)$ fulfils the conditions for vector-characterizing functions.

Example 5.10. Let us have a two-dimensional measurement, let $a = (3, 4)$ be the measurement result and let the uncertainty in the measurement be $u = \frac{3}{2}$. Then we can use the trapezoidal approach and the helping function $f: [0, \infty) \to [0, 1]$ is constructed as:

$$f(x) := \left\{ \begin{array}{ll} 1 & \text{for } x \in [0, \frac{3}{2}] \\ 0 & \text{for } x \in (\frac{3}{2}, \infty) \end{array} \right\} \qquad \forall x \in [0, \infty)$$

The resulting vector-characterizing function $\zeta: \mathbb{R}^2 \to [0, 1]$ constructed by using the Euclidean metric and the above defined function $f(\cdot)$ is defined as:

$$\zeta(x, y) = f\left(\sqrt{(x-3)^2 + (y-4)^2}\right) \qquad \forall (x, y) \in \mathbb{R}^2$$

More precisely:

$$\zeta(x, y) = \left\{ \begin{array}{ll} 1 & \text{for } \sqrt{(x-3)^2 + (y-4)^2} \in [0, \frac{3}{2}] \\ 0 & \text{for } \sqrt{(x-3)^2 + (y-4)^2} \in (\frac{3}{2}, \infty) \end{array} \right\} \qquad \forall (x, y) \in \mathbb{R}^2$$

This vector-characterizing function $\zeta(\cdot, \cdot)$ is displayed in Figure 5.11.

Example 5.11. Let us have a two-dimensional measurement, let $a = (3, 4)$ be the measurement result and let the uncertainties in the measurement be $u = 1$, $U = \frac{5}{2}$. Then we can use the trapezoidal approach and the following helping function $f: [0, \infty) \to [0, 1]$:

$$f(x) := \left\{ \begin{array}{ll} 1 & \text{for } x \in [0, 1] \\ \frac{5-2x}{3} & \text{for } x \in (1, \frac{5}{2}] \\ 0 & \text{for } x \in (\frac{5}{2}, \infty) \end{array} \right\} \qquad \forall x \in [0, \infty)$$

Again the resulting vector-characterizing function $\zeta: \mathbb{R}^2 \to [0, 1]$ constructed using the Euclidean metric and the above defined function $f(\cdot)$ is defined as:

$$\zeta(x, y) = f\left(\sqrt{(x-3)^2 + (y-4)^2}\right) \qquad \forall (x, y) \in \mathbb{R}^2$$

This vector-characterizing function $\zeta(\cdot, \cdot)$ is displayed in Figure 5.12.

Figure 5.11: Characterizing function $\zeta(\cdot,\cdot)$ according to Example 5.10

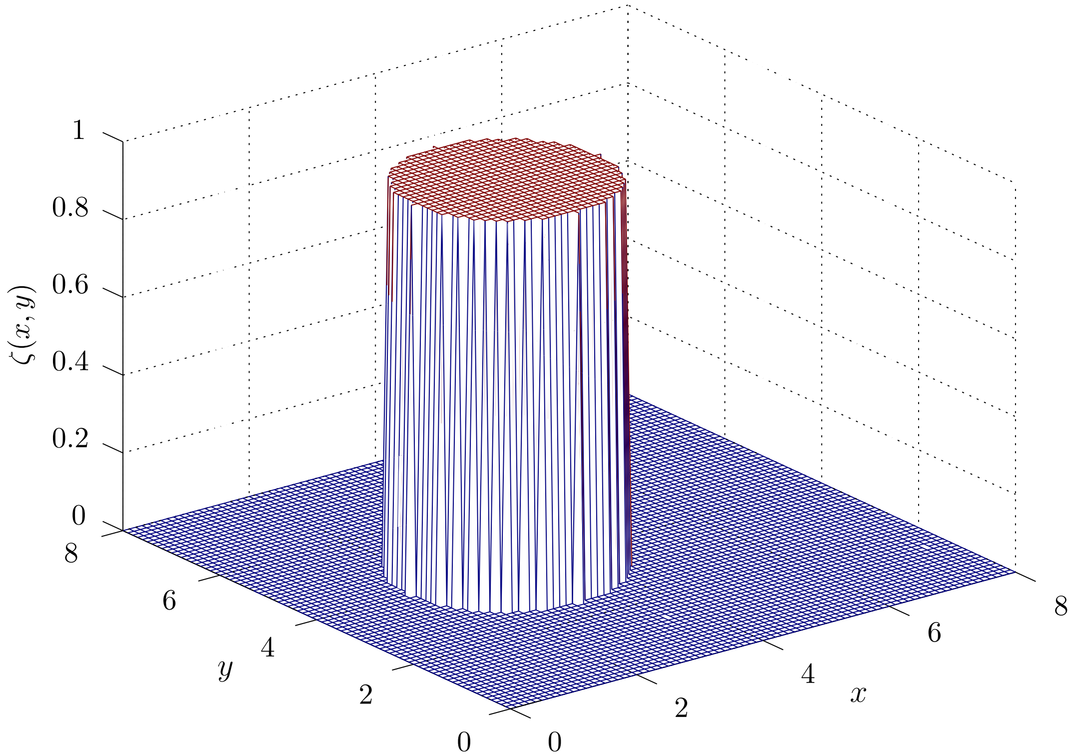
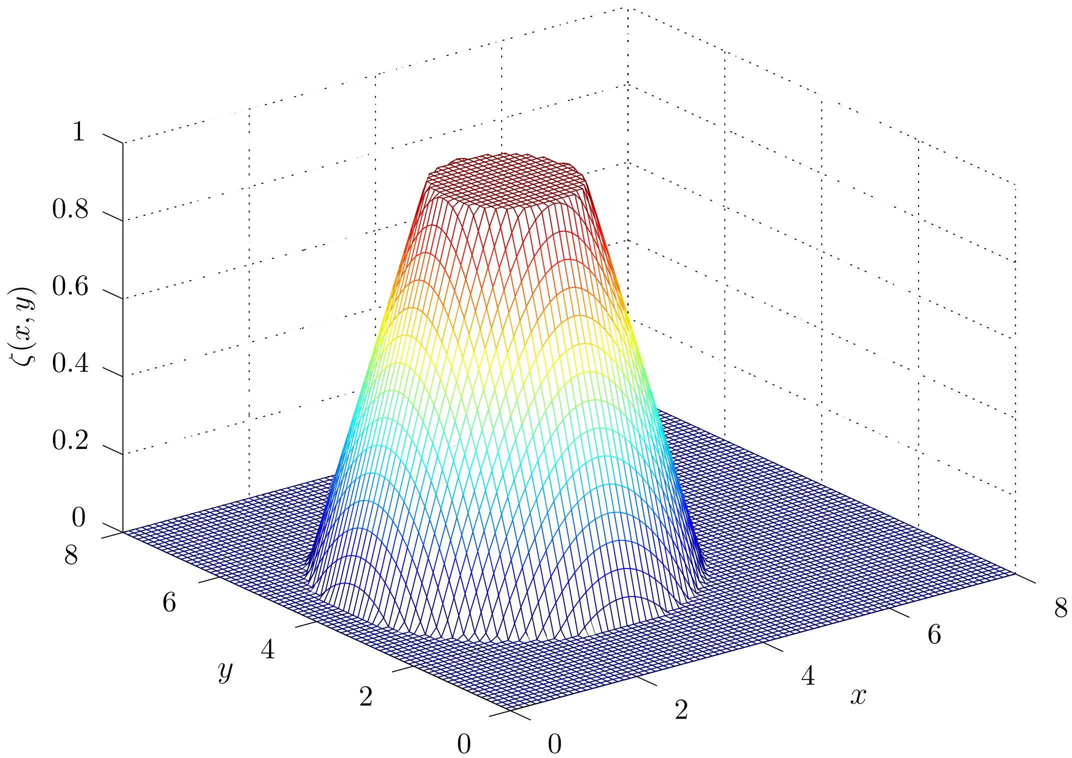


Figure 5.12: Characterizing function $\zeta(\cdot,\cdot)$ according to Example 5.11



5.2.5 Normal distribution approach

In the case, where in the measurement is some uncertainty and we know the distribution of this uncertainty, for example from the equipment specification, then a modified density function can be used. When the distribution of the uncertainty has a normal distribution with mean value 0 and with known variance, then the approach described in this section can be used.

Firstly we will investigate the construction of the characterizing function in one-dimensional case ($n = 1$). Let $a \in \mathbb{R}$ be a measurement result and let $u^2 \in [0, \infty)$ be the variance of the measurement uncertainty. Then the density function $f : \mathbb{R} \to \mathbb{R}$ of a normal distribution with mean a and variance u^2 is the following:

$$f(x) = \frac{1}{\sqrt{2\pi u^2}}\, e^{-\frac{(x-a)^2}{2u^2}} \qquad \forall x \in \mathbb{R}$$

The characterizing function of a fuzzy number (see Definition 2.1) should have its values in the interval [0,1], so we should omit the normalizing constant of the density function $f(\cdot)$. Another condition for characterizing functions is to have bounded support. Therefore we should use some interval, let say the confidence interval on the level $1-\alpha$ for some $\alpha \in (0,1)$. The confidence interval of the function $f(\cdot)$ is defined as $[a - u\Phi(\frac{\alpha}{2}), a + u\Phi(\frac{\alpha}{2})]$, where $\Phi : (0,1) \to (0,\infty)$ is the quantile function of the standard normal distribution.

Then we define the characterizing function $\xi : \mathbb{R} \to [0,1]$ using the density function $f(\cdot)$ and the interval $I = [a - u\Phi(\frac{\alpha}{2}), a + u\Phi(\frac{\alpha}{2})]$ in the following way:

$$\xi(x) := \left\{ \begin{array}{ll} e^{-\frac{(x-a)^2}{2u^2}} & \text{for } x \in I \\ 0 & \text{elsewhere} \end{array} \right\} \qquad \forall x \in \mathbb{R}$$

The function $\xi(\cdot)$ defined above fulfils the conditions for characterizing functions.

For the more-dimensional case ($n > 1$) the construction of a vector-characterizing function is analogue to the one-dimensional case. Let $\underline{a} \in \mathbb{R}^n$ be a measurement result and let $\Sigma \in \mathbb{R}^{n\times n}$ be a covariance matrix of measurement uncertainties in the following shape

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} u_1^2 & u_{1,2} & \cdots & u_{1,n} \\ u_{2,1} & u_2^2 & & u_{2,n} \\ \vdots & & \ddots & \vdots \\ u_{n,1} & u_{n,2} & & u_n^2 \end{pmatrix},$$

where $a_i \in \mathbb{R}$ are coordinates of the measurement result, $u_i^2 \in [0,\infty)$ are the uncertainties of the i^{th} coordinate and $u_{i,j} = u_{j,i} \in \mathbb{R}$ are covariances between uncertainty of the i^{th} and j^{th} coordinate, for all $i, j \in \{1, \ldots, n\}, i \neq j$.

The density function of a multivariate normal distribution with mean vector $\underline{a}$ and covariance matrix Σ is defined as

$$f(\underline{x}) = \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} e^{-\frac{1}{2}(\underline{x}-\underline{a})^T \Sigma^{-1} (\underline{x}-\underline{a})} \qquad \forall \underline{x} \in \mathbb{R}^n$$

where $\underline{x}$ is a column vector, $(\underline{x}-\underline{a})^T$ is a transposed vector, $|\Sigma|$ is the determinant of the matrix Σ, and Σ^{-1} is the inverse matrix of Σ.

Again, the vector-characterizing function of a fuzzy vector (see Definition 3.1) should have its values in the interval [0,1]. We should omit the normalizing constant of the density function $f(\cdot)$ by definition of a new function $\tilde{f} : \mathbb{R}^n \to [0,1]$ in the following way:

$$\tilde{f}(\underline{x}) = e^{-\frac{1}{2}(\underline{x}-\underline{a})^T\Sigma^{-1}(\underline{x}-\underline{a})} \qquad \forall \underline{x} \in \mathbb{R}^n$$

Another condition for vector-characterizing functions is to have bounded support. We should construct a confidence region, multi-dimensional version of confidence interval, or just select some $c \in (0,1)$ and restrict the non-zero values of the vector-characterizing function to the set, where the function $\tilde{f}(\cdot)$ has greater or equal values to c, denoted as $C := \{\underline{x} \in \mathbb{R}^n : \tilde{f}(\underline{x}) \geq c\}$. The set C is an ellipsoid.

Then the vector-characterizing function $\zeta : \mathbb{R}^n \to [0,1]$ is defined as:

$$\zeta(\underline{x}) = \left\{ \begin{array}{ll} e^{-\frac{1}{2}(\underline{x}-\underline{a})^T\Sigma^{-1}(\underline{x}-\underline{a})} & \text{for } \underline{x} \in C \\ 0 & \text{elsewhere} \end{array} \right\} \qquad \forall \underline{x} \in \mathbb{R}^n$$

The function $\zeta(\cdot, \ldots, \cdot)$ fulfils the conditions for vector-characterizing functions.

Example 5.12. Let us have a one-dimensional measurement, let $a = 2.5$ be the measurement result and let the standard deviation of the measurement uncertainty be $u = 0.7$. Let the 95% confidence interval and the 99% confidence interval be chosen, hence $\alpha_1 = 0.05$, $\alpha_2 = 0.01$.

Then the characterizing functions $\xi_1, \xi_2 : \mathbb{R} \to [0,1]$ are defined as

$$\xi_i(x) := \left\{ \begin{array}{ll} e^{-\frac{(x-a)^2}{2u^2}} & \text{for } x \in I_i \\ 0 & \text{elsewhere} \end{array} \right\} \qquad \forall x \in \mathbb{R},$$

where $I_i = [2.5 - 0.7\Phi(\frac{\alpha_i}{2}), 2.5 + 0.7\Phi(\frac{\alpha_i}{2})]$ is the support of the characterizing function $\xi_i(\cdot)$ for $i \in \{1,2\}$.

The resulting characterizing functions $\xi_1(\cdot)$ and $\xi_2(\cdot)$ are displayed in Figure 5.13 and Figure 5.14 respectively.

Example 5.13. Let us have two different two-dimensional measurements, let $\underline{a} = (3,4)^T$ be the measurement result and let the coordinates be independent in first case and correlated in second with covariance matrices $\Sigma_1, \Sigma_2 \in \mathbb{R}^{2\times 2}$ where:

$$\Sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix}$$

Let the constant $c = 0.04$ be chosen, then the support of vector-characterizing functions is selected as $C_i := \{\underline{x} \in \mathbb{R}^2 : e^{-\frac{1}{2}(\underline{x}-\underline{a})^T\Sigma_i^{-1}(\underline{x}-\underline{a})} \geq 0.04\}$ for $i \in \{1,2\}$.

The resulting vector-characterizing functions $\zeta_1, \zeta_2 : \mathbb{R}^2 \to [0,1]$ are constructed for $i \in \{1,2\}$ as

$$\zeta_i(\underline{x}) = \left\{ \begin{array}{ll} e^{-\frac{1}{2}(\underline{x}-\underline{a})^T\Sigma_i^{-1}(\underline{x}-\underline{a})} & \text{for } \underline{x} \in C_i \\ 0 & \text{elsewhere} \end{array} \right\} \qquad \forall \underline{x} \in \mathbb{R}^n$$

and are displayed in Figure 5.15 and Figure 5.16 respectively.

Figure 5.13: Characterizing function $\xi_1(\cdot)$ according to Example 5.12 for the 95% confidence interval

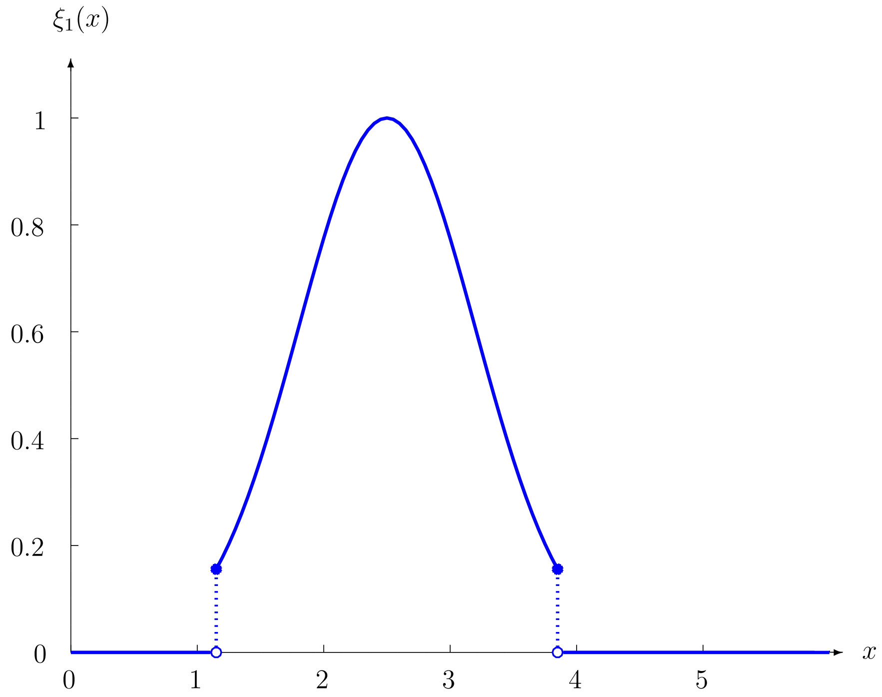


Figure 5.14: Characterizing function $\xi_2(\cdot)$ according to Example 5.12 for the 99% confidence interval

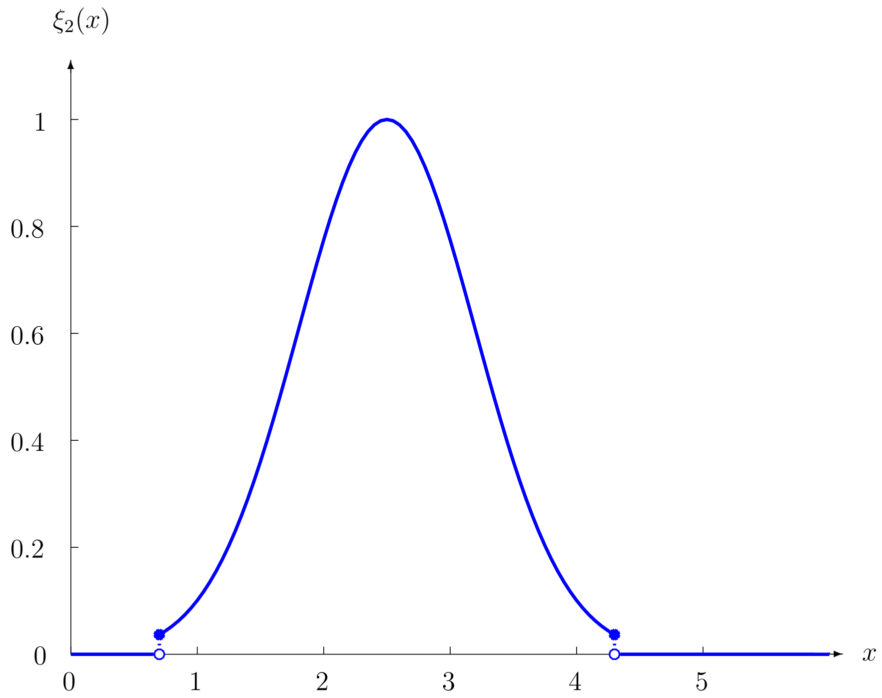


Figure 5.15: Characterizing function $\zeta_1(\cdot, \cdot)$ according to Example 5.13

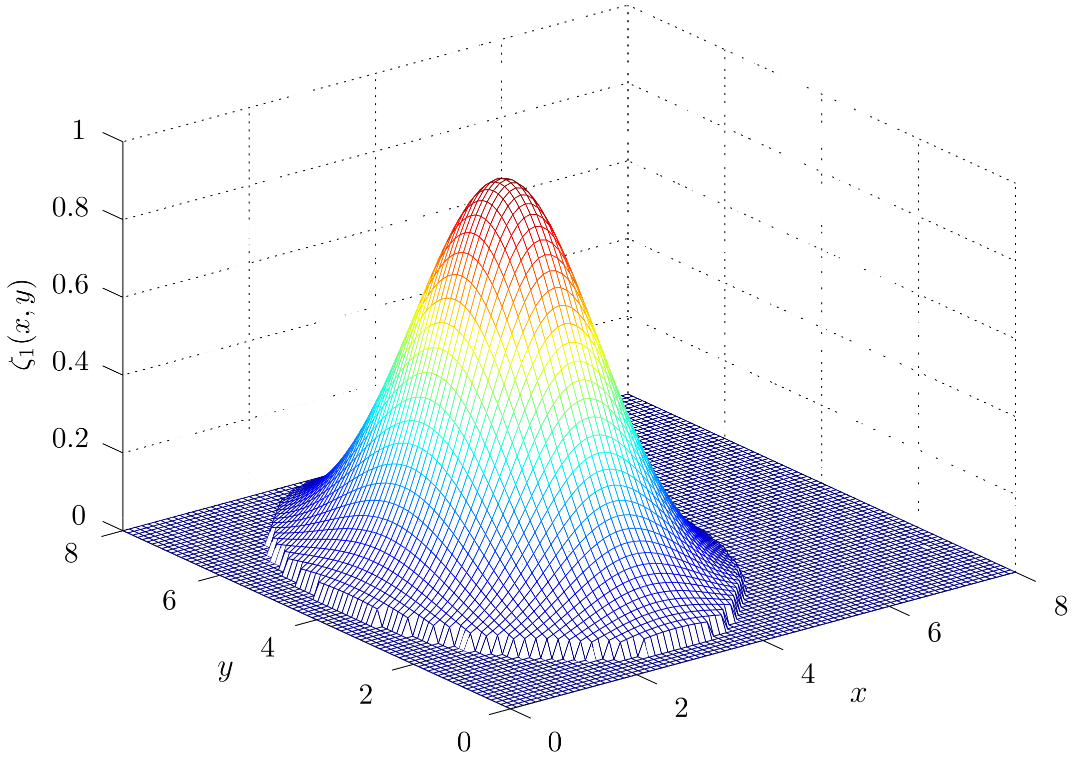
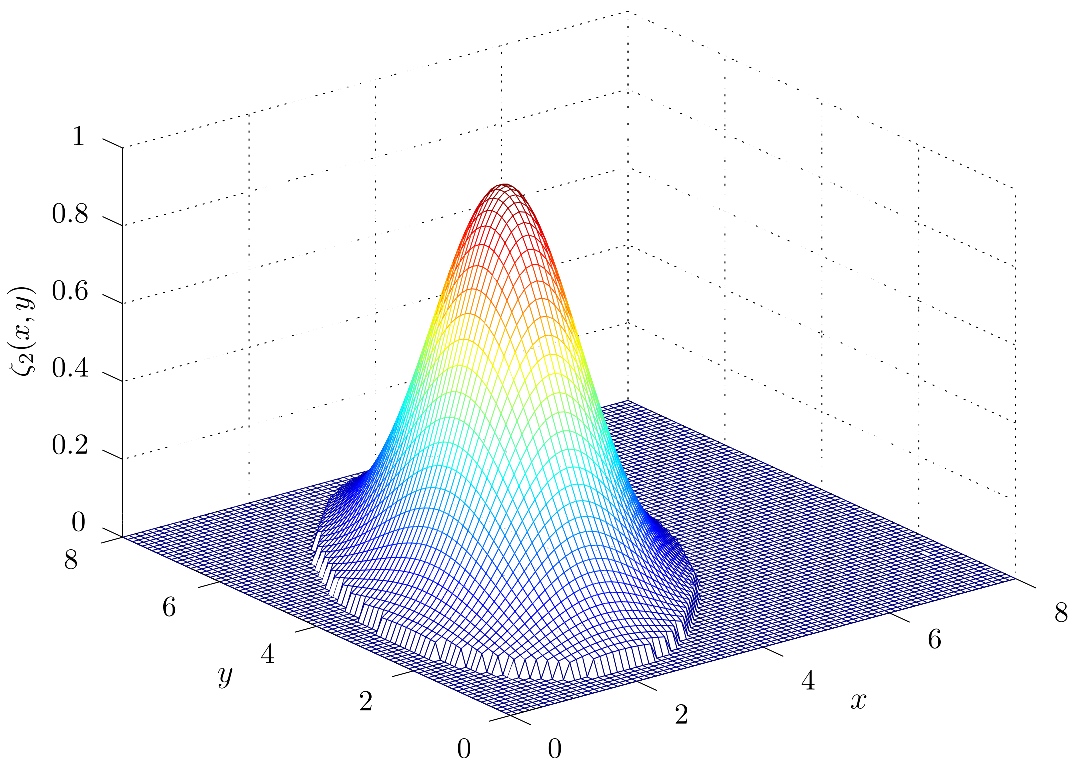


Figure 5.16: Characterizing function $\zeta_2(\cdot, \cdot)$ according to Example 5.13



5.3 Function h defined on equally spaced finite subsets of $\mathbb{R}$

If we have values of a function only in certain points, in order to obtain a characterizing function we have to extend the function to the whole space.

Let be $I = \{x_1, \ldots, x_n\}$ for $n \in \mathbb{N}$ and let $x_i \in \mathbb{R}$ be equally spaced and $x_1 < x_2 < \ldots < x_n$, more precisely

$$\exists \Delta \in (0, \infty) \quad \forall i \in \{2, \ldots, n\} \quad x_i - x_{i-1} = \Delta.$$

Let the function $h(\cdot)$ be defined on the set I. We define $x_0 := x_1 - \Delta$, and $x_{n+1} := x_n + \Delta$ and extend the function $h(\cdot)$ in the following way:

$$h(x_0) := 0 \quad \text{and} \quad h(x_{n+1}) := 0$$

Example 5.14. Let the set $I \subset \mathbb{R}$ be defined in following way:

$$I = \left\{\tfrac{1}{2}, 1\tfrac{1}{2}, 2\tfrac{1}{2}, 3\tfrac{1}{2}, 4\tfrac{1}{2}, 5\tfrac{1}{2}, 6\tfrac{1}{2}, 7\tfrac{1}{2}, 8\tfrac{1}{2}, 9\tfrac{1}{2}, 10\tfrac{1}{2}, 11\tfrac{1}{2}\right\}$$

Let the function $h(\cdot)$ be defined on I by the same equation as in Example 5.1. Let additionally $h(\frac{1}{2}) = h(11\frac{1}{2}) = 0$. Then the values of function $h(\cdot)$ are the following:

x_i	$\frac{1}{2}$	$1\frac{1}{2}$	$2\frac{1}{2}$	$3\frac{1}{2}$	$4\frac{1}{2}$	$5\frac{1}{2}$	$6\frac{1}{2}$	$7\frac{1}{2}$	$8\frac{1}{2}$	$9\frac{1}{2}$	$10\frac{1}{2}$	$11\frac{1}{2}$
$h(x_i)$	0	$\frac{3}{4}$	$2\frac{1}{4}$	3	3	$1\frac{1}{4}$	$-\frac{3}{4}$	$-\frac{3}{4}$	1	1	0	0

The function $h(\cdot)$ defined on the set I is depicted in Figure 5.17.

We want to extend the function $h\colon I \to \mathbb{R}$. We can define the function $g\colon \mathbb{R} \to \mathbb{R}$ as a partly constant function or as a partly linear function as will be explained in the following sections.

5.3.1 Partly constant approach

Firstly we will describe the partly constant approach. Let $I \subset \mathbb{R}$, the function $h\colon I \to \mathbb{R}$ and values $x_0, x_1, \ldots, x_n, x_{n+1}, \Delta \in \mathbb{R}$ be defined as in Section 5.3.

We can extend the function $h\colon I \to \mathbb{R}$ to a non-negative partially constant function $g\colon \mathbb{R} \to \mathbb{R}$ using the following formula:

$$g(x) := \left\{\begin{array}{ll} 0 & \text{for} \quad x < x_1 - \frac{\Delta}{2} \\ |h(x_1)| & \text{for} \quad x \in [x_1 - \frac{\Delta}{2}, x_1 + \frac{\Delta}{2}) \\ \max\{|h(x_1)|, |h(x_2)|\} & \text{for} \quad x = x_1 + \frac{\Delta}{2} \\ \cdots & \\ |h(x_i)| & \text{for} \quad x \in (x_i - \frac{\Delta}{2}, x_i + \frac{\Delta}{2}) \\ \max\{|h(x_i)|, |h(x_{i+1})|\} & \text{for} \quad x = x_i + \frac{\Delta}{2} \\ \cdots & \\ |h(x_n)| & \text{for} \quad x \in (x_n - \frac{\Delta}{2}, x_n + \frac{\Delta}{2}] \\ 0 & \text{for} \quad x > x_n + \frac{\Delta}{2} \end{array}\right\} \qquad \forall x \in \mathbb{R}$$

We can write this formula in another way, using the structure of I, as:

$$g(x) := \max\left(\left\{|h(x_i)| : |x - x_i| \le \tfrac{\Delta}{2}\right\} \cup \{0\}\right) \qquad \forall x \in \mathbb{R}$$

Then the characterizing function $\xi\colon \mathbb{R} \to [0,1]$ can be obtained from the function $g(\cdot)$ by normalization in the following way:

$$\xi(x) := \frac{g(x)}{\max\{g(z)\colon z \in \mathbb{R}\}} \qquad \forall x \in \mathbb{R}$$

The function $\xi(\cdot)$ defined above fulfils the conditions for characterizing functions.

Example 5.15. Let the set $I \subset \mathbb{R}$ and the function $h(\cdot)$ be defined in the same way as in Example 5.14 and depicted in Figure 5.17. The characterizing function $\xi(\cdot)$ obtained by the partly constant approach is shown in Figure 5.18.

5.3.2 Partly linear approach

Now we describe the partly linear approach. Let $I \subset \mathbb{R}$, the function $h\colon I \to \mathbb{R}$ and values $x_0, x_1, \ldots, x_n, x_{n+1}, \Delta \in \mathbb{R}$ be defined as in Section 5.3.

We can think about this approach as connecting points using line segments. More precisely we will extend the function $h : I \to \mathbb{R}$ to a continuous partially linear function $g\colon \mathbb{R} \to \mathbb{R}$ in the following way:

$$g(x) := \left\{\begin{array}{ll} 0 & \text{for } x < x_0 \\ \frac{x-x_1}{\Delta}\, h(x_2) & \text{for } x \in [x_1, x_2) \\ \cdots & \\ h(x_i) + \frac{x-x_i}{\Delta}\,(h(x_{i+1}) - h(x_i)) & \text{for } x \in [x_i, x_{i+1}) \\ \cdots & \\ h(x_n) - \frac{x-x_n}{\Delta}\, h(x_n) & \text{for } x \in [x_n, x_{n+1}] \\ 0 & \text{for } x > x_{n+1} \end{array}\right\} \qquad \forall\, x \in \mathbb{R}$$

The characterizing function $\xi\colon \mathbb{R} \to [0,1]$ can be obtained from the function $g(\cdot)$ by normalization in the following way:

$$\xi(x) := \frac{|g(x)|}{\max\{|g(z)|\colon z \in \mathbb{R}\}} \qquad \forall x \in \mathbb{R}$$

The function $\xi(\cdot)$ defined above fulfils the conditions for characterizing functions.

Example 5.16. Let the set $I \subset \mathbb{R}$ and the function $h(\cdot)$ be defined in the same way as in Example 5.14 and depicted in Figure 5.17. The characterizing function $\xi(\cdot)$ obtained by the partly linear approach is shown in Figure 5.19.

5.3.3 Differential approach

If we are more interested in the slope of the function $h(\cdot)$ instead of it's values, we can investigate the differences between neighbouring values. Let the set $I \subset \mathbb{R}$ and the function $h\colon I \to \mathbb{R}$ and values $x_0, x_1, \ldots, x_n, x_{n+1}, \Delta \in \mathbb{R}$ be defined as in Section 5.3.

We want to investigate the differences between the neighbourhood values of the function $h(\cdot)$. The standard way to do so is to compute the derivative. More precisely it would be the function $\dot{g}(\cdot)$ fulfilling the following:

$$\dot{g}(x) := \left\{ \begin{array}{ll} |h(x_{i+1}) - h(x_i)| & \text{for } x \in (x_i, x_{i+1}) \\ \max\{|h(x_i) - h(x_{i-1})|, |h(x_{i+1}) - h(x_i)|\} & \text{for } x = x_i \\ 0 & \text{for } x \notin [x_0, x_{n+1}] \end{array} \right\} \forall x \in \mathbb{R}$$

Actually, we can obtain the function $\dot{g}(\cdot)$ as the derivative of the function $g(\cdot)$ constructed by the partly linear approach (see Section 5.3.2), more precisely as the maximum of the derivatives of the function $g(\cdot)$ from the left and from the right direction (multiplied by constant $\frac{1}{\Delta}$):

$$\dot{g}(x) = \max\left\{ \left|g'^{-}(x)\right|, \left|g'^{+}(x)\right| \right\} \cdot \tfrac{1}{\Delta} \qquad \forall x \in \mathbb{R}$$

To obtain a fuzzy number we should make the normalization. The formula for the characterizing function for the fuzzy number is:

$$\xi(x) := \frac{|\dot{g}(x)|}{\max\{|\dot{g}(z)| \colon z \in \mathbb{R}\}} \qquad \forall x \in \mathbb{R}$$

The function $\xi(\cdot)$ defined above fulfils the conditions for characterizing functions.

Example 5.17. Let the set $I \subset \mathbb{R}$ and the function $h(\cdot)$ be defined in the same way as in Example 5.14 and depicted in Figure 5.17. Then the values of function $h(\cdot)$ are the following:

x_i	$\frac{1}{2}$	$1\frac{1}{2}$	$2\frac{1}{2}$	$3\frac{1}{2}$	$4\frac{1}{2}$	$5\frac{1}{2}$	$6\frac{1}{2}$	$7\frac{1}{2}$	$8\frac{1}{2}$	$9\frac{1}{2}$	$10\frac{1}{2}$	$11\frac{1}{2}$
$h(x_i)$	0	$\frac{3}{4}$	$2\frac{1}{4}$	3	3	$1\frac{1}{4}$	$-\frac{3}{4}$	$-\frac{3}{4}$	1	1	0	0

The function $\dot{g}(\cdot)$ has according the previous definition the following values:

x	$(-\infty, \frac{1}{2})$	$[\frac{1}{2}, 1\frac{1}{2})$	$[1\frac{1}{2}, 2\frac{1}{2}]$	$(2\frac{1}{2}, 3\frac{1}{2}]$	$(3\frac{1}{2}, 4\frac{1}{2})$	$[4\frac{1}{2}, 5\frac{1}{2})$
$\dot{g}(x)$	0	$\frac{3}{4}$	$1\frac{1}{2}$	$\frac{3}{4}$	0	$1\frac{3}{4}$

x	$[5\frac{1}{2}, 6\frac{1}{2}]$	$(6\frac{1}{2}, 7\frac{1}{2})$	$[7\frac{1}{2}, 8\frac{1}{2}]$	$(8\frac{1}{2}, 9\frac{1}{2})$	$[9\frac{1}{2}, 10\frac{1}{2}]$	$(10\frac{1}{2}, \infty)$
$\dot{g}(x)$	2	0	$1\frac{3}{4}$	0	1	0

The characterizing function $\xi(\cdot)$ obtained by the differential approach described in this section is defined in the following way:

$$\xi(x) = \frac{\dot{g}(x)}{2} \qquad \forall x \in \mathbb{R}$$

The characterizing function $\xi(\cdot)$ is shown in Figure 5.20.

Figure 5.17: Set I and function h(·) defined in Example 5.14

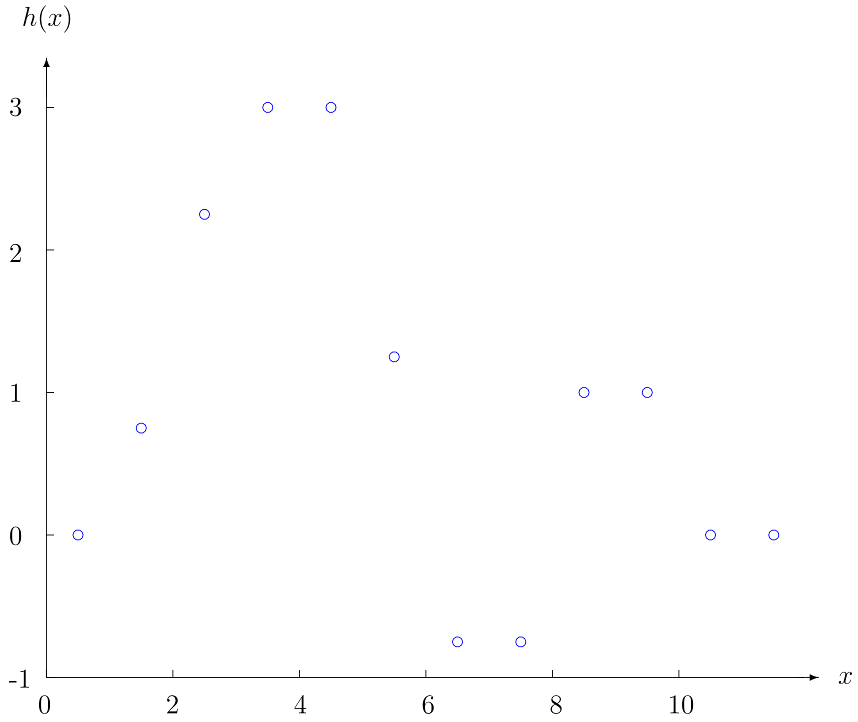


Figure 5.18: Characterizing function ξ(·) constructed from the function h(·) according to Example 5.15

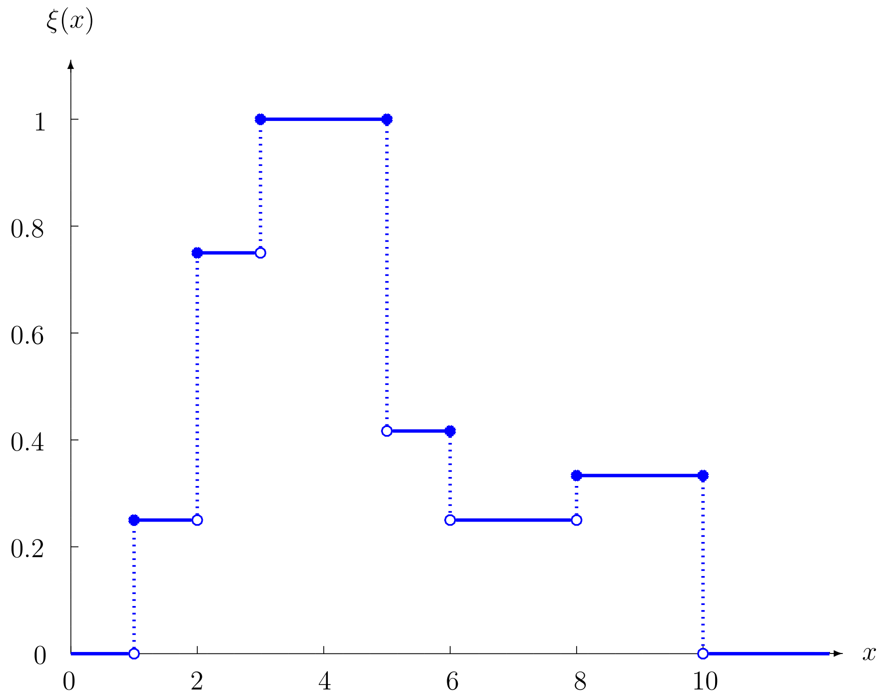


Figure 5.19: Characterizing function $\xi(\cdot)$ constructed from the function $h(\cdot)$ according to Example 5.16

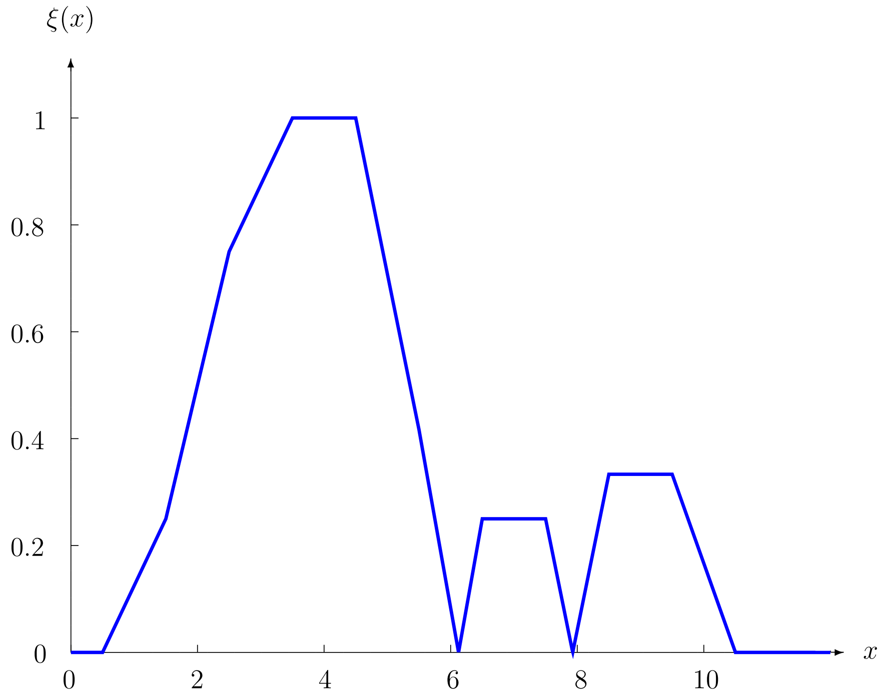
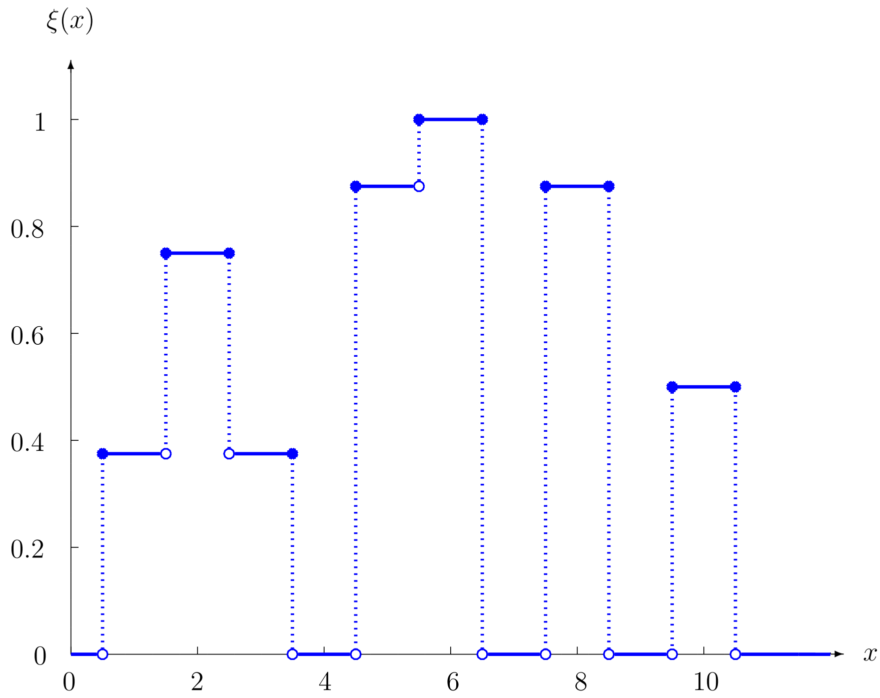


Figure 5.20: Characterizing function $\xi(\cdot)$ constructed from the function $h(\cdot)$ according to Example 5.17



5.4 Function h defined on equally spaced finite subsets of $\mathbb{R}^2$

This section describes a two-dimensional generalization of the one-dimensional approach described in Section 5.3.

Let be $I = \{(x_1, y_1), \ldots, (x_1, y_m), \ldots, (x_n, y_1), \ldots, (x_n, y_m)\}$ for $n, m \in \mathbb{N}$, x_i and y_i ordered, and let the points (x_i, y_j) be equally spaced in the plane, more precisely:

$$\exists \Delta_x \in (0, \infty) \quad \forall i \in \{2, 3, \ldots, n\} \quad x_i - x_{i-1} = \Delta_x$$

$$\exists \Delta_y \in (0, \infty) \quad \forall j \in \{2, 3, \ldots, m\} \quad y_j - y_{j-1} = \Delta_y$$

Let the function $h(\cdot, \cdot)$ be defined on such a set I.

We define values x_0, x_{n+1}, y_0, y_{m+1} as following:

$$x_0 := x_1 - \Delta_x, \;\; x_{n+1} := x_n + \Delta_x$$

$$y_0 := y_1 - \Delta_y, \;\; y_{m+1} := y_m + \Delta_y$$

Add to the set I all border points B defined as:

$$B := \bigcup_{i \in \{0, \ldots, n+1\}} \{(x_i, y_0), (x_i, y_{m+1})\} \;\cup \bigcup_{j \in \{1, \ldots, m\}} \{(x_0, y_j), (x_{n+1}, y_j)\}$$

We extend the function $h(\cdot, \cdot)$ in the following way:

$$\forall i \in \{0, \ldots, n+1\} \quad h(x_i, y_0) := 0, \;\; h(x_i, y_{m+1}) := 0$$

$$\forall j \in \{0, \ldots, m+1\} \quad h(x_0, y_j) := 0, \;\; h(x_{n+1}, y_j) := 0$$

Example 5.18. Let the set $I \subset \mathbb{R}^2$ be defined in the following way:

$$I = \{-2.96, -2.48, -2, \ldots, 2.32, 2.8\} \times \{-2.96, -2.48, -2, \ldots, 2.32, 2.8\}$$

Let the function $h(\cdot, \cdot)$ be defined on I by the same equation as was in Example 5.2. Let additionally $h(-2.96, y) = h(2.8, y) = h(x, -2.96) = h(x, 2.8) = 0$ $\forall (x, y) \in I$. The function $h(\cdot, \cdot)$ defined on the set I is depicted in Figure 5.21.

Example 5.19. Let the set $I \subset \mathbb{R}^2$ be defined in the following way:

$$I = \{-14.8, -13, -11.2, \ldots, 12.2, 14\} \times \{-14.8, -13, -11.2, \ldots, 12.2, 14\}$$

Let the function $h(\cdot, \cdot)$ be defined on I by the same equation as was in Example 5.3. Let additionally $h(-14.8, y) = h(14, y) = h(x, -14.8) = h(x, 14) = 0$ $\forall (x, y) \in I$. The function $h(\cdot, \cdot)$ defined on the set I is depicted in Figure 5.22.

We want to extend the function $h : I \to \mathbb{R}$. We can define the function $g : \mathbb{R}^2 \to \mathbb{R}$ as a partly constant function or as a partly linear function. This will be explained in the following sections.

Figure 5.21: Set I and function $h(\cdot,\cdot)$ defined in Example 5.18

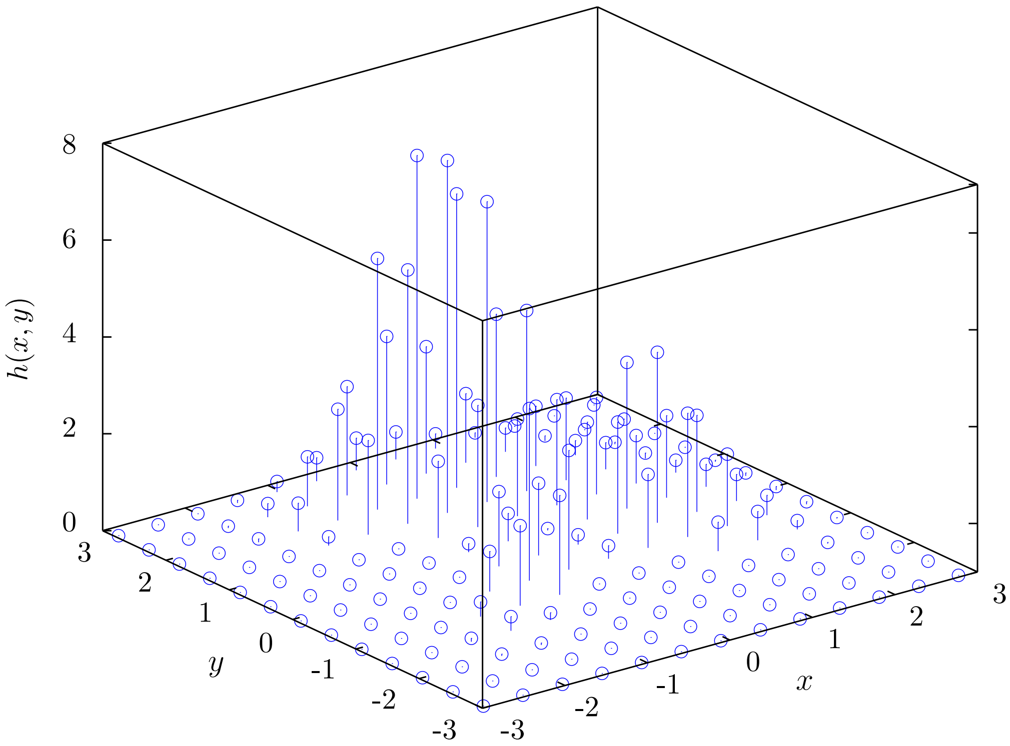
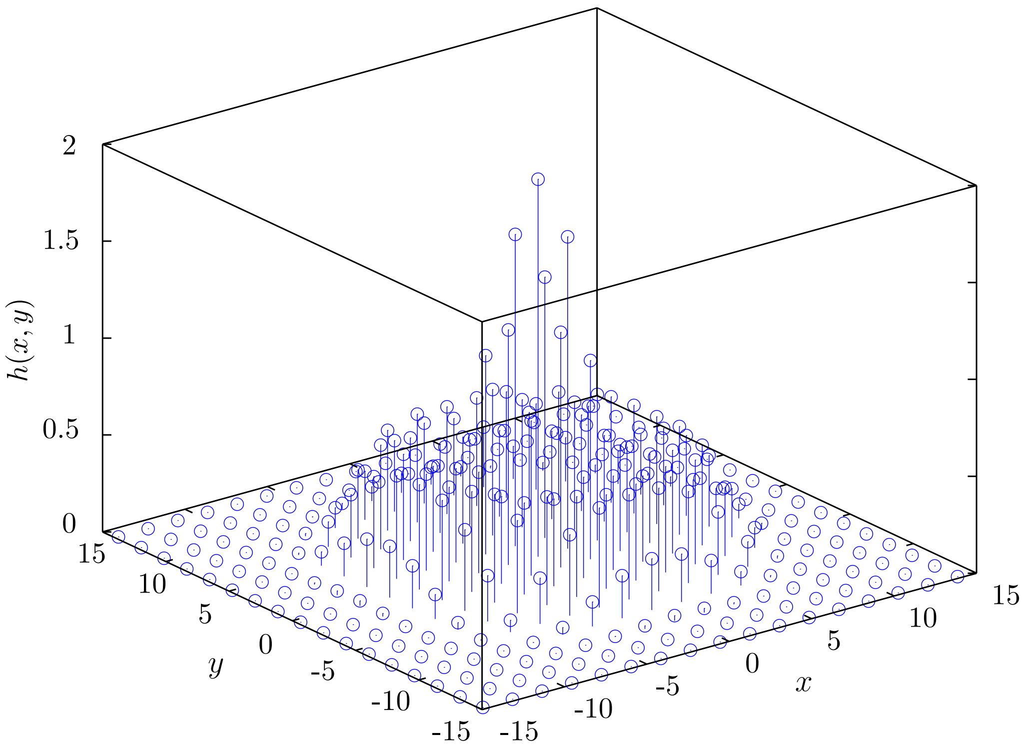


Figure 5.22: Set I and function $h(\cdot,\cdot)$ defined in Example 5.19



5.4.1 Partly constant approach

Firstly we describe the partly constant approach. Let the set $I \subset \mathbb{R}^2$, the function $h : I \to \mathbb{R}$ and values $x_0, x_1, \dots, x_n, x_{n+1}, y_0, y_1, \dots, y_m, y_{m+1}, \Delta_x, \Delta_y \in \mathbb{R}$ be defined as precedes in Section 5.4.

We want to create a partly constant function $g\colon \mathbb{R}^2 \to \mathbb{R}$, which extends the function $h : I \to \mathbb{R}$. There could be written two formulas similarly to the one-dimensional case described in Section 5.3.1. The first formula describing the function $g(\cdot,\cdot)$ is the following:

$$g(x,y) := \left\{ \begin{array}{l} |h(x_i,y_j)| \\ \qquad \text{for } x \in (x_i - \frac{1}{2}\Delta_x, x_i + \frac{1}{2}\Delta_x) \text{ and } y \in (y_j - \frac{1}{2}\Delta_y, y_j + \frac{1}{2}\Delta_y) \\ \\ \max\{|h(x_i,y_j)|, |h(x_{i+1},y_j)|\} \\ \qquad \text{for } x = x_i + \frac{1}{2}\Delta_x \text{ and } y \in (y_j - \frac{1}{2}\Delta_y, y_j + \frac{1}{2}\Delta_y) \\ \\ \max\{|h(x_i,y_j)|, |h(x_i,y_{j+1})|\} \\ \qquad \text{for } x \in (x_i - \frac{1}{2}\Delta_x, x_i + \frac{1}{2}\Delta_x) \text{ and } y = y_j + \frac{1}{2}\Delta_y \\ \\ \max\{|h(x_i,y_j)|, |h(x_{i+1},y_j)|, |h(x_i,y_{j+1})|, |h(x_{i+1},y_{j+1})|\} \\ \qquad \text{for } x = x_i + \frac{1}{2}\Delta_x \text{ and } y = y_j + \frac{1}{2}\Delta_y \\ \\ \qquad\qquad\qquad \text{where } i \in \{0,\dots,n\},\ j \in \{0,\dots,m\} \\ \\ 0 \quad \text{for } x \notin (x_0, x_{n+1}) \text{ or } y \notin (y_0, y_{m+1}) \end{array} \right\}$$

$$\forall (x,y) \in \mathbb{R}^2$$

The second formula for the function $g(\cdot,\cdot)$, which is equivalent to the first formula, is the following:

$$g(x,y) = \max_{\substack{i\in\{1,\dots,n\}\\ j\in\{1,\dots,m\}}} \left(\left\{ |h((x_i,y_j))| : |x - x_i| \le \tfrac{1}{2}\Delta_x, |y - y_j| \le \tfrac{1}{2}\Delta_y \right\} \cup \{0\} \right) \qquad \forall (x,y) \in \mathbb{R}^2$$

Then the vector-characterizing function $\zeta : \mathbb{R}^2 \to [0,1]$ can be obtained from the function $g(\cdot,\cdot)$ by normalization in the following way:

$$\zeta(x,y) := \frac{g(x,y)}{\max\{g(u,v)\colon (u,v) \in \mathbb{R}^2\}} \qquad \forall (x,y) \in \mathbb{R}^2$$

The function $\zeta(\cdot,\cdot)$ defined above fulfils the conditions for vector-characterizing functions.

Example 5.20. Let the set $I \subset \mathbb{R}^2$ and the function $h(\cdot,\cdot)$ be defined in the same way as in Example 5.18. The vector-characterizing function $\zeta(\cdot,\cdot)$ obtained by the partly constant approach is shown in Figure 5.23.

Example 5.21. Let the set $I \subset \mathbb{R}^2$ and the function $h(\cdot,\cdot)$ be defined in the same way as in Example 5.19. The vector-characterizing function $\zeta(\cdot,\cdot)$ obtained by the partly constant approach is shown in Figure 5.24.

Figure 5.23: Vector-characterizing function $\zeta(\cdot,\cdot)$ constructed from the function $h(\cdot,\cdot)$ according to Example 5.20

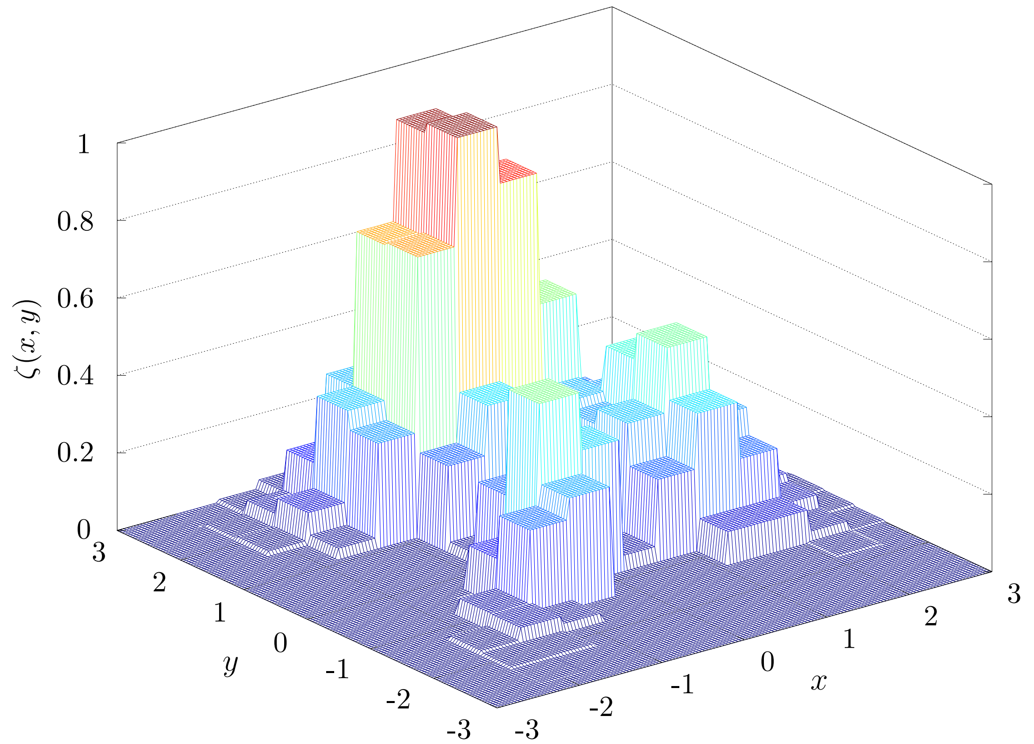
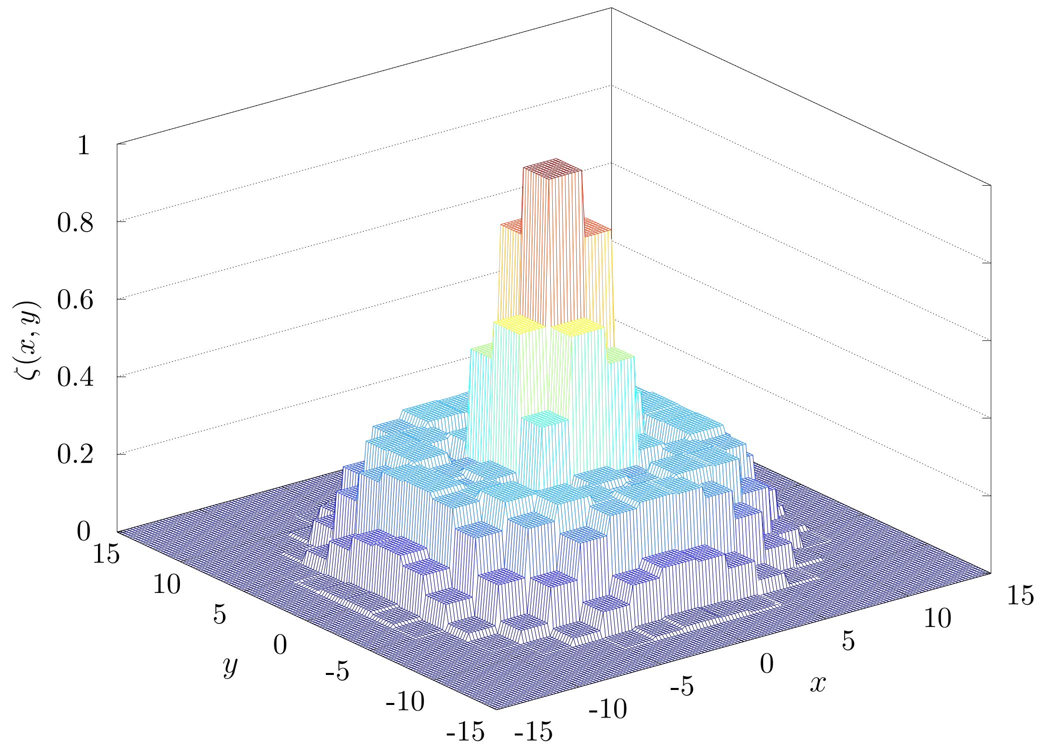


Figure 5.24: Vector-characterizing function $\zeta(\cdot,\cdot)$ constructed from the function $h(\cdot,\cdot)$ according to Example 5.21



5.4.2 Partly linear approach

In this section we describe the partly linear approach. Let the set $I \subset \mathbb{R}^2$, the function $h\colon I \to \mathbb{R}$ and values $x_0, x_1, \ldots, x_n, x_{n+1}, y_0, y_1, \ldots, y_m, y_{m+1}, \Delta_x, \Delta_y \in \mathbb{R}$ be defined as in Section 5.4.

The partly linear approach is a little more complicated. In the one-dimensional case described in Section 5.3.2 we have simply connected two neighbouring points. In the two dimensional case there are four points arranged in rectangle. We want to interpolate a plane through four edge points of this rectangle, but this isn't possible in general.

We create a new point $(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})$ in the centre of each rectangle and set the value of the function $h(\cdot,\cdot)$ at this new point as the mean of the values at the edges of the rectangle.

We define $x_{i+\frac{1}{2}} := x_i + \frac{1}{2}\Delta_x$ and $y_{j+\frac{1}{2}} := y_j + \frac{1}{2}\Delta_y$ and define the set J in the following way:

$$J := \left\{ \left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}} \right) : i \in \{1, \ldots, n\},\ j \in \{1, \ldots, m\} \right\} \subset \mathbb{R}^2$$

We extend the function $h(\cdot,\cdot)$ to the set J in the following way:

$$h(x,y) := \frac{1}{4} \left(\begin{array}{r} h(x - \frac{1}{2}\Delta_x, y - \frac{1}{2}\Delta_y) + h(x - \frac{1}{2}\Delta_x, y + \frac{1}{2}\Delta_y) + \\ + h(x + \frac{1}{2}\Delta_x, y - \frac{1}{2}\Delta_y) + h(x + \frac{1}{2}\Delta_x, y + \frac{1}{2}\Delta_y) \end{array} \right) \quad \forall\, (x,y) \in J$$

Then we create a partly linear function $g(\cdot,\cdot)$ in the following way:

$$g(x,y) := \left\{\begin{array}{l} h(x_i,y_j) + \frac{(x-x_i)\Delta_y-(y-y_j)\Delta_x}{\Delta_x\Delta_y}(h(x_{i+1},y_j)-h(x_i,y_j))+ \\ \qquad +\frac{2(y-y_j)}{\Delta_y}(h(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}})-h(x_i,y_j)) \\ \quad \text{for } x > x_i \text{ and } \frac{x-x_i}{x_{i+1}-x_i} \le \frac{y-y_j}{y_{j+1}-y_j} \text{ and } \frac{x-x_{i+1}}{x_i-x_{i+1}} > \frac{y-y_j}{y_{j+1}-y_j} \\ \\ h(x_{i+1},y_j) + \frac{(x-x_{i+1})\Delta_y+(y-y_j)\Delta_x}{\Delta_x\Delta_y}(h(x_{i+1},y_{j+1})-h(x_{i+1},y_j))- \\ \qquad -\frac{2(x-x_{i+1})}{\Delta_x}(h(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}})-h(x_{i+1},y_j)) \\ \quad \text{for } y \le y_{j+1} \text{ and } \frac{x-x_i}{x_{i+1}-x_i} \le \frac{y-y_j}{y_{j+1}-y_j} \text{ and } \frac{x-x_{i+1}}{x_i-x_{i+1}} \le \frac{y-y_j}{y_{j+1}-y_j} \\ \\ h(x_i,y_{j+1}) + \frac{(x-x_i)\Delta_y+(y-y_{j+1})\Delta_x}{\Delta_x\Delta_y}(h(x_{i+1},y_{j+1})-h(x_i,y_{j+1}))- \\ \qquad -\frac{2(y-y_{j+1})}{\Delta_y}(h(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}})-h(x_i,y_{j+1})) \\ \quad \text{for } x \le x_{i+1} \text{ and } \frac{x-x_i}{x_{i+1}-x_i} > \frac{y-y_j}{y_{j+1}-y_j} \text{ and } \frac{x-x_{i+1}}{x_i-x_{i+1}} \le \frac{y-y_j}{y_{j+1}-y_j} \\ \\ h(x_i,y_j) + \frac{-(x-x_i)\Delta_y+(y-y_j)\Delta_x}{\Delta_x\Delta_y}(h(x_i,y_{j+1})-h(x_i,y_j))+ \\ \qquad +\frac{2(x-x_i)}{\Delta_x}(h(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}})-h(x_i,y_j)) \\ \quad \text{for } y > y_i \text{ and } \frac{x-x_i}{x_{i+1}-x_i} > \frac{y-y_j}{y_{j+1}-y_j} \text{ and } \frac{x-x_{i+1}}{x_i-x_{i+1}} > \frac{y-y_j}{y_{j+1}-y_j} \\ \\ \qquad\qquad \text{where } i \in \{1,\dots,n\},\ j \in \{1,\dots,m\} \\ \\ 0 \quad \text{for } x \notin (x_0,x_{n+1}) \text{ or } y \notin (y_0,y_{m+1}) \end{array}\right\}$$

$$\forall (x,y) \in \mathbb{R}^2$$

Then the vector-characterizing function $\zeta\colon \mathbb{R}^2 \to [0,1]$ can be constructed by normalization of the function $g(\cdot,\cdot)$ in the following way:

$$\zeta(x,y) := \frac{|g(x,y)|}{\max\{|g(u,v)|\colon (u,v) \in \mathbb{R}^2\}} \qquad \forall (x,y) \in \mathbb{R}^2$$

The function $\zeta(\cdot,\cdot)$ defined above fulfils the conditions for vector-characterizing functions. Note that the function $g(\cdot,\cdot)$ is partly linear and continuous and hence also the vector-characterizing function $\zeta(\cdot,\cdot)$ is partly linear and continuous.

Example 5.22. Let the set $I \subset \mathbb{R}^2$ and the function $h(\cdot,\cdot)$ be defined in the same way as in Example 5.18. The vector-characterizing function $\zeta(\cdot,\cdot)$ obtained by the partly linear approach is shown in Figure 5.25.

Example 5.23. Let the set $I \subset \mathbb{R}^2$ and the function $h(\cdot,\cdot)$ be defined in the same way as in Example 5.19. The vector-characterizing function $\zeta(\cdot,\cdot)$ obtained by the partly linear approach is shown in Figure 5.26.

Figure 5.25: Vector-characterizing function $\zeta(\cdot,\cdot)$ constructed from the function $h(\cdot,\cdot)$ according to Example 5.22

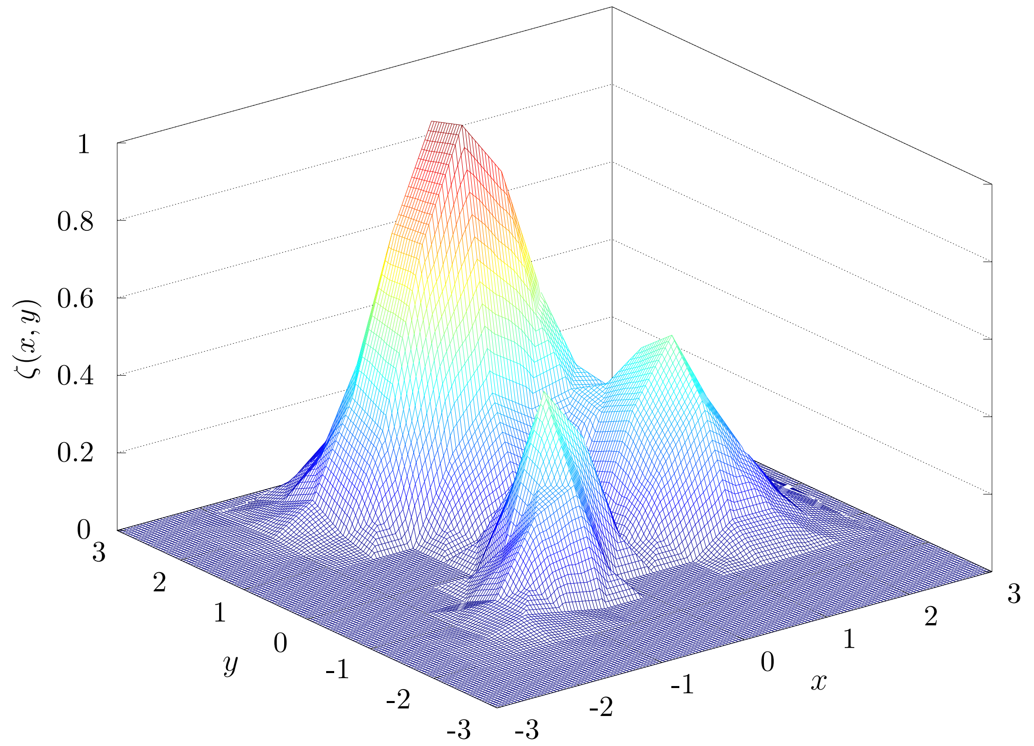
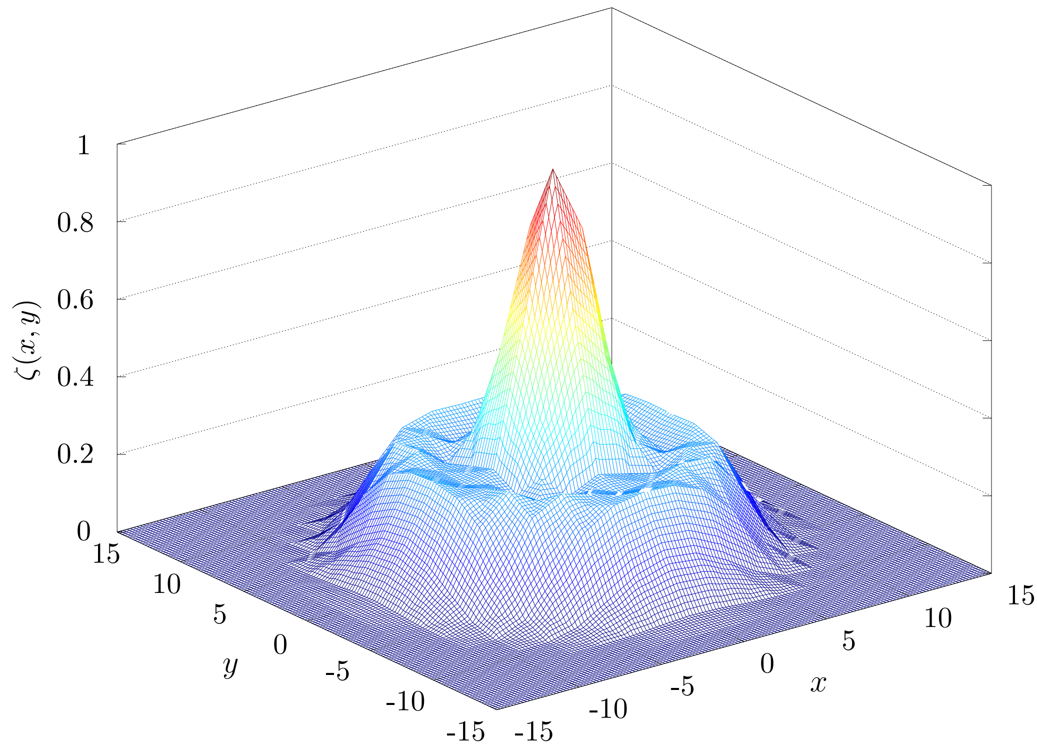


Figure 5.26: Vector-characterizing function $\zeta(\cdot,\cdot)$ constructed from the function $h(\cdot,\cdot)$ according to Example 5.23



5.4.3 Differential approach

If we are more interested in the slope of the function $h(\cdot,\cdot)$ instead of it's values, we can investigate the differences between neighbouring values and use the differential approach. Let the set $I \subset \mathbb{R}^2$, the function $h\colon I \to \mathbb{R}$, and the values $x_0, x_1, \ldots, x_n, x_{n+1}, y_0, y_1, \ldots, y_m, y_{m+1}, \Delta_x, \Delta_y \in \mathbb{R}$ be defined as in Section 5.4.

In the previous one-dimensional case we are able to use the derivative of the function constructed by the linear approach (see Section 5.3.3).

In the more-dimensional case the partial derivatives and the total differential can be used. The total differential of a function $g : \mathbb{R}^2 \to \mathbb{R}$ is defined in the following way:

$$\mathrm{Tot}(x,y) := \left(\frac{\partial\, g(x,y)}{\partial x}, \frac{\partial\, g(x,y)}{\partial y}\right) \qquad \forall (x,y)\mathbb{R}^2$$

Let the function $g : \mathbb{R}^2 \to \mathbb{R}$ be constructed from the function $h : I \to \mathbb{R}$ in the partly linear approach described in Section 5.4.2. Then we can construct the function $\dot{g} : \mathbb{R}^2 \to \mathbb{R}$ using the total differential $\mathrm{Tot} : \mathbb{R}^2 \to \mathbb{R}^2$ of the function $g(\cdot)$ and a metric $\rho : \mathbb{R}^2 \to [0,\infty) \cup \infty$ in the following way:

$$\dot{g}(x,y) := \rho\left(\mathrm{Tot}(x,y)\right) \qquad \forall (x,y) \in \mathbb{R}^2$$

In case where $\rho(\cdot,\cdot)$ is the Euclidean metric we can construct the function $\dot{g}(\cdot,\cdot)$ from the function $g(\cdot,\cdot)$ obtained by the partly linear approach in the following way:

$$\dot{g}(x,y) := \sqrt{\left(\frac{\partial\, g(x,y)}{\partial x}\right)^2 + \left(\frac{\partial\, g(x,y)}{\partial y}\right)^2} \qquad \forall (x,y) \in \mathbb{R}^2$$

Then the vector-characterizing function $\zeta\colon \mathbb{R}^2 \to [0,1]$ can be constructed by normalization of the function $\dot{g}(\cdot,\cdot)$ in the following way:

$$\zeta(x,y) := \frac{|\dot{g}(x,y)|}{\max\left\{|\dot{g}(u,v)|\colon (u,v) \in \mathbb{R}^2\right\}} \qquad \forall (x,y) \in \mathbb{R}^2$$

The function $\zeta(\cdot,\cdot)$ defined above fulfils the conditions for vector-characterizing functions.

Example 5.24. Let the set $I \subset \mathbb{R}^2$ and the function $h(\cdot,\cdot)$ be defined in the same way as in Example 5.18. The vector-characterizing function $\zeta(\cdot,\cdot)$ obtained by the differential approach is shown in Figure 5.27.

Example 5.25. Let the set $I \subset \mathbb{R}^2$ and the function $h(\cdot,\cdot)$ be defined in the same way as in Example 5.19. The vector-characterizing function $\zeta(\cdot,\cdot)$ obtained by the differential approach is shown in Figure 5.28.

Figure 5.27: Vector-characterizing function $\zeta(\cdot,\cdot)$ constructed from the function $h(\cdot,\cdot)$ according to Example 5.24

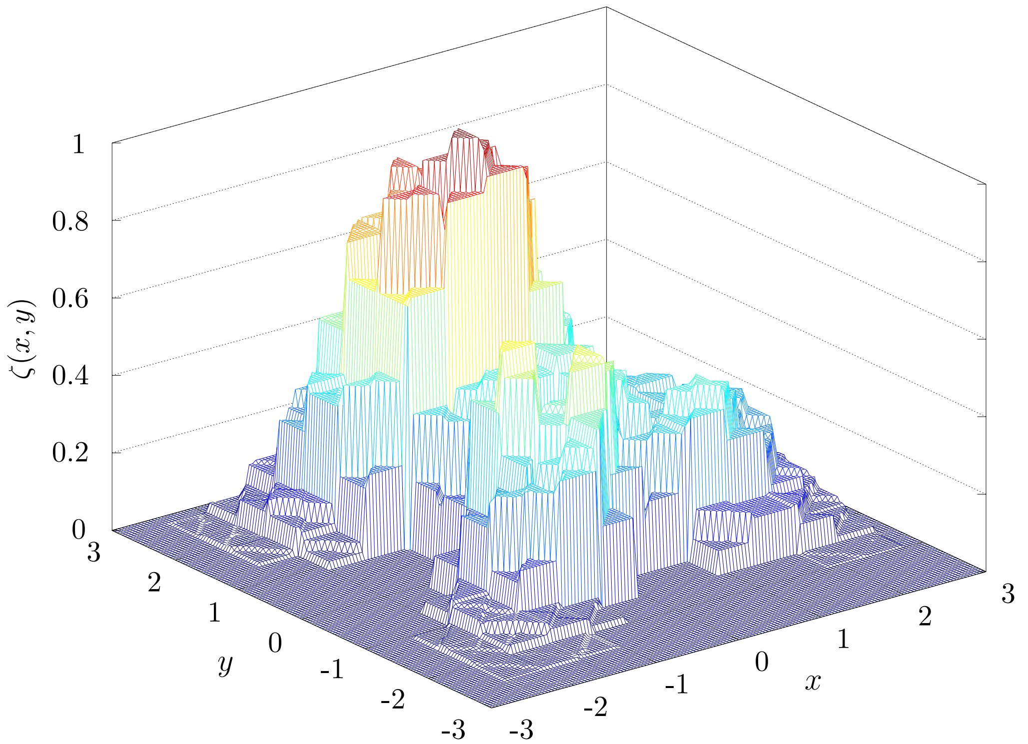
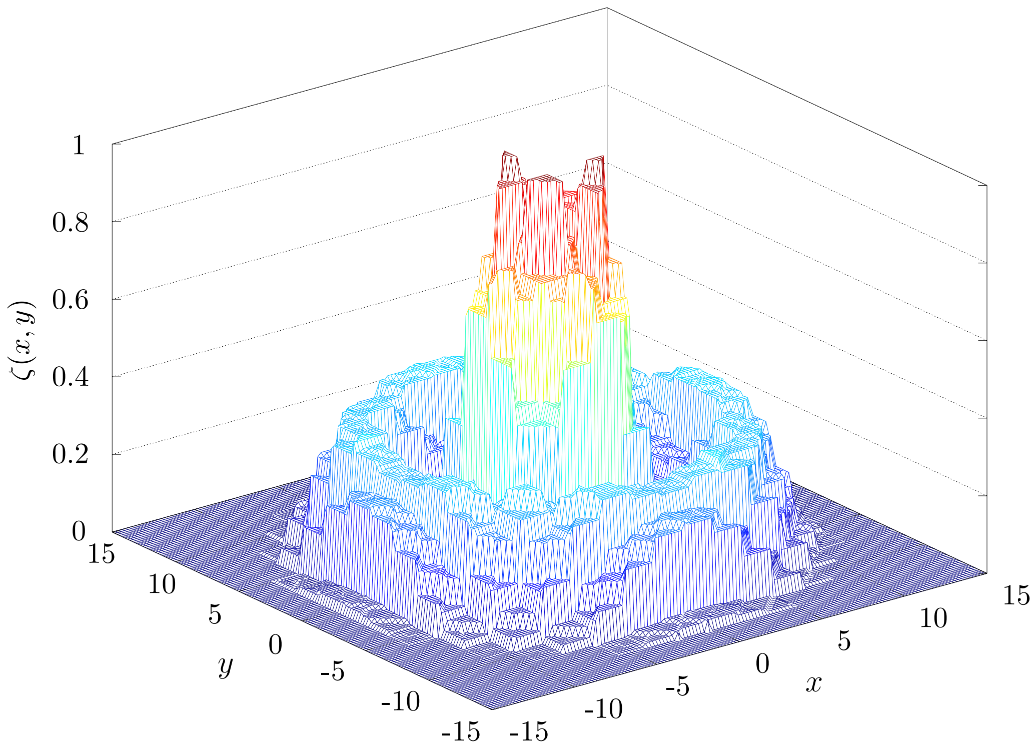


Figure 5.28: Vector-characterizing function $\zeta(\cdot,\cdot)$ constructed from the function $h(\cdot,\cdot)$ according to Example 5.25



6. Computing with Fuzzy Numbers

Example 6.1. Let x_1^* and x_2^* be fuzzy intervals with corresponding characterizing functions $\xi_1(\cdot)$ and $\xi_2(\cdot)$, and suppose that we want to add these two fuzzy numbers. The function for the sum of two numbers is:

$$g(x, y) = x + y \qquad \forall\, (x, y) \in \mathbb{R}^2$$

Firstly we create a vector of fuzzy numbers (x_1^*, x_2^*). In the next step we construct a fuzzy vector $(x_1, x_2)^*$ with vector-characterizing function $\zeta(\cdot, \cdot)$ from the characterizing functions of the fuzzy numbers x_1^* and x_2^* by applying a t-norm. Usually the minimum t-norm is used, but others are also possible.

$$\zeta(x, y) = \min\{\xi_1(x), \xi_2(y)\} \qquad \forall\, (x, y) \in \mathbb{R}^2$$

Now we can apply the extension principle in order to obtain the characterizing function $\psi(\cdot)$ for the addition of two fuzzy numbers based on the function for the addition of two real numbers, and based on the fuzzy vector $(x_1, x_2)^*$:

$$\psi(z) = \left\{ \begin{array}{ll} \sup\{\zeta(x, y) \colon g(x, y) = z\} & \text{if } \exists z \in \mathbb{R} : g(x, y) = z \\ 0 & \text{if } \nexists z \in \mathbb{R} : g(x, y) = z \end{array} \right\} \qquad \forall z \in \mathbb{R}$$

The function $\psi(\cdot)$ can be equivalently defined by a shorter formula:

$$\psi(z) = \sup(\{\zeta(x, y) \colon g(x, y) = z\} \cup \{0\}) \qquad \forall z \in \mathbb{R}$$

The result of applying the extension principle is the above defined characterizing function $\psi(\cdot)$ describing a fuzzy number $y^* = g((x_1, x_2)^*)$.

To be precise we should verify the presumptions of one of the extension principle theorems from Chapter 3.2, in this case Theorem 3.4. We will repeat here the wording of the Theorem.

Theorem 3.4. *Let $\underline{x}^*$ be a n-dimensional convex fuzzy interval with vector-characterizing function $\zeta(\cdot, \dots, \cdot)$. Let $M \subseteq \mathbb{R}^n$ be a set containing the support of $\underline{x}^*$. Let $g : M \to \mathbb{R}$ be a continuous and bounded function on M such that the image of an arbitrary convex set is an interval.*

We define a fuzzy set $y^* = g(\underline{x}^*)$ in $\mathbb{R}$ described by its membership function $\psi(\cdot)$ based on the extension principle by

$$\psi(y) = \sup(\{\zeta(\underline{x}) : \underline{x} \in \mathbb{R}^n, g(\underline{x}) = y\} \cup \{0\}) \quad \forall y \in \mathbb{R}\ .$$

Then y^* is a one-dimensional convex fuzzy interval and each δ-cut $\mathcal{C}_\delta(y^*)$ fulfils the following:

$$\mathcal{C}_\delta(y^*) = \left[\min_{\underline{x} \in \mathcal{C}_\delta(\underline{x}^*)} g(\underline{x}),\ \max_{\underline{x} \in \mathcal{C}_\delta(\underline{x}^*)} g(\underline{x})\right] \quad \forall \delta \in (0, 1]$$

In this example the fuzzy vector $\underline{x}^* = (x_1, x_2)^*$ was constructed by the minimum t-norm from fuzzy intervals x_1^* and x_2^*. The δ-cuts of the fuzzy vector $\underline{x}^*$ are in this case, where the minimum t-norm was used, the Cartesian products of δ-cuts of fuzzy intervals x_1^* and x_2^* (where for each $\delta \in (0,1]$ the δ-cuts $\mathcal{C}_\delta(x_1^*)$ and $\mathcal{C}_\delta(x_2^*)$ are compact intervals from definition of fuzzy interval):

$$\forall \delta \in (0,1] \qquad \mathcal{C}_\delta(\underline{x}^*) = \mathcal{C}_\delta(x_1^*) \times \mathcal{C}_\delta(x_2^*)$$

From that we can see, that each δ-cut $\mathcal{C}_\delta(\underline{x}^*)$ is a convex, closed and bounded set, hence $\underline{x}^*$ is a 2-dimensional convex fuzzy vector.

The set $M \subseteq \mathbb{R}^2$ can be choosen as the support of $\underline{x}^*$, $M := \operatorname{supp}(\underline{x}^*)$ where:

$$\operatorname{supp}(\underline{x}^*) = \{\underline{x} \in \mathbb{R}^2 \colon \zeta(\underline{x}) > 0\} = \{x_1 \in \mathbb{R} \colon \xi_1(x_1) > 0\} \times \{x_2 \in \mathbb{R} \colon \xi_2(x_2) > 0\}$$

Then the set M is a bounded set.

The function $g : M \to \mathbb{R}$ defined as $g(x,y) = x + y \quad \forall\, (x,y) \in \mathbb{R}^2$ is continuous on $\mathbb{R}^2$ and bounded on M. Now we will show, that the image of an arbitrary convex set is an interval. Let $N \subseteq M$ be an arbitrary convex set. Then from the definition of convex sets (Definition 3.7) the following holds true:

$$\forall x, y \in N \quad \forall \lambda \in [0,1] \quad \lambda x + (1-\lambda) y \in N$$

We want to show that the image of N, defined as $g(N) = \{g(x,y) \colon (x,y) \in N\}$, is an interval. Let $a, b \in g(N)$, $a < b$ be two arbitrary values from $g(N)$, then there exists $(x_a, y_a), (x_b, y_b) \in N$, such that $g(x_a, y_a) = a$ and $g(x_b, y_b) = b$. Now let $c \in \mathbb{R}$ be arbitrary such that $a < c < b$, we want to show that $c \in g(N)$. We set $\lambda = \frac{b-c}{b-a}$ and while N is a convex set and $(x_a, y_a), (x_b, y_b) \in N$ then $\lambda(x_a, y_a) + (1-\lambda)(x_b, y_b) = (\lambda x_a + (1-\lambda)x_b, \lambda y_a + (1-\lambda) y_b) \in N$. We compute

$$g(\lambda x_a + (1-\lambda)x_b, \lambda y_a + (1-\lambda)y_b) = \lambda x_a + (1-\lambda)x_b + \lambda y_a + (1-\lambda)y_b =$$

$$\lambda(x_a + y_a) + (1-\lambda)(x_b + y_b) = \lambda g(x_a, y_a) + (1-\lambda) g(x_b, y_b) = \lambda a + (1-\lambda) b =$$

$$= \tfrac{b-1}{b-a} a + (1 - \tfrac{b-c}{b-a}) b = \tfrac{b-c}{b-a} a + \tfrac{b-a-b+c}{b-a} b = \tfrac{ba-ca-ab+cb}{b-a} = \tfrac{c(b-a)}{b-a} = c$$

Then we have shown that $c \in g(N)$ and hence $g(N)$ is an interval.

Now we have verified all presumptions of Theorem 3.4 and hence we can use the conclusion of this theorem. This means that $z^* = g(\underline{x}^*)$ is described by its characterizing function $\psi(\cdot)$, where

$$\psi(y) = \sup(\{\zeta(x_1, x_2) : (x_1, x_2) \in \mathbb{R}^2, g(x_1, x_2) = y\} \cup \{0\}) \quad \forall y \in \mathbb{R}$$

is a fuzzy interval and each δ-cut $\mathcal{C}_\delta(y^*)$ is an interval fulfilling:

$$\mathcal{C}_\delta(y^*) = \left[\min_{\underline{x} \in \mathcal{C}_\delta(\underline{x}^*)} g(\underline{x}), \max_{\underline{x} \in \mathcal{C}_\delta(\underline{x}^*)} g(\underline{x}) \right] \quad \forall \delta \in (0,1]$$

In our case, where $g(x,y) = x + y$ and the minimum t-norm was used, the δ-cut $\mathcal{C}_\delta(y^*)$ can be rewritten in the following way:

$$\begin{aligned} \mathcal{C}_\delta(y^*) &= \left[\min_{(x,y) \in \mathcal{C}_\delta(x_1^*) \times \mathcal{C}_\delta(x_2^*)} x + y, \max_{(x,y) \in \mathcal{C}_\delta(x_1^*) \times \mathcal{C}_\delta(x_2^*)} x + y \right] \\ &= \left[\min \mathcal{C}_\delta(x_1^*) + \min \mathcal{C}_\delta(x_2^*), \max \mathcal{C}_\delta(x_1^*) + \max \mathcal{C}_\delta(x_2^*) \right] \quad \forall \delta \in (0,1] \end{aligned}$$

6.1 Fuzzy Arithmetic

To extend the arithmetic operations from real numbers to fuzzy numbers we need to realize, that the well known arithmetical operations can be represented as functions:

$$+, -, \cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad \text{for } / : \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$$

We can use the extension principle to these standard arithmetic functions, as was shown in Example 6.1, and get a new fuzzy number as a result of the arithmetic operation.

Let x_1^* and x_2^* be two fuzzy numbers with corresponding characterizing functions $\xi_1(\cdot)$ and $\xi_2(\cdot)$ and let $g \in \{g_+, g_-, g_\cdot, g_/\}$ be an arithmetic operation.

The well known functions for arithmetic operations $g_+, g_-, g_\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g_/ : \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$ are defined in the following way:

$$g_+(x_1, x_2) = x_1 + x_2, \quad g_-(x_1, x_2) = x_1 - x_2, \quad g_\cdot(x_1, x_2) = x_1 \cdot x_2 \quad \forall (x_1, x_2) \in \mathbb{R}^2$$

$$g_/(x_1, x_2) = \frac{x_1}{x_2} \qquad \forall (x_1, x_2) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$$

If the selected arithmetic operation g is division $g_/$, then we will additionally require that the support of the fuzzy number x_2^* will be contained in the set of positive numbers or in the set of negative numbers. The reason is that the function $g_/$ is not connected in 0 in the second argument and we want the image of an arbitrary convex subset of $\operatorname{supp}(x_1^*) \times \operatorname{supp}(x_2^*)$ to be a one-dimensional convex set, which is an interval.

The result of the selected arithmetic operation g is a fuzzy number x_{result}^* with characterizing function $\xi_{\text{result}}(\cdot)$ defined in the following way:

$$\xi_{\text{result}}(y) = \left\{ \begin{array}{ll} \sup\{\zeta(x_1, x_2) : (x_1, x_2) \in g^{-1}(\{y\})\} & \\ & \text{if } g^{-1}(\{y\}) \neq \emptyset \\ & \\ 0 & \text{if } g^{-1}(\{y\}) = \emptyset \end{array} \right\} \qquad \forall y \in \mathbb{R}$$

where x_1, x_2 are two real numbers, and where the vector-characterizing function $\zeta(\cdot, \cdot) : \mathbb{R}^2 \to [0, 1]$ is constructed by using one of the following t-norms, usually the minimum t-norm $\zeta_{\min}(\cdot, \cdot)$:

$$\zeta_{\min}(x_1, x_2) = \min\{\xi_1(x_1), \xi_2(x_2)\} \qquad \forall (x_1, x_2) \in \mathbb{R}^2$$

$$\zeta_{\text{prod}}(x_1, x_2) = \xi_1(x_1) \cdot \xi_2(x_2) \qquad \forall (x_1, x_2) \in \mathbb{R}^2$$

$$\zeta_{\text{lsum}}(x_1, x_2) = \max\{\xi_1(x_1) + \xi_2(x_2) - 1, 0\} \qquad \forall (x_1, x_2) \in \mathbb{R}^2$$

$$\zeta_{\text{dp}}(x_1, x_2) = \min\{\xi_1(x_1), \xi_2(x_2)\} \mathbb{1}_{\{\max\{\xi_1(x_1), \xi_2(x_2)\}\}}(1) \qquad \forall (x_1, x_2) \in \mathbb{R}^2$$

The fuzzy number x_{result}^* can be written as $x_1^* \oplus x_2^*$ if the selected arithmetical operation is addition, $x_1^* \ominus x_2^*$ if the selected arithmetical operation is subtraction, $x_1^* \odot x_2^*$ if the selected arithmetical operation is multiplication and $x_1^* \oslash x_2^*$ if the selected arithmetical operation is division.

Remark. Arithmetic operations defined in this way are an extension of arithmetic operations defined on real numbers. Furthermore, if you want to compute an arithmetic operation, where one operand is a fuzzy number x^* and the second operand is a real number a, then there can be constructed a fuzzy number a^* with characterizing function $\xi_a(\cdot) : \mathbb{R} \to [0,1]$ where:

$$\xi_a(x) = \left\{ \begin{array}{ll} 1 & \text{for } x = a \\ 0 & \text{for } x \in \mathbb{R} \setminus \{a\} \end{array} \right\} \qquad \forall x \in \mathbb{R}$$

Example 6.2. Let x_1^* and x_2^* be two fuzzy intervals with characterizing functions $\xi_1, \xi_2 : \mathbb{R} \to [0,1]$ defined in the following way:

$$\xi_1(x) = \left\{ \begin{array}{cl} x & \text{for } x \in [0,1) \\ 1 & \text{for } x \in [1,3] \\ 4-x & \text{for } x \in (3,4] \\ 0 & \text{for } x \in \mathbb{R} \setminus [0,4] \end{array} \right\} \qquad \forall x \in \mathbb{R}$$

$$\xi_2(x) = \left\{ \begin{array}{cl} \frac{x-1}{2} & \text{for } x \in [1,3) \\ 1 & \text{for } x \in [3,5] \\ \frac{7-x}{2} & \text{for } x \in (5,7] \\ 0 & \text{for } x \in \mathbb{R} \setminus [1,7] \end{array} \right\} \qquad \forall x \in \mathbb{R}$$

The characterizing functions $\xi_1(\cdot)$ and $\xi_2(\cdot)$ of the fuzzy numbers x_1^* and x_2^* are displayed in Figure 6.1 and Figure 6.2 respectively.

From the vector of fuzzy numbers (x_1^*, x_2^*) the fuzzy vector $(x_1, x_2)^*$ was constructed using the minimum, product, limited sum and drastic product t-norm with vector-characterizing functions $\zeta_{\text{min}}, \zeta_{\text{prod}}, \zeta_{\text{lsum}}, \zeta_{\text{dr}} : \mathbb{R} \times \mathbb{R} \to [0,1]$ where:

$$\begin{aligned} \zeta_{\text{min}}(x_1, x_2) &= \min\{\xi_1(x_1), \xi_2(x_2)\} && \forall (x_1, x_2) \in \mathbb{R}^2 \\ \zeta_{\text{prod}}(x_1, x_2) &= \xi_1(x_1) \cdot \xi_2(x_2) && \forall (x_1, x_2) \in \mathbb{R}^2 \\ \zeta_{\text{lsum}}(x_1, x_2) &= \max\{\xi_1(x_1) + \xi_2(x_2) - 1, 0\} && \forall (x_1, x_2) \in \mathbb{R}^2 \\ \zeta_{\text{dp}}(x_1, x_2) &= \min\{\xi_1(x_1), \xi_2(x_2)\} \mathbb{1}_{\{\max\{\xi_1(x_1), \xi_2(x_2)\}\}}(1) && \forall (x_1, x_2) \in \mathbb{R}^2 \end{aligned}$$

The resulting vector-characterizing functions $\zeta_{\text{min}}, \zeta_{\text{prod}}, \zeta_{\text{lsum}}, \zeta_{\text{dr}} : \mathbb{R} \times \mathbb{R} \to [0,1]$ of the fuzzy vector $(x_1, x_2)^*$ constructed from the vector of fuzzy numbers (x_1^*, x_2^*) by the minimum, product, limited sum and drastic product t-norm are shown in Figure 6.3, Figure 6.4, Figure 6.5 and Figure 6.6 respectively.

The characterizing functions $\psi_\oplus, \psi_\ominus, \psi_\odot, \psi_\oslash : \mathbb{R} \to [0,1]$ of the results of fuzzy addition $x_1^* \oplus x_2^*$, fuzzy subtraction $x_1^* \ominus x_2^*$, fuzzy multiplication $x_1^* \odot x_2^*$, and fuzzy division $x_1^* \oslash x_2^*$ using different t-norms are displayed in Figure 6.7, Figure 6.8, Figure 6.9, and Figure 6.10 respectively.

The values of different vector-characterizing functions fulfils the following:

$$\zeta_{\text{min}}(x_1, x_2) \geq \zeta_{\text{prod}}(x_1, x_2) \geq \zeta_{\text{lsum}}(x_1, x_2) \geq \zeta_{\text{dp}}(x_1, x_2) \qquad \forall (x_1, x_2) \in \mathbb{R}^2$$

Therefore the values of the characterizing functions giving the result of fuzzy arithmetic operations using fuzzy vectors obtained by different t-norms obeys the same:

$$\psi_{\text{min}}(x) \geq \psi_{\text{prod}}(x) \geq \psi_{\text{lsum}}(x) \geq \psi_{\text{dp}}(x) \qquad \forall x \in \mathbb{R}$$

Selected points used by the extension principle for different types of arithmetic operations and different t-norms are depicted in Figures 6.11 – 6.14.

Figure 6.1: Characterizing function $\xi_1(\cdot)$ of the fuzzy number x_1^* from Example 6.2

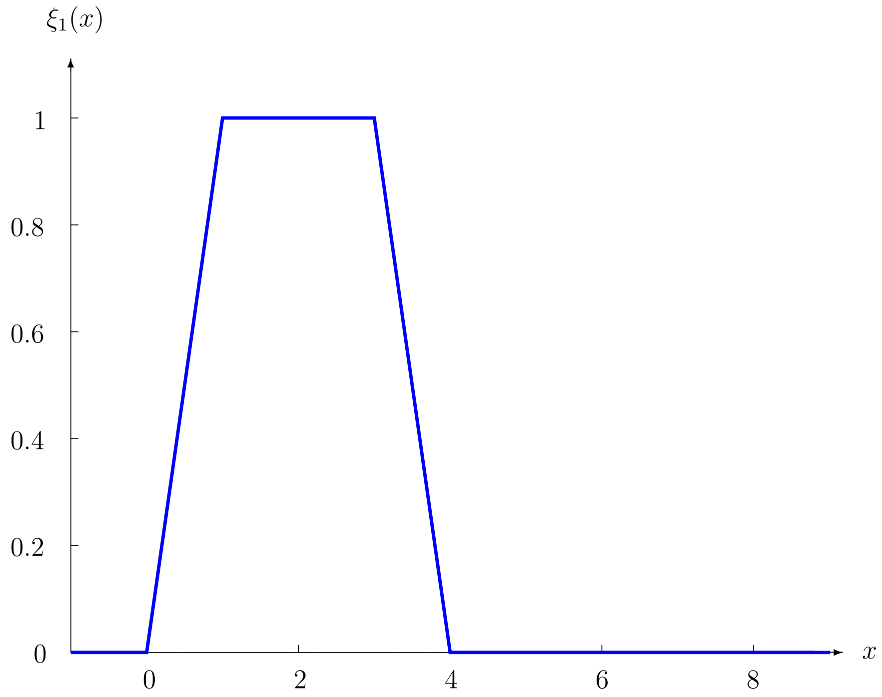


Figure 6.2: Characterizing function $\xi_2(\cdot)$ of the fuzzy number x_2^* from Example 6.2

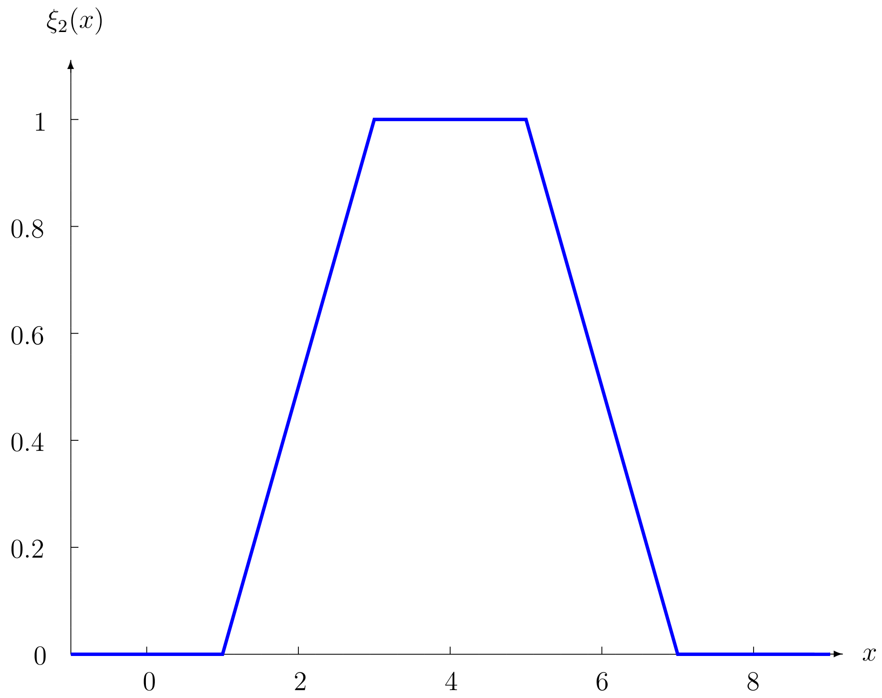


Figure 6.3: Vector-characterizing function $\zeta(\cdot,\cdot)$ of the fuzzy vector $(x_1,x_2)^*$, where the fuzzy vector was constructed using the minimum t-norm

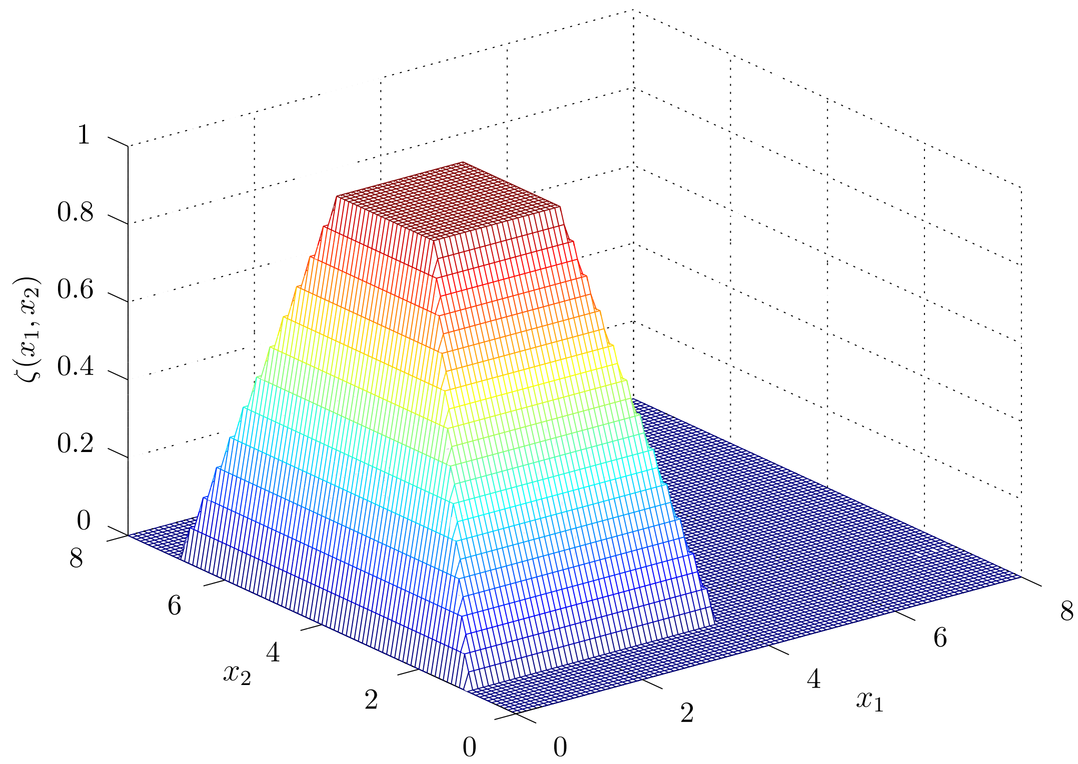


Figure 6.4: Vector-characterizing function $\zeta(\cdot,\cdot)$ of the fuzzy vector $(x_1,x_2)^*$, where the fuzzy vector was constructed using the product t-norm

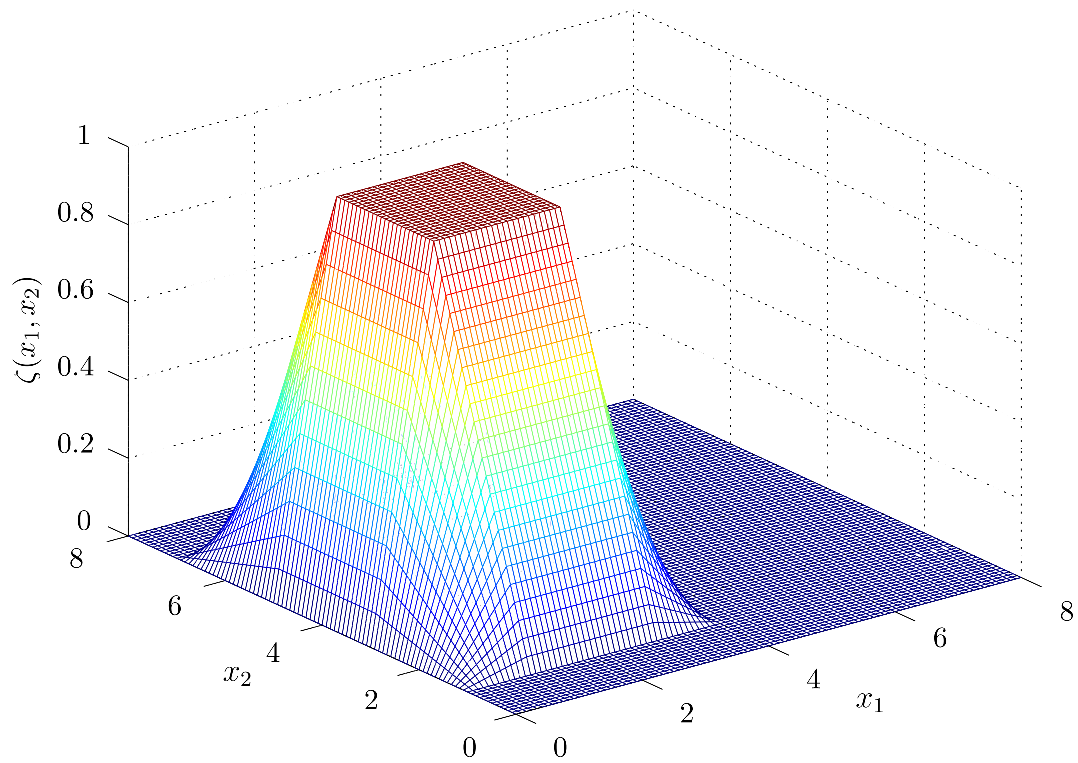


Figure 6.5: Vector-characterizing function $\zeta(\cdot, \cdot)$ of the fuzzy vector $(x_1, x_2)^*$, where the fuzzy vector was constructed using the limited sum t-norm

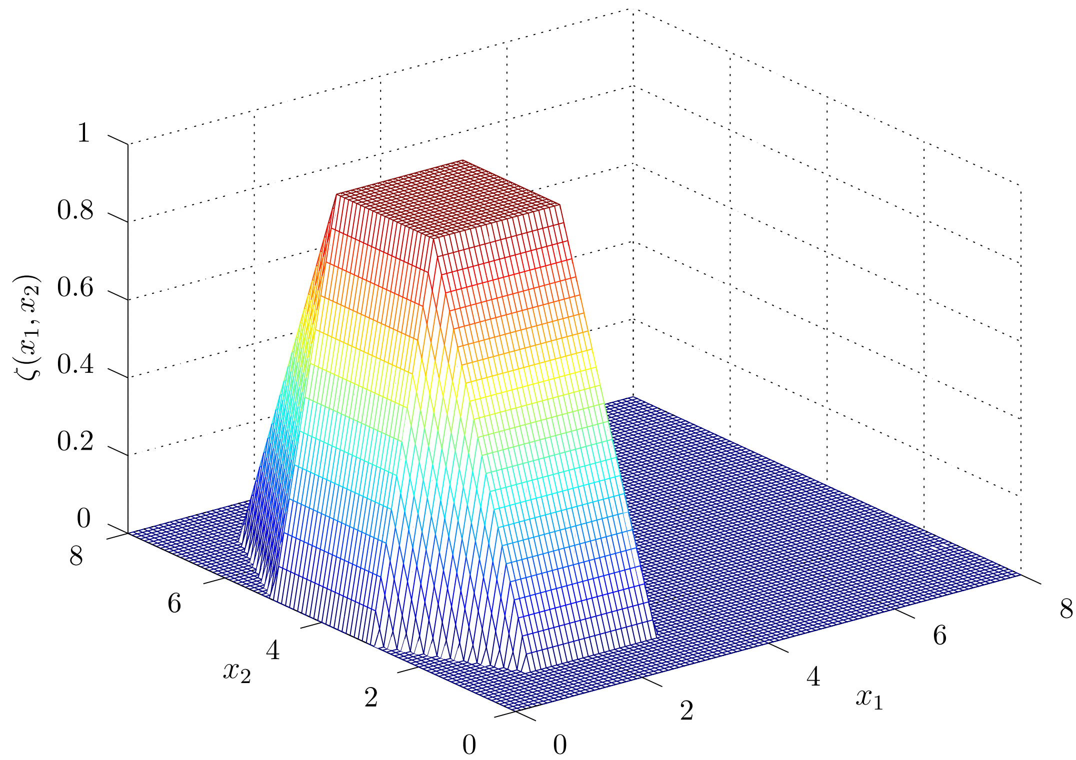


Figure 6.6: Vector-characterizing function $\zeta(\cdot, \cdot)$ of the fuzzy vector $(x_1, x_2)^*$, where the fuzzy vector was constructed using the drastic product t-norm

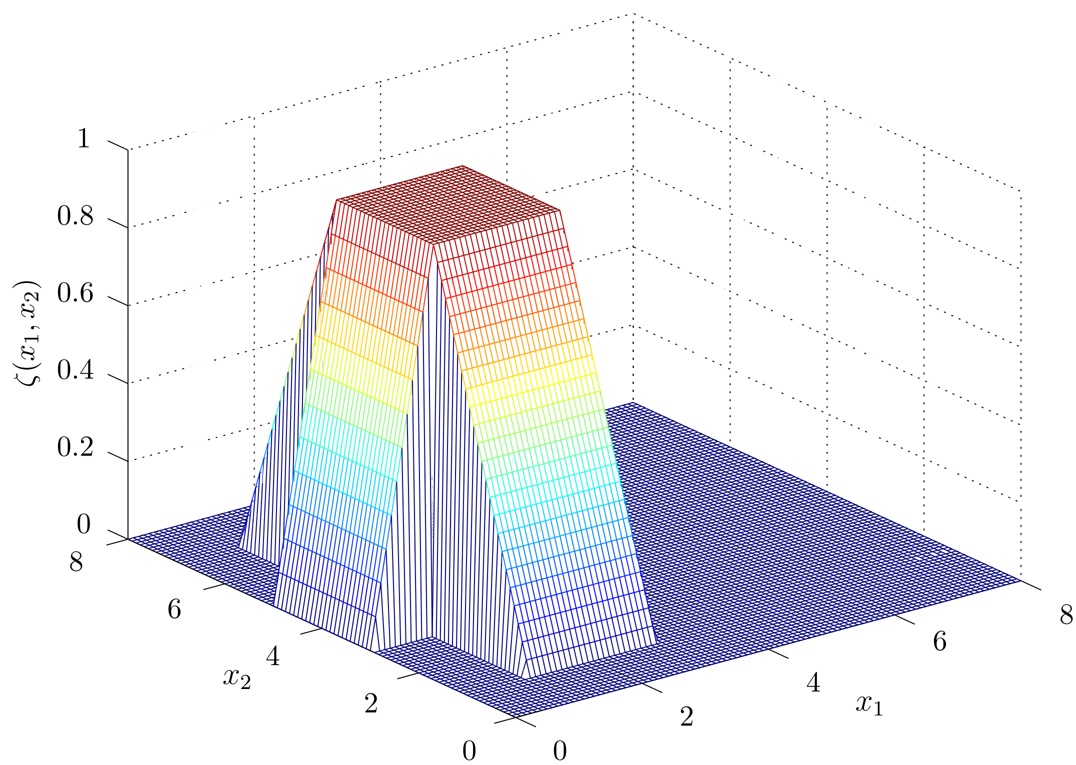


Figure 6.7: Characterizing function $\psi(\cdot)$ of the fuzzy number $x_1^* \oplus x_2^*$, where the fuzzy vector was constructed using different t-norms (— minimum, — product, — limited sum, and drastic product t-norm)

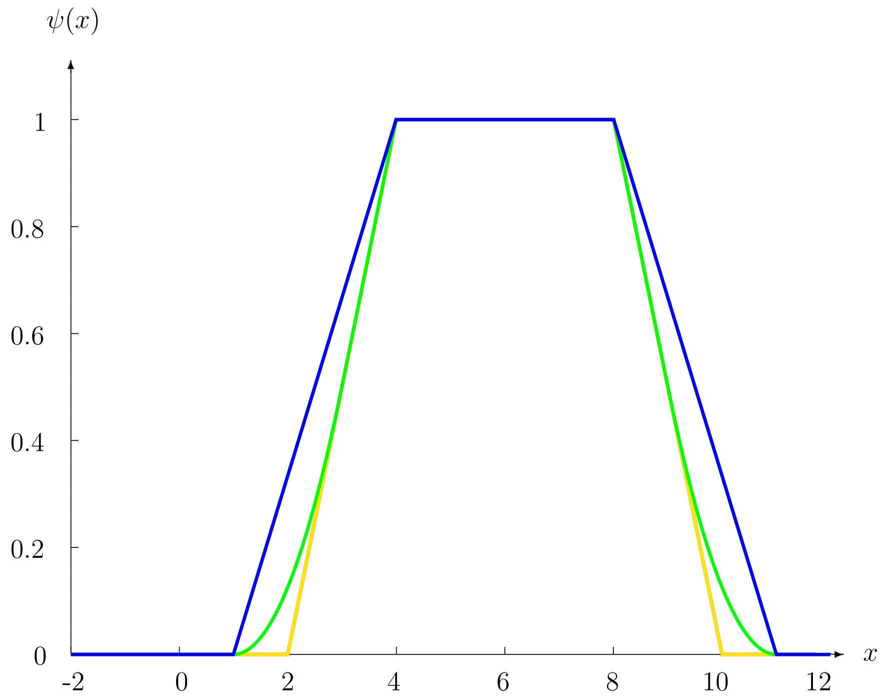


Figure 6.8: Characterizing function $\psi(\cdot)$ of the fuzzy number $x_1^* \ominus x_2^*$, where the fuzzy vector was constructed using different t-norms (— minimum, — product, — limited sum, and drastic product t-norm)

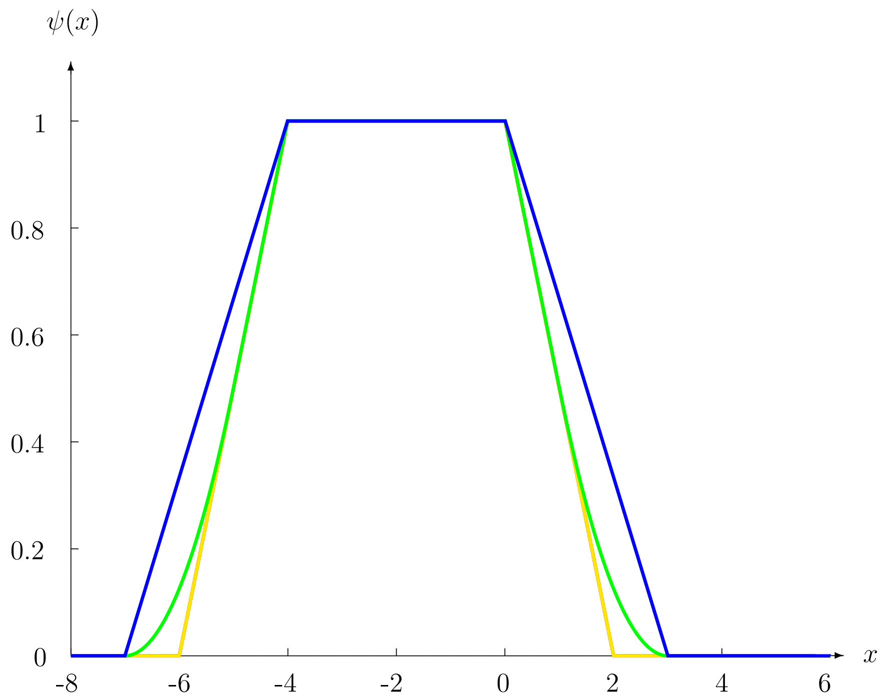


Figure 6.9: Characterizing function $\psi(\cdot)$ of the fuzzy number $x_1^* \odot x_2^*$, where the fuzzy vector was constructed using different t-norms (— minimum, — product, — limited sum, and — drastic product t-norm)

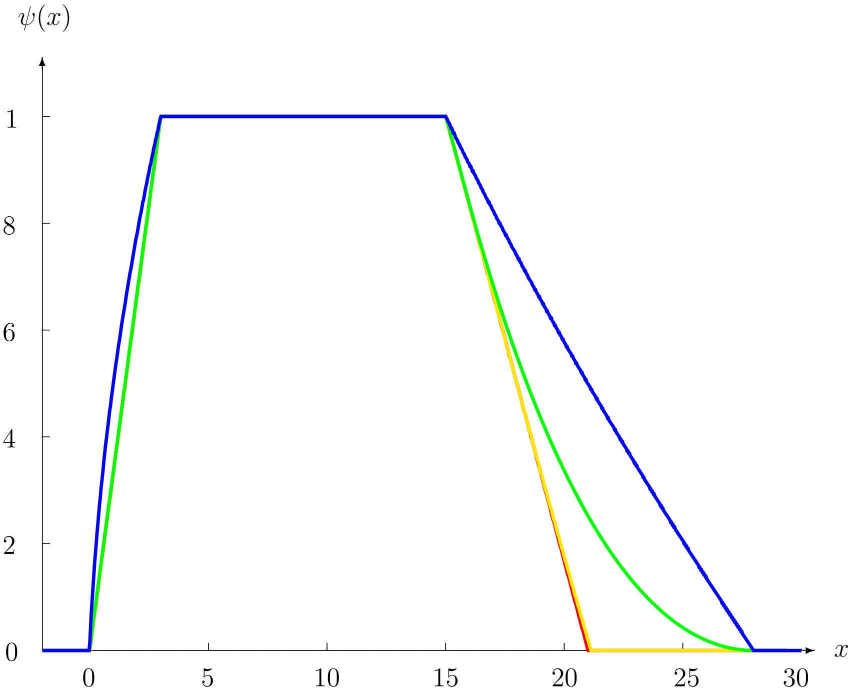


Figure 6.10: Characterizing function $\psi(\cdot)$ of the fuzzy number $x_1^* \oslash x_2^*$, where the fuzzy vector was constructed using different t-norms (— minimum, — product, — limited sum, and drastic product t-norm)

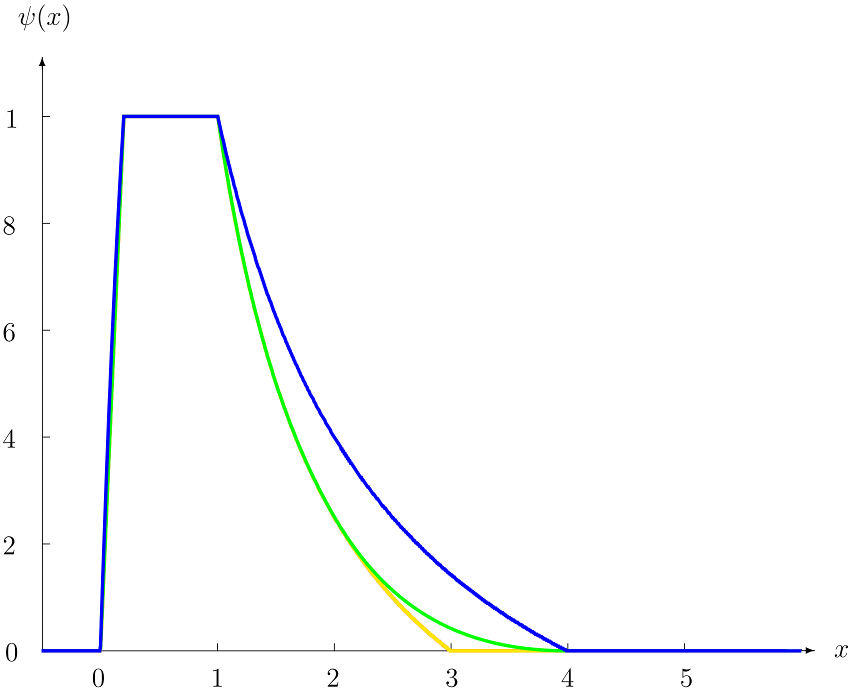


Figure 6.11: Construction of the fuzzy number $x_1^* \oplus x_2^*$

Vector-characterizing functions $\zeta(\cdot,\cdot)$ of the fuzzy vector $(x_1, x_2)^*$ constructed by using different t-norms with highlighted points for which the supremum selected by the extension principle during the computation of the fuzzy number $x_1^* \oplus x_2^*$ is obtained

— *points selected by the extension principle*

(a) minimum t-norm, (b) product t-norm, (c) limited sum t-norm, and (d) drastic product t-norm

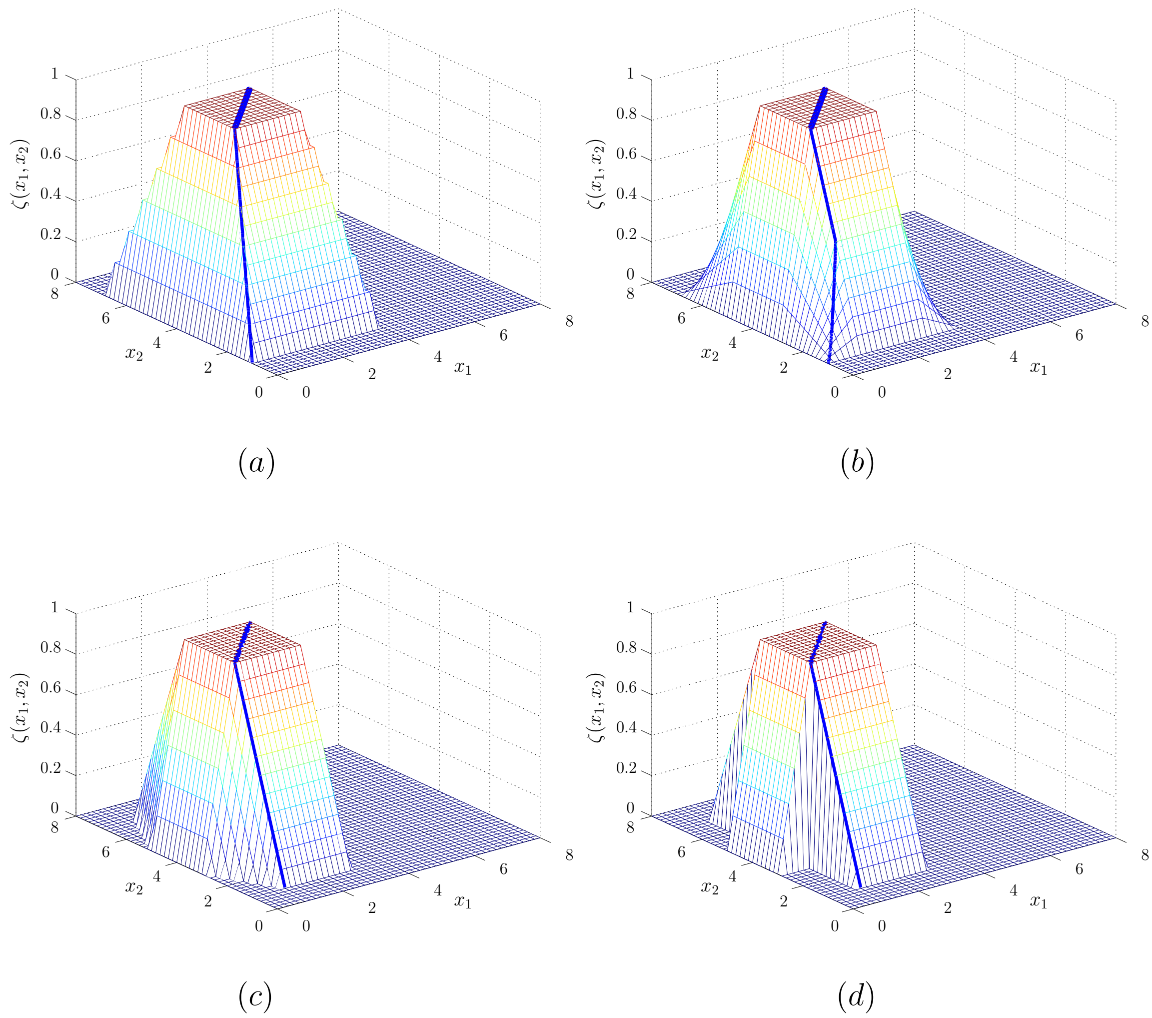


Figure 6.12: Construction of the fuzzy number $x_1^* \ominus x_2^*$

Vector-characterizing functions $\zeta(\cdot,\cdot)$ of the fuzzy vector $(x_1, x_2)^*$ constructed by using different t-norms with highlighted points for which the supremum selected by the extension principle during the computation of the fuzzy number $x_1^* \ominus x_2^*$ is obtained

— *points selected by the extension principle*

(a) minimum t-norm, (b) product t-norm, (c) limited sum t-norm, and (d) drastic product t-norm

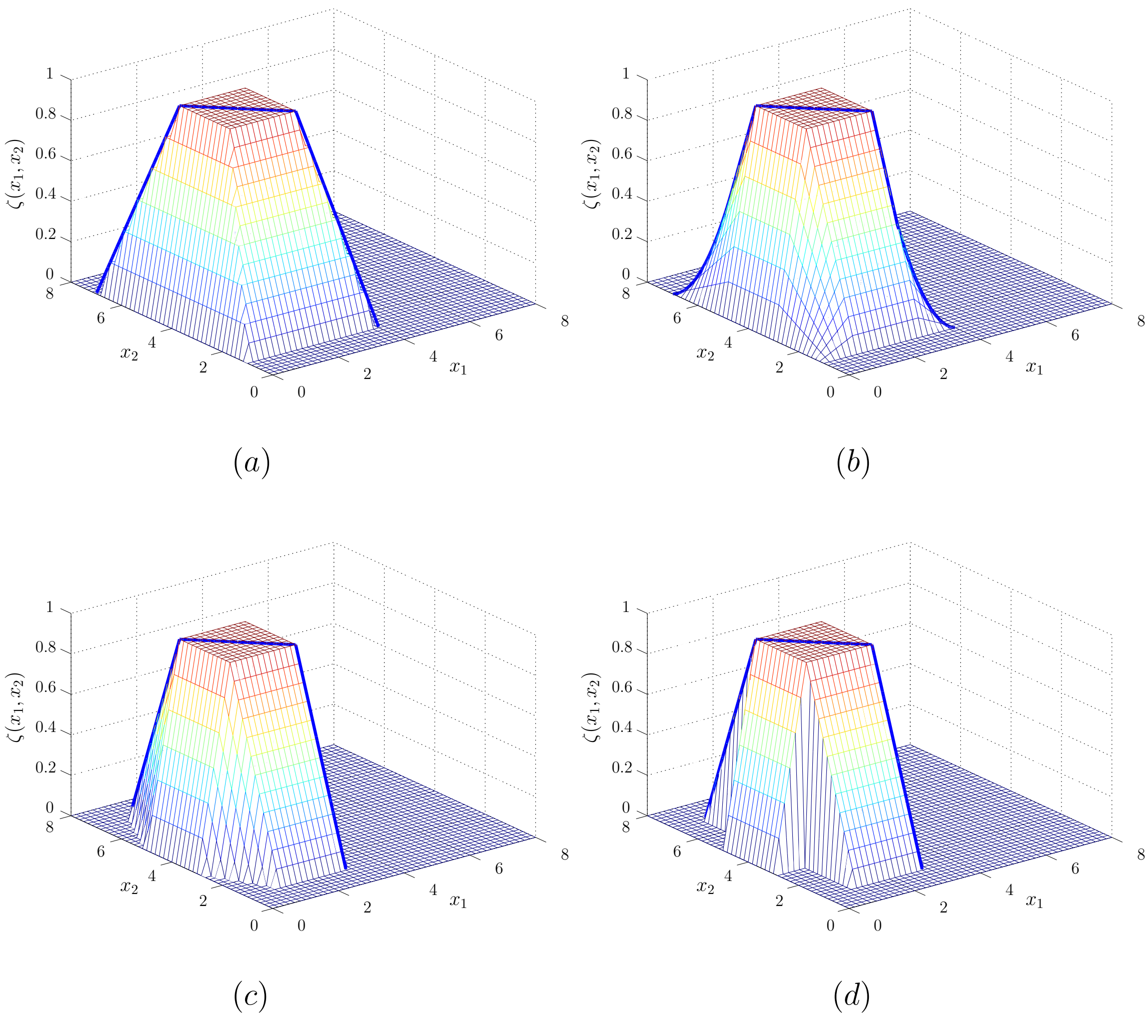


Figure 6.13: Construction of the fuzzy number $x_1^* \odot x_2^*$

Vector-characterizing functions $\zeta(\cdot,\cdot)$ of the fuzzy vector $(x_1, x_2)^*$ constructed by using different t-norms with highlighted points for which the supremum selected by the extension principle during the computation of the fuzzy number $x_1^* \odot x_2^*$ is obtained

— *points selected by the extension principle*

(a) minimum t-norm, (b) product t-norm, (c) limited sum t-norm, and (d) drastic product t-norm

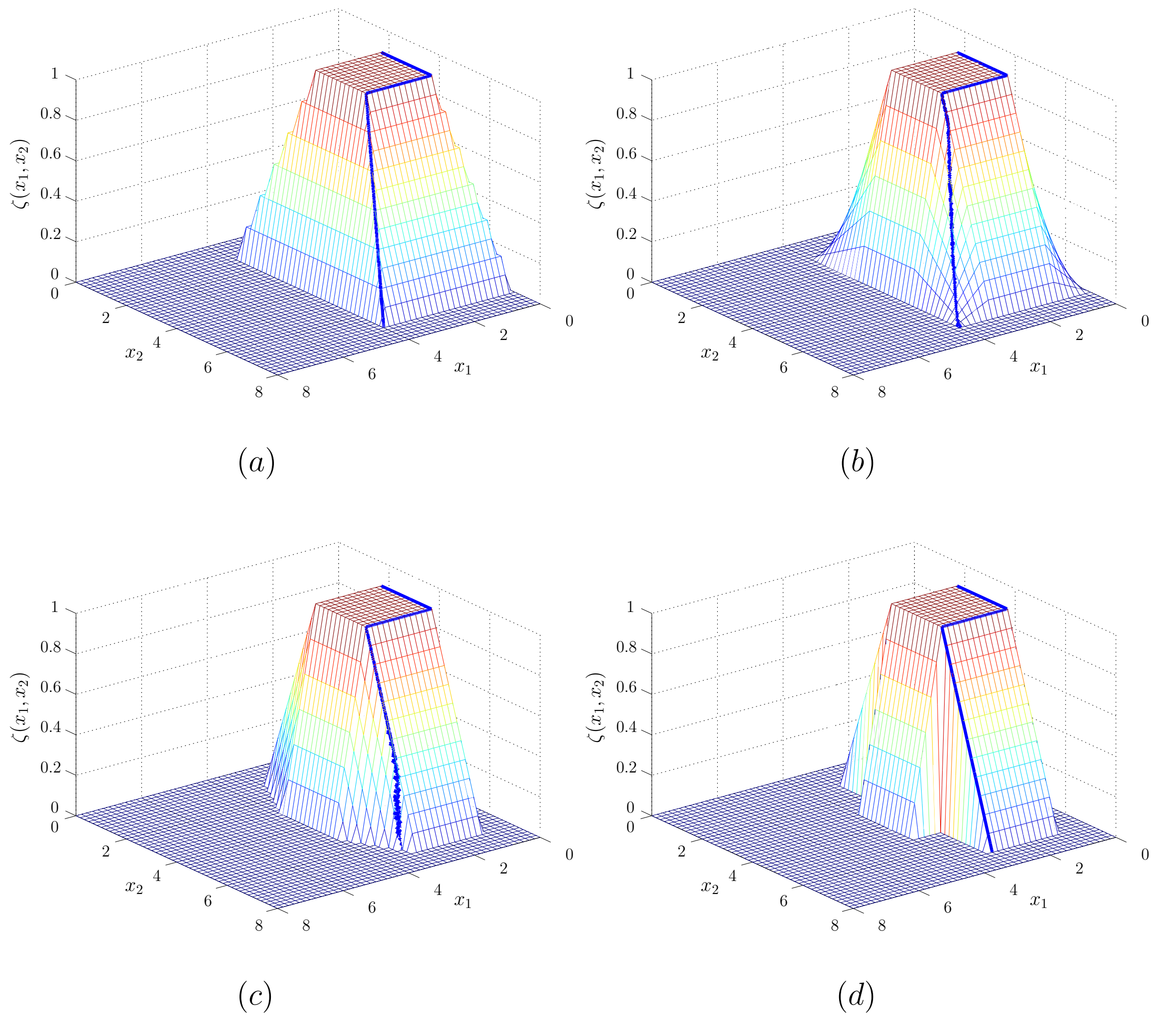
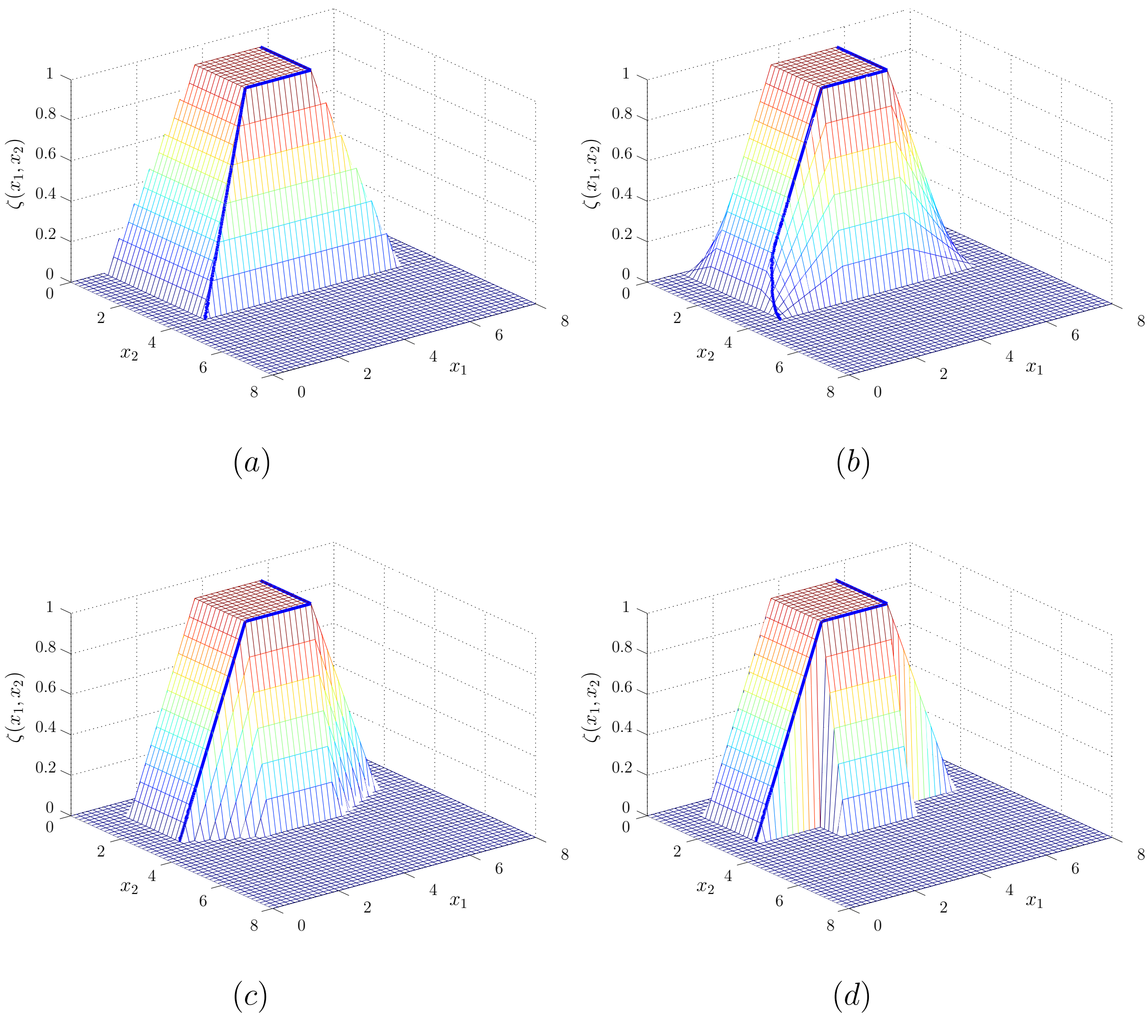


Figure 6.14: Construction of the fuzzy number $x_1^* \oslash x_2^*$

Vector-characterizing functions $\zeta(\cdot,\cdot)$ of the fuzzy vector $(x_1, x_2)^*$ constructed by using different t-norms with highlighted points for which the supremum selected by the extension principle during the computation of the fuzzy number $x_1^* \oslash x_2^*$ is obtained

— *points selected by the extension principle*

(a) minimum t-norm, (b) product t-norm, (c) limited sum t-norm, and (d) drastic product t-norm



6.2 Fuzzy Mean and Fuzzy Standard Deviation

In classical measurement analysis the mean value and the standard deviation of repeated measurements $x_1, \ldots, x_n$ are usually taken.

$$x_{\text{mean}} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$x_{\text{std.dev}} = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} \left[x_i - x_{\text{mean}}\right]^2}$$

For a vector of fuzzy numbers $(x_1^*, \ldots, x_n^*)$, where the fuzzy numbers $x_1^*, \ldots, x_n^*$ have corresponding characterizing functions $\xi_1(\cdot), \ldots, \xi_n(\cdot)$, this operation has to be generalized. This is possible based on the extension principle.

Let $x_1^*, \ldots, x_n^*$ be fuzzy numbers with corresponding characterizing functions $\xi_1(\cdot), \ldots, \xi_n(\cdot)$. Then the fuzzy mean value x_{mean}^* and the fuzzy standard deviation $x_{\text{std.dev}}^*$ have characterizing functions $\xi_{\text{mean}}(\cdot)$ and $\xi_{\text{std.dev}}(\cdot)$ respectively defined in the following way:

$$\xi_{\text{mean}}(y) = \left\{ \begin{array}{ll} \sup\left\{\zeta(\underline{x}_n) \colon \underline{x}_n \in g_{\text{mean}}^{-1}(\{y\})\right\} & \\ & \text{if } g_{\text{mean}}^{-1}(\{y\}) \neq \emptyset \\ & \\ 0 & \text{if } g_{\text{mean}}^{-1}(\{y\}) = \emptyset \end{array} \right\} \qquad \forall y \in \mathbb{R}$$

$$\xi_{\text{std.dev}}(y) = \left\{ \begin{array}{ll} \sup\left\{\zeta(\underline{x}_n) \colon \underline{x}_n \in g_{\text{std.dev}}^{-1}(\{y\})\right\} & \\ & \text{if } g_{\text{std.dev}}^{-1}(\{y\}) \neq \emptyset \\ & \\ 0 & \text{if } g_{\text{std.dev}}^{-1}(\{y\}) = \emptyset \end{array} \right\} \qquad \forall y \in \mathbb{R}$$

where $\underline{x}_n = (x_1, \ldots, x_n)$ is a vector of real numbers, and the vector-characterizing function $\zeta(\cdot, \ldots, \cdot) : \mathbb{R}^n \to [0, 1]$ is constructed by using a t-norm, usually the minimum t-norm:

$$\zeta(x_1, \ldots, x_n) = \min\{\xi_1(x_1), \ldots, \xi_n(x_n)\} \qquad \forall (x_1, \ldots, x_n) \in \mathbb{R}^n$$

The well known functions for computing the mean value $g_{\text{mean}}(\cdot, \ldots, \cdot) \colon \mathbb{R}^n \to \mathbb{R}$ and computing the standard deviation $g_{\text{st.dev}}(\cdot, \ldots, \cdot) \colon \mathbb{R}^n \to \mathbb{R}$ are defined in the following way:

$$g_{\text{mean}}(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i \qquad \forall (x_1, \ldots, x_n) \in \mathbb{R}^n$$

$$g_{\text{std.dev}}(x_1, \ldots, x_n) = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} \left[x_i - \left(\tfrac{1}{n} \textstyle\sum_{j=1}^{n} x_j\right)\right]^2} \qquad \forall (x_1, \ldots, x_n) \in \mathbb{R}^n$$

Example 6.3. The fuzzy mean and fuzzy standard deviation for fuzzy numbers x_1^* and x_2^* given in Example 6.2 was computed using different types of t-norms. Selected points used by the extension principle are depicted in Figure 6.17 and Figure 6.18. The resulting characterizing functions of fuzzy mean and fuzzy standard deviation are given in Figure 6.15 and Figure 6.16 respectively.

Figure 6.15: Characterizing function $\psi(\cdot)$ of the fuzzy mean of (x_1^*, x_2^*), where the fuzzy vector was constructed using different t-norms (— minimum, — product, — limited sum, and drastic product t-norm)

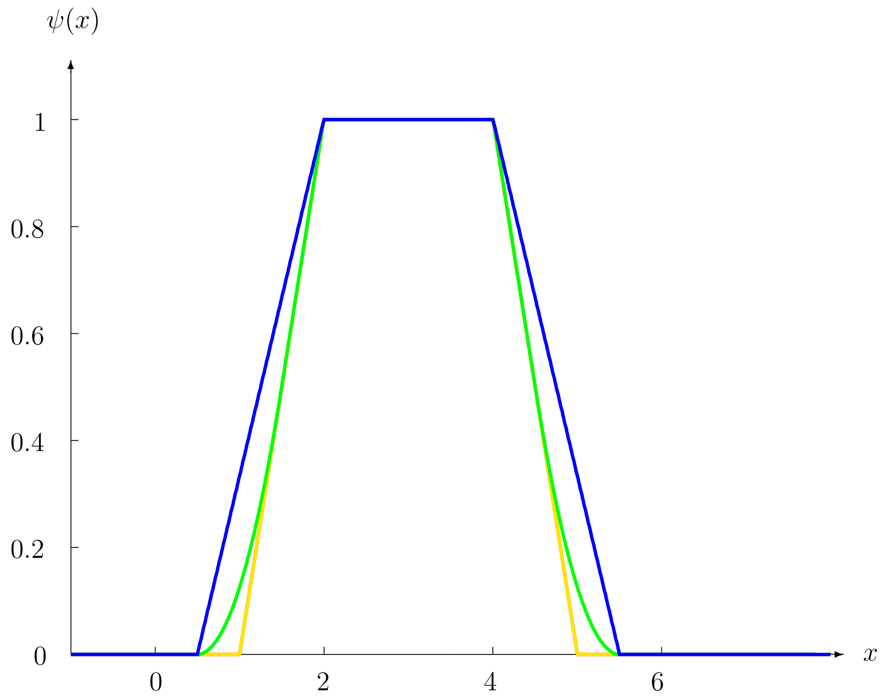


Figure 6.16: Characterizing function $\psi(\cdot)$ of the fuzzy std. deviation of (x_1^*, x_2^*), where the fuzzy vector was constructed using the different t-norms (— minimum, — product, — limited sum, and drastic product t-norm)

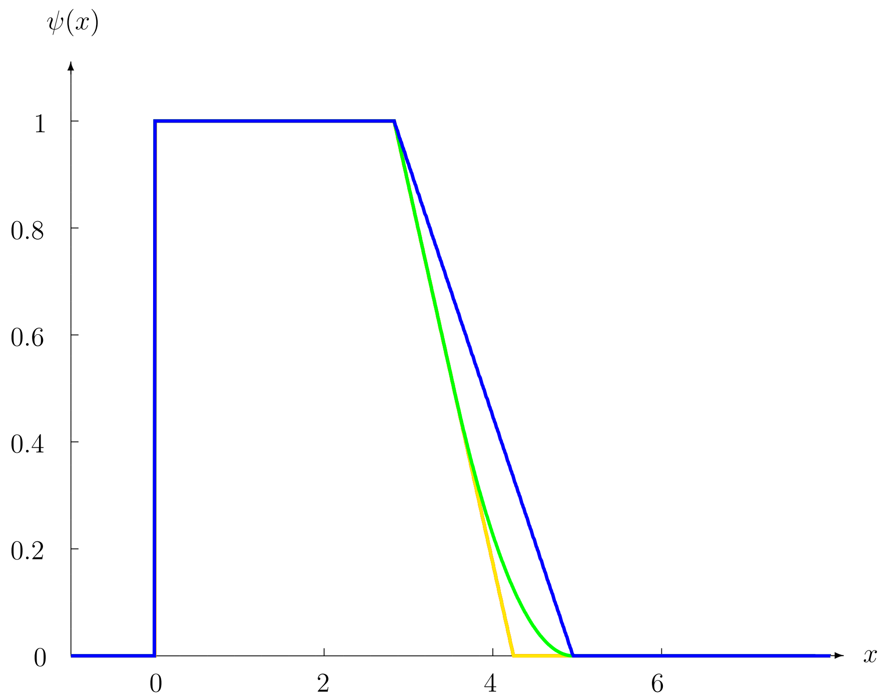
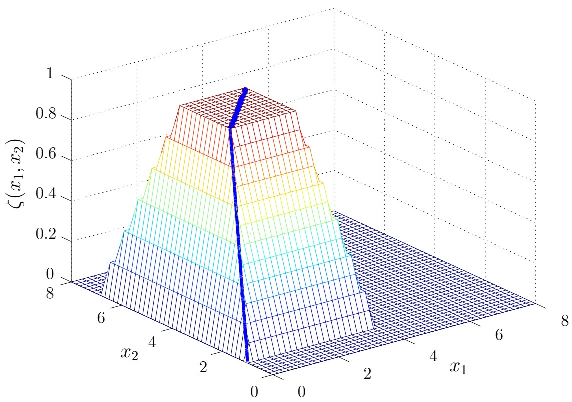


Figure 6.17: Construction of the fuzzy mean of fuzzy numbers x_1^* and x_2^*

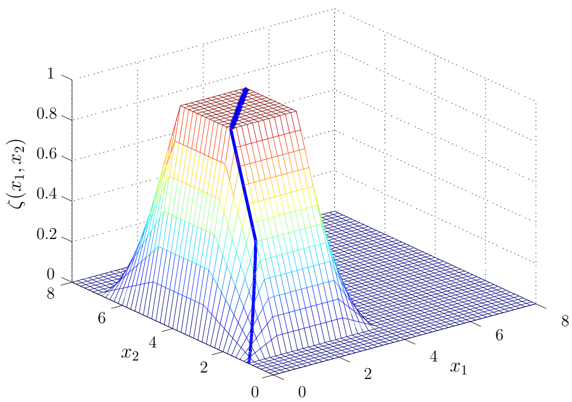
Vector-characterizing functions $\zeta(\cdot,\cdot)$ of the fuzzy vector $(x_1, x_2)^*$ constructed by using different t-norms with highlighted points for which the supremum selected by the extension principle during the computation of the fuzzy mean of numbers x_1^* and x_2^* is obtained

— *points selected by the extension principle*

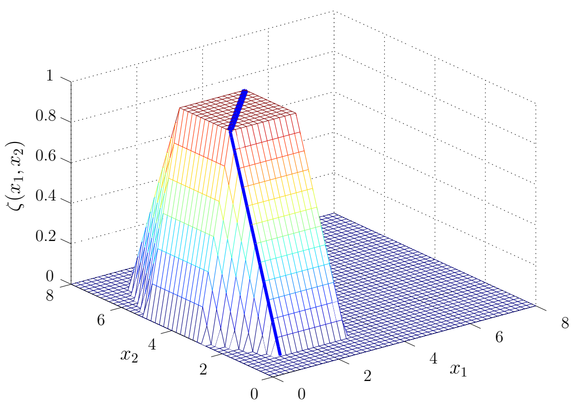
(a) minimum t-norm, (b) product t-norm, (c) limited sum t-norm, and (d) drastic product t-norm



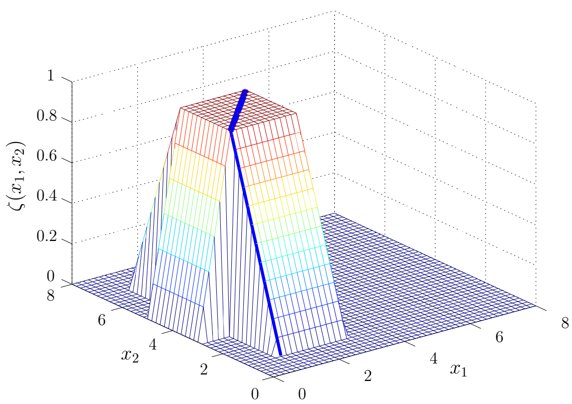
(a)



(b)



(c)



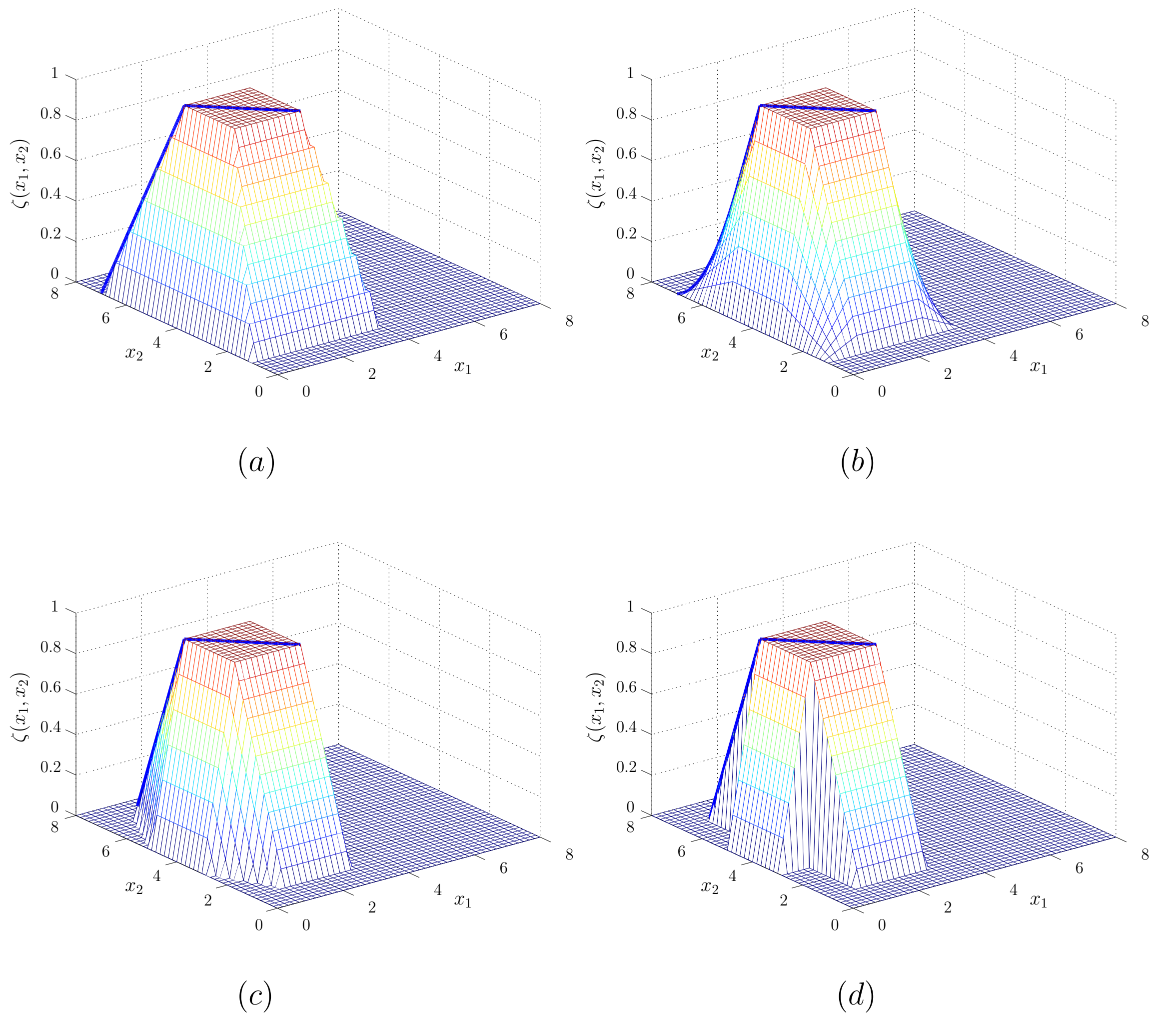
(d)

Figure 6.18: Construction of the fuzzy std. deviation of fuzzy numbers x_1^* and x_2^*

Vector-characterizing functions $\zeta(\cdot,\cdot)$ of the fuzzy vector $(x_1, x_2)^*$ constructed by using different t-norms with highlighted points for which the supremum selected by the extension principle during the computation of the fuzzy standard deviation of fuzzy numbers x_1^* and x_2^* is obtained

— *points selected by the extension principle*

(a) minimum t-norm, (b) product t-norm, (c) limited sum t-norm, and (d) drastic product t-norm



7. Fuzzy Valued Functions

Definition 7.1. *A fuzzy valued function* $f^*(\cdot)$ is a function whose range of values are fuzzy intervals, more precisely $f^* : M \to \mathcal{F}_{\mathcal{I}}(\mathbb{R})$.

Definition 7.2. Let $f^*(\cdot)$ be a fuzzy valued function with domain M and let $x \in M$. Then $f^*(x)$ is a fuzzy interval and we denote its δ-cuts by

$$\mathcal{C}_\delta(f^*(x)) = \left[\underline{f}_\delta(x), \overline{f}_\delta(x)\right] \quad \forall \delta \in (0,1],$$

where $\underline{f}_\delta(\cdot), \overline{f}_\delta(\cdot) : M \to \mathbb{R}$ are real valued functions called *δ-level functions*. In the case $M = \mathbb{R}$ the δ-level functions are called *δ-level curves*.

Example 7.1. An example of a fuzzy valued function $f^*(\cdot) : [-2,2] \to \mathcal{F}_{\mathcal{I}}(\mathbb{R})$ is given in Figure 7.4. The δ-level curves $\underline{f}_\delta(\cdot), \overline{f}_\delta(\cdot) \colon [-2,2] \to \mathbb{R}$ are the following:

$$\underline{f}_\delta(x) = \delta \cdot \left(x^4 - 8x^2 + 16\right), \qquad \overline{f}_\delta(x) = \left(\tfrac{5}{2} - \delta\right) \cdot \left(x^4 - 8x^2 + 16\right)$$

Some of the δ-level curves are given in Figure 7.2. The values of a fuzzy valued functions are fuzzy intervals, values $f^*(-1)$, $f^*(0)$, $f^*(1)$ are shown in Figure 7.3.

Figure 7.1: Fuzzy valued function $f^*(\cdot)$ from Example 7.1

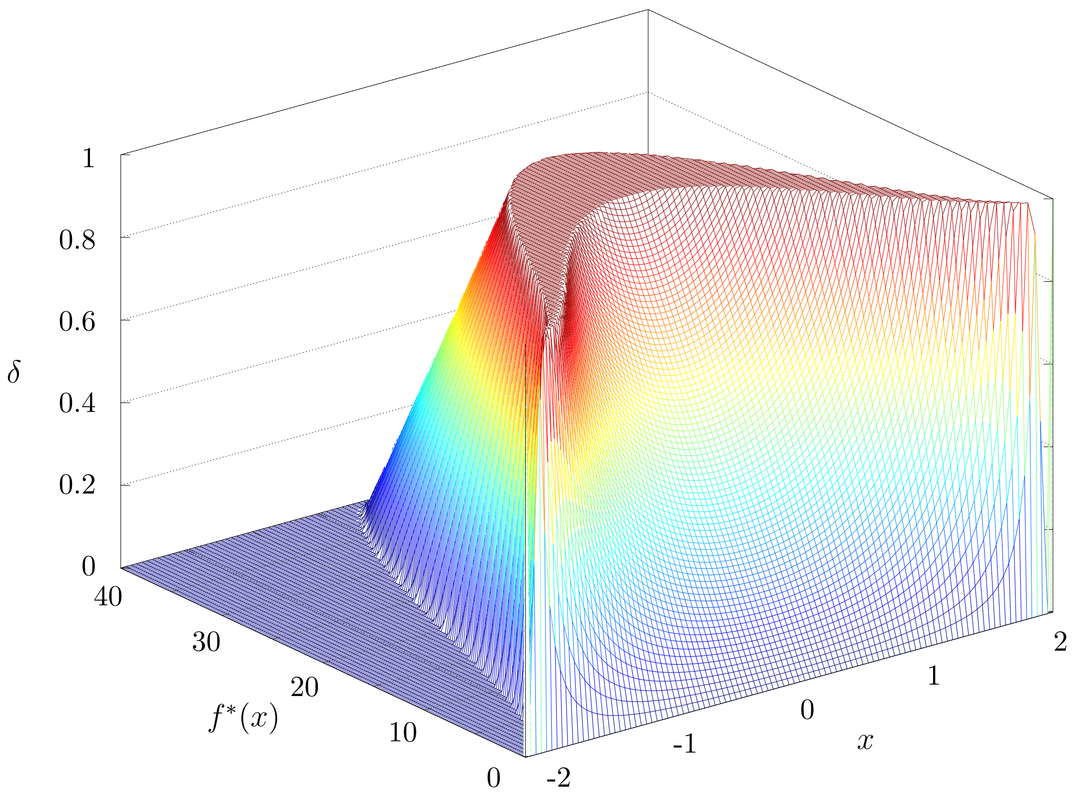


Figure 7.2: δ-level curves of the fuzzy valued function $f^*(\cdot)$ from Example 7.1 for $\delta \in \{ — \frac{1}{5}, — \frac{2}{5}, — \frac{3}{5}, — \frac{4}{5}, — 1\}$

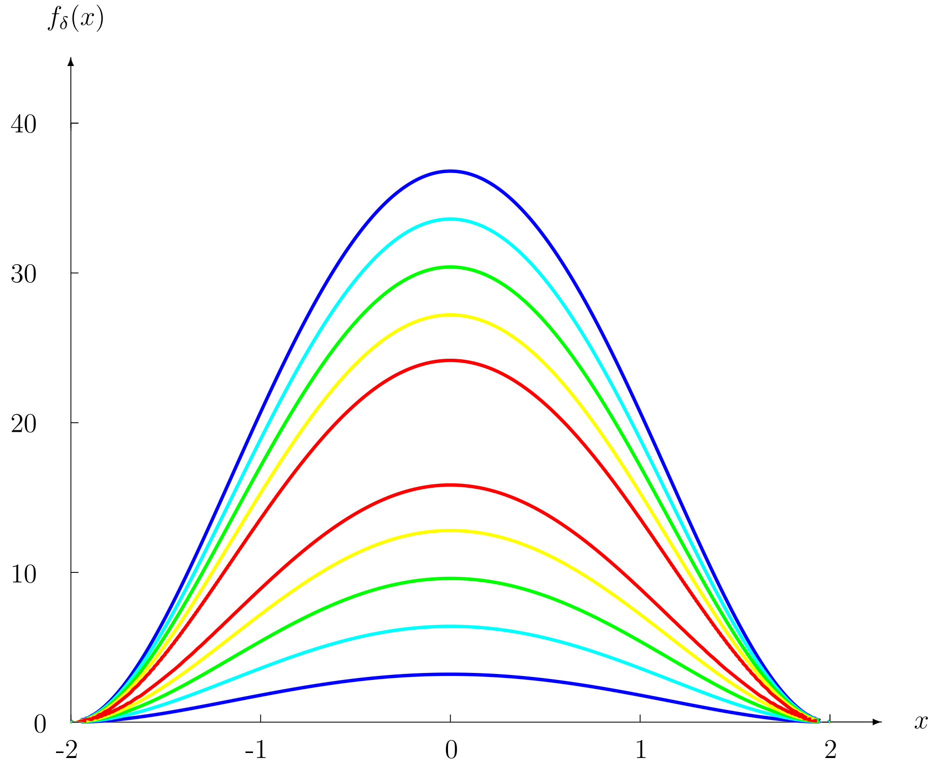
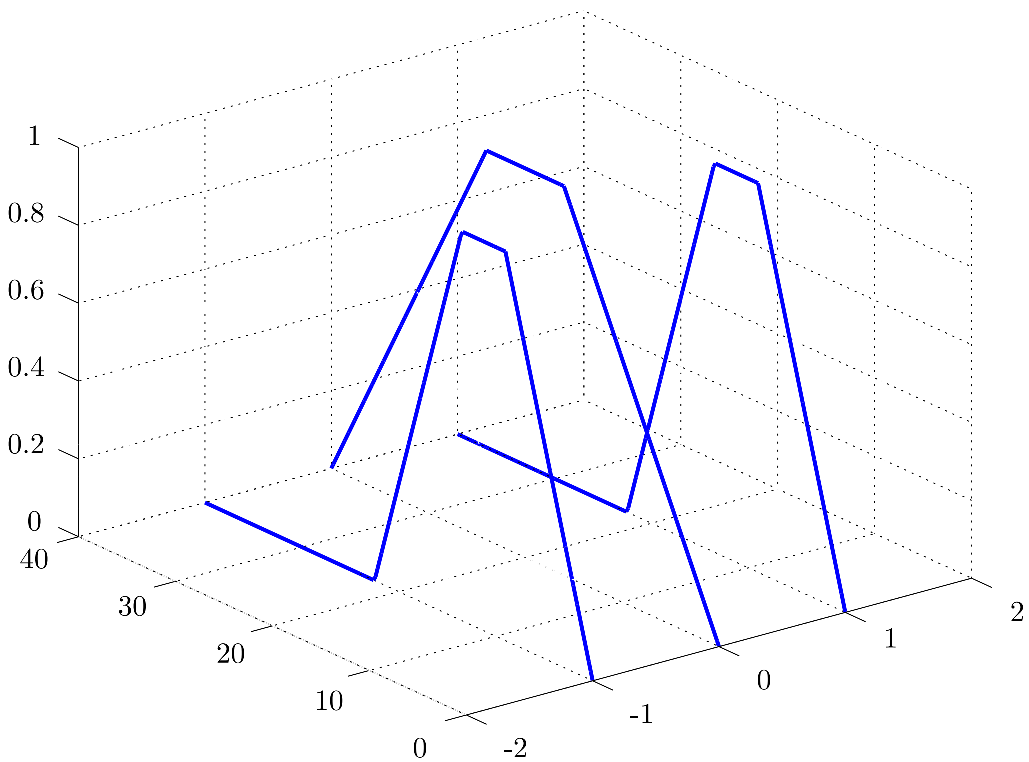


Figure 7.3: Characterizing functions of the fuzzy intervals $f^*(-1)$, $f^*(0)$, and $f^*(1)$ of the fuzzy valued function $f^*(\cdot)$ from Example 7.1



7.1 Integral of Fuzzy Valued Functions

Definition 7.3. Let $f^*(\cdot)$ be a fuzzy valued function defined on a measure space $(M, \mathcal{A}, \mu)$ and let all δ-level functions $\underline{f}_\delta(\cdot)$ and $\overline{f}_\delta(\cdot)$ be integrable with finite integrals:

$$-\infty < \int_M \underline{f}_\delta(x)d\mu(x) \leq \int_M \overline{f}_\delta(x)d\mu(x) < \infty$$

Then the *fuzzy integral* $\displaystyle\int_M f^*(x)d\mu(x)$ is the fuzzy number I^* whose generating family $(A_\delta; \delta \in (0,1])$ is defined by:

$$A_\delta(I^*) = \left[\underline{I}_\delta, \overline{I}_\delta\right] := \left[\int_M \underline{f}_\delta(x)d\mu(x), \int_M \overline{f}_\delta(x)d\mu(x)\right] \quad \forall \delta \in (0,1]$$

Remark. By the Generation Lemma 2.2 the characterizing function of the fuzzy integral I^* is given by:

$$\xi_{I^*}(x) := \sup\left\{\delta \cdot \mathbb{1}_{[\underline{I}_\delta, \overline{I}_\delta]}(x) : \delta \in [0,1]\right\} \quad \forall x \in \mathbb{R}$$

Theorem 7.1. *Let $f^* : M \to \mathcal{F}_\mathcal{I}(\mathbb{R})$ be a fuzzy valued function with non-negative integrable δ-level functions $\underline{f}_\delta(x), \overline{f}_\delta(x) : M \to \mathbb{R}\ \forall\, \delta \in (0,1]$ with finite integrals. We denote by I^* the fuzzy integral*

$$I^* := \int_M f^*(x)d\mu(x),$$

we denote by $(A_\delta; \delta \in (0,1])$ the family of intervals generating the fuzzy number I^*

$$A_\delta := \left[\int_M \underline{f}_\delta(x)d\mu(x), \int_M \overline{f}_\delta(x)d\mu(x)\right] \qquad \forall\, \delta \in (0,1],$$

and we denote by $\xi_{I^*}(\cdot)$ the characterizing function of I^*, defined by

$$\xi_{I^*}(x) := \sup\left\{\delta \cdot \mathbb{1}_{A_\delta}(x) : \delta \in (0,1]\right\} \quad \forall x \in \mathbb{R}.$$

Then for all $\delta \in (0,1]$ the following is equivalent:

- $\mathcal{C}_\delta(I^*) = A_\delta$
- $\lim_{\beta \uparrow \delta} \underline{f}_\beta(x) = \underline{f}_\delta(x)$ *and* $\lim_{\beta \uparrow \delta} \overline{f}_\beta(x) = \overline{f}_\delta(x)$ *almost everywhere on* M

Proof. Choose arbitrary $\delta \in (0,1]$. From Theorem 2.2 we know that:

$$\mathcal{C}_\delta(I^*) = A_\delta \qquad \Leftrightarrow \qquad \bigcap_{\beta < \delta} A_\beta = A_\delta$$

We can formulate the following sequence of equivalences:

$$\bigcap_{\beta < \delta} A_\beta = A_\delta \quad \Leftrightarrow$$

$$\Leftrightarrow \bigcap_{\beta < \delta} \left[\int_M \underline{f}_\beta(x)d\mu(x), \int_M \overline{f}_\beta(x)d\mu(x)\right] = \left[\int_M \underline{f}_\delta(x)d\mu(x), \int_M \overline{f}_\delta(x)d\mu(x)\right]$$

$$\Leftrightarrow \sup_{\beta<\delta} \int_M \underline{f}_\beta(x)d\mu(x) = \int_M \underline{f}_\delta(x)d\mu(x) \ \& \ \inf_{\beta<\delta} \int_M \overline{f}_\beta(x)d\mu(x) = \int_M \overline{f}_\delta(x)d\mu(x) \quad (*)$$

We denote by $\underline{g}(\cdot)$ and $\overline{g}(\cdot)$ real valued functions $\underline{g}(\cdot) : (0,1] \to \mathbb{R}$ and $\overline{g}(\cdot)\colon (0,1] \to \mathbb{R}$ defined by:

$$\underline{g}(\alpha) := \int_M \underline{f}_\alpha(x)d\mu(x) \quad \text{and} \quad \overline{g}(\alpha) := \int_M \overline{f}_\alpha(x)d\mu(x) \quad \forall \alpha \in (0,1] \qquad (\triangle)$$

We can reformulate the statement $(*)$ as:

$$\sup_{\beta<\delta} \underline{g}(\beta) = \underline{g}(\delta) \quad \text{and} \quad \inf_{\beta<\delta} \overline{g}(\beta) = \overline{g}(\delta) \qquad (**)$$

The conditions $(**)$ are fulfilled for $\underline{g}(\cdot)$ non-decreasing on $(0,1]$ and continuous from left at the point δ and $\overline{g}(\cdot)$ non-increasing on $(0,1]$ and continuous from left at the point δ.

Let $h : (0,1] \to \mathbb{R}$ be a function which is non-decreasing on $(0,1]$ and continuous from left at the point δ. We define a value s and set A as following:

$$s := h(\delta) \qquad A := \{h(\beta) : \beta \in (0,\delta)\}$$

Now we have to check by verifying the conditions of supremum that the value s is a supremum of the set A, with $\sup A = \sup_{\beta<\delta} h(\beta)$.

- $\forall x \in A \quad x \leq s$:
  We can find $\beta \in (0,\delta)$ such that $x = h(\beta)$ for all $x \in A$, since the function $h(\cdot)$ is non-decreasing and $\beta < \delta$, we have $x = h(\beta) \leq h(\delta) = s$.
- $\forall s' \in \mathbb{R}, s' < s \quad \exists x \in A \quad x > s'$
  We can reformulate this condition in the following way:

  $$\forall \varepsilon > 0 \quad \exists x \in A\colon \quad x > s - \varepsilon$$

  $$\forall \varepsilon > 0 \quad \exists \beta \in (0,\delta)\colon \quad h(\beta) > h(\delta) - \varepsilon$$

  We suppose, that $h(\cdot)$ is continuous from left at the point δ, which means:

  $$\forall \varepsilon > 0 \quad \exists \gamma > 0 \quad \forall \beta \in (\delta - \gamma, \delta] \quad h(\beta) \in (h(\delta) - \varepsilon, h(\delta) + \varepsilon)$$

  When adding the first condition of supremum, we get:

  $$\forall \varepsilon > 0 \quad \exists \gamma > 0 \quad \forall \beta \in (\delta - \gamma, \delta] \quad h(\beta) \in (h(\delta) - \varepsilon, h(\delta)]$$

We have proved that $\sup_{\beta<\delta} h(\beta) = h(\delta)$ for arbitrary function $h(\cdot)$ non-decreasing on $(0,1]$ and continuous from left at the point δ. Now we will investigate if the function $\underline{g}(\cdot)$ constructed according the $(\triangle)$ statement is non-decreasing on $(0,1]$ and continuous from left at the point δ.

Firstly we will prove that $\underline{g}(\cdot)$ is non-decreasing on $(0,1]$ and $\overline{g}(\cdot)$ is non-increasing on $(0,1]$ (for arbitrary fuzzy valued function f^*). This results from the structure of δ-level functions and δ-cuts, and monotonicity of Lebesgue integral.

We choose arbitrary $\alpha, \beta \in (0,1],\ \alpha < \beta$. Then we have:

$$[\underline{f}_\alpha(x), \overline{f}_\alpha(x)] = \mathcal{C}_\alpha(f^*(x)) \supseteq \mathcal{C}_\beta(f^*(x)) = [\underline{f}_\beta(x), \overline{f}_\beta(x)] \quad \forall x \in \mathbb{R}$$

$$\underline{f}_\alpha(x) \leq \underline{f}_\beta(x) \quad \text{and} \quad \overline{f}_\alpha(x) \geq \overline{f}_\beta(x) \quad \forall x \in \mathbb{R}$$

The δ-level functions are non-negative integrable functions, so we can use the monotonicity of Lebesgue integral A.7:

$$\underline{g}(\alpha) = \int_M \underline{f}_\alpha(x) d\mu(x) \leq \int_M \underline{f}_\beta(x) d\mu(x) = \underline{g}(\beta) \quad \text{and}$$

$$\overline{g}(\alpha) = \int_M \overline{f}_\alpha(x) d\mu(x) \geq \int_M \overline{f}_\beta(x) d\mu(x) = \overline{g}(\beta)$$

Secondly we will investigate the conditions which are required for the function $\underline{g}(\cdot)$ to be continuous from left at the point δ. From the definition of the function $\underline{g}(\cdot)$ we know the following:

$$\lim_{\beta \uparrow \delta} \int_M \underline{f}_\beta(x) d\mu(x) = \lim_{\beta \uparrow \delta} \underline{g}(\beta) \quad \text{and} \quad \underline{g}(\delta) = \int_M \underline{f}_\delta(x) d\mu(x)$$

According to the definition of continuous functions and Lemma A.1, the function $\underline{g}(\cdot)$ is continuous from left at the point δ if and only if $\lim_{\beta \uparrow \delta} \underline{g}(\beta) = \underline{g}(\delta)$, which can be formulated as:

$$\lim_{\beta \uparrow \delta} \int_M \underline{f}_\beta(x) d\mu(x) = \lim_{\beta \uparrow \delta} \underline{g}(\beta) = \underline{g}(\delta) = \int_M \underline{f}_\delta(x) d\mu(x) \qquad (\triangle\triangle)$$

Now we have to prove this equation.

For all $x \in M$ the function $\underline{f}_{(\cdot)}(x)$ is non-decreasing and bounded as a function of the parameter in the subscript index. According to Theorem A.4 there exists a pointwise limit of $\underline{f}_{(\cdot)}(x)$. We denote by $\underline{\tilde{f}}$ the function of pointwise limits for β going to δ from the left:

$$\underline{\tilde{f}}(x) := \lim_{\beta \uparrow \delta} \underline{f}_\beta(x) \quad \Big(\leq \underline{f}_\delta(x)\Big) \qquad \forall x \in M$$

Let $(a_n)_{n \in \mathbb{N}} \subseteq (0, \delta)$ be an arbitrary monotone sequence converging to δ, then $(\underline{f}_{a_n})_{n \in \mathbb{N}}$ is a non-decreasing sequence of non-negative measurable functions:

$$\underline{f}_{a_n}(x) \leq \underline{f}_{a_{n+1}}(x) \qquad \forall n \in \mathbb{N} \quad \forall x \in M$$

The pointwise limit of the sequence $(\underline{f}_{a_n})_{n \in \mathbb{N}}$ is the function $\underline{\tilde{f}}$ according to the Heine theorem A.3. We have fulfilled the prerequisites of the Lebesgue monotone convergence Theorem A.8 so we can use this theorem and get the following statement:

$$\lim_n \int_M \underline{f}_{a_n}(x) d\mu(x) = \int_M \underline{\tilde{f}}(x) d\mu(x)$$

According to Heine Theorem A.3 we can generalize this result as follows:

$$\lim_{\beta \uparrow \delta} \int_M \underline{f}_\beta(x) d\mu(x) = \int_M \underline{\tilde{f}}(x) d\mu(x)$$

We know that $\tilde{\underline{f}}(x) \leq \underline{f}_\delta(x)$ for all $x \in M$ from the construction of $\tilde{\underline{f}}$. From the monotonicity of Lebesgue integral A.7 we get:

$$\lim_{\beta \uparrow \delta} \int_M \underline{f}_\beta(x) d\mu(x) = \int_M \tilde{\underline{f}}(x) d\mu(x) \leq \int_M \underline{f}_\delta(x) d\mu(x)$$

We will investigate and denote by $\underline{N}$ the following set:

$$\underline{N} := \{x \in M : \tilde{\underline{f}}(x) < \underline{f}_\delta(x)\} = \{x \in M : \lim_{\beta \uparrow \delta} \underline{f}_\beta(x) < \underline{f}_\delta(x)\}$$

According to the linearity of Lebesgue integral A.6 and the fact $0 \leq \tilde{\underline{f}}(x) \leq \underline{f}_\delta(x)$, the following is fulfilled:

$$\lim_{\beta \uparrow \delta} \int_M \underline{f}_\beta(x) d\mu(x) = \int_{M \setminus \underline{N}} \underline{f}_\delta(x) d\mu(x) + \int_{\underline{N} \cap \mathrm{supp}(\underline{f}_\delta)} \tilde{\underline{f}}(x) d\mu(x) + \int_{\underline{N} \setminus \mathrm{supp}(\underline{f}_\delta)} 0 \, d\mu(x)$$

$$\int_M \underline{f}_\delta(x) d\mu(x) = \int_{M \setminus \underline{N}} \underline{f}_\delta(x) d\mu(x) + \int_{\underline{N} \cap \mathrm{supp}(\underline{f}_\delta)} \underline{f}_\delta(x) d\mu(x) + \int_{\underline{N} \setminus \mathrm{supp}(\underline{f}_\delta)} 0 \, d\mu(x)$$

These equations differ only on the set $\underline{N} \cap \mathrm{supp}(\underline{f}_\delta)$.

The set $\underline{N} \setminus \mathrm{supp}(\underline{f}_\delta)$ is empty by the fact that $0 \leq \tilde{\underline{f}}(x) \leq \underline{f}_\delta(x)$, from the construction of the set $\underline{N}$, where $\tilde{\underline{f}}(x) < \underline{f}_\delta(x)$, and from the complement of the support $\mathrm{supp}(\underline{f}_\delta)$, where $\underline{f}_\delta(x) = 0$, hence $0 \leq \tilde{\underline{f}}(x) < \underline{f}_\delta(x) = 0$ and therefore:

$$\underline{N} = \underline{N} \cap \mathrm{supp}(\underline{f}_\delta)$$

If the measure of the set $\underline{N}$ will be zero, $\mu(\underline{N}) = 0$, then according to Lemma A.5 the following is fulfilled:

$$\lim_{\beta \uparrow \delta} \int_M \underline{f}_\beta(x) d\mu(x) = \int_M \underline{f}_\delta(x) d\mu(x)$$

and according to the $(\triangle\triangle)$ statement the function $\underline{g}(\cdot)$ is continuous from left at the point δ.

Otherwise, for $\mu(\underline{N}) > 0$, from the similar reason the following is fulfilled:

$$\lim_{\beta \uparrow \delta} \int_M \underline{f}_\beta(x) d\mu(x) < \int_M \underline{f}_\delta(x) d\mu(x)$$

According to the $(\triangle\triangle)$ statement the function $\underline{g}(\cdot)$ is not continuous from left at the point δ.

We have just proved that the first part of statement $(**)$ is equivalent with the fact that the measure of the set $\underline{N}$ is zero:

$$\sup_{\beta < \delta} \underline{g}(\beta) = \underline{g}(\delta) \qquad \Leftrightarrow \qquad \mu(\underline{N}) = 0$$

The fact that the measure of the set $\underline{N}$ is zero can be also formulated as follows:

$$\lim_{\beta \uparrow \delta} \underline{f}_\beta(x) = \underline{f}_\delta(x) \text{ almost everywhere on } M$$

The last part of the proof is to investigate the conditions which are required for the function $\overline{g}(\cdot)$ to be continuous from left at the point δ. We denote by $\tilde{\overline{f}}$ the function of pointwise limits for β going to δ from the left:

$$\tilde{\overline{f}}(x) := \lim_{\beta \uparrow \delta} \overline{f}_\beta(x) \quad \left(\geq \overline{f}_\delta(x)\right) \qquad \forall x \in M$$

Let $(a_n)_{n\in\mathbb{N}} \subseteq (0,\delta)$ be a monotone increasing sequence converging to δ. Then $(\overline{f}_{a_n})_{n\in\mathbb{N}}$ is a non-increasing sequence of measurable functions less than 1 with pointwise limit equal to $\tilde{\overline{f}}$, and $(1-\overline{f}_{a_n})_{n\in\mathbb{N}}$ is a non-decreasing sequence of non-negative measurable functions with pointwise limit equal to $1-\tilde{\overline{f}}$.

According to the Lebesgue monotone convergence theorem A.8, Heine theorem A.3, monotonicity of Lebesgue integral A.7, linearity of Lebesgue integral A.6 and linearity of limit A.2 we get:

$$1 - \lim_{\beta\uparrow\delta} \int_M \overline{f}_\beta(x)d\mu(x) = 1 - \int_M \tilde{\overline{f}}(x)d\mu(x) \leq 1 - \int_M \overline{f}_\delta(x)d\mu(x)$$

From the construction of $\tilde{\overline{f}}$ we know that $1 \geq \tilde{\overline{f}}(x) \geq \overline{f}_\delta(x)$ for all $x \in M$ and we denote by $\overline{N}$ the following set:

$$\overline{N} := \{x \in M : \tilde{\overline{f}}(x) > \overline{f}_\delta(x)\} = \{x \in M : \lim_{\beta\uparrow\delta} \overline{f}_\beta(x) > \overline{f}_\delta(x)\}$$

We define the kernel of the function $\overline{f}_\delta(\cdot)$ in the following way:

$$\mathrm{kern}(\overline{f}_\delta) := \{x \in M : \overline{f}_\delta(x) = 1\}$$

According to the linearity of Lebesgue integral A.6 and the fact $1 \geq \tilde{\overline{f}}(x) \geq \overline{f}_\delta(x)$, the following is fulfilled:

$$\lim_{\beta\uparrow\delta} \int_M \overline{f}_\beta(x)d\mu(x) = \int_{M\setminus\overline{N}} \overline{f}_\delta(x)d\mu(x) + \int_{\overline{N}\setminus\mathrm{kern}(\overline{f}_\delta)} \tilde{\overline{f}}(x)d\mu(x) + \int_{\overline{N}\cap\mathrm{kern}(\overline{f}_\delta)} 1\, d\mu(x)$$

The set $\overline{N} \cap \mathrm{kern}(\overline{f}_\delta)$ is empty from the fact that $1 \geq \tilde{\overline{f}}(x) \geq \overline{f}_\delta(x)$, from the construction of the set $\overline{N}$, where $\tilde{\overline{f}}(x) > \overline{f}_\delta(x)$, and from the definition of the kernel $\mathrm{kern}(\overline{f}_\delta)$, where $\overline{f}_\delta(x) = 1$, hence $1 \geq \tilde{\overline{f}}(x) > \overline{f}_\delta(x) = 1$ and therefore:

$$\overline{N} = \overline{N} \setminus \mathrm{kern}(\overline{f}_\delta)$$

If the measure of the set $\overline{N}$ will be zero, $\mu\left(\overline{N}\right) = 0$, then will be fulfilled

$$\lim_{\beta\uparrow\delta} \int_M \overline{f}_\beta(x)d\mu(x) = \int_M \overline{f}_\delta(x)d\mu(x),$$

and the function $\overline{g}(\cdot)$ will be continuous from left at the point δ. Otherwise, for $\mu\left(\overline{N}\right) > 0$, will be fulfilled

$$\lim_{\beta\uparrow\delta} \int_M \overline{f}_\beta(x)d\mu(x) > \int_M \overline{f}_\delta(x)d\mu(x),$$

and the function $\overline{g}(\cdot)$ will not be continuous from left at the point δ.

We have just proved that the second part of statement (∗∗) is equivalent with the fact that the measure of the set $\overline{N}$ is zero:

$$\inf_{\beta<\delta} \overline{g}(\beta) = \overline{g}(\delta) \qquad \Leftrightarrow \qquad \mu\left(\overline{N}\right) = 0$$

The fact that the measure of the set $\overline{N}$ is zero can be also formulated as:

$$\lim_{\beta\uparrow\delta} \overline{f}_\beta(x) = \overline{f}_\delta(x) \text{ almost everywhere on } M$$

At the beginning of this proof we have shown, that statement (∗∗) is equivalent with $\mathcal{C}_\delta(I^*) = A_\delta$. □

Example 7.2. I this example we will compute a fuzzy integral I^*

$$I^* := \int_{-2}^{2} f^*(x)dx$$

for the fuzzy valued function $f^*(\cdot) : [-2,2] \to \mathcal{F_I}(\mathbb{R})$ given in Example 7.1. The δ-level curves $\underline{f}_\delta(\cdot), \overline{f}_\delta(\cdot) : [-2,2] \to \mathbb{R}$ of the fuzzy valued function $f^*(\cdot)$ are defined as follows:

$$\underline{f}_\delta(x) = \delta\cdot\left(x^4 - 8x^2 + 16\right), \qquad \overline{f}_\delta(x) = \left(\tfrac{5}{2} - \delta\right)\cdot\left(x^4 - 8x^2 + 16\right)$$

We can easily check, that $\lim_{\beta\to\delta} \underline{f}_\beta(x) = \underline{f}_\delta(x)$ and $\lim_{\beta\to\delta} \overline{f}_\beta(x) = \overline{f}_\delta(x)$ everywhere on $[-2,2]$, while δ-level functions are linear and continuous in δ.

The δ-level function are also non-negative and integrable on $[-2,2]$:

$$\int_{-2}^{2} \underline{f}_\delta(x)dx = \int_{-2}^{2} \delta\cdot\left(x^4 - 8x^2 + 16\right)dx = \left[\delta\cdot\left(\frac{x^5}{5} - 8\cdot\frac{x^3}{3} + 16x\right)\right]_{-2}^{2} =$$

$$= \delta\cdot\left(\left(\frac{2^5}{5} - 8\cdot\frac{2^3}{3} + 16\cdot 2\right) - \left(\frac{(-2)^5}{5} - 8\cdot\frac{(-2)^3}{3} + 16\cdot(-2)\right)\right) = \delta\cdot\tfrac{128}{5}$$

$$\int_{-2}^{2} \overline{f}_\delta(x)dx = \int_{-2}^{2} \left(\tfrac{5}{2} - \delta\right)\cdot\left(x^4 - 8x^2 + 16\right)dx = \left(\tfrac{5}{2} - \delta\right)\cdot\tfrac{128}{5} = 64 - \delta\cdot\tfrac{128}{5}$$

We can construct the generating family $(A_\delta; \delta \in (0,1])$ in the following way:

$$A_\delta = \left[\delta\cdot\tfrac{128}{5}, 64 - \delta\cdot\tfrac{128}{5}\right] \quad \forall\delta \in (0,1]$$

The integral $\int_{-2}^{2} f^*(x)dx$ is a fuzzy number I^* generated by the family $(A_\delta; \delta \in (0,1])$ according to Definition 7.3. The fuzzy number I^* is shown in Figure 7.5.

According to Theorem 7.1 we know that δ-cuts of the fuzzy number I^* are equal to the sets of the generating family $(A_\delta; \delta\in(0,1])$:

$$\mathcal{C}_\delta(I^*) = \left[\delta\cdot\tfrac{128}{5}, 64 - \delta\cdot\tfrac{128}{5}\right] \quad \forall\delta \in (0,1]$$

Figure 7.4: Fuzzy valued function $f^*(\cdot)$ from Example 7.1

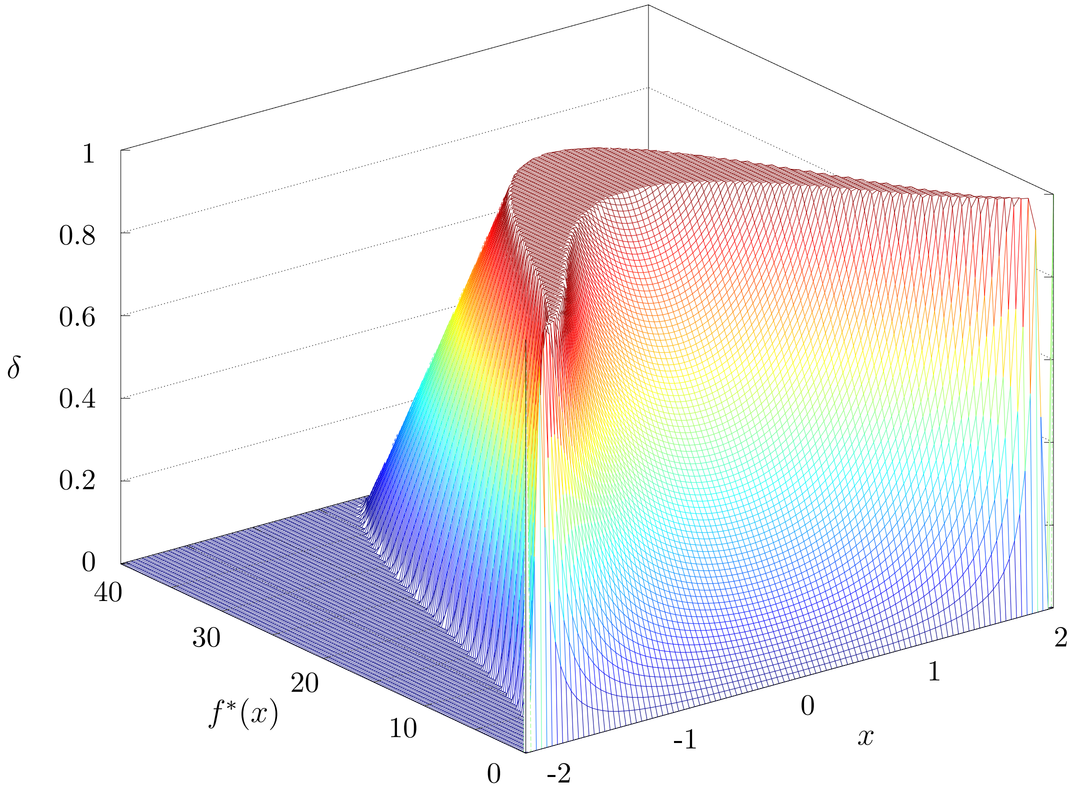
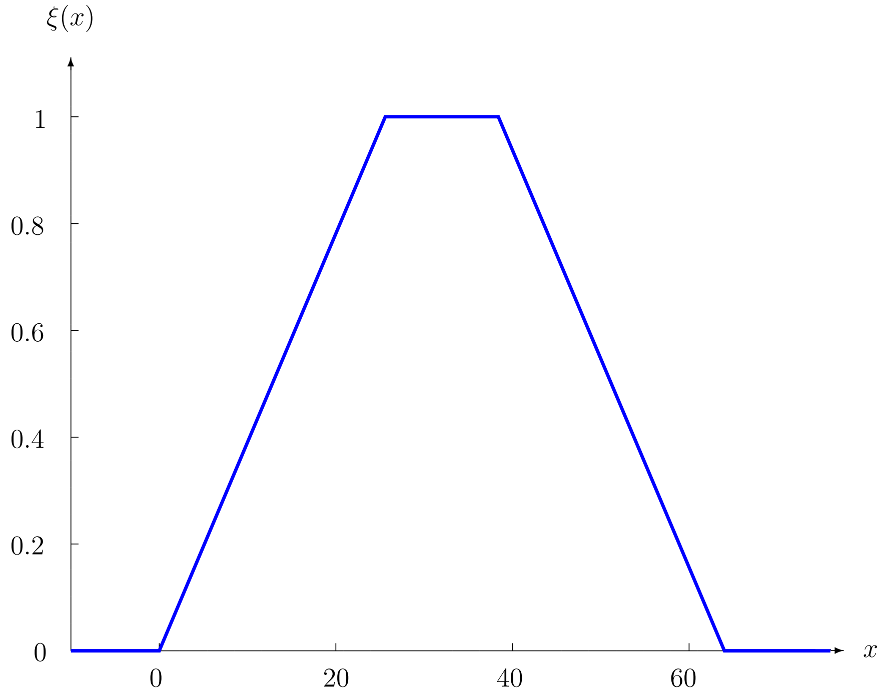


Figure 7.5: Characterizing function of the fuzzy integral $\int_{-2}^{2} f^*(x)dx$ of the fuzzy valued function $f^*(\cdot)$



8. Measurement and Fuzziness

Measurements are activities people were engaged in already long-ago, and with time various methods of measurement were continually refined, modified and cultivated. A persisting problem regarding measurements remained and remains an error handling, the errors caused by whether instrument or human deficiency. An importance of very precise measurements is increasing all the time.

When Galileo Galilei (1564 – 1642) wrote [1]: "Measure everything which is measurable, and the non-measurable make measurable" this had strong influence on the development of science, and still it is governing all kinds of quantitative scientific work. So-called exact sciences are based on the possibility of measurement of continuous quantities like time, length, volume, mass etc.

The fundamental question is: What is the result of a measurement? Already Julius Robert Mayer (1814 – 1878) wrote [6]: "Numbers are the fundament of exact scientific research". Moreover William Thomson Kelvin (1824 – 1907) expressed the importance of numbers by his words [10]: "When you cannot express it in numbers, your knowledge is of a meagre and non-satisfactory kind".

In 1951 Karl Menger published [7] the idea of generalizing indicator functions of subsets and created the term *ensembles flous* for generalized subsets. Later such generalized subsets were called *fuzzy sets* by Lotfi A. Zadeh [12]. This concept, applied to the set of real numbers $\mathbb{R}$ and specialized, is a suitable basis for the description of measurement results of one-dimensional continuous quantities.

As the exact sciences grew in importance in modern times, the measurements did the same as well as errors and errors handling. Contemporary science is dealing not only with clear cases as in the past but with the indistinct, blurred ones as well.

There is a guideline GUM "Guide to the expression of uncertainty in measurement" [2] released for unified approach to processing of measurements errors. Denomination of measurement error was replaced by a concept of measurement uncertainty, where all the circumstances causing indeterminacy in resulting exactness are considered as the causes of uncertainty, diverging thus the obtained value from the real one.

Measurements are connected with different kinds of uncertainty, the most important are *variability*, *errors* and *imprecision*. Whereas errors and variability are properly described by statistical methods, imprecision is not a statistical uncertainty. Imprecision is the unavoidable uncertainty coming from the impossibility to know the resulting numbers from measurement equipments exactly. The concept of fuzziness gives us suggestions how to handle this type of uncertainty and what to do in those cases.

8.1 Measurement Uncertainty

There are many reasons for measurement uncertainty and errors. These reasons can be sorted on accidental and systematic errors. The common reason of uncertainty are for example: an inappropriate measurement method, an unsuitable choice of measuring instrument, an improper selection of the processed sample, a miserable definition of measured value, rounding or rounding-off, an unknown influence of environment, a non-compliance of the same conditions by repeated measurements, a subjective influence of operating staff and many others.

The idea is that the result of a measurement of a one-dimensional continuous quantity is a real number times a measurement unit. Despite of all problems concerning the definition of measurement units a practical measurement result of a continuous variable can never be an exact real number. The reason is that it is impossible to obtain all infinitely many decimals of a real number. Therefore a measurement result can, also in principle, not realistically be identified with a precise real number.

The result of a measurement is viewed as an approximation of the value of the measurand and is complete only when it is accompanied by a statement of the uncertainty of that estimate. It is assumed that the result of a measurement has been corrected for all recognized significant systematic effects and that every effort has been made to identify such effects.

The uncertainty of the result of a measurement is defined as a parameter associated with the result of a measurement, that characterizes the dispersion of the values that could reasonably be attributed to the measurand, it reflects the lack of exact knowledge of the value of the measurand. The most important components of the uncertainty are variability, errors and imprecision.

The guideline GUM [2] is a standard specification which determines the general rules for evaluation and representation of the uncertainty in measurements. It handles well the variability and error components of uncertainty in the following way:

Type A standard uncertainty is calculated from series of repeated observations and is the familiar statistically estimated standard deviation obtained from a probability density function derived from an observed frequency distribution.

Type B standard uncertainty is usually based on a pool of comparatively reliable information as for example data of measurement units, components released by a manufacturer, knowledge of material behaviour, data obtained by calibration, uncertainty of data in reference manuals and so on.

In most cases, a measurand Y is not measured directly, but is determined from N other quantities $X_1, X_2, \ldots, X_N$ through a functional relationship f where $Y = f(X_1, X_2, \ldots, X_N)$.

An estimate of the measurand Y, called measurement result and denoted by y, is obtained using input estimates $x_1, x_2, \ldots, x_N$ for the values of the N quantities $X_1, X_2, \ldots, X_N$ by the same relationship f where $y = f(x_1, x_2, \ldots, x_N)$.

The combined standard uncertainty associated with the measurement result y, denoted by $u_c(y)$, is determined from the standard uncertainty associated with each input estimate x_i, denoted by $u(x_i)$ and evaluated by Type A or Type B evaluation. The combined standard uncertainty $u_c(y)$ characterizes the dispersion of the values that could reasonably be attributed to the measurand Y.

The combined standard uncertainty $u_c(y)$ associated with the result of a measurement y where all input quantities are independent is computed as

$$u_c^2(y) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i} \right)^2 u_i^2(x_i)$$

where f is the function giving relationship between measurement result and input quantities as $y = f(x_1, x_2, \ldots, x_N)$ and where $u_i(x_i)$ is the standard uncertainty associated with the input estimate x_i.

When some of the input quantities are correlated, the combined standard uncertainty $u_c(y)$ associated with the result of a measurement y is computed as

$$u_c^2(y) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i} \right)^2 u_i^2(x_i) + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u_{i,j}(x_i, x_j)$$

where $u_{i,j}(x_i, x_j) = u_{j,i}(x_j, x_i)$ is the estimated covariance associated with x_i and x_j.

Although the combined standard uncertainty $u_c(y)$ can be universally used to express the uncertainty of the result of a measurement result, in some commercial, industrial, regulatory, health and safety applications, it is often necessary to give a measure of uncertainty that defines an interval about the measurement result that may be expected to cover a large fraction of values that could reasonably be attributed to the measurand.

This type of uncertainty is called expanded uncertainty and is denoted by U. The result of a measurement is then expressed as $Y = y \pm U$, which is an interval $[y - U, y + U]$ that should cover a large fraction of values that could be attributed to Y and which is interpreted as the best estimate of that value.

The expanded uncertainty U is obtained by multiplying the combined standard uncertainty $u_c(y)$ by a so-called coverage factor k:

$$U = k \cdot u_c(y)$$

The value of the coverage factor k is chosen on the basis of the level of confidence required of the interval $[y - U, y + U]$. In general, k will be in the range 2 to 3. Ideally such that the interval $Y = y \pm U = y \pm k \cdot u_c(y)$ would be corresponding to a particular level of confidence as 95 or 99 percent.

However, this is not easy to do in practice because it requires extensive knowledge of the probability distribution characterized by the measurement result y and its combined standard uncertainty $u_c(y)$. Although these parameters are of critical importance, they are by themselves insufficient for the purpose of establishing intervals having exactly known levels of confidence.

8.2 Application of Fuzzy Models to Measurement Data

Statistics as a science of general data processing assumes a postulate that a given measurement process is possible to repeat many times over identical conditions, what needs not to agree with the reality all the times. On the other hand a fuzzy approach indicates the uncertainty of every single measurement. For the description of imprecision a generalization of real numbers, the fuzzy numbers are a suitable model (see Chapter 2, Definition 2.1).

Example 8.1. Our first example would include no imprecision. This is usually in the case, where the measurement result is a small natural number. An example of such measurement should be number of the apples on a table.

The best model to use in this case is a natural number $n \in \mathbb{N}$, but representation by a special fuzzy number is also possible (see Section 5.2.1). The characterizing function is given by the indicator function $\mathbb{1}_{\{n\}}(\cdot)$ of the number $n \in \mathbb{N}$.

Example 8.2. Some digital measurement equipments for continuous quantities produce decimal numbers with finitely many digits, but the idealized value is represented by a real number with infinitely many digits. Therefore a more general concept than precise numbers is necessary to model measurement results. The measurement result could be represented by an interval $[\underline{a}, \overline{a}]$, where the number $\underline{a}$ is the value given on the instrument, where the remaining infinitely many digits are set to be 0, the number $\overline{a}$ is obtained by setting the remaining infinitely many digits to be 9.

This approach can be represented by interval arithmetic and also by fuzzy numbers (see Section 5.2.2). The characterizing function is the indicator function $\mathbb{1}_{[\underline{a},\overline{a}]}(\cdot)$ of the interval $[\underline{a}, \overline{a}]$.

Example 8.3. Some digital measurement equipments for continuous quantities produce a decimal numbers with too many digits, where some of the digits may be incorrect, and the uncertainty of the measurement the measurement equipment is given. Let a be the result on the screen, and let u be the declared uncertainty of the measurement equipment. Then there are more ways how to handle the measurement imprecision of a single measurement.

One way is to use the interval approach as the guideline GUM advises. The expanded uncertainty $U = k \cdot u$ will be computed with the coverage factor $k \in [2, 3]$. The imprecision of the measurement will then be represented by the interval $[a - U, a + U]$. The corresponding fuzzy number can be constructed (see Section 5.2.2). The characterizing function is the indicator function $\mathbb{1}_{[a-U,a+U]}(\cdot)$ of the interval $[a - U, a + U]$.

Another way is to use the trapezoidal approach to represent the uncertainty of the measurement (see Section 5.2.3). The characterizing function of the measurement result is then represented using the value a, uncertainty u and the expanded uncertainty U in the following way:

$$\xi(x) = \left\{ \begin{array}{ll} \frac{x-(a-U)}{U-u} & \text{for } x \in [a-U, a-u) \\ 1 & \text{for } x \in [a-u, a+u] \\ \frac{(a+U)-x}{U-u} & \text{for } x \in (a+u, a+U] \\ 0 & \text{elsewhere} \end{array} \right\} \qquad \forall x \in \mathbb{R} \qquad (\triangle)$$

This type of characterizing function for $a = 5$, $u = 1$ and $U = 2$ is displayed in Figure 8.1.

While the uncertainty should be an estimation of standard deviation of the measurement, as is indicated in the guideline GUM [2], the normal distribution approach can be used (see Section 5.2.4). The characterizing function of the fuzzy measurement result would be a modified normal density with mean a and variance u^2, cropped to the confidence interval on the level $1 - \alpha$ for some $\alpha \in (0, 1)$.

The characterizing function $\xi(\cdot)$ is defined using the confidence interval $I = [a - u\Phi(\frac{\alpha}{2}), a + u\Phi(\frac{\alpha}{2})]$, where $\Phi(\cdot)$ is the quantile function of standardized normal distribution, in the following way:

$$\xi(x) = \left\{ \begin{array}{ll} e^{-\frac{(x-a)^2}{2u^2}} & \text{for } x \in I \\ 0 & \text{elsewhere} \end{array} \right\} \qquad \forall x \in \mathbb{R}$$

This type of characterizing function for $a = 5$, $u = 1$ and $\alpha = 0.05$ is displayed on Figure 8.2.

Example 8.4. For analogue measurement equipments the best one can do is to make a photograph of the pointer position on the measurement scale. This is a color intensity picture. From such pictures characterizing functions can be obtained using the colour intensity along the real axis for one-dimensional quantities. Each pixel of the photography on the measurement scale then represents one point of the function $h(\cdot)$ from which the characterizing function can be constructed according to Section 5.3. The partly constant approach (see Section 5.3.2) or partly linear approach (see Section 5.3.3) can be used.

Example 8.5. Let us have a set of five fuzzy measurements $x_1^*, \ldots, x_5^*$ with trapezoidal shape. Let the corresponding characterizing functions $\xi_1(\cdot), \ldots, \xi_5(\cdot)$ be defined according the formula $(\triangle)$ described in Example 8.3 using the following parameters:

Measurement result	Parameters a	u	U
1	7.0	0.8	2.0
2	7.0	1.0	3.0
3	4.2	0.4	0.8
4	8.0	0.4	1.2
5	6.0	0.4	1.2

The characterizing functions $\xi_1(\cdot), \ldots, \xi_5(\cdot)$ of the individual measurement results $x_1^*, \ldots, x_5^*$ are displayed in Figure 8.3.

In case of fuzzy individual measurements the calculation of an aggregate function is described in Chapter 6, especially the fuzzy mean and the fuzzy standard deviation can be computed in the way described in Section 6.2.

Let us have individual fuzzy measurement results $x_1^*, \ldots, x_n^*$ with corresponding characterizing functions $\xi_1(\cdot), \ldots, \xi_n(\cdot)$. We can construct a fuzzy vector $\underline{x}^*$ with corresponding vector-characterizing function $\zeta \colon \mathbb{R}^n \to [0,1]$ by using a t-norm (see Chapter 4). Usually the minimum t-norm is used:

$$\zeta(x_1, \ldots, x_n) = \min\{\xi_1(x_1), \ldots, \xi_n(x_n)\} \qquad \forall (x_1, \ldots, x_n) \in \mathbb{R}^n$$

Let $g \colon \mathbb{R}^n \to \mathbb{R}$ be a statistic. Usually the mean value $g_{\text{mean}}(\cdot, \ldots, \cdot)$, the sample variance $g_{\text{var}}(\cdot, \ldots, \cdot)$, and the sample standard deviation $g_{\text{std.dev}}(\cdot, \ldots, \cdot)$ are used, these statistics can be written in the following way:

$$g_{\text{mean}}(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i \qquad \forall (x_1, \ldots, x_n) \in \mathbb{R}^n$$

$$g_{\text{var}}(x_1, \ldots, x_n) = \frac{1}{n-1} \sum_{i=1}^{n} \left[x_i - \left(\tfrac{1}{n} \textstyle\sum_{j=1}^{n} x_j \right) \right]^2 \qquad \forall (x_1, \ldots, x_n) \in \mathbb{R}^n$$

$$g_{\text{std.dev}}(x_1, \ldots, x_n) = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} \left[x_i - \left(\tfrac{1}{n} \textstyle\sum_{j=1}^{n} x_j \right) \right]^2} \qquad \forall (x_1, \ldots, x_n) \in \mathbb{R}^n$$

Then we can generalize these statistics for fuzzy numbers, where the result would be a fuzzy number y^* with corresponding characterizing function $\xi(\cdot)$, using a fuzzy vector $\underline{x}^*$ with corresponding vector-characterizing function $\zeta(\cdot, \ldots, \cdot)$ in the following way:

$$\xi(y) = \left\{ \begin{array}{ll} \sup\left\{\zeta(x_1, \ldots, x_n) \colon (x_1, \ldots, x_n) \in g^{-1}(\{y\})\right\} & \\ & \text{if } g^{-1}(\{y\}) \neq \emptyset \\ & \\ 0 & \text{if } g^{-1}(\{y\}) = \emptyset \end{array} \right\} \qquad \forall y \in \mathbb{R}$$

The function $\xi(\cdot)$ defined above fulfils the conditions for characterizing functions.

Example 8.6. Let us have the measurement results $x_1^*, \ldots, x_5^*$ with corresponding characterizing functions $\xi_1(\cdot), \ldots, \xi_5(\cdot)$ according to Example 8.5.

The fuzzy mean value, fuzzy sample variance, and fuzzy sample standard deviation for fuzzy numbers $x_1^*, \ldots, x_5^*$ was computed using different types of t-norms. The resulting characterizing functions are given in Figure 8.4, Figure 8.5, and Figure 8.6 respectively.

By using fuzzy models it is possible to analyse realistic measurement results in a more adequate way, taking care of different types of uncertainty.

Figure 8.1: Trapezoidal shape of the characterizing function $\xi(\cdot)$ according to Example 8.3

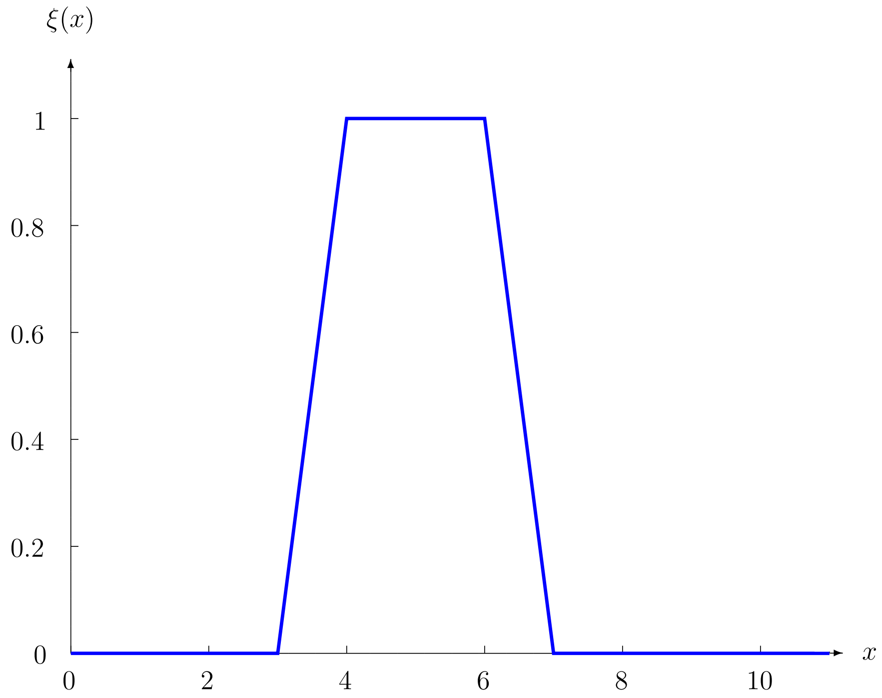


Figure 8.2: Modified normal distribution shape of the characterizing function $\xi(\cdot)$ according to Example 8.3

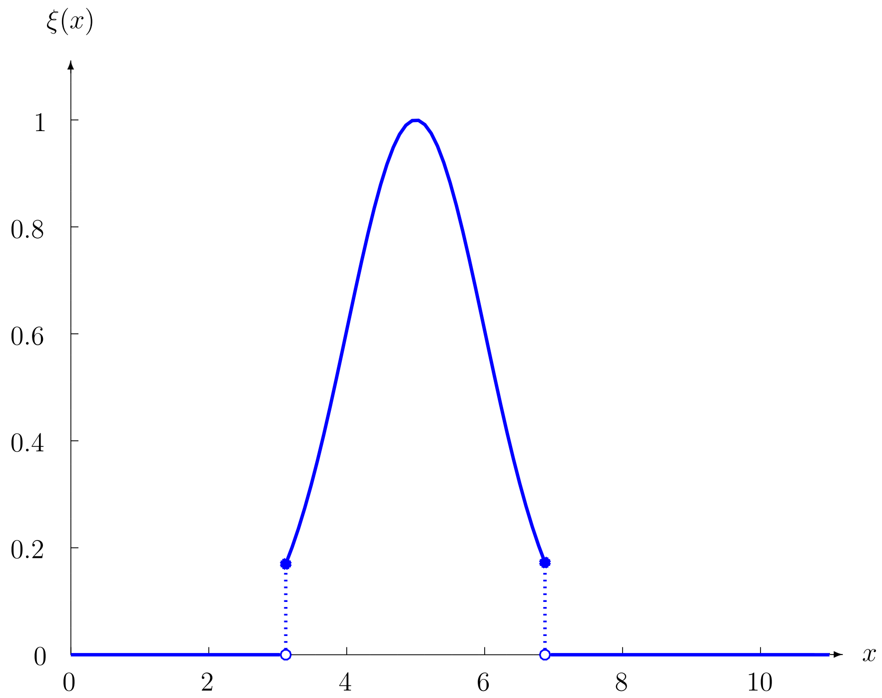


Figure 8.3: Characterizing functions — $\xi_1(\cdot)$, — $\xi_2(\cdot)$, — $\xi_3(\cdot)$, — $\xi_4(\cdot)$, — $\xi_5(\cdot)$ of the fuzzy numbers $x_1^*, x_2^*, x_3^*, x_4^*, x_5^*$ according to Example 8.5

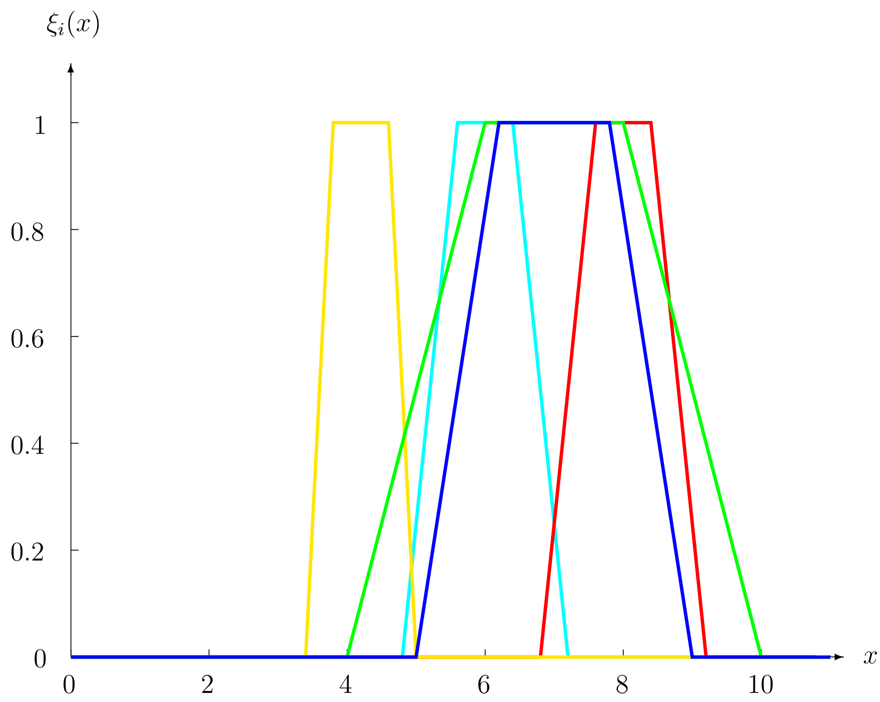


Figure 8.4: Characterizing function $\psi(\cdot)$ of the fuzzy mean from Example 8.6, where the fuzzy vector was constructed using different t-norms (— minimum, — product, — limited sum, and — drastic product t-norm)

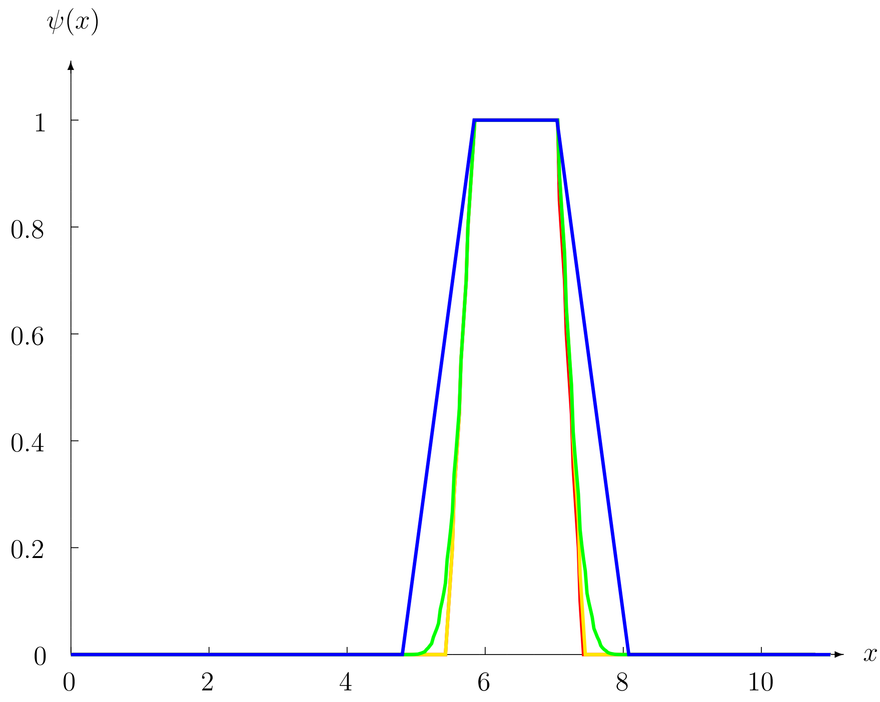


Figure 8.5: Characterizing function $\psi(\cdot)$ of the fuzzy variance from Example 8.6, where the fuzzy vector was constructed using different t-norms (— minimum, — product, — limited sum, and drastic product t-norm)

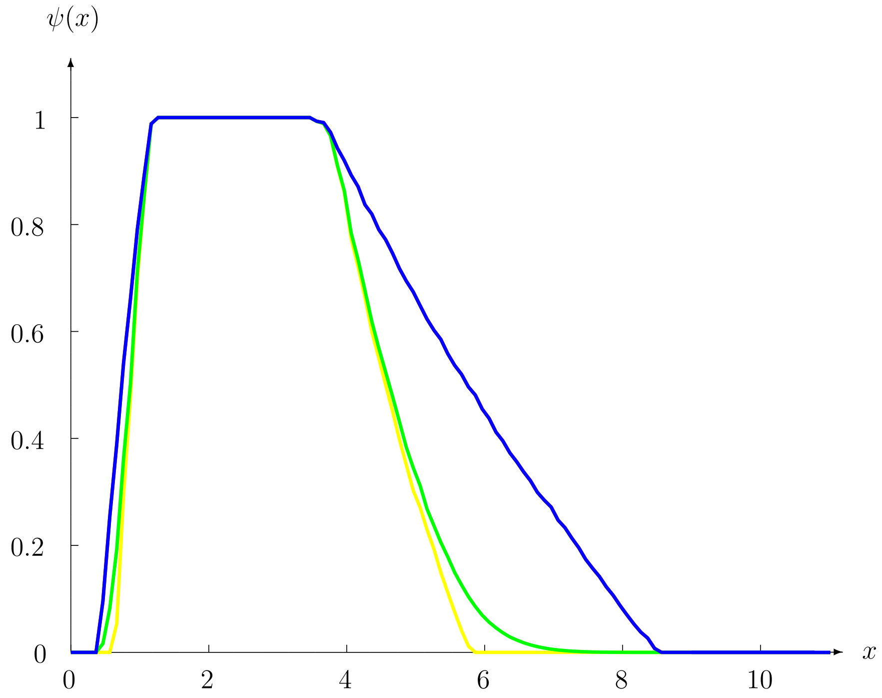
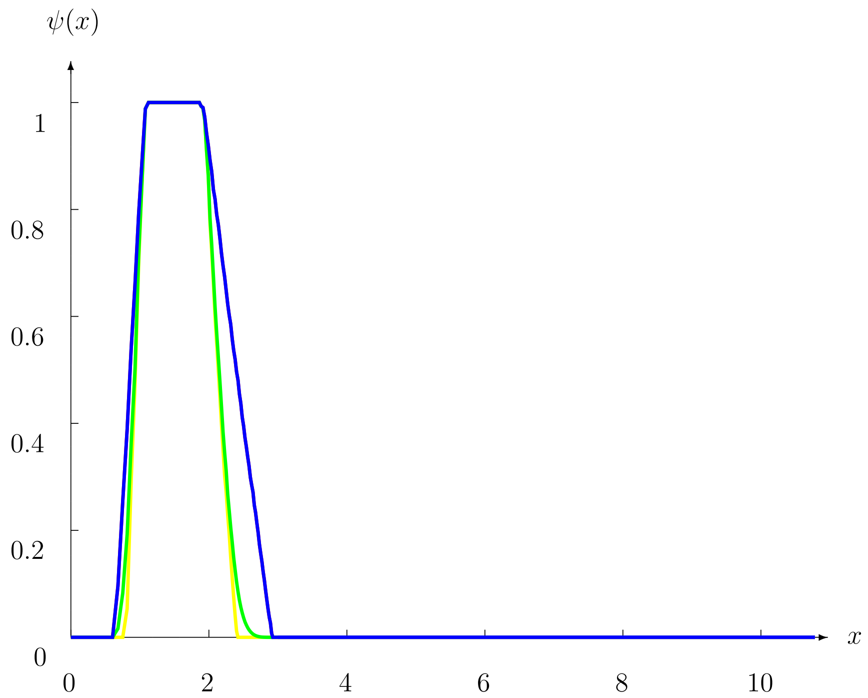


Figure 8.6: Characterizing function $\psi(\cdot)$ of the fuzzy sample standard deviation from Example 8.6, where the fuzzy vector was constructed using different t-norms (— minimum, — product, — limited sum, and drastic product t-norm)



A. Appendix

A.1 List of Definitions

Definition A.1. Let M be an ordered set. We call m the *maximum of* M if and only if:

$$m \in M \qquad \text{and} \qquad \forall x \in M \quad x \leq m$$

Definition A.2. Let $f : M \to N$ be a function. We call m the *global maximum of the function* f if and only if:

$$\exists x \in M \quad f(x) = m \qquad \text{and} \qquad \forall x \in M \quad f(x) \leq m$$

Definition A.3. Let $M \subset \mathbb{R}$ be a set. We call $s \in \mathbb{R} \cup \{-\infty, \infty\}$ the *supremum of* M and write $s = \sup(M)$ if and only if the following two conditions are fulfilled:

$$\forall x \in M \quad x \leq s,$$

$$\forall s' \in \mathbb{R},\ s' < s \quad \exists x \in M \quad s' < x$$

The supremum of empty set is $-\infty$ and the supremum of $\mathbb{R}$ is ∞.

Definition A.4. Let $M \subset \mathbb{R}$ be a set. We call $i \in \mathbb{R} \cup \{-\infty, \infty\}$ the *infimum of* M and write $i = \inf(M)$ if and only if $i = -\sup(\{-x \colon x \in M\})$.

Definition A.5. Let A, B be sets. We call the *Cartesian product* of sets A and B the set of all pairs (a, b) where $a \in A$ and $b \in B$. It is denoted $A \times B$.

Definition A.6. We denote a non-negative function $d : M \times M \to [0, \infty)$ describing the distance between two points for a given set M as *metric* if and only if it fulfils the following conditions:

- $\forall x, y \in M \quad [d(x, y) = 0] \ \Rightarrow\ [x = y]$
- $\forall x, y \in M \quad d(x, y) = d(y, x)$ (symmetry)
- $\forall x, y, z \in M \quad d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality)

Definition A.7. We call a *metric space* (M, d) a set M with a metric d defined on the set M.

Definition A.8. The metric space $(\mathbb{R}^n, d_{Euklid})$, where

$$d_{Euklid}(\underline{x}, \underline{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \qquad \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},\ \underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$$

is called *n-dimensional Euclidean space* and the metric d_{Euklid} is called *Euclidean metric.*

Definition A.9. Let (M,d) be a metric space and let $(x_n)_{n\in\mathbb{N}} \subseteq M$ be a sequence. We call $(x_n)_{n\in\mathbb{N}}$ a *Cauchy sequence* if and only if

$$\forall\varepsilon>0 \quad \exists n_0\in\mathbb{N} \quad \forall m\in\mathbb{N},\ m>n_0 \quad \forall n\in\mathbb{N},\ n>n_0: \quad d(x_m,x_n)<\varepsilon$$

Definition A.10. Let (M,d) be a metric space and let $(x_n)_{n\in\mathbb{N}} \subseteq M$ be a sequence. We call $(x_n)_{n\in\mathbb{N}}$ a *convergent sequence* if and only if there exist $a\in M$ such that:

$$\forall\varepsilon>0 \quad \exists n_0\in\mathbb{N} \quad \forall n\in\mathbb{N},\ n>n_0: \quad d(x_n,a)<\varepsilon$$

We write $x_n\to a$ and call a the *limit of the sequence* $(x_n)_{n\in\mathbb{N}}$.

Definition A.11. A *complete metric space* is a metric space in which every Cauchy sequence is a convergent sequence.

Definition A.12. Let $(M,d_M),(N,d_N)$ be metric spaces, let $f: M\to N$ be a function, $a\in M$ and $L\in N$. We say that the *limit of the function* f, as x approaches a, is L and write $\lim\limits_{x\to a} f(x)=L$ if and only if:

$$\forall\varepsilon>0 \quad \exists\delta>0 \quad \forall x\in M,\ x\neq a,\ d_M(x,a)<\delta: \quad d_N(f(x),L)<\varepsilon$$

Definition A.13. Let $(M,d_M),(N,d_N)$ be metric spaces, let $f: M\to N$ be a function, $a\in M$. We say that the function f is *continuous* in a if and only if:

$$\forall\varepsilon>0 \quad \exists\delta>0 \quad \forall x\in M,\ d_M(x,a)<\delta: \quad d_N(f(x),f(a))<\varepsilon$$

Definition A.14. Let M be a set in a metric space (M,d). Then M is a *bounded set* if and only if there exists a real number $r<\infty$ such that:

$$\forall x,y\in M \quad d(x,y)\leq r$$

Definition A.15. Let S be a set and (M,d) be a metric space, let $f: S\to M$ be a function. Then f is a *bounded function* if and only if the range of the function $\{f(x)\colon x\in S\}\subseteq M$ is a bounded set.

Definition A.16. Let (M,d) be a metric space, $S\subseteq M$ be a subset. Then S is called *open set* if and only if every point in S has a neighbourhood lying in the set.

$$\forall x\in S \quad \exists\varepsilon>0 \quad \{x'\in M\colon d(x',x)<\varepsilon\}\subset S$$

The set S is called *closed set* if and only if it is a complement of an open set.

Definition A.17. Let (M,d) be a metric space, $S\subseteq M$ be a subset. We define the *closure of a set* S, write $\overline{S}$, as the smallest closed set containing the set S.

Definition A.18. Let (M,d) be a metric space, $S\subseteq M$ be a subset. Then S is called *connected set* if and only if the set S cannot be partitioned into two non-empty subsets such that each subset has no points in common with the set closure of the other.

Definition A.19. Let (M,d) be a metric space, $S\subseteq M$ be a subset. We call S a *compact set* if and only if from each sequence of elements of S can be chosen a convergent subsequence with its limit in S.

Definition A.20. A function $f(\cdot) : \mathbb{R} \to \mathbb{R}$ is called *convex function* on the interval $[a, b] \subset \mathbb{R}$ if for any two points x and y in $[a, b]$ and any λ where $\lambda \in (0, 1)$:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

A function $f(\cdot, \ldots, \cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is called *convex function* on a convex set $M \subset \mathbb{R}^n$ if the function $f(\cdot, \ldots, \cdot)$ is convex in each coordinate.

Definition A.21. A function $f(\cdot, \ldots, \cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is called *concave function* on a convex set $M \subset \mathbb{R}^n$ if the opposite function $-f(\cdot, \ldots, \cdot)$ is convex on M.

Definition A.22. Let scalars be members of a field F, and let V be a space over F. In order for V to be a *vector space*, the following conditions must hold for all elements $X, Y, Z \in V$ and for all scalars $r, s \in F$:

- $X + Y = Y + X$ (commutativity)
- $(X + Y) + Z = X + (Y + Z)$ (associativity of vector addition)
- $\exists 0 \in V \ \forall X \in V : 0 + X = X + 0 = X$ (additive identity)
- $\forall X \in V \ \exists - X \in V : X + (-X) = 0$ (existence of additive inverse)
- $r \cdot (s \cdot X) = (r \cdot s) \cdot X$ (associativity of scalar multiplication)
- $(r + s) \cdot X = r \cdot X + s \cdot X$ (distributivity of scalar sums)
- $r \cdot (X + Y) = r \cdot X + r \cdot Y$ (distributivity of vector sums)
- $\exists 1 \in F \ \forall X \in V : 1 \cdot X = X$ (scalar multiplication identity)

Definition A.23. A *Banach space* is a complete vector space B with a norm.

Definition A.24. Let $(M, d_M), (N, d_N)$ be Banach spaces, let $f : M \to N$ be a function, $a \in M$. We define:

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

If the limit exists we call $f'(a)$ the *derivative* of the function $f(\cdot)$ at point a and we call function $f(\cdot)$ *differentiable* at point a.

Definition A.25. Let S be a set and (M, d) be a metric space, let $f_n : S \to M$ be a function $\forall n \in \mathbb{N}$. We call the sequence $(f_n(\cdot))_{n \in \mathbb{N}}$ *pointwise convergent* if and only if there exists a function $f : S \to M$ such that:

$$\forall x \in M \quad \forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall n \in \mathbb{N},\ n > n_0 : \quad d(f_n(x), f(x)) < \varepsilon$$

Definition A.26. Let S be a set and (M, d) be a metric space, let $f_n : S \to M$ be a function $\forall n \in \mathbb{N}$. We call the sequence $(f_n(\cdot))_{n \in \mathbb{N}}$ *uniformly convergent* if and only if there exists a function $f : S \to M$ such that:

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall x \in M \quad \forall n \in \mathbb{N},\ n > n_0 : \quad d(f_n(x), f(x)) < \varepsilon$$

Definition A.27. Let M be a set. We define a *σ-algebra* $\mathcal{A}$ as a non-empty collection of subsets of M such that the following holds true:

- $M \in \mathcal{A}$.
- If a set $A \subseteq M$ is in $\mathcal{A}$, then its complement A^c is in $\mathcal{A}$.
- If $A_n \subseteq M$ is a sequence of elements of $\mathcal{A}$, then its union $\bigcup\limits_n A_n$ is in $\mathcal{A}$.

The elements of $\mathcal{A}$ are called *measurable sets*.

Definition A.28. We call a *measurable space* $(M, \mathcal{A})$ a set M with σ-algebra $\mathcal{A}$ defined on M.

Definition A.29. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be two measurable spaces. A function $f : M \to N$ is called *measurable* if and only if for every set $B \in \mathcal{B}$ the inverse image $f^{(-1)}(B) = \{x \in M : f(x) \in B\}$ is an $\mathcal{A}$-measurable set, i.e $f^{-1}(B) \in \mathcal{A}$.

Definition A.30. Let M be a set, $\mathcal{A}$ be a σ-algebra on M. Then a function $\mu : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$ is called *measure* if it satisfies the following properties:

- $\forall A \in \mathcal{A} : \ \mu(A) \geq 0$ (non-negativity)
- $\mu(\emptyset) = 0$ (null empty set)
- For all countable collections $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{A}$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \qquad (\sigma\text{−additivity})$$

Definition A.31. We call *measure space* $(M, \mathcal{A}, \mu)$ a measurable space $(M, \mathcal{A})$ with a non-negative measure $\mu : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$ defined on $(M, \mathcal{A})$.

A.2 Selected Parts of Mathematical Analysis

Lemma A.1. *Let $(M, d_M), (N, d_N)$ be Banach spaces, let $f : M \to N$ be a function, $a \in M$. Then $f(\cdot)$ is continuous in a if and only if $\lim_{x \to a} f(x) = f(a)$.*

Proof. This lemma follows directly from the Definition A.12 and Definition A.13. □

Lemma A.2. *Let $(M, d_M), (N, d_N)$ be Banach spaces, $a \in M$, let $f, g : M \to N$ be functions with limit in point a, $\alpha, \beta \in \mathbb{R}$. Then the function $\alpha f + \beta g$ has limit in a and:*

$$\lim_{x \to a} (\alpha f + \beta g)(x) = \alpha \lim_{x \to a} f(x) + \beta \lim_{x \to a} g(x)$$

Proof. From the prerequisites we know there exist $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$. We will denote these limits by L_1 and L_2 respectively. Written by the definition it means that we know that the following holds true:

$$\forall \varepsilon_1 > 0 \quad \exists \delta_1 > 0 \quad \forall x \in M,\ x \neq a,\ d_M(x, a) < \delta_1 : \quad d_N(f(x), L_1) < \varepsilon_1$$

$$\forall \varepsilon_2 > 0 \quad \exists \delta_2 > 0 \quad \forall x \in M,\ x \neq a,\ d_M(x, a) < \delta_2 : \quad d_N(g(x), L_2) < \varepsilon_2$$

We want to prove that there exists $\lim_{x \to a} (\alpha f + \beta g)(x)$ with value $L := \alpha L_1 + \beta L_2$. For arbitrary $\varepsilon > 0$ we need to find $\delta > 0$ such that the following will hold:

$$\forall x \in M,\ x \neq a,\ d_M(x, a) < \delta : \quad d_N(\alpha f(x) + \beta g(x), L) < \varepsilon$$

From the triangle inequality of the metric space we have:

$$d_N(\alpha f(x) + \beta g(x), L) \leq \alpha \cdot d_N(f(x), L_1) + \beta \cdot d_N(g(x), L_2)$$

We can set $\varepsilon_1 := \frac{\varepsilon}{2\alpha}$, $\varepsilon_2 := \frac{\varepsilon}{2\beta}$ and find appropriate δ_1 and δ_2 fulfilling the formulas. Then we can set $\delta := \min\{\delta_1, \delta_2\}$ and from that the following is fulfilled:

$$d_N(\alpha f(x) + \beta g(x), L) \leq \alpha \cdot d_N(f(x), L_1) + \beta \cdot d_N(g(x), L_2) < \alpha\varepsilon_1 + \beta\varepsilon_2 = \varepsilon$$

□

Theorem A.3 (Heine)**.** *Let $(M, d_M), (N, d_N)$ be complete metric spaces, let $f : M \to N$ be a function, $a \in M$ and $L \in N$. Then the following conditions are equivalent:*

(∗) $\lim_{x \to a} f(x) = L$

(∗∗) *For all sequences $(x_n)_{n \in \mathbb{N}} \subseteq M$, such that $x_n \to a$ and $x_n \neq a\ \forall n \in \mathbb{N}$, it follows $f(x_n) \to L$, in symbols*

$$\forall (x_n)_{n \in \mathbb{N}} \subseteq M \quad \text{with} \quad [\, x_n \to a \quad \& \quad x_n \neq a\ \forall n \in \mathbb{N} \,] \quad \Rightarrow \quad f(x_n) \to L$$

Proof. The implication $(*) \Rightarrow (**)$ can be proved directly from definitions.

From $\lim_{x\to a} f(x) = L$ we know that:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in M,\ x \neq a,\ d_M(x,a) < \delta : \quad d_N(f(x), L) < \varepsilon \qquad (\triangle)$$

Let $(x_n)_{n\in\mathbb{N}} \subseteq M$ be a sequence, such that $x_n \to a$ and $\forall n \in \mathbb{N} : x_n \neq a$. From the convergence of the sequence we know that:

$$\forall \delta > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall n \in \mathbb{N}, n > n_0 : \quad d_M(x_n, a) < \delta \qquad (\triangle\triangle)$$

We want to prove that $f(x_n) \to L$, which is from definition:

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall n \in \mathbb{N}, n > n_0 : \quad d_N(f(x_n), L) < \varepsilon \qquad (\triangle\triangle\triangle)$$

For $\varepsilon > 0$ we will find $\delta > 0$ in $(\triangle)$, and for this δ we will find $n_0 \in \mathbb{N}$ in $(\triangle\triangle)$, which will fulfil statement $(\triangle\triangle\triangle)$.

We will prove implication $(**) \Rightarrow (*)$ as a reverse implication of negations $\neg(*) \Rightarrow \neg(**)$.

The negation of the $(*)$ statement is following:

$$\exists \varepsilon > 0 \quad \forall \delta > 0 \quad \exists x \in M,\ x \neq a,\ d_M(x,a) < \delta : \quad d_N(f(x), L) \geq \varepsilon \qquad (\circ)$$

We will find the $\varepsilon > 0$, choose $\delta = \frac{1}{n}$ for all $n \in \mathbb{N}$ and construct the sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ using the following reformulation of $(\circ)$:

$$\exists \varepsilon > 0 \quad \forall n \in \mathbb{N} \quad \exists \tilde{x}_n \in M,\ \tilde{x}_n \neq a,\ d_M(\tilde{x}_n, a) < \tfrac{1}{n} : \quad d_N(f(\tilde{x}_n), L) \geq \varepsilon$$

We have to construct a sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$, such that $f(\tilde{x}_n)$ is not converging to the value L. From the fact that $d_M(\tilde{x}_n, a) < \frac{1}{n}$ for all $n \in \mathbb{N}$, we can see that $\tilde{x}_n$ is converging to the value a.

We have just found the sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$, which we need for proving the negation of the $(**)$ statement:

$$\exists (x_n)_{n\in\mathbb{N}} \subseteq M : \quad x_n \to a \quad \& \quad \forall n \in \mathbb{N} \colon x_n \neq a \quad \& \quad \neg(f(x_n) \to L)$$

$\square$

Theorem A.4. *Let M, N be ordered Banach spaces, let $f : M \to N$ be a function, $a \in M$. If the function $f(\cdot)$ is monotonous and bounded, then exists $L \in N$ such that $\lim\limits_{x\to a} f(x) = L$.*

A.3 Remarks on Lebesgue Integral

In this section we will summarize the definition and some basic characteristics of the Lebesgue integral.

We start with a measure space $(M, \mathcal{A}, \mu)$ where M is a set, $\mathcal{A}$ is a σ-algebra of subsets of M and μ is a non-negative measure on M defined on sets of $\mathcal{A}$.

For example, M can be the Euclidean space $\mathbb{R}^n$ or some Lebesgue measurable subset of it, $\mathcal{A}$ will be the σ-algebra of all Lebesgue measurable subsets of M, and μ will be the Lebesgue measure.

In Lebesgue's theory, integrals are defined for a class of functions called measurable functions. A real-valued function f on M is measurable if the pre-image of every interval of the form (t, ∞) is in $\mathcal{A}$, $\{x \in M : f(x) > t\} \in \mathcal{A} \quad \forall t \in \mathbb{R}$.

Definition A.32. A finite linear combination of indicator functions $\sum_k a_k \mathbb{1}_{S_k}$ where $a_k \in \mathbb{R}$ are real numbers and the sets S_k are pairwise disjoint and measurable, is called a *simple function*.

Definition A.33. The *Lebesgue integral*

$$\int_M f \, d\mu = \int_M f(x) \; \mu(dx)$$

for a measurable real-valued function f defined on M is defined in the following steps:

Indicator functions: To assign a value to the integral of the indicator function $\mathbb{1}_S$ of a measurable set S consistent with the given measure μ, the only reasonable choice is to set:

$$\int \mathbb{1}_S \, d\mu = \mu(S).$$

Simple functions: We extend the integral by linearity to non-negative simple functions. When the coefficients a_k are non-negative, we set:

$$\int_M \left(\sum_k a_k \mathbb{1}_{S_k} \right) d\mu = \sum_k a_k \int_M \mathbb{1}_{S_k} \, d\mu = \sum_k a_k \, \mu(S_k)$$

The result may be infinite, the convention $0 \cdot \infty = 0$ is used.

Non-negative functions: Let $f : M \to [0, +\infty]$ be a non-negative measurable function on M, in other words f takes non-negative values in the extended real number line. We define:

$$\int_M f \, d\mu = \sup \left\{ \int_M s \, d\mu : s \leq f, \; s \text{ is simple function} \right\}$$

This integral coincides with the preceding one, defined on the set of simple functions. For some functions this integral $\int_M f \, d\mu$ will be infinite.

Signed functions: Let $f : M \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a measurable function, then we can write $f = f^+ - f^-$ where

$$f^+(x) = \left\{ \begin{array}{cc} f(x) & f(x) > 0 \\ 0 & \text{elsewhere} \end{array} \right\}, \; f^-(x) = \left\{ \begin{array}{cc} -f(x) & f(x) < 0 \\ 0 & \text{elsewhere} \end{array} \right\} \; \forall x \in \mathbb{R}.$$

Both f^+ and f^- are non-negative measurable functions and $|f| = f^+ + f^-$.

We say that the *Lebesgue integral of the measurable function f exists* if at least one of $\int f^+ \, d\mu$ and $\int f^- \, d\mu$ is finite. In this case we define:

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$$

If $\int |f| \, d\mu < \infty$, we say that f is *Lebesgue integrable.*

It turns out that this definition gives the desirable properties of the integral.

The Lebesgue integral does not distinguish between functions which differ only on a set of μ-measure zero.

Definition A.34. Functions f and g are said to be equal *almost everywhere* if

$$\mu(\{x \in M : f(x) \neq g(x)\}) = 0.$$

Lemma A.5. *Let f, g be non-negative measurable functions (possibly assuming the value $+\infty$) such that $f = g$ almost everywhere, then $\int f \, d\mu = \int g \, d\mu$.*

Lemma A.6 (Linearity of Lebesgue integral)**.** *Let f and g be Lebesgue integrable functions and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is a Lebesgue integrable function and*

$$\int (\alpha f + \beta g) \, d\mu = \alpha \int f \, d\mu + \beta \int g \, d\mu.$$

Lemma A.7 (Monotonicity of Lebesgue integral)**.** *Let f and g be non-negative measurable functions, $f \leq g$, then*

$$\int f \, d\mu \leq \int g \, d\mu.$$

Theorem A.8 (Lebesgue monotone convergence)**.** *Let $(f_k)_{k \in \mathbb{N}}$ be a sequence of non-negative measurable functions such that*

$$f_k(x) \leq f_{k+1}(x) \quad \forall k \in \mathbb{N}, \forall x \in M.$$

Then, the pointwise limit f of f_k is Lebesgue integrable and

$$\lim_k \int f_k \, d\mu = \int f \, d\mu.$$

The value of any of the integrals is allowed to be infinite.

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C. Published Paper

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THE GENERATION OF FUZZY SETS AND THE CONSTRUCTION OF CHARACTERIZING FUNCTIONS OF FUZZY DATA

L. KOVÁŘOVÁ AND R. VIERTL

Abstract. Measurement results contain different kinds of uncertainty. Besides systematic errors and random errors individual measurement results are also subject to another type of uncertainty, so-called *fuzziness*. It turns out that special fuzzy subsets of the set of real numbers $\mathbb{R}$ are useful to model fuzziness of measurement results. These fuzzy subsets x^* are called *fuzzy numbers*. The membership functions of fuzzy numbers have to be determined. In the paper first a characterization of membership function is given, and after that methods to obtain special membership functions of fuzzy numbers, so-called *characterizing functions* describing measurement results are treated.

1. Introduction

A critical point in fuzzy set theory is "how to obtain the membership function". There are some methods for special situations, where this is possible in a natural way. Especially, for special fuzzy subsets of the set of real numbers $\mathbb{R}$, and for special fuzzy subsets of $\mathbb{R}^2$.

Construction of membership functions of data is useful in many field of sciences [2, 4, 8]. There are methods based on probabilistic theory [1]. The fuzzy set concept was firstly introduced by L. Zadeh [7], description of the fuzzy set theory can be found in monographs, for example in [3].

2. δ-Cuts and Generating Families

Let A^* be a fuzzy subset of a universal set M, which means, that there exists a function, called *membership function* $\mu\colon M \to [0,1]$ associating each object in M with a real number in the interval [0,1]. Then for $\delta \in (0,1]$ the so-called δ-Cut $\mathcal{C}_\delta[A^*]$ is defined based on the membership function $\mu(\cdot)$ of A^* in the following way:

$$\mathcal{C}_\delta[A^*] := \{x \in M\colon\ \mu(x) \geq \delta\}$$

Denoting by $\mathbb{1}_A(\cdot)$ the indicator function of a classical set $A \subseteq M$, where

$$\mathbb{1}_A(x) := \left\{ \begin{array}{ll} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{array} \right\} \qquad \forall x \in M,$$

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the family of δ-Cuts $\big(\mathcal{C}_\delta\,[A^*]\,;\ \delta \in (0,1]\,\big)$ determines the membership function $\mu(\cdot)$ of A^* by the following lemma:

Lemma 2.1.

$$\mu(x) = \max\left\{\delta \cdot \mathbb{1}_{\mathcal{C}_\delta[A^*]}(x):\ \delta \in [0,1]\right\} \qquad \forall\, x \in M$$

This lemma is well known. Moreover the following holds true for all $\delta \in (0,1]$:

$$\mathcal{C}_\delta[A^*] = \bigcap_{0<\beta<\delta} \mathcal{C}_\beta[A^*]$$

Now one could think that all nested families $\big(A_\delta;\ \delta \in (0,1]\,\big)$ of classical subsets of M, i.e. $A_{\delta_1} \supseteq A_{\delta_2}$ for $\delta_1 < \delta_2$, are already the δ-Cuts of a fuzzy subset of M. This is not true as can be seen by simple examples (compare Example 2.3).

Definition 2.2. Given a nested family $\big(A_\delta;\ \delta \in (0,1]\,\big)$ of classical subsets of M, a fuzzy subset A^* of M is generated, whose membership function $\mu(\cdot)$ is defined in the following way:

$$\mu(x) := \sup\left\{\delta \cdot \mathbb{1}_{A_\delta}(x):\ \delta \in (0,1]\right\} \qquad \forall\, x \in M$$

Example 2.3. Let M be the set of real numbers and let the nested family $\big(A_\delta;\ \delta \in (0,1]\,\big)$ of classical subsets of M be defined in following way:

$$A_\delta := \left\{\begin{array}{cl} [0,\ 2] & \text{for } \delta \in \left(0, \frac{1}{2}\right) \\ \left[\frac{1}{2},\ \frac{3}{2}\right] & \text{for } \delta \in \left[\frac{1}{2}, 1\right] \end{array}\right\}$$

Then the generated fuzzy subset A^* has the following membership function $\mu(\cdot)$ according to Definition 2.2:

$$\mu(x) = \left\{\begin{array}{cl} 0 & \text{for } x \notin [0,2] \\ \frac{1}{2} & \text{for } x \in \left[0, \frac{1}{2}\right) \cup \left(\frac{3}{2}, 2\right] \\ 1 & \text{for } x \in \left[\frac{1}{2}, \frac{3}{2}\right] \end{array}\right\} \qquad \forall\, x \in M$$

The δ-Cuts for this fuzzy subset A^* are the following:

$$\mathcal{C}_\delta[A^*] = \left\{\begin{array}{cl} [0,\ 2] & \text{for } \delta \in \left(0, \frac{1}{2}\right] \\ \left[\frac{1}{2},\ \frac{3}{2}\right] & \text{for } \delta \in \left(\frac{1}{2}, 1\right] \end{array}\right\} \qquad \forall\, \delta \in (0,1]$$

Now the set $A_{\frac{1}{2}}$ is equal to the interval $\left[\frac{1}{2},\ \frac{3}{2}\right]$ and the δ-Cut $\mathcal{C}_{\frac{1}{2}}[A^*]$ is equal to the interval $[0,\ 2]$.

For the membership function generated by a nested family $\big(A_\delta;\ \delta \in (0,1]\,\big)$ the following holds:

Theorem 2.4. *Let $\mathcal{C}_\delta[A^*]$ be the δ-Cut of a fuzzy set A^* with membership function $\mu(\cdot)$ generated by a nested family of subsets $\big(A_\delta;\ \delta \in (0,1]\,\big)$ of a set M according to Definition 2.2. Then for any $\delta \in (0,1]$ the following holds true:*

$$\mathcal{C}_\delta[A^*] = A_\delta \qquad \textit{if and only if} \qquad A_\delta = \bigcap_{0<\beta<\delta} A_\beta$$

Proof. Firstly extend the nested family of subsets $\big(A_\delta;\ \delta \in (0,1]\big)$ by the element $A_0 := M$. Then the proof is made in three steps.

- $A_\delta \subseteq \mathcal{C}_\delta[A^*]$ is fulfilled $\forall\, \delta \in (0,1]$.
  For arbitrary $x \in A_\delta$ we have $\delta \cdot \mathbb{1}_{A_\delta} = \delta$ and thus $\sup\big\{\beta \cdot \mathbb{1}_{A_\beta}(x) \colon \beta \in (0,1]\big\} \geq \delta$. Using the definition of $\mu(\cdot)$ we have $\mu(x) \geq \delta$ and from the definition of the δ-Cut we obtain $x \in \mathcal{C}_\delta[A^*]$.
- $A_\delta = \bigcap\limits_{0<\beta<\delta} A_\beta \quad \Rightarrow \quad \mathcal{C}_\delta[A^*] = A_\delta$
  Choose $x \notin A_\delta$. From $A_\delta = \bigcap_{0<\beta<\delta} A_\beta$ we know that there exists $\alpha < \delta$, with $x \notin A_\alpha$ and from the nested structure of the generating family $\big(A_\delta;\ \delta \in (0,1]\big)$ we know that $x \notin A_\beta\ \forall\, \beta \in [\alpha, 1]$. We have
  $$\mu(x) = \sup\big\{\beta \cdot \mathbb{1}_{A_\beta}(x) \colon \beta \in [0,1]\big\} \leq \alpha < \delta$$
  and from that $x \notin \mathcal{C}_\delta[A^*]$.
- $\mathcal{C}_\delta[A^*] = A_\delta \quad \Rightarrow \quad A_\delta = \bigcap\limits_{0<\beta<\delta} A_\beta$
  $A_\delta \subseteq \bigcap_{0<\beta<\delta} A_\beta$ holds from the nested structure of the generating family $\big(A_\delta;\ \delta \in (0,1]\big)$. Choose $x \notin \mathcal{C}_\delta[A^*]$ and suppose that $A_\delta = \mathcal{C}_\delta[A^*]$. Then
  $$\mu(x) = \sup\big\{\beta \cdot \mathbb{1}_{A_\beta}(x) \colon \beta \in [0,1]\big\} < \delta.$$
  Choose $\alpha \in (\mu(x), \delta)$, then $x \notin A_\alpha$ and hence $x \notin \bigcap_{0<\beta<\delta} A_\beta$.

□

Remark 2.5. For a nested sequence of classical subsets $B_1 \subseteq B_2 \subseteq \ldots \subseteq B_n$ of a set M a nested family of subsets $\big(A_\delta;\ \delta \in (0,1]\big)$, where $A_{\delta_1} \supseteq A_{\delta_2}$ for $\delta_1 < \delta_2$, can be constructed in the following way:

$$A_\delta = B_i, \text{ where } \delta \in \big(1 - \tfrac{i}{n}, 1 - \tfrac{i-1}{n}\big] \qquad \forall \delta \in (0,1]$$

3. Fuzzy Numbers and Characterizing Functions

In applications, measurement results of continuous quantities do not result in *precise numbers times the measurement unit*, but are always more or less imprecise and we call them fuzzy. For one-dimensional continuous quantities real measurement data are best described by so-called fuzzy numbers.

Definition 3.1. A fuzzy subset x^* of the set of real numbers $\mathbb{R}$ is called *fuzzy number* if its membership function $\xi(\cdot)$ fulfils the following three conditions:

(1) $\xi : \mathbb{R} \to [0,1]$
(2) $\operatorname{supp}[\xi(\cdot)] := \big\{x \in \mathbb{R} \colon\ \xi(x) > 0\big\}$ is a bounded set
(3) $\forall\, \delta \in (0,1]$ the δ-Cut $\mathcal{C}_\delta[\xi(\cdot)]$ is non-empty and a finite union of compact intervals, i.e.

$$\mathcal{C}_\delta[\xi(\cdot)] = \bigcup_{j=1}^{k_\delta} [a_{\delta,j}, b_{\delta,j}]$$

Functions $\xi(\cdot)$ fulfilling conditions (1) – (3) are called *characterizing functions.*
In Figure 1 some examples of characterizing functions are depicted.

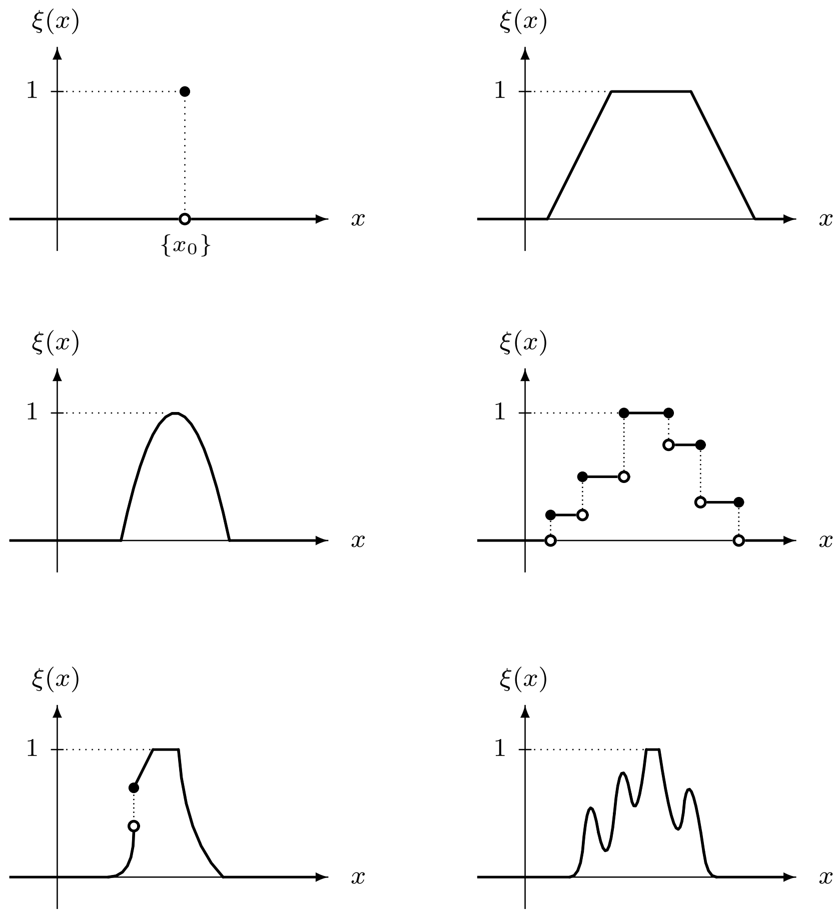


FIGURE 1. Characterizing Functions

An important topic is how to obtain the characterizing function of a fuzzy observed (measured) quantity. There is no general solution to this problem, but some measurement situations allow to generate the characterizing function. This problem is a special case of the problem obtaining the membership function of a fuzzy set. Four references for that are the following: [4] and [5] are good introductions to the problem, [2] and [8] are more recent application-oriented contributions to this topic.

Example 3.2. For a digital measurement equipment the measurement result is a decimal number with finitely many digits. About the remaining infinitely many digits nothing is known. Therefore the measurement result is an interval $[\underline{x}, \overline{x}]$, where the number $\underline{x}$ is the reading of the instrument and the remaining (infinitely many) decimals are all set to be 0. The number $\overline{x}$ is obtained if the remaining decimals are all set to be 9.

The characterizing function is the indicator function $\mathbb{1}_{[\underline{x},\overline{x}]}(\cdot)$ of the interval $[\underline{x}, \overline{x}]$. This is a special characterizing function.

Example 3.3. Let the result of the measurement of a one-dimensional continuous quantity be a light point on a screen. Then the light intensity $h(x)$ is a real valued, non-negative function of a real variable x. The characterizing function $\xi(\cdot)$ of the measurement result x^* is obtained by normalization in the following way:

$$\xi(x) := \frac{h(x)}{\max\left\{h\left(z\right):\ z \in \mathbb{R}\right\}} \quad \forall\, x \in \mathbb{R}$$

This function $\xi(\cdot)$ is a characterizing function.

Example 3.4. In case the measurement result is characterized by the (fuzzy) boundary of a color intensity picture, the color intensity $g(x)$ for $x \in \mathbb{R}$ can be used to generate the characterizing function $\xi(\cdot)$ of the observed quantity x^* in the following way:

The so-called *scaled rate of change* of the color intensity transition is used, i.e. the normalized derivative of the function $g(\cdot)$, where the derivative is denoted by $g'(\cdot)$:

$$\xi(x) := \frac{|g'(x)|}{\max\left\{\,|g'\left(z\right)|:\ z \in \mathbb{R}\right\}} \quad \forall\, x \in \mathbb{R}$$

Again by this definition a characterizing function $\xi(\cdot)$ is obtained, which is characterizing the fuzzy measurement result x^*.

Example 3.5. For analogue measurement equipments with pointers the best one can do is to make a photograph of the pointer position on the measurement scale. This is a color intensity picture. Taking the scale on a straight line the color intensity is a fuzzy point on the scale. This situation can be dealt with as in Example 3.3.

For the measurement of a one-dimensional quantity in discrete case (finitely many points) we have to extend it to the whole real line. We can create a partly constant characterizing function (Example 3.6) or a partly linear characterizing function (Example 3.7).

Example 3.6. Let the function $h(\cdot)$ be a non-negative function defined in finitely many points $\{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}$ and let the distance between $x_1 < x_2 < \ldots < x_n$ be constant and equal to Δ.

We can extend the function $h(\cdot)$ to a non-negative partially constant function $g(\cdot)$ in the following way:

$$g(x) := \left\{ \begin{array}{cl} 0 & \text{for } x < x_1 - \frac{\Delta}{2} \\ h(x_1) & \text{for } x \in \left[x_1 - \frac{\Delta}{2}, x_1 + \frac{\Delta}{2}\right) \\ \max\{h(x_1), h(x_2)\} & \text{for } x = x_1 + \frac{\Delta}{2} \\ \cdots & \\ h(x_i) & \text{for } x \in \left(x_i - \frac{\Delta}{2}, x_i + \frac{\Delta}{2}\right) \\ \max\{h(x_i), h(x_{i+1})\} & \text{for } x = x_i + \frac{\Delta}{2} \\ \cdots & \\ h(x_n) & \text{for } x \in \left(x_n - \frac{\Delta}{2}, x_n + \frac{\Delta}{2}\right] \\ 0 & \text{for } x > x_n + \frac{\Delta}{2} \end{array} \right\} \qquad \forall\, x \in \mathbb{R}$$

The function $g(\cdot)$ is defined on the whole real line, we can construct the characterizing function by normalization in the same way as in Example 3.3. In Figure 2(b) is shown the results of such construction.

Example 3.7. Let the function $h(\cdot)$ be a non-negative function defined in finitely many points $\{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}$, where $x_1 < x_2 < \ldots < x_n$ and $h(x_1) = h(x_n) = 0$.

We can extend the function $h(\cdot)$ to a non-negative partially linear function $g(\cdot)$ in the following way:

$$g(x) := \left\{ \begin{array}{cl} 0 & \text{for } x < x_0 \\ \frac{x-x_1}{x_2-x_1}\, h(x_2) & \text{for } x \in [x_1, x_2) \\ \cdots & \\ h(x_i) + \frac{x-x_i}{x_{i+1}-x_i}\left(h(x_{i+1}) - h(x_i)\right) & \text{for } x \in [x_i, x_{i+1}) \\ \cdots & \\ h(x_{n-1}) - \frac{x-x_{n-1}}{x_n-x_{n-1}}\, h(x_{n-1}) & \text{for } x \in [x_{n-1}, x_n] \\ 0 & \text{for } x > x_n \end{array} \right\} \qquad \forall\, x \in \mathbb{R}$$

This situation can be dealt with as in Example 3.3 or Example 3.4. In Figure 2(c) is shown the results of such construction (using the normalization approach).

Remark 3.8. Many measurement results are in one of the forms given by the above examples.

Another problem is measurement results of vector-valued quantities. This is explained in the next section.

4. **Fuzzy Vectors and Vector-characterizing Functions**

For vector-valued continuous quantities $\underline{x} = (x_1, \ldots, x_k)$, measurement results are again subject to variability, errors, and fuzziness. The mathematical models for errors are stochastic quantities, and that for fuzziness so-called *fuzzy vectors*.

Definition 4.1. A fuzzy subset $\underline{x}^*$ of the k-dimensional Euclidean space $\mathbb{R}^k$ is called *fuzzy vector* if its membership function $\zeta(\cdot, \ldots, \cdot)$ fulfils the following conditions:

($\underline{1}$) $\zeta : \mathbb{R}^k \to [0, 1]$

($\underline{2}$) $\operatorname{supp}\left[\zeta\left(\cdot, \ldots, \cdot\right)\right] := \left\{\underline{x} \in \mathbb{R}^k :\ \zeta\left(\underline{x}\right) > 0\right\}$ is a bounded subset of $\mathbb{R}^k$

($\underline{3}$) $\forall\ \delta \in (0, 1]$ the δ-Cut $\mathcal{C}_\delta\left[\zeta\left(\cdot, \ldots, \cdot\right)\right]$ is non-empty and a finite union of compact connected subsets of $\mathbb{R}^k$

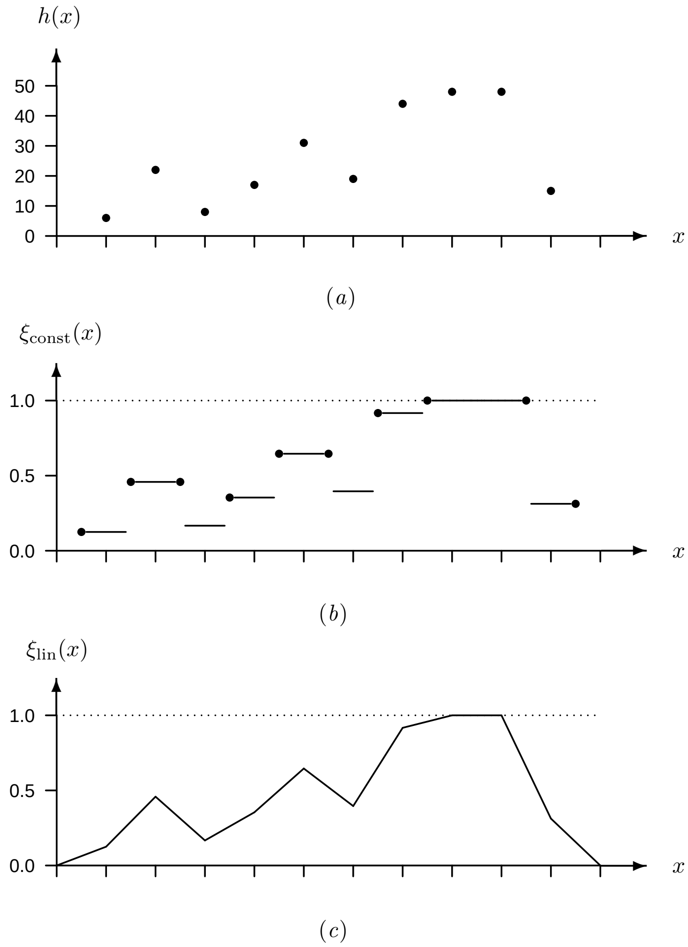


FIGURE 2. Construction of Characterizing Functions

(a) Selected points from the function $h(\cdot)$, from which the characterizing function is constructed in Examples 3.6 and 3.7
(b) Characterizing function constructed from the selected points in (a) according to Example 3.6
(c) Characterizing function constructed from the selected points in (a) according to Example 3.7

Functions $\zeta(\cdot, \ldots, \cdot)$ fulfilling the conditions in Definition 4.1 are called *vector-characterizing functions.*

In Figure 3 examples of vector-characterizing functions are depicted.

Again, it is important how to obtain the vector-characterizing function $\zeta(\cdot, \dots, \cdot)$ of a fuzzy vector.

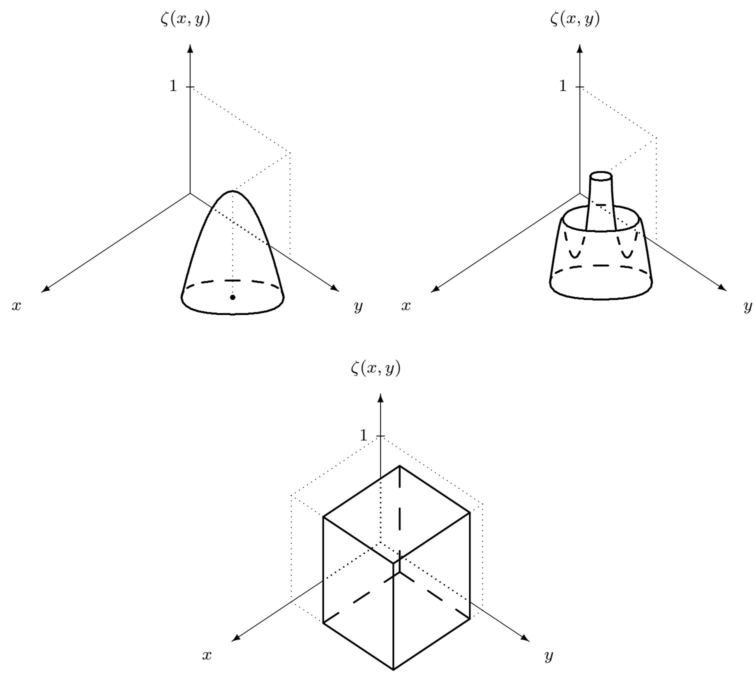


FIGURE 3. Vector-characterizing Functions

Example 4.2. In the situation of 2-dimensional continuous quantities where the so-called measurement results are presented on a screen, we can obtain the vector-characterizing function in the following way:

Let $h(x, y)$ be the light intensity of the light point $\underline{x}^* = (x, y)^*$. Then the function $h(\cdot, \cdot)$ is non-negative and bounded on the whole plane. Therefore, the values $\zeta(x, y)$ of the vector-characterizing function $\zeta(\cdot, \cdot)$ are given by

$$\zeta(x, y) := \frac{h(x, y)}{\max\left\{h(u, v) \colon (u, v) \in \mathbb{R}^2\right\}} \qquad \forall\, (x, y) \in \mathbb{R}^2.$$

The function $\zeta(\cdot, \cdot)$ defined above fulfils the conditions for vector-characterizing functions. In Figures $4(a)$ and $5(a)$ examples are shown.

Example 4.3. In case of 2-dimensional quantities, where the light intensities are defined on a finite number of points, the vector-characterizing function can be obtained in the following way:

Let $I = \{(x_1, y_1), \dots, (x_1, y_m),\ \dots,\ (x_n, y_1), \dots,\ (x_n, y_m)\} \subset \mathbb{R}^2$ be a set with $n, m \in \mathbb{N}$, where x_i and y_j are linearly ordered, and the points (x_i, y_j) are equally spaced in the plane:

$$\begin{array}{lll} \exists \Delta_x \in \mathbb{R}^+ & \forall\, i \in \{1,\dots,n-1\} & x_{i+1} = x_i + \Delta_x \\ \exists \Delta_y \in \mathbb{R}^+ & \forall\, j \in \{1,\dots,m-1\} & y_{j+1} = y_j + \Delta_y \end{array}$$

Let $h(\cdot,\cdot)$ be a non-negative function defined on the set I and let $h(x_1,\cdot) = h(\cdot,y_1) = h(x_n,\cdot) = h(\cdot,y_m) = 0$. Examples of such functions are depicted in Figures $4(b)$ and $5(b)$.

Then we can create a partly constant function $g(\cdot,\cdot)$ in the following way:

$$g(x,y) := \left\{ \begin{array}{l} h(x_i,y_j) \\ \quad \text{for } x \in (x_i - \frac{1}{2}\Delta_x, x_i + \frac{1}{2}\Delta_x) \text{ and } y \in (y_j - \frac{1}{2}\Delta_y, y_j + \frac{1}{2}\Delta_y) \\[1ex] \max\{h(x_i,y_j), h(x_{i+1},y_j)\} \\ \quad \text{for } x = x_i + \frac{1}{2}\Delta_x \text{ and } y \in (y_j - \frac{1}{2}\Delta_y, y_j + \frac{1}{2}\Delta_y) \\[1ex] \max\{h(x_i,y_j), h(x_i,y_{j+1})\} \\ \quad \text{for } x \in (x_i - \frac{1}{2}\Delta_x, x_i + \frac{1}{2}\Delta_x) \text{ and } y = y_j + \frac{1}{2}\Delta_y \\[1ex] \max\{h(x_i,y_j), h(x_{i+1},y_j), h(x_i,y_{j+1}), h(x_{i+1},y_{j+1})\} \\ \quad \text{for } x = x_i + \frac{1}{2}\Delta_x \text{ and } y = y_j + \frac{1}{2}\Delta_y \\[1ex] \qquad\qquad \text{where } i \in \{1,\dots,n-1\},\ j \in \{1,\dots,m-1\} \\[1ex] 0 \quad \text{for } x \notin (x_1,x_n) \text{ or } y \notin (y_1,y_m) \end{array} \right\}$$

$$\forall\,(x,y) \in \mathbb{R}^2$$

The vector-characterizing function $\zeta(\cdot,\cdot)$ can be obtained from the function $g(\cdot,\cdot)$ by normalization:

$$\zeta(x,y) := \frac{g(x,y)}{\max\left\{g(u,v) \colon (u,v) \in \mathbb{R}^2\right\}} \qquad \forall\,(x,y) \in \mathbb{R}^2$$

The function $\zeta(\cdot,\cdot)$ defined above again fulfils the conditions for vector-characterizing functions. In Figures $4(c)$ and $5(c)$ are shown the results of this construction.

Example 4.4. The so-called partly linear approach is a little more complicated. In the one-dimensional case we have simply connected two neighbouring points. In the two dimensional case there are four neighbouring points arranged in a rectangle. We want to interpolate a plane through four edge points of this rectangle, but this is not possible in all cases. We will create a new point in the centre of each rectangle and set it's value as mean of the values at the edges of the rectangle.

Let the set I and the function $h(\cdot,\cdot)$ be defined in the same way as in the previous example.

We define $x_{i+\frac{1}{2}} := x_i + \frac{1}{2}\Delta_x$, $y_{j+\frac{1}{2}} := y_j + \frac{1}{2}\Delta_y$, and define the set J in the following way:

$$J := \left\{ \left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}} \right) \colon i \in \{1,\dots,n-1\},\ j \in \{1,\dots,m-1\} \right\} \subset \mathbb{R}^2$$

We extend the function $h(\cdot,\cdot)$ to the set J in the following way:

$$h(x,y) = \frac{1}{4} \left(\begin{array}{l} h(x-\frac{1}{2}\Delta_x, y-\frac{1}{2}\Delta_y) + h(x-\frac{1}{2}\Delta_x, y+\frac{1}{2}\Delta_y) + \\ \qquad\qquad + h(x+\frac{1}{2}\Delta_x, y-\frac{1}{2}\Delta_y) + h(x+\frac{1}{2}\Delta_x, y+\frac{1}{2}\Delta_y) \end{array} \right)$$

$$\forall\,(x,y) \in J$$

Then we can create a partly linear function $g(\cdot,\cdot)$ in the following way:

$$g(x,y) := \left\{ \begin{array}{l} h(x_i,y_j) + \frac{(x-x_i)\Delta_y-(y-y_j)\Delta_x}{\Delta_x\Delta_y}(h(x_{i+1},y_j)-h(x_i,y_j))+ \\ \qquad +\frac{2(y-y_j)}{\Delta_y}(h(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}})-h(x_i,y_j)) \\ \qquad \text{for } x>x_i \text{ and } \frac{x-x_i}{x_{i+1}-x_i} \le \frac{y-y_j}{y_{j+1}-y_j} \text{ and } \frac{x-x_{i+1}}{x_i-x_{i+1}} > \frac{y-y_j}{y_{j+1}-y_j} \\ \\ h(x_{i+1},y_j) + \frac{(x-x_{i+1})\Delta_y+(y-y_j)\Delta_x}{\Delta_x\Delta_y}(h(x_{i+1},y_{j+1})-h(x_{i+1},y_j))- \\ \qquad -\frac{2(x-x_{i+1})}{\Delta_x}(h(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}})-h(x_{i+1},y_j)) \\ \qquad \text{for } y\le y_{j+1} \text{ and } \frac{x-x_i}{x_{i+1}-x_i} \le \frac{y-y_j}{y_{j+1}-y_j} \text{ and } \frac{x-x_{i+1}}{x_i-x_{i+1}} \le \frac{y-y_j}{y_{j+1}-y_j} \\ \\ h(x_i,y_{j+1}) + \frac{(x-x_i)\Delta_y+(y-y_{j+1})\Delta_x}{\Delta_x\Delta_y}(h(x_{i+1},y_{j+1})-h(x_i,y_{j+1}))- \\ \qquad -\frac{2(y-y_{j+1})}{\Delta_y}(h(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}})-h(x_i,y_{j+1})) \\ \qquad \text{for } x\le x_{i+1} \text{ and } \frac{x-x_i}{x_{i+1}-x_i} > \frac{y-y_j}{y_{j+1}-y_j} \text{ and } \frac{x-x_{i+1}}{x_i-x_{i+1}} \le \frac{y-y_j}{y_{j+1}-y_j} \\ \\ h(x_i,y_j) + \frac{-(x-x_i)\Delta_y+(y-y_j)\Delta_x}{\Delta_x\Delta_y}(h(x_i,y_{j+1})-h(x_i,y_j))+ \\ \qquad +\frac{2(x-x_i)}{\Delta_x}(h(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}})-h(x_i,y_j)) \\ \qquad \text{for } y>y_i \text{ and } \frac{x-x_i}{x_{i+1}-x_i} > \frac{y-y_j}{y_{j+1}-y_j} \text{ and } \frac{x-x_{i+1}}{x_i-x_{i+1}} > \frac{y-y_j}{y_{j+1}-y_j} \\ \\ \qquad\qquad \text{where } i\in\{1,\dots,n-1\},\ j\in\{1,\dots,m-1\} \\ \\ 0 \qquad \text{for } x\notin(x_1,x_n) \text{ or } y\notin(y_1,y_m) \end{array} \right\}$$

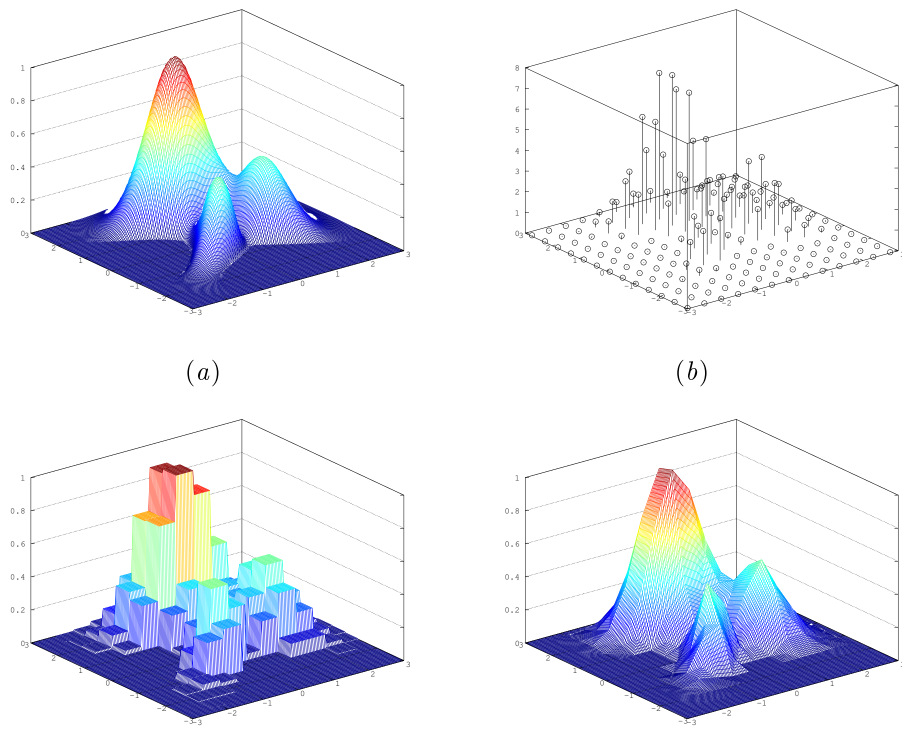
$$\forall\,(x,y)\in\mathbb{R}^2$$

The vector-characterizing function $\zeta(\cdot,\cdot)$ can be obtained again from the function $g(\cdot,\cdot)$ by normalization:

$$\zeta(x,y) := \frac{g(x,y)}{\max\left\{g(u,v)\colon (u,v)\in\mathbb{R}^2\right\}} \qquad \forall\,(x,y)\in\mathbb{R}^2$$

The function $\zeta(\cdot,\cdot)$ defined above fulfils the conditions for vector-characterizing functions. In Figures $4(d)$ and $5(d)$ the results of this construction are given.

$$h(x,y) = \max \left\{ \begin{array}{l} 3\,(1-x)^2\, e^{\left(-x^2-(y+1)^2\right)} - 10\left(\frac{1}{5}x - x^3 - y^5\right) e^{\left(-x^2-y^2\right)} \\ \qquad\qquad\qquad\qquad\qquad - \frac{1}{3} e^{\left(-(x+1)^2-y^2\right)} - \frac{1}{100} \\ 0 \end{array} \right\}$$



(a) (b) (c) (d)

FIGURE 4. Two-dimensional Vector-characterizing Functions

(a) Vector-characterizing function constructed from the function $h(\cdot,\cdot)$ according to Example 4.2
(b) Selected points from the function $h(\cdot,\cdot)$, from which the vector-characterizing function is constructed in Examples 4.3 and 4.4
(c) Vector-characterizing function constructed from the selected points in (b) according to Example 4.3
(d) Vector-characterizing function constructed from the selected points in (b) according to Example 4.4

$$h(x,y) = \left\{ \begin{array}{ll} 1 + \frac{1}{3}\pi & \text{for } x = y = 0 \\ \frac{\sin\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}} + \frac{1}{12}\left(4\pi - \sqrt{x^2+y^2}\right) & \text{for } \ 0 < \sqrt{x^2+y^2} \leq 4\pi \\ 0 & \text{for } 4\pi < \sqrt{x^2+y^2} \end{array} \right\}$$

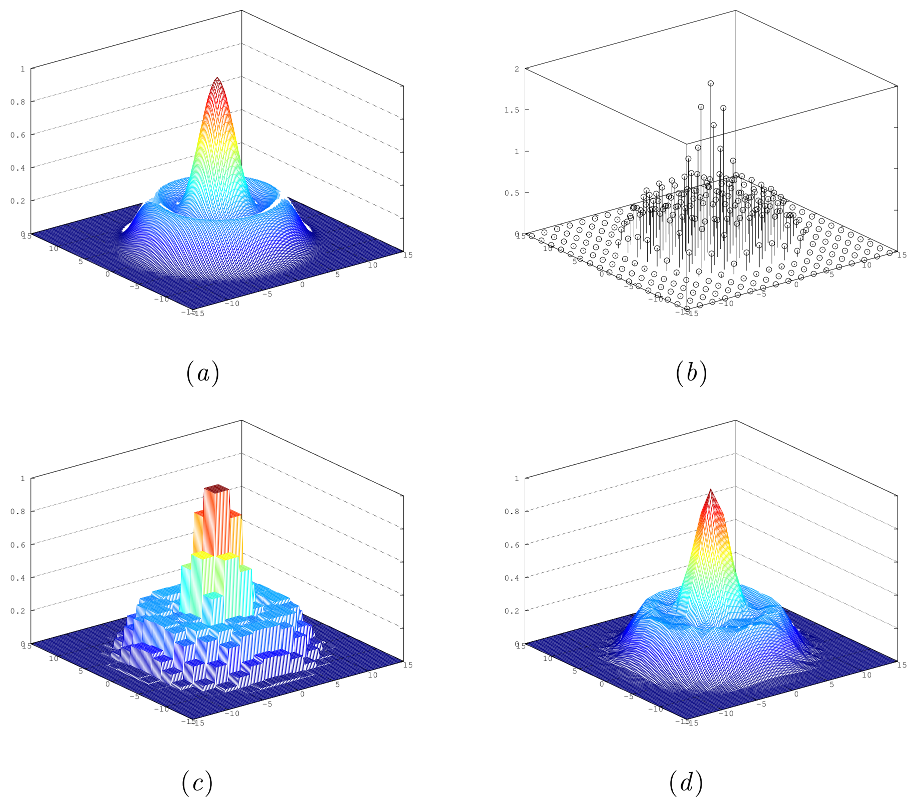


FIGURE 5. Two-dimensional Vector-characterizing Functions

(a) Vector-characterizing function constructed from the function $h(\cdot,\cdot)$ according to Example 4.2
(b) Selected points from the function $h(\cdot,\cdot)$, from which the vector-characterizing function is constructed in Examples 4.3 and 4.4
(c) Vector-characterizing function constructed from the selected points in (b) according to Example 4.3
(d) Vector-characterizing function constructed from the selected points in (b) according to Example 4.4

5. **Vectors of Fuzzy Numbers**

Definition 5.1. A n-dimensional *vector of fuzzy numbers* $(x_1^*, \ldots, x_n^*)$ is a vector containing n fuzzy numbers $x_1^*, \ldots, x_n^*$. It is determined by n characterizing functions $\xi_1(\cdot), \ldots, \xi_n(\cdot)$ belonging to the fuzzy numbers $x_1^*, \ldots, x_n^*$.

Definition 5.2. A function $T : [0,1] \times [0,1] \to [0,1]$ is called *triangular norm* or *t-norm*, if and only if $\forall\, x, y, z \in [0,1]$ the following conditions are fulfilled:

(1) $T(x, y) = T(y, x)$, i.e. T is commutative
(2) $T(T(x, y), z) = T(x, T(y, z))$, i.e. T is associative
(3) $T(x, 1) = x$, i.e. 1 is neutral to T
(4) $x \leq y \Rightarrow T(x, z) \leq T(y, z)$, i.e. T is monotone

Combination of fuzzy numbers into a fuzzy vector is possible based on t-norms. For two fuzzy numbers a^* and b^* with corresponding characterizing functions $\xi_{a^*}(\cdot)$, $\xi_{b^*}(\cdot)$ a fuzzy vector $(a, b)^*$ is given by its vector-characterizing function $\zeta_{(a,b)^*}(\cdot, \cdot)$, whose values $\zeta_{(a,b)^*}(x, y)$ are defined based on a t-norm T by:

$$\zeta_{(a,b)^*}(x, y) := T(\xi_{a^*}(x), \xi_{b^*}(y)) \qquad \forall\, (x, y) \in \mathbb{R}^2$$

In the general case of n fuzzy numbers $x_1^*, \ldots, x_n^*$ with corresponding characterizing functions $\xi_1(\cdot), \ldots, \xi_n(\cdot)$ a fuzzy vector $(x_1, \ldots, x_n)^*$ is given by its vector-characterizing function $\zeta_{(x_1,\ldots,x_n)^*}$ whose values $\zeta_{(x_1,\ldots,x_n)^*}(x_1, \ldots, x_n)$ are defined based on a t-norm T by using its associativity:

$$\zeta_{(x_1,\ldots,x_n)^*}(x_1, \ldots, x_n) := T(\xi_1(x_1), T(\ldots, T(\xi_{n-1}(x_{n-1}), \xi_n(x_n)) \ldots)) \qquad \forall\, (x_1, \ldots, x_n) \in \mathbb{R}^n$$

For details see [6].

Remark 5.3. A fuzzy vector can be obtained from a vector of fuzzy numbers using a t-norm. For statistical applications the most important t-norm is the minimum t-norm.

In Figure 6 examples for the above construction using the following t-norms are given:

Minimum t-norm:

$$T_{min}(x, y) = \min\{x, y\} \qquad \forall\, (x, y) \in [0,1]^2$$

Product t-norm:

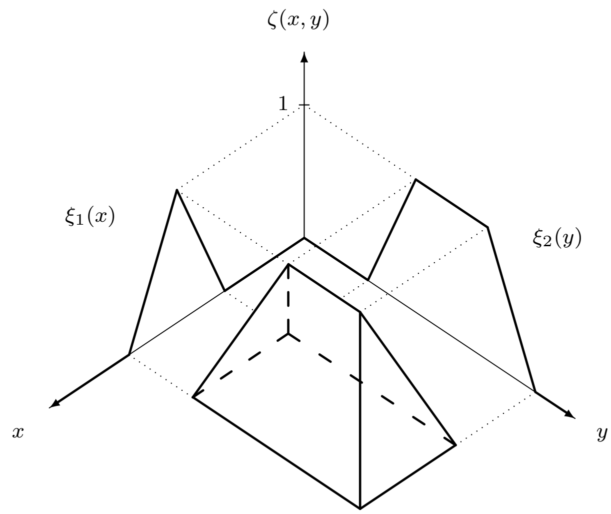
$$T_{prod}(x, y) = x \cdot y \qquad \forall\, (x, y) \in [0,1]^2$$

Limited sum t-norm:

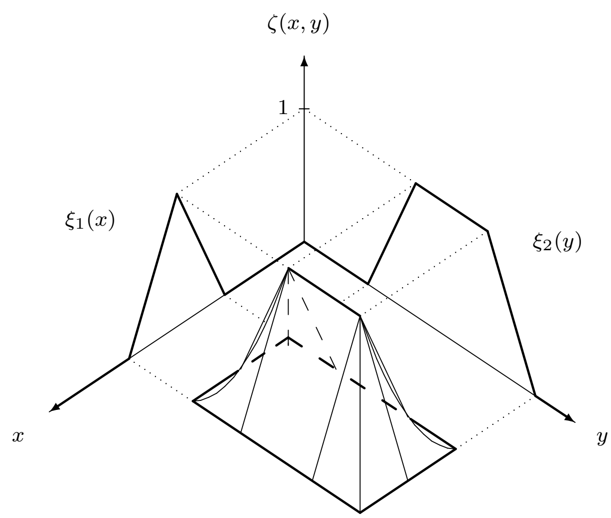
$$T_{lsum}(x, y) = \max\{x + y - 1, 0\} \qquad \forall\, (x, y) \in [0,1]^2$$

Drastic product t-norm:

$$T_{dp}(x, y) = \min\{x, y\} \cdot \mathbb{1}_{\{\max\{x,y\}\}}(1) \qquad \forall\, (x, y) \in [0,1]^2$$

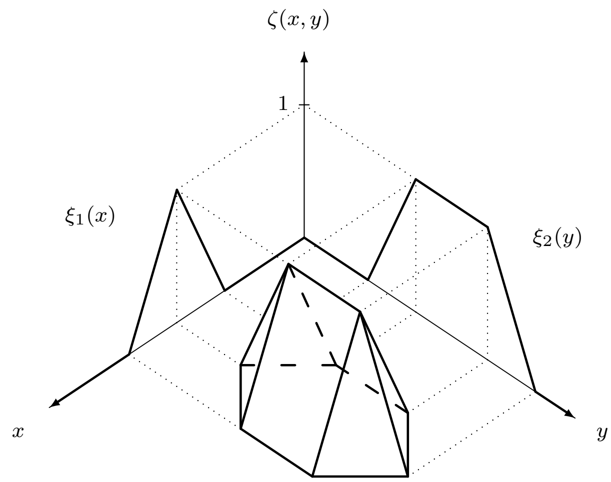


(a) Combination of Two Fuzzy Numbers Using the Minimum t-norm

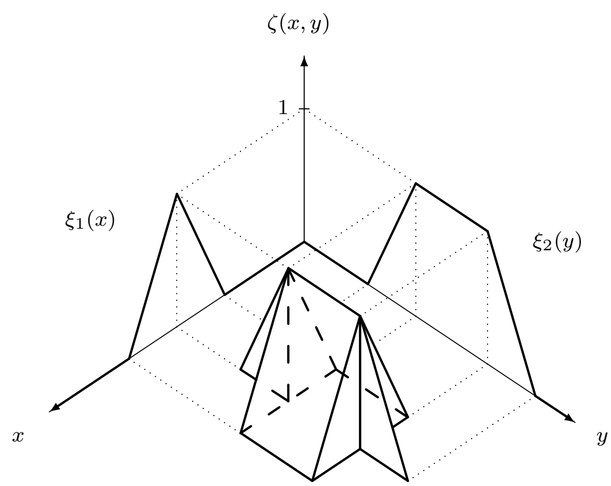


(b) Combination of Two Fuzzy Numbers Using the Product t-norm

FIGURE 6. Combinations of Fuzzy Numbers



(c) Combination of Two Fuzzy Numbers Using the Limited Sum t-norm



(d) Combination of Two Fuzzy Numbers Using the Drastic Product t-norm

FIGURE 6. Combinations of Fuzzy Numbers

6. Conclusion

There is no general rule of how to obtain the membership function of a fuzzy set. But in this paper for different important situations methods for the construction of the characterizing function of a fuzzy number as well as for the construction of vector-characterizing functions of fuzzy vectors are given.

In Section 2 of this article the δ-Cuts are introduced and an universal approach for the construction of the membership function based on a nested family of sets $\big(A_\delta;\ \delta \in (0,1]\big)$ is explained. In Theorem 2.4 a necessary and sufficient condition for the nested family of sets to be equal to the δ-Cuts of the generated fuzzy set is given.

In Section 3 fuzzy numbers are introduced and different methods are proposed how to obtain the characterizing function for one-dimensional measured quantities. Similarly in Section 4 fuzzy vectors are introduced, and different methods to obtain the vector-characterizing function for multi-dimensional data are proposed and graphically demonstrated. In Section 5 t-norms are used to describe how to combine fuzzy numbers into a fuzzy vector. Using the above described principles the characterizing function of a fuzzy number or the vector-characterizing function of a fuzzy vector can be obtained for different types of measurement data.

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