DIPLOMARBEIT

Extended CreditRisk^{+} with Guarantees

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Contents

Abstract

The current Extended CreditRisk$^+$-model has a rather rudimentary methodology to consider guarantees, which, for example, does not take any dependence between guarantors into account. This drastically limits its applicability on the market.

This thesis proposes several possible approaches how to incorporate guarantees – be it credit guarantees, reinsurance contracts or government subsidies – into the Extended CreditRisk$^+$ framework.

We first adapt the current notation of the model to allow for the securitisation of the exposure. Subsequently we propose three different methods to include the additional information in the computation of the potential portfolio loss. Finally we apply all these approaches to several exemplary portfolios and benchmark them agains known reference distributions.

Additionally we give a short presentation of a software library developed to model various distributions and in particular used to implement the proposed methods.

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1. Original model

Since this paper presents further extensions to the ECR^+-model[1] a short review of the existing results is necessary. This will also allow us to introduce the simplified notation used in this paper.

1.1. Input data

The model requires the following data:

- The number m of obligors,
- the number C of non-idiosyncratic default causes,
- the number K of independent risk factors,
- the parameters specifying the gamma distributions of the independent risk factors $R_1, \ldots, R_K$,
- a non-empty finite set $\mathcal{J}$ of dependence scenarios,
- a probability distribution on the set $\mathcal{J}$ of dependence scenarios,
- for each dependence scenario $j \in \mathcal{J}$ a matrix $A_j = (a^j_{c,k})_{c\in\{0,\ldots,C\},k\in\{0,\ldots,K\}}$ of size $(C + 1) \times (K + 1)$ with non-negative entries, where

$$a^j_{0,k} = 0 \qquad \text{for all } j \in \mathcal{J} \text{ and } k \in \{1, \ldots, K\},$$

- the collection G of nonempty subsets of all obligors $\{1, \ldots, m\}$, called the risk groups, which are subject to joint defaults.

For every group we need

- the default probability $p_g \in [0, 1]$

and then, for every dependence scenario $j \in \mathcal{J}$,

- the susceptibility $w_{0,g,j} \in [0, 1]$ to idiosyncratic default,
- the susceptibility $w_{c,g,j} \in [0, 1]$ to default causes $c \in \{1, \ldots, C\}$,

[1]This entire chapter is based on [8].

- the multivariate probability distributions $Q_{c,g,j} = (q_{c,g,j,\mu})_{\mu\in\mathbb{N}_0^g}$ describing the stochastic losses of all the obligors $i \in g$ in case the risk group g defaults due to cause $c \in \{0, \ldots, C\}$.

The original lecture notes are still evolving – in this paper we use the basic model structure from autumn 2014. The most important difference to the currently published model is that only a single time step is considered here.

1.2. Results

We begin with a reduction of the given multivariate loss distribution into a univariate distribution.

Definition 1.1 (Distribution of univariate group loss, [8, Equation 6.8])**.**
Let $Q^s_{c,g,j} = (q^s_{c,g,j,\nu})_{\nu\in\mathbb{N}_0}$ denote the distribution of the total loss of a group $g \in G$ due to the default cause $c \in \{0, \ldots, C\}$ in the scenario $j \in \mathcal{J}$, then

$$q^s_{c,g,j,\nu} = \sum_{\substack{\mu=(\mu_i)_{i\in g}\in\mathbb{N}_0^g \\ \sum_{i\in g}\mu_i=\nu}} q_{c,g,j,\mu}$$

describes the loss in terms of the underlying multivariate loss.

The corresponding random variable is defined in a natural fashion.

Definition 1.2 (Univariate group loss, [8, Remark 6.17])**.** Let $L_{c,g,j,n}$ denote the random variable representing the total loss of a group $g \in G$ due to the default cause $c \in \{0, \ldots, C\}$ in the scenario $j \in \mathcal{J}$, then

$$\mathbb{P}[L_{c,g,j,n} = \nu] = q^s_{c,g,j,\nu}$$

for all $\nu \in \mathbb{N}$.

One of the strongest assumptions in this model is the independence of group losses.

Assumption 1.3 (Independence of group losses, [8, Assumption 6.16])**.** The sequence of $\mathbb{N}_0$-valued random group losses $(L_{c,g,j,n})_{n\in\mathbb{N}}$ is i.i.d. and independent of all other random variables. (This means all other sequences of loss vectors, the scenario J, the default numbers $(N_{c,g})_{c\in\{0,\ldots,C\},g\in G}$ and the risk factors $R_1, \ldots, R_K$.)

Since the underlying idea of the model is a Poisson approximation, an intensity is needed:

Definition 1.4 (Default intensity [8, Section 6.2.3])**.** Several choices for the Poisson intensity λ_g for defaults of group $g \in G$ are possible, such as $\lambda_g = p_g(1 - p_g)$ or $\lambda_g = -\log(1 - p_g)$. We set

$$\lambda_g = p_g,$$

because with this choice the expected value of the model corresponds to the expected value of the data.

Further default causes are defined, which creates a dependence structure between the defaults of groups.

Definition 1.5 (Default causes, [8, Assumption 6.26])**.** For each $c \in \{0, \dots, C\}$ let

$$\Lambda_c = a^J_{c,0} + \sum_{k=1}^{K} a^J_{c,k} R_k.$$

be the default cause intensity of the default cause c.

The random variables $N_{c,g,j}$ – the number of defaults of risk group g due to the default cause c, given a scenario j – are the main source of randomness and carry the dependence structure within this model.

Definition 1.6 (Default numbers, [8, Assumption 6.29])**.** Let $c \in \{0, \dots, C\}$, $g \in G$, $j \in \mathscr{J}$ and $N_{c,g,j}$ such that

$$\mathscr{L}(N_{c,g,j} \mid J, R_1, \dots, R_K) = \text{Poisson}(\lambda_g w_{c,g,J} \Lambda_c)$$

then $N_{c,g,j}$ describes the number of defaults of risk group g due to the default cause c, given a scenario j.

Similarly to the group losses, default numbers are assumed to be independent:

Assumption 1.7 (Independence of default numbers, [8, Assumption 6.30])**.** Conditionally on $J, R_1, \dots, R_K$ the family

$$\{N_{c,g} \mid c \in \{0, \dots, C\}, g \in G\}$$

of default numbers is independent.

With all necessary pieces in place, we can define the main quantity of the model:

Definition 1.8 (Portfolio loss, [8, Equations 6.15, 6.17, 6.18 and 6.19])**.**
The random variable

$$L = \sum_{j \in \mathscr{J}} \mathbb{1}_{\{J=j\}} \sum_{g \in G} \sum_{c=0}^{C} \sum_{n=1}^{N_{c,g,j}} L_{c,g,j,n}.$$

describes the random loss of the entire portfolio.

For a complete introduction to the notation used within the model see [8, Chapter 6].

1.3. Notation

In order to discuss the additions to the model more efficiently we propose a slightly modified, condensed notation.

1.3.1. Scenario

The distribution of the dependence scenarios has not received a name, but as it will be used more frequently, we shall write

$$\pi_j := \mathbb{P}[J = j], \quad j \in \mathcal{J}.$$

The outermost sum in the definition of the portfolio loss shows us that L is a mixture distribution with component-weight pairs $(L_j, \pi_j)_{j\in\mathcal{J}}$. The computation of the mixture takes place at the very last moment of the algorithm (cf. last paragraph above [8, Exercise 6.53]), which allows for an individual consideration of each scenario $j \in \mathcal{J}$. Since all statements about the loss L_j for an arbitrary $j \in \mathcal{J}$ will hold for all dependence scenarios $j \in \mathcal{J}$, the random variable J shall be set for the remainder of this paper to an arbitrary, but fixed value $j \in \mathcal{J}$, unless otherwise noted.

The fixing of the the scenario $J = j$ affords us to omit the sub- or superscript j within the variables. As a consequence we will also call

$$L = \sum_{g\in G} \sum_{c=0}^{C} \sum_{n=1}^{N_{c,g}} L_{c,g,n} \tag{1.9}$$

the loss and refer to

$$\widehat{L} = \sum_{j\in\mathcal{J}} \mathbb{1}_{\{J=j\}} \sum_{g\in G} \sum_{c=0}^{C} \sum_{n=1}^{N_{c,g,j}} L_{c,g,j,n}$$

as the *portfolio loss*.

1.3.2. Groups

The input data allows each risk group to default due to several default causes and the proportion of the default intensity of a risk group $g \in G$ due to the default cause $c \in \{0, \dots, C\}$ is the susceptibility $w_{c,g}$. Assumption 1.3, however, stipulates that all loss distributions $L_{c,g}$ be independent.

Instead of considering each group and then, within the group, each possible default cause, it is possible to consider the actual risk group / default cause pairs as single entities. Not only will this reduce the number of empty summands in sums of the type $\sum_{g\in G} \sum_{c=0}^{C}$, but also allow us to eliminate the susceptibility constants altogether.

Definition 1.10. Let $\widehat{g} := (g, c)$ be a pair consisting of a risk group $g \in G$ and a default cause $c \in \{0, \dots, C\}$. Let

$$c_g c_{\widehat{g}} := c$$

denote the pair's default cause and

$$o_{\widehat{g}} := g$$

denote the pair's set of obligors. Further let all set related operations and relations on $\widehat{g}$ pertain to $o_{\widehat{g}} = g$, so that for example

$$\sum_{i \in \widehat{g}} L_i := \sum_{i \in g} L_i.$$

The pair $\widehat{g}$ shall be called a *portfolio group* or just *group*.
The set of all groups shall be $\widehat{G}$:

$$\widehat{G} = \{\widehat{g} = (g, c) \,|\, g \in G, c \in \{0, \dots, C\}, w_{c,g} > 0\} \subseteq G \times \{0, \dots, C\}.$$

Remark 1.11. It is important to note that the definition of $\widehat{G}$ depends on the current scenario $j \in \mathcal{J}$, because the susceptibility can be defined differently for each scenario.

Whenever a variable references both a risk group g and a default cause c, this reference can now be replaced with a single reference to a group $\widehat{g}$, e.g.

$$L_{\widehat{g},n} := L_{c,g,n}, \qquad N_{\widehat{g}} := N_{c,g}.$$

By extending the notation of the probability of default and the default cause as

$$p_{\widehat{g}} := p_g w_{c,g} \qquad \text{and} \qquad \Lambda_{\widehat{g}} := \Lambda_{c_{\widehat{g}}}$$

and following the same logic with the derived parameters as in [8], so that e.g. $\lambda_{\widehat{g}} = p_{\widehat{g}}$, the conditional distribution of the number of defaults of a portfolio group $N_{\widehat{g}}$ from definition (1.6) can be rewritten as

$$\mathcal{L}(N_{\widehat{g}} \,|\, R_1, \dots, R_K) = \text{Poisson}(\lambda_{\widehat{g}} \Lambda_{\widehat{g}})$$

and the definition of the total loss L in Equation (1.9) as

$$L = \sum_{\widehat{g} \in \widehat{G}} \sum_{n=1}^{N_{\widehat{g}}} L_{\widehat{g},n}.$$

Nowhere in the rephrased equations is the original set of risk groups G used; this allows the "hat" from $\widehat{G}$ to be dropped subsequently and write $g \in G$ for "a portfolio group".

1.3.3. Dependence matrix

The description of the input data has a constraint on the dependence matrix which stipulates for each scenario $j \in \mathcal{J}$, that

$$a^j_{0,k} = 0 \qquad \forall k \in \{1, \dots, K\}.$$

The reason for this condition is to guarantee the existence of a purely idiosyncratic default cause. An examination of the ensuing arguments in [8] reveals, however, that the existence of such a default cause is not necessary. It is, indeed, considered with special cases throughout the computation, but the algorithm also goes to great lengths to ensure its validity under degenerate risk factors – that is risk factors, whose variance σ_k^2 is 0. (See [8, Equation 6.94] or [8, Equation 6.100].) With these safeguards in place it is easy to generate an arbitrary number (including none!) of purely idiosyncratic default causes:

Example 1.12 (Custom idiosyncratic default cause, not Λ_0)**.** Let $R_\kappa, \kappa \in \{1, \dots, K\}$ be a degenerate risk factor, i.e. $\sigma_\kappa^2 = 0$. Further set for some $c \in \{1, \dots, C\}$

$$a_{c,k} = \mathbb{1}_{\{k=\kappa\}} \qquad \forall k \in \{1, \dots, K\}.$$

Then

$$\Lambda_c = a_{c,0}^J + \sum_{k=1}^{K} a_{c,k}^J R_k = R_\kappa \equiv e_\kappa$$

behaves exactly like the idiosyncratic default cause Λ_0.

Therefore it is in fact possible to further generalise the input data by omitting any particular condition for the scenario matrix and explicitly allowing degenerate distributions for the risk factors.

1.3.4. Revised notation for the input data

Summarising all the proposed simplifications and rephrasing the requirement for the input data of the ECR$^+$-model: (Here we deliberately specify the sub- or superscript j denoting the scenario, because we deal with a complete model.)

The model itself contains these pieces of information which are shared across all scenarios:

- The set $O = \{0, \dots, m\}$ of all obligors,
- the independent risk factors $R_1, \dots, R_K$ which are gamma distributed with expectation e_k and variance σ_k^2 (the degenerate case of $\sigma_k^2 = 0$ is explicitly allowed!),
- a non-empty finite set $\mathcal{J}$ of dependence scenarios and
- the probability distribution $(\pi_j)_{j \in \mathcal{J}}$ of the scenarios.

A single scenario contains

- the dependence matrix, a $C \times K$ matrix $A_j = (a_{c,k}^j)_{c \in \{1,\dots,C\}, k \in \{1,\dots,K\}}$ with non-negative entries and
- a set G_j of portfolio groups;

whereas each group $g \in G_j$ contains

- a set $o_g \subseteq O$ of obligors,
- its probability of default $p_g \in [0,1]$,
- its default cause $c_g \in \{1, \ldots, C\}$ and
- a multivariate probability distribution $Q_g = (q_{g,\mu})_{\mu \in \mathbb{N}_0^g}$ describing the stochastic losses of all the obligors $i \in g$ in case the portfolio group g defaults.

With this input data the building blocks of the model, as defined in section 1.2, can also be rewritten.

Each default cause intensity $\Lambda_c, c \in \{1, \ldots, C\}$ can be written as

$$\Lambda_c = \sum_{k=1}^{K} a_{c,k}^J R_k,$$

or even simpler, if we consider the vectors $\mathbf{\Lambda} = (\Lambda_1, \ldots, \Lambda_C)$ and $\mathbf{R} = (R_1, \ldots, R_K)$, as

$$\mathbf{\Lambda} = A_J \cdot \mathbf{R}.$$

The number of default causes $N_{g,j}$ has the conditional distribution

$$\mathscr{L}(N_g \mid J, R_1, \ldots, R_K) = \text{Poisson}(\lambda_g \Lambda_g)$$

and the portfolio loss can be written as

$$\widehat{L} = \sum_{j \in \mathcal{J}} \mathbb{1}_{\{J=j\}} \sum_{g \in G_j} \sum_{n=1}^{N_g} L_{g,n}. \tag{1.13}$$

1.3.5. Aggregated groups

The transformation of the multivariate group loss Q_g into the univariate group loss Q_g^s is one of the first steps performed in the model. If only the portfolio loss distribution and its properties are of interest, but not the additional results of the model, such as obligor contributions, this offers a further possibility to reduce the input data.

The model works equally as well when each group is defined with a single obligor and a corresponding univariate loss distribution Q_g. This removes the need to compute Q_g^s, which in some situations – such as a large, comonotonic group – may be computationally expensive.

1.3.6. Computation

While Equation (1.13) is the definition of the portfolio loss, a direct computation thereof usually will be prohibitively expensive (in terms of computational power and time), but [8] shows that the computation can be reduced to a recursive algorithm based on the Panjer-recursion.

Definition 1.14 (Mixture distribution)**.** Let $(X_i)_{i\in\mathcal{J}}$ be a family of random variables and let $(w_i)_{i\in\mathcal{J}} \in [0,1]^{\mathcal{J}}$ be a family of corresponding weights such that $\sum_{i\in\mathcal{J}} w_i = 1$.

We say that the random variable M is distributed according to a mixture of $(X_i)_{i\in\mathcal{J}}$ with weights $(w_i)_{i\in\mathcal{J}}$ if

$$\mathscr{L}(M) = \sum_{i\in\mathcal{J}} w_i \mathscr{L}(X_i).$$

In this case we can also write

$$\bigoplus_{i\in\mathcal{J}} w_i \cdot X_i := M$$

whenever we need this mixture of $(X_i)_{i\in\mathcal{J}}$ without explicitly defining M.

Definition 1.15 (Non-Zero transformation)**.** Let X be a discrete random variable with domain $\mathbb{N}$ such that $\mathbb{P}(X = 0) < 1$.

We call X^+ the non-zero transformation of X, which is a random variable such that

$$\mathbb{P}[X^+ = x] = \begin{cases} 0 & \text{if } x = 0 \text{ and} \\ \frac{\mathbb{P}[X=x]}{1-\mathbb{P}[X=0]} & \text{otherwise.} \end{cases}$$

If $\mathbb{P}[X = 0] = 0$ then X^+ is equal to X.

The portfolio loss can be computed with the following steps[2]:

$$
\begin{aligned}
\lambda_{j,k,g} &:= \frac{\lambda_g a^j_{c_g,k}}{\sum_{g'\in G_j} \lambda_{g'} a^j_{c_{g'},k}} & Y_{j,k} &:= \left(\bigoplus_{g\in G_j} \lambda_{j,k,g}\cdot Q^s_g\right)^+ \\
\overline{\lambda}_{j,k} &:= \sum_{g\in G_j} \lambda_g a^j_{c_g,k}(1-q^s_{g,0}) & p_{j,k} &:= \frac{\overline{\lambda}_{j,k}\sigma^2_k}{e_k+\overline{\lambda}_{j,k}\sigma^2_k} \\
M_{j,k} &\sim \begin{cases}\mathrm{Log}(p_{j,k}), & p>0\\ \mathrm{Dirac}(1), & p=0\end{cases} & S_{j,k} &:= \sum_{n=1}^{M_{j,k}} Y_{j,k,n} \\
c(p) &:= \begin{cases}-\frac{\log(1-p)}{p}, & p\in(0,1)\\ 1, & p=0\end{cases} & \widetilde{\lambda}_{j,k} &:= \frac{\overline{\lambda}_{j,k}e^2_k}{e_k+\overline{\lambda}_{j,k}\sigma^2_k}c(p_{j,k}) \\
\lambda_j &:= \sum_{k=1}^{K}\widetilde{\lambda}_{j,k} & Y_j &:= \bigoplus_{k=1}^{K}\frac{\widetilde{\lambda}_{j,k}}{\lambda_j}\cdot S_{j,k} \\
M_j &\sim \mathrm{Poi}(\lambda_j) & S_j &:= \sum_{n=1}^{M_j} Y_{j,n} \\
L &= \bigoplus_{j\in\mathcal{J}} \pi_j\cdot S_j
\end{aligned}
\tag{1.16}
$$

or in summary

$$
L = \bigoplus_{j\in\mathcal{J}} \pi_j \cdot \sum_{n=1}^{\mathrm{Poi}(\lambda_j)} \bigoplus_{k=1}^{K} \frac{\widetilde{\lambda}_{j,k}}{\lambda_j} \cdot \sum_{m=1}^{M_{j,k}} \left(\bigoplus_{g\in G_j} \lambda_{j,k,g}\cdot Q^s_g\right)^+.
$$

[2]This is a half-page summary of a 45 page long [8, Chapter 6]. It only contains the essential results and omits any explanations or considerations of special cases.

2. Guarantees

Risk reducing arrangements play an important role in the economy. Not only do they act as a kind of shock-absorber spreading the strain of a default, they also allow smaller market participants to combine their resources and take on bigger positions than they could on their own.

2.1. Definitions

This paper is not a legal text, nor is the author a lawyer; therefore simplified, albeit adequate definitions of some common terms will be used:

Definition 2.1. An *exposure* is a cash flow triggered by the default of an obligor.

In the the ECR^+-model we represent the exposures of the obligors in a group as the random variables $L_{g,i}$, or of the whole group as L_g

Definition 2.2. A *guarantee* is an arrangement whereby obligation to settle an exposure is passed from one obligor to another when the former defaults.

The receiving obligor is called a *guarantor*.

We thus disregard any legal subtleties such as the distinction between a contingent guarantee, a surety or an aval, since apart from their legal differences, their influence is equal from the lender's point of view. In fact, what we call a guarantee in this paper does not have to be a *credit* guarantee; it can be a reinsurance contract or a government subsidy – generally speaking *anything* which *reduces the exposure*.

An important property of guarantors is their ability to default, which we have to model.

Definition 2.3. We assign to each guarantor s

- a[1] Bernoulli random variable D_s indicating the default of the guarantor s such that $\mathbb{P}[D_s = 1] =: \pi_s$ and
- the guarantor's default cause $c_s \in \{1, \ldots, C\}$.

In analogy to groups we set

$$\Lambda_s := \Lambda_{c_s}, \qquad s \in S$$

as the default cause intensity of the guarantor s.

[1]Section 3.3 provides an explicit construction of these random variables.

Definition 2.4. Let $S = \{1, \ldots, s\}$ be a set representing guarantors. Using the individual default indicators of the guarantors we can construct composite events.

Let $\mathfrak{S} \subseteq S$ be a subset of S.

The event

$$D_{\mathfrak{S}} := \left\{ \prod_{s \in \mathfrak{S}} D_s = 1 \right\}$$

describes the situation that all of the guarantors of the set $\mathfrak{S}$ have defaulted, with the corresponding probability

$$\pi_{\mathfrak{S}} := \mathbb{P}(D_{\mathfrak{S}}).$$

Further let

$$U_{\mathfrak{S}} := \left\{ \sum_{s \in \mathfrak{S}} D_s = 0 \right\}$$

be the event that none of the guarantors of the set $\mathfrak{S}$ have defaulted, with the probability

$$\rho_{\mathfrak{S}} := \mathbb{P}(U_{\mathfrak{S}}).$$

Combining the two events, we define

$$D^S_{\mathfrak{S}} := D_{\mathfrak{S}} \cap U_{S \setminus \mathfrak{S}} = \left\{ \prod_{s \in \mathfrak{S}} D_s = 1, \sum_{s \in S \setminus \mathfrak{S}} D_s = 0 \right\}$$

as the event that from the set S exactly the guarantors from $\mathfrak{S}$ default. Correspondingly we set

$$\pi^S_{\mathfrak{S}} := \mathbb{P}(D^S_{\mathfrak{S}})$$

to be the probability of this event.

Remark 2.5 (Independence of guarantors)**.** If we assume that all guarantors are independent, the probabilities of the $\mathfrak{S}$-events can be computed by elementary means as

$$\pi_{\mathfrak{S}} = \prod_{s \in \mathfrak{S}} \pi_s, \qquad \rho_{\mathfrak{S}} = \prod_{s \in \mathfrak{S}} (1 - \pi_s) \qquad \text{and} \qquad \pi^S_{\mathfrak{S}} = \pi_{\mathfrak{S}} \cdot \rho_{S \setminus \mathfrak{S}}.$$

In order to strengthen the security of a guarantee multiple guarantors may take part in a guarantee. In this case we assume that the guarantors have a defined order – it has to be known who has to shoulder the liability in case of a default.

Definition 2.6. A sequence $(s_r)_{r=1,\ldots,R}$ of guarantors of a guarantee is called the *chain of guarantors*. The index r of a guarantor s_r is called his *rank*.

The order of this list shall be such, that in case of a default of the underlying obligor the sequence will be traversed sequentially starting from $r = 1$ and the first guarantor not in default shall take on the liability. If no such guarantor can be found the guarantee (and its underlying exposure) will be considered in default.

The guarantor with the rank 1 is therefore called *primary*.

Assumption 2.7. A guarantor can not appear in a chain of guarantors more than once.

Since we model the order of guarantors *when the preceding ones default*, it is pointless to insert a guarantor *who is defaulted already* again with a higher rank.

Assumption 2.8. All chains of guarantors are finite.

Since repeating guarantors have already been ruled out an infinite guarantor chain would require an infinite number of unique guarantors – a scenario our current understanding of physics does not allow.

A guarantee exposure and its chain of guarantors are inextricably connected:

Definition 2.9. A tuple $h := \left(L, s_{h,0}, (s_{h,r})_{r=1,\dots,R_h}\right)$ of a univariate loss distribution L (the exposure), its obligor $s_{h,0}$ and its corresponding chain of guarantors $(s_{h,r})_{r=1,\dots,R_h}$ is called a *guarantee block*.

Remark 2.10. The definition of a guarantee block also encompasses unsecured exposures, i.e. exposures without any guarantors: for such a block h we set $R_h = 0$.

The guarantee blocks do not make any assumptions about the joint distribution of their exposures. This can be used to represent even complex guarantee structures containing multiple guarantors with unequal covers. In fact, pretty much any customary guarantee, which does not depend on any external factors (such as the outcome of an other guarantee), can be represented with guarantee blocks.

The structure of a guarantee block allows them to be compared and – to some extent – to be combined.

Definition 2.11. Two guarantee blocks are called *reducible* if they have the same obligor and chain of guarantors. A set of blocks is called irreducible if it does not contain any reducible pairs of guarantee blocks.

Let $\left(L, o, (s_r)_{r=1,\dots,R}\right)$ and $\left(M, o, (s_r)_{r=1,\dots,R}\right)$ be two reducible guarantee blocks. They can be represented as a single guarantee block by $\left(L + M, o, (s_r)_{r=1,\dots,R}\right)$.

Assumption 2.12. All sets of guarantee blocks encountered in this paper will be assumed to be irreducible.

Since we will not only deal with single guarantors, but entire sets of them, some notational shortcuts will prove useful:

Definition 2.13. Let h be a guarantee block, then let

$$S_h := \bigcup_{r=\{1,\dots,R_h\}} \{s_{h,r}\}$$

be the set of all guarantors within the block h. This can be extended naturally to any set of guarantees $\mathfrak{H}$ by

$$S_{\mathfrak{H}} := \bigcup_{h \in \mathfrak{H}} S_h.$$

It is also important to know which guarantees will default if a certain set of guarantors defaults. This is described by the set

$$H_{\mathfrak{H}}^{\mathfrak{S}} := \{h \in \mathfrak{H} : S_h \subseteq \mathfrak{S}\},$$

where $\mathfrak{H}$ is the set of guarantees to consider and $\mathfrak{S}$ is the set of defaulted guarantors.

2.2. Example

Let us consider the following situation:

> Company A receives a loan of €100,000 from the bank K. In order to receive this loan company A has secured a guarantee from the bank L for €70,000. Further there is a credit insurance for €50,000 provided by the insurer S.

On the assumption that the bank guarantee from L takes precedence over the credit insurance S, there are essentially three ways to distribute the guarantees and thus build the guarantee blocks from K's point of view: (cf. Figure 2.1)

(a) The insurer's guarantee covers only a sum already covered by the guaranteeing bank. In this case the blocks are

- $h_1 = (€\,50{,}000, A, (L, S))$
- $h_2 = (€\,20{,}000, A, (L))$
- $h_3 = (€\,30{,}000, A, ())$

(b) The insurer's guarantee overlaps with the bank's guarantee, but also leaves a part of the underlying uncovered. In this case the blocks could be

- $h_1 = (€\,40{,}000, A, (L))$
- $h_2 = (€\,30{,}000, A, (L, S))$
- $h_3 = (€\,20{,}000, A, (S))$
- $h_4 = (€\,10{,}000, A, ())$

(c) The insurer's guarantee overlaps with the bank's guarantee and both cover the entire underlying. In this case the blocks are

- $h_1 = (€\,50{,}000, A, (L))$
- $h_2 = (€\,20{,}000, A, (L, S))$
- $h_3 = (€\,30{,}000, A, (S))$

Which of these to set up in the model can only be determined by the language of the guarantees' contracts.

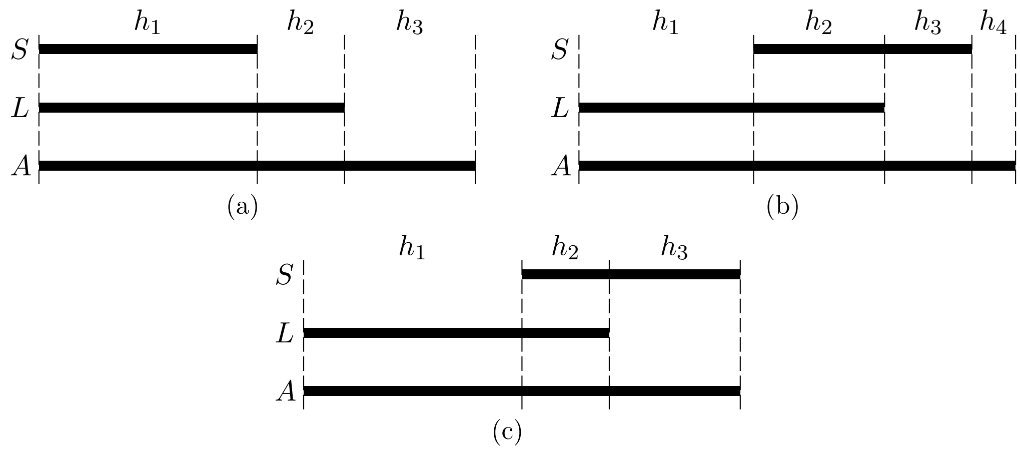


Figure 2.1.: Possible block configurations

2.3. Input data

The current input data does not contain any information on guarantees and has to be amended accordingly.

To the global data of the model will be added

- the set $S := \{1, \ldots, s\}$ representing the guarantors and
- for each guarantor $s \in S$
  - a Bernoulli random variable D_s indicating the guarantor's default such that $\mathbb{P}[D_s = 1] =: \pi_s$ and
  - the guarantor's default cause $c_s \in \{1, \ldots, C\}$.

Within each group $g \in G$ a set $H_g = \{(L_{g,h}, s_{h,0}, (s_{h,r})_{r \in \{1,\ldots,R_h\}})\}$ of guarantee blocks has to be stated. On the other hand the loss distribution Q_g of the group does not have to be given any more, as all of the proposed approaches will compute it on their own.

Note that no requirements on the dependencies between two losses of guarantee blocks or defaults of guarantors have been put forward. This is because the following approaches will either impose their own assumptions on the dependence structure, or accept any dependence structure.

3. Conditioning

While considering the guarantees of a portfolio we will encounter questions like "What does the portfolio look like, if a given guarantor defaults?" Since the defaults of guarantors are driven by the same default causes as the default of groups, they are dependent. This, in turn, allows us – or even forces us – to update the risk factors to accommodate the additional information of the default or non-default of guarantors.

3.1. Gamma distribution

The risk factors of the ECR$^+$-model as it is discussed in this paper are independent and gamma distributed. This section therefore gives the basic properties and some results on the gamma distribution.

Definition 3.1. We denote the non-negative real numbers with $\mathbb{R}_+ := [0, \infty)$. Similarly we denote the positive real numbers with $\mathbb{R}_+^* := (0, \infty)$.

Definition 3.2 (Gamma function)**.** For any number $s \in \mathbb{R}_+$ we call

$$\Gamma(s) := \int_0^\infty t^{s-1}\,\mathrm{e}^{-t}\,\mathrm{d}t.$$

the gamma function at s.

Lemma 3.3. *The gamma function satisfies the relation*

$$\Gamma(x+1) = x\,\Gamma(x),$$

which for natural numbers $n \in \mathbb{N}$ implies

$$\Gamma(n+1) = n!,$$

making the gamma function an extension of the factorial function to real positive numbers.

Proof. The first statement follows by integrating the defining integral of the gamma function by parts, while the second statement follows by induction, starting with $\Gamma(1) = 1$. ∎

Definition 3.4 (Gamma distribution)**.** Let $\alpha \in \mathbb{R}_+$ be the *shape* and $\beta \in \mathbb{R}_+$ the *rate* of a gamma distribution, then its density is given by

$$f(x) := \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\,\mathrm{e}^{-\beta x}, \quad x > 0.$$

The gamma distribution is denoted by $\Gamma(\alpha, \beta)$.

Lemma 3.5. *Let $X \sim \Gamma(\alpha, \beta)$. Its cumulative distribution function is*

$$\mathbb{P}(X \leq x) = \frac{1}{\Gamma(\alpha)} \gamma(\alpha, \beta x), \quad x > 0$$

where

$$\gamma(s, x) := \int_0^x t^{s-1} \operatorname{e}^{-t} \mathrm{d}t, \quad s, t > 0$$

is the lower incomplete gamma function.

Proof. The result follows from a direct integration of the density and a substitution:

$$\begin{aligned}\mathbb{P}(X \leq x) &= \int_0^x f(t) \, \mathrm{d}t = \int_0^x \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} \operatorname{e}^{-\beta t} \mathrm{d}t \\ &= \frac{1}{\Gamma(\alpha)} \int_0^x \beta(\beta t)^{\alpha-1} \operatorname{e}^{-\beta t} \mathrm{d}t,\end{aligned}$$

substituting $u = \beta t$ yields

$$\begin{aligned} &= \frac{1}{\Gamma(\alpha)} \int_0^{\beta x} u^{\alpha-1} \operatorname{e}^{-u} \mathrm{d}u \\ &= \frac{1}{\Gamma(\alpha)} \gamma(\alpha, \beta x).\end{aligned}$$

■

Both the regular as well as the exponential moments of a gamma distribution can be computed.

Lemma 3.6 ([8, Section 4.3.1])**.** *Let $X \sim \Gamma(\alpha, \beta)$ be a random variable. For $r \in (-\alpha, \infty)$ and $s \in (-\infty, \beta)$*

$$\mathbb{E}\Big[X^r \operatorname{e}^{sX}\Big] = \mathbb{E}\big[X^r\big] \cdot \mathbb{E}\Big[\operatorname{e}^{sX}\Big] = \frac{\Gamma(\alpha + r)}{\beta^r \Gamma(\alpha)} \cdot \left(1 - \frac{s}{\beta}\right)^{-(\alpha+r)}.$$

Proof. The expression can be computed directly using the distribution's density:

$$\begin{aligned}\mathbb{E}\Big[X^r \operatorname{e}^{sX}\Big] &= \int_0^\infty x^r \operatorname{e}^{sx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \operatorname{e}^{-\beta x} \mathrm{d}x \\ &= \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha+r-1} \operatorname{e}^{-(\beta-s)x} \mathrm{d}x.\end{aligned}$$

By expanding the fractions and moving some parts out of the integral we get

$$\mathbb{E}\Big[X^r \operatorname{e}^{sX}\Big] = \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)} \cdot \frac{\beta^\alpha}{(\beta - s)^{\alpha+r}} \int_0^\infty \frac{(\beta - s)^{\alpha+r}}{\Gamma(\alpha + r)} x^{\alpha+r-1} \operatorname{e}^{-(\beta-s)x} \mathrm{d}x$$

The term inside the integral is a density of the $\Gamma(\alpha + r, \beta - s)$ distribution, therefore the integral itself is 1. Expanding the second fraction with β^r and rearranging the terms yields the proposition. ■

Remark 3.7. Setting either $s = 0$ or $r = 0$ in the expression above yields the known expressions for the moments and exponential moments of the gamma distribution:

$$\mathbb{E}[X^r] = \frac{\Gamma(\alpha + r)}{\beta^r \Gamma(\alpha)}, \quad r \in (-\alpha, \infty),$$

and

$$\mathbb{E}\left[\mathrm{e}^{sX} \right] = \left(1 - \frac{s}{\beta}\right)^{-\alpha} = \frac{\beta^\alpha}{(\beta - s)^\alpha}, \quad s \in (-\infty, \beta).$$

Definition 3.8 (Biased probability measure – [8, Definition 2.10])**.**
Let X be a $[0, \infty)$-valued random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $0 < \mathbb{E}[X] < \infty$. Then the X-biased probability measure $\mathbb{P}_X$ on $(\Omega, \mathcal{F})$ is defined by

$$\mathbb{P}_X[A] := \frac{\mathbb{E}[X \mathbb{1}_A]}{\mathbb{E}[X]}, \qquad A \in \mathcal{F}.$$

Theorem 3.9 (cf. [8, Lemma 4.23])**.** *Let $X \sim \Gamma(\alpha, \beta)$ be a random variable, $r \in (-\alpha, \infty)$ and $s \in (-\beta, \infty)$. Then*

$$\mathbb{P}_{X^r \mathrm{e}^{-sX}} X^{-1} = \Gamma(\alpha + r, \beta + s),$$

which means that the distribution of X under the $X^r \mathrm{e}^{-sX}$-biased probability measure is the $\Gamma(\alpha + r, \beta + s)$ distribution.

Proof. A density of the biased probability measure can be computed directly using the definition and lemma above:

$$\frac{\mathrm{d}\mathbb{P}_{X^r \mathrm{e}^{-sX}}}{\mathrm{d}\mathbb{P}} = \frac{\beta^r \Gamma(\alpha)}{\Gamma(\alpha + r)} \left(1 + \frac{s}{\beta}\right)^{\alpha + r} X^r \mathrm{e}^{-sX} = \frac{(\beta + s)^{\alpha + r}}{\beta^\alpha} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha + r)} X^r \mathrm{e}^{-sX}.$$

For the distribution of X under the biased measure, we will use the Radon–Nikodym chain rule. To that end, let λ be the Lebesgue measure on $\mathbb{R}$ and f a probability density of X, then for λ-almost all $x > 0$

$$\begin{aligned}
\frac{\mathrm{d}(\mathbb{P}_{X^r \mathrm{e}^{-sX}} X^{-1})}{\mathrm{d}\lambda}(x) &= \frac{\mathrm{d}(\mathbb{P}_{X^r \mathrm{e}^{-sX}} X^{-1})}{\mathrm{d}(\mathbb{P}X^{-1})}(x) \cdot \frac{\mathrm{d}(\mathbb{P}X^{-1})}{\mathrm{d}\lambda}(x) \\
&= \frac{(\beta + s)^{\alpha + r}}{\beta^\alpha} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha + r)} x^r \mathrm{e}^{-sx} f(x) \\
&= \frac{(\beta + s)^{\alpha + r}}{\Gamma(\alpha + r)} x^{\alpha + r - 1} \mathrm{e}^{-(\beta + s)x}.
\end{aligned}$$

This, in turn, is a density of a $\Gamma(\alpha + r, \beta + s)$ distribution, proving the theorem. ∎

Definition 3.10. Let μ and ν be probability measures defined on a σ-algebra $\mathcal{A}$. Then

$$\mathrm{d}_{\mathrm{TV}}(\mu, \nu) := \sup_{A \in \mathcal{A}} |\mu(A) - \nu(A)|$$

defines the total variation distance between μ and ν.

Lemma 3.11 ([8, Lemma 3.18 and Exercise 3.19])**.** *Let μ and ν be probability measures defined on a σ-algebra $\mathscr{A}$ and let λ be a non-negative σ-finite measure such that $\mu \ll \lambda$ and $\nu \ll \lambda$. There exist corresponding Radon-Nikodym densities $f := \mathrm{d}\mu/\,\mathrm{d}\lambda$ and $g := \mathrm{d}\nu/\,\mathrm{d}\lambda$ and*

$$\mathrm{d_{TV}}(\mu,\nu) = \frac{1}{2}\|f-g\|_{L^1(\lambda)}.$$

Definition 3.12 ([4, Chapter 9.3])**.** Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space and let $(P_n)_{n\in\mathbb{N}}$ be a sequence of probability measures. We say that $P_n \to P$ in total variation if $\mathrm{d_{TV}}(P_n, P) \to 0$ as $n \to \infty$.

Lemma 3.13. *Let $X \sim \Gamma(\alpha,\beta)$ be a random variable and $s \in (0,\infty)$. The $(1-\mathrm{e}^{-sX})$-biased moment generating function of X is*

$$\mathbb{E}_{(1-\mathrm{e}^{-sX})}\big[\,\mathrm{e}^{tX}\,\big] = \frac{\beta^\alpha(\beta+s)^\alpha\big((\beta-t)^{-\alpha}-(\beta+s-t)^{-\alpha}\big)}{(\beta+s)^\alpha-\beta^\alpha}, \quad t \in (-\infty,\beta).$$

For $s \searrow 0$ the distributions of the biased random variable converge in total variation to the $\Gamma(\alpha+1,\beta)$ distribution.

Proof. The moment generating function can be computed directly:

$$\mathbb{E}_{(1-\mathrm{e}^{-sX})}\big[\,\mathrm{e}^{tX}\,\big] = \frac{\mathbb{E}\Big[(1-\mathrm{e}^{-sX})\,\mathrm{e}^{tX}\,\Big]}{\mathbb{E}\big[1-\mathrm{e}^{-sX}\,\big]} = \frac{\mathbb{E}\big[\,\mathrm{e}^{tX}\,\big]-\mathbb{E}\big[\,\mathrm{e}^{(t-s)X}\,\big]}{1-\mathbb{E}\big[\,\mathrm{e}^{-sX}\,\big]}.$$

Applying Lemma 3.6 and Remark 3.7 the expected values can be solved, thus

$$\mathbb{E}_{(1-\mathrm{e}^{-sX})}\big[\,\mathrm{e}^{tX}\,\big] = \frac{\frac{\beta^\alpha}{(\beta-t)^\alpha}-\frac{\beta^\alpha}{(\beta-t+s)^\alpha}}{1-\frac{\beta^\alpha}{(\beta+s)^\alpha}}.$$

Rearranging the terms yields the desired expression.

To prove the convergence to the limit we use Lemma 3.11. First we note that $\Gamma(\alpha+1,\beta)$ is the X-biased distribution of X – see Theorem 3.9. We apply the lemma with the following mappings:

$$\lambda = \nu = \mathbb{P}_X X^{-1} \qquad \text{and} \qquad \mu = \mathbb{P}_{\frac{1-\mathrm{e}^{-sX}}{sX}} X^{-1}.$$

Thus

$$f = \frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = \frac{s\mathbb{E}[X]}{\mathbb{E}\big[1-\mathrm{e}^{-sX}\,\big]}\cdot\frac{1-\mathrm{e}^{-sX}}{sX}, \qquad g = \frac{\mathrm{d}\nu}{\mathrm{d}\lambda} = 1$$

and

$$\mathrm{d_{TV}}(\mu,\nu) = \frac{1}{2}\widetilde{\mathbb{E}}\left[\left|\frac{s\mathbb{E}[X]}{\mathbb{E}\big[1-\mathrm{e}^{-sX}\,\big]}\cdot\frac{1-\mathrm{e}^{-sX}}{sX}-1\right|\right], \tag{3.14}$$

where $\widetilde{\mathbb{E}}$ denotes the X-biased expectation.

Let us consider the function $h : \mathbb{R} \to \mathbb{R}_+$

$$h(a) := \frac{1 - \mathrm{e}^{-a}}{a}$$

By comparing the derivatives of the numerator and denominator we can see (by l'Hôpital's rule) that $\lim_{a \to 0} h(a) = 1$ and that h is strictly monotonically falling.

The first fraction in Equation (3.14) can be rewritten as

$$\begin{aligned} \frac{s\mathbb{E}[X]}{\mathbb{E}\left[1 - \mathrm{e}^{-sX}\right]} &= \left(\mathbb{E}\left[\frac{X}{\mathbb{E}[X]} \cdot \frac{1 - \mathrm{e}^{-sX}}{sX}\right]\right)^{-1} \\ &= \left(\widetilde{\mathbb{E}}\left[\frac{1 - \mathrm{e}^{-sX}}{sX}\right]\right)^{-1} = \left(\widetilde{\mathbb{E}}\left[h(sX)\right]\right)^{-1}. \end{aligned} \tag{3.15}$$

Since for non-negative values $h(\cdot)$ is bounded from above by 1, the dominated convergence theorem allows the computation of the limit as

$$\lim_{s \searrow 0} \frac{s\mathbb{E}[X]}{\mathbb{E}\left[1 - \mathrm{e}^{-sX}\right]} = \left(\widetilde{\mathbb{E}}\left[\lim_{s \searrow 0} h(sX)\right]\right)^{-1} = 1.$$

If we consider only s up to a fixed point, e.g. $s < 1$, then the entire expression in Equation (3.15) is also bounded. Combined with the upper bound of 1 for $h(sX)$ dominated convergence can also be used in Equation (3.14), which yields

$$\lim_{s \searrow 0} \mathrm{d}_{\mathrm{TV}}(\mu, \nu) = \frac{1}{2}\widetilde{\mathbb{E}}\left[|1 \cdot 1 - 1|\right] = 0.$$

■

3.2. Conditioned risk factors

With all prerequisites in place we can compute the conditional distribution of risk factors given the default of guarantors.

3.2.1. Single risk factor

Theorem 3.16. *Let $R \sim \Gamma(\alpha, \beta)$ be a risk factor, $\Lambda := R$ a default cause intensity consisting only of R, $\pi \in [0, 1]$ a parameter of default and N the number of defaults of a guarantor with default cause intensity Λ such that $\mathscr{L}(N \,|\, R) = \mathrm{Poisson}(\pi\Lambda)$.*

(a) Conditioned on $\{N = 0\}$ the distribution of R is an $\mathrm{e}^{-\pi R}$-biased distribution of the unconditioned R:

$$\mathscr{L}(R \mid N = 0) = \mathbb{P}_{\mathrm{e}^{-\pi R}} R^{-1} = \Gamma(\alpha, \beta + \pi)$$

(b) If $\pi > 0$, then conditioned on $\{N \geq 1\}$ the distribution of R is an $\left(1 - \mathrm{e}^{-\pi R}\right)$-biased distribution of the unconditioned R:

$$\mathscr{L}(R \mid N \geq 1) = \mathbb{P}_{(1 - \mathrm{e}^{-\pi R})} R^{-1}$$

$$= \frac{1}{\mathbb{P}(N \geq 1)} \mathscr{L}(R) - \frac{\mathbb{P}(N=0)}{\mathbb{P}(N \geq 1)} \mathscr{L}(R \mid N=0) \qquad (3.17)$$

$$= \mathscr{L}(R) + \frac{\mathbb{P}(N=0)}{\mathbb{P}(N \geq 1)} \big(\mathscr{L}(R) - \mathscr{L}(R \mid N=0)\big). \qquad (3.18)$$

Remark 3.19. While Equation (3.17) is the expression which should be used for the actual computation of values, since each underlying distribution is evaluated only once, Equation (3.18) will prove itself useful for some theoretical results.

Proof. First we have to determine the probability of the terms we are conditioning on:

$$\mathbb{P}(N=0) = \mathbb{E}\big[\mathbb{P}(N=0 \mid R)\big] = \mathbb{E}\big[\,\mathrm{e}^{-\pi R}\,\big]$$

and

$$\mathbb{P}(N \geq 1) = 1 - \mathbb{P}(N=0).$$

In order to determine the conditional distribution, consider $h : \mathbb{R} \to \mathbb{C}$ a bounded and measurable function[1]. By the law of iterated expectations (cf. [4, Theorem 10.1.3]) we can write

$$\mathbb{E}\big[h(R) \mid N=0\big] = \frac{\mathbb{E}\big[h(R)\,\mathbb{1}_{\{N=0\}}\big]}{\mathbb{P}(N=0)} = \frac{\mathbb{E}\Big[\mathbb{E}\big[h(R)\,\mathbb{1}_{\{N=0\}} \mid R\big]\Big]}{\mathbb{P}(N=0)}.$$

Since $h(R)$ is R-measurable it can be taken out of the inner expectation. Using further that $\mathbb{E}\big[\mathbb{1}_A\big] = \mathbb{P}(A)$ and the conditional distribution of N, we get

$$\mathbb{E}\big[h(R) \mid N=0\big] = \frac{\mathbb{E}\big[h(R)\,\mathbb{P}(N=0 \,|\, R)\big]}{\mathbb{P}(N=0)} = \frac{\mathbb{E}\big[h(R)\ \mathrm{e}^{-\pi R}\,\big]}{\mathbb{E}\big[\,\mathrm{e}^{-\pi R}\,\big]}.$$

According to Definition 3.8 this is the $\mathrm{e}^{-\pi R}$-biased expectation of $h(R)$. Substituting $h(R)$ with indicator functions of Borel-measurable sets yields the conditioned distribution of R and Theorem 3.9 supplies an explicit representation of this distribution.

Since for $\pi = 0$ the number of defaults N has a degenerate Poisson distribution with $\mathbb{P}(N=0) = 1$, making the event $\{N \geq 1\}$ impossible, we only consider the case $\pi > 0$. As above, let $h : \mathbb{R} \to \mathbb{C}$ be a bounded and measurable function.

$$\mathbb{E}\big[h(R) \mid N \geq 1\big] = \frac{\mathbb{E}\big[h(R)\mathbb{1}_{\{N \geq 1\}}\big]}{\mathbb{P}(N \geq 1)} = \frac{\mathbb{E}\Big[h(R)\big(1 - \mathbb{1}_{\{N=0\}}\big)\Big]}{\mathbb{P}(N \geq 1)}. \qquad (3.20)$$

Conditioning the right parenthesis of the expectation on R – $h(R)$ is R-measurable – and substitution the probabilities yields

$$\mathbb{E}\big[h(R) \mid N \geq 1\big] = \frac{\mathbb{E}\Big[h(R)\big(1 - \mathbb{P}(N=0 \,|\, R)\big)\Big]}{\mathbb{P}(N \geq 1)} = \frac{\mathbb{E}\Big[h(R)\big(1 - \mathrm{e}^{-\pi R}\,\big)\Big]}{\mathbb{E}\big[1 - \mathrm{e}^{-\pi R}\,\big]}.$$

[1] It can be helpful to think of h as either $R \mapsto \mathrm{e}^{\pm tR}$, or $R \mapsto \mathrm{e}^{\pm itR}$ for some $t \in \mathbb{R}$ – which results in some kind of characteristic function – or simply $R \mapsto \mathbb{1}_A(R)$ for some measurable set A – which gives the measure itself.

Instead of conditioning on R we can simply expand the product in Equation (3.20), which gives us the desired expression:

$$\begin{aligned}\mathbb{E}\big[h(R) \mid N \geq 1\big] &= \frac{1}{\mathbb{P}(N \geq 1)}\, \mathbb{E}\big[h(R)\big] - \frac{\mathbb{P}(N = 0)}{\mathbb{P}(N \geq 1)} \cdot \frac{\mathbb{E}\big[h(R)\, \mathbb{1}_{\{N=0\}}\big]}{\mathbb{P}(N = 0)} \\ &= \frac{1}{\mathbb{P}(N \geq 1)}\, \mathbb{E}\big[h(R)\big] - \frac{\mathbb{P}(N = 0)}{\mathbb{P}(N \geq 1)}\, \mathbb{E}\big[h(R) \mid N = 0\big]\end{aligned}$$

Using the same arguments as for the event $\{N = 0\}$ the result for the event $\{N \geq 1\}$ follows immediately. Equation (3.18) follows from the decomposition of 1 as

$$1 = \mathbb{P}(N \geq 1) + \mathbb{P}(N = 0). \qquad \blacksquare$$

Remark 3.21. Lemma 3.13 provides an explicit representation of the moment generating function of $\mathscr{L}(R \mid N \geq 1)$ as well as the limit for $\pi \searrow 0$.

Definition 3.22 (Stochastic dominance – cf. [5, Section 2.4])**.** Let μ and ν be two probability measures on $\mathbb{R}$. We say that μ is dominated stochastically by ν, or $\mu \leq_{\text{st}} \nu$, if $F_\mu(x) \geq F_\nu(x)$ for all $x \in \mathbb{R}$.

Lemma 3.23. *Let R, N and π be defined as in Theorem 3.16, then*

(a) It holds that

$$\mathscr{L}(R \mid N = 0) \leq_{\text{st}} \mathscr{L}(R)$$

with equality if and only if $\pi = 0$.

(b) If $\pi > 0$, then[2]

$$\mathscr{L}(R) <_{\text{st}} \mathscr{L}(R \mid N \geq 1).$$

Here equality can not be achieved.

Proof. (a) Due to Theorem 3.16 it has to be shown that

$$\Gamma(\alpha, \beta + \pi) \leq_{\text{st}} \Gamma(\alpha, \beta).$$

This can be directly seen from the distributions' cumulative functions for $r \in \mathbb{R}_+$:

$$\begin{aligned}F_{R \mid \{N=0\}}(r) = \frac{1}{\Gamma(\alpha)}\gamma(\alpha, (\beta + \pi)r) &= \frac{1}{\Gamma(\alpha)} \int_0^{(\beta+\pi)r} t^{\alpha-1}\, \mathrm{e}^{-t}\, \mathrm{d}t \\ &\geq \frac{1}{\Gamma(\alpha)} \int_0^{\beta r} t^{\alpha-1}\, \mathrm{e}^{-t}\, \mathrm{d}t = \frac{1}{\Gamma(\alpha)}\gamma(\alpha, \beta r) = F_R(r).\end{aligned}$$

Obviously only for $\pi = 0$ both distributions are equal.

[2] Since we compare two distributions with same support, we require for strict stochastic dominance $\mu <_{\text{st}} \nu \Leftrightarrow F_\mu(x) > F_\nu(x)$ for all x in the interior of $\operatorname{supp}(F_\mu) = \operatorname{supp}(F_\nu)$.

(b) For $\pi > 0$ we know from Equation (3.18) that

$$F_{R\,|\,\{N\geq 1\}}(r) = F_R(r) + \frac{\mathbb{P}(N=0)}{\mathbb{P}(N\geq 1)}\big(F_R(r) - F_{R\,|\,\{N=0\}}(r)\big), \quad r \in \mathbb{R}.$$

The fraction of probabilities is certainly positive and the first part of this lemma shows that the difference in the parentheses is negative. Therefore the second summand of the expression above is negative; thus for all $r \in \mathbb{R}_+^*$

$$F_{R\,|\,\{N\geq 1\}}(r) < F_R(r) \qquad \text{and} \qquad \mathscr{L}(R \mid N \geq 1) >_{\text{st}} \mathscr{L}(R).$$

For both distributions to be equal the second summand would have to be zero, which is equivalent to either

$$\mathbb{P}(N=0) = 0 \qquad \text{or} \qquad F_R(r) - F_{R\,|\,\{N=0\}}(r) = 0 \quad \forall r \in \mathbb{R}_+^*,$$

neither of which is possible. ■

Corollary 3.24. *For $\pi > 0$ the expected values of the conditioned and unconditioned risk factor follow a natural ordering:*

$$\mathbb{E}[R \mid N=0] < \mathbb{E}[R] < \mathbb{E}[R \mid N \geq 1].$$

Proof. This follows immediately from the lemma above by considering that for any non-negative random variable X it holds that $\mathbb{E}[X] = \int_0^\infty \big(1 - F_X(t)\big)\,\mathrm{d}t$. ■

Having discussed the distribution of risk factors conditioned on the default of a guarantor, who depends on a single risk factor, the next theorem expands the proposition for an arbitrary number of risk factors.

3.2.2. Single condition

Theorem 3.25. *Let $R := (R_1, \dots, R_K)$ be a random vector of independent risk factors with $R_k \sim \Gamma(\alpha_k, \beta_k)$ and for $k = 1, \dots, K$ let $w_k \in \mathbb{R}_+^*$ be some weight. Let $R_0 \in \mathbb{R}_+$ be the*[3] *idiosyncratic risk factor and $w_0 \in \mathbb{R}_+$ its weight. Further define $\Lambda = \sum_{k=0}^K w_k R_k$ as a default cause intensity, $\pi \in [0,1]$ as the default parameter of a guarantor and N with $\mathscr{L}(N \mid R_1, \dots, R_K) = \text{Poisson}(\pi\Lambda)$ as the number of defaults of the guarantor.*

(a) For each $k \in \{1, \dots, K\}$

$$\mathscr{L}(R_k \mid N=0) = \mathbb{P}_{\mathrm{e}^{-\pi w_k R_k}} R_k^{-1} = \Gamma(\alpha_k, \beta_k + \pi w_k)$$

and the risk factors remain independent when conditioned on $\{N=0\}$.

[3] If there are multiple degenerate risk factors their respective values and weights can be combined by summation.

(b) If $\pi > 0$, then the joint distribution of the risk factors conditioned on the event $\{N \geq 1\}$ is given by

$$\begin{aligned}\mathscr{L}(R \mid N \geq 1) &= \mathbb{P}_{(1-\mathrm{e}^{-\pi\Lambda})} R^{-1} \\ &= \frac{1}{\mathbb{P}(N \geq 1)}\,\mathscr{L}(R) - \frac{\mathbb{P}(N=0)}{\mathbb{P}(N \geq 1)}\,\mathscr{L}(R \mid N=0).\end{aligned} \tag{3.26}$$

Proof. Using the conditional distribution of N we can write

$$\mathbb{P}(N = 0 \mid R) = \mathrm{e}^{-\pi\Lambda} = \mathrm{e}^{-\pi\sum_{k=0}^{K} w_k R_k} = \prod_{k=0}^{K} \mathrm{e}^{-\pi w_k R_k} \tag{3.27}$$

and thus, with the independence of the risk factors,

$$\mathbb{P}(N = 0) = \mathbb{E}\big[\mathbb{P}(N = 0 \mid R)\big] = \mathrm{e}^{-\pi w_0 R_0} \prod_{k=1}^{K} \mathbb{E}\big[\,\mathrm{e}^{-\pi w_k R_k}\,\big]. \tag{3.28}$$

Obviously

$$\mathbb{P}(N \geq 1) = 1 - \mathbb{P}(N = 0).$$

Let $h_k : \mathbb{R} \to \mathbb{C}$ be a bounded and measurable function for every $k = 1, \ldots, K$. Define $h : \mathbb{R}^K \to \mathbb{C}$ as $h(x_1, \ldots, x_K) = \prod_{k=1}^{K} h_k(x_k)$. As a product of bounded and measurable functions the function h is itself bounded and measurable.

For the joint distribution of the conditioned risk factors we compute again the conditional expectation of $h(R)$.

$$\mathbb{E}\big[h(R) \mid N = 0\big] = \frac{\mathbb{E}\big[h(R)\mathbb{1}_{\{N=0\}}\big]}{\mathbb{P}(N = 0)} = \frac{\mathbb{E}\big[h(R)\mathbb{P}(N = 0 \mid R)\big]}{\mathbb{P}(N = 0)}$$

Applying Equations (3.27) and (3.28) and using the independence of the risk factors yields (the factors for R_0 cancel out)

$$\mathbb{E}\big[h(R) \mid N = 0\big] = \frac{\mathbb{E}\Big[\prod_{k=1}^{K} h_k(R_k) \prod_{k=0}^{K} \mathrm{e}^{-\pi w_k R_k}\Big]}{\prod_{k=0}^{K} \mathbb{E}\big[\,\mathrm{e}^{-\pi w_k R_k}\,\big]} = \prod_{k=1}^{K} \frac{\mathbb{E}\big[h_k(R_k)\,\mathrm{e}^{-\pi w_k R_k}\,\big]}{\mathbb{E}\big[\,\mathrm{e}^{-\pi w_k R_k}\,\big]}.$$

Utilising the same arguments as in the proof of Theorem 3.16 we can see that each factor represents a biased distribution and the representation of the joint distribution as a product of marginal distributions proves their independence.

For the event $\{N \geq 1\}$ we follow – ceteris paribus – the final part of the proof of the sibling Theorem 3.16.

$$\begin{aligned}\mathbb{E}\big[h(R) \mid N \geq 1\big] &= \frac{\mathbb{E}\big[h(R)\mathbb{1}_{\{N\geq 1\}}\big]}{\mathbb{P}(N \geq 1)} \\ &= \frac{\mathbb{E}\Big[h(R)\big(1 - \mathbb{1}_{\{N=0\}}\big)\Big]}{\mathbb{P}(N \geq 1)} = \frac{\mathbb{E}\Big[h(R)\big(1 - \mathrm{e}^{-\pi\Lambda}\,\big)\Big]}{\mathbb{E}\big[1 - \mathrm{e}^{-\pi\Lambda}\,\big]}\end{aligned}$$

$$= \frac{1}{\mathbb{P}(N \geq 1)}\ \mathbb{E}\big[h(R)\big] - \frac{\mathbb{P}(N = 0)}{\mathbb{P}(N \geq 1)} \cdot \frac{\mathbb{E}\big[h(R)\,\mathbb{1}_{\{N=0\}}\big]}{\mathbb{P}(N = 0)}$$

$$= \frac{1}{\mathbb{P}(N \geq 1)}\ \mathbb{E}\big[h(R)\big] - \frac{\mathbb{P}(N = 0)}{\mathbb{P}(N \geq 1)}\ \mathbb{E}\big[h(R) \mid N = 0\big].$$

■

The difference in Equation (3.26) means that conditioned on the event $\{N \geq 1\}$ the risk factors are not independent anymore. They can, however, still be used for the computation of the ECR$^+$-model.

Lemma 3.29. *Let L be the portfolio loss of an extended ECR$^+$-model as described in Chapter 1 with independent risk factors and some default cause intensity as in Theorem 3.25.*

(a) The loss distribution $\mathscr{L}(L \mid N = 0)$ can be computed as described in [8] by substituting the risk factors with their conditioned counterparts.

(b) The loss distribution $\mathscr{L}(L \mid N \geq 1)$ can be computed in two steps as

$$\mathscr{L}(L \mid N \geq 1) = \frac{1}{\mathbb{P}(N \geq 1)}\ \mathscr{L}(L) - \frac{\mathbb{P}(N = 0)}{\mathbb{P}(N \geq 1)}\ \mathscr{L}(L \mid N = 0).$$

Proof. In both cases it is important to note that the conditioned distributions of the risk factors are biased distributions of the unconditioned risk factors. Since the respective bias is only a function of the risk factors themselves, their biased counterparts inherit any independence towards other random variables.

(a) The case $\{N = 0\}$ is straightforward since the updated risk factors are again gamma-distributed and independent and thus comply with the model's assumptions.

(b) For the case $\{N \geq 1\}$ it is important to note, that the distribution itself is computed via its probability generating function $\mathbb{E}\big[s^L\big]$. When computing the expected value over the risk factors – [8, Equation 6.89] – we can use the linearity of the Lebesgue–Stieltjes integral with respect to the integrator. After computing the distribution for the original risk factors and the risk factors conditioned on the event $\{N = 0\}$, the portfolio loss conditioned on $\{N \geq 1\}$ will be the weighted difference of these two distributions with the weights $1/\mathbb{P}(N \geq 1)$ and $\mathbb{P}(N = 0)/\mathbb{P}(N \geq 1)$ respectively. ■

3.2.3. Homogeneous conditions

Definition 3.30. Let $\mathcal{N}$ be a set of random variables. We define two events

$$\{\mathcal{N} = 0\} := \bigcap_{N \in \mathcal{N}} \{N = 0\} \quad \text{and} \quad \{\mathcal{N} \geq 1\} := \bigcap_{N \in \mathcal{N}} \{N \geq 1\}.$$

Since $\bigcap_{\emptyset} = \Omega$, it follows that for $\mathcal{N} = \emptyset$ we get $\{\mathcal{N} = 0\} = \{\mathcal{N} \geq 1\} = \Omega$.

Theorem 3.31. *Let $R := (R_1, \dots, R_K)$ be a random vector of independent risk factors with $R_k \sim \Gamma(\alpha_k, \beta_k)$. Let $R_0 \in \mathbb{R}_+$ be the idiosyncratic risk factor. Further let G be a finite set of guarantors. For each $g \in G$ and each $k = 0, \dots, K$ let $w_{g,k} \in \mathbb{R}_+$ be some weight, while for each $g \in G$ let $\Lambda_g := \sum_{k=0}^{K} w_{g,k} R_k$ be the default cause intensity, $\pi_g \in [0,1]$ the default parameter, and N_g the number of defaults of the guarantor g such that $\mathscr{L}(N_g \,|\, R) = \text{Poisson}(\pi_g \Lambda_g)$. Furthermore let the numbers of default be independent of each other when conditioned on R. Let $N_{\mathscr{G}} := \bigcup_{g\in\mathscr{G}} \{N_g\}$ for any $\mathscr{G} \subseteq G$ and let $N := N_G$.*

(a) For each $k = 1, \dots, K$

$$\mathscr{L}(R_k \,|\, N = 0) = \Gamma\Big(\alpha_k, \beta_k + \sum_{g\in G} \pi_g w_{g,k}\Big)$$

and the risk factors remain independent when conditioned on $\{N = 0\}$.

(b) If $\mathbb{P}(N \geq 1) > 0$, then the joint conditional distribution of the risk factors is given by

$$\mathscr{L}(R \mid N \geq 1) = \sum_{n=0}^{|G|} (-1)^n \sum_{\substack{\mathscr{G}\subseteq G \\ |\mathscr{G}|=n}} \frac{\mathbb{P}(N_{\mathscr{G}} = 0)}{\mathbb{P}(N \geq 1)} \mathscr{L}(R \mid N_{\mathscr{G}} = 0).$$

Proof.
(a) For $G = \emptyset$ trivially $\mathbb{P}(N_\emptyset = 0) = 1$ and $\mathscr{L}(R_k \mid N = 0) = \mathscr{L}(R_k)$.

Consider a $\mathscr{G} \neq \emptyset$. Since the support of the Poisson distribution are the (non-negative) natural numbers,

$$\{N_{\mathscr{G}} = 0\} = \Big\{\sum_{g\in\mathscr{G}} N_g = 0\Big\}. \tag{3.32}$$

Conditioned on R the distribution of this sum is a single Poisson distribution

$$\mathscr{L}\Big(\sum_{g\in\mathscr{G}} N_g \,\Big|\, R\Big) = \text{Poisson}\Big(\sum_{g\in\mathscr{G}} \pi_g \sum_{k=0}^{K} w_{g,k} R_k\Big).$$

Rearranging the sums in the distribution's parameter yields

$$\mathscr{L}\Big(\sum_{g\in\mathscr{G}} N_g \,\Big|\, R\Big) = \text{Poisson}\Big(\sum_{k=0}^{K} R_k \sum_{g\in\mathscr{G}} \pi_g w_{g,k}\Big). \tag{3.33}$$

The event $\{N_{\mathscr{G}} = 0\}$ is equal to the event of the sum of N_g being 0; and since the distribution conditioned on R of the sum is a Poisson distribution whose parameter is a linear combination of the risk factors, it turns out that this case is equivalent to the first statement of Theorem 3.25 with the weights of the linear combination $\sum_{g\in\mathscr{G}} \pi_g w_{g,k}$ instead of πw_k.

Therefore, with $\widetilde{R}_k \sim \Gamma\big(\alpha_k, \beta_k + \sum_{g\in\mathscr{G}} \pi_g w_{g,k}\big)$,

$$\mathbb{E}\big[h(R) \mid N_{\mathscr{G}} = 0\big] = \prod_{k=1}^{K} \mathbb{E}\big[h_k(\widetilde{R}_k)\big]$$

for any function $h : \mathbb{R}^K \to \mathbb{C}$ such that $h(x_1, \dots, x_K) = \prod_{k=1}^{K} h_k(x_k)$, where for each $k = 1, \dots, K$ the function $h_k : \mathbb{R} \to \mathbb{C}$ is bounded and measurable.

Like before, the product indicates the independence of each conditioned risk factor and a judicious substitution of h yields the desired distributions.

(b) For the probability of all groups having at least one default we use the independence of the conditioned numbers of default and the inclusion-exclusion principle.

$$\begin{aligned}\mathbb{P}(N \geq 1 \mid R) &= \prod_{g\in G} \mathbb{P}(N_g \geq 1 \mid R) = \prod_{g\in G} \big(1 - \mathbb{P}(N_g = 0 \mid R)\big)\\ &= \sum_{n=0}^{|G|} (-1)^n \sum_{\substack{\mathscr{G}\subseteq G\\ |\mathscr{G}|=n}} \prod_{g\in\mathscr{G}} \mathbb{P}(N_g = 0 \mid R).\end{aligned}$$

Using the conditional independence of the numbers of default, we get

$$\mathbb{P}(N \geq 1 \mid R) = \sum_{n=0}^{|G|} (-1)^n \sum_{\substack{\mathscr{G}\subseteq G\\ |\mathscr{G}|=n}} \mathbb{P}(N_{\mathscr{G}} = 0 \mid R).$$

The distribution of the risk factors can, as in the theorems before, be computed via the conditional expectation of a transformation. Let h be a bounded and R-measurable function.

$$\begin{aligned}\mathbb{E}\big[h(R) \mid N \geq 1\big] &= \frac{\mathbb{E}\big[h(R)\mathbb{1}_{\{N\geq 1\}}\big]}{\mathbb{P}(N \geq 1)} = \frac{\mathbb{E}\big[h(R)\mathbb{P}(N \geq 1 \mid R)\big]}{\mathbb{P}(N \geq 1)}\\ &= \frac{\mathbb{E}\Big[h(R)\sum_{n=0}^{|G|}(-1)^n \sum_{\substack{\mathscr{G}\subseteq G\\ |\mathscr{G}|=n}} \mathbb{P}(N_{\mathscr{G}} = 0 \mid R)\Big]}{\mathbb{P}(N \geq 1)}\\ &= \sum_{n=0}^{|G|} (-1)^n \sum_{\substack{\mathscr{G}\subseteq G\\ |\mathscr{G}|=n}} \frac{\mathbb{P}(N_{\mathscr{G}} = 0)}{\mathbb{P}(N \geq 1)} \cdot \frac{\mathbb{E}\big[h(R)\mathbb{P}(N_{\mathscr{G}} = 0 \mid R)\big]}{\mathbb{P}(N_{\mathscr{G}} = 0)}.\end{aligned}$$

The second fraction is, of course, a conditional expectation again, therefore

$$\mathbb{E}\big[h(R) \mid N \geq 1\big] = \sum_{n=0}^{|G|} (-1)^n \sum_{\substack{\mathscr{G}\subseteq G\\ |\mathscr{G}|=n}} \frac{\mathbb{P}(N_{\mathscr{G}} = 0)}{\mathbb{P}(N \geq 1)}\, \mathbb{E}\big[h(R) \mid N_{\mathscr{G}} = 0\big].$$

Appropriate substitutions for h yield the stated result. ■

3.2.4. Mixed conditions

In this final theorem we combine both defaulted and non-defaulted guarantors and consider the total information of both sets.

Theorem 3.34. *Let $R := (R_1, \dots, R_K)$ be a random vector of independent risk factors with $R_k \sim \Gamma(\alpha_k, \beta_k)$. Let $R_0 \in \mathbb{R}_+$ be the idiosyncratic risk factor. Further let G and H be disjoint finite sets of guarantors and for each $g \in G \cup H$ let $w_{g,k} \in \mathbb{R}_+$ be some weight for each $k = 0, \dots, K$; let $\Lambda_g := \sum_{k=0}^{K} w_{g,k} R_k$ be the default cause intensity, π_g the default probability and N_g the number of defaults of the guarantor g such that $\mathscr{L}(N_g \,|\, R) = \text{Poisson}(\pi_g \Lambda_g)$. Furthermore let the numbers of default be independent of each other when conditioned on R. Let $N_{\mathscr{G}} := \bigcup_{g \in \mathscr{G}} \{N_g\}$ for any $\mathscr{G} \subseteq G \cup H$. If $\mathbb{P}(N_G = 0, N_H \geq 1) > 0$, then*

$$\mathscr{L}(R \mid N_G = 0, N_H \geq 1) = \sum_{n=0}^{|H|} (-1)^n \sum_{\substack{\mathscr{H} \subseteq H \\ |\mathscr{H}| = n}} \frac{\mathbb{P}(N_{G \cup \mathscr{H}} = 0)}{\mathbb{P}(N_G = 0, N_H \geq 1)} \mathscr{L}(R \mid N_{G \cup \mathscr{H}} = 0).$$

Proof. As before, we begin with the conditional probability of the event, which will be conditioned on. Using the conditional independence of the numbers of default and the arguments from the proof of Theorem 3.31 we get

$$\begin{aligned}
\mathbb{P}(N_G = 0, N_H \geq 1 \mid R) &= \mathbb{P}(N_G = 0 \mid R) \cdot \mathbb{P}(N_H \geq 1 \mid R) \\
&= \mathbb{P}(N_G = 0 \mid R) \cdot \Big(\sum_{n=0}^{|H|} (-1)^n \sum_{\substack{\mathscr{H} \subseteq H \\ |\mathscr{H}| = n}} \mathbb{P}(N_{\mathscr{H}} = 0 \mid R) \Big) \\
&= \sum_{n=0}^{|H|} (-1)^n \sum_{\substack{\mathscr{H} \subseteq H \\ |\mathscr{H}| = n}} \mathbb{P}(N_{G \cup \mathscr{H}} = 0 \mid R), \qquad (3.35)
\end{aligned}$$

where the last step again uses the conditional independence of the numbers of default.

Let $h_k : \mathbb{R} \to \mathbb{C}$ be a bounded and measurable function for every $k = 1, \dots, K$. Define the function $h : \mathbb{R}^K \to \mathbb{C}$ as $h(x_1, \dots, x_K) = \prod_{k=1}^{K} h_k(x_k)$, so that as a product of bounded and measurable functions the function itself is still bounded and measurable.

$$\begin{aligned}
&\mathbb{E}\big[h(R) \mid N_G = 0, N_H \geq 1\big] \\
= &\frac{\mathbb{E}\big[h(R) \mathbb{1}_{\{N_G = 0, N_H \geq 1\}}\big]}{\mathbb{P}(N_G = 0, N_H \geq 1)} \\
= &\frac{\mathbb{E}\big[h(R) \mathbb{P}(N_G = 0, N_H \geq 1 \mid R)\big]}{\mathbb{P}(N_G = 0, N_H \geq 1)}
\end{aligned}$$

$$= \frac{\mathbb{E}\Big[h(R) \sum_{n=0}^{|H|} (-1)^n \sum_{\substack{\mathscr{H} \subseteq H \\ |\mathscr{H}|=n}} \mathbb{P}(N_{G \cup \mathscr{H}} = 0 \mid R)\Big]}{\mathbb{P}(N_G = 0, N_H \geq 1)}$$

$$= \sum_{n=0}^{|H|} (-1)^n \sum_{\substack{\mathscr{H} \subseteq H \\ |\mathscr{H}|=n}} \frac{\mathbb{P}(N_{G \cup \mathscr{H}} = 0)}{\mathbb{P}(N_G = 0, N_H \geq 1)} \cdot \frac{\mathbb{E}\big[h(R)\mathbb{P}(N_{G \cup \mathscr{H}} = 0 \mid R)\big]}{\mathbb{P}(N_{G \cup \mathscr{H}} = 0)}$$

The second fraction is a conditional expectation, which simplifies the entire expression into

$$= \sum_{n=0}^{|H|} (-1)^n \sum_{\substack{\mathscr{H} \subseteq H \\ |\mathscr{H}|=n}} \frac{\mathbb{P}(N_{G \cup \mathscr{H}} = 0)}{\mathbb{P}(N_G = 0, N_H \geq 1)} \, \mathbb{E}\big[h(R) \mid N_{G \cup \mathscr{H}} = 0\big].$$

Applying this computation for $h(x) = \mathbb{1}_A(x)$ for all measurable Borel-sets in $\mathbb{R}^K$ – or at least all rectangles in $\mathbb{R}^K$ – yields the desired statement on the joint distribution of the conditioned risk factors. ■

Remark 3.36 (Computation of probabilities)**.** For the actual implementation of Theorem 3.34 the mixed probabilities of defaults have to be computed. Taking the expectation of Equation (3.35) yields

$$\mathbb{P}(N_G = 0, N_H \geq 1) = \sum_{n=0}^{|H|} (-1)^n \sum_{\substack{\mathscr{H} \subseteq H \\ |\mathscr{H}|=n}} \mathrm{e}^{-R_0 \sum\limits_{g \in G \cup \mathscr{H}} \pi_g w_{g,0}} \prod_{k=1}^{K} \left(1 + \frac{\sum\limits_{g \in G \cup \mathscr{H}} \pi_g w_{g,k}}{\beta_k} \right)^{-\alpha_k}.$$

3.3. Bernoulli default indicators

In all the pertinent theorems only the events $\{N = 0\}$ and $\{N \geq 1\}$ were considered. This lends itself to the construction of dependent default indicators for all defaultable elements of an ECR$^+$-portfolio.

In this section we use the notation defined in Chapters 1 and 2.

Definition 3.37. Let i be either a group or a guarantor with default probability p_i and default cause c_i. Let ρ_i solve the equation

$$1 - p_i = \mathbb{E}\Big[\, \mathrm{e}^{-\rho_i \sum_{k=1}^{K} a_{c_i,k} R_k} \,\Big] \tag{3.38}$$

and let B_i be an indicator of i's default such that

$$\mathscr{L}(B_i \,|\, R) = \mathrm{Bern}\left(1 - \mathrm{e}^{-\rho_i \sum_{k=1}^{K} a_{c_i,k} R_k}\right).$$

Finally let B_i be independent of all other random variables when conditioned on R.

Remark 3.39. The marginal distribution of a single B_i is consistent with the given data in the sense that

$$\mathbb{P}(B_i = 1) = p_i \qquad \text{and} \qquad \mathbb{P}(B_i = 0) = 1 - p_i,$$

but in connection with other indicators $B_.$ the dependence structures described with the scenario matrix A^j is maintained.

Remark 3.40. If we consider N_i, the Poisson-based default indicator as used in the model, but with default probability ρ_i, such that

$$\mathscr{L}(N_i \,|\, R) = \text{Poisson}\Big(\rho_i \sum_{k=1}^{K} a_{c_i,k} R_k\Big),$$

it is clear that

$$\{B_i = 0\} = \{N_i = 0\} \qquad \text{and} \qquad \{B_i = 1\} = \{N_i \geq 1\},$$

which allows us to use the entire apparatus described in this chapter to compute joint probabilities of the B_is.

Remark 3.41. The defining equation of ρ_i – Equation (3.38) – can be immediately solved if only a single non-degenerate risk factor or exclusively degenerate risk factors are involved. If $R_k \sim \Gamma(\alpha_k, \beta_k)$ is this only non-degenerate risk factor, then

$$\rho_i = \frac{\beta}{a_{c_i,k}}\Big((1 - p_i)^{-1/\alpha_k} - 1\Big).$$

Otherwise let $\theta := \sum_{k=1}^{K} a_{c_i,k} R_k$, where $R_k \in \mathbb{R}_+$ for all $k = 1, \dots, K$. In this case it holds that

$$\rho_i = -\frac{\log(1 - p_i)}{\theta}.$$

In the general case we reorder the risk factors in such a way that for $k = 1, \dots, L$ all risk factors R_k are degenerate and for $k = L + 1, \dots, K$ all risk factors are non-degenerate. Then ρ_i solves the equation

$$1 - p_i = \mathrm{e}^{-\rho_i \sum_{k=1}^{L} a_{c_i,k} R_k} \prod_{k=L+1}^{K} \left(1 + \frac{\rho_i a_{c_i,k}}{\beta_k}\right)^{-\alpha_k},$$

which has to be solved numerically.

For guarantors these Bernoulli variables are in fact an explicit construction of the variables D_s described in Definition 2.3.

4. Approaches

4.1. Layers of the model

The first assumption stipulated in this paper is the independence of the loss distributions of each group (Assumption 1.3). This assumption paired with the independence of the number of defaults (Assumption 1.7) make it virtually impossible to model the entire dependence structure of guarantees within a simple ECR$^+$-model.

As soon as a guarantor guarantees two groups they should not be considered independent. Therefore it would be necessary to combine all groups which share a guarantor into one and model its loss distribution accordingly. Considering a realistic portfolio of a bank or insurer and taking into account the fact that nowadays most financial operations are interconnected, this would very easily lead to a portfolio of a single group, where all individual losses have to be aggregated by the user – negating the very point of this model.

Before we propose some approaches to solve this problem, let us take a look at the ECR$^+$-model. It can be roughly divided into three "layers":

1. The groups: A group on its own has no (in)dependence assumptions, which allows us to model the loss of a group however we see fit, as long as we stay within its boundaries.

2. The scenarios: The loss of each group within a scenario as well as the number of defaults of every group have to be independent from each other and any other random variables of the scenario. This leaves the set of its groups itself as the only modifiable parameter of a scenario.

3. The model: While the loss distributions of the scenarios are linked with each other through the common risk factors, the construction of the portfolio loss as a mixture distribution removes this dependence from the final result. Similar to the scenarios this leaves the set of all scenarios as the only modifiable parameter.

In line with the points above we will present three approaches – one for each layer of the model – to incorporate guarantees into the model's loss distribution.

After presenting all approaches we will provide numerical results underlying each approach's method and compare them with each other.

Throughout this chapter we will use the notation introduced in Sections 3.2 and 3.3. Thus for any defaultable object i – be it a group of a guarantor – N_i denotes the Poisson-based number of defaults, whereas B_i is the Bernoulli-based default indicator with the recalibrated probability of default ρ_i.

4.2. Mixed group

We begin with the first level: the groups. From within a group we have no access to data from other groups, let alone other scenarios. The eliminates any temptation to create cross-dependent loss distributions which would rely on the results of other groups or scenarios, thus violating the independence assumptions.

4.2.1. Algorithm

Using Definition 2.13 as a foundation we extend the notation for guarantees and guarantors to groups.

Definition 4.1. Let $g \in G$ be a group with its guarantees H_g. Let

$$S_g := S_{H_g}$$

be the set of all guarantors involved in guarantees of the group g.

In analogy to $H_{\mathfrak{H}}^{\mathfrak{S}}$ we define

$$H_g^{\mathfrak{S}} := \{h \in H_g : S_h \subseteq \mathfrak{S}\}$$

as those guarantees which default if the set $\mathfrak{S}$ of guarantors defaults.

Since the default of a group does carry information on the underlying risk factors, we take this information into account, by setting

$$\pi_g^{\mathfrak{S}} := \mathbb{P}\Big(\bigcap_{s \in \mathfrak{S}} \{B_s = 1\} \cap \bigcap_{s \in S_g \setminus \mathfrak{S}} \{B_s = 0\} \,\Big|\, N_g \geq 1 \Big)$$

to be the probability that exactly the given subset $\mathfrak{S}$ of the group's guarantors will default, given a default of the group g.

Approach 4.2 (Mixed group)**.** Combining the notation from above the loss distribution of a group $g \in G$ can be set to

$$Q_g := \sum_{\mathfrak{S} \subseteq S_g} \mathscr{L}\Big(\sum_{h \in H_g^{\mathfrak{S}}} L_{g,h} \Big) \cdot \pi_g^{\mathfrak{S}}. \tag{4.3}$$

Lemma 4.4. *Equation* (4.3) *represents a mixture distribution.*

Proof. To make (4.3) a mixture distribution it has to be a convex combination of probability distributions.

The terms $\mathscr{L}(\sum_{h \in H_{\mathfrak{S}}^g} L_{g,h})$ clearly represent probability distributions.

The term $\pi_{\mathfrak{S}}^g$ represents the probability that all guarantors in the set $\mathfrak{S}$ default, while all the remaining guarantors from the set S_g do not. Since we iterate with $\mathfrak{S}$ over all possible subsets of S_g (including the empty set and the entire S_g), we enumerate the entire probability space spanned by the guarantors' default indicators. Therefore the sum of all these probabilities adds up to the total probability of the entire space, which is 1. ■

4.2.2. Further considerations

Computation

A simple implementation of (4.3) might iterate over all $2^{|S_g|}$ subsets of S_g to calculate the probability measure if Q_g. This is, however, unnecessary since many terms of the sum will either be 0 or equal to other terms of the sum.

Example 4.5. As the most extreme example consider a group containing a single block which is secured by $n \in \mathbb{N}_+$ different guarantors. This group only produces a loss if all the guarantors default, but a naïve implementation might check all 2^n possibilities for each computation.

If the implementation environment is capable of comparing distributions without evaluation them in their entirety, a single pass through all candidates can determine the loss distribution for each set of guarantors. The resulting distributions can then be grouped and the corresponding probabilities added together to create a more efficient mixture distribution.

4.2.3. Criticism

While this method does reduce the risk metrics of a portfolio in line with its guarantees, it does not take into account the interplay between guarantees. In contrast to this approach's assumptions two guarantees with the same guarantor but in two different groups can not default independently, but only if their common guarantor has defaulted. This leads to the mixture method underestimating the concentration risk of the portfolio and resulting in values for high-level quantiles of the loss distribution which are too low.

4.3. Iterated group

We want all guarantees with the same primary guarantor to default simultaneously; we also know that from within a model we can only guarantee a concurrent default inside a group. Why not put all guarantees of a guarantor into an own group? This is, in fact, the leading thought of the following method.

4.3.1. Algorithm

As always we consider a fixed scenario $j \in \mathcal{J}$.

Definition 4.6. Define by

$$S_h^1 := \begin{cases} \{s_{h,1}\} & \text{if } R_h > 0, \\ \emptyset & \text{if } R_h = 0, \end{cases}$$

the primary guarantor of the guarantee block h, by

$$S_g^1 := \bigcup_{h \in H_g} S_h^1$$

the set of all primary guarantors of the group g and by

$$S^1 := \bigcup_{g \in G_j} S_g^1$$

the set of all primary guarantors of the current scenario.

Further let

$$H_g^s := \{h \in H_g \mid S_h^1 = s\}$$

be the set of all guarantee blocks whose primary guarantor is s and finally let

$$G^s := \{g \in G_j \mid H_g^s \neq \emptyset\}$$

be the set of all groups having s as the primary guarantor on at least one block.

The following algorithm gathers all the guarantee blocks covered by a given guarantor and places them into a new ECR$^+$-model. The loss distribution of this model is then used as the loss distribution of a new group representing the risk associated with the given guarantor.

Inside each guarantor's sub-model the default of this guarantor is a given fact, which should be used to recalibrate the risk factors accordingly. Theorem 3.25 gives the distribution of the updated risk factors. It is, however, important to note that the sub-model will have to be computed twice – as outlined in Lemma 3.29.

Algorithm 4.7 (Iterated groups for scenario j)**.** Perform the following steps for each primary guarantor s in S^1:

1. Create a new scenario and set its dependence matrix to a normalised version of the original scenario's dependence matrix, such that each row sums up to 1.

2. For each group g in G^s:

    a) Create a new group and set its probability of default and default cause to those of the original group.

    b) For each block h in H_g^s:

        i. Remove the block from its original group.

        ii. Remove the primary guarantor from the block, advancing any remaining guarantors.

        iii. Add the block to the new group.

    c) Add the new group to the new scenario.

3. Define N as the number of defaults of the guarantor s, constructed as in Theorem 3.25 using the guarantor's default cause c_s and probability of default π_s.

4. Create two new models m_1 and m_2, both consisting of only a copy of the new scenario.

5. For m_1 set the risk factors to a copy of the parent's risk factors.

6. For m_2 set the risk factors to their conditioned on $\{N = 0\}$ – the *non-default* of the guarantor s – counterparts of the parent's risk factors, as described in Theorem 3.25.

7. Apply this algorithm to each of the sub-models' scenarios.

8. Create a new group g and set its probability of default and default cause to those of the guarantor.

9. Set the loss distribution of the new group to a weighted difference of the loss distributions of m_1 and m_2:

$$\mathscr{L}(L_g) := \frac{1}{\mathbb{P}(N \geq 1)}\,\mathscr{L}(L_{m_1}) - \frac{\mathbb{P}(N = 0)}{\mathbb{P}(N \geq 1)}\,\mathscr{L}(L_{m_2}).$$

10. Add the new group to the original scenario.

This algorithm is a recursion (step 7). All guarantor chains are, however, finite (Assumption 2.8) which ensures that the algorithm halts in finite time.

Remark 4.8. In step 5 it is necessary to give each sub-model a fresh copy of risk guarantors in order to decouple them from their parent. Otherwise the loss distribution of such a sub-model would not be independent "of all other random variables". (cf. Assumption 1.3)

After applying the algorithm neither the portfolio nor any of its sub-models will contain any true guarantor block, which are blocks with a non-empty guarantor chain. Further each scenario will contain two kinds of groups: either consisting of a guarantee block without guarantors or containing a single sub-model.

4.3.2. Further considerations

Switching of combining algorithms

In order to improve tractability the proposed algorithm omits a major opportunity of optimisation. As proposed a sub-model with all its overhead is created for each guarantor regardless of the structure and number of guarantees dependent on the guarantor. Any real-life implementation should take into account that the underlying Poisson-approximation only increases its accuracy with increasing number of summands, whereas any somewhat complex guarantee-structure is bound to generate sub-models with very few guarantee blocks. (Even up to single-block models!)

A possible solution is to define a certain threshold for the number of blocks a sub-model has to contain and in case the threshold is undercut to fall back on convolutions.

Order of guarantors

If a guarantee block has at least two guarantors which are dependent the result will depend on the order of the guarantors.

Worst case number of sub-groups

An interesting question is the number of created sub-groups in the worst case.

For the worst case to happen all possible combinations in all possible permutations of all available guarantors have to be present in the guarantor chains of the guarantor blocks of each scenario.

A well known result of combinatorics is that the number of possible partial permutations of k elements from a set of n elements is

$$P(n,k) = \frac{n!}{(n-k)!}.$$

We are interested in all partial permutations of all sizes except for the empty permutation and our base set is the set S of all guarantors, therefore[1]

$$\widehat{M} = \sum_{k=1}^{|S|} \frac{|S|!}{(|S|-k)!} = \sum_{k=1}^{|S|} \frac{|S|!}{k!}.$$

The sub-groups are created within each scenario, which means that for the final result we have to multiply by the number of scenarios:

$$M = |\mathcal{J}|\ \widehat{M} = |\mathcal{J}| \cdot |S|! \sum_{k=1}^{|S|} \frac{1}{k!}.$$

4.3.3. Criticism

This approach does take into account the dependence between guarantees and forces all guarantees with the same primary guarantor to default simultaneously. In exchange, however, it decouples the guarantees from their underlying exposures. This in turn leads to an over estimation of the risk, since a disengaged guarantee may be considered in default despite an upstanding underlying.

Further the computational overhead of multiple sub-models has to be taken into account.

[1] The number $\widehat{M}$ is also called "the number of (nonnull) variations of n distinct objects" and the sequence for increasing base sets can be found on OEIS.org under the index A007526.

Finally, the higher the probabilities of default of the groups the more repeat-defaults the Poisson-approximation of the model generates. Iterating the entire model – and thus the Poisson-approximation – compounds these repeat-defaults resulting in exaggerated loss numbers.

4.4. Scenarios

As it was already argued in the introduction to this chapter, the first two approaches cannot possibly model the interconnecting nature of guarantees due to the assumed independence of the group losses. Mixed groups bind the guarantees to their obligor, but underestimate the concentration risk of a guarantor's default, whereas iterated groups correctly model a guarantor's concentration risk for the initial guarantor, but overestimate the guarantee's overall risk, by decoupling it from its original obligor.

Therefore a third approach, which tries to unify the advantages of both approaches described before, is proposed. Here – like in the mixed groups – the idea of mixture distributions is used, but unlike the previous procedure, which mixed group losses, entire dependence scenarios are to be mixed.

4.4.1. Algorithm

As it was already noted in section 1.3.1, the entire ECR$^+$-model is a mixture distribution of portfolio losses conditional on the selected scenario.

We expand on the idea of scenarios as components of a mixture distribution and duplicate the existing scenarios while setting varying guarantors as defaulted or standing.

For this approach the risk factors are not defined globally for the entire model anymore, but rather assigned to each scenario. This does not change anything for the computation, since each scenario is computed on its own, but it allows to assign different risk factors for each scenario.

Algorithm 4.9 (Scenario approach)**.** For each scenario j perform the following steps:

1. Gather all guarantors relevant to the scenario j into the set S_j.
2. For each subset $\mathfrak{S} \subseteq S_j$:
   a) Create a new scenario $j_{\mathfrak{S}}$.
   b) Set its probability to
   $$\pi_{j_{\mathfrak{S}}} := \pi_j \cdot \pi_{\mathfrak{S}}^{S_j}.$$
   c) Set its risk factors to their conditioned counterparts conditioned on the default of all guarantors in $\mathfrak{S}$ and non-default of all guarantors in $S_j \setminus \mathfrak{S}$ according to Theorem 3.34.

d) Set the dependence matrix identical to the original scenario,

$$A_{j_{\mathfrak{S}}} := A_j.$$

e) For each group $g' \in G_j$ add a group g to $G_{j_{\mathfrak{S}}}$ with the same obligors, probability of default and default cause,

$$o_g := o_{g'}, \qquad p_g := p_{g'}, \qquad c_g := c_{g'}$$

and the loss set to

$$L_g := \sum_{h \in H_g^{\mathfrak{S}}} L_h,$$

where – like in Section 4.2 – $H_{\mathfrak{S}}^g$ is the set of all guarantor blocks which default, when the set $\mathfrak{S}$ of guarantors defaults.

f) Add the scenario $j_{\mathfrak{S}}$ to the model.

3. Remove the original scenario j.

Analogous to the mixed distributions approach no assumptions about the joint distribution of the losses within each group are necessary.

4.4.2. Further considerations

Partial evaluation of overlapping scenarios

A naïve implementation of this approach might simply perform an entire ECR$^+$-calculation for the base model while only switching certain guarantors on and off and combine the results in a convex sum with the corresponding probabilities of guarantor defaults.

While mathematically correct such procedure will be usually quite inefficient. In a realistic setting only a handful of groups is affected by the (non-)default of a certain guarantor. Instead of recomputing each group for each sub-scenario, some form of book-keeping should keep track of the groups which actually change, in order to evaluate only those and to combine the final result afterwards.

4.4.3. Criticism

With this approach we have reached the limits of what can be done within the constraints of the current ECR$^+$-model. All guarantees of a given guarantor default at the same time; the guarantees are still tied to their underlying exposure; and guarantors can have an arbitrary dependence structure.

The biggest limitation of this approach is the number of scenarios created by it. Since a sub-scenario is created for every subset of the guarantors, the number of sub-scenarios doubles for each additional guarantor – resulting in an explosive growth of $|J| \cdot 2^{|S|}$ sub-scenarios.

4.5. Hybrid approach

While all the proposed approaches achieve their goal, choosing one of them is a balance act between speed ("mixed groups") and accuracy ("scenarios"). Each method is designed to model the combined default of guarantees of a single guarantor (cluster risk). In a portfolio of hundreds or even thousands of exposures and guarantees the impact of a particular guarantor covering only two small guarantees will be minuscule, whereas the impact of a guarantor covering half of the guarantees at hand will be tremendous.

We can, however, mix all three approaches to achieve a "good enough" result while keeping the computational time reasonable.

One such possible mixture will be discussed here.

4.5.1. Algorithm

We want to divide all guarantors into three groups: the high-impact guarantors, the low-impact guarantors and the rest. To do so we introduce a new coefficient for each guarantor, which should measure the impact of said guarantor's default on the final loss distribution of the portfolio.

In order to compute an "impact-statistic" of each guarantor, we devise a weighted number of guarantee block which depend on a given guarantor. To that end we use the Bernoulli default indicators of groups (B_g) and guarantors (B_s) as defined in Section 3.3.

Definition 4.10 (Weighted number of guarantees of a guarantor)**.** Let $s \in S$ be a guarantor. For each guarantee block h let

$$R_h^s := \begin{cases} \arg_{r \in \{1,\dots,R_h\}} s_{h,r} = s, & \text{if } s \in S_h, \\ 0, & \text{else.} \end{cases}$$

be the rank of the guarantor s in the block h. (We do not use arg min or arg max, since each guarantor can occur in a guarantor chain only once – see Assumption 2.7.)

Let

$$\delta_s := \sum_{j \in \mathcal{J}} \pi_j \sum_{g \in G_j} \sum_{\substack{h \in H_g \\ R_h^s > 0}} \mathbb{P}\Big(\{B_g = 1\} \cap \bigcap_{r=1}^{R_h^s - 1} \{B_{s_{h,r}} = 1\}\Big)$$

be the *weighted number of guarantees* of guarantor s.

For guarantees, where the guarantor s is the primary guarantor, δ_s adds a 1 and weights it with the probabilities of its containing group and scenario. For all other guarantees containing s the 1 is further weighted with the probability that the guarantor s will have to cover the guarantee – i.e. the probability that all preceding guarantors default.

With these coefficients for each guarantor at hand we proceed as follows:

1. Sort all guarantors in S according to their expected number of guarantees in descending order. Break ties according to the raw (unweighted) number of guarantees or any other appropriate measure.

2. Select the top t guarantors and set them aside as the high-impact guarantors.

3. For each scenario $j \in \mathcal{J}$

   a) Select those guarantors, who do not cover guarantees across different groups and set them aside as the low-impact guarantors.

   b) Apply the "iterated group"-method for the scenario j and the remaining guarantors.

4. Apply the "scenarios"-method for the high-impact guarantors.

5. Apply the "mixed group"-method for the remaining low-impact guarantors.

4.5.2. Further considerations

Number of high-impact guarantors

For the number of high-impact guarantors t either 3 or 4 is recommended, since each additional high-impact guarantor *doubles* the number of computed scenarios.

Portfolio dependence

It is important to note that both the algorithm above as well as the criterion of expected number of guarantees are just examples of an entire possible family of hybrid approaches. Depending on the portfolio and its dependence structure other choices may be better suited.

4.6. Reference model

All the approaches above are in the end just approximations of the "true" loss distribution of the model. Thanks to the Bernoulli default indicators of both groups and guarantors, we can, however, write down and in some cases even compute the exact loss distribution.

4.6.1. Algorithm

The driving factors of this model are the defaults of groups and guarantors. Once these are fixed, all there is left to do is to add up the corresponding losses.

Let

$$L_g^{\mathfrak{S}} := \sum_{h \in H_g^{\mathfrak{S}}} L_h$$

denote the loss of group g when the set $\mathfrak{S}$ of guarantors defaults.

By writing $\bigoplus$ for a mixture distribution as defined in Definition 1.14, the portfolio loss can be written as

$$L = \bigoplus_{j \in \mathcal{J}} \pi_j \cdot \bigoplus_{\substack{\mathcal{G} \subseteq G_j \\ \mathfrak{S} \subseteq S}} \mathbb{P}\Big(\bigcap_{g \in \mathcal{G} \cup \mathfrak{S}} \{B_g = 1\} \cap \bigcap_{g \in (\mathcal{G} \cup \mathfrak{S})^C} \{B_g = 0\} \Big) \cdot \sum_{g \in \mathcal{G}} L_g^{\mathfrak{S}}, \tag{4.11}$$

with the complement taken over the set $G_j \cup S$. Since the loss of each group is independent (Assumption 1.3) the last sum is a convolution.

4.6.2. Further considerations

Computation

Similar to the mixed group approach, the reference model iterates over all possible combinations of groups and guarantors. In a realistic portfolio this will lead to many unnecessary steps: Consider a simple portfolio with one group – g_1 – containing only unsecured exposures and another – g_2 – with many guarantors. A direct implementation of Equation (4.11) will at one point consider the default of the single group $\{g_1\} \subseteq G_j$ and iterate over all possible constellations of guarantor-defaults despite the group g_1 is not being affected by them.

A better implementation should therefore try to predict which combinations of group- and guarantor-defaults will have an impact on the final result.

4.6.3. Criticism

Even though this method provides the best results in terms of accuracy, its appeal is mostly limited to minuscule portfolios and theoretical considerations, due to its calculating time: For each scenario j there are $2^{|G_j \cup S|}$ convolutions of varying complexity to compute and the computation of the probability of each group/guarantor-combination is also non-trivial.

4.7. Expected value

The original Extended CreditRisk$^+$ framework goes to great lengths to ensure a consistent expectation of the loss distribution. By introducing guarantees, however, this desirable property is generally lost due to the fact that each approach uses a different combination of Poisson-distributed numbers of default and Bernoulli-distributed default indicators. On their own they are both calibrated to yield the same expected value – cf. Remark 3.39. In a general portfolio, however, they will be used together and due to their dependence on mutual risk factors the expected value of the entire distribution will be skewed.

Only if the guarantors induce no dependence in the portfolio – i.e. no guarantor has guarantees in more than one group and each guarantor depends on a set of risk factors disjoint for all the other groups' and guarantors' sets – can the expected value be written as

$$\mathbb{E}[\widehat{L}] = \sum_{j \in \mathcal{J}} \pi_j \sum_{g \in G_j} \lambda_g \mathbb{E}[L_g] = \sum_{j \in \mathcal{J}} \pi_j \sum_{g \in G_j} \lambda_g \sum_{h \in H_g} \pi_{S_h} \mathbb{E}[L_h].$$

5. Comparison

5.1. Preliminaries

In order to visualise the different approaches I have implemented each of them in Java using a stochastics library which I have developed. (See Appendix A for a description of the library.)

Several portfolios of increasing complexity will be proposed, with each highlighting a different aspect of the computation. In order to benchmark each approach we will also calculate a reference distribution as described in Section 4.6. In some cases we will even be able to provide an explicit formula for the loss.

Throughout this chapter we make the following simplifying assumption:

> The guarantee blocks within each group default comonotonously.

This allows us to compute the loss distribution of a group without providing any additional information regarding the dependence structure within the group's guarantee blocks.

5.1.1. Wasserstein distance

In order to quantify how "good" any of the discussed methods is, we will compare each of them to a reference distribution representing the "true" distribution of the portfolio loss. This comparison is done by taking the Wasserstein distance between the two distributions.

For continuous distributions we follow the definition of the Wasserstein distance by [4, Section 11.8], but restrict ourselves to the real-valued case, as it is the only one we will use.

Definition 5.1 (Wasserstein distance)**.** Let $\mathscr{M}$ denote the set of all probability measures on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} |x| \, \mathrm{d}\mu(x) < \infty \quad \text{for each } \mu \in \mathscr{M}$$

and let P and Q be measures from $\mathscr{M}$. Further let $\mathscr{C}(P,Q)$ be the set of all couplings of P and Q, that is the set of all laws on $\mathbb{R}^2$ with marginals P and Q. The Wasserstein distance $\mathrm{d_W}(P,Q)$ is defined by

$$\mathrm{d_W}(P,Q) := \inf_{\mu \in \mathscr{C}(P,Q)} \int_{\mathbb{R}^2} |x-y| \, \mathrm{d}\mu(x,y).$$

The infimum over the entire set of couplings may prove cumbersome, but the following theorem allows us to replace the search over probability measures with a search over a class of functions.

Definition 5.2 (Lipschitz continuity)**.** We call a function f *Lipschitz continuous with constant* c if

$$\|f\|_{\mathrm{L}} := \sup_{\substack{x,y\in\mathbb{R}\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|} \leq c.$$

With $\mathcal{F}_{\mathrm{W}}$ we denote the set of Lipschitz continuous functions with constant at most 1.

Theorem 5.3 (Kantorovich–Rubinstein)**.** *For any two laws $P, Q \in \mathcal{M}$ it holds that*

$$\mathrm{d_W}(P,Q) = \sup_{f\in\mathcal{F}_{\mathrm{W}}} \left| \int_{\mathbb{R}} f \,\mathrm{d}(P-Q) \right| = \sup_{f\in\mathcal{F}_{\mathrm{W}}} \int_{\mathbb{R}} f \,\mathrm{d}P - \int_{\mathbb{R}} f \,\mathrm{d}Q$$

and there exists a law S in $\mathcal{C}(P,Q)$ such that

$$\int |x-y| \,\mathrm{d}S(x,y) = \mathrm{d_W}(P,Q)$$

so that the infimum in Definition 5.1 is attained.

Proof. The absolute value can be omitted, because for every $f \in \mathcal{F}_{\mathrm{W}}$ also $-f \in \mathcal{F}_{\mathrm{W}}$.
For the proof of the theorem proper see [4, Section 11.8]. ■

If the distributions which are compared are discrete, the metric is also known as the Earth Mover's Distance (EMD) – see [7] for a proof of equivalence.

We only encounter non-negative discrete distributions and for those the EMD can be explicitly written down:

Theorem 5.4 (Earth Mover's Distance)**.** *Let X and Y be two random variables with domain $\mathbb{N}$. Then*

$$\mathrm{EMD}(\mathcal{L}(X), \mathcal{L}(Y)) = \sum_{i=0}^{\infty} |\mathbb{P}(X \leq i) - \mathbb{P}(Y \leq i)|$$

computes the Earth Mover's Distance between the laws of the two variables.

Proof. See [3, Section 4.3.2]. ■

Remark 5.5. The EMD can also be computed recursively using the probabilities of the random variables X and Y without using the cumulative distribution function:

Step -1
: Set $\mathrm{EMD}_{-1} := 0$.

Step $i = 0, \dots$
: Set $\mathrm{EMD}_i = \mathrm{EMD}_{i-1} + \mathbb{P}[X = i] - \mathbb{P}[Y = i]$.
Break if both variables' support is below i or if the desired accuracy has been reached.

Finalize
: Set $\mathrm{EMD}(\mathcal{L}(X), \mathcal{L}(Y)) = \sum_{i=0,\dots} |\mathrm{EMD}_i|$.

5.1.2. Computed values

Each presented portfolio will be computed in four ways: the "true" reference model, the mixture approach, the iterated groups approach and the scenario approach.

For each of the resulting distributions the following values will be calculated:

- the expected value,
- the probability mass function at the points $x = 0, \dots, 49$,
- the value at risk (quantile) at levels 0.95 to 0.999 in steps of 0.001,
- the expected shortfall at levels 0.95 to 0.999 in steps of 0.001 and
- the Wasserstein distance between all distributions.

The computed quantities allow us to grasp the overall shape of the loss distribution and to perform some qualitative comparisons between the various approaches.

5.2. Portfolios

The portfolios presented here can be divided into two parts. In the first half – Sections 5.2.1 to 5.2.6 – none of the portfolios uses stochastic risk factors. Here the point is to gradually create more and more complex portfolios and to compare the approaches. The second half uses the same portfolios, but adds one or more stochastic risk factors. This allows us to study the effects of stochastic risk factors in comparison to the deterministic portfolios.

All portfolios discussed have only one scenario, because all the desired effects manifest themselves using a single scenario already and adding further scenarios would only interfere with them.

Each portfolio also has only one common obligor – this will not be explicitly mentioned with each portfolio.

In each portfolio we will call L^r, L^m, L^g and L^s the loss of the reference, mixture, group and scenario model respectively.

5.2.1. Minimal portfolio

Structure

Number of guarantors		1	Number of groups			2
s	π_s	10%	g_1	p_g		4%
	c_s	idiosync.		c_g		idiosync.
				h_1	Exposure	Dirac(1)
					Guarantor	s
				h_2	Exposure	Dirac(1)
					Guarantor	—

The minimal portfolio which illustrates the differences between the guarantor approaches consists of two groups such that one group has a partial cover from the guarantor, whereas the other is fully covered.

The second group is necessary, because with only one group the mixture method is equivalent to the scenario method.

The uncovered guarantee block is necessary to distinguish between the group method and the scenario method.

Reference distribution

For this portfolio the reference distribution can be determined explicitly:

```
<EmpiricalDistribution>
  <value index="0">9.6e-1;50</value>
  <value index="1">3.6e-2;50</value>
  <value index="2">4e-3;50</value>
</EmpiricalDistribution>
```

Listing 5.1: Minimal portfolio – reference distribution

A minimal portfolio is wont to have a minimal reference distribution. In this case we only consider three cases: no defaults, only the group defaults, and both the group and the guarantor default.

To illustrate how the three approaches to guarantees work we discuss the generated loss distributions for this portfolio.

Mixture method

```
<CompoundDistribution>
  <summand>
    <EmpiricalDistribution>
      <value index="1">9e-1;51</value>
      <value index="2">1e-1;51</value>
    </EmpiricalDistribution>
  </summand>
  <number>
    <PoissonDistribution>
      <lambda>4e-2;50</lambda>
    </PoissonDistribution>
  </number>
</CompoundDistribution>
```

Listing 5.2: Minimal portfolio – mixture method

The compound Poisson-distribution corresponds to the random variable S_j in (1.16) and the `NonZeroDistribution` to $Y_{j,k}$.

The two innermost mixture distributions (`ZeroSwitch` is also a mixture distribution) are the result of this approach's method: the guarantor has a default probability of 10%; with this probability the higher loss occurs and the lower loss with the counter-probability.

Group method

```
<CompoundDistribution>
  <summand>
    <NonZeroDistribution>
      <MixtureDistribution>
        <p>
          <number>7.14285714285714285714285714285714285714285714285714285e
              -1;50</number>
          <number>2.85714285714285714285714285714285714285714285714285714e
              -1;50</number>
        </p>
        <v>
          <PoissonDistribution>
            <lambda>4e-2;50</lambda>
          </PoissonDistribution>
          <DiracDistribution>1</DiracDistribution>
        </v>
      </MixtureDistribution>
    </NonZeroDistribution>
  </summand>
  <number>
    <PoissonDistribution>
      <lambda>4.3921056084767679056078930867675411397202790628208343e-2;50</
          lambda>
```

```
    </PoissonDistribution>
  </number>
</CompoundDistribution>
```

Listing 5.3: Minimal portfolio – group method

Here we find, like in the mixture method, a compound Poisson as the outermost distribution, which again corresponds to S_j.

The inner compound distribution is in fact the inner sub-model generated for the guarantor s and contains the two guarantees with values 2 and 5. To increase the efficiency during computation the mixture distribution representing $Y_{j,k}$ of the inner model has been replaced with an `EmpiricalDistribution`.

Scenario method

```
<MixtureDistribution>
  <p>
    <number>9e-1;51</number>
    <number>1e-1;51</number>
  </p>
  <v>
    <PoissonDistribution>
      <lambda>4e-2;50</lambda>
    </PoissonDistribution>
    <LatticeDistribution step="2">
      <PoissonDistribution>
        <lambda>4e-2;50</lambda>
      </PoissonDistribution>
    </LatticeDistribution>
  </v>
</MixtureDistribution>
```

Listing 5.4: Minimal portfolio – scenario method

Here we see the outer mixture distribution corresponding to the two default outcomes of the guarantor mixing entire CreditRisk$^+$ models. The second one – related to the case where the guarantor does not default – has again been optimised. In this case only the first group generates a loss of constant value 2. Instead of computing the more expensive Panjer-recursion for the comound Poisson distribution with a constant summand of 2, we simply stretch the Poisson distribution by a factor of 2.

Numerical results

All four methods return (as expected) the same expected value of 0.108. This is the correct value, because

$$(2 + 2 \cdot 10\%) \cdot 4\% + 5 \cdot 10\% \cdot 4\% = 0.108.$$

The next pages contain the raw results of each distribution.

x	$\mathbb{P}[L^m = x]$	$\mathbb{P}[L^g = x]$	$\mathbb{P}[L^s = x]$	$\mathbb{P}[L^r = x]$
0	0.9607894	0.9570295	0.960789	0.96
1	0.0345884	0.0419592	0.034588	0.036
2	0.0044657	9.934E-04	0.004535	0.004
3	1.458E-04	1.765E-05	9.224E-06	0
4	1.024E-05	2.737E-07	7.696E-05	0
5	3.071E-07	3.976E-09	7.379E-10	0
6	1.550E-08	5.528E-11	1.025E-06	0
7	4.307E-10	7.379E-13	2.811E-14	0
8	1.744E-11	9.478E-15	1.025E-08	0
9	4.526E-13	1.177E-16	6.247E-19	0
10	1.558E-14	1.422E-18	8.199E-11	0
11	3.801E-16	1.677E-20	9.086E-24	0
12	1.153E-17	1.936E-22	5.466E-13	0
13	2.658E-19	2.193E-24	9.319E-29	0
14	7.271E-21	2.440E-26	3.123E-15	0
15	1.592E-22	2.672E-28	7.100E-34	0
16	3.994E-24	2.882E-30	1.562E-17	0
17	8.339E-26	3.067E-32	4.177E-39	0
18	1.942E-27	3.221E-34	6.941E-20	0
19	3.879E-29	3.344E-36	1.954E-44	0
20	8.465E-31	3.432E-38	2.776E-22	0
21	1.623E-32	3.486E-40	7.444E-50	0
22	3.344E-34	3.506E-42	1.010E-24	0
23	6.168E-36	3.493E-44	0	0
24	1.207E-37	3.449E-46	3.365E-27	0
25	2.148E-39	3.376E-48	0	0
26	4.012E-41	3.279E-50	1.035E-29	0
27	6.898E-43	0	0	0
28	1.235E-44	0	2.958E-32	0
29	2.056E-46	0	0	0
30	3.540E-48	0	7.889E-35	0
31	5.718E-50	0	0	0
32	0	0	1.972E-37	0
33	0	0	0	0
34	0	0	4.641E-40	0
35	0	0	0	0
36	0	0	1.031E-42	0
37	0	0	0	0
38	0	0	2.171E-45	0
39	0	0	0	0
40	0	0	4.342E-48	0
41	0	0	0	0
42	0	0	0	0
43	0	0	0	0
44	0	0	0	0
45	0	0	0	0
46	0	0	0	0
47	0	0	0	0
48	0	0	0	0
49	0	0	0	0

q	VaR_q^m	VaR_q^g	VaR_q^s	VaR_q^r	ES_q^m	ES_q^g	ES_q^s	ES_q^r
0.950	0	0	0	0	0.88	0.88	0.88	0.88
0.951	0	0	0	0	0.897959	0.897959	0.897959	0.897959
0.952	0	0	0	0	0.916667	0.916667	0.916667	0.916667
0.953	0	0	0	0	0.93617	0.93617	0.93617	0.93617
0.954	0	0	0	0	0.956522	0.956522	0.956522	0.956522
0.955	0	0	0	0	0.977778	0.977778	0.977778	0.977778
0.956	0	0	0	0	1.0	1.0	1.0	1.0
0.957	0	0	0	0	1.023256	1.023256	1.023256	1.023256
0.958	0	1	0	0	1.047619	1.024512	1.047619	1.047619
0.959	0	1	0	0	1.073171	1.02511	1.073171	1.073171
0.960	0	1	0	0	1.1	1.025738	1.1	1.1
0.961	1	1	1	1	1.122806	1.026398	1.122806	1.102564
0.962	1	1	1	1	1.126038	1.027092	1.126038	1.105263
0.963	1	1	1	1	1.129444	1.027824	1.129444	1.108108
0.964	1	1	1	1	1.13304	1.028597	1.13304	1.111111
0.965	1	1	1	1	1.136841	1.029414	1.136841	1.114286
0.966	1	1	1	1	1.140866	1.03028	1.140866	1.117647
0.967	1	1	1	1	1.145135	1.031197	1.145135	1.121212
0.968	1	1	1	1	1.14967	1.032172	1.14967	1.125
0.969	1	1	1	1	1.154498	1.03321	1.154498	1.129032
0.970	1	1	1	1	1.159648	1.034317	1.159648	1.133333
0.971	1	1	1	1	1.165153	1.0355	1.165153	1.137931
0.972	1	1	1	1	1.171051	1.036768	1.171051	1.142857
0.973	1	1	1	1	1.177387	1.03813	1.177387	1.148148
0.974	1	1	1	1	1.184209	1.039596	1.184209	1.153846
0.975	1	1	1	1	1.191578	1.04118	1.191578	1.16
0.976	1	1	1	1	1.19956	1.042896	1.19956	1.166667
0.977	1	1	1	1	1.208236	1.044761	1.208236	1.173913
0.978	1	1	1	1	1.217702	1.046796	1.217702	1.181818
0.979	1	1	1	1	1.228069	1.049024	1.228069	1.190476
0.980	1	1	1	1	1.239472	1.051475	1.239472	1.2
0.981	1	1	1	1	1.252076	1.054185	1.252076	1.210526
0.982	1	1	1	1	1.26608	1.057195	1.26608	1.222222
0.983	1	1	1	1	1.281732	1.060559	1.281732	1.235294
0.984	1	1	1	1	1.29934	1.064344	1.29934	1.25
0.985	1	1	1	1	1.319296	1.068634	1.319296	1.266667
0.986	1	1	1	1	1.342103	1.073536	1.342103	1.285714
0.987	1	1	1	1	1.368418	1.079193	1.368418	1.307692
0.988	1	1	1	1	1.39912	1.085792	1.39912	1.333333
0.989	1	1	1	1	1.435404	1.093591	1.435404	1.363636
0.990	1	1	1	1	1.478944	1.102951	1.478944	1.4
0.991	1	1	1	1	1.53216	1.11439	1.53216	1.444444
0.992	1	1	1	1	1.59868	1.128688	1.59868	1.5
0.993	1	1	1	1	1.684206	1.147072	1.684206	1.571429
0.994	1	1	1	1	1.79824	1.171584	1.79824	1.666667
0.995	1	1	1	1	1.957888	1.205901	1.957888	1.8
0.996	2	1	2	1	2.041825	1.257377	2.041825	2.0
0.997	2	1	2	2	2.055766	1.343169	2.055766	2.0
0.998	2	1	2	2	2.083649	1.514753	2.083649	2.0
0.999	2	2	2	2	2.167298	2.018208	2.167298	2.0

Table 5.1.: Some probabilistic values of the minimal portfolio

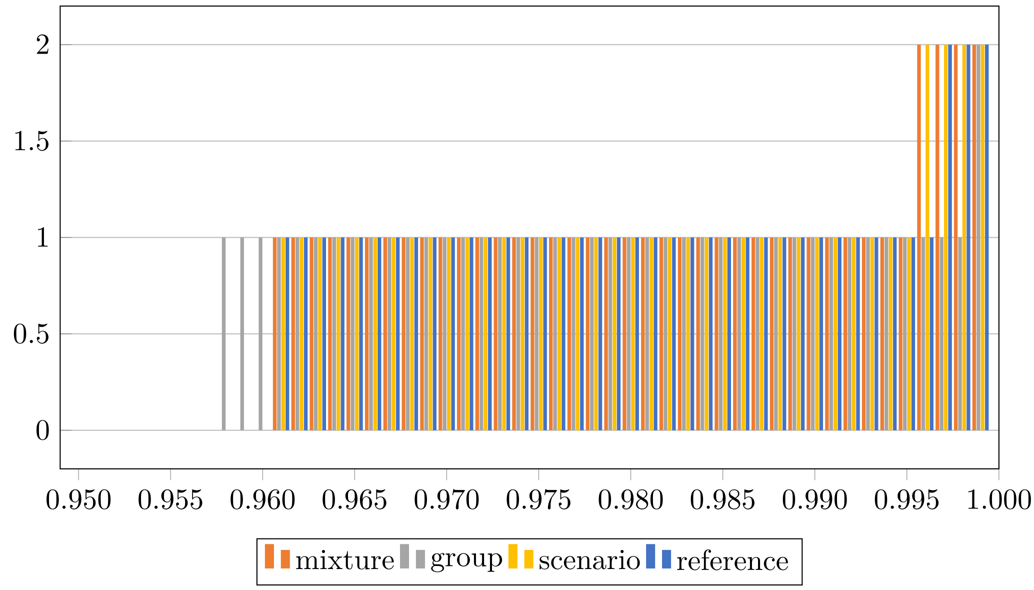


Figure 5.1.: Value-at-Risk of the minimal portfolio

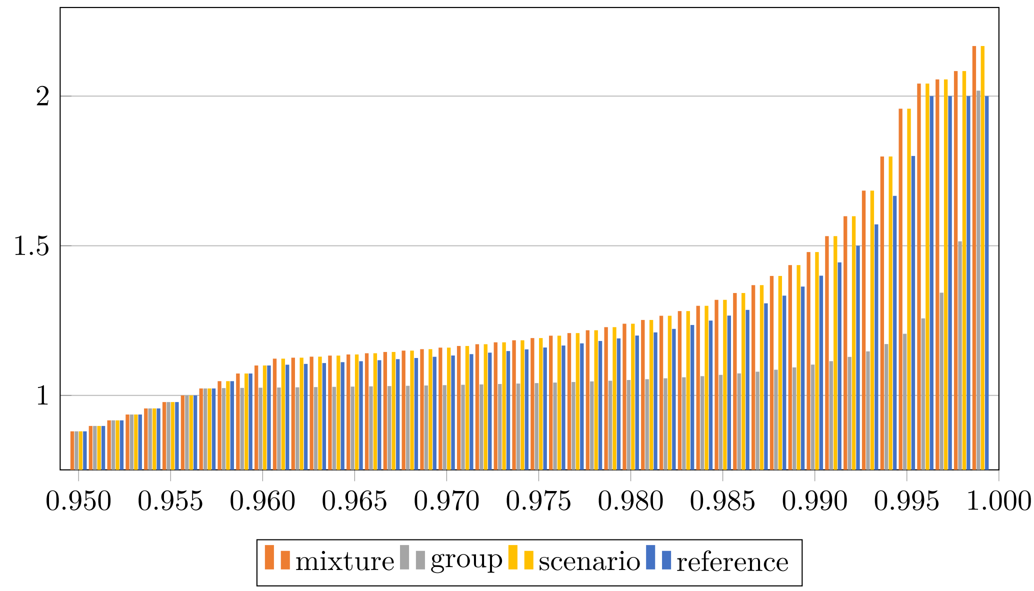


Figure 5.2.: Expected shortfall of the minimal portfolio

mixture	group	scenario
0.001568	0.005977	0.001499

Table 5.2.: Wasserstein distances of the minimal portfolio

As we can see the group method has by far the worst result.

The main culprit are – as already mentioned in the criticism of the group approach in Section 4.3.3 – the decoupling of the guarantees from their main exposure and the ECR$^+$-model's possibility of repeat defaults. This inflates the tail of the distribution in comparison to the target distribution. Evidence of this is the inflated expected shortfall in the highest quantiles. As soon as the quantiles reach the ends of the reference distribution – which is around the 0.995 level – the expected shortfall (which measures the mass of the tail) increases markedly.

The same phenomenon can be observed for the scenario distribution, but here the heavier tail is compensated by more accurate values within the true range of the loss.

5.2.2. Multiple groups

Structure

Number of guarantors		1	Number of groups			10
s	π_s	10%	$g_{1,\dots,5}$	p_g		4%
	c_s	idiosync.		c_g		idiosync.
				h_1	Exposure	Dirac(2)
					Guarantor	s
				h_2	Exposure	Dirac(1)
					Guarantor	—
			$g_{6,\dots,10}$	p_g		4%
				c_g		idiosync.
				h_1	Exposure	Dirac(2)
					Guarantor	s

This portfolio is very similar to the minimal example, but increases the number of groups. This allows us to observe the influence of the number of groups on the quality of the ECR$^+$-model.

Numerical results

The expected value all four methods return is 0.32:

$$5 \cdot (1 + 2 \cdot 10\%) \cdot 4\% + 5 \cdot 2 \cdot 4\% \cdot 10\% = 0.28.$$

The next pages contain the raw results of each distribution.

x	$\mathbb{P}[L^m = x]$	$\mathbb{P}[L^g = x]$	$\mathbb{P}[L^s = x]$	$\mathbb{P}[L^r = x]$
0	0.8025188	0.7921789	0.80389	0.800319
1	0.1444534	0.1584358	0.147372	0.152882
2	0.0290512	0.0370841	0.028144	0.026591
3	0.0197195	0.0053043	0.014389	0.014382
4	0.0033447	0.0050105	0.00139	0.001165
5	6.268E-04	9.370E-04	0.002683	0.002886
6	2.383E-04	7.750E-04	0.00143	0.001202
7	3.838E-05	1.427E-04	2.681E-04	2.405E-04
8	6.756E-06	9.872E-05	2.726E-04	2.415E-04
9	1.894E-06	1.788E-05	1.073E-04	5.811E-05
10	2.914E-07	1.135E-05	2.699E-05	2.005E-05
11	4.851E-08	2.037E-06	1.877E-05	1.023E-05
12	1.117E-08	1.256E-06	6.262E-06	1.837E-06
13	1.649E-09	2.246E-07	1.823E-06	8.367E-07
14	2.610E-10	1.372E-07	9.833E-07	2.261E-07
15	5.221E-11	2.451E-08	2.991E-07	4.314E-08
16	7.423E-12	1.480E-08	9.296E-08	1.754E-08
17	1.123E-12	2.640E-09	4.174E-08	2.464E-09
18	2.018E-13	1.567E-09	1.204E-08	7.248E-10
19	2.772E-14	2.790E-10	3.814E-09	1.510E-10
20	4.021E-15	1.626E-10	1.493E-09	1.510E-11
21	6.637E-16	2.889E-11	4.165E-10	6.040E-12
22	8.833E-17	1.657E-11	1.312E-10	1.258E-13
23	1.233E-17	2.938E-12	4.619E-11	1.258E-13
24	1.899E-18	1.662E-12	1.260E-11	0
25	2.454E-19	2.943E-13	3.887E-12	1.049E-15
26	3.308E-20	1.646E-13	1.260E-12	0
27	4.804E-21	2.910E-14	3.376E-13	0
28	6.040E-22	1.610E-14	1.013E-13	0
29	7.883E-23	2.842E-15	3.072E-14	0
30	1.089E-23	1.556E-15	8.102E-15	0
31	1.334E-24	2.744E-16	2.357E-15	0
32	1.689E-25	1.488E-16	6.773E-16	0
33	2.233E-26	2.620E-17	1.759E-16	0
34	2.671E-27	1.408E-17	4.956E-17	0
35	3.288E-28	2.476E-18	1.362E-17	0
36	4.183E-29	1.319E-18	3.482E-18	0
37	4.890E-30	2.317E-19	9.509E-19	0
38	5.864E-31	1.225E-19	2.517E-19	0
39	7.208E-32	2.149E-20	6.332E-20	0
40	8.246E-33	1.127E-20	1.678E-20	0
41	9.647E-34	1.976E-21	4.302E-21	0
42	1.150E-34	1.029E-21	1.064E-21	0
43	1.288E-35	1.802E-22	2.742E-22	0
44	1.473E-36	9.322E-23	6.833E-23	0
45	1.706E-37	1.631E-23	1.663E-23	0
46	1.875E-38	8.381E-24	4.170E-24	0
47	2.097E-39	1.465E-24	1.014E-24	0
48	2.368E-40	7.481E-25	2.426E-25	0
49	2.555E-41	1.307E-25	5.934E-26	0

q	VaR_q^m	VaR_q^g	VaR_q^s	VaR_q^r	ES_q^m	ES_q^g	ES_q^s	ES_q^r
0.950	2	1	1	1	2.58982	2.443579	2.677794	2.606374
0.951	2	2	1	1	2.601857	2.465176	2.712034	2.639157
0.952	2	2	2	1	2.614395	2.474867	2.73231	2.673306
0.953	2	2	2	1	2.627468	2.484971	2.747891	2.708908
0.954	2	2	2	2	2.641108	2.495514	2.76415	2.728691
0.955	2	2	2	2	2.655355	2.506525	2.781131	2.744884
0.956	2	2	2	2	2.67025	2.518037	2.798884	2.761813
0.957	2	2	2	2	2.685837	2.530085	2.817463	2.779529
0.958	2	2	2	2	2.702166	2.542706	2.836926	2.79809
0.959	2	2	2	2	2.719292	2.555942	2.857339	2.817555
0.960	2	2	2	2	2.737274	2.569841	2.878773	2.837994
0.961	2	2	2	2	2.756179	2.584452	2.901305	2.859481
0.962	2	2	2	2	2.776078	2.599833	2.925024	2.882099
0.963	2	2	2	2	2.797054	2.616044	2.950024	2.90594
0.964	2	2	2	2	2.819194	2.633157	2.976414	2.931105
0.965	2	2	2	2	2.842599	2.651247	3.004311	2.957708
0.966	2	2	2	2	2.867382	2.670401	3.03385	2.985875
0.967	2	2	2	2	2.893666	2.690716	3.065179	3.01575
0.968	2	2	2	2	2.921593	2.712301	3.098466	3.047493
0.969	2	2	2	2	2.951322	2.735279	3.1339	3.081283
0.970	2	2	2	2	2.983033	2.759788	3.171697	3.117325
0.971	2	2	2	2	3.01693	2.785988	3.2121	3.155854
0.972	2	2	2	2	3.053249	2.814059	3.255389	3.197134
0.973	2	2	2	2	3.092259	2.844209	3.301885	3.241473
0.974	2	2	2	2	3.134268	2.876678	3.351958	3.289222
0.975	2	2	2	2	3.179639	2.911746	3.406036	3.340791
0.976	2	2	2	2	3.228791	2.949735	3.464621	3.396657
0.977	3	2	2	2	3.239754	2.991028	3.5283	3.457381
0.978	3	2	2	2	3.250652	3.036074	3.597768	3.523626
0.979	3	2	2	2	3.262588	3.085411	3.673852	3.596179
0.980	3	2	3	3	3.275717	3.139682	3.727784	3.665586
0.981	3	2	3	3	3.290229	3.199665	3.766088	3.700617
0.982	3	2	3	3	3.306352	3.266313	3.808648	3.73954
0.983	3	2	3	3	3.324373	3.340802	3.856216	3.783042
0.984	3	2	3	3	3.344646	3.424602	3.90973	3.831982
0.985	3	2	3	3	3.367623	3.519576	3.970378	3.887448
0.986	3	2	3	3	3.393882	3.628117	4.039691	3.950837
0.987	3	2	3	3	3.42418	3.753357	4.119667	4.023978
0.988	3	3	3	3	3.459528	3.874372	4.212973	4.10931
0.989	3	3	3	3	3.501304	3.95386	4.323243	4.210156
0.990	3	3	3	3	3.551434	4.049247	4.455567	4.331171
0.991	3	3	3	3	3.612705	4.165829	4.617297	4.479079
0.992	3	3	3	3	3.689293	4.311558	4.819459	4.663964
0.993	3	3	3	3	3.787763	4.498924	5.079382	4.901673
0.994	3	4	4	3	3.919057	4.582606	5.391554	5.218619
0.995	3	4	4	4	4.102868	4.699128	5.669865	5.497037
0.996	4	4	5	5	4.314299	4.873909	5.883184	5.705985
0.997	4	4	5	5	4.419065	5.165213	6.177579	5.941314
0.998	4	4	6	5	4.628598	5.747819	6.699698	6.41197
0.999	4	6	6	6	5.257196	6.459968	7.399396	7.048346

Table 5.3.: Some probabilistic values of the multiple groups portfolio

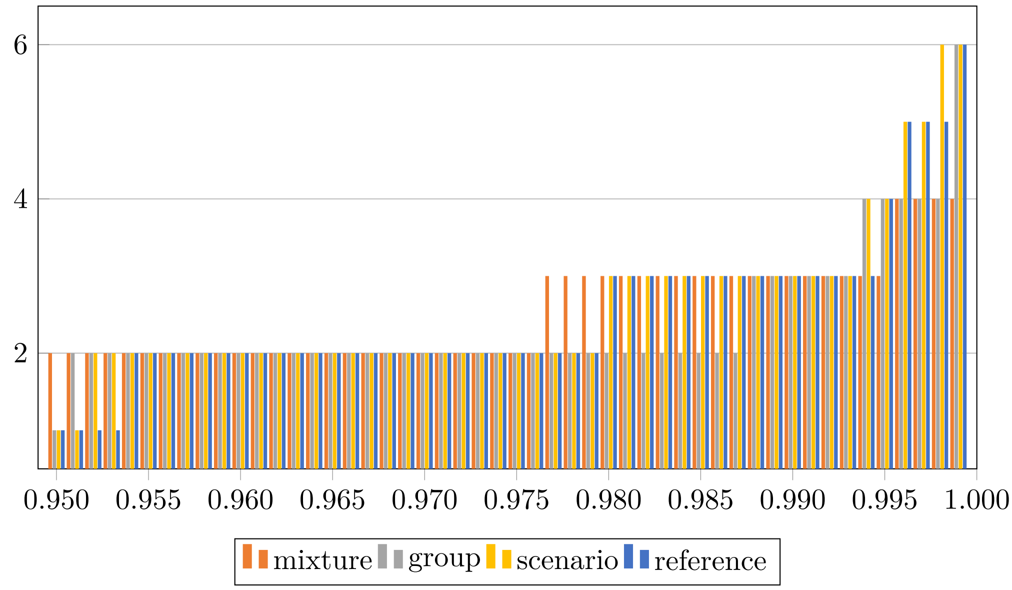


Figure 5.3.: Value-at-Risk of the multiple groups portfolio

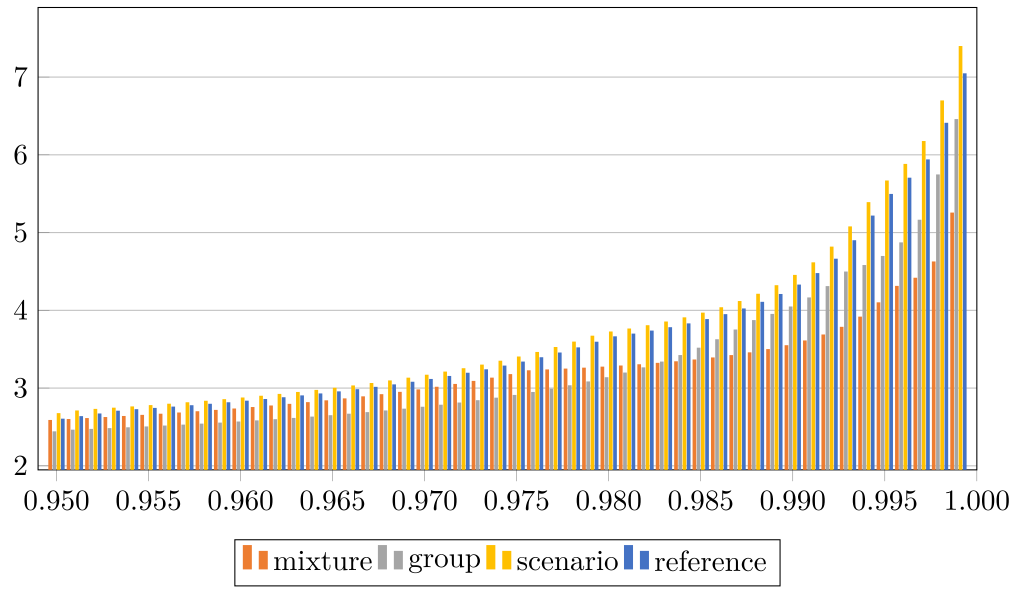


Figure 5.4.: Expected shortfall of the multiple groups portfolio

mixture	group	scenario
0.019995	0.023793	0.007142

Table 5.4.: Wasserstein distances of the multiple groups portfolio

It turns out that the problems we have seen in the minimal example vanish when we increase the number groups we consider. The group method, while still inferior to the scenario method, closes its gap to the mixture method.

A look on the quantile graphs underscores the initial criticism of the mixture method that it greatly underestimates the risk of clustering. At higher levels the expected shortfall of the mixture loss is almost a third smaller than the expected shortfall of the reference loss.

5.2.3. Multiple guarantors

Structure

Number of guarantors		5	Number of groups			10
$s_{i=1,\dots,5}$	π_s	i%	$g_{i=1,\dots,5}$	p_g		4%
	c_s	idiosync.		c_g		idiosync.
				h_1	Exposure	Dirac(2)
					Guarantor	s_i
				h_2	Exposure	Dirac(1)
					Guarantor	—
			$g_{6,\dots,10}$	p_g		4%
				c_g		idiosync.
				h_1	Exposure	Dirac(3)
					Guarantor	—

The first two examined examples have only used a single guarantor. In this portfolio each group with a guarantee has its own independent guarantor.

Numerical results

The expected value all four methods return is 0.812:

$$5 \cdot 3 \cdot 4\% + \sum_{i=1}^{5} (1 + 2 \cdot i\%) \cdot 4\% = 0.812.$$

The next pages contain the raw results of each distribution.

x	$\mathbb{P}[L^m = x]$	$\mathbb{P}[L^g = x]$	$\mathbb{P}[L^s = x]$	$\mathbb{P}[L^r = x]$
0	0.67032	0.666389	0.67032	0.664833
1	0.130042	0.133278	0.130042	0.134352
2	0.012614	0.017169	0.012692	0.01086
3	0.138902	0.134935	0.138916	0.143101
4	0.026828	0.026865	0.026675	0.028672
5	0.0026	0.003458	0.002589	0.002303
6	0.014391	0.013663	0.014469	0.01251
7	0.002767	0.002708	0.002748	0.002476
8	2.680E-04	3.483E-04	2.650E-04	1.971E-04
9	9.940E-04	9.224E-04	0.001011	5.630E-04
10	1.903E-04	1.820E-04	1.897E-04	1.093E-04
11	1.841E-05	2.339E-05	1.816E-05	8.576E-06
12	5.149E-05	4.671E-05	5.332E-05	1.353E-05
13	9.815E-06	9.173E-06	9.876E-06	2.539E-06
14	9.487E-07	1.178E-06	9.378E-07	1.941E-07
15	2.134E-06	1.892E-06	2.268E-06	1.608E-07
16	4.050E-07	3.699E-07	4.142E-07	2.803E-08
17	3.911E-08	4.750E-08	3.896E-08	2.016E-09
18	7.367E-08	6.390E-08	8.114E-08	7.498E-10
19	1.392E-08	1.243E-08	1.459E-08	1.044E-10
20	1.344E-09	1.595E-09	1.358E-09	5.884E-12
21	2.180E-09	1.850E-09	2.513E-09	1.548E-12
22	4.104E-10	3.583E-10	4.445E-10	1.648E-13
23	3.956E-11	4.594E-11	4.086E-11	6.351E-15
24	5.647E-11	4.685E-11	6.887E-11	1.525E-15
25	1.058E-11	9.034E-12	1.196E-11	1.139E-16
26	1.019E-12	1.158E-12	1.085E-12	2.246E-18
27	1.300E-12	1.055E-12	1.699E-12	7.046E-19
28	2.426E-13	2.025E-13	2.894E-13	2.810E-20
29	2.335E-14	2.593E-14	2.585E-14	0
30	2.693E-14	2.139E-14	3.822E-14	1.258E-22
31	5.005E-15	4.085E-15	6.376E-15	0
32	4.812E-16	5.230E-16	5.602E-16	0
33	5.071E-16	3.941E-16	7.932E-16	0
34	9.386E-17	7.494E-17	1.294E-16	0
35	9.017E-18	9.588E-18	1.116E-17	0
36	8.754E-18	6.659E-18	1.532E-17	0
37	1.614E-18	1.260E-18	2.441E-18	0
38	1.549E-19	1.612E-19	2.064E-19	0
39	1.395E-19	1.039E-19	2.777E-19	0
40	2.561E-20	1.957E-20	4.314E-20	0
41	2.456E-21	2.501E-21	3.570E-21	0
42	2.064E-21	1.505E-21	4.752E-21	0
43	3.773E-22	2.821E-22	7.189E-22	0
44	3.616E-23	3.604E-23	5.816E-23	0
45	2.850E-23	2.034E-23	7.722E-23	0
46	5.190E-24	3.797E-24	1.136E-23	0
47	4.968E-25	4.848E-25	8.972E-25	0
48	3.689E-25	2.579E-25	1.197E-24	0
49	6.691E-26	4.791E-26	1.711E-25	0

q	VaR_q^m	VaR_q^g	VaR_q^s	VaR_q^r	ES_q^m	ES_q^g	ES_q^s	ES_q^r
0.950	3	3	3	3	4.913168	4.897844	4.914718	4.721212
0.951	3	3	3	3	4.952212	4.936575	4.953794	4.756339
0.952	4	4	4	3	4.990339	4.972147	4.993861	4.792929
0.953	4	4	4	3	5.01141	4.992831	5.015007	4.831077
0.954	4	4	4	4	5.033397	5.014414	5.037072	4.852287
0.955	4	4	4	4	5.056361	5.036957	5.060118	4.871227
0.956	4	4	4	4	5.080369	5.060524	5.084212	4.891028
0.957	4	4	4	4	5.105494	5.085187	5.109426	4.911749
0.958	4	4	4	4	5.131816	5.111025	5.135841	4.933457
0.959	4	4	4	4	5.159421	5.138123	5.163544	4.956225
0.960	4	4	4	4	5.188406	5.166577	5.192633	4.98013
0.961	4	4	4	4	5.218878	5.196489	5.223213	5.005262
0.962	4	4	4	4	5.250954	5.227975	5.255403	5.031716
0.963	4	4	4	4	5.284764	5.261164	5.289333	5.0596
0.964	4	4	4	4	5.320452	5.296196	5.325147	5.089034
0.965	4	4	4	4	5.358179	5.33323	5.363009	5.120149
0.966	4	4	4	4	5.398125	5.372443	5.403097	5.153094
0.967	4	4	4	4	5.440493	5.414032	5.445615	5.188037
0.968	4	4	4	4	5.485508	5.458221	5.490791	5.225163
0.969	4	4	4	4	5.533428	5.50526	5.538881	5.264684
0.970	4	4	4	4	5.584542	5.555435	5.590177	5.30684
0.971	4	4	4	4	5.639181	5.609071	5.645011	5.351904
0.972	4	4	4	4	5.697723	5.666538	5.703761	5.400186
0.973	4	4	4	4	5.760602	5.728262	5.766863	5.452045
0.974	4	4	4	4	5.828318	5.794733	5.83482	5.507893
0.975	4	4	4	4	5.90145	5.866522	5.908212	5.568208
0.976	4	4	4	4	5.980677	5.944294	5.987721	5.63355
0.977	4	4	4	4	6.066794	6.028829	6.074144	5.704574
0.978	4	4	4	4	6.160739	6.121048	6.168423	5.782055
0.979	5	5	5	4	6.249636	6.204698	6.254727	5.866915
0.980	5	5	5	4	6.312117	6.264933	6.317464	5.960261
0.981	5	5	5	4	6.381176	6.331508	6.386804	6.063432
0.982	6	5	6	5	6.41936	6.405481	6.421229	6.167862
0.983	6	6	6	5	6.444028	6.434859	6.446007	6.236559
0.984	6	6	6	5	6.47178	6.462038	6.473882	6.313844
0.985	6	6	6	6	6.503231	6.49284	6.505474	6.342743
0.986	6	6	6	6	6.539177	6.528043	6.54158	6.367224
0.987	6	6	6	6	6.580652	6.568662	6.58324	6.395472
0.988	6	6	6	6	6.629039	6.61605	6.631843	6.428428
0.989	6	6	6	6	6.686225	6.672055	6.689283	6.467376
0.990	6	6	6	6	6.754847	6.73926	6.758212	6.514114
0.991	6	6	6	6	6.838719	6.8214	6.842457	6.571238
0.992	6	6	6	6	6.943559	6.924075	6.947765	6.642643
0.993	6	6	6	6	7.078353	7.056086	7.08316	6.734449
0.994	6	6	6	6	7.258079	7.232101	7.263686	6.856857
0.995	6	6	6	6	7.509694	7.478521	7.516423	7.028228
0.996	7	7	7	6	7.811384	7.787314	7.820971	7.285285
0.997	7	7	7	7	8.081845	8.049752	8.094627	7.590338
0.998	7	7	7	7	8.622768	8.574629	8.641941	7.885507
0.999	9	9	9	7	9.442391	9.426536	9.447883	8.771015

Table 5.5.: Some probabilistic values of the multiple guarantors portfolio

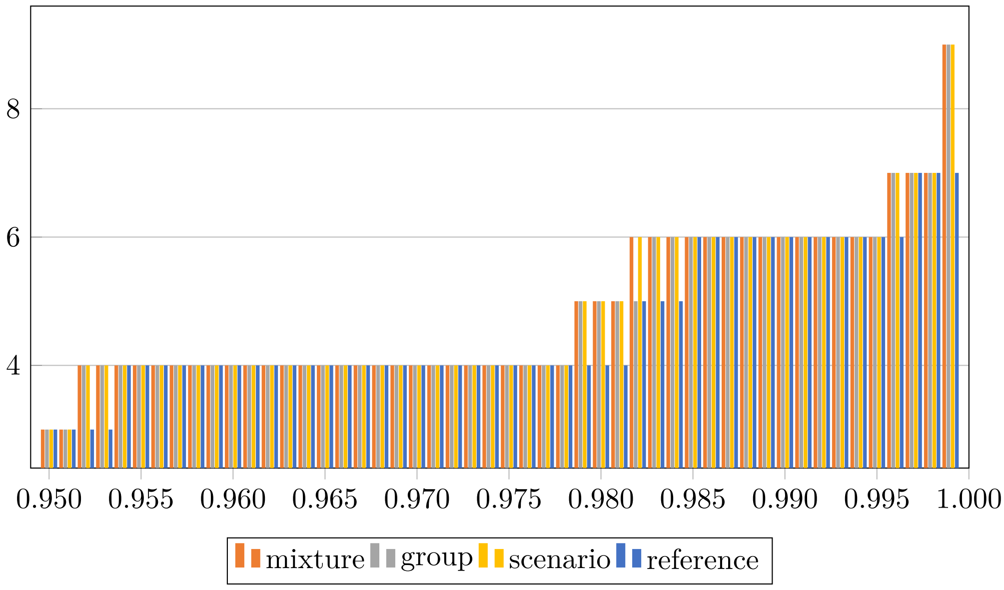


Figure 5.5.: Value-at-Risk of the multiple guarantors portfolio

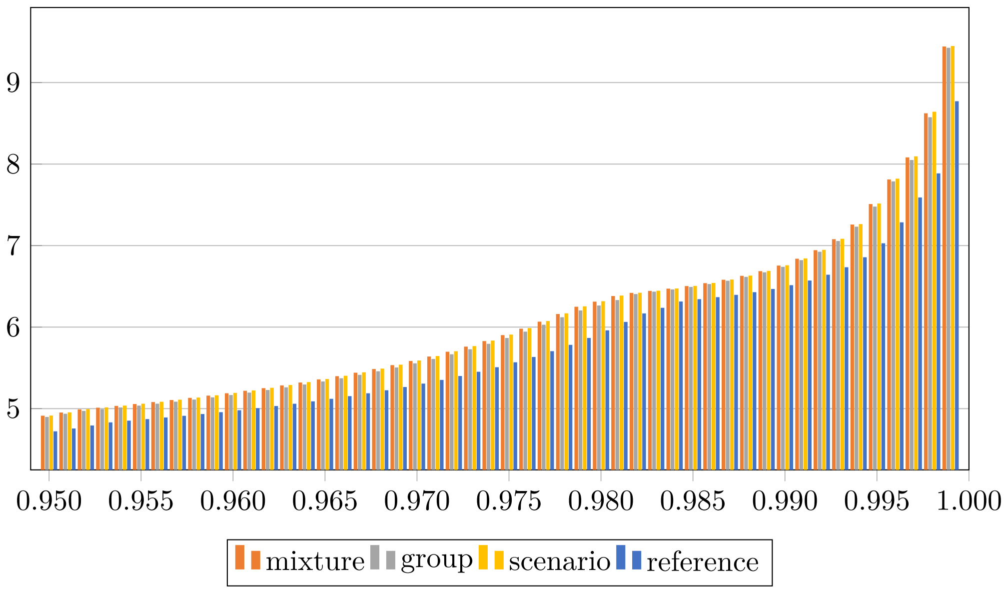


Figure 5.6.: Expected shortfall of the multiple guarantors portfolio

mixture	group	scenario
0.019196	0.017663	0.019351

Table 5.6.: Wasserstein distances of the multiple guarantors portfolio

This portfolio is interesting, because here the group method turns out to be the best approximation.

While it is not as much better compared to the others as it was worse in the other examples, it is nevertheless – according to the Wasserstein distance – the most accurate approximation.

Here the scenario approach has the same problem as the groups approach: it binds unrelated guarantees in repeat defaults. Since the scenario model operates "from outside" the model the assumptions which guarantor has defaulted and which is standing are fixed once a scenario is chosen. Thus in a scenario with a high number of defaulted guarantors any repeat default will be forced to generate the higher loss again.

At first it might seem that this portfolio should be perfect for the mixture method, since all the guarantees are independent. The group approach is better here, however, because the defaults of the guarantees are modelled by two Poisson distributions.

We know that for each group the Poisson intensity λ_g is chosen such that the expected value of the entire model agrees with the natural expected value of the data, i.e. $\lambda_g = p_g$. Since $\Lambda_g \equiv 1$ in this portfolio, because the only default cause in use is the idiosyncratic one, we get

$$\mathbb{P}[N_g = 0] = \frac{(\lambda_g \Lambda_g)^0 \, \mathrm{e}^{-\lambda_g \Lambda_g}}{(\lambda_g \Lambda_g)!} = \mathrm{e}^{-\lambda_g} > 1 - \lambda_g.$$

Both the mixture model and the group model have a compound Poisson sum driving the defaults of the groups, but in the group approach the number of defaults of each guarantor is Poisson(λ_s), while in the mixture approach it is Bern(λ_s). In case of a repeat default of the outer compound sum the probability that no loss is generated is higher in the group approach than in the mixture approach. As a result the tail of the group model is lighter than the tail of the mixture model and since the reference distribution does not have any tail at all a lighter tail is a better match.

This phenomenon, however, only manifests itself in portfolios with low dependence between groups. This particular portfolio is in fact an extereme example of such a portfolio, because there is no dependence at all between all the groups.

5.2.4. High dependence

Structure

Number of guarantors		7	Number of groups			301
$s_{i=0,\ldots,6}$	π_s	10%	$g_{k=1,\ldots,301}$	p_g		3%
	c_s	idiosync.		c_g		idiosync.
				$h_{i=1,\ldots,7}$	Exposure	Dirac(i)
					Guarantor	$s_{(i+k) \bmod 7}$
				h_8	Exposure	Dirac(1)
					Guarantor	—

This portfolio can be seen as the opposite construction of the previous one.

Here we have 7 guarantors and their guarantees are spread over the entire portfolio and each guarantor has a guarantee in every group.

Numerical results

The expected value all four methods return is 34.314:

$$301 \cdot (3\% \cdot (1 + \frac{7 \cdot 8}{2} \cdot 10\%)) = 34.314$$

The next pages contain the raw results of each distribution.

x	$\mathbb{P}[L^m = x]$	$\mathbb{P}[L^g = x]$	$\mathbb{P}[L^s = x]$	$\mathbb{P}[L^r = x]$
0	1.198E-04	5.948E-05	1.198E-04	1.043E-04
1	5.173E-04	5.371E-04	5.173E-04	4.644E-04
2	0.0011745	0.002425	0.002393	0.002206
3	0.0019138	0.0072994	0.007087	0.006693
4	0.0025983	0.0164789	0.01597	0.015393
5	0.0032345	0.0297623	0.028798	0.028238
6	0.0039208	0.0447959	0.043331	0.043071
7	0.0047585	0.0577941	0.055909	0.056151
8	0.0058141	0.0652494	0.063173	0.063879
9	0.0070681	0.0654926	0.063438	0.064371
10	0.0084105	0.059183	0.057488	0.058322
11	0.0097305	0.0486531	0.047394	0.047912
12	0.010998	0.0367174	0.036025	0.036153
13	0.012254	0.0256602	0.025461	0.02528
14	0.013546	0.0167724	0.017021	0.01666
15	0.014888	0.0104029	0.01097	0.010565
16	0.01625	0.0062827	0.007119	0.006743
17	0.017581	0.0038784	0.004864	0.004561
18	0.018835	0.0026427	0.003814	0.003574
19	0.019992	0.0021369	0.003341	0.003161
20	0.021058	0.002059	0.003362	0.003214
21	0.022044	0.0022226	0.003525	0.003404
22	0.022948	0.0025225	0.003877	0.003765
23	0.023759	0.0029028	0.004215	0.004115
24	0.02446	0.0033344	0.004657	0.004559
25	0.025038	0.0038017	0.005027	0.004941
26	0.025491	0.004295	0.005516	0.005432
27	0.025822	0.0048069	0.005886	0.005814
28	0.026039	0.0053305	0.006366	0.0063
29	0.026144	0.0058591	0.006714	0.006663
30	0.026139	0.006386	0.007177	0.007133
31	0.026027	0.0069042	0.007503	0.007477
32	0.025811	0.007407	0.007936	0.007919
33	0.025498	0.0078877	0.008218	0.00822
34	0.025095	0.0083401	0.008613	0.008624
35	0.024612	0.0087585	0.008834	0.008864
36	0.024057	0.0091379	0.00919	0.009228
37	0.023436	0.009474	0.009335	0.009391
38	0.022759	0.0097633	0.009616	0.00968
39	0.022032	0.0100034	0.009693	0.009773
40	0.021263	0.0101924	0.009907	0.009993
41	0.020462	0.0103297	0.009913	0.010013
42	0.019635	0.0104154	0.010058	0.010161
43	0.01879	0.0104505	0.009983	0.010098
44	0.017933	0.0104367	0.010075	0.010189
45	0.017071	0.0103764	0.009937	0.01006
46	0.016208	0.0102727	0.009966	0.010086
47	0.015351	0.0101292	0.009765	0.00989
48	0.014505	0.0099497	0.009736	0.009854
49	0.013673	0.0097385	0.009489	0.009611

q	VaR_q^m	VaR_q^g	VaR_q^s	VaR_q^r	ES_q^m	ES_q^g	ES_q^s	ES_q^r
0.950	62	99	99	99	71.506362	122.934592	126.526217	126.031397
0.951	63	99	99	99	71.681953	123.423053	127.087976	126.583058
0.952	63	100	100	100	71.862828	123.922227	127.656327	127.147425
0.953	63	100	101	100	72.051398	124.431211	128.240635	127.72503
0.954	63	101	101	101	72.248168	124.94934	128.832823	128.308684
0.955	64	101	102	102	72.451482	125.481548	129.436967	128.908376
0.956	64	102	103	102	72.643561	126.019852	130.057827	129.51993
0.957	64	103	103	103	72.844574	126.577099	130.687078	130.141524
0.958	64	103	104	104	73.055159	127.138458	131.33073	130.780158
0.959	64	104	105	104	73.276017	127.717879	131.992208	131.433332
0.960	65	104	105	105	73.490334	128.310826	132.667013	132.097384
0.961	65	105	106	106	73.708035	128.914519	133.35527	132.779896
0.962	65	106	107	107	73.937194	129.53831	134.062958	133.481634
0.963	65	106	108	107	74.17874	130.17448	134.790138	134.197354
0.964	66	107	108	108	74.414612	130.826223	135.534309	134.930555
0.965	66	108	109	109	74.655029	131.499284	136.29598	135.685207
0.966	66	108	110	110	74.909589	132.19044	137.079771	136.46171
0.967	67	109	111	111	75.178839	132.896794	137.886968	137.261239
0.968	67	110	112	111	75.434427	133.626533	138.718231	138.081903
0.969	67	111	113	112	75.706506	134.380274	139.575131	138.925911
0.970	67	112	114	113	75.996723	135.158775	140.458738	139.797302
0.971	68	112	115	114	76.283207	135.957354	141.371035	140.697422
0.972	68	113	115	115	76.579036	136.783062	142.312857	141.628454
0.973	68	114	116	116	76.896778	137.638152	143.288185	142.592348
0.974	69	115	118	117	77.213737	138.524315	144.299476	143.591958
0.975	69	116	119	118	77.542286	139.443579	145.348349	144.630038
0.976	69	117	120	119	77.898215	140.398381	146.438861	145.710371
0.977	70	118	121	120	78.245845	141.391647	147.575431	146.836889
0.978	70	119	122	122	78.620657	142.426911	148.762704	148.010961
0.979	71	120	123	123	79.011771	143.508448	150.006826	149.236869
0.980	71	122	125	124	79.41236	144.640713	151.310203	150.524509
0.981	72	123	126	125	79.847287	145.821075	152.676153	151.88236
0.982	72	124	127	127	80.283247	147.06428	154.122186	153.304158
0.983	73	125	129	128	80.76416	148.379688	155.642724	154.81524
0.984	73	127	131	130	81.24942	149.760066	157.262323	156.410522
0.985	74	129	132	132	81.781653	151.233763	158.982047	158.114155
0.986	74	130	134	134	82.337485	152.793336	160.820865	159.937114
0.987	75	132	136	135	82.932193	154.465109	162.798249	161.893945
0.988	76	134	138	138	83.583804	156.263378	164.93609	164.00686
0.989	76	136	141	140	84.27324	158.207876	167.263513	166.303478
0.990	77	138	143	142	85.023475	160.32738	169.806974	168.826005
0.991	78	141	146	145	85.85201	162.659965	172.624109	171.610643
0.992	79	143	149	148	86.772578	165.248301	175.773909	174.727469
0.993	80	146	153	152	87.808431	168.166677	179.348931	178.259278
0.994	81	150	157	156	89.000119	171.509621	183.471228	182.335949
0.995	83	154	162	161	90.382488	175.438401	188.346474	187.154642
0.996	85	159	168	167	92.069713	180.208906	194.296887	193.033592
0.997	87	165	175	174	94.194891	186.301762	201.946224	200.583786
0.998	90	174	186	185	97.148193	194.779398	212.658484	211.148685
0.999	95	189	205	203	102.069895	208.9965	230.800975	229.027756

Table 5.7.: Some probabilistic values of the high dependence portfolio

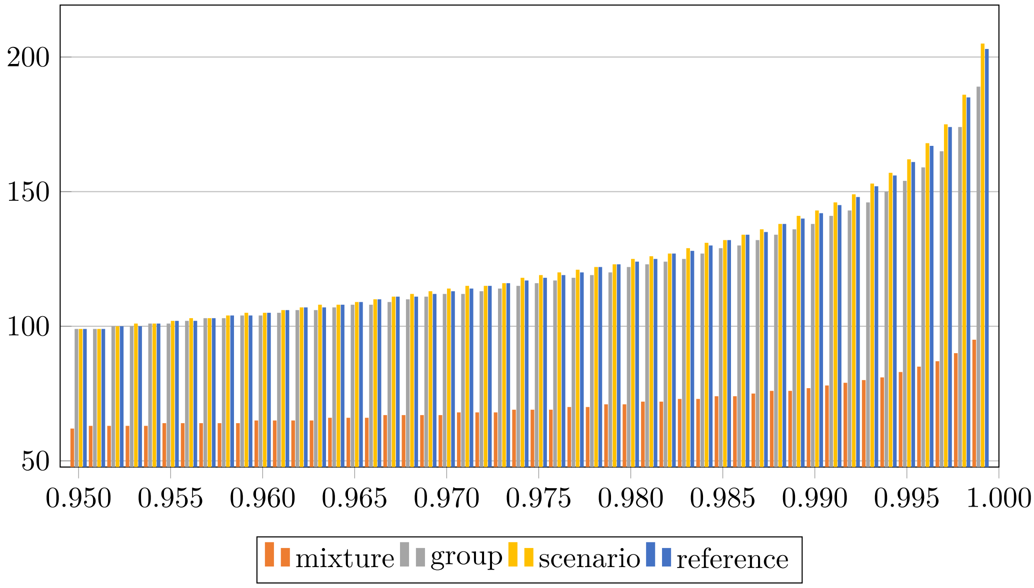


Figure 5.7.: Value-at-Risk of the high dependence portfolio

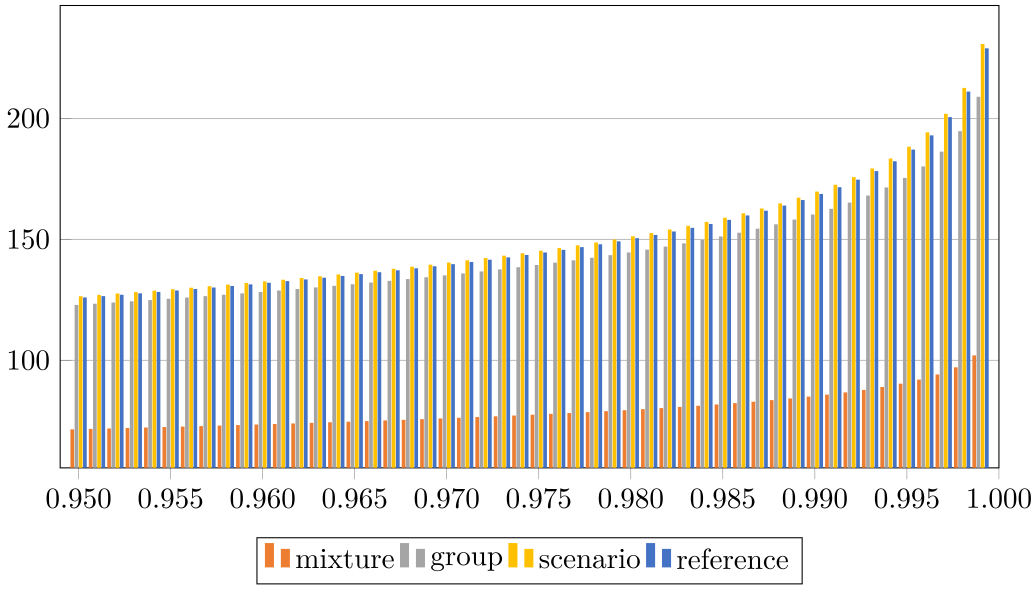


Figure 5.8.: Expected shortfall of the high dependence portfolio

mixture	group	scenario
13.458131	0.654849	0.099850

Table 5.8.: Wasserstein distances of the high dependence portfolio

Unsurprisingly this is a portfolio where the scenario method can showcase its strengths. It fits the reference distribution almost perfectly both in the quantiles and in the expected shortfall.

The mixture approach on the other hand is completely out of its depth providing no usable results other than the correct expected value, since it does not take into account the dependence structure of this portfolio at all.

To see how bad the approximation of the mixture approach is, let us plot the probabilities for loss sizes 0 to 50:

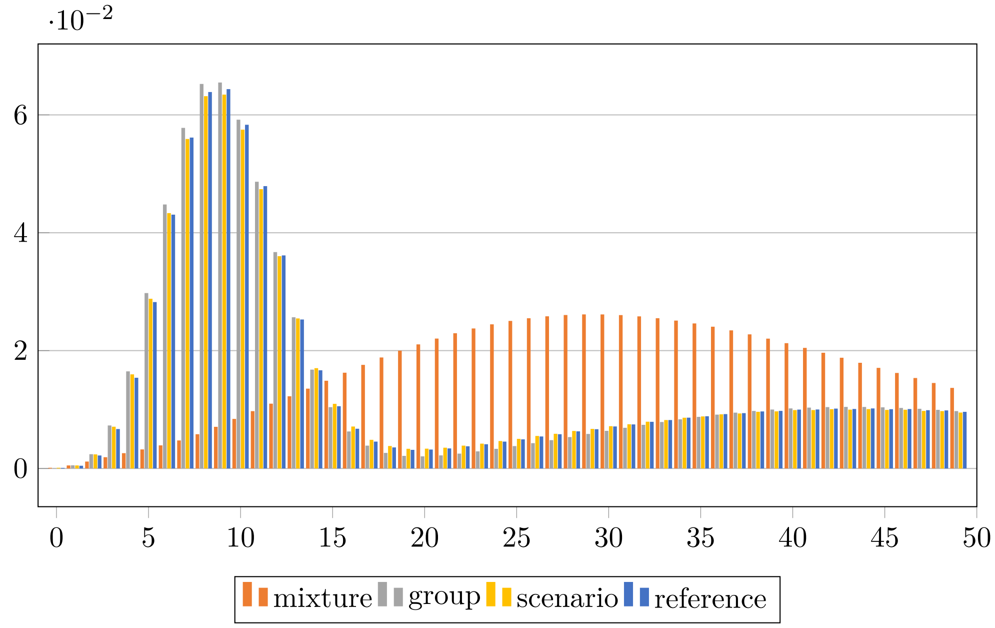


Figure 5.9.: Probabilities of the high dependence portfolio

The results of the group approach on the other hand are quite good. While not as accurate as the scenario method, we have to consider that in contrast to the scenario approach the group method does not increase the time of computation exponentially.

This portfolio is therefore a nice example why a hybrid approach combining the accuracy of the scenario approach with the speed of the group approach should be considered.

5.2.5. Massive guarantors

Structure

Number of guarantors		7	Number of groups			7000
$s_{1,\dots,7}$	π_s	10%	$g_{i=1,\dots,7000}$	p_g		3%
	c_s	idiosync.		c_g		idiosync.
				h_1	Exposure	Dirac(1)
					Guarantor	$s_{\lceil i/1000 \rceil}$
				h_2	Exposure	Dirac(1)
					Guarantor	—

This portfolio is similar to the previous one since it has a high degree of dependence between the guarantees, but instead of having multiple guarantors in each group we have a substantial amount of groups being guaranteed by single guarantors.

Numerical results

The expected value all four methods return is 231.

$$7000 \cdot 3\% \cdot (1 + 1 \cdot 10\%) = 231.$$

The next pages contain the raw results of each distribution.

q	VaR_q^m	VaR_q^g	VaR_q^s	VaR_q^r	ES_q^m	ES_q^g	ES_q^s	ES_q^r
0.950	259	286	285	285	265.814954	304.347751	302.632103	302.322029
0.951	259	286	285	285	265.954034	304.722195	302.991942	302.675539
0.952	259	286	286	285	266.09891	305.11224	303.359738	303.04378
0.953	259	287	286	286	266.249951	305.504593	303.729094	303.411034
0.954	259	287	286	286	266.407558	305.906867	304.114509	303.789535
0.955	259	288	287	287	266.572171	306.319025	304.49626	304.176138
0.956	260	288	287	287	266.732437	306.735366	304.893903	304.566505
0.957	260	289	288	288	266.889006	307.167586	305.295264	304.972619
0.958	260	289	288	288	267.05303	307.600147	305.707057	305.376729
0.959	260	290	289	288	267.225055	308.052859	306.128363	305.800551
0.960	260	290	289	289	267.405681	308.50418	306.556572	306.222733
0.961	261	291	290	289	267.586849	308.978044	306.998356	306.664341
0.962	261	291	290	290	267.760187	309.451151	307.445681	307.107655
0.963	261	292	291	290	267.942894	309.947128	307.90873	307.570024
0.964	261	292	291	291	268.135753	310.44566	308.378417	308.035415
0.965	261	293	292	291	268.339631	310.965108	308.863856	308.522141
0.966	262	293	292	292	268.535866	311.493493	309.359852	309.01094
0.967	262	294	293	293	268.733922	312.038298	309.869251	309.524102
0.968	262	294	293	293	268.944357	312.601994	310.396415	310.040481
0.969	262	295	294	294	269.168369	313.174756	310.93195	310.579404
0.970	263	296	295	294	269.391761	313.772054	311.492325	311.13205
0.971	263	296	295	295	269.612166	314.384884	312.061026	311.699638
0.972	263	297	296	296	269.848315	315.015067	312.651618	312.292903
0.973	263	298	297	296	270.101956	315.671371	313.268223	312.896344
0.974	264	298	297	297	270.34813	316.351039	313.893924	313.523909
0.975	264	299	298	298	270.602055	317.048364	314.547501	314.179122
0.976	264	300	299	298	270.87714	317.775143	315.229672	314.853252
0.977	265	301	299	299	271.161469	318.532533	315.93531	315.552207
0.978	265	302	300	300	271.441536	319.322037	316.665855	316.282852
0.979	265	303	301	301	271.748276	320.145574	317.430154	317.046647
0.980	266	303	302	302	272.068605	321.002852	318.230178	317.845475
0.981	266	304	303	302	272.388005	321.900608	319.06845	318.679447
0.982	266	305	304	303	272.742895	322.844298	319.948191	319.552962
0.983	267	306	305	304	273.099423	323.839172	320.873522	320.473786
0.984	267	308	306	305	273.480637	324.885481	321.849743	321.447403
0.985	268	309	307	307	273.890822	325.99428	322.883724	322.478063
0.986	268	310	308	308	274.311595	327.176412	323.984454	323.570199
0.987	269	311	309	309	274.777802	328.445813	325.163861	324.738796
0.988	269	313	311	310	275.259285	329.80089	326.426621	325.999404
0.989	270	314	312	312	275.789757	331.27547	327.789157	327.35542
0.990	270	316	314	313	276.368733	332.872916	329.275048	328.836228
0.991	271	318	316	315	276.98501	334.63391	330.909915	330.453432
0.992	272	320	317	317	277.680391	336.591525	332.714755	332.250452
0.993	273	322	320	319	278.463073	338.800435	334.742875	334.27089
0.994	274	325	322	322	279.350028	341.326218	337.057376	336.578818
0.995	275	328	325	325	280.375448	344.29415	339.767464	339.277257
0.996	276	332	328	328	281.611862	347.89225	343.046956	342.525564
0.997	278	337	333	332	283.179946	352.47659	347.188454	346.65754
0.998	280	343	339	338	285.324965	358.826188	352.913982	352.352359
0.999	284	354	349	348	288.85315	369.426966	362.410032	361.79443

Table 5.9.: Some quantile values of the massive guarantors portfolio

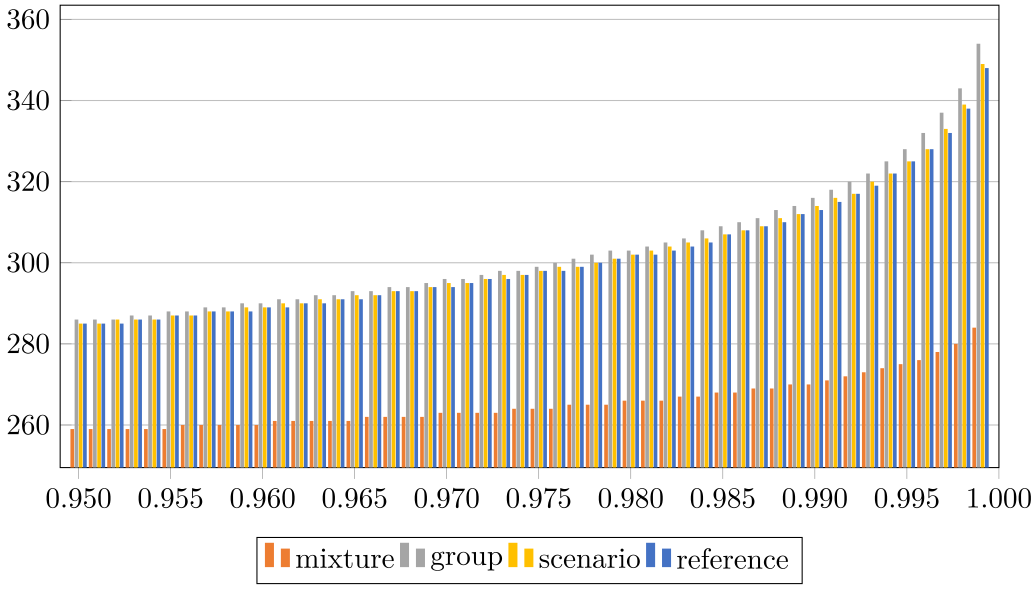


Figure 5.10.: Value-at-Risk of the massive guarantors portfolio

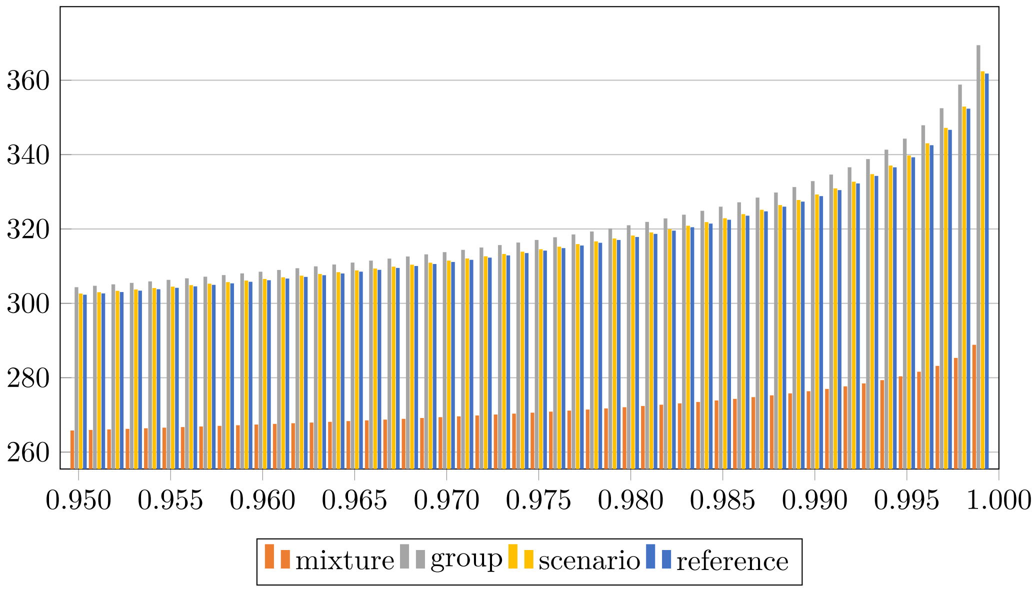


Figure 5.11.: Expected shortfall of the massive guarantors portfolio

mixture	group	scenario
10.014764	0.319311	0.105173

Table 5.10.: Wasserstein distances of the massive guarantors portfolio

Once again we see how ill-suited the mixture approach is for interdependent guarantees.

Similar to the previous portfolio the scenario distribution has the best fit, but here the group distribution is even closer than in the high dependence portfolio, because there is no dependence between the guarantors through shared groups anymore.

5.2.6. Complex portfolio with guarantor chains

Structure

Number of guarantors		3	Number of groups			5
s_1	π_s	2%	$g_{i=1,\dots,5}$	p_g		i%
	c_s	idiosync.		c_g		idiosync.
s_2	π_s	3%		$h_{k=1,\dots,15}$	Exposure	Dirac(k)
	c_s	idiosync.			Guarantor	$S(i,k)$
s_3	π_s	5%				
	c_s	idiosync.				

where $S(i,k)$ returns the following sequence[1]:

$$\left(s_1 \mathbb{1}_{\{15(i-1)+k \equiv 0\,(2)\}}, s_2 \mathbb{1}_{\{k \equiv 0\,(3)\}}, s_3 \mathbb{1}_{\{k \equiv 0\,(5)\}}\right) \tag{5.6}$$

This portfolio contains many different combinations of exposures, guarantors and even guarantor chains. It aims to simulate a somewhat more realistic, albeit still small portfolio.

The construction of $S(i,k)$ guarantees that every possible combination of the guarantors is present in the portfolio.

Numerical results

The expected value all four methods return is 4.770756.
The next pages contain the raw results of each distribution.

[1] Here we use the shorthand notation of parentheses for the modulo-operation.

x	$\mathbb{P}[L^m = x]$	$\mathbb{P}[L^g = x]$	$\mathbb{P}[L^s = x]$	$\mathbb{P}[L^r = x]$
0	0.860708	0.848545	0.860708	0.858278
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	0	0	0	0
5	0	0.003324	0	0
6	0	0	0	0
7	0	0	0	0
8	0	0	0	0
9	0	0	0	0
10	0	0.002403	0	0
11	0	0	0	0
12	0	0.001986	0	0
13	0	0	0	0
14	0	0	0	0
15	0	3.053E-04	0	0
16	0	0	0	0
17	0	7.836E-06	0	0
18	0	0.001342	0	0
19	0	0	0	0
20	0	8.201E-05	0	0
21	0	0	0	0
22	0	5.711E-06	0	0
23	0	5.345E-06	0	0
24	0	9.137E-05	0	0
25	0	7.516E-06	0	0
26	0	0	0	0
27	0	8.860E-06	0	0
28	0.046637	0.052217	0.046637	0.048113
29	0	3.598E-07	0	0
30	0	1.292E-04	0	0
31	0	0	0	0
32	0.069955	0.077236	0.069955	0.072595
33	0	2.139E-04	0	0
34	0	2.616E-07	0	0
35	0	6.908E-07	0	0
36	0	4.475E-05	0	0
37	0.003682	3.048E-04	0.003682	0.003821
38	0.002455	1.531E-04	0.002455	0.002532
39	0	4.175E-07	0	0
40	0	1.246E-04	0	0
41	0	1.795E-07	0	0
42	0	2.283E-04	0	0
43	0	1.926E-05	0	0
44	0.002164	1.821E-04	0.002164	0.002245
45	0	1.208E-06	0	0
46	0.001442	8.575E-05	0.001442	0.001488
47	0	2.811E-05	0	0
48	0	9.569E-06	0	0
49	0	7.291E-07	0	0

q	VaR_q^m	VaR_q^g	VaR_q^s	VaR_q^r	ES_q^m	ES_q^g	ES_q^s	ES_q^r
0.950	32	32	32	32	41.999167	38.833187	41.999167	40.561927
0.951	32	32	32	32	42.203231	38.97264	42.203231	40.73666
0.952	32	32	32	32	42.415799	39.117903	42.415799	40.918674
0.953	32	32	32	32	42.637411	39.269348	42.637411	41.108433
0.954	32	32	32	32	42.86866	39.427377	42.86866	41.306442
0.955	32	32	32	32	43.110185	39.59243	43.110185	41.513252
0.956	32	32	32	32	43.36269	39.764985	43.36269	41.729462
0.957	32	32	32	32	43.626938	39.945566	43.626938	41.955729
0.958	32	32	32	32	43.90377	40.134746	43.90377	42.19277
0.959	32	32	32	32	44.194106	40.333155	44.194106	42.441374
0.960	32	32	32	32	44.498958	40.541483	44.498958	42.702409
0.961	32	32	32	32	44.819445	40.760496	44.819445	42.976829
0.962	32	32	32	32	45.156798	40.991035	45.156798	43.265693
0.963	32	32	32	32	45.512388	41.234036	45.512388	43.570171
0.964	32	32	32	32	45.887732	41.490537	45.887732	43.891565
0.965	32	32	32	32	46.284524	41.761695	46.284524	44.231324
0.966	32	32	32	32	46.704657	42.048804	46.704657	44.591069
0.967	32	32	32	32	47.150253	42.353313	47.150253	44.972616
0.968	32	32	32	32	47.623698	42.676854	47.623698	45.378011
0.969	32	32	32	32	48.127688	43.021269	48.127688	45.809559
0.970	32	32	32	32	48.665278	43.388645	48.665278	46.269878
0.971	32	32	32	32	49.239943	43.781356	49.239943	46.761943
0.972	32	32	32	32	49.855655	44.202119	49.855655	47.289155
0.973	32	32	32	32	50.516975	44.65405	50.516975	47.85542
0.974	32	32	32	32	51.229167	45.140744	51.229167	48.465244
0.975	32	32	32	32	51.998334	45.666374	51.998334	49.123854
0.976	32	32	32	32	52.831597	46.235806	52.831597	49.837348
0.977	32	32	32	32	53.737319	46.854754	53.737319	50.612884
0.978	37	32	37	32	54.566267	47.52997	54.566267	51.458925
0.979	37	32	37	37	55.402756	48.269492	55.402756	52.382151
0.980	37	32	37	37	56.322894	49.082967	56.322894	53.151259
0.981	38	32	38	37	57.338929	49.98207	57.338929	54.001325
0.982	38	32	38	37	58.413314	50.981074	58.413314	54.945843
0.983	38	32	38	38	59.614097	52.097608	59.614097	55.990101
0.984	44	32	44	38	60.7536	53.353709	60.7536	57.114482
0.985	44	32	44	38	61.870507	54.777289	61.870507	58.388781
0.986	46	32	46	44	63.089813	56.404238	63.089813	59.561761
0.987	46	32	46	44	64.404414	58.281488	64.404414	60.758819
0.988	56	37	56	46	65.140001	60.42006	65.140001	62.086058
0.989	60	46	60	46	65.746061	62.245042	65.795376	63.548426
0.990	60	56	60	60	66.320667	63.039133	66.374913	64.299114
0.991	60	60	60	60	67.022964	63.78288	67.083237	64.776793
0.992	60	60	60	60	67.900834	64.25574	67.968642	65.373893
0.993	60	60	60	60	69.029525	64.863702	69.107019	66.141592
0.994	60	60	60	60	70.534445	65.67432	70.624855	67.16519
0.995	64	60	60	60	72.28258	66.809183	72.749826	68.598228
0.996	64	64	64	60	74.353225	68.272543	75.0313	70.747785
0.997	64	64	64	64	77.8043	69.696724	78.7084	74.035982
0.998	69	64	64	64	84.075637	72.545086	86.062601	79.053973
0.999	84	64	88	75	94.717647	81.090173	100.481119	92.15438

Table 5.11.: Some probabilistic values of the complex portfolio

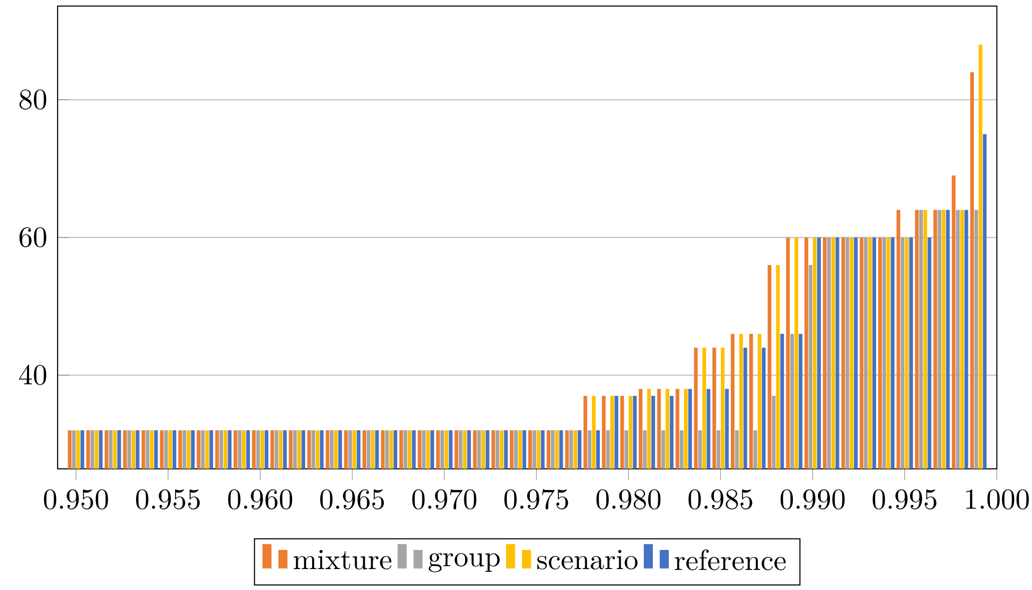


Figure 5.12.: Value-at-Risk of the complex portfolio

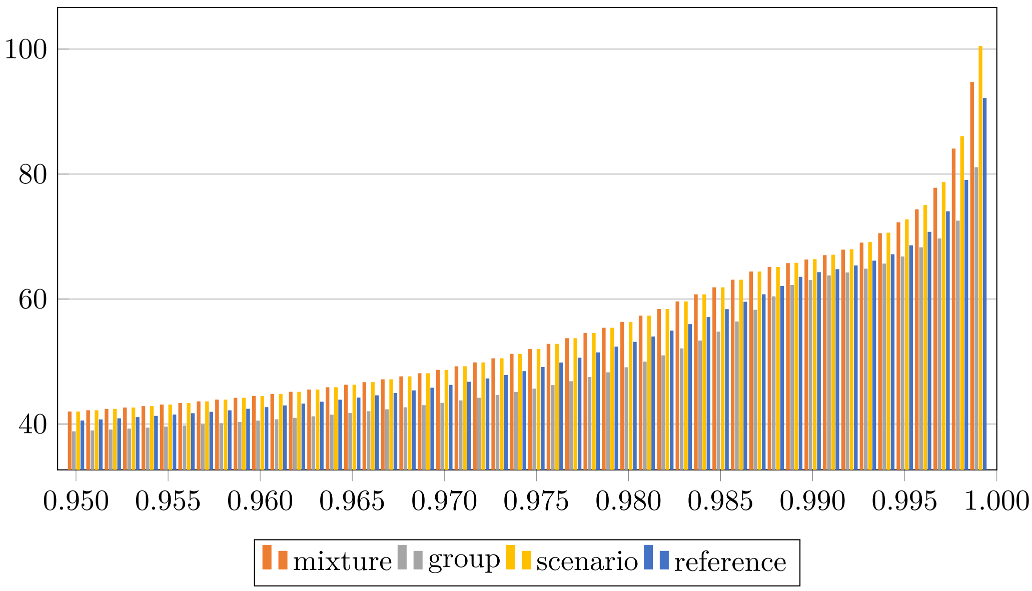


Figure 5.13.: Expected shortfall of the complex portfolio

mixture	group	scenario
0.149157	0.213156	0.143724

Table 5.12.: Wasserstein distances of the complex portfolio

Here we see again the tendency of the group approach to generate losses with values which according to the structure of the portfolio are impossible. However, due to the size of the portfolio – 75 blocks – this is compensated by a lighter tail.

The scenario approximation is again the best approximation of the actual distribution according to the Wasserstein distance, but we can see in Figure 5.12 and Figure 5.13 that both the mixture method and the scenario method tend to overestimate the Value-at-Risk and even more so the expected shortfall.

5.2.7. Stochastic minimal portfolio

Until this point all portfolios discussed did not use any non-degenerate risk factors and thus were deterministic. We begin our foray into stochastic portfolios with a minimal viable portfolio, which has been pared down even in comparison to the minimal portfolio discussed before.

Structure

Number of guarantors		1	Number of groups			1
s	π_s	10%	g	p_g		4%
	c_s	R		c_g		R
				h_1	Exposure	Dirac(1)
Number of risk factors		1			Guarantor	s
R	$\mathbb{E}[R]$	1		h_2	Exposure	Dirac(1)
	$\mathbb{V}[R]$	$1/4$			Guarantor	—

Expected value

The simplicity of this portfolio allows us to find an explicit expression for the expected values for some approaches and thus to show that each approach yields a slightly different expected value.

To calculate the expected value of this portfolio we will use the notation of Section 3.3. Thus N_g and N_s are the Poisson-distributed numbers of default, whereas B_g and B_s are the Bernoulli-distributed default indicators.

Definition 5.7. Let $\mathscr{M}$ be the *stochastic minimal portfolio* – an ECR$^+$-model with

- a single scenario,
- a single risk factor $R \sim \Gamma(\alpha, \beta)$,
- a single default cause c,
- a trivial dependency matrix $A = (a_{c,1}) = (1)$,
- a single guarantor s with probability of default p_s,

- a single group g with probability of default p_g, some obligor o and two guarantee blocks $\big(\mathrm{Dirac}(1), o, ()\big)$ and $\big(\mathrm{Dirac}(1), o, (s)\big)$.

We define the Bernoulli parameters of default ρ_g and ρ_s as in Section 3.3.

Lemma 5.8. *If $\mathbb{E}[R] = 1$, then in the scenario approach the stochastic minimal portfolio $\mathcal{M}$ has expected value*

$$\mathbb{E}[L^s] = p_g\Big(2 - (1-p_s)^{\frac{\alpha+1}{\alpha}}\Big). \tag{5.9}$$

Proof. In the scenario approach the default intensities of guarantors are set to the Bernoulli parameters ρ_s, but the groups are modelled with the full Poisson distribution. Therefore the loss can be written as

$$L = N_g\big(1 + \mathbb{1}_{\{N_s \geq 1\}}\big).$$

Taking the expectation we get

$$\mathbb{E}[L] = \mathbb{E}[N_g] + \mathbb{E}\big[N_g \mathbb{1}_{\{N_s \geq 1\}}\big]$$

Conditioning on R and using the conditional independence of N_g and N_s

$$= p_g + \mathbb{E}\big[\mathbb{E}[N_g \,|\, R]\,\mathbb{P}(N_s \geq 1 \,|\, R)\big]$$

Applying the conditional distributions

$$\begin{aligned} &= p_g + \mathbb{E}\big[p_g R(1 - \mathrm{e}^{-\rho_s R})\big] \\ &= p_g + p_g\mathbb{E}[R] - p_g\mathbb{E}\big[R\,\mathrm{e}^{-\rho_s R}\big] \end{aligned}$$

Using Lemma 3.6 and that $\mathbb{E}[R] = \frac{\alpha}{\beta} = 1$

$$= 2p_g - p_g\left(1 + \frac{\rho_s}{\beta}\right)^{-(\alpha+1)}.$$

Since by Remark 3.41 $(1 + \frac{\rho_s}{\beta})^\alpha = \frac{1}{1-p_s}$, the result (5.9) follows. ∎

Lemma 5.10. *In the reference model the stochastic minimal portfolio $\mathcal{M}$ has expected value*

$$\mathbb{E}[L] = 2p_g - (1-p_s) + \big((1-p_s)^{-1/\alpha} + (1-p_g)^{-1/\alpha} - 1\big)^{-\alpha}. \tag{5.11}$$

Proof. In the reference model the default intensities of both the groups and the guarantors are set to the Bernoulli parameters and the loss can be expressed with default indicators as

$$L = \mathbb{1}_{\{N_g \geq 1\}}\big(1 + \mathbb{1}_{\{N_s \geq 1\}}\big).$$

Taking the expectation yields

$$\mathbb{E}[L] = \mathbb{P}(N_g \geq 1) + \mathbb{P}(N_g \geq 1, N_s \geq 1)$$

Conditioning on R and using the conditional independence of N_g and N_s

$$= \mathbb{P}(N_g \geq 1) + \mathbb{E}\big[\mathbb{P}(N_g \geq 1 \mid R)\mathbb{P}(N_s \geq 1 \mid R)\big]$$

Using the conditional distribution of N_g and N_s

$$\begin{aligned} &= 1 - \mathbb{E}\Big[\mathrm{e}^{-\rho_g R}\Big] + \mathbb{E}\Big[\Big(1 - \mathrm{e}^{-\rho_g R}\Big)\Big(1 - \mathrm{e}^{-\rho_s R}\Big)\Big] \\ &= 2 \cdot \Big(1 - \mathbb{E}\Big[\mathrm{e}^{-\rho_g R}\Big]\Big) - \mathbb{E}\Big[\mathrm{e}^{-\rho_s R}\Big] + \mathbb{E}\Big[\mathrm{e}^{-(\rho_g + \rho_s) R}\Big] \end{aligned}$$

Substituting the definition of p_g and p_s, and applying Lemma 3.6

$$= 2p_g - (1 - p_s) + \left(1 + \frac{\rho_g + \rho_s}{\beta}\right)^{-\alpha}.$$

Using the explicit representation of ρ_g and ρ_s from Remark 3.41 allows to eliminate the variable β and yields the stated result. ■

Applying these lemmata to the parameters of this portfolio yields

$$\mathbb{E}[L^s] = 0.0449359 \qquad \text{and} \qquad \mathbb{E}[L^r] = 0.0449128,$$

which are exactly the values the implementation provides.

Since the mixture approach and group approach update the risk factors within their groups, the expected loss of these models can be written down, but it is far less succinct than the two presented. At the same time the two Equations (5.9) and (5.11) already demonstrate that each approach will return a different expectation, as each approach conditions the default numbers differently.

The expected values of the remaining two approaches are

$$\mathbb{E}[L^m] = 0.0449134 \qquad \text{and} \qquad \mathbb{E}[L^g] = 0.0449398,$$

Numerical results

The next pages contain the raw results of each distribution.

x	$\mathbb{P}[L^m = x]$	$\mathbb{P}[L^g = x]$	$\mathbb{P}[L^s = x]$	$\mathbb{P}[L^r = x]$
0	0.96098	0.95643	0.96098	0.96
1	0.034253	0.042238	0.033405	0.035087
2	0.004569	0.001295	0.005459	0.004913
3	1.832E-04	3.583E-05	1.554E-05	0
4	1.417E-05	9.653E-07	1.367E-04	0
5	6.009E-07	2.570E-08	4.049E-09	0
6	3.608E-08	6.745E-10	3.113E-06	0
7	1.564E-09	1.741E-11	8.073E-13	0
8	8.169E-11	4.424E-13	6.087E-08	0
9	3.552E-12	1.108E-14	1.377E-16	0
10	1.706E-13	2.743E-16	1.071E-09	0
11	7.369E-15	6.721E-18	2.120E-20	0
12	3.354E-16	1.632E-19	1.746E-11	0
13	1.434E-17	3.934E-21	3.035E-24	0
14	6.295E-19	9.418E-23	2.683E-13	0
15	2.661E-20	2.240E-24	4.115E-28	0
16	1.139E-21	5.299E-26	3.935E-15	0
17	4.758E-23	1.247E-27	5.349E-32	0
18	1.998E-24	2.921E-29	5.559E-17	0
19	8.261E-26	6.811E-31	6.723E-36	0
20	3.418E-27	1.582E-32	7.611E-19	0
21	1.400E-28	3.661E-34	8.222E-40	0
22	5.724E-30	8.443E-36	1.015E-20	0
23	2.324E-31	1.941E-37	9.828E-44	0
24	9.414E-33	4.448E-39	1.324E-22	0
25	3.793E-34	1.017E-40	1.152E-47	0
26	1.524E-35	2.317E-42	1.694E-24	0
27	6.099E-37	5.269E-44	0	0
28	2.434E-38	1.195E-45	2.130E-26	0
29	9.681E-40	2.706E-47	0	0
30	3.841E-41	0	2.639E-28	0
31	1.519E-42	0	0	0
32	5.997E-44	0	3.228E-30	0
33	2.361E-45	0	0	0
34	9.275E-47	0	3.901E-32	0
35	3.635E-48	0	0	0
36	1.422E-49	0	4.665E-34	0
37	0	0	0	0
38	0	0	5.526E-36	0
39	0	0	0	0
40	0	0	6.489E-38	0
41	0	0	0	0
42	0	0	7.560E-40	0
43	0	0	0	0
44	0	0	8.744E-42	0
45	0	0	0	0
46	0	0	1.005E-43	0
47	0	0	0	0
48	0	0	1.147E-45	0
49	0	0	0	0

q	VaR_q^m	VaR_q^g	VaR_q^s	VaR_q^r	ES_q^m	ES_q^g	ES_q^s	ES_q^r
0.950	0	0	0	0	0.88	0.898796	0.898717	0.898257
0.951	0	0	0	0	0.897959	0.917138	0.917058	0.916588
0.952	0	0	0	0	0.916667	0.936245	0.936164	0.935684
0.953	0	0	0	0	0.93617	0.956165	0.956082	0.955592
0.954	0	0	0	0	0.956522	0.976952	0.976867	0.976366
0.955	0	0	0	0	0.977778	0.998662	0.998575	0.998063
0.956	0	0	0	0	1.0	1.021359	1.02127	1.020746
0.957	0	1	0	0	1.023256	1.031855	1.04502	1.044484
0.958	0	1	0	0	1.047619	1.032614	1.069902	1.069353
0.959	0	1	0	0	1.073171	1.033409	1.095997	1.095435
0.960	0	1	0	0	1.1	1.034244	1.123397	1.122821
0.961	1	1	1	1	1.127701	1.035122	1.151698	1.12597
0.962	1	1	1	1	1.131062	1.036047	1.15569	1.129285
0.963	1	1	1	1	1.134604	1.037021	1.159898	1.132779
0.964	1	1	1	1	1.138343	1.038049	1.164339	1.136468
0.965	1	1	1	1	1.142296	1.039136	1.169035	1.140367
0.966	1	1	1	1	1.146481	1.040287	1.174006	1.144495
0.967	1	1	1	1	1.15092	1.041508	1.179279	1.148874
0.968	1	1	1	1	1.155636	1.042805	1.184882	1.153526
0.969	1	1	1	1	1.160656	1.044186	1.190845	1.158478
0.970	1	1	1	1	1.166011	1.045659	1.197207	1.163761
0.971	1	1	1	1	1.171736	1.047233	1.204007	1.169408
0.972	1	1	1	1	1.177869	1.04892	1.211293	1.175458
0.973	1	1	1	1	1.184457	1.050732	1.219119	1.181957
0.974	1	1	1	1	1.191552	1.052683	1.227547	1.188955
0.975	1	1	1	1	1.199214	1.054791	1.236648	1.196513
0.976	1	1	1	1	1.207514	1.057074	1.246509	1.204701
0.977	1	1	1	1	1.216537	1.059555	1.257227	1.213601
0.978	1	1	1	1	1.226379	1.062262	1.268919	1.223311
0.979	1	1	1	1	1.237159	1.065227	1.281724	1.233944
0.980	1	1	1	1	1.249017	1.068488	1.29581	1.245642
0.981	1	1	1	1	1.262123	1.072093	1.311379	1.25857
0.982	1	1	1	1	1.276686	1.076098	1.328678	1.272935
0.983	1	1	1	1	1.292961	1.080575	1.348012	1.28899
0.984	1	1	1	1	1.311272	1.08561	1.369763	1.307052
0.985	1	1	1	1	1.332023	1.091318	1.394414	1.327522
0.986	1	1	1	1	1.355739	1.097841	1.422586	1.350917
0.987	1	1	1	1	1.383103	1.105367	1.455093	1.37791
0.988	1	1	1	1	1.415029	1.114147	1.493017	1.409403
0.989	1	1	1	1	1.452759	1.124524	1.537837	1.446621
0.990	1	1	1	1	1.498034	1.136977	1.591621	1.491283
0.991	1	1	1	1	1.553372	1.152196	1.657357	1.54587
0.992	1	1	1	1	1.622543	1.171221	1.739526	1.614104
0.993	1	1	1	1	1.711478	1.195681	1.845173	1.701833
0.994	1	1	1	1	1.830057	1.228295	1.986035	1.818805
0.995	1	1	2	1	1.996069	1.273954	2.060369	1.982567
0.996	2	1	2	2	2.053363	1.342442	2.075461	2.0
0.997	2	1	2	2	2.071151	1.456589	2.100615	2.0
0.998	2	1	2	2	2.106727	1.684884	2.150922	2.0
0.999	2	2	2	2	2.213454	2.037841	2.301844	2.0

Table 5.13.: Some probabilistic values of the stochastic minimal portfolio

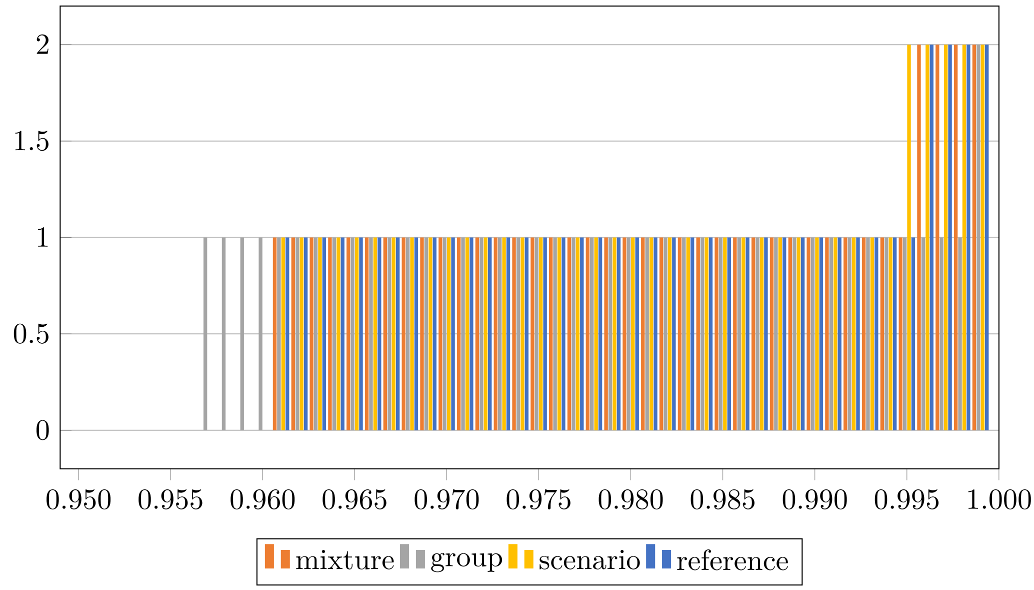


Figure 5.14.: Value-at-Risk of the stochastic minimal portfolio

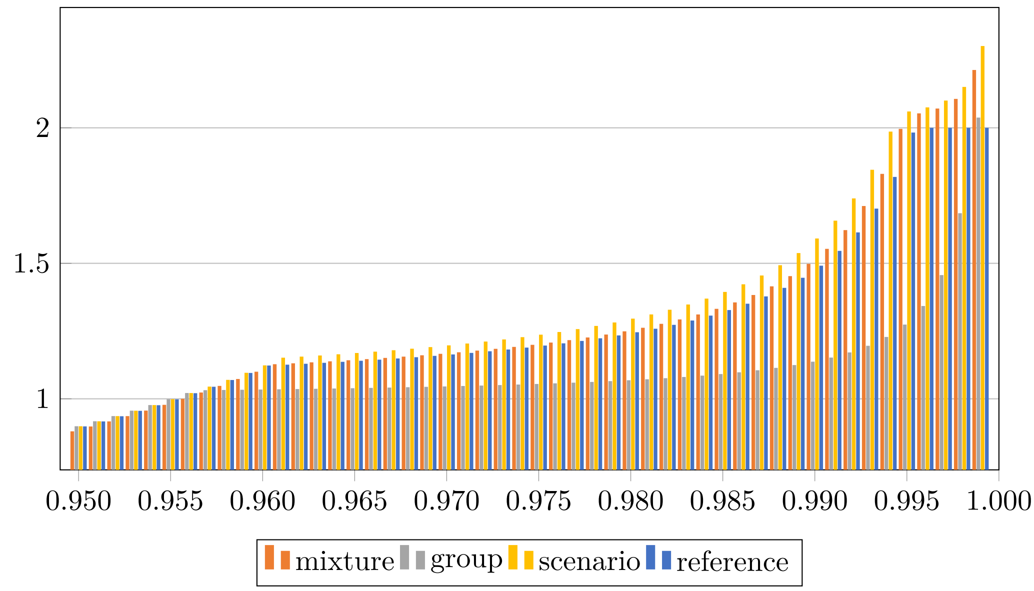


Figure 5.15.: Expected shortfall of the stochastic minimal portfolio

mixture	group	scenario
0.001940	0.007188	0.001837

Table 5.14.: Wasserstein distances of the stochastic minimal portfolio

Similar to the deterministic minimal portfolio, the group approach struggles with such a simplistic portfolio. It models the single guarantee with two Poisson-approximations in series, which only exacerbates the model error due to repeat defaults.

The fact that the mixture approach is nearer to the reference distribution than the scenario approach is again due to the fact that repeat defaults in the scenario approach are forced to repeat their losses, thus increasing the chances of more expensive losses due to repeat defaults.

5.2.8. Stochastic multiple guarantors

We skip a "stochastic multiple groups" portfolio, since the differences to the deterministic portfolio would be too minuscule to warrant a separate paragraph.

Instead we proceed immediately to multiple guarantors in multiple groups.

Structure

Number of guarantors		5	Number of groups			10
$s_{i=1,\dots,5}$	π_s	i%	$g_{i=1,\dots,5}$	p_g		4%
	c_s	R		c_g		R
				h_1	Exposure	Dirac(2)
Number of risk factors		1			Guarantor	s_i
R	$\mathbb{E}[R]$	1		h_2	Exposure	Dirac(1)
	$\mathbb{V}[R]$	v			Guarantor	—
			$g_{6,\dots,10}$	p_g		4%
				c_g		R
				h_1	Exposure	Dirac(3)
					Guarantor	—

Simply replacing the idiosyncratic risk factor with a stochastic one merely varies the resulting distribution, but does not affect to overall shape of the loss distribution.

Therefore we do not reproduce all possible distributions with all their results. Instead we evaluate the group model (as it had the best results in the deterministic model) with various levels for the risk factor's variance.

$\mathbb{P}$	$\mathbb{V}[R] = 1$	$\mathbb{V}[R] = 1/2$	$\mathbb{V}[R] = 1/4$	$\mathbb{V}[R] = 1/8$	$\mathbb{V}[R] = 1/50$	$\mathbb{V}[R] = 0$
0	0.7086527	0.6895073	0.6785313	0.672619	0.667409	0.666389
1	0.1004377	0.1145086	0.1231664	0.128018	0.132407	0.133278
2	0.0195087	0.0189838	0.0183058	0.017807	0.017282	0.017169
3	0.1039501	0.1172638	0.1254177	0.129983	0.134114	0.134935
4	0.0294123	0.0290922	0.0283262	0.027702	0.027015	0.026865
5	0.0077496	0.0060105	0.0048429	0.004177	0.003577	0.003458
6	0.0160735	0.0153310	0.0146542	0.014205	0.013757	0.013663
7	0.0065504	0.0049578	0.0039191	0.003334	0.002811	0.002708
8	0.0021699	0.0012211	7.560E-04	5.412E-04	3.773E-04	3.483E-04
9	0.0026386	0.0018328	0.0013871	0.001156	9.600E-04	9.224E-04
10	0.0013260	7.107E-04	4.233E-04	2.947E-04	1.988E-04	1.820E-04
11	5.167E-04	2.018E-04	9.065E-05	5.117E-05	2.705E-05	2.339E-05
12	4.584E-04	2.116E-04	1.164E-04	7.813E-05	5.126E-05	4.671E-05
13	2.580E-04	9.270E-05	3.936E-05	2.135E-05	1.074E-05	9.173E-06
14	1.138E-04	2.956E-05	9.232E-06	3.942E-06	1.481E-06	1.178E-06
15	8.330E-05	2.410E-05	9.055E-06	4.635E-06	2.233E-06	1.892E-06
16	4.944E-05	1.142E-05	3.310E-06	1.343E-06	4.732E-07	3.699E-07
17	2.388E-05	4.000E-06	8.405E-07	2.624E-07	6.610E-08	4.750E-08
18	1.562E-05	2.735E-06	6.689E-07	2.497E-07	8.261E-08	6.390E-08
19	9.440E-06	1.357E-06	2.590E-07	7.601E-08	1.769E-08	1.243E-08
20	4.863E-06	5.126E-07	7.048E-08	1.563E-08	2.503E-09	1.595E-09
21	2.985E-06	3.100E-07	4.764E-08	1.249E-08	2.670E-09	1.850E-09
22	1.805E-06	1.574E-07	1.922E-08	3.958E-09	5.773E-10	3.583E-10
23	9.719E-07	6.315E-08	5.553E-09	8.529E-10	8.265E-11	4.594E-11
24	5.767E-07	3.510E-08	3.301E-09	5.890E-10	7.691E-11	4.685E-11
25	3.463E-07	1.795E-08	1.369E-09	1.928E-10	1.677E-11	9.034E-12
26	1.920E-07	7.553E-09	4.164E-10	4.337E-11	2.430E-12	1.158E-12
27	1.120E-07	3.967E-09	2.239E-10	2.648E-11	2.006E-12	1.055E-12
28	6.667E-08	2.026E-09	9.439E-11	8.889E-12	4.409E-13	2.025E-13
29	3.767E-08	8.837E-10	3.002E-11	2.079E-12	6.461E-14	2.593E-14
30	2.181E-08	4.469E-10	1.492E-11	1.144E-12	4.792E-14	2.139E-14
31	1.288E-08	2.267E-10	6.344E-12	3.915E-13	1.061E-14	4.085E-15
32	7.358E-09	1.017E-10	2.096E-12	9.490E-14	1.572E-15	5.230E-16
33	4.248E-09	5.016E-11	9.798E-13	4.780E-14	1.059E-15	3.941E-16
34	2.493E-09	2.522E-11	4.175E-13	1.659E-14	2.360E-16	7.494E-17
35	1.434E-09	1.154E-11	1.424E-13	4.152E-15	3.535E-17	9.588E-18
36	8.275E-10	5.606E-12	6.352E-14	1.940E-15	2.184E-17	6.659E-18
37	4.833E-10	2.792E-12	2.701E-14	6.798E-16	4.893E-18	1.260E-18
38	2.792E-10	1.296E-12	9.464E-15	1.752E-16	7.406E-19	1.612E-19
39	1.612E-10	6.237E-13	4.070E-15	7.681E-17	4.228E-19	1.039E-19
40	9.383E-11	3.080E-13	1.722E-15	2.707E-17	9.518E-20	1.957E-20
41	5.430E-11	1.444E-13	6.170E-16	7.160E-18	1.455E-20	2.501E-21
42	3.137E-11	6.909E-14	2.581E-16	2.974E-18	7.729E-21	1.505E-21
43	1.823E-11	3.385E-14	1.085E-16	1.051E-18	1.747E-21	2.821E-22
44	1.056E-11	1.598E-14	3.957E-17	2.846E-19	2.698E-22	3.604E-23
45	6.106E-12	7.622E-15	1.621E-17	1.129E-19	1.340E-22	2.034E-23
46	3.543E-12	3.710E-15	6.758E-18	3.991E-20	3.041E-23	3.797E-24
47	2.053E-12	1.759E-15	2.503E-18	1.104E-20	4.740E-24	4.848E-25
48	1.188E-12	8.375E-16	1.009E-18	4.213E-21	2.215E-24	2.579E-25
49	6.889E-13	4.054E-16	4.172E-19	1.486E-21	5.039E-25	4.791E-26

ES_q	$\mathbb{V}[R] = 1$	$\mathbb{V}[R] = 1/2$	$\mathbb{V}[R] = 1/4$	$\mathbb{V}[R] = 1/8$	$\mathbb{V}[R] = 1/50$	$\mathbb{V}[R] = 0$
0.950	6.04938	5.492506	5.211723	5.071943	4.931126	4.897844
0.951	6.091204	5.522965	5.236452	5.09382	4.970537	4.936575
0.952	6.134771	5.554693	5.262212	5.116607	4.995157	4.972147
0.953	6.180191	5.587772	5.289067	5.140365	5.01633	4.992831
0.954	6.227587	5.622289	5.31709	5.165156	5.038424	5.014414
0.955	6.277089	5.65834	5.346359	5.191048	5.0615	5.036957
0.956	6.328841	5.696029	5.376958	5.218117	5.085625	5.060524
0.957	6.383	5.735472	5.40898	5.246446	5.110873	5.085187
0.958	6.439738	5.776792	5.442527	5.276123	5.137322	5.111025
0.959	6.499244	5.820129	5.477711	5.307248	5.165061	5.138123
0.960	6.561725	5.865632	5.514654	5.339929	5.194188	5.166577
0.961	6.62741	5.913469	5.553491	5.374286	5.224808	5.196489
0.962	6.695543	5.963823	5.594372	5.410452	5.25704	5.227975
0.963	6.741368	6.0169	5.637464	5.448572	5.291014	5.261164
0.964	6.78974	6.072925	5.682949	5.48881	5.326876	5.296196
0.965	6.840875	6.132151	5.731033	5.531347	5.364786	5.33323
0.966	6.895018	6.194861	5.781946	5.576387	5.404927	5.372443
0.967	6.952443	6.261372	5.835944	5.624156	5.447501	5.414032
0.968	7.013457	6.33204	5.893317	5.674911	5.492735	5.458221
0.969	7.078407	6.407267	5.954392	5.728941	5.540888	5.50526
0.970	7.138061	6.466034	6.019538	5.786572	5.592251	5.555435
0.971	7.177304	6.516587	6.089178	5.848178	5.647156	5.609071
0.972	7.219351	6.57075	6.163791	5.914184	5.705983	5.666538
0.973	7.264512	6.628926	6.243932	5.98508	5.769167	5.728262
0.974	7.313147	6.691577	6.320525	6.061429	5.837212	5.794733
0.975	7.365673	6.75924	6.373347	6.143886	5.910701	5.866522
0.976	7.422576	6.806134	6.430569	6.233215	5.990313	5.944294
0.977	7.484427	6.841184	6.492768	6.292464	6.076849	6.028829
0.978	7.551901	6.879419	6.560621	6.351212	6.171251	6.121048
0.979	7.625801	6.921297	6.615433	6.415556	6.237806	6.204698
0.980	7.707091	6.967361	6.646204	6.486334	6.299697	6.264933
0.981	7.796938	7.018275	6.680215	6.528073	6.368102	6.331508
0.982	7.896768	7.074846	6.718005	6.55741	6.433172	6.405481
0.983	8.008342	7.138072	6.760241	6.590199	6.458653	6.434859
0.984	8.133864	7.209202	6.807756	6.627086	6.487319	6.462038
0.985	8.276121	7.289815	6.861606	6.668892	6.519807	6.49284
0.986	8.42332	7.381945	6.923149	6.71667	6.556936	6.528043
0.987	8.532806	7.488248	6.994161	6.771798	6.599777	6.568662
0.988	8.660539	7.612269	7.077007	6.836115	6.649759	6.61605
0.989	8.811498	7.758839	7.174917	6.912125	6.708828	6.672055
0.990	8.992647	7.934723	7.292409	7.003338	6.77971	6.73926
0.991	9.214053	8.11605	7.43601	7.11482	6.866345	6.8214
0.992	9.490809	8.255556	7.615511	7.254172	6.974638	6.924075
0.993	9.751648	8.434922	7.846298	7.43334	7.113872	7.056086
0.994	10.04359	8.674075	8.02812	7.67223	7.299517	7.232101
0.995	10.3513	9.00889	8.233743	7.909083	7.559421	7.478521
0.996	10.689125	9.424873	8.542179	8.136354	7.839223	7.787314
0.997	11.252167	9.858553	9.056239	8.515138	8.118964	8.049752
0.998	11.94501	10.287829	9.626024	9.195891	8.678445	8.574629
0.999	13.32556	11.284637	10.252049	9.779339	9.47522	9.426536

Table 5.15.: Probabilities and expected shortfall for various variances

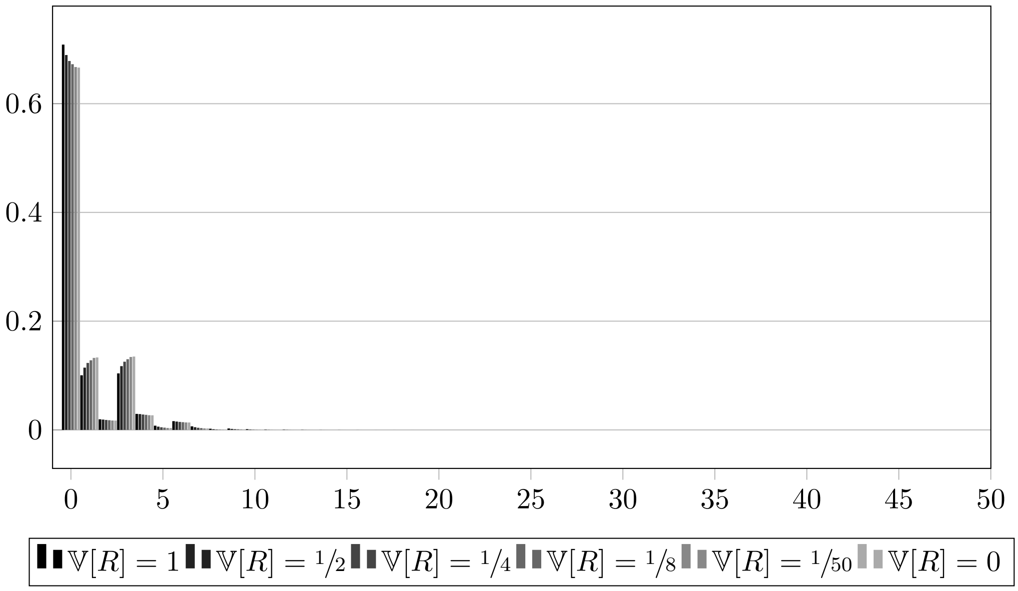


Figure 5.16.: Probability of default for various variances

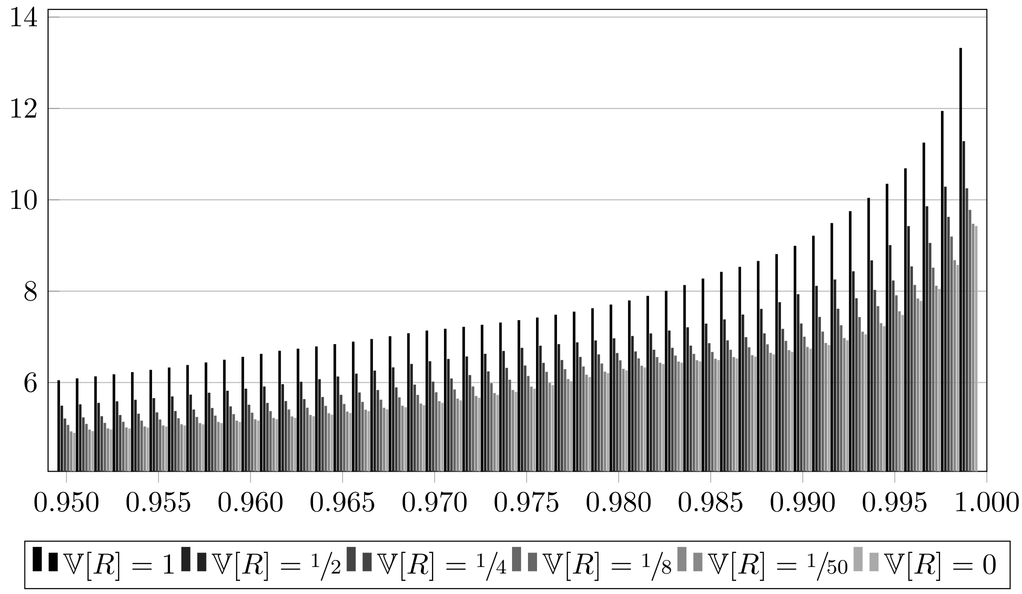


Figure 5.17.: Expected shortfall for various variances

$\mathbb{V}[R] = 1$	$\mathbb{V}[R] = 1/2$	$\mathbb{V}[R] = 1/4$	$\mathbb{V}[R] = 1/8$	$\mathbb{V}[R] = 1/50$	$\mathbb{V}[R] = 0$
1.07799	1.08517	1.08550	1.08441	1.08265	1.08221

Table 5.16.: The entropy of the loss distribution for various variances

The main pattern is clearly visible: the higher the variance, the higher the probability of repeat defaults and thus a fatter tail of the loss distribution, as evidenced by a markedly higher expected shortfall. To compensate this, the probability of no loss increases alongside the variance. Further the entropy[2] of the distribution decreases with increasing variance. This means that as the risk factor becomes more unpredictable, the guarantors and groups, which depend on it, have a higher chance of clustering.

5.2.9. Stochastic high dependence

Structure

Number of guarantors		7	Number of groups			301
$s_{i=0,\dots,6}$	π_s	10%	$g_{k=1,\dots,301}$	p_g		3%
	c_s	idiosync.		c_g		idiosync.
				$h_{i=1,\dots,7}$	Exposure	Dirac(i)
Number of risk factors		1			Guarantor	$s_{(i+k) \bmod 7}$
R	$\mathbb{E}[R]$	1		h_8	Exposure	Dirac(1)
	$\mathbb{V}[R]$	$1/4$			Guarantor	—

This portfolio is the stochastic counterpart to Section 5.2.4.

Numerical results

As discussed above, the expected value depends on the method of the computation:

reference	mixture	scenario	groups
39.906931	40.122733	40.254331	40.229603

The next pages contain the raw results of each distribution.

[2] See Definition A.3

x	$\mathbb{P}[L^m = x]$	$\mathbb{P}[L^g = x]$	$\mathbb{P}[L^s = x]$	$\mathbb{P}[L^r = x]$
0	0.008881	0.0072079	0.008881	0.008427
1	0.0104855	0.0189669	0.018628	0.017941
2	0.0089962	0.0311962	0.031262	0.030422
3	0.007684	0.0410536	0.04075	0.040041
4	0.0074179	0.0472814	0.046875	0.046443
5	0.0078888	0.0498005	0.049317	0.049222
6	0.0089189	0.0491963	0.048795	0.048983
7	0.0102599	0.0463153	0.046084	0.046482
8	0.0119331	0.0420075	0.042109	0.042601
9	0.0125723	0.0369958	0.036742	0.037291
10	0.0127869	0.031829	0.032107	0.032593
11	0.0130027	0.0268836	0.027267	0.027678
12	0.013455	0.0223893	0.023114	0.023407
13	0.013995	0.0184623	0.01938	0.019576
14	0.014599	0.015138	0.016313	0.016399
15	0.015075	0.0123983	0.013688	0.013703
16	0.015446	0.0101939	0.011655	0.011593
17	0.01571	0.0084599	0.009909	0.009814
18	0.015924	0.0071266	0.008848	0.00871
19	0.016143	0.0061266	0.007773	0.007626
20	0.016346	0.0053979	0.00716	0.007
21	0.016517	0.0048863	0.006585	0.006433
22	0.01665	0.0045453	0.006296	0.006146
23	0.016735	0.0043364	0.005994	0.005862
24	0.016776	0.004228	0.005906	0.005782
25	0.016785	0.0041949	0.005728	0.005626
26	0.016768	0.004217	0.005812	0.005718
27	0.016727	0.0042783	0.005717	0.005642
28	0.016659	0.0043667	0.005813	0.005747
29	0.016564	0.0044725	0.005747	0.005699
30	0.016444	0.0045884	0.005847	0.005808
31	0.016301	0.0047088	0.005803	0.005779
32	0.016137	0.0048296	0.005907	0.00589
33	0.015955	0.0049475	0.005859	0.005856
34	0.015755	0.0050603	0.005965	0.005967
35	0.01554	0.0051663	0.005905	0.005919
36	0.01531	0.0052643	0.006014	0.006031
37	0.015066	0.0053534	0.005932	0.00596
38	0.014811	0.0054331	0.006011	0.006041
39	0.014546	0.0055031	0.005919	0.005959
40	0.014271	0.0055634	0.005981	0.006022
41	0.013989	0.0056138	0.005878	0.005928
42	0.013699	0.0056547	0.005923	0.005974
43	0.013404	0.0056862	0.005799	0.005857
44	0.013105	0.0057085	0.005841	0.005899
45	0.012801	0.005722	0.005715	0.005781
46	0.012495	0.0057272	0.005743	0.005807
47	0.012187	0.0057243	0.005609	0.005679
48	0.011878	0.0057139	0.005622	0.00569
49	0.011568	0.0056963	0.005488	0.005561

q	VaR_q^m	VaR_q^g	VaR_q^s	VaR_q^r	ES_q^m	ES_q^g	ES_q^s	ES_q^r
0.950	91	137	147	145	111.090519	178.978279	216.022621	209.697568
0.951	92	138	148	147	111.484316	179.83107	217.422115	210.999769
0.952	92	139	149	148	111.890239	180.70078	218.854036	212.327187
0.953	93	140	151	149	112.303843	181.587989	220.315622	213.684667
0.954	93	140	152	151	112.723492	182.492076	221.81176	215.072853
0.955	94	141	154	152	113.157136	183.415413	223.343796	216.489446
0.956	94	142	155	153	113.592526	184.358846	224.910709	217.94112
0.957	95	143	157	155	114.04662	185.323261	226.518284	219.424683
0.958	95	144	158	156	114.500111	186.309641	228.163788	220.945404
0.959	96	145	160	158	114.975286	187.319074	229.853252	222.502022
0.960	96	147	161	159	115.449668	188.351765	231.586848	224.09979
0.961	97	148	163	161	115.946832	189.408757	233.366178	225.735822
0.962	97	149	165	163	116.445433	190.492209	235.196291	227.417173
0.963	98	150	166	164	116.965856	191.603745	237.078759	229.143905
0.964	98	151	168	166	117.492686	192.745182	239.014229	230.916959
0.965	99	152	170	168	118.038105	193.918553	241.008668	232.741697
0.966	99	154	172	170	118.598049	195.124335	243.065282	234.620633
0.967	100	155	174	172	119.17082	196.362043	245.18769	236.55662
0.968	101	156	176	174	119.767805	197.638499	247.379995	238.552883
0.969	101	158	178	176	120.373218	198.955744	249.646892	240.613096
0.970	102	159	180	178	121.001926	200.310239	251.993792	242.74148
0.971	103	160	183	180	121.656215	201.713308	254.425066	244.942943
0.972	103	162	185	182	122.322509	203.160042	256.946591	247.223215
0.973	104	163	187	185	123.015525	204.661769	259.568745	249.587698
0.974	105	165	190	187	123.7369	206.213333	262.294614	252.040101
0.975	105	167	193	190	124.486376	207.826935	265.137785	254.592761
0.976	106	168	195	192	125.257857	209.505037	268.103595	257.247417
0.977	107	170	198	195	126.062909	211.248655	271.204764	260.017101
0.978	108	172	201	198	126.903508	213.068379	274.454293	262.910801
0.979	109	174	204	201	127.782125	214.970473	277.866746	265.939055
0.980	110	176	208	204	128.701842	216.962651	281.456481	269.115012
0.981	111	178	211	207	129.66651	219.054458	285.244696	272.45497
0.982	112	181	215	211	130.680956	221.255586	289.252272	275.97294
0.983	113	183	219	215	131.751267	223.57582	293.505981	279.692323
0.984	114	186	223	219	132.885179	226.03732	298.036313	283.633657
0.985	115	188	228	223	134.092619	228.647603	302.88066	287.824525
0.986	117	191	233	227	135.370648	231.434664	308.08329	292.301556
0.987	118	194	238	232	136.745208	234.422917	313.696482	297.099702
0.988	120	197	243	238	138.225667	237.645581	319.792557	302.269478
0.989	121	201	250	244	139.833149	241.135894	326.453221	307.873557
0.990	123	205	257	250	141.581922	244.9525	333.792858	313.984178
0.991	125	209	264	257	143.512156	249.161428	341.955216	320.704007
0.992	127	214	273	265	145.666196	253.851424	351.14008	328.161976
0.993	130	220	283	274	148.08922	259.155785	361.629606	336.533825
0.994	133	226	294	285	150.880996	265.252801	373.833094	346.072415
0.995	136	233	308	297	154.1661	272.438098	388.393465	357.143105
0.996	140	242	325	312	158.165653	281.193543	406.394146	370.331987
0.997	146	254	348	331	163.289053	292.424178	429.88161	386.637418
0.998	153	270	380	358	170.440229	308.141212	463.50301	408.010338
0.999	165	297	436	400	182.528559	334.747092	522.336013	439.253144

Table 5.17.: Some probabilistic values of the stochastic high dependence portfolio

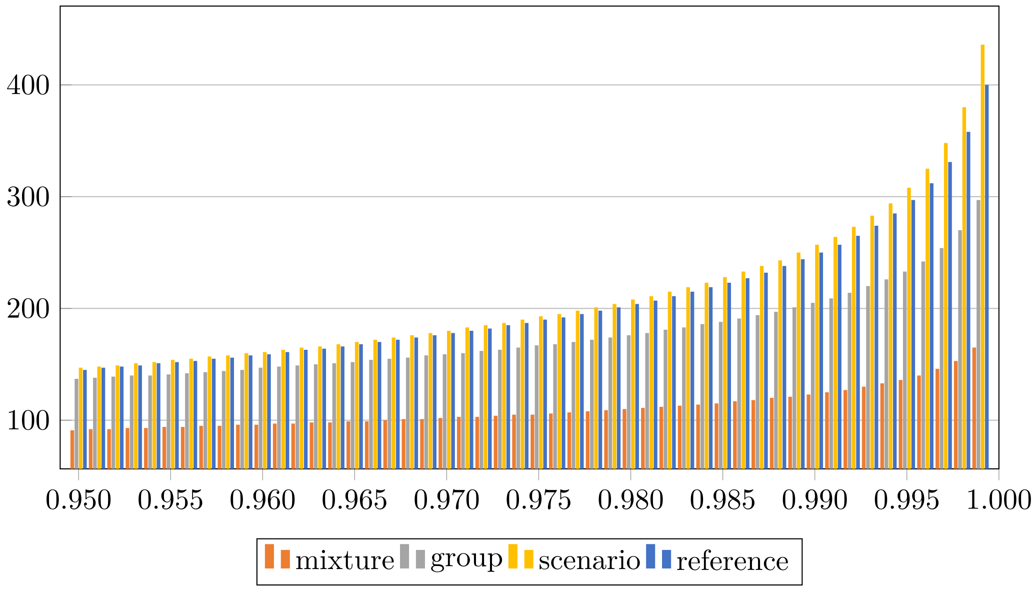


Figure 5.18.: Value-at-Risk of the stochastic high dependence portfolio

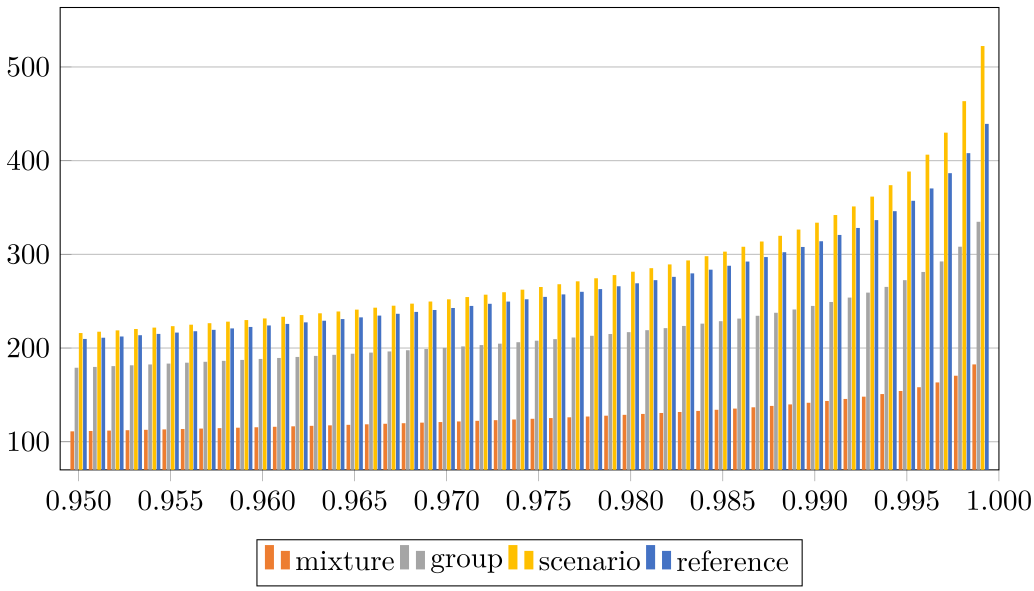


Figure 5.19.: Expected shortfall of the stochastic high dependence portfolio

mixture	group	scenario
17.181829	3.699885	0.430873

Table 5.18.: Wasserstein distances of the stochastic high dependence portfolio

The most notable difference to the idiosyncratic version of this portfolio is the nearly fivefold increase of the Wasserstein distance between the best approach - the scenarios – and the reference distribution. The added risk factor creates a dependence between the guarantors and groups, thus increasing the clustering of the loss.

The two charts above illustrate again the tendency of the scenario approach to overestimate high quantiles because of the high-valued repeat defaults. The group approach's quicker decline of accuracy highlights again the pitfalls of removing the guarantees from their respective groups, whereas the mixture approach is still completely unable to capture the structure of the loss.

In order to compare with the idiosyncratic case we also include a chart of the probability mass functions of each loss distribution:

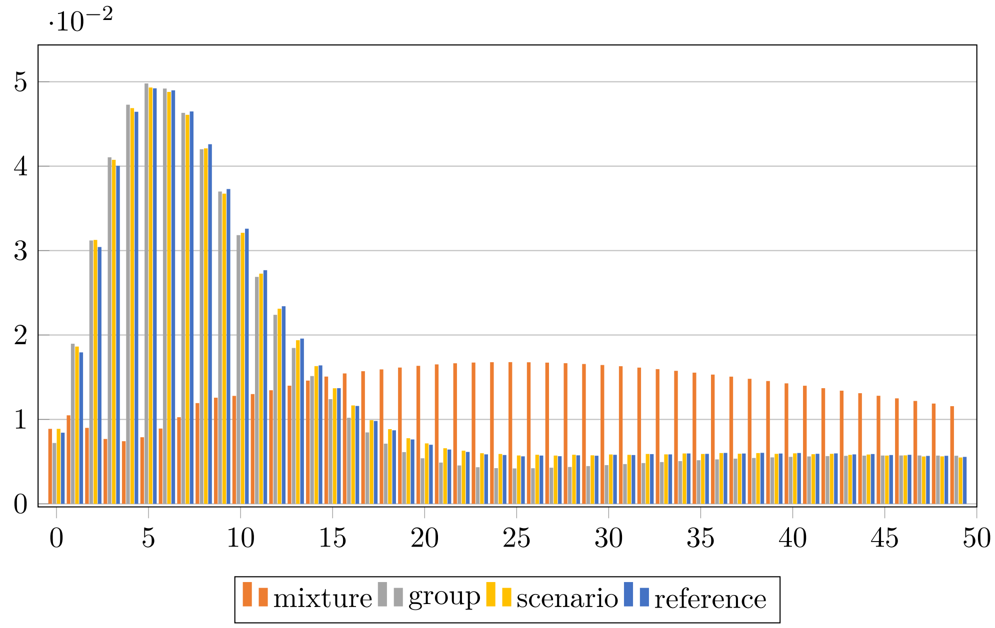


Figure 5.20.: Probabilities of the stochastic high dependence portfolio

5.2.10. Stochastic complex portfolio

Structure

For our final portfolio we expand the previous “complex portfolio” by adding multiple risk factors.

Number of guarantors		3	Number of groups			5
s_1	π_s	2%	$g_{i=1,\dots,5}$	p_g		i%
	c_s	6		c_g		i
s_2	π_s	3%		$h_{k=1,\dots,15}$	Exposure	Dirac(k)
	c_s	7			Guarantor	$S(i,k)$
s_3	π_s	5%				
	c_s	8				

Number of risk factors		3
R_1	Expected value	1
	Variance	1/2
R_2	Expected value	1
	Variance	1/3
R_3	Expected value	1
	Variance	1/4

Scenario-Matrix A

c	R_1	R_2	R_3
1	1/2	1/2	0
2	1/2	0	1/2
3	0	1/2	1/2
4	1/3	1/3	1/3
5	1/2	1/3	1/6
6	1	0	0
7	0	1	0
8	0	0	1

The function $S(i,k)$ is defined as in Equation (5.6).

Numerical results

As described before, the computed expected value depends on the choice of method:

reference	mixture	scenario	groups
4.79776	4.79777	4.79838	4.79839

The next pages contain the raw results of each distribution.

x	$\mathbb{P}[L^m = x]$	$\mathbb{P}[L^g = x]$	$\mathbb{P}[L^s = x]$	$\mathbb{P}[L^r = x]$
0	0.8610686	0.8477453	0.861069	0.859139
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	0	0	0	0
5	0	0.0034707	0	0
6	0	0	0	0
7	0	0	0	0
8	0	0	0	0
9	0	0	0	0
10	0	0.002604	0	0
11	0	0	0	0
12	0	0.0021982	0	0
13	0	0	0	0
14	0	0	0	0
15	0	3.515E-04	0	0
16	0	0	0	0
17	0	9.084E-06	0	0
18	0	0.0014124	0	0
19	0	0	0	0
20	0	1.039E-04	0	0
21	0	0	0	0
22	0	6.876E-06	0	0
23	0	5.911E-06	0	0
24	0	1.207E-04	0	0
25	0	1.017E-05	0	0
26	0	0	0	0
27	0	1.154E-05	0	0
28	0.045943	0.0521034	0.04601	0.047016
29	0	4.980E-07	0	0
30	0	1.552E-04	0	0
31	0	0	0	0
32	0.068809	0.0768142	0.068909	0.071094
33	0	2.254E-04	0	0
34	0	3.763E-07	0	0
35	0	9.472E-07	0	0
36	0	5.588E-05	0	0
37	0.003859	3.178E-04	0.003826	0.00395
38	0.002656	1.672E-04	0.002635	0.002695
39	0	6.527E-07	0	0
40	0	1.384E-04	0	0
41	0	2.372E-07	0	0
42	0	2.495E-04	0	0
43	0	2.236E-05	0	0
44	0.002418	2.011E-04	0.00239	0.002467
45	0	1.655E-06	0	0
46	0.001526	9.120E-05	0.001506	0.001541
47	0	3.238E-05	0	0
48	0	1.280E-05	0	0
49	0	8.485E-07	0	0

q	VaR_q^m	VaR_q^g	VaR_q^s	VaR_q^r	ES_q^m	ES_q^g	ES_q^s	ES_q^r
0.950	32	32	32	32	42.714838	39.165621	42.732399	41.565218
0.951	32	32	32	32	42.933508	39.311858	42.951427	41.760427
0.952	32	32	32	32	43.161289	39.464189	43.179582	41.963769
0.953	32	32	32	32	43.398764	39.623001	43.417446	42.175764
0.954	32	32	32	32	43.646563	39.788719	43.665651	42.396976
0.955	32	32	32	32	43.905375	39.961801	43.924888	42.62802
0.956	32	32	32	32	44.175952	40.142751	44.195908	42.869566
0.957	32	32	32	32	44.459114	40.332118	44.479534	43.122347
0.958	32	32	32	32	44.755759	40.530501	44.776665	43.387164
0.959	32	32	32	32	45.066875	40.738562	45.088291	43.6649
0.960	32	32	32	32	45.393547	40.957026	45.415499	43.956523
0.961	32	32	32	32	45.736971	41.186694	45.759486	44.2631
0.962	32	32	32	32	46.098471	41.428449	46.121578	44.585813
0.963	32	32	32	32	46.47951	41.683272	46.503242	44.92597
0.964	32	32	32	32	46.881719	41.952252	46.90611	45.285025
0.965	32	32	32	32	47.306911	42.236602	47.331998	45.664597
0.966	32	32	32	32	47.757114	42.537678	47.78294	46.066497
0.967	32	32	32	32	48.234603	42.857002	48.26121	46.492755
0.968	32	32	32	32	48.741934	43.196283	48.769373	46.945653
0.969	32	32	32	32	49.281996	43.557454	49.310321	47.427771
0.970	32	32	32	32	49.858063	43.942702	49.887332	47.94203
0.971	32	32	32	32	50.473858	44.354519	50.504136	48.491755
0.972	32	32	32	32	51.133639	44.795752	51.164998	49.080747
0.973	32	32	32	32	51.842292	45.269669	51.874813	49.713367
0.974	32	32	32	32	52.605457	45.780041	52.639229	50.39465
0.975	32	32	32	32	53.429675	46.331242	53.464798	51.130436
0.976	37	32	37	32	54.285164	46.928377	54.356655	51.927538
0.977	37	32	37	32	55.036692	47.577437	55.111292	52.793952
0.978	37	32	37	37	55.856542	48.285503	55.934533	53.568451
0.979	37	32	37	37	56.754473	49.061003	56.836177	54.357425
0.980	38	32	38	37	57.726169	49.914053	57.818678	55.225297
0.981	38	32	38	37	58.764388	50.856898	58.861767	56.184523
0.982	38	32	38	38	59.917965	51.904503	60.020754	57.205809
0.983	44	32	44	38	60.972865	53.075356	61.121587	58.335562
0.984	44	32	44	44	62.033669	54.392566	62.191686	59.566675
0.985	46	32	46	44	63.20304	55.885404	63.382978	60.604453
0.986	46	32	46	44	64.431829	57.591504	64.62462	61.790485
0.987	56	32	56	46	65.29544	59.560081	65.553555	63.060638
0.988	60	40	60	56	65.96468	61.530631	66.313295	64.400514
0.989	60	56	60	60	66.506924	62.841108	66.887231	65.054224
0.990	60	56	60	60	67.157616	63.525219	67.575954	65.559646
0.991	60	60	60	60	67.952907	64.187159	68.417727	66.177385
0.992	60	60	60	60	68.94702	64.710553	69.469943	66.949558
0.993	60	60	60	60	70.225166	65.38349	70.822791	67.942352
0.994	60	60	60	60	71.92936	66.280738	72.62659	69.266077
0.995	64	60	64	60	73.602992	67.536885	74.882319	71.119292
0.996	64	64	64	60	76.00374	68.872836	77.602899	73.899115
0.997	64	64	64	64	80.004986	70.497115	82.137198	77.61983
0.998	70	64	64	64	86.554677	73.745672	91.205797	84.429745
0.999	88	64	88	75	96.33958	83.491344	104.891108	99.182248

Table 5.19.: Some probabilistic values of the stochastic complex portfolio

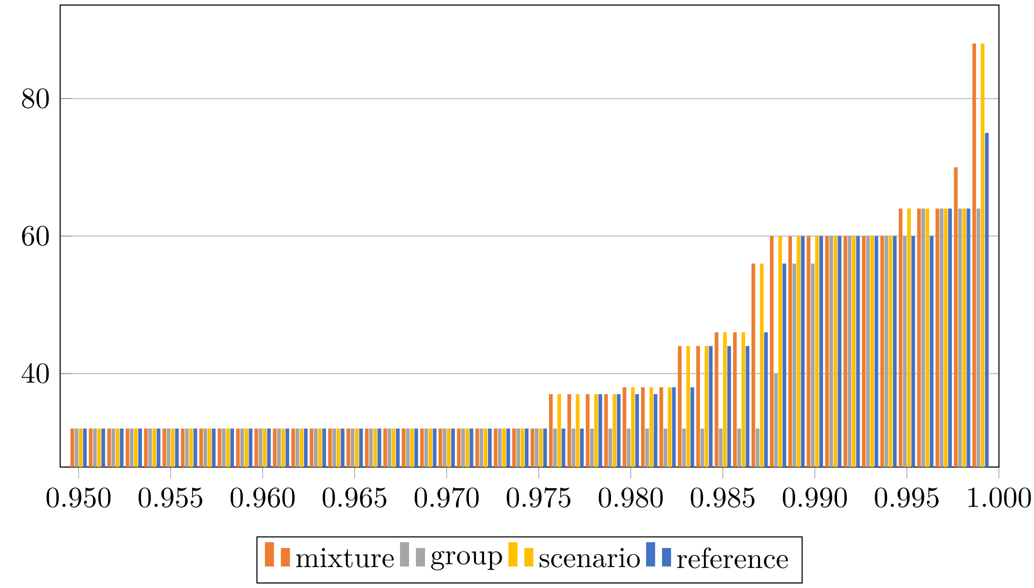


Figure 5.21.: Value-at-Risk of the stochastic complex portfolio

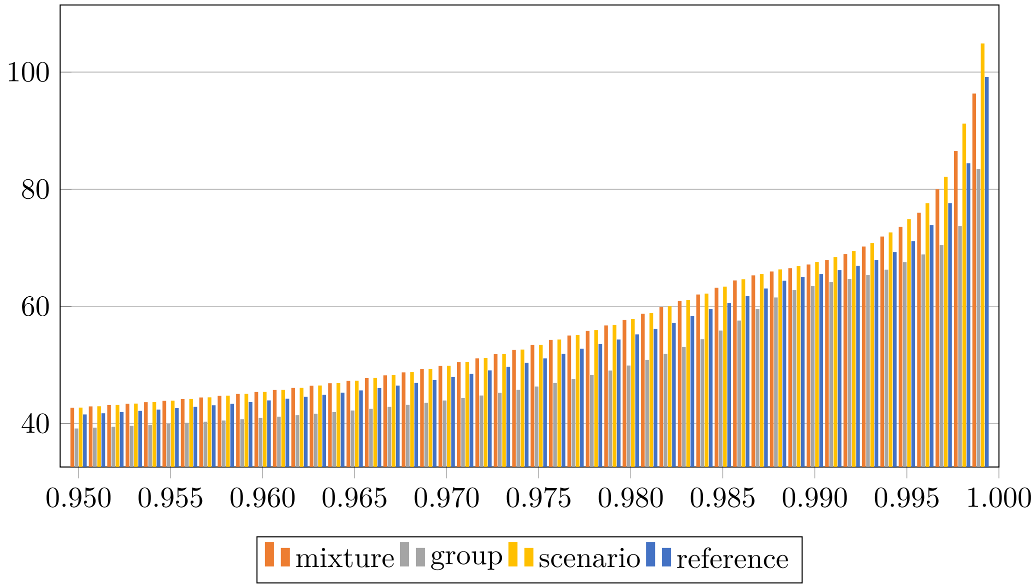


Figure 5.22.: Expected shortfall of the stochastic complex portfolio

mixture	group	scenario
0.124676	0.280103	0.116095

Table 5.20.: Wasserstein distances of the stochastic complex portfolio

5.3. Conclusions

Multiple portfolios have been presented, discussed and compared.

Each portfolio tried to underscore either a specific aspect of one of the approaches introduced in this paper or some general characteristics common to many actual credit portfolios. (Such as guarantors who guarantee a significant portion of the portfolio or having multiple guarantor covering a single exposure.)

With the exception of the multiple guarantor portfolio the scenario method proved to be the best approximation throughout, which given the construction of each approach is hardly surprising. None of the portfolios was able to throw it off entirely. This does not, however, mean that the scenario method is the be-all and end-all of guarantor-computing methods. As good as it is – its exponentially increasing runtime renders it prohibitively slow for any portfolio of realistic dimensions with a non-negligible number of guarantors.

The group method has proven to be a good contender. While certainly off from the reference distribution more often than not and even underperforming the mixture method several times in these examples, it never suffered such terrible runaway results as the mixture approach. Therefore this approach would be advisable if there are too many guarantors for the scenario approach and they are too connected to use the mixture method. Usually at least the order of magnitude of the quantiles should be correct.

Finally we examine the mixture method. As long as the guarantors and their guarantees are not too much interconnected this method provides fairly good results, but it falls completely flat as soon as the dependence structure becomes more complex and especially when the structure becomes more intertwined. (See Figure 5.20 for an illustration of such an outlier of result.) In realistic portfolios – especially when the actual dependence structure of the guarantees is not clear – this approach should only be used as a last resort.

All these results only reinforce the point made in Section 4.5 about the importance of the hybrid approach. Only by combining the strengths of each of the approaches to compensate for the weaknesses of the other will a method of computing a good approximation of the loss distribution emerge. Unfortunately the decision regarding which guarantors or groups to compute with which method is far from a trivial question and in fact still a topic of research.

6. Outlook

As already noted in the previous section, the biggest open topic in the area of guarantees within the CreditRisk$^+$ framework is the sensible automated choice of a hybrid approach to the computation.

Possible solutions range from more sophisticated indices of interdependence of guarantors (e.g. expected volume of loss to cover per guarantor) to novel measures on distributions for the quantification of the approximation error to iterative methods trying to improve the approximation with each pass.

Further the computation of each of the approaches can be optimised. In each section introducing one of the methods is a subsection with further considerations. Some of them – such as the merging of equal distributions in the mixture distributions of the mixture and scenario approach – are already implemented in the current stochastics library, but other still remain open.

Finally we have to acknowledge that the proposed methods of computing an ECR$^+$-model with guarantors were developed under the self-imposed constraint of maintaining the current model and trying to solve the problem with the current tools at hand. Perhaps another extension of the model – as was the case with the introduction of the scenarios – will allow a much more succinct and precise modelling of guarantors and guarantees.

A. The stochastics library

In the course of a joint project of the TU Vienna and the Oesterreichische Kontrollbank AG I was tasked with the reimplementation of the the current CreditRisk$^+$ model. Having the opportunity of a clean slate I decided against a direct implementation of the ECR$^+$algorithm as set laid out in Section 1.3.6, but instead to develop a general framework to manipulate discrete distributions. With such a library at hand the implementation of the ECR$^+$algorithm itself is reduced to the correct application of the objects and methods of the library. (Correct data-sourcing notwithstanding.)

This appendix gives a short overview over the main aspects of the library, some optimisations employed and its data interface, which is also used in this paper.

A.1. APFloat

In order to simplify the numerical part of the library I have decided to use an open source library allowing computations with arbitrary precision. (In fact the precision can be infinite as long as one stays within the rational numbers!) The library I have chosen is called APFloat[1] and is maintained by Mikko Tommila[2]. This library uses a modified version of the official APFloat library, which has been extended with some convenience methods and is hosted on GitHub[3].

A.2. Power Series

The fundamental idea of this library is to represent all distributions as their probability generating functions. Since we are dealing here with discrete distributions whose domain is (a subset of) $\mathbb{N}$, these pgf-s can be represented as power series.

A simple approach to store a power series – assuming it is finite – are arrays or any kind of list with the indices corresponding to $n \in \mathbb{N}$ and their value corresponding to $\mathbb{P}[X = n]$. While certainly feasible even preliminary tests with industry-provided data showed that the loss distributions encountered in reality are far from being dense. As a result during all computations a lot of computing power would be wasted on summing zeroes and multiplying ones.

A far more sensible approach is to store the power series in a map[4] with each key corresponding to a n an its value to its probability. For those numbers without an

[1] `http://www.apfloat.org`
[2] Mikko.Tommila@apfloat.org
[3] `https://github.com/TriangularIT/extended-apfloat`
[4] also known as a dictionary

entry in the map a probability of 0 will be implicitly assumed.

Not only does this save memory, it also allows to perform operation over the support of the probability generating function without constant filtering of zeroes. This is especially helpful for such manipulations as convolutions.

A power series does not have to be known in advance – some, like infinite ones, even can not – therefore the library also provides *lazy* power series. These are power series whose coefficients are only computed when needed but then cached for future reference. Combined with the ability to iterate over the support of the power series (if appropriately declared) allows the library to efficiently compute powers and convolutions of infinite power series and to evaluate them up to arbitrary precision.

A.3. Distributions

The centre-piece of the library are, of course, the implementations of various discrete and a handful of continuous distributions.

Currently the following distributions are implemented:

- Binomial
- Dirac
- Empirical
- Logarithmic
- Negative binomial
- Poisson
- Exponential
- Gamma
- Uniform

These distributions are fully parametrised and often accept even degenerated parameters.

While these distributions are nice as they are, there is only so much that can be done with single distributions. Therefore several operations on discrete distributions have been implemented:

Comonotonic sum
: Represents the distribution of the sum of other distributions under the assumption that all summands are comonotonic.

Compound sum

Accepts two distributions $\mathcal{N}$ and $\mathcal{X}$ and models the distribution of a random sum S such that

$$S = \sum_{n=1}^{N} X_n,$$

where $N \sim \mathcal{N}$ and $X_n \sim \mathcal{X}$ are all independent of each other.

Currently the following distributions are supported for $\mathcal{N}$:

- Dirac,
- Poisson,
- Logarithmic and
- Negative binomial.

Compressed distribution

Performs stochastic rounding[5] on a distribution. This may be considered the counterpart of a lattice distribution.

Convolution

Represents the distribution of the sum of other distributions under the assumption that all summands are independent.

The library provides a way to iterate over the support of a convolution without computing the entire convolution beforehand.

Lattice distribution

This distribution is a scaled and shifted transformation of some other distribution. If X and Y are random variables then the distribution of Y is a lattice distribution of X if

$$Y = max\{aX + b, 0\}, \qquad \text{for } a \in \mathbb{N}_+, b \in \mathbb{Z}$$

such that $\mathbb{P}(aX + b \geq 0) > 0$.

Mixture distribution

A distribution where the distribution function is a convex sum of afinite number of other distribution functions.
See Definition 1.14.

Non-Zero distribution

A distribution conditioned not to attain the value 0.
See Definition 1.15. Can only be applied to distributions, which do not concentrate their entire mass on 0.

[5] See [8, Section 6.2.2].

Every transformation returns again a discrete distributions, which allows to use the result in further computations.

All distributions and transformations are based on lazy power series. Therefore arbitrary combinations of distributions and transformations can be quickly defined without any evaluation of the underlying values. Evaluation happens only if, when and just as far as necessary.

Every distribution itself has the following operations:

- Probability of a point $n \in \mathbb{N}$
- Entire probability generating function as power series
- Cumulative distribution at a point $n \in \mathbb{N}$
- Quantile at a level $q \in [0, 1]$
- Expected shortfall at a level $q \in [0, 1]$
- Expected value

These quantities are computed using the APFloat library which means that they can be computed to an arbitrary precision.

A.4. Random generator

In order to facilitate the implementation of a Monte Carlo simulation every distribution – including the continuous and transformed ones – has the ability to generate random values according to the given distribution.

To this end the uniform distribution implements the fairly new xoroshiro128+ generator developed by Sebastiano Vigna in collaboration with David Blackman – [9]. The generator returns a random 64-bit number which is then transformed into a positive APRational. As the APFloat library provides arbitrary precision we can even use the sign bit in the creation of the random number.

For the gamma distribution two different generators are used depending on the shape parameter $\alpha > 0$:

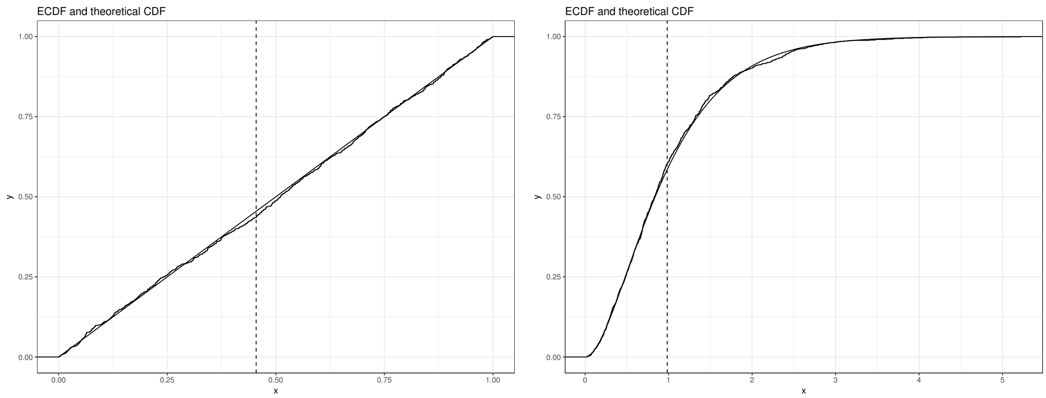
- if $\alpha < 1$ Algorithm RGS by D. J. Best ([1]) is used;
- if $\alpha = 1$ the random generator for the exponential distribution is used;
- if $\alpha > 1$ Algorithm GB by R. C. H. Cheng ([2]) is used.

The exponential distribution generates its random number by the simple and efficient method of inversion:

Theorem A.1. *Let $U \sim U(0, 1)$. Set*

$$X = -\log(U)/\lambda,$$

then $X \sim \mathrm{Exp}(\lambda)$.



(a) $U(0,1)$ (b) $\Gamma(2,2)$

Figure A.1.: KS-tests of random generators

Proof. This follows directly from the fact that $f(x) = -\log(x)/\lambda$ is the inverse compound distribution function of an $\mathrm{Exp}(\lambda)$-distribution and that for $U \sim U(0,1)$ the transformed $1 - U$ is as well $U(0,1)$. ∎

All other distributions generate their random values via the inversion method as well using their provided quantile function.

The quality of both the uniform random generator as well as of the gamma distribution generators has been tested with iterated Kolmogorov–Smirnov tests as described by Knuth in [6, Section 3.3.1.B]. Figures A.1a and A.1b show exemplary results of a single iteration of a KS-test with sample size 1000. The dashed lines mark the point where the maximal difference between the empirical and expected cdf happens. The corresponding statistics and p-values of the tests are $(0.020283, 0.8052)$ and $(0.019764, 0.8296)$.

A.5. Divergences

Apart from distributions and transformations thereon the library also provides several divergences and metrics to measure differences between distributions.

Currently these three are implemented:

Kullback–Leibler divergence

Also known as *information divergence* or *relative entropy* this divergence measures the expectation of the logarithmic difference between two probabilities P and Q.

Definition A.2. Let P and Q be discrete probability distributions such that

$$P \gg Q, \quad \text{that is } P(n) = 0 \;\Rightarrow\; Q(n) = 0 \quad \forall n \in \mathbb{N},$$

then define

$$\mathrm{d_{KL}}(P,Q) := \sum_{n\in\mathbb{N}} P(n)\log\frac{P(n)}{Q(n)}$$

as the Kullback–Leibler divergence from Q to P.

The algorithm used to compute the Kullback–Leibler divergence can also be used to calculate a related quantity – the entropy.

Definition A.3. Let P be a discrete probability distribution, then

$$H(P) := -\sum_{n\in\mathbb{N}} P(n)\log P(n)$$

is called the *entropy* of P.

Total variation distance

See Definition 3.10.

Since all considered distributions are discrete and $\mathbb{N}$ is countable, the library uses a simplified method to compute the distance.

Theorem A.4 ([8, Lemma 3.18])**.** *Let $S \neq \emptyset$ be a finite or countably infinite set. Then for all probability measures P and Q on $(S, \mathfrak{P}(S))$*

$$\mathrm{d_{TV}}(P,Q) = \frac{1}{2}\sum_{s\in S}|P(s)-Q(s)|.$$

Proof. We denote with $e_s := P(s) - Q(s)$ the difference between P and Q for every $s \in S$. Then for $A \subseteq S$

$$\frac{1}{2}\sum_{s\in S}|e_s| \geq \frac{1}{2}\sum_{s\in A} e_s - \frac{1}{2}\sum_{s\in S\setminus A} e_s = \sum_{s\in A} e_s - \frac{1}{2}\underbrace{\sum_{s\in S} e_s}_{=0} = P(A) - Q(A).$$

The inequality becomes an equality if and only if $|e_s| = e_s$ for every $s \in A$ and $|e_s| = -e_s$ for every $s \in S \setminus A$. ∎

Wasserstein distance

See Section 5.1.1.

A.6. XML-Interface

In order to facilitate the use of the library it is not necessary to define the desired distributions and transformations programmatically in Java. The library provides a graphical interface which allows to load an XML-file describing the desired distribution and to select which computations to perform. Figure A.2 shows a screenshot of the user interface after calculating one of the example portfolios.

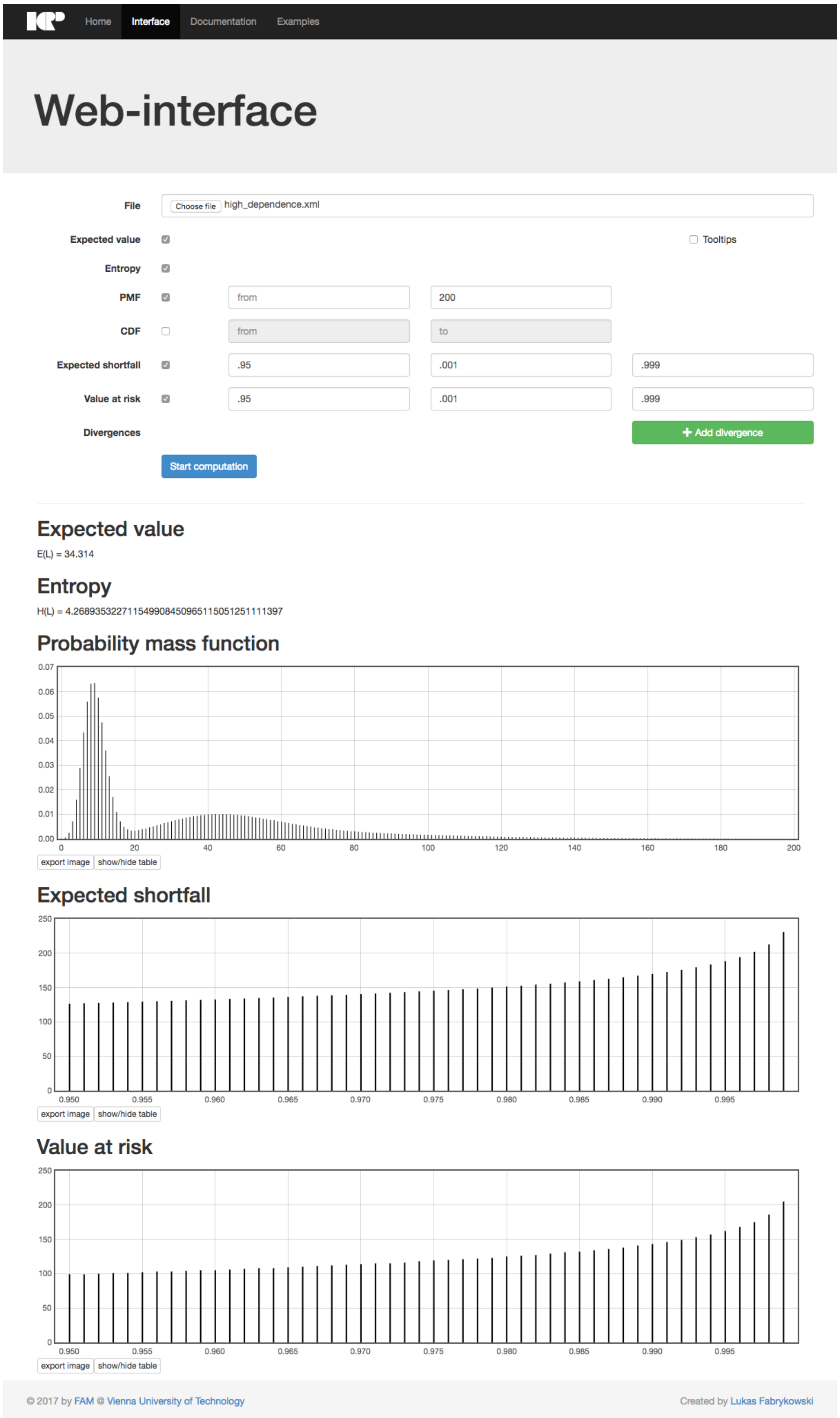


Figure A.2.: A screenshot of the graphical user interface of the library

Each distribution from this library and its parameters can be represented as an XML tag and through combination of these tags further distributions can be created.

This XML notation is also used in this paper to describe the loss distributions of some portfolios.

The following list contains a short description of the XML tag for each distribution and transformation available. For most only an XML example is provided, since the names of the parameters are self-explanatory.

number

```
<number>3.14159265358979323846264338327950</number>
<number>1/3</number>
<number><sqrt>2</sqrt></number>
```

Represents a numerical value. This can be either a plain number, a fraction or one of several functions:

- `add`: sum of two values
- `sum`: difference of two values
- `div`: quotient of two values
- `mul`: product of two values
- `pow`: one value raised to the other
- `abs`: absolute value
- `acos`: inverse cosine
- `acosh`: inverse hyperbolic cosine
- `asin`: inverse sine
- `asinh`: inverse hyperbolic sine
- `atan`: inverse tangent
- `atanh`: inverse hyperb. tangent
- `cbrt`: cube root
- `ceil`: ceiling function
- `cos`: cosine
- `cosh`: hyperbolic cosine
- `exp`: exponent function
- `floor`: floor function
- `frac`: fractional part
- `log`: natural logarithm
- `negate`: negative value
- `sin`: sine
- `sinh`: hyperbolic sine
- `sqrt`: square root
- `tan`: tangent
- `tanh`: hyperbolic tangent
- `toDegrees`: radians to degrees
- `toRadians`: degrees to radians
- `truncate`: truncate fraction
- `w`: Lambert W function

BinomialDistribution

```
<BinomialDistribution>
    <p>{APFloat∈[0,1]}</p>
    <m>{integer>0}</m>
</BinomialDistribution>
```

ComonotonicSum

```
<ComonotonicSum>
    {distribution}
    {distribution}
    ...
</ComonotonicSum>
```

CompoundDistribution

```
<CompoundDistribution>
    <size>{distribution}</size>
    <number>{distribution}</number>
</CompoundDistribution>
```

A compound distribution with $N \sim$`<number/>` and $X_n \sim$`<size/>`.

CompressedDistribution

```
<CompressedDistribution step="{integer>0}">
    {distribution}
</CompressedDistribution>
```

Convolution

```
<Convolution>
    {distribution}
    {distribution}
    ...
</Convolution>
```

DiracDistribution

```
<DiracDistribution>{integer≥0}</DiracDistribution>
```

EmpiricalDistribution

```
<EmpiricalDistribution>
    <value>{APFloat∈[0,1]}</value>
    <value index="{integer}">{APFloat∈[0,1]}</value>
    <value>{APFloat∈[0,1]}</value>
    ...
</EmpiricalDistribution>
```

Each `<value/>` describes the probability of its index value. If no index is given, the previous index is increased by 1. The first element has the implied index 0.

Naturally all values have to add up to 1.

LatticeDistribution

```
<LatticeDistribution step="{integer>0}" lag="{integer}">
    {distribution}
</LatticeDistribution>
```

LogDistribution

```
<LogDistribution>
    <p>{APFloat∈(0,1)}</p>
</LogDistribution>
```

MixtureDistribution

```
<MixtureDistribution>
    <p>
        {APFloat∈[0,1]}
        {APFloat∈[0,1]}
        ...
    </p>
    <v>
        {distribution}
        {distribution}
        ...
    </v>
</MixtureDistribution>
```

A mixture distribution with weights `<p/>` and components `<v/>`. The weights have to add up to 1.

NegBinomialDistribution

```
<NegBinomialDistribution>
    <number>{APFloat>0}</number>
    <success>{APFloat∈(0,1)}</success>
</NegBinomialDistribution>
```

NonZeroDistribution

```
<NonZeroDistribution>
    {distribution}
</NonZeroDistribution>
```

PoissonDistribution

```
<PoissonDistribution>
    <lambda>{APFloat>0}</lambda>
</PoissonDistribution>
```

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