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A Mass Conserving Mixed Stress Formulation For Incompressible Flows

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Kurzfassung

Diese Arbeit befasst sich mit der Einführung und der Analyse einer neuen Finite Elemente Methode zur Berechnung von inkompressiblen Strömungen. Im Wesentlichen liegt hierbei der Fokus auf den linearen inkompressiblen Stokesgleichungen, welche die physikalischen Zusammenhänge – hergeleitet aus den grundlegenden Newtonschen Gesetzten – zwischen der Geschwindigkeit des Fluids und dem vorherrschenden Druck(gradienten) wiedergibt. Während die zugehörige Standard-Varitionsformulierung den Sobolevraum erster Ordnung als Funktionenraum für die Geschwindigkeit in Betracht zieht, zeigen wir in dieser Arbeit, dass es auch möglich ist, eine Variationsformulierung zu definieren, die weniger Regularität an die Geschwindigkeit fordert.

Dazu wird eine formal äquivalente Formulierung der Stokesgleichungen in Betracht gezogen und ein neuer Funktionenraum für den Gradienten der Geschwindigkeit definiert. Die resultierende Variationsformulierung ist nun wohldefiniert für den Fall, dass die (schwache) Divergenz der Geschwindigkeit quadratisch integrierbar ist, und nicht, wie es die Standard-Formulierung verlangt, alle partielle Ableitungen. Wir präsentieren wichtige Eigenschaften des neu definierten Funktionenraums, wie zum Beispiel die Definition eines stetigen Spuroperators und die Dichtheit von glatten Funktionen.

Motiviert durch diese neue Formulierung befasst sich der Rest der Arbeit mit der Herleitung und der Analyse einer neuen zugehörigen Finite Elemente Methode. Für die Approximation der Geschwindigkeit kann nun ein passender konformer Raum gewählt werden, welcher zur exakten (physikalisch korrekten) Einhaltung der Inkompressibilitätsbedingung führt. Für die Diskretisierung des Geschwindigkeitsgradienten definieren wir neue matrixwertige Finite Elemente-Basisfunktionen, deren Normal-Tangentialkomponenten stetig über Elementgrenzen hinweg ist. Wir präsentieren eine ausführliche Stabilitätsanalyse und beweisen optimale Konvergenzraten des Diskretisierungsfehlers.

Abstract

This work deals with the introduction and the analysis of a new finite element method for the discretization of incompressible flows. The main focus essentially lies on the discussion of the linear incompressible Stokes equations. These equations describe the physical behaviour and the relation – derived from the fundamental Newtonian laws – between the fluid velocity and the pressure (-gradient). Where the standard variational formulation of the Stokes equations demand a Sobolev regularity of order one for the velocity, we give an answer to the question if it is possible to define a variational formulation demanding a weaker regularity property of the velocity.

With respect to a formally equivalent representation of the Stokes equations, we answer this question by the introduction of a new function space used for the definition of the gradient of the velocity. The resulting variational formulation is well-posed if we assume that the divergence of the velocity is square integrable. Thereby, with respect to the standard formulation, where all partial derivatives have to be square integrable, this is a reduced regularity property. We present certain properties of the new defined function space and discuss a proper continuous trace operator and the density of smooth functions.

Motivated by this new variational formulation, we present and analyse a new finite element method in the rest of this work. For the approximation of the velocity we can now choose a conforming discrete space. This results in a (physically correct) incompressibility of the velocity field, thus exact mass conservation is provided. For the approximation of the gradient of the velocity we define new matrix-valued finite element shape functions, which are normal-tangential continuous across element interfaces. We present a detailed stability analysis and prove optimal convergence order of the discretization error.

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Vienna, February, 2019 — Philip Lederer

Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Wien, am 12. Februar 2019

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1 Introduction

Computational fluid dynamics (CFD) is an ongoing research topic and is widely used in modern industry. Whereas the principles of fluid motion are well known – in the sense of the governing equations – a numerical approximation of these equations is needed in order to satisfy the demands of modern applications. The aim of this thesis is to contribute to the CFD-community by presenting a new method for the descritization of flow fields. In particular, the presented results not only provide a new numerical scheme but also allow a new insight on the mathematical structure of the basic equations.

This thesis considers the unsteady incompressible Stokes equations given by the following set of partial differential equations

$$\begin{cases} -\operatorname{div}(\nu\nabla u) + \nabla p = f & \text{in } \Omega, \\ \operatorname{div}(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where u and p are the velocity and the pressure respectively, $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, is the computational domain, f is an external force and ν is the kinematic viscosity. Note that we also discuss other types of boundary conditions in this thesis, but for the ease we now only discuss the simplest case of homogeneous Dirichlet values for the velocity. The question of solvability and well-posedness of the Stokes equations is well analyzed. In this work we particularly aim to discuss the existence of solutions only in a weak sense, thus we seek solutions of variational formulations. With respect to the standard velocity-pressure setting of the Stokes equations as denoted above, a weak formulation now demands the spaces as follows. Whereas the pressure only needs to be an element of the space of square integrable functions with a zero meanvalue, denoted by $L^2_0(\Omega, \mathbb{R})$, the appropriate velocity space is given by the (vector-valued) Sobolev space $H^1_0(\Omega, \mathbb{R}^d)$. Assuming $f \in L^2(\Omega, \mathbb{R}^d)$, the resulting weak formulation is thus given by: find $u \in H^1_0(\Omega, \mathbb{R}^d)$ and $p \in L^2_0(\Omega, \mathbb{R})$ s.t.

$$\begin{aligned} \int_\Omega \nu\nabla u : \nabla v \,\mathrm{d}x - \int_\Omega \operatorname{div}(u) p \,\mathrm{d}x &= \int_\Omega f \cdot v \,\mathrm{d}x \qquad \forall v \in H^1_0(\Omega, \mathbb{R}^d), \\ - \int_\Omega \operatorname{div}(v) q \,\mathrm{d}x &= 0 \qquad \forall q \in L^2_0(\Omega, \mathbb{R}). \end{aligned} \tag{1.2}$$

This formulation has – especially from the mathematical point of view – a very interesting structure: it is a saddle point problem. In contrast to standard symmetric and coercive variational formulations, this makes the solvability analysis very challenging. In particular, one has to show that the inf-sup condition for the divergence constraint is fulfilled. This was proven in the famous work of Nečas, see [88], and many subsequent works [18, 19, 15]. To be precise, in the work [88] the author presented a proof for the estimate

$$\|f\|_{L^2(\Omega)} \leq \|f\|_{H^{-1}(\Omega)} + \|\nabla f\|_{H^{-1}(\Omega)} \quad \text{for all} \quad f \in L^2(\Omega, \mathbb{R}),$$

which was later realized to be equivalent to the inf-sup condition of the divergence constraint.

Scientists of different fields developed various numerical schemes to approximate the exact solutions of equations (1.1). The finite element method, which is based on the variational formulation of the Stokes equations given by (1.2), has shown to be especially well suited for this problem. Similarly as in the continuous setting, the saddle point structure makes things complicated: the finite dimensional velocity and pressure spaces need to be compatible, by means that the corresponding discrete inf-sup condition for the divergence constraint can not be inherited from the continuous setting and needs to be proven for each method separately. However, many different stable pairs were found, as for example the famous Taylor-Hood element, see [116, 53, 85] and its high order extension [20], or the Scott-Vogelius element, [119]. Beside these conforming methods, many other conforming and also non-conforming (with respect to the velocity space $H^1_0(\Omega, \mathbb{R}^d)$) finite element pairs can be found, for example in [11, 53, 72, 40].

Talking of non-conforming schemes, we shall particularly discuss discontinuous Galerkin (DG) methods. Since their introduction by Reed and Hill in [100], DG schemes got popular for parabolic and second-order elliptic equations, see for example [7, 67, 103], and particularly also for the discretization of flow problems, see for example [113, 108, 54, 29, 31, 30, 35, 71, 8]. Where continuity with respect to $H^1(\Omega, \mathbb{R})$ is only imposed in a weak sense, it is this discontinuity that allows to use certain stabilization techniques, which are needed to attain stable methods in the case of dominating convective forces. For flow fields this is specifically of interest if we not only consider the Stokes setting, but the Navier-Stokes case, as it includes the non-linear convection term $(u \cdot \nabla)u$. In recent years, there has been a growing body of literature particularly discussing DG methods (and also their hybridized versions), where the discrete velocity space V_h is chosen to be conforming with respect to $H(\operatorname{div}, \Omega)$, see for example in the works [30, 31, 29, 80, 81, 76, 77, 49, 51, 101, 109, 74, 73, 38]. In this case, weak conformity with respect to $H^1(\Omega, \mathbb{R}^d)$ only needs to be imposed for the tangential component of the velocity field as the normal component is continuous. The motivation of this choice is given by the incompressibility constraint $\operatorname{div}(u) = 0$. Obviously, $H(\operatorname{div})$-conformity is tailored to approximate this constraint in a proper sense. To be precise, using standard (well known) $H(\operatorname{div})$-conforming finite elements for the approximation of the velocity, the discrete pressure space can be chosen as $Q_h = \operatorname{div}(V_h)$. This implies that weakly divergence free velocity fields are exactly divergence free:

$$\int_\Omega \operatorname{div}(v_h) q_h \,\mathrm{d}x = 0 \quad \forall q_h \in Q_h \quad \Leftrightarrow \quad \operatorname{div}(v_h) = 0. \tag{1.3}$$

This exact mass conservation property of discrete velocity solutions was identified as a crucial advantage compared to other methods, as it leads to energy stability of discretizations of the Navier-Stokes equations (using for example an upwind-stabilization for the convection term), and to pressure robust error estimates in the Stokes setting. The latter property was first discussed in the literature in [83] and lead to several subsequent works [79, 70, 16, 82, 75] and more. Standard finite element methods for the velocity pressure formulation of the Stokes equations usually provide a priori estimates for the discrete velocity u_h, which read as

$$\|u - u_h\|_{H^1(\Omega)} \leq C\Big(\inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)} + \frac{1}{\nu} \inf_{q_h \in Q_h} \|p - q_h\|_{L^2(\Omega)} \Big),$$

where p_h is the discrete pressure solution, and $C > 0$ is a constant independent of ν. Thus, the velocity error is not only bounded by the best approximation of the velocity space, but also depends on the approximation properties of the pressure space. Further, the best approximation of the pressure includes the scaling $1/\nu$. Assuming that the exact pressure is not included in the discrete pressure space (thus, the second term is not equal to zero), this scaling can lead to a bad velocity approximation if the viscosity tends to get small $\nu \to 0$. In the literature this phenomenon is also associated with a property called *poor mass conservation* and is inherited from the weak treatment of the incompressibilty constraint. In contrast to this, $H(\operatorname{div})$-conformity of the discrete velocity space leads to exactly divergence free velocity fields, see equation (1.3). This then allows us to derive error estimates that are independent of the pressure, and thus in particular do not show the bad behavior as described above if the viscosity ν gets small.

It seems as if $H(\operatorname{div})$-conformity of the velocity space is the right, or at least a very good, choice in the discrete setting, rising the question if this might also be an appropriate choice in the continuous setting. We give an answer to this question by introducing a formally equivalent formulation of the Stokes equations. Introducing a viscous stress variable $\sigma = \nu \nabla u$, elementary manipulations of the standard Stokes equations yield the mixed form

$$\begin{aligned} \frac{1}{\nu}\sigma - \nabla u &= 0 \quad && \text{in} \quad \Omega, \\ \operatorname{div}(\sigma) - \nabla p &= -f \quad && \text{in} \quad \Omega, \\ \operatorname{div}(u) &= 0 \quad && \text{in} \quad \Omega, \\ u &= 0 \quad && \text{on} \quad \Gamma_D. \end{aligned}$$

Based on this set of equations it is then possible to derive a variational formulation, where the proper velocity space is given by $H_0(\operatorname{div})$ (including zero normal boundary conditions). The stability analysis of this new formulation is one of the main issues of this thesis, and includes the introduction of a new function space for the viscous stress variable σ. As we aim to require as little regularity as possible, we define this new space as

$$H(\operatorname{curl}\operatorname{div}) := \{\sigma \in L^2(\Omega, \mathbb{R}^{d \times d}) : \operatorname{div}(\sigma) \in H_0(\operatorname{div})^*\},$$

which reads as the set of matrix-valued square integrable functions, whose divergence can continuously act (in the sense of a linear functional) on functions in $H_0(\operatorname{div})$. The definition of this new function space allows a detailed analysis and leads to new findings of the mathematical structure of the Stokes equations. Motivated by this new continuous setting, we then aim to define a new discrete variational formulation. Where the approximation spaces for the velocity space and the pressure space are based on (standard) conforming finite elements, the approximation of functions in $H(\operatorname{curl}\operatorname{div})$ demands the introduction of a new finite element. Using these elements we then define a discrete method, which we call the mass conserving mixed stress method (MCS).

Various publications have assessed the numerical treatment of the mixed formulation above, see for example [47, 48, 45, 62]. In particular, we want to mention the work [46] where the authors claim that their method locally conserves mass (and momentum). Their velocity approximation is given by element-wise (with respect to a triangulation) constants and additional (constant) Lagrange multipliers $\hat{u}_n, \hat{u}_t$ used for the approximation of the velocity in normal and tangential direction at element interfaces, respectively. The resulting velocity solution then provides the property

$$\int_{\partial T} \hat{u}_n \, \mathrm{d}s = 0 \quad \text{for all} \quad T \in \mathcal{T}_h,$$

where T is an element of the triangulation $\mathcal{T}_h$. The equation above can therefore be understood as a local conservation of mass as the "netto inflow equals the netto outflow" on each element boundary. However, the resulting method is not pressure robust as it does not provide exactly divergence free velocity test functions. In contrast, the schemes presented in this work lead to exactly divergence free, hence pressure robust, velocity approximations that have no computational overhead (in the sense of coupling degrees of freedom) and further provide optimal convergence orders of the finite element error.

1.1 Structure of this thesis

In this thesis we first examine the continuous setting and derive new insights on the mathematical structure of the Stokes equations. The later chapters are dedicated to the introduction of new finite element methods. More precisely, this thesis is structured as follows:

- In chapter 2 we derive the governing equations of fluid motion. We follow the standard procedures known in the literature using the fundamental theorems of Newton given by *the conservation of mass* and the *conservation of momentum*. We discuss the special case of Newtonian fluids, and conclude with the derivation of the Stokes and Navier-Stokes equations.
- Chapter 3 introduces the notation that shall be used within this work. We define several well known function spaces and present the basic theorems proving well-posedness of variational formulations in an abstract setting.
- Chapter 4 is dedicated to the derivation of a new variational formulation of the Stokes equations including the definition of the function space $H(\operatorname{curl}\operatorname{div})$. In a first step we provide an equivalent definition and prove that this new function space can be approximated by smooth functions. Thereby, we can apply standard density arguments to define an appropriate trace operator, which is continuous and surjective with respect to a corresponding (newly defined) trace space. Based on these findings we discuss well-posedness of the standard Stokes equations and their symmetric version using the theory of saddle point problems. The analysis includes a Korn-type inequality (for the deviator) that is proven in detail.
- Beside the definition of standard well known finite elements used for the approximation of the velocity and the pressure solution, chapter 5 is committed to the definition

of new matrix-valued finite elements that shall be used for the approximation of functions in $H(\operatorname{curl}\operatorname{div})$. By the definition of a distributional divergence (with respect to a given triangulation), we motivate the incorporated normal-tangential continuity of discrete stress functions. We define an appropriate finite element mapping and present the construction of arbitrary high order basis functions.

- Using these new stress finite elements, we introduce a discrete variational formulation, in the chapter 6. The analysis of this method is split into several parts. In a first step we present some norm equivalences that are based on standard scaling arguments and the structure of the considered finite elements. Based on these results we then present an inf-sup condition proving discrete stability. We conclude the chapter with the introduction of several interpolation operators and present an a priori error analysis proving optimal convergence order of the finite element error. The stability analysis as well as the error estimates are presented in two different versions: whereas the first case uses discrete norms inspired by the original standard velocity-pressure setting of the Stokes equations, the second version uses *natural* norms with respect to the new continuous setting derived in chapter 3.

- Chapter 7 discusses the discretization of the symmetric case. As the discrete stress space is not suited to approximate symmetric matrices, we introduce a new constraint and impose symmetry only in a weak sense. In order to prove well-posedness, we present an enrichment of the stress space that is motivated by the work of Stenberg, see [111]. We present a detailed stability analysis and show optimal convergence of the finite element errors.

- We conclude the work with chapter 8, in which we present several numerical examples to validate the findings of the previous chapters.

Some of the presented results have been published in the works [69, 68]. However, in this thesis we extend these results and add several additional comments.

2 The equations of fluid motion

This chapter is devoted to the basic principles of fluid mechanics and the derivation of the governing equations. We follow the same ideas as provided in standard literature on fluid dynamics, see [93, 10, 114, 3].

In the following we consider an Euclidean space with the independent three-dimensional variable $x = (x_1, x_2, x_3)$ and assume that the time t proceeds independently. Using the unit vectors e_1, e_2 and e_3 along the x_1, x_2 and x_3 axes, respectively, we define the vector velocity field by

$$u := u_1 e_1 + u_2 e_2 + u_3 e_3,$$

with the scalar-valued components $u_1 = u_1(x_1, x_2, x_3, t)$, $u_2 = u_2(x_1, x_2, x_3, t)$ and $u_3 = u_3(x_1, x_2, x_3, t)$. Similarly, the scalar density field and the scalar pressure is given by $\rho := \rho(x_1, x_2, x_3, t)$ and $p := p(x_1, x_2, x_3, t)$. We speak of a two-dimensional flow field, when the fluid motion is restricted to parallel planes. In this case the the velocity component, which is perpendicular to the plane is equal to zero at each point. Further, the flow is independent of deformations that are parallel to the flow. In this work a two dimensional flow is always considered in the x_1-x_2 plane, thus the velocity field is given by $u := u_1 e_1 + u_2 e_2$. Note that in order to speak of the above defined physical quantities we assumed that the *continuum assumption* holds true. This means that the physical quantities of interest of the liquid contained in a given small volume are imagined to be uniformly distributed over that volume. We can then also talk about fluid particles at a specific point, when we keep in mind that this particle is actually sufficiently large to contain enough molecules of the liquid such that an averaging, for example of the velocity, makes sense.

For the derivation of the governing equations of fluid mechanics we are using the concept of (finite) control volumes and their associated control surfaces. The main purpose of using a control volume is to focus the attention on physical events and quantities only in a small region and its boundary in order to be able to keep track of all effects. We can distinguish between two different types. A fixed control volume is specified by a given (fixed) location in space, thus the fluid passes into and out off the volume through the surface. The second type is called a material control volume. The idea is that the control volume is moving with the liquid such that the fluid particles stay inside and do not pass the surface. This leads to two different aspects. A *Lagrangian viewpoint* focuses on the flow of fluid particles. Each particle is identified by its initial position at a specific given (start) time. When time passes all particles move and change their position. This position (trajectory) now is a function that depends on the original location and the time. Similarly, all other physical quantities only depend on the initial position and time, thus refer to one specific fluid particle. In contrast to this, the *Eulerian viewpoint* deals with fixed points in space. At a given time we can evaluate physical quantities at each point to retrieve local information on the fluid.

In this work we always use the Eulerian viewpoint. The close relation of the two different viewpoints is given by the substantial derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} := \frac{\partial}{\partial t} + (u \cdot \nabla), \tag{2.1}$$

which can be interpreted as the time rate of change following a fluid particle. It consists of the local time derivative at a fixed point $\partial/\partial t$ and the convective derivative $(u \cdot \nabla)$, which describes the time rate of change induced by the movement of the particle.

2.1 Fundamental laws

2.1.1 The continuity equation

The fundamental physical principle that is considered in this section is the conservation of mass. To this end, let ω be an arbitrary fixed control volume, hence we assume that it is not moving with the flow. The principle of mass conservation then reads as

$$\text{Netto mass flow through the surface } \partial\omega = \text{time rate of decrease of mass inside } \omega. \tag{2.2}$$

In the following we translate (2.2) into an explicit equation including functions and variables. We first deal with the left hand side of this equation. The mass that is transported through an infinitesimal small surface area is given by the density times the size of this area times the velocity that is perpendicular to the surface. Thus, we have, using the Gaussian theorem,

$$\text{Netto mass flow through the surface } \partial\omega := \int_{\partial\omega} \rho u \cdot n \,\mathrm{d}s = \int_\omega \operatorname{div}(\rho u)\,\mathrm{d}x\,.$$

The right hand side of (2.2) is given by the negative derivation with respect to time of the mass inside of ω, thus

$$\text{time rate of decrease of mass inside } \omega := -\frac{\partial}{\partial t}\int_\omega \rho \,\mathrm{d}x\,.$$

Note that the control volume is fixed in time, allowing us to change the order of integration and differentiation. Combining the last two results then leads to

$$\int_\omega \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u)\,\mathrm{d}x = 0.$$

Taking into account that the control volume ω was arbitrary, the equation inside the integral has to be fulfilled at each point and so we finally derive the continuity equation given by

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0. \tag{2.3}$$

This means that the time rate of change at a specific point equals the negative netto flow of the mass out of an infinitesimal small volume area (a fluid particle). Before we proceed with the next chapter we try to give a physical interpretation of (2.3) following one specific fluid particle. For this we use the product rule for the second term and use the definition of the substantial derivative (2.1) to get

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\,\mathrm{div}(u).$$

The left hand side is the time rate of change of the density of one fluid particle as it follows the flow, and the right hand side is equivalent to the mass per unit volume (the density ρ) times the expansion rate of the fluid particle. Note that this version of the continuity equation can also be derived by starting with a material control volume $\omega(t)$, which follows the flow, and using the principle of mass conservation. In fact, there are four equivalent versions of the continuity equation, which can all be converted into each other. For more details we refer to [3, chapter 2.5].

2.1.2 The momentum equation

In the following chapter we derive the momentum equation, which is based on Newton's second law: The time rate of change of the momentum of a particle is proportional to the force acting on it. For the derivation we choose a material control volume $\omega(t)$, which is moving with the flow. Then we have, according to Newton's second law applied on the fluid seen as a continuum,

$$\text{time rate of change of momentum of } \omega(t) = \text{netto forces acting on } \omega(t). \tag{2.4}$$

For the computation of the momentum we first focus on the physical effects in the x_1-direction. The product ρu_1 is equivalent to the momentum in the direction of e_1 per unit volume,

$$\text{time rate of change of momentum in } x_1\text{-direction of } \omega(t) = \frac{\partial}{\partial t}\int_{\omega(t)} \rho u_1 \,\mathrm{d}x\,.$$

Using Leibnitz's theorem to change the order of integration and differentiation (the control volume depends on time and is moving with velocity u), and using Gauss's theorem on the appearing surface integral, we can further write

$$\frac{\partial}{\partial t}\int_{\omega(t)} \rho u_1 \,\mathrm{d}x = \int_{\omega(t)} \frac{\partial}{\partial t}(\rho u_1)\,\mathrm{d}x + \int_{\partial\omega(t)} (\rho u_1) u \cdot n \,\mathrm{d}s = \int_{\omega(t)} \frac{\partial}{\partial t}(\rho u_1) + \mathrm{div}(\rho u_1 u)\,\mathrm{d}x\,.$$

For the right hand side of (2.4) we first consider an example. Imagine a bubble consisting of oil that is surrounded by water and moves with the flow. There are two different forces acting on this bubble. Firstly, the gravity acts on each oil particle inside the bubble, and secondly, there is a force induced by the surrounding water that acts on the surface of the bubble. Although this is an example of a two-phase flow and many different physical effects are going to occur, we can apply the same ideas on our material control volume, where we consider a volume force f and a surface force s. Thus, again restricting on the x_1-direction, we have

$$\text{net forces in } x_1\text{-direction acting on } \omega(t) = \int_{\omega(t)} \rho f_1 \,\mathrm{d}x + \int_{\partial\omega(t)} s_1 \,\mathrm{d}s\,.$$

Note that there is no density included for the boundary forces as the infinitesimal small areas contain no mass. The same arguments are now also applied to the x_2-direction and the x_3-direction. Combining all results we have for $f = (f_1, f_2, f_3)$ and $s = (s_1, s_2, s_3)$ the equation

$$\int_{\omega(t)} \frac{\partial}{\partial t}(\rho u) + \operatorname{div}(\rho u \otimes u)\,\mathrm{d}x = \int_{\omega(t)} \rho f\,\mathrm{d}x + \int_{\partial\omega(t)} s\,\mathrm{d}s\,. \tag{2.5}$$

To proceed with our derivation we take a closer look on the idea of the surface force. Let us go back to our example and assume that the considered material control volume $\omega(t)$ is the oil bubble. Next, imagine that the surrounding water is removed. Then the surface force s has to be equivalent to the forces that were induced by the surrounding water. Thus, we could just consider the oil bubble and think of s to be a function depending on the position and the orientation of the surface. This idea then leads to the introduction of a stress tensor T such that $s = Tn$. The derivation of this stress tensor is complicated and can be found for example in [93, chapter 5.4].

Putting back the surrounding water, we now assume that the bubble is not moving. In this case the surrounding weight of the water creates a force that presses the oil bubble together. This is the pressure force p. However, when the surrounding water is moving, there will not only be a pressure but also some viscous effects. The velocity at the interface of the bubble has to be continuous, thus the moving, surrounding water creates some stresses at the surface. This leads to the introduction of the viscous stress tensor τ. These two different forces are combined by the relation $T = -\mathrm{Id}p + \tau$. Using this relation and the idea of introducing a stress tensor in equation (2.5) leads to

$$\int_{\omega(t)} \frac{\partial}{\partial t}(\rho u) + \operatorname{div}(\rho u \otimes u)\,\mathrm{d}x = \int_{\omega(t)} \rho f + \operatorname{div}(-\mathrm{Id}p + \tau)\,\mathrm{d}x,$$

where we applied Gauss's theorem for the boundary integral. Similar as in the previous section this equation has to be fulfilled at each point as the material control volume $\omega(t)$ was arbitrary, leading to

$$\frac{\partial}{\partial t}(\rho u) + \operatorname{div}(\rho u \otimes u) = \rho f - \nabla p + \operatorname{div}(\tau). \tag{2.6}$$

This is not the final form of the momentum equation, however we can not proceed without making some assumptions about the physical properties of the considered fluid.

2.2 Incompressible Newtonian fluids

Supposing a constant density ρ, the continuity equation (2.3) simplifies to

$$\operatorname{div}(u) = 0. \tag{2.7}$$

This divergence constraint plays a very important role, especially for numerical simulations. If the divergence of the velocity is not approximated well enough, this may result in a poor (numerical) mass conservation and can lead to big discretization errors.

The second assumption concerns the definition of the stress tensor T and is called Newton's viscosity law. It states that the stress tensor linearly depends on the rate of strain. This relation, also called *constitutive equation*, has a big influence on the behaviour of the fluid. We refer to chapter 6 in [93] for a detailed discussion on this topic. Newton's viscosity law – we then speak of a Newtonian fluid – leads to the relation

$$T = -\mathrm{Id}p + \tau \quad \text{with} \quad \tau = \lambda \mathrm{Id}\,\mathrm{div}(u) + 2\mu\,\varepsilon(\mathrm{u}), \tag{2.8}$$

with the first and second viscosity coefficients λ and μ respectively, and the symmetric gradient given by

$$\varepsilon(\mathrm{u}) = \frac{1}{2}\left(\nabla u + (\nabla u)^{\mathrm{T}}\right).$$

Another common assumption considering Newtonian fluids is that they further fulfill the Stoke's assumption, which reads as $\lambda = -2/3\mu$.

2.2.1 The Navier-Stokes equations

Using relation (2.8) and the incompressibility constraint (2.7), we can simplify the divergence of the viscous stress tensor

$$\mathrm{div}(\tau) = \mathrm{div}\left(\frac{2}{3}\mu \mathrm{Id}\,\mathrm{div}(u) + 2\mu\,\varepsilon(\mathrm{u})\right) = \mathrm{div}(2\mu\,\varepsilon(\mathrm{u})).$$

The set of partial differential equations given by the momentum equation (2.6) (using the simplification of the previous step), and the incompressibility constraint (2.7) are called the incompressible Navier-Stokes equations

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u) - \mathrm{div}(2\mu\,\varepsilon(\mathrm{u})) + \mathrm{div}(\rho u \otimes u) + \nabla p &= \rho f, \\ \mathrm{div}(u) &= 0. \end{aligned}$$

Making the further assumption that the second viscosity μ is constant in space and time we can use the identity

$$\mathrm{div}(2\mu\,\varepsilon(\mathrm{u})) = 2\mu\,\mathrm{div}(\varepsilon(\mathrm{u})) = \mu\left(\Delta u + \nabla\,\mathrm{div}(u)\right) = \mu\Delta u.$$

Dividing by ρ then leads to

$$\frac{\partial u}{\partial t} - \nu\Delta u + \mathrm{div}(u \otimes u) + \nabla p = f, \tag{2.9a}$$

$$\mathrm{div}(u) = 0, \tag{2.9b}$$

with the kinematic viscosity $\nu := \mu/\rho$ and the scaled pressure (again denoted by p) $p := p/\rho$.

2.2.2 Creeping flows - The steady Stokes equations

A import characteristic number in fluid dynamics is the Reynolds number. It is defined by

$$\mathrm{Re} := \frac{UL}{\nu}, \tag{2.10}$$

where U and L are characteristic length and velocity scales. The Reynolds number is important as it can be interpreted as the ratio between inertia and viscous forces. For example a very small viscosity (compared to U and L) leads to a high Reynols number. The friction between fluid particles is then small and the acceleration initiated by inertia forces dominates. However, in a flow characterized by a small Reynolds number, the viscous effects are crucial. Such flows are often called creeping flows and are of practical importance. This has a great impact on the governing equations of fluid motion. Using a dimension analysis for the case when $\mathrm{Re} \to 0$ shows that the nonlinear term in (2.9a) vanishes, thus $\mathrm{div}(u \otimes u) \to 0$. The resulting set of partial equations is called the *unsteady Stokes equations*. If we make the further assumption that the flow is steady, thus does not change when time passes, we get the *steady Stokes equations* given by

$$\begin{aligned} -\nu \Delta u + \nabla p &= f, \\ \mathrm{div}(u) &= 0. \end{aligned} \tag{2.11}$$

These equations are of great interest as they fit in the mathematical concept of a saddle point problem. Although the full nonlinear setting of the Navier-Stokes equations (2.9) is generally applied, a proper (numerical) treatment of (2.11) is needed in order to understand the principles and effects that appear. This thesis only focuses on the steady Stokes equation and presents a new analysis of the mathematical structure of (2.11).

2.2.3 Boundary conditions

For the systems of partial differential equations introduced above we need suitable boundary conditions. Note that we only consider the steady case. In this work deal with two different types of boundary conditions. The first one is the case, where the fluid comes in contact with a wall. Obviously, we require that no fluid is going to pass through the wall. Assuming the wall moves with velocity u_{w} we impose the condition

$$u \cdot n = u_{\mathrm{w}} \cdot n. \tag{2.12}$$

This condition only influences the normal component of the velocity, but has no impact on the tangential velocity. This is mainly due to the different physical effects that are responsible for the boundary condition. If we think of a wall that comes in contact with the fluid, a common approach is to consider the wall as a part of the fluid. In chapter 2.1.2 we used a control volume and its surface to determine the forces produced by the fluid from the outside. Beside the pressure force viscous effects also occurred. Considering a wall, similar observations can be made. The viscous effects close to the wall create a force that holds the fluid particles and the wall together. They are, in other words, glued together

$$u \times n = u_{\mathrm{w}} \times n. \tag{2.13}$$

Today this *no-slip* condition is commonly accepted, although many different approaches were considered in the past. For a detailed discussion we refer to [93, chapter 6.4]. These two conditions together are called Dirichlet conditions.

The second type is called a Neumann boundary condition. At this point we do not want to have a lengthy discussion on the physical interpretation of these conditions and refer to the literature. A Neumann boundary condition is denoted by

$$(-\mathrm{Id}p + \tau) \cdot n = -P, \tag{2.14}$$

with an given force P. For more details we refer to [59, chapter 1.2].

3 Variational framework

3.1 Preliminaries, basic notations and functional spaces

In the following, we introduce the notation and establish properties of certain Sobolev spaces that we use throughout this work. For a more detailed discussion on this topic we refer to [2, 88, 11] and [53]. First, we introduce the notation $A \sim B$ to indicate that there exists constants $c, C > 0$ independent of the mesh size (as defined in chapter 5) h and the viscosity ν such that $cA \leq B \leq CA$. We also use $A \lesssim B$ when there exists a $C > 0$ independent of h and ν such that $A \leq CB$. In a similar manner we also define the symbol $\gtrsim$.

For the rest of the work let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be an open bounded subset such that the boundary $\Gamma := \partial\Omega$ is either

- smooth, i.e. $\Gamma \in C^\infty$,
- or piecewise smooth, i.e. we assume that there exists a (finite) decomposition of Γ into smooth Lipschitz boundary parts Γ_i such that $\Gamma = \bigcup_i \overline{\Gamma}_i$. Further, for each component Γ_i there exists an open Lipschitz domain $\Omega_i \subset \mathbb{R}^d$ such that

$$\overline{\Omega_i} \cap \overline{\Omega} = \Gamma_i \quad \text{and} \quad \Omega_i \cap \Omega = \emptyset,$$

  and Ω_i and Ω_j have a positive distance for $i \neq j$. Finally, we assume that the interior of $\overline{\Omega} \cup \overline{\Omega}_1 \cup \ldots$ is also a Lipschitz domain. Those are the same assumptions as in [65] and fit into the setting of [56].

Let $\mathcal{C}^k(\Omega, \mathbb{R})$ be the function space consisting of real-valued k-times continuously differentiable functions on Ω. Then we define $\mathcal{D}(\Omega, \mathbb{R}) := \mathcal{C}_0^\infty(\Omega, \mathbb{R})$ as the set of infinitely differentiable, compactly supported, real-valued functions on Ω and denote by $\mathcal{D}'(\Omega)$ the space of distributions. To inidicate vector and matrix-valued functions we include the range in the notation, thus $\mathcal{D}(\Omega, \mathbb{R}^d) := \{\phi : \Omega \to \mathbb{R}^d \text{ with } \phi_i \in \mathcal{D}(\Omega, \mathbb{R})\}$ and $\mathcal{D}(\Omega, \mathbb{R}^{d\times d}) := \{\phi : \Omega \to \mathbb{R}^{d\times d} \text{ with } \phi_{ij} \in \mathcal{D}(\Omega, \mathbb{R})\}$ indicate vector and matrix-valued infinitely differentiable, compactly supported, real-valued functions, respectively. This notation is extended to other functions spaces as needed. Whereas

$$L^2(\Omega, \mathbb{R}) := \{f : \int_\Omega |f|^2 \,\mathrm{d}x < \infty\} \tag{3.1}$$

denotes the space of square integrable functions with the inner product and the norm

$$(f, g)_{L^2(\Omega)} := \int_\Omega fg \,\mathrm{d}x, \qquad \|f\|^2_{L^2(\Omega)} := (f, f)_{L^2(\Omega)}, \qquad \forall f, g \in L^2(\Omega), \tag{3.2}$$

the spaces $L^2(\Omega, \mathbb{R}^d)$ and $L^2(\Omega, \mathbb{R}^{d\times d})$ denote its vector and matrix-valued versions. We also define the closed subspace

$$L_0^2(\Omega, \mathbb{R}) := \{f \in L^2(\Omega, \mathbb{R}) : \int_\Omega f \, \mathrm{d}x = 0\}.$$

At several points in the later chapters we make use of the local L^2-norm defined on subsets $\omega \subset \Omega$. For a better readability we introduce the following notation

$$\| \cdot \|_\omega := \| \cdot \|_{L^2(\omega)}.$$

Certain differential operators have different definitions depending on the context. We define the "curl" operator by

$$\begin{aligned}
&\mathrm{curl}(\phi) = (-\partial_2\phi, \partial_1\phi)^{\mathrm{T}}, && \text{for } \phi \in \mathcal{D}'(\Omega, \mathbb{R}) \text{ and } d = 2,\\
&\mathrm{curl}(\phi) = -\partial_2\phi_1 + \partial_1\phi_2, && \text{for } \phi \in \mathcal{D}'(\Omega, \mathbb{R}^2) \text{ and } d = 2,\\
&\mathrm{curl}(\phi) = (\partial_2\phi_3 - \partial_3\phi_2, \partial_3\phi_1 - \partial_1\phi_3, \partial_1\phi_2 - \partial_2\phi_1)^{\mathrm{T}} && \text{for } \phi \in \mathcal{D}'(\Omega, \mathbb{R}^3) \text{ and } d = 3,
\end{aligned}$$

where $(\cdot)^{\mathrm{T}}$ denotes the transpose and ∂_i abbreviates ∂/∂_i. Similarly, $\nabla\phi$ has different meanings depending on the context and results either in a vector $[\nabla\phi]_i = \partial_i\phi$ for $\phi \in \mathcal{D}'(\Omega, \mathbb{R})$ or in a matrix $[\nabla\phi]_{ij} = \partial_i\phi_j$ for $\phi \in \mathcal{D}'(\Omega, \mathbb{R}^d)$. Finally, we denote by $\mathrm{div}(\phi) = \sum_{i=1}^3 \partial_i\phi_i$ the standard divergence operator for $\phi \in \mathcal{D}'(\Omega, \mathbb{R}^d)$ and by $[\mathrm{div}(\phi)]_j = \sum_{i=1}^3 \partial_i\phi_{ji}$ the vector-valued divergence operator applied to $\phi \in \mathcal{D}'(\Omega, \mathbb{R}^{d\times d})$.

Let $\tilde{d} := d(d-1)/2$ (such that $\tilde{d} = 1$ and $\tilde{d} = 3$ for $d = 2$ and $d = 3$, respectively). The standard Sobolev spaces are denoted by

$$\begin{aligned}
H^1(\Omega, \mathbb{R}) &:= \{u \in L^2(\Omega, \mathbb{R}) : \|\nabla u\|_{L^2(\Omega)} < \infty\},\\
H^1(\Omega, \mathbb{R}^d) &:= \{u \in L^2(\Omega, \mathbb{R}^d) : \|\nabla u\|_{L^2(\Omega)} < \infty\},\\
H(\mathrm{div}, \Omega) &:= \{u \in L^2(\Omega, \mathbb{R}^d) : \|\mathrm{div}(u)\|_{L^2(\Omega)} < \infty\},\\
H(\mathrm{curl}, \Omega) &:= \{u \in L^2(\Omega, \mathbb{R}^d) : \|\mathrm{curl}(u)\|_{L^2(\Omega)} < \infty\},
\end{aligned}$$

with the associated norms given by $\| \cdot \|_{H^1(\Omega)}$, $\| \cdot \|_{H(\mathrm{div},\Omega)}$ and $\| \cdot \|_{H(\mathrm{curl},\Omega)}$, respectively. Note that we will not distinguish between the dimension of the ordinary Sobolev space in the definition of the norm, thus we use $\| \cdot \|_{H^1(\Omega)}$ as the symbol for the norm on $H^1(\Omega, \mathbb{R})$ and $H^1(\Omega, \mathbb{R}^d)$. In the same fashion we also denote the seminorms by $| \cdot |_{H^1(\Omega)}$, $| \cdot |_{H(\mathrm{div},\Omega)}$ and $| \cdot |_{H(\mathrm{curl},\Omega)}$. Sobolev spaces with higher regularity are similarly given by

$$\begin{aligned}
H^m(\Omega, \mathbb{R}) &:= \{u \in L^2(\Omega, \mathbb{R}) : \|\nabla^m u\|_{L^2(\Omega)} < \infty\},\\
H^m(\Omega, \mathbb{R}^d) &:= \{u \in L^2(\Omega, \mathbb{R}^d) : \|\nabla^m u\|_{L^2(\Omega)} < \infty\},\\
H^m(\mathrm{div}, \Omega) &:= \{u \in H^m(\Omega, \mathbb{R}^d) : \|\mathrm{div}(u)\|_{L^2(\Omega)} < \infty\},\\
H^m(\mathrm{curl}, \Omega) &:= \{u \in H^m(\Omega, \mathbb{R}^d) : \|\mathrm{curl}(u)\|_{L^2(\Omega)} < \infty\},
\end{aligned}$$

and we use the notation $\| \cdot \|_{H^m(\Omega)}$, $\| \cdot \|_{H^m(\mathrm{div},\Omega)}$ and $\| \cdot \|_{H^m(\mathrm{curl},\Omega)}$ for the corresponding norms. Note that the Sobolev spaces above can also be defined as the closure of $\mathcal{C}^\infty(\overline{\Omega}, \cdot)$

(for sufficiently smooth boundaries) with the according norms, see for example in [64] for spaces with more regularity and for the standard spaces [57, 58, 53]. The equivalence of those definitions is not trivial and goes back to the famous theorem of N. Meyers and J. Serrin, see [86]. A detailed proof can also be found in the book [44, 2].

We continue with the definition of appropriate Sobolev spaces on the boundary. As we are also dealing with non-smooth boundaries, we follow the notations and definitions given in [24] and [84]. In the case of a partially smooth domain and $d = 3$ we denote by e_{ij} the open edges of Γ. When Γ_i and Γ_j are two adjacent faces then e_{ij} is the common edge. By n we denote the piecewise smooth, outward unit normal on Ω. On the edge e_{ij} we define the parallel unit tangential vector by t_{ij}. Note that on each Γ_i, the three vectors given by $n_i := n|_{\Gamma_i}$, t_{ij} and $t_i = t_{ij} \times n_i$ form an orthonormal basis of $\mathbb{R}^3$ and the vectors t_i and t_{ij} are an orthonormal basis of the tangential plane. We denote by $\mathcal{F}(\Gamma_i)$ the set of indices j such that Γ_i and Γ_j have a common edge e_{ij}. In two dimensions the tangential vector t_i is simply given by rotating the normal vector by 90 degrees. Further, e_{ij} is the common vertex (and $\mathcal{F}(\Gamma_i)$ is defined correspondingly). Using the notations from above the space of square integrable functions on the boundary Γ is denoted by $L^2(\Gamma, \mathbb{R})$. Next, on each part of the boundary we define the space

$$H^{1/2}(\Gamma_i, \mathbb{R}) := \overline{\mathcal{C}^\infty(\overline{\Gamma_i})}^{\|\cdot\|_{H^{1/2}(\Gamma_i)}},$$

with

$$\|u\|_{H^{1/2}(\Gamma_i)}{}^2 := \int_{\Gamma_i}\int_{\Gamma_i} \frac{|u(x) - u(y)|^2}{|x - y|^d}\, \mathrm{d}s(x)\, \mathrm{d}s(y) + \int_{\Gamma_i} u^2\, \mathrm{d}s\,.$$

Similarly, we define the space $H^{1/2}(\Gamma, \mathbb{R})$. Using the notation

$$u_i \overset{1/2}{=} u_j \text{ at } e_{ij} \Leftrightarrow \int_{\Gamma_i}\int_{\Gamma_j} \frac{|u(x) - u(y)|^2}{|x - y|^d}\, \mathrm{d}s(x)\, \mathrm{d}s(y) < \infty,$$

for $u_i \in H^{1/2}(\Gamma_i, \mathbb{R})$ and $u_j \in H^{1/2}(\Gamma_j, \mathbb{R})$, there holds

$$u \in H^{1/2}(\Gamma, \mathbb{R}) \Leftrightarrow \begin{cases} u \in H^{1/2}(\Gamma_i, \mathbb{R}) & \forall i, \\ u|_{\Gamma_i} \overset{1/2}{=} u|_{\Gamma_j} & \forall i \neq j \text{ s.t. } \overline{\Gamma_i} \cap \overline{\Gamma_j} \neq \emptyset. \end{cases}$$

Beside the (equivalently defined) vector-valued functions spaces $L^2(\Gamma, \mathbb{R}^d), H^{1/2}(\Gamma_i, \mathbb{R}^d)$, $H^{1/2}(\Gamma, \mathbb{R}^d)$, we now further have for $d = 3$:

$$\begin{aligned}
L^2_t(\Gamma, \mathbb{R}^d) &:= \{u \in L^2(\Gamma, \mathbb{R}^d) : u \cdot n = 0\}, \\
H^{1/2}_-(\Gamma, \mathbb{R}^d) &:= \{u \in L^2_t(\Gamma, \mathbb{R}^d) : u|_{\Gamma_i} \in H^{1/2}(\Gamma_i, \mathbb{R}^d)\}, \\
H^{1/2}_\parallel(\Gamma, \mathbb{R}^d) &:= \{u \in H^{1/2}_-(\Gamma, \mathbb{R}^d) : u|_{\Gamma_j} \cdot t_{ij} \overset{1/2}{=} u|_{\Gamma_i} \cdot t_{ij} \forall j, \forall i \in \mathcal{F}(\Gamma_j)\}, \\
H^{1/2}_\perp(\Gamma, \mathbb{R}^d) &:= \{u \in H^{1/2}_-(\Gamma, \mathbb{R}^d) : u|_{\Gamma_j} \cdot t_j \overset{1/2}{=} u|_{\Gamma_i} \cdot t_i \forall i \in \mathcal{F}(\Gamma_j), \forall j\}.
\end{aligned}$$

The compatibility equation $u|_{\Gamma_j} \cdot t_{ij} \overset{1/2}{=} u|_{\Gamma_i} \cdot t_{ij}$ assures "continuity" of the tangential components across the edge of vector functions in the tangential planes on Γ_i and Γ_j.

Similarly, the compatibility equation $u|_{\Gamma_j} \cdot t_j \overset{1/2}{=} u|_{\Gamma_i} \cdot t_i$ assures "continuity" of the (in plane) normal component across the edge. The last two function spaces can also be defined on each Γ_i separately by $H_{\parallel}^{1/2}(\Gamma_i, \mathbb{R}^d) := \{u|_{\Gamma_i} : u \in H_{\parallel}^{1/2}(\Gamma, \mathbb{R}^d)\}$. In two dimensions there is no equivalent definition of the tangential vector t_{ij}, thus we simply set $H_{\parallel}^{1/2}(\Gamma, \mathbb{R}^d) = H_{-}^{1/2}(\Gamma, \mathbb{R}^d)$.

Next we introduce the following trace operators for smooth functions

$$\begin{aligned}
\gamma\phi &:= \phi|_\Gamma && \text{for } \phi \in C^1(\overline{\Omega}, \mathbb{R}), & \gamma_n\phi &:= \phi|_\Gamma \cdot n && \text{for } \phi \in C^1(\overline{\Omega}, \mathbb{R}^d),\\
\gamma_t\phi &:= \phi|_\Gamma \times n && \text{for } \phi \in C^1(\overline{\Omega}, \mathbb{R}^d), & \pi_t\phi &:= (\phi|_\Gamma - (\phi|_\Gamma \cdot n)n) && \text{for } \phi \in C^1(\overline{\Omega}, \mathbb{R}^d),\\
\gamma_{nn}\phi &:= \gamma_n(\phi|_\Gamma n)|_\Gamma && \text{for } \phi \in C^1(\overline{\Omega}, \mathbb{R}^{d\times d}), & \pi_{nt}\phi &:= \pi_t(\phi|_\Gamma n) && \text{for } \phi \in C^1(\overline{\Omega}, \mathbb{R}^d).
\end{aligned}$$

Note that in three dimensions there holds $\pi_t\phi = n \times (\phi \times n)|_\Gamma$ and that in two dimensions γ_t does not exist. In a similar manner we define the corresponding trace operators also on a part of the boundary symbolizing it with further indices, e.g. the tangential projection on the boundary Γ_i is denoted by $\pi_{t,i}$. For the ease of notation we omit the symbols of the corresponding trace operator if it is clear from the context, e.g. where ϕ_n, ϕ_t represent the normal part and the tangential projection (with respect to π_t) of a vector-valued function. Similarly, ϕ_{nn} and ϕ_{nt} are the normal-normal and the normal-tangential projection of a matrix-valued function.

In the following we introduce trace operators for Sobolev spaces. First, recall that γ can be extended to the Sobolev space $H^1(\Omega, \mathbb{R})$ such that

$$\gamma : H^1(\Omega, \mathbb{R}) \to H^{1/2}(\Gamma, \mathbb{R}),$$

is a linear, continuous and surjective operator. Keeping this in mind, we omit the notation including the symbol γ for a better readability. Note that the trace operator allows us to define an equivalent norm by

$$\|u\|_{H^{1/2}(\Gamma)} \sim \inf_{\substack{\phi \in H^1(\Omega,\mathbb{R})\\ \phi = u \text{ on } \Gamma}} \|\phi\|_{H^1(\Omega)}.$$

Next, we define the closed subspaces with vanishing trace

$$\begin{aligned}
H_0^1(\Omega, \mathbb{R}) &:= \{u \in H^1(\Omega, \mathbb{R}) : u = 0 \text{ on } \partial\Omega\},\\
H_{0,\Gamma_i}^1(\Omega, \mathbb{R}) &:= \{u \in H^1(\Omega, \mathbb{R}) : u = 0 \text{ on } \partial\Gamma_i\},
\end{aligned}$$

and similarly the vector-valued versions $H_0^1(\Omega, \mathbb{R}^d)$ and $H_{0,\Gamma_i}^1(\Omega, \mathbb{R}^d)$. Note that there holds a similar density argument

$$H_0^1(\Omega, \mathbb{R}) = \overline{\mathcal{C}_c^\infty(\Omega, \mathbb{R})}^{\|\cdot\|_{H^1(\Omega)}} \quad \text{and} \quad H_{0,\Gamma_i}^1(\Omega, \mathbb{R}) = \overline{\mathcal{C}_{0,\Gamma_i}^\infty(\Omega, \mathbb{R})}^{\|\cdot\|_{H^1(\Omega)}},$$

where $\mathcal{C}_{0,\Gamma_i}^\infty(\Omega, \cdot)$ denotes infinitely differentiable functions with compact support on $\overline{\Omega} \setminus \Gamma_i$. For the definition of further trace operators we first need some dual spaces. We use the superscript * in the case of a Hilbert space, whereas the dual spaces of the above defined

Sobolev spaces are simply defined using the well known notation with negative indices. Thus we have for example

$$H^{-1}(\Omega,\mathbb{R}) := [H_0^1(\Omega,\mathbb{R})]^* \quad \text{and} \quad H_{\Gamma_i}^{-1}(\Omega,\mathbb{R}) := [H_{0,\Gamma_i}^1(\Omega,\mathbb{R})]^*,$$

and similarly on the boundary

$$H^{-1/2}(\Gamma,\mathbb{R}) := [H^{1/2}(\Gamma,\mathbb{R})]^*.$$

Further we introduce the following notation: the action of a continuous linear functional f on an element g belonging to a topological space X is denoted by $\langle f,g\rangle_X$. We omit the subscript in $\langle\cdot,\cdot\rangle$ when it is obvious from the context. Then the corresponding dual norms are defined in the common way, as for example

$$\|u\|_{H^{-1/2}(\Gamma)} := \sup_{v\in H^{1/2}(\Gamma,\mathbb{R})} \frac{\langle u,v\rangle}{\|v\|_{H^{1/2}(\Gamma)}}.$$

For the Soblev space $H(\mathrm{div},\Omega)$ the appropriate trace operator is given by γ_n such that

$$\gamma_n : H(\mathrm{div},\Omega) \to H^{-1/2}(\Gamma,\mathbb{R}),$$

is a linear, continuous and surjective operator. We define the closed subspaces with vanishing normal trace

$$H_0(\mathrm{div},\Omega) := \{u \in H(\mathrm{div},\Omega) : \langle u\cdot n,\phi\rangle = 0 \;\forall\phi \in H^1(\Omega,\mathbb{R})\},$$
$$H_{0,\Gamma_i}(\mathrm{div},\Omega) := \{u \in H(\mathrm{div},\Omega) : \langle u\cdot n,\phi\rangle = 0 \;\forall\phi \in H^1_{0,\Gamma\setminus\overline{\Gamma}_i}(\Omega,\mathbb{R})\}.$$

For the trace operator of $H(\mathrm{curl},\Omega)$ we need further (dual) Sobolev spaces with even less regularity. For this let ∇_Γ, div_Γ and curl_Γ be the corresponding differential operators on the boundary (see chapter 3.1 in [24]). Then we define

$$H_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma) := \{u \in H_\parallel^{-1/2}(\Gamma,\mathbb{R}^d) : \mathrm{div}_\Gamma(u) \in H^{-1/2}(\Gamma,\mathbb{R})\},$$
$$H_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) := \{u \in H_\perp^{-1/2}(\Gamma,\mathbb{R}^d) : \mathrm{curl}_\Gamma(u) \in H^{-1/2}(\Gamma,\mathbb{R})\}.$$

The operators γ_t and π_t can be extended such that

$$\gamma_t : H(\mathrm{curl},\Omega) \to H_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma),$$
$$\pi_t : H(\mathrm{curl},\Omega) \to H_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma),$$

are linear, continuous and surjective operators. Notem that γ_t and π_t can also be applied to functions in $H^1(\Omega,\mathbb{R}^d)$ such that $\gamma_t : H^1(\Omega,\mathbb{R}^d) \to H_\perp^{1/2}(\Gamma,\mathbb{R}^d)$ and $\pi_t : H^1(\Omega,\mathbb{R}^d) \to H_\parallel^{1/2}(\Gamma,\mathbb{R}^d)$ are linear, continuous and surjective.

Trace operators for Sobolev spaces with higher regularity were also analyzed in the literature. In the case of $H^m(\Omega,\mathbb{R})$ we refer for example to [57, 58]. However, in this work a special interest lies on the traces of functions in $H^1(\mathrm{curl},\Omega)$, which were, to the best knowledge of the author, so far not analyzed in detail. Obviously, as this is a subspace of $H^1(\Omega, \mathbb{R}^d)$, the above trace operators γ and π_t (and γ_t for $d = 3$) are all linear and continuous. Next we proceed similar to chapter 3.1 in [24] to define another trace. To this end let $\operatorname{curl}_{\pi_{t,j}} := \pi_{t,j}(\operatorname{curl}(u))$ for all $u \in H^1(\operatorname{curl}, \Omega)$, where $\pi_{t,j} u := u - (u \cdot n_j) n_j$ with the unit normal vector n_j of the domain Ω on Γ_j. This yields $\operatorname{curl}_{\pi_{t,j}} : H^1(\operatorname{curl}, \Omega) \to H^{1/2}(\Gamma_j, \mathbb{R}^3)$. Next we define the operator $\operatorname{curl}_{\pi_t}$ as

$$\operatorname{curl}_{\pi_t} : H^1(\operatorname{curl}, \Omega) \to H_-^{1/2}(\Gamma, \mathbb{R}^d), \quad \operatorname{curl}_{\pi_t}(x) = \operatorname{curl}_{\pi_{t,j}}(x) \quad \text{a.e.} \quad x \in \Gamma_j. \tag{3.3}$$

Then there holds the following identity

$$\operatorname{curl}_{\pi_t} u = \pi_t(\operatorname{curl}(u)).$$

In the case of a smooth boundary the tangential projections π_t maps functions from $H^1(\Omega, \mathbb{R}^d)$ onto $H_\parallel^{1/2}(\Gamma, \mathbb{R}^3)$. This also assures the continuity (but not surjectivity!)

$$\operatorname{curl}_{\pi_t} : H^1(\operatorname{curl}, \Omega) \to H_\parallel^{1/2}(\Gamma, \mathbb{R}^d).$$

In the case of a piecewise smooth boundary we only have that the operator $\operatorname{curl}_{\pi_t}$ maps functions from $H^1(\operatorname{curl}, \Omega)$ onto $H_-^{1/2}(\Gamma_i, \mathbb{R}^d)$.

Finally, similarly to the differential operators above, we define the operator $\operatorname{skw}(\cdot)$ depending on the context. To this end let $\phi \in \mathcal{D}'(\Omega, \mathbb{R})$ and $\psi \in \mathcal{D}'(\Omega, \mathbb{R}^3)$ then we have

$$\operatorname{skw}(\phi) = \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix}, \quad \text{and} \quad \operatorname{skw}(\psi) = \begin{pmatrix} 0 & \psi_3 & -\psi_2 \\ -\psi_3 & 0 & \psi_1 \\ \psi_2 & -\psi_1 & 0 \end{pmatrix}.$$

For matrix valued functions $\phi \in \mathcal{D}'(\Omega, \mathbb{R}^{d \times d})$ we simply $\operatorname{skw}(\phi) = \frac{1}{2}\phi - \phi^{\mathrm{T}}$.

Next we introduce an important regular decomposition result for the space $H(\operatorname{div}, \Omega)$ including different boundary conditions. The result is well known and can be found for example in [56]. Note that in the two-dimensional case the result follows from the corresponding decomposition of $H(\operatorname{curl}, \Omega)$ functions and by the equivalence $\operatorname{curl}(\phi) = \nabla^\perp \phi$ for $\phi \in \mathcal{D}'(\Omega, \mathbb{R})$.

Theorem 1 (Regular decomposition of $H(\operatorname{div})$-functions)**.** *Let $\Gamma_i \subset \Gamma$ be a part of the boundary Γ defined at the beginning of this chapter. For $u \in H_{0,\Gamma_i}(\operatorname{div}, \Omega)$ there exist functions $\phi \in H^1_{0,\Gamma_i}(\Omega, \mathbb{R}^{\tilde{d}})$ and $z \in H^1_{0,\Gamma_i}(\Omega, \mathbb{R}^d)$ such that*

$$u = \operatorname{curl}(\phi) + z,$$

with the stability estimates

$$\|\phi\|_{H^1(\Omega)} \le c \|u\|_{H(\operatorname{div}, \Omega)} \quad \textit{and} \quad \|z\|_{H^1(\Omega)} \le c \|\operatorname{div}(u)\|_{L^2(\Omega)}.$$

We conclude this section by introducing some important inequalities.

Theorem 2 (Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be an arbitrary bounded and connected Lipschitz domain with* $\operatorname{diam}(\Omega) = 1$. *For a function $u \in H^1(\Omega)$ there holds*

$$\|u\|^2_{H^1(\Omega)} \le c_P \left(|u|^2_{H^1(\Omega)} + \left(\int_\Omega u \, \mathrm{d}x \right)^2 \right),$$

where c_p only depends on the shape of Ω.

Theorem 3 (Friedrichs inequality)**.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be an arbitrary bounded and connected Lipschitz domain with $\operatorname{diam}(\Omega) = 1$. Let $\Gamma_D \subset \partial\Omega$ be of positive measure $|\Gamma_D| > 0$. For functions $u \in H^1_{0,\Gamma_D}(\Omega)$ and $v \in H^2_{0,\Gamma_D}(\Omega)$ there holds*

$$\|u\|_{H^1(\Omega)} \leq c_F |u|_{H^1(\Omega)} \quad \text{and} \quad \|v\|_{H^2(\Omega)} \leq c_F |v|_{H^2(\Omega)},$$

where c_F only depends on the shape of Ω.

Theorem 4 (Korn inequality)**.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be an arbitrary bounded and connected Lipschitz domain. For $u \in H^1(\Omega, \mathbb{R}^d)$ there holds*

$$\|\varepsilon(u)\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \geq c_k \|u\|^2_{H^1(\Omega)},$$

where the constant c_k depends on the domain Ω. Now let $\Gamma_D \subset \partial\Omega$ be of positive measure $|\Gamma_D| > 0$, and let $u \in H^1_{0,\Gamma_D}(\Omega, \mathbb{R}^d)$, then

$$\|\varepsilon(u)\|^2_{L^2(\Omega)} \geq c_k \|\nabla u\|^2_{L^2(\Omega)}.$$

Proof. For a detailed proof for a smooth boundary we refer to chapter 3.3 in [39], and for non-smooth boundaries see [15]. □

3.2 Abstract theory

In this section we discuss the abstract theory of coercive and saddle point problems. We derive necessary and sufficient conditions yielding existence and uniqueness. For coercive problems this leads to the theorem of Lax-Milgram. For saddle point problems, often encountered when applying the method of Lagrangian multipliers, we quote the well known *LBB-condition* (named after Olga Alexandrowna Ladyschenskaja, Ivo Babuška and Franco Brezzi), and Brezzi's theorem.

Let V be a Hilbert space with the inner product and norm given by $(\cdot,\cdot)_V$ and $||\cdot||_V$, respectively. On the space V we define the bilinear form $a : V \times V \to \mathbb{R}$ and the linear form $F : V \to \mathbb{R}$. We want to solve the problem: Find $u \in V$ such that

$$a(u, v) = F(v) \quad \forall v \in V. \tag{3.4}$$

Theorem 5 (Lax-Milgram)**.** *Let V be a Hilbert space and let $a : V \times V \to \mathbb{R}$ be a symmetric, continuous and coercive bilinear form, thus there exist constants α and β such that*

$$\begin{aligned} a(u, v) &\leq \beta ||u||_V ||v||_V \quad && \forall u, v \in V, \\ a(u, u) &\geq \alpha ||u||^2_V && \forall u \in V. \end{aligned}$$

Then, for every $F \in V^*$ there exists a unique solution fulfilling (3.4), and there holds the stability estimate

$$||u||_V \leq \frac{1}{\alpha} ||F||_{V^*}.$$

Next, we present the conditions needed to guarantee existence and uniqueness of saddle point problems. These problems often appear if we deal with mixed finite element methods and can also be interpreted as a minimization problem with certain constraints. One of the most famous examples is given by the variational formulation of the steady Stokes equations (2.11). Again, we focus on a more general setting. Beside the Hilbert space V we further define the Hilbert space Q with the scalar product $(\cdot,\cdot)_Q$ and the norm $||\cdot||_Q$. Further set $X := V \times Q$ and $\|(u,p)\|_X^2 := \|u\|_V^2 + \|p\|_Q^2$ for all $(u,p) \in X$. Next, we define the bilinear form $b : V \times Q \to \mathbb{R}$ and the bilinear form

$$B : X \times X \to \mathbb{R}, \quad B((u,p),(v,q)) := a(u,v) + b(v,p) + b(u,q).$$

Finally, let $G : Q \to \mathbb{R}$ be a linear form on Q. Then we have the abstract problem: Find $(u,p) \in X$ such that

$$B((u,p),(v,q)) = F(v) + G(q) \quad \forall (v,q) \in X. \tag{3.5}$$

Of course, an equivalent formulation is given by: Find $(u,p) \in V \times Q$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= F(v) \qquad \forall v \in V, \\ b(u,q) &= G(v) \qquad \forall q \in Q. \end{aligned}$$

There are several different conditions (depending on the structure and the properties of the system), which lead to existence and uniqueness of solutions of equation 3.5 and equation 3.6. For a detailed discussion on this topic we refer to chapter 4.2 in [11].

In this work we only consider the case where the bilinear form B (respectively a) is symmetric. In the famous paper [19], F. Brezzi gives sharp conditions on the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ that lead to existence and uniqueness. For this we define the kernel K_b with respect to the bilinear form b by $K_b := \{u \in V : b(u,q) = 0 \ \forall q \in Q\}$.

Theorem 6 (Brezzi)**.** *Let V and Q be two Hilbert spaces. Let $a : V \times V \to \mathbb{R}$ be a symmetric and continuous bilinear form, and let $b : V \times Q \to \mathbb{R}$ be a continuous bilinear form, thus*

$$\begin{aligned} a(u,v) &\le c_1 \|u\|_V \|v\|_V \quad \forall u,v \in V, \\ b(u,q) &\le c_2 \|u\|_V \|q\|_Q \quad \forall u \in V, \forall q \in Q. \end{aligned}$$

Further assume that $a(\cdot,\cdot)$ is coercive on the kernel K_b

$$a(u,u) \ge \alpha \|u\|_V^2 \quad \forall u \in K_b,$$

and that the LBB-condition is fulfilled, thus for all $q \in Q$ with $q \neq 0$ there holds

$$\sup_{\substack{u \in V \\ u \neq 0}} \frac{b(u,q)}{\|u\|_V \|q\|_Q} \ge \beta > 0. \tag{3.6}$$

Then equation 3.6 has a unique solution and there holds

$$\|u\|_V \le \frac{1}{\alpha}\|F\|_{V^*} + \frac{2c_1}{\alpha\beta}\|G\|_{Q^*} \quad \textit{and} \quad \|q\|_Q \le \frac{2c_1}{\alpha\beta}\|F\|_{V^*} + \frac{2c_1^2}{\alpha\beta^2}\|G\|_{Q^*}.$$

4 A new variational formulation of the Stokes equations

In the first chapter we introduced a variational formulation of the steady Stokes equations (2.11) in a velocity pressure setting. This is the most common formulation and was analyzed in detail over the last decades. The velocity pressure formulation includes the Sobolev space $H_0^1(\Omega, \mathbb{R}^d)$ as the corresponding space for the velocity and the space $L_0^2(\Omega, \mathbb{R})$ for the pressure (in the case of homogeneous Dirichlet boundary conditions). The goal of this chapter is to derive a new formulation, which demands less regularity for the velocity space. For this we introduce a new function space, the $H(\operatorname{curl}\operatorname{div})$, and present a detailed analysis. The derivation of the new method and the introduction of the new function space is based on the work [55].

Before we start with the derivation of a formally equivalent definition of the Stokes system we present the problem we consider for the rest of this work, including all different types of boundary conditions. To keep the problem as generic as possible, we assume that the boundary Γ can be split into four parts. To this end let $\Gamma_{D,n}, \Gamma_{D,t}, \Gamma_{N,n}, \Gamma_{N,t} \subset \Gamma$ such that

$$\Gamma_{D,n} \cup \Gamma_{N,n} = \Gamma \quad \text{and} \quad \Gamma_{D,t} \cup \Gamma_{N,t} = \Gamma,$$

and further assume that either $\Gamma_{D,n} = \Gamma$ or $\Gamma_{D,n} \cap \Gamma_{D,t} \neq \emptyset$. Then, the four different parts are given by

$$\Gamma = (\Gamma_{D,n} \cap \Gamma_{D,t}) \cup (\Gamma_{D,n} \cap \Gamma_{N,t}) \cup (\Gamma_{D,t} \cap \Gamma_{N,n}) \cup (\Gamma_{N,n} \cap \Gamma_{N,t}). \tag{4.1}$$

For a better understanding we illustrated an example in Figure 4.1. Note however, that we do not assume that the parts $\Gamma_{D,n}, \Gamma_{D,t}, \Gamma_{N,n}$ and $\Gamma_{N,t}$ are connected.

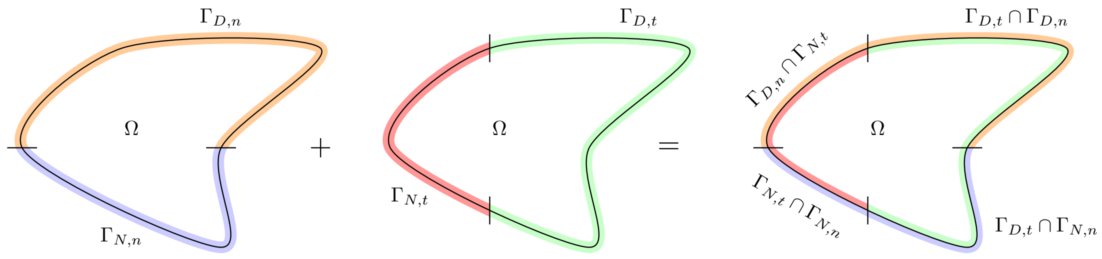

Figure 4.1: Illustration of the assumed boundary splitting.

Similar to the natural splitting of the Dirichlet boundary conditions into a normal and a tangential part as described in section 2.2.3, we can also split the Neumann-like boundary conditions. To this end note that there holds the identity

$$\mathrm{Id}_{nt} = \pi_t(\mathrm{Id}n) = \pi_t(n) = n - (n \cdot n)n = n - n = 0, \tag{4.2}$$

thus equation (2.14) allows the splitting

$$-\nu(\nabla u)_{nn} + p = P_n, \quad \text{and} \quad -\nu(\nabla u)_{nt} = P_t.$$

Based on this findings, let f be a given volume force, let $g_{D,n}$ and $g_{D,t}$ be given Dirichlet boundary values on $\Gamma_{D,n}$ and $\Gamma_{D,t}$, respectively, and similarly let $g_{N,n}$ and $g_{N,t}$ be given Neumann boundary values on $\Gamma_{D,n}$ and $\Gamma_{D,t}$. Then, we seek for a solution u and p of the problem

$$-\nu\Delta u + \nabla p = f \quad \text{in} \quad \Omega, \tag{4.3a}$$
$$\mathrm{div}(u) = 0 \quad \text{in} \quad \Omega, \tag{4.3b}$$
$$u_n = g_{D,n} \quad \text{on} \quad \Gamma_{D,n}, \tag{4.3c}$$
$$u_t = g_{D,t} \quad \text{on} \quad \Gamma_{D,t}, \tag{4.3d}$$
$$-\nu(\nabla u)_{nn} + p = g_{N,n} \quad \text{on} \quad \Gamma_{N,n}, \tag{4.3e}$$
$$-\nu(\nabla u)_{nt} = g_{N,t} \quad \text{on} \quad \Gamma_{N,t}. \tag{4.3f}$$

Defining the spaces

$$X_D := \{u \in H^1(\Omega, \mathbb{R}^d) : u_n = g_{D,n} \text{ on } \Gamma_{D,n}, u_t = g_{D,t} \text{ on } \Gamma_{D,t}\},$$
$$X_0 := \{u \in H^1(\Omega, \mathbb{R}^d) : u_n = 0 \text{ on } \Gamma_{D,n}, u_t = 0 \text{ on } \Gamma_{D,t}\},$$

and assuming $f \in L^2(\Omega, \mathbb{R}^d)$ and enough regularity of boundary data, the classical variational formulation of (4.3) then reads as: Find (u, p) in $X_D \times L^2(\Omega)$ such that

$$\left\{ \begin{aligned} \int_\Omega \nu\nabla u : \nabla v \,\mathrm{d}x - \int_\Omega \mathrm{div}(v) p \,\mathrm{d}x &= \int_\Omega f \cdot v \,\mathrm{d}s - \int_{\Gamma_{N,n}} g_{N,n} v_n \,\mathrm{d}s - \int_{\Gamma_{N,t}} g_{N,t} \cdot v_t \,\mathrm{d}s, && \forall v \in X_0, \\ \int_\Omega \mathrm{div}(u) q \,\mathrm{d}x &= 0, && \forall q \in L^2(\Omega). \end{aligned} \right. \tag{4.4}$$

Note, that in the case $\Gamma_{D,n} = \Gamma$, the pressure space is exchanged with $L^2_0(\Omega)$. For existence and uniqueness of solutions of (4.4) we refer to [18, 19, 15, 88, 43, 53].

4.1 A stress formulation for the Stokes equations

In the following we apply several modifications to the equations (4.3). In a first step we proceed similarly as in chapter 2 and introduce a new variable for the viscous fluid stress by $\sigma := \nu\nabla u$. Then, equation (4.3a) reformulates to (including a scaling by -1)

$$\mathrm{div}(\nu\sigma) - \nabla p = -f \quad \text{in} \quad \Omega.$$

Next, we use the trace of a matrix $\operatorname{tr}(\tau) := \sum_{i+1}^{d} \tau_{ii}$ to define the deviatoric part by

$$\operatorname{dev}(\tau) := \tau - \frac{\operatorname{tr}(\tau)}{d}\mathrm{Id}.$$

We observe that due to equation (4.3b) we have the identity

$$\operatorname{dev}(\sigma) = \operatorname{dev}(\nu\nabla u) = \nu\nabla u - \frac{\nu}{d}\operatorname{tr}(\nabla u)\mathrm{Id} = \nu(\nabla u - \frac{1}{d}\operatorname{div}(u)\mathrm{Id}) = \nu\nabla u, \tag{4.5}$$

thus σ actually only represents the deviatoric part of the velocity gradient. Hence, we can reformulate (4.3) to define the mixed stress formulation of the Stokes equations given by

$$\frac{1}{\nu}\operatorname{dev}(\sigma) - \nabla u = 0 \quad \text{in} \quad \Omega, \tag{4.6a}$$
$$\operatorname{div}(\sigma) - \nabla p = -f \quad \text{in} \quad \Omega, \tag{4.6b}$$
$$\operatorname{div}(u) = 0 \quad \text{in} \quad \Omega, \tag{4.6c}$$
$$u_n = g_{D,n} \quad \text{on} \quad \Gamma_{D,n}, \tag{4.6d}$$
$$u_t = g_{D,t} \quad \text{on} \quad \Gamma_{D,t}, \tag{4.6e}$$
$$-\sigma_{nn} + p = g_{N,n} \quad \text{on} \quad \Gamma_{N,n}, \tag{4.6f}$$
$$-\sigma_{nt} = g_{N,t} \quad \text{on} \quad \Gamma_{N,t}. \tag{4.6g}$$

There are several observations we can make studying the structure of this set of equations. Whereas formulations (4.6) and (4.3) are formally equivalent, the respective variational formulations demand a different regularity of the velocity. For the ease of notation we assume in the following discussion that $\Gamma_{D,n} = \Gamma_{D,t} = \Gamma$ and $g_{D,n} = g_{D,t} = 0$. Testing equation (4.3a) with a velocity test function v, integrating over the domain and integrating by parts leads to the well known bilinear form

$$\int_\Omega \nu\nabla u : \nabla v \,\mathrm{d}x,$$

which also appears in the variational formulation (4.4). As it is well known, the adequate space for the velocity, such that the above integral is well defined, is given by $H_0^1(\Omega, \mathbb{R}^d)$. In a similar manner we proceed with formulation (4.6). To this end we test the first equation (4.6a) with a stress function τ, integrate over the domain and integrate by parts in the second integral. As all boundary terms vanish, the second integral then reads as

$$\int_\Omega \operatorname{div}(\tau) \cdot u \,\mathrm{d}x\,. \tag{4.7}$$

Including the weak formulation of the divergence constraint (4.6c)

$$\int_\Omega \operatorname{div}(u) q \,\mathrm{d}x,$$

where q is an appropriate pressure test function, we conclude that an appropriate space for the velocity is given by $H_0(\operatorname{div}, \Omega)$. However, proceeding this way, the reduced regularity property of the velocity space was only *shifted* to the stress space, as well-posedness of the integral (4.7) demands that the divergence applied to each row of τ has to be in $L^2(\Omega)$. This is the motivation to define a new function space that interprets (4.7) in a less regular setting. The construction of this space is the topic of the next section.

We desire less regularity for the velocity space, because the space $H_0(\mathrm{div}, \Omega)$ seems to naturally fit the incompressibility constraint, especially in a discrete setting. This is discussed in more detail in chapter 5. There we see, that it is possible to define the discrete velocity space conformingly with respect to $H_0(\mathrm{div}, \Omega)$. This then leads to exactly divergence free velocity approximations eliminating an effect in the literature known as "poor mass conservation".

For the rest of this chapter we always assume that $\Gamma_{D,n} = \Gamma$, but we allow the case $\Gamma_{D,t} \neq \Gamma$, see equation (4.1). This assumption is needed in order to present a rigorous stability analysis of the variational formulation introduced in section 4.3. The solvability (and precise definition) in the continuous setting in the case $\Gamma_{D,n} \neq \Gamma$ is an open question and is discussed in the outlook chapter 9. Note however, that above assumption is **not** applied for the discrete methods introduced in chapter 6 and 7. In chapter 8 we further present several numerical examples providing optimal convergence rates including all different types of boundary conditions as discussed above.

Remark 1. *We want to give another motivation why the regularity assumptions* $\mathrm{div}(\tau) \in L^2(\Omega, \mathbb{R}^d)$ *and* $u \in H(\mathrm{div}, \Omega)$ *are not optimal. An appropriate discrete* ***conforming*** *approximation would demand normal continuity of each row of the discrete approximations* τ_h *and of* u_h*. Thus, in the lowest order case and in two dimensions the resulting number of coupling degrees of freedoms equals* 3*, and not* 2 *as it is demanded for example by the (optimal) method in [81, 76]. Note, that the discrete method introduced in chapter 6 is optimal with respect to the number of coupling degrees of freedoms.*

4.2 The stress space $H(\mathrm{curl}\,\mathrm{div})$

This section is dedicated to the derivation of a new function space, which is needed to formulate an appropriate variational formulation of (4.6) as discussed in section 4.1. Our aim is to reduce the increased regularity of a stress function as it would be needed such that (4.7) is well-posed. We reduce this regularity by demanding that $\mathrm{div}(\sigma)$ can only continuously act – in the sense of a linear functional – on functions in $H_0(\mathrm{div}, \Omega)$. Thus, for given functions σ and $v \in H_0(\mathrm{div}, \Omega)$ the term $\langle \sigma, v \rangle_{H_0(\mathrm{div},\Omega)}$ should be well defined. To this end we introduce the new function space

$$H(\mathrm{curl}\,\mathrm{div}, \Omega) := \{\sigma \in L^2(\Omega, \mathbb{R}^{d \times d}) : \mathrm{div}(\sigma) \in H_0(\mathrm{div}, \Omega)^*\}. \tag{4.8}$$

Obviously, this definition allows us to define a natural norm on $H(\mathrm{curl}\,\mathrm{div}, \Omega)$ by

$$\begin{aligned} \|\sigma\|^2_{\mathrm{cd}} :&= \|\sigma\|^2_{L^2(\Omega)} + \|\,\mathrm{div}(\sigma)\|^2_{H_0(\mathrm{div})^*} \\ &= \|\sigma\|^2_{L^2(\Omega)} + \left(\sup_{v \in H_0(\mathrm{div})} \frac{\langle \mathrm{div}(\sigma), v \rangle_{H_0(\mathrm{div})}}{\|v\|_{H_0(\mathrm{div})}} \right)^2. \end{aligned} \tag{4.9}$$

In the following three sections, 4.2.1, 4.2.2 and 4.2.3, we discuss certain properties of the $H(\operatorname{curl}\operatorname{div},\Omega)$ space. Based on these findings we then finally introduce a new variational formulation in section 4.3.

4.2.1 An equivalent definition

In this section we discuss an equivalent definition of the $H(\operatorname{curl}\operatorname{div},\Omega)$ space. The equivalence is based on proving that the dual space of $H_0(\operatorname{div})$ is topologically and algebraically equivalent to the space

$$H^{-1}(\operatorname{curl},\Omega) := \{\phi \in H^{-1}(\Omega,\mathbb{R}^d) : \operatorname{curl}(\phi) \in H^{-1}(\Omega,\mathbb{R}^{\tilde{d}})\}. \tag{4.10}$$

The proof was first given in [69] and is strongly related to the decomposition of functions in $H_0(\operatorname{div})$, see theorem 1. We start by showing the following lemma.

Lemma 1. *Let $F \in H_0(\operatorname{div},\Omega)^*$ be an arbitrary functional. Then F is also an element of $H^{-1}(\operatorname{curl},\Omega)$ and for all $v \in H_0^1(\Omega,\mathbb{R}^{\tilde{d}})$ there holds the equivalence*

$$\langle \operatorname{curl}(F), v\rangle_{H_0^1(\Omega,\mathbb{R}^{\tilde{d}})} = \langle F, \operatorname{curl}(v)\rangle_{H_0(\operatorname{div},\Omega)}.$$

Proof. For any $F \in H_0(\operatorname{div},\Omega)^*$, by the Riesz representation theorem, there exists a function $q^F \in H_0(\operatorname{div},\Omega)$ such that for all $v \in H_0(\operatorname{div},\Omega)$ we have

$$\langle F, v\rangle_{H_0(\operatorname{div},\Omega)} = (q^F, v)_{L^2(\Omega)} + (\operatorname{div}(q^F), \operatorname{div}(v))_{L^2(\Omega)}. \tag{4.11}$$

If we replace v by a function in $\mathcal{D}(\Omega,\mathbb{R}^d)$, we conclude that F is actually the distribution $F = q^F - \nabla\operatorname{div}(q^F) \in H^{-1}(\Omega,\mathbb{R}^d)$. This implies that $\operatorname{curl}(F) = \operatorname{curl}(q^F) \in H^{-1}(\Omega,\mathbb{R}^{\tilde{d}})$, and thus $F \in H^{-1}(\operatorname{curl},\Omega)$. Now let $\phi \in \mathcal{D}(\Omega,\mathbb{R}^{\tilde{d}})$, then we have

$$\langle \operatorname{curl}(F), \phi\rangle_{H_0^1(\Omega,\mathbb{R}^{\tilde{d}})} = \langle \operatorname{curl}(q^F), \phi\rangle_{H_0^1(\Omega,\mathbb{R}^{\tilde{d}})} = \langle \operatorname{curl}(q^F), \phi\rangle_{\mathcal{D}(\Omega,\mathbb{R}^{\tilde{d}})} = (q^F, \operatorname{curl}(\phi))_{L^2(\Omega)}.$$

As smooth functions $\mathcal{D}(\Omega,\mathbb{R}^{\tilde{d}})$ are dense in $H_0^1(\Omega,\mathbb{R}^{\tilde{d}})$ we obtain

$$\langle \operatorname{curl}(F), v\rangle_{H_0^1(\Omega,\mathbb{R}^{\tilde{d}})} = (q^F, \operatorname{curl}(v))_{L^2(\Omega)}$$

for all $v \in H_0^1(\Omega,\mathbb{R}^{\tilde{d}})$. As $\operatorname{curl}(v) \in H_0(\operatorname{div},\Omega)$ for $v \in H_0^1(\Omega,\mathbb{R}^{\tilde{d}})$, we conclude by (4.11). □

Using this lemma we can now show the desired equivalence.

Theorem 7. *The equality*

$$H_0(\operatorname{div},\Omega)^* = H^{-1}(\operatorname{curl},\Omega) \tag{4.12}$$

holds algebraically and topologically.

Proof. In lemma 1 we have already proven that $H_0(\mathrm{div},\Omega)^* \subseteq H^{-1}(\mathrm{curl},\Omega)$. To show that $H^{-1}(\mathrm{curl},\Omega) \subseteq H_0(\mathrm{div},\Omega)^*$, let $g \in H^{-1}(\mathrm{curl},\Omega)$. Using the regular decomposition, see theorem 1, we find for each function $v \in H_0(\mathrm{div},\Omega)$ functions $\phi_v \in H_0^1(\Omega,\mathbb{R}^{\tilde{d}})$ and $z_w \in H_0^1(\Omega,\mathbb{R}^d)$ such that $v = \mathrm{curl}(\phi_v) + z_v$ and

$$\|\phi_v\|_{H^1(\Omega)} \lesssim \|u\|_{H(\mathrm{div},\Omega)} \quad \text{and} \quad \|z_v\|_{H^1(\Omega)} \lesssim \|\mathrm{div}(u)\|_{L^2(\Omega)}.$$

By this stability result we can define the functional $G \in H_0(\mathrm{div},\Omega)^*$ by

$$\langle G, v\rangle_{H_0(\mathrm{div},\Omega)} := \langle \mathrm{curl}(g), \phi_v\rangle_{H_0^1(\Omega,\mathbb{R}^{\tilde{d}})} + \langle g, z_v\rangle_{H_0^1(\Omega,\mathbb{R}^d)}. \tag{4.13}$$

Note that by lemma 1 it follows that G is also an element of $H^{-1}(\mathrm{curl},\Omega)$. It suffices to show that G coincides with g as an element of $H^{-1}(\Omega,\mathbb{R}^d)$ as this then trivially also implies that they coincide in $H^{-1}(\mathrm{curl},\Omega)$. To this end, let $w \in H_0^1(\Omega,\mathbb{R}^d)$. Since $H_0^1(\Omega,\mathbb{R}^d)$ is continuously embedded in $H_0(\mathrm{div},\Omega)$, we have the equality $\langle G, w\rangle_{H_0^1(\Omega,\mathbb{R}^d)} = \langle G, w\rangle_{H_0(\mathrm{div},\Omega)}$. Next, as $w \in H_0^1(\Omega,\mathbb{R}^d) \subset H_0(\mathrm{div},\Omega)$, we can also use the regular decomposition for w. Since both w and z_w are in $H_0^1(\Omega,\mathbb{R}^d)$ the equality $w = \mathrm{curl}(\phi_w) + z_w$ implies that $\mathrm{curl}(\phi_w) \in H_0^1(\Omega,\mathbb{R}^d)$, thus there is a $C > 0$ independent of w such that

$$\|\mathrm{curl}(\phi_w)\|_{H^1(\Omega)} \le C\|w\|_{H^1(\Omega)}. \tag{4.14}$$

Using this decomposition we have

$$\langle G, w\rangle_{H_0^1(\Omega,\mathbb{R}^d)} = \langle \mathrm{curl}(g), \phi_w\rangle_{H_0^1(\Omega,\mathbb{R}^{\tilde{d}})} + \langle g, z_w\rangle_{H_0^1(\Omega,\mathbb{R}^d)}.$$

Now let $w_n \in \mathcal{D}(\Omega,\mathbb{R}^d)$ converge to w in $H_0^1(\Omega,\mathbb{R}^d)$ as $n \to \infty$, and further define the regular decomposition $w_n = \mathrm{curl}(\phi_{w_n}) + z_{w_n}$. By the construction of the regular decomposition components (see e.g., [36]), ϕ_{w_n} is an element of $\mathcal{D}(\Omega,\mathbb{R}^{\tilde{d}})$. Moreover we have the equivalence

$$\begin{aligned}\langle \mathrm{curl}(g), \phi_{w_n}\rangle_{H_0^1(\Omega,\mathbb{R}^{\tilde{d}})} &= \langle \mathrm{curl}(g), \phi_{w_n}\rangle_{\mathcal{D}(\Omega,\mathbb{R}^{\tilde{d}})} \\ &= \langle g, \mathrm{curl}(\phi_{w_n})\rangle_{\mathcal{D}(\Omega,\mathbb{R}^d)} = \langle g, \mathrm{curl}(\phi_{w_n})\rangle_{H_0^1(\Omega,\mathbb{R}^d)}.\end{aligned}$$

Since $\mathrm{curl}(g)$ is in $H^{-1}(\Omega,\mathbb{R}^d)$, the left-most term converges to $\langle \mathrm{curl}(g), \phi_w\rangle_{H_0^1(\Omega,\mathbb{R}^{\tilde{d}})}$. The right-most term must converge to $\langle g, \mathrm{curl}(\phi_w)\rangle_{H_0^1(\Omega,\mathbb{R}^{\tilde{d}})}$ because equation (4.14) implies that $\|\mathrm{curl}(\phi_{w_n} - \phi_w)\|_{H^1(\Omega)} \to 0$. Thus $\langle \mathrm{curl}(g), \phi_w\rangle_{H_0^1(\Omega,\mathbb{R}^{\tilde{d}})} = \langle g, \mathrm{curl}(\phi_w)\rangle_{H_0^1(\Omega,\mathbb{R}^d)}$ and consequently,

$$\langle G, w\rangle_{H_0^1(\Omega,\mathbb{R}^d)} = \langle g, \mathrm{curl}(\phi_w) + z_w\rangle_{H_0^1(\Omega,\mathbb{R}^d)} = \langle g, w\rangle_{H_0^1(\Omega,\mathbb{R}^d)}.$$

This proves that $G = g$ in $H^{-1}(\Omega,\mathbb{R}^d)$ and so $g \in H_0(\mathrm{div},\Omega)^*$.

Finally, we prove that $\|f\|_{H_0(\mathrm{div},\Omega)^*} \sim \|f\|_{H^{-1}(\mathrm{curl},\Omega)}$, implying equivalence of the resulting topologies. Note that by the stability of the regular decomposition and the triangle inequality we have $\|\phi_v\|_{H^1(\Omega)} + \|z_v\|_{H^1(\Omega)} \sim \|v\|_{H(\mathrm{div},\Omega)}$. For any $f \in H_0(\mathrm{div},\Omega)^*$ we then have, using the equivalence proven in lemma 1, that

$$\begin{aligned}\|f\|_{H_0(\mathrm{div},\Omega)^*} &= \sup_{v\in H_0(\mathrm{div},\Omega)} \frac{\langle f, v\rangle_{H_0(\mathrm{div},\Omega)}}{\|v\|_{H(\mathrm{div},\Omega)}} \\ &\sim \sup_{\phi\in H_0^1(\Omega,\mathbb{R}^{\tilde d}),\, z\in H_0^1(\Omega,\mathbb{R}^d)} \frac{\langle f, \mathrm{curl}(\phi)+z\rangle_{H_0(\mathrm{div},\Omega)}}{\|\phi\|_{H^1(\Omega)}+\|z\|_{H^1(\Omega)}} \\ &= \sup_{\phi\in H_0^1(\Omega,\mathbb{R}^{\tilde d}),\, z\in H_0^1(\Omega,\mathbb{R}^d)} \frac{\langle \mathrm{curl}(f), \phi\rangle_{H_0^1(\Omega,\mathbb{R}^{\tilde d})}+\langle f, z\rangle_{H_0^1(\Omega,\mathbb{R}^d)}}{\|\phi\|_{H^1(\Omega)}+\|z\|_{H^1(\Omega)}} \\ &\sim \|f\|_{H^{-1}(\Omega)} + \|\mathrm{curl}(f)\|_{H^{-1}(\Omega)}.\end{aligned}$$

Thus, the $H_0(\mathrm{div},\Omega)^*$-norm and the $H^{-1}(\mathrm{curl},\Omega)$-norm are equivalent. □

By theorem 7 it follows that the requirement $\mathrm{div}(\sigma) \in H_0(\mathrm{div},\Omega)^*$ is equivalent to $\mathrm{div}(\sigma) \in H^{-1}(\mathrm{curl},\Omega)$. As σ is a square integrable matrix-valued function, $\sigma \in L^2(\Omega,\mathbb{R}^{d\times d})$, we immediately have that $\mathrm{div}(\sigma) \in H^{-1}(\Omega,\mathbb{R}^d)$. Therefore, the only non-redundant requirement emerging from $\mathrm{div}(\sigma) \in H^{-1}(\mathrm{curl},\Omega)$ is that $\mathrm{curl}(\mathrm{div}(\sigma)) \in H^{-1}(\Omega,\mathbb{R}^{\tilde d})$. By this we have the equivalences

$$H(\mathrm{curl}\,\mathrm{div},\Omega) = \{\sigma \in L^2(\Omega,\mathbb{R}^{d\times d}) : \mathrm{curl}(\mathrm{div}(\sigma)) \in H^{-1}(\Omega,\mathbb{R}^{\tilde d})\}, \tag{4.15}$$

and

$$\|\sigma\|_{\mathrm{cd}}^2 \simeq \|\sigma\|_{L^2(\Omega)}^2 + \left(\sup_{v\in H_0^1(\Omega,\mathbb{R}^{\tilde d})} \frac{\langle \mathrm{curl}(\mathrm{div}(\sigma)), v\rangle}{\|v\|_{H^1(\Omega)}}\right)^2. \tag{4.16}$$

Obviously, this equivalent definition was also the motivation for the name of the new function space $H(\mathrm{curl}\,\mathrm{div},\Omega)$.

At this point we want to mention that similar spaces, in the sense of the definition, were introduced in the works [97, 96, 95, 110]. Therein the authors introduced a function space for the analysis of the elasticity problem given by symmetric square integrable matrix-valued functions whose div div is in $H^{-1}(\Omega,\mathbb{R})$, thus

$$H(\mathrm{div}\,\mathrm{div},\Omega) := \{\sigma \in L^2(\Omega,\mathbb{R}^{d\times d}) : \sigma = \sigma^{\mathrm{T}}, \mathrm{div}(\mathrm{div}(\sigma)) \in H^{-1}(\Omega,\mathbb{R})\}.$$

4.2.2 Density of smooth functions

Density results are well known for standard Sobolev spaces and are used for example in the definition of appropriate trace operators. One of the first works considering an approximation by smooth functions is the famous paper of Meyers and Serrin [86].

The main idea is the following: In a first step one constructs a series of smooth functions with simple mollification techniques. Whereas this idea can be applied without any constraints in the interior of the domain, we have to add a shift (mapping) from the boundary into the interior of the domain, if we want to apply a similar technique at $\partial\Omega$. In order to prove convergence in the appropriate Sobolev norm one then shows that the convolution commutes with the according differential operator and that the series converges in L^2.

A classical pullback (at the boundary) works in the case of $H^1(\Omega)$, but one has to use appropriate mappings in the case of $H(\mathrm{div}, \Omega)$ and $H(\mathrm{curl}, \Omega)$. This was for example proven in [106] and [42]. In this section we show that smooth functions can also be used to approximate functions in $H(\mathrm{curl}\,\mathrm{div}, \Omega)$ by proving that the space $\mathcal{C}^\infty(\overline{\Omega}, \mathbb{R}^{d\times d})$ is dense. Similar as for standard spaces the proof is based on a convolution, but this time convergence has to be proven with respect to the norm $\|\cdot\|_{\mathrm{cd}}$.

We start with the introduction of a smooth mapping as it is defined in [42], which is based on several results from [66]. Due to the assumptions on $\partial\Omega$, see section 3.1, there exists a vector field $j \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$, whose restriction to $\partial\Omega$ is globally transversal (has a zero tangential component) and has an Euclidian norm equal to one, thus $(j(x))_t = 0$ and $||j(x)||_{l^2} = 1$ for all $x \in \partial\Omega$. Using this smooth function we define the mapping

$$\phi^\varepsilon : \mathbb{R}^d \to \mathbb{R}^d, x \mapsto x - \varepsilon j(x).$$

Applying the function ϕ^ε on Ω shrinks the domain and each point on the boundary is moved into the interior by the length ε. Note that one can show (using the uniform cone property, see pp. 599-600 [66]) that there exists a $r > 0$ such that $\phi^\varepsilon(\Omega) + B(0, \varepsilon r) \subset \Omega$ for all $\varepsilon \in [0, 1]$. Further we have the following properties:

Lemma 2. *There holds:*

- $\phi^\varepsilon \in \mathcal{C}^\infty$ *for all* $\varepsilon \in [0, 1]$.
- *For all* $l \in \mathbb{N}$ *there exists a constant* c *such that* $\|D^l\phi^\varepsilon(x) - D^l x\|_\infty \leq c\varepsilon$ *for all* $\varepsilon \in [0, 1]$, *where* D^l *denotes the Fréchet derivative of order* l.
- *The Jacobian determinant* $J^\varepsilon := \det(\phi^\varepsilon)'$ *converges* $\|J^\varepsilon\|_\infty \to 1$ *for* $\varepsilon \to 0$.

Proof. For (i) and (ii) we refer to [42]. Point (iii) follows from (ii). □

The inverse of the shrinking function ϕ^ε will enlarge the domain. For this let $\varepsilon^* > 0$ such that ϕ^ε is invertible for all $\varepsilon \in [0, \varepsilon^*]$ and define the inverse $\phi^{-\varepsilon} := (\phi^\varepsilon)^{-1} : \mathbb{R}^d \to \mathbb{R}^d$.

Lemma 3. *There holds:*

- $\phi^{-\varepsilon} \in \mathcal{C}^\infty$.
- *For all* $l \in \mathbb{N}$ *there exists a constant* c *such that* $\|D^l\phi^{-\varepsilon}(x) - D^l x\|_\infty \leq c\varepsilon$ *for all* $\varepsilon \in [0, \varepsilon^*]$.
- $J^{-\varepsilon} := \det(\phi^{-\varepsilon})'$ *converges* $\|J^{-\varepsilon}\|_\infty \to 1$ *for* $\varepsilon \to 0$.

Proof. Follows from lemma 2 and the implicit function theorem. □

Next, we define the standard mollifier by

$$\psi(y) := \begin{cases} c\exp\left(-1/(1 - \|y\|_{l^2}^2)\right) & \text{for } \|y\|_{l^2} < 1, \\ 0 & \text{for } \|y\|_{l^2} \geq 1, \end{cases}$$

with a constant $c \in \mathbb{R}$ chosen such that $\int_B \psi = 1$, and the ball in zero with radius 1 denoted $B := B(0,1)$. Using the shrinking function ϕ^ε and setting $F(x) = (\phi^\varepsilon)'(x)$, we define for all $\varepsilon \in [0,1]$ the smoothing operator

$$(S^\varepsilon\sigma)(x) := \int_B \Psi(y)J(x)F^{\mathrm{T}}(x)\sigma(\phi^\varepsilon(x) + (\varepsilon r)y)F^{-\mathrm{T}}(x)\,\mathrm{d}y \quad \forall\sigma \in L^2(\Omega, \mathbb{R}^{d\times d}),$$

where $r > 0$ is fixed and chosen as stated above. The function $S^\varepsilon\sigma$ reads as the convolution of σ with the smooth function $\Psi(y)$ including the mapping ϕ^ε. This allows us to evaluate the function $S^\varepsilon\sigma$ also on the boundary $\partial\Omega$, because $\phi^\varepsilon(x) + (\varepsilon r)y \in \Omega$ for all $y \in B$ and all $x \in \partial\Omega$. There holds the following approximation result.

Lemma 4. *Let $\sigma \in L^2(\Omega, \mathbb{R}^{d\times d})$. There holds $S^\varepsilon\sigma \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R}^{d\times d})$ and the approximation result $\|S^\varepsilon\sigma - \sigma\|_{L^2(\Omega)} \to 0$ for $\varepsilon \to 0$.*

Proof. The proof follows the same step as the proof of lemma 3.1 in [42]. Let x and z be two arbitrary points in Ω. Using the transformation rule for integrals we have

$$(S^\varepsilon\sigma)(x) - (S^\varepsilon\sigma)(z) \lesssim \frac{1}{(\varepsilon r)^d}\int_\Omega \Psi(\frac{y-\phi^\varepsilon(x)}{\varepsilon r}) - \Psi(\frac{y-\phi^\varepsilon(z)}{\varepsilon r})\sigma(y)\,\mathrm{d}y,$$

where we used that J and F are uniformly bounded and that Ψ is zero outside of $B(0,1)$. As Ψ and ϕ^ε are uniformly Lipschitz continuous, we can bound

$$\Psi(\frac{y-\phi^\varepsilon(x)}{\varepsilon r}) - \Psi(\frac{y-\phi^\varepsilon(z)}{\varepsilon r}) \lesssim \frac{1}{\varepsilon r}\|x-z\|_{l^2},$$

and thus $|(S^\varepsilon\sigma)(x) - (S^\varepsilon\sigma)(z)| \lesssim 1/(\varepsilon r)^{d+1}\|\sigma\|_{L^2(\Omega)}\|x-z\|_{l^2}$ from which we conclude continuity of $S^\varepsilon\sigma$ in Ω and the existence of a continuous extension up to the boundary. Continuing with a partial derivative we have, using the product rule,

$$\begin{aligned}\partial_i(S^\varepsilon\sigma)(x) = &\int_B \Psi(y)J(x)\partial_i F^{\mathrm{T}}(x)\sigma(\phi^\varepsilon(x)+(\varepsilon r)y)F^{-\mathrm{T}}(x)\,\mathrm{d}y\\ &+\int_B \Psi(y)J(x)F^{\mathrm{T}}(x)\partial_i\sigma(\phi^\varepsilon(x)+(\varepsilon r)y)F^{-\mathrm{T}}(x)\,\mathrm{d}y\\ &+\int_B \Psi(y)J(x)F^{\mathrm{T}}(x)\sigma(\phi^\varepsilon(x)+(\varepsilon r)y)\partial_i F^{-\mathrm{T}}(x)\,\mathrm{d}y\,.\end{aligned}$$

As ϕ^ε is smooth, the first and the third integral are continuous with respect to x with the same arguments as above. Using the chain rule we observe that for each component σ_{lm} with $0 \le l, m \le d$ we have

$$\begin{aligned}\int_B \Psi(y)\partial_i\sigma_{lm}(\phi^\varepsilon(x)+(\varepsilon r)y)\,\mathrm{d}y &\lesssim \int_B \Psi(y)(\nabla\sigma_{lm})(\phi^\varepsilon(x)+(\varepsilon r)y)\partial_i\phi^\varepsilon\,\mathrm{d}y\\ &\lesssim \frac{1}{\varepsilon r}\partial_i\phi^\varepsilon(x)\int_B -\nabla\Psi(y)\sigma_{lm}(\phi^\varepsilon(x)+(\varepsilon r)y)\,\mathrm{d}y\,.\end{aligned}$$

Because $-(\varepsilon r)^{-1}\partial_i\phi^\varepsilon(x)\nabla\Psi(y)$ is also a mollifier, we have, using that J and F are uniformly bounded and the same arguments as above, that $\partial_i(S^\varepsilon\sigma)(x)$ is continuous. By induction we conclude that $S^\varepsilon\sigma \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R}^{d\times d})$ and the first statement is proven.

Similarly, continuity with respect to the L^2 norm follows by applying the transformation and using that J, J^{-1} and F are uniformly bounded. Let $\sigma_s \in \mathcal{C}^{0,1}(\Omega, \mathbb{R}^{d\times d})$ with Liptschitz-constant L. As $\int_B \psi = 1$, we can write

$$\begin{aligned}(S^\varepsilon \sigma_s)(x) - \sigma_s(x) &\lesssim \int_B \Psi(y)\sigma_s(\phi^\varepsilon(x) + (\varepsilon r)y)\,\mathrm{d}y - \sigma_s(x)\\ &= \int_B \Psi(y)\sigma_s(\phi^\varepsilon(x) + (\varepsilon r)y) - \sigma_s(x)\,\mathrm{d}y \lesssim \varepsilon L,\end{aligned}$$

as $|\sigma_s(\phi^\varepsilon(x) + (\varepsilon r)y) - \sigma_s(x)| \leq L|\phi^\varepsilon(x) + (\varepsilon r)y - x| \lesssim \varepsilon L$. By the density of Lipschitz functions $\mathcal{C}^{0,1}(\Omega, \mathbb{R}^{d\times d})$ in L^2 we can find for a given $\sigma \in L^2(\Omega, \mathbb{R}^{d\times d})$ such a smooth function $\sigma_s \in \mathcal{C}^{0,1}(\Omega, \mathbb{R}^{d\times d})$ with $\|\sigma - \sigma_s\|_{L^2(\Omega)} \leq \delta$ and $\|\sigma_s\|_{\mathcal{C}^{0,1}} \leq L$. Then by the triangle inequality and the continuity of S^ε with respect to the L^2 norm we have

$$\begin{aligned}\|\sigma - S^\varepsilon\sigma\|_{L^2(\Omega)} &\leq \|\sigma - \sigma_s\|_{L^2(\Omega)} + \|\sigma_s - S^\varepsilon\sigma_s\|_{L^2(\Omega)} + \|S^\varepsilon(\sigma - \sigma_s)\|_{L^2(\Omega)}\\ &\lesssim (1 + \|S^\varepsilon\|)\delta + \varepsilon L.\end{aligned}$$

The term on the right side can be made arbitrarily small: first, one chooses δ small enough resulting in a corresponding function σ_s with a (potentially great) Lipschitz-constant L, and then one chooses ε small enough. This concludes the proof. □

In the last lemma we have proven that the operator S^ε can be used to approximate well with respect to the L^2-norm. This is important as the L^2-norm is one part of the introduced norm $\|\cdot\|_{\mathrm{cd}}$. The second term is given by the dual norm of $\operatorname{div}(\sigma)$ in $H_0(\operatorname{div}, \Omega)^*$. Although we do not generally assume that $\operatorname{div}(\sigma) \in L^2(\Omega, \mathbb{R}^d)$ for a $\sigma \in H(\operatorname{curl}\operatorname{div}, \Omega)$, the next lemma deals with the case where we have this enhanced regularity and shows that in this case the operator S^ε is also suitable to approximate the divergence.

Lemma 5. *Let $\sigma \in H(\operatorname{curl}\operatorname{div}, \Omega)$ and assume that $\operatorname{div}(\sigma) \in L^2(\Omega, \mathbb{R}^d)$. Then*

$$\|\operatorname{div}(S^\varepsilon\sigma) - \operatorname{div}(\sigma)\|_{L^2(\Omega)} \to 0 \quad \textit{for} \quad \varepsilon \to 0.$$

Proof. We present the proof only in the three-dimensional case as the two-dimensional case follows similarly. For the ease of notation we define for all $y \in B$ the mapping $\eta(x) := \phi^\varepsilon(x) + (\varepsilon r)y$. Note that there holds $\eta' = (\phi^\varepsilon)' = F$. Defining further $\tilde{F} := [\eta^{-1}]'$, we have with $\hat{x} = \eta(x)$ the following identities

$$\begin{aligned}F(x)^{-1} &= \tilde{F}(\eta(x)) = \tilde{F}(\hat{x}),\\ \tilde{F}^{-1}(\hat{x}) &= F(\eta^{-1}(\hat{x})),\\ \tilde{J}(\hat{x}) &= 1/J(x),\\ [F(\eta^{-1}(\hat{x}))]^{-\mathrm{T}}(\nabla w)(\eta^{-1}(\hat{x})) &= \tilde{F}(\hat{x})^{\mathrm{T}}(\nabla w)(\eta^{-1}(\hat{x})) = \nabla[w(\eta^{-1}(\hat{x}))].\end{aligned} \tag{4.17}$$

We use the following notation: For a matrix $A(x)$ let $A_{i\cdot}$ be the $i-th$ row. Then we define

$$\nabla A_{i\cdot} := \begin{pmatrix}\partial_{x_1}A_{i1} & \partial_{x_2}A_{i1} & \partial_{x_3}A_{i1}\\ \partial_{x_1}A_{i2} & \partial_{x_2}A_{i2} & \partial_{x_3}A_{i2}\\ \partial_{x_1}A_{i3} & \partial_{x_2}A_{i3} & \partial_{x_3}A_{i3}\end{pmatrix},$$

thus we use the same notation of the gradient as for column vectors. Let $\varphi \in \mathcal{C}_0^\infty(\Omega, \mathbb{R}^d)$, then the weak divergence of σ is given by

$$\langle \operatorname{div}(S^\varepsilon \sigma), \varphi \rangle_{H_0(\operatorname{div},\Omega)} = -\int_\Omega \int_B \Psi(y) J(x) F^T(x) \sigma(\phi^\varepsilon(x) + (\varepsilon r) y) F^{-\mathrm{T}}(x)\, \mathrm{d}y : (\nabla\varphi)(x)\, \mathrm{d}x\,.$$

Next we apply Fubinis theorem to change the order of integration and then apply the transformation theorem. Together with the identities (4.17) this yields

$$\begin{aligned}
\langle \operatorname{div}(S^\varepsilon \sigma), \varphi \rangle_{H_0(\operatorname{div},\Omega)} &= -\int_B \Psi(y) \int_\Omega J(x) F^T(x) \sigma(\phi^\varepsilon(x) + (\varepsilon r) y) F^{-\mathrm{T}}(x) : (\nabla\varphi)(x)\, \mathrm{d}x\, \mathrm{d}y \\
&= -\int_B \Psi(y) \int_{\eta(\Omega)} F^T(\eta^{-1}(\hat{x})) \sigma(\hat{x}) F^{-\mathrm{T}}(\eta^{-1}(\hat{x})) : (\nabla\varphi)(\eta^{-1}(\hat{x}))\, \mathrm{d}\hat{x}\, \mathrm{d}y \\
&= -\int_B \Psi(y) \int_{\eta(\Omega)} \tilde{F}^{-T}(\hat{x}) \sigma(\hat{x}) \tilde{F}^{\mathrm{T}}(\hat{x}) : (\nabla\varphi)(\eta^{-1}(\hat{x}))\, \mathrm{d}\hat{x}\, \mathrm{d}y \\
&= -\int_B \Psi(y) \int_{\eta(\Omega)} \tilde{F}^{-T}(\hat{x}) \sigma(\hat{x}) : (\nabla\varphi)(\eta^{-1}(\hat{x})) \tilde{F}(\hat{x})\, \mathrm{d}\hat{x}\, \mathrm{d}y \\
&= -\int_B \Psi(y) \int_{\eta(\Omega)} \tilde{F}^{-T}(\hat{x}) \sigma(\hat{x}) : \nabla[\varphi(\eta^{-1}(\hat{x}))]\, \mathrm{d}\hat{x}\, \mathrm{d}y\,.
\end{aligned}$$

As $\operatorname{div}(\sigma) \in L^2(\Omega, \mathbb{R}^d)$, we can integrate by parts. The product rule then yields

$$\begin{aligned}
\langle \operatorname{div}(S^\varepsilon \sigma), \varphi \rangle_{H_0(\operatorname{div},\Omega)} &= \int_B \Psi(y) \int_{\eta(\Omega)} \operatorname{div}(\tilde{F}^{-T}(\hat{x}) \sigma(\hat{x})) \cdot \varphi(\eta^{-1}(\hat{x}))\, \mathrm{d}\hat{x}\, \mathrm{d}y \\
&= \int_B \Psi(y) \int_{\eta(\Omega)} \tilde{F}^{-T}(\hat{x}) \operatorname{div}(\sigma(\hat{x})) \cdot \varphi(\eta^{-1}(\hat{x}))\, \mathrm{d}\hat{x}\, \mathrm{d}y \\
&\quad + \int_B \Psi(y) \int_{\eta(\Omega)} \begin{pmatrix} \nabla(\tilde{F}_{1\cdot}^{-T}(\hat{x})) : \sigma(\hat{x}) \\ \nabla(\tilde{F}_{2\cdot}^{-T}(\hat{x})) : \sigma(\hat{x}) \\ \nabla(\tilde{F}_{3\cdot}^{-T}(\hat{x})) : \sigma(\hat{x}) \end{pmatrix} \cdot \varphi(\eta^{-1}(\hat{x}))\, \mathrm{d}\hat{x}\, \mathrm{d}y\,.
\end{aligned}$$

Using (4.17) we can write

$$\begin{pmatrix} \nabla(\tilde{F}_{1\cdot}^{-T}(\hat{x})) : \sigma(\hat{x}) \\ \nabla(\tilde{F}_{2\cdot}^{-T}(\hat{x})) : \sigma(\hat{x}) \\ \nabla(\tilde{F}_{3\cdot}^{-T}(\hat{x})) : \sigma(\hat{x}) \end{pmatrix} = \begin{pmatrix} \nabla(F_{1\cdot}^{T}(x)) : \sigma(\eta(x)) \\ \nabla(F_{2\cdot}^{T}(x)) : \sigma(\eta(x)) \\ \nabla(F_{3\cdot}^{T}(x)) : \sigma(\eta(x)) \end{pmatrix}.$$

Thus, again by the transformation rule and Fubinis theorem we have

$$\begin{aligned}
\langle \operatorname{div}(S^\varepsilon \sigma), \varphi \rangle_{H_0(\operatorname{div},\Omega)} &= \int_B \Psi(y) \int_\Omega F^T(x) \operatorname{div}(\sigma(\eta(x))) \cdot \varphi(x)\, \mathrm{d}x\, \mathrm{d}y \\
&\quad + \int_B \Psi(y) \int_\Omega \begin{pmatrix} \nabla(F_{1\cdot}^{T}(x)) : \sigma(\eta(x)) \\ \nabla(F_{2\cdot}^{T}(x)) : \sigma(\eta(x)) \\ \nabla(F_{3\cdot}^{T}(x)) : \sigma(\eta(x)) \end{pmatrix} \cdot \varphi(x)\, \mathrm{d}x\, \mathrm{d}y \\
&= \int_\Omega S_{d1}^\varepsilon \operatorname{div}(\sigma) \cdot \varphi(x)\, \mathrm{d}x + \int_\Omega S_{d2}^\varepsilon \sigma \cdot \varphi(x)\, \mathrm{d}x,
\end{aligned}$$

with the new smoothing operators defined by

$$S^{\varepsilon}_{d1}\operatorname{div}(\sigma) := \int_B \Psi(y) F^T(x)(\operatorname{div}(\sigma))(\phi^{\varepsilon}(x) + (r\varepsilon)y)\,\mathrm{d}y,$$

$$S^{\varepsilon}_{d2}\sigma := \int_B \Psi(y)\begin{pmatrix}\nabla(F^T_{1\cdot}(x)) : \sigma(\phi^{\varepsilon}(x) + (r\varepsilon)y)\\ \nabla(F^T_{2\cdot}(x)) : \sigma(\phi^{\varepsilon}(x) + (r\varepsilon)y)\\ \nabla(F^T_{3\cdot}(x)) : \sigma(\phi^{\varepsilon}(x) + (r\varepsilon)y)\end{pmatrix}\mathrm{d}y\,.$$

Following the same steps as in the proof of lemma 4 one shows that these operators are continuous and that $\|f - S^{\varepsilon}_{d1} f\|_{L^2} \to 0$ for functions $f \in L^2(\Omega, \mathbb{R}^d)$. As $\operatorname{div}(\sigma) \in L^2(\Omega, \mathbb{R}^d)$ and $\|\nabla(F^T(x))\|_{\infty} \to 0$ (see lemma 2), we conclude

$$\begin{aligned}\|\operatorname{div}(S^{\varepsilon}\sigma) - \operatorname{div}(\sigma)\|_{L^2(\Omega)} &= \|S^{\varepsilon}_{d1}\operatorname{div}(\sigma) + S^{\varepsilon}_{d2}\sigma - \operatorname{div}(\sigma)\|_{L^2(\Omega)}\\ &\le \|S^{\varepsilon}_{d1}\operatorname{div}(\sigma) - \operatorname{div}(\sigma)\|_{L^2(\Omega)} + \|S^{\varepsilon}_{d2}\sigma\|_{L^2(\Omega)}\\ &\le \|S^{\varepsilon}_{d1}\operatorname{div}(\sigma) - \operatorname{div}(\sigma)\|_{L^2(\Omega)} + \|\nabla(F^T(x))\|_{\infty}\|\sigma\|_{L^2(\Omega)} \to 0.\end{aligned}$$

□

In contrast to lemma 5, the next lemma considers the case where we do not assume enhanced regularity of $\operatorname{div}(\sigma)$ but that $\operatorname{curl}(\operatorname{div}(\sigma))$ is the null functional.

Lemma 6. *Assume $\sigma \in H(\operatorname{curl}\operatorname{div}, \Omega)$ with $\operatorname{curl}(\operatorname{div}(\sigma)) = 0 \in H^{-1}(\Omega, \mathbb{R}^{\tilde{d}})$. There holds*

$$\|\operatorname{curl}(\operatorname{div}(S^{\varepsilon}\sigma))\|_{H^{-1}(\Omega)} \to 0.$$

Proof. We only present the proof in the three-dimensional case, as the two-dimensional case follows similarly. Note that in two dimensions the curl operator maps slightly different but yields the same results. With the same steps as in the proof of lemma 5 we have for a $\varphi \in \mathcal{C}^{\infty}_0(\Omega, \mathbb{R}^{\tilde{d}})$

$$\begin{aligned}&\langle \operatorname{curl}(\operatorname{div}(S^{\varepsilon}\sigma)), \varphi\rangle_{H^{-1}(\Omega,\mathbb{R}^{\tilde{d}})}\\ &= -\int_{\Omega}\int_B \Psi(y) J(x) F^T(x)\sigma(\phi^{\varepsilon}(x) + (\varepsilon r)y) F^{-\mathrm{T}}(x)\,\mathrm{d}y : (\nabla\operatorname{curl}(\varphi))(x)\,\mathrm{d}x\\ &= -\int_B \Psi(y)\int_{\eta(\Omega)} F^T(\eta^{-1}(\hat{x}))\sigma(\hat{x}) F^{-\mathrm{T}}(\eta^{-1}(\hat{x})) : (\nabla\operatorname{curl}(\varphi))(\eta^{-1}(\hat{x}))\,\mathrm{d}\hat{x}\,\mathrm{d}y\\ &= -\int_B \Psi(y)\int_{\eta(\Omega)} \tilde{F}^{-T}(\hat{x})\sigma(\hat{x})\tilde{F}^{\mathrm{T}}(\hat{x}) : (\nabla\operatorname{curl}(\varphi))(\eta^{-1}(\hat{x}))\,\mathrm{d}\hat{x}\,\mathrm{d}y\,.\end{aligned}$$

We continue with estimating the inner integral. Using the chain rule for the curl operator and the transformation properties (4.17) we get

$$\begin{aligned}&\int_{\eta(\Omega)} \tilde{F}^{-T}(\hat{x})\sigma(\hat{x})\tilde{F}^{\mathrm{T}}(\hat{x}) : (\nabla\operatorname{curl}(\varphi))(\eta^{-1}(\hat{x}))\,\mathrm{d}\hat{x}\\ &\qquad = \int_{\eta(\Omega)} \sigma(\hat{x}) : \tilde{F}^{-1}(\hat{x})(\nabla\operatorname{curl}(\varphi))(\eta^{-1}(\hat{x}))\tilde{F}(\hat{x})\,\mathrm{d}\hat{x}\\ &\qquad = \int_{\eta(\Omega)} \sigma(\hat{x}) : \tilde{F}^{-1}(\hat{x})\nabla[(\operatorname{curl}(\varphi))(\eta^{-1}(\hat{x}))]\,\mathrm{d}\hat{x}\end{aligned}$$

Using the product rule for the gradient, the integral on the right side can be split into two integrals which yields

$$\begin{aligned}
&\int_{\eta(\Omega)} \tilde{F}^{-T}(\hat{x})\sigma(\hat{x})\tilde{F}^{\mathrm{T}}(\hat{x}) : (\nabla \operatorname{curl}(\varphi))(\eta^{-1}(\hat{x}))\,\mathrm{d}\hat{x} \\
&\qquad = \int_{\eta(\Omega)} \sigma(\hat{x}) : \nabla[\tilde{F}^{-1}(\hat{x})(\operatorname{curl}(\varphi))(\eta^{-1}(\hat{x}))]\,\mathrm{d}\hat{x} \\
&\qquad\quad - \int_{\eta(\Omega)} \sigma(\hat{x}) : \begin{pmatrix} \nabla[\tilde{F}^{-1}_{1\cdot}(\hat{x})]^{\mathrm{T}}(\operatorname{curl}(\varphi))(\eta^{-1}(\hat{x})) \\ \nabla[\tilde{F}^{-1}_{2\cdot}(\hat{x})]^{\mathrm{T}}(\operatorname{curl}(\varphi))(\eta^{-1}(\hat{x})) \\ \nabla[\tilde{F}^{-1}_{2\cdot}(\hat{x})]^{\mathrm{T}}(\operatorname{curl}(\varphi))(\eta^{-1}(\hat{x})) \end{pmatrix} \mathrm{d}\hat{x}.
\end{aligned}$$

For the first integral we have by the product rule of the curl operator,

$$\begin{aligned}
&\int_{\eta(\Omega)} \sigma(\hat{x}) : \nabla[\tilde{F}^{-1}(\hat{x})(\operatorname{curl}(\varphi))(\eta^{-1}(\hat{x}))]\,\mathrm{d}\hat{x} \\
&\qquad = \int_{\eta(\Omega)} \sigma(\hat{x}) : \nabla\left[\frac{\tilde{J}(\hat{x})}{\tilde{J}(\hat{x})}\tilde{F}^{-1}(\hat{x})(\operatorname{curl}(\varphi))(\eta^{-1}(\hat{x}))\right]\mathrm{d}\hat{x} \\
&\qquad = \int_{\eta(\Omega)} \sigma(\hat{x}) : \nabla\left[\frac{1}{\tilde{J}(\hat{x})}\operatorname{curl}[\tilde{F}^{\mathrm{T}}(\hat{x})\varphi(\eta^{-1}(\hat{x}))]\right]\mathrm{d}\hat{x} \\
&\qquad = \int_{\eta(\Omega)} \sigma(\hat{x}) : \nabla\left[\operatorname{curl}\left[\frac{1}{\tilde{J}(\hat{x})}\tilde{F}^{\mathrm{T}}(\hat{x})\varphi(\eta^{-1}(\hat{x}))\right]\right]\mathrm{d}\hat{x} \\
&\qquad\quad - \int_{\eta(\Omega)} \sigma(\hat{x}) : \nabla\left[\nabla\left(\frac{1}{\tilde{J}(\hat{x})}\right) \times \tilde{F}^{\mathrm{T}}(\hat{x})\varphi(\eta^{-1}(\hat{x}))\right]\mathrm{d}\hat{x}\,.
\end{aligned}$$

Summing up all terms, this yields

$$\langle \operatorname{curl}(\operatorname{div}(S^{\varepsilon}\sigma)), \varphi\rangle_{H^{-1}(\Omega,\mathbb{R}^{\tilde{d}})} = -\int_B \Psi(y)(A+B+C)\,\mathrm{d}y,$$

with

$$\begin{aligned}
A &:= \int_{\eta(\Omega)} \sigma(\hat{x}) : \nabla\left[\operatorname{curl}\left[\frac{1}{\tilde{J}(\hat{x})}\tilde{F}^{\mathrm{T}}(\hat{x})\varphi(\eta^{-1}(\hat{x}))\right]\right]\mathrm{d}\hat{x}, \\
B &:= -\int_{\eta(\Omega)} \sigma(\hat{x}) : \begin{pmatrix} \nabla[\tilde{F}^{-1}_{1\cdot}(\hat{x})]^{\mathrm{T}}(\operatorname{curl}(\varphi))(\eta^{-1}(\hat{x})) \\ \nabla[\tilde{F}^{-1}_{2\cdot}(\hat{x})]^{\mathrm{T}}(\operatorname{curl}(\varphi))(\eta^{-1}(\hat{x})) \\ \nabla[\tilde{F}^{-1}_{2\cdot}(\hat{x})]^{\mathrm{T}}(\operatorname{curl}(\varphi))(\eta^{-1}(\hat{x})) \end{pmatrix} \mathrm{d}\hat{x}, \\
C &:= -\int_{\eta(\Omega)} \sigma(\hat{x}) : \nabla\left[\nabla\left(\frac{1}{\tilde{J}(\hat{x})}\right) \times \tilde{F}^{\mathrm{T}}(\hat{x})\varphi(\eta^{-1}(\hat{x}))\right]\mathrm{d}\hat{x}\,.
\end{aligned}$$

As $\varphi \in \mathcal{C}_0^\infty(\Omega)$ it can be trivially extended by zero on $\mathbb{R}^d$. Further, as η and the boundary $\partial\Omega$ is smooth (Lipschitz) it can also be extended such that $\tilde{F}(\hat{x})$ can be evaluated for

$\hat{x} \in \Omega \setminus \eta(\Omega)$. By this we have for the first term

$$
\begin{aligned}
-\int_B \Psi(y) A \,\mathrm{d}y &= -\int_B \Psi(y) \int_{\eta(\Omega)} \sigma(\hat{x}) : \nabla \left[\operatorname{curl}\left[\frac{1}{\tilde{J}(\hat{x})} \tilde{F}^{\mathrm{T}}(\hat{x}) \varphi(\eta^{-1}(\hat{x}))\right]\right] \mathrm{d}\hat{x}\,\mathrm{d}y \\
&= -\int_\Omega \sigma(\hat{x}) : \nabla \left[\operatorname{curl}\left[\int_B \Psi(y) \frac{1}{\tilde{J}(\hat{x})} \tilde{F}^{\mathrm{T}}(\hat{x}) \varphi(\eta^{-1}(\hat{x}))\,\mathrm{d}y\right]\right] \mathrm{d}\hat{x} = 0,
\end{aligned}
$$

where we used that $\operatorname{curl}(\operatorname{div}(\sigma)) = 0$ in $H^{-1}(\Omega, \mathbb{R}^{\tilde{d}})$. Using the Cauchy Schwarz inequality we get with similar arguments and using that $\nabla \tilde{F}(\hat{x}) = \nabla F(\eta^{-1}(x))$ and $1/\tilde{J}(\hat{x}) = J(\eta^{-1}(x))$ the estimates

$$
\begin{aligned}
&-\int_B \Psi(y) B \,\mathrm{d}y - \int_B \Psi(y) C \,\mathrm{d}y \\
&\qquad \le \|\sigma\|_{L^2(\Omega)} \|\nabla \tilde{F}^{-1}\|_\infty \|\operatorname{curl}(\varphi)\|_{L^2(\Omega)} + \|\sigma\|_{L^2(\Omega)} \|\nabla^2 \frac{1}{\tilde{J}(\hat{x})}\|_\infty \|\tilde{F}^{\mathrm{T}}\|_\infty \|\varphi\|_{L^2(\Omega)} \\
&\qquad \le \|\sigma\|_{L^2(\Omega)} \|\nabla F\|_\infty \|\operatorname{curl}(\varphi)\|_{L^2(\Omega)} + \|\sigma\|_{L^2(\Omega)} \|\nabla^2 J(x)\|_\infty \|\varphi\|_{L^2(\Omega)}.
\end{aligned}
$$

By the definition of the dual norm this yields

$$
\begin{aligned}
&\|\operatorname{curl}(\operatorname{div}(S^\varepsilon \sigma))\|_{H^{-1}(\Omega, \mathbb{R}^{\tilde{d}})} \\
&\qquad = \sup_{\varphi \in \mathcal{C}_0^\infty(\Omega)} \frac{\langle \operatorname{curl}(\operatorname{div}(S^\varepsilon \sigma)), \varphi \rangle}{\|\varphi\|_{H^1(\Omega)}} \\
&\qquad \le \sup_{\varphi \in \mathcal{C}_0^\infty(\Omega)} \frac{\|\sigma\|_{L^2(\Omega)} \|\nabla F\|_\infty \|\operatorname{curl}(\varphi)\|_{L^2(\Omega)} + \|\sigma\|_{L^2(\Omega)} \|\nabla^2 J(x)\|_\infty \|\varphi\|_{L^2(\Omega)}}{\|\varphi\|_{H^1(\Omega)}} \\
&\qquad \le \|\sigma\|_{L^2(\Omega)} \|\nabla F\|_\infty + \|\sigma\|_{L^2(\Omega)} \|\nabla^2 J\|_\infty \to 0,
\end{aligned}
$$

where we used the properties given in lemma 2 in the last step. Using the denisty $\mathcal{C}_0^\infty(\Omega, \mathbb{R}^{\tilde{d}})$ in $H_0^1(\Omega, \mathbb{R}^{\tilde{d}})$ we conclude the proof. □

The last lemma needed for the final result is a decomposition result for functions in $H(\operatorname{curl}\operatorname{div}, \Omega)$. The idea is to split a given function into a part with enhanced regularity and a part whose $\operatorname{curl}\operatorname{div}$ vanishes.

Lemma 7 (Decomposition)**.** *Let $\sigma \in H(\operatorname{curl}\operatorname{div}, \Omega)$ arbitrary. There exist functions $\tilde{\sigma} \in H(\operatorname{curl}\operatorname{div}, \Omega)$ and θ, with $\theta \in H_0(\operatorname{curl}, \Omega)$ for $d = 3$ and $\theta \in H_0^1(\Omega)$ for $d = 2$, such that*

$$
\sigma = \operatorname{skw}(\theta) - \tilde{\sigma}, \quad \textit{and} \quad \operatorname{curl}(\operatorname{div}(\sigma)) = \operatorname{curl}(\operatorname{div}(\operatorname{skw}(\theta))).
$$

Proof. By the definition of the space $H(\operatorname{curl}\operatorname{div}, \Omega)$ we have $\operatorname{curl}(\operatorname{div}(\sigma)) \in H^{-1}(\Omega, \mathbb{R}^{\tilde{d}})$. We start with the three-dimensional case. In a first step we show that

$$
\|\operatorname{curl}(\operatorname{div}(\sigma))\|_{H^{-1}(\Omega, \mathbb{R}^3)} \simeq \|\operatorname{curl}(\operatorname{div}(\sigma))\|_{H(\operatorname{curl}, \Omega)^*}.
$$

For this note that by density of $\mathcal{C}_0^\infty(\Omega,\mathbb{R}^3)$ in $H_0^1(\Omega,\mathbb{R}^3)$ we have

$$\begin{aligned}\|\operatorname{curl}(\operatorname{div}(\sigma))\|_{H^{-1}(\Omega,\mathbb{R}^3)} &= \sup_{u\in H_0^1(\Omega,\mathbb{R}^3)} \frac{\langle \operatorname{curl}(\operatorname{div}(\sigma)), u\rangle}{\|u\|_{H^1(\Omega)}} = \sup_{\varphi\in \mathcal{C}_0^\infty(\Omega,\mathbb{R}^3)} \frac{\langle \operatorname{curl}(\operatorname{div}(\sigma)), \varphi\rangle}{\|\varphi\|_{H^1(\Omega)}} \\ &\leq \sup_{\varphi\in \mathcal{C}_0^\infty(\Omega,\mathbb{R}^3)} \frac{\langle \operatorname{curl}(\operatorname{div}(\sigma)), \varphi\rangle}{\|\varphi\|_{H(\operatorname{curl},\Omega)}} = \|\operatorname{curl}(\operatorname{div}(\sigma))\|_{H(\operatorname{curl},\Omega)^*}.\end{aligned}$$

Next, we use for each $u \in H_0(\operatorname{curl},\Omega)$ a regular decomposition, see for example in [63, 94], thus there exist a $\psi \in H_0^1(\Omega,\mathbb{R})$ and $z \in H_0^1(\Omega,\mathbb{R}^d)$ such that $u = \nabla\psi + z$ and

$$\|\psi\|_{H^1(\Omega)} \leq \|u\|_{H(\operatorname{curl},\Omega)} \quad \text{and} \quad \|z\|_{H^1(\Omega)} \leq \|u\|_{H(\operatorname{curl},\Omega)}.$$

By a density argument and integration by parts for the duality pair including $\nabla\psi$ this then yields

$$\begin{aligned}\|\operatorname{curl}(\operatorname{div}(\sigma))\|_{H(\operatorname{curl})^*} &= \sup_{u\in H(\operatorname{curl},\Omega)} \frac{\langle \operatorname{curl}(\operatorname{div}(\sigma)), u\rangle}{\|u\|_{H(\operatorname{curl},\Omega)}} \\ &= \sup_{\substack{\psi\in H_0^1(\Omega,\mathbb{R})\\ z\in H_0^1(\Omega,\mathbb{R}^3)}} \frac{\langle \operatorname{curl}(\operatorname{div}(\sigma)), \nabla\psi + z\rangle}{\|\psi\|_{H^1(\Omega)} + \|z\|_{H^1(\Omega)}} \approx \sup_{z\in H_0^1(\Omega,\mathbb{R}^3)} \frac{\langle \operatorname{curl}(\operatorname{div}(\sigma)), z\rangle}{\|z\|_{H^1(\Omega)}}.\end{aligned}$$

Due to the proven norm equivalence $\operatorname{curl}(\operatorname{div}(\sigma))$ is an admissible functional for $H_0(\operatorname{curl},\Omega)$, and we can solve the problem: Find $\theta \in H_0(\operatorname{curl},\Omega)/\nabla H_0^1(\Omega)$ such that

$$\int_\Omega \operatorname{curl}(\theta)\cdot\operatorname{curl}(\xi)\,\mathrm{d}x = \langle \operatorname{curl}(\operatorname{div}(\sigma)), \xi\rangle \quad \forall \xi \in H_0(\operatorname{curl},\Omega)/\nabla H_0^1(\Omega).$$

This problem is solvable applying the Lax-Milgram theorem 5, and a Friedrich-type inequality (see [87]). In two-dimensions we similarly solve the problem: Find $H_0^1(\Omega)$ such that

$$\int_\Omega \operatorname{curl}(\theta)\cdot\operatorname{curl}(\xi)\,\mathrm{d}x = \langle \operatorname{curl}(\operatorname{div}(\sigma)), \xi\rangle \quad \forall \xi \in H_0^1(\Omega).$$

As the curl is just the rotated gradient, solvability is agaim guaranteed by the Lax-Milgram theorem 5 and the standard Friedrich inequality, theorem 3. In both dimensions we have that $\operatorname{div}(\operatorname{skw}(\theta)) = \operatorname{curl}(\theta)$, thus $\operatorname{curl}(\operatorname{div}(\sigma)) = \operatorname{curl}(\operatorname{div}(\operatorname{skw}(\theta)))$. Defining $\tilde\sigma = \sigma - \operatorname{skw}(\theta)$ concludes the proof. □

Theorem 8. *The space $\mathcal{C}^\infty(\overline{\Omega},\mathbb{R}^{d\times d})$ is dense in $H(\operatorname{curl}\operatorname{div},\Omega)$.*

Proof. Choose an arbitrary $\sigma \in H(\operatorname{curl}\operatorname{div},\Omega)$. For $\varepsilon \to 0$, the functions $S^\varepsilon\sigma$ define a sequenc of smooth functions in $\mathcal{C}^\infty(\overline{\Omega},\mathbb{R}^{d\times d})$, and by lemma 4 we have $\|S^\varepsilon\sigma - \sigma\|_{L^2(\Omega)} \to 0$.

Next, we use the decomposition lemma 7 to get a $\tilde\theta := \operatorname{skw}(\theta)$ with $\theta \in H_0(\operatorname{curl},\Omega)$ for $d = 3$ and $\theta \in H_0^1(\Omega)$ for $d = 2$ and a $\tilde\sigma \in H(\operatorname{curl}\operatorname{div},\Omega)$ such that $\sigma = \tilde\theta + \tilde\sigma$ and

$\operatorname{curl}(\operatorname{div}(\tilde\sigma)) = 0$. Density arguments then yield

$$\begin{aligned}
&\sup_{v\in H_0^1(\Omega,\mathbb{R}^{\tilde d})} \frac{\langle \operatorname{curl}(\operatorname{div}(\sigma)) - \operatorname{curl}(\operatorname{div}(S^\varepsilon\sigma)), v\rangle}{\|v\|_{H^1(\Omega)}} \\
&\quad = \sup_{\varphi\in \mathcal{C}_0^\infty(\Omega,\mathbb{R}^{\tilde d})} \frac{\langle \operatorname{curl}(\operatorname{div}(\sigma)) - \operatorname{curl}(\operatorname{div}(S^\varepsilon\sigma)), \varphi\rangle}{\|\varphi\|_{H^1(\Omega)}} \\
&\quad = \sup_{\varphi\in \mathcal{C}_0^\infty(\Omega,\mathbb{R}^{\tilde d})} \frac{\langle \operatorname{curl}(\operatorname{div}(\tilde\theta)) - \operatorname{curl}(\operatorname{div}(S^\varepsilon\tilde\theta)), \varphi\rangle}{\|\varphi\|_{H^1(\Omega)}} - \sup_{\varphi\in \mathcal{C}_0^\infty(\Omega,\mathbb{R}^{\tilde d})} \frac{\langle \operatorname{curl}(\operatorname{div}(S^\varepsilon\tilde\sigma)), \varphi\rangle}{\|\varphi\|_{H^1(\Omega)}}.
\end{aligned}$$

As $\tilde\theta$ is also an element of $H(\operatorname{curl}\operatorname{div},\Omega)$, lemma 5 yields

$$\begin{aligned}
&\sup_{\varphi\in \mathcal{C}_0^\infty(\Omega,\mathbb{R}^{\tilde d})} \frac{\langle \operatorname{curl}(\operatorname{div}(\tilde\theta)) - \operatorname{curl}(\operatorname{div}(S^\varepsilon\tilde\theta)), \varphi\rangle}{\|\varphi\|_{H^1(\Omega)}} \\
&\qquad = \sup_{\varphi\in \mathcal{C}_0^\infty(\Omega,\mathbb{R}^{\tilde d})} \frac{\int_\Omega (\operatorname{div}(\tilde\theta) - \operatorname{div}(S^\varepsilon\tilde\theta)) \cdot \operatorname{curl}(\varphi)}{\|\varphi\|_{H^1(\Omega)}} \\
&\qquad \le \sup_{\varphi\in \mathcal{C}_0^\infty(\Omega,\mathbb{R}^{\tilde d})} \frac{\|\operatorname{div}(\tilde\theta) - \operatorname{div}(S^\varepsilon\tilde\theta)\|_{L^2(\Omega)} \|\operatorname{curl}(\varphi)\|_{L^2(\Omega)}}{\|\varphi\|_{H^1(\Omega)}} \\
&\qquad \le \|\operatorname{div}(\tilde\theta) - \operatorname{div}(S^\varepsilon\tilde\theta)\|_{L^2(\Omega)} \to 0.
\end{aligned}$$

By lemma 6 we further have

$$\sup_{\varphi\in \mathcal{C}_0^\infty(\Omega,\mathbb{R}^{\tilde d})} \frac{\langle \operatorname{curl}(\operatorname{div}(S^\varepsilon\tilde\sigma)), \varphi\rangle}{\|\varphi\|_{H^1(\Omega)}} \to 0.$$

Using the norm equivalence (4.16) concludes the proof. □

The denisty allows us to give another equivalent definition of the $H(\operatorname{curl}\operatorname{div},\Omega)$ space by

$$H(\operatorname{curl}\operatorname{div},\Omega) = \overline{\mathcal{C}^\infty(\overline{\Omega},\mathbb{R}^{d\times d})}^{\|\cdot\|_{\mathrm{cd}}}. \tag{4.18}$$

4.2.3 A trace space for $H(\operatorname{curl}\operatorname{div},\Omega)$

The proven density of smooth functions in the last section is the key ingredient needed for the definition of an appropriate trace operator for functions in $H(\operatorname{curl}\operatorname{div},\Omega)$. This is done in the usual way. First, we define the trace operator for smooth functions, and then we use a density argument. To this end let $\Gamma_i \subseteq \Gamma$ be an arbitrary subset of the boundary. Then we define the following spaces

$$\begin{aligned}
W(\Gamma_i) &:= \{w \in H^1(\operatorname{curl},\Omega) : w = 0 \text{ on } \Gamma, \operatorname{curl}_{\pi_t} w = 0 \text{ on } \Gamma\setminus\Gamma_i\} && \text{for } d = 3,\\
W(\Gamma_i) &:= \{w \in H^2(\Omega,\mathbb{R}) : w = 0 \text{ on } \Gamma, \operatorname{curl}_{\pi_t} w = 0 \text{ on } \Gamma\setminus\Gamma_i\} && \text{for } d = 2,
\end{aligned}$$

where in three dimensions the operator $\operatorname{curl}_{\pi_t}$ is defined according to equation (3.3) and in two dimensions by

$$\operatorname{curl}_{\pi_t} w = \pi_t \circ (\operatorname{curl}(w)) = \pi_t((-\partial_2 w, \partial_1 w)^{\mathrm{T}}).$$

Based on this space we then further define the space

$$TW(\Gamma_i) := \{\mathrm{curl}_{\pi_t} w : w \in W(\Gamma_i)\}, \quad \text{with} \quad \|\mu\|_{TW(\Gamma_i)} := \inf_{\substack{w \in W(\Gamma_i) \\ \mathrm{curl}_{\pi_t} w = \mu}} \|w\|_{H^1(\mathrm{curl},\Omega)}.$$

As we see in the following, the "normal-tangential" trace of functions in $H(\mathrm{curl}\,\mathrm{div},\Omega)$ on Γ_i lies in the dual space of $TW(\Gamma_i)$. To see this, we choose a smooth function $\phi \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R}^{d\times d})$ and a function $w \in W(\Gamma_i)$. As $\mathrm{curl}_{\pi_t} w = 0$ on $\Gamma \setminus \Gamma_i$, we first have the identity

$$\langle (\phi)_{nt}, \mathrm{curl}_{\pi_t} w\rangle_{TW(\Gamma_i)} = \int_{\Gamma_i} \phi_{nt} \cdot \mathrm{curl}_{\pi_t} w \,\mathrm{d}s = \int_{\Gamma} \phi_{nt} \cdot \mathrm{curl}_{\pi_t} w \,\mathrm{d}s\,.$$

Then, using integration by parts, we can write

$$\begin{aligned} \langle \phi_{nt}, \mathrm{curl}_{\pi_t} w\rangle_{TW(\Gamma_i)} &= \int_\Omega \mathrm{div}(\phi) \cdot \mathrm{curl}(w)\,\mathrm{d}x + \int_\Omega \phi : \nabla\,\mathrm{curl}(w)\,\mathrm{d}x - \int_\Gamma \phi_{nn} \cdot (\mathrm{curl}(w))_n \,\mathrm{d}s \\ &= \int_\Omega \mathrm{div}(\phi) \cdot \mathrm{curl}(w)\,\mathrm{d}x + \int_\Omega \phi : \nabla\,\mathrm{curl}(w)\,\mathrm{d}x, \end{aligned} \tag{4.19}$$

where we used the identity (see for example [11])

$$(\mathrm{curl}(w))_n = \mathrm{curl}_t(\gamma_t w) = 0,$$

as all functions $w \in W(\Gamma_i)$ have a vanishing trace $w = 0$ on the boundary. Here, curl_t is the curl-operator applied on the tangential plane. Note that this identity and the trivial property $\mathrm{div}(\mathrm{curl}(w)) = 0$ further shows that the function $\mathrm{curl}(w)$ belongs to $H_0(\mathrm{div},\Omega)$. As smooth functions are dense in $H(\mathrm{curl}\,\mathrm{div},\Omega)$ (see section 4.2.2), this allows us to define the normal-tangential trace operator by

$$\begin{aligned} \gamma_{nt} &: H(\mathrm{curl}\,\mathrm{div},\Omega) \to TW(\Gamma_i)^*, \\ \gamma_{nt}(\sigma) &:= \langle \mathrm{div}(\sigma), \mathrm{curl}(w)\rangle_{H_0(\mathrm{div},\Omega)} + (\sigma, \nabla\,\mathrm{curl}(w))_\Omega. \end{aligned} \tag{4.20}$$

Similarly as for the other trace operators we omit the operator γ_{nt} whenever it is clear from the context. Integrating the first term of the second line of equation (4.19) by parts, we further see

$$\int_\Omega \mathrm{div}(\phi) \cdot \mathrm{curl}(w)\,\mathrm{d}x + \int_\Omega \phi : \nabla\,\mathrm{curl}(w)\,\mathrm{d}x = \int_\Omega \mathrm{curl}(\mathrm{div}(\phi)) \cdot w\,\mathrm{d}x + \int_\Omega \phi : \nabla\,\mathrm{curl}(w)\,\mathrm{d}x,$$

where we used that w vanishes on the boundary. Thus, equivalence (4.15) yields the integration by parts formula

$$\langle (\sigma)_{nt}, \mathrm{curl}_{\pi_t} w\rangle_{TW(\Gamma_i)} = \int_\Omega \sigma : \nabla\,\mathrm{curl}(w)\,\mathrm{d}x + \langle \mathrm{curl}(\mathrm{div}(\sigma)), w\rangle_{H^{-1}(\Omega)} \quad \forall w \in W(\Gamma_i). \tag{4.21}$$

Theorem 9 (Normal-tangential trace)**.** *The normal-tangential operator defined by equation (4.20) is linear, continuous and surjective.*

Proof. The linearity is clear due to the definition. Now let $\sigma \in H(\operatorname{curl}\operatorname{div},\Omega)$. Then we have by the definition of the normal-tangential trace

$$\begin{aligned}\|\sigma_{nt}\|_{TW(\Gamma_i)^*} &= \sup_{\mu\in TW(\Gamma_i)} \frac{\langle \sigma_{nt},\mu\rangle_{TW(\Gamma_i)}}{\|\mu\|_{TW(\Gamma_i)}} \leq \sup_{w\in W(\Gamma_i)} \frac{\langle \sigma_{nt},\operatorname{curl}_{\pi_t} w\rangle_{TW(\Gamma_i)}}{\|w\|_{H^1(\operatorname{curl},\Omega)}} \\ &= \sup_{w\in W(\Gamma_i)} \frac{\langle \operatorname{div}(\sigma),\operatorname{curl}(w)\rangle_{H_0(\operatorname{div},\Omega)}}{\|w\|_{H^1(\operatorname{curl},\Omega)}} + \sup_{w\in W(\Gamma_i)} \frac{(\sigma,\nabla\operatorname{curl}(w))_\Omega}{\|w\|_{H^1(\operatorname{curl},\Omega)}}.\end{aligned}$$

Using the Cauchy Schwarz inequality and the definition of the $H^1(\operatorname{curl})$-norm, we can bound the second term by

$$\sup_{w\in W(\Gamma_i)} \frac{(\sigma,\nabla\operatorname{curl}(w))_{L^2(\Omega)}}{\|w\|_{H^1(\operatorname{curl},\Omega)}} \leq \sup_{w\in W(\Gamma_i)} \frac{\|\sigma\|_{L^2(\Omega)}\|\nabla\operatorname{curl}(w)\|_{L^2(\Omega)}}{\|w\|_{H^1(\operatorname{curl},\Omega)}} \leq \|\sigma\|_{L^2(\Omega)}.$$

Similarly, we have for the first term

$$\sup_{w\in W(\Gamma_i)} \frac{\langle \operatorname{div}(\sigma),\operatorname{curl}(w)\rangle_{H_0(\operatorname{div},\Omega)}}{\|w\|_{H^1(\operatorname{curl},\Omega)}} \leq \sup_{w\in W(\Gamma_i)} \frac{\|\operatorname{div}(\sigma)\|_{H_0(\operatorname{div},\Omega)^*}\|\operatorname{curl}(w)\|_{H(\operatorname{div},\Omega)}}{\|w\|_{H^1(\operatorname{curl},\Omega)}}.$$

As $\|\operatorname{curl}(w)\|_{H(\operatorname{div},\Omega)} = \|\operatorname{curl}(w)\|_{L^2(\Omega)} \leq \|w\|_{H^1(\operatorname{curl},\Omega)}$, we conclude

$$\|\sigma_{nt}\|_{TW(\Gamma_i)^*} \leq \|\operatorname{div}(\sigma)\|_{H_0(\operatorname{div},\Omega)^*} + \|\sigma\|_{L^2(\Omega)} \leq \|\sigma\|_{\mathrm{cd}}.$$

Now let $g \in TW(\Gamma_i)^*$. We solve the following problem:

$$\begin{aligned} -\operatorname{curl}(\operatorname{div}(\nabla\operatorname{curl}(\tilde{w}))) - \Delta\tilde{w} + \tilde{w} &= 0 && \text{in } \Omega,\\ \tilde{w} &= 0 && \text{on } \Gamma,\\ \operatorname{curl}_{\pi_t}\tilde{w} &= 0 && \text{on } \Gamma\setminus\Gamma_i,\\ (\nabla\operatorname{curl}(\tilde{w}))_{nt} &= g && \text{on } \Gamma_i.\end{aligned}$$

The variational formulation of this problem reads as: Find $\tilde{w} \in W(\Gamma_i)$ such that

$$\int_\Omega \nabla\operatorname{curl}(\tilde{w}) : \nabla\operatorname{curl}(\tilde{v}) + \nabla\tilde{w} : \nabla\tilde{v} + \tilde{w}\cdot\tilde{v}\,\mathrm{d}x = \langle g,\operatorname{curl}_{\pi_t}\tilde{v}\rangle_{TW(\Gamma_i)}, \quad \forall \tilde{v}\in W(\Gamma_i).$$

Due to the theory provided by Lax-Milgram, see theorem 5, this problem is solvable as the right hand side is an admissible functional and the bilinearform on the left is (trivially) coercive with respect to the $H^1(\operatorname{curl})$-norm. Further, the solution fulfills the stability estimate

$$\|\tilde{w}\|_{H^1(\operatorname{curl},\Omega)} \leq \|g\|_{TW(\Gamma_i)}.$$

Now set $\sigma = \nabla\operatorname{curl}(\tilde{w})$. In the following we show that $\sigma \in H(\operatorname{curl}\operatorname{div})$ and $\sigma_{nt} = g$ in $TW(\Gamma_i)^*$. By the above stability estimate, we first have

$$\|\sigma\|_{L^2(\Omega)} = \|\nabla\operatorname{curl}(\tilde{w})\|_{L^2(\Omega)} \leq \|\tilde{w}\|_{H^1(\operatorname{curl},\Omega)} \leq \|g\|_{TW(\Gamma)},$$

thus $\sigma \in L^2(\Omega, \mathbb{R}^{d\times d})$. Next, let $\phi \in \mathcal{C}_0^\infty(\Omega, \mathbb{R}^{\tilde{d}})$. By the definition of the distributional curl div, and as ϕ is an admissible test function of the variational formulation above we have

$$\begin{aligned} \langle \operatorname{curl}(\operatorname{div}(\sigma)), \phi\rangle_{H_0^1(\Omega,\mathbb{R}^{\tilde{d}})} &= -\int_\Omega \sigma : \nabla \operatorname{curl}(\phi) \,\mathrm{d}x = \int_\Omega \nabla\tilde{w} : \nabla\phi \,\mathrm{d}x + \int_\Omega \tilde{w}\cdot\phi \,\mathrm{d}x \\ &\lesssim \|\tilde{w}\|_{H^1(\operatorname{curl},\Omega)} \|\phi\|_{H^1(\Omega)}, \end{aligned} \tag{4.22}$$

thus by a density argument

$$\begin{aligned} \|\operatorname{curl}(\operatorname{div}(\sigma))\|_{H^{-1}(\Omega,\mathbb{R}^{\tilde{d}})} &= \sup_{u\in H_0^1(\Omega,\mathbb{R}^{\tilde{d}})} \frac{\langle \operatorname{curl}(\operatorname{div}(\sigma)), u\rangle_{H_0^1(\Omega,\mathbb{R}^{\tilde{d}})}}{\|u\|_{H^1(\Omega)}} \\ &\lesssim \sup_{u\in H_0^1(\Omega,\mathbb{R}^{\tilde{d}})} \frac{\|\tilde{w}\|_{H^1(\operatorname{curl},\Omega)}\|u\|_{H^1(\Omega)}}{\|u\|_{H^1(\Omega)}} \leq \|g\|_{TW(\Gamma_i)}. \end{aligned}$$

By equation (4.16), we conclude that $\sigma \in H(\operatorname{curl}\operatorname{div}, \Omega)$. This allows us to apply the normal tangential trace. Then, the integration by parts formula (4.21) and the equality given by the first line of (4.22) shows that we have for all $\tilde{w} \in W(\Gamma_i)$ the identity

$$\begin{aligned} \langle (\sigma)_{nt}, \operatorname{curl}_{\pi_t}\tilde{v}\rangle_{TW(\Gamma_i)} &= \int_\Omega \sigma : \nabla \operatorname{curl}(w)\,\mathrm{d}x + \langle \operatorname{curl}(\operatorname{div}(\sigma)), w\rangle_{H^{-1}(\Omega)} \\ &= \int_\Omega \sigma : \nabla \operatorname{curl}(w)\,\mathrm{d}x + \int_\Omega \nabla\tilde{w} : \nabla\tilde{v}\,\mathrm{d}x + \int_\Omega \tilde{w}\cdot\tilde{v}\,\mathrm{d}x \\ &= \langle g, \operatorname{curl}_{\pi_t}\tilde{v}\rangle_{TW(\Gamma_i)}. \end{aligned}$$

By the above results we conclude that the normal-tangential trace operator is surjective and continuous. □

Using these trace operators it is now possible to define subspaces of $H(\operatorname{curl}\operatorname{div}, \Omega)$ including certain boundary conditions. This is needed in the definition of the variational formulation in the next section.

4.3 The MCS formulation

By the definition of the new function space we are now able to derive a new variational formulation of the system (4.6). We start by the introduction of the velocity space. As denoted at the beginning of this chapter our aim is to define a variational formulation demanding less regularity. To this end we define the space

$$V := H_0(\operatorname{div}, \Omega),$$

thus in contrast to the standard velocity-pressure formulation we do not aim for the velocity space $H_0^1(\Omega, \mathbb{R}^d)$, but only demand that the divergence is square integrable. Note that this (reduced) regularity property still allows us to consider normal traces of velocities in V. By this we can incorporate the Dirichlet boundary conditions (4.6d) as an essential boundary condition in the space V, so we further define

$$V_D := \{v \in H(\mathrm{div}, \Omega) : v_n = g_{D,n} \text{ on } \Gamma\}.$$

The tangential velocity $g_{D,t}$ on $\Gamma_{D,t}$ will be given as a natural boundary condition. Finally, we further define the subset of divergence free velocities by

$$V^0 := \{v \in H(\mathrm{div}, \Omega) : \mathrm{div}(v) = 0\}.$$

The appropriate space for the pressure is given by the set of square integrable functions. Note that we have to assume a zero mean value (for the uniqueness of the pressure) as we assumed that $\Gamma_{D,n} = \Gamma$, thus we set

$$Q := L_0^2(\Omega).$$

Finally, we define the stress space with a homogeneous normal-tangential trace on $\Gamma_{N,t}$ as

$$\Sigma := \{\sigma \in H(\mathrm{curl\,div}, \Omega) : \mathrm{tr}(\sigma) = 0, \sigma_{nt} = 0 \text{ on } \Gamma_{N,t}\}. \tag{4.23}$$

By the findings of section 4.2.3, the normal-tangential trace is continuous, thus Σ is a closed subspace of $H(\mathrm{curl\,div}, \Omega)$. The property $\mathrm{tr}(\sigma) = 0$ is motivated by the equivalence (4.5), hence as σ only approximates the deviatoric part of ∇u it can be chosen such that the matrix trace equals zero. Similarly as for the velocity space, one part of the Neumann boundary conditions can be incorporated as an essential boundary condition. To this end we define the space

$$\Sigma_N := \{\sigma \in H(\mathrm{curl\,div}, \Omega) : \mathrm{tr}(\sigma) = 0, \sigma_{nt} = g_{N,t} \text{ on } \Gamma_{N,t}\}.$$

Based on these function spaces we can now derive a new variational formulation using the standard approach. Multiplying (4.6c) with a test function $q \in Q$ and integrating over the domain Ω, we obtain the familiar weak incompressibility constraint

$$\int_\Omega \mathrm{div}(u) q \,\mathrm{d}x = 0. \tag{4.24}$$

Next, we test equation (4.6a) with a test function $\tau \in \Sigma$ and integrate over the domain Ω. Assuming enhanced regularity of the exact solution u, a density argument shows that we can write

$$\int_\Omega \tau : \nabla u \,\mathrm{d}x = \langle \mathrm{div}(\tau), u \rangle_V - \langle \tau_{nt}, g_{D,t} \rangle_{TW(\Gamma_{D,t})},$$

where we used that $u_n = 0$ on $\Gamma_{D,n}$ and $\tau_{nt} = 0$ on $\Gamma_{N,t}$. Thus, we get the equation

$$\int_\Omega \frac{1}{\nu} \sigma : \tau \,\mathrm{d}x + \langle \mathrm{div}(\tau), u \rangle_V = \langle \tau_{nt}, g_{D,t} \rangle_{TW(\Gamma_{D,t})}. \tag{4.25}$$

Finally, we also test equation (4.6b) with a test function $v \in V$, and integrate over the domain and by parts in the term including the pressure. This yields

$$\langle \operatorname{div}(\sigma), u\rangle_V + \int_\Omega \operatorname{div}(v) p \,\mathrm{d}x = -\int_\Omega f \cdot v \,\mathrm{d}x\,.$$

Now assume a given $f \in V^*$ and $g_{D,t} \in TW(\Gamma_{D,t})$. Further assume the increased regularity of the essential boundary data $g_{D,n} \in [H^{1/2}(\Gamma_D)]_n$ and $g_{N,t} \in [H^{-1/2}(\Gamma_N, \mathbb{R}^d)]_t$. The mass conserving mixed stress formulation (MCS) now reads as: Find $(\sigma, u, p) \in \Sigma_N \times V_D \times Q$ such that

$$\begin{cases} \displaystyle\int_\Omega \frac{1}{\nu}\sigma : \tau \,\mathrm{d}x + \langle \operatorname{div}(\tau), u\rangle_V = \langle \tau_{nt}, g_{D,t}\rangle_{TW(\Gamma_{D,t})} & \text{for all } \tau \in \Sigma, \\ \displaystyle\langle \operatorname{div}(\sigma), v\rangle_V + \int_\Omega \operatorname{div}(v) p \,\mathrm{d}x = -\langle f, v\rangle_V & \text{for all } v \in V, \\ \displaystyle\int_\Omega \operatorname{div}(u) q \,\mathrm{d}x = 0 & \text{for all } q \in Q. \end{cases} \tag{4.26}$$

In the following we check if the exact solution (σ, u, p) of the system (4.26), assuming enough regularity, fulfills the boundary conditions (4.3c),(4.3d) and (4.3f). Incorporated as essential boundary conditions into the spaces V_D and Σ_N, we already have

$$u_n = g_{D,n} \quad \text{and} \quad \sigma_{nt} = g_{N,t},$$

on Γ and $\Gamma_{N,t}$, respectively. In the following we assume enough regularity of the given right hand side. The second line of (4.26) then reads as

$$\int_\Omega \operatorname{div}(\sigma) \cdot v \,\mathrm{d}x + \int_\Omega \operatorname{div}(v) p \,\mathrm{d}x = -\int_\Omega f \cdot v \,\mathrm{d}x\,.$$

Using integration by parts for the third term we have, as $v_n = 0$ on Γ,

$$\int_\Omega \operatorname{div}(\sigma) \cdot v \,\mathrm{d}x - \int_\Omega v \cdot \nabla p \,\mathrm{d}x + \int_{\Gamma_N} p v_n = -\int_\Omega f \cdot v \,\mathrm{d}x\,.$$

Choosing test functions with a compact support in Ω we conclude that in the volume we have $\operatorname{div}(\sigma) - \nabla p = -f$. In the same manner we proceed with the first equation of (4.26). Using integration by parts for the volume integral we have

$$\int_\Omega \frac{1}{\nu}\sigma : \tau \,\mathrm{d}x - \int_\Omega \tau : \nabla u \,\mathrm{d}x + \int_\Gamma \tau_n \cdot u \,\mathrm{d}s = \int_{\Gamma_D} \tau_{nt} \cdot g_{D,t} \,\mathrm{d}s\,.$$

Splitting the third integral into a normal-normal part and a normal-tangential part, thus

$$\int_\Gamma \tau_n \cdot u \,\mathrm{d}s = \int_\Gamma \tau_{nn} u_n \,\mathrm{d}s + \int_\Gamma \tau_{nt} \cdot u_t \,\mathrm{d}s,$$

and using $u \in V_D$ and $\tau \in \Sigma$ we have

$$\int_\Omega \frac{1}{\nu}\sigma : \tau \,\mathrm{d}x - \int_\Omega \tau : \nabla u \,\mathrm{d}x + \int_{\Gamma_D} \tau_{nt} \cdot u_t \,\mathrm{d}s = \int_{\Gamma_D} \tau_{nt} \cdot g_{D,t} \,\mathrm{d}s\,.$$

With the same arguments as above we conclude $\sigma = \nu \nabla u$ in the volume and $u_t = g_{D,t}$ on the boundary.

We want to make a final comment on the increased regularity of the essential boundary data. As usual, in the case of non-homogeneous essential boundary conditions, one solves equations (4.26) using a homogenization step, thus one splits the solutions into two parts

$$\begin{aligned} u &= u_0 + u_g \quad \text{with} \quad u_0 \in V, u_g \in V_D, \\ \sigma &= \sigma_0 + \sigma_g \quad \text{with} \quad \sigma_0 \in \Sigma, \sigma_g \in \Sigma_N, \end{aligned}$$

where u_g and σ_g are found using the surjectivity of the trace operators. In particular, due to the increased regularity of the essential boundary data, one finds functions $u_g \in H^1(\Omega, \mathbb{R}^d)$ and $\sigma_g \in [H(\mathrm{div}, \Omega)]^d$ such that

$$(u_g)_n = g_{D,n} \text{ on } \Gamma_{D,n} \quad \text{and} \quad (\sigma_g)_{nt} = g_{N,t} \text{ on } \Gamma_{N,t}.$$

To find the homogeneous solutions we then solve the problem: Find $(\sigma_0, u_0, p) \in \Sigma \times V \times Q$ such that

$$\left\{ \begin{aligned} \int_\Omega \frac{1}{\nu} \sigma : \tau \, \mathrm{d}x + \langle \mathrm{div}(\tau), u \rangle_V &= \langle \tau_{nt}, g_{D,t} \rangle_{TW(\Gamma_D)} - \langle \mathrm{div}(\tau), u_g \rangle_V && \forall \tau \in \Sigma, \\ \langle \mathrm{div}(\sigma), v \rangle_V + \int_\Omega \mathrm{div}(v) p \, \mathrm{d}x &= -\langle f, v \rangle_V - \langle \mathrm{div}(\sigma_g), v \rangle_V && \forall v \in V, \\ \int_\Omega \mathrm{div}(u) q \, \mathrm{d}x &= 0 && \forall q \in Q. \end{aligned} \right.$$

Note that the terms $\langle \mathrm{div}(\tau), u_g \rangle_V$ and $\langle \mathrm{div}(\sigma_g), v \rangle_V$ are well defined, as the increased regularity of u_g and σ_g yields that the duality pairs can be written as

$$\langle \mathrm{div}(\tau), u_g \rangle_V = -\int_\Omega \tau : \nabla u_g \, \mathrm{d}x, \quad \text{and} \quad \langle \mathrm{div}(\sigma_g), v \rangle_V = \int_\Omega \mathrm{div}(\sigma_g) \cdot v \, \mathrm{d}x \, .$$

If we would only assume $g_{D,n} \in H^{-1/2}(\Gamma_D)$ and $g_{N,t} \in [TW(\Gamma_N)]^*$, the surjectivity of the trace operators would imply $u_g \in H(\mathrm{div}, \Omega)$ and $\sigma_g \in H(\mathrm{curl}\,\mathrm{div}, \Omega)$, thus the duality pairs above would not be well defined, and a homogenization process would not be possible.

4.3.1 Stability analysis

In this section we present a detailed stability analysis of the mass conserving mixed stress formulation given by the set of equations (4.26). Analyzing the structure of the left hand side, we realize that the resulting system is a saddle point problem where σ and p are the primal variable, and u is the dual variable. Similarly as in the mixed formulation of a Poisson problem, the role of the variables switched, as in the classical velocity-pressure formulation of the Stokes equations the velocity u is the primal variable and the pressure p is the dual variable. In order to show stability we aim to use the theory of saddle point problems as discussed in section 3.2. As usual, we only consider the case of homogeneous essential boundary values of the normal component u_n and the normal-tangential trace σ_{nt} on Γ and $\Gamma_{N,t}$, respectively. The case of non-homogeneous essential values follows again by homogenization.

In the set of equations (4.26) we identify the following bilinear forms. In the top left we have the L^2-inner product of the primal variable σ with a test function τ, thus we define

$$\begin{cases} a : L^2(\Omega, \mathbb{R}^{d\times d}) \times L^2(\Omega, \mathbb{R}^{d\times d}) \to \mathbb{R}, \\ a(\sigma, \tau) := \int_\Omega \frac{1}{\nu} \sigma : \tau \,\mathrm{d}x \,. \end{cases} \tag{4.27}$$

Note, that although the bilinear form a is continuous with respect to the L^2-norm on $L^2(\Omega, \mathbb{R}^{d\times d})$, for functions $\sigma, \tau \in \Sigma$ it is equivalent to

$$a(\sigma, \tau) = \int_\Omega \frac{1}{\nu} \mathrm{dev}(\sigma) : \mathrm{dev}(\tau) \,\mathrm{d}x \,.$$

The next bilinear form is given by the incompressibility constraint (4.24)

$$\begin{cases} b_1 : H(\mathrm{div}, \Omega) \times L^2(\Omega, \mathbb{R}) \to \mathbb{R}, \\ b_1(u, q) := \int_\Omega \mathrm{div}(u) q \,\mathrm{d}x \,. \end{cases} \tag{4.28}$$

Finally, we have the divergence constraint for the stress variable σ. As discussed above, the velocity u is now a dual variable, and can be interpreted as a Lagrange multiplier to enforce the constraint on σ. We define

$$\begin{cases} b_2 : H(\mathrm{curl}\,\mathrm{div}, \Omega) \times H_0(\mathrm{div}, \Omega) \to \mathbb{R}, \\ b_2(\sigma, u) := \langle \mathrm{div}(\sigma), u \rangle_{H_0(\mathrm{div}, \Omega)}. \end{cases} \tag{4.29}$$

Adding up both constraints we further define the bilinear form

$$\begin{cases} b : H(\mathrm{curl}\,\mathrm{div}, \Omega) \times L^2(\Omega, \mathbb{R}) \times H_0(\mathrm{div}, \Omega) \to \mathbb{R}, \\ b(\sigma, q; u) := b_1(u, q) + b_2(\sigma, q), \end{cases} \tag{4.30}$$

and the corresponding kernel

$$K_b := \{(\sigma, p) \in \Sigma \times Q : b(\sigma, q; u) = 0 \text{ for all } u \in V\}. \tag{4.31}$$

Beside the bilinear forms we define for all $\sigma \in \Sigma$, $u \in V$ and $q \in Q$ the following norms

$$\|\sigma\|_\Sigma := \|\sigma\|_{\mathrm{cd}}, \quad \|u\|_V := \|u\|_{H(\mathrm{div}, \Omega)}, \quad \|q\|_Q := \|q\|_{L^2(\Omega)}.$$

The first result shows that the introduced bilinear forms are all continuous.

Lemma 8 (Continuity)**.** *The bilinear forms a, b_1 and b_2 are continuous:*

$$\begin{aligned} a(\sigma, \tau) &\lesssim \frac{1}{\sqrt{\nu}} \|\sigma\|_\Sigma \frac{1}{\sqrt{\nu}} \|\tau\|_\Sigma && \forall \sigma, \tau \in \Sigma, \\ b_1(u, q) &\lesssim \|u\|_V \|q\|_Q && \forall u \in Q, q \in Q, \\ b_2(\sigma, u) &\lesssim \|\sigma\|_\Sigma \|u\|_V && \forall \sigma \in \Sigma, u \in V. \end{aligned}$$

Proof. The continuity of a and b_1 follows with a Cauchy Schwarz argument for the appearing integrals. For b_2 it follows by the continuity of the duality pair with the corresponding norms and $\|\operatorname{div}(\sigma)\|_{V^*} \leq \|\sigma\|_\Sigma$. □

We continue with proving that the bilinear form a is coercive on the kernel.

Lemma 9 (Coercivity on the kernel)**.** *The bilinearform a is coercive on the kernel K_b*

$$\frac{1}{\nu}(\|\sigma\|_\Sigma + \|p\|_Q)^2 \lesssim a(\sigma,\sigma) \quad \textit{for all } (\sigma,p) \in K_b.$$

Proof. Let $(\sigma,p) \in K_b$ be arbitrary. Using the Stokes-LBB, lemma 4.9 in [43], we find a function $u \in H^1_0(\Omega,\mathbb{R}^d) \subset V$ such that $\operatorname{div}(u) = p$ and $\|u\|_{H^1(\Omega)} \leq \|p\|_Q$. Thus, we can bound

$$\begin{aligned}\|p\|_Q &\leq \frac{(\operatorname{div}(u),p)_\Omega}{\|u\|_{H^1(\Omega)}} \leq \sup_{v\in H^1_0(\Omega,\mathbb{R}^d)} \frac{(\operatorname{div}(v),p)_\Omega}{\|v\|_{H^1(\Omega)}} \\ &= \sup_{v\in H^1_0(\Omega,\mathbb{R}^d)} \frac{\langle \operatorname{div}(\sigma),v\rangle_V}{\|v\|_{H^1(\Omega)}} = \sup_{v\in H^1_0(\Omega,\mathbb{R}^d)} \frac{(\sigma,\nabla u)}{\|v\|_{H^1(\Omega)}} \leq \|\sigma\|_{L^2(\Omega)}.\end{aligned}$$

Similarly, we further have

$$\sup_{v\in V} \frac{\langle \operatorname{div}(\sigma),v\rangle_V}{\|v\|_V} = \sup_{v\in V} \frac{(\operatorname{div}(v),p)_\Omega}{\|v\|_V} \leq \|p\|_Q \leq \|\sigma\|_{L^2(\Omega)}.$$

This yields for all $(\sigma,p) \in K_b$ that $(\|\sigma\|_\Sigma + \|p\|_Q) \leq \nu a(\sigma,\sigma)$. A division by ν gives the desired result. □

Lemma 10. *For all $u \in V$ there exists a function $q \in Q$ such that*

$$b_1(u,q) \gtrsim \|\operatorname{div}(u)\|^2_{L^2(\Omega)} \quad \textit{and} \quad \|q\|_Q \lesssim \|u\|_V.$$

Proof. This follows immediately by choosing $q := \operatorname{div}(u)$. This is an admissible choice as

$$\int_\Omega \operatorname{div}(u)\,\mathrm{d}x = \int_\Gamma u_n\,\mathrm{d}s = 0,$$

thus $q \in Q$. Now we have

$$b_1(u,q) = \int_\Omega \operatorname{div}(u) q\,\mathrm{d}x = \|\operatorname{div}(u)\|^2_{L^2(\Omega)},$$

and trivially $\|q\|_Q = \|\operatorname{div}(u)\|_{L^2(\Omega)} \leq \|u\|_V$. □

To prove a similar inf-sup stability result for the bilinear form b_2, we are going to solve a symmetric auxiliary problem on the subspace $H^1(\Omega,\mathbb{R}^d) \cap V$. In order to solve this auxiliary problem we need to show that it is well-posed and solvable. The next two results are dedicated to this question. The first theorem shows a very similar estimate as given by Korn's first inequality, see theorem 4. The second result then handles the case with (partly) homogeneous boundary conditions.

Theorem 10 (A Korn-like inequality for the deviator)**.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be an arbitrary bounded and connected Lipschitz domain. For $u \in H^1(\Omega, \mathbb{R}^d)$ there holds*

$$\|\mathrm{dev}(\nabla u)\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \geq c_d \|u\|^2_{H^1(\Omega)},$$

where c_d only depends on Ω.

Proof. The proof follows similar steps as the proof of Korn's inequality, theorem 3.1 in [39]. For the ease of notation let $\mathbb{D}(u) := \mathrm{dev}(\nabla u)$, then we define the subspace

$$D := \{u \in L^2(\Omega, \mathbb{R}^d) : \mathbb{D}(u) \in L^2(\Omega, \mathbb{R}^{d\times d})\}. \tag{4.32}$$

Note that D is a Hilbert space with the norm $\|u\|^2_D := \|\mathbb{D}(u)\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}$. Let $u \in D$, then elementary calculations show that in three dimensions we can write for all $i, j, k \in \{1, 2, 3\}$

$$\frac{\partial^2 u_i}{\partial x_j \partial x_k} = \begin{cases} \frac{\partial}{\partial x_j}\mathbb{D}(u)_{i,k} & \text{for } i \neq k, \\ \frac{\partial}{\partial x_k}\mathbb{D}(u)_{i,j} & \text{for } i \neq j, \\ 3\frac{\partial}{\partial x_i}\mathbb{D}(u)_{i,i} + \frac{\partial}{\partial x_{i+1}}\mathbb{D}(u)_{i+1,i} + \frac{\partial}{\partial x_{i+2}}\mathbb{D}(u)_{i+2,i} & \text{for } i = j = k, \end{cases}$$

where the indices $i + 1$ and $i + 2$ are taken modulo 3. Similarly, we can write in two dimensions for all $i, j, k \in \{1, 2\}$

$$\frac{\partial^2 u_i}{\partial x_j \partial x_k} = \begin{cases} \frac{\partial}{\partial x_i}\mathbb{D}(u)_{i,k} & \text{for } i \neq k, \\ \frac{\partial}{\partial x_k}\mathbb{D}(u)_{i,j} & \text{for } i \neq j, \\ 2\frac{\partial}{\partial x_i}\mathbb{D}(u)_{i,i} + \frac{\partial}{\partial x_{i+1}}\mathbb{D}(u)_{i+1,i} & \text{for } i = j = k, \end{cases},$$

where the index $i + 1$ is taken modulo 2. Therefore, we have, as $u \in D$, that

$$\frac{\partial^2 u_i}{\partial x_j \partial x_k} \in H^{-1}(\Omega) \quad \text{for all } i, j, k \in \{1, \ldots, d\}. \tag{4.33}$$

Applying the famous result from Nečas (see [88], and for non-smooth boundaries, see [15]) which reads as

$$\|f\|_{L^2(\Omega)} \leq \|f\|_{H^{-1}(\Omega)} + \|\nabla f\|_{H^{-1}(\Omega)} \quad \text{for all} \quad f \in L^2(\Omega, \mathbb{R}),$$

to the partial derivatives $\partial u_i/\partial x_k$, equation (4.33) implies that $\partial u_i/\partial x_k$ is an element of $L^2(\Omega)$ for all i, k, and therefore $u \in H^1(\Omega, \mathbb{R}^d)$. This shows the algebraic equivalence $D = H^1(\Omega, \mathbb{R}^d)$ (as each element of the right side is also in D), and that the injection of $H^1(\Omega, \mathbb{R}^d) \to D$ is continuous, and – as just proven – also surjective. Applying the closed graph theorem, we conclude that this injection is an isomorphism, thus we have proven norm equivalence. □

Lemma 11. *Let $\Gamma_i \subseteq \partial\Omega$ and $\Gamma_j \subset \partial\Omega$. Assume that either $\Gamma_i = \partial\Omega$ or $\Gamma_j \cap \Gamma_i \neq \emptyset$. Let $u \in \{v \in H^1(\Omega, \mathbb{R}^d), v_n = 0 \text{ on } \Gamma_i\}$. There holds*

$$\int_\Omega \mathrm{dev}(\nabla \tilde{u}) : \mathrm{dev}(\nabla \tilde{v})\,\mathrm{d}x + \int_{\Gamma_j} \tilde{u}_t \cdot \tilde{v}_t\,\mathrm{d}s \geq \|u\|^2_1.$$

Proof. By theorem 10 it suffices to show that $\|u\|_{L^2(\Omega)} \leq \|\mathbb{D}(u)\|^2_{L^2(\Omega)} + \|u_t\|^2_{L^2(\Gamma_j)}$, where again $\mathbb{D}(u) = \operatorname{dev}(\nabla u)$. We prove this by contradiction. To this end we assume that there exists no $\alpha > 0$ such that

$$\|\mathbb{D}(u)\|^2_{L^2(\Omega)} + \|u_t\|^2_{L^2(\Gamma_j)} \geq \alpha \|u\|_{L^2(\Omega)}.$$

Then for all $l \in \mathbb{N}^*$ there exists a function $v_l \in \{v \in H^1(\Omega, \mathbb{R}^d), v_n = 0 \text{ on } \Gamma_i\}$ such that

$$\|\mathbb{D}(v_l)\|^2_{L^2(\Omega)} + \|(v_l)_t\|^2_{L^2(\Gamma_j)} \leq \frac{1}{l}\|v_l\|_{L^2(\Omega)}. \tag{4.34}$$

Further, without loss of generality, we can choose $\|v_l\|_{L^2(\Omega)} = 1$. By theorem 10 and estimate (4.34), we further have

$$\|v_l\|_{H^1(\Omega)} \leq \|v_l\|_{L^2(\Omega)} + \|\mathbb{D}(v_l)\|_{L^2(\Omega)} \leq \|v_l\|_{L^2(\Omega)},$$

thus v_l is bounded in $H^1(\Omega, \mathbb{R}^d)$. Applying the theorem of Rellich, see chapter VI in [2], this yields that v_l is also relatively compact in $L^2(\Omega, \mathbb{R}^d)$, hence there exists a sub sequence (again denoted by v_l) such that $v_l \to v$ with $v \in L^2(\Omega, \mathbb{R}^d)$. Further, estimate (4.34) yields $\|\mathbb{D}(v_l)\|_{L^2(\Omega)} \to 0$, thus $\mathbb{D}(v) = 0$ and by theorem 10

$$\|v - v_l\|_{H^1(\Omega)} \leq \|v - v_l\|_{L^2(\Omega)} + \|\mathbb{D}(v_l)\|_{L^2(\Omega)} \to 0, \tag{4.35}$$

thus $v \in H^1(\Omega, \mathbb{R}^d)$. By the continuity of the trace operator we conclude that $v_n = 0$ on Γ_i as $v_l \in \{v \in H^1(\Omega, \mathbb{R}^d), v_n = 0 \text{ on } \Gamma_i\}$ for all $l \in \mathbb{N}^*$. Equation (4.34) further shows that also $v_t = 0$ on Γ_j. As $\mathbb{D}(v) = 0$, it is either zero or has the form $v = a + bx$ with $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$. Assuming $\Gamma_j \cap \Gamma_i \neq \emptyset$ implies that $v = 0$ on $\Gamma_j \cap \Gamma_i$, which yields $v = 0$. For the other case, $\Gamma_i = \partial\Omega$, the identity $\operatorname{div}(u) = db$ first shows the equivalence

$$b|\Omega|d = \int_\Omega \operatorname{div}(\tilde{u}) \,\mathrm{d}x = \int_\Omega \operatorname{div}(\tilde{u}) \,\mathrm{d}x = \int_{\partial\Omega} \tilde{u} \cdot n \,\mathrm{d}s = 0,$$

implying $b = 0$. As $v_n = 0$ on Γ, we similarly as before conclude that $a = 0$. In both cases equation (4.35) then further shows $\|v_l\|_{L^2(\Omega)} \to 0$, which is a contradiction, thus the lemma is proven. □

Using the two previous results we are finally ready to show an inf-sup result for b_2.

Lemma 12. *For all $u \in V$ there exists a $\sigma \in \Sigma$ such that*

$$b_2(\sigma, u) \gtrsim \|u\|^2_V \quad \text{and} \quad \|\sigma\|_\Sigma \lesssim \|u\|_V.$$

Proof. Let $u \in V$ be arbitrary. We solve the following auxiliary problem: Find $\tilde{u} \in \tilde{V}$ with $\tilde{V} := H^1(\Omega, \mathbb{R}^d) \cap V$, such that

$$\int_\Omega \operatorname{dev}(\nabla \tilde{u}) : \operatorname{dev}(\nabla \tilde{v}) \,\mathrm{d}x = \int_\Omega u \cdot \tilde{v} \,\mathrm{d}x + \int_\Omega \operatorname{div}(u) \operatorname{div}(\tilde{v}) \,\mathrm{d}x \quad \forall \tilde{v} \in \tilde{V}. \tag{4.36}$$

Applying lemma 11, we immediately see that the bilinear form on the left hand side is coercive with respect to the H^1-norm. A simple Cauchy Schwarz argument then further shows the continuity, thus theorem 5 (Lax-Milgram) yields that there exists a unique solution, which is continuously bounded by the right hand side:

$$\|\tilde{u}\|_{H^1(\Omega)} \lesssim \|u\|_V. \tag{4.37}$$

Now set $\sigma := -\mathrm{dev}(\nabla\tilde{u})$. In the following we show that $\sigma \in \Sigma$. The above continuity of the solution $\tilde{u}$, proves that σ is square integrable as

$$\|\sigma\|_{L^2(\Omega)} = \|\mathrm{dev}(\nabla\tilde{u})\|_{L^2(\Omega)} \leq \|\tilde{u}\|_{H^1(\Omega)} \lesssim \|u\|_V.$$

Next, let $\phi \in \mathcal{C}_0^\infty(\Omega, \mathbb{R}^d)$. Using the definition of the distributional divergence we see that

$$\langle \mathrm{div}(\sigma), \phi\rangle_V = -\int_\Omega \sigma : \nabla\phi \,\mathrm{d}x = \int_\Omega \mathrm{dev}(\nabla\tilde{u}) : \nabla\phi \,\mathrm{d}x = \int_\Omega \mathrm{dev}(\nabla\tilde{u}) : \mathrm{dev}(\nabla\phi)\,\mathrm{d}x\,.$$

As ϕ is an admissible test function for the weak formulation (4.36), we observe

$$\langle \mathrm{div}(\sigma), \phi\rangle_V = \int_\Omega u\cdot\phi\,\mathrm{d}x + \int_\Omega \mathrm{div}(u)\,\mathrm{div}(\phi)\,\mathrm{d}x \lesssim \|u\|_V\|\phi\|_V.$$

The density of $\mathcal{C}_0^\infty(\Omega, \mathbb{R}^d)$ in V then yields

$$\sup_{v\in V} \frac{\langle \mathrm{div}(\sigma), v\rangle_V}{\|v\|_V} \lesssim \|u\|_V,$$

thus summing up both bounds proves $\|\sigma\|_\Sigma \lesssim \|u\|_V$. Further note that the above arguments show the identity

$$b_2(\sigma, u) = \|u\|^2_{L^2(\Omega)} + \|\,\mathrm{div}(u)\|^2_{L^2(\Omega)} \sim \|u\|^2_V. \tag{4.38}$$

Finally, as $\mathrm{tr}(\sigma) = 0$, we conclude that $\sigma \in H(\mathrm{curl}\,\mathrm{div}, \Omega)$, which allows us to apply the normal-tangential trace operator. Now let $tw \in TW(\Gamma_{N,t})$ be arbitrary. By definition of the trace space $TW(\Gamma_{N,t})$, there exists a $w \in W(\Gamma_{N,t})$ such that $(\mathrm{curl}(w))_t = tw$ on $\Gamma_{N,t}$. The definition of the trace operator then gives

$$\langle \gamma_{nt}(\sigma), tw\rangle_{TW(\Gamma_{N,t})} = \langle \mathrm{div}(\sigma), \mathrm{curl}(w)\rangle_{H_0(\mathrm{div},\Omega)} + (\sigma, \nabla\,\mathrm{curl}(w))_\Omega.$$

Next, as $w \in W(\Gamma_{N,t})$ we have $(\mathrm{curl}(w))_n = \mathrm{curl}_t(\gamma_t(w)) = 0$ on Γ, thus $\mathrm{curl}(w)$ is an admissible test function for the weak formulation (4.36). Using the same density arguments as above yields

$$\begin{aligned}\langle \gamma_{nt}(\sigma), tw\rangle_{TW(\Gamma_{N,t})} &= \langle \mathrm{div}(\sigma), \mathrm{curl}(w)\rangle_{H_0(\mathrm{div},\Omega)} + (-\mathrm{dev}(\nabla\tilde{u}), \nabla\,\mathrm{curl}(w))_\Omega \\ &= \langle \mathrm{div}(\sigma), \mathrm{curl}(w)\rangle_{H_0(\mathrm{div},\Omega)} - (u, \mathrm{curl}(w))_\Omega - (\mathrm{div}(u), \mathrm{div}(\mathrm{curl}(w)))_\Omega = 0,\end{aligned}$$

thus $\sigma \in \Sigma$ and the lemma is proven. □

In the next theorem we combine both result to prove the inf-sup estimate of the bilinear form b.

Theorem 11 (Inf-sup estimate of b)**.** *For all $u \in V$ there exists a $(\sigma, p) \in \Sigma \times Q$ such that*

$$\sup_{(\tau,q)\in\Sigma\times Q} \frac{b(\tau,q;u)}{\|\tau\|_\Sigma + \|q\|_Q} \gtrsim \|u\|_V. \tag{4.39}$$

Proof. Let $\sigma \in \Sigma$ and $p \in Q$ be the variables given by lemma 10 and lemma 12, respectively. Then there holds

$$\sup_{(\tau,q)\in\Sigma\times Q} \frac{b(\tau,q;u)}{\|\tau\|_\Sigma + \|q\|_Q} \geq \frac{b_1(u,p) + b_2(\sigma,u)}{\|p\|_Q + \|\sigma\|_\Sigma} = \frac{\|\operatorname{div}(u)\|^2_{L^2(\Omega)} + \|u\|^2_V}{\|p\|_Q + \|\sigma\|_\Sigma} \gtrsim \|u\|_V,$$

where we used that $\|\operatorname{div}(u)\|^2_{L^2(\Omega)} + \|u\|^2_V \sim \|u\|^2_V$ and $\|p\|_Q + \|\sigma\|_\Sigma \lesssim \|u\|_V$. □

Corollary 1. *Assume a given $f \in V^*$ and $g_{D,t} \in TW(\Gamma_D)$. Further assume the increased regularity of the essential boundary data $g_{D,n} \in H^{1/2}(\Gamma_D)$ and $g_{N,t} \in H^{-1/2}(\Gamma_N, \mathbb{R}^d)$. There exists a unique solution $(\sigma, u, p) \in \Sigma_N \times V_D \times Q$ of equations (4.26) such that*

$$\begin{aligned}\frac{1}{\sqrt{\nu}}\big(\|\sigma\|_\Sigma + \|p\|_Q\big) + \sqrt{\nu}\|u\|_V \lesssim \frac{1}{\sqrt{\nu}}\Big(\|f\|_{V^*} + \|g_{N,t}\|_{H^{-1/2}(\Gamma_N)}\Big)\\ + \sqrt{\nu}\Big(\|g_{D,t}\|_{TW(\Gamma_D)} + \|g_{D,n}\|_{H^{1/2}(\Gamma_D)}\Big).\end{aligned}$$

Proof. This is a direct consequence of theorem 6, and the results of lemma 8, lemma 9 and theorem 11. In the case of non-homogeneous essential boundary conditions a standard homogenization method is used as discussed at the end of section 4.3. □

4.4 The symmetric version of Stokes equations

A different version of the Stokes equations is given if we replace the gradient of the velocity u by its symmetric version $\varepsilon(u)$, hence equation (4.3a) then reads as

$$-\operatorname{div}(\nu\varepsilon(u)) + \nabla p = f \quad \text{in} \quad \Omega.$$

This formulation is considered for example in [11], and is particulary of interest in two-phase flows, see for example in [59]. There it is important that the matching conditions between the two phases is formulated with the symmetric gradient. Assuming $f \in L^2(\Omega, \mathbb{R}^d)$ and enough regularity of boundary data, the classical variational formulation then reads as: Find (u, p) in $X_D \times L^2(\Omega)$ such that

$$\left\{\begin{aligned}\int_\Omega \nu\varepsilon(u) : \varepsilon(v)\,\mathrm{d}x - \int_\Omega \operatorname{div}(v)p\,\mathrm{d}x &= \int_\Omega f\cdot v\,\mathrm{d}s - \int_{\Gamma_{N,n}} g_{N,n}v_n\,\mathrm{d}s - \int_{\Gamma_{N,t}} g_{N,t}v_t\,\mathrm{d}s, && \forall v \in X_0,\\ \int_\Omega \operatorname{div}(u)q\,\mathrm{d}x &= 0, && \forall q \in L^2(\Omega).\end{aligned}\right. \tag{4.40}$$

Note that in the case $\Gamma_{D,n} = \Gamma$ the pressure space is exchanged to $L^2_0(\Omega)$. In the following we want to derive a similar stress formulation as it is given by equation (4.6) using the symmetric gradient. Introducing $\sigma = \nu\varepsilon(u)$ and using the same manipulation as in the beginning of section 4.1 we derive the equations

$$
\begin{aligned}
\frac{1}{\nu}\operatorname{dev}(\sigma) - \varepsilon(u) &= 0 && \text{in} \quad \Omega, && \text{(4.41a)}\\
\operatorname{div}(\sigma) - \nabla p &= -f && \text{in} \quad \Omega, && \text{(4.41b)}\\
\operatorname{div}(u) &= 0 && \text{in} \quad \Omega, && \text{(4.41c)}\\
u_n &= g_{D,n} && \text{on} \quad \Gamma_{D,n}, && \text{(4.41d)}\\
u_t &= g_{D,t} && \text{on} \quad \Gamma_{D,t}, && \text{(4.41e)}\\
-\sigma_{nn} + p &= g_{N,n} && \text{on} \quad \Gamma_{N,n}, && \text{(4.41f)}\\
-\sigma_{nt} &= g_{N,t} && \text{on} \quad \Gamma_{N,t}. && \text{(4.41g)}
\end{aligned}
$$

Although formulated in an arbitrary setting (used for the discrete method introduced in chapter 7), we assume again that $\Gamma_{D,n} = \Gamma$.

Based on (4.41) we now want to derive a new variational formulation where we include the findings from above. Thus, again we aim for a formulation where the velocity space is given by $V = H_0(\operatorname{div}, \Omega)$ and the pressure space by $Q = L^2_0(\Omega)$. Note however that we have to change the stress space. Equation (4.41a) shows that the stress variable now approximates the deviatoric part of the symmetric gradient. To this end we define the closed subspaces of functions in $H(\operatorname{curl}\operatorname{div}, \Omega)$ that are trace free and symmetric:

$$
\begin{aligned}
\Sigma^{\text{sym}} &:= \{\sigma \in H(\operatorname{curl}\operatorname{div}, \Omega) : \operatorname{tr}(\sigma) = 0, \sigma + \sigma^{\mathrm{T}} = 0, \sigma_{nt} = 0 \text{ on } \Gamma_{N,t}\},\\
\Sigma^{\text{sym}}_N &:= \{\sigma \in H(\operatorname{curl}\operatorname{div}, \Omega) : \operatorname{tr}(\sigma) = 0, \sigma + \sigma^{\mathrm{T}} = 0, \sigma_{nt} = g_{N,t} \text{ on } \Gamma_{N,t}\}.
\end{aligned}
$$

Following the same steps as before we test equation (4.41a) with a test function $\tau \in \Sigma^{\text{sym}}$ and integrate over the domain Ω. Due to the symmetry of τ there holds the identity

$$
\begin{aligned}
\int_\Omega \varepsilon(u) : \tau \,\mathrm{d}x &= \frac{1}{2}\int_\Omega \nabla u : \tau \,\mathrm{d}x + \frac{1}{2}\int_\Omega (\nabla u)^{\mathrm{T}} : \tau \,\mathrm{d}x\\
&= \frac{1}{2}\int_\Omega \nabla u : \tau \,\mathrm{d}x + \frac{1}{2}\int_\Omega \nabla u : \tau \,\mathrm{d}x = \int_\Omega \nabla u : \tau \,\mathrm{d}x,
\end{aligned} \tag{4.42}
$$

thus again using the enhanced regularity of the velocity field we have the equation

$$
\int_\Omega \frac{1}{\nu}\sigma : \tau \,\mathrm{d}x + \langle \operatorname{div}(\tau), u\rangle_V = \langle \tau_{nt}, g_{D,t}\rangle_{TW(\Gamma_D)}.
$$

Note that the second term on the left is the bilinearform b_2 that was introduced in the standard formulation. For the rest we follow the same steps as in section 4.3. Now let $f \in V^*$ and $g_{D,t} \in TW(\Gamma_D)$. Further assume the increased regularity of the essential boundary data $g_{D,n} \in [H^{1/2}(\Gamma_D)]_n$ and $g_{N,t} \in [H^{-1/2}(\Gamma_N, \mathbb{R}^d)]_t$. The symmetric MCS formulation now reads as: Find $(\sigma, u, p) \in \Sigma^{\text{sym}}_N \times V_D \times Q$ such that

$$\begin{cases} \displaystyle\int_\Omega \frac{1}{\nu}\sigma : \tau \,\mathrm{d}x + \langle \operatorname{div}(\tau), u\rangle_V = \langle \tau_{nt}, g_{D,t}\rangle_{TW(\Gamma_D)} & \text{for all } \tau \in \Sigma^{\mathrm{sym}}, \\ \langle \operatorname{div}(\sigma), v\rangle_V + \displaystyle\int_\Omega \operatorname{div}(v) p \,\mathrm{d}x = -\langle f, v\rangle_V + \langle v_n, g_{N,n}\rangle_{H^{1/2}(\Gamma_N)} & \text{for all } v \in V, \\ \displaystyle\int_\Omega \operatorname{div}(u) q \,\mathrm{d}x = 0 & \text{for all } q \in Q. \end{cases} \tag{4.43}$$

4.4.1 Stability analysis

Again we only consider the case of homogeneous essential boundary conditions as the non-homogeneous case follows with the techniques discussed at the end of section 4.3. Further, comparing the symmetric weak formulation (4.43) with the standard formulation (4.26) we realize that we have only exchanged the stress space, thus we aim to derive a stability analysis using the same norms and the same bilinear forms as we have defined in section 4.3.1.

By the definition of the symmetric stress space we immediately see that $\Sigma^{\mathrm{sym}} \subset \Sigma$, hence continuity of the bilinear forms follows from lemma 8. Similarly, also coercivity is a direct consequence of lemma 9 due the property $K_b^{\mathrm{sym}} \subset K_b$ where

$$K_b^{\mathrm{sym}} := \{(\sigma, p) \in \Sigma^{\mathrm{sym}} \times Q : b(\sigma, q; u) = 0 \text{ for all } u \in V\}.$$

In order to prove existence and uniqueness of a solution of equation 4.43 it is sufficient to show inf-sup stability given by the following theorem.

Theorem 12 (Inf-sup of b)**.** *For all $u \in V$ there exists a $(\sigma, p) \in \Sigma^{\mathrm{sym}} \times Q$ such that*

$$\sup_{(\tau,q)\in\Sigma^{\mathrm{sym}}\times Q} \frac{b(\tau, q; u)}{\|\tau\|_\Sigma + \|q\|_Q} \gtrsim \|u\|_V. \tag{4.44}$$

Proof. The proof follows with similar steps as in the proof of lemma 12 and lemma 10. Let $u \in V$ be fixed, then we define the auxiliary problem: Find $(\tilde{u}, \tilde{p}) \in \tilde{V} \times Q$ with $\tilde{V} := H^1(\Omega, \mathbb{R}^d) \cap V$ such that

$$\begin{aligned} \int_\Omega \varepsilon(\tilde{u}) : \varepsilon(\tilde{v}) \,\mathrm{d}x + \int_\Omega \operatorname{div}(\tilde{v})\tilde{p} \,\mathrm{d}x &= \int_\Omega u \cdot \tilde{v} \,\mathrm{d}x + \int_\Omega \operatorname{div}(u) \operatorname{div}(\tilde{v}) \,\mathrm{d}x \quad && \forall \tilde{v} \in \tilde{V}, \\ \int_\Omega \operatorname{div}(\tilde{u})\tilde{q} \,\mathrm{d}x &= 0 && \forall \tilde{q} \in Q. \end{aligned}$$

This problem is solvable due to the LBB-condition of the standard Stokes problem, as seen in lemma 4.9 in [43], and a similar result as given by lemma 11, including Korn's inequality, see theorem 4. Brezzi's theorem then further yields the continuity estimate

$$\|\tilde{u}\|_{H^1(\Omega)} + \|\tilde{p}\|_{L^2(\Omega)} \lesssim \|u\|_V. \tag{4.45}$$

For $\sigma := -\varepsilon(\tilde{u})$ we immediately observe that $\sigma + \sigma^{\mathrm{T}} = 0$ and $\operatorname{tr}(\sigma) = \operatorname{div}(\tilde{u}) = 0$ and due to the stability estimate (4.45) also $\|\sigma\|_{L^2(\Omega)} \lesssim \|u\|_V$. Following the same arguments as in lemma 12 we have for a smooth function $\phi \in \mathcal{C}_0^\infty(\Omega, \mathbb{R}^d)$ the identity

$$\langle \operatorname{div}(\sigma), \phi \rangle_V = \int_\Omega u \cdot \phi \, \mathrm{d}x + \int_\Omega \operatorname{div}(u) \operatorname{div}(\phi) \, \mathrm{d}x - \int_\Omega \operatorname{div}(\phi) \tilde{p} \, \mathrm{d}x \,. \tag{4.46}$$

Using estimate (4.45) we conclude by a density argument that

$$\sup_{v \in V} \frac{\langle \operatorname{div}(\sigma), v \rangle_V}{\|v\|_V} \lesssim \sup_{v \in V} \frac{\|u\|_V \|v\|_V + \|\tilde{p}\|_{L^2(\Omega)} \|v\|_V}{\|v\|_V} \lesssim \|u\|_V,$$

and further

$$\langle \operatorname{div}(\sigma), u \rangle_V \gtrsim \|u\|_V^2 - \|\operatorname{div}(u)\|_{L^2(\Omega)} \|\tilde{p}\|_{L^2(\Omega)} \gtrsim \|u\|_V^2 - \|\operatorname{div}(u)\|_{L^2(\Omega)} \|u\|_V. \tag{4.47}$$

By the above estimates we observe that $\sigma \in H(\operatorname{curl} \operatorname{div}, \Omega)$, which allows us to apply the normal-tangential trace operator. Then, with the same steps as in the proof of lemma 12, we also have $\sigma_{nt} = 0$ on $\Gamma_{N,t}$, thus with the findings from above we conclude $\sigma \in \Sigma^{\mathrm{sym}}$.

Now choose $p := \alpha \operatorname{div}(u)$, with $\alpha > 2$. This is an admissible choice with the same argument as in the proof of lemma 10. We observe that there holds the equivalence

$$b_1(u, p) = \alpha \|\operatorname{div}(u)\|_{L^2(\Omega)}^2,$$

and hence by (4.47)

$$\begin{aligned} \sup_{(\tau, q) \in \Sigma^{\mathrm{sym}} \times Q} \frac{b(\tau, q; u)}{\|\tau\|_\Sigma + \|q\|_Q} &\geq \frac{b(\sigma, p; u)}{\|\sigma\|_\Sigma + \|p\|_Q} \\ &\geq \frac{\|u\|_V^2 - \|\operatorname{div}(u)\|_{L^2(\Omega)} \|u\|_V + \alpha \|\operatorname{div}(u)\|_{L^2(\Omega)}^2}{\|\sigma\|_\Sigma + \|p\|_Q}. \end{aligned}$$

Using Young's inequality with a scaling factor $1/2$ we get

$$\|u\|_V^2 - \|\operatorname{div}(u)\|_{L^2(\Omega)} \|u\|_V + \alpha \|\operatorname{div}(u)\|_{L^2(\Omega)}^2 \geq \frac{1}{2} \|u\|_V^2 + (\alpha - 2) \|\operatorname{div}(u)\|_{L^2(\Omega)}^2 \sim \|u\|_V^2.$$

By (4.45) we further have

$$\|\sigma\|_\Sigma + \|p\|_Q \lesssim \|u\|_V + \alpha \|\operatorname{div}(u)\|_{L^2(\Omega)} \sim \|u\|_V,$$

and the statement is proven. □

Corollary 2. *Assume a given $f \in V^*$ and $g_{D,t} \in TW(\Gamma_D)$. Further assume the increased regularity of the essential boundary data $g_{D,n} \in H^{1/2}(\Gamma_D)$ and $g_{N,t} \in H^{-1/2}(\Gamma_N, \mathbb{R}^d)$. There exists a unique solution $(\sigma, u, p) \in \Sigma_N^{\mathrm{sym}} \times V_D \times Q$ of equation (4.43) such that*

$$\begin{aligned} \frac{1}{\sqrt{\nu}} \big(\|\sigma\|_\Sigma + \|p\|_Q \big) + \sqrt{\nu} \|u\|_V \lesssim & \frac{1}{\sqrt{\nu}} \Big(\|f\|_{V^*} + \|g_{N,t}\|_{H^{-1/2}(\Gamma_N)} \Big) \\ & + \sqrt{\nu} \Big(\|g_{D,t}\|_{TW(\Gamma_D)} + \|g_{D,n}\|_{H^{1/2}(\Gamma_D)} \Big). \end{aligned}$$

Proof. This is a direct consequence of theorem 6, and the results of lemma 8, lemma 9 and theorem 12. In the case of non-homogeneous essential boundary conditions a standard homogenization method is used. □

5 Finite Elements - a discrete stress space

5.1 Triangulation and preliminaries

We start with the introduction of several preliminaries that we shall use within this work. Given a domain $\Omega \subset \mathbb{R}^d$ with $d = 2$ or 3 with a Lipschitz boundary as described in chapter 3, let $\mathcal{T}_h$ be a partition of Ω into triangles and tetrahedrons in two and three dimensions, respectively. Throughout this work we assume that the triangulation $\mathcal{T}_h$ is

- shape regular: There exists a constant $c_s > 0$ such that

$$\max_{T \in \mathcal{T}_h} \frac{\operatorname{diam}(T)^d}{|T|} \leq c_s \quad \text{for all } T \in \mathcal{T}_h,$$

  and

- quasi-uniform: There exists a constant $c_q > 0$ such that

$$\operatorname{diam}(T) \geq c_q h \quad \text{for all } T \in \mathcal{T}_h,$$

  where $h := \max_{T \in \mathcal{T}_h} \operatorname{diam}(T)$.

For a given element $T \in \mathcal{T}_h$ we denote by $\mathcal{V}_h(T)$ the set of vertices of the element T, and by $\mathcal{F}_h(T)$ the set of faces, so the $d-1$ subsimplices, of the element T. In a similar manner we then denote by $\mathcal{F}_h$ the set of all element interfaces and boundaries of the given triangulation $\mathcal{T}_h$. This set can further be split into two parts. The first part is denoted by $\mathcal{F}_h^{\text{ext}}$ and is given by all facets that lie on the boundary of the domain, thus $\mathcal{F}_h^{\text{ext}} := \{F \in \mathcal{F}_h : F \cap \Gamma \neq \emptyset\}$. The second part, denoted by $\mathcal{F}_h^{\text{int}}$, contains all facets that are in the interior of the domain, thus $\mathcal{F}_h^{\text{int}} = \mathcal{F}_h \setminus \mathcal{F}_h^{\text{ext}}$.

With a slight abuse of notation, we use the same symbol n for the outward unit normal vector on each element boundary ∂T and for the normal vector defined on the boundary Γ. Then, the corresponding normal and tangential traces of smooth vector-valued functions, and the normal-normal and normal-tangential traces of smooth matrix-valued functions on element boundaries and facets are equivalently defined as in section 3.1.

At several points in the definition of the finite elements and also in the numerical analysis we make use of a mapping from a physical element $T \in \mathcal{T}_h$ to a so called reference element denoted by $\widehat{T}$. To this end we define

$$\widehat{T} := \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1, x_2 \text{ and } x_1 + x_2 \leq 1\} \qquad \text{for} \qquad d = 2,$$
$$\widehat{T} := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leq x_1, x_2, x_3 \text{ and } x_1 + x_2 + x_3 \leq 1\} \qquad \text{for} \qquad d = 3.$$

Although one could define a different reference element, it is important that the diameter is approximately one, thus $\operatorname{diam}(\widehat{T}) = \mathcal{O}(1)$. On these reference elements we denote the vertices by

$$V_0 := (0,0), \quad V_1 := (1,0), \quad V_2 := (0,1),$$

and

$$V_0 := (0,0,0), \quad V_1 := (1,0,0), \quad V_2 := (0,1,0), \quad V_3 := (0,0,1),$$

for two and three dimensions, respectively. Next, we further define the follwoing reference faces and the associated normal and tangential vectors. In two dimensions we have

$$\begin{aligned}
\hat{F}_0 &:= \{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x_1, x_2 \le 1, x_1 + x_2 = 1\},\\
\hat{F}_1 &:= \{(0,x_2) \in \mathbb{R}^2 : 0 \le x_2 \le 1\},\\
\hat{F}_2 &:= \{(x_1,0) \in \mathbb{R}^2 : 0 \le x_1 \le 1\},
\end{aligned}$$

with

$$\begin{aligned}
\hat{n}_0 &:= \frac{1}{\sqrt{2}}(1,1)^{\mathrm{T}}, \quad \hat{n}_1 := (-1,0)^{\mathrm{T}}, \quad \hat{n}_2 := (0,-1)^{\mathrm{T}},\\
\hat{t}_0 &:= \frac{1}{\sqrt{2}}(-1,1)^{\mathrm{T}}, \quad \hat{t}_1 := (0,-1)^{\mathrm{T}}, \quad \hat{t}_2 := (1,0)^{\mathrm{T}}.
\end{aligned}$$

For the three dimensional case we have

$$\begin{aligned}
\hat{F}_0 &:= \{(x_1,x_2,x_3) \in \mathbb{R}^3 : 0 \le x_1, x_2, x_3 \le 1, x_1 + x_2 + x_3 = 1\},\\
\hat{F}_1 &:= \{(0,x_2,x_3) \in \mathbb{R}^3 : 0 \le x_2, x_3 \le 1, 0 \le x_2 + x_3 \le 1\},\\
\hat{F}_2 &:= \{(x_1,0,x_3) \in \mathbb{R}^2 : 0 \le x_1, x_3 \le 1, 0 \le x_1 + x_3 \le 1\},\\
\hat{F}_3 &:= \{(x_1,x_2,0) \in \mathbb{R}^2 : 0 \le x_1, x_2 \le 1, 0 \le x_1 + x_2 \le 1\},
\end{aligned}$$

with

$$\begin{aligned}
\hat{n}_0 &:= \frac{1}{\sqrt{3}}(1,1,1)^{\mathrm{T}}, & \hat{t}_{01} &:= \frac{1}{\sqrt{2}}(-1,1,0)^{\mathrm{T}}, & \hat{t}_{02} &:= \frac{1}{\sqrt{2}}(0,1,-1)^{\mathrm{T}},\\
\hat{n}_1 &:= (-1,0,0)^{\mathrm{T}}, & \hat{t}_{11} &:= (0,-1,0)^{\mathrm{T}}, & \hat{t}_{12} &:= (0,0,-1)^{\mathrm{T}},\\
\hat{n}_2 &:= (0,-1,0)^{\mathrm{T}}, & \hat{t}_{21} &:= (1,0,0)^{\mathrm{T}}, & \hat{t}_{22} &:= (0,0,-1)^{\mathrm{T}},\\
\hat{n}_3 &:= (0,0,-1)^{\mathrm{T}}, & \hat{t}_{31} &:= (1,0,0)^{\mathrm{T}}, & \hat{t}_{32} &:= (0,-1,0)^{\mathrm{T}}.
\end{aligned}$$

In figure 5.1 we illustrated the reference elements in both dimensions.

By the definition of the reference element we are now able to define the associated element mappings. For an arbitrary element $T \in \mathcal{T}_h$ let $\phi_T : \widehat{T} \to T$ be an affine homeomorphism, with the Jacobi matrix denoted by $F_T := \phi_T'$. As we assumed that the triangulation $\mathcal{T}_h$ is shape regular and quasi-uniform we have

$$||F_T||_\infty \approx h \quad \text{and} \quad ||F_T^{-1}||_\infty \approx h^{-1} \quad \text{and} \quad |\det(F_T)| \approx h^d. \tag{5.1}$$

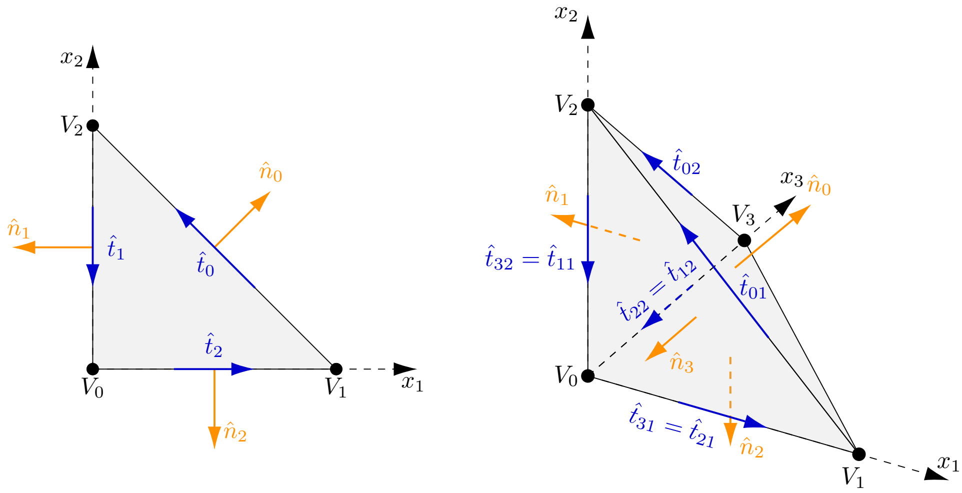

Figure 5.1: The reference element $\hat{T}$ and the corresponding normal and tangential vectors in two dimensions (left) and in three dimensions (right).

Similarly, we can restrict the mapping ϕ_T to a reference face $\hat{F} \in \mathcal{F}_h(\widehat{T})$ and reference edge $\hat{E} \subset \partial\hat{F}$ (in three dimensions) whose gradients are then denoted by $F_T^F := (\phi_T|_{\hat{F}})'$ and $F_T^E := (\phi_T|_{\hat{E}})'$. Using these quantities the unit normals and tangents of the reference element and its mapped configurations on the physical element T are related by

$$n = \frac{\det(F_T)}{\det(F_T^F)} F_T^{-\mathrm{T}} \hat{n} \quad \text{and} \quad t = \frac{1}{\det(F_T^E)} F_T \hat{t}, \tag{5.2}$$

where in two dimensions we have to replace F_T^E by F_T^F.

On each face $F \in \mathcal{F}_h^{\mathrm{int}}$ we make use of the standard notations for the jump and the average of a functions. To this end let $T_1, T_2 \in \mathcal{T}_h$ and let $F = T_1 \cap T_2$ be its common element interface. Further let n_1 and n_2 be the corresponding outward normal vectors. For a function $\phi \in \mathcal{C}^0(T_1, \mathbb{R}) \cup \mathcal{C}^0(T_1, \mathbb{R})$ we define the mean value and the jump on F as

$$\{\phi\} := \frac{1}{2}(\phi|_{T_1} + \phi|_{T_2}) \quad \text{and} \quad [\![\phi]\!] := \phi|_{T_1} - \phi|_{T_2}.$$

For a facet $F \in \mathcal{F}_h^{\mathrm{ext}}$ the average and the jump is just defined as the identity.

We continue with the definition of polynomial spaces. For a given element $T \in \mathcal{T}_h$ we denote by $\mathbb{P}^k(T)$ the space of polynomials defined on T whose total order is less or equal k. Again, we use the same notation as for function spaces for non scalar-valued polynomial spaces, e.g. where $\mathbb{P}^k(T, \mathbb{R}^d)$ denotes the space of vector-valued polynomials, we use $\mathbb{P}^k(T, \mathbb{R}^{d\times d})$ for the space of matrix-valued polynomials. Using these notations we further define polynomials on the triangulation by

$$\mathbb{P}^k(\mathcal{T}_h, \mathbb{R}) := \prod_{T \in \mathcal{T}_h} \mathbb{P}^k(T, \mathbb{R}),$$

and similarly $\mathbb{P}^k(\mathcal{T}_h, \mathbb{R}^d)$ and $\mathbb{P}^k(\mathcal{T}_h, \mathbb{R}^{d\times d})$. Beside this we make use of homogeneous polynomials denoted by $\mathbb{P}^k_{\text{hom}}(\mathcal{T}_h, \mathbb{R})$ and the space of matrix-valued skew symmetric polynomials defined by

$$\mathbb{P}^k_{\text{skw}}(\mathcal{T}_h, \mathbb{R}^{d\times d}) := \{\eta \in \mathbb{P}^k(\mathcal{T}_h, \mathbb{R}^{d\times d}) : (\eta + \eta^{\mathrm{T}})|_T = 0 \text{ on all } T \in \mathcal{T}_h\}.$$

Finally, we introduce the space of rigid displacements by

$$\mathrm{RM}(\mathcal{T}_h) := \{a + Bx : a \in \mathbb{P}^0(T, \mathbb{R}^d), B \in \mathbb{P}^0_{\text{skw}}(T, \mathbb{R}^{d\times d})\}. \tag{5.3}$$

At several points in the analysis we make use of polynomials defined in the tangent plane of a face of a given element T. To this end let $F \in \mathcal{F}_h(T)$, then with a slight abuse of notation we do not distinguish between the tangent plane parallel to the facet F and the isomorphic $\mathbb{R}^{d-1}$ and write instead $\mathbb{P}^k(F, \mathbb{R}^{d-1})$. Note that for example the tangential projection of a polynomial $\mu \in \mathbb{P}^k(T, \mathbb{R}^d)$ is in this space, thus $\mu_t \in \mathbb{P}^k(F, \mathbb{R}^{d-1})$.

With respect to a triangulation we introduce for each element $T \in \mathcal{T}_h$ the local element-wise L^2-projection on polynomials of order k by Π^k_T. Note that we do not distinguish between scalar-, vector- or matrix-valued functions, but always use the same symbol. Following the notations from above the corresponding global L^2-projection onto the space $\mathbb{P}^k(\mathcal{T}_h)$ is given by $\Pi^k_{\mathcal{T}_h}$. Similarly, on each facet $F \in \mathcal{F}_h$, let Π^k_F denote the L^2-projection onto the space of polynomials of order k on F. Again, we use the same symbols for projections with different ranges. For example, the projection into the tangent plane of F is also given by Π^k_F, i.e., with the notation from above we have for any vector-valued function $v \in L^2(F, \mathbb{R}^{d-1})$ that the projection $\Pi^k_F v \in \mathbb{P}^k(F, \mathbb{R}^{d-1})$ satisfies $(\Pi^k_F v, q)_F = (v, q)_F$ for all $q \in \mathbb{P}^k(F, \mathbb{R}^{d-1})$. Finally, the local projection onto the space of rigid displacements is given by

$$\Big| \int_T \Pi^{\mathrm{RM}} v - v \,\mathrm{d}x \Big| = 0,$$
$$\Big| \int_T \operatorname{curl}\big(\Pi^{\mathrm{RM}} v - v\big) \,\mathrm{d}x \Big| = 0,$$

for all functions $v \in H^1(\mathcal{T}_h, \mathbb{R}^d)$ (see definition below). This projection fulfills the following properties (see [18])

$$\|\nabla(v_h - \Pi^{\mathrm{RM}} v_h)\|_T \lesssim \|\varepsilon(v_h)\|_T \qquad \forall T \in \mathcal{T}_h, \tag{5.4}$$
$$\|[\![v_h - \Pi^{\mathrm{RM}} v_h]\!]\|_F^2 \lesssim \sum_{T: T\cap F \neq \emptyset} h\|\varepsilon(v_h)\|_T^2 \quad \forall F \in \mathcal{F}_h. \tag{5.5}$$

Similarly, we also define function spaces with respect to the triangulation $\mathcal{T}_h$, e.g.

$$H^m(\mathcal{T}_h, \mathbb{R}) := \{u \in L^2(\Omega, \mathbb{R}) : u|_T \in H^m(T, \mathbb{R}) \text{ for all } T \in \mathcal{T}_h\},$$

denotes the broken Sobolev space of order m. Note that we use the same symbols for a broken differential operator applied on each element for functions in a broken Sobolev space and the continuous operator applied on functions in the corresponding standard Sobolev space, e.g. we write $(\nabla u)|_T = \nabla(u|_T)$ for functions $u \in H^1(\mathcal{T}_h, \mathbb{R})$.

We conclude this section with the definition of several local polynomial basis functions. For a detailed discussion we refer to [1, 4]. To this end let $I_{\mathcal{V}_h(T)}$ be the index set of the vertices $\mathcal{V}_h(T)$, then we use the standard notation for the barycentric coordinate functions given by λ_i, thus we have

$$\lambda_i \in \mathbb{P}^1(T,\mathbb{R}) \quad \text{such that} \quad \lambda_i(V_j) = \delta_{ij} \quad \forall i,j \in I_{\mathcal{V}_h(T)},$$

where δ_{ij} is the Kronecker delta. Next, let $l_i(x_1)$ be the Legendre polynomial of order i and let $l_i^S(x_1,x_2) := x_2^i l_i(x_1/x_2)$ be the scaled Legendre polynomial of order i. Further let $p_i^j(x_1)$ be the Jacobi polynomial of order i with coefficients $\alpha = j$, $\beta = 0$. Using these one-dimensional polynomials, we define in two dimensions

$$\hat{r}_{ij}(\lambda_\alpha,\lambda_\beta,\lambda_\gamma) := l_i^S(\lambda_\beta - \lambda_\alpha, \lambda_\alpha + \lambda_\beta) p_j^{2i+1}(\lambda_\gamma - \lambda_\alpha - \lambda_\beta). \tag{5.6}$$

The set

$$\{\hat{r}_{ij}(\lambda_\alpha,\lambda_\beta,\lambda_\gamma) : 0 \le i+j \le k \text{ and } (\alpha,\beta,\gamma) \text{ permutation of } \{0,1,2\}\}$$

form a basis of the polynomial space $\mathbb{P}^k(\hat{T},\mathbb{R})$. Next note, as p_0^{2i+1} is constant, there holds the equality $\hat{r}_{ij}(\lambda_\alpha,\lambda_\beta,\lambda_\gamma) = \hat{r}_{i0}(\lambda_\alpha,\lambda_\beta)$. Then the set

$$\{\hat{r}_{i0}(\lambda_{j+1},\lambda_{j+2})|_{\hat{F}_j} : 0 \le i \le k\},$$

where the indices $j+1$ and $j+2$ of the barycentric coordinate functions are taken modulo 3, form a basis of the polynomial space $\mathbb{P}^k(\hat{F}_j,\mathbb{R})$ (see chapter 3.2 in [72], or in [37] and [107]). In three dimensions a similar results holds true. We define

$$\begin{aligned} \hat{r}_{ijl}(\lambda_\alpha,&\lambda_\beta,\lambda_\gamma,\lambda_\delta) \\ &:= l_i^S(\lambda_\beta - \lambda_\alpha, \lambda_\alpha + \lambda_\beta) \\ &\quad p_j^{2i+1,S}(\lambda_\gamma - \lambda_\alpha - \lambda_\beta, \lambda_\gamma + \lambda_\alpha + \lambda_\beta) p_l^{2i+2j+2,S}(\lambda_\delta - \lambda_\alpha - \lambda_\beta - \lambda_\gamma), \end{aligned} \tag{5.7}$$

where $p_i^{j,S}(x_1,x_2) := x_2^i p_i^j(x_1/x_2)$ is the scaled Jacobi polynomial. Again, we have that the set $\hat{r}_{ijl}(\lambda_\alpha,\lambda_\beta,\lambda_\gamma,\lambda_\delta)$ with $0 \le i+j+l \le k$ and an arbitrary permutation $(\alpha,\beta,\gamma,\delta)$ of $(0,1,2,3)$, defines a basis for $\mathbb{P}^k(\hat{T},\mathbb{R})$ and that for $0 \le i+l \le k$ the restriction on a facet $\hat{F}_j$ given by $\hat{r}_{il0}(\lambda_{j+1},\lambda_{j+2},\lambda_{j+3})|_{\hat{F}_j}$ is a basis of $\mathbb{P}^k(\hat{F}_j,\mathbb{R})$, where the indices of the barycentric coordinate functions are now taken modulo 4.

5.2 A discrete space for the approximation of $H(\mathrm{div})$

In the previous chapter we derived the MCS formulation, equation (4.26), where the velocity solution u is an element of the Sobolev space $H(\mathrm{div},\Omega)$. An appropriate discretization, in the sense of conformity, of the space $H(\mathrm{div},\Omega)$ is well known in the literature. In the following we introduce two spaces with several properties that shall be used in this work. For a detailed discussion on this topic we refer for example to the book [11].

The first space we introduce is based on the work [99] and [89] and is called the Raviart-Thomas space. To this end let $T \in \mathcal{T}_h$, then we define

$$\mathcal{RT}^k(T) := \{u = a + bx : a \in \mathbb{P}^k(T, \mathbb{R}^d), b \in \mathbb{P}^k_{\text{hom}}(T, \mathbb{R})\},$$

and the global space

$$\mathcal{RT}^k(\mathcal{T}_h) := \{u \in H(\text{div}, \Omega) : u|_T \in \mathcal{RT}^k(T) \text{ for all } T \in \mathcal{T}_h\}. \tag{5.8}$$

The space $\mathcal{RT}^k(\mathcal{T}_h)$ contains, locally on each element, the full polynomial space $\mathbb{P}^k(T, \mathbb{R}^d)$ and certain polynomials of order $k+1$. Due to this enrichment the (global) divergence is equivalent to $\mathbb{P}^k(\mathcal{T}_h, \mathbb{R})$. The space is constructed such that we have for all functions $u \in \mathcal{RT}^k(\mathcal{T}_h)$ and on each facet $F \in \mathcal{F}_h$ that the normal component $u_n \in \mathbb{P}^k(T, \mathbb{R})$, thus the high order polynomials, have a zero normal trace. Further, due to the $H(\text{div})$-conformity, the normal jump is zero $[\![u_n]\!] = 0$ on all element interfaces $F \in \mathcal{F}_h^{\text{int}}$. Note that in the original work [99] the space was defined on the reference element and was mapped appropriately, given by equation (5.10) (the Piola transformation) below.

In contrast to the Raviart-Thomas space the corresponding discrete space that contains, locally on each element, all polynomials up to a certain total order is given by the Brezzi-Douglas-Marini space (see [23, 21, 22]), denoted by

$$\begin{aligned} \mathcal{BDM}^k(\mathcal{T}_h) :=&\{u \in H(\text{div}, \Omega) : u|_T \in \mathbb{P}^k(T, \mathbb{R}^d) \text{ for all } T \in \mathcal{T}_h\} \\ =&\{u \in \mathbb{P}^k(\mathcal{T}_h, \mathbb{R}^d) : [\![u_n]\!] = 0 \text{ on all } F \in \mathcal{F}_h\}. \end{aligned} \tag{5.9}$$

Again, due to the $H(\text{div})$-conformity, we can apply the (global) divergence to all functions in $\mathcal{BDM}^k(\mathcal{T}_h)$ to get a polynomial of order $k-1$. To be precise, the divergence is surjective, and we have the following relation between the two spaces

$$\text{div}(\mathcal{RT}^k(\mathcal{T}_h)) = \text{div}(\mathcal{BDM}^{k+1}(\mathcal{T}_h)) \subset \mathbb{P}^k(\mathcal{T}_h, \mathbb{R}),$$

and

$$\ldots \subset \mathcal{RT}^{k-1}(\mathcal{T}_h) \subset \mathcal{BDM}^k(\mathcal{T}_h) \subset \mathcal{RT}^k(\mathcal{T}_h) \subset \mathcal{BDM}^{k+1}(\mathcal{T}_h) \subset \ldots$$

Whereas the standard pullback preserves continuity for standard H^1-conforming finite elements, the proper mapping for $H(\text{div})$-conforming finite elements is given by the Piola mapping. To this end let $\hat{u} \in L^2(\widehat{T}, \mathbb{R}^d)$, then we define on a physical element $T \in \mathcal{T}_h$ the function

$$\mathcal{P}(\hat{u}) := \frac{1}{\det F_T} F_T \hat{u} \circ \phi^{-1}. \tag{5.10}$$

The Piola mapping preserves the normal component and maps the divergence one-to-one, thus if $\hat{u} \in H(\text{div}, \widehat{T})$, then

$$\text{div}(\mathcal{P}(\hat{u})) = \frac{1}{\det F_T} \widehat{\text{div}}(\hat{u}) \circ \phi^{-1},$$

where $\widehat{\operatorname{div}}$ is the divergence operator with respect to the reference coordinates on $\widehat{T}$. At several points in the analysis we use scaling arguments for the gradient of a Piola mapped function. Then, by the chain rule we have for a function $\hat{u} \in H^1(\Omega, \mathbb{R}^d)$ on each element $T \in \mathcal{T}_h$ the identity

$$\nabla(\mathcal{P}(\hat{u})) = \frac{1}{\det F_T} F_T \widehat{\nabla} \hat{u} F_T^{-1}. \tag{5.11}$$

We conclude this section with the introduction of appropriate degrees of freedoms for the Raviart-Thomas space $\mathcal{RT}^k(T)$. Note that similar results can be found for $\mathcal{BDM}^k(T)$ in [11, section 2.3.1].

Lemma 13. *For any $k \geq 0$, $T \in \mathcal{T}_h$ and for any $u \in \mathcal{RT}^k(T)$ the following equations,*

$$\int_F u_n q \, \mathrm{ds} = 0 \quad \textit{for all } q \in \mathbb{P}^k(F, \mathbb{R}), \quad \textit{for all } F \in \mathcal{F}_h(T),$$
$$\int_T u \cdot r \, \mathrm{ds} = 0 \quad \textit{for all } r \in \mathbb{P}^{k-1}(F, \mathbb{R}^d),$$

imply that $u = 0$.

Proof. See proof of Proposition 2.3.4 in [11]. □

5.3 A discrete space for the approximation of $H(\operatorname{curl}\operatorname{div}, \Omega)$

In this section we introduce a new finite element and a corresponding new finite dimensional discrete space in which we can approximate the solution σ of the system (4.26). As σ is an element of the (matrix) trace-free subspace $\Sigma \subset H(\operatorname{curl}\operatorname{div}, \Omega)$ we proceed as follows: First, we discuss the definition of a discrete space to approximate arbitrary functions in $H(\operatorname{curl}\operatorname{div}, \Omega)$. Then, we show that the introduced construction of this space further allows a simple splitting such that we can also only approximate functions in the subspace Σ.

The discrete space that approximates the space $H(\operatorname{curl}\operatorname{div}, \Omega)$ should fulfill several properties: First of all, the discrete space should provide approximations with optimal convergence results (with respect to the mesh size h). To this end we demand that the finite element space should include matrix-valued polynomials up to a fixed, but arbitrary, given total order k. Secondly, we demand a certain continuity of the finite element basis functions across element interfaces such that the divergence can be continuously applied to discrete stress functions in a proper sense.

We first tackle the second question. As we have already seen in section 4.2.3, the normal-tangnetial trace operator is continuous for functions in $H(\operatorname{curl}\operatorname{div}, \Omega)$. Similarly as for the Sobolev space $H^1(\Omega)$, where the trace operator γ is continuous and results in a $\mathcal{C}^0$-continuity of the discrete basis functions across element interfaces, the continuity of γ_{nt} already gives us a hint how the new stress finite element basis functions should be constructed. The following theorem should further motivate this insight.

Theorem 13. *Suppose σ is in $H^1(\mathcal{T}_h, \mathbb{R}^{d\times d})$ and $\sigma_{nn}|_{\partial T} \in H^{1/2}(\partial T)$ for all elements $T \in \mathcal{T}_h$. Assume that the normal-tangential trace σ_{nt} is continuous across element interfaces. Then σ is in $H(\operatorname{curl}\operatorname{div}, \Omega)$ and moreover*

$$\langle \operatorname{div}(\sigma), v\rangle_{H_0(\operatorname{div},\Omega)} = \sum_{T\in\mathcal{T}_h} \left[\int_T \operatorname{div}(\sigma)\cdot v \,\mathrm{d}x - \langle v_n, \sigma_{nn}\rangle_{H^{1/2}(\partial T)}\right] \tag{5.12}$$

for all $v \in H_0(\operatorname{div}, \Omega)$.

Proof. Let $\phi \in \mathcal{D}(\Omega, \mathbb{R}^d)$. By the definition of the distributional divergence and using integration by parts on each element T separately, we have

$$\langle \operatorname{div}(\sigma), \phi\rangle = -\int_\Omega \sigma : \nabla\phi \,\mathrm{d}x = \sum_{T\in\mathcal{T}_h} \int_T \operatorname{div}(\sigma)\cdot\phi \,\mathrm{d}x - \int_{\partial T} \sigma_n\cdot\phi \,\mathrm{d}s\,.$$

Next, we split the boundary integral into two parts given by the normal and tangential direction, and we obtain

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} -\int_{\partial T} \sigma_n\cdot\phi \,\mathrm{d}s &= \sum_{T\in\mathcal{T}_h} -\int_{\partial T} \sigma_{nn}\phi_n \,\mathrm{d}s - \int_{\partial T} \sigma_{nt}\cdot\phi_t \,\mathrm{d}s \\ &= \sum_{T\in\mathcal{T}_h} -\int_{\partial T} \sigma_{nn}\phi_n \,\mathrm{d}s - \sum_{F\in\mathcal{F}_h^{\text{int}}} \int_F [\![\sigma_{nt}]\!]\cdot\phi_t \,\mathrm{d}s,\end{aligned}$$

where we used that ϕ vanishes at the boundary Γ. By the continuity of the normal-tangential trace of σ we get $[\![\sigma_{nt}]\!] = 0$ on all facets $F \in \mathcal{F}_h^{\text{int}}$, and the second sum vanishes. This yields

$$\begin{aligned}\langle \operatorname{div}(\sigma), \phi\rangle &= \sum_{T\in\mathcal{T}_h} \int_T \operatorname{div}(\sigma)\cdot\phi \,\mathrm{d}x - \int_{\partial T} \sigma_{nn}\phi_n \,\mathrm{d}s \qquad (5.13)\\ &\le \sum_{T\in\mathcal{T}_h} \|\operatorname{div}(\sigma)\|_{L^2(T)}\|\phi\|_{L^2(T)} + \|\sigma_{nn}\|_{H^{1/2}(\partial T)}\|\phi_n\|_{H^{-1/2}(\partial T)} \\ &\le c(\sigma)\|\phi\|_{H(\operatorname{div},\Omega)}, \qquad (5.14)\end{aligned}$$

where $c(\sigma)$ is a constant depending on σ. Since $\mathcal{D}(\Omega, \mathbb{R}^d)$ is dense in $H_0(\operatorname{div}, \Omega)$, we conclude that $\operatorname{div}(\sigma)$ is in $H_0(\operatorname{div}, \Omega)^*$, hence by definition, $\sigma \in H(\operatorname{curl}\operatorname{div}, \Omega)$. The identity (5.12) also follows from (5.13) and a density argument. □

Theorem 13 shows that normal-tangential continuity, assuming enough regularity of σ on each element T, is enough to guarantee that the divergence of σ lies in the dual space of $H_0(\operatorname{div}, \Omega)$, thus $\sigma \in H(\operatorname{curl}\operatorname{div}, \Omega)$. However, the regularity assumption $\sigma_{nn}|_{\partial T} \in H^{1/2}(\partial T)$ makes the construction of finite element basis functions much more complicated. Although it is possible to show that it would be enough to assume only an increased regularity property at the vertices of each element (as the normal-normal component jumps there), this would still make the construction much more complex. In contrast to this, we define a finite element basis that is *slightly non-conforming* with respect to the space

$H(\operatorname{curl}\operatorname{div},\Omega)$, but is much easier to construct. Note that this non-conformity is taken into account in the stability analysis of the next chapters.

With the findings from above we define a discrete finite dimensional space that shall be used for the approximation for functions in $H(\operatorname{curl}\operatorname{div},\Omega)$

$$\Xi^k(\mathcal{T}_h) := \{\sigma_h \in \mathbb{P}^k(\mathcal{T}_h, \mathbb{R}^{d\times d}) : [\![(\sigma_h)_{nt}]\!] = 0 \text{ on all } F \in \mathcal{F}_h^{\text{int}}\}.$$

The rest of this chapter focuses on the definition of a finite element and the construction of proper shape functions for $\Xi^k(\mathcal{T}_h)$. First note that the definition of trace free matrices,

$$\mathbb{D} := \{M \in \mathbb{R}^{d\times d} : (M, \mathrm{Id}) = 0\},$$

allows us to decompose

$$\Xi^k(\mathcal{T}_h) = \Xi^k_{\mathbb{D}}(\mathcal{T}_h) \oplus \Xi^k_{\mathrm{Id}}(\mathcal{T}_h).$$

Here $\Xi^k_{\mathbb{D}}(\mathcal{T}_h)$ and $\Xi^k_{\mathrm{Id}}(\mathcal{T}_h)$ are defined by an orthogonal decomposition with respect to $\mathbb{D}$,

$$\begin{aligned} \Xi^k_{\mathbb{D}}(\mathcal{T}_h) :&= \{\sigma_h \in \mathbb{P}^k(\mathcal{T}_h, \mathbb{D}) : [\![(\sigma_h)_{nt}]\!] = 0 \text{ on all } F \in \mathcal{F}_h^{\text{int}}\}, \\ \Xi^k_{\mathrm{Id}}(\mathcal{T}_h) :&= \{\sigma_h \in \mathbb{P}^k(\mathcal{T}_h, \mathrm{Id}) : [\![(\sigma_h)_{nt}]\!] = 0 \text{ on all } F \in \mathcal{F}_h^{\text{int}}\} \\ &= \mathbb{P}^k(\mathcal{T}_h, \mathrm{Id}), \end{aligned} \tag{5.15}$$

where we used that the normal-tangential trace of a polynomial in $\mathbb{P}^k(\mathcal{T}_h, \mathrm{Id})$ trivially vanishes on each facet $F \in \mathcal{F}_h^{\text{int}}$ due to (4.2).

5.3.1 Lowest order basis functions and normal-tangential bubbles

In a first step we are going to define lowest order constant matrix-valued basis functions for the space $\mathbb{R}^{d\times d}$, which are suited to study normal-tangential components on facets $F \in \mathcal{F}_h$. In the latter sections these basis functions are then used to define a stress finite element and to give an explicit representation of (high order) shape functions for the space $\Xi^k(\mathcal{T}_h)$.

Now let $T \in \mathcal{T}_h$ be an arbitrary element with the vertices $\mathcal{V}(T) := \{V_i\}_{i=0}^d$ with the index set $I_{\mathcal{V}(T)} := \{0, \ldots, |\mathcal{V}(T)|\}$. Then, as defined in section 5.1, the corresponding barycentric coordinate functions on T are given by $\{\lambda_i\}_{i=0}^d$. Further let $F_i \in \mathcal{F}_h(T)$ denote the face opposite to the vertex V_i, with the associated normal vector n_i. In two dimensions we define the tangential vector on F_i as the rotated normal vector $t_i = (n_i)^\perp$, and in three dimensions we denote by $t_{ij} = (V_i - V_j)/|v_i - V_j|$ the unit tangential vector along the edge between the vertices V_i and V_j.

For $d = 2$ we define for all $i \in I_{\mathcal{V}(T)}$ the constant matrix functions

$$S^i := \operatorname{dev}(\nabla\lambda_{i+1} \otimes \operatorname{curl}(\lambda_{i+2})), \tag{5.16}$$

where the indices $i+1$ and $i+2$ are taken modulo 3. In three dimensions we define similarly on each face $F_i \in \mathcal{F}_h(T)$ two constant matrix functions by

$$S^i_0 := \operatorname{dev}\big(\nabla\lambda_{i+1} \otimes (\nabla\lambda_{i+2} \times \nabla\lambda_{i+3})\big), \quad S^i_1 := \operatorname{dev}\big(\nabla\lambda_{i+2} \otimes (\nabla\lambda_{i+3} \times \nabla\lambda_{i+1})\big), \tag{5.17}$$

where the indices $i+1$, $i+2$ and $i+3$ are now taken modulo 4.

Lemma 14. *The sets* $\{S^i : i \in I_{\mathcal{V}(T)}\}$ *and* $\{S^i_q : i \in I_{\mathcal{V}(T)}$ *and* $q = 0,1\}$ *form a basis of* $\mathbb{D}$ *for* $d = 2$ *and* 3, *respectively. Moreover, the normal-tangential component of* S^i *and* S^i_q *vanishes everywhere on the element boundary except on* F_i,

$$S^i_{nt}|_{F_j} = 0, \qquad (S^i_q)_{nt}|_{F_j} = 0, \qquad \text{for} \quad i \neq j, \; F_j \in \mathcal{F}_h(T), \; i,j \in I_{\mathcal{V}(T)},$$

When $i = j \in I_{\mathcal{V}(T)}$ *and* $d = 3$,

$$t^T_{i+2,i+3} S^i_0 n_i = 0, \quad t^T_{i+1,i+2} S^i_0 n_i \neq 0, \quad t^T_{i+3,i+1} S^i_0 n_i \neq 0, \tag{5.18a}$$

$$t^T_{i+2,i+3} S^i_1 n_i \neq 0, \quad t^T_{i+1,i+2} S^i_1 n_i \neq 0, \quad t^T_{i+3,i+1} S^i_1 n_i = 0. \tag{5.18b}$$

Proof. By construction, all functions in the sets $\{S^i : i \in I_{\mathcal{V}(T)}\}$ and $\{S^i_q : i \in I_{\mathcal{V}(T)}$ and $q = 0,1\}$ are trace free and constant. Thus, in order to show that they are a basis of $\mathbb{D}$ it is sufficient to show that they are linearly independent and that the dimension matches. To prove linear independence we provide a proof for the second statement of the lemma.

We start with the two-dimensional case. First note that by equation (4.2) the normal-tangential trace of a function S^i is equivalent to

$$S^i_{nt} = (\operatorname{dev}(\nabla\lambda_{i+1} \otimes \operatorname{curl}(\lambda_{i+2})))_{nt} = (\nabla\lambda_{i+1} \otimes \operatorname{curl}(\lambda_{i+2}))_{nt}\,.$$

Using the abbreviation $s_{ij} := \nabla\lambda_{i+1} \otimes \operatorname{curl}(\lambda_{i+2})$, thus $s_{i+1,i+2} = S^i$, the normal-tangential trace on the facet F_l with $l \in I_{\mathcal{V}(T)}$ is given by

$$t^{\mathrm{T}}_l s_{i,j} n_l = t^{\mathrm{T}}_l \left[\nabla\lambda_i \otimes \operatorname{curl}(\lambda_j)\right] n_l = (\nabla\lambda_i \cdot t_l)(\nabla\lambda_j \cdot t_l).$$

As $n_i \sim \nabla\lambda_i$ we further have

$$(\nabla\lambda_i \cdot t_l)(\nabla\lambda_j \cdot t_l) \sim (n_i \cdot t_l)(n_j \cdot t_l).$$

In the case $l \neq i,j$ this leads to $S^i_{nt}|_{F_i} \sim (n_i \cdot t_{i+1})(n_{i+2} \cdot t_l) \neq 0$ as the triangle T is not degenerated. Similarly, for $l = i$ or $l = q$ the normal-tangential trace is zero as $(n_i \cdot t_l) = (n_i \cdot t_i) = 0$ or $(n_j \cdot t_l) = (n_j \cdot t_j) = 0$, respectively.

In three dimensions we define as above the quantity $s_{i,j,k} = \operatorname{dev}\big(\nabla\lambda_i \otimes (\nabla\lambda_j \times \nabla\lambda_k)\big)$. If i,j,k,l is any permutation of $I_{\mathcal{V}(T)}$, we see that for any $p \in I_{\mathcal{V}(T)}$ and any $t_p \in n_p^{\perp}$, where $n_p^{\perp}$ is the tangent plane of F_p, we have

$$t^{\mathrm{T}}_p s_{i,j,k} n_p \sim (n_i \cdot t_p)(t_{il} \cdot n_p). \tag{5.19}$$

This can be derived with elementary manipulations. An example is given in figure 5.2, where we can easily see $\nabla\lambda_j \times \nabla\lambda_k \sim n_j \times n_k \sim t_{il}$. Due to this we have on any facet F_p

$$t^{\mathrm{T}}_p (S^i_0) n_p = t^{\mathrm{T}}_p (s_{i+1,i+2,i+3}) n_p \sim (n_{i+1} \cdot t_p)(t_{i+1,i} \cdot n_p),$$

which vanishes for all $p \neq i$ since $n_{i+1} \cdot t_{i+1} = 0$ and $t_{i+1,i} \cdot n_{i+2} = t_{i+1,i} \cdot n_{i+3} = 0$. Similarly, we conclude that $(S^i_1)_{nt} = 0$ on all facets except F_i. By equation (5.19) we further have

$$t^{\mathrm{T}}_{jk} s_{i,j,k} n_l = 0, \quad t^{\mathrm{T}}_{ki} s_{i,j,k} n_l \neq 0, \quad t^{\mathrm{T}}_{ji} s_{i,j,k} n_l \neq 0,$$

hence the statements in (5.18) also follow. This proves that the sets $\{S^i : i \in I_{\mathcal{V}(T)}\}$ and $\{S^i_q : i \in I_{\mathcal{V}(T)}$ and $q = 0,1\}$ are linearly independent, and we conclude the proof with a simple counting argument. □

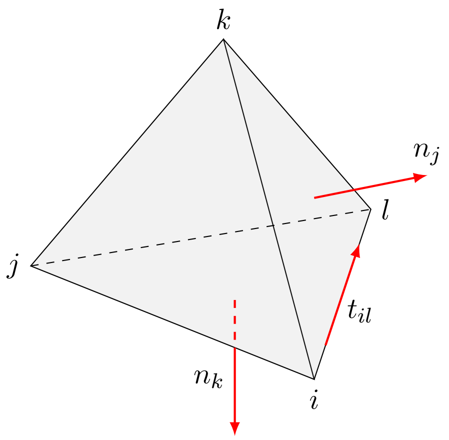

Figure 5.2: An example of an element configuration.

Corollary 3. *The sets* $\{S^i : i \in I_{\mathcal{V}(T)}\} \cup \{\mathrm{Id}\}$ *and* $\{S^i_q : i \in I_{\mathcal{V}(T)}$ *and* $q = 0, 1\} \cup \{\mathrm{Id}\}$ *form a basis of* $\mathbb{R}^{d\times d}$ *for* $d = 2$ *and* 3*, respectively.*

Proof. As the functions defined by (5.16) and (5.17) are all trace free, and as $(\mathrm{Id})_{nt}|_{F_j} = 0$ for all $F_j \in \mathcal{F}_h(T)$, the sets $\{S^i : i \in I_{\mathcal{V}(T)}\} \cup \{\mathrm{Id}\}$ and $\{S^i_q : i \in I_{\mathcal{V}(T)}$ and $q = 0, 1\} \cup \{\mathrm{Id}\}$ are linearly independent. We conclude the proof with a simple counting argument and lemma 14. □

Beside the trivial normal-tangential bubble given by the identity matrix we can further define other higher order bubbles. To this end we define the local space

$$\mathbb{B}^k_{nt}(T) := \left\{ b \in \mathbb{P}^k(T, \mathbb{D}) : b_{nt} = 0 \right\}.$$

Using the above defined low order functions, we can give a representation to these bubbles in the following lemma.

Lemma 15. *Any normal-tangential bubble* $b \in \mathbb{B}^k_{nt}(T)$ *can be represented as either*

$$b = \sum_{i \in I_{\mathcal{V}}(T)} \mu_i \lambda_i S^i \quad \text{or} \quad b = \sum_{q=0}^{1} \sum_{i \in I_{\mathcal{V}}(T)} \mu^q_i \lambda_i S^i_q, \tag{5.20}$$

for $d = 2$ *or* 3*, respectively, where* $\mu_i, \mu^0_i, \mu^1_i \in \mathbb{P}^{k-1}(T)$*. Consequently,*

$$\dim \mathbb{B}^k_{nt}(T) = \begin{cases} \frac{3}{2} k(k+1), & \text{if } d = 2, \\ \frac{8}{6} k(k+1)(k+2), & \text{if } d = 3. \end{cases}$$

Proof. We start with the proof in two dimensions. By lemma 14 we know that the set $\{S^i : i \in I_{\mathcal{V}(T)}\}$ is a basis for $\mathbb{D}$. Therefore, the matrix $b(x)$ can be represented as

$$b(x) = \sum_{i \in I_{\mathcal{V}}(T)} a_i(x) S^i, \tag{5.21}$$

with polynomials $a_i \in \mathbb{P}^k(T, \mathbb{R})$. Again, by lemma 14 there exists a nonzero constant c_i that equals the value of $S^i_{nt}|_{F_i}$. Then we have

$$b_{nt}(x) = t_i^{\mathrm{T}} b(x) n_i = c_i a_i(x) = 0,$$

for all $x \in F_i$ as $b \in \mathbb{B}^k_{nt}(T)$. As c_i is a constant, this implies that $a_i(x)$ vanishes on F_i, and we can use a factorization with respect to λ_i (which is a linear polynomial and zero on F_i), thus a_i can be equivalently written as

$$a_i(x) = \mu_i(x)\lambda_i(x) \quad \text{with} \quad \mu_i \in \mathbb{P}^{k-1}(T),$$

which proves equation (5.20). In three dimensions we have equivalently

$$b(x) = \sum_{q=0}^{1} \sum_{i \in I_{\mathcal{V}}(T)} a_i^q(x) S_q^i,$$

with $a_i^q(x) \in \mathbb{P}^k(T, \mathbb{R})$, and with lemma 14 we find constants c_i^0 and c_i^1 such that

$$t_{i+2,i+3}^{\mathrm{T}} b(x) n_i = c_i^1 a_i^1(x) = 0 \quad \text{and} \quad t_{i+3,i+1}^{\mathrm{T}} b(x) n_i = c_i^0 a_i^0(x) = 0,$$

which also proves equation (5.20) with the same steps as above.

The dimension follows from the representation (5.20): In two dimensions lemma 14 yields that the basis functions $\{S^i : i \in I_{\mathcal{V}(T)}\}$ are linearly independent, and thus representation (5.20) shows that $\dim \mathcal{B}_k(T)$ equals 3 times the dimensions $\dim \mathbb{P}^{k-1}(T)$. The same argument can be applied in three dimensions. □

5.3.2 The covariant Piola mapping

For the definition of a finite element for the discrete stress space we need to define an appropriate transformation that preserves normal-tangential continuity. In section 5.2 we presented the Piola transformation that preserves the normal components on facets, thus it is a suitable transformation for functions in $H(\mathrm{div}, \Omega)$. The transformation that preserves tangential continuity is known in the literature (see [11] and [87]) as the covariant transformation, and is given by

$$\mathcal{C}(\hat{u}_h) := F_T^{-\mathrm{T}} \hat{u}_h \circ \phi^{-1}.$$

This mapping is particularly of interest for discretizations of the space $H(\mathrm{curl}, \Omega)$. The idea now is to combine those two transformations to define a mapping that preserves normal-tangential continuity. To motivate this let S^i be an arbitrary constant matrix function in the two dimensional case defined by equation (5.16), thus we have the representation

$$S^i = \mathrm{dev}(\nabla\lambda_{i+1} \otimes \mathrm{curl}(\lambda_{i+2})).$$

As analyzed in the last chapter, this constant matrix is designed to study normal-tangential traces and may help to understand how to define a suitable mapping. The barycentric coordinate function λ_{i+1} is an element of $H^1(\Omega, \mathbb{R})$, thus $\nabla\lambda_{i+1}$ is an element of $H(\mathrm{curl}, \Omega)$

(although – or exactly because – $\nabla\lambda_{i+1}$ is equivalent to a normal vector). With a similar argument, $\mathrm{curl}(\lambda_{i+2})$ is an element of $H(\mathrm{div},\Omega)$. Ignoring the deviator, the definition of S^i suggests that the *proper* transformation is given by a Piola mapping from the right and a covariant mapping from the left. We define the covariant Piola mapping by

$$\mathcal{M}(\hat{\sigma}_h) := \frac{1}{\det(F_T)} F_T^{-\mathrm{T}} \hat{\sigma}_h \circ \phi^{-1} F_T^{\mathrm{T}}, \tag{5.22}$$

where $\hat{\sigma}_h \in \mathbb{P}^k(\widehat{T}, \mathbb{R}^{d\times d})$.

Lemma 16. *For an arbitrary* $\hat{\tau} \in \mathbb{P}^k(\widehat{T}, \mathbb{R}^{d\times d})$*, let* $\tau = \mathcal{M}(\hat{\tau})$*. We have*

$$c\,t^T \tau n = \hat{t}^T \hat{\tau} \hat{n}, \qquad \text{where } c = \begin{cases} \det(F_T^F)^2 & \text{if } d = 2, \\ \det(F_T^F)\det(F_T^E) & \text{if } d = 3. \end{cases}$$

Moreover, there holds the relation $\mathrm{tr}(\hat{\tau}) = 0 \Leftrightarrow \mathrm{tr}(\tau) = 0$*, thus the covariant Piola mapping and the deviator commute*

$$\mathrm{dev}(\mathcal{M}(\hat{\tau})) = \mathcal{M}(\mathrm{dev}(\hat{\tau})).$$

Proof. Using the relation between the physical and the reference unit normal and tangential vectors, see equation (5.2), we immediately see

$$t^{\mathrm{T}} \tau n = \frac{1}{\det(F_T^E)} \hat{t}^{\mathrm{T}} F_T^{\mathrm{T}} \frac{1}{\det(F_T)} F_T^{-\mathrm{T}} \hat{\tau} F_T^{\mathrm{T}} \frac{\det(F_T)}{\det(F_T^F)} F_T^{-\mathrm{T}} \hat{n} = \frac{1}{\det(F_T^E)\det(F_T^F)} \hat{t}^{\mathrm{T}} \hat{\tau} \hat{n}.$$

Next, note that for arbitrary matrices $A, B \in \mathbb{R}^{d\times d}$ we have the relation $\mathrm{tr}(A^{-1}BA) = \mathrm{tr}(B)$, and thus $\mathrm{tr}(F_T^{-\mathrm{T}} \hat{\tau} F_T^{\mathrm{T}}) = \mathrm{tr}(\hat{\tau})$. Then we see

$$\begin{aligned} \mathrm{dev}(\mathcal{M}(\hat{\tau})) &= \frac{1}{\det(F_T)} \left(F_T^{-\mathrm{T}} \hat{\tau} F_T^{\mathrm{T}} - \frac{1}{d} \mathrm{tr}(F_T^{-\mathrm{T}} \hat{\tau} F_T^{\mathrm{T}}) \mathrm{Id} \right) \\ &= \frac{1}{\det(F_T)} \left(F_T^{-\mathrm{T}} \hat{\tau} F_T^{\mathrm{T}} - \frac{1}{d} \mathrm{tr}(\hat{\tau}) \mathrm{Id} \right) \\ &= \frac{1}{\det(F_T)} \left(F_T^{-\mathrm{T}} \hat{\tau} F_T^{\mathrm{T}} - \frac{1}{d} \mathrm{tr}(\hat{\tau}) F_T^{-\mathrm{T}} \mathrm{Id} F_T^{\mathrm{T}} \right) \\ &= \frac{1}{\det(F_T)} F_T^{-\mathrm{T}} \left(\hat{\tau} - \frac{1}{d} \mathrm{tr}(\hat{\tau}) \mathrm{Id} \right) F_T^{\mathrm{T}} = \mathcal{M}(\mathrm{dev}(\hat{\tau})). \end{aligned}$$

□

5.3.3 A stress finite element

With the findings from above we can finally define a local stress finite element in the formal style of [25] (also adopted in other texts, e.g., [43, 13]) as a triple $(T, \Xi^k(T), \Phi(T))$, where the element T is a simplex, thus either a triangle or a tetrahedron, the space $\Xi^k(T) = \mathbb{P}^k(T, \mathbb{R}^{d\times d})$, and $\Phi(T)$ is a set of linear functionals representing the degrees of freedom defined in the following. The first group of degrees of freedom is associated to the set of element facets $\mathcal{F}_h(T)$. We define for each $F \in \mathcal{F}_h(T)$,

$$\Phi^F(\tau) := \left\{ \int_F \tau_{nt} \cdot r \,\mathrm{d}s : \; r \in \mathbb{P}^k(F, \mathbb{R}^{d-1}) \right\}. \tag{5.23}$$

The next group is associated to the interior of the element T and given by

$$\Phi^T_{\mathbb{D}}(\tau) := \left\{ \int_T \tau : F_T \hat{\eta} F_T^{-1} \,\mathrm{d}x : \; \hat{\eta} \in \mathbb{B}^k_{nt}(\hat{T}) \right\}. \tag{5.24}$$

As $\operatorname{tr}(F_T \hat{\eta} F_T^{-1}) = 0$, for $\eta \in \mathbb{B}^k_{nt}(\hat{T})$, every functional in $\Phi^T_{\mathbb{D}}$ tests a function τ only with a trace free bubble. Thus, in order to also test the matrix trace we further define the degrees of freedom

$$\Phi^T_{\mathrm{Id}}(\tau) := \left\{ \int_T \operatorname{tr}(\tau) : \hat{\mu} \,\mathrm{d}x : \; \hat{\mu} \in \mathbb{P}^k(\hat{T}, \mathbb{R}) \right\}. \tag{5.25}$$

All together, we define the set

$$\Phi(T) := \Phi^T_{\mathbb{D}} \cup \Phi^T_{\mathrm{Id}} \cup \{\Phi^F : \; F \in \mathcal{F}_h(T)\}, \tag{5.26}$$

and proceed to prove that this set of degrees of freedom is unisolvent and that the number of degrees of freedom matches the dimension of $\Xi^k(T)$.

Theorem 14. *The triple $(T, \Xi^k(T), \Phi(T))$ defines a finite element.*

Proof. To prove the unisolvency of the degrees of freedom, consider a $\tau_h \in \Xi^k(T)$ satisfying $\phi(\tau_h) = 0$ for all $\phi \in \Phi(T)$. In the following we show that this implies that $\tau_h = 0$ proving unisolvency. As $(\tau_h)_{nt} \in \mathbb{P}^k(F, \mathbb{R}^{d-1})$, the facet degrees of freedom $\phi(\tau_h) = 0$ imply that $\tau_h \in \{\sigma \in \mathbb{P}^k(T, \mathbb{R}^{d \times d}) : \sigma_{nt} = 0 \text{ on } \partial T\}$. Next, as $\operatorname{tr}(\tau_h) \in \mathbb{P}^k(T, \mathbb{R})$ the second group Φ^T_{Id} of the interior degrees of freedom $\phi(\tau_h) = 0$, implies that τ_h is trace free, thus $\tau_h \in \mathbb{B}^k_{nt}(T)$. Finally, the first group of the interior degrees of freedom then yields

$$0 = \int_T \tau_h : F_T \hat{\eta} F_T^{-1} = \int_T F_T^{\mathrm{T}} \tau_h F_T^{-\mathrm{T}} : \hat{\eta} = \int_T (\det F_T)^{-1} \mathcal{M}^{-1}(\tau_h) : \hat{\eta} = \int_{\hat{T}} \mathcal{M}^{-1}(\tau_h) : \hat{\eta}$$

for all $\hat{\eta} \in \mathbb{B}^k_{nt}(\hat{T})$. By lemma 16 we have that $\mathcal{M}^{-1}(\tau_h)$ is in $\mathbb{B}^k_{nt}(\hat{T})$, yielding $\mathcal{M}^{-1}(\tau_h) = 0$ and $\tau_h = 0$.

It only remains to prove that the dimensions match. The dimension of $\Xi^k(T)$ is given by $\dim \mathbb{P}^k(T, \mathbb{R}^{d \times d})$ (which equals $d^d \dim \mathbb{P}^k(T, \mathbb{R})$). Using lemma 15 to count the number of degrees of freedom in $\Phi(T)$, we see in two dimensions

$$\begin{aligned}
\#\Phi(T) &= \#\Phi^T_{\mathbb{D}} + \#\Phi^T_{\mathrm{Id}} + \#\{\Phi^F : \; F \in \mathcal{F}_h(T)\} \\
&= \frac{3}{2} k(k+1) + \frac{1}{2}(k+1)(k+2) + 3(k+1) \\
&= \frac{3}{2}(k+1)(k+2) + \frac{1}{2}(k+1)(k+2) = 4 \dim \mathbb{P}^k(T, \mathbb{R}).
\end{aligned}$$

With the same argument in three dimensions we conclude the proof. $\square$

As mentioned in the beginning of this chapter, the space $\Xi^k(\mathcal{T}_h)$ and similarly also the space $\Xi^k(T)$ allow a natural splitting into two parts, see equation (5.15). To this end we define the spaces $\Xi^k_{\mathbb{D}}(T) = \mathbb{P}^k(T, \mathbb{D})$ and $\Xi^k_{\mathrm{Id}}(T) = \mathbb{P}^k(T, \mathrm{Id})$. The construction of the above degrees of freedoms allows us to define finite elements for each of the subsets separately.

Lemma 17. *The triple* $(T, \Xi^k_{\mathbb{D}}(T), \Phi_{\mathbb{D}}(T) \cup \{\Phi^F : F \in \mathcal{F}_h(T)\})$ *and* $(T, \Xi^k_{\mathrm{Id}}(T), \Phi_{\mathrm{Id}}(T))$ *define a finite element, and*

$$\dim \Xi^k_{\mathbb{D}}(T) = (d^d - 1) \dim \mathbb{P}^k(T, \mathbb{R}). \tag{5.27}$$

Proof. The proofs follow the same steps as the proof of Theorem 14. □

We conclude this section by defining another set of unisolvent linear functionals for the space $\Xi^k(T)$, used to define an interpolation operator for the stress space in section 6.3.1. To this end we define the degrees of freedom associated to the interior of T by

$$\tilde{\Phi}^T_{\mathbb{D}}(\tau) := \left\{ \int_T \tau : F_T \hat{\eta} F_T^{-1} \,\mathrm{d}x : \; \hat{\eta} \in \mathbb{P}^{k-1}(\hat{T}, \mathbb{D}) \right\}. \tag{5.28}$$

In contrast to $\Phi^T_{\mathbb{D}}$, the functionals in $\tilde{\Phi}^T_{\mathbb{D}}$ do not test with normal-tangential bubbles but with trace free polynomials of order $k-1$. Now we similarly define the sets

$$\tilde{\Phi}(T) := \tilde{\Phi}^T_{\mathbb{D}} \cup \Phi^T_{\mathrm{Id}} \cup \{\Phi^F : \; F \in \mathcal{F}_h(T)\}, \tag{5.29}$$

$$\tilde{\Phi}_{\mathbb{D}}(T) := \tilde{\Phi}^T_{\mathbb{D}} \cup \{\Phi^F : \; F \in \mathcal{F}_h(T)\}. \tag{5.30}$$

Theorem 15. *The set* $\tilde{\Phi}(T)$ *is unisolvent for the space* $\Xi^k(T)$*. The set* $\tilde{\Phi}_{\mathbb{D}}(T)$ *is unisolvent for* $\Xi^k_{\mathbb{D}}(T)$*.*

Proof. To prove the unisolvency of the degrees of freedom we again consider a $\tau_h \in \Xi^k(T)$ satisfying $\phi(\tau_h) = 0$ for all $\phi \in \tilde{\Phi}^T_{\mathbb{D}} \cup \Phi^T_{\mathrm{Id}} \cup \{\Phi^F : \; F \in \mathcal{F}_h(T)\}$. Similar as in the proof of theorem 14, the facet and the second group of interior degrees of freedoms show that $\tau_h \in \mathbb{B}^k_{nt}(T)$. Now, lemma 15 yields that τ_h can be written as

$$\tau_h = \sum_{i \in I_{\mathcal{V}}(T)} \mu_i \lambda_i S^i, \text{ for } d = 2, \quad \text{and} \quad \tau_h = \sum_{q=0}^{1} \sum_{i \in I_{\mathcal{V}}(T)} \mu_i^q \lambda_i S^i_q, \text{ for } d = 3,$$

with polynomials $\mu_i, \mu_i^0, \mu_i^1 \in \mathbb{P}^{k-1}(T)$. According to the definition of the element degress of freedom $\tilde{\Phi}^T_{\mathbb{D}}$ we can choose $F \hat{\eta} F^{-1} = \sum_{i \in I_{\mathcal{V}}(T)} \mu_i S^i$ and $F \hat{\eta} F^{-1} = \sum_{q=0}^{1} \sum_{i \in I_{\mathcal{V}}(T)} \mu_i^q S^i_q$ for $d = 2$ and 3, respectively. As we assume that the degrees of freedom applied on τ_h vanish, we have in two dimensions

$$\int_T \sum_{\imath \in \mathcal{V}} \mu_i \lambda_i S^i : \sum_{\imath \in \mathcal{V}} \mu_i S^i \,\mathrm{d}x = \int_T \lambda_i \Big| \sum_{\imath \in \mathcal{V}} \mu_i S^i \Big|^2 \,\mathrm{d}x = 0,$$

yielding $\mu_i = 0$, and thus $\tau_h = 0$. In three dimensions the argument is the same. We conclude the proof with a simple counting argument similar as in the proof of theorem 14. □

5.3.4 Arbitrary order shape basis functions

Using the lowest order basis functions defined in section 5.3.1 we can write down shape functions on an element T using barycentric coordinates. In two dimensions we immediately see that the set

$$\lambda_{i+1}^{\alpha_1}\lambda_{i+2}^{\alpha_2}S^i, \qquad \lambda_i^{\beta_0}\lambda_{i+1}^{\beta_1}\lambda_{i+2}^{\beta_2}(\lambda_i S^i), \qquad \lambda_i^{\gamma_0}\lambda_{i+1}^{\gamma_1}\lambda_{i+2}^{\gamma_2}\mathrm{Id}, \tag{5.31}$$

for all $i \in \mathcal{V}$, and all multi-indices (α_1, α_2), $(\beta_0, \beta_1, \beta_2)$ and $(\gamma_0, \gamma_1, \gamma_2)$, with $\alpha_i \geq 0$, $\beta_i \geq 0$, $\gamma_i \geq 0$ having length $\alpha_1 + \alpha_2 = \gamma_0 + \gamma_1 + \gamma_2 = k$ and $\beta_0 + \beta_1 + \beta_2 = k - 1$, form a basis for $\Xi^k(T)$. Similarly, when T is a tetrahedron, the following set is a basis for $\Xi_k(T)$:

$$\lambda_{i+1}^{\alpha_1}\lambda_{i+2}^{\alpha_2}\lambda_{i+3}^{\alpha_3}S_q^i, \qquad \lambda_i^{\beta_0}\lambda_{i+1}^{\beta_1}\lambda_{i+2}^{\beta_2}\lambda_{i+3}^{\beta_3}(\lambda_i S_q^i), \qquad \lambda_i^{\gamma_0}\lambda_{i+1}^{\gamma_1}\lambda_{i+2}^{\gamma_2}\lambda_{i+3}^{\gamma_3}\mathrm{Id}, \tag{5.32}$$

for all $i \in \mathcal{V}$, $q = 0, 1$, and all multi-indices $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_0, \beta_1, \beta_2, \beta_3)$ and $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$, with $\alpha_i \geq 0$, $\beta_i \geq 0$ $\gamma_i \geq 0$ having length $\alpha_1 + \alpha_2 + \alpha_3 = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 = k$, and $\beta_1 + \beta_2 + \beta_3 = k - 1$. Note that if the last set of basis functions is removed, then the resulting set of functions forms a basis for $\Xi_{\mathbb{D}}^k(T)$.

Although it is easy to prove that the functions defined by equation (5.31) or (5.32) are linearly independent, we opt to do so for another set of reference element shape functions. Using a Dubiner basis instead of barycentric monomials, the ensuing construction produces better conditioned matrices. For a better understanding of this topic we refer the reader for example to the works [52, 115].

In the following we are going to define arbitrary high order shape basis functions on the reference element. In section 5.3.1 we gave an explicit construction of the lowest order basis functions by equations (5.16) and (5.17). Using these definitions on the reference element (including a scaling with a proper constant) we derive for $d = 2$ the matrices given by

$$\hat{S}^0 := \sqrt{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \hat{S}^1 := \begin{pmatrix} 0.5 & 0 \\ 1 & -0.5 \end{pmatrix} \quad \text{and} \quad \hat{S}^2 := \begin{pmatrix} 0.5 & -1 \\ 0 & -0.5 \end{pmatrix}, \tag{5.33}$$

and for $d = 3$ the matrices

$$\begin{gathered}
\hat{S}_0^0 = \sqrt{6}\begin{pmatrix} \frac{-2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \hat{S}_0^1 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 1 & \frac{-2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \hat{S}_0^2 = \begin{pmatrix} \frac{-2}{3} & 1 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \hat{S}_0^3 = \begin{pmatrix} \frac{-2}{3} & 0 & 1 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \\
\hat{S}_1^0 = \sqrt{6}\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{-2}{3} \end{pmatrix}, \hat{S}_1^1 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 1 & 0 & \frac{-2}{3} \end{pmatrix}, \hat{S}_1^2 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{-2}{3} \end{pmatrix}, \hat{S}_1^3 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{-2}{3} & 1 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.
\end{gathered} \tag{5.34}$$

Note that we made the particular choice of the numbering of the vertices of $\hat{T}$ and the corresponding tangential vectors as it is given in section 5.1. With the same techniques as in lemma 14, one easily sees that

$$\begin{aligned}
&\hat{t}_j^{\mathrm{T}}\hat{S}^i\hat{n}_j = \delta_{ij} && \text{and} && \hat{t}_j^{\mathrm{T}}\lambda_i\hat{S}^i\hat{n}_j = 0 && \text{for} && i, j = 0, 1, 2, \\
&\hat{t}_{jl}^{\mathrm{T}}\hat{S}_q^i\hat{n}_j = \delta_{ij}\delta_{ql} && \text{and} && \hat{t}_{jl}^{\mathrm{T}}\lambda_i\hat{S}_q^i\hat{n}_j = 0 && \text{for} && i, j = 0, 1, 2, 3 \quad \text{and} \quad q, l = 0, 1,
\end{aligned} \tag{5.35}$$

and that $\{\hat{S}^i : i = 0,1,2\}$ and $\{\hat{S}^i_q : i = 0,1,2,3; q = 0,1\}$ are basis for $\mathbb{D}$ in two and three dimensions, respectively. Based on these constant matrices and the identity matrix we now construct shape functions for the local stress space $\Xi^k(\hat{T})$.

Using the definition of the Dubiner basis in two dimensions, see equation (5.6), we define a local basis by

$$\begin{aligned}\hat{\Psi}^k_F &:= \{\hat{S}^j \hat{r}_{i0}(\lambda_{j+1}, \lambda_{j+2}) : j = 0,1,2 \text{ and } 0 \le i \le k\},\\ \hat{\Psi}^k_{\mathbb{D}} &:= \{\lambda_j \hat{S}^j \hat{r}_{il}(\lambda_0, \lambda_1, \lambda_2) : j = 0,1,2 \text{ and } 0 \le i + l \le k-1\},\\ \hat{\Psi}^k_{\mathrm{Id}} &:= \{\mathrm{Id}\hat{r}_{il}(\lambda_0, \lambda_1, \lambda_2) : 0 \le i + l \le k\},\end{aligned}$$

and similar in three dimensions, using the definition (5.7), we define

$$\begin{aligned}\hat{\Psi}^k_F &:= \{\hat{S}^j_q \hat{r}_{il0}(\lambda_{j+1}, \lambda_{j+2}, \lambda_{j+3}) : j = 0,1,2,3 \text{ and } q = 0,1 \text{ and } 0 \le i + l \le k\},\\ \hat{\Psi}^k_{\mathbb{D}} &:= \{\lambda_j \hat{S}^j_q \hat{r}_{ilg}(\lambda_0, \lambda_1, \lambda_2, \lambda_3) : j = 0,1,2,3 \text{ and } q = 0,1 \text{ and } 0 \le i + l + g \le k-1\},\\ \hat{\Psi}^k_{\mathrm{Id}} &:= \{\mathrm{Id}\hat{r}_{ilg}(\lambda_0, \lambda_1, \lambda_2, \lambda_3) : 0 \le i + l + g \le k\}.\end{aligned}$$

Theorem 16. *The set of functions* $\{\hat{\Psi}^k_F \cup \hat{\Psi}^k_{\mathbb{D}} \cup \hat{\Psi}^k_{\mathrm{Id}}\}$ *is a basis for* $\Xi^k(\hat{T})$*, and* $\{\hat{\Psi}^k_F \cup \hat{\Psi}^k_{\mathbb{D}}\}$ *is a basis for* $\Xi^k_{\mathbb{D}}(\hat{T})$.

Proof. We start with the two-dimensional case. First note that the functions $\lambda_i \hat{S}^i$ with $i = 0,1,2$ are linearly independent. This follows with elementary calculations and using the property that $(\hat{S}^i)_{nt}$ vanishes also on lines parallel to F_j with $j \ne i$. Now let $\alpha^j_i \in \mathbb{R}$,$\beta^j_{il} \in \mathbb{R}$, and $\gamma_{il} \in \mathbb{R}$ be arbitrary coefficients and define $\hat{S}^j_i := \hat{S}^j \hat{r}_{i0}(\lambda_{j+1}, \lambda_{j+2})$, $\hat{B}^j_{il} := \lambda_j \hat{S}^j \hat{r}_{il}(\lambda_0, \lambda_1, \lambda_2)$, and $\hat{D}_{il} := \mathrm{Id}\hat{r}_{il}(\lambda_0, \lambda_1, \lambda_2)$. We assume that

$$\sum_{j=0}^{2}\sum_{i=0}^{k} \alpha^j_i \hat{S}^j_i + \sum_{j=0}^{2}\sum_{i=0}^{k-1}\sum_{l=i}^{k-1} \beta^j_{il}\hat{B}^j_{il} + \sum_{i=0}^{k}\sum_{l=i}^{k} \gamma_{il}\hat{D}_{il} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and show that this induces that all coefficients are equal to zero proving linear independency of the basis functions. Let $\hat{F}_g$ with $g = 0,1,2$ be an arbitrary reference face. Due to (5.35) and (4.2) there holds

$$\begin{aligned}\hat{t}^{\mathrm{T}}_g \Big(\sum_{j=0}^{2}\sum_{i=0}^{k} \alpha^j_i \hat{S}^j_i + \sum_{j=0}^{2}\sum_{i=0}^{k-1}\sum_{l=i}^{k-1} \beta^j_{il}\hat{B}^j_{il} + \sum_{i=0}^{k}\sum_{l=i}^{k} \gamma_{il}\hat{D}_{il}\Big)\hat{n}_g \\ = \hat{t}^{\mathrm{T}}_g \left(\sum_{i=0}^{k} \alpha^g_i \hat{S}^g_i\right) \hat{n}_g = \hat{t}^{\mathrm{T}}_g \left(\sum_{i=0}^{k} \alpha^g_i \hat{S}^g \hat{r}_{i0}(\lambda_{g+1}, \lambda_{g+2})\right) \hat{n}_g = 0.\end{aligned}$$

As $\hat{r}_{i0}(\lambda_{g+1}, \lambda_{g+2})$ is a polynomial basis on $\hat{F}_g$, and as $\hat{S}^g, \hat{n}_g$ and $\hat{t}_g$ are constant, it follows that all coefficients α^g_i have to be zero. As g was arbitrary we conclude $\alpha^j_i = 0$ for $j = 0,1,2$ and $0 \le i \le k$.

Next, as $\mathrm{tr}(\lambda_i \hat{S}^i = 0)$, we have

$$\mathrm{tr}\Big(\sum_{j=0}^{2}\sum_{i=0}^{k-1}\sum_{l=i}^{k-1} \beta^j_{il}\hat{B}^j_{il} + \sum_{i=0}^{k}\sum_{l=i}^{k} \gamma_{il}\hat{D}_{il}\Big) = \mathrm{tr}\Big(\sum_{i=0}^{k}\sum_{l=i}^{k} \gamma_{il}\hat{D}_{il}\Big) = \sum_{i=0}^{k}\sum_{l=i}^{k} \gamma_{il}\hat{r}_{il}(\lambda_0, \lambda_1, \lambda_2),$$

and thus with the same arguments as before also $\gamma_{il} = 0$ for $0 \leq i + l \leq k$.

Finally, as the functions $\lambda_i \hat{S}^i$ are linearly independent, we have for each $g = 0, 1, 2$ (due to the assumption at the beginning)

$$\sum_{i=0}^{k-1} \sum_{l=i}^{k-1} \beta_{il}^g \hat{B}_{il}^g = \sum_{i=0}^{k-1} \sum_{l=i}^{k-1} \beta_{il}^g \hat{r}_{il} \lambda_g \hat{S}^g = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

As $\hat{r}_{il}\lambda_g$ is a basis for $\lambda_g \mathbb{P}^{k-1}(\hat{T})$, and the last equation holds true for all points in $\hat{T}$, we conclude $\beta_{il}^g = 0$ for $0 \leq i + l \leq k - 1$. As g was arbitrary, this yields that all coefficients are equal to zero. With a simple counting argument similar as in the proof of theorem 14, and with the same steps in the three dimensional case the first statement of the theorem is proven.

For the second statement note that due to $\operatorname{tr}(S^i) = 0$ all shape functions in $\{\hat{\Psi}_{\mathbb{D}}^k \cup \hat{\Psi}_F^k\}$ are trace free and are further matrix-valued polynomials up to order k. Then the proof follows the same lines as above. □

Remark 2. *In the work [69] a slightly different space with respect to the polynomial orders was defined. Due to the construction of the shape functions and the degrees of freedom in theorem 14 it is possible to combine different orders for the different groups such that same spaces as in [69] can be constructed. To this end we define*

$$\Xi^{k,r,s}(T) := \{\sigma \in \mathbb{P}^k(T, \mathbb{R}^{d\times d}) : \operatorname{tr}(\sigma) \in \mathbb{P}^r(T, \mathbb{R}), \sigma_{nt} \in \mathbb{P}^s(F, \mathbb{R}^{d-1}) \text{ for all } F \in \mathcal{F}_h(T)\}. \tag{5.36}$$

With the same arguments as above one shows that in two dimensions

$$\begin{aligned}
\hat{\Psi}_F^k &:= \{\hat{S}^j \hat{r}_{i0}(\lambda_{j+1}, \lambda_{j+2}) : j = 0, 1, 2 \text{ and } 0 \leq i \leq s\}, \\
\hat{\Psi}_{\mathbb{D}}^k &:= \{\lambda_j \hat{S}^j \hat{r}_{il}(\lambda_0, \lambda_1, \lambda_2) : j = 0, 1, 2 \text{ and } 0 \leq i + l \leq k - 1\}, \\
\hat{\Psi}_{\mathrm{Id}}^k &:= \{\mathrm{Id}\hat{r}_{il}(\lambda_0, \lambda_1, \lambda_2) : 0 \leq i + l \leq r\},
\end{aligned}$$

is a basis for $\Xi^{k,r,s}(\widehat{T})$, and that the set of functionals

$$\begin{aligned}
\Phi^{F,s}(\tau) &:= \left\{ \int_F \tau_{nt} \cdot r \, \mathrm{d}s : \; r \in \mathbb{P}^s(F, \mathbb{R}^{d-1}) \right\}, \\
\Phi_{\mathbb{D}}^{T,k}(\tau) &:= \left\{ \int_T \tau : F_T \hat{\eta} F_T^{-1} \, \mathrm{d}x : \; \hat{\eta} \in \mathbb{B}_{nt}^k(\hat{T}) \right\}, \\
\Phi_{\mathrm{Id}}^{T,r}(\tau) &:= \left\{ \int_T \operatorname{tr}(\tau) : \hat{\mu} \, \mathrm{d}x : \; \hat{\mu} \in \mathbb{P}^r(\hat{T}, \mathbb{R}) \right\},
\end{aligned}$$

is unisolvent for $\Xi^{k,r,s}(T)$. A similar observation can be made in three dimensions.

5.3.5 A global basis

Using the local basis on the reference triangle $\hat{T}$ we can now simply define a global basis for the stress space $\Xi^k(\mathcal{T}_h)$. This is done in the usual way. Using the mapping $\mathcal{M}$ and a basis function $\hat{S} \in \{\hat{\Psi}^k_{\mathbb{D}} \cup \hat{\Psi}^k_{\mathrm{Id}} \cup \hat{\Psi}^k_F\}$ we define the restriction of a global shape function S (with support on a patch) on an arbitrary physical element $T \in \mathcal{T}_h$ by

$$S := \mathcal{M}(\hat{S}).$$

Next, we identify all topological entities, vertices and faces of the physical element T with the corresponding entities of the global mesh. This identification is needed as faces and vertices coincide for adjacent physical elements. Note that the global orientation of the faces (and edges) plays an important role in order to assure (normal-tangential) continuity. This is a well known difficulty, and we refer for example to the work [118] for a detailed discussion regarding this topic. By this, we construct global basis functions which are, restricted on a physical element $T \in \mathcal{T}_h$, always a mapped basis function of the basis defined on the reference element $\hat{T}$.

Further note that due to lemma 16 the resulting basis functions are normal-tangential continuous, thus $[\![S_{nt}]\!] = 0$. To see this, let ϕ_1 be the mapping of an arbitrary element T_1, and let ϕ_2 be the mapping of an element T_2 such that $F = T_1 \cap T_2$. There exists a reference face $\hat{F} \subset \partial\hat{T}$ such that $F = \phi_1(\hat{F}) = \phi_1(\hat{F})$ (in the sense of a set) and $\phi_1|_{\hat{F}} = \phi_2|_{\hat{F}}$ (in the sense of equivalent functions). By this, and the same ideas for a reference edge $\hat{E}$ in the three-dimensional case, the constant c in lemma 16 is the same for both mappings. In two dimensions we have the identity $S_{nt} = (t^{\mathrm{T}} S n)t$, thus lemma 16 implies normal-tangential continuity of S because S was a mapped basis function of the reference element. In three dimensions S_{nt} is a tangent vector in F. Each tangent vector can be represented as a linear combination of two arbitrary edge tangent vectors $t_i \subset \partial F$. By lemma 16, we deduce that the scalar values ${t_i}^{\mathrm{T}} S n$ are preserved, thus again, we have normal-tangential continuity. Taking all functions in $\{\hat{\Psi}^k_{\mathbb{D}} \cup \hat{\Psi}^k_{\mathrm{Id}} \cup \hat{\Psi}^k_F\}$ and mapping them to each element separately results in a basis for $\Xi^k(\mathcal{T}_h)$. In the same manner one also maps functions $\hat{S} \in \{\hat{\Psi}^k_{\mathbb{D}} \cup \hat{\Psi}^k_F\}$ to each physical element separately and identifies them with the restriction of a global shape function S to define a basis for $\Xi^k_{\mathbb{D}}(\mathcal{T}_h)$.

6 The MCS method

This chapter is dedicated to the numerical approximation of the solution of the system (4.26), and is structured as follows: At the beginning we focus on the derivation of a new variational formulation, which is based on discrete finite dimensional spaces. These spaces are chosen such that they fit to the formulation (4.26). Afterwards, we present two different versions of a stability analysis. In a first setting we show solvability using a discrete DG-like H^1-norm for the velocity space and a L^2-norm for the stress space. Obviously, this choice is motivated by the primal velocity-pressure formulation of the Stokes equations, where the velocity is an element of $H^1(\Omega, \mathbb{R}^d)$. The second version then discusses well-posedness in *natural* norms. In this case, the velocity is measured in the $H(\mathrm{div})$-norm, which fits to the continuous setting given by (4.26). Similarly, we then also choose a discrete stress norm that fits to the space $H(\mathrm{curl}\,\mathrm{div}, \Omega)$. We conclude the chapter with several error estimates proving optimal convergence of the finite element error.

For the discrete setting we allow all different kinds of boundary conditions as it is given in equation (4.6). To this end we assume that the boundary is divided according to (4.1).

6.1 A new variational formulation

We begin with the definition of the discrete spaces. For the ease of representation we assume that the essential boundary conditions $g_{N,t}$ and $g_{D,n}$ lie in the trace spaces of the corresponding polynomial spaces as defined below. Now, let $k \geq 1$, and define

$$\Sigma_h := \Xi^k_{\mathbb{D}}(\mathcal{T}_h) = \{\tau_h \in \mathbb{P}^k(\mathcal{T}_h, \mathbb{D}) : [\![(\tau_h)_{nt}]\!] = 0 \text{ for all } F \in \mathcal{F}_h^{\mathrm{int}} \cup \Gamma_{N,t}\}, \tag{6.1}$$

$$V_h := \mathcal{RT}^k(\mathcal{T}_h) \cap V, \tag{6.2}$$

and for given boundary conditions $g_{D,n}$ and $g_{N,t}$ the spaces

$$\Sigma_{h,N} := \{\tau_h \in \mathbb{P}^k(\mathcal{T}_h, \mathbb{D}) : [\![(\tau_h)_{nt}]\!] = 0 \text{ for all } F \in \mathcal{F}_h^{\mathrm{int}}, (\tau_h)_{nt} = g_{N,t} \text{ on } \Gamma_{N,t}\},$$
$$V_{h,D} := \{v_h \in \mathcal{RT}^k(\mathcal{T}_h) : v_n = g_{D,n} \text{ on } \Gamma_{D,n}\}.$$

The pressure space is given by

$$Q_h := \begin{cases} \mathbb{P}^k(\mathcal{T}_h, \mathbb{R}) \cap L^2_0(\Omega, \mathbb{R}) & \text{if } \Gamma_{D,n} = \Gamma, \\ \mathbb{P}^k(\mathcal{T}_h, \mathbb{R}) & \text{else .} \end{cases} \tag{6.3}$$

Note that the discrete stress space only uses the subset of trace free polynomials, see equation (5.15). As discussed in chapter 5, the velocity and the pressure spaces are chosen to be conforming, but the stress space is slightly non-conforming $\sigma_h \not\subset H(\mathrm{curl}\,\mathrm{div}, \Omega)$. Still, all elements in the discrete stress space are square integrable functions, thus we have the conformity $\Sigma_h \subset L^2(\Omega, \mathbb{R}^{d\times d})$. With these findings we realize that the bilinear forms defined in the continuous setting by equations (4.27) and (4.28) can also be used in the discrete setting. We have

$$a(\sigma_h, \tau_h) = \int_\Omega \frac{1}{\nu}\sigma_h : \tau_h \,\mathrm{d}x \quad \text{and} \quad b_1(u_h, q_h) = \int_\Omega \operatorname{div}(u_h) q_h \,\mathrm{d}x,$$

for all functions $\sigma_h, \tau_h \in \Sigma_h$ and $u_h \in V_h, q_h \in Q_h$.

Handling the terms with the divergence of stress variables we can not proceed in the same way. To this end we define the discrete bilinear form

$$\begin{aligned} &b_{2h} : \{\tau \in H^1(\mathcal{T}_h, \mathbb{R}^{d\times d}) : [\![\tau_{nt}]\!] = 0\} \times \{v \in H^1(\mathcal{T}_h, \mathbb{R}^d) : [\![v_n]\!] = 0\} \to \mathbb{R}, \\ &b_{2h}(\tau, v) := \sum_{T\in\mathcal{T}_h} \int_T \operatorname{div}(\tau)\cdot v \,\mathrm{d}x - \sum_{F\in\mathcal{F}_h} \int_F [\![\tau_{nn}]\!] v_n \,\mathrm{d}s\,. \end{aligned} \tag{6.4}$$

This definition is motivated by identity (5.12) of theorem 13. Moreover, using an integration by parts argument, we find the equivalent representation

$$b_{2h}(\tau, v) = -\sum_{T\in\mathcal{T}_h} \int_T \tau : \nabla v \,\mathrm{d}x + \sum_{F\in\mathcal{F}_h} \int_F \tau_{nt}\cdot [\![v_t]\!] \,\mathrm{d}s, \tag{6.5}$$

since $[\![\tau_{nt}]\!] = 0$ and $[\![v_n]\!] = 0$. Using above definitions, and assuming enough regularity of the right hand side, the discrete counterpart of the weak form (4.26) is given by the (MCS) method that reads as: Find $(\sigma_h, u_h, p_h) \in \Sigma_{h,N} \times V_{h,D} \times Q_h$ such that

$$\begin{cases} a(\sigma_h, \tau_h) + b_{2h}(\tau_h, u_h) = (g_{D,t}, (\tau_h)_{nt})_{\Gamma_{D,t}} & \text{for all } \tau_h \in \Sigma_h, \\ b_{2h}(\sigma_h, v_h) + b_1(v_h, p_h) = -(f, v_h)_\Omega + (g_{N,n}, (v_h)_n)_{\Gamma_{N,n}} & \text{for all } v_h \in V_h, \\ b_1(u_h, q_h) = 0 & \text{for all } q_h \in Q_h. \end{cases} \tag{6.6}$$

As discussed in section 5.2, the velocity and the pressure space fulfill the property $\operatorname{div}(V_h) = Q_h$. Therefore, any weakly divergence-free velocity field is also strongly divergence free:

$$\int_\Omega \operatorname{div}(u_h) q_h \,\mathrm{d}x = 0,\ \forall q_h \in Q_h \quad \Leftrightarrow \quad \operatorname{div}(u_h) = 0 \quad \text{in } \Omega. \tag{6.7}$$

Thus, the velocity solution u_h of the system (6.6) is exactly divergence free.

Remark 3. *Similarly as in the infinite dimensional setting, system (6.6) is solved using a homogenization process.*

6.2 Discrete inf-sup stability

In this section we discuss discrete inf-sup stability of the MCS method (6.6). Although we only consider the case of homogeneous boundary conditions,

$$g_{D,n} = g_{D,t} = g_{N,t} = g_{N,n} = 0, \tag{6.8}$$

we want to mention that with the usual techniques all results can be extended also to the non-homogeneous case. The stability analysis that we present in this section is based on norms that might seem unnatural with respect to the continuous setting of problem (4.26). To this end we define

$$
\begin{aligned}
&||\tau_h||^2_{\Sigma_h} := ||\tau_h||^2_{L^2(\Omega)} = ||\mathrm{dev}(\tau_h)||^2_{L^2(\Omega)}, && \tau_h \in \Sigma_h,\\
&||v_h||^2_{V_h} := ||v_h||^2_{1,h} := \sum_{T\in\mathcal{T}_h} ||\nabla v_h||^2_T + \sum_{F\in\mathcal{F}_h} \frac{1}{h} ||[\![(v_h)_t]\!]||^2_F, && v_h \in V_h,\\
&||q_h||^2_{Q_h} := ||q_h||^2_{L^2(\Omega)}, && q_h \in Q_h.
\end{aligned}
$$

This choice is motivated by the primal velocity-pressure formulation of the Stokes system, hence with respect to the variational formulation given by (4.4). Note that the L^2-like norm for the space Σ_h is also related to an H^1-like norm of the velocity since we expect σ_h to be an approximation of $\nu\nabla u$. The norm for the velocity space is the same as it occurs in discontinuos Galerkin settings, see for example in [7, 67, 102], and for flow problems in [113, 108, 54, 29, 31, 30] and [78, 76, 77].

Although the analysis that we present in this section follows very similar lines as the results given in [69], there are some crucial differences. In [69] the velocity space was chosen as the conforming polynomial space $\mathcal{BDM}^{k+1}(\mathcal{T}_h)$. In order to show discrete inf-sup stability, the authors used a stress space that was locally enriched with normal-tangential bubbles, thus using the notation of this thesis given by equation (5.36), the authors used the space $\Xi^{k+1,0,k}(\mathcal{T}_h)$. Note that a similar choice of the velocity space as in this thesis was also made in the work [49]. Therein, the authors presented a hybrid DG method for solving the Brinkman problem, which is based on the work of Cockburn et al. [32]. Their discretization results in a stress approximation with a similar normal-tangential continuity and can be seen as a hybridized version of the MCS method. Note that the choice $\mathcal{RT}^k(\mathcal{T}_h)$ leads to a less accurate velocity approximation (compared to an approximation with $\mathcal{BDM}^{k+1}$), thus in order to retain the optimal convergence order of the velocity (measured in a discrete H^1-norm) a local element-wise post processing has to be introduced. With the reconstruction operators introduced in [76, 77], this post processing can be done retaining the exact divergence-free property.

Before we start, we show several norm equivalences that we shall use within this thesis.

6.2.1 Norm equivalences

For the discrete analysis we are going to use several norm equivalences proven in the following. Due to quasi-uniformity of the triangulation $\mathcal{T}_h$ these results are all based on standard scaling arguments.

Lemma 18. *For any $\hat{\tau} \in \Xi^k_{\mathbb{D}}(\hat{T})$, letting $\tau = \mathcal{M}(\hat{\tau})$, we have*

$$h^d \|\tau_h\|^2_T \sim \|\hat{\tau}_h\|^2_{\hat{T}}, \tag{6.9}$$

and on any $F \in \mathcal{F}_h(T)$

$$h^{d+1} \left\|t^T \tau_h n\right\|^2_F \sim \left\|\hat{t}^T \hat{\tau}_h \hat{n}\right\|^2_{\hat{F}}. \tag{6.10}$$

Proof. This follows with a scaling argument, lemma 16, and equations (5.1) and (5.2). Note that a similar scaling result is presented in the work [112]. □

Lemma 19. *For all $\tau_h \in \Sigma_h$,*

$$||\tau_h||^2_{\Sigma_h} \sim \sum_{T\in\mathcal{T}_h} ||\mathrm{dev}(\tau_h)||^2_T + \sum_{F\in\mathcal{F}_h} h\big\|(\tau_h)_{nt}\big\|^2_F.$$

Proof. Using norm equivalence for finite dimensional spaces on the reference element $\widehat{T}$, we have for any face $\hat{F} \in \mathcal{F}_{\hat{T}}$ and for all functions and $\hat{\tau} \in \Xi^k_{\mathbb{D}}(\widehat{T})$

$$\|\hat{t}^{\mathrm{T}}\hat{\tau}_h\hat{n}\|^2_{\hat{F}} \lesssim \|\hat{\tau}_h\|^2_{\hat{T}}.$$

Due to (6.10) and lemma 18, this yields

$$\sum_{F\in\mathcal{F}_h} h\big\|(\tau_h)_{nt}\big\|^2_F \lesssim \sum_{T\in\mathcal{T}_h} \|\tau_h\|^2_T, \qquad \text{for all } \tau_h \in \Xi^k_{\mathbb{D}}(T).$$

This proves one side of the stated equivalence. The other side is obvious. □

Lemma 20. *For all $v_h \in V_h$ there holds*

$$||v_h||^2_{V_h} \sim \sum_{T\in\mathcal{T}_h} ||\nabla v_h||^2_T + \sum_{F\in\mathcal{F}_h} \frac{1}{h}\big\|\Pi^0_F[\![(v_h)_t]\!]\big\|^2_F.$$

Proof. One side of the equivalence is obvious from the continuity of Π^0_F. For the other direction first note that

$$||v_h||^2_{V_h} \leq \sum_{T\in\mathcal{T}_h} ||\nabla v_h||^2_T + \sum_{F\in\mathcal{F}_h} \frac{2}{h}\big\|\Pi^0_F[\![(v_h)_t]\!]\big\|^2_F + \frac{2}{h}\big\|[\![(v_h)_t]\!] - \Pi^0_F[\![(v_h)_t]\!]\big\|^2_F. \tag{6.11}$$

Now, on each facet $F \in \mathcal{F}_h(T)$, using a Poincaré type inequality on the boundary and an inverse inequality for polynomials, see [117], we get the standard estimate

$$\big\|(v_h)_t - \Pi^0_F(v_h)_t\big\|_F \lesssim h\|\nabla_t v_h\|_F \lesssim h^{1/2}\|\nabla v_h\|_T,$$

and the estimate is proven. □

Lemma 21. *Let $v_h \in \mathcal{RT}^k(\mathcal{T}_h)$, and $\mathrm{div}(v_h) = 0$. Then v_h is in $\mathcal{BDM}^k(\mathcal{T}_h)$ and has the local representation*

$$v_h|_T = a_T \quad \textit{with} \quad a_T \in \mathbb{P}^k(T, \mathbb{R}^d).$$

Proof. By the definition of the Raviart-Thomas space $\mathcal{RT}^k(\mathcal{T}_h)$ there exist for each v_h and for each element $T \in \mathcal{T}_h$ a function $a_T \in \mathbb{P}^k(T, \mathbb{R}^d)$ and $b_T \in \mathbb{P}^k_{\mathrm{hom}}(T, \mathbb{R})$ such that the restriction on T is given by $v_h|_T = a_T + x b_T$. Now assume $b_T \neq 0$, then we have as $\mathrm{div}(v_h) = 0$ the identity

$$\mathrm{div}(b_T x) = d b_T + \nabla b_T x = \mathrm{div}(a_T).$$

As $\mathrm{div}(a_T) \in \mathbb{P}^{k-1}(T, \mathbb{R})$ and $b_T \in \mathbb{P}^k_{\mathrm{hom}}(T, \mathbb{R})$, this leads to a contradiction, and thus $b_T = 0$, and $v_h|_T = a_T$. As we further have $v_h \in H(\mathrm{div}, \Omega)$, this yields $v_h \in \mathcal{BDM}^k(\mathcal{T}_h)$. □

Lemma 22. *For all $T \in \mathcal{T}_h$ and $v_h \in \mathcal{RT}^k(T)$ there holds the equivalence*

$$\|\nabla v_h\|_T^2 \sim \|\Pi_T^{k-1}[\operatorname{dev}(\nabla v_h)]\|_T^2 + \|\operatorname{div}(v_h)\|_T^2,$$

and the estimate

$$\|(\mathrm{Id} - \Pi_T^{k-1})\nabla v_h\|_T + \frac{1}{\sqrt{h}}\|(\mathrm{Id} - \Pi_F^k)(v_h)_t\|_{\partial T} \lesssim \|\operatorname{div}(v_h)\|_T.$$

Proof. One side of the equivalence is obvious by the continuity of the Π_T^{k-1}. For the other direction let $T \in \mathcal{T}_h$ be arbitrary and define $\hat{v}_h = \mathcal{P}^{-1}(v_h|_T)$. We solve the following problem: Find $\hat{b} \in \mathbb{P}^k(\hat{T}, \mathbb{R})$ such that

$$\int_{\hat{T}} \hat{\operatorname{div}}(\hat{x}\hat{b})\hat{\operatorname{div}}(\hat{x}\hat{q})\, \mathrm{d}x = \int_{\hat{T}} \hat{\operatorname{div}}(\hat{v}_h)\hat{\operatorname{div}}(\hat{x}\hat{q})\, \mathrm{d}x \quad \forall \hat{q} \in \mathbb{P}^k(\hat{T}, \mathbb{R}).$$

Note that this problem is solvable by the Lax-Milgram theorem 5, as there holds the norm equivalence

$$\|\hat{\nabla}(\hat{q}\hat{x})\| \sim \|\hat{\operatorname{div}}(\hat{q}\hat{x})\|,$$

see for example in the appendix of the work [79] (also known as Euler identity). Since $\hat{\operatorname{div}}(\hat{x}\mathbb{P}^k(\hat{T}, \mathbb{R})) = \mathbb{P}^k(\hat{T}, \mathbb{R})$, the solution fulfills the property $\hat{\operatorname{div}}(\hat{x}\hat{b}) = \hat{\operatorname{div}}(\hat{v}_h)$. Now set $w_h := \mathcal{P}(\hat{x}\hat{b})$. Due to the properties of the Piola mapping, the function w_h is in $\mathcal{RT}^k(T)$ and $\operatorname{div}(w_h) = \operatorname{div}(v_h)$. Further a standard scaling argument shows that

$$\|\nabla w_h\|_T \sim \|\operatorname{div}(w_h)\|_T. \tag{6.12}$$

Using lemma 21, the function v_h can now be written as $v_h = a + w_h$, with $a \in \mathbb{P}^k(T, \mathbb{R}^d)$ (as $\operatorname{div}(w_h - v_h) = 0$). This then yields

$$\begin{aligned}
\|\nabla v_h\|_T = \|\nabla(a + w_h)\|_T &\leq \|\operatorname{dev}(\nabla a)\|_T + \|\operatorname{dev}(\nabla w_h)\|_T + \|\operatorname{div}(v_h)\|_T \\
&\leq \|\operatorname{dev}(\nabla a)\|_T + \|\nabla w_h\|_T + \|\operatorname{div}(v_h)\|_T \\
&\lesssim \|\operatorname{dev}(\nabla a)\|_T + \|\operatorname{div}(v_h)\|_T \\
&= \|\Pi_T^{k-1}\operatorname{dev}(\nabla a)\|_T + \|\operatorname{div}(v_h)\|_T.
\end{aligned}$$

As $a = v_h - w_h$ we conclude

$$\begin{aligned}
\|\nabla v_h\|_T &\lesssim \|\Pi_T^{k-1}\operatorname{dev}(\nabla v_h)\|_T + \|\Pi_T^{k-1}\operatorname{dev}(\nabla w_h)\|_T + \|\operatorname{div}(v_h)\|_T \\
&\lesssim \|\Pi_T^{k-1}\operatorname{dev}(\nabla v_h)\|_T + \|\nabla w_h\|_T + \|\operatorname{div}(v_h)\|_T \\
&\lesssim \|\Pi_T^{k-1}\operatorname{dev}(\nabla v_h)\|_T + \|\operatorname{div}(v_h)\|_T.
\end{aligned}$$

It remains to show the second statement. Using the local representation of v_h the triangle inequality yields

$$\|(\mathrm{Id} - \Pi_T^{k-1})\nabla v_h\|_T \leq \|(\mathrm{Id} - \Pi_T^{k-1})\nabla a\|_T + \|(\mathrm{Id} - \Pi_T^{k-1})\nabla w_h\|_T.$$

As $a \in \mathbb{P}^k(T, \mathbb{R}^d)$, the first term vanishes, and thus by the findings above and the continuity of Π_T^{k-1} we have

$$\|(\mathrm{Id} - \Pi_T^{k-1})\nabla v_h\|_T \le \|\nabla w_h\|_T \sim \|\operatorname{div}(w_h)\|_T = \|\operatorname{div}(v_h)\|_T.$$

On each facet $F \in \mathcal{F}_h(T)$ with the same steps as above we also see that

$$\frac{1}{\sqrt{h}}\|(\mathrm{Id} - \Pi_F^k)(v_h)_t\|_F \le \frac{1}{\sqrt{h}}\|(\mathrm{Id} - \Pi_F^k)(w_h)_t\|_F,$$

and similar as in the proof of lemma 20, an inverse inequality for polynomials, see [117], yields

$$\frac{1}{\sqrt{h}}\|(\mathrm{Id} - \Pi_F^k)(v_h)_t\|_F \lesssim \sqrt{h}\|\nabla_t w_h\|_F \lesssim \|\nabla w_h\|_T \sim \|\operatorname{div}(w_h)\|_T = \|\operatorname{div}(v_h)\|_T.$$

□

6.2.2 Stability analysis in a discrete H^1-norm

To prove discrete inf-sup stability we are aiming to use Brezzi's theorem 6. To this end we prove all the conditions needed in the following.

Lemma 23 (Continuity of a, b_1 and b_{2h})**.** *The bilinear forms a, b_1 and b_{2h} are continuous:*

$$\begin{aligned}
a(\sigma_h, \tau_h) &\lesssim \frac{1}{\sqrt{\nu}}||\sigma_h||_{\Sigma_h}\frac{1}{\sqrt{\nu}}||\tau_h||_{\Sigma_h} && \textit{for all } \sigma_h, \tau_h \in \Sigma_h,\\
b_1(v_h, p_h) &\lesssim ||v_h||_{V_h}||p_h||_{Q_h} && \textit{for all } v_h \in V_h, p_h \in Q_h,\\
b_{2h}(\sigma_h, v_h) &\lesssim ||\sigma_h||_{\Sigma_h}||v_h||_{V_h} && \textit{for all } \sigma_h \in \Sigma_h, v_h \in V_h.
\end{aligned}$$

Proof. The continuity for the bilinear forms a and b_1 follows with the Cauchy-Schwarz inequality. For b_{2h} we have, using representation (6.5), that

$$b_{2h}(\sigma_h, v_h) = -\sum_{T\in\mathcal{T}_h}\int_T \sigma_h : \nabla v_h \,\mathrm{d}x + \sum_{F\in\mathcal{F}_h}\int_F (\sigma_h)_{nt}\cdot [\![(v_h)_t]\!]\,\mathrm{d}s\,.$$

Since $(\sigma_h)_{nt} = (\operatorname{dev}(\sigma_h))_{nt}$, we conclude the proof by the Cauchy-Schwarz inequality including a scaling with $\sqrt{h}/\sqrt{h}$ for the facet integral and lemma 19. □

Lemma 24 (Coercivity of a on the kernel)**.** *Let $K_{b_h} := \{(\tau_h, q_h) \in \Sigma_h \times Q_h : b_1(v_h, q_h) + b_{2h}(\sigma_h, v_h) = 0$ for all $v_h \in V_h\}$. For all $(\sigma_h, p_h) \in K_{b_h}$ there holds the estimate*

$$\frac{1}{\nu}\big(\,||\sigma_h||_{\Sigma_h} + ||p_h||_{Q_h}\big)^2 \lesssim\; a(\sigma_h, \sigma_h).$$

Proof. Let $(\sigma_h, p_h) \in K_{b_h}$ be arbitrary. As $\nu^{-1}||\sigma_h||^2_{\Sigma_h} = a(\sigma_h, \sigma_h)$, it is sufficient to bound only the norm of p_h by $||\sigma_h||_{\Sigma_h}$. Using the discrete Stokes-LBB of a velocity-pressure formulation with V_h and Q_h, we find for any $p_h \in Q_h$ a discrete velocity $v_h \in V_h$ such that

$$\operatorname{div}(v_h) = p_h, \quad \text{and} \quad ||v_h||_{V_h} \lesssim ||p_h||_{Q_h}. \tag{6.13}$$

In the case of $\Gamma_{D,n} = \Gamma$ this can be found e.g. in [11] and [78]. For the other case one uses the infinite dimensional Stokes-LBB, see lemma 4.9 in [41], and the Fortin interpolator constructed in [78]. As $(\sigma_h, p_h) \in K_{b_h}$, this yields

$$2||p_h||^2_{Q_h} = \sum_{T\in\mathcal{T}_h} \int_T p_h p_h \,\mathrm{d}x = \sum_{T\in\mathcal{T}_h} \int_T \operatorname{div}(v_h) p_h \,\mathrm{d}x = b_1(v_h, p_h). = -b_{2h}(\sigma_h, v_h).$$

Using representation (6.5), norm equivalence lemma 19 and a Cauchy-Schwarz argument then further yields

$$2||p_h||^2_{Q_h} = \sum_{T\in\mathcal{T}_h} \int_T \sigma_h : \nabla v_h \,\mathrm{d}x - \sum_{F\in\mathcal{F}_h} \int_F (\sigma_h)_{nt} \cdot [\![(v_h)_t]\!] \,\mathrm{d}s \le \|\sigma\|_{\Sigma_h} \|v_h\|_{V_h},$$

thus by (6.13) we conclude the proof. □

Next, we proceed to verify the discrete LBB-condition (see theorem 17 below). To this end we define the subspace of divergence free velocities $V_h^0 := \{w_h \in V_h : \operatorname{div}(w_h) = 0\}$, and the norm

$$\|v_h\|_{1,\Pi,h} := \left(\sum_{K\in\mathcal{T}_h} \|\Pi_T^{k-1} \operatorname{dev}(\nabla v_h)\|^2_T + \sum_{F\in\mathcal{F}_h} \frac{1}{h} \left\|\Pi_F^0 [\![(v_h)_t]\!]\right\|^2_F \right)^{1/2}.$$

As $\|\nabla v_h\|^2_T \sim \|\Pi_T^{k-1} \operatorname{dev}(\nabla v_h)\|^2_T + \|\operatorname{div}(v_h)\|^2_T$ on any $T \in \mathcal{T}_h$ (see lemma 22), we have together with lemma 20

$$\|v_h\|_{1,\Pi,h} \sim \|v_h\|_{V_h} \qquad \text{for all } v_h \in V_h^0. \tag{6.14}$$

A first step towards proving the LBB-condition is the construction of a specific stress function τ_h, which only depends on $\Pi_T^{k-1} \operatorname{dev}(\nabla v_h)$ for any $v_h \in V_h^0$. Using this τ_h we prove an LBB-condition for b_{2h} on V_h^0, which is the content of the next lemma. As $\tau_h \in \Sigma_h$ has a zero trace, we cannot in general control the divergence of a general $v_h \in V_h$ solely using such a τ_h. Therefore, to complete the proof of the full inf-sup condition (in the proof of theorem 17 below) we utilize an appropriate pressure test function as well.

Lemma 25. *For any nonzero $v_h \in V_h$ there exists a nonzero stress function $\tau_h \in \Sigma_h$ satisfying $b_{2h}(\tau_h, v_h) \gtrsim \|v_h\|^2_{1,\Pi,h}$ and $\|\tau_h\|_{\Sigma_h} \lesssim \|v_h\|_{1,\Pi,h}$. Equation (6.14) implies discrete inf-sup stability of b_{2h} on V_h^0,*

$$\|v_h\|_{V_h} \lesssim \sup_{\tau_h\in\Sigma_h} \frac{b_{2h}(\tau_h, v_h)}{\|\tau_h\|_{\Sigma_h}} \qquad \textit{for all } v_h \in V_h^0.$$

Proof. Since the ideas are the same for $d = 2$ and 3, for ease of exposition, we give the details of the proof only in the two-dimensional case. Because of the decomposition of the degrees of freedom into face and interior degrees of freedom given by equation (5.23) and (5.24) (note that the set (5.25) is not used as $\Sigma_h = \Xi^k_{\mathbb{D}}(\mathcal{T}_h)$), we can decompose the stress space as

$$\Sigma_h = \Sigma_h^0 \oplus \Sigma_h^1,$$

where $\Sigma_h^0 = \oplus_{T\in\mathcal{T}_h}\mathbb{B}_{nt}^k(T)$ and Σ_h^1 is the span of facet shape functions. In particular, Σ_h^1 contains the lowest order shape functions S^F with the property that $S_{nt}^F \in n^\perp$ (thus in the tangent plane of F) and $||S_{nt}^F||_{l^2} = 1$ on the facet F and equals $(0,0)$ on all other facets in $\mathcal{F}_h$. S^F can be explicity written down by mapping (5.33) or by appropriately scaling (5.16). Given any $v_h \in V_h$, define

$$\begin{aligned}\tau_h^0 &:= \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_h(T)} -(S^F : \Pi_T^{k-1}\mathrm{dev}(\nabla v_h))\lambda_T^F S^F,\\ \tau_h^1 &:= \sum_{F\in\mathcal{F}_h}\frac{1}{\sqrt{h}}(\Pi_F^0[\![(v_h)_t]\!])S^F,\end{aligned}\tag{6.15}$$

where λ_T^F is the barycentric coordinate of T that vanishes on F. Note that τ_h^0 and τ_h^1 can also be defined using a projection that is based on the degrees of freedoms (5.24) and (5.23), respectively. By lemma 18 and (6.10), a scaling argument (like in lemma 19) shows that there is a mesh-independent C_1 such that

$$\|\tau_h^1\|_{\Sigma_h}^2 \le C_1 \sum_{F\in\mathcal{F}_h}\frac{1}{h}\|\Pi_F^0[\![(v_h)_t]\!]\|_F^2.\tag{6.16}$$

A similar scaling argument also shows that

$$||\tau_h^0||_{\Sigma_h}^2 \lesssim \sum_{T\in\mathcal{T}_h}\|\Pi_T^{k-1}\mathrm{dev}(\nabla v_h)\|_T^2.\tag{6.17}$$

By construction we have $(\tau_h^0)_{nt} = (0,0)$, which yields

$$\begin{aligned}b_{2h}(\tau_h^0, v_h) &= \sum_{T\in\mathcal{T}_h} -\int_T \tau_h^0 : \nabla v_h \,\mathrm{d}x\\ &= \sum_{T\in\mathcal{T}_h}\int_T\sum_{F\in\mathcal{F}_h(T)}(S^F : \Pi_T^{k-1}\mathrm{dev}(\nabla v_h))\lambda_T^F S^F : \nabla v_h\,\mathrm{d}x\\ &= \sum_{T\in\mathcal{T}_h}\int_T\sum_{F\in\mathcal{F}_h(T)}(S^F : \Pi_T^{k-1}\mathrm{dev}(\nabla v_h))^2\lambda_T^F\,\mathrm{d}x\,.\end{aligned}$$

Since the functions S^F form a basis for $\mathbb{D}$, see lemma 14, and as the barycentric coordinate functions are greater than or equal to, $\lambda_T^F \ge 0$ on T, a scaling argument shows that

$$b_{2h}(\tau_h^0, v_h) \gtrsim \sum_{T\in\mathcal{T}_h}\|\Pi_T^{k-1}\mathrm{dev}(\nabla v_h)\|_T^2.\tag{6.18}$$

We continue to define a linear combination to prove the results. To this end set $\tau_h = \gamma_0\tau_h^0 + \gamma_1\tau_h^1$, where $\gamma_0 > 0$ and $\gamma_1 > 0$ are constants to be chosen. Stability (6.18) yields

$$b_{2h}(\tau_h, v_h) \gtrsim \gamma_0\sum_{T\in\mathcal{T}_h}\|\Pi_T^{k-1}\mathrm{dev}(\nabla v_h)\|_T^2 + \gamma_1 b_2(\tau_h^1, v_h).$$

By definition (6.15) we then further have

$$
\begin{aligned}
&b_{2h}(\tau_h, v_h) \\
&\gtrsim \gamma_0 \sum_{T\in\mathcal{T}_h} \|\Pi_T^{k-1}\mathrm{dev}(\nabla v_h)\|_T^2 + \gamma_1 \left(\sum_{T\in\mathcal{T}_h} -\int_T \tau_h^1 : \nabla v_h \,\mathrm{d}x + \sum_{F\in\mathcal{F}_h} \int_F (\tau_h^1)_{nt} \cdot [\![(v_h)_t]\!] \,\mathrm{d}s \right) \\
&= \gamma_0 \sum_{T\in\mathcal{T}_h} \|\Pi_T^{k-1}\mathrm{dev}(\nabla v_h)\|_T^2 - \gamma_1 \sum_{T\in\mathcal{T}_h} \int_T \tau_h^1 : \Pi_T^{k-1}\mathrm{dev}(\nabla v_h) \,\mathrm{d}x + \gamma_1 \sum_{F\in\mathcal{F}_h} \frac{1}{h} \left\|\Pi_F^0 [\![(v_h)_t]\!]\right\|_F^2.
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality for the second sum and using (6.16), we obtain

$$
\begin{aligned}
-\gamma_1 &\sum_{T\in\mathcal{T}_h} \int_T \tau_h^1 : \Pi_T^{k-1}\mathrm{dev}(\nabla v_h) \,\mathrm{d}x \\
&\geq -\gamma_1 \|\tau_h^1\|_{\Sigma_h} \sqrt{\sum_{T\in\mathcal{T}_h} \|\Pi_T^{k-1}\mathrm{dev}(\nabla v_h)\|_T^2} \\
&\gtrsim -\gamma_1 \sqrt{C_1 \sum_{F\in\mathcal{F}_h} \frac{1}{h} \left\|\Pi_F^0 [\![(v_h)_t]\!]\right\|_F^2} \sqrt{\sum_{T\in\mathcal{T}_h} \|\Pi_T^{k-1}\mathrm{dev}(\nabla v_h)\|_T^2},
\end{aligned}
$$

and thus by Young's inequality with $\delta > 0$ we get

$$
b_{2h}(\tau_h, v_h) \gtrsim \left(\gamma_0 - \frac{\gamma_1 \delta}{2}\right) \sum_{T\in\mathcal{T}_h} \|\Pi_T^{k-1}\mathrm{dev}(\nabla v_h)\|_T^2 + \left(1 - \frac{C_1}{2\delta}\right) \frac{\gamma_1}{h} \sum_{F\in\mathcal{F}_h} \left\|\Pi_F^0 [\![(v_h)_t]\!]\right\|_F^2.
$$

Choosing $\delta = C_1$, $\gamma_1 = 1/\delta = 1/C_1$, and $\gamma_0 = 1$ yields

$$
b_{2h}(\tau_h, v_h) \gtrsim \sum_{T\in\mathcal{T}_h} \|\Pi_T^{k-1}\mathrm{dev}(\nabla v_h)\|_T^2 + \sum_{F\in\mathcal{F}_h} \frac{1}{h} \left\|\Pi_F^0 [\![(v_h)_t]\!]\right\|_F^2. \tag{6.19a}
$$

By (6.16) and (6.17) we further obtain

$$
\|\tau_h\|_{\Sigma_h} \lesssim \sum_{T\in\mathcal{T}_h} \|\Pi_T^{k-1}\mathrm{dev}(\nabla v_h)\|_T^2 + \sum_{F\in\mathcal{F}_h} \frac{1}{h} \left\|\Pi_F^0 [\![(v_h)_t]\!]\right\|_F^2. \tag{6.19b}
$$

The estimates (6.19) and the norm equivalences of (6.14) complete the proof. □

Theorem 17 (Discrete LBB-condition)**.** *For all $v_h \in V_h$ there holds*

$$
\sup_{(\tau_h, q_h)\in\Sigma_h\times Q_h} \frac{b_1(v_h, q_h) + b_{2h}(\tau_h, v_h)}{||\tau_h||_{\Sigma_h} + ||q_h||_{Q_h}} \gtrsim ||v_h||_{V_h}. \tag{6.20}
$$

Proof. By lemma 25 for any $v_h \in V_h$ there exists a $\tau_h \in \Sigma_h$ satisfying $b_1(\tau_h, v_h) \gtrsim \|v_h\|_{1,\Pi,h}^2$ and $\|\tau_h\|_{\Sigma_h} \lesssim \|v_h\|_{1,\Pi,h}$. Next, we choose the pressure variable $q_h = \mathrm{div}(v_h)$, which is an admissible choice as $\mathrm{div}(V_h) = Q_h$, yielding $b_1(v_h, q_h) = \|\mathrm{div}(v_h)\|_{Q_h}^2$. With these choices of τ_h and q_h we have

$$
\frac{b_1(v_h, q_h) + b_{2h}(\tau_h, v_h)}{||\tau_h||_{\Sigma_h} + ||q_h||_{Q_h}} \geq \frac{\|v_h\|_{1,\Pi,h}^2 + \|\mathrm{div}(v_h)\|_{Q_h}^2}{||\tau_h||_{\Sigma_h} + ||q_h||_{Q_h}} \gtrsim \|v_h\|_{V_h}.
$$

□

For the ease of notation we now define the norm on the product space $V_h \times \Sigma_h \times Q_h$,

$$\|(u_h, \sigma_h, p_h)\|_* := \sqrt{\nu}||u_h||_{V_h} + \frac{1}{\sqrt{\nu}}(||\sigma_h||_{\Sigma_h} + ||p_h||_{Q_h}),$$

and the bilinear form

$$B(u_h, \sigma_h, p_h; v_h, \tau_h, q_h) := a(\sigma_h, \tau_h) + b_1(u_h, q_h) + b_1(v_h, p_h) + b_{2h}(\sigma_h, v_h) + b_{2h}(\tau_h, u_h).$$

Corollary 4. *The bilinearform B is inf-sup stable with respect to $\|\cdot\|_*$, thus there exists a constant $\beta > 0$ such that for all nonzero functions $(u_h, \sigma_h, p_h) \in V_h \times \Sigma_h \times Q_h$ it holds*

$$\sup_{(v_h,\tau_h,q_h)\in V_h\times\Sigma_h\times Q_h} \frac{B(u_h, \sigma_h, p_h; v_h, \tau_h, q_h)}{||(v_h, \tau_h, q_h)||_*} \geq \beta||(u_h, \sigma_h, p_h)||_*.$$

Let $f \in L^2(\Omega, \mathbb{R}^d)$ and assume homogeneous boundary conditions (6.8). There exists a unique solution $(u_h, \sigma_h, p_h) \in V_h \times \Sigma_h \times Q_h$ of the MCS method (6.6) with the stability estimate

$$\|u_h, \sigma_h, p_h\|_* \lesssim \frac{1}{\sqrt{\nu}}\|f\|_{L^2(\Omega)}.$$

Proof. This is a direct consequence from lemma 23, lemma 24, theorem 17 and Brezzi's theorem 6. ☐

Remark 4. *To avoid technical details the stability analysis was based on the assumption of homogeneous boundary conditions (6.8). With the usual techniques one can extend the results also to the case where we have the regularity properties*

$$g_{D,n} \in [H^{1/2}(\Gamma_{D,n}, \mathbb{R}^d)]_n, \qquad g_{D,t} \in [H^{1/2}(\Gamma_{D,t}, \mathbb{R}^d)]_t,$$
$$g_{N,n} \in H^{-1/2}(\Gamma_{D,n}, \mathbb{R}), \qquad g_{N,t} \in H^{-1/2}(\Gamma_{D,t}, \mathbb{R}^{d-1}).$$

We conclude this section with two consistency results that shall be needed in the next section.

Theorem 18 (Consistency)**.** *The MCS method (6.6) is consistent in the following sense. If the exact solution of the mixed Stokes problem (4.6) fulfills the regularity property $u \in H^1(\Omega, \mathbb{R}^d)$, $\sigma \in H^1(\Omega, \mathbb{R}^{d\times d})$ and $p \in L^2(\Omega, \mathbb{R})$, then*

$$B(u, \sigma, p; v_h, \tau_h, q_h) = (-f, v_h)_\Omega + (g_{D,t}, (\tau_h)_{nt})_{\Gamma_{D,t}} + (g_{N,n}, v_n)_{\Gamma_{N,n}},$$

for all $v_h \in V_h, q_h \in Q_h$, and $\tau_h \in \Sigma_h$.

Proof. As the exact solutions σ and u are continuous, we have $[\![\sigma_{nn}]\!] = 0$ and $[\![u_t]\!] = 0$ on all faces $F \in \mathcal{F}_h^{\text{int}}$, and thus using representations (6.4) and (6.5) we have

$$\begin{aligned} b_{2h}(\sigma, v_h) &= \sum_{T\in\mathcal{T}_h} \int_T \operatorname{div}(\sigma) \cdot v_h \,\mathrm{d}x - \sum_{F\in\mathcal{F}_h} \int_F [\![\sigma_{nn}]\!](v_h)_n \,\mathrm{d}s \\ &= \sum_{T\in\mathcal{T}_h} \int_T \operatorname{div}(\sigma) \cdot v_h \,\mathrm{d}x - \int_{\Gamma_{N,n}} \sigma_{nn}(v_h)_n \,\mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} b_{2h}(\tau_h, u) &= -\sum_{T\in\mathcal{T}_h}\int_T \tau_h : \nabla u \,\mathrm{d}x + \sum_{F\in\mathcal{F}_h}\int_F (\tau_h)_{nt}\cdot [\![u_t]\!]\,\mathrm{d}s \\ &= -\sum_{T\in\mathcal{T}_h}\int_T \tau_h : \nabla u \,\mathrm{d}x + \int_{\Gamma_{D,t}} (\tau_n)_{nt} u_t \,\mathrm{d}s\,. \end{aligned}$$

Using $\operatorname{div}(u) = 0$ we further get that $b_1(u, q_h) = 0$, so all together this yields

$$\begin{aligned} &a(\sigma,\tau_h) + b_{2h}(\tau_h, u) + b_{2h}(\sigma, v_h) + b_1(v_h, p) + b_1(u, q_h) \\ &\quad= \int_\Omega \frac{1}{\nu}\operatorname{dev}(\sigma) : \operatorname{dev}(\tau_h)\,\mathrm{d}x - \sum_{T\in\mathcal{T}_h}\int_T \tau_h : \nabla u\,\mathrm{d}x + \sum_{T\in\mathcal{T}_h}\int_T \operatorname{div}(\sigma)\cdot v_h\,\mathrm{d}x \\ &\quad\quad+ \int_\Omega \operatorname{div}(v_h) p\,\mathrm{d}x + \int_{\Gamma_{D,t}} (\tau_n)_{nt} u_t\,\mathrm{d}s - \int_{\Gamma_{N,n}} \sigma_{nn}(v_h)_n\,\mathrm{d}s\,. \end{aligned}$$

For the exact solution we have $\operatorname{dev}(\sigma) = \nu\nabla u$. Further, as $\operatorname{div}(u) = 0$, a simple calculation shows that $\tau_h : \nabla u = \tau_h : \operatorname{dev}(\nabla u) = \operatorname{dev}(\tau_h) : \nabla u$, thus the first two volume integrals vanish. Using integration by parts for the last volume integral we observe

$$\int_\Omega \operatorname{div}(v_h) p\,\mathrm{d}x = -\int_\Omega v_h \nabla p\,\mathrm{d}x + \int_{\Gamma_{N,n}} (v_h)_n p\,\mathrm{d}s\,.$$

For the exact solutions we have $u_t = g_{D,t}$ on $\Gamma_{D,t}$, $-\sigma_{nn} + p = g_{N,n}$ on $\Gamma_{N,n}$. Adding up all results from above yields

$$\begin{aligned} &a(\sigma,\tau_h) + b_{2h}(\tau_h, u) + b_{2h}(\sigma, v_h) + b_1(v_h, p) + b_1(u, q_h) \\ &\quad= \int_\Omega \big[\operatorname{div}(\sigma) - \nabla p\big]\cdot v_h\,\mathrm{d}x + \int_{\Gamma_{D,t}} (\tau_n)_{nt} u_t\,\mathrm{d}s - \int_{\Gamma_{N,n}} \sigma_{nn}(v_h)_n\,\mathrm{d}s + \int_{\Gamma_{N,n}} (v_h)_n p\,\mathrm{d}s \\ &\quad= \int_\Omega -f v_h\,\mathrm{d}x + \int_{\Gamma_{D,t}} (\tau_n)_{nt}\cdot g_{D,t}\,\mathrm{d}s + \int_{\Gamma_{N,n}} g_{N,n}(v_h)_n\,\mathrm{d}s\,. \end{aligned}$$

□

Theorem 19 (Consistency on V_h^0)**.** *The MCS method (6.6) is consistent on the subspace of divergence free velocity fields, thus if the exact solution of the mixed Stokes problem (4.6) is such that $u \in H^1(\Omega, \mathbb{R}^d)$, $\sigma \in H^1(\Omega, \mathbb{R}^{d\times d})$ and $p \in L^2(\Omega, \mathbb{R})$, then there holds*

$$B(u, \sigma, 0; v_h, \tau_h, 0) = (-f, v_h)_\Omega + (g_{D,t}, (\tau_h)_{nt})_{\Gamma_{D,t}} + (g_{N,n}, v_n)_{\Gamma_{N,n}},$$

for all $v_h \in V_h^0$ and $\tau_h \in \Sigma_h$.

Proof. With the same steps as above we end up with the equation

$$\begin{aligned} &a(\sigma,\tau_h) + b_{2h}(\tau_h, u) + b_{2h}(\sigma, v_h) \\ &\quad= \int_\Omega \operatorname{div}(\sigma)\cdot v_h\,\mathrm{d}x + \int_{\Gamma_{D,t}} (\tau_n)_{nt} u_t\,\mathrm{d}s - \int_{\Gamma_{,n}N} \sigma_{nn}(v_h)_n\,\mathrm{d}s \\ &\quad= \int_\Omega \big[-f + \nabla p\big] v_h\,\mathrm{d}x + \int_{\Gamma_{D,t}} (\tau_n)_{nt}\cdot g_{D,t}\,\mathrm{d}s + \int_{\Gamma_{N,n}} \big[g_{N,n} - p\big](v_h)_n\,\mathrm{d}s\,. \end{aligned}$$

As $v_h \in V_h^0$, we conclude the proof by

$$\int_\Omega \nabla p \cdot v_h \,\mathrm{d}x = -\int_\Omega p \,\mathrm{div}(v_h)\,\mathrm{d}x + \int_{\Gamma_{N,n}} p(v_h)_n.$$

□

6.2.3 Stability analysis in natural norms

This section is dedicated to prove discrete inf-sup stability using norms that fit to the continuous formulation given by (4.26). As mentioned at the beginning of this chapter, the velocity space and the pressure space are chosen conformingly with respect to V and Q, thus natural norms are simply given by $\|\cdot\|_{H(\mathrm{div},\Omega)}$ and $\|\cdot\|_{L^2(\Omega)}$, respectively. The stress space, however, is slightly non-conforming. To this end we define the discrete stress $H(\mathrm{curl}\,\mathrm{div},\Omega)$-norm by

$$\|\sigma_h\|_{\mathrm{cd},h}^2 := \|\sigma_h\|_{L^2(\Omega)}^2 + \Big(\sup_{v_h \in \mathcal{RT}^k(\mathcal{T}_h)} \frac{b_{2h}(\sigma_h, v_h)}{\|v_h\|_{H(\mathrm{div},\Omega)}} \Big)^2. \tag{6.21}$$

This definition is again motivated by the identity (5.12) of theorem 13, thus the above definition can be seen as a discrete version of the stress norm $\|\cdot\|_{\mathrm{cd}}$. Note, however, that in contrast to $\|\cdot\|_{\mathrm{cd}}$ the supremum in (6.21) is taken with respect to the smaller set $V_h \subset V$, hence even in the case of enough regularity of a given τ we do not have norm equivalence

$$\|\tau_h\|_{\mathrm{cd},h} \not\sim \|\tau_h\|_{\mathrm{cd}}.$$

We follow the same steps as in the last chapter proving the results needed to apply Brezzi's theorem 6.

Lemma 26. *The bilinear forms a, b_1 and b_{2h} are continuous:*

$$\begin{aligned}
a(\sigma_h, \tau_h) &\lesssim \frac{1}{\sqrt{\nu}}\|\sigma_h\|_{\mathrm{cd},h} \frac{1}{\sqrt{\nu}}\|\tau_h\|_{\mathrm{cd},h} && \text{for all } \sigma_h, \tau_h \in \Sigma_h,\\
b_1(v_h, p_h) &\lesssim \|v_h\|_{H(\mathrm{div},\Omega)}\|p_h\|_{L^2(\Omega)} && \text{for all } v_h \in V_h, p_h \in Q_h,\\
b_{2h}(\sigma_h, v_h) &\lesssim \|\sigma_h\|_{\mathrm{cd},h}\|v_h\|_{H(\mathrm{div},\Omega)} && \text{for all } \sigma_h \in \Sigma_h, v_h \in V_h.
\end{aligned}$$

Proof. The continuity of a and b_1 follow using the Cauchy-Schwarz inequality. The continuity of b_{2h} follows by

$$b_{2h}(\sigma_h, v_h) = \frac{b_{2h}(\sigma_h, v_h)}{\|v_h\|_{H(\mathrm{div},\Omega)}}\|v_h\|_{H(\mathrm{div},\Omega)} \le \|\sigma_h\|_{\mathrm{cd},h}\|v_h\|_{H(\mathrm{div},\Omega)}.$$

□

Lemma 27 (Coercivity of a on the kernel)**.** *For all $(\sigma_h, p_h) \in K_{b_h}$ there holds*

$$\frac{1}{\nu}\big(\|\sigma_h\|_{\mathrm{cd},h} + \|p_h\|_{L^2(\Omega)}\big)^2 \lesssim a(\sigma_h, \sigma_h).$$

Proof. Let $(\sigma_h, p_h) \in K_{b_h}$ be arbitrary. With the same steps as in the in the proof of the discrete coercivity with non-natural norms, see lemma 24, we find a velocity field $u_h \in V_h$ such that

$$\operatorname{div}(u_h) = q_h \quad \text{and} \quad \|u_h\|_{V_h} \leq \|p_h\|_{L^2(\Omega)},$$

and there holds the estimate $\|p_h\|_{L^2(\Omega)} \leq \|\sigma_h\|_{L^2(\Omega)}$. Next we observe

$$\sup_{v_h \in \mathcal{RT}^k(\mathcal{T}_h)} \frac{b_{2h}(\sigma_h, v_h)}{\|v_h\|_{H(\operatorname{div},\Omega)}} = \sup_{v_h \in \mathcal{RT}^k(\mathcal{T}_h)} \frac{(\operatorname{div}(v_h), p_h)_\Omega}{\|v_h\|_{H(\operatorname{div},\Omega)}} \leq \|p_h\|_{L^2(\Omega)}.$$

This yields for all $(\sigma_h, p_h) \in K_{b_h}$ the estimate $(\|\sigma_h\|_{\mathrm{cd},h} + \|p_h\|_{L^2(\Omega)})^2 \lesssim \nu a(\sigma_h, \sigma_h)$. We conclude the proof with a division by ν. □

Theorem 20 (LBB)**.** *For all $v_h \in V_h$ there holds*

$$\sup_{(\tau_h, q_h) \in \Sigma_h \times Q_h} \frac{b_1(v_h, q_h) + b_{2h}(\tau_h, v_h)}{\|\tau_h\|_{\mathrm{cd},h} + \|q_h\|_{L^2(\Omega)}} \gtrsim \|v_h\|_{H(\operatorname{div},\Omega)}. \tag{6.22}$$

Proof. Let $w_h \in V_h$ be arbitrary. Due to the solvability of the discrete system in non-natural norms, see corollary 4, there exists a unique solution (u_h, σ_h, p_h) of the following problem

$$\begin{aligned} a(\sigma_h, \tau_h) + b_{2h}(\tau_h, u_h) &= 0 && \text{for all } \tau_h \in \Sigma_h, \\ b_{2h}(\sigma_h, v_h) + b_1(v_h, p_h) &= (w_h, v_h)_{L^2(\Omega)} + (\operatorname{div}(w_h), \operatorname{div}(v_h))_{L^2(\Omega)} && \text{for all } v_h \in V_h, \\ b_1(u_h, q_h) &= 0 && \text{for all } q_h \in Q_h, \end{aligned} \tag{6.23}$$

and the solution fulfills

$$\sqrt{\nu}||u_h||_{V_h} + \frac{1}{\sqrt{\nu}}(||\sigma_h||_{\Sigma_h} + ||p_h||_{Q_h}) \lesssim \frac{1}{\sqrt{\nu}}\|w_h\|_{H(\operatorname{div},\Omega)}. \tag{6.24}$$

By the second equation of the system (6.23), the equation $\|\sigma_h\|_{\Sigma_h} = \|\sigma_h\|_{L^2}$ and estimate (6.24), we get

$$\begin{aligned} \|\sigma_h\|^2_{\mathrm{cd},h} + \|p_h\|^2_{L^2(\Omega)} &= \|\sigma_h\|^2_{L^2(\Omega)} + \|p_h\|^2_{L^2(\Omega)} + \Big(\sup_{v_h \in V_h} \frac{b_{2h}(\sigma_h, v_h)}{\|v_h\|_{H(\operatorname{div},\Omega)}} \Big)^2 \\ &\lesssim \|w_h\|^2_{H(\operatorname{div},\Omega)} + \Big(\sup_{v_h \in V_h} \frac{b_{2h}(\sigma_h, v_h) + b_1(v_h, p_h)}{\|v_h\|_{H(\operatorname{div},\Omega)}} \Big)^2 \\ &= \|w_h\|^2_{H(\operatorname{div},\Omega)} + \Big(\sup_{v_h \in V_h} \frac{(w_h, v_h)_{L^2(\Omega)} + (\operatorname{div}(w_h), \operatorname{div}(v_h))_{L^2(\Omega)}}{\|v_h\|_{H(\operatorname{div},\Omega)}} \Big)^2 \\ &\leq \|w_h\|^2_{H(\operatorname{div},\Omega)}, \end{aligned}$$

and thus

$$\sup_{(\tau_h,q_h)\in\Sigma_h\times Q_h}\frac{b_1(w_h,q_h)+b_{2h}(\tau_h,w_h)}{\|\tau_h\|_{\mathrm{cd},h}+\|q_h\|_{L^2(\Omega)}}\geq\frac{b_1(w_h,p_h)+b_{2h}(\sigma_h,w_h)}{\|\sigma_h\|_{\mathrm{cd},h}+\|p_h\|_{L^2(\Omega)}}$$
$$\gtrsim\frac{(w_h,w_h)_{L^2(\Omega)}+(\operatorname{div}(w_h),\operatorname{div}(w_h))_{L^2(\Omega)}}{\|w_h\|_{H(\operatorname{div},\Omega)}}$$
$$=\|w_h\|_{H(\operatorname{div},\Omega)}.$$

□

Similarly as before we define the norm on the product space $V_h\times\Sigma_h\times Q_h$,

$$\|(u_h,\sigma_h,p_h)\|_{**}:=\sqrt{\nu}\|u_h\|_{H(\operatorname{div},\Omega)}+\frac{1}{\sqrt{\nu}}(\|\sigma_h\|_{\mathrm{cd},h}+\|p_h\|_{L^2(\Omega)}),$$

and have the following result:

Corollary 5. *The bilinear form B is inf-sup stable with respect to $\|\cdot\|_{**}$, thus there exists a constant $\beta>0$ such that for all nonzero functions $(u_h,\sigma_h,p_h)\in V_h\times\Sigma_h\times Q_h$ there holds*

$$\sup_{(v_h,\tau_h,q_h)\in V_h\times\Sigma_h\times Q_h}\frac{B(u_h,\sigma_h,p_h;v_h,\tau_h,q_h)}{||(v_h,\tau_h,q_h)||_{**}}\geq\beta||(u_h,\sigma_h,p_h)||_{**}.$$

Let $f\in L^2(\Omega,\mathbb{R}^d)$ and assume homogeneous boundary conditions (6.8). There exists a unique solution $(u_h,\sigma_h,p_h)\in V_{h,D}\times\Sigma_{h,N}\times Q_h$ of the MCS method (6.6) with the stability estimate

$$\sqrt{\nu}\|u_h\|_{\mathrm{div}}+\frac{1}{\sqrt{\nu}}(\|\sigma_h\|_{\mathrm{cd},h}+\|p_h\|_\Omega)\lesssim\frac{1}{\sqrt{\nu}}\|f\|_{L^2(\Omega)}.$$

Proof. This follows with lemma 26, lemma 27, theorem 20 and Brezzi's theorem 6. □

6.3 Error estimates

In this section we discuss several error estimates of the MCS method. First, we define proper interpolation operators for the discrete velocity, stress and pressure spaces and provide approximation results in different norms. Using these interpolation operators we are then going to discuss the following error estimates: First we focus on a standard error estimate in section 6.3.2. We start with the analysis using the norms defined in section 6.2.2, and prove that the errors of the stress and the pressure converge with optimal order. Note, however, that we could only expect a lower convergence order of the velocity error measured in a discrete H^1-norm. This is mainly due to the bad approximation properties of the Raviart-Thomas interpolator, see lemma 28 below. It will be crucial in the analysis that this degenerated convergence order of the velocity error does not effect the results of the stress and the pressure variable. This is achieved by showing that the velocity error *measured in a discrete sense* is still of optimal order. After this we provide also an error estimate in the case of natural norms, i.e. with the norms defined in section 6.2.3. In this case we can also provide optimal convergence of the velocity error as the Raviart-Thomas interpolator has good approximation poperties with respect to the natural velocity $H(\text{div})$-norm.

Section 6.3.3 deals with an error estimate that is known in the literature as *pressure robustness*: In the original work [83], the author showed that the proper scaling of irrotational and rotational forces with respect to the viscosity ν – as it appears in the continuous setting – might be disturbed in the discrete case. As a consequence, the velocity error scales with the factor $1/\nu$, which gets big in the case of vanishing viscosity. We show that the MCS method does not only provide optimal convergence errors, but is also pressure robust, hence we provide error estimates that are independent of ν and the pressure approximation.

We conclude with section 6.3.4, where we provide a local post processing procedure. Thereby, we retrieve optimal convergence of the error of a lifted velocity approximation (measured in a discrete H^1-norm). Using the results of [76, 77] it is possible that this post processing can be defined in such a way that the lifted velocity is still exactly divergence free.

6.3.1 Interpolation operators

We start with the definition of interpolation operators for the velocity and the pressure space. To this end we denote by $\mathcal{I}_{\mathcal{RT}^k}$ the Raviart-Thomas interpolator of order k that is locally based on the degrees of freedoms given by lemma 13. There holds the following approximation results:

Lemma 28. *For any $m \geq 1$ and any smooth $u \in H^1(\Omega, \mathbb{R}^d) \cap H^m(\mathcal{T}_h, \mathbb{R}^d)$ the interpolation operator $\mathcal{I}_{\mathcal{RT}^k}$ is well defined and there holds the interpolation result*

$$\|u - \mathcal{I}_{\mathcal{RT}^k} u\|_{1,h} \lesssim h^{s-1} \|u\|_{H^s(\mathcal{T}_h)},$$

for all $s \leq min(k+1, m)$. Similarly, if $u \in H^m(\text{div}, \Omega)$, we have the approximation result

$$\|u - \mathcal{I}_{\mathcal{RT}^k} u\|_{H(\text{div},\Omega)} \lesssim h^s \|u\|_{H^s(\text{div},\Omega)},$$

for all $s \leq min(k+1, m)$.

Proof. We first show the continuity of $\mathcal{I}_{\mathcal{RT}^k}$ in the norm $\|\cdot\|_{1,h}$. By definition, we have

$$\|\mathcal{I}_{\mathcal{RT}^k} u\|_{1,h}^2 = \sum_{T \in \mathcal{T}_h} \|\nabla \mathcal{I}_{\mathcal{RT}^k} u\|_T^2 + \sum_{F \in \mathcal{F}_h} \frac{1}{h} \|[\![(\mathcal{I}_{\mathcal{RT}^k} u)_t]\!]\|_F^2.$$

First, we bound the element terms by a triangle inequality,

$$\|\nabla \mathcal{I}_{\mathcal{RT}^k} u\|_T = \|\nabla \mathcal{I}_{\mathcal{RT}^k} u + \nabla u - \nabla u\|_T \leq \|\nabla (u - \mathcal{I}_{\mathcal{RT}^k} u)\|_T + \|u\|_{H^1(T)}.$$

As proposition 2.5.3 from [11] gives the estimates

$$\|(u - \mathcal{I}_{\mathcal{RT}^k} u)\|_T \leq h\|u\|_{H^1(T)} \quad \text{and} \quad \|\nabla(u - \mathcal{I}_{\mathcal{RT}^k} u)\|_T \leq \|u\|_{H^1(T)}, \tag{6.25}$$

the stability of the element terms is proven. Similarly, we add and subtract the function u on the facets to split the jump into two terms, thus

$$\sum_{F \in \mathcal{F}_h} \frac{1}{h} \| [\![(\mathcal{I}_{\mathcal{RT}^k} u)_t]\!] \|_F^2 \leq \sum_{T \in \mathcal{T}_h} \frac{1}{h} \| (u - \mathcal{I}_{\mathcal{RT}^k} u)_t \|_{\partial T}^2.$$

Next, using standard scaling arguments and a multiplicative trace inequality, see theroem 1.6.6 in [17], we get

$$\begin{aligned} \| (u - \mathcal{I}_{\mathcal{RT}^k} u)_t \|_{\partial T}^2 &\lesssim h^{d-1} \| (u - \mathcal{I}_{\mathcal{RT}^k} u)_t \|_{\partial \widehat{T}}^2 \\ &\lesssim (h^{d/2} \| u - \mathcal{I}_{\mathcal{RT}^k} u \|_{\widehat{T}})(h^{(d-2)/2} \| u - \mathcal{I}_{\mathcal{RT}^k} u \|_{H^1(\widehat{T})}) \\ &\lesssim \| (u - \mathcal{I}_{\mathcal{RT}^k} u) \|_T \big(\frac{1}{h} \| (u - \mathcal{I}_{\mathcal{RT}^k} u) \|_T + \| \nabla (u - \mathcal{I}_{\mathcal{RT}^k} u) \|_T \big). \end{aligned}$$

Again using (6.25) we get $1/h \| (u - \mathcal{I}_{\mathcal{RT}^k} u)_t \|_{\partial T}^2 \leq \| u \|_{H^1(T)}$ implying continuity,

$$\| \mathcal{I}_{\mathcal{RT}^k} u \|_{1,h}^2 \lesssim \sum_{T \in \mathcal{T}_h} \| u \|_{H^1(T)}^2 \leq \| u \|_{H^1(\Omega)}^2.$$

The rest follows with a standard Bramble-Hilbert argument (see lemma 4.3.8 in [17] or in the original work [14]) using that $(\mathrm{Id} - \mathcal{I}_{\mathcal{RT}^k}) q = 0$ for $q \in \mathbb{P}^k(\mathcal{T}_h, \mathbb{R}^d)$. The proof for the second statement follows from proposition 2.5.3 and 2.5.1 in [11]. □

Similarly, we define the standard $\mathcal{BDM}^k$-interpolator by $\mathcal{I}_{\mathcal{BDM}^k}$ and there holds the following approximation result.

Lemma 29. *For any $m \geq 1$ and any smooth $u \in H^1(\Omega, \mathbb{R}^d) \cap H^m(\mathcal{T}_h, \mathbb{R}^d)$ the interpolation operator $\mathcal{I}_{\mathcal{BDM}^k}$ is well defined and there holds the interpolation result*

$$\| u - \mathcal{I}_{\mathcal{BDM}^k} u \|_{1,h} \lesssim h^{s-1} \| u \|_{H^s(\mathcal{T}_h)},$$

for all $s \leq min(k+1, m)$. Similarly, if $u \in H^m(\mathrm{div}, \Omega)$ we have

$$\| u - \mathcal{I}_{\mathcal{BDM}^k} u \|_{H(\mathrm{div}, \Omega)} \lesssim h^s \| u \|_{H^s(\mathrm{div}, \Omega)},$$

for all $s \leq min(k, m)$.

Proof. Follows with the same steps as the proof of lemma 28. □

For the pressure space we use the standard element-wise L^2-projection given by $\Pi_{\mathcal{T}_h}^k$.

Lemma 30. *For any $m \geq 1$ and any smooth $u \in H^m(\mathcal{T}_h, \mathbb{R}^d)$ the operator $\Pi_{\mathcal{T}_h}^k$ is well defined, and there holds the interpolation result*

$$\| u - \Pi_{\mathcal{T}_h}^k u \|_\Omega \lesssim h^s \| u \|_{H^s(\mathcal{T}_h)},$$

for all $s \leq min(k+1, m)$.

Proof. See for example in [17], [11] or [43]. □

We conclude this section with the definition of an interpolation operator for the discrete stress space. Using the global degrees of freedom of $\Xi^k(\mathcal{T}_h)$, a canonical interpolation operator $\mathcal{I}_{\Xi^k}$ can be defined as usual. On each $T \in \mathcal{T}_h$, the interpolant $(\mathcal{I}_{\Xi^k}\sigma)|_T$ coincides with the canonical local interpolant $I_{\Xi^k(T)}(\sigma|_T)$ defined using the local degrees of freedom given by the set $\tilde{\Phi}(T)$, see equation (5.29). Note that these degrees of freedoms are unisolvent, see theorem 15, thus are appropriate for the definition of an interpolation operator. Then we have

$$\phi(\sigma - I_{\Xi^k(T)}\sigma) = 0 \quad \text{for all } \phi \in \tilde{\Phi}(T). \tag{6.26}$$

Recalling the map $\mathcal{M}$ from (5.22), note that $\mathcal{M}^{-1}(\sigma) = \det(F_T^F)F_F^{\mathrm{T}}\sigma F_T^{-\mathrm{T}}$. In a first step we show that interpolation and mapping with respect to $\mathcal{M}$ commutes.

Lemma 31. *For any $\sigma \in H^1(T, \mathbb{R}^{d\times d})$ there holds*

$$\mathcal{M}^{-1}(I_T\sigma) = I_{\hat{T}}(\mathcal{M}^{-1}(\sigma)).$$

Proof. Since both the left and right hand sides are in the local space $\Xi^k(\widehat{T})$, it suffices to prove that

$$\hat{\phi}(\mathcal{M}^{-1}(I_T\sigma) - I_{\hat{T}}(\mathcal{M}^{-1}\sigma)) = 0 \quad \text{for all } \hat{\phi} \in \tilde{\Phi}(\hat{T}). \tag{6.27}$$

For the first type of interior degrees of freedom on $\hat{T}$, given by definition (5.28), we have for all $\hat{\eta} \in \mathbb{P}^{k-1}(\hat{T}, \mathbb{D})$ (due to definition (6.26)) the identity

$$\int_{\hat{T}} I_{\Xi^k(\hat{T})}(\mathcal{M}^{-1}\sigma) : F_{\hat{T}}\hat{\eta}F_{\hat{T}}^{-1}\,\mathrm{d}\hat{x} = \int_{\hat{T}} \mathcal{M}^{-1}\sigma : F_{\hat{T}}\hat{\eta}F_{\hat{T}}^{-1}\,\mathrm{d}\hat{x}\,.$$

As $F_{\hat{T}} = \mathrm{Id}$, this yields

$$\begin{aligned}\int_{\hat{T}} \left[\mathcal{M}^{-1}(I_{\Xi^k(T)}\sigma) - I_{\Xi^k(\hat{T})}(\mathcal{M}^{-1}\sigma)\right] : F_{\hat{T}}\hat{\eta}F_{\hat{T}}^{-1}\,\mathrm{d}\hat{x} &= \int_{\hat{T}} \left[\mathcal{M}^{-1}(I_{\Xi^k(T)}\sigma) - \mathcal{M}^{-1}\sigma\right] : \hat{\eta}\,\mathrm{d}\hat{x} \\ &= \int_T (I_{\Xi^k(T)}\sigma - \sigma) : F_T\hat{\eta}F_T^{-1}\,\mathrm{d}x = 0,\end{aligned}$$

where we used the equality of interior degrees of freedom on T in (6.26). With the same argument we see for all polynomials $\hat{\mu} \in \mathbb{P}^k(\hat{T}, \mathbb{R})$ that the second type of interior degrees of freedom, see definition (5.25), vanishes, as

$$\begin{aligned}\int_{\hat{T}} \mathrm{tr}(\mathcal{M}^{-1}(I_{\Xi^k(T)}\sigma) - I_{\Xi^k(\hat{T})}(\mathcal{M}^{-1}\sigma)) : \hat{\mu}\,\mathrm{d}\hat{x} &= \int_{\hat{T}} \mathrm{tr}(\mathcal{M}^{-1}(I_{\Xi^k(T)}\sigma) - \mathcal{M}^{-1}\sigma) : \hat{\mu}\,\mathrm{d}\hat{x} \\ &= \int_T \mathrm{tr}(I_{\Xi^k(T)}\sigma - \sigma) : \mu\,\mathrm{d}x = 0,\end{aligned}$$

where we used a standard pullback $\mu \circ \phi_T = \hat{\mu}$ and the properties of the mapping $\mathcal{M}$, see lemma 16 and (6.26).

We continue with the facet degrees of freedom. We only consider the three-dimensional case as the two-dimensional case is simpler and follows then with very similar steps. On any arbitrary facet $\hat{F} \in \mathcal{F}_h(\widehat{T})$ choose two arbitrary edges $\hat{E}_1, \hat{E}_2$ with unit tangential vectors $\hat{t}_1$ and $\hat{t}_2$. Using a dual tangential basis $\hat{s}_1$ and $\hat{s}_2$ such that $\hat{s}_i \cdot \hat{t}_i = \delta_{ij}$, we expand

$$[\mathcal{M}^{-1}(I_{\Xi^k(T)}(T)\sigma - \sigma)]_{nt} = [\hat{t}_1^{\mathrm{T}} \mathcal{M}^{-1}(I_{\Xi^k(T)}\sigma - \sigma)\hat{n}]\hat{s}_1 + [\hat{t}_2^{\mathrm{T}} \mathcal{M}^{-1}(I_{\Xi^k(T)}\sigma - \sigma)\hat{n}]\hat{s}_2.$$

Next, we choose arbitrary $\hat{r}_1, \hat{r}_2 \in \mathbb{P}^k(\hat{F}, \mathbb{R})$ and define

$$\hat{r} := \frac{\hat{r}_1}{\det(F_{E_1})}\hat{t}_1 + \frac{\hat{r}_2}{\det(F_{E_2})}\hat{t}_2.$$

Let $r_i = \hat{r}_i \circ \phi_T$. Using again a biorthogonal basis s_1, s_2 with respect to unit tangents t_1 and t_2 of the mapped edges E_1 and E_2, we have $r := r_1 t_1 + r_2 t_2$. Thus, the proper mapping of the normal-tangential trace, see lemma 16, yields

$$\begin{aligned}[\mathcal{M}^{-1}(I_{\Xi^k(T)}\sigma - \sigma)]_{nt} = \det(F_T^F)\det(F_{E_1})[t_1^{\mathrm{T}}(I_{\Xi^k(T)}\sigma - \sigma)n]\hat{s}_1 \\ + \det(F_T^F)\det(F_{E_2})[t_2^{\mathrm{T}}(I_{\Xi^k(T)}\sigma - \sigma)n]\hat{s}_2.\end{aligned}$$

Then, this identity shows with similar steps as above

$$\begin{aligned}\int_{\hat{F}} &[\mathcal{M}^{-1}(I_{\Xi^k(T)}\sigma) - I_{\Xi^k(\widehat{T})}\mathcal{M}^{-1}\sigma)]_{nt} \cdot \hat{r} \,\mathrm{d}\hat{s} \\ &= \int_{\hat{F}} [\mathcal{M}^{-1}(I_{\Xi^k(T)}\sigma - \sigma)]_{nt} \cdot \hat{r} \,\mathrm{d}\hat{s} \\ &= \int_F [t_1^{\mathrm{T}}(I_{\Xi^k(T)}\sigma - \sigma)n] r_1 s_1 \cdot t_1 \,\mathrm{d}x + \int_F [t_2^{\mathrm{T}}(I_T\sigma - \sigma)n] r_2 s_2 \cdot t_2 \,\mathrm{d}s \\ &= \int_F [(t_1^{\mathrm{T}}(I_T\sigma - \sigma)n)s_1 + (t_2^{\mathrm{T}}(I_T\sigma - \sigma)n)s_2] \cdot [r_1 t_1 + r_2 t_2] \,\mathrm{d}s \\ &= \int_F (I_T\sigma - \sigma)_{nt} \cdot r \,\mathrm{d}s = 0,\end{aligned}$$

where the last step is due to the equality of the facet degrees of freedom in (6.26). □

Next, we show that the interpolation operator on the reference element is continuous.

Lemma 32. *For any $\sigma \in H^s(\widehat{T}, \mathbb{R}^{d\times d})$ with $s \geq 1$ there holds*

$$\|I_{\Xi^k(\widehat{T})}\sigma\|_{\widehat{T}} + \sqrt{\sum_{\hat{F}\in\mathcal{F}_h(\widehat{T})} \|(I_{\Xi^k(\widehat{T})}\sigma)_{nt}\|_{\hat{F}}^2} \lesssim \|\sigma\|_{H^s(\widehat{T})}.$$

Proof. As $I_{\Xi^k(\widehat{T})}\sigma$ is an element of $\Xi^k(\widehat{T})$, lemma 19 shows that it is sufficient to estimate only the volume term. The estimate then follows directly using a nodal basis with respect to the degrees of freedom given by $\tilde{\Phi}_{\mathbb{D}}^T, \Phi_{\mathrm{Id}}^T$ and $\{\Phi^F : F \in \mathcal{F}_h(T)\}$ and that the L^2-norms on elements and facets of those polynomial basis functions are uniformly bounded. One then derives an estimate

$$\|I_{\Xi^k(\widehat{T})}\sigma\|_{\widehat{T}} \lesssim \|\sigma\|_{\widehat{T}} + \sum_{\hat{F}\in\mathcal{F}_h(\widehat{T})} \|\sigma_{nt}\|_{\hat{F}}.$$

By the continuity of the trace we conclude the proof. □

We can now finally show the following approximation result.

Theorem 21. *For any $m \geq 1$ and any $\sigma \in \{\tau \in H^m(\mathcal{T}_h, \mathbb{R}^{d\times d}) : [\![\tau_{nt}]\!] = 0\}$ the interpolant $I_{\Xi^k}\sigma$ is well defined and there holds the approximation result*

$$||\sigma - I_{\Xi^k}\sigma||_{L^2(\Omega)} + \sqrt{\sum_{F\in\mathcal{F}_h} h||(\sigma - I_{\Xi^k}\sigma)_{nt}||_F^2} \lesssim h^s||\sigma||_{H^s(\mathcal{T}_h)}, \tag{6.28}$$

for all $s \leq \min(k, m)$.

Proof. Let $\hat{\sigma} = \mathcal{M}^{-1}(\sigma|_T)$. By lemma 31, $\mathcal{M}^{-1}(\sigma - I_{\Xi^k(T)}\sigma) = \hat{\sigma} - I_{\Xi^k(\widehat{T})}\hat{\sigma}$. Thus, for each element $T \in \mathcal{T}_h$ we have, using lemma 16, that

$$\|\sigma - I_{\Xi^k(T)}\sigma||_T^2 + h\|(\sigma - I_{\Xi^k(T)}\sigma)_{nt}||_{\partial T}^2 \sim \frac{1}{h^d}\Big(\|\hat{\sigma} - I_{\Xi^k(\widehat{T})}\hat{\sigma}||_{\hat{T}}^2 + \|(\hat{\sigma} - I_{\Xi^k(\widehat{T})}\hat{\sigma})_{nt}||_{\partial\hat{T}}^2\Big).$$

Next, note that the unisolvency of the reference element degrees of freedom, see theorem 15, shows that

$$\hat{\sigma} - I_{\Xi^k(\widehat{T})}\hat{\sigma} = 0 \quad \text{for all } \hat{\sigma} \in \mathbb{P}^k(\hat{T}, \mathbb{R}^{d\times d}).$$

Lemma 32 further shows continuity of the operator $\mathrm{Id} - I_{\Xi^k(\widehat{T})}$, thus we apply the Bramble-Hilbert lemma (lemma 4.3.8 in [17]), and we get

$$\frac{1}{h^d}\Big(\|\hat{\sigma} - I_{\Xi^k(\widehat{T})}\hat{\sigma}||_{\hat{T}}^2 + \|(\hat{\sigma} - I_{\Xi^k(\widehat{T})}\hat{\sigma})_{nt}||_{\partial\hat{T}}^2\Big) \leq \frac{1}{h^d}|\hat{\sigma}|_{H^s(\widehat{T})}^2.$$

A standard scaling argument shows that

$$|\hat{\sigma}|_{H^s(\widehat{T})}^2 \sim h^{2s+d}|\sigma|_{H^s(T)}^2.$$

□

We conclude this section with an approximation result of the stress interpolator I_{Ξ^k} in natural norms.

Theorem 22. *For any $m \geq 1$ and any $\sigma \in \{\tau \in H^m(\mathcal{T}_h, \mathbb{R}^{d\times d}) : [\![\tau_{nt}]\!] = 0\}$, the interpolant $I_{\Xi^k}\sigma$ is well defined and there holds the approximation result*

$$||\sigma - I_{\Xi^k}\sigma||_{\mathrm{cd},h} \lesssim h^s||\sigma||_{H^s(\mathcal{T}_h)}, \tag{6.29}$$

for all $s \leq \min(k, m)$.

Proof. Theorem 21 yields that the term including the L^2-norm of $\|\cdot\|_{\mathrm{cd},h}$ converges with optimal order. It remains to estimate the term including the bilinear form b_{2h}. To this end let $v_h \in \mathcal{RT}^k(\mathcal{T}_h)$ be arbitrary. Adding and subtracting $\Pi_T^{k-1}\nabla v_h$ and $\Pi^k[\![(v_h)_t]\!]$ on the element and facet terms, respectively, we have

$$\begin{aligned} b_{2h}(\sigma - I_{\Xi^k}\sigma, v_h) = \\ &- \sum_{T\in\mathcal{T}_h}\int_T(\sigma - I_{\Xi^k}\sigma) : (\mathrm{Id} - \Pi_T^{k-1})\nabla v_h \,\mathrm{d}x - \sum_{T\in\mathcal{T}_h}\int_T(\sigma - I_{\Xi^k}\sigma) : \Pi_T^{k-1}\nabla v_h \,\mathrm{d}x \\ &+ \sum_{F\in\mathcal{F}_h}\int_F(\sigma - I_{\Xi^k}\sigma)_{nt}\cdot(\mathrm{Id} - \Pi_F^k)[\![(v_h)_t]\!] \,\mathrm{d}s + \sum_{F\in\mathcal{F}_h}\int_F(\sigma - I_{\Xi^k}\sigma)_{nt}\cdot\Pi_F^k[\![(v_h)_t]\!] \,\mathrm{d}s\,. \end{aligned}$$

By the local definition of the interpolator I_{Ξ^k}, see equation 6.26, the second and the fourth sum vanish. By the Cauchy-Schwarz inequality and rewriting the jump with its contribution from both element sides we then get

$$b_{2h}(\sigma - I_{\Xi^k}\sigma, v_h) \lesssim \left(||\sigma - I_{\Xi^k}\sigma||_{L^2(\Omega)} + \sqrt{\sum_{F\in\mathcal{F}_h} h||(\sigma - I_{\Xi^k}\sigma)_{nt}||_F^2} \right) \left(\sqrt{\sum_{T\in\mathcal{T}_h} \|(\mathrm{Id} - \Pi_T^{k-1})\nabla v_h\|_T^2} + \sqrt{\sum_{T\in\mathcal{T}_h} \frac{1}{h}\|(\mathrm{Id} - \Pi_F^k)(v_h)_t\|_{\partial T}^2} \right).$$

Using lemma 22, we obtain

$$b_{2h}(\sigma - I_{\Xi^k}\sigma, v_h) \lesssim \left(||\sigma - I_{\Xi^k}\sigma||_{L^2(\Omega)} + \sqrt{\sum_{F\in\mathcal{F}_h} h||(\sigma - I_{\Xi^k}\sigma)_{nt}||_F^2} \right) \|v_h\|_{H(\mathrm{div},\Omega)},$$

and thus, again by theorem 21,

$$\sup_{v_h\in\mathcal{RT}^k(\mathcal{T}_h)} \frac{b_{2h}(\sigma - I_{\Xi^k}\sigma, v_h)}{\|v_h\|_{H(\mathrm{div},\Omega)}} \lesssim h^s||\sigma||_{H^s(\mathcal{T}_h)}.$$

□

6.3.2 A standard error estimate

With the interpolation operators defined in the last section we can now show optimal convergence of the finite element errors. For the ease of notation we define

$$I_{V_h} := \mathcal{I}_{\mathcal{RT}^k}, \quad I_{\Sigma_h} := I_{\Xi^k}, \quad I_{Q_h} := \Pi_{\mathcal{T}_h}^k. \tag{6.30}$$

Theorem 23 below shows that the solutions of the system (6.6) provide optimal convergence of the stress and the pressure errors. Further, as mentioned at the beginning of this section, we have optimal convergence of a discrete velocity error given by the difference of the solution and the corresponding velocity interpolation of the exact solution. Note that the standard velocity error $\|u - u_u\|$ is only of order $\mathcal{O}(h^k)$ (assuming a smooth exact solution). This can be easily seen, as the according velocity interpolation operator I_{V_h} only provides approximation results in a discrete H^1-norm of order k, see lemma 28.

Theorem 23 (Optimal convergence)**.** *Assuming homogeneous boundary conditions* (6.8)*, let* $u \in H^1(\Omega, \mathbb{R}^d) \cap H^m(\mathcal{T}_h, \mathbb{R}^d)$*,* $\sigma \in H^1(\Omega, \mathbb{D}) \cap H^{m-1}(\mathcal{T}_h, \mathbb{D})$ *and* $p \in Q \cap H^{m-1}(\mathcal{T}_h, \mathbb{R})$ *be the exact solution of the mixed Stokes problem* (4.6)*. Further let* u_h*,* σ_h *and* p_h *be the solutions of the MCS method* (6.6)*. For* $s = \min(m-1, k+1)$ *there holds*

$$\frac{1}{\nu}(||\sigma - \sigma_h||_{\Sigma_h} + \|p - p_h\|_{Q_h}) + \|u_h - I_{V_h}u\|_{V_h} \lesssim h^s(\frac{1}{\nu}\|\sigma\|_{H^s(\mathcal{T}_h)} + \frac{1}{\nu}\|p\|_{H^s(\mathcal{T}_h)}).$$

Proof. In a first step we use the triangle inequality to divide the error into an interpolation error and a discrete measure of error

$$\begin{aligned}
&\frac{1}{\nu}(||\sigma - \sigma_h||_{\Sigma_h} + \|p - p_h\|_{Q_h}) + \|u_h - I_{V_h}u\|_{V_h} \\
&\qquad \lesssim \frac{1}{\nu}(||\sigma - I_{\Sigma_h}\sigma||_{\Sigma_h}\|p - I_{Q_h}p\|_{Q_h}) \\
&\qquad + \frac{1}{\nu}(||I_{\Sigma_h}\sigma - \sigma_h||_{\Sigma_h} + \|I_{Q_h}p - p_h\|_{Q_h}) + \|u_h - I_{V_h}u\|_{V_h}.
\end{aligned}$$

By the approximation properties of the L^2-projection, see lemma 30, and the interpolation operator I_{Σ_h}, see theorem 21, we already have for $s = \min(m-1, k+1)$,

$$\frac{1}{\nu}(||\sigma - I_{\Sigma_h}\sigma||_{\Sigma_h} + \|p - I_{Q_h}p\|_{Q_h}) \lesssim h^s \left(\frac{1}{\nu}\|\sigma\|_{H^s(\mathcal{T}_h)} + \frac{1}{\nu}\|p\|_{H^s(\mathcal{T}_h)}\right).$$

Next, corollary 4 and theorem 18 yield

$$\begin{aligned}
&\frac{1}{\sqrt{\nu}}\|(I_{V_h}u - u_h, I_{\Sigma_h}\sigma - \sigma_h, I_{Q_h}p - p_h)\|_* \\
&\qquad \lesssim \sup_{\substack{v_h \in V_h \\ \tau_h \in \Sigma_h, q_h \in Q_h}} \frac{B(I_{V_h}u - u_h, I_{\Sigma_h}\sigma - \sigma_h, I_{Q_h}p - p_h; v_h, \tau_h, q_h)}{\sqrt{\nu}\|(v_h, \tau_h, q_h)\|_*} \\
&\qquad \lesssim \sup_{\substack{v_h \in V_h \\ \tau_h \in \Sigma_h, q_h \in Q_h}} \frac{B(I_{V_h}u - u, I_{\Sigma_h}\sigma - \sigma, I_{Q_h}p - p; v_h, \tau_h, q_h)}{\sqrt{\nu}\|(v_h, \tau_h, q_h)\|_*}.
\end{aligned}$$

We continue estimating each term of B separately. First note that due to the polynomial order of the space Σ_h we have that locally $\operatorname{div}(\tau_h) \in \mathbb{P}^{k-1}(T, \mathbb{R}^d)$ and $\tau_{nn} \in \mathbb{P}^k(F, \mathbb{R})$ for all $T \in \mathcal{T}_h, F \in \mathcal{F}_h$. By the definition of the Raviart-Thomas interpolator, based on the degrees of freedom given in lemma 13, and the element wise L^2-projection we then have

$$b_{2h}(\tau_h, I_{V_h}u - u) = \sum_{T \in \mathcal{T}_h} \int_T \operatorname{div}(\tau_h) \cdot (I_{V_h}u - u) - \sum_{F \in \mathcal{F}_h} \int_F [\![(\tau_h)_{nn}]\!](I_{V_h}u - u)_n \,\mathrm{d}s = 0.$$

Similarly, we have

$$b_{2h}(I_{\Sigma_h}\sigma - \sigma, v_h) = -\sum_{T \in \mathcal{T}_h} \int_T (I_{\Sigma_h}\sigma - \sigma) : \nabla v_h \,\mathrm{d}x + \sum_{F \in \mathcal{F}_h} \int_F (I_{\Sigma_h}\sigma - \sigma)_{nt} \cdot [\![(v_h)_t]\!] \,\mathrm{d}s\,.$$

Applying the Cauchy-Schwarz inequality then yields

$$b_{2h}(I_{\Sigma_h}\sigma - \sigma, v_h) \lesssim \frac{1}{\sqrt{\nu}}\left(\|I_{\Sigma_h}\sigma - \sigma\|_{\Sigma_h} + \sqrt{\sum_{F \in \mathcal{F}_h} h\|(I_{\Sigma_h}\sigma - \sigma)_{nt}\|_F^2}\right)\sqrt{\nu}\|v_h\|_{V_h}.$$

For the terms including the bilinear form b_1 we observe with the same arguments as above

$$\begin{aligned}
b_1(I_{V_h}u - u, q_h) &= \sum_{T \in \mathcal{T}_h} \int_T \operatorname{div}(I_{V_h}u - u)q_h = 0, \\
b_1(v_h, I_{Q_h}p - p) &\lesssim \sqrt{\nu}\|v_h\|_{V_h}\frac{1}{\sqrt{\nu}}\|I_{Q_h}p - p\|_{Q_h}.
\end{aligned}$$

Together with the continuity of a these findings yield

$$\begin{aligned}&B(I_{V_h}u-u, I_{\Sigma_h}\sigma-\sigma, I_{Q_h}p-p; v_h,\tau_h,q_h)\\&\lesssim\left(\|(0,I_{\Sigma_h}\sigma-\sigma,I_{Q_h}p-p)\|_*+\frac{1}{\sqrt{\nu}}\sqrt{\sum_{F\in\mathcal{F}_h}h\|(I_{\Sigma_h}\sigma-\sigma)_{nt}\|_F^2}\right)\|(v_h,\tau_h,0)\|_*.\end{aligned}$$

Thereby, theorem 21 and the approximation properties of the L^2-projection (30), conclude

$$\frac{1}{\sqrt{\nu}}\|(I_{V_h}u-u_h, I_{\Sigma_h}\sigma-\sigma_h, I_{Q_h}p-p_h)\|_*\lesssim h^s\Big(\frac{1}{\nu}\|\sigma\|_{H^s(\mathcal{T}_h)}+\frac{1}{\nu}\|p\|_{H^s(\mathcal{T}_h)}\Big).$$

□

We also aim for a similar result in natural norms. The proof follows very similar steps including some slightly different estimates. As the natural norm on the velocity space is given by the $H(\mathrm{div})$-norm, lemma 28 shows that we now have optimal rates also for the velocity error.

Theorem 24 (Optimal convergence in natural norms)**.** *Assuming homogeneous boundary conditions (6.8), let $u\in H^1(\Omega,\mathbb{R}^d)\cap H^m(\mathcal{T}_h,\mathbb{R}^d)$, $\sigma\in H^1(\Omega,\mathbb{D})\cap H^{m-1}(\mathcal{T}_h,\mathbb{D})$ and $p\in Q\cap H^{m-1}(\mathcal{T}_h,\mathbb{R})$ be the exact solution of the mixed Stokes problem (4.6). Further let u_h, σ_h and p_h be the solution of the MCS method (6.6). For $s=\min(m-1,k+1)$ there holds*

$$\frac{1}{\nu}(\|\sigma-\sigma_h\|_{\mathrm{cd},h}+\|p-p_h\|_{Q_h})+\|u-u_h\|_{H(\mathrm{div},\Omega)}\lesssim h^s(\frac{1}{\nu}\|\sigma\|_{H^s(\mathcal{T}_h)}+\frac{1}{\nu}\|p\|_{H^s(\mathcal{T}_h)}).$$

Proof. We use the triangle inequality to divide the error into an interpolation error and a discrete measure of the error. By lemma 30, lemma 28 and theorem 22, the interpolation error already converges with optimal order

$$\begin{aligned}\frac{1}{\nu}(\|\sigma-I_{\Sigma_h}\sigma\|_{\mathrm{cd},h}+\|p-I_{Q_h}p\|_{L^2(\Omega)})+\|u-I_{V_h}\|_{H(\mathrm{div},\Omega)}\\\lesssim h^s\left(\frac{1}{\nu}\|\sigma\|_{H^s(\mathcal{T}_h)}+\frac{1}{\nu}\|p\|_{H^s(\mathcal{T}_h)}\right).\end{aligned}$$

Next, we use discrete inf-sup stability, see corollary 5, to get

$$\begin{aligned}&\frac{1}{\sqrt{\nu}}\|(I_{V_h}u-u_h, I_{\Sigma_h}\sigma-\sigma_h, I_{Q_h}p-p_h)\|_{**}\\&\qquad\lesssim\sup_{\substack{v_h\in V_h\\\tau_h\in\Sigma_h,q_h\in Q_h}}\frac{B(I_{V_h}u-u_h, I_{\Sigma_h}\sigma-\sigma_h, I_{Q_h}p-p_h; v_h,\tau_h,q_h)}{\sqrt{\nu}\|(v_h,\tau_h,q_h)\|_{**}}\\&\qquad\lesssim\sup_{\substack{v_h\in V_h\\\tau_h\in\Sigma_h,q_h\in Q_h}}\frac{B(I_{V_h}u-u, I_{\Sigma_h}\sigma-\sigma, I_{Q_h}p-p; v_h,\tau_h,q_h)}{\sqrt{\nu}\|(v_h,\tau_h,q_h)\|_{**}}.\end{aligned}$$

We estimate each term separately. In contrast to the proof of theorem 23, the Cauchy-Schwarz inequality and the same steps as in the proof of theorem 22 now yields

$$b_{2h}(I_{\Sigma_h}\sigma - \sigma, v_h) \lesssim \frac{1}{\sqrt{\nu}} \left(\|I_{\Sigma_h}\sigma - \sigma\|_{\Sigma_h} + \sqrt{\sum_{F\in\mathcal{F}_h} h\|(I_{\Sigma_h}\sigma - \sigma)_{nt}\|_F^2} \right) \sqrt{\nu}\|v_h\|_{H(\mathrm{div},\Omega)},$$

$$b_1(v_h, I_{Q_h}p - p) \lesssim \sqrt{\nu}\|v_h\|_{H(\mathrm{div},\Omega)} \frac{1}{\sqrt{\nu}}\|I_{Q_h}p - p\|_{L^2(\Omega)}.$$

Adding up all terms we conclude with the same steps as in the proof of the previous theorem. □

We conclude this section by proving that the L^2-norm error of the velocity u_h, again measured in a proper sense, has an improved convergence rate. This results is in particular needed in section 6.3.4. The proof of the enhanced convergence rate is based on the standard Aubin-Nitsche technique. To this end let $z \in H^1(\Omega, \mathbb{R}^d)$ and $\mu \in L^2(\Omega, \mathbb{R})$ be the solution of a standard variational formulation in a velocity-pressure setting (4.4). Then we need the following regularity assumption (shift theorem)

$$\nu\|z\|_{H^2(\Omega,\mathbb{R}^d)} + \|p\|_{H^1(\Omega,\mathbb{R})} \lesssim \|f\|_{L^2}. \tag{6.31}$$

Theorem 25 (Optimal order of the L^2-norm error)**.** *Assuming homogeneous boundary conditions (6.8), let $u \in H^1(\Omega, \mathbb{R}^d) \cap H^m(\mathcal{T}_h, \mathbb{R}^d)$, $\sigma \in H^1(\Omega, \mathbb{D}) \cap H^{m-1}(\mathcal{T}_h, \mathbb{D})$ and $p \in Q \cap H^{m-1}(\mathcal{T}_h, \mathbb{R})$ be the exact solution of the mixed Stokes problem (4.6). Further let u_h, σ_h and p_h be the solution of the MCS method (6.6). Further assume that the solutions z and μ of the problem (4.4) fulfill the regularity estimate (6.31). For $s = \min(m-1, k+1)$ there holds*

$$\|I_{V_h}u - u_h\|_{L^2(\Omega)} \lesssim h^{s+1}\frac{1}{\nu}\|\sigma\|_{H^s(\mathcal{T}_h)}.$$

Proof. The proof follows along the same lines as for example in [111] and is based on a standard Aubin-Nitsche technique. To this end let z and μ be the solutions of problem (4.4) with the right hand side $f = -(I_{V_h}u - u_h)$, and define $\pi := \nu\nabla z$. With standard techniques (testing with smooth functions with compact support), the solution fulfills the equivalence

$$-\operatorname{div}(\nu\pi) + \nabla\mu = -(I_{V_h}u - u_h) \quad \text{in} \quad L^2(\Omega, \mathbb{R}^d).$$

Testing this equation with $-(I_{V_h}u - u_h)$ and integrating by parts in the pressure term (note that $u_h \in H(\mathrm{div})$) we have

$$\|I_{V_h}u - u_h\|_{L^2(\Omega)}^2 = \int_\Omega \operatorname{div}(\pi)\cdot(I_{V_h}u - u_h)\,\mathrm{d}x + \int_\Omega \operatorname{div}(I_{V_h}u - u_h)\mu\,\mathrm{d}x\,.$$

As the Raviart-Thomas interpolator I_{V_h} preserves the divergence, and the discrete solution u_h is exactly divergence free, the second term vanishes. Next, due to assumption (6.31), the viscous stress function π is continuous across element interfaces, thus there holds on all facets $F \in \mathcal{F}_h$ that $[\![(\pi)_{nn}]\!] = 0$, and above equation can be written as

$$\begin{aligned}
&\|I_{V_h}u - u_h\|^2_{L^2(\Omega)} \\
&= \int_\Omega \operatorname{div}(\pi) \cdot (I_{V_h}u - u_h)\,\mathrm{d}x \\
&= \sum_{T\in\mathcal{T}_h} \int_T \operatorname{div}(\pi) \cdot (I_{V_h}u - u_h)\,\mathrm{d}x - \sum_{F\in\mathcal{F}_h} \int_F [\![(\pi)_{nn}]\!](I_{V_h}u - u_h)_n\,\mathrm{d}s \\
&= b_{2h}(\pi, I_{V_h}u - u_h).
\end{aligned} \tag{6.32}$$

Similarly, as $\pi = \nu\nabla z$ in $L^2(\Omega, \mathbb{R}^{d\times d})$, we can test this equation with $\sigma - \sigma_h$, and get

$$\int_\Omega \frac{1}{\nu}\pi : (\sigma - \sigma_h)\,\mathrm{d}x - \int_\Omega (\sigma - \sigma_h) : \nabla z = 0.$$

As z is continuous, we have that $[\![z_t]\!] = 0$ on all facets $F \in \mathcal{F}_h$, yielding

$$\int_\Omega \frac{1}{\nu}\pi : (\sigma - \sigma_h)\,\mathrm{d}x + b_{2h}(\sigma - \sigma_h, z) = 0. \tag{6.33}$$

Now, let $\tilde{\pi} := I_{\Sigma_h}\pi$ and $\tilde{z} = I_{V_h}z$. The regularity assumption of the solutions σ and u and the consistency theorem 18 shows that we have the Galerkin like orthogonality

$$a(\sigma - \sigma_h, \tilde{\pi}) + b_{2h}(\tilde{\pi}, u - u_h) + b_{2h}(\sigma - \sigma_h, \tilde{z}) + b_1(\tilde{z}, p - p_h) + b_1(u - u_h, q_h) = 0, \tag{6.34}$$

where q_h can be chosen arbitrarily. Next, note that by the definition of the interpolator I_{V_h} we have that

$$b_{2h}(\tilde{\pi}, I_{V_h}u - u) = \sum_{T\in\mathcal{T}_h} \int_T \operatorname{div}(\tilde{\pi}) \cdot (I_{V_h}u - u)\,\mathrm{d}x - \sum_{F\in\mathcal{F}_h} \int_F [\![(\tilde{\pi})_{nn}]\!](I_{V_h}u - u)_n\,\mathrm{d}s = 0,$$

and thus

$$b_{2h}(\tilde{\pi}, u - u_h) = b_{2h}(\tilde{\pi}, u - I_{V_h}u + I_{V_h}u - u_h) = b_{2h}(\tilde{\pi}, I_{V_h}u - u_h). \tag{6.35}$$

As $\operatorname{div}(z) = \operatorname{div}(\tilde{z}) = \operatorname{div}(u) = \operatorname{div}(u_h) = 0$, summing up equations (6.32), (6.33), (6.34) and (6.35), a Cauchy-Schwarz estimate including a scaling for the facet terms similarly as in the proof of theorem (23), yields

$$\begin{aligned}
&\|I_{V_h}u - u_h\|^2_{L^2(\Omega)} \\
&\quad\lesssim \Big(\|\pi - \tilde{\pi}\|_{L^2(\Omega)} + \sqrt{\sum_{F\in\mathcal{F}_h} h\|(\pi - \tilde{\pi})_{nt}\|^2_F} + \nu\|z - \tilde{z}\|_{V_h}\Big) \\
&\quad\Big(\frac{1}{\nu}\|\sigma - \sigma_h\|_{L^2(\Omega)} + \sqrt{\sum_{F\in\mathcal{F}_h} h\|(\sigma - \sigma_h)_{nt}\|^2_F} + \|I_{V_h}u - u_h\|_{V_h}\Big).
\end{aligned}$$

Thereby, equation (6.31) and the properties of the interpolation operators show that

$$\|\pi - \tilde{\pi}\|_{L^2(\Omega)} + \sqrt{\sum_{F\in\mathcal{F}_h} h\|(\pi - \tilde{\pi})_{nt}\|_F^2} \lesssim h\|\pi\|_{H^1(\Omega)} \lesssim h\|I_{V_h}u - u_h\|,$$

and similarly also $\nu\|z - \tilde{z}\|_{V_h} \lesssim h\nu\|z\|_{H^2(\Omega)} \lesssim h\|I_{V_h}u - u_h\|$. Lemma 19 then yields

$$\begin{aligned}\sqrt{\sum_{F\in\mathcal{F}_h} h\|(\sigma - \sigma_h)_{nt}\|_F^2} &\lesssim \sqrt{\sum_{F\in\mathcal{F}_h} h\|(\sigma - I_{\Sigma_h}\sigma)_{nt}\|_F^2} + \sqrt{\sum_{F\in\mathcal{F}_h} h\|(I_{\Sigma_h}\sigma - \sigma_h)_{nt}\|_F^2}\\ &\lesssim h^{k+1}\|\sigma\|_{H^s(\mathcal{T}_h)} + \|I_{\Sigma_h}\sigma - \sigma\|_{L^2(\Omega)} + \|\sigma - \sigma_h\|_{L^2(\Omega)},\end{aligned}$$

thus, we conclude the proof by theorem 24 and theorem 23. □

6.3.3 A pressure robust error estimate

We define the continuous Helmholtz projector $\mathbb{H}$ as the rotational part of a Helmholtz decomposition (see [53]) of a given load $f \in L^2(\Omega, \mathbb{R}^d)$

$$f = \nabla\theta + \xi =: \nabla\theta + \mathbb{H}(f),$$

with $\theta \in H^1(\Omega)/\mathbb{R}$ and $\xi =: \mathbb{H}(f) \in \{v \in H_0(\mathrm{div}, \Omega) : \mathrm{div}(v) = 0\}$. For the following abstract we assume that $\Gamma_{D,n} = \Gamma$. Testing the second line of (4.26) with an arbitrary divergence free test function $v \in \{v \in H_0(\mathrm{div}, \Omega) : \mathrm{div}(v) = 0\}$, we see that

$$\langle \mathrm{div}(\sigma), v\rangle_{H_0(\mathrm{div},\Omega)} = -\langle \mathbb{H}(f), v\rangle_{H_0(\mathrm{div},\Omega)},$$

hence $\sigma = \nu\nabla u$ is steered only by a part of f, namely $\mathbb{H}(f)$. If the right hand side is perturbed by a gradient field $\nabla\alpha$, then σ and u should not change because $\mathbb{H}(f + \nabla\alpha) = \mathbb{H}(f)$. In the work [83] this relation was discussed in a discrete setting. If a discrete method fulfills this property, it is called pressure robust because one can then deduce an H^1-velocity error that is independent of the pressure. In particular, standard inf-sup stable methods of a velocity-pressure formulation – as for example the Taylor-Hood method, see [11, 85] – show error estimates that read as

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \lesssim \inf_{v_h\in V_h} \|\nabla(u - v_h)\|_{L^2(\Omega)} + \frac{1}{\nu} \inf_{q_h\in Q_h} \|p - q_h\|_{L^2(\Omega)}. \tag{6.36}$$

Here we see that the velocity error might get big if the viscosity tends to get very small $\nu \to 0$, as the scaled term including the best approximation of the pressure might blow up.

In recent works, as for example in [16, 82, 79, 70], it was shown that a modification of the right hand side allows to obtain pressure robust error estimates, which read as

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \lesssim \inf_{v_h\in V_h} \|\nabla(u - v_h)\|_{L^2(\Omega)} + h^{k+1}|u|_{H^{k+1}(\Omega)}.$$

The second term on the right hand side is induced by a consistency error due to the modified right hand side. As we can see, the velocity error does not depend on the pressure discretization and the viscosity ν anymore.

The convergence results of theorem 23 and theorem 24 also include the scaled term $1/\nu||p||_{H^s(\mathcal{T}_h)}$. However, in recent works it was proven that $H(\mathrm{div})$-conforming methods are pressure robust, see for example [30, 81], and a similar result also holds true for the MCS method (6.6). To this end we provide optimal error estimates of the stress and the velocity error that are independent of the pressure, thus we show pressure robustness.

Theorem 26 (Pressure robustness)**.** *Assuming homogeneous boundary conditions (6.8), let $u \in H^1(\Omega, \mathbb{R}^d) \cap H^m(\mathcal{T}_h, \mathbb{R}^d)$, $\sigma \in H^1(\Omega, \mathbb{D}) \cap H^{m-1}(\mathcal{T}_h, \mathbb{D})$ and $p \in Q \cap H^{m-1}(\mathcal{T}_h, \mathbb{R})$ be the exact solution of the mixed Stokes problem (4.6). Further let u_h, σ_h and p_h be the solution of MCS method (6.6). For $s = \min(m-1, k+1)$ there holds*

$$\frac{1}{\nu}||\sigma - \sigma_h||_{\Sigma_h} + \|u_h - I_{V_h} u\|_{V_h} \lesssim h^s \frac{1}{\nu} \|\sigma\|_{H^s(\mathcal{T}_h)},$$

and

$$\frac{1}{\nu}\|\sigma - \sigma_h\|_{\mathrm{cd},h} + \|u - u\|_{H(\mathrm{div},\Omega)} \lesssim h^s \frac{1}{\nu} \|\sigma\|_{H^s(\mathcal{T}_h)}.$$

Proof. The proof follows the same steps as the proof of theorem 23 and theorem 24. Again, one first splits the error using a triangle inequality into an interpolation error and a discrete measure of the error. The first part can be bounded with the corresponding interpolation operators. Next, note that due to lemma 25 we can similarly as in corollary 5 derive inf-sup stability of $B(u_h, \sigma_h, 0; v_h, \tau_h, 0)$ with respect to the norm $\|(\cdot,\cdot,0)\|_*$ on the product space $V_h^0 \times \Sigma_H \times \{0\}$. Together with the consistency theorem 19, this yields

$$\begin{aligned}\frac{1}{\sqrt{\nu}}\|(I_{V_h}u - u_h, I_{\Sigma_h}\sigma - \sigma_h, 0)\|_* &\lesssim \sup_{\substack{v_h \in V_h^0 \\ \tau_h \in \Sigma_h}} \frac{B(I_{V_h}u - u_h, I_{\Sigma_h}\sigma - \sigma_h, 0; v_h, \tau_h, 0)}{\sqrt{\nu}\|(v_h, \tau_h, 0)\|_*} \\ &\lesssim \sup_{\substack{v_h \in V_h^0 \\ \tau_h \in \Sigma_h}} \frac{B(I_{V_h}u - u, I_{\Sigma_h}\sigma - \sigma, 0; v_h, \tau_h, 0)}{\sqrt{\nu}\|(v_h, \tau_h, 0)\|_*}.\end{aligned}$$

The rest follows with the same arguments as before. For the estimate in natural norms one also show inf-sup stability of $B(u_h, \sigma_h, 0; v_h, \tau_h, 0)$ with respect to the norm $\|(\cdot,\cdot,0)\|_{**}$ using similar steps as in theorem 20. □

6.3.4 An exactly divergence free post processing

We conclude this section with the definition of a local post processing. As we have discussed in section 6.3.2, the velocity error $\|u - u_h\|_{V_h}$ does not converge with optimal order. However, in the following we show that it is possible to define a new velocity field u_h^* with an enhanced accuracy. Note that u_h^* is given by solving local element-wise minimization problems, thus no global problem has to be solved. The goal is to obtain an improved convergence order $\mathcal{O}(h^{k+1})$ of the velocity error measured in a discrete H^1-norm. Our post processing is motivated by the ideas of the work [111, 112], thus we use the accurate solution σ_h and the optimal order of $\|u_h - I_{V_h} u\|$ for the definition of a velocity lifting. Note that such post processing schemes are particularly known for hybridized DG methods, see for example [9, 33, 26, 90, 28, 34, 98].

In this thesis a crucial ingredient is that the solution of the post processing scheme satisfies the property $\operatorname{div}(u_h^*) = 0$ exactly. To this end we use a reconstruction operator that was introduced in the works [76, 77]. Let $V_h^* := \mathcal{BDM}^{k+1}(\mathcal{T}_h)$ be the standard $\mathcal{BDM}$-space as defined by equation (5.9). Then we define the *relaxed* Brezzi-Douglas-Marini space of order $k+1$ as

$$V_h^{*,-} := \{u \in \mathbb{P}^{k+1}(\mathcal{T}_h, \mathbb{R}^d) : \Pi_F^k [\![u_n]\!] = 0 \text{ on all } F \in \mathcal{F}_h\}. \tag{6.37}$$

Functions in the space $V_h^{*,-}$ are "almost normal continuous". Only the highest order modes can be discontinuous. The $\mathcal{BDM}$-space and its relaxed counterpart are connected by a reconstruction operator with the following properties.

Lemma 33. *There exists an operator $\mathcal{R} : V_h^{*,-} \to V_h^*$ such that*

$$\|\mathcal{R} v_h\|_{1,h} \lesssim \|v_h\|_{1,h},$$

and $\mathcal{R} v_h^* = v_h^*$ for all $v_h^* \in V_h^*$. Further, if the element wise property $\operatorname{div}(v_h|_T) = 0$ is fulfilled for all $T \in \mathcal{T}_h$, then $\operatorname{div}(\mathcal{R} v_h) = 0$ globally.

Proof. Follows by lemma 3.3 and lemma 4.8 in [76]. Further, one simply sets the hybrid variable on the skeleton as the mean value of the two neighbouring elements to replace the corresponding HDG-norm with the DG-norm. The projection property follows from the definition of $\mathcal{R}$ in [76]. □

Remark 5. *A simple choice of $\mathcal{R}$ is given by a (DG) generalization of the classical $\mathcal{BDM}^k$-interpolator as it has been used in [61]. As this interpolation operator might produce a big computational overhead, a different version of $\mathcal{R}$ was introduced in [76] that is based on an averaging of coefficients. This is possible due to a special basis for $\mathcal{BDM}^{k+1}(\mathcal{T}_h)$, whose normal components form a hierarchical L^2-orthogonal basis on faces.*

On $V_h^{*,-}$ we now define the following minimization problem:

$$u_h^{*,-} := \underset{\substack{v_h^{*,-} \in V_h^{*,-} \\ I_{V_h}(v_h^{*,-} - u_h) = 0}}{\operatorname{argmin}} \|\nu \nabla (v_h^{*,-}) - \sigma_h\|_T^2. \tag{6.38}$$

Note that there exists a solution of this minimization problem as for example u_h fulfills the constraints (thus the admissible set is not empty) and the functional is convex. The final solution of the post processing step is now given by $u_h^* := \mathcal{R}(u_h^{*,-})$, and there holds the following pressure robust error estimate.

Theorem 27. *Assuming homogeneous boundary conditions (6.8), let $u \in H^1(\Omega, \mathbb{R}^d) \cap H^m(\mathcal{T}_h, \mathbb{R}^d)$ and $\sigma \in H^1(\Omega, \mathbb{D}) \cap H^{m-1}(\mathcal{T}_h, \mathbb{D})$ be the exact solution of the mixed Stokes problem (4.6). Further let u_h be the solution of the MCS method (6.6) and let u_h^* be the post processed solution defined as above. There holds $u_h^* \in V_h^*$ and $\operatorname{div}(u_h^*) = 0$. Further, for $s = \min(m-1, k+1)$ there holds*

$$\|u - u_h^*\|_{1,h} \lesssim h^s \frac{1}{\nu} \|\sigma\|_{H^s(\mathcal{T}_h)}.$$

If we further assume that the solution of the dual problem (6.6) fulfills the regularity assumption 6.31, we have

$$\|u - u_h^*\|_{L^2(\Omega)} \lesssim h^{s+1}\frac{1}{\nu}\|\sigma\|_{H^s(\mathcal{T}_h)}.$$

Proof. As $I_{V_h}(u_h^{*,-} - u_h) = 0$, we have by the definition of I_{V_h} (based on the degrees of freedom given in lemma 13), that for all $q_h \in \mathbb{P}^k(T, \mathbb{R})$ and $T \in \mathcal{T}_h$

$$\int_T \operatorname{div}(u_h^{*,-}) q_h \,\mathrm{d}x = -\int_T u_h^{*,-} \cdot \nabla q_h \,\mathrm{d}x + \int_{\partial T} u_h^{*,-} \cdot n q_h \,\mathrm{d}s =$$
$$= -\int_T u_h \cdot \nabla q_h \,\mathrm{d}x + \int_{\partial T} u_h \cdot n q_h \,\mathrm{d}s = \int_T \operatorname{div}(u_h) q_h \,\mathrm{d}x = 0.$$

Here we used that $\operatorname{div}(u_h) = 0$. Thus, the first two statements follow from lemma 33. Now let $I_{V_h^*}$ be the standard $\mathcal{BDM}^{k+1}$-interpolator, then we have by the continuity of $\mathcal{R}$ and the identity $\mathcal{R} I_{V_h^*} u = I_{V_h^*} u$ that

$$\|u - u_h^*\|_{1,h} \leq \|u - I_{V_h^*} u\|_{1,h} + \|\mathcal{R}(I_{V_h^*} u - u_h^{*,-})\|_{1,h} \leq \|u - I_{V_h^*} u\|_{1,h} + \|u - u_h^{*,-}\|_{1,h}.$$

Thereby, the approximation properties of the interpolation operator $I_{V_h^*}$, see lemma 29, show that the first term already converges with the proper order. A triangle inequality (where we add and subtract different functions in the element and facet terms) yields

$$\|u - u_h^{*,-}\|_{1,h}^2 \leq \frac{1}{\nu^2}\left(\sum_{T \in \mathcal{T}_h} \|\nu \nabla u - \sigma_h\|_T^2 + \sum_{T \in \mathcal{T}_h} \|\sigma_h - \nu \nabla u_h^{*,-}\|_T^2\right) \tag{6.39}$$
$$+ \sum_{F \in \mathcal{F}_h} \frac{1}{h}\|[\![(u - I_{V_h^*} u)_t]\!]\|_F^2 + \sum_{F \in \mathcal{F}_h} \frac{1}{h}\|[\![(I_{V_h^*} u - u_h^{*,-})_t]\!]\|_F^2.$$

In the following we bound each element term and facet term separately. We start with the element terms and define $w_h := I_{V_h^*} u + u_h - I_{V_h} u \in V_h^*$. Due to the properties of the interpolation operators, we have on each element $T \in \mathcal{T}_h$ that $\operatorname{div}(w_h|_T) = 0$, and

$$\Pi_F^k (w_h - u_h) \cdot n = \left(\Pi_F^k I_{V_h^*} u + \Pi_F^k u_h - \Pi_F^k I_{V_h} u - \Pi_F^k u_h\right) \cdot n$$
$$= \left(\Pi_F^k I_{V_h^*} u - \Pi_F^k I_{V_h} u\right) \cdot n = \left(\Pi_F^k u - \Pi_F^k u\right) \cdot n = 0.$$

This shows that w_h is an admissible function for the minimization problem (6.38), and so we have for the solution $u_h^{*,-}$ by a triangle inequality,

$$\|\sigma_h - \nu \nabla u_h^{*,-}\|_T \leq \|\sigma_h - \nu \nabla w_h\|_T^2 \leq \|\sigma_h - \nu \nabla I_{V_h^*} u\|_T^2 + \|\nu \nabla u_h - \nu \nabla I_{V_h} u\|_T^2$$
$$\leq \|\sigma_h - \sigma\|_T^2 + \|\sigma - \nu \nabla I_{V_h^*} u\|_T^2 + \|\nu \nabla u_h - \nu \nabla I_{V_h} u\|_T^2.$$

As $\sigma = \nu \nabla u$, this yields for the element terms of (6.39) the estimate

$$\begin{aligned}\frac{1}{\nu^2}\Big(\sum_{T\in\mathcal{T}_h} \|\nu\nabla u - \sigma_h\|_T^2 + \sum_{T\in\mathcal{T}_h} \|\sigma_h - \nu\nabla u_h^{*,-}\|_T^2\Big) \\ &\leq \frac{1}{\nu^2}\left(\sum_{T\in\mathcal{T}_h} \|\sigma - \sigma_h\|_T^2 + \|\nu\nabla u - \nu\nabla I_{V_h^*} u\|_T^2 + \|\nu\nabla u_h - \nu\nabla I_{V_h} u\|_T^2 \right) \\ &\leq \frac{1}{\nu^2}\|\sigma - \sigma_h\|_{\Sigma_h}^2 + \|u - I_{V_h^*} u\|_{1,h}^2 + \|u_h - I_{V_h} u\|_{1,h}^2.\end{aligned}$$

We continue with the facet terms of (6.39). The first sum can be bounded by $\|u - I_{V_h^*} u\|_{1,h}$ and converges with the proper order. For the second sum we have

$$\begin{aligned}\sum_{F\in\mathcal{F}_h} \frac{1}{h}\|[\![(I_{V_h^*} u - u_h^{*,-})_t]\!]\|_F^2 \\ &\leq \sum_{F\in\mathcal{F}_h} \frac{1}{h}\|[\![(I_{V_h}(I_{V_h^*} u - u_h^{*,-}))_t]\!]\|_F^2 + \sum_{F\in\mathcal{F}_h} \frac{1}{h}\|[\![((\mathrm{Id} - I_{V_h})(I_{V_h^*} u - u_h^{*,-}))_t]\!]\|_F^2 \\ &\leq \sum_{F\in\mathcal{F}_h} \frac{1}{h}(\|[\![(I_{V_h} u - u_h)_t]\!]\|_F^2 + \sum_{F\in\mathcal{F}_h} \frac{1}{h}\|[\![((\mathrm{Id} - I_{V_h})(I_{V_h^*} u - u_h^{*,-}))_t]\!]\|_F^2.\end{aligned}$$

Here we used that $I_{V_h} I_{V_h^*} u = I_{V_h} u$ (as $V_h \subset V_h^*$) and $I_{V_h} u_h^{*,-} = I_{V_h} u_h = u_h$ (as $u_h^{*,-}$ is the solution of (6.38)) in the last step. Thereby, theorem 26 shows that the first sum again has the proper convergence rate. Next, as I_{V_h} preserves element wise constant functions, we have for each $F \in \mathcal{F}_h$

$$\frac{1}{h}\|[\![((\mathrm{Id} - I_{V_h})(I_{V_h^*} u - u_h^{*,-}))_t]\!]\|_F^2 = \frac{1}{h}\|[\![((\mathrm{Id} - I_{V_h})(\mathrm{Id} - \Pi_T^0)(I_{V_h^*} u - u_h^{*,-}))_t]\!]\|_F^2.$$

The jump can be estimated by the contributions from each side, thus a trace inequality (with a scaling argument) yields for $a := (\mathrm{Id} - I_{V_h})(\mathrm{Id} - \Pi_T^0)(I_{V_h^*} u - u_h^{*,-})$ the estimate

$$\frac{1}{h}\|[\![(a)_t]\!]\|_F^2 \leq \sum_{\substack{T\in\mathcal{T}_h\\ F\cap\partial T\neq\emptyset}} \frac{1}{h}\|(a)_t\|_{\partial T}^2 \leq \sum_{\substack{T\in\mathcal{T}_h\\ F\cap\partial T\neq\emptyset}} \|\nabla a\|_T^2 + \frac{1}{h^2}\|a\|_T^2.$$

As I_{V_h} is continuous, we have

$$\begin{aligned}\|\nabla a\|_T^2 + \frac{1}{h^2}\|a\|_T^2 &\leq \|\nabla(\mathrm{Id} - \Pi_T^0)(I_{V_h^*} u - u_h^{*,-})\|_T^2 + \frac{1}{h^2}\|(\mathrm{Id} - \Pi_T^0)(I_{V_h^*} u - u_h^{*,-})\|_T^2 \\ &\leq \|\nabla(I_{V_h^*} u - u_h^{*,-})\|_T^2,\end{aligned}$$

where we used a Poincaré inequality for the L^2-norm on T. As

$$\begin{aligned}\|\nabla(I_{V_h^*} u - u_h^{*,-})\|_T^2 &\leq \|\nabla I_{V_h^*} u - \sigma_h\|_T^2 + \|\sigma_h - u_h^{*,-}\|_T^2 \\ &\leq \|\nabla I_{V_h^*} u - \sigma\|_T^2 + \|\sigma - \sigma_h\|_T^2 + \|\sigma_h - u_h^{*,-}\|_T^2,\end{aligned}$$

we can bound the facet terms with the same estimates as for the element contributions. Finally, we conclude the proof by the approximation properties of $I_{V_h^*}$ and theorem 26.

For the estimate of the L^2-norm error first note that we have, similarly as before,

$$\begin{aligned}\|u - u_h^*\|_{L^2(\Omega)} &\leq \|u - I_{V_h^*}u\|_{L^2(\Omega)} + \|\mathcal{R}(I_{V_h^*}u - u_h^{*,-})\|_{L^2(\Omega)} \\ &\leq \|u - I_{V_h^*}u\|_{L^2(\Omega)} + \|I_{V_h^*}u - u_h^{*,-}\|_{L^2(\Omega)}.\end{aligned}$$

The first term already has the proper convergence order. Next, we have

$$\begin{aligned}\|I_{V_h^*}u - u_h^{*,-}\|_{L^2(\Omega)} &\lesssim \|I_{V_h}(I_{V_h^*}u - u_h^{*,-})\|_{L^2(\Omega)} + \|(\mathrm{Id} - I_{V_h})(I_{V_h^*}u - u_h^{*,-})\|_{L^2(\Omega)} \\ &= \|I_{V_h}u - u_h\|_{L^2(\Omega)} + \|(\mathrm{Id} - I_{V_h})(I_{V_h^*}u - u_h^{*,-})\|_{L^2(\Omega)}.\end{aligned}$$

For the second term we get by the properties of I_{V_h} the local estimate

$$\begin{aligned}&\|(\mathrm{Id} - I_{V_h})(I_{V_h^*}u - u_h^{*,-})\|_{L^2(\Omega)} \\ &\qquad \lesssim \sum_T h\|\nabla I_{V_h^*}u - \nabla u_h^{*,-}\|_T \\ &\qquad \lesssim \sum_T h\Big(\|\nabla I_{V_h^*}u - \frac{1}{\nu}\sigma\|_T + \frac{1}{\nu}\|\sigma - \sigma_h\|_T + \|\frac{1}{\nu}\sigma_h - \nabla u_h^{*,-}\|_T\Big).\end{aligned}$$

Now, using the same steps as above to bound $\|\sigma_h - \nabla u_h^{*,-}\|_T$, we get

$$\|(\mathrm{Id} - I_{V_h})(I_{V_h^*}u - u_h^{*,-})\|_{L^2(\Omega)} \lesssim h^{s+1}(\frac{1}{\nu}\|\sigma\|_{H^s(\mathcal{T}_h)} + \|u\|_{H^{s+1}(\mathcal{T}_h)}).$$

We conclude the proof by using theorem 25 for the term $\|I_{V_h}u - u_h\|_{L^2(\Omega)}$. □

We want to make a short comment on the implementation: In order to solve the minimization problem (6.38) we solve local problems defined as follows. Let

$$V_h^*(T) = \mathbb{P}^{k+1}(T, \mathbb{R}^d), \quad M_h^*(T) := \mathbb{P}^{k-1}(T, \mathbb{R}^2), \quad \Lambda_h^*(T) := \mathbb{P}^k(\mathcal{F}_h(T), \mathbb{R}),$$

then we solve on each element $T \in \mathcal{T}_h$ the problem: Find $u_T^*, p_h^*, \lambda_h^* \in V_h^*(T) \times M_h^*(T) \times \Lambda_h^*$ such that

$$\begin{aligned}\int_T \nu\nabla(u_T^*) : \nabla(v_h^*)\,\mathrm{d}x + \int_T p_h^* \cdot v_h^*\,\mathrm{d}x + \int_{\partial T} \lambda_h^*(v_h^*)_n\,\mathrm{d}s &= \int_T \sigma_h : \nabla(v_h^*)\,\mathrm{d}x, \\ \int_T u_T^* \cdot q_h^*\,\mathrm{d}x &= \int_T u_h \cdot q_h^*\,\mathrm{d}x, \\ \int_{\partial T} (u_T^*)_n \mu_h^*\,\mathrm{d}s &= \int_{\partial T} (u_h)_n \mu_h^*\,\mathrm{d}s,\end{aligned}$$

holds true for all $v_h^* \in V_h^*(T)$, $q_h^* \in M_h^*(T)$ and $\mu_h^* \in \Lambda_h^*(T)$. Then, we define the global function $u_h^{*,-}$ such that its restriction is given by the local solutions u_T^*, thus $u_h^{*,-}|_T := u_T^*$ for all $T \in \mathcal{T}_h$. Note that the constraints are given by the functionals of the Raviart-Thomas interpolator I_{V_h}. By the definition of the jump it is then easy to see that $\Pi^k[\![(u_h^{*,-})_n]\!] = 0$, hence $u_h^{*,-} \in V_h^{*,-}$. A simple integration by parts argument and the local element-wise orthogonality on $\mathbb{P}^{k-1}(T, \mathbb{R}^2)$ further shows $\mathrm{div}(u_h^{*,-}|_T) = 0$, leading to a point-wise divergence free function $\mathcal{R}(u_h^{*,-})$.

7 The MCS method with weakly imposed symmetry

This chapter is dedicated to defining a discrete formulation for the symmetric Stokes formulation given by equation (4.41). As we have seen in section 4.4, there exists a weak formulation of this set of equations that is based on the new function space $H(\operatorname{curl}\operatorname{div},\Omega)$. In particular, we used the subspace of trace free, symmetric and matrix-valued functions denoted by Σ^{sym}, and showed stability with respect to the corresponding norms.

The discrete method that we present in this chapter is based on similar findings as in the previous chapter, thus our goal is to use again an $H(\operatorname{div})$-conforming discrete velocity space and the new developed stress space $\Xi^k(\mathcal{T}_h)$. The definition of the stress space (see section 5.3) is based on the construction of low order constant basis functions, designed to study normal-tangential traces, and the definition of the covariant Piola mapping, given by equation (5.22). Although the mapping $\mathcal{M}$ is designed to preserve normal-tangential continuity, it does not retain symmetry. Thus, assuming a given matrix $\hat{\sigma}$ is symmetric, the mapped function $\mathcal{M}(\hat{\sigma})$ might not be symmetric. By these observations we realize that the finite element space $\Xi^k(\mathcal{T}_h)$ is not well suited to discretize the symmetric stress space Σ^{sym}. This motivates us to incorporate symmetry of a discrete stress variable $\sigma_h \in \Xi^k(\mathcal{T}_h)$ only in a weak sense.

Imposing symmetry in a weak sense is a well known technique and is especially known for the discretization of the elasticity problem, see for example [5, 6, 12, 48]. In particular, we want to mention the work of Stenberg [111], as the analysis is based on similar norms that we aim to use in this thesis. To provide solvability, Stenberg enriched the stress space by curls of local element bubbles. By construction, these enrichment functions lie in the kernel of the divergence operator and are only "seen" by the weak-symmetry constraint allowing to prove discrete inf-sup stability. For the definition of the MCS method with weakly imposed symmetry we also use a local element-wise enrichment of the stress space as it is discussed in the next section.

Before we introduce the method we define a new formulation of the equations (4.41). To this end we introduce the vorticity function by

$$\omega(u) := \frac{1}{2}\begin{pmatrix} 0 & -\operatorname{curl}(u) \\ \operatorname{curl}(u) & 0 \end{pmatrix} \text{ for } d = 2, \quad \text{and}$$

$$\omega(u) := \frac{1}{2}\begin{pmatrix} 0 & -(\operatorname{curl}(u))_3 & (\operatorname{curl}(u))_2 \\ (\operatorname{curl}(u))_3 & 0 & -(\operatorname{curl}(u))_1 \\ -(\operatorname{curl}(u))_2 & (\operatorname{curl}(u))_1 & 0 \end{pmatrix} \text{ for } d = 3,$$

and derive the following identity

$$\nabla u = \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{T}}) + \frac{1}{2}(\nabla u - (\nabla u)^{\mathrm{T}}) = \varepsilon(u) + \omega(u).$$

As $\operatorname{tr}(\omega(u)) = 0$ and $\operatorname{tr}(\nabla u) = 0$, this leads to the relation $\operatorname{dev}(\sigma) := \nu\nabla u - \nu\omega(u)$. Note that in the continuous setting this relation is equivalent to (4.41a). Finally, due to $\operatorname{skw}(\sigma) = \operatorname{skw}(\varepsilon(u)) = 0$, we get the following set of equations

$$
\begin{aligned}
\frac{1}{\nu}\operatorname{dev}(\sigma) - \nabla u + \omega(u) &= 0 && \text{in} \quad \Omega, && (7.1a)\\
\operatorname{div}(\sigma) - \nabla p &= -f && \text{in} \quad \Omega, && (7.1b)\\
\operatorname{div}(u) &= 0 && \text{in} \quad \Omega, && (7.1c)\\
\operatorname{skw}(\sigma) &= 0 && \text{in} \quad \Omega, && (7.1d)\\
u_n &= g_{D,n} && \text{on} \quad \Gamma_{D,n}, && (7.1e)\\
u_t &= g_{D,t} && \text{on} \quad \Gamma_{D,t}, && (7.1f)\\
-\sigma_{nn} + p &= g_{N,n} && \text{on} \quad \Gamma_{N,n}, && (7.1g)\\
-\sigma_{nt} &= g_{N,t} && \text{on} \quad \Gamma_{N,t}. && (7.1h)
\end{aligned}
$$

7.1 An enrichment for the stress space - the symmetric bubble matrix

As a motivation of the local enrichment, we first discuss the techniques Stenberg introduced in [111]. To this end we define the local bubble function

$$
b_T := \prod_{i \in I_{\mathcal{V}_h(T)}} \lambda_i \quad \text{for all } T \in \mathcal{T}_h. \tag{7.2}
$$

Then, the matrix-valued enrichment space of Stenberg was given by

$$
\{\sigma \in \mathbb{P}^{k+(d-1)}(\mathcal{T}_h, \mathbb{R}^{d\times d}) : (\sigma_{i1}, \dots, \sigma_{id}) = \operatorname{curl}(b_T q) \text{ for } i = 0, \dots, d\},
$$

with $q \in \mathbb{P}^{k-1}(\mathcal{T}_h, \mathbb{R}^{d(d-1)/2})$. As $\operatorname{div}(\operatorname{curl}(b_T q)) = 0$, the additional basis functions lie in the kernel of the momentum equation, thus are only seen by the constraint imposing weak symmetry. Note, however, that the polynomial orders of the resulting enrichment functions are $k+1$ and $k+2$ in two and three space dimensions, respectively, thus with respect to computational costs, the three-dimensional case is less efficient.

The enrichment we use is motivated by the additional functions that were first introduced by Cockburn, Gopalakrishnan and Guzmán in [27]. Therein, the authors tackled the question if it is possible to retain the good convergence properties of Stenbergs method in [111], but to reduce the number of additional basis functions and the polynomial order in three dimensions. Introducing the so called *symmetric bubble matrix* the authors of [27] were able to define certain curl-bubbles and achieved the desired properties.

It turns out that a slight modification of the bubbles defined in [27] can also be used to enrich the discrete stress space $\Xi^k(\mathcal{T}_h)$. This is astonishing as these additional functions need to fulfill the following properties: First, they need to lie in the kernel of the *distributional divergence* given by equation (5.12) of theorem 13, and second, we use a different mapping for the discrete stress space as in the original work [27]. Using these bubbles it is possible to prove discrete inf-sup stability with the classical approach based on scaling arguments with respect to a given reference element. This is done with the same norms used in Stenbergs work [111] and is the key result of section 7.3.

Following the lines of [27] we define for $d = 3$ the *symmetric bubble matrix* by

$$B := \sum_{i \in I_{\mathcal{V}_h(T)}} (\lambda_{i-3}\lambda_{i-2}\lambda_{i-1}) \nabla\lambda_i \otimes \nabla\lambda_i \in \mathbb{P}^3(T, \mathbb{R}^{d\times d}), \tag{7.3}$$

where the indices on the barycentric coordinate functions are always calculated modulo 4. In two dimensions the bubble function is simply given by $B := b_T$, see equation (7.2). Next, we denote by $\mathbb{P}^k_{\text{skw},\perp}(T, \mathbb{R}^{d\times d})$ the L^2-orthogonal complement of the set $\mathbb{P}^{k-1}_{\text{skw}}(T, \mathbb{R}^{d\times d})$ in $\mathbb{P}^k_{\text{skw}}(T, \mathbb{R}^{d\times d})$, thus

$$\mathbb{P}^k_{\text{skw},\perp}(T, \mathbb{R}^{d\times d}) := \{r_h \in \mathbb{P}^k_{\text{skw}}(T, \mathbb{R}^{d\times d}) : (r_h, s_h)_T = 0 \text{ for all } s_h \in \mathbb{P}^{k-1}_{\text{skw}}(T, \mathbb{R}^{d\times d})\}.$$

This leads to the enrichment space

$$\delta\Sigma_h := \{\text{dev}(\text{curl}(\text{curl}(r_h)B)) : r_h \in \prod_{T\in\mathcal{T}_h} \mathbb{P}^k_{\text{skw},\perp}(T, \mathbb{R}^{d\times d})\}.$$

In contrast to the original work [27], we defined our enrichment bubbles including the deviator, which is motivated by the equivalence (7.1a). Further, note that in two dimensions the inner curl is applied on each row of r_h, and the outer curl is applied on each scalar component of the vector $\text{curl}(r_h)B$. Thus, if we have for example

$$r_h = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0, \end{pmatrix}, \quad \text{this gives} \quad \text{curl}(\text{curl}(r_h)B) = \begin{pmatrix} -\frac{\partial}{\partial x_2}(\frac{\partial}{\partial x_1}\alpha B) & \frac{\partial}{\partial x_1}(\frac{\partial}{\partial x_1}\alpha B) \\ -\frac{\partial}{\partial x_2}(\frac{\partial}{\partial x_2}\alpha B) & \frac{\partial}{\partial x_1}(\frac{\partial}{\partial x_2}\alpha B) \end{pmatrix}.$$

The definition of the symmetric bubble matrix motivates to define a transformation by

$$\mathcal{CC}(\hat{B}) := F_T^{-\mathrm{T}} \hat{B} \circ \phi_T^{-1} F_T^{-1}, \tag{7.4}$$

which reads as a covariant transformation (see section 5.3.2) from the left and from the right. The definition is motivated by the following findings. First, note that the barycentric coordinate functions λ_i are mapped by the standard pullback, thus the gradient can be written in terms of the gradient of a barycentric coordinate function on the reference element:

$$\nabla\lambda_l = \hat{\nabla}\hat{\lambda}_l F_T^{-1}.$$

Hence, by the definition of the symmetric bubble matrix (7.3) we have

$$B = \sum_{i \in I_{\mathcal{V}_h(T)}} (\lambda_{i-3}\lambda_{i-2}\lambda_{i-1}) \nabla\lambda_i \otimes \nabla\lambda_i = \sum_{i \in I_{\mathcal{V}_h(T)}} (\hat{\lambda}_{i-3}\hat{\lambda}_{i-2}\hat{\lambda}_{i-1}) F_T^{-\mathrm{T}} \hat{\nabla}\hat{\lambda}_i \otimes \hat{\nabla}\hat{\lambda}_i F_T^{-1} = \mathcal{CC}(\hat{B}).$$

For two dimensions the bubble B is simply mapped by a pullback. Using the above mapping we can preserve the proper trace of the matrix bubble as stated in the following lemma.

Lemma 34. *Let $\hat{B}$ be the symmetric bubble matrix on the reference element and let q be an arbitrary matrix. The products $q\hat{B}$ and $\hat{B}q$ have vanishing tangential trace on $\partial\hat{T}$. Similarly, for any $T \in \mathcal{T}_h$ let $B = \mathcal{CC}(\hat{B})$. Then the products qB and Bq have vanishing tangential trace on the boundary ∂T. By these findings we further observe that the function* $\text{curl}(qB)$ *has a vanishing normal trace on ∂T.*

Proof. For the result on the reference element we refer to lemma 2.3 in [27]. The proof on ∂T follows the same lines using $\nabla \lambda_l = \nabla \hat{\lambda}_l F_T^{-1}$. Now note that on the surface ∂T we have the identity $\operatorname{curl}(qB) \cdot n = \operatorname{curl}_t((qB)_t)$, where curl_t is the surface curl (see e.g. [11]). By this we conclude that $(\operatorname{curl}(qB))_n = 0$. □

Corollary 6. *Let $\sigma \in \delta\Sigma_h$, there holds*

$$[\![(\sigma)_{nt}]\!] = 0 \quad \text{on all} \quad F \in \mathcal{F}_h.$$

Proof. This is a direct consequence of lemma 34 and equation (4.2). □

7.2 A discrete variational formulation

Using the enrichment space $\delta\Sigma_h$ we can now derive a discrete variational formulation of equation (7.1). To this end let $k \geq 1$, then we define the stress space by

$$\Sigma_h^\dagger := \Sigma_h \oplus \delta\Sigma_h,$$

and similarly $\Sigma_{h,N}^\dagger := \Sigma_{h,N} \oplus \delta\Sigma_h$. Note that due to corollary 6, all functions in $\Sigma_h^\dagger$ are normal-tangential continuous and are polynomials up to order $k+1$, thus we have

$$\Sigma_h^\dagger \subset \Xi_{\mathbb{D}}^{k+1}. \tag{7.5}$$

Whereas the pressure in equation (7.1) can be interpreted as a Lagrange multiplier for the incompressibility constraint $\operatorname{div}(u) = 0$, the function $\gamma := \omega(u)$ is responsible for the symmetry constraint (7.1d). In order to discretize γ we introduce the space

$$W_h := \mathbb{P}^k_{\text{skw}}(\mathcal{T}_h, \mathbb{R}^{d \times d}). \tag{7.6}$$

Functions in W_h can be mapped from the reference element such that skew symmetry is retained, thus for a given $\gamma_h \in W_h$ there exists a function $\hat{\gamma}_h \in \mathbb{P}^k_{\text{skw}}(\hat{T}, \mathbb{R}^{d \times d})$ such that

$$\gamma_h = F_T^{-\mathrm{T}} \hat{\gamma}_h \circ \phi_T^{-1} F_T^{-1}. \tag{7.7}$$

Again, we use the Raviart-Thomas space of order k as the velocity space, $V_h = \mathcal{RT}^k(\mathcal{T}_h)$, and define the product space $U_h := V_h \times W_h$. Further, we use the same pressure space Q_h as it was chosen for the standard MCS method, see equation (6.3).

We follow the same steps as in section 6.1 to derive a new discrete variational formulation. The discrete velocity and pressure space are chosen conformingly with respect to V and Q, thus we can use the bilinear forms defined in section 4.3, see equations (4.27) and (4.28). Motivated by identity (5.12) of theorem 13 we further realize that we can use the bilinear form b_{2h} defined by equation (6.4) to use it as an approximation of the bilinear form b_2 as it appears in the continuous setting (4.43). It remains to define a bilinear form for the symmetry constraint. As the discrete velocity and the discrete vorticity can both be interpreted as a dual variable, thus they act as a Lagrange multiplier to fulfill certain constraints on the discrete stress, we define the bilinear form

$$b_{2h}^\varepsilon : \{\tau \in H^1(\mathcal{T}_h, \mathbb{R}^{d \times d}) : [\![\tau_{nt}]\!] = 0\} \times \Big(\{v \in H^1(\mathcal{T}_h, \mathbb{R}^d) : [\![v_n]\!] = 0\} \times L^2(\Omega, \mathbb{R}^{d \times d})\Big) \to \mathbb{R}$$

by

$$b_{2h}^{\varepsilon}(\tau,(v,\eta)) := \sum_{T\in\mathcal{T}_h}\int_T \operatorname{div}(\tau)\cdot v\,\mathrm{d}x + \sum_{T\in\mathcal{T}_h}\int_T \tau : \eta\,\mathrm{d}x - \sum_{F\in\mathcal{F}_h}\int_F [\![\tau_{nn}]\!] v_n\,\mathrm{d}s\,. \tag{7.8}$$

Thus, b_{2h}^{ε} is a combination of b_{2h} and the symmetry constraint. Using these bilinear forms the discrete MCS method with weakly imposed symmetry now finds: $\sigma_h, u_h, \gamma_h, p_h \in \Sigma_{h,N}^{\dagger} \times V_{h,D} \times W_h \times Q_h$ such that

$$\begin{cases} a(\sigma_h,\tau_h) + b_{2h}^{\varepsilon}(\tau_h,(u_h,\gamma_h)) = (g_{D,t},(\tau_h)_{nt})_{\Gamma_{D,t}} & \text{for all } \tau_h \in \Sigma_h^{\dagger}, \\ b_{2h}^{\varepsilon}(\sigma_h,(v_h,\eta_h)) + b_1(v_h,p_h) = -(f,v_h)_\Omega + (g_{N,n},v_n)_{\Gamma_{N,n}} & \text{for all } (v_h,\eta_h) \in V_h \times W_h, \\ b_1(u_h,q_h) = 0 & \text{for all } q_h \in Q_h. \end{cases} \tag{7.9}$$

7.3 Discrete inf-sup stability

As in section 6.2 we only consider the case of homogeneous boundary conditions (6.8) for the stability analysis. The analysis we discuss in this section is based on the same norms used in the work [111]. To this end let $v_h \in V_h$ and $\eta_h \in W_h$, then we define

$$\begin{aligned} \|v_h\|_{1,h,sym}^2 &:= \sum_{T\in\mathcal{T}_h} \|\varepsilon(v_h)\|_T^2 + \sum_{F\in\mathcal{F}_h} \frac{1}{h}\|[\![(v_h)_t]\!]\|_F^2, \\ \|(v_h,\eta_h)\|_{U_h}^2 &:= \|v_h\|_{1,h,sym}^2 + \sum_{T\in\mathcal{T}_h} \|\omega(v_h)-\eta_h\|_T^2. \end{aligned}$$

Whereas the first norm reads as a symmetric H^1-like DG norm, the second norm further measures the differences between the discrete vorticity of a velocity v_h and a given field η_h. In lemma 40 below it is shown how the product-space norm $\|(\cdot,\cdot)\|_{U_h}$ is related to norms defined on the parts V_h and W_h separately. Beside that we choose the norms

$$\|\tau_h\|_{\Sigma_h^{\dagger}}^2 := \|\tau_h\|_{\Sigma_h}^2 = \|\tau_h\|_{L^2(\Omega)}^2 \quad \forall \tau_h \in \Sigma_h^{\dagger}, \qquad \text{and} \qquad ||q_h||_{Q_h}^2 := \|q_h\|_{L^2(\Omega)}^2 \quad \forall q_h \in Q_h.$$

Before we prove discrete inf-sup stability of the system (7.9) we provide some norm equivalences in the next section.

7.3.1 Norm equivalences

The first equivalence is similar to the result of lemma 20. Note that the projection on the facets is of one order higher.

Lemma 35. *For all $v_h \in V_h$,*

$$\|(v_h,\eta_h)\|_{U_h}^2 \sim \sum_{T\in\mathcal{T}_h} \left(\|\varepsilon(v_h)\|_T^2 + \|\omega(v_h)-\eta_h\|_T^2\right) + \sum_{F\in\mathcal{F}_h} \frac{1}{h}\left\|\Pi_F^1[\![(v_h)_t]\!]\right\|_F^2.$$

Proof. One side of the equivalence is obvious by the continuity of the Π_F^1. For the other direction first note that

$$\frac{1}{h}\|[\![(v_h)_t]\!]\|_F^2 \le \frac{2}{h}\|\Pi_F^1[\![(v_h)_t]\!]\|_F^2 + \frac{2}{h}\|[\![(v_h - \Pi_F^1 v_h)_t]\!]\|_F^2.$$

As $\Pi^{\mathrm{RM}} v_h \in \mathbb{P}^1(T, \mathbb{R}^d)$, we have again by the continuity of Π_F^1

$$\|[\![(v_h - \Pi_F^1 v_h)_t]\!]\|_F^2 = \|(\mathrm{Id} - \Pi_F^1)[\![(v_h - \Pi^{\mathrm{RM}} v_h)_t]\!]\|_F^2 \le \|[\![(v_h - \Pi^{\mathrm{RM}} v_h)_t]\!]\|_F^2.$$

We conclude the proof using the estimate (5.5). □

Lemma 36. *For all $T \in \mathcal{T}_h$ and $v_h \in \mathcal{RT}^k(T)$ there holds the equivalence*

$$\begin{aligned}\|\varepsilon(v_h)\|_T^2 &\sim \|\Pi^{k-1}[\mathrm{dev}(\varepsilon(v_h))]\|_T^2 + \|\operatorname{div}(v_h)\|_T^2 \\ &= \|\Pi^{k-1}[\mathrm{dev}(\nabla v_h - \omega(v_h))]\|_T^2 + \|\operatorname{div}(v_h)\|_T^2,\end{aligned}$$

and the estimate

$$\|(\mathrm{Id} - \Pi^{k-1})\omega(v_h)\|_T^2 \lesssim \|\operatorname{div}(v_h)\|_T^2, \quad \text{and} \quad \|(\mathrm{Id} - \Pi^{k-1})\nabla v_h\|_T^2 \lesssim \|\operatorname{div}(v_h)\|_T^2.$$

Proof. The proof follows along the same lines as the proof of lemma 22. Let $v_h \in V_h$ be arbitrary. In a first step we solve the same problem (6.12) to find a function $w_h \in \mathcal{RT}^k(T)$ such that $\operatorname{div}(w_h) = \operatorname{div}(v_h)$. Further v_h, can locally be written as $v_h = a + w_h$ with $a \in \mathbb{P}^k(T, \mathbb{R}^d)$ and the estimate

$$\|\nabla w_h\|_T \sim \|\operatorname{div}(w_h)\|_T.$$

This yields

$$\begin{aligned}\|\varepsilon(v_h)\|_T = \|\varepsilon(a + w_h)\|_T &\le \|\mathrm{dev}(\varepsilon(a))\|_T + \|\mathrm{dev}(\varepsilon(w_h))\|_T + \|\operatorname{div}(v_h)\|_T \\ &\le \|\mathrm{dev}(\varepsilon(a))\|_T + \|\nabla w_h\|_T + \|\operatorname{div}(v_h)\|_T \\ &\lesssim \|\mathrm{dev}(\varepsilon(a))\|_T + \|\operatorname{div}(v_h)\|_T \\ &= \|\Pi^{k-1}\mathrm{dev}(\varepsilon(a))\|_T + \|\operatorname{div}(v_h)\|_T.\end{aligned}$$

As $a = v_h - w_h$, we conclude

$$\begin{aligned}\|\varepsilon(v_h)\|_T &\lesssim \|\Pi^{k-1}\mathrm{dev}(\varepsilon(v_h))\|_T + \|\Pi^{k-1}\mathrm{dev}(\varepsilon(w_h))\|_T + \|\operatorname{div}(v_h)\|_T \\ &\lesssim \|\Pi^{k-1}\mathrm{dev}(\varepsilon(v_h))\|_T + \|\nabla w_h\|_T + \|\operatorname{div}(v_h)\|_T \\ &\lesssim \|\Pi^{k-1}\mathrm{dev}(\varepsilon(v_h))\|_T + \|\operatorname{div}(v_h)\|_T.\end{aligned}$$

It remains to show the second statement. First, note that due to the definition of the function $\omega(\cdot)$ we have $\|\omega(v_h)\|_T \sim \|\operatorname{curl}(v_h)\|_T$. Thus, using the representation from above, we see

$$\|(\mathrm{Id} - \Pi^{k-1})\omega(v_h)\|_T^2 \lesssim \|(\mathrm{Id} - \Pi^{k-1})\operatorname{curl}(a)\|_T^2 + \|(\mathrm{Id} - \Pi^{k-1})\operatorname{curl}(w_h)\|_T^2.$$

As $a \in \mathbb{P}^k(T, \mathbb{R}^d)$, the first term vanishes, thus with the arguments from above

$$\|(\mathrm{Id} - \Pi^{k-1})\operatorname{curl}(w_h)\|_T^2 \le \|\nabla w_h\|_T^2 \lesssim \|\operatorname{div}(v_h)\|_T.$$

The estimate for the gradient also follows with the same steps and the local representation of v_h, see also in the proof of lemma 22. □

Lemma 37. *Let $\mu_h \in \mathbb{P}^k_{skw}(\hat{T}, \mathbb{R}^{d\times d})$ and set and $\tau_h = \operatorname{dev}(\widehat{\operatorname{curl}}(\widehat{\operatorname{curl}}(\mu_h)\hat{B}))$. Then there holds the norm equivalence*

$$\|\tau_h\|_{\hat{T}} \sim \|\widehat{\operatorname{curl}}(\mu_h)\|_{\hat{T}}.$$

Proof. If $\widehat{\operatorname{curl}}(\mu_h) = 0$, then obviously also the left side is zero. We claim that the converse is also true. Indeed, if $\tau_h = 0$, then putting $s = d^{-1}\operatorname{tr}(\widehat{\operatorname{curl}}(\widehat{\operatorname{curl}}(\mu_h)\hat{B}))$, we have

$$\widehat{\operatorname{curl}}(\widehat{\operatorname{curl}}(\mu_h)\hat{B}) = s\,\mathrm{Id}. \tag{7.10}$$

Taking the divergence on both sides, we find that $\nabla s = 0$, so s must be a constant on T. Then, applying the normal trace operator on both sides of (7.10) and on each facet, we realize that $sn = 0$, which yields $s = 0$. Hence, we have the equation $\widehat{\operatorname{curl}}(\widehat{\operatorname{curl}}(\mu_h)\hat{B}) = 0$, which in turn implies that $0 = (\widehat{\operatorname{curl}}(\widehat{\operatorname{curl}}(\mu_h)\hat{B}, \mu_h)_T = (\operatorname{curl}(\mu_h)\hat{B}, \widehat{\operatorname{curl}}(\mu_h))_T = 0$. Therefore, lemma 2.2 in [27] proves $\widehat{\operatorname{curl}}(\mu_h) = 0$. □

Lemma 38. *For all $T \in \mathcal{T}_h$ and $\eta_h \in W_h$ there holds*

$$\|\nabla\eta_h\|_T \sim \|\operatorname{curl}(\eta_h)\|_T.$$

Proof. The proof is based on a scaling argument and norm equivalence on finite dimensional spaces on the reference element. To this end let $\hat{\eta}_h = F_T^{\mathrm{T}}\eta_h F_T$. We start with the three-dimensional case. As F_T^{T} is constant, a simple calculation shows that $\widehat{\operatorname{curl}}(\hat{\eta}_h) = \widehat{\operatorname{curl}}(F_T^{\mathrm{T}}\eta_h F) = F_T^{\mathrm{T}}\widehat{\operatorname{curl}}(\eta_h F)$. Using the well known formula of the curl of a covariant mapped function ($\eta_h F$ reads as a function where each row is mapped with the covariant transformation) we conclude for $d = 3$

$$\widehat{\operatorname{curl}}(F_T^{\mathrm{T}}\eta_h F) = F_T^{\mathrm{T}}\widehat{\operatorname{curl}}(\eta_h^i F) = F_T^{\mathrm{T}}\operatorname{curl}(\eta_h)F_T^{-\mathrm{T}}\det F_T.$$

In two dimensions a simple calculation first shows that

$$\widehat{\operatorname{curl}}(F_T^{\mathrm{T}}\gamma_h^i F_T) = F_T^{\mathrm{T}}(\widehat{\operatorname{curl}}(\gamma_h^i F_T)).$$

Next, again $\gamma_h^i F_T$ results in a matrix where each row of γ_h^i is mapped with the covariant mapping. In two dimensions this then yields the identity

$$F_T^{\mathrm{T}}\widehat{\operatorname{curl}}(\gamma_h^i F_T) = \det F_T F_T^{\mathrm{T}}\operatorname{curl}(\gamma_h^i).$$

A scaling argument then gives in both dimensions

$$\|\nabla\eta_h\|_T^2 \sim h^{d-6}\|\hat{\nabla}\hat{\eta}_h\|_{\hat{T}}^2 \quad\text{and}\quad \|\operatorname{curl}(\eta_h)\|_T^2 \sim h^{d-6}\|\widehat{\operatorname{curl}}(\hat{\eta}_h)\|_{\hat{T}}^2.$$

In the following we show the norm equivalence $\|\widehat{\operatorname{curl}}(\hat{\eta}_h)\|_{\hat{T}} \sim \|\hat{\nabla}\hat{\eta}_h\|_{\hat{T}}$ by proving that both sides only contain constant functions in the kernel. This obviously holds true for the right side. In three dimensions the curl of a skew symmetric matrix reads as

$$\begin{aligned}\widehat{\operatorname{curl}}(\hat{\eta}_h) &= \widehat{\operatorname{curl}}\begin{pmatrix} 0 & -(\hat{\eta}_h)_3 & (\hat{\eta}_h)_2 \\ (\hat{\eta}_h)_3 & 0 & -(\hat{\eta}_h)_1 \\ -(\hat{\eta}_h)_2 & (\hat{\eta}_h)_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \partial_{\hat{x}_2}(\hat{\eta}_h)_2 + \partial_{\hat{x}_3}(\hat{\eta}_h)_3 & -\partial_{\hat{x}_1}(\hat{\eta}_h)_2 & -\partial_{\hat{x}_1}(\hat{\eta}_h)_3 \\ -\partial_{\hat{x}_2}(\hat{\eta}_h)_1 & \partial_{\hat{x}_1}(\hat{\eta}_h)_1 + \partial_{\hat{x}_3}(\hat{\eta}_h)_3 & -\partial_{\hat{x}_2}(\hat{\eta}_h)_3 \\ -\partial_{\hat{x}_3}(\hat{\eta}_h)_1 & -\partial_{\hat{x}_3}(\hat{\eta}_h)_2 & \partial_{\hat{x}_1}(\hat{\eta}_h)_1 + \partial_{\hat{x}_2}(\hat{\eta}_h)_2 \end{pmatrix}.\end{aligned}$$

Thus, if we assume $\|\widehat{\operatorname{curl}}(\hat{\eta}_h)\|_{\hat{T}} = 0$, the off-diagonal entries above show that

$$
\begin{aligned}
(\hat{\eta}_h)_1(\hat{x}_1, \hat{x}_2, \hat{x}_3) &= (\hat{\eta}_h)_1(\hat{x}_1), \\
(\hat{\eta}_h)_2(\hat{x}_1, \hat{x}_2, \hat{x}_3) &= (\hat{\eta}_h)_2(\hat{x}_2), \\
(\hat{\eta}_h)_2(\hat{x}_1, \hat{x}_2, \hat{x}_3) &= (\hat{\eta}_h)_3(\hat{x}_3),
\end{aligned}
$$

and the entries on the diagonal yield

$$
\partial_{\hat{x}_2}(\hat{\eta}_h)_2 + \partial_{\hat{x}_3}(\hat{\eta}_h)_3 = 0, \quad \partial_{\hat{x}_1}(\hat{\eta}_h)_1 + \partial_{\hat{x}_3}(\hat{\eta}_h)_3 = 0, \quad \partial_{\hat{x}_1}(\hat{\eta}_h)_1 + \partial_{\hat{x}_2}(\hat{\eta}_h)_2 = 0.
$$

From the above equations we see that the entries $(\hat{\eta}_h)_i$, $i = 1, 2, 3$, have to be constant. In two dimensions the equivalence is obvious as the curl operator is simply given by the rotated gradient. □

Lemma 39. *For all $T \in \mathcal{T}_h$ and $(v_h, \eta_h) \in U_h$ there holds*

$$
\begin{aligned}
&\|\varepsilon(v_h)\|_T^2 + \|\omega(v_h) - \eta_h\|_T^2 \\
&\qquad \sim \|\Pi^{k-1}[\operatorname{dev}(\nabla v_h - \eta_h)]\|_T^2 + h^2\|\operatorname{curl}(\eta_h)\|_T^2 + \|\operatorname{div}(v_h)\|_T^2.
\end{aligned}
$$

Proof. We start by showing a Pythagoras-Theorem like equivalence. Adding and subtracting $\omega(v_h)$ we have the identity

$$
\begin{aligned}
\int_T |\Pi^{k-1}[\operatorname{dev}(\nabla v_h - \eta_h)]|^2 \, dx &= \int_T |\Pi^{k-1}[\operatorname{dev}(\nabla v_h - \omega(v_h) + \omega(v_h) - \eta_h)]|^2 \, dx \\
&= \int_T |\Pi^{k-1}(\operatorname{dev}(\nabla v_h - \omega(v_h)))|^2 \\
&\quad + 2(\Pi^{k-1}[\operatorname{dev}(\nabla v_h - \omega(v_h))] : \Pi^{k-1}[\omega(v_h) - \eta_h] \\
&\quad + |\Pi^{k-1}[\omega(v_h) - \eta_h]|^2 \, dx \,.
\end{aligned}
$$

Due to the orthogonality of symmetric and skew symmetric matrices the mixed term vanishes, yielding

$$
\|\Pi^{k-1}[\operatorname{dev}(\nabla v_h - \eta_h)]\|_T^2 = \|\Pi^{k-1}[\operatorname{dev}(\nabla v_h - \omega(v_h))]\|_T^2 + \|\Pi^{k-1}[\omega(v_h) - \eta_h]\|_T^2. \tag{7.11}
$$

For the proof of the lemma we first show that the left side can be bounded by the right side. Using lemma 36 and equation (7.11) we have

$$
\|\varepsilon(v_h)\|_T^2 \leq \|\Pi^{k-1}[\operatorname{dev}(\nabla v_h - \eta_h)]\|_T^2 + \|\operatorname{div}(v_h)\|_T^2.
$$

Next, note that

$$
\|\omega(v_h) - \eta_h\|_T^2 \leq \|\Pi^{k-1}(\omega(v_h) - \eta_h)\|_T^2 + \|(\mathrm{Id} - \Pi^{k-1})(\omega(v_h) - \eta_h)\|_T^2.
$$

Using identity (7.11), the first term can trivially be bounded by $\|\Pi^{k-1}[\operatorname{dev}(\nabla v_h - \eta_h)]\|_T^2$. For the second term we have with lemma 36 and lemma 38

$$
\begin{aligned}
\|(\mathrm{Id} - \Pi^{k-1})(\omega(v_h) - \eta_h)\|_T^2 &\leq \|(\mathrm{Id} - \Pi^{k-1})\omega(v_h)\|_T^2 + \|(\mathrm{Id} - \Pi^{k-1})\eta_h\|_T^2 \\
&\leq \|\operatorname{div}(v_h)\|_T^2 + h^2\|\nabla \eta_h\|_T^2 \\
&\leq \|\operatorname{div}(v_h)\|_T^2 + h^2\|\operatorname{curl}(\eta_h)\|_T^2.
\end{aligned}
$$

To bound the right side we use (7.11), lemma 36 and the continuity of the projection,

$$\|\Pi^{k-1}[\operatorname{dev}(\nabla v_h - \eta_h)]\|_T^2 + \|\operatorname{div}(v_h)\|_T^2 \leq \|\varepsilon(v_h)\|_T^2 + \|\omega(u) - \eta_h\|_T^2.$$

As $\operatorname{curl}(\omega(\Pi^{RM} v_h)) = 0$, we obtain, using an inverse inequality for polynomials (see [117]) and estimate (5.4),

$$\begin{aligned} h^2\|\operatorname{curl}(\eta_h)\|_T^2 = h^2\|\operatorname{curl}(\eta_h - \omega(\Pi^{RM} v_h))\|_T^2 &\lesssim \|\eta_h - \omega(\Pi^{RM} v_h)\|_T^2 \\ &\leq \|\eta_h - \omega(v_h)\|_T^2 + \|\omega(v_h) - \omega(\Pi^{RM} v_h)\|_T^2 \\ &\sim \|\eta_h - \omega(v_h)\|_T^2 + \|\operatorname{curl}(v_h - \Pi^{RM} v_h)\|_T^2 \\ &\lesssim \|\eta_h - \omega(v_h)\|_T^2 + \|\varepsilon(v_h)\|_T^2, \end{aligned}$$

which concludes the proof. □

Lemma 40. *For all $T \in \mathcal{T}_h$ and $(u_h, \gamma_h) \in U_h$ with $\gamma_h \perp \mathbb{P}^0(T, \mathbb{R}^{d\times d})$ there holds*

$$\begin{aligned} \|\gamma_h\|_{L^2(\Omega)} &\sim \inf_{v_h \in V_h} \|(v_h, \gamma_h)\|_{U_h}, \\ \|u_h\|_{1,h,sym} &\sim \inf_{\eta_h \in W_h} \|(u_h, \eta_h)\|_{U_h}. \end{aligned}$$

Proof. We start with the first equivalence. It is easy to see that the right side can be bounded by the left side by choosing $v_h = 0$. For the other direction we have, using lemma 39 and lemma 38,

$$\inf_{v_h \in V_h} \|(v_h, \gamma_h)\|_{U_h} \gtrsim h\|\operatorname{curl}(\gamma_h)\|_{L^2(\Omega)} \sim h\|\nabla\gamma_h\|_{L^2(\Omega)} \sim \|\gamma_h\|_{L^2(\Omega)},$$

where we used $\gamma_h \perp \mathbb{P}^0(T, \mathbb{R}^{d\times d})$ in the last step. For the other equivalence, again the right side can be bounded by the left side by choosing $\eta_h = \omega(u_h)$. For the other direction note that

$$\inf_{\eta_h \in W_h} \|(u_h, \eta_h)\|_{U_h}^2 = \|u_h\|_{1,h,sym}^2 + \sum_{T\in\mathcal{T}_h} \|\omega(u_h) - \eta_h\|_T^2 \geq \|v_h\|_{1,h,sym}^2,$$

thus the other direction follows trivially. □

7.3.2 Stability analysis

Similarly as in section 6.2.2 we are aiming to use Brezzi's theorem 6 to show discrete inf-sup stability. To this end we prove all the conditions needed in the following.

Lemma 41 (Continuity of a and b_1 and b_{2h}^{ε})**.** *The bilinear forms a, b_1 and b_{2h}^{ε} are continuous:*

$$\begin{aligned} a(\sigma_h, \tau_h) &\lesssim \nu^{-1}\|\sigma_h\|_{\Sigma_h^\dagger}\nu^{-1}\|\tau_h\|_{\Sigma_h^\dagger}, && \textit{for all } \sigma_h, \tau_h \in \Sigma_h^\dagger, \\ b_{2h}^{\varepsilon}(\tau_h, (v_h, \eta_h)) &\lesssim \|\tau_h\|_{\Sigma_h^\dagger}\|(v_h, \eta_h)\|_{U_h}, && \textit{for all } \tau_h \in \Sigma_h^\dagger, (v_h, \eta_h) \in U_h, \\ b_1(v_h, q_h) &\lesssim \|(v_h, 0)\|_{U_h}\|q_h\|_{Q_h}, && \textit{for all } (v_h, 0) \in U_h, q_h \in Q_h. \end{aligned}$$

Proof. The continuity of a and b_1 follow by the Cauchy-Schwarz inequality. For b^{ε}_{2h} we have

$$b^{\varepsilon}_{2h}(\tau_h, (v_h, \eta_h)) = -\sum_{T\in\mathcal{T}_h}\int_T \tau : (\nabla v_h - \eta_h)\,\mathrm{d}x + \sum_{F\in\mathcal{F}_h}\int_F \tau_{nt}\cdot [\![(v_h)_t]\!]\,\mathrm{d}s\,.$$

Using the estimate

$$\|\nabla v_h - \eta_h\|_T^2 \leq \|\nabla v_h - \omega(v_h)\|_T^2 + \|\omega(v_h) - \eta\|_T^2,$$

we conclude the proof by the Cauchy-Schwarz inequality and lemma 19. □

With respect to the bilinear forms b_1 and b^{ε}_{2h} we define the kernel as

$$K_{b^{\varepsilon}_h} := \{(\tau_h, q_h) \in \Sigma_h \times Q_h : b_1(v_h, q_h) + b^{\varepsilon}_{2h}(\sigma_h, (v_h, \eta_h)) = 0 \text{ for all } (v_h, \eta_h) \in U_h\}.$$

Lemma 42 (Coercivity of a on the kernel)**.** *For all $(\sigma_h, p_h) \in K_{b^{\varepsilon}_h}$ there holds*

$$\frac{1}{\nu}\big(\|\sigma_h\|_{\Sigma_h^\dagger} + ||p_h||_{Q_h}\big)^2 \lesssim a(\sigma_h, \sigma_h).$$

Proof. The proof follows with the same steps as the proof of lemma 24. Again, bounding the symmetric norm trivially by the full norm, we find for all $p_h \in Q_h$ a $v_h \in V_h$ with

$$\operatorname{div}(v_h) = p_h, \quad \text{and} \quad \|v_h\|^2_{1,h,sym} \lesssim \|p_h\|_{Q_h}.$$

For $\eta_h := \omega(v_h) \in W_h$ we then immediately also have the estimate

$$\|(v_h, \eta_h)\|_{U_h} \lesssim \|p_h\|_{Q_h},$$

thus

$$2||p_h||^2_{Q_h} = b_1(v_h, p_h) = -b^{\varepsilon}_{2h}(\sigma_h, v_h) \lesssim \|\sigma\|_{\Sigma_h^\dagger}\|(v_h, \eta_h)\|_{U_h} \lesssim \|\sigma\|_{\Sigma_h^\dagger}||p_h||_{Q_h}.$$

The rest of the proof follows similarly as in the proof of lemma 24. □

We continue with the proof of the discrete LBB-condition. For this we proceed similarly as in section 6.2.2 and first show two lemmas needed for the final result. Whereas the first lemma proves a LBB-condition for the symmetry constraint, the second lemma discusses inf-sup stability of the discrete divergence of stress variables.

Lemma 43. *Let $\gamma_h \in W_h$ be arbitrary. There exists a $\tau_h \in \Sigma_h^\dagger$ such that*

$$\int_\Omega \tau_h : \gamma_h \gtrsim h\|\operatorname{curl}(\gamma_h)\|_{L^2(\Omega)}\|\tau_h\|_{\Sigma_h^\dagger}.$$

Further let $u_h \in V_h$ be arbitrary, then additionally we have

$$b^{\varepsilon}_{2h}(\tau_h, (u_h, \gamma_h)) \gtrsim \big(h\|\operatorname{curl}(\gamma)\|_{L^2(\Omega)} - \|\operatorname{div}(u_h)\|_{L^2(\Omega)}\big)\,\|\tau_h\|_{\Sigma_h^\dagger}.$$

Proof. Let $\gamma_h \in W_h$ be arbitrary. Using the proper transformation for skew symmetric matrices we define for each $T \in \mathcal{T}_h$ the function

$$\hat{\tau}_h^T := \operatorname{dev}(\widehat{\operatorname{curl}}(\widehat{\operatorname{curl}}(F_T^{\mathrm{T}} \gamma_h F)\hat{B})),$$

where $\widehat{\operatorname{curl}}$ is the curl operator with respect to the coordinates on the reference element. Using these locally defined functions we set

$$\tau_h := \sum_{T \in \mathcal{T}_h} \mathcal{M}(\hat{\tau}_h^T).$$

In the following we show that this is an admissible choice, thus that τ_h is an element of $\Sigma_h^\dagger$. We start with the three-dimensional case. By the definition of the covariant Piola transformation $\mathcal{M}$, see equation (5.22), and the proper mapping for the bubble matrix, see equation (7.4), we observe on a fixed element $T \in \mathcal{T}_h$ that

$$\mathcal{M}\Big(\widehat{\operatorname{curl}}(\widehat{\operatorname{curl}}(F_T^{\mathrm{T}} \gamma_h F)\hat{B})\Big) = \frac{1}{\det(F_T)} F_T^{-\mathrm{T}} \widehat{\operatorname{curl}}(\widehat{\operatorname{curl}}(F_T^{\mathrm{T}} \gamma_h F) F_T^{\mathrm{T}} B F_T) F_T^{\mathrm{T}}.$$

With the same argument as in the proof of lemma 38 we can write the curl on the reference element $\hat{T}$ in terms of the curl on the physical element T. Using this identity twice we deduce

$$\begin{aligned}
F_T^{-\mathrm{T}} \widehat{\operatorname{curl}}(\widehat{\operatorname{curl}}(F_T^{\mathrm{T}} \gamma_h F) F_T^{\mathrm{T}} B F_T) F_T^{\mathrm{T}} &= F_T^{-\mathrm{T}} \widehat{\operatorname{curl}}(F_T^{\mathrm{T}} \operatorname{curl}(\gamma_h) F_T^{-\mathrm{T}} \det F_T F_T^{\mathrm{T}} B F_T) F_T^{\mathrm{T}} \\
&= \det F_T F_T^{-\mathrm{T}} \widehat{\operatorname{curl}}(F_T^{\mathrm{T}} \operatorname{curl}(\gamma_h) B F_T) F_T^{\mathrm{T}} \\
&= (\det F_T)^2 F_T^{-\mathrm{T}} F_T^{\mathrm{T}} \operatorname{curl}(\operatorname{curl}(\gamma_h) B) F_T^{-\mathrm{T}} F_T^{\mathrm{T}} \\
&= (\det F_T)^2 \operatorname{curl}(\operatorname{curl}(\gamma_h) B).
\end{aligned}$$

As the mapping $\mathcal{M}$ and the deviator commutes, see lemma 16, this shows

$$\tau_h|_T = (\det F_T) \operatorname{dev}(\operatorname{curl}(\operatorname{curl}(\gamma_h) B)).$$

Further, we have that $\Pi^k(\tau_h|_T) \in \Sigma$ and $(\mathrm{Id} - \Pi^k)(\tau_h|_T) \in \delta\Sigma$, thus by corollary 6 and lemma 34 we conclude that $\tau_h \in \Sigma_h^\dagger$.

Next, note that due to the skew symmetry of γ_h we have on each element $T \in \mathcal{T}_h$

$$\begin{aligned}
\int_T \tau_h : \gamma_h \, \mathrm{d}x &= \int_T \det F_T \operatorname{dev}(\operatorname{curl}(\operatorname{curl}(\gamma_h) B)) : \gamma_h \, \mathrm{d}x \\
&= \int_T \det F_T \operatorname{curl}(\operatorname{curl}(\gamma_h) B) : \gamma_h \, \mathrm{d}x \,.
\end{aligned}$$

Integrating by parts and using lemma 34 this yields

$$\int_T \tau_h : \gamma_h \, \mathrm{d}x = \int_T \det F_T \operatorname{curl}(\gamma_h) B : \operatorname{curl}(\gamma_h) \, \mathrm{d}x \,.$$

As $|\det F_T| \sim h^d$ and $\|B\|_\infty \sim \| \sum_{l \in \mathcal{V}} \nabla \lambda_l \otimes \nabla \lambda_l \|_\infty \sim h^{-2}$, we conclude

$$\int_T \tau_h : \gamma_h \, \mathrm{d}x \gtrsim h \| \operatorname{curl}(\gamma_h) \|_T^2.$$

Next, lemma 37, lemma 18 and a standard scaling argument further show that

$$\|\tau_h\|_{\Sigma_h^\dagger} \sim \|\operatorname{curl}(\gamma_h)\|_{L^2(\Omega)},$$

which proves the first part of the lemma for $d = 3$.

In two dimensions we first have with similar steps as in the proof of lemma 38 that

$$\widehat{\operatorname{curl}}(\widehat{\operatorname{curl}}(F_T^{\mathrm{T}}\gamma_h F_T)\hat{B}) = F_T^{\mathrm{T}}\widehat{\operatorname{curl}}(\widehat{\operatorname{curl}}(\gamma_h F_T)\hat{B}) = (\det F_T)F_T^{\mathrm{T}}\widehat{\operatorname{curl}}(\operatorname{curl}(\gamma_h)\hat{B}).$$

According to the definition of the enrichment space, the outer curl operator is applied on each scalar component of the vector $\operatorname{curl}(\gamma_h)\hat{B}$. By the properties of the discrete de Rham complex (in two dimension), the resulting field is mapped with the (inverse) Piola mapping, which yields

$$(\det F_T)F_T^{\mathrm{T}}\widehat{\operatorname{curl}}(\operatorname{curl}(\gamma_h)\hat{B}) = (\det F)^2 F_T^{\mathrm{T}}\operatorname{curl}(\operatorname{curl}(\gamma_h)\hat{B})F_T^{-\mathrm{T}}.$$

Summing up the above results and using the classical pullback for the bubble we have similarly as in three dimensions that

$$\mathcal{M}\big(\widehat{\operatorname{curl}}(\widehat{\operatorname{curl}}(F^{\mathrm{T}}\gamma_h F)\hat{B})\big) = (\det F)\operatorname{curl}(\operatorname{curl}(\gamma_h)B).$$

With the same arguments as in the three-dimensional case this shows that $\tau_h \in \Sigma_h^\dagger$, and an integration by parts argument yields

$$\int_T \tau_h : \gamma_h \,\mathrm{d}x \gtrsim h^2\|\operatorname{curl}(\gamma_h)\|_T^2.$$

A scaling argument as above (using $\|B\|_\infty \sim \mathcal{O}(1)$) further shows $\|\tau_h\|_{\Sigma_h^\dagger} \sim h\|\operatorname{curl}(\gamma_h)\|_{L^2(\Omega)}$, from which we conclude the first statement of the lemma.

Now let $u_h \in V_h$ be arbitrary. For $d = 3$ we observe as $[\![(\tau_h)_{nt}]\!] = 0$ (see lemma 34) that

$$b_{2h}^{\varepsilon}(\tau_h,(u_h,\gamma_h)) = -\sum_{T\in\mathcal{T}_h}\int_T \tau_h : \nabla u_h \,\mathrm{d}x + \sum_{T\in\mathcal{T}_h}\int_T \tau_h : \gamma_h \,\mathrm{d}x\,.$$

With the above results the second sum can be bounded from below by $h\|\operatorname{curl}(\gamma_h)\|_{L^2(\Omega)}^2$. By the definition of τ_h, we can split the first sum into two parts

$$\begin{aligned}-\sum_{T\in\mathcal{T}_h}\int_T \tau_h : \nabla u_h \,\mathrm{d}x = &-\sum_{T\in\mathcal{T}_h}\det F_T\int_T \operatorname{curl}(\operatorname{curl}(\gamma_h)B) : \nabla u_h \,\mathrm{d}x \\ &+ \frac{\det F_T}{d}\sum_{T\in\mathcal{T}_h}\int_T \operatorname{Id}\operatorname{tr}(\operatorname{curl}(\operatorname{curl}(\gamma_h)B)) : \nabla u_h \,\mathrm{d}x\,.\end{aligned}$$

For the first sum we observe by an integration by parts argument on each element $T \in \mathcal{T}_h$ separately that

$$\begin{aligned}&\int_T \operatorname{curl}(\operatorname{curl}(\gamma_h)B) : \nabla u_h \,\mathrm{d}x \\ &\quad= \int_T (\operatorname{curl}(\gamma_h)B) : \operatorname{curl}(\nabla u_h)\,\mathrm{d}x - \int_{\partial T}(\operatorname{curl}(\gamma_h)B)\times n : (\nabla u_h)_t \,\mathrm{d}s = 0,\end{aligned}$$

where lemma 34 was used to prove that the boundary terms vanish. Thus, all together this yields

$$\begin{aligned} b_{2h}^{\varepsilon}(\tau_h,(u_h,\gamma_h)) &\geq \sum_{T\in\mathcal{T}_h}\frac{\det F_T}{d}\int_T \operatorname{Id}\operatorname{tr}(\operatorname{curl}(\operatorname{curl}(\gamma_h)B)) : \nabla u_h \,\mathrm{d}x + h\|\operatorname{curl}(\gamma_h)\|^2_{L^2(\Omega)} \\ &= \sum_{T\in\mathcal{T}_h}\frac{\det F_T}{d}\int_T \operatorname{tr}(\operatorname{curl}(\operatorname{curl}(\gamma_h)B))\operatorname{div}(u_h)\,\mathrm{d}x + h\|\operatorname{curl}(\gamma_h)\|^2_{L^2(\Omega)} \\ &\gtrsim -\|\operatorname{curl}(\gamma_h)\|_{L^2(\Omega)}\|\operatorname{div}(u_h)\|_{L^2(\Omega)} + h\|\operatorname{curl}(\gamma_h)\|^2_{L^2(\Omega)}, \end{aligned}$$

where we used an inverse inequality for polynomials in the last step. Using $\|\tau_h\|_{\Sigma_h^\dagger} \sim \|\operatorname{curl}(\gamma_h)\|_{L^2(\Omega)}$ we conclude the proof (the two dimensional case follows similarly). $\square$

Lemma 43 states that it is possible to choose a stress function τ_h, which lies almost in the kernel of the distributional divergence operator. The facet contributions vanish completely, and the (low order) element contributions are bounded from below (with a minus) only by the divergence of the velocity u_h. This is one of the key results needed to show stability of the global system.

Next, we define the following norm on the product space U_h

$$\|(v_h,\eta_h)\|^2_{U_h,\Pi} := \sum_{T\in\mathcal{T}_h}\|\Pi_T^{k-1}\operatorname{dev}(\nabla v_h - \eta_h)\|_T^2 + \sum_{F\in\mathcal{F}_h}\frac{1}{h}\|\Pi_F^1[\![(v_h)_t]\!]\|_F^2.$$

For the discrete solution $u_h \in V_h$ and $\gamma_h \in W_h$ of (7.9) we have that $\omega(u_h) \approx \gamma_h$ and $\operatorname{div}(u_h) = 0$, and thus the first term in this norm is approximately $\Pi^{k-1}\varepsilon(u)$. Therefore, for divergence free velocity functions and a proper η_h the norm $\|(v_h,\eta_h)\|_{U_h,*}$ can be interpreted as a (projected) symmetric discrete H^1-norm of the velocity similarly as the norm $\|\cdot\|_{1,h,\Pi}$ defined it in section 6.2.2.

Lemma 44. *Let $(u_h,\gamma_h) \in U_h$ be arbitrary. There exists a $\tau_h \in \Sigma_h^\dagger$ such that*

$$b_{2h}^{\varepsilon}(\tau_h,(u_h,\gamma_h)) \gtrsim \|(u_h,\gamma_h)\|^2_{U_h,\Pi},$$

and $\|\tau_h\|_{\Sigma_h^\dagger} \lesssim \|(u_h,\gamma_h)\|_{U_h,\Pi}$.

Proof. The proof follows along the same lines as the proof of lemma 25, and is based on a decomposition of the space $\Sigma_h = \Sigma_h^F \oplus \Sigma_h^T$, where Σ_h^T contains element-wise normal-tangential bubbles, and Σ_h^F contains basis functions associated with the facets given by S^F with the property that $S^F_{nt} \in \mathbb{P}^0(F, n_T^\perp)$ and $\|S^F_{nt}\|_2 = 1$ on the facet F and equal $(0,0)$ on all other facets. Given any $(u_h,\gamma_h) \in U_h$, we now define similarly to equation (6.15), the function

$$\begin{aligned} \tau_h^0 &:= \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_h} -(S^F : \Pi^{k-1}\operatorname{dev}(\nabla u_h - \gamma_h))\lambda_T^F S^F, \\ \tau_h^1 &:= \sum_{F\in\mathcal{F}_h}\frac{1}{\sqrt{h}}(\Pi^1[\![(u_h)_t]\!])S^F. \end{aligned}$$

Note that the choice of τ_h^0 in terms of $(S^F : \Pi^{k-1}\mathrm{dev}(\nabla u_h - \gamma_h))\lambda_T^F S^F$ is admissible as $\Pi^{k-1}\mathrm{dev}(\nabla u_h - \gamma_h))$ is a polynomial of degree $k-1$, and $\Sigma_h^\dagger = \Sigma_h \oplus \delta\Sigma_h$ contains polynomials up to order k. Using the norm equivalences stated in lemma 19 and the proper mapping for u_h and γ_h, we easily see that

$$\|\tau_h^0\|^2_{\Sigma_h^\dagger} \lesssim \sum_{T\in\mathcal{T}_h} \|\Pi^{k-1}\mathrm{dev}(\nabla u_h - \gamma_h))\|_T^2, \quad \text{and} \quad \|\tau_h^1\|^2_{\Sigma_h^\dagger} \lesssim \sum_{F\in\mathcal{F}_h} \frac{1}{h}\|\Pi^1[\![(u_h)_t]\!]\|_F^2.$$

We conclude the proof by a proper linear combination of τ_h^0 and τ_h^1, and the same lines as in the proof of lemma 25. □

Theorem 28 (Discrete LBB-condition)**.** *Let $(u_h, \gamma_h) \in W_h$ be arbitrary. There holds*

$$\sup_{(\tau_h, q_h)\in\Sigma_h^\dagger\times Q_h} \frac{b_1(u_h, q_h) + b_{2h}^\varepsilon(\tau_h, (u_h, \gamma_h))}{\|\tau_h\|_{\Sigma_h^\dagger} + \|q_h\|_{Q_h}} \gtrsim \|(u_h, \eta_h)\|_{U_h}.$$

Proof. We choose $\tau_h^1, \tau_h^2 \in \Sigma_h^\dagger$ according to lemma 43 and lemma 44, respectively, and scale them such that

$$\|\tau_h^1\|_{\Sigma_h^\dagger} = h\|\operatorname{curl}(\gamma_h)\|_{L^2(\Omega)} \quad \text{and} \quad \|\tau_h^2\|_{\Sigma_h^\dagger} = \|(u_h, \gamma_h)\|_{U_h,\Pi}.$$

Next, we further choose $q_h = \operatorname{div}(u_h)$, which is possible due to the choice of the velocity space V_h and the pressure space Q_h. This yields that $b_1(u_h, q_h) = \|\operatorname{div}(u_h)\|^2_{Q_h}$. For $\tau_h := \alpha\tau_h^1 + \tau_h^2$ and βq_h, where $\alpha, \beta \in \mathbb{R}$ are constants to be chosen later, it follows

$$\begin{aligned} b_1(u_h, q_h) + b_{2h}^\varepsilon(\tau_h, (u_h, \gamma_h)) =& \beta\|\operatorname{div}(u_h)\|^2_{Q_h} + \alpha b_{2h}^\varepsilon(\tau_h^1, (u_h, \gamma_h)) + b_{2h}^\varepsilon(\tau_h^2, (u_h, \gamma_h)) \\ \gtrsim& \beta\|\operatorname{div}(u_h)\|^2_{Q_h} + \alpha h^2\|\operatorname{curl}(\gamma_h)\|^2_{L^2(\Omega)} \\ &- \alpha h\|\operatorname{curl}(\gamma_h)\|_{L^2(\Omega)}\|\operatorname{div}(u_h)\|_{Q_h} + \|(u_h, \gamma_h)\|^2_{U_h,\Pi}. \end{aligned}$$

Using Young's inequality for the mixed terms and choosing $\alpha > 1$ and $\beta > \alpha^2/2$ we get

$$b_1(u_h, q_h) + b_{2h}^\varepsilon(\tau_h, (u_h, \gamma_h)) \gtrsim \|\operatorname{div}(u_h)\|^2_{Q_h} + h^2\|\operatorname{curl}(\gamma_h)\|^2_{L^2(\Omega)} + \|(u_h, \gamma_h)\|^2_{U_h,\Pi},$$

and $\|\tau_h\|_{\Sigma_h^\dagger} + \|q_h\|_{Q_h} \lesssim \|\operatorname{div}(u_h)\|_{Q_h} + h\|\operatorname{curl}(\gamma_h)\|_{L^2(\Omega)} + \|(u_h, \gamma_h)\|_{U_h,\Pi}$. We conclude the proof by using lemma 39 (separately on each element) and lemma 35, which yields

$$\begin{aligned} &\|\operatorname{div}(u_h)\|^2_{Q_h} + h^2\|\operatorname{curl}(\gamma_h)\|^2_{L^2} + \|(u_h, \gamma_h)\|^2_{U_h,\Pi} \\ &= \sum_{T\in\mathcal{T}_h} \|\Pi_T^{k-1}\mathrm{dev}(\nabla v_h - \gamma_h)\|_T^2 + h^2\|\operatorname{curl}(\gamma_h)\|_T^2 + \|\operatorname{div}(u_h)\|_T^2 + \sum_{F\in\mathcal{F}_h} \frac{1}{h}\|\Pi_F^1[\![(v_h)_t]\!]\|_F^2 \\ &\sim \sum_{T\in\mathcal{T}_h} \|\varepsilon(u_h)\|_T^2 + \|\omega(u_h) - \gamma_h\|_T^2 + \sum_{F\in\mathcal{F}_h} \frac{1}{h}\|\Pi_F^1[\![(v_h)_t]\!]\|_F^2 \sim \|(u_h, \gamma_h)\|^2_{U_h}. \end{aligned}$$

□

We are now able to show existence and uniqueness of the discrete solution. To this end we define the bilinear form

$$\begin{aligned} B_\varepsilon(u_h, \gamma_h, &\sigma_h, p_h; v_h, \eta_h, \tau_h, q_h) \\ &:= a(\sigma_h, \tau_h) + b_1(u_h, q_h) + b_1(v_h, p_h) + b_{2h}^\varepsilon(\sigma_h, (v_h, \eta_h)) + b_{2h}^\varepsilon(\tau_h, (u_h, \gamma_h)), \end{aligned}$$

and the norm $\|(u_h, \gamma_h, \sigma_h, p_h)\|_{*,\varepsilon} := \sqrt{\nu}\|(u_h, \gamma_h)\|_{U_h} + \frac{1}{\sqrt{\nu}}(\|\sigma_h\|_{\Sigma_h^\dagger} + \|p_h\|_{Q_h})$.

Corollary 7. *The bilinearform B_ε is inf-sup stable with respect to $\|\cdot\|_{*,\varepsilon}$, thus there exists a constant $\beta > 0$ such that for all nonzero functions $(\sigma_h, u_h, \gamma_h, p_h) \in \Sigma_h^\dagger \times V_h \times W_h \times Q_h$ there holds*

$$\|(u_h, \gamma_h, \sigma_h, p_h)\|_{*,\varepsilon} \lesssim \sup_{\substack{(v_h,\eta_h)\in V_h\times W_h \\ \tau_h\in\Sigma_h^\dagger, q_h\in Q_h}} \frac{B_\varepsilon(u_h, \gamma_h, \sigma_h, p_h; v_h, \eta_h, \tau_h, q_h)}{\|(v_h, \eta_h, \tau_h, q_h)\|_{*,\varepsilon}}.$$

Let $f \in L^2(\Omega, \mathbb{R}^d)$ and assume homogeneous boundary conditions (6.8). There exists a unique solution $(\sigma_h, u_h, \gamma_h, p_h) \in \Sigma_{h,N}^\dagger \times V_{h,D} \times W_h \times Q_h$ of the MCS method with weakly imposed symmetry (7.9) with the stability estimate

$$\|u_h, \gamma_h, \sigma_h, p_h\|_{*,\varepsilon} \lesssim \frac{1}{\sqrt{\nu}}\|f\|_{L^2(\Omega)}.$$

Proof. This follows along the same lines as the proof of corollary 4 using lemma 41, lemma 42, theorem 28 and Brezzi's theorem 6. ☐

Theorem 29 (Consistency)**.** *The mass conserving mixed stress formulation with weakly imposed symmetry (7.9) is consistent in the following sense. If the exact solution of the mixed Stokes problem (7.1) fulfills the regularity property $u \in H^1(\Omega, \mathbb{R}^d)$, $\sigma \in H^1(\Omega, \mathbb{R}^{d\times d})$, $\gamma = \omega(u) \in L^2(\Omega, \mathbb{R}^d)$ and $p \in L^2(\Omega, \mathbb{R})$, then*

$$B_\varepsilon(u, \gamma, \sigma, p; v_h, \eta_h, \tau_h, q_h) = (-f, v_h)_\Omega + (g_{D,t}, (\tau_h)_{nt})_{\Gamma_{D,t}} + (g_{N,n}, v_n)_{\Gamma_{N,n}},$$

for all $(v_h, \eta_h) \in V_h \times W_h, q_h \in Q_h$, and $\tau_h \in \Sigma_h$. Further, there holds consistency on the subspace of divergence free velocity fields

$$B(u, \gamma, \sigma, 0; v_h, \eta_h, \tau_h, 0) = (-f, v_h)_\Omega + (g_{D,t}, (\tau_h)_{nt})_{\Gamma_{D,t}} + (g_{N,n}, v_n)_{\Gamma_{N,n}},$$

for all $(v_h, \eta_h) \in V_h^0 \times W_h$ and $\tau_h \in \Sigma_h$.

Proof. Follows with the same steps as in the proof of theorem 18 and theorem 19. ☐

7.4 Error estimates and post processing

Beside the operators defined by equation (6.30) we further set $I_{W_h} := \Pi^k_{\mathcal{T}_h}$. As we also aim to prove an enhanced convergence rate of the L^2-norm error measured in a proper sense, we again assume that the solution $z \in H^1(\Omega, \mathbb{R}^d)$ and $\mu \in L^2(\Omega, \mathbb{R})$ of a symmetric version of the standard variational formulation in a velocity-pressure setting (4.40) fulfills the regularity property equation (6.31).

There holds the following error estimate.

Theorem 30 (Optimal convergence)**.** *Assuming homogeneous boundary conditions* (6.8)*, let* $u \in H^1(\Omega, \mathbb{R}^d) \cap H^m(\mathcal{T}_h, \mathbb{R}^d)$*,* $\sigma \in H^1(\Omega, \mathbb{D}) \cap H^{m-1}(\mathcal{T}_h, \mathbb{D})$*,* $p \in Q \cap H^{m-1}(\mathcal{T}_h, \mathbb{R})$ *and* $\gamma = \omega(u) \in L^2(\Omega, \mathbb{R}) \cap H^{m-1}(\mathcal{T}_h, \mathbb{R})$ *be the exact solutions of the symmetric mixed Stokes equations* (7.1)*. Further, let* u_h*,* σ_h*,* γ_h *and* p_h *be the solution of the MCS method with weakly imposed symmetry* (7.9)*. For* $s = \min(m-1, k+1)$ *there holds*

$$\frac{1}{\nu}(\|\sigma - \sigma_h\|_{\Sigma_h^\dagger} + \|p - p_h\|_{Q_h}) + \|u_h - I_{V_h}u, \gamma - \gamma_h\|_{U_h} \lesssim h^s(\frac{1}{\nu}\|\sigma\|_{H^s(\mathcal{T}_h)} + \frac{1}{\nu}\|p\|_{H^s(\mathcal{T}_h)} + \|\gamma\|_{H^s(\mathcal{T}_h)}).$$

Assume that the solution (z, μ) *of problem* (4.40) *fulfills the regularity estimate* (6.31)*. Then, for* $s = \min(m-1, k+1)$*, there holds*

$$\|I_{V_h}u - u_h\|_{L^2(\Omega)} \lesssim h^{s+1}(\frac{1}{\nu}\|\sigma\|_{H^s(\mathcal{T}_h)} + \frac{1}{\nu}\|p\|_{H^s(\mathcal{T}_h)} + \|\gamma\|_{H^s(\mathcal{T}_h)}).$$

Proof. This follows with the same steps as the proof of theorem 23. First, we split the error into an interpolation error and a discrete measure of the error

$$\begin{aligned}\frac{1}{\nu}(||\sigma - \sigma_h||_{\Sigma_h} + \|p - p_h\|_{Q_h}) + \|u_h - I_{V_h}u, \gamma - \gamma_h\|_{U_h} \\ \lesssim \frac{1}{\nu}(||\sigma - I_{\Sigma_h}\sigma||_{\Sigma_h} + \|p - \Pi^k p\|_{Q_h}) + \|\gamma - I_{W_h}\gamma\|_{L^2(\Omega)} \\ + \frac{1}{\nu}(||I_{\Sigma_h}\sigma - \sigma_h||_{\Sigma_h} + \|\Pi^k p - p_h\|_{Q_h}) + \|u_h - I_{V_h}u, I_{W_h}\gamma - \gamma_h\|_{U_h}.\end{aligned}$$

By the approximation results of the interpolation operators, the first three terms already converge with optimal order. Next, we use discrete inf-sup stability theorem 7 and consistency, see corollary 29, to get

$$\begin{aligned}\frac{1}{\sqrt{\nu}}\|(I_{V_h}u - u_h, I_{W_h}\gamma - \gamma_h, I_{\Sigma_h}\sigma - \sigma_h, \Pi^k p - p_h)\|_* \\ \lesssim \sup_{\substack{v_h,\eta_h \in V_h \times W_h \\ \tau_h \in \Sigma_h^\dagger, q_h \in Q_h}} \frac{B_\varepsilon(I_{V_h}u - u_h, I_{W_h}\gamma - \gamma_h, I_{\Sigma_h}\sigma - \sigma_h, \Pi^k p - p_h; v_h, \eta_h, \tau_h, q_h)}{\sqrt{\nu}\|(v_h, \eta_h, \tau_h, q_h)\|_*} \\ \lesssim \sup_{\substack{v_h,\eta_h \in V_h \times W_h \\ \tau_h \in \Sigma_h^\dagger, q_h \in Q_h}} \frac{B_\varepsilon(I_{V_h}u - u, I_{W_h}\gamma - \gamma, I_{\Sigma_h}\sigma - \sigma, \Pi^k p - p; v_h, \eta_h, \tau_h, q_h)}{\sqrt{\nu}\|(v_h, \eta_h, \tau_h, q_h)\|_*}.\end{aligned}$$

Next, each term of the bilinear form B_ε is estimated separately. For the terms including the bilinear form b_{2h}^ε we have by the properties of I_{V_h}

$$\begin{aligned}&b_{2h}^\varepsilon(\tau_h, (I_{V_h}u - u, I_{W_h}\gamma - \gamma)) \\ &= \sum_{T \in \mathcal{T}_h} \int_T \operatorname{div}(\tau_h) \cdot (I_{V_h}u - u) + \tau_h : (I_{W_h}\gamma - \gamma)\,\mathrm{d}x - \sum_{F \in \mathcal{F}_h} \int_F [\![(\tau_h)_{nn}]\!](I_{V_h}u - u)_n\,\mathrm{d}s \\ &\lesssim \|\tau_h\|_{\Sigma_h}\|I_{W_h}\gamma - \gamma\|_{L^2(\Omega)} \lesssim \frac{1}{\sqrt{\nu}}\|\tau_h\|_{\Sigma_h^\dagger}\sqrt{\nu}\|(0, I_{W_h}\gamma - \gamma)\|_{U_h},\end{aligned}$$

and for the other term, adding and subtracting $\Pi_{\mathcal{T}_h}^{k-1} \nabla v_h$,

$$
\begin{aligned}
&b_{2h}^{\varepsilon}(I_{\Sigma_h}\sigma - \sigma, (v_h, \eta_h)) \\
&= - \sum_{T \in \mathcal{T}_h} \int_T I_{\Sigma_h}\sigma - \sigma : (\mathrm{Id} - \Pi_T^{k-1}) \nabla v_h \, \mathrm{d}x - \sum_{T \in \mathcal{T}_h} \int_T I_{\Sigma_h}\sigma - \sigma : (\mathrm{Id} - \Pi_T^{k-1}) \nabla v_h \, \mathrm{d}x \\
&\quad + \sum_{F \in \mathcal{F}_h} \int_F (I_{\Sigma_h}\sigma - \sigma)_{nt} \cdot [\![(v_h)_t]\!] \, \mathrm{d}s + \sum_{T \in \mathcal{T}_h} \int_T (I_{\Sigma_h}\sigma - \sigma) : \eta_h \, \mathrm{d}x \, .
\end{aligned}
$$

By the definition of the interpolator I_{Σ_h} the second sum vanishes. Further, as $I_{\Sigma_h}\sigma - \sigma$ is orthogonal on constants, we have the equality

$$
\sum_{T \in \mathcal{T}_h} \int_T (I_{\Sigma_h}\sigma - \sigma) : \eta_h \, \mathrm{d}x = \sum_{T \in \mathcal{T}_h} \int_T (I_{\Sigma_h}\sigma - \sigma) : (\mathrm{Id} - \Pi_T^0) \eta_h \, \mathrm{d}x \, .
$$

Next, lemma 40 shows that $\|(\mathrm{Id} - \Pi_T^0)\eta_h\| \lesssim \|v_h, \eta_h\|_{U_h}$, thus by lemma 36 and the Cauchy-Schwarz inequality this yields

$$
b_{2h}^{\varepsilon}(I_{\Sigma_h}\sigma - \sigma, (v_h, \eta_h)) \lesssim \frac{1}{\sqrt{\nu}} \left(\|I_{\Sigma_h}\sigma - \sigma\|_{\Sigma_h} + \sqrt{\sum_{F \in \mathcal{F}_h} h \|(I_{\Sigma_h}\sigma - \sigma)_{nt}\|_F^2} \right) \sqrt{\nu} \|v_h, \eta_h\|_{U_h} .
$$

For the terms including the bilinear forms a and b_1 we proceed as in the proof of theorem 23. Adding up all results from above yields

$$
\begin{aligned}
&B_{\varepsilon}(I_{V_h}u - u, I_{W_h}\gamma - \gamma, I_{\Sigma_h}\sigma - \sigma, \Pi^k p - p; v_h, \eta_h, \tau_h, q_h) \\
&\lesssim \left(\|(0, I_{W_h}\gamma - \gamma, I_{\Sigma_h}\sigma - \sigma, \Pi^k p - p)\|_* + 1/\sqrt{\nu} \|I_{\Sigma_h}\sigma - \sigma\|_{\Sigma_h^\dagger, nt} \right) \|(v_h, \eta_h, \tau_h, 0)\|_* .
\end{aligned}
$$

We conclude with the interpolation properties given by lemma 28, theorem 21 and the optimal approximation properties of the L^2-projection, see lemma 30. The improved convergence order of the L^2-norm error follows with an Aubin-Nitsche technique similarly as in the proof of theorem 25. □

Theorem 31 (Pressure robustness). *Assuming homogeneous boundary conditions* (6.8), *let* $u \in H^1(\Omega, \mathbb{R}^d) \cap H^m(\mathcal{T}_h, \mathbb{R}^d)$, $\sigma \in H^1(\Omega, \mathbb{D}) \cap H^{m-1}(\mathcal{T}_h, \mathbb{D})$ *and* $\gamma = \omega(u) \in L^2(\Omega, \mathbb{R}) \cap H^{m-1}(\mathcal{T}_h, \mathbb{R})$ *be the exact solutions of the symmetric mixed Stokes equations* (7.1)*. Further, let* u_h, σ_h *and* γ_h *be the solutions of the MCS method with weakly imposed symmetry* (7.9)*. For* $s = \min(m-1, k+1)$ *there holds*

$$
\frac{1}{\nu} \|\sigma - \sigma_h\|_{\Sigma_h^\dagger} + \|u_h - I_{V_h}u, \gamma - \gamma_h\|_{U_h} \lesssim h^s \left(\frac{1}{\nu} \|\sigma\|_{H^s(\mathcal{T}_h)} + \|\gamma\|_{H^s(\mathcal{T}_h)}\right).
$$

Proof. The proof follows along the lines of the proof of theorem 26. Note that discrete inf-sup stability with respect to the subspace of divergence free velocity functions V_h^0 is given by lemma 43, lemma 44 and lemma 39. □

Remark 6. *Using a Korn inequality as given by lemma 1 in [50] there exists a constant c_K such that*

$$\|v_h\|_{1,h} \leq c_K \|v_h\|_{1,h,sym} \quad \textit{for all } v_h \in V_h.$$

This yields for all $(v_h, \gamma_h) \in U_h$ the estimate

$$\|\gamma_h\|^2_{L^2(\Omega)} = \sum_{T \in \mathcal{T}_h} \|\gamma_h\|^2_T \leq \sum_{T \in \mathcal{T}_h} \|\omega(v_h) - \gamma_h\|^2_T + \sum_{T \in \mathcal{T}_h} \|\omega(v_h)\|^2_T \leq c_K \|u_h - I_{V_h} u, \gamma - \gamma_h\|^2_{U_h}.$$

Thus, the results of theorem 30 and theorem 31 can be extended to prove optimal convergence of $\|\gamma - \gamma_h\|_{L^2(\Omega)}$. However, in contrast to the results given by theorem 30 and theorem 31, the constant then depends on the Korn constant c_K.

We conclude this section with the definition of a local element-wise post processing as it is defined in section 6.3.4. To this end let $V_h^* = \mathcal{BDM}^{k+1}(\mathcal{T}_h)$ and $V_h^{*,-}$ as in (6.37). We define the minimization problem

$$u_h^{*,-} := \operatorname*{argmin}_{\substack{v_h^{*,-} \in V_h^{*,-} \\ I_{V_h}(v_h^{*,-} - u_h) = 0}} \|\nu \varepsilon(v_h^{*,-}) - \sigma_h\|^2_T, \tag{7.12}$$

and set $u_h^* := \mathcal{R}(u_h^{*,-})$. There holds the following result.

Theorem 32. *Assuming homogeneous boundary conditions (6.8), let $u \in H^1(\Omega, \mathbb{R}^d) \cap H^m(\mathcal{T}_h, \mathbb{R}^d)$ and $\sigma \in H^1(\Omega, \mathbb{D}) \cap H^{m-1}(\mathcal{T}_h, \mathbb{D})$ be the exact solutions of the symmetric mixed Stokes equations (7.1). Further, let u_h be the solutions of the MCS method with weakly imposed symmetry (7.9) and let u_h^* be the post processed solution defined as above. There holds $u_h^* \in V_h^*$ and $\operatorname{div}(u_h^*) = 0$. Further, for $s = \min(m-1, k+1)$ there holds*

$$\|u - u_h^*\|_{1,h,sym} \lesssim h^s \frac{1}{\nu} \|\sigma\|_{H^s(\mathcal{T}_h)}.$$

If we further assume that the solution of the dual problem (7.9) fulfills the regularity assumption 6.31, we have

$$\|u - u_h^*\|_{L^2(\Omega)} \lesssim h^{s+1} \frac{1}{\nu} \|\sigma\|_{H^s(\mathcal{T}_h)}.$$

Proof. Follows with the same steps as in theorem 27. Note that this includes stability of the reconstruction operator $\mathcal{R}$ in the norm $\|\cdot\|_{1,h,sym}$. This follows with the same steps as the proof of lemma 3.3 in [76] changing the gradient to the symmetric gradient. □

8 Numerical examples

In this chapter we present three numerical examples to verify our methods and to validate the findings of section 6.3 and section 7.4. All numerical examples were implemented within the finite element library NGSolve/Netgen, see [105, 104] and www.ngsolve.org. We added the finite element basis of $\Xi^k(\mathcal{T}_h)$ to the library and used the Python interface NGS-Py for the implementation of several examples.

To this end we define the computational domain by $\Omega = [0,1]^d$. We want to explore the velocity field driven by the volume force determined by $f = -\operatorname{div}(\sigma) + \nabla p$ with the exact solutions

$$u = (\operatorname{curl}(\psi_2)), \quad \text{and} \quad p := x^5 + y^5 - \frac{1}{3} \quad \text{for } d = 2, \tag{8.1}$$

$$u = (\operatorname{curl}(\psi_3, \psi_3, \psi_3)), \quad \text{and} \quad p := x^5 + y^5 + z^5 - \frac{1}{2} \quad \text{for } d = 3. \tag{8.2}$$

Here, $\psi_2 := x^2(x-1)^2y^2(y-1)^2$ and $\psi_3 := x^2(x-1)^2y^2(y-1)^2z^2(z-1)^2$ define a given potentials in two and three dimensions, respectively. Further, we choose the viscosity $\nu = 10^{-3}$. In both space dimensions the velocity solution fulfills homogeneous Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. Depending on the choice of σ we either solve the standard Stokes problem, see equation (4.6), or the symmetric version, see equation (4.41).

8.1 Optimal convergence rates

The first example is dedicated to the analysis of the convergence rates. In table 8.1 and table 8.2 we listed several errors for varying polynomial orders $k = 1, 2, 3, 4$ in the two-dimensional case for the choices $\sigma = \nu\nabla u$ and $\sigma = \nu\varepsilon(u)$, respectively. For the symmetric version we have further added the error of the vorticity approximation. Beside the errors the experimentally determined orders of convergence are given in brackets. As predicted by theorem 23 and theorem 30, the L^2-norm error of the stress variable σ_h, the pressure p_h and the vorticity ω_h (in the symmetric case, see also remark 6) converge with optimal order. In contrast to this, the L^2-norm error and the discrete H^1-seminorm error of the velocity solution u_h show a reduced order as the velocity was only approximated in the Raviart-Thomas space of order k. Note, however, that the post processed velocity solution u_h^* shows the expected improved accuracy as proven in theorem 27 and theorem 32. The same observation can be made for the three-dimensional case for varying polynomial orders $k = 1, 2, 3$ in table 8.3 and table 8.4. Note that all calculations were done with respect to the choice $\Gamma_{D,n} = \Gamma$ and $\Gamma_{N,t} = \emptyset$. Thus, whereas the normal part of the homogeneous boundary conditions of the velocity is incorporated as an essential boundary condition in the discrete velocity space, the homogeneous tangential velocity is given as a natural boundary condition.

8.2 Pressure robustness

The second example is dedicated to the analysis of pressure robustness. To this end we choose the same example as before, thus the exact solution is given by equation (8.1). Due to the construction of this example the right hand side naturally splits into two different parts given by

$$f = -\operatorname{div}(\sigma) + \nabla p.$$

Using the Helmholtz projector (see section 6.3.3) we immediately see that

$$\mathbb{H}(f) = -\operatorname{div}(\sigma) = -\operatorname{div}(\nu \nabla u),$$

thus the discrete velocity solution should only be steered by $-\operatorname{div}(\nu\nabla u)$ and the discrete pressure should (obviously) only depend on ∇p. In the following we study the behaviour of the presented methods in the case of varying viscosities. In particular, we are interested in what happens in the limit $\nu \to 0$. Then, the two parts of the force f scale differently as it was discussed in section 6.3.3. To validate our findings we compare the H^1-seminorm error of the velocity solution of our methods with the errors of the velocity solution of the standard Taylor-Hood method, see e.g., [20] and [53]. In order to compare the results we choose the polynomial orders such that the post processed solution u_h^*, which is defined according to section 6.3.4, and the solution of the Taylor-Hood method, denoted by u_h^{TH}, have the same convergence order. Further, we only present the results for the non-symmetric case $\sigma = \nu \nabla u$, as the symmetric case behaves similarly. In figure 8.1 we can observe that the error of the Taylor-Hood method increases as $\nu \to 0$ and behaves as if it was scaled by a factor $1/\nu$ for small values of ν. This is the phenomenon we discussed in section 6.3.3: Clearly, the Taylor-Hood method is not pressure robust (and does not provide exactly divergence-free numerical velocity). This results in a velocity error estimate as it is given by equation (6.36). In contrast, the velocity errors of the MCS method (see equation (6.6)) appear not to be influenced by varying values of ν. This behaviour is observed for several polynomial orders and matches the predictions of theorem 26.

8.3 Mixed boundary conditions

In the last example we want to show that the MCS method is well suited for all different kinds of boundary conditions as given in (4.6). To this end we again solve the Stokes equations with the exact solutions denoted by (8.1), but with a different choice of the boundaries. In particular, we set in two dimensions

$$\Gamma_{D,n} = \big((0,1)\times\{0\}\big) \cup \big(\{1\}\times(0,1)\big) \quad \text{and} \quad \Gamma_{D,t} = \big((0,1)\times\{1\}\big) \cup \big(\{1\}\times(0,1)\big), \tag{8.3}$$

and in three dimensions

$$\begin{aligned} \Gamma_{D,n} =& \big((0,1)\times(0,1)\times\{0\}\big) \cup \big(\{1\}\times(0,1)\times(0,1)\big) \\ & \cup \big((0,1)\times\{0\}\times(0,1)\big) \cup \big((0,1)\times\{1\}\times(0,1)\big) \\ \Gamma_{D,t} =& \cup \big(\{1\}\times(0,1)\times(0,1)\big) \cup \big((0,1)\times(0,1)\times\{1\}\big). \end{aligned} \tag{8.4}$$

The Neumann boundaries are then given by

$$\Gamma_{N,n} = \Gamma \setminus \Gamma_{D,n} \quad \text{and} \quad \Gamma_{N,t} = \Gamma \setminus \Gamma_{D,t}.$$

According to (4.1) this results in all four different types of boundary conditions. As discussed in chapter 5, our finite element spaces are suited to incorporate the essential boundary conditions (4.6d) and (4.6g) into the finite element spaces V_h and Σ_h, respectively. On $\Gamma_{N,n}$ we add the right hand side term

$$\int_{\Gamma_{N,n}} (-\sigma_{nn} + p) v_n \, ds,$$

where σ and p are the exact solutions given by (8.1). In table 8.5 we present the finite element error for the symmetric case in two dimensions for the polynomial orders $k = 2, 3, 4$. Similarly as in the first example the L^2-norm error of the discrete stress, the pressure and the vorticity converge with optimal order. Further, the H^1-seminorm error of the post processed velocity solution u_h^* shows the enhanced accuracy. This matches the predictions of theorem 27. The same observation can be made also for the non-symmetric case in three dimensions for the polynomial orders $k = 2, 3$, see table 8.6. Note, however that in both cases the L^2-norm error of the post processed solution does not show the increased convergence rate $\mathcal{O}(h^{k+2})$. It seems that the regularity assumption (6.31) is not fulfilled. In the work [60] the authors discuss the regularity of Stokes flows on polygonal domains in two dimensions. In particular, they quote the works [91, 92], saying that the velocity solution of the Stokes problem shows a reduced regularity $H^r(\Omega, \mathbb{R}^d)$, with $r = 2$ for pure Dirichlet boundary conditions and $r < 2$ with other boundary conditions. Hence, we can not apply the Aubin-Nitsche techniques used to prove the increased convergence rate of $\|u - u_h^*\|_0$. Nevertheless, by the findings from above, we conclude that the presented methods are capable of handling all different kinds of boundary conditions.

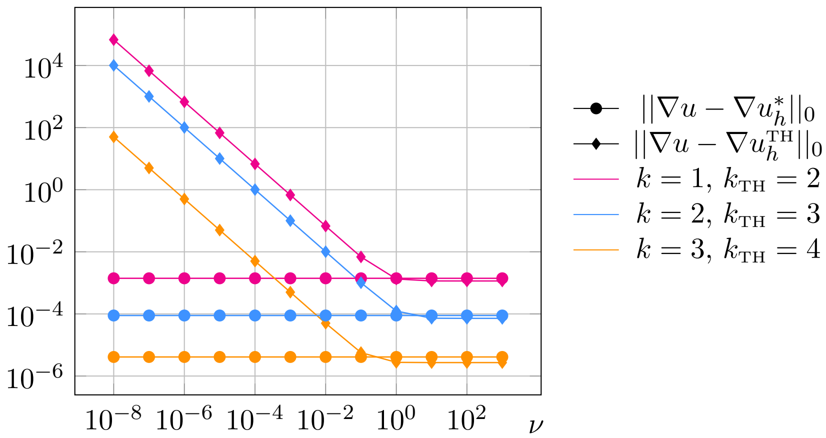

Figure 8.1: The H^1-seminorm error for the MCS method and a Taylor-Hood approximation for $k = 2, 3, 4$ and varying viscosity ν.

$\lvert\mathcal{T}\rvert$	$\lVert\nabla(u-u_h^*)\rVert_0$	(eoc)	$\lVert u-u_h^*\rVert_0$	(eoc)	$\lVert\sigma-\sigma_h\rVert_0$	(eoc)	$\lVert p-p_h\rVert_0$	(eoc)	$\lVert\nabla u-\nabla u_h\rVert_0$	(eoc)	$\lVert u-u_h\rVert_0$	(eoc)
						$k=1$						
20	$2.09\cdot 10^{-2}$	(–)	$1.04\cdot 10^{-3}$	(–)	$1.71\cdot 10^{-2}$	(–)	$3.44\cdot 10^{-2}$	(–)	$3.88\cdot 10^{-2}$	(–)	$1.83\cdot 10^{-3}$	(–)
80	$7.13\cdot 10^{-3}$	(1.6)	$1.97\cdot 10^{-4}$	(2.4)	$4.94\cdot 10^{-3}$	(1.8)	$9.36\cdot 10^{-3}$	(1.9)	$2.64\cdot 10^{-2}$	(0.6)	$7.59\cdot 10^{-4}$	(1.3)
320	$1.81\cdot 10^{-3}$	(2.0)	$2.65\cdot 10^{-5}$	(2.9)	$1.30\cdot 10^{-3}$	(1.9)	$2.38\cdot 10^{-3}$	(2.0)	$1.32\cdot 10^{-2}$	(1.0)	$1.85\cdot 10^{-4}$	(2.0)
1280	$4.65\cdot 10^{-4}$	(2.0)	$3.47\cdot 10^{-6}$	(2.9)	$3.32\cdot 10^{-4}$	(2.0)	$5.97\cdot 10^{-4}$	(2.0)	$6.62\cdot 10^{-3}$	(1.0)	$4.60\cdot 10^{-5}$	(2.0)
5120	$1.18\cdot 10^{-4}$	(2.0)	$4.45\cdot 10^{-7}$	(3.0)	$8.39\cdot 10^{-5}$	(2.0)	$1.49\cdot 10^{-4}$	(2.0)	$3.31\cdot 10^{-3}$	(1.0)	$1.15\cdot 10^{-5}$	(2.0)
						$k=2$						
20	$3.87\cdot 10^{-3}$	(–)	$9.63\cdot 10^{-5}$	(–)	$2.42\cdot 10^{-3}$	(–)	$3.72\cdot 10^{-3}$	(–)	$1.71\cdot 10^{-2}$	(–)	$5.52\cdot 10^{-4}$	(–)
80	$8.09\cdot 10^{-4}$	(2.3)	$1.13\cdot 10^{-5}$	(3.1)	$5.15\cdot 10^{-4}$	(2.2)	$5.31\cdot 10^{-4}$	(2.8)	$4.91\cdot 10^{-3}$	(1.8)	$8.12\cdot 10^{-5}$	(2.8)
320	$1.16\cdot 10^{-4}$	(2.8)	$7.84\cdot 10^{-7}$	(3.8)	$6.74\cdot 10^{-5}$	(2.9)	$6.75\cdot 10^{-5}$	(3.0)	$1.27\cdot 10^{-3}$	(2.0)	$1.02\cdot 10^{-5}$	(3.0)
1280	$1.48\cdot 10^{-5}$	(3.0)	$5.02\cdot 10^{-8}$	(4.0)	$8.52\cdot 10^{-6}$	(3.0)	$8.47\cdot 10^{-6}$	(3.0)	$3.20\cdot 10^{-4}$	(2.0)	$1.28\cdot 10^{-6}$	(3.0)
5120	$1.85\cdot 10^{-6}$	(3.0)	$3.16\cdot 10^{-9}$	(4.0)	$1.07\cdot 10^{-6}$	(3.0)	$1.06\cdot 10^{-6}$	(3.0)	$8.01\cdot 10^{-5}$	(2.0)	$1.61\cdot 10^{-7}$	(3.0)
						$k=3$						
20	$8.25\cdot 10^{-4}$	(–)	$1.48\cdot 10^{-5}$	(–)	$4.01\cdot 10^{-4}$	(–)	$7.20\cdot 10^{-5}$	(–)	$3.04\cdot 10^{-3}$	(–)	$4.26\cdot 10^{-5}$	(–)
80	$9.71\cdot 10^{-5}$	(3.1)	$8.05\cdot 10^{-7}$	(4.2)	$3.71\cdot 10^{-5}$	(3.4)	$5.70\cdot 10^{-6}$	(3.7)	$7.36\cdot 10^{-4}$	(2.0)	$6.04\cdot 10^{-6}$	(2.8)
320	$5.91\cdot 10^{-6}$	(4.0)	$2.49\cdot 10^{-8}$	(5.0)	$2.40\cdot 10^{-6}$	(3.9)	$3.65\cdot 10^{-7}$	(4.0)	$9.48\cdot 10^{-5}$	(3.0)	$3.91\cdot 10^{-7}$	(4.0)
1280	$3.70\cdot 10^{-7}$	(4.0)	$7.81\cdot 10^{-10}$	(5.0)	$1.53\cdot 10^{-7}$	(4.0)	$2.29\cdot 10^{-8}$	(4.0)	$1.18\cdot 10^{-5}$	(3.0)	$2.44\cdot 10^{-8}$	(4.0)
5120	$2.32\cdot 10^{-8}$	(4.0)	$2.45\cdot 10^{-11}$	(5.0)	$9.66\cdot 10^{-9}$	(4.0)	$1.44\cdot 10^{-9}$	(4.0)	$1.48\cdot 10^{-6}$	(3.0)	$1.52\cdot 10^{-9}$	(4.0)
						$k=4$						
20	$5.85\cdot 10^{-5}$	(–)	$6.52\cdot 10^{-7}$	(–)	$2.27\cdot 10^{-5}$	(–)	$8.79\cdot 10^{-6}$	(–)	$6.77\cdot 10^{-4}$	(–)	$8.17\cdot 10^{-6}$	(–)
80	$5.18\cdot 10^{-6}$	(3.5)	$3.20\cdot 10^{-8}$	(4.3)	$2.41\cdot 10^{-6}$	(3.2)	$4.36\cdot 10^{-7}$	(4.3)	$7.59\cdot 10^{-5}$	(3.2)	$4.15\cdot 10^{-7}$	(4.3)
320	$1.71\cdot 10^{-7}$	(4.9)	$5.08\cdot 10^{-10}$	(6.0)	$7.45\cdot 10^{-8}$	(5.0)	$1.36\cdot 10^{-8}$	(5.0)	$4.63\cdot 10^{-6}$	(4.0)	$1.25\cdot 10^{-8}$	(5.1)
1280	$5.41\cdot 10^{-9}$	(5.0)	$7.97\cdot 10^{-12}$	(6.0)	$2.33\cdot 10^{-9}$	(5.0)	$4.26\cdot 10^{-10}$	(5.0)	$2.86\cdot 10^{-7}$	(4.0)	$3.84\cdot 10^{-10}$	(5.0)
5120	$1.70\cdot 10^{-10}$	(5.0)	$1.25\cdot 10^{-13}$	(6.0)	$7.29\cdot 10^{-11}$	(5.0)	$1.33\cdot 10^{-11}$	(5.0)	$1.78\cdot 10^{-8}$	(4.0)	$1.20\cdot 10^{-11}$	(5.0)

Table 8.1: Finite element errors for the solutions of the MCS method (6.6) (non-symmetric case) for the the problem (8.1) in the two-dimensional case with $\nu = 1e^{-3}$ and polynomial orders $k = 1, 2, 3, 4$.

$\|\mathcal{T}\|$	$\|\varepsilon(u-u_h^*)\|_0$	(eoc)	$\|u-u_h^*\|_0$	(eoc)	$\|\sigma-\sigma_h\|_0$	(eoc)	$\|p-p_h\|_0$	(eoc)	$\|\omega-\omega_h\|_0$	(eoc)	$\|\varepsilon(u)-\varepsilon(u_h)\|_0$	(eoc)	$\|u-u_h\|_0$	(eoc)
							$k=1$							
20	$3.53\cdot 10^{-3}$	(–)	$1.65\cdot 10^{-4}$	(–)	$3.58\cdot 10^{-3}$	(–)	$9.36\cdot 10^{-3}$	(–)	$3.22\cdot 10^{-3}$	(–)	$2.09\cdot 10^{-2}$	(–)	$7.63\cdot 10^{-4}$	(–)
80	$9.50\cdot 10^{-4}$	(1.9)	$2.39\cdot 10^{-5}$	(2.8)	$9.41\cdot 10^{-4}$	(1.9)	$2.38\cdot 10^{-3}$	(2.0)	$9.24\cdot 10^{-4}$	(1.8)	$1.05\cdot 10^{-2}$	(1.0)	$1.86\cdot 10^{-4}$	(2.0)
320	$2.52\cdot 10^{-4}$	(1.9)	$3.40\cdot 10^{-6}$	(2.8)	$2.46\cdot 10^{-4}$	(1.9)	$5.97\cdot 10^{-4}$	(2.0)	$2.57\cdot 10^{-4}$	(1.8)	$5.26\cdot 10^{-3}$	(1.0)	$4.61\cdot 10^{-5}$	(2.0)
1280	$6.53\cdot 10^{-5}$	(1.9)	$4.61\cdot 10^{-7}$	(2.9)	$6.30\cdot 10^{-5}$	(2.0)	$1.49\cdot 10^{-4}$	(2.0)	$6.86\cdot 10^{-5}$	(1.9)	$2.63\cdot 10^{-3}$	(1.0)	$1.15\cdot 10^{-5}$	(2.0)
5120	$3.53\cdot 10^{-3}$	(−5.8)	$1.65\cdot 10^{-4}$	(−8.5)	$3.58\cdot 10^{-3}$	(−5.8)	$9.36\cdot 10^{-3}$	(−6.0)	$3.22\cdot 10^{-3}$	(−5.6)	$2.09\cdot 10^{-2}$	(−3.0)	$7.63\cdot 10^{-4}$	(−6.1)
							$k=2$							
20	$2.22\cdot 10^{-3}$	(–)	$1.00\cdot 10^{-4}$	(–)	$1.81\cdot 10^{-3}$	(–)	$3.72\cdot 10^{-3}$	(–)	$1.46\cdot 10^{-3}$	(–)	$1.35\cdot 10^{-2}$	(–)	$5.53\cdot 10^{-4}$	(–)
80	$5.03\cdot 10^{-4}$	(2.1)	$1.06\cdot 10^{-5}$	(3.2)	$3.72\cdot 10^{-4}$	(2.3)	$5.31\cdot 10^{-4}$	(2.8)	$2.77\cdot 10^{-4}$	(2.4)	$3.85\cdot 10^{-3}$	(1.8)	$8.11\cdot 10^{-5}$	(2.8)
320	$6.66\cdot 10^{-5}$	(2.9)	$7.74\cdot 10^{-7}$	(3.8)	$5.06\cdot 10^{-5}$	(2.9)	$6.75\cdot 10^{-5}$	(3.0)	$4.14\cdot 10^{-5}$	(2.7)	$1.00\cdot 10^{-3}$	(1.9)	$1.02\cdot 10^{-5}$	(3.0)
1280	$8.36\cdot 10^{-6}$	(3.0)	$4.93\cdot 10^{-8}$	(4.0)	$6.37\cdot 10^{-6}$	(3.0)	$8.47\cdot 10^{-6}$	(3.0)	$5.20\cdot 10^{-6}$	(3.0)	$2.53\cdot 10^{-4}$	(2.0)	$1.28\cdot 10^{-6}$	(3.0)
5120	$1.04\cdot 10^{-6}$	(3.0)	$3.08\cdot 10^{-9}$	(4.0)	$7.96\cdot 10^{-7}$	(3.0)	$1.06\cdot 10^{-6}$	(3.0)	$6.45\cdot 10^{-7}$	(3.0)	$6.34\cdot 10^{-5}$	(2.0)	$1.61\cdot 10^{-7}$	(3.0)
							$k=3$							
20	$4.15\cdot 10^{-4}$	(–)	$1.44\cdot 10^{-5}$	(–)	$2.38\cdot 10^{-4}$	(–)	$7.20\cdot 10^{-5}$	(–)	$2.24\cdot 10^{-4}$	(–)	$2.22\cdot 10^{-3}$	(–)	$4.30\cdot 10^{-5}$	(–)
80	$4.78\cdot 10^{-5}$	(3.1)	$8.44\cdot 10^{-7}$	(4.1)	$2.70\cdot 10^{-5}$	(3.1)	$5.70\cdot 10^{-6}$	(3.7)	$2.63\cdot 10^{-5}$	(3.1)	$5.76\cdot 10^{-4}$	(1.9)	$6.08\cdot 10^{-6}$	(2.8)
320	$2.96\cdot 10^{-6}$	(4.0)	$2.61\cdot 10^{-8}$	(5.0)	$1.73\cdot 10^{-6}$	(4.0)	$3.65\cdot 10^{-7}$	(4.0)	$1.71\cdot 10^{-6}$	(3.9)	$7.33\cdot 10^{-5}$	(3.0)	$3.91\cdot 10^{-7}$	(4.0)
1280	$1.86\cdot 10^{-7}$	(4.0)	$8.30\cdot 10^{-10}$	(5.0)	$1.12\cdot 10^{-7}$	(4.0)	$2.29\cdot 10^{-8}$	(4.0)	$1.13\cdot 10^{-7}$	(3.9)	$9.12\cdot 10^{-6}$	(3.0)	$2.44\cdot 10^{-8}$	(4.0)
5120	$1.17\cdot 10^{-8}$	(4.0)	$2.63\cdot 10^{-11}$	(5.0)	$7.10\cdot 10^{-9}$	(4.0)	$1.44\cdot 10^{-9}$	(4.0)	$7.32\cdot 10^{-9}$	(4.0)	$1.14\cdot 10^{-6}$	(3.0)	$1.52\cdot 10^{-9}$	(4.0)
							$k=4$							
20	$2.58\cdot 10^{-5}$	(–)	$9.67\cdot 10^{-7}$	(–)	$2.24\cdot 10^{-5}$	(–)	$8.79\cdot 10^{-6}$	(–)	$8.41\cdot 10^{-6}$	(–)	$4.96\cdot 10^{-4}$	(–)	$8.17\cdot 10^{-6}$	(–)
80	$2.96\cdot 10^{-6}$	(3.1)	$4.37\cdot 10^{-8}$	(4.5)	$1.98\cdot 10^{-6}$	(3.5)	$4.36\cdot 10^{-7}$	(4.3)	$1.03\cdot 10^{-6}$	(3.0)	$5.71\cdot 10^{-5}$	(3.1)	$4.14\cdot 10^{-7}$	(4.3)
320	$9.00\cdot 10^{-8}$	(5.0)	$7.29\cdot 10^{-10}$	(5.9)	$6.20\cdot 10^{-8}$	(5.0)	$1.36\cdot 10^{-8}$	(5.0)	$3.28\cdot 10^{-8}$	(5.0)	$3.48\cdot 10^{-6}$	(4.0)	$1.24\cdot 10^{-8}$	(5.1)
1280	$2.79\cdot 10^{-9}$	(5.0)	$1.15\cdot 10^{-11}$	(6.0)	$1.93\cdot 10^{-9}$	(5.0)	$4.26\cdot 10^{-10}$	(5.0)	$1.03\cdot 10^{-9}$	(5.0)	$2.14\cdot 10^{-7}$	(4.0)	$3.84\cdot 10^{-10}$	(5.0)
5120	$8.68\cdot 10^{-11}$	(5.0)	$1.80\cdot 10^{-13}$	(6.0)	$6.01\cdot 10^{-11}$	(5.0)	$1.33\cdot 10^{-11}$	(5.0)	$3.20\cdot 10^{-11}$	(5.0)	$1.33\cdot 10^{-8}$	(4.0)	$1.20\cdot 10^{-11}$	(5.0)

Table 8.2: Finite element errors for the solutions of the MCS method (7.1) (symmetric case) for the problem (8.1) in the two-dimensional case with $\nu = 1e^{-3}$ and polynomial orders $k = 1, 2, 3, 4$.

$\lvert\mathcal{T}\rvert$	$\lVert\nabla(u-u_h^*)\rVert_0$	(eoc)	$\lVert u-u_h^*\rVert_0$	(eoc)	$\lVert\sigma-\sigma_h\rVert_0$	(eoc)	$\lVert p-p_h\rVert_0$	(eoc)
				$k=1$				
28	$2.46\cdot 10^{-3}$	(–)	$1.49\cdot 10^{-4}$	(–)	$2.23\cdot 10^{-3}$	(–)	$7.45\cdot 10^{-2}$	(–)
224	$2.03\cdot 10^{-3}$	(0.3)	$9.38\cdot 10^{-5}$	(0.7)	$1.31\cdot 10^{-3}$	(0.8)	$3.11\cdot 10^{-2}$	(1.3)
1792	$9.19\cdot 10^{-4}$	(1.1)	$2.21\cdot 10^{-5}$	(2.1)	$4.57\cdot 10^{-4}$	(1.5)	$9.52\cdot 10^{-3}$	(1.7)
14336	$2.61\cdot 10^{-4}$	(1.8)	$3.20\cdot 10^{-6}$	(2.8)	$1.27\cdot 10^{-4}$	(1.8)	$2.53\cdot 10^{-3}$	(1.9)
114688	$6.71\cdot 10^{-5}$	(2.0)	$4.10\cdot 10^{-7}$	(3.0)	$3.26\cdot 10^{-5}$	(2.0)	$6.44\cdot 10^{-4}$	(2.0)
				$k=2$				
28	$1.46\cdot 10^{-3}$	(–)	$6.10\cdot 10^{-5}$	(–)	$1.20\cdot 10^{-3}$	(–)	$6.75\cdot 10^{-3}$	(–)
224	$5.38\cdot 10^{-4}$	(1.4)	$1.14\cdot 10^{-5}$	(2.4)	$2.49\cdot 10^{-4}$	(2.3)	$1.55\cdot 10^{-3}$	(2.1)
1792	$1.75\cdot 10^{-4}$	(1.6)	$1.90\cdot 10^{-6}$	(2.6)	$4.91\cdot 10^{-5}$	(2.3)	$2.62\cdot 10^{-4}$	(2.6)
14336	$2.46\cdot 10^{-5}$	(2.8)	$1.33\cdot 10^{-7}$	(3.8)	$6.75\cdot 10^{-6}$	(2.9)	$3.53\cdot 10^{-5}$	(2.9)
114688	$3.24\cdot 10^{-6}$	(2.9)	$8.84\cdot 10^{-9}$	(3.9)	$8.85\cdot 10^{-7}$	(2.9)	$4.50\cdot 10^{-6}$	(3.0)
				$k=3$				
28	$3.26\cdot 10^{-4}$	(–)	$1.16\cdot 10^{-5}$	(–)	$2.34\cdot 10^{-4}$	(–)	$2.37\cdot 10^{-3}$	(–)
224	$1.39\cdot 10^{-4}$	(1.2)	$2.24\cdot 10^{-6}$	(2.4)	$5.61\cdot 10^{-5}$	(2.1)	$2.51\cdot 10^{-4}$	(3.2)
1792	$2.10\cdot 10^{-5}$	(2.7)	$1.56\cdot 10^{-7}$	(3.8)	$6.39\cdot 10^{-6}$	(3.1)	$2.98\cdot 10^{-5}$	(3.1)
14336	$1.83\cdot 10^{-6}$	(3.5)	$6.65\cdot 10^{-9}$	(4.6)	$5.33\cdot 10^{-7}$	(3.6)	$2.05\cdot 10^{-6}$	(3.9)
114688	$1.24\cdot 10^{-7}$	(3.9)	$2.21\cdot 10^{-10}$	(4.9)	$3.54\cdot 10^{-8}$	(3.9)	$1.31\cdot 10^{-7}$	(4.0)

Table 8.3: Finite element errors for the solutions of the MCS method (6.6) (non-symmetric case) for the problem (8.1) in the three-dimensional case with $\nu = 1e^{-3}$ and polynomial orders $k = 1, 2, 3$.

$\lvert\mathcal{T}\rvert$	$\lVert\varepsilon(u-u_h^*)\rVert_0$	(eoc)	$\lVert u-u_h^*\rVert_0$	(eoc)	$\lVert\sigma-\sigma_h\rVert_0$	(eoc)	$\lVert p-p_h\rVert_0$	(eoc)	$\lVert\omega-\omega_h\rVert_0$	(eoc)
					$k=1$					
28	$1.53\cdot 10^{-3}$	(–)	$1.36\cdot 10^{-4}$	(–)	$1.46\cdot 10^{-3}$	(–)	$7.45\cdot 10^{-2}$	(–)	$1.06\cdot 10^{-3}$	(–)
224	$8.11\cdot 10^{-4}$	(0.9)	$5.42\cdot 10^{-5}$	(1.3)	$8.15\cdot 10^{-4}$	(0.8)	$3.11\cdot 10^{-2}$	(1.3)	$6.70\cdot 10^{-4}$	(0.7)
1792	$3.17\cdot 10^{-4}$	(1.4)	$1.32\cdot 10^{-5}$	(2.0)	$3.16\cdot 10^{-4}$	(1.4)	$9.52\cdot 10^{-3}$	(1.7)	$3.17\cdot 10^{-4}$	(1.1)
14336	$9.20\cdot 10^{-5}$	(1.8)	$1.93\cdot 10^{-6}$	(2.8)	$8.98\cdot 10^{-5}$	(1.8)	$2.53\cdot 10^{-3}$	(1.9)	$9.05\cdot 10^{-5}$	(1.8)
114688	$2.38\cdot 10^{-5}$	(1.9)	$2.48\cdot 10^{-7}$	(3.0)	$2.31\cdot 10^{-5}$	(2.0)	$6.44\cdot 10^{-4}$	(2.0)	$2.34\cdot 10^{-5}$	(1.9)
					$k=2$					
28	$5.01\cdot 10^{-4}$	(–)	$4.30\cdot 10^{-5}$	(–)	$5.76\cdot 10^{-4}$	(–)	$6.75\cdot 10^{-3}$	(–)	$4.88\cdot 10^{-4}$	(–)
224	$2.08\cdot 10^{-4}$	(1.3)	$9.65\cdot 10^{-6}$	(2.2)	$1.58\cdot 10^{-4}$	(1.9)	$1.55\cdot 10^{-3}$	(2.1)	$1.35\cdot 10^{-4}$	(1.9)
1792	$5.70\cdot 10^{-5}$	(1.9)	$1.51\cdot 10^{-6}$	(2.7)	$3.87\cdot 10^{-5}$	(2.0)	$2.62\cdot 10^{-4}$	(2.6)	$3.57\cdot 10^{-5}$	(1.9)
14336	$7.87\cdot 10^{-6}$	(2.9)	$1.06\cdot 10^{-7}$	(3.8)	$5.42\cdot 10^{-6}$	(2.8)	$3.53\cdot 10^{-5}$	(2.9)	$5.24\cdot 10^{-6}$	(2.8)
114688	$1.04\cdot 10^{-6}$	(2.9)	$7.00\cdot 10^{-9}$	(3.9)	$7.15\cdot 10^{-7}$	(2.9)	$4.50\cdot 10^{-6}$	(3.0)	$7.02\cdot 10^{-7}$	(2.9)
					$k=3$					
28	$1.76\cdot 10^{-4}$	(–)	$1.28\cdot 10^{-5}$	(–)	$1.67\cdot 10^{-4}$	(–)	$2.37\cdot 10^{-3}$	(–)	$1.27\cdot 10^{-4}$	(–)
224	$5.75\cdot 10^{-5}$	(1.6)	$2.42\cdot 10^{-6}$	(2.4)	$4.43\cdot 10^{-5}$	(1.9)	$2.51\cdot 10^{-4}$	(3.2)	$2.98\cdot 10^{-5}$	(2.1)
1792	$6.81\cdot 10^{-6}$	(3.1)	$1.68\cdot 10^{-7}$	(3.8)	$4.95\cdot 10^{-6}$	(3.2)	$2.98\cdot 10^{-5}$	(3.1)	$3.62\cdot 10^{-6}$	(3.0)
14336	$5.74\cdot 10^{-7}$	(3.6)	$7.31\cdot 10^{-9}$	(4.5)	$4.11\cdot 10^{-7}$	(3.6)	$2.05\cdot 10^{-6}$	(3.9)	$3.02\cdot 10^{-7}$	(3.6)
114688	$3.98\cdot 10^{-8}$	(3.9)	$2.46\cdot 10^{-10}$	(4.9)	$2.76\cdot 10^{-8}$	(3.9)	$1.31\cdot 10^{-7}$	(4.0)	$2.03\cdot 10^{-8}$	(3.9)

Table 8.4: Finite element errors for the solutions of the MCS method (7.1) (symmetric case) for the problem (8.1) in the three-dimensional case with $\nu = 1e^{-3}$ and polynomial orders $k = 1, 2, 3$.

$\lvert\mathcal{T}\rvert$	$\lVert\varepsilon(u-u_h^*)\rVert_0$	(eoc)	$\lVert u-u_h^*\rVert_0$	(eoc)	$\lVert\sigma-\sigma_h\rVert_0$	(eoc)	$\lVert p-p_h\rVert_0$	(eoc)	$\lVert\omega-\omega_h\rVert_0$	(eoc)
					$k=2$					
28	$7.62\cdot 10^{-2}$	(–)	$5.16\cdot 10^{-3}$	(–)	$6.44\cdot 10^{-2}$	(–)	$3.72\cdot 10^{-3}$	(–)	$5.02\cdot 10^{-2}$	(–)
224	$3.67\cdot 10^{-3}$	(4.4)	$1.23\cdot 10^{-4}$	(5.4)	$3.00\cdot 10^{-3}$	(4.4)	$5.31\cdot 10^{-4}$	(2.8)	$2.43\cdot 10^{-3}$	(4.4)
1792	$1.73\cdot 10^{-4}$	(4.4)	$2.82\cdot 10^{-6}$	(5.4)	$1.43\cdot 10^{-4}$	(4.4)	$6.75\cdot 10^{-5}$	(3.0)	$1.15\cdot 10^{-4}$	(4.4)
14336	$1.09\cdot 10^{-5}$	(4.0)	$7.75\cdot 10^{-8}$	(5.2)	$8.92\cdot 10^{-6}$	(4.0)	$8.47\cdot 10^{-6}$	(3.0)	$6.88\cdot 10^{-6}$	(4.1)
114688	$1.07\cdot 10^{-6}$	(3.3)	$3.30\cdot 10^{-9}$	(4.6)	$8.63\cdot 10^{-7}$	(3.4)	$1.06\cdot 10^{-6}$	(3.0)	$6.32\cdot 10^{-7}$	(3.4)
					$k=3$					
28	$1.97\cdot 10^{-3}$	(–)	$9.90\cdot 10^{-5}$	(–)	$1.46\cdot 10^{-3}$	(–)	$7.19\cdot 10^{-5}$	(–)	$1.01\cdot 10^{-3}$	(–)
224	$5.83\cdot 10^{-5}$	(5.1)	$1.35\cdot 10^{-6}$	(6.2)	$4.17\cdot 10^{-5}$	(5.1)	$5.70\cdot 10^{-6}$	(3.7)	$3.65\cdot 10^{-5}$	(4.8)
1792	$2.97\cdot 10^{-6}$	(4.3)	$2.83\cdot 10^{-8}$	(5.6)	$1.91\cdot 10^{-6}$	(4.4)	$3.65\cdot 10^{-7}$	(4.0)	$1.77\cdot 10^{-6}$	(4.4)
14336	$1.82\cdot 10^{-7}$	(4.0)	$8.25\cdot 10^{-10}$	(5.1)	$1.15\cdot 10^{-7}$	(4.1)	$2.29\cdot 10^{-8}$	(4.0)	$1.08\cdot 10^{-7}$	(4.0)
114688	$1.15\cdot 10^{-8}$	(4.0)	$2.73\cdot 10^{-11}$	(4.9)	$7.19\cdot 10^{-9}$	(4.0)	$1.44\cdot 10^{-9}$	(4.0)	$6.85\cdot 10^{-9}$	(4.0)
					$k=4$					
28	$2.49\cdot 10^{-5}$	(–)	$1.07\cdot 10^{-6}$	(–)	$2.46\cdot 10^{-5}$	(–)	$8.79\cdot 10^{-6}$	(–)	$9.69\cdot 10^{-6}$	(–)
224	$3.04\cdot 10^{-6}$	(3.0)	$4.74\cdot 10^{-8}$	(4.5)	$2.16\cdot 10^{-6}$	(3.5)	$4.36\cdot 10^{-7}$	(4.3)	$1.11\cdot 10^{-6}$	(3.1)
1792	$9.08\cdot 10^{-8}$	(5.1)	$7.46\cdot 10^{-10}$	(6.0)	$6.54\cdot 10^{-8}$	(5.0)	$1.36\cdot 10^{-8}$	(5.0)	$3.33\cdot 10^{-8}$	(5.1)
14336	$2.78\cdot 10^{-9}$	(5.0)	$2.57\cdot 10^{-11}$	(4.9)	$2.00\cdot 10^{-9}$	(5.0)	$4.26\cdot 10^{-10}$	(5.0)	$1.01\cdot 10^{-9}$	(5.0)
114688	$9.67\cdot 10^{-11}$	(4.8)	$2.37\cdot 10^{-11}$	(0.1)	$6.76\cdot 10^{-11}$	(4.9)	$1.33\cdot 10^{-11}$	(5.0)	$4.38\cdot 10^{-11}$	(4.5)

Table 8.5: Finite element errors for the solutions of the MCS method (7.1) (symmetric case) for the problem (8.1) in the two-dimensional case with $\nu = 1e^{-3}$ and polynomial orders $k = 2, 3, 4$, with boundary conditions according to the splitting (8.3).

$\lvert\mathcal{T}\rvert$	$\lVert\nabla(u-u_h^*)\rVert_0$	(eoc)	$\lVert u-u_h^*\rVert_0$	(eoc)	$\lVert\sigma-\sigma_h\rVert_0$	(eoc)	$\lVert p-p_h\rVert_0$	(eoc)
				$k=2$				
20	$1.53\cdot 10^{-3}$	(–)	$7.77\cdot 10^{-5}$	(–)	$1.26\cdot 10^{-3}$	(–)	$6.75\cdot 10^{-3}$	(–)
80	$5.59\cdot 10^{-4}$	(1.5)	$1.28\cdot 10^{-5}$	(2.6)	$2.57\cdot 10^{-4}$	(2.3)	$1.55\cdot 10^{-3}$	(2.1)
320	$1.78\cdot 10^{-4}$	(1.6)	$2.02\cdot 10^{-6}$	(2.7)	$5.03\cdot 10^{-5}$	(2.4)	$2.62\cdot 10^{-4}$	(2.6)
1280	$2.48\cdot 10^{-5}$	(2.8)	$1.59\cdot 10^{-7}$	(3.7)	$6.88\cdot 10^{-6}$	(2.9)	$3.53\cdot 10^{-5}$	(2.9)
5120	$3.26\cdot 10^{-6}$	(2.9)	$1.47\cdot 10^{-8}$	(3.4)	$8.98\cdot 10^{-7}$	(2.9)	$4.50\cdot 10^{-6}$	(3.0)
				$k=3$				
28	$3.30\cdot 10^{-4}$	(–)	$1.55\cdot 10^{-5}$	(–)	$2.38\cdot 10^{-4}$	(–)	$2.37\cdot 10^{-3}$	(–)
224	$1.49\cdot 10^{-4}$	(1.1)	$3.21\cdot 10^{-6}$	(2.3)	$5.87\cdot 10^{-5}$	(2.0)	$2.51\cdot 10^{-4}$	(3.2)
1792	$2.14\cdot 10^{-5}$	(2.8)	$2.44\cdot 10^{-7}$	(3.7)	$6.53\cdot 10^{-6}$	(3.2)	$2.98\cdot 10^{-5}$	(3.1)
14336	$1.85\cdot 10^{-6}$	(3.5)	$1.50\cdot 10^{-8}$	(4.0)	$5.40\cdot 10^{-7}$	(3.6)	$2.05\cdot 10^{-6}$	(3.9)
114688	$1.23\cdot 10^{-7}$	(3.9)	$9.25\cdot 10^{-10}$	(4.0)	$3.57\cdot 10^{-8}$	(3.9)	$1.31\cdot 10^{-7}$	(4.0)

Table 8.6: Finite element errors for the solutions of the MCS method (6.6) (non-symmetric case) for the problem (8.1) in the three-dimensional case with $\nu = 1e^{-3}$ and polynomial orders $k = 2, 3$, with boundary conditions according to the splitting (8.4).

9 Open questions

In this thesis we tried to establish a new understanding of the mathematical structure of the incompressible Stokes equations. Based on the new results in the continuous setting we introduced two finite element methods and presented a detailed a priori analysis. Nevertheless, the variational formulation and the stability analysis (in the continuous setting) that we presented in chapter 4 are based on the assumption that we impose Dirichlet boundary conditions for the velocity in normal direction on the whole boundary, thus $\Gamma_{D,n} = \Gamma$. However, as illustrated by the numerical examples in section 8, the corresponding finite element methods are suitable to deal with all different kinds of boundary conditions. As the discrete methods were motivated by the continuous setting, this suggests that the stability analysis for the continuous formulation can also be extended to the case $\Gamma_{D,n} \neq \Gamma$. In the following we try to motivate this setting and lead the reader to the point where, to the best knowledge of the authors, the (standard) theory can not be applied.

In order to incorporate partial boundary conditions the obvious choice of the velocity space is now given by $V = H_{0,\Gamma_{D,n}}(\mathrm{div}, \Omega)$. With similar findings as in chapter 4 we want that the divergence of stress variables can continuously act on velocity functions. Thus, the appropriate stress space is given by

$$H(\mathrm{curl}\,\mathrm{div}, \Omega)(\Gamma_{D,n}) := \{\sigma \in L^2(\Omega, \mathbb{R}^{d\times d}) : \mathrm{div}(\sigma) \in H_{0,\Gamma_{D,n}}(\mathrm{div}, \Omega)^*\},$$

with the norm

$$\|\sigma\|_{\mathrm{cd}}^2 := \|\sigma\|_{L^2(\Omega)}^2 + \left(\sup_{v \in H_{0,\Gamma_{D,n}}(\mathrm{div})} \frac{\langle \sigma, v\rangle_{H_{0,\Gamma_{D,n}}(\mathrm{div})}}{\|v\|_{H_{0,\Gamma_{D,n}}(\mathrm{div})}} \right)^2.$$

Then, similarly as before we define the stress space as the matrix trace-free subspace

$$\Sigma := \{\sigma \in H(\mathrm{curl}\,\mathrm{div}, \Omega)(\Gamma_{D,n}) : \mathrm{tr}(\sigma) = 0, \sigma_{nt} = 0 \text{ on } \Gamma_{N,t}\}.$$

Next, we aim to follow along the same steps as in section 4.3 to define a variational formulation. To this end we test equation (4.6a) with a test function $\tau \in \Sigma$. Assuming that τ can be approximated by a series of smooth functions τ_l with $l \in \mathbb{N}$ this then leads (assuming enough regularity of the exact solution u) to

$$\int_\Omega \frac{1}{\nu}\sigma : \tau_l \,\mathrm{d}x + \int_\Omega \mathrm{div}(\tau_l)\cdot u \,\mathrm{d}x - \int_{\Gamma_{N_n}} (\tau_l)_{nn} u_n \,\mathrm{d}s = 0 \quad \forall l \in \mathbb{N}.$$

The crucial question now is how the resulting equation looks like if we pass to the limit $l \to \infty$. In contrast to the derivation as in section 4.3, where the limit was given by equation (4.25), it is not obvious if we get a similar result, which would read as

$$\int_\Omega \frac{1}{\nu}\sigma : \tau_l \,\mathrm{d}x + \int_\Omega \mathrm{div}(\tau_l)\cdot u \,\mathrm{d}x - \int_{\Gamma_{N_n}} (\tau_l)_{nn} v_n \,\mathrm{d}s \overset{?}{\to} \int_\Omega \frac{1}{\nu}\sigma : \tau \,\mathrm{d}x + \langle \mathrm{div}(\tau), u\rangle_{H_{0,\Gamma_{D,n}}(\mathrm{div})}.$$

In particular, this is not trivial as there exists no continuous normal-normal trace operator that can be applied for functions in $H(\operatorname{curl}\operatorname{div},\Omega)(\Gamma_{D,n})$. In fact, this question is related to the definition of the *distributional divergence*. To this end let $\phi \in \{\psi \in \mathcal{C}^\infty(\Omega,\mathbb{R}^d) : \psi_n = 0 \text{ on } \Gamma_{D,n}\}$, then we would like to define

$$\langle \operatorname{div}(\tau), \phi\rangle_{H_{0,\Gamma_{D,n}}(\operatorname{div})} \overset{!}{:=} -\int \tau : \nabla\phi \,\mathrm{d}x + \int_{\Gamma\setminus\Gamma_{D,n}} \tau_{nt}\cdot\phi_t \,\mathrm{d}s, \tag{9.1}$$

where the last integral has to be understood as a duality pair. Then, assuming enough regularity, an integration by parts argument yields the identity that is needed to pass to the limit above. Although it seems feasible that the functional $\operatorname{div}(\tau) \in H_{0,\Gamma_{D,n}}(\operatorname{div})^*$ can be decomposed into a part which continuously acts on u on the domain Ω, and a part that continuously acts on u_n on the boundary $\Gamma\setminus\Gamma_{D,n}$, we are not aware of any decomposition results such that the above results can be defined rigorously.

Such a result would be the key ingredient that is needed to define a well-posed variational formulation and to give a precise stability analysis. If we assume that functions in $H(\operatorname{curl}\operatorname{div},\Omega)(\Gamma_{D,n})$ can be approximated by smooth functions and that the distributional divergence $\operatorname{div}(\tau)$ is given according to definition (9.1), the stability analysis would follow with very similar steps as in section 4.3.1. Continuity of the bilinear forms and kernel coercivity of $a(\cdot,\cdot)$ follow as in the proofs of lemma 8 and lemma 9, respectively. An inf-sup result for the term $\langle\operatorname{div}(\tau), u\rangle_{H_{0,\Gamma_{D,n}}(\operatorname{div})}$ can be proven as in the proof of lemma 12 and would include to solve the auxiliary problem: Find $\tilde{u} \in \tilde{V} := H^1(\Omega,\mathbb{R}^d)\cap H_{0,\Gamma_{D,n}}(\operatorname{div})$ such that

$$\int_\Omega \operatorname{dev}(\nabla\tilde{u}) : \operatorname{dev}(\nabla\tilde{v})\,\mathrm{d}x + \int_{\Gamma_{D,t}} \tilde{u}_t\cdot\tilde{v}_t\,\mathrm{d}s = \int_\Omega u\cdot\tilde{v}\,\mathrm{d}x + \int_\Omega \operatorname{div}(u)\operatorname{div}(\tilde{v})\,\mathrm{d}x \quad \forall \tilde{v}\in\tilde{V}.$$

According to lemma 11 this problem is solvable if $\Gamma_{D,n}\cap\Gamma_{D,t} \neq \emptyset$, which was one of the assumptions at the beginning of chapter 6. The Robin-type boundary conditions on the boundary $\Gamma_{D,t}$ are needed to show that the choice $\sigma = \operatorname{dev}(\nabla\tilde{u})$ is admissible, thus to prove $\langle\sigma, tw\rangle_{TW(\Gamma_{N,t})} = 0$ for all $tw \in TW(\Gamma_{N,t})$ and $\sigma\in\Sigma$. We hope that future contributions will tackle the analysis of dual spaces for Sobolev spaces with vanishing traces only on a part of the boundary such that a stability analysis in the case $\Gamma_{D,n}\neq\Gamma$ can be presented.

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Wissenschaftliche Publikationen

- J. Gopalakrishnan, P. L. Lederer, and J. Schöberl: *A mass conserving mixed stress formulation for Stokes flow with weakly imposed stress symmetry*, (arXiv:1901.04648)
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