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D I S S E R T A T I O N

# Computable structure theory with respect to equivalence relations

Ausgeführt zum Zwecke der Erlangung des akademischen Grades eines Doktors der Technischen Wissenschaften unter Leitung von

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# Kurzfassung

Die berechenbare Strukturtheorie beschäftigt sich mit dem Verhältnis zwischen strukturellen und algorithmischen Eigenschaften mathematischer Objekte. Diese Dissertation leistet Beiträge zu mehreren Forschungsrichtungen in diesem Feld, mit einem Fokus auf dem Verhältnis zwischen strukturellen und algorithmischen Eigenschaften von Strukturen, wenn wir sie unter schwachen strukturellen Äquivalenzrelationen betrachten.

Das erste Kapitel enthält eine Einleitung und eine Zusammenfassung der restlichen Kapitel. Im zweiten Kapitel untersuchen wir aufzählbare Funktoren, eine neue Reduktion zwischen Strukturen, welche es uns erlaubt Strukturen in Bezug auf ihre algorithmischen Eigenschaften zu vergleichen. Wir untersuchen die Verbindung von unserer Reduktion mit anderen bekannten Reduktionen. Unser wichtigstes Resultat in diesem Kapitel zeigt, dass es einen aufzählbaren Funktor zwischen einer Struktur $\mathcal{A}$ und einer Struktur $\mathcal{B}$ genau dann gibt, wenn $\mathcal{B}$ sich in $\mathcal{A}$ effektiv interpretieren lässt. Wir zeigen weiters, dass unter der üblichen Einschränkung, dass alle Strukturen als Universum die natürlichen Zahlen besitzen, aufzählbare und berechenbare Funktoren äquivalent sind.

Im dritten Kapitel untersuchen wir Gradspektren unter Bi-Einbettbarkeit und elementarer Bi-Einbettbarkeit. Das Gradspektrum einer gegebenen Struktur $\mathcal{A}$ unter einer Äquivalenzrelation $E$ ist die Menge der Turing grade der zu $\mathcal{A}$ E-äquivalenten Strukturen. Bi-Einbettbarkeit und elementare Bi-Einbettbarkeit sind wesentlich schwächere Äquivalenzrelationen als, die üblicherweise betrachtete, Isomorphie. Wir finden verschieden Beispiele von Familien von Turing graden die Bi-Einbettbarkeit und elementare Bi-Einbettbarkeitspektren sind. Unser Hauptresultat über Bi-Einbettbarkeitspektren ist eine komplette Klassifikation der Spektren von linearen Ordnungen und einer Unterklasse der stark lokal-endlichen Graphen. Für elementare Bi-Einbettbarkeitspektren zeigen wir, dass es Familien von Turing graden gibt, welche elementare Bi-Einbettbarkeitspektren sind aber keine Theoriespektren und vice versa.

Im vierten und letzten Kapitel beschäftigen wir uns mit der Komplexität von Einbettungen zwischen bi-einbettbaren Strukturen. Die wichtigsten Eigenschaften hier sind die berechenbare bi-einbettbare Kategorizität und der Grad der bi-einbettbaren Kategorizität. Wir zeigen, dass jede Äquivalenzstruktur Grad der bi-einbettbaren Kategorizität

$\mathbf{0}$, $\mathbf{0}'$ oder $\mathbf{0}''$ hat und geben eine vollständige Charakterisierung der Äquivalenzrelationen die einen dieser Grade als Grad der bi-einbettbaren Kategorizität haben. Weiters zeigen wir, dass die berechenbar bi-einbettbar kategorischen linearen Ordnungen und Booleschen Algebren genau die endlichen sind und geben ein Beispiel eines Graphen der $\mathbf{0}'$ berechenbar kategorisch aber nicht hyperarithmetisch bi-einbettbar kategorisch ist. Auch allgemeine Resultate über bi-einbettbare Kategorizität werden präsentiert. Wir zeigen, dass für alle $\alpha < \omega_1^{\mathrm{CK}}$, jeder Turinggrad $\mathbf{d}$, d-c.e. über $\mathbf{0}^{(\alpha)}$, der Grad der bi-einbettbaren Kategorizität einer Struktur ist und, dass die Indexmenge der $\mathbf{0}'$ berechenbar bi-einbettbar kategorischen Strukturen $\Pi_1^1$ vollständig ist.

# Abstract

Computable structure theory studies the relationship between structural and algorithmic properties of mathematical objects. This thesis contributes to several research directions in this field with a focus on the relation between structural and algorithmic properties of structures when we classify them under structural equivalences weaker than isomorphism.

In the first chapter, we give an introduction to the thesis and a summary of our results. In the second chapter, we study enumerable functors, a new method of algorithmic reduction between structures. We are specifically interested in its relation to existing methods of reduction. Our main result says that the existence of an enumerable functor between two structures is equivalent to one being interpretable in the other using a restricted version of effective interpretability. We also show that under the usual assumption that all structures have as their universes the whole set of natural numbers, computable functors and enumerable functors are equivalent.

In the third chapter, we study degree spectra under bi-embeddability and elementary bi-embeddability. The degree spectrum of a structure under a given equivalence relation $E$ is the set of Turing degrees of structures $E$-equivalent to it. Bi-embeddability and elementary bi-embeddability are equivalence relations on structures which are much weaker than isomorphism. We find several examples of families of Turing degrees which are bi-embeddability/elementary bi-embeddability spectra. Our main results about bi-embeddability spectra give a complete classification of the bi-embeddability spectra of linear orderings and a subclass of strongly locally finite graphs. For elementary bi-embeddability we show that there are elementary bi-embeddability spectra which are known not to be theory spectra and vice versa.

In the fourth and last chapter of this thesis, we investigate the complexity of embeddings between bi-embeddable structures. The main notions of study are computable bi-embeddable categoricity and degrees of bi-embeddable categoricity. We show that every equivalence structure has degree of bi-embeddable categoricity either $\mathbf{0}$, $\mathbf{0}'$, or $\mathbf{0}''$ and provide a complete structural characterization of the equivalence structures having one of those degrees. We also characterize the computably bi-embeddably categorical

linear orderings and Boolean algebras by showing that a structure in one of these classes is computably bi-embeddably categorical if and only if it is finite. We then give an example of a strongly locally finite graph which is $\mathbf{0}'$ computably categorical but not hyperarithmetically bi-embeddably categorical. On the more general side, we prove that for every $\alpha < \omega_1^{CK}$ every degree $\mathbf{d}$ d-c.e. over $\mathbf{0}^{(\alpha)}$ is the degree of bi-embeddable categoricity of a structure and show that the index set of $\mathbf{0}'$ computably bi-embeddably categorical structures is $\Pi^1_1$ complete.

# Acknowledgments

Over the last four years, I met many amazing people who had an impact on my scientific development and therefore also on this thesis. Giving a complete list seems like an impossible task but I want to mention a few people without whom my scientific experience would have been a lot duller.

First, I want to thank my adviser Ekaterina Fokina for introducing me to this beautiful area of research, giving me the freedom to develop my own ideas but also providing me with guidance when necessary. I am also deeply grateful to my coauthors Nikolay Bazhenov and Luca San Mauro for the many hours of scientific discussion which led to part of the results presented here and for sharing their ideas with me.

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# Contents

# 1 Introduction

Computable structure theory is the study of the relationship between structural and computational properties of mathematical objects. Two mathematical structures are considered to be structurally the same if they are isomorphic. But computationally they can be quite different. As an example take the natural numbers under their canonical ordering $\omega = 0 \leq 1 \leq 2 \leq 3 \leq \dots$ and the linear ordering $L = x_0 \leq y_0 \leq_L x_1 \leq_L y_1 \leq_L x_2 \leq_L y_2 \dots$ where $x_i = 2i$, $y_i = 2i + 1$ if $\varphi_i(i) \downarrow$[1] and $x_i = 2i + 1$, $y_i = 2i$ if $\varphi_i(i) \uparrow$. Clearly, $\omega$ is computable and $L$ and $\omega$ are isomorphic. However, $L$ is not computable. Assume towards a contradiction that it was, then to determine whether $\varphi_i(i) \downarrow$ we would just have to ask whether $2i \leq_L 2i + 1$ and thus we would have found an algorithm to decide the halting problem which is absurd.

Staying with our example we see that if we allow the halting problem $K$ as an oracle then $L$ becomes computable, i.e., $L \leq_T K$. From our argument above it is clear that the converse is also true, i.e., $K \leq_T L$. Thus $L$ has Turing degree $\mathbf{0}'$, the degree of $K$. The study of Turing degrees of isomorphic copies of a given countable structure is one of the central topics of computable structure theory. It was initiated by Richter [Ric81] and Knight [Kni86] who defined the notion of the *degree spectrum of a structure*. Given a countable structure $\mathcal{A}$, its degree spectrum is the set of subsets $X$ of the natural numbers such that there exists an isomorphic copy $\tilde{\mathcal{A}}$ with $\tilde{\mathcal{A}} \equiv_T X$. Similarly to this, we could consider $\mathcal{A}$'s degree spectrum as the set of degrees of such $X$. We will fluently switch between these two views.

The fundamental result about degree spectra is due to Knight [Kni86] and says that in non-trivial cases the degree spectrum of a structure is upwards closed. Since then the question which families of Turing degrees are realizable as the degree spectra of structures has seen much interest.

Knight's result shows that most computable structures have isomorphic copies in all Turing degrees. This is true even for very natural examples of structures such as our example of the ordering of the natural numbers above. But what if $\mathcal{A}$ and $\mathcal{B}$ are isomor-

[1] Here $\varphi_i$ is the $i^{\text{th}}$ Turing machine in the standard enumeration of algorithms, or, to be precise, Turing machines.

phic computable structures. Do they behave the same for other algorithmic questions? For instance, on $\omega$ we can clearly compute the successor relation. Can we compute it on every computable copy of $\omega$? Indeed this is not the case. Consider a computable enumeration of the halting set $K$ and let $k_s$ be the $s^{\text{th}}$ element in the enumeration. Now we can construct a linear ordering $L$ as follows. Order even numbers in the canonical way, i.e., $2m \leq_L 2m+2$; let $2k_s \leq_L 2s+1 \leq_L 2k_s+2$ and take the closure of the relation under transitivity. This ordering is clearly computable because to check whether $2m \leq_L 2n+1$ we just have to wait for $k_n$, the $n^{\text{th}}$ number in our enumeration, and check whether $2m \leq_L 2k_n$. A similar argument works to compare two odd numbers. However, in $L$ the successor relation is not computable. Assume it was, then to decide membership of $k \in K$ we just ask whether the successor of $2k$ is odd or even, a contradiction. Furthermore, $\omega$ and $L$ are clearly of the same order type and thus isomorphic. However, they can not be computably isomorphic because if that was the case we could compute the successor relation of $L$ by considering the successors of the images under a computable isomorphism.

It is not hard to conclude that if two structures are computably isomorphic then they must have the same algorithmic properties. This leads to the study of the complexity of isomorphisms of a structure. This study has a long tradition in computable structure theory which originated from work of Fröhlich and Shepherdson [FS56] and, independently, from work of Maltsev [Mal62]. The most important concept in this line of work is computable categoricity, or, in the Russian terminology, autostability. A structure with a computable copy is *computably categorical* (autostable) if every two computable copies of it are isomorphic by computable isomorphisms. Thus, if we only consider computable structures the structural properties of a computably categorical structure completely determine its algorithmic properties.

Apart from being good examples, linear orderings have interesting model theoretic and algorithmic properties. Montalbán showed that every hyperarithmetic linear ordering is bi-embeddable with a computable one. Two structures are bi-embeddable if there is an embedding from either in the other. Montalbán's result states that for every hyperarithmetic linear ordering $L$ there is a computable linear ordering $J$ which is bi-embeddable, but not necessarily isomorphic to it. It is quite easy to come up with linear orderings which have hyperarithmetic but no computable copies, see for instance [Fro+10] for a list of results. Montalbán's result also holds for Boolean algebras, compact metric spaces and Abelian p-groups [GM08] and later in this thesis we will prove that every equivalence structure is bi-embeddable with a computable one. It is surprising that for very complicated structures which are far from having a computable copy there exists a structure which is computable and looks very similar. Indeed if we look only at finite pieces of the structure, then the two structures appear to be the same and we can only find differences after having seen an infinite substructure. This motivates the study of the relation between algorithmic and structural properties when we consider weaker equivalence relations than isomorphism — the main topic of this thesis.

Before we give an overview of the contents in this thesis let us point out that we are not the first to study structural properties with respect to other equivalence relations than isomorphism. Andrews and J. Miller [AM15] initiated the study of theory spectra, the collection of degrees of models of a complete theory $T$. We can consider this as the spectrum of a countable model of $T$ under elementary equivalence. Fokina, Semukhin, and Turetsky [FST18] studied degree spectra under $\Sigma_n$ equivalence. Two structures are $\Sigma_n$ equivalent if they satisfy the same $\Sigma_n$ sentences. This can be seen as an approximation to elementary equivalence since two structures are $\Sigma_n$ equivalent for all $n \in \omega$ if and only if they are elementary equivalent.

## 1.1 Outline of the thesis

Apart from the introduction this thesis consists of three chapters. All of them are based on published work of the author, some of the publications are with co-authors and some alone. The chapters all study different algorithmic properties of structures under equivalence relations other than isomorphism. Therefore each of the chapters might stand on its own and contains a detailed introduction. Let us give a quick preview of their content.

**Chapter 2** In this chapter, we introduce and study enumerable functors, a notion of effective reduction between structures, and related notions of reduction between classes of structures. Enumerable functors are an effective version of functors related to other effective notions as studied in [Har+17; Mil+18; HMM16]. We study the connection between enumerable functors and effective interpretability, a syntactic notion of reduction introduced by [Mon12]. Our main results show that a restricted version of effective interpretability and enumerable functors are equivalent and that if we restrict our attention to structures which have as their universe the set of all natural numbers, enumerable functors and computable functors are equivalent. Most of the results in this chapter have been published in

[Ros17] Dino Rossegger. "On Functors Enumerating Structures". In: *Siberian Electronic*

*Mathematical Reports* 14 (2017), pp. 690–702.

However, the second of our main results is new and answers a question raised in [Ros17]. There we only showed that the existence of an enumerable functor from a structure $\mathcal{A}$ to a structure $\mathcal{B}$ implies the existence of a computable functor. In Section 2.3 we improve this result by showing that when we only consider structures having universe $\omega$ the converse also holds, i.e., that in this case, the existence of a computable functor implies the existence of an enumerable functor.

**Chapter 3** This chapter contains the contents of two articles:

[FRM18] Ekaterina Fokina, Dino Rossegger, and Luca San Mauro. "Bi-Embeddability Spectra and Bases of Spectra". In: *to appear in Mathematical Logic Quarterly* (2018). arXiv: 1808.05451
[Ros18] Dino Rossegger. "Elementary Bi-Embeddability Spectra of Structures". In: *Conference on Computability in Europe.* Lecture Notes in Computer Science. Springer, 2018, pp. 349–358.

The results from [FRM18], obtained jointly with Ekaterina Fokina and Luca San Mauro, are presented in Section 3.1 and the presentation of the results is similar to that in the paper. In Section 3.2 we present the results of [Ros18]. The presentation is more detailed than the one given in the paper where many proofs were omitted or only sketched due to reasons of space. We give a brief overview of the two sections.

In Section 3.1 we investigate bi-embeddability spectra. Two structures are bi-embeddable if either is embeddable in the other. Given a structure $\mathcal{A}$ its bi-embeddability spectrum is the family of degrees of structures bi-embeddable with it. We show that, like classical degree spectra, bi-embeddability spectra are upwards closed for non-trivial structures. Using the concept of b.e. triviality we show that several families of degrees known to be degree spectra are bi-embeddability spectra. We then introduce the notion of the basis of a spectrum and investigate bi-embeddability spectra of linear orders and strongly locally finite graphs using this notion. We give a complete characterization of the bi-embeddability spectra of linear orderings and study the bi-embeddability spectra of strongly locally finite graphs.

The topic of Section 3.2 is elementary bi-embeddability spectra. The definition is similar to the one of bi-embeddability spectra except that this time we require the embeddings to be elementary. We show that graphs are universal for elementary bi-embeddability spectra, i.e., that every elementary bi-embeddability spectrum can be realized by a graph and show for several families of degrees that they are or are not elementary bi-embeddability spectra of structures.

**Chapter 4** At last, we investigate the complexity of embeddings between bi-embeddable structures. Towards this, we define the notions of (relative) computable bi-embeddable categoricity and the related notion of degree of bi-embeddable categoricity. These notions are based on the classical notions of (relative) computable categoricity and degree of categoricity for isomorphism.

In Section 4.1 we give a complete characterization of the degrees of bi-embeddable categoricity of equivalence structures, highlight differences to the case for isomorphism and calculate the complexity of index sets related to bi-embeddability for equivalence structures. This results came out of joint work with Nikolay Bazhenov, Ekaterina Fokina, and Luca San Mauro and have been published in [Baz+18b].

[Baz+18b] Nikolay Bazhenov, Ekaterina Fokina, Dino Rossegger, and Luca San Mauro. "Degrees of Bi-Embeddable Categoricity of Equivalence Structures". In: *Archive for Mathematical Logic* (Nov. 2018). arXiv: `1710.10927`

In Section 4.2 we investigate other classes of structures and present general results. We show that linear orderings and Boolean algebras are computably bi-embeddably categorical if and only if they are finite, that for all successor ordinals $\alpha$, every degree d.c.e. above $\mathbf{0}^{(\alpha)}$ is a degree of bi-embeddable categoricity and that the index set of relatively $\Delta^0_2$ bi-embeddable categorical structures is $\Pi^1_1$-complete. As a corollary of this result, we get that there is a relatively $\Delta^0_2$ categorical strongly locally finite graph which is not hyperarithmetically bi-embeddably categorical. These results have been communicated in [Baz+18a].

[Baz+18a] Nikolay Bazhenov, Ekaterina Fokina, Dino Rossegger, and Luca San Mauro. "Computable Bi-Embeddable Categoricity". In: *Algebra and Logic* 57.5 (2018), pp. 392–396

## 1.2 Preliminaries

Our computability theoretic notations are standard and as in [Soa16] or [Coo03]. We will need model theoretic notions as discussed in most introductory textbooks on model theory. We refer the reader to [Mar02] or [Hod97] for a full fledged discussion. An excellent background reading for computable structure theory is the upcoming book by

Montalbán. At the date of writing a draft can be found on his website [Mon18]. Our notation mostly follows his. We quickly review some of the basic notions. Everything else will be defined whenever we need it.

We only consider countable structures in computable languages with equality. Throughout this thesis we will furthermore assume that all our languages are relational. We use calligraphic letters $\mathcal{A}$, $\mathcal{B}$,... for structures and capital letters $A$, $B$,... for their universes. Without loss of generality we assume that structures have computable universes. We could also assume that all structures have universe $\omega$ if we want to exclude finite structures from our considerations. Indeed we do so in Chapter 3 and Chapter 4 when it helps with the presentation of our results.

Let $\mathcal{A}$ be a structure in the language $L$ and let $\langle \varphi_i \rangle_{i\in\omega}$ be an enumeration of the atomic $L$ formulas with variables a subset of $\{x_i : i \in \omega\}$. Fix a computable $1-1$ enumeration $f$ of $A$.

**Definition 1.2.1.** The *atomic diagram* of $\mathcal{A}$ is the infinite binary string $D(\mathcal{A}) \in 2^\omega$ defined as

$$D(\mathcal{A})(i) = \begin{cases} 1 & \text{if } \mathcal{A} \models \varphi_i[x_j \mapsto f(j) : j \in \omega] \\ 0 & \text{otherwise} \end{cases}$$

For $X \subseteq \omega$, we say that a structure $\mathcal{A}$ is *$X$-computable*, or computable from $X$, if $D(\mathcal{A}) \leq_T X$. A structure $\mathcal{A}$ is computable if it is computable from the empty set. We usually identify structures with their atomic diagram and thus write $\mathcal{A}$ instead of $D(\mathcal{A})$. A common example is $\Phi_e^{\mathcal{A}}$ instead of $\Phi_e^{D(\mathcal{A})}$.

# 2 Functors that enumerate structures

One goal in computable structure theory is to compare structures, or classes of structures with respect to their computability theoretic properties. This is usually achieved by using reductions. Several different notions of reduction between structures are known, most notably Muchnik reducibility, Medvedev reducibility, computable functors, Σ-definability, and effective interpretability. The first three notions are computational, while the other two are syntactic, based on the model theoretic notion of interpretability. The study of computable functors was recently intiated by R. Miller, Poonen, Schoutens, and Shlapentokh [Mil+18]. They are a strengthening of Medvedev reducibility. Harrison-Trainor, Melnikov, R. Miller, and Montalbán [HH17] showed that computable functors are equivalent to effective interpretability first studied by Montalbán [Mon12]. In [HMM16], Harrison-Trainor, R. Miller, and Montalbán proved a similar result for Baire measurable functors and infinitary interpretability. Σ-definability was introduced by Ershov [Ers] and has since been heavily studied by Russian researchers [Kal09; Puz09; MK08; Stu07; Stu08; Stu13]. Effective interpretability is equivalent to Σ-definability without parameters [Mon12].

Between classes of structures the most notable notions are computable embeddings, Turing computable embeddings, uniform transformations, HKSS interpretations and reduction by effective bi-interpretability. Turing computable embeddings [KMV07] are an analogue of Medvedev reducibility for classes of structures, while computable embeddings [Cal04] use enumeration reducibility, a well studied notion of reducibility in computability theory. Uniform transformations are based on computable functors and reduction by effective bi-interpretability [Mon14] on effective interpretations. It was shown in [HH17] that these two notions are equivalent. Effective bi-interpretability is closely related to HKSS interpretations [Hir+02]. Hirschfeldt, Khoussainov, Shore, and Slinko [Hir+02] gave such interpretations of graphs in several classes of structures. It turns out that with minor modifications of these interpretations one can obtain effective interpretations [Mon14; Ros15]. As computable functors are a strengthening of Medvedev reducibility, uniform transformations are a strengthening of Turing computable embeddings.

In this chapter we study *enumerable functors*. Enumerable functors are a strengthening of computable embeddings. We prove that enumerable functors are at least as strong as computable functors and show that under the usual assumption that all structure have as universe the set of natural numbers they are equivalent. We obtain similar results for the related notions on classes of structures. We furthermore show that even without the assumption that all structures have universe all the natural numbers the existence of an enumerable functor between a structure $\mathcal{A}$ and a structure $\mathcal{B}$ is equivalent to $\mathcal{B}$ being interpretable in $\mathcal{A}$ with a restricted version of effective interpretability.

**Notation** Since in this chapter our arguments often require pairs of non isomorphic structures $\mathcal{A}$ ,$\mathcal{B}$ we write $\tilde{\mathcal{A}}, \hat{\mathcal{A}}$ for isomorphic copies of $\mathcal{A}$. We denote categories by fraktal letters $\mathfrak{C}, \mathfrak{D}, \ldots$. We write $\mathcal{A} \in \mathfrak{C}$ to say that $\mathcal{A}$ is an object of $\mathfrak{C}$ and $f \in \mathfrak{C}$ means that $f$ is an arrow in $\mathfrak{C}$. We will introduce all further category theoretic notions when needed.

### 2.0.1 Enumerable functors

Recall the notion of a functor between categories. In our setting the categories are classes of countable structures, i.e., collections of structures closed under isomorphism with isomorphisms as arrows.

**Definition 2.0.1.** A *functor from* $\mathfrak{C}$ *to* $\mathfrak{D}$ is a map $F$ that assigns to each structure $\mathcal{A} \in \mathfrak{C}$ a structure $F(\mathcal{A}) \in \mathfrak{D}$, and assigns to each arrow $f : \mathcal{A} \to \mathcal{B} \ \in \mathfrak{C}$ a morphism $F(f) : F(\mathcal{A}) \to F(\mathcal{B}) \in \mathfrak{D}$ so that the following two properties hold.

(i) $F(id_{\mathcal{A}}) = id_{F(\mathcal{A})}$ for every $\mathcal{A} \in \mathfrak{C}$, and

(ii) $F(f \circ g) = F(f) \circ F(g)$ for all morphisms $f, g \in \mathfrak{C}$.

We abuse notation and write $F : \mathcal{A} \to \mathcal{B}$ for a functor $F$ between the isomorphism classes of $\mathcal{A}$ and $\mathcal{B}$. The *isomorphism class* of $\mathcal{A}$ denoted by $Iso(\mathcal{A})$ has as objects all structures isomorphic to $\mathcal{A}$ and as arrows all the isomorphisms between copies of $\mathcal{A}$. Enumeration reducibility is a well studied notion in classic computability theory that has also been studied in the context of computable structure theory, see [SS17] for a survey. For $A, B \subseteq \omega$, $B$ is *enumeration reducible* to $A$ if there is an enumeration operator, i.e., a c.e. set $\Psi$ of pairs $(\alpha, b)$ where $\alpha$ is a finite subset of $\omega$ and $b \in \omega$, such that

$$B = \{b \mid (\exists \alpha \subseteq A)(\alpha, b) \in \Psi\}.$$

We may write $B$ as $\Psi^A$ because $B$ is unique given $\Psi$ and $A$. Using an enumeration operator and a Turing operator we now define enumerable functors.

**Definition 2.0.2.** An *enumerable functor* is a functor $F : \mathfrak{C} \to \mathfrak{D}$ together with an enumeration operator $\Psi$ and a Turing operator $\Phi_*$ such that

(i) for every $\mathcal{A} \in \mathfrak{C}$, $\Psi^{\mathcal{A}} = F(\mathcal{A})$,

(ii) for every morphism $f : \mathcal{A} \to \mathcal{B} \in \mathfrak{C}$, $\Phi_*^{\mathcal{A} \oplus f \oplus \mathcal{B}} = F(f)$.

As for computable functors we often identify enumerable functors with their pair $(\Psi, \Phi_*)$ of operators.

This effective version of functors is inspired by computable embeddings, investigated in [Cal04]. There, a computable embedding from a class $\mathfrak{C}$ to a class $\mathfrak{D}$ is an enumeration operator $\Psi$ as defined in (i) of Definition 2.0.2 and the property that $\mathcal{A} \cong \mathcal{B}$ if and only if $\Psi^{\mathcal{A}} \cong \Psi^{\mathcal{B}}$. Our definition is stronger than this, since we additionally require isomorphisms $F(f) : F(\mathcal{A}) \to F(\mathcal{B})$ to be uniformly computable from $\mathcal{A} \oplus f \oplus \mathcal{B}$.

The authors of [Cal04] showed that substructures are preserved by computable embeddings. The same observation can be made for enumerable functors. The proof is exactly the same, for sake of completeness we state it here.

**Proposition 2.0.1.** *[Substructure preservation] Let $F : \mathfrak{C} \to \mathfrak{D}$ be an enumerable functor witnessed by $(\Psi, \Phi_*)$. If $\mathcal{A}_1, \mathcal{A} \in \mathfrak{C}$ and $\mathcal{A}_1 \subseteq \mathcal{A}$, then $F(\mathcal{A}_1) \subseteq F(\mathcal{A})$.*

*Proof.* Assume $\mathcal{A}_1 \subseteq \mathcal{A}$. If $\varphi \in D(F(\mathcal{A}_1))$, then there is a finite set of formulas $\alpha \subseteq D(\mathcal{A}_1)$ such that $(\alpha, \varphi) \in \Psi$ and since $D(\mathcal{A}_1) \subseteq D(\mathcal{A})$, $\varphi \in D(F(\mathcal{A}))$. □

As part of this article is concerned with the relationship between enumerable functors and computable functors we recall the notion of a computable functor first investigated in [**miller2015**].

**Definition 2.0.3.** A *computable functor* is a functor $F : \mathfrak{C} \to \mathfrak{D}$ together with two Turing operators $\Phi$ and $\Phi_*$ such that

(i) for every $\mathcal{A} \in \mathfrak{C}$, $\Phi^{\mathcal{A}} = F(\mathcal{A})$,

(ii) for every morphism $f : \mathcal{A} \to \mathcal{B} \in \mathfrak{C}$, $\Phi_*^{\mathcal{A} \oplus f \oplus \mathcal{B}} = F(f)$.

We often identify a computable functor with its pair $(\Phi, \Phi_*)$ of Turing operators witnessing its computability.

The following notions originated in [HH17].

**Definition 2.0.4.** A functor $F : \mathfrak{C} \to \mathfrak{D}$ is *effectively (naturally) isomorphic* to a functor $G : \mathfrak{C} \to \mathfrak{D}$ if there is a Turing functional $\Lambda$ such that for every $\mathcal{A} \in \mathfrak{C}$, $\Lambda^{\mathcal{A}}$ is an isomorphism from $F(\mathcal{A})$ to $G(\mathcal{A})$ and the following diagram commutes for every $\mathcal{A}, \mathcal{B} \in \mathfrak{C}$ and every morphism $h : \mathcal{A} \to \mathcal{B}$:

$$\begin{array}{ccc} F(\mathcal{A}) & \xrightarrow{\Lambda^{\mathcal{A}}} & G(\mathcal{A}) \\ {\scriptstyle F(h)}\big\downarrow & & \big\downarrow{\scriptstyle G(h)} \\ F(\mathcal{B}) & \xrightarrow{\Lambda^{\mathcal{B}}} & G(\mathcal{B}) \end{array}$$

Note that in the above definition it does not matter whether $F$ and $G$ are both computable functors or enumerable functors. Hence, it is legal to say that an enumerable functor is effectively isomorphic to an computable functor. Intuitively, two functors are effectively naturally isomorphic if they are equivalent up to computable isomorphism. Using this idea one can generalize the idea of an inverse.

Let $F : \mathfrak{C} \to \mathfrak{D}$ and $G : \mathfrak{D} \to \mathfrak{C}$ be functors such that $G \circ F$ and $F \circ G$ are effectively isomorphic to the identity functors $id_{\mathfrak{C}}$ and $id_{\mathfrak{D}}$ respectively. Let $\Lambda_{\mathfrak{C}}$ be the Turing functional witnessing the effective isomorphism between $G \circ F$ and the identity functor $id_{\mathfrak{C}}$, i.e., for any $\mathcal{A} \in \mathfrak{C}$, $\Lambda_{\mathfrak{C}}^{\mathcal{A}} : \mathcal{A} \to G(F(\mathcal{A}))$. Define $\Lambda_{\mathfrak{D}}$ similarly, i.e., $\Lambda_{\mathfrak{D}}^{\mathcal{B}} : \mathcal{B} \to F(G(\mathcal{B}))$ for any $\mathcal{B} \in \mathfrak{D}$. Then there are maps $\Lambda_{\mathfrak{D}}^{F(\mathcal{A})} : F(\mathcal{A}) \to F(G(F(\mathcal{A})))$ and $F(\Lambda_{\mathfrak{C}}^{\mathcal{A}}) : F(\mathcal{A}) \to F(G(F(\mathcal{A})))$. If these two maps, and the similarly defined maps for $\mathfrak{D}$, agree for every $\mathcal{A} \in \mathfrak{C}$ and $\mathcal{B} \in \mathfrak{D}$, then we say that $F$ and $G$ are *pseudo-inverses*.

**Definition 2.0.5.** Two structures $\mathcal{A}$ and $\mathcal{B}$ are *enumerably bi-transformable* if there exist enumerable functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ which are pseudo-inverses.

Harrison-Trainor, Melnikov, R. Miller, and Montalbán [HH17] defined *computably bi-transformability* which is analogous to our definition except that it uses computable functors.

### 2.0.2 Reduction between classes

**Definition 2.0.6.** A class $\mathfrak{C}$ is *uniformly enumerably transformally reducible*, or *u.e.t. reducible*, to a class $\mathfrak{D}$ if there exists a subclass $\mathfrak{D}' \subseteq \mathfrak{D}$, enumerable functors $F : \mathfrak{C} \to \mathfrak{D}'$ and $G : \mathfrak{D}' \to \mathfrak{C}$, and $F$, $G$ are pseudo-inverses. We say that a class is *complete for u.e.t. reducibility* if for every computable language $L$, the class of $L$-structures u.e.t. reduces to it.

The authors of [HH17] gave an analogous definition using computable functors called reduction by uniform transformation. In this chapter we call it *reduction by uniform computable transformations*, or short *reduction by u.c.t.* to distinguish it from reduction by uniform enumerable transformation. It is obtained by swapping the enumerable functors in the definition by computable functors.

## 2.1 On the relation between enumerable and computable functors

Kalimullin and Greenberg independently showed that if a class is computably embeddable in another class, then it is also Turing computably embeddable, see [KMV07, Proposition 1.4]. Using a similar procedure to the one given in their proof one can construct a computable functor from an enumerable functor.

**Theorem 2.1.1.** *Let $F : \mathfrak{C} \to \mathfrak{D}$ be an enumerable functor, then there is a computable functor $G$ effectively isomorphic to $F$.*

*Proof.* Let $F$ be witnessed by $(\Psi, \Phi_*)$. Given some $\mathcal{A} \in \mathfrak{C}$, let $\mathcal{B} = F(\mathcal{A}) = \Psi^{\mathcal{A}}$. We first show that there is a Turing functional $\Phi'$ transforming every such $\mathcal{A}$ into a structure $\tilde{\mathcal{B}}$ isomorphic to $\mathcal{B}$, i.e., $\Phi'^{\mathcal{A}} = \tilde{\mathcal{B}}$.

Let $\langle \cdot, \cdot \rangle : \omega \times \omega \to \omega$ be the standard computable pairing function. The universe of $\tilde{\mathcal{B}}$ is

$$\tilde{B} = \{\langle b, s\rangle \mid b \in B \wedge s = \mu x[b = b \in \Psi_x^{\mathcal{A}}]\}.$$

Here $\Psi_x$ is the approximation of $\Psi$ at stage $x$ of the enumeration. $\tilde{B}$ is computable relative to $\mathcal{A}$ as computing membership can be done by enumerating $\Psi$ until stage $s$ and checking if $s$ is the first step such that $b = b \in \Psi_s^{\mathcal{A}}$. For each $R_i$ with arity $r_i$ in the language of $\mathfrak{D}$ define $R_i^{\tilde{\mathcal{B}}}$ as

$$\begin{aligned} (\langle x_1, s_1\rangle, \dots, \langle x_{r_i}, s_{r_i}\rangle) \in R_i^{\tilde{\mathcal{B}}} &\iff \quad R_i(x_1, \dots, x_{r_i}) \in \Psi^{\mathcal{A}}, \\ (\langle x_1, s_1\rangle, \dots, \langle x_{r_i}, s_{r_i}\rangle) \notin R_i^{\tilde{\mathcal{B}}} &\iff \neg R_i(x_1, \dots, x_{r_i}) \in \Psi^{\mathcal{A}}. \end{aligned}$$

Since for all relations $R_i$ and tuples $(\langle x_1, s_1\rangle, \dots, \langle x_{r_i}, s_{r_i}\rangle) \in \tilde{B}^{r_i}$, either $R_i(x_1, \dots, x_{r_i}) \in \Psi^{\mathcal{A}}$ or $\neg R_i(x_1, \dots, x_{r_i}) \in \Psi^{\mathcal{A}}$ and $\tilde{B}$ is computable from $\mathcal{A}$, $\tilde{\mathcal{B}} \leq_T \mathcal{A}$. Furthermore the computation of $\tilde{\mathcal{B}}$ from $\mathcal{A}$ is uniform, hence there is a Turing functional $\Phi'$ that given $\mathcal{A} \in \mathfrak{C}$ as oracle computes $\tilde{\mathcal{B}}$. Set $G(\mathcal{A}) = \tilde{\mathcal{B}}$, then $\Phi'$ is the first partial witness of computability of $G$.

For $\mathcal{A} \in \mathfrak{C}$ let $\theta^{\mathcal{A}} : F(\mathcal{A}) \to G(\mathcal{A})$ be defined by $x \to \langle x, s \rangle$. $\theta^{\mathcal{A}}$ is uniformly computable from $\mathcal{A}$ and is an isomorphism between $F(\mathcal{A})$ and $G(\mathcal{A})$ by construction of $G(\mathcal{A})$. For the second partial witness consider $f : \mathcal{A} \to \tilde{\mathcal{A}} \in \mathfrak{C}$, then $F(f) : F(\mathcal{A}) \to F(\tilde{\mathcal{A}})$. Set $G(f) = \theta^{\tilde{\mathcal{A}}} \circ F(f) \circ (\theta^{\mathcal{A}})^{-1}$. As $\theta^{\mathcal{A}}$, $\theta^{\tilde{\mathcal{A}}}$ are uniformly computable from $\mathcal{A}$, respectively $\tilde{\mathcal{A}}$ in $\mathfrak{C}$ and $F(f)$ is uniformly computable from $\mathcal{A} \oplus f \oplus \tilde{\mathcal{A}}$, there is a Turing operator, say $\Phi'_*$, such that

$$\Phi'^{\mathcal{A} \oplus f \oplus \tilde{\mathcal{A}}}_* = G(f).$$

It follows that $\Phi'_*$ qualifies as the second partial witness of computability of $G$. $G$ is a functor as for $\mathcal{A} \in \mathfrak{C}$, $G(id_{\mathcal{A}}) = \theta^{\mathcal{A}} \circ F(id_{\mathcal{A}}) \circ (\theta^{\mathcal{A}})^{-1} = \theta^{\mathcal{A}} \circ (\theta^{\mathcal{A}})^{-1} = id_{G(\mathcal{A})}$ and for $f : \mathcal{A} \to \tilde{\mathcal{A}}, g : \tilde{\mathcal{A}} \to \hat{\mathcal{A}} \in \mathfrak{C}$,

$$G(g \circ f) = \theta^{\hat{\mathcal{A}}} \circ F(g \circ f) \circ (\theta^{\mathcal{A}})^{-1} = \theta^{\hat{\mathcal{A}}} \circ F(g) \circ (\theta^{\tilde{\mathcal{A}}})^{-1} \circ \theta^{\tilde{\mathcal{A}}} \circ F(f) \circ (\theta^{\mathcal{A}})^{-1} = G(g) \circ G(f).$$

As argued above, the function $\theta^{\mathcal{A}}$, which induces the isomorphism between $F(\mathcal{A})$ and $G(\mathcal{A})$ is uniformly computable in $\mathfrak{C}$ from $\mathcal{A}$. Hence, there is a Turing functional $\Lambda$ such that $\Lambda^{\mathcal{A}} = \theta^{\mathcal{A}}$. It witnesses the effective isomorphism between $F$ and $G$. □

A similar result as Theorem 2.1.1 holds for enumerable bi-transformability and computable bi-transformability.

**Theorem 2.1.2.** *Let $\mathcal{A}$ and $\mathcal{B}$ be enumerably bi-transformable, then they are also computably bi-transformable.*

*Proof.* Let $F : Iso(\mathcal{A}) \to Iso(\mathcal{B})$, $G : Iso(\mathcal{B}) \to Iso(\mathcal{A})$ be enumerable functors witnessing the enumerable bi-transformability between $\mathcal{A}, \mathcal{B}$. By Theorem 2.1.1 there are computable functors

$$F' : Iso(\mathcal{A}) \to Iso(\mathcal{B}) \quad \text{and} \quad G' : Iso(\mathcal{B}) \to Iso(\mathcal{A}).$$

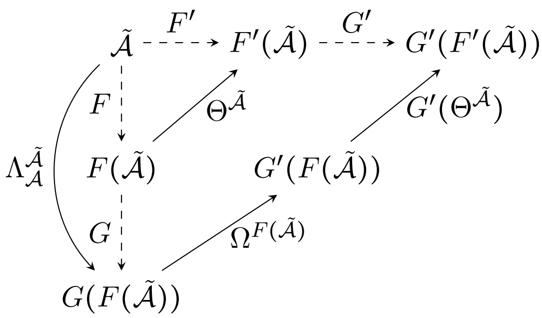
Furthermore there are Turing operators $\Theta$ and $\Omega$ inducing the effective isomorphisms between $F$ and $F'$ and $G$ and $G'$ respectively, i.e.,

$$\Theta^{\tilde{\mathcal{A}}} : F(\tilde{\mathcal{A}}) \to F'(\tilde{\mathcal{A}}) \quad \text{and} \quad \Omega^{\tilde{\mathcal{B}}} : G(\tilde{\mathcal{B}}) \to G'(\tilde{\mathcal{B}}).$$

Recall the Turing operators $\Lambda_{\mathcal{A}}$ and $\Lambda_{\mathcal{B}}$ witnessing that $F$ and $G$ are pseudo-inverses. For any $\tilde{\mathcal{A}} \in Iso(\mathcal{A})$ and $\tilde{\mathcal{B}} \in Iso(\mathcal{B})$,

$$\Lambda^{\tilde{\mathcal{A}}}_{\mathcal{A}} : \tilde{\mathcal{A}} \to G(F(\tilde{\mathcal{A}})) \quad \text{and} \quad \Lambda^{\tilde{\mathcal{B}}}_{\mathcal{B}} : \tilde{\mathcal{B}} \to F(G(\tilde{\mathcal{B}})).$$

Observe that the isomorphisms computed by $\Omega$ given $\tilde{\mathcal{B}} \in Iso(\mathcal{B})$ as oracle are uniformly computable in $Iso(\mathcal{A})$ because $Iso(\mathcal{B})$ is uniformly computable in $Iso(\mathcal{A})$ since $F'$ is a computable functor, and that the analogous statement holds for $\Theta$. Consider the following diagram for any $\tilde{\mathcal{A}} \in Iso(\mathcal{A})$.




Note that $\Theta^{\tilde{\mathcal{A}}} : F(\tilde{\mathcal{A}}) \to F'(\tilde{\mathcal{A}}) \in \mathfrak{D}$ and hence $G'(\Theta^{\tilde{\mathcal{A}}})$ is an isomorphism from $G'(F(\tilde{\mathcal{A}}))$ to $G'(F'(\tilde{\mathcal{A}}))$. Analogous diagrams can be drawn for any $\tilde{\mathcal{B}} \in Iso(\tilde{\mathcal{B}})$. We therefore define $\Gamma^{\tilde{\mathcal{A}}}_{\mathcal{A}}$ and $\Gamma^{\tilde{\mathcal{B}}}_{\mathcal{B}}$ as

$$\Gamma^{\tilde{\mathcal{A}}}_{\mathcal{A}} = G'(\Theta^{\tilde{\mathcal{A}}}) \circ \Omega^{F(\tilde{\mathcal{A}})} \circ \Lambda^{\tilde{\mathcal{A}}}_{\mathcal{A}} \quad \text{and} \quad \Gamma^{\tilde{\mathcal{B}}}_{\mathcal{B}} = F'(\Omega^{\tilde{\mathcal{B}}}) \circ \Theta^{G(\tilde{\mathcal{B}})} \circ \Lambda^{\tilde{\mathcal{B}}}_{\mathcal{B}}.$$

It is easy to see from the above diagram that they induce the wanted isomorphisms $\Gamma^{\tilde{\mathcal{A}}}_{\mathcal{A}} : \mathcal{A} \to G'(F'(\tilde{\mathcal{A}}))$, respectively $\Gamma^{\tilde{\mathcal{B}}}_{\mathcal{B}} : \mathcal{B} \to F'(G'(\tilde{\mathcal{B}}))$ for all $\tilde{\mathcal{A}} \in Iso(\mathcal{A})$ and all $\tilde{\mathcal{B}} \in Iso(\mathcal{B})$. Since all functions in their definition are uniformly computable in $\mathcal{A}$, respectively $\mathcal{B}$, $\Gamma_{\mathcal{A}}$, $\Gamma_{\mathcal{B}}$ witness that $G' \circ F'$ and $F' \circ G'$ are effectively isomorphic to the identity functors $id_{\mathfrak{C}}$, respectively $id_{\mathfrak{D}}$. It remains to show that $\Gamma^{F'(\tilde{\mathcal{A}})}_{\mathcal{B}} = F'(\Gamma^{\tilde{\mathcal{A}}}_{\mathcal{A}})$ and $\Gamma^{G'(\tilde{\mathcal{B}})}_{\mathcal{A}} = G'(\Gamma^{\tilde{\mathcal{B}}}_{\mathcal{B}})$. We will prove the first statement, the proof of the second statement is analogous.

First recall that by the construction of $F', G'$ in Theorem 2.1.1, for any isomorphism $f : \tilde{\mathcal{A}} \to \hat{\mathcal{A}}$ between two copies $\tilde{\mathcal{A}}, \hat{\mathcal{A}}$ of $\mathcal{A}$ and for any isomorphism $g : \tilde{\mathcal{B}} \to \hat{\mathcal{B}}$ between two copies $\tilde{\mathcal{B}}, \hat{\mathcal{B}}$ of $\mathcal{B}$

$$F'(f) = \Theta^{\hat{\mathcal{A}}} \circ F(f) \circ (\Theta^{\tilde{\mathcal{A}}})^{-1} \quad \text{and} \quad G'(g) = \Omega^{\hat{\mathcal{B}}} \circ G(g) \circ (\Omega^{\tilde{\mathcal{B}}})^{-1}$$

because $F, F'$ and $G, G'$ are effectively isomorphic. Now, let $\tilde{\mathcal{A}} \in Iso(\mathcal{A})$, then

$$G'(\Theta^{\tilde{\mathcal{A}}}) = \Omega^{F'(\tilde{\mathcal{A}})} \circ G(\Theta^{\tilde{\mathcal{A}}}) \circ (\Omega^{F(\tilde{\mathcal{A}})})^{-1}.$$

Therefore $\Gamma_{\mathcal{A}}^{\tilde{\mathcal{A}}} = \Omega^{F'(\tilde{\mathcal{A}})} \circ G(\Theta^{\mathcal{A}}) \circ \Lambda_{\mathcal{A}}^{\tilde{\mathcal{A}}}$ and

$$F'(\Gamma_{\mathcal{A}}^{\tilde{\mathcal{A}}}) = F'(\Omega^{F'(\tilde{\mathcal{A}})}) \circ F'(G(\Theta^{\tilde{\mathcal{A}}})) \circ F'(\Lambda_{\mathcal{A}}^{\tilde{\mathcal{A}}}).$$

Furthermore, $F'(\Lambda_{\mathcal{A}}^{\tilde{\mathcal{A}}}) = \Theta^{G(F(\tilde{\mathcal{A}}))} \circ F(\Lambda_{\mathcal{A}}^{\tilde{\mathcal{A}}}) \circ (\Theta^{\tilde{\mathcal{A}}})^{-1}$. It follows that

$$F'(\Gamma_{\mathcal{A}}^{\tilde{\mathcal{A}}}) = F'(\Omega^{F'(\tilde{\mathcal{A}})}) \circ F'(G(\Theta^{\tilde{\mathcal{A}}})) \circ \Theta^{G(F(\tilde{\mathcal{A}}))} \circ F(\Lambda_{\mathcal{A}}^{\tilde{\mathcal{A}}}) \circ (\Theta^{\tilde{\mathcal{A}}})^{-1}.$$

Notice that $F'(G(\Theta^{\tilde{\mathcal{A}}})) = \Theta^{G(F'(\tilde{\mathcal{A}}))} \circ F(G(\Theta^{\tilde{\mathcal{A}}})) \circ (\Theta^{G(F(\tilde{\mathcal{A}}))})^{-1}$, hence

$$F'(\Gamma_{\mathcal{A}}^{\tilde{\mathcal{A}}}) = F'(\Omega^{F'(\tilde{\mathcal{A}})}) \circ \Theta^{G(F'(\tilde{\mathcal{A}}))} \circ F(G(\Theta^{\tilde{\mathcal{A}}})) \circ F(\Lambda_{\mathcal{A}}^{\tilde{\mathcal{A}}}) \circ (\Theta^{\tilde{\mathcal{A}}})^{-1}.$$

Consider $\Gamma_{\mathcal{B}}^{F'(\tilde{\mathcal{A}})} = F'(\Omega^{F'(\tilde{\mathcal{A}})}) \circ \Theta^{G(F'(\tilde{\mathcal{A}}))} \circ \Lambda_{\mathcal{B}}^{F'(\tilde{\mathcal{A}})}$ and recall that $\Lambda_{\mathcal{B}}$ witnesses the effective isomorphism between $id_{Iso(\mathcal{B})}$ and $F \circ G$. As $F'(\tilde{\mathcal{A}})$ and $F(\tilde{\mathcal{A}})$ are both in $Iso(\mathcal{B})$ we have by Definition 2.0.4 that $\Lambda_{\mathcal{B}}^{F'(\tilde{\mathcal{A}})} \circ \Theta^{\tilde{\mathcal{A}}} = F(G(\Theta^{\tilde{\mathcal{A}}})) \circ \Lambda_{\mathcal{B}}^{F(\tilde{\mathcal{A}})}$ and thus

$$\Lambda_{\mathcal{B}}^{F'(\tilde{\mathcal{A}})} = F(G(\Theta^{\tilde{\mathcal{A}}})) \circ \Lambda_{\mathcal{B}}^{F(\tilde{\mathcal{A}})} \circ (\Theta^{\tilde{\mathcal{A}}})^{-1}.$$

Because $\Lambda_{\mathcal{B}}^{F(\tilde{\mathcal{A}})} = F(\Lambda_{\mathcal{A}}^{\tilde{\mathcal{A}}})$

$$\Gamma_{\mathcal{B}}^{F'(\tilde{\mathcal{A}})} = F'(\Omega^{F'(\tilde{\mathcal{A}})}) \circ \Theta^{G(F'(\mathcal{A}))} \circ F(G(\Theta^{\tilde{\mathcal{A}}})) \circ F(\Lambda_{\mathcal{A}}^{\tilde{\mathcal{A}}}) \circ \left(\Theta^{\tilde{\mathcal{A}}}\right)^{-1} = F'(\Gamma_{\mathcal{A}}^{\tilde{\mathcal{A}}}).$$

By the same argument $G'(\Gamma_{\mathcal{B}}^{\tilde{\mathcal{B}}}) = \Gamma_{\mathcal{A}}^{G'(\tilde{\mathcal{B}})}$ for all $\tilde{\mathcal{B}} \in Iso(\mathcal{B})$. It follows that $F'$ and $G'$ are pseudo-inverses. □

By adapting the proof of Theorem 2.1.2 we get the same result for u.e.t. reduction.

**Corollary 2.1.3.** *Let $\mathfrak{C}$ be uniformly enumerably transformally reducible to $\mathfrak{D}$, then $\mathfrak{C}$ is uniformly computably transformally reducible to $\mathfrak{D}$.*

## 2.2 Enumerable Functors and Effective Interpretability

The main goal of this section is to prove the following theorem which provides a syntactic characterization of enumerable functors.

**Theorem 2.2.1.** *A structure $\mathcal{A}$ is effectively interpretable in $\mathcal{B}$ with $\sim$ computable if and only if there is an enumerable functor $F : Iso(\mathcal{B}) \to Iso(\mathcal{A})$.*

We prove Theorem 2.2.1 constructively and furthermore show that given a functor

$F$, the functor $I^F$ obtained by using the procedures we give in the proof is effectively isomorphic to $F$.

**Proposition 2.2.2.** *Let $F : Iso(\mathcal{B}) \to Iso(\mathcal{A})$ be an enumerable functor. Then $F$ and $I^F$ are effectively isomorphic.*

We also prove statements analogous to Theorem 2.2.1 for enumerable bi-transformability and effective bi-interpretability, and reducibility by uniform enumerable transformations and reducibility via effective bi-interpretability.

The authors of [HH17] proved similar results for computable functors and effective interpretability.

Before we give the proofs we recall the necessary definitions.

### 2.2.1 Effective Interpretability

**Definition 2.2.1.** A relation $R$ is *uniformly intrinsically computable, short u.r.i. computable,* in $\mathcal{A}$ if there is a Turing operator $\Phi$ such that $\Phi^{\tilde{\mathcal{A}}} = R^{\tilde{\mathcal{A}}}$ for any $\tilde{\mathcal{A}} \in Iso(\mathcal{A})$.
A relation $R$ is *uniformly intrinsically computably enumerable, short u.r.i.c.e.,* in $\mathcal{A}$ if there is a Turing operator $\Phi$ such that $R^{\tilde{\mathcal{A}}} = \text{range}(\Phi^{\tilde{\mathcal{A}}})$ for any $\tilde{\mathcal{A}} \in Iso(\mathcal{A})$.

We say that a relation is $\Sigma^c_1$-definable in the language $L$ if it is definable by a $\Sigma_1$ computable infinitary formula without parameters in $L$. A relation is $\Delta^c_1$-definable if it and its corelation are definable by a $\Sigma^c_1$ formula. That a relation $R$ is $\Sigma^c_1$-definable ($\Delta^c_1$-definable) in a structure $\mathcal{A}$ is strongly connected to it being u.r.i.c.e. (u.r.i. computable). Ash, Knight, and Slaman [AKS93], building on work by Ash, Knight, Manasse, and Slaman [Ash+89] and Chisholm [Chi90], proved that a relation $R$ is u.r.i.c.e. (u.r.i. computable) in $\mathcal{A}$ iff it is $\Sigma^c_1$-definable ($\Delta^c_1$-definable) in $\mathcal{A}$.

In [Mon12] Montalbán studied the algorithmic complexity of sequences of relations. Following his definition, a sequence of relations $(R_i)_{i\in\omega}$ is u.r.i. computable in $\mathcal{A}$ if the set $\oplus_{i\in\omega} R_i$ is u.r.i. computable in $\mathcal{A}$. By the work of Ash, Knight, and Slaman [AKS93] this is the case iff $\oplus_{i\in\omega} R_i$ is $\Delta^c_1$-definable. Thus, a sequence of relations $(R_i)_{i\in\omega}$ is $\Delta^c_1$-definable iff $\oplus_{i\in\omega} R_i$ is.

**Definition 2.2.2.** A structure $\mathcal{A} = (A, P^{\mathcal{A}}_0, P^{\mathcal{A}}_1, \cdots)$ is *effectively interpretable* in $\mathcal{B}$ if there exists a $\Delta^c_1$-definable sequence of relations (in $\mathcal{B}$) $(\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}, \sim, R_0, R_1, \cdots)$ such that

(i) $\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}} \subseteq B^{<\omega}$,
(ii) $\sim$ is an equivalence relation on $\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}$,
(iii) $R_i \subseteq (B^{<\omega})^{a_{R_i}}$ is closed under $\sim$ within $\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}$,

and there exists a function $f_{\mathcal{A}}^{\mathcal{B}} : \mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} \to \mathcal{A}$, the *effective interpretation* of $\mathcal{A}$ in $\mathcal{B}$, which induces an isomorphism:

$$(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}, R_0, R_1, \cdots)/_{\sim} \cong (A, P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, \cdots)$$

We use the definition from [HH17], in the literature [Mon12; Mon14], effective interpretability is sometimes defined differently with $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$ required to be $\Sigma_1^c$-definable instead of $\Delta_1^c$-definable. These two definitions are equivalent; in our proof of Proposition 2.2.5 we demonstrate how to transform an effective interpretation where $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$ is $\Sigma_1^c$-definable into one where it is $\Delta_1^c$-definable.

Several possibilities to define an equivalence between structures based on effective interpretations exist. One is the notion of $\Sigma$-equivalence investigated in [Stu13], where two structures are $\Sigma$-equivalent if they are $\Sigma$-definable in each other. We will look at a stronger notion, effective bi-interpretability, which additionally requires the composition of the interpretations to be computable in the respective structures. This was first studied by Montalbán [Mon14].

**Definition 2.2.3.** Two structures $\mathcal{A}$ and $\mathcal{B}$ are *effectively bi-interpretable* if there are effective interpretations of one in the other such that the compositions

$$f_{\mathcal{B}}^{\mathcal{A}} \circ \tilde{f}_{\mathcal{A}}^{\mathcal{B}} : \mathcal{D}om_{\mathcal{B}}^{\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}} \to \mathcal{B} \quad \text{and} \quad f_{\mathcal{A}}^{\mathcal{B}} \circ \tilde{f}_{\mathcal{B}}^{\mathcal{A}} : \mathcal{D}om_{\mathcal{A}}^{\mathcal{D}om_{\mathcal{B}}^{\mathcal{A}}} \to \mathcal{A}$$

are uniformly relatively intrinsically computable in $\mathcal{B}$ and $\mathcal{A}$ respectively. (Here the function $\tilde{f}_{\mathcal{A}}^{\mathcal{B}} : (\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}})^{<\omega} \to \mathcal{A}^{<\omega}$ is the canonic extension of $f_{\mathcal{A}}^{\mathcal{B}} : \mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} \to \mathcal{A}$ mapping $\mathcal{D}om_{\mathcal{B}}^{\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}}$ to $\mathcal{D}om_{\mathcal{B}}^{\mathcal{A}}$.)

In line with the definition of $\tilde{f}_{\mathcal{A}}^{\mathcal{B}}$ in Definition 2.2.3, for a function $f : \mathcal{A} \to \mathcal{B}$, $\tilde{f}$ is the canonic extension of $f$ to tuples, i.e.,

$$\tilde{f} : \mathcal{A}^{<\omega} \to \mathcal{B}^{<\omega} \text{ with } \tilde{f}((x_1, \dots)) = (f(x_1), \dots).$$

In Theorem 2.1.1 we do not only use effective interpretability but we also require the equivalence relation in the definition to be computable. The following proposition shows that this is justified.

**Proposition 2.2.3.** *Let $(R_i)_{i\in\omega}$ be $\Delta_1^c$-definable in $\mathcal{A}$ and let $X$ be a computable set. Then $(X, R_1, R_2, \dots)$ is $\Delta_1^c$-definable in $\mathcal{A}$.*

*Proof.* Let $(R_i)_{i\in\omega}$ be $\Delta_1^c$-definable in $\mathcal{A}$ and let $X$ be a computable set, say it is computed by $\varphi_e$, and let $\Phi$ be the Turing operator witnessing that $(R_i)_{i\in\omega}$ is u.r.i. computable.

Now define a new operator by

$$\Phi'^S(\langle i,x\rangle) = \begin{cases} \Phi^S(\langle i-1,x\rangle) & i>1 \\ \varphi_e(x) & \text{otherwise} \end{cases}.$$

Clearly, $\Phi'$ is a computable operator and witnesses that the sequence $(X, R_1, R_2, \dots)$ is u.r.i. computable in $\mathcal{A}$. □

As for computable and enumerable functors one gets a method of reduction of classes of structures based on effective bi-interpretability [Mon14].

**Definition 2.2.4.** A class $\mathfrak{C}$ *is reducible to* $\mathfrak{D}$ *via effective bi-interpretability* if there are $\Delta_1^c$-formulas such that for every $\mathcal{A} \in \mathfrak{C}$, there is a $\mathcal{B} \in \mathfrak{D}$ such that $\mathcal{A}$ and $\mathcal{B}$ are effectively bi-interpretable using those formulas and the formulas are independent of the choice of $\mathcal{A}, \mathcal{B}$. A class $\mathfrak{C}$ is *complete for effective bi-interpretability*, if for every computable language $L$, the class of $L$-structures is reducible to $\mathfrak{C}$ via effective bi-interpretability.

### 2.2.2 Proof of Theorem 2.2.1 and Proposition 2.2.2

We prove the two directions of the equivalence in Theorem 2.2.1 separately in Proposition 2.2.4 and Proposition 2.2.5.

**Proposition 2.2.4.** *If $\mathcal{A}$ is effectively interpretable in $\mathcal{B}$ with $\sim$ computable, then there is an enumerable functor $F : Iso(\mathcal{B}) \to Iso(\mathcal{A})$.*

*Proof.* Let $\mathcal{A}$ be effectively interpretable in $\mathcal{B}$ and $\sim$ computable using the same notation as in Definition 2.2.2. We will construct $F$ by giving two witnesses $(\Psi, \Phi_*)$ for it.

By assumption the languages $L_{\mathcal{A}}$ and $L_{\mathcal{B}}$ are computable. Hence the set $\mathcal{D}_{L_{\mathcal{B}}} \subseteq \omega^{<\omega}$ of all possible finite atomic diagrams in $L_{\mathcal{B}}$ is computable. For $(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}, R_1, \dots)$, let their defining $\Sigma_1^c$ formulas be $(\varphi_{\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}}, \varphi_{\neg\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}}, \varphi_{R_1}, \varphi_{\neg R_1}, \dots)$; notice that we also use $\varphi_{\neg R_i}$, the defining formula of the complement of $R_i$. Fix a computable bijection $\sigma : \omega \to \omega^{<\omega}$ and define the function $h : \omega^{<\omega} \to \omega$ by

$$h(\overline{y}) = \mu x[\sigma(x) \sim \overline{y}] = \mu x \le \sigma^{-1}(\overline{y})[\sigma(x) \sim \overline{y}]. \tag{1}$$

Intuitively $h$ maps any tuple $\overline{y}$ to a fixed presentation of it under $\sim$. We use the minimal presentation in the order induced by $\sigma$ to make $h$ computable. We now build $\Psi$ using $h$ in the following way.

$$(\alpha, x = x) \in \Psi \Leftrightarrow \exists \overline{y}\ x = h(\overline{y}) \wedge \alpha \in \mathcal{D}_{L_{\mathcal{B}}} \wedge \alpha \models \varphi_{\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}}(\overline{y}) \tag{2}$$

Let $p_i$ be the arity of the relation $P_i$, then

$$\begin{aligned}(\alpha, P_i(x_1,\dots,x_{p_i})) \in \Psi \Leftrightarrow \exists \overline{y}_1,\dots,\overline{y}_{p_i} \left( \bigwedge_{i\in\{1\dots p_i\}} x_i = h(\overline{y}_i) \right) \\ \wedge\, \alpha \in \mathcal{D}_{L_{\mathcal{B}}} \wedge \alpha \vDash \varphi_{R_i}(\overline{y}_1,\dots,\overline{y}_{p_i})\end{aligned} \tag{3}$$

$$\begin{aligned}(\alpha, \neg P_i(x_1,\dots,x_{p_i})) \in \Psi \Leftrightarrow \exists \overline{y}_1,\dots,\overline{y}_{p_i} \left( \bigwedge_{i\in\{1\dots p_i\}} x_i = h(\overline{y}_i) \right) \\ \wedge\, \alpha \in \mathcal{D}_{L_{\mathcal{B}}} \wedge \alpha \vDash \varphi_{\neg R_i}(\overline{y}_1,\dots,\overline{y}_{p_i})\end{aligned} \tag{4}$$

Notice that the problem of deciding whether a finite structure in a computable language is a model of a $\Sigma^c_1$-formula is c.e. It follows that $\Psi$ defined by equations (2) to (4) is c.e.

We now show that for $\tilde{\mathcal{B}} \in Iso(\mathcal{B})$, its image $\Psi^{\tilde{\mathcal{B}}}$ is in the isomorphism class of $\mathcal{A}$. Define $(\mathcal{D}om^{\tilde{\mathcal{B}}}_{\mathcal{A}}, \sim, R^{\tilde{\mathcal{B}}}_1, \dots)$ in the obvious way using the formulas of the effective interpretation of $\mathcal{A}$ in $\mathcal{B}$. Say $g$ is an isomorphism from $\mathcal{B}$ to $\tilde{\mathcal{B}}$, then $(\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}, R^{\mathcal{B}}_1, \dots)/_\sim \cong_{\tilde{g}} (\mathcal{D}om^{\tilde{\mathcal{B}}}_{\mathcal{A}}, R^{\tilde{\mathcal{B}}}_1, \dots)/_\sim$. Recall the function $h$ used in the definition of $\Psi$. Let the function $\xi_{\tilde{\mathcal{B}}} : (\mathcal{D}om^{\tilde{\mathcal{B}}}_{\mathcal{A}}, R^{\tilde{\mathcal{B}}}_1, \dots)/_\sim \to \Psi^{\tilde{\mathcal{B}}}$ be the canonical restriction of $h$ to the quotient of the domain, i.e., $\xi_{\tilde{\mathcal{B}}}([\overline{y}]_\sim) = h(\overline{y})$. It follows from equation (1) that $\xi_{\tilde{\mathcal{B}}}$ is well defined. We will show that $\xi_{\tilde{\mathcal{B}}}$ is an isomorphism.

- $\xi_{\tilde{\mathcal{B}}}$ is $1-1$ because by equation (1) if $h(\overline{y}) = h(\overline{x})$ then $\overline{x} \sim \overline{y}$. Hence, $[\overline{y}]_\sim = [\overline{x}]_\sim$.
- $\xi_{\tilde{\mathcal{B}}}$ is onto because by equation (2) if $x \in \Psi^{\tilde{\mathcal{B}}}$, then $\exists \overline{y} \in \mathcal{D}om^{\tilde{\mathcal{B}}}_{\mathcal{A}}$ such that $x = h(\overline{y})$.

It follows that $\xi_{\tilde{\mathcal{B}}}$ is bijective. By equations (3) and (4) $\xi_{\tilde{\mathcal{B}}}$ is an homomorphism and therefore by the above arguments also an isomorphism. Hence, $\Psi^{\tilde{\mathcal{B}}} \in Iso(\mathcal{A})$ as $\xi_{\tilde{\mathcal{B}}} \circ \tilde{g} \circ (f^{\mathcal{B}}_{\mathcal{A}})^{-1}$ is an isomorphism from $\mathcal{A}$ to $\Psi^{\tilde{\mathcal{B}}}$. Notice that $\xi_{\tilde{\mathcal{B}}}$ is computable from $\tilde{\mathcal{B}}$ and that the computation is uniform.

We now build $\Phi_*$. Assume $\tilde{\mathcal{B}} \cong_f \hat{\mathcal{B}}$; we use the extension of $f : \omega \to \omega$, $\tilde{f} : \omega^{<\omega} \to \omega^{<\omega}$ and set $F(f) = \xi_{\hat{\mathcal{B}}} \circ \tilde{f} \circ \xi^{-1}_{\tilde{\mathcal{B}}}$. Because $\xi_{\hat{\mathcal{B}}}$, and $\xi^{-1}_{\tilde{\mathcal{B}}}$ are uniformly computable in $Iso(\mathcal{B})$ and $\tilde{f}$ is uniformly computable from $f$, there is a Turing operator $\Phi_*$ such that $\Phi^{\tilde{\mathcal{B}}\oplus f \oplus \hat{\mathcal{B}}}_* = F(f)$. Furthermore, $F(f)$ is a bijection because so are the functions it is composed of. Moreover, $F(\tilde{\mathcal{B}}) \cong_{\xi_{\hat{\mathcal{B}}} \circ \tilde{f} \circ \xi^{-1}_{\tilde{\mathcal{B}}}} F(\hat{\mathcal{B}})$ because for $Q \in (\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}, R_1, \neg R_1, \dots)$, $\tilde{\mathcal{B}} \vDash \varphi_Q(\overline{x})$ if and only if $\hat{\mathcal{B}} \vDash \varphi_Q(\tilde{f}(\overline{x}))$. □

**Proposition 2.2.5.** *If there is an enumerable functor $F : Iso(\mathcal{B}) \to Iso(\mathcal{A})$, then $\mathcal{A}$ is effectively interpretable in $\mathcal{B}$ with the restriction that $\sim$ is computable.*

*Proof.* Assume $F$ is witnessed by $(\Psi, \Phi_*)$. We will first provide definitions of $\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}$ and relations $R_i$ which are $\Sigma^c_1$-definable in $\mathcal{B}$. We then use these to build an interpretation having the desired properties, i.e., the sequence of relations is $\Delta_1^c$-definable and the equivalence relation is computable. We apply the standard argument that the definition of effective interpretability which requires $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$ to be $\Sigma_1^c$-definable is equivalent to the definition we use.

In what follows we write $\mathcal{B}\restriction_A$ for the substructure of $\mathcal{B}$ induced by the restriction of its universe to elements in the set $A$. Similarly $\mathcal{B}\restriction_{\overline{a}}$ is the substructure of $\mathcal{B}$ induced by the restriction of its universe to elements in the tuple $\overline{a}$.

The $\Sigma_1^c$-definable interpretation made of $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}, \sim$ and the sequence of relations $(R_1, \dots)$ are defined as follows.

$\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$: The domain is a subset of $\omega^{<\omega} \times \omega$ such that

$$(\overline{a}, i) \in \mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} \Leftrightarrow i = i \in \Psi^{\mathcal{B}\restriction_{\overline{a}}}.$$

Since $\Psi$ is c.e. and the restriction of $\mathcal{B}$ to $\overline{a}$ is computable relative to $\mathcal{B}$, $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$ is uniformly r.i.c.e. and therefore also $\Sigma_1^c$-definable in $\mathcal{B}$.

$\sim$: For all $(\overline{a}, i), (\overline{b}, j) \in \omega^{<\omega} \times \omega$,

$$(\overline{a}, i) \sim (\overline{b}, j) \Leftrightarrow i = j.$$

By definition $\sim$ is computable, reflexive, symmetric, and transitive.

$R_i$: Let $P_i$ have arity $p_i$. Then for all $(\overline{a}_1, x_1), \dots, (\overline{a}_{p_i}, x_{p_i}) \in \mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$ we define $R_i$ as follows.

$$((\overline{a}_1, x_1), \dots, (\overline{a}_{p_i}, x_{p_i})) \in R_i \Leftrightarrow P_i(x_1, \dots, x_{p_i}) \in \Psi^{\mathcal{B}}$$

$$((\overline{a}_1, x_1), \dots, (\overline{a}_{p_i}, x_{p_i})) \notin R_i \Leftrightarrow \neg P_i(x_1, \dots, x_{p_i}) \in \Psi^{\mathcal{B}}$$

For $(\overline{a}_1, x_1), \dots, (\overline{a}_{p_i}, x_{p_i}) \in \mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$ either $P_i(x_1, \dots, x_n)$ or $\neg P_i(x_1, \dots, x_n)$ is in $\Psi^{\mathcal{B}}$ and $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$ is $\Sigma_1^c$-definable. Therefore $R_i$ is also $\Sigma_1^c$-definable uniformly in $i$.

Because $\sim$ is computable the restriction to the domain is trivially $\Sigma_1^c$-definable. Hence, also the sequence $(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}, \sim, R_1, \dots)$ is $\Sigma_1^c$-definable.

**Claim 2.2.5.1.** *The equivalence relation $\sim$ is compatible with the definition of $R_i$, i.e., if for all $(\overline{a}_1, k_1), \dots, (\overline{a}_{p_i}, k_{p_i}), (\overline{b}_1, l_1), \dots, (\overline{b}_{p_i}, l_{p_i}) \in \mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$, $(\overline{a}_1, k_1) \sim (\overline{b}_1, l_1), \dots, (\overline{a}_{p_i}, k_{p_i}) \sim (\overline{b}_{p_i}, l_{p_i})$, then $((\overline{a}_1, k_1), \dots, (\overline{a}_{p_i}, k_{p_i})) \in R_i$ iff $((\overline{b}_1, l_1), \dots (\overline{b}_{p_i}, l_{p_i})) \in R_i$.*

*Proof.* The claim follows from the definitions of $R_i$ and $\sim$ because for $i \in \{1, \dots, p_i\}$, $(\overline{a}_i, k_i) \sim (\overline{b}_i, l_i)$ if and only $k_i$ is equal to $l_i$. □

Consider the function $f : (\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}, R_1, \dots)/_\sim \to F(\mathcal{B})$ defined as $f([(\overline{a}, x)]_\sim) = x$. We claim that $(\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}, R_1, \dots)/_\sim \cong_f F(\mathcal{B})$. The function $f$ is a bijection by the definition of $\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}$ and $\sim$. It follows from the definition of $R_i$ and Claim 2.2.5.1 that $f$ is an isomorphism. We defined everything needed for an effective interpretation with the exception that $(\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}, \sim, R_1, \dots)$ is not $\Delta^{\mathrm{c}}_1$-definable.

We now define a sequence of relations $(\mathcal{D}om^{*\mathcal{B}}_{\mathcal{A}}, \sim^*, R^*_1, \dots)$ $\Delta^{\mathrm{c}}_1$-definable in $\mathcal{B}$ such that the structure $(\mathcal{D}om^{*\mathcal{B}}_{\mathcal{A}}, R^*_1, \dots)/_{\sim^*}$ is isomorphic to $(\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}, R_1, \dots)/_\sim$. This sequence of relations is an effective interpretation of $\mathcal{A}$ in $\mathcal{B}$.

$\mathcal{D}om^{*\mathcal{B}}_{\mathcal{A}}$: Since the original domain $\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}$ is $\Sigma^{\mathrm{c}}_1$-definable, every element $(\overline{a}, i)$ satisfies a finitary existential formula $\exists \overline{y} \varphi_j(\overline{a}, i, \overline{y})$ in the infinite disjunction defining $\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}$ where $j$ is the index of the formula in some computable enumeration. The new domain $\mathcal{D}om^{*\mathcal{B}}_{\mathcal{A}}$ is a subset of $\omega^{<\omega} \times \omega \times \omega^{<\omega} \times \omega$ defined as follows.

$$(\overline{a}, i, \overline{y}, j) \in \mathcal{D}om^{*\mathcal{B}}_{\mathcal{A}} \Leftrightarrow \mathcal{B} \vDash \varphi_j(\overline{a}, i, \overline{y})$$

$\mathcal{D}om^{*\mathcal{B}}_{\mathcal{A}}$ is clearly uniformly r.i. computable and thus $\Delta^{\mathrm{c}}_1$-definable in the language of $\mathcal{B}$.

$\sim^*$: For all $(\overline{a}, i, \overline{y}, j), (\overline{b}, k, \overline{z}, l) \in \omega^{<\omega} \times \omega \times \omega^{<\omega} \times \omega$

$$(\overline{a}, i, \overline{y}, j) \sim^* (\overline{b}, k, \overline{z}, l) \Leftrightarrow \overline{a} = \overline{b} \wedge i = k.$$

This is by definition a computable equivalence relation.

$R^*_i$: As above let $P_i$ have arity $p_i$. Then for all $(\overline{a}_1, k_1, \overline{y}_1, j_1), \dots, (\overline{a}_{p_i}, k_{p_i}, \overline{y}_{p_i}, j_{p_i}) \in \mathcal{D}om^{*\mathcal{B}}_{\mathcal{A}}$, $R^*_i$ is defined as follows.

$$((\overline{a}_1, k_1, \overline{y}_1, j_1), \dots, (\overline{a}_{p_i}, k_{p_i}, \overline{y}_{p_i}, j_{p_i})) \in R^*_i \Leftrightarrow P_i(k_1, \dots, k_{p_i}) \in \Psi^{D(\mathcal{B})}$$

$$((\overline{a}_1, k_1, \overline{y}_1, j_1), \dots, (\overline{a}_{p_i}, k_{p_i}, \overline{y}_{p_i}, j_{p_i})) \notin R^*_i \Leftrightarrow \neg P_i(k_1, \dots, k_{p_i}) \in \Psi^{D(\mathcal{B})}$$

By the same arguments as for $R_i$, $R^*_i$ is uniformly relatively intrinsically computable from $\mathcal{B}$ and therefore $\Delta^{\mathrm{c}}_1$-definable in the language of $\mathcal{B}$.

The sequence of relations $(\mathcal{D}om^{*\mathcal{B}}_{\mathcal{A}}, \sim^*, R^*_1, \dots)$ is $\Delta^{\mathrm{c}}_1$-definable by an argument similar to the argument that the sequence $(\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}, \sim, R_1, \dots)$ is $\Sigma^{\mathrm{c}}_1$-definable.

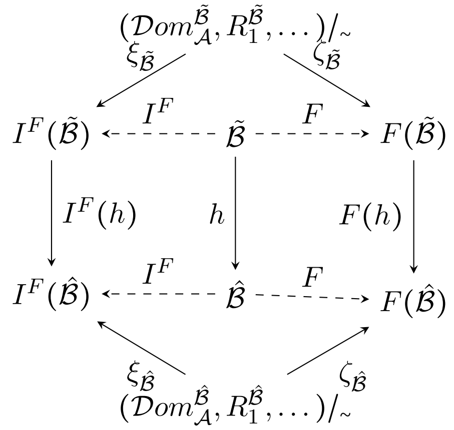
**Claim 2.2.5.2.** *The equivalence relation $\sim^*$ is compatible with the definition of $R_i$.*

*Proof.* The claim follows from an argument analogous to that given in Claim 2.2.5.1. □

Define $f^* : (\mathcal{D}om^{*\mathcal{B}}_{\mathcal{A}}, R_1^*, \dots)/_{\sim^*} \to (\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}, R_1, \dots)/_{\sim}$ as $f^*([(\overline{a}, x, \overline{y}, j)]_{\sim^*}) = [(\overline{a}, x)]_{\sim}$. It is not hard to see that $f^*$ is well defined on the quotient structure $(\mathcal{D}om^{*\mathcal{B}}_{\mathcal{A}}, R_1^*, \dots)/_{\sim^*}$ and induces an isomorphism between it and $(\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}, R_1, \dots)/_{\sim}$. Therefore the structure $(\mathcal{D}om^{*\mathcal{B}}_{\mathcal{A}}, R_1^*, \dots)/_{\sim^*}$ is isomorphic to $F(\mathcal{B})$ by $f \circ f^*$. Since $(\mathcal{D}om^{*\mathcal{B}}_{\mathcal{A}}, R_1^*, \dots)$ is $\Delta_1^c$-definable and $\sim^*$ is computable, the theorem follows. □

*Proof of Proposition 2.2.2.* Let $F : \mathcal{B} \to \mathcal{A}$ be an enumerable functor, $(\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}, R_1, \dots)/_{\sim}$ be the effective interpretation one gets by applying the procedure described in Proposition 2.2.5 to $F$, and let $\zeta_{\mathcal{B}}$ be the effective interpretation, i.e., using the definition given in the proof, $\zeta_{\mathcal{B}} = f \circ f^*$. The function $\zeta_{\mathcal{B}}$ is computable relative to $\mathcal{B}$ using projection.

We now transform the interpretation back to an enumerable functor using the procedure described in the proof of Proposition 2.2.4. We get a functor $I^F : \mathcal{B} \to \mathcal{A}$, such that $I^F(\mathcal{B}) \cong_{\xi_{\mathcal{B}}} (\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}, R_1, \dots)/_{\sim}$ by the $\xi_{\mathcal{B}}$ defined in the proof of Proposition 2.2.4. The following diagram shows the relation between two presentations $\tilde{\mathcal{B}}, \hat{\mathcal{B}} \in Iso(\mathcal{B})$ isomorphic by $h$ under the two functors.




By the above diagram for every presentation $\tilde{\mathcal{B}}$ of $\mathcal{B}$, $I^F(\tilde{\mathcal{B}})$ and $F(\tilde{\mathcal{B}})$ are isomorphic by $\xi^{-1}_{\tilde{\mathcal{B}}} \circ \zeta_{\tilde{\mathcal{B}}}$. Also the squares as given in Definition 2.0.4 for any two presentations $\tilde{\mathcal{B}}, \hat{\mathcal{B}}$ can be seen to commute by the above diagram. Hence, $I^F$ and $F$ are naturally isomorphic. Since the functions $\xi_{\mathcal{B}}$ and $\zeta_{\mathcal{B}}$ are both uniformly computable in $Iso(\mathcal{B})$, $I^F$ and $F$ are effectively isomorphic. □

### 2.2.3 Enumerable bi-transformability and u.e.t. reductions

**Theorem 2.2.6.** *$\mathcal{A}$ and $\mathcal{B}$ are enumerably bi-transformable iff they are effectively bi-interpretable.*

Using the proof of Proposition 2.2.4 and Proposition 2.2.5 the proof of this theorem is similar to the proof of the statement that computable bi-transformability and effective bi-interpretability are equivalent [HH17, Theorem 1.9]. We therefore omit it here and refer the reader to [HH17, Section 4] for a detailed proof.

Since the proof given in [HH17] is uniform we get that the same holds for reduction by uniform enumerable transformation and reduction by effective bi-interpretability.

**Corollary 2.2.7.** *$\mathfrak{C}$ is uniformly enumerably transformally reducible to $\mathfrak{D}$ iff $\mathfrak{C}$ is reducible by effective bi-interpretability to $\mathfrak{D}$.*

## 2.3 On $\omega$, computable and enumerable functors are equivalent

All the results in this chapter so far were proven without any assumptions on the structures except that their languages are relational. For what follows we need to assume that all structures have as universes the set of all natural numbers. It turns out that under this assumption the notions of a computable functor and enumerable functor are equivalent.

**Theorem 2.3.1.** *Let $\mathfrak{C}$ and $\mathfrak{D}$ be classes of structures such that all structures in them have universe $\omega$ and assume that there is a computable functor $F : \mathfrak{C} \to \mathfrak{D}$. Then there is an enumerable functor $G : \mathfrak{C} \to \mathfrak{D}$ effectively isomorphic to $F$.*

*Proof.* Let $L_1$ and $L_2$ be the languages of $\mathfrak{C}$ and $\mathfrak{D}$, respectively. Say $F$ is given by the pair of operators $(\Phi, \Phi_*)$. By the use principle we may assume that $\Phi \subset 2^{<\omega} \times \omega \times \omega$ is upwards closed, i.e.,

$$\forall \sigma, \tau, n, m \left( (\tau \geq \sigma \wedge (\sigma, n, m) \in \Phi) \to (\tau, n, m) \in \Phi \right).$$

Since $\Phi$ is a Turing operator it is c.e. We now build a new c.e. set $\Psi$ which will serve as the first operator of $G$. Let $(\varphi_i)_{i\in\omega}$ be the standard enumeration of the $L_1$ formulas with variables a subset of $\{x_i : i \in \omega\}$ as in our definition of the atomic diagram of an $L_1$ structure. Given an index $i$ let $i'$ be the index for $\neg\varphi_i$ in this enumeration. Now, given a string $\sigma \in 2^{<\omega}$ such that $\sigma$ is the atomic diagram of a finite $L_1$ structure let $X_\sigma$ be the finite set defined by

$$i \in X_\sigma \Leftrightarrow \sigma(i) = 1.$$

As $\sigma$ is the atomic diagram of a finite $L_1$ structure $\sigma(i) = 1$ implies that $\sigma(i') \downarrow= 0$ and therefore $X_\sigma$ is the set coded by $\sigma$.

We can finally define $\Psi$. Towards this computably enumerate $\Phi$. Whenever you see $(\sigma, i, 1) \searrow \Phi$ with $\sigma$ an atomic diagram of a finite $L_1$ structure and $i \in \omega$ enumerate $(X_\sigma, i)$ into $\Psi$. For $\mathcal{A} \in K_1$ we have that $\Phi(\mathcal{A})$ and $\Psi(\mathcal{A})$ produce the same structure. To see this observe that for all $i \in \omega$, $\Phi(\mathcal{A})(i) = 1$ if and only if there is $\sigma \prec D(\mathcal{A})$ such that $(\sigma, i, 1) \in \Phi$, in which case $(X_\sigma, i) \in \Psi$ and thus $i \in \Psi(\mathcal{A})$. On the other hand if $\Phi(\mathcal{A})(i) = 0$ there is $\sigma \prec D(\mathcal{A})$ such that $(\sigma, i', 1) \in \Phi$ and therefore $i' \in \Psi(\mathcal{A})$. Therefore $\Psi(\mathcal{A})$ is the set having as its characteristic function $D(\mathcal{A})$.

Now let $G$ be given by $(\Psi, \Phi_*)$. We have that $G$ is an enumerable functor and that $G = F$ in the sense that $F(\mathcal{A}) = G(\mathcal{A})$ for all $\mathcal{A} \in K_1$ and $F(f) = G(f)$ for all $f : \mathcal{A}_1 \cong \mathcal{A}_2$ with $\mathcal{A}_1, \mathcal{A}_2 \in K_1$. The functors are thus trivially effectively naturally isomorphic. □

**Corollary 2.3.2.** *Assume that all structures in $\mathfrak{C}$ and $\mathfrak{D}$ have universe $\omega$. Then there exists a computable functor from $\mathfrak{C}$ to $\mathfrak{D}$ if and only if there exists an enumerable functor from $\mathcal{C}$ to $\mathfrak{D}$.*

*Proof.* The direction from left to right follows directly from Theorem 2.3.1. The direction from right to left is proven similarly to Theorem 2.1.1. The only exception that we do not have to take care of the universes. □

# 3 Spectra of structures under equivalence relations

The study of degrees realized by structures is a central topic in computable structure theory initiated by Richter [Ric81] who was the first to study the degrees of isomorphic copies of structures. She observed that the family of degrees of structures may not have a least element, and that if a structure in a certain class, such as for example linear orderings or abelian groups, has a least degree computing one of its presentations, then this structure must have a computable copy. This motivated the systematic study of the family of degrees of a given structure. Towards this Knight [Kni86] introduced the notion of the degree spectrum of a structure.

**Definition 3.0.1** ([Kni86])**.** The degree spectrum of a structure $\mathcal{A}$ is

$$DgSp(\mathcal{A}) = \{deg(\mathcal{B}) : \mathcal{B} \cong \mathcal{A}\}.$$

In the same paper Knight proved the main theorem about degree spectra: that in non-trivial cases, they are upwards closed. Since then degree spectra have seen a lot of interest with the main focus of researchers on the question which families of degrees can be realized as degree spectra of structures. It follows already from Richter's result [Ric81] that there are degree spectra of structures without a least degree. Soskov [Sos04] and Knight et al. [folklore] showed that if that is the case then the degree spectrum must contain uncountably many incomparable elements. A widely studied question is which large classes of degrees can be realized as degree spectra of structures. We can define a class of degrees to be large using category or measure. The most famous example of a large class known to be a degree spectrum of a structure is the class of non-computable degrees $\{\mathbf{d} : \mathbf{d} > \mathbf{0}\}$. This was shown to be a spectrum by Slaman [Sla98] and, independently, Wehner [Weh98]. One research direction which has emerged in recent years is to study degree spectra with respect to other equivalence relations than isomorphism. Instead of considering the family of degrees of isomorphic copies of structures one could look at the family of degrees of structures equivalent to a given structure under different equivalence relations. This research direction was independently proposed by Yu Lian [Yu15], Montalbán [Mon15], and Fokina, Semukhin, and Turetsky [FST18]. The latter gave the following definition.

**Definition 3.0.2** ( [FST18])**.** Given a structure $\mathcal{A}$ and an equivalence relation $\sim$, the *degree spectrum of* $\mathcal{A}$ *under* $\sim$ is

$$DgSp_{\sim}(\mathcal{A}) = \{deg(\mathcal{B}) : \mathcal{B} \sim \mathcal{A}\}.$$

Under this notion the classical degree spectrum of a structure $\mathcal{A}$ is $DgSp_{\cong}(\mathcal{A})$. Fokina, Semukhin, and Turetsky [FST18] investigated degree spectra under $\Sigma_n$ equivalence, $DgSp_{\equiv_n}(\mathcal{A})$. Two structures are $\Sigma_n$ equivalent, $\mathcal{A} \equiv_n \mathcal{B}$, if every first order $\Sigma_n$ sentence true of $\mathcal{A}$ is true of $\mathcal{B}$ and vice versa. Andrews and Miller [AM15] studied the spectra of theories, the family of degrees of models of a complete theory $T$. In terms of the above definition the theory spectrum of $T$ is the spectrum of $\mathcal{A} \models T$ under elementary equivalence, $DgSp_{\equiv}(\mathcal{A})$.

In this chapter we contribute to this direction of research by studying degree spectra under bi-embeddability and elementary bi-embeddability. Our main goal is to compare the degree spectra under these two notions with spectra under other equivalence relations. In Section 3.1 we present our work on bi-embeddability spectra. This section is based on joint work with Ekaterina Fokina and Luca San Mauro. All of our results here can be found in [FRM18]. In Section 3.2 we investigate elementary bi-embeddability spectra. The results of this section have been published in [Ros18], however, the proofs in this section are considerably more detailed then the ones given there.

## 3.1 Bi-embeddability spectra and bases of spectra

Two structures $\mathcal{A}$ and $\mathcal{B}$ are *bi-embeddable*, written $\mathcal{A} \approx \mathcal{B}$, if either is embeddable in the other. The degree spectrum of $\mathcal{A}$ under bi-embeddability, or short, bi-embeddability spectrum of $\mathcal{A}$, is then

$$DgSp_{\approx}(\mathcal{A}) = \{deg(\mathcal{B}) : \mathcal{B} \approx \mathcal{A}\}.$$

Obtaining examples of sets of degrees which are, or are not, bi-embeddability spectra of structures is in general difficult, since the bi-embeddability relation does not seem to possess strong combinatorial properties one could use to construct such examples. However for many of the examples constructed for classical degree spectra a thorough analysis of their construction shows that their isomorphism spectrum coincides with their bi-embeddability spectrum. Either because the structure is *b.e. trivial*, i.e., its isomorphism type and bi-embeddability type coincide, or every bi-embeddable copy computes an isomorphic copy, in which case we say that the structure is a *basis* for its bi-embeddability spectrum. More formally, given a single structure $\mathcal{A}$ we say that $\mathcal{A}$ is a $\sim$ basis of $\mathcal{B}$ if $\mathcal{A} \sim \mathcal{B}$ and $DgSp_{\cong}(\mathcal{A}) = DgSp_{\sim}(\mathcal{B})$.

Apart from the above observation another motivation to study bases of spectra arises from the comparison of degree spectra under different equivalence relations. Given two equivalence relations $\sim_0, \sim_1$ on structures and a structure $\mathcal{A}$, a common question is if there is a structure $\mathcal{B}$ such that $DgSp_{\sim_1}(\mathcal{B}) = DgSp_{\sim_0}(\mathcal{A})$. In general this structure $\mathcal{B}$ might look very different than $\mathcal{A}$ from a structural point of view. Thus, given $\mathcal{A}$ it might be hard to find $\mathcal{B}$. Therefore it is useful to restrict $\mathcal{B}$ to some specific class of structures. The notion of a basis captures this question nicely for the most restrictive class of structures one could want $\mathcal{B}$ to be in, the $\sim_1$ type of $\mathcal{A}$. Note that while our definition of a basis only captures the case where $\sim_0$ is isomorphism, it can be adapted to capture the general case without much effort.

In the present section we study the phenomenon of b.e. triviality and bi-embeddability bases of structures. Thus, if we say that $\mathcal{A}$ is a basis of $\mathcal{B}$ we mean that $\mathcal{A}$ is a bi-embeddability basis. In Section 3.1.1 we give some examples of b.e. trivial structures and use these to obtain examples of well known families of degrees that are realized as bi-embeddability spectra. In Section 3.1.2 we give a more general definition of a basis where we allow families of structures. This definition is motivated by the notion of basis in topology and linear algebra. In Section 3.1.2 we give a complete characterization of the bi-embeddability spectra of linear orderings and in Section 3.1.2 we show that in a subclass of strongly locally finite graphs every structure has a basis consisting of a single structure.

### 3.1.1 B.e. triviality

**Definition 3.1.1.** A structure $\mathcal{A}$ is *b.e. trivial* if any bi-embeddable copy $\mathcal{B}$ of $\mathcal{A}$ is isomorphic to $\mathcal{A}$.

A stronger condition that implies b.e. triviality is that any epimorphism of a structure is an automorphism. However, there is no connection between the number of automorphisms of a structure and b.e. triviality. Recall that a structure is *rigid* if its only automorphism is the identity. It is not hard to see that the two definitions are independent.

**Proposition 3.1.1.** *There is a b.e. trivial structure that is not rigid and there is a rigid structure that is not b.e. trivial.*

*Proof.* Consider a tree in the language of graphs, where the number of successors of a vertex is strictly monotonic in the canonic lexicographical ordering on the tree. This tree is rigid as any automorphism must map a vertex to a vertex with the same number of children. It is however not b.e. trivial as it is bi-embeddable with two disjoint copies of itself. For an example of a b.e. trivial structure that is not rigid consider any finite structure with more than one automorphism. □

Recall that a structure is *automorphically trivial* if there is a finite subset of its universe such that every permutation of its universe that fixes this subset pointwise is an automorphism.

**Proposition 3.1.2.** *Automorphically trivial structures are b.e. trivial.*

*Proof.* Let $\mathcal{A}$ be automorphically trivial and $\mathcal{B} \approx \mathcal{A}$. Assume $\mu : \mathcal{A} \to \mathcal{B}$ and $\nu : \mathcal{B} \to \mathcal{A}$ are embeddings, and that $S_0$ is a finite substructure of $\mathcal{A}$ such that every permutation of $A$ fixing $S_0$ pointwise is an automorphism. We have that $\mathcal{B}$ is isomorphic to a substructure of $\mathcal{A}$ by $\nu$ and thus every permutation that fixes $\nu(\mathcal{B}) \cap S_0$ is an automorphism of $\nu(\mathcal{B})$. Let $S_1$ be the pullback of $\nu(\mathcal{B}) \cap S_0$ along $\nu$. Then $S_1$ witnesses that $\mathcal{B}$ is automorphically trivial. We can inductively define $S_{n+1}$ switching the roles of $\mathcal{A}$, $\mathcal{B}$ and $\mu$, $\nu$ when $n$ is odd. Observe that for all $n$, $S_{n+1}$ is isomorphic to a substructure of $S_n$. Because $S_0$ was finite we will find a fixpoint, i.e., there is an $n$ such that $S_{n+1} \cong S_n$. Let $k$ be the first even number such that $S_{k+1} \cong S_k$. Since we constructed $S_{k+1}$ by pulling back $S_k$ along $\nu$ we have that $\nu$ is an isomorphism between $S_{k+1}$ and $S_k$.

We can now build an isomorphism $f : \mathcal{B} \to \mathcal{A}$. At stage 0 let $f$ be $\nu\restriction_{S_{k+1}}$, the isomorphism between $S_{k+1}$ and $S_k$. At stage $s$, if $f(s)$ is already defined or not in $B$ proceed to the next stage. Otherwise take the least $x \in A$ that is not in the range of $f$ and let $f(s) = x$. Then proceed to the next stage.

Clearly in the limit $f$ will be a bijection between $\mathcal{B}$ and $\mathcal{A}$. To see that it is an isomorphism let $T = dom(f)$ at some stage $s$. We have that $\nu(T) \cap f(T) \supseteq S_k$ and thus there is a permutation $\pi$ of $A$ fixing $S_k$ pointwise such that $\pi(f(T)) = \nu(T)$. By automorphic triviality of $\mathcal{A}$ we have that

$$f(T) \cong \pi(f(T)) = \nu(T) \cong T.$$

Thus, at every stage $s$, $f$ is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ and therefore in the limit an isomorphism. □

Knight [Kni86] showed that if a structure is automorphically trivial, then its degree spectrum is a singleton, and that otherwise it is upwards closed. By the above proposition also the bi-embeddability spectrum of automorphically trivial structures is a singleton. Clearly every bi-embeddability spectrum of a structure is the union of the degree spectra of structures in its bi-embeddability type. Thus, Knight's result also holds for bi-embeddability spectra.

**Corollary 3.1.3.** *If $\mathcal{A}$ is automorphically trivial, then its bi-embeddability spectrum is a singleton. Otherwise it is upwards closed.*

We now look at examples of b.e. trivial structures that appear in the literature. The following definition appears in [CK10].

**Definition 3.1.2.** Let $X \subseteq \omega$ and $n \in \omega$. The graph $G(\{n\} \oplus X)$ is an $\omega$ chain with an $n+5$ cycle attached to 0, a 3 cycle attached to $m$ if $m \in X$ and a 4 cycle attached to $m$ if $m \notin X$.

**Proposition 3.1.4.** *Let $X \subseteq \omega$, $\mathfrak{F}$ be a family of sets and $\mathcal{G}$ be the disjoint union of the graphs $G(\{n\} \oplus F)$ for $F \in \mathfrak{F}$ and $n \in X$. Then $\mathcal{G}$ is b.e. trivial.*

*Proof.* It is easy to see that for any set $Y$ and $n \in \omega$, $G(\{n\} \oplus Y)$ is b.e. trivial as cycles of length $m$ only embed into cycles of length $m$.

Now, say $\mathcal{G}$ is bi-embeddable with $\mathcal{A}$, say $f : \mathcal{G} \to \mathcal{A}$ and $g : \mathcal{A} \to \mathcal{G}$. Let $G(\{n\} \oplus F)$ be a component of $\mathcal{G}$, then $g(f(G(\{n\} \oplus F)))$ must be in a component containing a substructure isomorphic to $G(\{n\} \oplus F)$. By construction the only component like this is $G(\{n\} \oplus F)$ and as it is b.e. trivial we get that $g$ is the inverse of $f$ on $G(\{n\} \oplus F)$. We have that for every $n \in X$ and $F \in \mathfrak{F}$, $\mathcal{G}$ contains exactly one component isomorphic to $G(\{n\} \oplus F)$ and no other components. Therefore, $g$ is the inverse of $f$, and thus, $f$ is an isomorphism. □

Graphs of the form required in Proposition 3.1.4 were used in [CK10] to show that the class of non computable degrees and the class of hyperimmune degrees are isomorphism spectra. We now get the same result for bi-embeddability spectra.

**Theorem 3.1.5.** *(1) For every Turing degree $\mathbf{a}$ there is a graph $\mathcal{G}$ such that $DgSp_{\approx}(\mathcal{G}) = \{\mathbf{d} : \mathbf{d} \geq \mathbf{a}\}$.*

*(2) There is a graph $\mathcal{G}$ such that $DgSp_{\approx}(\mathcal{G}) = \{\mathbf{d} : \mathbf{d} > \mathbf{0}\}$.*

*(3) There is a graph $\mathcal{G}$ such that $DgSp_{\approx}(\mathcal{G}) = \{\mathbf{d} : \mathbf{d}$ is hyperimmune$\}$.*

*Proof.* For (1), given a set $X \in \mathbf{a}$ consider the graph using $\{0\} \oplus X$. It is not hard to see that $G(\{0\} \oplus X)$ is b.e. trivial and $DgSp_{\approx}(G(\{0\} \oplus X)) = \{\mathbf{d} : \mathbf{d} \geq \mathbf{a}\}$. Items (2) and (3) follow directly from Proposition 3.1.4 and the results in [CK10]. The proofs given there follow the ideas of Wehner's proof that the non-computable degrees are the spectrum of a structure [Weh98] but with some differences.

We sketch the proof of (2). Wehner considered the family of finite sets $\mathfrak{F} = \{\{n\} \oplus F :$ F finite $\wedge F \neq W_n\}$. He showed that this family is $X$-computably enumerable if and only if $X$ is not computable and coded this family into a structure $\mathcal{H}$ such that $\mathcal{H}$ is $X$-computable if and only if the family is $X$-c.e. The structure $\mathcal{H}$ is not b.e. trivial and thus we do not know what its bi-embeddability spectrum is. However, if we consider the graph $\mathcal{G}$ obtained by taking the disjoint unions of the graphs $G(\{n\} \oplus F)$ in the family $\mathfrak{F}$ this is b.e. trivial by Proposition 3.1.4. Csima and Kalimullin showed that $\mathcal{G}$ is $X$-computable if and only if there is $Y \equiv_T X$ such that for all $e \in \omega$, $Y^{[e]}$ is finite and $Y^{[e]} \neq W_e$.[1] They then showed that the degrees with this property are exactly the non-computable degrees. □

There are also other spectra known to be bi-embeddability spectra. We will get as a corollary of results in Section 3.2 that for all computable successor ordinals $\alpha$ and $\beta$, $\{\mathbf{d} : \mathbf{d}^{(\alpha)} \geq \mathbf{0}^{(\beta)}\}$ is the bi-embeddability spectrum of a structure. It is doubtful whether this result can be extended to include limit ordinals. Soskov [Sos13] gave an example of an isomorphism spectrum of a structure $\mathcal{A}$ such that $DgSp_{\cong}(\mathcal{A}) \subseteq \{\mathbf{d} : \mathbf{d} \geq \mathbf{0}^{(\omega)}\}$ and showed that no structure has $\{\mathbf{d} : \mathbf{d}^{(\omega)} \in DgSp_{\cong}(\mathcal{A})\}$ as its isomorphism spectrum. Faizrahmanov, Kach, Kalimullin, and Montalbán [Fai+18] recently showed that no structure realizes the family $\{\mathbf{d} : \mathbf{d}^{(\omega)} \geq \mathbf{a}^{(\omega)}\}$ for $\mathbf{a} \geq \mathbf{0}^{(\omega)}$ as its isomorphism spectrum.

### 3.1.2 Basis

**Definition 3.1.3.** Given a structure $\mathcal{A}$ and an equivalence relation $\sim$ we say that a family $\mathfrak{B}$ of structures is a $\sim$ *basis* for $\mathcal{A}$ if

(1) $\forall \mathcal{B} \in \mathfrak{B}\ \mathcal{B} \sim \mathcal{A}$,
(2) $\forall \mathcal{B}, \mathcal{C} \in \mathfrak{B}\ DgSp_{\cong}(\mathcal{B}) \notin DgSp_{\cong}(\mathcal{C})$,
(3) and $DgSp_{\sim}(\mathcal{A}) = \bigcup_{\mathcal{B} \in \mathfrak{B}} DgSp_{\cong}(\mathcal{B})$.

Recall the notion of Muchnik reducibility; a set of reals $P$ is *Muchnik reducible* to a set of reals $Q$, $P \leq_w Q$, if every real in $Q$ computes a real in $P$. In terms of struc-

[1] Here $Y^{[e]}$ denotes the $e^{\text{th}}$ column of $Y$, i.e., $Y^{[e]} = \{y : \langle e, y \rangle \in Y\}$.

tures one usually says that $\mathcal{A} \leq_w \mathcal{B}$ if every structure in the isomorphism type of $\mathcal{B}$ computes a structure in the isomorphism type of $\mathcal{A}$, which is equivalent to saying that $DgSp_{\cong}(\mathcal{B}) \subseteq DgSp_{\cong}(\mathcal{A})$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be families of structures. Muchnik reducibility extends naturally to such families.

$$\mathfrak{A} \leq_w \mathfrak{B} :\Leftrightarrow \bigcup_{\mathcal{B}\in\mathfrak{B}} DgSp_{\cong}(\mathcal{B}) \subseteq \bigcup_{\mathcal{A}\in\mathfrak{A}} DgSp_{\cong}(\mathcal{A})$$

Using this we get the following characterization of a $\sim$ basis.

**Proposition 3.1.6.** *Let $\mathfrak{A}$ be the family of structures bi-embeddable with $\mathcal{A}$. The family $\mathfrak{B} \subseteq \mathfrak{A}$ is a $\sim$ basis of $\mathcal{A}$ if and only if $\mathfrak{B}$ is a minimum with respect to inclusion such that $\mathfrak{B} \leq_w \mathfrak{A}$.*

### Equivalence structures

An equivalence structure is a structure in the language $E/2$ where $E$ is an equivalence relation. We will study the complexity of the embeddings between equivalence structures in Chapter 4.

**Theorem 3.1.7.** *Every countable equivalence structure is bi-embeddable with a computable one.*

*Proof.* Let $\mathcal{A}$ be an equivalence structure. We distinguish three cases. First, if $\mathcal{A}$ contains infinitely many infinite equivalence classes, then it is bi-embeddable with the computable equivalence relation $(\omega, E)$ where $\langle x_0, x_1\rangle E \langle y_0, y_1\rangle$ if and only if $x_0 = y_0$. Second, if $\mathcal{A}$ has $n$ infinite equivalence classes for $n \in \omega$ and there is no bound on the size of the finite classes, then it is bi-embeddable with the equivalence structure consisting of $n$ infinite classes and one finite class for each $m \in \omega$. This equivalence structure is clearly computable. Third, if $\mathcal{A}$ has finitely many infinite classes and the sizes of the finite classes are bounded, then it is describable by a finite set of parameters and thus has a computable copy. □

### Linear orderings

Montalbán [Mon05] showed that all hyperarithmetic linear orderings are bi-embeddable with a computable one, and thus their bi-embeddability spectrum contains all Turing degrees. The following is a relativization of his theorem [Mon05, Theorem 1.2].

**Theorem 3.1.8.** *Let $X \subseteq \omega$. If a linear ordering is hyperarithmetic in $X$ then it is bi-embeddable with an $X$-computable linear ordering.*

This theorem implies that every linear ordering has a singleton bi-embeddability basis.

The proof of the original theorem is involved and most of it is not computability theoretic. Its relativization, Theorem 3.1.8, can be obtained by relativizing the computability theoretic part.

As a corollary we obtain a characterization of the bi-embeddability spectra of linear orderings in terms of their Hausdorff rank. Before we state the corollary we introduce the required notions.

**Definition 3.1.4.** Let $\mathcal{L} = (L, \leq)$ be a linear ordering. For $x, y \in L$ let $x \sim_0 y$ if $x = y$, for $\alpha$ a countable limit ordinal $x \sim_\alpha y$ if $x \sim_\gamma y$ for some $\gamma < \alpha$ and for $\alpha = \beta + 1$ $x \sim_\alpha y$ if the intervals $[[x]_{\sim_\beta}, [y]_{\sim_\beta}]$ or $[[x]_{\sim_\beta}, [y]_{\sim_\beta}]$ are finite.

The *Hausdorff rank* of $\mathcal{L}$, $r(\mathcal{L})$, is the least countable ordinal $\alpha$ such that $\mathcal{L}/{\sim_\alpha}$ is finite.

Hausdorff [Hau08] showed that a linear ordering is *scattered*, i.e., it does not embed a copy of $\eta$, if and only if it has countable Hausdorff rank. Clearly, if $\mathcal{L}$ is not scattered then it is bi-embeddable with $\eta$, and thus has a computable bi-embeddable copy. In [Mon05] it was shown that a scattered linear ordering is bi-embeddable with a computable one if and only if it has computable Hausdorff rank. We again give a relativization of this result and delay the proof until the end of this section.

Given a set $X \subseteq \omega$ we write $\omega_1^X$ for the first non $X$-computable ordinal. We give the relativized version of Montalbán's theorem.

**Theorem 3.1.9.** *Let $X \subseteq \omega$. A scattered linear ordering $\mathcal{L}$ has an $X$-computable bi-embeddable copy if and only if $r(\mathcal{L}) < \omega_1^X$.*

In other words, $\mathcal{L}$ has an $X$-computable copy if and only if it computes its Hausdorff rank, i.e., $X \geq_T r(\mathcal{L})$. This combined with Theorem 3.1.8 yields the following characterization of bi-embeddability spectra of linear orderings.

**Corollary 3.1.10.** *Let $\mathcal{L}$ be a linear ordering.*

*(1) If $\eta \hookrightarrow \mathcal{L}$, then $\eta$ is a b.e. basis for $\mathcal{L}$, i.e., $DgSp_{\approx}(\mathcal{L}) = DgSp_{\cong}(\eta) = \{\mathbf{d} : \mathbf{d} \geq \mathbf{0}\}$,*

*(2) if $\mathcal{L}$ is scattered, then $DgSp_{\approx}(\mathcal{L}) = DgSp_{\cong}(r(\mathcal{L})) = \{deg(X) : X \geq_T r(\mathcal{L})\}$.*

### Strongly locally finite graphs

A graph $\mathcal{G}$ is *strongly locally finite* if it is the disjoint union of finite graphs, or, equivalently, if all of its connected components are finite. In what follows let $\mathfrak{F} = \langle F_i \rangle_{i \in \omega}$ be a Friedberg enumeration of the finite connected graphs. We may assume without loss of generality that $\mathfrak{F}$ is such that we can compute the size $|F_i|$ of every graph $F_i$ uniformly in $i$. Given $x \in \mathcal{G}$, let $[x]_{\mathcal{G}}$ be the atomic diagram of the component of $x$ and denote by $\ulcorner [x]_{\mathcal{G}} \urcorner$ the number $i$ such that $\ulcorner [x]_{\mathcal{G}} \urcorner = F_i$ (if $\mathcal{G}$ is clear from the context we omit the subscript). The *trace* of a graph is the set of indices of finite graphs embeddable into $\mathcal{G}$, i.e.,

$$tr(\mathcal{G}) = \{i : F_i \hookrightarrow \mathcal{G}\}.$$

The components of $\mathcal{G}$ form a pre-ordering $P_{\mathcal{G}}$ under embeddability, i.e., for $x, y \in \mathcal{G}$

$$[x] \leq_{P_{\mathcal{G}}} [y] :\Leftrightarrow [x] \hookrightarrow [y].$$

We denote by $c(\mathcal{G})$ the *set of components* of $\mathcal{G}$, i.e.,

$$c(\mathcal{G}) = \{i : F_i \text{ is isomorphic to a component of } \mathcal{G}\}.$$

A component of $\mathcal{G}$ is *open* if it belongs to an infinite ascending chain of $P_{\mathcal{G}}$, and open$(\mathcal{G})$ is the subset of $c(\mathcal{G})$ containing all open components of $\mathcal{G}$.

We first state some computability theoretic facts about the relations introduced above.

**Proposition 3.1.11.** *Given a strongly locally finite graph $\mathcal{G}$ and $x, y \in G$,*

*(1) $y \in [x]_{\mathcal{G}}$, $tr(\mathcal{G})$ are $\Sigma_1^{\mathcal{G}}$,*
*(2) and $[x] \hookrightarrow [y]$, $|[x]| \leq |[y]|$, $[x] \cong [y]$, $c(\mathcal{G})$ are $\Sigma_2^{\mathcal{G}}$.*

*Proof.* Ad (1): For $x \in \mathcal{G}$, $[x]_{\mathcal{G}}$ is definable by the following $\Sigma_1$ formula.

$$y \in [x]_{\mathcal{G}} \Leftrightarrow \bigvee_{n \in \omega} \exists u_1, \dots u_n \bigwedge_{1 \leq i \neq j \leq n} u_i E u_j$$

Given $x \in \mathcal{G}$ with $|[x]| = n$, let $D([x])(x_1, \dots, x_n)$ be the formula obtained by replacing every constant in the atomic diagram of $[x]$ by a variable. Note that given $n$ we can computably define $D([x])(x_1, \dots, x_n)$ and that for $F_i$ we can obtain $n$ computably. Thus the trace of $\mathcal{G}$ is definable by the following $\Sigma_1$ formula.

$$x \in tr(\mathcal{G}) \Leftrightarrow \exists x_1, \dots, x_n \; D(F_x)(x_1 \dots x_n)$$

Ad (2): In general, given $x \in \mathcal{G}$ the size of its component $[x]$ is $\Sigma_2$ as

$$|[x]| = n \Leftrightarrow \exists x_1, \dots x_n \bigvee_{1 \leq i \leq n} x_i \in [x] \wedge \forall y ( \bigvee_{1 \leq i \leq n} x_i \neq y \to \bigvee_{1 \leq i \leq n} \neg x_i E y).$$

Then

$$[x] \hookrightarrow [y] \Leftrightarrow \bigvee_{n\in\omega} |[x]| = n \wedge \exists y_1, \dots y_n D_\exists([x])(y_1, \dots, y_n),$$

which is $\Sigma_2$. Thus also $|[x]| \leq |[y]|$ is $\Sigma_2$ and $[x] \cong [y] \Leftrightarrow [x] \hookrightarrow [y] \wedge [y] \hookrightarrow [x]$ as $[x]$ and $[y]$ are finite; hence, it is also $\Sigma_2$. By definition, $x \in c(\mathcal{G})$ if and only if $\exists y \in \mathcal{G}\ F_x \cong [y]$ which by the above arguments is $\Sigma_2$. □

**Definition 3.1.5.** A graph $\mathcal{G}$ is *open-ended* if every component of $\mathcal{G}$ is open.

We say that a graph $S^{\mathcal{G}}$ is the *skeleton* of $\mathcal{G}$ if $S^{\mathcal{G}} \cong \bigcup_{i\in tr(\mathcal{G})} F_i$. It is not hard to see that two bi-embeddable graphs $\mathcal{A}$, $\mathcal{B}$ have the same trace, and thus the same skeleton. For open-ended strongly locally finite graphs the skeletons form a basis.

**Theorem 3.1.12.** *Let $\mathcal{G}$ be an open-ended strongly locally finite graph, then $S^{\mathcal{G}}$ is a b.e. basis of $\mathcal{G}$.*

*Proof.* We first show that $\mathcal{G}$ and $S^{\mathcal{G}}$ are bi-embeddable given that $\mathcal{G}$ is open-ended. Given enumerations of the components of $\mathcal{G}$ and $S^{\mathcal{G}}$, say we have defined an embedding $\mu$ on the first $s$ components of the enumeration of $\mathcal{G}$ and want to define it for the component with index $s+1$ in the enumeration. As $\mathcal{G}$ is open-ended, so is $S^{\mathcal{G}}$; thus, there is a component which is disjoint from the range of $\mu$ and in which the component with index $s+1$ embeds; define $\mu$ accordingly. It is then not hard to see that in the limit $\mu$ is an embedding of $\mathcal{G}$ in $S^{\mathcal{G}}$. By the same argument we can embed $S^{\mathcal{G}}$ in $\mathcal{G}$.

By Proposition 3.1.6, it remains to show that $S^{\mathcal{G}}$ is minimal with respect to Muchnik reducibility, i.e., that every $\mathcal{A} \approx \mathcal{G}$ computes a copy of $S^{\mathcal{G}}$. By Proposition 3.1.11, $tr(\mathcal{A})$ is $\Sigma_1^{\mathcal{A}}$. Let $W_e^{\mathcal{A}} = tr(\mathcal{A})$ and $W_{e,s}^{\mathcal{A}}$ the approximation to $W_e^{\mathcal{A}}$ at stage $s$. We construct the copy of $S_{\mathcal{G}}$ in stages. At every stage $s$ check if any $i < s$ enters $W_{e,s}^{\mathcal{A}}$ and if so build a component isomorphic to $F_i$ using elements bigger than $s$ not yet used during the construction.[2] As the construction is $\mathcal{A}$-computable and $tr(\mathcal{A}) = tr(\mathcal{G})$, the constructed structure is an $\mathcal{A}$-computable copy of $S^{\mathcal{G}}$. □

Notice that we can reformulate Theorem 3.1.12 as follows. For any open-ended graph $\mathcal{G}$, we have that

$$\mathrm{DgSp}_{\approx}(\mathcal{G}) = \{\deg(Y) :\ tr(\mathcal{G}) \text{ is c.e. in } Y\}.$$

This is close to the definition of *enumeration degree* of a structure $\mathcal{S}$ as given by Montalbán [Mon17] in the spirit of Knight [Kni98].

[2] We assume without loss of generality that no $i$ may enter $W_{e,s}$ at a stage $s$ smaller than $i$

**Definition 3.1.6.** A structure $\mathcal{S}$ has *enumeration degree* $X \subseteq \omega$ if the following holds

$$\mathrm{DgSp}_{\cong}(\mathcal{S}) = \{\deg(Y) : \ X \text{ is c.e. in } Y\}.$$

Related to this is the notion of the jump degree of a structure.

**Definition 3.1.7.** A structure $\mathcal{S}$ has jump degree $X \subseteq \omega$ if $\deg(X)$ is the least degree in

$$\mathrm{DgSp}'_{\cong}(\mathcal{S}) = \{\mathbf{d}' : \mathbf{d} \in \mathrm{DgSp}(\mathcal{A})\}$$

The set $\mathrm{DgSp}'_{\cong}(\mathcal{S})$ is often called the *jump spectrum* of $\mathcal{S}$.

Coles, Downey, and Slaman [CDS00] showed that for any set $X \subseteq \omega$ the set $\{\mathbf{d}' : X \text{ is c.e. in } \mathbf{d}\}$ has a minimum. It follows from this that a structure has jump degree if it has enumeration degree.

Examples of classes of structures always having an enumeration degree are algebraic fields (see Frolov, Kalimullin, and Miller [FKM09]) and connected, finite-valence, pointed graphs (see Steiner [Ste13]). Bi-embeddability spectra of open-ended graphs are therefore similar to isomorphism spectra of structures in these classes.

**Theorem 3.1.13.** *(1) For every $X \subseteq \omega$ there is an open ended graph $\mathcal{G}$ such that $tr(\mathcal{G}) \equiv_e X$.*
*(2) For all open-ended $\mathcal{G}$, $\mathrm{DgSp}'_{\approx}(\mathcal{G}) = \{\mathbf{d}' : \ \mathbf{d} \in \mathrm{DgSp}_{\approx}(\mathcal{G})\}$ is a cone of degrees.*

*Proof.* The idea of the proof is similar to that given in [FKM09, Corollary 1].

(1) Let $X \subseteq \omega$ and define $\mathcal{G}$ to be the graph consisting of a cycle of length $n$ for every $n \in X$.
We have $tr(\mathcal{G}) \equiv_e X$. Indeed, to enumerate $X$ from an enumeration of $tr(\mathcal{G})$, enumerate $tr(\mathcal{G})$ and for every $x \in tr(\mathcal{G})$ check in a c.e. way if $F_x$ is a cycle. If so enumerate the length of the cycle. Clearly this is an enumeration of $X$. On the other hand given an element $x \in X$, consider the trace of the cycle of length $x$ and enumerate it. By Proposition 3.1.11 this is c.e. Thus, given an enumeration of $X$ we can produce an enumeration of $tr(\mathcal{G})$.
(2) Given an open-ended $\mathcal{G}$, by the above mentioned result by Coles, Downey, and Slaman [CDS00], the set of jumps of degrees enumerating $tr(\mathcal{G})$ has a minimum. By Theorem 3.1.12 this is $\mathrm{DgSp}'_{\approx}(\mathcal{G})$.

□

**Corollary 3.1.14.** *There is a open-ended graph such such that* $\mathrm{DgSp}_{\approx}(\mathcal{G})$ *does not have a least element.*

*Proof.* Take $X \subseteq \omega$ to be non-total. It follows that the set of Turing degrees enumerating $X$ does not have a least element. Then by (1) of Theorem 3.1.13 and the observation after Theorem 3.1.12 we get that there is $\mathcal{G}$ such that

$$DgSp_{\approx}(\mathcal{G}) = \{deg(Y) : tr(\mathcal{G}) \text{ is c.e. in Y}\} = \{deg(Y) : X \text{ is c.e. in Y }\}.$$

Therefore $\mathrm{DgSp}_{\approx}(\mathcal{G})$ does not have a least element. □

It is immediate from the construction in (1) that $\mathcal{G} \equiv_T tr(\mathcal{G}) \equiv_T D$. Thus $\mathcal{G}$ has enumeration degree with respect to its bi-embeddability type and, as it is b.e. trivial, also with respect to its isomorphism type.

## 3.2 Elementary bi-embeddability spectra of structures

Two structures $\mathcal{A}$ and $\mathcal{B}$ are *elementary bi-embeddable*, $\mathcal{A} \approxeq \mathcal{B}$, if either is embeddable in the other by embeddings that preserve the first order type of elements. The elementary bi-embeddability spectrum of a structure $\mathcal{A}$ is then

$$DgSp_{\approxeq}(\mathcal{A}) = \{\deg(\mathcal{B}) : \mathcal{B} \approxeq \mathcal{A}\}.$$

In this section we give several examples of collections of degrees which are or are not elementary bi-embeddability spectra. One of the goals of this research is to separate elementary bi-embeddability spectra with spectra under other equivalence relations that have been investigated. Using our examples we obtain that there are collections of degrees that are elementary bi-embeddability spectra but are not theory spectra, $\Sigma_n$ spectra or spectra of atomic theories and vice versa. As a corollary to one of our proofs we get that there is a bi-embeddability spectrum which is not the theory spectrum of any complete theory.

### 3.2.1 Elementary bi-embeddability spectra

For many arguments presented in this section it is useful to assume that our languages are finite. Indeed in our scenario this assumption is justified since graphs are universal for elementary bi-embeddability spectra.

**Theorem 3.2.1.** *Given a countable structure $\mathcal{A}$ in any language we can compute a graph $\mathcal{G}_{\mathcal{A}}$ such that $DgSp_{\approxeq}(\mathcal{A}) = DgSp_{\approxeq}(\mathcal{G}_{\mathcal{A}})$.*

*Proof.* We use the coding given in [AM15, Proposition 2.2]. We assume without loss of generality that $\mathcal{A}$ is in a relational language $\langle R_i \rangle_{i\in I}$ where each $R_i$ has arity $i$. Given $\mathcal{A}$ the graph $\mathcal{G}_{\mathcal{A}}$ consists of 3 vertices $a, b, c$ where to $a$ we connect the unique 3-cycle in the graph, to $b$ the unique 5-cycle, and to $c$ the unique 7-cycle. For each element $x \in A$ we add a vertex $v_x$ and an edge $a \to v_x$. For every $i$ tuple $x_1, \dots, x_i \in A$ we add chains of length $i + k$ for $1 \le k \le i$ where for each such chain $y_1, \dots, y_{i+k}$ the last vertex $y_{i+k}$ is the same. We add an edge $v_{x_k} \to y_1$ only if $y_1$ is the first element of the chain of length $i + k$. If $\mathcal{A} \vDash R_i(x_1, \dots, x_i)$ then we add an edge $y_{i+k} \to b$ and if $\mathcal{A} \vDash \neg R_i(x_1, \dots, x_i)$ we add $y_{i+k} \to c$.

It is easy to see that any copy of $\mathcal{G}_{\mathcal{A}}$ computes a copy of $\mathcal{A}$ and vice versa. Andrews and Miller [AM15] showed that $\mathcal{A} \equiv \mathcal{B}$ if and only if $\mathcal{G}_{\mathcal{A}} \equiv \mathcal{G}_{\mathcal{B}}$; we need to show that $\mathcal{A} \approxeq \mathcal{B}$ if and only if $\mathcal{G}_{\mathcal{A}} \approxeq \mathcal{G}_{\mathcal{B}}$.

($\Rightarrow$). Assume that $\mathcal{A} \approxeq \mathcal{B}$ and that $\mathcal{A} \preccurlyeq \mathcal{B}$. We thus may assume without less of generality that $\mathcal{G}_{\mathcal{A}} \subseteq \mathcal{G}_{\mathcal{B}}$. We will show that for all $n \in \omega$ and any $\overline{a} \in G_{\mathcal{A}}^{<\omega}$ player II has a winning strategy for the $n$ turn Ehrenfeucht-Fraïssé game $G_n((\mathcal{G}_{\mathcal{A}}, \overline{a}), (\mathcal{G}_{\mathcal{B}}, \overline{a}))$. Assume towards a contradiction that $n$ is the least such that player II has no winning strategy for $G_n((\mathcal{G}_{\mathcal{A}}, \overline{a}), (\mathcal{G}_{\mathcal{B}}, \overline{a}))$. Then either there is $\overline{v} \in G_{\mathcal{A}}^n$ such that for all $\overline{u} \in G_{\mathcal{B}}^n$, $\langle \overline{a}, \overline{v} \rangle^{\mathcal{G}_{\mathcal{A}}} \not\cong \langle \overline{a}, \overline{u} \rangle^{\mathcal{G}_{\mathcal{B}}}$, or there is $\overline{u} \in G_{\mathcal{B}}^n$ such that for all $\overline{v} \in G_{\mathcal{B}}^n$, $\langle \overline{a}, \overline{u} \rangle^{\mathcal{G}_{\mathcal{B}}} \not\cong \langle \overline{a}, \overline{v} \rangle^{\mathcal{G}_{\mathcal{A}}}$. We will prove the second case, it is easy to prove the first case using the same techniques.

Notice that $\overline{au}$ is in a substructure of $\mathcal{G}_{\mathcal{B}}$ coding a finite substructure of $\mathcal{B}$ in a finite part $\mathcal{L}_1$ of the language of $\mathcal{B}$. Extend $\langle \overline{a}, \overline{u} \rangle^{\mathcal{G}_{\mathcal{B}}}$ so that it codes such a substructure $\mathcal{B}_1$ of $\mathcal{B}$. Now consider the conjunction $\varphi$ of atomic formulas or negations thereof true of $\mathcal{B}_1$ in $\mathcal{L}_1$. Let $\overline{a}'$ be the elements in $B_1 \cap A$ and $\overline{u}'$ the elements in $B_1 \setminus A$. Then $\mathcal{B} \vDash \varphi(\overline{a}', \overline{u}')$ and Tarski-Vaught gives us elements $\overline{v}'$ in $\mathcal{A}$ such that $\mathcal{A} \vDash \varphi(\overline{a}', \overline{v}')$. It follows that we have a partial isomorphism between $\langle \overline{a}', \overline{u}' \rangle^{\mathcal{B}}$ and $\langle \overline{a}', \overline{v}' \rangle^{\mathcal{A}}$ in $\mathcal{L}_1$. This induces an isomorphism between the subgraph coding $\mathcal{B}_1$ and the subgraph coding $\langle \overline{a}', \overline{v}' \rangle^{\mathcal{A}}$. But $\langle \overline{a}, \overline{u} \rangle^{\mathcal{G}_{\mathcal{B}}}$ is a subgraph of the graph coding $\mathcal{B}_1$ and thus it is isomorphic to a substructure $\langle \overline{a}, \overline{v} \rangle^{\mathcal{G}_{\mathcal{A}}}$ of the structure coding $\langle \overline{a}', \overline{v}' \rangle^{\mathcal{A}}$, a contradiction.

($\Leftarrow$). Let $\mathcal{G}_{\mathcal{A}} \approxeq \mathcal{G}$ and assume without loss of generality that $\mathcal{G}$ is an elementary substructure of $\mathcal{G}_{\mathcal{A}}$. Then for all $\overline{v} \in G^{<\omega}$ $tp_{\mathcal{G}}(\overline{v}) = tp_{\mathcal{G}_{\mathcal{A}}}(\overline{v})$. $\mathcal{G}$ codes a structure $\mathcal{B}$ in the language of $\mathcal{A}$ by the following argument. For every element $v$ with a formula saying that $a \to v$ in its type we add an element $x_v$. Assume that $\mathcal{A} \vDash R_i(x_1, \dots, x_i)$; then our construction gives a finite graph coding that $R_i$ holds on the elements $x_1, \dots, x_i$. It is easy to see that this graph is definable in $\mathcal{A}$, say by $\varphi$. Thus, to define $R_i$ in the structure $\mathcal{B}$ we construct, we say that $\mathcal{B} \models R_i(x_{v_1}, \ldots, x_{v_i})$ if the formula $\varphi \in tp_{\mathcal{G}}(v_1, \ldots, v_i)$ where the $v_k$, $0 < k \leq i$, are elements with $a \to v_k$ in their type. By a similar argument there is a formula $\overline{\varphi}$ defining the structure coding $\neg R_i$, and we let $\mathcal{B} \models \neg R_i(x_{v_1}, \ldots x_{v_i})$ if $\overline{\varphi} \in tp_{\mathcal{G}}(v_1, \ldots, v_i)$.

As $\mathcal{G}$ is an elementary substructure of $\mathcal{G}_{\mathcal{A}}$ we have that $\mathcal{B}$ is well defined. Furthermore the constructions of $\mathcal{B}$ of $\mathcal{G}$ and $\mathcal{G}_{\mathcal{A}}$ of $\mathcal{A}$ give us injective $g : \mathcal{B} \to \mathcal{G}$ $x_v \mapsto v$ and $f : \mathcal{A} \to \mathcal{G}_{\mathcal{A}}$ $x_v \mapsto v$ with the property that $\mathcal{B}$ is embeddable in $\mathcal{A}$ by $f^{-1}g$. An easy induction over the complexity of formulas shows that for $\overline{a} \in A^{<\omega}$ and $\varphi$ in the language of $\mathcal{A}$

$$\mathcal{A} \models \varphi(\overline{a}) \Leftrightarrow \mathcal{B} \models \varphi(\overline{a}).$$

Thus $\mathcal{A} \preccurlyeq \mathcal{B}$. □

As a corollary of Proposition 3.1.2 which says that two bi-embeddable automorphically trivial structures are isomorphic and Knight's result [Kni86] we get the following.

**Corollary 3.2.2.** *The elementary bi-embeddability spectrum of a structure is either a singleton or upwards closed.*

*Proof.* It is a simple observation that the elementary b.e. spectrum of a structure $\mathcal{A}$ is the union of isomorphism spectra of the structures elementary bi-embeddable with it, i.e.,

$$DgSp_{\approxeq}(\mathcal{A}) = \bigcup_{\mathcal{B} \approxeq \mathcal{A}} DgSp_{\cong}(\mathcal{B}).$$

Thus, if $\mathcal{A}$ is automorphically trivial we get by Proposition 3.1.2 that its elementary b.e. spectrum is a singleton and that otherwise it is upwards closed. □ □

Fokina, Semukhin and Turetsky [FST18] showed the following.

**Theorem 3.2.3** ([FST18])**.** *(1) There is a class $\mathcal{F}$ of degrees such that for all $n \in \omega$ and all structures $\mathcal{A}$, $DgSp_{\equiv_n}(\mathcal{A}) \neq \mathcal{F}$.*

*(2) There is a structure $\mathcal{A}$ such that $DgSp_{\equiv}(\mathcal{A}) = DgSp_{\cong}(\mathcal{A}) = \mathcal{F}$.*

By the same argument as in Corollary 3.2.2 we get that there is a structure $\mathcal{A}$ with $DgSp_{\approxeq}(\mathcal{A}) = \mathcal{F}$ and thus obtain an example of an elementary bi-embeddability spectrum that is not a $\Sigma_n$ spectrum for any $n \in \omega$.

Knight [Kni86] showed that a set $X$ is c.e. in all isomorphic copies of a structure $\mathcal{A}$ if and only if it is enumeration reducible to the existential type of a tuple in $A$. In fact this holds for elementary bi-embeddable copies.

**Lemma 3.2.4.** *Let $X \subseteq \omega$ and $\mathcal{A}$ be a structure then the following are equivalent.*

*(1) $X$ is c.e. in every isomorphic copy of $\mathcal{A}$,*
*(2) $X$ is e-reducible to $\exists$-$tp_{\mathcal{A}}(\overline{a})$ for some $\overline{a} \in A^{<\omega}$,*
*(3) $X$ is c.e. in every elementary bi-embeddable copy of $\mathcal{A}$.*

*Proof.* The equivalence of (1) and (2) was proven in [Kni86]. To see the equivalence with (3) let $\mathcal{B}$ be an elementary bi-embeddable copy of $\mathcal{A}$ and $f : \mathcal{A} \to \mathcal{B}$ be elementary. Say $X$ is e-reducible to $\exists$-$tp_{\mathcal{A}}(\overline{a})$, then by elementarity of $f$ $\exists$-$tp_{\mathcal{A}}(\overline{a}) = \exists$-$tp_{\mathcal{B}}(f(\overline{a}))$ and thus $X$ is e-reducible to $\exists$-$tp_{\mathcal{B}}(f(\overline{a}))$. (3) now follows from the equivalence between (1) and (2). □

**Theorem 3.2.5.** *For $n > 1$ let $\mathbf{a}_1, \ldots \mathbf{a}_n$ be incomparable enumeration degrees. Then for any structure $\mathcal{A}$, $DgSp_{\approxeq}(\mathcal{A}) \neq \bigcup_{i<n}\{\mathbf{d} : \mathbf{d} \geq \mathbf{a}_i\}$.*

**Corollary 3.2.6.** *No elementary bi-embeddability spectrum is the union of finitely or countably many non degenerate cones of Turing degrees.*

Using Lemma 3.2.4 the proofs of the above Theorem and Corollary are similar to those for isomorphism spectra as presented by Montalbán [Mon18, Theorem V.3.1]. We therefore omit them here.

Corollary 3.2.6 also holds for isomorphism spectra [Sos04], spectra of atomic theories [AM15], and $\Sigma_1$ spectra [FST18]. On the other hand spectra of non-atomic theories and $\Sigma_n$ spectra for $n > 1$ can be the union of two cones as was shown in [AM15], respectively, [FST18].

Given a set $X$, the *strong flower graph*[3] $\mathcal{G}_X^s$ is the graph containing one vertex to which for every $x \in \omega$ a cycle of length $2x + 1$ is attached if $x \in X$ and a cycle of length $2x + 2$ if $x \notin X$. It is not hard to see that any copy of $\mathcal{G}_X^s$ computes $X$ and thus the isomorphism spectrum of $\mathcal{G}_X^s$ is $\{deg(Y) : Y \geq_T X\}$. Furthermore any bi-embeddable copy $\mathcal{A}$ of $\mathcal{G}_X^s$ must be isomorphic to it as any cycle in $\mathcal{G}_X^s$ must be mapped to a cycle of the same length in $\mathcal{A}$, all cycles must intersect in a single point, and $\mathcal{G}_X$ contains at most one cycle of length $n$ for every $n \in \omega$. We thus obtain a similar result for elementary bi-embeddability spectra.

**Proposition 3.2.7.** *For every Turing degree $\mathbf{d}$, $\{\mathbf{e} : \mathbf{e} \geq \mathbf{d}\}$ is an elementary bi-embeddability spectrum.*

[3] In the literature strong flower graphs are sometimes called flower graphs or daisy graphs. We use the term strong flower graph as we will use a similar construction only coding positive membership information of $X$ below.

**Definition 3.2.1.** Two sets $A$, $B$ form a $\Sigma_1$ *minimal pair* if every set that is both $A$-c.e. and $B$-c.e. is c.e.

**Definition 3.2.2.** A structure $\mathcal{A}$ has the *c.e. extension property (ceep)* if every existential type of a finite tuple of $A$ is c.e.

**Proposition 3.2.8** ([AM15, Proposition 3.6])**.** *Let $Y$ be any set and $P$ be a non-empty $\Pi^0_1$ class. Then there is $X \in P$ such that $X$ and $Y$ form a $\Sigma_1$-minimal pair.*

**Proposition 3.2.9.** *$DgSp_{\approxeq}(\mathcal{A})$ contains a $\Sigma_1$ minimal pair if and only if every $\mathcal{B}$ elementary bi-embeddable with $\mathcal{A}$ has the ceep.*

*Proof.* ($\Rightarrow$). Let $\mathbf{a}, \mathbf{b} \in DgSp_{\approxeq}(\mathcal{A})$ be a $\Sigma_1$ minimal pair and $\mathcal{A}$ be $\mathbf{a}$-computable, $\mathcal{B}$ be $\mathbf{b}$-computable. As the existential types realized in elementary b.e. copies coincide and existential types realized in a structure are computably enumerable from it, the existential types of $\mathcal{A}$ and $\mathcal{B}$ are c.e.

($\Leftarrow$). This follows from the same Proposition for isomorphism spectra, i.e., it holds that given $\mathcal{A}$ with the ceep there exists a $\mathcal{B} \cong \mathcal{A}$ such that $deg(\mathcal{A})$ and $deg(\mathcal{B})$ form a $\Sigma_1$ minimal pair [AM15, Proposition 3.5]. □

A $\Pi^0_1$ class is *special* if it does not have a computable member.

**Proposition 3.2.10** ([AM15, Proposition 3.8])**.** *Let $\mathcal{A}$ be a structure with the ceep, then there is an isomorphic copy of $\mathcal{A}$ that does not compute a member of any special $\Pi^0_1$ class.*

Putting Propositions 3.2.8 to 3.2.10 together we get that no elementary bi-embeddability spectrum is the upward closure of a special $\Pi^0_1$ class.

**Theorem 3.2.11.** *For all structures $\mathcal{A}$ and special $\Pi^0_1$ classes $P$, $DgSp_{\approxeq}(\mathcal{A}) \neq \{deg(X) : \exists p \in P\ X \geq_T p\}$.*

The class of *diagonally non-computable functions*, short $DNC$, form a special $\Pi^0_1$ class. Thus, their upward closure can not be an elementary bi-embeddability spectrum, and, furthermore, any elementary bi-embeddability spectrum contains a degree that does not compute a $DNC$. Combining this with the result by Jockusch and Soare [JS72], and Solovay [**unpublished**] that a degree computes a complete extension of Peano arithmetic (short, is a PA degree) if and only if it computes a two valued $DNC$ function we obtain the following corollary.

**Corollary 3.2.12.** *The class of* PA *degrees is not the elementary bi-embeddability spectrum of a structure.*

In contrast to elementary bi-embeddability spectra, it was shown in [AM15] that there is a theory spectrum that consists of the PA degrees.

Slaman [Sla98] and, independently, Wehner [Weh98] showed that there is a structure whose isomorphism spectrum is all the non-computable degrees. Wehner used the following result.

**Theorem 3.2.13** ([Weh98])**.** *There is a family $\mathfrak{F}$ of finite sets $(F_i)_{i\in\omega}$ such that $(F_i)$ is uniformly $X$-c.e. for any non computable set $X$ but not c.e.*

We discussed this result already in the proof of Theorem 3.1.5. For elementary bi-embeddability spectra we use a different coding than the one discussed there. This is due to the fact that this coding works better with generalizations of Wehner's result as given for example by Kalimullin [Kal07] which use families of c.e. sets instead of families of finite sets.

Given a set $X$ the *weak flower graph* $\mathcal{G}_X^w$ is defined as the strong flower graph apart from the fact that if $x \notin X$ we do not attach a cycle to the central vertex. Then, for a family $\mathfrak{F}$ the *bouquet graph* $\mathcal{G}_{\mathfrak{F}}^\infty$ is the disjoint union of infinitely many copies of the graphs $\mathcal{G}_X^w$ for any $X \in \mathfrak{F}$. It is not hard to see that $\mathcal{G}_{\mathfrak{F}}^\infty$ has an $X$-computable copy if and only if $\mathfrak{F}$ is $X$-c.e. Taking $\mathfrak{F}$ as the family given by Wehner, one gets a graph whose degree spectrum is the set of non-computable degrees.

The same strategy was later used by Kalimullin [Kal07; Kal08] to show that for any *low* or c.e. degree $\mathbf{a}$ there is a structure having degree spectrum $\{\mathbf{x} : \mathbf{x} \nleq \mathbf{a}\}$. This was later improved by Andrews, Cai, Kalimullin, Lempp, J. Miller, and Montalbán [AM15] who showed that for any degree $\mathbf{a}$ that is low over a c.e. degree $\mathbf{g}$ (a degree $\mathbf{a}$ is low over a degree $\mathbf{g}$ if $\mathbf{g} \le \mathbf{a}$ and $\mathbf{a}' \le \mathbf{g}'$) there is a structure having degree spectrum $\{\mathbf{x} : \mathbf{x} \nleq \mathbf{a}\}$. As for flower graphs we have that if a structure is elementary bi-embeddable with a bouquet graph, then it is isomorphic to it. Note that this is not true for bi-embeddability, as the bouquet graph may contain flower graphs coding infinite chains of subsets of natural numbers $A_0 \subset A_1 \subset \dots$. However, for elementary bi-embeddability spectra we get that the above examples provide examples of elementary bi-embeddability spectra.

**Proposition 3.2.14.** *For every family $\mathfrak{F}$ of finite sets and every graph $\mathcal{A} \approxeq \mathcal{G}_{\mathfrak{F}}^\infty$, $\mathcal{A} \cong \mathcal{G}_{\mathfrak{F}}^\infty$.*

**Corollary 3.2.15.** *Suppose that for $\mathbf{a}$ there is c.e. $\mathbf{g} \le \mathbf{a}$ such that $\mathbf{a}' \le \mathbf{g}'$, then there is $\mathcal{A}$ such that $DgSp_{\approxeq}(\mathcal{A}) = \{\mathbf{d} : \mathbf{d} \nleq \mathbf{a}\}$.*

For $\Sigma_1$ spectra the above corollary does not hold. Fokina, Semukhin, and Turetsky [FST18] showed that the collection of non-computable Turing degrees is not a $\Sigma_1$ spectrum.

We now work towards an analog of Theorem 3.1.5 for elementary bi-embeddability spectra. The following follows directly from Proposition 3.1.4.

**Proposition 3.2.16.** *For every family $\mathfrak{F}$ of finite sets there is a structure $\mathcal{A}$ such that $X$ computes an elementary bi-embeddable copy of $\mathcal{A}$ if and only if $X$ uniformly computes $\mathfrak{F}$.*

Csima and Kalimullin showed that there is a family of finite sets $\mathfrak{F}$ such that $\mathfrak{F}$ is $X$-computable if and only if $deg(X)$ is hyperimmune. Diamondstone, Greenberg, and Turetsky [DGT13] constructed a family $\mathfrak{F}$ of sets such that $\mathfrak{F}$ is $X$-computable if and only if the degree of $X$ is array non-computable, and they showed that the degrees who uniformly enumerate this family are exactly the non-jump traceable degrees. Using Proposition 3.2.16 we get that these collections are elementary bi-embeddability spectra of structures.

**Corollary 3.2.17.** *The hyperimmune degrees, the array non-computable degrees and the non-jump traceable degrees are all elementary bi-embeddability spectra of structures.*

### 3.2.2 Towards jump inversion for elementary bi-embeddability spectra

Goncharov, Harizanov, Knight, McCoy, R. Miller, and Solomon [Gon+05] showed that if $\mathcal{F}$ is the isomorphism spectrum of a structure, then so is $\{\mathbf{d} : \mathbf{d}^{(\alpha)} \in \mathcal{F}\}$ for successor ordinals $\alpha < \omega_1^{CK}$. This result can be seen as an analogue of the classical jump inversion results in computability theory. Andrews and J. Miller [AM15] showed that $\{\mathbf{d} : \mathbf{d}^{(\omega+1)} \geq_T \mathbf{0}^{(\omega\cdot 2+2)}\}$ is not the spectrum of a theory but by the above it is the isomorphism spectrum of a structure. Thus, in general we can not do transfinite jump inversion for theory spectra. In this section we obtain some positive examples for jump inversion of elementary bi-embeddability spectra, and, among other things give an example of an elementary bi-embeddability spectrum that is not the spectrum of a theory.

To obtain the results in this section we "invert" graphs. Given a graph $\mathcal{G}$ we create a structure $\mathcal{G}^{-\alpha}$ by replacing every edge in $\mathcal{G}$ by a copy of a structure $\mathcal{S}_{\alpha,0}$ and associating a structure $\mathcal{S}_{\alpha,1}$ with every pair of non-adjacent vertices. These two structures have the property that it is $\Delta^0_\alpha$-complete to check whether a structure is a copy of $\mathcal{S}_{\alpha,0}$ or $\mathcal{S}_{\alpha,1}$. We then get that

$$DgSp_{\approxeq}(\mathcal{G}^{-\alpha}) = \{\mathbf{d} : \mathbf{d}^{(\alpha)} \in DgSp_{\approxeq}(\mathcal{G})\}.$$

Formally, $\mathcal{G}^{-\alpha}$ is in the language consisting of relation symbols $V/1, R/3$ union the language of $\mathcal{S}_{\alpha,0}$, $\mathcal{S}_{\alpha,1}$. The relation $V$ is true of elements representing the vertices of

$\mathcal{G}$ and $R$ partitions the remaining elements into infinitely many infinite sets where for $a, b \in V$, $R(a,b,-) \cong \mathcal{S}_{\alpha,0}$ if $a$ and $b$ are adjacent in $\mathcal{G}$, and $R(a,b,-) \cong S_{\alpha,1}$ otherwise.

In the following proofs we use pairs of linear orderings for $\mathcal{S}_{\alpha,0}$ and $\mathcal{S}_{\alpha,1}$. Formally a pair of linear orderings $(L_1, L_2)$ is in the language $(T/1, \leq/2)$ where $\leq$ restricted to $T$ is isomorphic to $L_1$ and $\leq$ restricted to $\neg T$ is isomorphic to $L_2$. If we do jump inversion for even ordinals, i.e., ordinals of the form $2\alpha+2$ we let

$$S_{2\alpha+2,0} \cong (\omega^{\alpha+1} + \omega^{\alpha}, \omega^{\alpha+1}) \text{ and } S_{2\alpha+2,1} \cong (\omega^{\alpha+1}, \omega^{\alpha+1} + \omega^{\alpha}).$$

And for jump inversion of odd ordinals, i.e., of the form $2\alpha+1$ we use

$$S_{2\alpha+1,0} \cong (\omega^{\alpha} \cdot 2, \omega^{\alpha}) \text{ and } S_{2\alpha+1,1} \cong (\omega^{\alpha}, \omega^{\alpha} \cdot 2).$$

To get that for every copy $\mathcal{H}$ of $\mathcal{G}^{-\alpha}$, $\mathcal{H}^{(\alpha)}$ computes a copy of $\mathcal{G}$ we need the following Lemma.

**Lemma 3.2.18.** *(1) It is $\Delta^0_{2\alpha+2}$-complete to check whether $L \cong (\omega^{\alpha+1} + \omega^{\alpha}, \omega^{\alpha+1})$ or $L \cong (\omega^{\alpha+1}, \omega^{\alpha+1} + \omega^{\alpha})$.*

*(2) It is $\Delta^0_{2\alpha+1}$-complete to check whether $L \cong (\omega^{\alpha} \cdot 2, \omega^{\alpha})$ or $L \cong (\omega^{\alpha}, \omega^{\alpha} \cdot 2)$.*

To prove the Lemma we will use the relation $\sim_\alpha$ we defined in our definition of the Hausdorff rank, Definition 3.1.4. The relation $\sim_1$ is commonly referred to as the *block* relation. For $\alpha < \omega_1^{\mathrm{CK}}$ we will call $\sim_\alpha$ the *$\alpha$-block relation.*

*Lemma 3.2.18.* The $\alpha$ block relation $\sim_\alpha$ for $\alpha = \beta+1$ is definable by the following computable $\Sigma_{2\alpha}$ formula.

$$x \sim_\alpha y \Leftrightarrow \bigvee_{n \in \omega} \forall y_1, \dots, y_n \left( x < y_1 < \dots < y_n < y \to \bigvee_{1 \leq i < j \leq n} y_i \sim_\beta y_j \right).$$

For $\lambda$ a limit ordinal the defining formula is the disjunction of all formulas defining $\sim_\beta$ for $\beta < \lambda$. This is then clearly $\Sigma_\lambda$ and $2\lambda = \lambda$. Goncharov et al. [Gon+05] proved the following.

**Claim 3.2.18.1** ([Gon+05, Lemma 5.1])**.** *Let $\alpha$ be a computable successor ordinal and $\mathcal{A}_1, \mathcal{A}_2$ such that*

*(1) $\{\mathcal{A}_1, \mathcal{A}_2\}$ is $\alpha$-friendly,*

*(2) $\mathcal{A}_1$, $\mathcal{A}_2$ satisfy the same infinitary $\Pi_{2\beta}$ formulas for $\beta < \alpha$,*

*(3) and each* $\mathcal{A}_i$, $i \in \{1,2\}$, *satisfies a computable* $\Pi_{2\alpha}$ *sentence not satisfied by the other.*

*Then for any* $\Delta^0_\alpha$ *set* $S$ *there is a uniformly computable sequence* $(\mathcal{C}_i)_{i\in\omega}$ *such that*

$$\mathcal{C}_i \cong \begin{cases} \mathcal{A}_1 & \text{if } n \in S, \\ \mathcal{A}_2 & \text{otherwise.} \end{cases}$$

Fix $\alpha$. That the pairs $\mathcal{S}_{2\alpha+1,0}$ and $\mathcal{S}_{2\alpha+1,1}$, respectively, $\mathcal{S}_{2\alpha+2,0}$ and $\mathcal{S}_{2\alpha+2,1}$ satisfy (1) and (2) follows from [AK00, Lemma 15.10]. To see that (3) is satisfied consider the following facts.

(1) $(\omega^\alpha 2, \omega^\alpha) \vDash \forall x, y\,((\neg T(x) \wedge \neg T(y)) \rightarrow x \sim_\alpha y)$ which is $\Pi_{2\alpha+1}$,
(2) $(\omega^\alpha, \omega^\alpha 2) \vDash \forall x, y\,((T(x) \wedge T(y)) \rightarrow x \sim_\alpha y)$ which is $\Pi_{2\alpha+1}$,
(3) $(\omega^{\alpha+1} + \omega^\alpha, \omega^{\alpha+1}) \vDash \forall x\,(\neg T(x) \rightarrow (\exists y\,(y > x \wedge \neg T(y) \wedge y \nsim_\alpha x)))$ which is $\Pi_{2\alpha+2}$,
(4) $(\omega^{\alpha+1}, \omega^{\alpha+1} + \omega^\alpha) \vDash \forall x\,(T(x) \rightarrow (\exists y\,(y > x \wedge T(y) \wedge y \nsim_\alpha x)))$ which is $\Pi_{2\alpha+2}$.

Neither of the above sentences satisfied by one of the structures is satisfied by its partner. Hence, the conditions in the Claim are satisfied and the Lemma follows. □

**Definition 3.2.3.** A degree $\mathbf{d}$ is non-low$_\alpha$ if $\mathbf{d}^{(\alpha)} > \mathbf{0}^{(\alpha)}$.

**Theorem 3.2.19.** *For every* $n < \omega$ *the non-low*$_n$ *degrees,* $\{\mathbf{d} : \mathbf{d}^{(n)} > \mathbf{0}^{(n)}\}$, *are the elementary bi-embeddability spectrum of a structure.*

*Proof.* Let $\mathcal{G}$ be the bouquet graph of the Wehner family relativized to $\varnothing^{(n)}$, then $DgSp_{\approx}(G) = \{\mathbf{d} : \mathbf{d} > \mathbf{0}^{(n)}\}$. To obtain the inverted graph $\mathcal{G}^{-n}$ we use the construction described above with $\mathcal{S}_{n,0}$ and $\mathcal{S}_{n,1}$ as our structures. We get from Lemma 3.2.18 that $DgSp_{\cong}(\mathcal{G}^{-n}) = \{\mathbf{d} : \mathbf{d}^{(n)} > \mathbf{0}^{(n)}\}$.

By Proposition 3.2.14, $DgSp_{\approx}(\mathcal{G}) = DgSp_{\cong}(\mathcal{G})$. We show that the degree spectra of the inverted graphs are also the same, i.e., $DgSp_{\approx}(\mathcal{G}^{-n}) = DgSp_{\cong}(\mathcal{G}^{-n})$. The proof relies on the fact that for ordinals $\alpha, \beta < \omega^\omega$, $\alpha \preccurlyeq \beta$ if and only if $\alpha \cong \beta$ [DMT78]. Let $\mathcal{H} \approx \mathcal{G}^{-n}$ and $\mu : \mathcal{H} \rightarrow \mathcal{G}^{-n}$ be an elementary embedding. Clearly $R$ holds only on triples with elements in the first and second column satisfying $V$ and by the above mentioned fact for all $a, b \in H$ such that $V(a)$ and $V(b)$, $R(a, b, -) \cong R(\mu(a), \mu(b), -)$. Thus we can construct a graph $\mathcal{H}^{+n}$ from $\mathcal{H}$ such that $\mathcal{H}^{+n} \approx \mathcal{G}$ and hence, $\mathcal{H}^{+n} \cong \mathcal{G}$. But this implies that $\mathcal{H} \cong \mathcal{G}^{-n}$ and thus $deg(\mathcal{H}) \in DgSp_{\cong}(\mathcal{G}^{-n})$. □

**Theorem 3.2.20.** *For all successor ordinals* $\alpha, \beta < \omega_1^{\mathrm{CK}}$, $\{\mathbf{d} : \mathbf{d}^{(\alpha)} \geq_T \mathbf{0}^{(\beta)}\}$ *is the elementary bi-embeddability spectrum of a structure.*

*Proof.* We start with the strong flower graph $\mathcal{G}$ coding $\varnothing^{(\beta)}$ and produce a structure $\mathcal{G}_e$ in the language $E/2, P/2, S/1$. We interpret $E$ as the edge relation in $\mathcal{G}$, $P$ as the successor relation on $\omega$ and let $S$ hold of the single vertex which is the first element in $P$. It is easy to see that $\mathcal{G}_e$ computes $\varnothing^{(\beta)}$ and that from $\varnothing^{(\beta)}$ we can compute a copy of $\mathcal{G}_e$.

Now we obtain a structure $\mathcal{G}^{-\alpha}$ by inverting $\mathcal{G}$. We get the structure $\mathcal{G}_e^{-\alpha}$ by adding relations $P/2, S/1$ and interpreting them so that the canonical bijection between $\mathcal{G}_e$ and the elements for which $V$ holds in $\mathcal{G}_e^{-\alpha}$ is structure preserving on $P$ and $S$. Let this structure be $\mathcal{G}_e^{-\alpha}$. By Lemma 3.2.18 $DgSp_{\cong}(\mathcal{G}_e^{-\alpha})$ is the desired spectrum and clearly for $\mathcal{G}_e$ and all $\mathcal{H}$, $\mathcal{H} \approxeq \mathcal{G}_e$ if and only if $\mathcal{H} \cong \mathcal{G}_e$.

We show that $\mathcal{H} \approxeq \mathcal{G}_e^{-\alpha}$ if and only if $\mathcal{H} \cong \mathcal{G}_e^{-\alpha}$. Let $\mathcal{H} \approxeq \mathcal{G}_e^{-\alpha}$ and $\mu : \mathcal{H} \to \mathcal{G}^{-n}$, $\nu : \mathcal{G}^{-n} \to \mathcal{H}$ be elementary embeddings. Then, as in the proof of Theorem 3.2.19, $R$ holds only on triples with elements in the first and second column satifying $V$. Let $a$ be the single element in $H$ such that $\mathcal{H} \models S(a)$. Then $\nu(\mu(a)) = a$ and by induction we get the same for any $u \in H$ such that $V(u)$. Thus we get that for all $a, b \in H$ satisfying $V$ that $R(a, b, -) \cong R(\mu(a), \mu(b), -)$. Hence, we can construct a structure $\mathcal{H}^{+\alpha}$ from $\mathcal{H}$ such that $\mathcal{H}^{+\alpha} \approxeq \mathcal{G}_e$, and therefore $\mathcal{H}^{+\alpha} \cong \mathcal{G}_e$. This implies that $\mathcal{H} \cong \mathcal{G}_e^{-\alpha}$. □ □

Notice that in the proof of Theorem 3.2.20 we never used the fact that our embeddings are elementary, therefore the analogue of this theorem also holds for bi-embeddability spectra.

**Corollary 3.2.21.** *For all successor ordinals $\alpha, \beta < \omega_1^{\mathrm{CK}}$, $\{\mathbf{d} : \mathbf{d}^{(\alpha)} \geq_T \mathbf{0}^{(\beta)}\}$ is the bi-embeddability spectrum of a structure.*

Andrews and J. Miller [AM15] showed that $\{\mathbf{d} : \mathbf{d}^{(\omega+1)} \geq \mathbf{0}^{(\omega\cdot 2+2)}\}$ is not the spectrum of a theory but by the above it is both a bi-embeddability spectrum and an elementary bi-embeddability spectrum. We have thus found an example of an elementary bi-embeddability spectrum that is not the spectrum of a theory.

Corollary 3.2.6 and Theorem 3.2.20 show that there are theory spectra that are not elementary bi-embeddability spectra and vice versa. From Theorem 3.2.3 we get an elementary bi-embeddability spectrum that is not a $\Sigma_n$ spectrum for any $n$ and Corollary 3.2.6 again shows that there are $\Sigma_n$ spectra for $n > 1$ that are not elementary bi-embeddability spectra. Whether the same holds for $\Sigma_1$ spectra and the relationship between elementary bi-embeddability spectra and isomorphism spectra is still unknown.

**Question 1.** *Is every isomorphism spectrum the elementary bi-embeddability spectrum of a structure and vice versa?*

# 4 The complexity of embeddings between bi-embeddable structures

The systematic study of the complexity of isomorphisms between computable copies of structures was initiated in the 1950's by Fröhlich and Shepherdson [FS56] and independently by Maltsev [Mal62]. The notions of computable categoricity (in the Russian tradition also called autostability) and relative computable categoricity are probably the most prominent in this line of research. A computable structure $\mathcal{A}$ is *computably categorical* if for every computable copy $\mathcal{B}$ there is a computable isomorphism from $\mathcal{B}$ to $\mathcal{A}$. A structure $\mathcal{A}$ is *relatively computably categorical* if for every copy $\mathcal{B}$ of $\mathcal{A}$ there is a $deg(\mathcal{A} \oplus \mathcal{B})$-computable isomorphism from $\mathcal{B}$ to $\mathcal{A}$. For a survey of this topic see Fokina, Harizanov, and Melnikov [FHM14].

Fokina, Kalimullin, and R. Miller [Fok+10] introduced the notion of categoricity spectra and (strong) degree of categoricity of a structure. The *categoricity spectrum* of $\mathcal{A}$, $CatSpec(\mathcal{A})$, is the collection of degrees computing isomorphisms between two computable copies of $\mathcal{A}$. If $CatSpec(\mathcal{A})$ contains a least degree then this is the *degree of categoricity* of $\mathcal{A}$. A degree of categoricity of $\mathcal{A}$ is strong if there are two computable copies $\mathcal{A}_0$ and $\mathcal{A}_1$ such that for any $f : \mathcal{A}_0 \cong \mathcal{A}_1$, $f \geq_T \mathbf{d}$. Note that $\mathcal{A}$ might not have a (strong) degree of categoricity [Mil09; FFK+16]. A Turing degree $\mathbf{d}$ is a (strong) degree of categoricity if there exists a structure having $\mathbf{d}$ as its (strong) degree of categoricity. This notion has seen a lot of interest over the last years. Fokina, Kalimullin, and R. Miller [Fok+10] showed that all strong degrees of categoricity are hyperarithmetical. Csima, Franklin, and Shore [CFS+13] extended this result by showing that all degrees of categoricity are hyperarithmetical. R. Miller [Mil09] exhibited a field that does not have a degree of categoricity, and Fokina, Frolov, and Kalimullin [FFK+16] gave an example of a rigid structure without degree of categoricity. Recently, Csima and Stephenson [**csima2017a**], and independently, Bazhenov, Kalimullin, and Yamaleev [Baz16; BKY18] found examples of structures that have degree of categoricity but no strong degree of categoricity. The question whether there exists a degree of categoricity that is not strong is still open.

We define notions analogous to the ones discussed above but for bi-embeddability.

**Definition 4.0.1.**

- A computable structure $\mathcal{A}$ is $\mathbf{d}$-*computably bi-embeddably categorical* if any computable bi-embeddable copy of $\mathcal{A}$ is bi-embeddable with $\mathcal{A}$ by $\mathbf{d}$-computable embeddings. For the case $\mathbf{d} = \mathbf{0}^{(\alpha)}$, we usually use the term "$\Delta^0_\beta$ *bi-embeddably categorical*" where $\beta = \alpha + 1$ if $\alpha$ is finite, and $\beta = \alpha$ otherwise.
- A countable (not necessarily computable) structure $\mathcal{A}$ is *relatively* $\Delta^0_\alpha$ *bi-embeddably categorical* if for any bi-embeddable copy $\mathcal{B}$, $\mathcal{A}$ and $\mathcal{B}$ are bi-embeddable by $\Delta^{\mathcal{A}\oplus\mathcal{B}}_\alpha$ embeddings. A structure is relatively computably bi-embeddably categorical if $\alpha = 1$.

**Definition 4.0.2.** The *bi-embeddable categoricity spectrum* of $\mathcal{A}$ is the set

$$CatSpec_{\approx}(\mathcal{A}) = \{\mathbf{d} : \mathcal{A} \text{ is } \mathbf{d}\text{-computably bi-embeddably categorical}\}.$$

A degree $\mathbf{c}$ is the *degree of bi-embeddable categoricity* of $\mathcal{A}$ if $\mathbf{c}$ is the least degree in $CatSpec_{\approx}(\mathcal{A})$. It is a *strong* degree of bi-embeddable categoricity if there are $\mathcal{A}_0 \approx \mathcal{A}_1$ such that for all $\mu : \mathcal{A}_0 \hookrightarrow \mathcal{A}_1$ and $\nu : \mathcal{A}_1 \hookrightarrow \mathcal{A}_0$, $\mathcal{A}_0 \oplus \mathcal{A}_1 \geq_T \mathbf{d}$.

Montalbán [Mon05] showed that any hyperarithmetic linear ordering is bi-embeddable with a computable one and together with Greenberg [GM08] they showed the same for hyperarithmetic Boolean algebras, trees, compact metric spaces, and Abelian p-groups. We have seen in Theorem 3.1.7 that any countable equivalence structure is bi-embeddable with a computable one. For this reason, the study of the algorithmic complexity of embeddings is particularly interesting for this classes of structures.

In Section 4.1 we study equivalence structures in this setting. We give a complete characterization of the degrees of bi-embeddable categoricity of equivalence structures.

In Section 4.2 we present general results as well as results in other classes of structures. We show that all degrees d.c.e. above $\mathbf{0}^{(\alpha)}$ are strong degrees of categoricity and that the index set of relatively $\Delta^0_2$ b.e. categorical structures is $\Pi^1_1$ complete. We also characterize the computably b.e. categorical linear orderings and Boolean algebras.

## 4.1 Degrees of b.e. categoricity of equivalence structures

Calvert, Cenzer, Harizanov, and Morozov [CKM06] initiated the study of computable categoricity for equivalence structures. Given a structure $\mathcal{A}$ and a structure $\mathcal{B}$ bi-embeddable with $\mathcal{A}$, we say that $\mathcal{B}$ is a *bi-embeddable copy* of $\mathcal{A}$. We study the complexity of embeddings through the following notions analogous to computable categoricity and relative computable categoricity.

Csima and Ng [**unpublished**] showed that a computable equivalence structure has strong degree of categoricity $\mathbf{0}$, $\mathbf{0}'$, or $\mathbf{0}''$. Our main result reflects theirs in our setting.

**Theorem 4.1.1.** *Let $\mathcal{A}$ be a computable equivalence structure.*

*(1) If $\mathcal{A}$ has bounded character and finitely many infinite equivalence classes, then its degree of bi-embeddable categoricity is $\mathbf{0}$.*
*(2) If $\mathcal{A}$ has unbounded character and finitely many infinite equivalence classes, then its degree of bi-embeddable categoricity is $\mathbf{0}'$.*
*(3) If $\mathcal{A}$ has infinitely many infinite equivalence classes, then its degree of bi-embeddable categoricity is $\mathbf{0}''$.*

*Thus, the degree of bi-embeddable categoricity of equivalence structures is either $\mathbf{0}$, $\mathbf{0}'$, or $\mathbf{0}''$. Furthermore, the degrees of bi-embeddable categoricity of equivalence structures are strong.*

The proof of Theorem 4.1.1 combines various theorems proved in Sections 4.1.1 and 4.1.2. In these sections we also obtain results on the relation between classical notions of categoricity and bi-embeddable categoricity, summarized in Fig. 4.1.

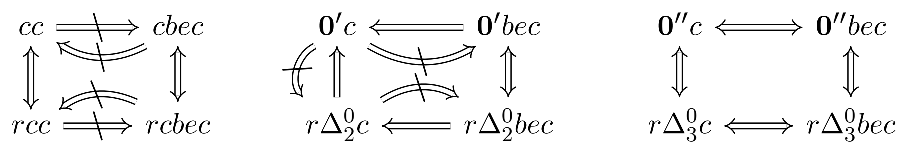



Figure 4.1: Relations between categoricity for computable equivalence structures

In Section 4.1.1 we characterize the computably bi-embeddably categorical equivalence structures. In Section 4.1.2 we study $\Delta^0_2$ and $\Delta^0_3$ bi-embeddably categorical and relatively $\Delta^0_2$ and $\Delta^0_3$ bi-embeddably categorical equivalence structures. We show that all equivalence structures are relatively $\Delta^0_3$ categorical. We prove (2) and (3) of Theorem 4.1.1 and study the relations between those notions summarized in Fig. 4.1.

In Section 4.1.3 we obtain results on the complexity of the index sets of equivalence structures with degrees of bi-embeddable categoricity $\mathbf{0}, \mathbf{0}'$, and $\mathbf{0}''$.

### 4.1.1 Computable bi-embeddable categoricity

Given an equivalence structure $\mathcal{A}$ and $a \in A$ we write $[a]^{\mathcal{A}}$ for the equivalence class of $a$; if it is clear from the context which structure is meant, we omit the superscript. The following notions are central to our analysis.

**Definition 4.1.1.** Let $\mathcal{A}$ be an equivalence structure. A set $T \subseteq A$ is a *transversal* of $\mathcal{A}$ if

(1) for $x, y \in T$, if $x \neq y$, then $x \notin [y]^{\mathcal{A}}$,
(2) and $A = \bigcup_{x \in T} [x]^{\mathcal{A}}$.

**Proposition 4.1.2.** *Let $\mathcal{A}$ be an equivalence structure, then there is a transversal $T$ of $\mathcal{A}$ such that $T \leq_T \mathcal{A}$.*

*Proof.* For each equivalence class, we choose the least element in the class. We can do this computably in (the atomic diagram of) $\mathcal{A}$. □

**Definition 4.1.2** (Calvert, Cenzer, Harizanov, and Morozov [CKM06])**.** Let $\mathcal{A}$ be an equivalence structure.

(1) We say that $\mathcal{A}$ *has bounded character*, or simply *is bounded*, if there is some finite $k$ such that all finite equivalence classes of $\mathcal{A}$ have size at most $k$. If $\mathcal{A}$ has bound $k$ on the sizes of its finite equivalence classes, we say that $\mathcal{A}$ is *$k$-bounded*.
(2) $$Inf^{\mathcal{A}} = \{a \in A : [a]^{\mathcal{A}} \text{ is infinite}\} \qquad Fin^{\mathcal{A}} = \{a \in A : [a]^{\mathcal{A}} \text{ is finite}\}$$

We will use the following relativization of [CKM06, Lemma 2.2].

**Lemma 4.1.3.** *Let $\mathcal{A}$ be an equivalence structure, then*

*(1) For $k \in \omega$, $|[a]^{\mathcal{A}}| \leq k$ is $\Pi^{\mathcal{A}}_1$, $|[a]^{\mathcal{A}}| \geq k$ is $\Sigma^{\mathcal{A}}_1$, $|[a]^{\mathcal{A}}| = k$ is $\Delta^{\mathcal{A}}_2$,*
*(2) $Inf^{\mathcal{A}}$ is $\Pi^{\mathcal{A}}_2$, and $Fin^{\mathcal{A}}$ is $\Sigma^{\mathcal{A}}_2$,*

*Proof.* Ad (1). A $\Pi^{\mathcal{A}}_1$ definition for $|[a]^{\mathcal{A}}| \leq k$ is

$$|[a]^{\mathcal{A}}| \leq k \Leftrightarrow \forall x_1, \dots, x_{k+1} \bigwedge_{1 \leq i \leq k+1} x_i E a \rightarrow \bigvee_{1 \leq i < j \leq k+1} x_i = x_j,$$

a $\Sigma^{\mathcal{A}}_1$ definition for $|[a]^{\mathcal{A}}| \geq k$ is

$$|[a]^{\mathcal{A}}| \geq k \Leftrightarrow \exists x_1, \dots, x_k \bigwedge_{1 \leq i \leq k} x_i E a \wedge \bigwedge_{1 \leq i < j \leq k} x_i \neq x_j,$$

and a $\Delta^{\mathcal{A}}_2$ definition for $|[a]^{\mathcal{A}}| = k$ is then just the conjunction of $|[a]^{\mathcal{A}}| \leq k$ and $|[a]^{\mathcal{A}}| \geq k$.

Ad (2). The property $a \in Inf^{\mathcal{A}}$ has a $\Pi_2^{\mathcal{A}}$ definition by $\forall k \ |[a]^{\mathcal{A}}| \geq k$. It follows immediately that $a \in Fin^{\mathcal{A}}$ has a $\Sigma_2^{\mathcal{A}}$ definition. □

Our first goal is to characterize computably bi-embeddably categorical equivalence structures. In [CKM06] the following characterization of computably categorical equivalence structures was given.

**Theorem 4.1.4** (Calvert, Cenzer, Harizanov, and Morozov)**.** *Let $\mathcal{A}$ be a computable equivalence structure, then $\mathcal{A}$ is computably categorical if and only if*

*(1) $\mathcal{A}$ has finitely many finite equivalence classes,*
*(2) or $\mathcal{A}$ has finitely many infinite classes, bounded character, and at most one finite $k$ such that there are infinitely many classes of size $k$.*

**Theorem 4.1.5.** *An equivalence structure $\mathcal{A}$ is computably bi-embeddably categorical if and only if it has finitely many infinite equivalence classes and bounded character.*

*Proof.* ($\Leftarrow$). Let $\mathcal{A}$ be $k$-bounded and let $l$ be the size of the largest equivalence class such that $\mathcal{A}$ has infinitely many equivalence classes of size $l$ (notice that $l$ might be 0). Then the restriction $\mathcal{A}_{>l}$ of $\mathcal{A}$ to equivalence classes of size larger than $l$ is computably categorical, as the number of equivalence classes in $\mathcal{A}_{>l}$ is finite, i.e., the bi-embeddability type and the isomorphism type of $\mathcal{A}_{>l}$ coincide. Hence, if $\mathcal{B}$ is a bi-embeddable copy of $\mathcal{A}$, then $\mathcal{B}_{>l}$ is isomorphic to $\mathcal{A}_{>l}$. Non-uniformly fix a computable isomorphism $f : \mathcal{A}_{>l} \to \mathcal{B}_{>l}$.

Let $T_{\mathcal{A}_{>l}}$ be a transversal of $\mathcal{A}_{>l}$, and $T_{\mathcal{B}_{>l}}$ one of $\mathcal{B}_{>l}$. Clearly both $T_{\mathcal{A}_{>l}}$ and $T_{\mathcal{B}_{>l}}$ are finite and hence computable. Furthermore the equivalence classes of size $l$ have a c.e. transversal as

$$|[a]^{\mathcal{A}}| \leq l \Leftrightarrow \forall x \in T_{\mathcal{A}_{>l}} \ a \notin [x]^{\mathcal{A}}$$

and $|[a]^{\mathcal{A}}| \geq l$ is $\Sigma_1$. Let $(b_i)_{i\in\omega}$ be an enumeration of the transversal of the equivalence classes of size $l$ in $\mathcal{B}$ and let $(a_i)_{i\in\omega}$ be a computable enumeration of $A$. We can define a computable embedding $\nu : \mathcal{A} \hookrightarrow \mathcal{B}$ by recursion as follows.

$$\nu(a_i) = \begin{cases} f(a_i) & \exists y \in T_{\mathcal{A}_{>l}} \ a_i \in [y]^{\mathcal{A}} \\ b_k, \ \ k = \mu l[\forall j < i \ \nu(a_j) \notin [b_l]^{\mathcal{B}}] & \forall j < i \ a_i \notin [a_j]^{\mathcal{A}} \\ \mu x \in B[x \in [\nu(a_j)]^{\mathcal{B}} \wedge \forall l < i \ x \neq \nu(a_l)] & \exists j < i \ a_i \in [a_j]^{\mathcal{A}} \end{cases}$$

The embedding of $\mathcal{B}$ in $\mathcal{A}$ is defined similarly.

($\Rightarrow$). We show that computable equivalence structures with unbounded character and without infinite equivalence classes are not computably bi-embeddably categorical. The proof for equivalence structures with finitely many infinite equivalence classes is analogous. By Corollary 4.1.12 below, equivalence structures with infinitely many infinite equivalence classes are not even $\Delta^0_2$ bi-embeddably categorical.

Note that any two equivalence structures with unbounded character and the same number of infinite equivalence classes are bi-embeddable and that any embedding needs to map elements to elements in equivalence classes of at least the same size. Consider the equivalence structure $\mathcal{A}$ with universe $\bigcup_{i\in\omega}\{\langle i,n\rangle : n \leq i\}$ where all elements with the same left column are in the same equivalence class. This structure is clearly a computable equivalence structure with computable size function $|\cdot|$. We build a computable equivalence structure $\mathcal{B} = (\omega, E^{\mathcal{B}})$ in stages such that no partial computable function is an embedding of $\mathcal{B}$ in $\mathcal{A}$. We want to satisfy the following requirements.

$$P_e : \quad \varphi_e \text{ is not an embedding of } \mathcal{B} \text{ in } \mathcal{A}$$

We say that a requirement $P_e$ needs attention at stage $s$ if the restriction of the approximation $\varphi_{e,s}$ of $\varphi_e$ to elements in the structure $\mathcal{B}_s$ is a partial embedding of $\mathcal{B}$ in $\mathcal{A}$ not equal to $\varnothing$. The structure $\mathcal{B}$ is the limit of the structures $\mathcal{B}_s$ constructed as follows.
*Construction:*
*Stage $s = 0$:* $\mathcal{B}_0$ is the singleton $\langle 0,0\rangle$.
*Stage $s+1$:* Check if there is a requirement $P_e$, $e \leq s$ that needs attention. If such $P_e$ exists, do the following. Choose the least requirement $P_e$ that needs attention. Then $\varphi_{e,s}(\langle s,0\rangle)\downarrow = \langle i,n\rangle$ for some $\langle i,n\rangle$. Let $\mathcal{B}_{s+1} = \mathcal{B}_s \cup \{\langle s,s+j\rangle : 0 \leq j \leq s\} \cup \{\langle s+1,0\rangle\}$; put all elements with $s$ in the left column in the equivalence class of $\langle s,0\rangle$, and let $\langle s+1,0\rangle$ be a singleton.

If no $P_e$ needs attention set $\mathcal{B}_{s+1} = \mathcal{B}_s \cup \{\langle s+1,0\rangle\}$ and let $\langle s+1,0\rangle$ be a singleton.

*Verification:*
Assume towards a contradiction that $\varphi_e$ is an embedding of $\mathcal{B}$ in $\mathcal{A}$ and no $\varphi_j$, $j < e$ is an embedding. Then there is a stage $s$ such that $\varphi_{e,s}$ is a partial embedding of $\mathcal{B}_s$ in $\mathcal{A}$ and no $P_j$, $j < e$, needs attention. Thus, $P_e$ receives attention at stage $s+1$ and $\varphi_{e,s}(\langle s,0\rangle)\downarrow$; say $\varphi_{e,s}(\langle s,0\rangle) = \langle i,n\rangle$. Then, by construction, the equivalence class of $\langle s,0\rangle$ in $\mathcal{B}_{s+1}$ is bigger than the one of $\langle i,s\rangle$. Thus, $\varphi_e$ can not be an embedding. However, by construction of $\mathcal{B}$, every equivalence class is grown only once and has the size of an equivalence in $\mathcal{A}$ plus one. Thus, $\mathcal{B}$ is unbounded without infinite equivalence classes and hence, is bi-embeddable with $\mathcal{A}$. □

The following corollaries follow directly from Theorem 4.1.4 and Theorem 4.1.5.

**Corollary 4.1.6.** *There is a computably bi-embeddably categorical equivalence structure that is not computably categorical.*

**Corollary 4.1.7.** *There is a computably categorical equivalence structure that is not computably bi-embeddably categorical.*

Calvert, Cenzer, Harizanov, and Morozov [CKM06] showed that a computable equivalence structure is computably categorical if and only if it is relatively computably categorical. The analogous result holds in the context of bi-embeddability.

**Proposition 4.1.8.** *Let $\mathcal{A}$ be a computable equivalence structure. Then $\mathcal{A}$ is computably bi-embeddably categorical if and only if it is relatively computably bi-embeddably categorical.*

*Proof.* Relativization of the proof of Theorem 4.1.5 ensures the result. □

### 4.1.2 $\Delta^0_2$ and $\Delta^0_3$ bi-embeddable categoricity

In this section we characterize $\Delta^0_2$ and $\Delta^0_3$ bi-embeddably categorical equivalence structures. We will show that a computable equivalence structure is $\Delta^0_2$ ($\Delta^0_3$) bi-embeddably categorical if and only if it relatively so. We will also see that all equivalence structures are relatively $\Delta^0_3$ bi-embeddably categorical. This, together with the fact that by Theorem 3.1.7 any countable equivalence structure is bi-embeddable with a computable one gives a complete structural characterization of bi-embeddable categoricity for equivalence structures. We also establish the remaining parts of Theorem 4.1.1. At first we characterize $\Delta^0_2$ bi-embeddably categorical equivalence structures. We start by exhibiting a class of equivalence structures that is relatively $\Delta^0_2$ bi-embeddably categorical.

**Theorem 4.1.9.** *If $\mathcal{A}$ has finitely many infinite equivalence classes, then $\mathcal{A}$ is relatively $\Delta^0_2$ bi-embeddably categorical.*

*Proof.* By Theorem 4.1.5 and Proposition 4.1.8 equivalence structures with bounded character are relatively computably bi-embeddably categorical and thus relatively $\Delta^0_2$ bi-embeddably categorical. It remains to show that equivalence structures with unbounded character and finitely many infinite equivalence classes are relatively $\Delta^0_2$ bi-embeddably categorical.

Let $\mathcal{A}$ have finitely many infinite equivalence classes and unbounded character, and let $\mathcal{B}$ be a bi-embeddable copy of $\mathcal{A}$. Note that $\mathcal{B}$ must have the same number of infinite equivalence classes as $\mathcal{A}$. Fix transversals $T^{\mathcal{A}}$ and $T^{\mathcal{B}}$ of $Inf^{\mathcal{A}}$ and $Inf^{\mathcal{B}}$, respectively. Let $f : T^{\mathcal{A}} \to T^{\mathcal{B}}$ be a bijection. As $T^{\mathcal{A}}$ and $T^{\mathcal{B}}$ are finite sets, they are computable. We define a $\Delta_2^{\mathcal{A}\oplus\mathcal{B}}$ embedding $\nu : \mathcal{A} \hookrightarrow \mathcal{B}$ by recursion. Let $(a_i)_{i\in\omega}$ be a computable enumeration of $A$.

$$\nu(a_0) = \begin{cases} f(t) & \text{if } \exists t \in T^{\mathcal{A}}\ a_0 \in [t]^{\mathcal{A}} \\ \mu x \in B[|[x]^{\mathcal{B}}| \geq |[a_0]^{\mathcal{A}}| \wedge \forall t \in T^{\mathcal{B}}\ x \notin [t]^{\mathcal{B}}] & \text{otherwise} \end{cases}$$

Assume $\nu$ has been defined for $a_0, \dots a_s$. We define $\nu(a_{s+1})$; there are three cases.

*Case 1:* $a_{s+1}$ is equivalent to an element for which $\nu$ has already been defined, i.e., $\exists j \leq s\ a_{s+1} \in [a_j]^{\mathcal{A}}$. Then

$$\nu(a_{s+1}) = \mu x \in B[x \in [\nu(a_j)]^{\mathcal{B}} \wedge \forall i \leq s\ x \neq \nu(a_i)].$$

*Case 2:* $a_{s+1}$ is not equivalent to any element for which $\nu$ has been defined and its equivalence class is infinite, i.e., $\exists t \in T^{\mathcal{A}}\ a_{s+1} \in [t]^{\mathcal{A}}$, then

$$\nu(a_{s+1}) = \mu x \in B[x \in [f(t)]^{\mathcal{B}} \wedge \forall i \leq s\ x \neq \nu(a_i)].$$

*Case 3:* $a_{s+1}$ is not equivalent to any element for which $\nu$ has been defined and its equivalence class is finite. Then

$$\nu(a_{s+1}) = \mu x \in B[|[x]^{\mathcal{B}}| \geq |[a_{s+1}]^{\mathcal{A}}| \wedge \forall t \in T^{\mathcal{B}}\ x \notin [t]^{\mathcal{B}} \wedge \forall i \leq s\ x \notin [\nu(a_i)]^{\mathcal{B}}].$$

As $\mathcal{A}$ and $\mathcal{B}$ are both unbounded, at any stage $s$ of the construction we can find an element in B with an equivalence class greater than or equal to the one of $a_s$ in $\mathcal{A}$. Therefore, $\nu$ is an embedding. As $T^{\mathcal{A}}, T^{\mathcal{B}}$, and $f$ are computable and comparing the size of two equivalence classes is $\Delta_2^{\mathcal{A}\oplus\mathcal{B}}$, $\nu$ is $\Delta_2^{\mathcal{A}\oplus\mathcal{B}}$. □

The following is the relativization of the classical computability theoretic concepts of immune and simple sets to $\mathbf{0}'$.

**Definition 4.1.3.** An infinite set $A$ is $\mathbf{0}'$-*immune* if it contains no infinite set which is computably enumerable in $\mathbf{0}'$. A $\Sigma_2^0$ set $A$ is $\mathbf{0}'$-*simple* if it is the complement of a $\mathbf{0}'$-immune set.

**Theorem 4.1.10.** *There is a computable equivalence structure $\mathcal{A}$ with infinitely many infinite equivalence classes such that $Fin^{\mathcal{A}}$ is $\mathbf{0}'$-simple. Hence, $Inf^{\mathcal{A}}$ is $\mathbf{0}'$-immune.*

*Proof.* We build $\mathcal{A}$ with universe $\omega$ such that $Fin^{\mathcal{A}}$ is $\mathbf{0}'$-simple, i.e., for every infinite $\Sigma_2$ set $S$ the intersection of $Fin^{\mathcal{A}}$ and $S$ is nonempty. It has to satisfy the following requirements.

$$P_e: \quad |W_e^{\varnothing'}| = \infty \quad \Rightarrow \quad W_e^{\varnothing'} \cap Fin^{\mathcal{A}} \neq \varnothing$$

and the overall requirement that any transversal $T_{\mathrm{Inf}^{\mathcal{A}}}$ of $\mathrm{Inf}^{\mathcal{A}}$ is infinite.

$$G: \quad T_{\mathrm{Inf}^{\mathcal{A}}} \text{ is infinite.}$$

Our strategy to satisfy a requirement $P_e$ is to pick a witness $x_e$ for $W_e^{\varnothing'}$ and prevent the equivalence class of $x_e$ from growing any further.

We will construct $\mathcal{A}$ in stages. Elements and equivalence classes can be in one of three states. An element is *blocked by $P_e$* if it is equivalent to a witness picked by $P_e$. During the construction we also *designate* unblocked elements for expansion, i.e., we allow the equivalence class of such elements to grow in a later stage. Elements which are neither designated nor blocked are *fresh*, these elements have equivalence classes of size 1. The set $W_{e,s}^{\varnothing'}$ is the $\Sigma_2$ approximation of $W_e^{\varnothing'}$ at stage $s$ and the set $Fin_s^{\mathcal{A}}$ is the set of blocked elements at stage $s$; we will have that $Fin^{\mathcal{A}} = \lim_s Fin_s^{\mathcal{A}}$. A strategy $P_e$ *needs attention* at stage $s+1$ if

$$W_{e,s}^{\varnothing'} \cap Fin_s^{\mathcal{A}} = \varnothing \qquad \& \qquad \exists x > e^3 \ (x \in W_{e,s}^{\varnothing'}). \tag{$\star$}$$

*Construction:*
*Stage $s = 0$:* Let $A = \omega$ and $E^{\mathcal{A}} = \{(x,x) : x \in \omega\}$. Define $Fin_s^{\mathcal{A}} = \varnothing$.
*Stage $s+1$:* Assume we have built $\mathcal{A}_s$.

(1) Choose the least $e < s$ such that $P_e$ needs attention. Take the least $x > e^3$ satisfying the second part of the matrix in equation $(\star)$. Check if $x \leq s$. If so, then take the element $y$, $e^3 < y \leq s$ which has been in the approximation the longest without interruption and declare its equivalence class as blocked by $P_e$. If not, then declare the equivalence class of $x$ as blocked by $P_e$. If $P_e$ receives attention for the first time, designate the least fresh element.
(2) Add to all designated equivalence classes a fresh element bigger than $s$.
(3) Check if for any $e < s$ there is an element $x$ blocked by $P_e$ that is not blocked by any $P_j$, $j < s$, and $x \in W_{e,s-1}^{\varnothing'}$ but $x \notin W_{e,s}^{\varnothing'}$. If so, declare the equivalence class of $x$ as designated.

*Verification*: It is clear from the construction that $\mathcal{A} = \lim_s \mathcal{A}_s$ is a computable equivalence structure. The following two claims establish that it has the desired properties.

**Claim 4.1.10.1.** *Every requirement $P_e$ is eventually satisfied.*

*Proof.* By construction, no requirement $P_e$ injures a requirement $P_j$, $j \neq e$, so it is sufficient to consider them in isolation. As $W_{e,s}^{\emptyset'}$ is a $\Sigma_2$ approximation we have that $x \in W_e^{\emptyset'}$ iff there is a stage $s_0$ such that for all $t > s_0$, $x \in W_{e,t}^{\emptyset'}$. In particular, there is a stage after which an element $x_0 \in W_e^{\emptyset'}$, $x_0 > e^3$ will be in the approximation longer than any element $y \notin W_e^{\emptyset'}$. This element will be chosen by our strategy the next time that $P_e$ receives attention (which it will if its current witness is not in $W_e^{\emptyset'}$). Hence, $P_e$ is satisfied in the limit. □

**Claim 4.1.10.2.** *The requirement $G$ is satisfied.*

*Proof.* Assume that $P_e$ is the maximum requirement that acted at some stage in the construction. At most $e$ equivalence classes are blocked at this stage and because at every stage every designated equivalence class grows by one element, at most $e^2$ out of $e^3$ elements are blocked. By the same reasoning at most $e^2$ elements are designated for expansion. Hence, at least $e^3 - 2e^2$ fresh elements are left to expand the equivalence classes of designated elements and so, for $e > 2$ there are enough fresh elements smaller than $e^3$ left to expand the designated elements. As every requirement that receives attention for the first time designates one fresh element, in the limit there are infinitely many infinite equivalence classes. □

□

It is immediate that two equivalence structures $\mathcal{A}$ and $\mathcal{B}$ with infinitely many infinite equivalence classes are bi-embeddable. To obtain an embedding of $\mathcal{A}$ in $\mathcal{B}$ just map all equivalence classes of $\mathcal{A}$ to infinite equivalence classes of $\mathcal{B}$.

**Proposition 4.1.11.** *If an equivalence structure $\mathcal{A}$ has infinitely many infinite equivalence classes, then it is not $\Delta_2^0$ bi-embeddably categorical.*

*Proof.* Let $\mathcal{A}$ be a computable equivalence structure with infinitely many infinite equivalence classes and no finite equivalence classes, and take $\mathcal{B}$ as in Theorem 4.1.10. Then, by the above argument, they are bi-embeddable and every embedding of $\mathcal{A}$ in $\mathcal{B}$ has as range an infinite subset of $Inf^{\mathcal{B}}$.

Now, assume that $\nu : \mathcal{A} \hookrightarrow \mathcal{B}$ is $\mathbf{0}'$-computable. Then, its range is a $\Sigma^0_2$ set and an infinite subset of $Inf^{\mathcal{B}}$. But $Inf^{\mathcal{B}}$ is $\mathbf{0}'$-immune, a contradiction. Hence, no embedding of $\mathcal{A}$ in $\mathcal{B}$ is $\mathbf{0}'$-computable and therefore $\mathcal{A}$ is not $\Delta^0_2$ bi-embeddably categorical. □

**Corollary 4.1.12.** *An equivalence structure $\mathcal{A}$ is $\Delta^0_2$ bi-embeddably categorical if and only if it has finitely many infinite equivalence classes.*

Calvert, Cenzer, Harizanov, and Morozov [CKM06] characterized relatively $\Delta^0_2$ categorical computable equivalence relations. Their result relativizes.

**Proposition 4.1.13** (Relativization of [CKM06, Corollary 4.8])**.** *A countable equivalence structure $\mathcal{A}$ is relatively $\Delta^0_2$ categorical if and only if $\mathcal{A}$ has finitely many infinite equivalence classes or $\mathcal{A}$ has bounded character.*

**Corollary 4.1.14.** *An equivalence structure with bounded character and infinitely many infinite equivalence classes is relatively $\Delta^0_2$ categorical but not relatively $\Delta^0_2$ bi-embeddably categorical.*

**Corollary 4.1.15.** *Let $\mathcal{A}$ be a computable equivalence structure. Then $\mathcal{A}$ is $\Delta^0_2$ bi-embeddably categorical if and only if it is relatively $\Delta^0_2$ bi-embeddably categorical.*

*Proof.* By Theorem 4.1.9 computable equivalence structures with finitely many infinite equivalence classes are relatively $\Delta^0_2$ bi-embeddably categorical and therefore, also $\Delta^0_2$ bi-embeddably categorical. It follows from Proposition 4.1.11 that these are all equivalence structures which are $\Delta^0_2$ bi-embeddably categorical. As relatively $\Delta^0_2$ bi-embeddably categorical equivalence structures have the same characterization the result follows. □

The analogue of Corollary 4.1.15 does not hold for isomorphisms. Kach and Turetsky [KT09] gave an example of a $\Delta^0_2$ categorical but not relatively $\Delta^0_2$ categorical equivalence structure. Downey, Melnikov, and Ng [Dow+15] showed that an equivalence structure $\mathcal{A}$ is $\Delta^0_2$ categorical iff the structure containing only one equivalence class of $\mathcal{A}$ for each finite size and all its infinite equivalence classes is $\Delta^0_2$-computably categorical.

We now proceed with the study of possible degrees of categoricity for equivalence structures.

**Definition 4.1.4.** A function $f$ is *limitwise monotonic* if there is a computable approximation function $h_f(\cdot,\cdot)$ such that

(1) $f(x) = \lim_s h_f(x,s)$
(2) for all $x, s$ $h_f(x,s) \leq h_f(x,s+1)$

It is not hard to see that for any limitwise monotonic function $f$, $f \leq_T \mathbf{0}'$. For more on limitwise monotonic functions and their applications see Downey, Kach, and Turetsky [DKT11].

**Theorem 4.1.16.** *The degree of bi-embeddable categoricity of computable equivalence structures with unbounded character and finitely many infinite equivalence classes is* $\mathbf{0}'$.

*Proof.* First notice that there are countably many bi-embeddability types of equivalence structures with unbounded character and finitely many infinite equivalence classes. Namely exactly one for each number of equivalence classes of infinite size. We prove the theorem for equivalence structures with no infinite equivalence classes. However, the argument can be easily modified to accomodate equivalence structures with finitely many infinite equivalence classes.

We define the following function.

$$f(x) := 1 + \sum_{0 \leq i \leq x, \varphi_i(x)\downarrow} \varphi_i(x)$$

Clearly $f$ is limitwise monotonic and dominates every partial computable function. By the domination theorem (see [Soa16, Theorem 4.5.4]) it holds that for any set $D$ such that $f \leq_T D$, $D \geq_T \mathbf{0}'$; hence, in particular, $f \equiv_T \mathbf{0}'$.

We build a computable equivalence structure $\mathcal{A}_f$ with universe $\omega$ and no infinite equivalence classes in stages such that

$$|[\langle x, 0\rangle]^{\mathcal{A}_f}| = f(x).$$

Let $h_f$ be the computable approximation for $f$. At stage 0 of the construction, let the universe of the approximation $\mathcal{A}_f$ be $\omega$ and put $\langle 0, n\rangle$ in the equivalence class of $\langle 0, 0\rangle$ for $n < h_f(0,0)$. At stage $s+1$ check if for any $\langle x, 0\rangle$ $x \leq s$, $|[\langle x, 0\rangle]| < h_f(x, s+1)$. If so, add $\langle x, s+1\rangle$ to the equivalence class.

Now consider the equivalence structure $\mathcal{A}$ with universe $\bigcup_{i\in\omega}\{\langle i, n\rangle : n \leq i\}$ and where all elements with the same left column are in the same equivalence class. This structure is clearly a computable equivalence structure bi-embeddable with $\mathcal{A}_f$ and computable size function $|\cdot|$. Any embedding $\nu : \mathcal{A}_f \to \mathcal{A}$ must map $[\langle x, 0\rangle]^{\mathcal{A}_f} \mapsto [\langle y, 0\rangle]^{\mathcal{A}}$ with $|[\langle y, 0\rangle]^{\mathcal{A}}| \geq [\langle x, 0\rangle]^{\mathcal{A}_f} = f(x)$. Consider the function $g(x) = |[\nu(\langle x, 0\rangle)]^{\mathcal{A}}|$; as $|\cdot|$ is computable, $g \equiv_T \nu$ and as $\forall x\ g(x) \geq f(x)$, $g \equiv_T \nu \geq_T \mathbf{0}'$ by the domination theorem. As by Corollary 4.1.12 every computable equivalence structure with finitely many infinite equivalence classes is $\Delta^0_2$ bi-embeddably categorical the theorem follows. □

**Theorem 4.1.17.** *Equivalence structures are relatively $\Delta^0_3$ bi-embeddably categorical.*

*Proof.* By Theorem 4.1.9, equivalence structures with finitely many infinite equivalence classes are relatively $\Delta^0_2$ bi-embeddably categorical. Thus, it suffices to show that equivalence structures with infinitely many infinite classes are relatively $\Delta^0_3$ bi-embeddably categorical. Let $\mathcal{A}$ and $\mathcal{B}$ be equivalence structures with infinitely many infinite classes. Recall that any two such equivalence structures are bi-embeddable. There is an embedding of $\mathcal{A}$ in $\mathcal{B}$ that maps every equivalence class in $\mathcal{A}$ to an infinite equivalence class in $\mathcal{B}$. As $\mathrm{Inf}^{\mathcal{B}}$ is $\Pi^{\mathcal{B}}_2$ there is at least one such embedding which is $\Delta^{\mathcal{A}\oplus\mathcal{B}}_3$. By the same argument there is a $\Delta^{\mathcal{A}\oplus\mathcal{B}}_3$ embedding of $\mathcal{B}$ in $\mathcal{A}$. □

The analogous result about classical relative $\Delta^0_3$ categoricity of equivalence structures is also true [CKM06], as every equivalence structure has a $\Sigma^c_3$ Scott family.

We close by proving the remaining parts of Theorem 4.1.1.

**Theorem 4.1.18.** *The degree of bi-embeddable categoricity of computable equivalence structures with infinitely many infinite equivalence classes is* $\mathbf{0}''$.

*Proof.* We first build a computable equivalence structure $\mathcal{A}$ with the property that any infinite partial transversal of $Inf^{\mathcal{A}}$ computes $\mathbf{0}''$. Let $(\sigma_i)_{i\in\omega}$ be a computable $1-1$ enumeration of $2^{<\omega}$ and associate to every $\sigma_i$ an infinite set of witnesses $\{\langle i,x,y\rangle : x,y \in \omega\}$. Elements of the form $\langle i,x,0\rangle$ will serve as witnesses while all other elements will be used to grow the equivalence classes. We will build $\mathcal{A}$ using a $\Pi_2$ approximation to $\overline{\varnothing''}$. Let $\overline{\varnothing''}_s$ be the $\Pi_2$ approximation at stage $s$ of our construction.
*Construction:*
*Stage s=0:* Let $A=\omega$ and $E^{\mathcal{A}} = \{(x,x) : x\in\omega\}$. Furthermore, for all strings $\sigma_i$ in our computable enumeration of $2^{<\omega}$ designate witnesses $\langle i,0,0\rangle$.
*stage s+1:* Assume we have built $\mathcal{A}_s$.

(1) For all witnesses with left column $i < s+1$ check if for some $x$ with $\sigma_i(x) = 0$, $x \in \overline{\varnothing''}_{s+1}$. If $\langle i,j,0\rangle$ is a witness for such $\sigma_i$, discard it (never touch its equivalence class again during the construction) and designate the witness $\langle i,s+1,0\rangle$.
(2) For any $\sigma_i$, $i<s+1$ grow the equivalence class of its designated witness $\langle i,j,0\rangle$ to match $\min\{|\{t : t\le s, x\in \overline{\varnothing''}_t\}| : \sigma_i(x)=1\}$ using fresh elements $\langle i,j,r\rangle$ with $r>s$.

*Verification:* We have to show that any infinite partial transversal of $\mathrm{Inf}^{\mathcal{A}}$ computes $\mathbf{0}''$. The following claim establishes the crucial part.

**Claim 4.1.18.1.** $\sigma_i \prec \overline{\varnothing''} \Leftrightarrow \exists y \langle i,y,0\rangle \in Inf^{\mathcal{A}}$

*Proof.* (⇒). Assume $\sigma_i \prec \overline{\varnothing''}$. As we have a $\Pi_2$ approximation to $\overline{\varnothing''}$ there is a stage $s$ such that no $x < |\sigma_i|$, $x \notin \overline{\varnothing''}$ enters $\overline{\varnothing''}_t$ at any stage $t > s$. Hence, by construction there is a $j < s$ such that $\langle i, j, 0\rangle$ has infinite equivalence class.

(⇐). Assume $|[\langle i, j, 0\rangle]| = \omega$ and let $\tau \prec \overline{\varnothing''}$ with $|\tau| = |\sigma_i|$. Assume, for some $x$, $\tau(x) = 0$ and $\sigma_i(x) = 1$. Then there are only finitely many stages $t$ such that $x \in \overline{\varnothing''}_t$. Thus, by construction, no equivalence class of elements with $i$ in the left column can become infinite. Therefore, if $\tau(x) = 0$, then $\sigma_i(x) = 0$ as well. Now assume that $\tau(x) = 1$ and $\sigma_i(x) = 0$. Then there are infinitely many stages $s$ such that $x \in \overline{\varnothing''}_s$. Hence, by construction, there can not be finite $j$ such that $|[\langle i, j, 0\rangle]| = \omega$ because for any $j$ if $\langle i, j, 0\rangle$ is designated at step $s$ there is a step $t > s$ such that $x \in \overline{\varnothing''}_t$. Let $t_0$ be the least such step, then $\langle i, j+1, 0\rangle$ will be designated at step $t_0$ and no new elements will be added to the equivalence class of $[\langle i, j, 0\rangle]$ at any stage $t \geq t_0$. Thus $[\langle i, j, 0\rangle]$ is finite, $\tau(x) = 1 \Rightarrow \sigma_i(x) = 1$ and hence $\tau = \sigma$. □

Consider an infinite partial transversal $T$ of $Inf^{\mathcal{A}}$. To check whether some fixed $x \in \mathbf{0}''$, consider an enumeration of $T$. As all elements in $T$ code an initial segment of $\overline{\varnothing''}$ in their left column there is a finite stage $t$ and $y, z$, such that $\langle i, y, z\rangle \in T_t$ and $|\sigma_i| \geq x$. Hence, $x \in \mathbf{0}'' \Leftrightarrow \sigma_i(x) = 0$ and we can find $\sigma_i$ uniformly. Therefore, $T \geq_T \mathbf{0}''$.

Now consider the structure $\mathcal{B}$ with universe $\omega$ and where $\forall x \forall n\ \langle x, n\rangle \in [\langle x, 0\rangle]^{\mathcal{B}}$. It is a computable equivalence structure consisting only of infinite equivalence classes and it clearly embeds into $\mathcal{A}$. To compute $\mathbf{0}''$ from any embedding $\nu : \mathcal{B} \hookrightarrow \mathcal{A}$ look at the strings coded by the left column of the images of elements of the form $\langle x, 0\rangle$. By the above argument, after enumerating a finite number of images of such elements we can decide whether $x \in \mathbf{0}''$, hence $\nu \geq_T \mathbf{0}''$. Since, by Theorem 4.1.17, any equivalence structure is relatively $\Delta^0_3$ bi-embeddably categorical the theorem follows. □

At last we put together the pieces that prove Theorem 4.1.1.

*Proof of Theorem 4.1.1.* (1) follows directly from Theorem 4.1.5. Theorem 4.1.16 proves that $\mathbf{0}'$ is the degree of bi-embeddable categoricity of equivalence structures with unbounded character and finitely many infinite equivalence classes. The two equivalence structures $\mathcal{A}_f$ and $\mathcal{A}$ constructed in the proof witness that $\mathbf{0}'$ is a strong degree of bi-embeddable categoricity for equivalence structures with unbounded character and no infinite equivalence classes. Similar structures can be easily constructed for equivalence structures with any finite number of infinite classes. This proves (2). Theorem 4.1.18 shows that $\mathbf{0}''$ is the degree of bi-embeddable categoricity of equivalence structures with infinitely many infinite classes. To see that it is a strong degree of bi-embeddable categoricity, consider the structures $\mathcal{A}$ and $\mathcal{B}$ constructed in the proof. Any embedding $\nu : \mathcal{B} \hookrightarrow \mathcal{A}$ computes $\mathbf{0}''$, hence, for any $\mu : \mathcal{A} \hookrightarrow \mathcal{B}$, $\mu \oplus \nu \geq_T \mathbf{0}''$. □

### 4.1.3 Index sets

Let $(\mathcal{C}_e)_{e\in\omega}$ be an enumeration of the partial computable equivalence structures, i.e., given a computable function $\varphi_e : \omega \times \omega \to \{0,1\}$, $\mathcal{C}_e$ has universe $\omega$ and

$$xE^{\mathcal{C}_e}y :\Leftrightarrow \varphi_e(x,y) = 1.$$

Note that it is $\Pi^0_1$ to check whether $\mathcal{C}_e$ is indeed an equivalence structure.

We say that a set is $D^0_n$ if it is the difference of two $\Sigma^0_n$ sets, or equivalently, the intersection of a $\Sigma^0_n$ and a $\Pi^0_n$ set. We start by recalling a simple observation.

**Lemma 4.1.19.** *For computably bi-embeddably categorical equivalence structures $\mathcal{A}$ we have that $\mathcal{B} \approx \mathcal{A}$ if and only if*

*(1) $\mathcal{B}$ has the same number of infinite equivalence classes as $\mathcal{A}$,*
*(2) $\mathcal{B}$ has the same bound as $\mathcal{A}$,*
*(3) and if $\mathcal{A}$ has infinitely many equivalence classes of size $n$ and for all $k > n$ there are only finitely many equivalence classes of size $k$, then for every $m \geq n$, $\mathcal{B}$ has the same number of equivalence classes of size $m$ as $\mathcal{A}$.*

**Theorem 4.1.20.** *Let $\mathcal{A}$ be a computable, computably bi-embeddably categorical equivalence structure.*

*(1) If $\mathcal{A}$ is finite then the index set $\{\mathcal{C}_e : \mathcal{C}_e \approx \mathcal{A}\}$ is $D^0_1$-complete.*
*(2) If $\mathcal{A}$ has infinitely many equivalence classes of size $n$ for some $n < \omega$, and no infinite equivalence classes, then the index set $\{\mathcal{C}_e : \mathcal{C}_e \approx \mathcal{A}\}$ is $\Pi^0_2$-complete.*
*(3) If $\mathcal{A}$ has $r > 0$ infinite equivalence classes, then the index set $\{\mathcal{C}_e : \mathcal{C}_e \approx \mathcal{A}\}$ is $\Pi^0_2$-complete.*

*Proof.* Ad (1). Assume $\mathcal{A}$ is finite, say $|A| = m$. Let $\theta_{\mathcal{A}}$ be the formula obtained from the atomic diagram by replacing the constants from $A$ by variables. Then the index set is definable by the following $D^0_1$ formula.

$$\mathcal{C}_e \approx \mathcal{A} \Leftrightarrow \mathcal{C}_e \cong \mathcal{A} \Leftrightarrow \exists x_1, \dots x_m(\theta_{\mathcal{A}}(x_1, \dots, x_m)) \\ \wedge \forall x_1, \dots x_{m+1}(\bigvee_{1\leq i<j\leq m+1} x_i = x_j).$$

To see that the index set is $D_1^0$-hard, we define a computable function $g$ such that $\mathcal{C}_{g(e,i)} \approx \mathcal{A}$ if and only if $e \in \varnothing'$ and $i \in \overline{\varnothing'}$. Fix an element $a$ of $\mathcal{A}$. Let $\mathcal{B}$ be a computable equivalence structure isomorphic to $\mathcal{A} \smallsetminus [a]^{\mathcal{A}}$; say $|[a]^{\mathcal{A}}| = n$. We build a computable function $f$ and a structure $\mathcal{E}_{f(e,i)}$ disjoint from $\mathcal{B}$ in stages such that $\mathcal{C}_{g(e,i)} = \mathcal{B} \oplus \mathcal{E}_{f(e,i)}$.

Let $\varnothing'_s, \mathcal{E}_{f(e,i),s}$ be the approximations to $\varnothing'$ and $\mathcal{E}_{f(e,i)}$, respectively, at stage $s$. Let $\mathcal{E}_{f(e,i),0} = \varnothing$ and assume we have defined $\mathcal{E}_{f(e,i),s}$. To define $\mathcal{E}_{f(e,i),s+1}$ check if (i) $e \searrow \varnothing'_s$ and (ii) $i \searrow \varnothing'_s$. The structure $\mathcal{E}_{f(e,i),s+1}$ extends $\mathcal{E}_{f(e,i),s}$ as follows. If (i), add a new equivalence class of size $n$ to $\mathcal{E}_{f(e,i),s+1}$ by using the elements $2(s+j)$ for $j \in 1,\dots,n$. If (ii), add a new equivalence class of size $n+1$ by using the elements $2(s+j)+1$ for $j \in 1,\dots n+1$.

Let $\mathcal{E}_{f(e,i)} = \lim_s \mathcal{E}_{f(e,i),s}$. It is now easy to see that $\mathcal{C}_{g(e,i)} = \mathcal{B} \oplus \mathcal{E}_{f(e,i)}$ is bi-embeddable with $\mathcal{A}$ (in this case even isomorphic) if $e \in \varnothing'$ and $i \in \overline{\varnothing'}$.

Ad (2). Assume $n$ is the maximal size such that $\mathcal{A}$ has infinitely many equivalence classes of size $n$. Let $\mathcal{A}_{>n}$ be the substructure of $\mathcal{A}$ restricted to classes bigger than $n$. Then $\mathcal{A}_{>n}$ is finite. The index set is definable by the following $\Pi_2^0$ formula.

$$\mathcal{C}_e \approx \mathcal{A} \Leftrightarrow \forall x \exists y > x([y]^{\mathcal{C}_e} \geq n \wedge \forall z < y \ \neg z E y) \wedge \mathcal{C}_{e,>n} \cong \mathcal{A}_{>n}$$

To see that it is hard, consider the $\Pi_2^0$ complete set $\mathrm{Inf} = \{e : W_e \text{ is infinite}\}$. We will build a computable function $g$ such that $\mathcal{C}_{g(e)} \approx \mathcal{A} \Leftrightarrow e \in \mathrm{Inf}$. Fix a computable equivalence structure $\mathcal{B}$ isomorphic to $\mathcal{A}_{>n}$. Our desired structure $\mathcal{C}_{g(e)}$ will be the disjoint union of $\mathcal{B}$ with the structure $\mathcal{E}_{f(e)}$, where $f$ is a computable function constructed as follows. Let $W_{e,s}$ be the computable approximation to $W_e$ after $s$ stages of our construction; we make the standard assumption that $x \in W_{e,s} \Rightarrow x < s$. Assume we have defined $\mathcal{E}_{f(e),s}$ and are at stage $s+1$ of the construction. The structure $\mathcal{E}_{f(e),s+1}$ extends $\mathcal{E}_{f(e),s}$ by a new equivalence class of size $n$ for every $x \searrow W_{e,s}$, i.e., if $x \searrow W_{e,s}$ let $\langle x, s\rangle, \dots, \langle x, s+n\rangle \in \mathcal{E}_{f(e),s+1}$ and set them to be equivalent. This finishes the construction; let $\mathcal{E}_{f(e)} = \lim_s \mathcal{E}_{f(e),s}$.

By construction $\mathcal{E}_{f(e)}$ has only equivalence classes of size $n$ and it has infinitely many of those if and only if $W_e$ is infinite. As $\mathcal{B} \cong \mathcal{A}_{>n}$, we have that

$$\mathcal{C}_{g(e)} = \mathcal{B} \oplus \mathcal{E}_{f(e)} \approx \mathcal{A} \Leftrightarrow e \in \mathrm{Inf}\,.$$

Thus the index set is $\Pi_2^0$ complete.

Ad (3). To see that it is in $\Pi_2^0$ one has to consider two cases. Either the finite part of $\mathcal{A}$ is as in (1) or as in (2). In any case, let $k$ be the bound. If we are in case (1), let $m$ be the number of elements in the finite part of $\mathcal{A}$. If we are in case (2), let $m$ be the number of elements in the finite part of $\mathcal{A}$ restricted to equivalence classes bigger than $n$, where $n$ is as above.

If we are in case (1) we can define the index set by

$$\begin{aligned}\mathcal{C}_e \approx \mathcal{A} \Leftrightarrow \forall y_1,\dots,y_{m+1}(\bigwedge_{1\le i\le m+1}[y_i]\le k \to \bigvee_{1\le i<j\le m+1} y_i = y_j)\\ \wedge \forall y_1,\dots,y_{r+1}(\bigwedge_{1\le i\le r+1}[y_i]>k \to \bigvee_{1\le i<j\le r+1} y_iEy_j)\\ \wedge \exists x_1,\dots,x_m(\theta_{\mathcal{A}_{fin}}(x_1,\dots,x_m))\\ \wedge \forall y_1,\dots,y_m((\bigwedge_{1\le i\le m}[y_i]\le k \wedge \bigwedge_{1\le i<j\le m} y_i\ne y_j)\to \Theta_{\mathcal{A}_{fin}}(y_1,\dots,y_m))\\ \wedge \exists x_1,\dots,x_r(\bigwedge_{1\le i\le r}[x_i]>k \wedge \bigwedge_{1\le i<j\le r}\neg x_iEx_j)\\ \wedge \forall x([x]>k \to \exists y>x\ yEx) \quad (*)\end{aligned}$$

where $\theta_{\mathcal{A}_{fin}}$ is the formula obtained from the atomic diagram of the finite part of $\mathcal{A}$ by replacing the constants by variables and $\Theta_{\mathcal{A}_{fin}}$ is the disjunction over $\theta_{\mathcal{A}_{fin}}(x_1,\dots,x_m)$ permuting over all variables. Let the formula in equation $(*)$ be $\varphi_{\mathcal{A}}$. If we are in case (2) the defining formula is

$$\mathcal{C}_e \approx \mathcal{A} \Leftrightarrow \forall x\exists y>x([y]^{\mathcal{C}_e}\ge n)\wedge \varphi'_{\mathcal{A}_{>n}}$$

where $\varphi'_{\mathcal{A}_{>n}}$ is as above with the slight difference that the second and third universal quantifiers now range over all $x$ where $n<[x]\le k$ instead of only $[x]\le k$. The two formulas are easily seen to be $\Pi^0_2$.

For the hardness consider the $\Pi^0_2$ complete set $\text{Inf}=\{e : W_e \text{ is infinite}\}$ and fix a computable structure $\mathcal{B}$ without infinite equivalence classes isomorphic to the finite part of $\mathcal{A}$. We build a computable function $g$ such that

$$\mathcal{C}_{g(e)} = \mathcal{B}\oplus\mathcal{E}_{f(e)} \approx \mathcal{A} \Leftrightarrow e\in \text{Inf}$$

where $f$ is again a computable function. We prove the hardness for the case that $\mathcal{A}$ has one infinite equivalence class, the case for $r>1$ is similar. The construction of $\mathcal{E}_{f(e)}$ is in stages, at stage 0 the universe of $\mathcal{E}_{f(e),0}$ is empty. Assume we have defined $\mathcal{E}_{f(e),s}$ and are at stage $s+1$ of the construction.

For any $x<s$ such that $x \searrow W_{e,s}$, add $\langle x, s+1\rangle$ to $\mathcal{E}_{f(e),s+1}$ and make it equivalent to all elements already in $\mathcal{E}_{f(e),s}$. It is easy to see that the structure $\mathcal{E}_{f(e)}=\lim_s\mathcal{E}_{f(e),s}$ has an infinite equivalence class if and only if $e\in\text{Inf}$ and thus $\mathcal{C}_{g(e)}\approx\mathcal{A}$ if and only if

$e \in \mathrm{Inf}$. □

**Theorem 4.1.21.** *Let $\mathcal{A}$ be an equivalence structure with degree of categoricity $\mathbf{0}'$. Then the following holds.*

*(1) If $\mathcal{A}$ has no infinite equivalence classes, then the index set $\{\mathcal{C}_e : \mathcal{C}_e \approx \mathcal{A}\}$ is $\Pi^0_3$-complete.*
*(2) If $\mathcal{A}$ has $0 < k < \omega$ infinite equivalence classes, then the index set $\{\mathcal{C}_e : \mathcal{C}_e \approx \mathcal{A}\}$ is $D^0_3$-complete.*

*Proof.* Recall that an equivalence structure has degree of cateogricity $\mathbf{0}'$ if and only if it has finitely many infinite classes and is unbounded. Notice that two unbounded equivalence structures are bi-embeddable if and only if they have the same number of infinite classes and assume that $\mathcal{A}$ has $0 < k < \omega$ infinite equivalence classes. Then the index set is definable by

$$\begin{aligned}\mathcal{C}_e \approx \mathcal{A} \Leftrightarrow \forall x \left(x \in \mathrm{Fin}^{\mathcal{C}_e} \to \exists y \left(y \in \mathrm{Fin}^{\mathcal{C}_e} \wedge [y] \geq [x]\right)\right) \\ \wedge\, \exists x_1, \dots, x_k \in \mathrm{Inf}^{\mathcal{C}_e} \left( \bigwedge_{1 \leq i < j \leq k} \neg x_i E x_j \right) \\ \wedge\, \forall x_0, x_1, \dots, x_k \in \mathrm{Inf}^{\mathcal{C}_e} \left( \bigvee_{1 \leq i < j \leq k} x_i E x_j \right).\end{aligned}$$

The part of the formula defining the finite equivalence classes is $\Pi^0_3$ and the part defining the infinite equivalence classes is $\Sigma^0_3$. Thus, the formula is $D^0_3$. If $\mathcal{A}$ has no infinite equivalence classes then the part of the defining formula defining the infinite classes becomes $\forall x\ x \in Fin^{\mathcal{C}_e}$, a $\Pi^0_3$ formula. Hence, in this case the above formula is $\Pi^0_3$.

To show the completeness of (2) we will define a computable function $f$ such that for every $\Pi^0_3$ set $P$ and every $\Sigma^0_3$ set $S$

$$\mathcal{C}_{f(p,e)} \approx \mathcal{A} \Leftrightarrow p \in P \wedge e \in S.$$

By $\Pi^0_2$ completeness of $\mathrm{Inf} = \{e : W_e \text{ is infinite}\}$ we have that there is a computable function $g$ such that

$$e \in S \Leftrightarrow \exists x\ W_{g(x,e)} \text{ is infinite}.$$

In the same vein, as $\mathrm{Fin} = \overline{\mathrm{Inf}}$ is $\Sigma^0_2$ complete we have that there is a computable function $h$ such that

$$p \in P \Leftrightarrow \forall x\ W_{h(x,p)} \text{ is finite}.$$

We may assume without loss of generality that

$$e \in S \Leftrightarrow \exists! x \; W_{g(x,e)} \text{ is infinite.}$$

In other words, we may assume that there is a unique $x$ witnessing that $e \in S$. See for instance [Soa16, Theorem 4.3.11] for a proof of this fact. We build the structure $\mathcal{C}_{f(p,e)}$ in stages.

*Construction:*
*Stage* 0*:* The universe of $\mathcal{C}_{f(p,e)}$ is $\omega$ and $\mathcal{C}_{f(p,e)}$ has exactly one equivalence class for each finite size, i.e., for each $x \in \omega$, we set $\langle 2x, 0, 0\rangle E \langle 2x, 0, i\rangle$ for $i \leq x$. All other elements are singletons.
*Stage* $s+1 = 2j$*:* For every $x < j$, if for some $t < j$, $t \searrow W_{h(x,p),j}$, then put $\langle 2x, i, j\rangle$ into the equivalence class of $\langle 2x, i, 0\rangle$ for all $i$ with $0 \leq i \leq k$.
*Stage* $s+1 = 2j+1$*:* For every $x < j$, if for some $t < j$, $t \searrow W_{g(x,e),j}$, then put $\langle 2x+1, i, j\rangle$ into the equivalence class of $\langle 2x+1, i, 0\rangle$ for all $i$ with $0 \leq i < k$.

*Verification:*
$p \in P, e \in S$: The construction at even stages, taking care of the $\Pi^0_3$ part, will not contribute any infinite equivalence classes and $\mathcal{C}_{f(p,e)}$ is unbounded from the beginning. To see that it has $k$ infinite equivalence classes just notice that by our assumption above there is exactly one $x$ such that $W_{g(x,e)}$ is infinite. Thus, the construction at odd stages guarantees that the equivalence classes of elements $\langle 2x+1, i, 0\rangle$ with $0 \leq i < k$ become infinite in the limit.
$p \notin P, e \in S$: Then there is an $x$ such that $W_{h(x,p)}$ is infinite. By construction the equivalence class of $\langle 2x, 0, 0\rangle$ is infinite and our strategy for $S$ builds $k$ infinite equivalence classes. Thus, $\mathcal{C}_{f(p,e)}$ has more than $k$ infinite equivalence classes and hence, $\mathcal{C}_{f(p,e)}$ is not bi-embeddable with $\mathcal{A}$.
$p \in P, e \notin S$: Then for no $x$ the set $W_{g(x,e)}$ is infinite. So, by construction, no equivalence class of elements with an odd number in the left column will grow to be infinite and all equivalence classes with an even number in the left column have finite equivalence classes as $W_{h(x,p)}$ is finite for all $x$.
$p \notin P, s \notin S$: There will be some $x$ such that $W_{h(p,x)}$ is infinite. By construction the equivalence classes of $\langle x, i, 0\rangle$ with $0 \leq i \leq k$ will be infinite and thus $C_{f(p,e)}$ will have $k+1$ infinite equivalence classes.

To prove completeness for (1), at every stage, we apply the strategy described above for even stages. □

**Theorem 4.1.22.** *The index set* $\{e : \mathcal{C}_e$ *is computably bi-embeddably categorical*$\}$ *is* $\Sigma^0_2$ *complete.*

*Proof.* Recall that an equivalence structure is computably bi-embeddably categorical if and only if it has bounded character and finitely many infinite equivalence classes. Thus the index set is definable by the following computable $\Sigma_2$ formula.

$$\mathcal{C}_e \text{ is computably bi-embeddably categorical} \\ \Leftrightarrow \exists k \bigvee_{r\in\omega} (\forall x_1, \dots, x_{r+1} ( \bigwedge_{1\le i\le r+1} |[x_i]| \ge k \to \bigvee_{1\le i<j\le r+1} x_i E x_j )).$$

To see that it is hard consider the classical $\Sigma^0_2$ complete set Fin = $\{e : W_e \text{ is finite}\}$. We build a computable function $f$ such that

$$\mathcal{C}_{f(e)} \text{ is computably bi-embeddably categorical} \Leftrightarrow W_e \text{ is finite}.$$

Let $W_{e,s}$ be the computable approximation of $W_e$ at stage $s$. We construct $\mathcal{C}_{f(e)}$ in stages. At stage 0, $\mathcal{C}_{f(e),0}$ has universe $\omega$ and the equivalence relation is the identity relation. Assume we have defined $\mathcal{C}_{f(e),s}$. To define $\mathcal{C}_{f(e),s+1}$ check for $x < s$ if $x \searrow W_{e,s}$. If so declare $\langle x, s+i\rangle$ for $i \in 0, \dots, x$ to be equivalent. This finishes the construction. Let $\mathcal{C}_{f(e)} = \lim_s \mathcal{C}_{f(e),s}$. It follows directly from the construction that $\mathcal{C}_{f(e)}$ has bounded character if and only if $W_e$ is finite. This completes the proof. □

**Theorem 4.1.23.** *The index set* $\{e : \mathcal{C}_e$ *has degree of b.e. categoricity* $\mathbf{0}'\}$ *is* $\Sigma^0_4$ *complete.*

To prove that Theorem 4.1.23 is $\Sigma^0_4$ hard we use the function constructed in the proof of Theorem 4.1.25. We will state and prove this theorem after proving the following representation lemma for $\Pi^0_4$ sets.

**Lemma 4.1.24.** *Let* $P$ *be a* $\Pi^0_4$ *set, then there is a computable function* $g$ *such that the following two conditions hold.*

$$x \in P \Leftrightarrow \forall y \exists! z\ W_{g(x,y,z)} \text{ is infinite} \tag{1a}$$

$$\forall x, y \left(\exists z\ W_{g(x,y,z)} \text{ is infinite} \to \exists! z\ W_{g(x,y,z)} \text{ is infinite}\right) \tag{1b}$$

$$\forall x, y \left(\forall z\ W_{g(x,y,z)} \text{ is finite} \to \forall z\ W_{g(x,y+1,z)} \text{ is finite}\right) \tag{2}$$

*Proof.* Using a proof similar to the one of [Soa16, Theorem 4.3.11] we get a computable function $h$ satisfying (1a) and (1b) if we replace $g$ by $h$. For $y > 0$ let

$$f(x,y,s) = \begin{cases} \mu(z < s)[|W_{h(x,y-1,z),s}| > |W_{h(x,y-1,z),s-1}|] & \text{such } z < s \text{ exists} \\ s & \text{otherwise} \end{cases}$$

Now at stage $s+1$, for $y = 0$ let $W_{g(x,y,z),s} = W_{h(x,y,z),s}$ and otherwise let $u = f(x,y,s)$ and enumerate into $W_{g(x,y,u),s}$ all elements in $W_{h(x,y,z),s}$.

*Verification:*

$x \in P$: Then for all $y$ there is one $z_y$ such that $W_{h(x,y,z_y)}$ is infinite. Hence, there is a stage $s$ such that for given $y$, $u < z_{y-1}$ and $t > s$, $W_{h(x,y-1,u),t} = W_{h(x,y-1,u),s}$. Thus, for infinitely many stages $t$, $z_{y-1}$ is the least such that $|W_{h(x,y-1,z_{y-1}),t}| > |W_{h(x,y-1,z_{y-1}),t-1}|$ and therefore by construction there is $z$ such that $W_{g(x,y,z)}$ is infinite. By definition of $h$ there can be at most one such $z$.

$x \notin P$: Then there is a least $y_0$ such that for all $z$, $W_{h(x,y_0,z)}$ is finite. By construction $\lim_{s\to\infty} f(x, y_0+1, s)$ does not exist and thus no $W_{g(x,y_0+1,z)}$ can be infinite. By induction the same holds for all $y > y_0$. □

**Theorem 4.1.25.** *The index set* $\{e : \mathcal{C}_e$ *has degree of b.e. categoricity* $\mathbf{0}''\}$ *is* $\Pi^0_4$*-complete.*

*Proof.* By Theorem 4.1.18 the index set of the equivalence structures having degree of bi-embeddability categoricity $\mathbf{0}''$ is the same as the index set

$$\{e : \mathcal{C}_e \text{ has infinitely many infinite equivalence classes}\}.$$

This index set is clearly $\Pi^0_4$. To see that it is complete consider a $\Pi^0_4$ set $P$, then there is a computable function $g$ such that

$$e \in P \Leftrightarrow \forall x \exists y\ W_{g(e,x,y)} \text{ is infinite.}$$

We may assume that $g$ satisfies the matrices in Lemma 4.1.24. We build a computable function $f$ such that

$$\mathcal{C}_{f(e)} \text{ has infinitely many infinite equivalence classes.} \Leftrightarrow e \in P$$

The construction is in stages; the universe of $\mathcal{C}_{f(e),0}$ is $\omega$ and $E^{\mathcal{C}_{f(e),0}} = \{(x,x) : x \in \omega\}$. Assume we have defined $\mathcal{C}_{f(e),s}$ and are at stage $s+1$ of the construction. The structure $\mathcal{C}_{f(e),s+1}$ extends $\mathcal{C}_{f(e),s}$ as follows. For each $x, y \le s+1$, if there is $u < s+1$ such that $u \searrow W_{g(e,x,y),s+1}$, then add $\langle x, y, s+1\rangle$ to the equivalence class of $\langle x, y, 0\rangle$. Then proceed to the next stage.

The desired structure $\mathcal{C}_{f(e)}$ is the structure in the limit of the construction. If $e \in P$, then $\mathcal{C}_{f(e)}$ has infinitely many infinite equivalence classes as for every $x$ there is a $y$ such that $W_{g(e,x,y)}$ is infinite and by construction the elements having the same first and second column are in the same equivalence class. Assume $e \notin P$, then there exists an $x_0$ such that for no $y_0$ the above set is infinite; so no equivalence class of elements with $x_0$ in the left column will be infinite. Then by Lemma 4.1.24 the same holds for all $x > x_0$ and for all $x_1 < x$ there is exactly one $y$ with $W_{g(e,x_1,y)}$ infinite. Hence, $\mathcal{C}_{f(e)}$ has only finitely many infinite equivalence classes. □

*Proof of Theorem 4.1.23.* The index set is definable by the $\Sigma^0_4$ formula

$$\forall k(\exists x(x \in Fin^{\mathcal{C}_e} \wedge [x] \geq k)) \wedge \exists r(T_{\mathrm{Inf}^{\mathcal{C}_e}} \leq r).$$

where $T_{\mathrm{Inf}^{\mathcal{C}_e}}$ is a $\Pi^0_2$ transversal of $\mathrm{Inf}^{\mathcal{C}_e}$. For the hardness we use the strategy used in the proof of Theorem 4.1.25. There, given a $\Pi^0_4$ set $P$, we define a computable function $f$ such that

$$\mathcal{C}_{f(e)} \text{ has degree of b.e. categoricity } \mathbf{0}'' \Leftrightarrow e \in P.$$

Clearly, $\overline{P}$ is $\Sigma^0_4$ and using the same function we have that

$$\mathcal{C}_{f(e)} \text{ has degree of b.e. categoricity } \mathbf{0} \text{ or } \mathbf{0}' \Leftrightarrow e \notin P$$

since if $e \notin P$, $f$ produces an equivalence structure with finitely many infinite equivalence classes. Notice that $\mathcal{C}_{f(e)}$ need not have unbounded character and thus might have degree of categoricity $\mathbf{0}$. Therefore, define a computable unbounded equivalence structure $\mathcal{B}$ and a function $g$ such that

$$\mathcal{C}_{g(e)} = \mathcal{C}_{f(e)} \oplus \mathcal{B}.$$

The function $g$ is clearly computable and

$$\mathcal{C}_{g(e)} \text{ has degree of b.e. categoricity } \mathbf{0}' \Leftrightarrow e \in \overline{P}.$$

This proves that the index set is $\Sigma^0_4$ hard. □

## 4.2 General results and other classes of structures

Our first result gives examples of degrees of bi-embeddable categoricity. Recall that a structure $\mathcal{A}$ is called *bi-embeddably trivial* (or *b.e. trivial* for short) if for any $\mathcal{B}$ bi-embeddable with $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A}$ are isomorphic.

**Theorem 4.2.1.** *Let $\alpha$ be a computable non-limit ordinal. Suppose that $\mathbf{d}$ is a Turing degree such that $\mathbf{d}$ is d.c.e. in $\mathbf{0}^{(\alpha)}$ and $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$. There is a computable, bi-embeddably trivial structure $\mathcal{S}$ with degree of bi-embeddable categoricity $\mathbf{d}$.*

*Proof.* We build two b.e. trivial computable structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \cong \mathcal{B}$, $\mathcal{A}$ is $\mathbf{d}$-computably categorical, and any embedding from $\mathcal{A}$ into $\mathcal{B}$ must compute $\mathbf{d}$. Here we give a construction for the case when $\mathbf{d}$ is d.c.e. over $\mathbf{0}^{(2\beta+1)}$, where $\beta$ is an infinite ordinal.

Ash's characterization of the back-and-forth relations for linear orders and his pairs of structures theorem, see Chapters 11 and 16 in [AK00], tells us that for any $\Sigma^0_{2\beta+1}$ set $S$, there is a computable sequence $(C_e)_{e\in\omega}$ of linear orders such that

$$C_e \cong \begin{cases} \omega^\beta \cdot 2, & \text{if } e \in S, \\ \omega^\beta, & \text{if } e \notin S. \end{cases} \tag{4.1}$$

A relativized version of the argument from [FKM10, Theorem 3.1] shows that one can choose a set $D \in \mathbf{d}$ such that $D$ is d.c.e. in $\mathbf{0}^{(2\beta+1)}$ and for any oracle $X$, we have:

$$(\overline{D} \text{ is c.e. in } X) \ \Rightarrow \ D \leq_T X \oplus \mathbf{0}^{(2\beta+1)}.$$

The language of our structures contains an equivalence relation $\sim$, a partial order $\leq$, a unary predicate $T$, and a unary predicate $P_e$ for $e \in \omega$. We have that $D = U \setminus V$ for $U$ and $V$ c.e. in $\mathbf{0}^{(2\beta+1)}$. We first describe the construction of $\mathcal{A}$. For every $e$, we choose elements $a_e$ and $b_e$ in $\mathcal{A}$, and for every $P_e$, $P_e(A)$ is infinite and includes $a_e, b_e$.

Fix $e$, we give a construction for the substructure on $P_e(A)$. We let $P_e(A)$ consist of two infinite equivalence classes (with respect to $\sim$) such that $a_e \not\sim b_e$. The two classes $[a_e]$ and $[b_e]$ will both contain pairs of linear orders, i.e., structures of the form $(L_1, L_2)$ where $L_1$ and $L_2$ are linear orders (with respect to $\leq$), any $x \in L_1$ and $y \in L_2$ are incomparable, and $T([a_e]) = L_1$.

If $e = 2m$, then we encode membership of $m \in D$ in $P_e(A)$. There are three cases:

(1) $m \notin U$: we build $T([a_e]), \neg T([a_e]), T([b_e]) \cong \omega^\beta$, and $\neg T([b_e]) \cong \omega^\beta \cdot 2$,

(2) $m \in U \setminus V$: we build $T([b_e]) \cong \omega^\beta$ and $T([a_e]), \neg T([a_e]), \neg T([b_e]) \cong \omega^\beta \cdot 2$,
(3) $m \in V$: we build $T([a_e]), T([b_e]), \neg T([a_e]), \neg T([b_e]) \cong \omega^\beta \cdot 2$.

Analyzing this construction, we see that

$$[a_e] \cong \begin{cases} (\omega^\beta \cdot 2, \omega^\beta \cdot 2), & \text{if } m \in U, \\ (\omega^\beta, \omega^\beta), & \text{if } m \notin U; \end{cases} \quad \text{and} \quad [b_e] \cong \begin{cases} (\omega^\beta \cdot 2, \omega^\beta \cdot 2), & \text{if } m \in V, \\ (\omega^\beta, \omega^\beta \cdot 2) & \text{if } m \notin V. \end{cases}$$

If $e = 2m+1$, then we let $[b_e] \cong (\omega^\beta, \omega^\beta \cdot 2)$, and for $[a_e]$ we let

$$[a_e] \cong \begin{cases} (\omega^\beta \cdot 2, \omega^\beta \cdot 2), & \text{if } m \in \varnothing^{(2\beta+1)}, \\ (\omega^\beta, \omega^\beta), & \text{if } m \notin \varnothing^{(2\beta+1)}. \end{cases}$$

The existence of the uniformly computable sequence of structures $(C_e)_{e\in\omega}$ from (4.1) implies that we can do the construction computably.

For $\mathcal{B}$, we again choose elements $\hat{a}_e$, $\hat{b}_e$ for every $e$, and we build $\mathcal{B}$ like $\mathcal{A}$ with the difference that the roles of $\hat{a}_e$ and $\hat{b}_e$ are switched. Clearly, $\mathcal{B}$ and $\mathcal{A}$ are isomorphic and computable. It is not hard to show that they are b.e. trivial: Indeed, every embedding of $\mathcal{A}$ into a bi-embeddable copy $\hat{\mathcal{A}}$ must map elements in $P_e(A)$ to elements in $P_e(\hat{A})$, for every $e \in \omega$. Every $P_e(\hat{A})$ must have exactly 2 equivalence classes as otherwise $P_e(\hat{A}) \not\hookrightarrow P_e(A)$. Moreover, the pairs of structures that we use are pairs of well-orders, and thus b.e. trivial.

Following the lines of the proof of [**Bazh17**], it is not hard to obtain that $\mathcal{A}$ is **d**-computably categorical. It remains to show that for every $f\colon \mathcal{A} \hookrightarrow \mathcal{B}$, $f \geq_T D$. We have that $f \geq_T \mathbf{0}^{(2\beta+1)}$ because

$$m \in \varnothing^{(2\beta+1)} \Leftrightarrow f(a_{2m+1}) \sim \hat{b}_{2m+1} \quad \text{and} \quad m \notin \varnothing^{(2\beta+1)} \Leftrightarrow f(a_{2m+1}) \sim \hat{a}_{2m+1}.$$

Similarly, we have that

$$m \notin U \setminus V \Leftrightarrow (f(a_{2m}) \sim \hat{a}_{2m}) \text{ or } (m \in V).$$

Thus, $\overline{D}$ is c.e. in $f \oplus \mathbf{0}^{(2\beta+1)}$. Hence, $D \leq_T (f \oplus \mathbf{0}^{(2\beta+1)}) \equiv_T f$.

The construction for the case $\alpha = 2\beta+2$ is nearly the same. The only difference is that in place of (4.1), we use the following fact: For any $\Sigma^0_{2\beta+2}$ set $S$, there is a computable sequence $(C_e)_{e\in\omega}$ of linear orders such that

$$C_e \cong \begin{cases} \omega^{\beta+1} + \omega^{\beta}, & \text{if } e \in S, \\ \omega^{\beta+1}, & \text{if } e \notin S. \end{cases}$$

The proof for finite $\alpha$ can be obtained by minor modifications. □

Theorem 4.2.1 shows that every degree of categoricity known in the literature [FKM10; CFS+13] can be realized as a degree of bi-embeddable categoricity.

The rest of this chapter is devoted to bi-embeddable categoricity for structures from familiar algebraic classes.

**Theorem 4.2.2.** *(a) A computable Boolean algebra is computably bi-embeddably categorical if and only if it is finite.*

*(b) A computable linear order is computably bi-embeddably categorical if and only if it is finite.*

*Proof. (a)* Alaev [Ala07] proved the following: If $\mathcal{B}$ is a computable atomic Boolean algebra, then there is a computable copy $\mathcal{A} \cong \mathcal{B}$ such that every c.e. ideal of $\mathcal{A}$ is principal.

We show that every computable infinite Boolean algebra $\mathcal{B}$ is not computably bi-embeddably categorical. In order to do this, we consider two cases.

*Case 1.* Suppose that $\mathcal{B}$ is superatomic. W.l.o.g., we may assume that the set of atoms $Atom(\mathcal{B})$ is computable. Let $\mathcal{A}$ be an isomorphic copy of $\mathcal{B}$ such that every c.e. ideal of $\mathcal{A}$ is principal.

Assume that there is a computable embedding $g: \mathcal{B} \hookrightarrow \mathcal{A}$. Then the set

$$I = \{x \in \mathcal{A} : (\exists b \in \mathcal{B})(b \text{ is a finite sum of atoms and } x \leq_{\mathcal{A}} g(b))\}$$

is a non-principal c.e. ideal in $\mathcal{A}$; this is a contradiction.

*Case 2.* Suppose that $\mathcal{B}$ is not superatomic. Then $\mathcal{B}$ is bi-embeddable with the atomless Boolean algebra $Int(\eta)$. We prove that $Int(\eta)$ is not hyperarithmetically bi-embeddably categorical.

Fix a computable copy $\mathcal{H}$ of the Harrison linear order $\omega_1^{\mathrm{CK}}(1+\eta)$ such that $\mathcal{H}$ has no infinite hyperarithmetic descending chains (such a copy will be built in the proof of the second part of the theorem). Let $\mathcal{L} := 1 + \mathcal{H}^*$, and note that $\mathcal{L}$ has no hyperarithmetic ascending chains. We define $\mathcal{A} = Int(\mathcal{L})$.

Suppose that $g$ is an embedding from a standard copy of $Int(\eta)$ into $\mathcal{A}$. Let

$$I = \left\{ \frac{m}{2^n} : n \in \omega,\ 0 \leq m \leq 2^n \right\}.$$

We choose a computable sequence $\{a_q\}_{q \in I}$ of elements from $\eta$ such that $a_q <_\eta a_r$ iff $q < r$. For $q < r \in I$, let $b(q,r)$ denote the leftmost (with respect to $\leq_{\mathcal{L}}$) point of the set $g([a_q; a_r[)$. Note that $b(q,r)$ is $deg_T(g)$-computable, uniformly in $q, r$. Furthermore, it is easy to prove that for any $q < r$, we have either

$$b(q, (q+r)/2) >_{\mathcal{L}} b(q,r) \text{ or } b((q+r)/2, r) >_{\mathcal{L}} b(q,r).$$

Now we build a sequence $(q_n, r_n)_{n \in \omega}$ using a dichotomy procedure: Set $q_0 = 0$ and $r_0 = 1$. If $b(q_n, (q_n + r_n)/2) >_{\mathcal{L}} b(q_n, r_n)$, then let $q_{n+1} = q_n$ and $r_n = (q_n + r_n)/2$. Otherwise, define $q_{n+1} = (q_n + r_n)/2$ and $r_{n+1} = r_n$. Hence, we obtain a $deg_T(g)$-computable strictly increasing sequence

$$b(q_0, r_0) <_{\mathcal{L}} b(q_1, r_1) <_{\mathcal{L}} b(q_2, r_2) <_{\mathcal{L}} \dots.$$

This implies that $g$ cannot be hyperarithmetic.

*(b)* We need to prove that any computable infinite linear order $\mathcal{L}$ is not computably bi-embeddably categorical. Again, we consider two cases.

*Case 1.* Suppose that $\eta \hookrightarrow \mathcal{L}$. Since $\mathcal{L}$ is countable, we have $\eta \approx \mathcal{L}$. Thus, we show that $\eta$ is not hyperarithmetically bi-embeddably categorical.

Take a computable ill-founded tree $T \subseteq \omega^{<\omega}$ with no hyperarithmetic paths. Harrison [**harrison1968a**] proved that the Kleene–Brouwer ordering of $T$ (denoted by $KB(T)$) is isomorphic to $\omega_1^{\mathrm{CK}}(1+\eta) + \alpha$, where $\alpha$ is a computable ordinal. Let $\mathcal{S} := KB(T)$, w.l.o.g., we may assume that $\mathcal{S} \cong \omega_1^{\mathrm{CK}}(1+\eta)$.

Assume that $\mathcal{S}$ has a hyperarithmetic descending chain $\sigma_0 >_{\mathcal{S}} \sigma_1 >_{\mathcal{S}} \sigma_2 >_{\mathcal{S}} \dots$. Then for $i \in \omega$, set $k_i = \lim_s \sigma_s(i)$ and $\tau_i = \langle k_0, k_1, \dots, k_i \rangle$. It is not hard to show that $(\tau_i)_{i \in \omega}$ is a hyperaritmetic path in $T$; this contradicts the choice of $T$. Thus, $\mathcal{S}$ has no hyperarithmetic descending chains.

Note that given an embedding $g : \eta \hookrightarrow \mathcal{S}$, one can produce a $deg_T(g)$-computable descending chain in $\mathcal{S}$. Therefore, there are no hyperarithmetic embeddings from $\eta$ into $\mathcal{S}$.

*Case 2.* Suppose that $\eta \not\hookrightarrow \mathcal{L}$, i.e. $\mathcal{L}$ is *scattered.*

A linear order $\mathcal{S}$ is *indecomposable* if the condition $\mathcal{S} = \mathcal{A} + \mathcal{B}$ implies that either $\mathcal{S} \hookrightarrow \mathcal{A}$ or $\mathcal{S} \hookrightarrow \mathcal{B}$. We first give a proof for an indecomposable $\mathcal{L}$, and after that we explain how to extend this to the case of an arbitrary scattered $\mathcal{L}$.

*(2.1)* In the proof of [Mon05, Theorem 1.2], Montalbán obtained the following result: If $\mathcal{S}$ is an infinite, hyperarithmetic, indecomposable scattered linear order, then there is a computable sequence of linear orders $\{\mathcal{A}_k\}_{k\in\omega}$ such that $\mathcal{S}$ is bi-embeddable with one of the two orders:

- $\mathcal{A}^+ := \mathcal{A}_{0,0} + (\mathcal{A}_{1,0} + \mathcal{A}_{1,1}) + (\mathcal{A}_{2,0} + \mathcal{A}_{2,1} + \mathcal{A}_{2,2}) + \dots$, or
- $\mathcal{A}^- := \dots + (\mathcal{A}_{2,2} + \mathcal{A}_{2,1} + \mathcal{A}_{2,0}) + (\mathcal{A}_{1,1} + \mathcal{A}_{1,0}) + \mathcal{A}_{0,0}$,

where $\mathcal{A}_{i,j} = \mathcal{A}_j$, for $j \leq i$.

We give a proof for the case when our $\mathcal{L}$ is bi-embeddable with $\mathcal{A}^+$ (in the case that $\mathcal{L} \approx \mathcal{A}^-$ one can work with the order $\mathcal{L}^* \approx \mathcal{A}^+$ instead). W.l.o.g., we may assume that $\mathcal{L} = \mathcal{A}^+$.

Now we build a computable $\mathcal{M} \approx \mathcal{L}$ satisfying the following requirements:

$$\mathcal{R}_e : \quad \varphi_e \text{ is not an isomorphic embedding from } \mathcal{M} \text{ into } \mathcal{L}$$

The order $\mathcal{M}$ is built as a sum

$$\mathcal{M} = \sum_{e\in\omega} \mathcal{B}_e.$$

For $e \in \omega$, the $\mathcal{R}_e$-requirement is satisfied as follows:

(1) Choose witnesses $a_e$ from $\mathcal{A}_{e+1,0}$ and $b_e$ from $\mathcal{A}_{e+1,e+1}$. Put $a_e$ and $b_e$ into $\mathcal{B}_e$, and set $a_e <_{\mathcal{M}} b_e$.

(2) While we are waiting for the values $\varphi_e(a_e)$ and $\varphi_e(b_e)$ to be defined, we build $\mathcal{B}_e$ as a copy of $(\mathcal{A}_{e+1,0} + \mathcal{A}_{e+1,1} + \dots + \mathcal{A}_{e+1,e+1})$.

(3) When $\varphi_e(a_e)\downarrow$ and $\varphi_e(b_e)\downarrow$, we consider two cases:

   (3.a) If $\varphi_e(a_e) \geq_{\mathcal{L}} \varphi_e(b_e)$, then continue building $\mathcal{B}_e$ as in (2).

   (3.b) Suppose that $\varphi_e(a_e) <_{\mathcal{L}} \varphi_e(b_e)$. Then find indices $i_0, j_0$ such that inside $\mathcal{L}$, we have $\varphi_e(b_e) \in \mathcal{A}_{i_0,j_0}$. Build $\mathcal{B}_e$ as $(\mathcal{C} + \mathcal{C} + \mathcal{C})$, where

$$\mathcal{C} := \sum_{i=0}^{\max(i_0,e+1)} (\mathcal{A}_{i,0} + \mathcal{A}_{i,1} + \dots + \mathcal{A}_{i,i}).$$

   We arrange $\mathcal{B}_e$ in such a way that the element $b_e$ belongs to the rightmost copy of $\mathcal{C}$.

It is easy to show that our requirements do not injure each other, and the constructed

$\mathcal{M}$ is a computable structure. Note that each of the orders $\mathcal{L}$ and $\mathcal{M}$ has a form

$$\sum_{k\in\omega} \mathcal{D}_k,$$

where each $\mathcal{D}_k$ is isomorphic to some $\mathcal{A}_j$, $j \in \omega$. Furthermore, for every $\mathcal{A}_j$, there are infinitely many summands $\mathcal{D}_k$ isomorphic to $\mathcal{A}_j$. Using this observation, it is straightforward to prove that $\mathcal{L}$ and $\mathcal{M}$ are bi-embeddable.

Assume that $\varphi_e$ is a computable embedding from $\mathcal{M}$ into $\mathcal{L}$. Then the $\mathcal{R}_e$-requirement must go through the stage (3.b). In particular, the order $(\mathcal{C} + \mathcal{C})$ (with only *two* copies of $\mathcal{C}$, *not three*) is embeddable via $\varphi_e$ into

$$\sum_{i=0}^{i_0} (\mathcal{A}_{i,0} + \mathcal{A}_{i,1} + \dots + \mathcal{A}_{i,i}) \subseteq \mathcal{C}.$$

Hence, $\mathcal{C} + \mathcal{C} \hookrightarrow \mathcal{C}$. This implies that $\eta \hookrightarrow \mathcal{C}$, thus, $\mathcal{C}$ is not scattered. This contradicts the scatteredness of $\mathcal{L}$.

*(2.2)* Now suppose that $\mathcal{L}$ is not indecomposable. Jullien (see [Mon06, Section 3.2] for a detailed discussion) proved that in this case, there exists a minimal non-zero number $n \in \omega$ such that there are indecomposable orders $\mathcal{I}_0, \mathcal{I}_1, \dots, \mathcal{I}_n$ satisfying

$$\mathcal{L} \approx \mathcal{I}_0 + \mathcal{I}_1 + \dots + \mathcal{I}_n.$$

Furthermore, this *minimal decomposition* of $\mathcal{L}$ is unique, up to bi-embeddability. In particular, we may assume that every $\mathcal{I}_j$ is a computable structure. Let $k_0$ be the maximal number $\leq n$ such that $\mathcal{I}_{k_0}$ is infinite.

For simplicity, we sketch a proof for the case when $n = 2$ and $k_0 = 1$. In other words, we consider a minimal decomposition

$$\mathcal{L} = \mathcal{I}_0 + \mathcal{I}_1 + \mathcal{I}_2,$$

where $\mathcal{I}_2 \cong 1$ and $\mathcal{I}_1$ is infinite. Again, here we describe a construction only for the case when $\mathcal{I}_1 \approx \mathcal{A}^+$ (we re-use notations from the previous construction).

We satisfy the same series of requirements $\mathcal{R}_e$, and the order $\mathcal{M}$ is built as a sum

$$\mathcal{I}_0 + (\sum_{i\in\omega} \mathcal{B}_i) + \mathcal{I}_2.$$

Here the $\mathcal{R}_e$-requirement is satisfied as follows:

(1) Choose a fresh number $k(e)$. Find fresh witnesses $a_e$ from $\mathcal{A}_{k(e)+1,0}$ and $b_e$ from $\mathcal{A}_{k(e)+1,k(e)+1}$. Set $a_e <_{\mathcal{M}} b_e$.

(2) While waiting for $\varphi_e(a_e)$ and $\varphi_e(b_e)$ to converge, build $\mathcal{B}_{k(e)}$ as a copy of $(\mathcal{A}_{k(e)+1,0} + \mathcal{A}_{k(e)+1,1} + \dots + \mathcal{A}_{k(e)+1,k(e)+1})$.

(3) When $\varphi_e(a_e)\downarrow$ and $\varphi_e(b_e)\downarrow$, consider the cases:

(3.a') If $\varphi_e(a_e) \geq_{\mathcal{L}} \varphi_e(b_e)$ or $\varphi_e(a_e) \in \mathcal{I}_2$ or $\varphi_e(b_e) \in \mathcal{I}_2$, then continue building $\mathcal{B}_{k(e)}$ as in (2). Declare $\mathcal{R}_e$ satisfied.

(3.b') If $\varphi_e(a_e) <_{\mathcal{L}} \varphi_e(b_e)$ and both $\varphi_e(a_e)$ and $\varphi_e(b_e)$ are in $\mathcal{I}_1$, then do actions similar to (3.b). Declare $\mathcal{R}_e$ satisfied.

(3.c') Otherwise, go to (1).

Now we verify the described construction. Again, it is easy to see that the constructed $\mathcal{M}$ is a computable bi-embeddable copy of $\mathcal{L}$.

Assume that $\varphi_e$ is a computable embedding from $\mathcal{M}$ into $\mathcal{L}$. Consider the strategy $\mathcal{R}_e$. There are three possible variants of the behavior of the strategy:

(i) $\mathcal{R}_e$ is eventually declared satisfied in (3.a'). Then this implies that $\varphi_e$ embeds the infinite interval $[b_e; \infty)_{\mathcal{M}}$ into $\mathcal{I}_2 \cong 1$. This gives a contradiction.

(ii) $\mathcal{R}_e$ is eventually declared satisfied in (3.b'). Then using an argument similar to that of the indecomposable case, one can show that $\mathcal{L}$ is not scattered, a contradiction again.

(iii) $\mathcal{R}_e$ infinitely often goes through (3.c'). Notice the following: if $a_e[s] \neq a_e[s+1]$, then $a_e[s+1] >_{\mathcal{M}} b_e[s]$. Since $\varphi_e\colon \mathcal{M} \hookrightarrow \mathcal{L}$, this observation implies the following: for any $i \in \omega$, there is $j \geq i$ and an element $c \in \mathcal{B}_j$ with $\varphi_e(c) \in \mathcal{I}_0$ (just consider a stage $s$ such that $k(e)[s] > i$ and choose $c \in \{a_e[s], b_e[s]\}$ such that $c \notin \mathcal{I}_1$).
Since $\varphi_e\colon \mathcal{M} \hookrightarrow \mathcal{L}$ and $\mathcal{I}_1 \approx \mathcal{A}^+$, we obtain that one can embed the sum

$$(\mathcal{I}_0 + \mathcal{I}_1) \approx \left(\mathcal{I}_0 + \sum_{i\in\omega} \mathcal{B}_i\right)$$

into $\mathcal{I}_0$. This means that $(\mathcal{I}_0 + \mathcal{I}_1 + \mathcal{I}_2)$ and $(\mathcal{I}_0 + \mathcal{I}_2)$ are bi-embeddable, hence, it contradicts the choice of the *minimal* decomposition of $\mathcal{L}$.

This concludes the proof of Theorem 4.2.2. □

Note that Theorem 4.2.2 contrasts with the characterizations of computably categorical Boolean algebras [GD80; Rem81a] and computably categorical linear orders [GD80; Rem81b]: In particular, a computable Boolean algebra is computably categorical iff its set of atoms is finite.

An undirected graph is *strongly locally finite* if each of its components is finite. It is easy to show that every computable, strongly locally finite graph is $\mathbf{0}'$-computably categorical.

**Theorem 4.2.3.** *(a) There exists a computable, strongly locally finite graph which is not hyperarithmetically bi-embeddably categorical.*

*(b) The index set of $\mathbf{0}'$-computably bi-embeddably categorical, strongly locally finite graphs is $\Pi^1_1$-complete.*

*Proof. Ad (a).* Let $H \subseteq \omega^{<\omega}$ be a computable tree without hyperarithmetic paths. We build a strongly locally finite graph $G_H$ such that the partial ordering under embeddability of its components is computably isomorphic to $H$.

For any $\sigma \in H$, $G_H$ contains the component $C_\sigma$: A ray of length $|\sigma| + 1$ where the first vertex has a loop connected to it and the $(i+1)^{th}$ vertex for $i \leq |\sigma|$ has a cycle of length $\sigma(i) + 1$ attached. Clearly the partial ordering of the components is computably isomorphic to $H$ by $C_\sigma \mapsto \sigma$. Now $G_H$ has a bi-embeddable copy $\tilde{G}$ that skips a fixed $C_\sigma$ such that $\sigma$ lies on a path in $H$. Now consider embeddings $\mu : G_H \to G$ and $\nu : G \to G_H$, then $C_\sigma \subset \mu(C_\sigma) \subset \nu(\mu(C_\sigma))\dots$ and thus there is $f \in [H]$ hyperarithmetic in $\mu \oplus \nu$. Hence, $\mu \oplus \nu$ itself can not be hyperarithmetic.

*Ad (b).* Let $(T_i)_{i\in\omega}$ be a uniformly computable sequence of trees such that $T_i$ is well-founded iff $i \in \mathcal{O}$. For two strings $\sigma, \tau$ of the same length let $\sigma \star \tau = \sigma_0\tau_0\sigma_1\tau_1\dots\sigma_{|\sigma|-1}\tau_{|\tau|-1}$, and consider the sequence of trees $(S_i)_{i\in\omega}$

$$S_i = \{\xi : \xi \subseteq \sigma \star \tau,\ |\sigma| = |\tau|,\ \sigma \in T_i,\ \tau \in H\}.$$

Clearly, it is uniformly computable, and $S_i$ is well-founded iff $i \in \mathcal{O}$. Furthermore, no path in $[S_i]$ is hyperarithmetical. Using the same coding as above we get that if $i \in \mathcal{O}$, then $G_{S_i}$ is b.e. trivial and thus $\mathbf{0}'$-computably bi-embeddably categorical. If $i \notin \mathcal{O}$, then $G_{S_i}$ is not $\mathbf{0}^{(\alpha)}$-computably bi-embeddably categorical for $\alpha < \omega_1^{\mathrm{CK}}$. □

Note that in [Dow+15], it was shown that the index set of computably categorical structures is $\Pi^1_1$-complete. We leave open whether a similar result can be obtained for computably bi-embeddably categorical structures.

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