TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

DIPLOMARBEIT

Gitik's Model or A model of ZF where all uncountable cardinals are singular

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Inhaltsverzeichnis

1 Abstract

In my Master's Thesis I want to show the following result by Gitik [G]: Assuming the consistency of arbitrarily large strongly compact cardinals, we show the consistency of ZF $+ \forall\alpha \in \mathrm{Lim}\colon \mathrm{cf}\,\alpha = \omega$, where Lim is the class of all limit ordinals.

To this end, we will start with a countable transitive model M of

$$\mathrm{ZFC} + \text{`}\forall\alpha \in \mathrm{On}\,\exists\kappa > \alpha\colon \kappa\,\text{is strongly compact'},$$

force with a proper class forcing to get a model $M[G]$ satisfying $\mathrm{ZF}^- + \text{`}\forall x\colon x$ is countable', where ZF^- is ZF without Power Set but Collection included, and finally define a symmetric submodel N_G, which will have the required properties.

The logic behind the consistency result can be found in Kunen [K].

2 The forcing and other prerequisites

We start with a ctm M of ZFC + '$\forall \alpha \in \mathrm{On}\, \exists \kappa > \alpha \colon \kappa$ is strongly compact'. W.l.o.g. we can assume, that there is no regular limit of strongly compact cardinals in M, since if there were one, the smallest such α, named α', would be strongly inaccessible, and the set $\{x \in M \colon \mathrm{rank}^M(x) < \alpha'\}$ would be a model with the required properties. Furthermore, we can assume, that M has a predicate WO_M, which is a global well-order of M, and the model satisfies Replacement with respect to the predicate WO_M. This can be easily seen, as we can always add such a global well-order by class forcing (see Felgner [F]).

Let $(\kappa_\alpha)_{\alpha \in \mathrm{On}}$ list the strongly compact cardinals in M, where $\kappa_0 = \omega$. Now we consider $\alpha \in \mathrm{Reg}$, the class of regular cardinals, and want to distinguish 3 cases:

- $\alpha < \kappa_1$: Let $\Phi_\alpha = \{X \subseteq \alpha \colon |\alpha - X| < \alpha\}$ be the co-bounded filter on α.
- There exist a maximal strongly compact cardinal $\kappa \leq \alpha$: Let Φ_α be the least κ- complete uniform ultrafilter on α. By least we refer to the well-order WO_M and by uniform we mean that $X \in \Phi_\alpha$ implies $|X| = \alpha$.
- There is no such κ: Let $\beta = \sup\{\kappa \colon \kappa < \alpha \wedge \kappa \,\text{is strongly compact}\}$. By our earlier assumption β must be singular. Let $\gamma = \mathrm{cf}\,\beta$. Let $(\kappa_\nu)_{\nu \in \gamma}$ be the least γ-sequence of strongly compact cardinals cofinal in β. We now define $\Phi_{\alpha,\nu}$ to be the least κ_ν-complete uniform ultrafilter on α for all $\nu < \gamma$.

In the second case we define $\mathrm{cf'}\,\alpha := \alpha$, and in the third case we shall say $\mathrm{cf'}\,\alpha := \gamma$.

We now consider the class $\mathrm{Reg} \times \omega \times \mathrm{On}$. For $x \subseteq \mathrm{Reg} \times \omega \times \mathrm{On}$ we shall now define $\mathrm{dom}_1(x) := \{\alpha \colon \exists n\, \exists \beta\, (\alpha, n, \beta) \in x\}$ and $\mathrm{dom}_{1,2}(x) := \{(\alpha, n) \colon \exists \beta\, (\alpha, n, \beta) \in x\}$. Furthermore, we define

$$P_1 := \{p \subseteq \mathrm{Reg} \times \omega \times \mathrm{On} \colon$$

$$p \colon \mathrm{Reg} \overset{\mathrm{par}}{\to} (\omega \overset{\mathrm{par}}{\to} \mathrm{On}) \wedge |\mathrm{dom}_{1,2}(p)| < \omega \wedge \forall \alpha \in \mathrm{dom}_1(p)\, [p(\alpha) \text{is 1-1} \wedge \mathrm{ran}(p(\alpha)) \subseteq \alpha]\}.$$

For $p_1, p_2 \in P_1$ we shall say $p_1 \approx p_2$, if $p_1|(\mathrm{Reg} - \kappa_1) = p_2|(\mathrm{Reg} - \kappa_1)$.

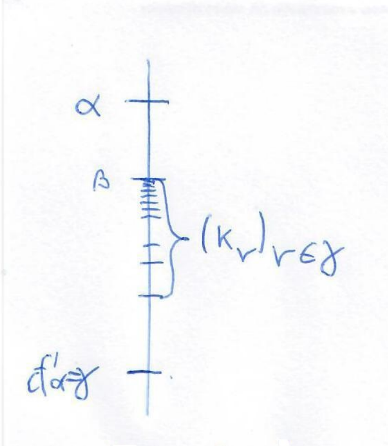
For technical reasons we are only going to use a subclass of P_1. Define P_2 as the class of $p \in P_1$ such that the following conditions hold:

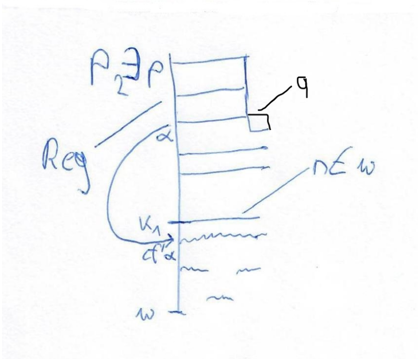
- $\forall \alpha \in \mathrm{dom}_1(p) \colon \mathrm{cf'}\,\alpha \in \mathrm{dom}_1(p)$.
- $\forall \alpha \in \mathrm{dom}_1(p) \colon \mathrm{dom}(p(\alpha)) \subseteq \mathrm{dom}(p(\mathrm{cf'}\,\alpha))$.
- $\exists \alpha \in \mathrm{dom}_1(p)\,, \alpha \geq \kappa_1, \exists n \in \omega \colon$
  $\forall \alpha' \in (\mathrm{dom}_1(p) - \alpha)\ \mathrm{dom}(p(\alpha')) = n \wedge \forall \alpha' \in \mathrm{dom}_1(p) \cap (\alpha - \kappa_1)\ \mathrm{dom}(p(\alpha')) = n+1$.

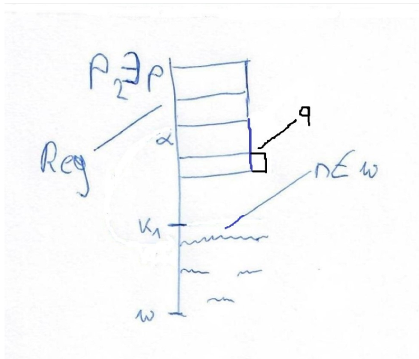
Since α and n are obviously uniquely determined, we shall set $\alpha(p) = \alpha$ and $n(p) = n$. $(\alpha(p), n(p))$ will be the first coordinate we will have to fill, if we want to extend p.

Definition 2.1. We shall call $p \in P_2$ <u>extendable</u>, iff $\exists q \in P_2\colon \mathrm{dom}_1(q) = \mathrm{dom}_1(p) \wedge q|\,\kappa_1 = p|\,\kappa_1 \wedge p \subsetneq q$. Therefore we see that a function p is extendable iff either $\mathrm{cf'}\,\alpha(p) = \alpha(p)$ or $(\mathrm{cf'}\,\alpha(p), n(p)) \in \mathrm{dom}_{1,2}(p)$.

So we see that if $\mathrm{cf'}\,\alpha(p) \geq \kappa_1$, p is extendable.







Definition 2.2. We can now define the forcing. We set P_3 to be the class of pairs (p, U) such that the following conditions hold:

(P1) $p \in P_2$.

(P2) $U \subseteq P_2$.

(P3) $p \in U$.

(P4) $\forall q \in U\colon p \subseteq q \wedge \mathrm{dom}_1(p) = \mathrm{dom}_1(q)$.

(P5) $\forall r \in U, r \approx p, \forall(\alpha, n)\colon$
$(\alpha, n) \in (\mathrm{dom}_1(p) \cap \kappa_1) \times \omega - \mathrm{dom}_{1,2}(r) \Rightarrow \{\beta\colon r \cup \{(\alpha, n, \beta)\} \in U\} \in \Phi_\alpha$.

(P6) $\forall r_1, r_2 \in U,\ r_1, r_2 \approx p\colon (r_1 \cup r_2 \in P_1 \Rightarrow r_1 \cup r_2 \in U) \wedge (r_1 \subseteq r_2 \Rightarrow T_{r_1} \subseteq T_{r_2})$, where $T_r = T_r^U := \{q \in U\colon q|\kappa_1 = r|\kappa_1\}$ for $r \approx p$. The sign $\subseteq$ should be interpreted as the embedding $\iota\colon T_{r_1} \ni q \mapsto q \cup r_2$.

(P7) $\forall q \in U\, \forall a \subseteq \kappa_1 \times \omega \times \kappa_1\colon p \cup (q \cap a) \in U$.

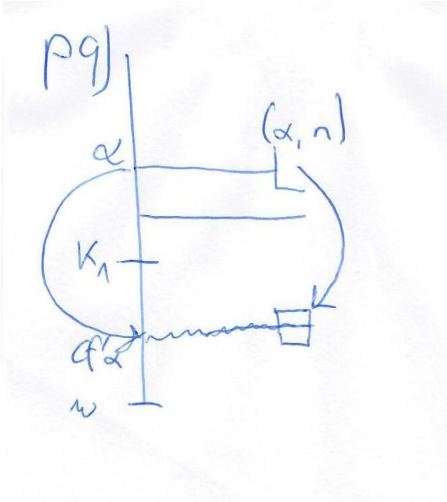
(P8) $\forall q \in U, \mathrm{cf'}\,\alpha(q) = \alpha(q)\colon \{\beta\colon q \cup \{(\alpha(q), n(q), \beta)\} \in U\} \in \Phi_{\alpha(q)}$.

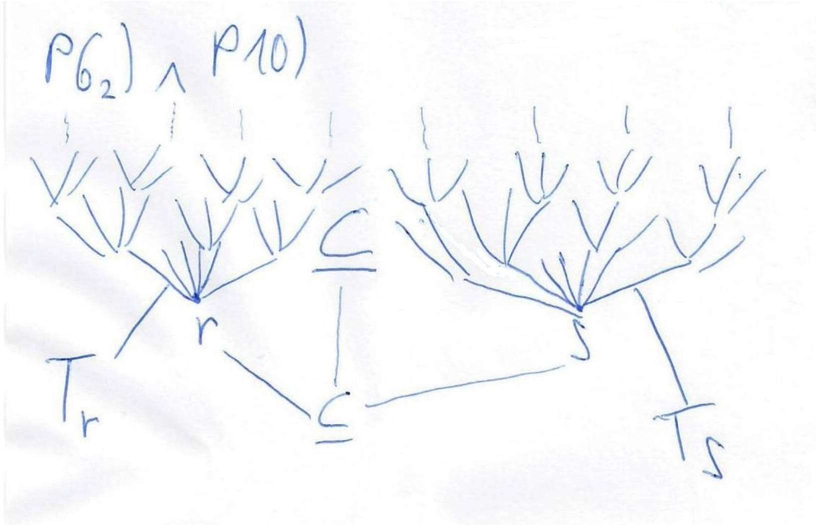
(P9) $\forall q \in U, \mathrm{cf'}\,\alpha(q) < \alpha(q)\colon$
$n(q) \in \mathrm{dom}\ (q(\mathrm{cf'}\,\alpha(q))) \Rightarrow \{\beta\colon q \cup \{(\alpha(q), n(q), \beta)\} \in U\} \in \Phi_{\alpha(q),\, q(\mathrm{cf'}\,\alpha(q))\,(n(q))}$.

(P10) $\forall q \in U, q \not\approx p, \exists q' \in U\, \exists(\alpha, n) \in \mathrm{dom}_{1,2}(q)\, \exists \beta \in \mathrm{On}\colon$
$q = q' \cup \{(\alpha, n, \beta)\} \wedge (\alpha(q'), n(q')) = (\alpha, n)$. We shall denote this q' by q^-.

The partial order will be defined at the end of this chapter.

If q is extendable, then we shall denote the corresponding ultrafilter by Φ_q.





Lemma 1. *Let (p, U) be a forcing condition and $a \subseteq \mathrm{Reg}$ such that $\alpha \in a \Rightarrow \mathrm{cf'}\,\alpha \in a$, then $(p|\,a, U|\,a)$, where $U|\,a = \{q|\,a\colon q \in U\}$, is a forcing condition too.*

Proof. As can be easily seen, only conditions (P5), (P6) and (P10) are non-trivial. For (P5) let $q \in U|\,a$, $q = r|\,a$ with some $r \in U$ and $q \approx p|\,a$. By (P7) we can assume that $r \approx p$. Let $(\alpha, n) \in (\mathrm{dom}_1(p|\,a) \cap \kappa_1) \times \omega - \mathrm{dom}_{1,2}(q)$, and it follows that also $(\alpha, n) \in (\mathrm{dom}_1(p) \cap \kappa_1) \times \omega - \mathrm{dom}_{1,2}(r)$. So we see that $\{\beta\colon r \cup \{(\alpha, n, \beta)\} \in U\} \subseteq \{\beta\colon q \cup \{(\alpha, n, \beta)\} \in U|\,a\} \in \Phi_\alpha$.
For (P6) let $q_1, q_2 \in U|\,a$ with $q_i = r_i|\,a$ for $i = 1, 2$. Again by (P7) and $q_i \approx p|\,a$ we can assume that $r_i = p \cup q_i$. Now if $q_1 \cup q_2 \in P_1$ then also $r_1 \cup r_2 \in P_1$, and since U satisfies (P6), we have $q_1 \cup q_2 = (r_1 \cup r_2)|\,a \in U|\,a$. Since $T_{r_1}|\,a \subseteq T_{r_2}|\,a$, if $r_1 \subseteq r_2$, it follows that $T^{U|a}_{q_1} = \bigcup_{r \in U\colon r \approx p \wedge r|\,a = q_1} T_r|\,a \subseteq \bigcup_{r \in U\colon r \approx p \wedge r|\,a = q_1} T_{r \cup r_2}|\,a \subseteq T^{U|a}_{q_2}$, if $q_1 \subseteq q_2$.
For (P10) let $q \in U|\,a$ with $q = t|\,a$ and t minimal with respect to cardinality. It must be that $t \not\approx p$, since $q \not\approx p|\,a$. Now $t = t^- \cup \{(\alpha, n, \beta)\}$. We see that $(\alpha, n, \beta) \in q$ due to the minimality of t. It easily follows that $(\alpha, n) = (\alpha(q), n(q))$ and $q = t^-|\,a \cup \{(\alpha, n, \beta)\}$, so that $q^- = t^-|\,a \in U|\,a$. □

Lemma 2. *If (p, U) is a forcing condition and $s \in U$, then (s, U_s) is a forcing condition too, where $U_s = \{t \in U\colon s \subseteq t\}$.*

Proof. Only (P5)-(P7) are non-trivial. For (P5) let $q \in U_s$ and $q \approx s$. Since U satisfies (P7), we have $p \cup q|\,\kappa_1 \in U$. Let $(\alpha, n) \in (\mathrm{dom}_1(p) \cap \kappa_1) \times \omega - \mathrm{dom}_{1,2}(q)$. Then we have that $E = \{\beta\colon p \cup q|\,\kappa_1 \cup \{(\alpha, n, \beta)\} \in U\} \in \Phi_\alpha$. Now since $\forall \beta \in E\colon T_{p \cup q|\,\kappa_1} \subseteq T_{p \cup q|\,\kappa_1 \cup \{(\alpha, n, \beta)\}}$, we see that $\forall \beta' \in E\colon q \cup \{(\alpha, n, \beta')\} \in U$. Therefore it holds that $E \subseteq \{\beta\colon q \cup \{(\alpha, n, \beta)\} \in U_s\} \in \Phi_\alpha$.

For (P6) let $r_1, r_2 \in U_s$ and $r_i \approx s$ for $i = 1, 2$. Again by (P7), $p \cup r_i | \kappa_1 \in U$ for $i = 1, 2$. It follows that $(p \cup r_1 | \kappa_1) \cup (p \cup r_2 | \kappa_1) \in P_1$, since we assume that $r_1 \cup r_2 \in P_1$. Therefore $(p \cup r_1 | \kappa_1) \cup (p \cup r_2 | \kappa) \in U$. By (P6) $r_1 \cup r_2 \in T_{p \cup (r_1 \cup r_2) | \kappa_1}$, so that $r_1 \cup r_2 \in T^{U_s}_{s \cup (r_1 \cup r_2) | \kappa_1}$. It easily follows that if $r_1 \subseteq r_2$, then $T^{U_s}_{r_1} \subseteq T^{U_s}_{r_2}$.
For (P7) let $q \in U_s$ and let $a \subseteq \kappa_1 \times \omega \times \kappa_1$. We have that $p \cup (q \cap a) \in U$ and therefore also $p \cup (q \cap a) \cup s | \kappa_1 \in U$. By (P6) it follows that $s \cup (q \cap a) \in T_{p \cup (q \cap a) \cup s | \kappa_1}$ and so it follows that $s \cup (q \cap a) \in U_s$. $\square$

Definition 2.3. If $p \in P_2$, $b \supseteq \mathrm{dom}_1(p)$ such that b is closed under cf' and $b \subseteq \mathrm{Reg}$ finite, then we call $p' \in P_2$ a <u>b-extension</u> of p, if $\mathrm{dom}_1(p') = b$ and $p' | \mathrm{dom}_1(p) = p$.

Lemma 3. *Let $(p, U) \in P_3$, $b \supseteq \mathrm{dom}_1(p)$, closed under* cf'*, and $b \subseteq \mathrm{Reg}$ finite. Let p' be a b-extension of p and set $U' := \{q' \in P_2 \colon q' \supseteq p' \wedge \exists q \in U\, [q' \text{ is a } b\text{-extension of } q]\}$. Then (p', U') is a condition.*

Proof. Straightforward checking of the conditions (P1)-(P10). $\square$

Lemma 4. *If (p, U), $(p, V) \in P_3$ then $(p, U \cap V)$ is a condition too.*

Proof. We note that $q \in P_2$ being extendable only depends on $\mathrm{dom}_{1,2}(q)$. All the conditions follow, since filters are closed under intersection. $\square$

We can now define the partial order of the forcing.

Definition 2.4. Let (p, U), $(q, V) \in P_3$. We say that (q, V) is stronger than (p, U), in terms $(q, V) \geq (p, U)$, if $V | \mathrm{dom}_1(p) \subseteq U$.

3 The symmetric extension

We consider the group of partial permutations of $\mathrm{Reg} \times \omega \times \mathrm{On}$. We define a subclass Gr as the permutations π satisfying:

1. $|\mathrm{dom}_1(\mathrm{dom}(\pi))| < \omega$.
2. For every $\alpha \in \mathrm{dom}_1(\mathrm{dom}(\pi))$ there is a permutation π^α of α with finite domain, such that $\forall n < \omega$: 'If $\beta \in \mathrm{dom}(\pi^\alpha)$ then $\pi((\alpha, n, \beta)) = (\alpha, n, \pi^\alpha(\beta))$, and $\pi((\alpha, n, \beta)) = (\alpha, n, \beta)$ otherwise'.

If $a \subseteq \mathrm{Reg}$ finite we define

$$H_a := \{\pi \in Gr \colon \forall \alpha \in a \cap \mathrm{dom}_1(\mathrm{dom}(\pi))\; [\pi^\alpha \text{ is the identity function}]\}.$$

We easily see that H_a is a normal subgroup of Gr. Furthermore, for each $\pi \in Gr$ we define a dense subclass $P^\pi \subseteq P_3$ as the forcing conditions (p, U) with the following properties:

1. $\mathrm{dom}_1(p) \supseteq \mathrm{dom}_1(\mathrm{dom}(\pi))$.

2. $\forall \alpha \in \mathrm{dom}_1(p)\colon \mathrm{dom}(p(\mathrm{cf}'\,\alpha)) = \mathrm{dom}(p(\alpha))$.

3. $\forall \alpha \in \mathrm{dom}_1(\mathrm{dom}(\pi))\colon \mathrm{rng}(p(\alpha)) \supseteq \{\beta \in \mathrm{dom}(\pi^\alpha)\colon \exists q \in U \beta \in \mathrm{rng}(q(\alpha))\}$.

The density follows easily.

For a $(p, U) \in P^\pi$ we define $\pi((p, U))$ to be $(\pi p, \pi U)$ with

$$\pi p := \pi[p|\,\mathrm{dom}_1(\mathrm{dom}(\pi))] \cup p - p|\,\mathrm{dom}_1(\mathrm{dom}(\pi)) \quad \text{and}$$

$$\pi U := \{\pi[q|\,\mathrm{dom}_1(\mathrm{dom}(\pi))] \cup q - q|\,\mathrm{dom}_1(\mathrm{dom}(\pi))\colon q \in U\}.$$

The reason why we restrict ourselves to P^π is the following lemma.

Lemma 5. *For every $\pi \in Gr$ the mapping $(p, U) \mapsto (\pi p, \pi U)$ is an automorphism of $(P^\pi, \geq)$.*

Proof. First we need to check that $(\pi p, \pi U)$ is a forcing condition. Only condition (P9) is non-trivial. Let $q \in U$ be extendable and we note that $(\alpha(q), n(q)) = (\alpha(\pi q), n(\pi q))$. Now let $\mathrm{cf}'\,\alpha(q) < \alpha(q)$ and assume that $\gamma = q(\mathrm{cf}'\,\alpha(q))(n(q)) \in \mathrm{dom}(\pi^{\mathrm{cf}'\,\alpha(q)})$. Now it follows that $\gamma \in \mathrm{ran}(p(\mathrm{cf}'\,\alpha(q)))$ and that $(\mathrm{cf}'\,\alpha(q), n(q)) \notin \mathrm{dom}_{1,2}(p)$. But this is a contradiction to $q(\mathrm{cf}'\,\alpha(q))$ being 1-1. Therefore $\gamma \notin \mathrm{dom}(\pi^{\mathrm{cf}'\,\alpha(q)})$ and we see that

$$\{\beta\colon \pi q \cup (\alpha(q), n(q), \beta) \in \pi U\} \supseteq \{\beta\colon q \cup (\alpha(q), n(q), \beta) \in U\} \setminus \mathrm{dom}(\pi^{\alpha(q)}) \in \Phi_{\alpha(q),\gamma} = \Phi_{\pi q}.$$

Since $\mathrm{dom}(\pi) = \mathrm{ran}(\pi)$ we immediately see that $(\pi p, \pi U) \in P^\pi$. Similarly, it follows that $P^{\pi^{-1}} = P^\pi$ and therefore the mapping is an automorphism. $\square$

We note that P_3 has a unique Boolean completion $\mathrm{RO}(P_3)$, since all M-definable antichains are sets in M, which we will show later. Therefore, every $\pi \in Gr$ uniquely extends to an automorphism of $\mathrm{RO}(P_3)$.

Now let G be a M-generic subclass of P_3, i.e. G meets all M-definable dense subclasses, and denote the generic extension by $M[G]$.

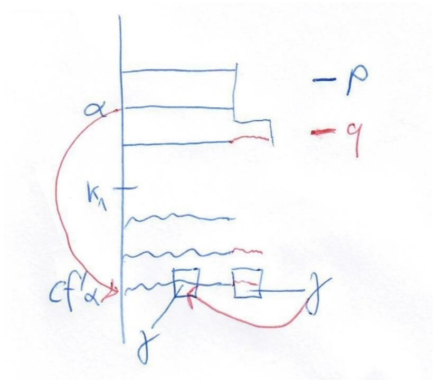
Definition 3.1. By N_G we shall denote the symmetric extension generated by the filter base $\{H_a\colon a \subseteq \mathrm{Reg} \text{ finite}\}$. In more detail: Call a name $\mathbf{x}$ symmetric iff $\exists a \subseteq \mathrm{Reg}$ finite such that $\mathrm{sym}(\mathbf{x}) = \{\pi \in Gr\colon \widetilde{\pi}(\mathbf{x}) = \mathbf{x}\} \supseteq H_a$, where $\widetilde{\pi}$ is defined recursively by $\widetilde{\pi}(\mathbf{x}) := \{(\widetilde{\pi}(\boldsymbol{\sigma}), \pi(p))\colon (\boldsymbol{\sigma}, p) \in \mathbf{x}\}$. Define HS as the class of all hereditarily symmetric names. Set the symmetric extension $N_G := \{\mathbf{x}^G\colon \mathbf{x} \in HS\}$ (see Jech[J]).

It can be easily seen that $M[G]$ and N_G are models of Extensionality, Pairing, Union and Infinity.

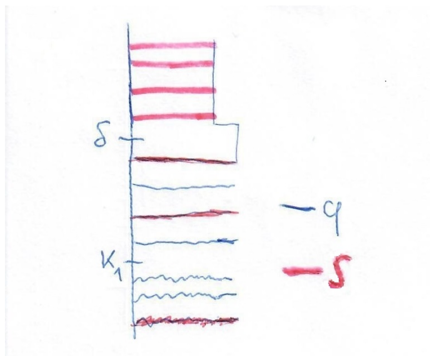
For $\alpha \in \mathrm{Reg}$ we shall set $P_\alpha := \{(p, U) \in P_3\colon \mathrm{dom}_1(p) \subseteq \alpha\}$ and $G_\alpha = G \cap P_\alpha$.

Lemma 6. $\forall \alpha \in \mathrm{Reg}$: P_α *is a complete subforcing of* P_3, $M[G] = \bigcup_{\alpha \in \mathrm{Reg}} M[G_\alpha]$ *and* $N_G = \bigcup_{\alpha \in \mathrm{Reg}} N_{G_\alpha}$.

Proof. We shall show that every maximal antichain in P_α, $\alpha \in \mathrm{Reg}$ arbitrary, is also maximal in P_3. Let A be a maximal antichain in P_α and $(p, U) \in P_3$. W.l.o.g. $(p|\,\alpha, U|\,\alpha) \neq (\emptyset, \{\emptyset\})$ and it follows that $(p|\,\alpha, U|\,\alpha)$ is compatible with some element $r \in A$. Let $(q, V) \geq (p|\,\alpha, U|\,\alpha), r$ with $(q, V) \in P_\alpha$ and w.l.o.g. we can assume that $\exists m \in \omega\, \forall \alpha' \in \mathrm{dom}_1(q)\colon \mathrm{dom}(q(\alpha')) = m$. Let $t = q|\,(\mathrm{dom}_1(p) \cap \alpha) \in U|\,\alpha$ and let $s \in U$ with $s|\,\alpha = t$. We note that $(s, U_s) \geq (p, U)$. Now $s \cup q \in P_2$, $(s \cup q)|\,\mathrm{dom}_1(q) = q$ and $(s \cup q)|\,\mathrm{dom}_1(s) = s$, since $s|\,\alpha = q|\,\mathrm{dom}_1(s)$. Therefore $(s \cup q, W) \geq (q, V)$ as a b-extension of (q, V) and $(s \cup q, X) \geq (s, U_s)$ as a b-extension of (s, U_s). So we see that $(s \cup q, W \cap X) \geq (q, V)$, $(s, U_s) \geq r$, (p, U).
The second statement follows easily, since every (symmetric) P_3-name is a (symmetric) P_α-name for some $\alpha \in \mathrm{Reg}$, and G_α is an M-generic filter of P_α. $\square$



Lemma 5



Lemma 6

Note that the definability of the forcing relation and the forcing theorem are non-trivial, but we will take care of these technicalities later. The following lemma is a generalization of the Symmetry lemma. It refers to $\Vdash$ as well as to $\Vdash^{HS}$.

Lemma 7. *Let* $\varphi(\mathbf{x_1}, ..., \mathbf{x_n})$ *be a formula with* $\mathbf{x_1}, ..., \mathbf{x_n} \in HS$. *Let* $a \subseteq \mathrm{Reg}$ *finite such that* a *is closed under* cf' *and* $\mathrm{sym}(\mathbf{x_1}, ..., \mathbf{x_n}) \supseteq H_a$. *If* $(p, U) \Vdash \varphi(\mathbf{x_1}, ..., \mathbf{x_n})$, *then already* $(p|\,a, U|\,a) \Vdash \varphi(\mathbf{x_1}, ..., \mathbf{x_n})$.

Proof. Suppose not. Then there is $(q, V) \geq (p|\,a, U|\,a)$ and $(q, V) \Vdash_{\mathrm{RO}(P_3)} \neg\varphi(\mathbf{x_1}, ..., \mathbf{x_n})$. Again, let q be of the form $\exists m \in \omega\, \forall \alpha \in \mathrm{dom}_1(q)\colon \mathrm{dom}(q(\alpha)) = m$. It will now suffice to show that there are conditions $(p', U') \geq (p, U)$ and $(q', V') \geq (q, V)$ and a permutation $\pi \in H_a$ with $(p', U') \in P^\pi$ such that $\pi((p', U')) = (q', V')$. This will yield a contradiction, since $\pi((p', U')) \Vdash_{\mathrm{RO}(P_3)} \varphi(\mathbf{x_1}, ..., \mathbf{x_n})$.
Since $(q, V) \geq (p|\,a, U|\,a)$ it follows that $q|\,(a \cap \mathrm{dom}_1(p)) \in U|\,a$. Let $t \in U$ with $t|\,a = q|\,(a \cap \mathrm{dom}_1(p))$ and such that $t \cup q|\,a \in P_2$. We set $p^\star = t \cup q|\,a$ and let $(p^\star, U^\star) \geq (p, U)$,

as a b-extension, with $p^\star|\, a = q|\, a$. We can now extend $(p^\star, U^\star)$ and (q, V) to (p_1, U_1) and (q_1, V_1) such that $\mathrm{dom}_1(p_1) = \mathrm{dom}_1(q_1)$,

$$\forall\alpha \in \mathrm{dom}_1(p_1)\colon \mathrm{dom}(p_1(\alpha)) = \mathrm{dom}(p_1(\mathrm{cf'}\,\alpha)) = \mathrm{dom}(q_1(\mathrm{cf'}\,\alpha)) = \mathrm{dom}(q_1(\alpha)),$$

and still $p_1|\, a = q_1|\, a$, since the filters are closed under intersection. We define

$$U_1' := \{t \in U_1 \colon \forall\alpha \in \mathrm{dom}_1(p_1) - a \quad [\mathrm{ran}(t(\alpha)) - \mathrm{ran}(p_1(\alpha)) \cap \mathrm{ran}(q_1(\alpha)) = \emptyset]\} \text{ and}$$

$$V_1' := \{t \in V_1 \colon \forall\alpha \in \mathrm{dom}_1(q_1) - a \quad [\mathrm{ran}(t(\alpha)) - \mathrm{ran}(q_1(\alpha)) \cap \mathrm{ran}(p_1(\alpha)) = \emptyset]\}.$$

We can now define the permutation π: For every $\alpha \in \mathrm{dom}_1(p_1) - a$ and $n \in \omega$ we set $\pi^\alpha(p_1(\alpha)(n)) = q_1(\alpha)(n)$, if defined. Since $q_1(\alpha)$ is 1-1, we can extend every π^α to a finite permutation. Of course, it holds that the resulting $\pi \in Gr$. Furthermore, one can easily show that $(p_1, U_1') \in P^\pi$. We set $(q', V') := (\pi p_1', \pi U_1' \cap V_1') \geq (q_1, V_1')$ and $(p', U') := \pi^{-1}((\pi p_1', \pi U_1' \cap V_1')) \geq (p_1', U_1')$. □

4 Separation and Replacement

Now we want to introduce three new predicates for the model $M[G]$: $B(x_1)$ will assert that $x_1 \in M$, $A(x_1, x_2)$ will assert that $(x_1, x_2) \in G$ and $WO(x_1, x_2)$ will assert that $x_1, x_2 \in M$ and $(x_1, x_2) \in WO_M$.

Furthermore, we extend the forcing language by the following predicates:

- $t \Vdash \check{B}(\mathbf{x_1})$ iff $\forall t' \geq t\, \exists t'' \geq t'\, \exists y \colon t'' \Vdash \mathbf{x_1} = \check{y}$.
- $t \Vdash \check{A}(\mathbf{x_1}, \mathbf{x_2})$ iff
  $\forall t' \geq t\, \exists t'' \geq t'\, \exists y_1, y_2 \colon (y_1, y_2) \in P_3 \wedge t'' \Vdash \mathbf{x_1} = \check{y_1} \wedge t'' \Vdash \mathbf{x_2} = \check{y_2} \wedge t'' \geq (y_1, y_2)$.
- $t \Vdash \check{WO}(\mathbf{x_1}, \mathbf{x_2})$ iff
  $\forall t' \geq t\, \exists t'' \geq t'\, \exists y_1, y_2 \colon WO_M(y_1, y_2) \wedge t'' \Vdash \mathbf{x_1} = \check{y_1} \wedge t'' \Vdash \mathbf{x_2} = \check{y_2}$.

Lemma 8. *The forcing relation for the expanded language is definable. Furthermore, let $\varphi(x_1, ..., x_n)$ be a formula in the language $(\in, =, A, B, WO)$. Then $M[G] \vDash \varphi(\mathbf{x_1}^G, ..., \mathbf{x_n}^G)$ iff $\exists t \in G \colon t \Vdash \check{\varphi}(\mathbf{x_1}, ..., \mathbf{x_n})$.*

Proof. Define the forcing relation for atomic formulas as follows: $p \Vdash \mathbf{x} \in \mathbf{y}$ iff

$$\exists\alpha \in \mathrm{Reg}\, \forall\alpha' \geq \alpha \colon p \in P_{\alpha'} \wedge p \Vdash_{P'_\alpha} \mathbf{x} \in \mathbf{y}.$$

Similarly for $\mathbf{x} = \mathbf{y}$. This definition works, because Δ_0-formulas are absolute between transitive models. The general case is defined by the usual induction over the number of quantifiers, e.g. $p \Vdash \forall x\, \check{\varphi}(x)$ iff $\forall \mathbf{x} \in M^{P_3} \colon p \Vdash \check{\varphi}(\mathbf{x})$. For more details see Shoenfield [S] and Zarach [Z]. □

Lemma 9. *$M[G]$ can be definably well-ordered.*

Proof. First we note that $\forall\alpha \in \mathrm{Reg}\colon M[G_\alpha]$ is definable in $M[G]$. The definition is the same formula defining '$M[G_\alpha]$ as the forcing extension of M with poset P_α and filter G_α' in $M[G_\alpha]$, i.e.

$M[G_\alpha] = \{y\colon \exists \mathbf{z} \in M^{P_\alpha}\, \phi_{G_\alpha}(\mathbf{z}, y)\}$ where $\phi_{G_\alpha}(\mathbf{z}, y)$ is the formula for $\mathbf{z}^{G_\alpha} = y$ in $M[G_\alpha]$.

Note that $\phi_{G_\alpha}(\mathbf{z}, y)$ is of the form $\exists x\, \psi(x, \mathbf{z}, y)$ and $\psi(x, \mathbf{z}, y)$ is Δ_0. By the upward absoluteness of Σ_1-formulas $M[G] \vDash \forall \mathbf{z} \in M^{P_\alpha}\, \exists y\colon \phi_{G_\alpha}(\mathbf{z}, y)$. Finally, one can easily check by induction that $M[G] \vDash \forall \mathbf{z} \in M^{P_\alpha}\, \exists! y\colon \phi_{G_\alpha}(\mathbf{z}, y)$ and that the recursion is absolute.
For $x \in M[G]$ we define $\Delta(x) := \min\{\alpha \in \mathrm{Reg}\colon x \in M[G_\alpha]\}$. For $x, y \in M[G]$ we say $x < y$ iff either $\Delta(x) < \Delta(y)$ or

$$\Delta(x) = \Delta(y) \text{ and } \min_{WO}\{\mathbf{z} \in M^{P_3}\colon \mathbf{z}^G = x\} <_{WO} \min_{WO}\{\mathbf{z} \in M^{P_3}\colon \mathbf{z}^G = y\}.$$

□

Theorem 10. *$M[G]$ satisfies Collection, i.e.*

$$\forall a\colon \forall x \in a\, \exists y\, \varphi(x, y) \to \exists b\, \forall x \in a\, \exists y \in b\ \varphi(x, y),$$

for any formula $\varphi(x, y)$ in the language $(\in, =, A, B, WO)$.

Proof. First we show that every definable antichain of P_3 is a set in M. Suppose not. Let $((p_\alpha, U_\alpha))_{\alpha\in\mathrm{On}}$ be an M-definable antichain. W.l.o.g. we can assume that $|\mathrm{dom}_1(p_\alpha)|$ is independent of α and $\mathrm{dom}(p_\alpha(\alpha')) \subseteq \omega$ is independent of α and α'. Let $f_i(\alpha)$ denote the ith $\alpha' \in \mathrm{dom}_1(p_\alpha)$ for $i < |\mathrm{dom}_1(p_\alpha)|$. Now there must be a i such that $\{f_i(\alpha)\colon \alpha \in \mathrm{On}\}$ is unbounded. Choose i minimal. Let $\beta = \sup\{f_j(\alpha)\colon j < i \wedge \alpha \in \mathrm{On}\}$ and choose $((p'_\alpha, U'_\alpha))_{\alpha\in\mathrm{On}}$ such that $\mathrm{dom}_1(p'_\alpha)$ are pairwise disjoint above β. Now choose $((p''_\alpha, U''_\alpha))_{\alpha\in\mathrm{On}}$ such that $p''_\alpha|\,\beta^+$ are identical. But this is a contradiction, since now all (p''_α, U''_α) are compatible.

Now to Collection: Let $\varphi(x, y)$ be a given formula, and $a \in M[G]$. Let us assume that $M[G] \vDash \forall x \in a\, \exists y\colon \varphi(x, y)$. For $x \in a$ let $\psi(x, \alpha)$ denote 'α is the least regular cardinal such that $\exists y \in R_\alpha^{M[G_\alpha]}\colon \varphi(x, y)$'. Note that for every $\mathbf{x} \in \mathrm{dom}(\mathbf{a})$ there is a $\beta_{\mathbf{x}}$ such that $\forall \gamma \geq \beta_{\mathbf{x}}\, \forall t \in P_3\colon t \Vdash \neg\check{\psi}(\mathbf{x}, \check{\gamma})$, since otherwise we would obtain a M - definable antichain which would not be a set in M. Now we use Replacement in M to define $\beta = \sup\{\beta_{\mathbf{x}}\colon \mathbf{x} \in \mathrm{dom}(\mathbf{a})\}$. Let α be the least regular cardinal $\geq \beta$. Now $M[G] \vDash \forall x \in a\, \exists y \in R_\alpha^{M[G_\alpha]}\colon \varphi(x, y)$. It follows that $R_\alpha^{M[G_\alpha]} \in M[G_\alpha] \subseteq M[G]$, since $M[G_\alpha]$ satisfies ZFC. □

Theorem 11. *$M[G]$ satisfies Separation for any formula in the language $(\in, =, A, B, WO)$.*

Proof. Let $\varphi(x)$ be a given formula, and $a \in M[G]$. For $(\mathbf{x}, p) \in \mathbf{a}$ we define $A_{(\mathbf{x},p)}$ to be a maximal antichain below p such that $\forall t \in A_{(\mathbf{x},p)} \colon t \Vdash \check{\varphi}(\mathbf{x})$. We set $\tau := \bigcup_{(\mathbf{x},p)\in\mathbf{a}} \{\mathbf{x}\} \times A_{(\mathbf{x},p)}$. We are left to show $M[G] \models \forall x \colon x \in \tau^G \leftrightarrow x \in a \wedge \varphi(x)$. Let $x \in \tau^G$. This means that there is a $(\mathbf{x}, q) \in \tau$ with $q \in G$ and $(\mathbf{x}, p) \in \mathbf{a}$ with $q \geq p$ and $q \Vdash \check{\varphi}(\mathbf{x})$. It follows that $x \in a$ and $\varphi(x)$. Now let $x \in a$ and $\varphi(x)$. It follows that there is a $(\mathbf{x}, p) \in \mathbf{a}$ with $p \in G$. Therefore, G must meet with $A_{(\mathbf{x},p)}$, so that $x \in \tau^G$. □

Now Replacement easily follows from Collection and Separation in $M[G]$.

Replacement in N_G can be shown similarly, although one has to work for Separation to show that the name τ is symmetric. A more elegant proof uses replacement in $M[G]$ and the fact that $N_G \models$ Power Set, which we will show without using any axioms in N_G.

Theorem 12. *N_G satisfies Replacement for any formula in the language $(\in, =)$.*

Proof. Let $\varphi(x, y)$ be a given formula, $a \in N_G$ and $N_G \models \forall x \in a \, \exists! y \colon \varphi(x, y)$. Now $N_G = \bigcup_{\alpha\in\mathrm{On}}(R_\alpha^{N_G})$ is a definable subclass of $M[G]$. Note that $R_\alpha^{N_G} \in N_G$ only because $N_G \models$ Power Set and if λ is a limit ordinal then $R_\lambda^{N_G} \in M[G]$ and $R_\lambda^{N_G} \subseteq N_{G_\alpha}$ for some $\alpha \in \mathrm{Reg}$. Since $N_{G_\alpha} \models$ ZF and rank is absolute, it follows that $R_\lambda^{N_G} \in N_{G_\alpha} \subseteq N_G$. Since $M[G] \models$ Replacement, we can now use Reflection. So we get an α such that $R_\alpha^{N_G} \preccurlyeq_{\varphi,\psi} N_G$ and $a \in R_\alpha^{N_G}$, where $\psi(z)$ is the formula $\forall x \in z \, \exists! y \colon \varphi(x, y)$. Let α' be the least regular cardinal such that $R_\alpha^{N_G} \in N_{G_{\alpha'}}$. Finally we use Separation in $N_{G_{\alpha'}}$ to get $b = \{y \in R_\alpha^{N_G} \colon \exists x \in a \, \varphi^{R_\alpha^{N_G}}(x, y)\} \in N_{G_{\alpha'}} \subseteq N_G$. □

5 The axiom of Power Set in $\mathrm{N_G}$

The problem with Power set is that it does not hold in $M[G]$. Indeed, since all limit ordinals have cofinality ω and AC holds within $M[G]$, every set must be countable in $M[G]$. So even the reals form a proper class, and so do all the R_α's for $\alpha > \omega$.

Let $\mathbf{x} \in HS$ be a hereditarily symmetric name. First we define the support of such a name: $\mathrm{supp}(\mathbf{x}) := \min_\subseteq\{a \subseteq \mathrm{Reg} \text{ finite} \colon \mathrm{sym}(\mathbf{x}) \supseteq H_a\}$. Note that this minimum exists, since if $\pi \in H_{a_1\cap a_2}$ then $\pi = \pi_2 \circ \pi_1$ with $\pi_i \in H_{a_i}$. So if $\mathbf{x}$ is supported by a_1 and a_2, then it is also supported by $a_1 \cap a_2$.

Let $x \in N_G$. In contrast to the other axioms, it might be that the supports of names $\mathbf{y}$ for subsets of x are cofinal in the ordinals, such that at each α new subsets arise. Therefore, we shall show that $\mathcal{P}^{N_G}(x) \in M[G]$, which will yield Power Set, since with Replacement in $M[G]$ we get an α with $\mathcal{P}^{N_G}(x) \subseteq N_{G_\alpha}$ and $x \in N_{G_\alpha}$, and since $N_{G_\alpha} \models$ ZF and being a subset of x is absolute for transitive models, we get $\mathcal{P}^{N_G}(x) \in N_{G_\alpha} \subseteq N_G$.

Let $\mathbf{x}$ be a symmetric name for x. We define $\delta := \sup\{\bigcup \mathrm{supp}(\mathbf{z}) \cup \bigcup \mathrm{supp}(\mathbf{x})\colon \mathbf{z} \in \mathrm{dom}(\mathbf{x})\}$. Let κ be the smallest strongly compact cardinal $> \delta^+$. We shall call $\alpha < \kappa$ small coordinates and $\alpha \geq \kappa$ big ones.

Lemma 13. *Let (p, U) be a forcing condition. Then there is a $(q, V) \geq (p, U)$ such that the following conditions hold:*

- $\forall t \in V\colon \alpha(t) \geq \kappa \Rightarrow \Phi_t$ *is κ-complete.*
- *for every $s \in V$ such that s is extendable and $\alpha(s) < \kappa$ the set*

  $$E_s := \{\beta \in \alpha(s)\colon s \cup \{(\alpha(s), m(s), \beta)\} \in V\}$$

  of possible extensions is independent of the values at the big coordinates, i.e. if $q_1| \kappa = q_2| \kappa$ then $E_{q_1} = E_{q_2}$.

Proof. First we want to find $(p', W) \geq (p, U)$ with $p' \in U$ and $n \in \omega$ such that $\forall \alpha \in \mathrm{dom}_1(p')\colon \mathrm{dom}(p'(\alpha)) = n$ and $\forall t \in W\colon \alpha(t) \geq \kappa \Rightarrow \Phi_t$ is κ-complete. Find such a p' and consider the condition $(p', U_{p'})$. For $\alpha \in \mathrm{dom}_1(p')$ with $\gamma = \mathrm{cf'}\,\alpha < \alpha$ and $\alpha \geq \kappa$ let $c_{\gamma,\alpha} := \min\{\nu \in \gamma\colon \gamma_\alpha(\nu) \geq \kappa\}$. Note that γ_α was the WO_M minimal cofinal sequence of length γ in $\sup\{\kappa'\colon \kappa' < \alpha \wedge \kappa' \text{is strongly compact}\}$. For $\gamma \in \mathrm{dom}_1(p')$ we define $c_\gamma := \max\{c_{\gamma,\alpha}\colon \alpha \in \mathrm{dom}_1(p') \wedge \alpha \geq \kappa \wedge \gamma = \mathrm{cf'}\,\alpha < \alpha\}$. We define $W := \{t \in U_{p'}\colon \forall \gamma \in \mathrm{dom}_1(p')\, \forall m \geq n\, t(\gamma)(m) \geq c_\gamma\}$. (p', W) is now a condition with the required property. So w.l.o.g we can assume that the condition (p, U) also has this property.

Choose $q \in U$ with $\alpha(q) \geq \kappa$. We will now work with the condition (q, U_q). Let $r \in U_q$, $r \approx q$. Let $k_r \leq \omega$ denote the number of levels of the tree T_r. For $i < k_r$ let T_r^i denote the tree, where we cut off T_r above the i'th level. We set $F_r^i := T_r^i \backslash T_r^{i-1}$. Let $x_r(i) := \mathrm{dom}_{1,2}(s)$ for $s \in F_r^i$. This is obviously well defined. We fix i and shall inductively define $T_r^{i,j}$ for $1 \leq j \leq i$.

We set $T_r^{i,i} := T_r^i$. Now let $T_r^{i,j}$ be defined. We set

$$V_r^{i,j} := \{s \in T_r^{i-1}\colon \forall n \in \{j, ..., i-1\}\, s| x_r(n) \in T_r^{i,n}\}.$$

We assume as induction hypothesis that for $s, t \in V_r^{i,j}$ and $s| x_r(j) \in F_r^j$ if $s| x_r(j-1) = t| x_r(j-1)$, $s| \kappa = t| \kappa$ and $\alpha(s) < \kappa$, then we have $E_s = E_t$.

Now let $s \in F_r^{j-1}$. If $\alpha(s^-) < \kappa$ we set $T_r^{i,j-1} := T_r^{j-1}$. Obviously, for $t, u \in V_r^{i,j-1}$ and $t| x_r(j-1) \in F_r^{j-1}$ if $t| x_r(j-2) = u| x_r(j-2)$, $t| \kappa = u| \kappa$ and $\alpha(t) < \kappa$, then we have $E_t = E_u$, since we can deduce $t| x_r(j-1) = u| x_r(j-1)$ and use the induction hypothesis. Note that if $t \in F_r^{j-1}$ then $t = u$.

If $\alpha(s^-) \geq \kappa$ we do the following: First fix

$$f \in C_{s^-} := \{g \in P_2\colon \exists h \in V_r^{i,j}\, [h \supseteq s^- \wedge h| \kappa = g]\}.$$

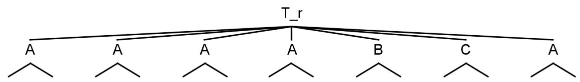
Let $t \supseteq s^-$, $t \in V_r^{i,j}$ with $t|\, x_r(j-1) \in F_r^{j-1}$ and $t|\, \kappa = f$. We know that if $\alpha(t) < \kappa$, then E_t only depends on $t(\alpha(s^-))\,(n(s^-))$. Since $|\mathcal{P}(\alpha(t))| < \kappa$ and Φ_{s^-} is κ-complete, there exists an $A \in \Phi_t$ such that

$$D_{s^-,f} := \{\beta \in \alpha(s^-) \colon \forall u \supseteq s^- \cup \{(\alpha(s^-), n(s^-), \beta)\}\, [u|\, \kappa = f \wedge \alpha(u) < \kappa \Rightarrow E_u = A]\} \in \Phi_{s^-}.$$

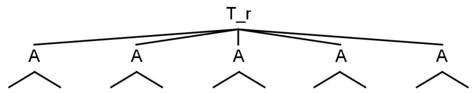
Note that A, $(\alpha(t), n(t))$ and Φ_t only depend on f. We define $D_{s^-} := \bigcap_{f \in C_{s^-}} D_{s^-,f} \in \Phi_{s^-}$. We set $T_r^{i,j-1} := T_r^{j-2} \cup \bigcup_{s \in F_r^{j-1}} \{s^- \cup \{(\alpha(s^-), n(s^-), \beta)\} \colon \beta \in D_{s^-}\}$. Again we see that for $t, u \in V_r^{i,j-1}$ and $t|\, x_r(j-1) \in F_r^{j-1}$ if $t|\, x_r(j-2) = u|\, x_r(j-2)$, $t|\, \kappa = u|\, \kappa$ and $\alpha(t) < \kappa$, then we have $E_t = E_u$, since we can deduce that $\exists s \in F_r^{j-1} \colon t, u \supseteq s^-$ and for $f = t|\, \kappa = u|\, \kappa$ the set $D_{s^-} \subseteq D_{s^-,f}$ is homogeneous.

We set $V_r' := \{t \in T_r \colon \forall i < k_r\, [t \in T_r^i \Rightarrow t \in V_r^{i,1}]\}$ and $V' := \bigcup_{r \in U_q \colon r \approx q} V_r'$. Now let $s, t \in V'$ with $s|\, \kappa = t|\, \kappa$ and $\alpha(s) < \kappa$. Suppose that $E_s \neq E_t$. Then there exists a $r \in U_q$, $r \approx q$ and an i such that $s, t \in V_r^{i,1}$, which contradicts the homogeneity of $V_r^{i,1}$. (q, V') satisfies all conditions except possibly (P6). Therefore, we set $T_r^V := \bigcap_{s \supseteq r \colon s \approx q} V_s'$ and $V := \bigcup_{r \in U_q \colon r \approx q} T_r^V$. Obviously, if $s \supseteq r$ then $T_s^V \supseteq T_r^V$. So (q, V) is the desired condition. $\square$

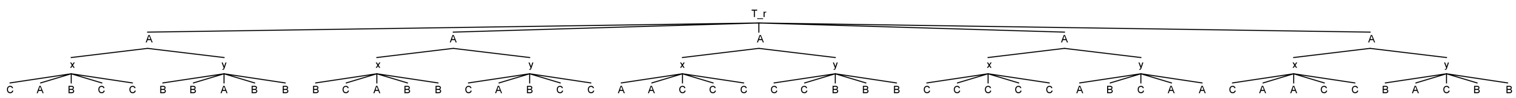
The following graphics show the process from the previous lemma with a condition whose domain consists only of one small and one big coordinate:



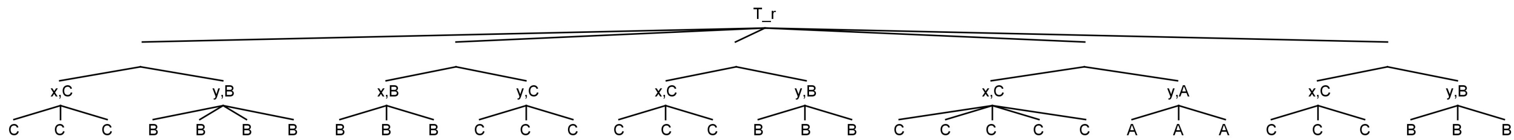
Tree at level 1



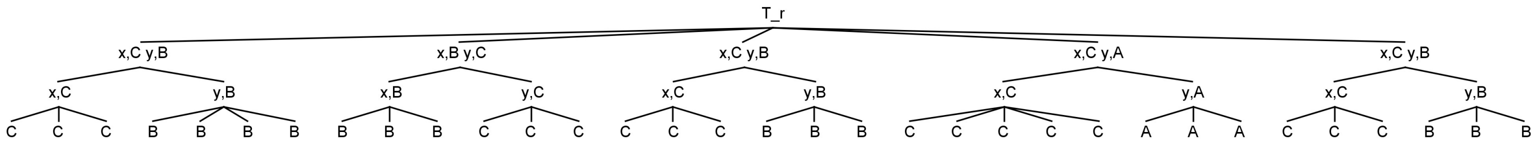
Tree made homogeneous at level 1



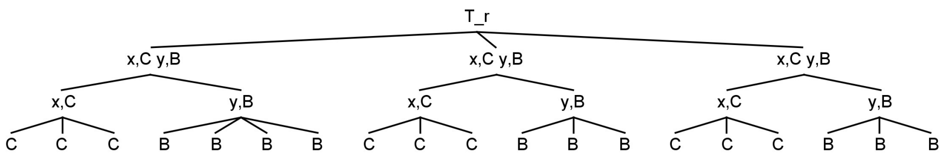
Tree at level 3, small extension at level 2, homogeneous at level 1



Tree made homogeneous at level 3, depending on small extension at level 2



Coloring level 1 with the possible small extensions and corresponding homogeneous sets



Tree made homogeneous at level 3 independent of small extension at level 2

Note that in the previous lemma, as well as in all the following lemmas the strengthened condition (q, V) has the following property: if $t \in V$ with $\alpha(t) < \kappa$, then $\{\beta \in \alpha(t) \colon t \cup \{(\alpha(t), n(t), \beta)\} \in U\} = \{\beta \in \alpha(t) \colon t \cup \{(\alpha(t), n(t), \beta)\} \in V\}$.

The following lemma will give an idea why Power Set might hold.

Lemma 14. *The strongly compact cardinals κ are not collapsed in N_G, i.e. they remain cardinals.*

Proof. Let κ be strongly compact and let τ be a symmetric name such that $(p, U) \Vdash \tau \colon \lambda \to \kappa$ onto, with $\lambda < \kappa$, $\mathrm{dom}_1(p) \subseteq \mathrm{supp}(\tau)$ and (p, U) has the properties from the previous lemma.

Let $\varphi_{n,\beta}$ be the formula $(\check{n}, \beta) \in \tau$ for $n \in \lambda$ and $\beta \in \kappa$. Now we want to make V homogeneous: Find a $W \subseteq V$ such that for $s, t \in W$, $s|\,\kappa = t|\,\kappa$ and $\mathrm{dom}_{1,2}(s) = \mathrm{dom}_{1,2}(t)$ the following holds:

- $\exists X \subseteq W_s \colon (s, X) \Vdash \neg\varphi_{n,\beta}$ iff $\exists X \subseteq W_s \colon (t, X) \Vdash \neg\varphi_{n,\beta}$
- $\exists Y \subseteq W_s \colon (s, Y) \Vdash \varphi_{n,\beta}$ iff $\exists Y \subseteq W_s \colon (t, Y) \Vdash \varphi_{n,\beta}$.

Note that (s,X) and (s,Y) cannot contradict each other, since they are compatible. We use the notation from the previous lemma. Let $r \in V$ with $r \approx p$. We fix $i < k_r$ and shall define $T_r^{i,j}$ for $0 \leq j \leq i$. Again we set $T_r^{i,i} := T_r^i$. Now let $T_r^{i,j}$ be defined. We set $W_r^{i,j} := \{s \in T_r^i \colon \forall n \in \{j, ..., i\}\, s|\, x_r(n) \in T_r^{i,n}\}$. Set $F_{n,\beta}(s)$ to be

- 0 if $\exists X \subseteq W_s \colon (s,X) \Vdash \neg\varphi_{n,\beta}$
- 1 if $\exists Y \subseteq W_s \colon (s,Y) \Vdash \varphi_{n,\beta}$
- 2 otherwise.

For $s,t \in W_r^{i,j}$, $s|\,\kappa = t|\,\kappa$, $\mathrm{dom}_{1,2}(s) = \mathrm{dom}_{1,2}(t)$ and $s|\,x_r(j) \in F_r^j$ we assume inductively that $F_{n,\beta}(s) = F_{n,\beta}(t)$.

Now let $s \in F_r^{j-1}$. If $\alpha(s) < \kappa$ we set $T_r^{i,j-1} := T_r^j$. Obviously, for $t,u \in W_r^{i,j-1}$ and $t|\,x_r(j-1) \in F_r^{j-1}$ if $t|\,x_r(j-1) = u|\,x_r(j-1)$, $t|\,\kappa = u|\,\kappa$ and $\mathrm{dom}_{1,2}(t) = \mathrm{dom}_{1,2}(u)$, then we have $F(t) = F(u)$, since we can prove $t|\,x_r(j) = u|\,x_r(j)$ and use the induction hypothesis. Note that if $t \in F_r^{j-1}$ then $t = u$.

If $\alpha(s) \geq \kappa$ we do the following: Let us fix $f \in C_s := \{g \in P_2 \colon \exists h \in W_r^{i,j}\, [h \supseteq s \wedge h|\,\kappa = g]\}$. Let $t \supseteq s$, $t \in W_r^{i,j}$ with $t|x_r(j) \in F_r^j$ and $t|\,\kappa = f$. We know that $F(t)$ only depends on $t(\alpha(s))(n(s))$ and $\mathrm{dom}_{1,2}(t)$. Therefore, for every $m \in \omega$ there is $l < 3$ such that

$D_{s,f,m} := \{\beta \in \alpha(s) \colon \forall u \supseteq s \cup \{(\alpha(s), n(s), \beta)\}\, [u|\,\kappa = f \wedge |\mathrm{dom}_{1,2}(u)| = m \Rightarrow F_{n,\beta}(u) = l]\}$

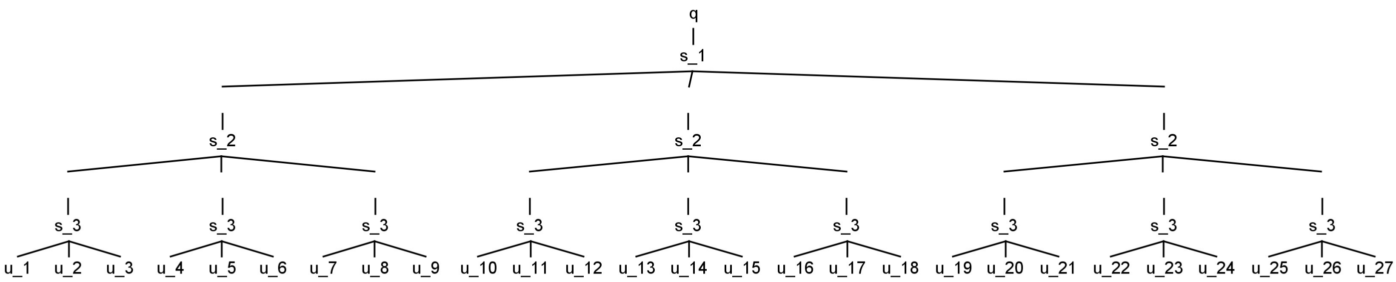
is in the filter $\in \Phi_s$. So $D_s := \bigcap_{f \in C_s} \bigcap_{m \in \omega} D_{s,f,m} \in \Phi_s$. We set

$$T_r^{i,j-1} := T_r^{j-1} \cup \bigcup_{s \in F_r^{j-1}} \{s \cup \{(\alpha(s), n(s), \beta)\} \colon \beta \in D_s\}.$$

We also set $W'_r := \{t \in T_r \colon \forall i < k_r\, [t \in T_r^i \Rightarrow t \in W_r^{i,0}]\}$ and $W'_{n,\beta} := \bigcup_{r \in V \colon r \approx q} W'_r$. Obviously, $W'_{n,\beta}$ is homogeneous with respect to $F_{n,\beta}$.

Choose $\nu \in \mathrm{Reg}$ with $\kappa > \nu > \lambda, |(V|\,\kappa) \times \aleph_0|$. Let $(p,W) \geq (p,V)$ be a condition which is homogeneous with respect to $F_{n,\beta}$ for every $n \in \lambda$ and $\beta \in \nu$. We get such a condition by intersecting $W'_{n,\beta}$ for $n \in \lambda$ and $\beta \in \nu$ and fixing (P6). It follows that there are only $|(V|\,\kappa) \times \aleph_0|$ equivalence classes (for $s,t \in W$ define $s \approx_{\Vdash} t$ iff for every $n \in \lambda$ and every $\beta \in \nu$ $F_{n,\beta}(s) = F_{n,\beta}(t)$ holds) of what can be forced below (p,W) . But then τ cannot even be onto ν, because for every $n \in \lambda$ and $\beta \in \nu$ the formula $(n,\beta) \in \tau$ must be decided below (p,W). $\square$

Definition 5.1. For $q \in V$ and $s \in V|\,\kappa$ with $q|\,\kappa \subseteq s$ we say that $q \cup s$ reaches V if either $q \cup s \in V$ or the set $V^\star_{q\cup s} := \{t \in T_{p \cup s|\,\kappa_1} \colon \exists u \in V\, [u \supseteq q \cup s \wedge u|\,\kappa = s \wedge t \subseteq u]\}$ satisfies the following property: $V^\star_{q\cup s}$ is non-empty and for $t \in V^\star_{q\cup s}$ if $q \subseteq t$, $\alpha(t) \geq \kappa$ and $t|\,\kappa \neq s$, then $\{\beta \in \alpha(t) \colon t \cup \{(\alpha(t), n(t), \beta)\} \in V^\star_{q\cup s}\} \in \Phi_t$.



A pruned tree: Only the corresponding values of s are allowed at the small extensions

Lemma 15. *Let (p, V) be a condition as in the lemma 13. Then there exists a $W \subseteq V$ such that (p, W) is a condition and for $s \in W|\,\kappa$ and $q \in W$ with $q|\,\kappa \subseteq s$ we have that $q \cup s$ reaches W.*

Proof. First we shall show that $\forall q \in V\colon p \cup q|\,\kappa$ reaches V. If $q = r$ with $r \approx p$ this is obvious. Now let $q \in V$ such that $p \cup q|\,\kappa$ reaches V: If $\alpha(q) \geq \kappa$ and q is extendable then obviously

$$\forall \beta \in \alpha(q)\colon q \cup \{(\alpha(q), n(q), \beta)\} \in V \Rightarrow p \cup (q \cup \{(\alpha(q), n(q), \beta)\})|\,\kappa$$

reaches V. If $\alpha(q) < \kappa$ and q is extendable, then let $\beta \in \alpha(q)$ with $q \cup \{(\alpha(q), n(q), \beta)\} \in V$ be arbitrary. If $p \cup q|\,\kappa \in V$ then $p \cup q|\,\kappa = q$, so that trivially $\cup(q \cup \{(\alpha(q), n(q), \beta)\})|\,\kappa \in V$. Otherwise let $u \supseteq p \cup q|\,\kappa$ with $u|\,\kappa = q|\,\kappa$, $u \in V$ and u is not extendable in $V^{\star}_{p \cup q|\,\kappa}$. It follows that $(\alpha(u), n(u)) = (\alpha(q), n(q))$. From lemma 13 we may assume that $E_u = E_q$, so that $u \cup \{(\alpha(q), n(q), \beta)\} \in V$. This shows that $V^{\star}_{p \cup q|\,\kappa} \subseteq V^{\star}_{p \cup (q \cup \{(\alpha(q), n(q), \beta)\})|\,\kappa}$. If $t \in V^{\star}_{p \cup (q \cup \{(\alpha(q), n(q), \beta)\})|\,\kappa}$ with $t|\,\kappa \neq (q \cup \{(\alpha(q), n(q), \beta)\})|\,\kappa$ and $\alpha(t) \geq \kappa$, then it easily follows that $t \in V^{\star}_{p \cup q|\,\kappa}$. Now if $t|\,\kappa \neq q|\,\kappa$ then $\{\beta \in \alpha(t)\colon t \cup \{(\alpha(t), n(t), \beta)\} \in V^{\star}_{p \cup q|\,\kappa}\} \in \Phi_t$ by definition. If $p \cup q|\,\kappa \subseteq t$ then $\{\beta \in \alpha(t)\colon t \cup \{(\alpha(t), n(t), \beta)\} \in V^{\star}_{p \cup q|\,\kappa}\} \in \Phi_t$ by definition of the forcing. So that $p \cup (q \cup \{(\alpha(t), n(t), \beta)\})|\,\kappa$ reaches V.

Inductively, we shall now define W' and show that $\forall q \in W' \, \forall s \in V|\,\kappa\colon q|\,\kappa \subseteq s \Rightarrow q \cup s$ reaches V. We put $r \approx p$ with $r \in V$ in W'. Let $s \in V|\,\kappa$ with $r|\,\kappa \subseteq s$. Since $r \cup s = p \cup s$ the statement follows from what we have shown before.

Now let $q \in W'$ and assume $\forall s \in V|\,\kappa\colon q|\,\kappa \subseteq s \Rightarrow q \cup s$ reaches V as the induction hypothesis. If q is extendable and $\alpha(q) < \kappa$ we put $q \cup \{(\alpha(q), n(q), \beta)\}$ with $q \cup \{(\alpha(q), n(q), \beta)\} \in V$ in W'. For $s \supseteq (q \cup \{(\alpha(q), n(q), \beta)\})|\,\kappa$, $s \in V|\,\kappa$ we see that $q \cup \{(\alpha(q), n(q), \beta)\} \cup s = q \cup s$ and we can us the induction hypothesis to show that $q \cup \{(\alpha(q), n(q), \beta)\} \cup s$ reaches V.

If $\alpha(q) \geq \kappa$, we fix $s \supseteq q|\,\kappa$ with $s \in V|\,\kappa$. Since $q \cup s$ reaches V, if $q \cup s \notin V$ then we have

$$C_s := \{\beta \in \alpha(q)\colon q \cup \{\alpha(q), n(q), \beta)\} \in V \wedge (q \cup \{(\alpha(q), n(q), \beta)\}) \cup s|\,\kappa_1 \in V^{\star}_{q \cup s}\} \in \Phi_q$$

by definition. It easily follows that $\forall\beta \in C_s\colon (q \cup \{(\alpha(q), n(q), \beta)\}) \cup s$ reaches V. Therefore, if we set $C := \bigcap_{s\in V|\,\kappa} C_s \in \Phi_q$, it follows that

$$\forall\beta \in C\,\forall s \in V|\,\kappa\colon (q \cup \{(\alpha(q), n(q), \beta)\}) \cup s$$

reaches V. Note that if $q\cup s \in V$, then $q|\,\kappa\cup s|\,\kappa_1 = s$ so that $(q\cup\{(\alpha(q), n(q), \beta)\})\cup s = (q\cup\{(\alpha(q), n(q), \beta)\})\cup s|\,\kappa_1 \in V$, if $q\cup\{(\alpha(q), n(q), \beta)\} \in V$. As in the previous lemma, (p, W') satisfies all conditions except possibly (P6). But as before, we can fix this to get a $W \subseteq W'$.

Now we show $\forall q \in W\,\forall s \in W|\,\kappa\colon q|\,\kappa \subseteq s \Rightarrow q \cup s$ reaches W. Let $q \in W$ and $s \in W|\,\kappa$ be arbitrary. If $q \cup s \in V$ and $\alpha(q) < \kappa$, we note that $q \cup s|\,\kappa_1 \in W$ and that $q \cup s|\,\kappa_1$ only has to be extended at small coordinates to become $q \cup s$, so with induction on $|q \cup s - q \cup (s|\,\kappa_1)|$ it can be shown that $q \cup s \in W'$. If $r \in V$ with $r \approx p$ and $r|\,\kappa_1 \supseteq (q \cup s)|\,\kappa_1$, then $q \cup r \in W \subseteq W'$. With the same argument as before, one can inductively show that $(q\cup s)\cup r = (q\cup r)\cup s \in W'$. It follows that $q\cup s \in W$.

If $q \cup s \notin V$ we shall show that $W^\star_{q\cup s} = V^\star_{q\cup s} \cap W$ which will obviously yield that $q\cup s$ reaches W. The $\subseteq$ inclusion is trivial. Now let $t \in V^\star_{q\cup s} \cap W$. We shall show that $\exists u \in W\colon u \supseteq q \cup s \wedge u|\,\kappa = s \wedge t \subseteq u$. If $t \supseteq q \cup s$ we are done. Otherwise we shall inductively define increasing $t_0, ..., t_n \in V^\star_{q\cup s} \cap W$ such that $t_n \supseteq q \cup s \wedge t_n|\,\kappa = s$. W.l.o.g. $t = q \cup s|\,\kappa_1$ and we set $t_0 = t$. Let t_i be defined, and assume that $t_i \not\supseteq q \cup s$, since otherwise we are done. Then we shall define t_{i+1} as follows: If $\alpha(t_i) \geq \kappa$ we set $t_{i+1} = t_i \cup \{(\alpha(t_i), n(t_i), \beta\}$ for some β which belongs to

$$\{\beta' \in \alpha(t_i)\colon t_i\cup\{(\alpha(t_i), n(t_i), \beta')\} \in V^\star_{q\cup s}\}\cap\{\beta' \in \alpha(t_i)\colon t_i\cup\{(\alpha(t_i), n(t_i), \beta')\} \in W\} \in \Phi_{t_i}.$$

If $\alpha(t_i) < \kappa$ then we see that $t_i \cup \{(\alpha(t_i), n(t_i), s(\alpha(t_i))\,(n(t_i))\,)\} \in V$, so that $t_{i+1} = t_i \cup \{(\alpha(t_i), n(t_i), s(\alpha(t_i))\,(n(t_i))\,)\} \in W$. $t_{i+1} \in V^\star_{q\cup s}$ follows trivially. Therefore (p, W) is the required condition.

□

In $M[G]$ we shall define a 1-1 mapping from $\mathcal{P}^{N_G}(x)$ onto $3^{\mathrm{dom}(\mathbf{x})\times P_2|\,\kappa}$. For $y \in \mathcal{P}^{N_G}(x)$ let $\mathbf{y}$ be the smallest symmetric name for y. For convenience in the proofs we will assume that $\max(\mathrm{supp}(\mathbf{y}) \cap \kappa) \geq \delta^+$ and $\kappa_1 \in \mathrm{supp}(\mathbf{y})$, where δ was defined at the beginning of this chapter.

Lemma 16. *Let (p, U) be a condition with $\mathrm{dom}_1(p) = \mathrm{supp}(\mathbf{x}, \mathbf{y})$, $(p, U) \Vdash \mathbf{y} \subseteq \mathbf{x}$ and the property from the previous lemma. Then there is a $V \subseteq U$ such that (p, V) is a condition and the following property holds: For every $\mathbf{z} \in \mathrm{dom}(\mathbf{x})$ and for every $s \in P_2|\,(\max(\mathrm{supp}(\mathbf{y}) \cap \kappa))^+$ the set V is homogeneous meaning that exclusively one statement holds:*

- $\exists q \in V\,\exists W \subseteq P_2\colon q|\,\kappa \subseteq s \wedge P_3 \ni (q \cup s, W) \geq (p, V) \wedge (q \cup s, W) \Vdash \neg\mathbf{z} \in \mathbf{y}$

- *or* $\exists q \in V\, \exists W \subseteq P_2 \colon q|\,\kappa \subseteq s \wedge P_3 \ni (q \cup s, W) \geq (p, V) \wedge (q \cup s, W) \Vdash \mathbf{z} \in \mathbf{y}$
- *or neither holds.*

Proof. Let us fix $\mathbf{z} \in \operatorname{dom}(\mathbf{x})$ and $s \in P_2|\,(\max(\operatorname{supp}(\mathbf{y}) \cap \kappa))^+$. First we note that $(q \cup s, X)$ and $(q \cup s, Y)$ cannot contradict each other, since they are compatible. Next we check if $s|\operatorname{dom}_1(p) \in U|\,\kappa$. If not, then we are done, since neither of the first two statements can hold. In this case we set $V_{\mathbf{z},s} := U$. Otherwise we define

$$U^\star_{\mathbf{z},s} := \{t \in T_{p\cup s|\,(\operatorname{dom}_1(p)\cap\kappa_1)} \colon \exists u \in U\, [u \supseteq t \wedge u|\,\kappa \subseteq s \wedge u \cup (s|\operatorname{dom}_1(p)) \in U]\}.$$

Now for every $t \in U^\star_{\mathbf{z},s}$, if $\alpha(t) \geq \kappa$ and t is extendable, then

$$\{\beta \in \alpha(t) \colon t \cup \{(\alpha(t), n(t), \beta)\} \in U^\star_{\mathbf{z},s}\} \in \Phi_t,$$

since $t\cup s|\operatorname{dom}_1(p)$ reaches U. Note that if $q \in U$, $q|\,\kappa \subseteq s$ and $\exists W \colon P_3 \ni (q \cup s, W) \geq (p, V)$, then $q \in U^\star_{\mathbf{z},s}$.

For $q \in \{t \in U^\star_{\mathbf{z},s} \colon t \text{ is not extendable in } U^\star_{\mathbf{z},s}\}$ we define $F(q)$ to be

- 0 iff $\exists W \subseteq P_2 \colon q|\,\kappa \subseteq s \wedge P_3 \ni (q \cup s, W) \geq (p, V) \wedge (q \cup s, W) \Vdash \neg\mathbf{z} \in \mathbf{y}$,
- 1 iff $\exists W \subseteq P_2 \colon q|\,\kappa \subseteq s \wedge P_3 \ni (q \cup s, W) \geq (p, V) \wedge (q \cup s, W) \Vdash \mathbf{z} \in \mathbf{y}$
- 2 otherwise.

We can make $U^\star_{\mathbf{z},s}$ homogeneous with respect to $F(q)$. The homogeneous set we shall call $U'_{\mathbf{z},s}$. We define

$$V_{\mathbf{z},p\cup s|\,(\operatorname{dom}_1(p)\cap\kappa_1)} := \{t \in T_{p\cup s|\,(\operatorname{dom}_1(p)\cap\kappa_1)} \colon t \in U'_{\mathbf{z},s} \vee [t \notin U^\star_{\mathbf{z},s} \wedge \exists t' \in U'_{\mathbf{z},s}\, (t \supseteq t' \vee t\cup s \notin P_1)]\}.$$

We set $V_{\mathbf{z},s} := \bigcup_{r\in U \colon r\approx p\wedge r\neq p\cup s|\,(\operatorname{dom}_1(p)\cap\kappa_1)} T_r \cup V_{\mathbf{z},p\cup s|\,(\operatorname{dom}_1(p)\cap\kappa_1)}$. Next we intersect with respect to $\mathbf{z}$ and s: $V' := \bigcap_{\mathbf{z}\in\operatorname{dom}(\mathbf{x})} \bigcap_{s\in P_2|\,(\max(\operatorname{supp}(\mathbf{y})\cap\kappa))^+} V_{\mathbf{z},s}$. Furthermore, we fix condition (P6) and get a forcing condition (p, V).

Now we must show that (p, V) satisfies the required property. Let us fix $\mathbf{z} \in \operatorname{dom}(\mathbf{x})$ and $s \in P_2|\,(\max(\operatorname{supp}(\mathbf{y}) \cap \kappa))^+$. Assume that there exist $t_1, t_2 \in V$ and $W_1, W_2 \subseteq P_2$ such that $t_i|\,\kappa \subseteq s$, $(t_i \cup s, W_i)$ are conditions stronger than (p, U) and they contradict each other. First we note that w.l.o.g. we can assume $t_i \in T_{p\cup s|\,(\operatorname{dom}_1(p)\cap\kappa_1)}$, since $t_i \cup s = t_i \cup s|\,(\operatorname{dom}_1(p) \cap \kappa_1) \cup s$. Next we may assume that t_i are not extendable in $U^\star_{\mathbf{z},s}$. This follows because one can show by induction on $|\operatorname{dom}_{1,2}(t') - \operatorname{dom}_{1,2}(t_i)|$ for some $t' \in U^\star_{\mathbf{z},s}$, which is not extendable in $U^\star_{\mathbf{z},s}$, that

$$\{t \in U^\star_{\mathbf{z},s} \colon t \,\text{is not extendable in} U^\star_{\mathbf{z},s}\} \cap W_i|\operatorname{dom}_1(p)$$

is non-empty, since the t_i's need only be extended at big coordinates and the filters are closed under intersection. Let q_i witness that the intersection is non-empty. It follows that $q_i \cup s \in W_i$ so that $(q_i \cup s, W_{i\,q_i\cup s})$ contradict each other. So w.l.o.g. $q_i = t_i$. But if the t_i's are not extendable in $U^\star_{\mathbf{z},s}$, they contradict the homogeneity of $U'_{\mathbf{z},s}$. $\square$

Now we can define the mapping: For $y \in \mathcal{P}^{N_G}(x)$ let $(p(\mathbf{y}), U(\mathbf{y}))$ be the least condition in G with $\mathrm{dom}_1(p(\mathbf{y})) = \mathrm{supp}(\mathbf{x}, \mathbf{y})$, $(p(\mathbf{y}), U(\mathbf{y})) \Vdash \mathbf{y} \subseteq \mathbf{x}$ and the property from the previous lemma. We define $f(y)\colon \mathrm{dom}(\mathbf{x}) \times P_2|\,\kappa \to 3$ as follows:

- $(\mathbf{z}, s) \mapsto 0$ iff $s \in P_2|\,(\max(\mathrm{supp}(\mathbf{y}) \cap \kappa))^+ \wedge \exists q \in U(\mathbf{y})\, \exists W \subseteq P_2\colon$ $q|\,\kappa \subseteq s \wedge P_3 \ni (q \cup s, W) \geq (p, V) \wedge (q \cup s, W) \Vdash \neg \mathbf{z} \in \mathbf{y}$
- $(\mathbf{z}, s) \mapsto 1$ iff $s \in P_2|\,(\max(\mathrm{supp}(\mathbf{y}) \cap \kappa))^+ \wedge \exists q \in U(\mathbf{y})\, \exists W \subseteq P_2\colon$ $q|\,\kappa \subseteq s \wedge P_3 \ni (q \cup s, W) \geq (p, V) \wedge (q \cup s, W) \Vdash \mathbf{z} \in \mathbf{y}$
- and $(\mathbf{z}, s) \mapsto 2$ otherwise.

Note that only due to the previous lemma the mapping is well defined. Assuming that $f(\cdot)$ is 1-1, we see that $\mathcal{P}^{N_G}(x) = f^{-1}[3^{\mathrm{dom}(\mathbf{x}) \times P_2|\,\kappa}] \in M[G]$.

Theorem 17. *The mapping $f(\cdot)\colon \mathcal{P}^{N_G}(x) \to 3^{\mathrm{dom}(\mathbf{x}) \times P_2|\,\kappa}$ is 1-1, so the Power Set axiom holds in N_G.*

Proof. Let $y_1, y_2 \in \mathcal{P}^{N_G}(x)$ with $y_1 \neq y_2$. Case 1: $\max(\mathrm{supp}(\mathbf{y}_1) \cap \kappa) > \max(\mathrm{supp}(\mathbf{y}_2) \cap \kappa)$. W.l.o.g. let y_1 be non-empty and $\mathbf{z} \in \mathrm{dom}(\mathbf{x})$ such that $\mathbf{z}^G \in \mathbf{y}_1^G$. Let $G \ni (t, V) \geq (p(\mathbf{y}_1), U(\mathbf{y}_1))$ with $\mathrm{dom}_1(t) = \mathrm{dom}_1(p(\mathbf{y}_1)) \cup \mathrm{supp}(\mathbf{z})$ and $(t, V) \Vdash \mathbf{z} \in \mathbf{y}_1$. If we set $s = t|\,\kappa$, $W = V$ and $q = t|\,\mathrm{dom}_1(p(\mathbf{y}_1)) \in U(\mathbf{y}_1)$ we see that $(t, V) = (q \cup s, W) \Vdash \mathbf{z} \in \mathbf{y}_1$. So it follows that $f(y_1)(\mathbf{z}, s) = 1$ while $f(y_2)(\mathbf{z}, s) = 2$, since $s \notin P_2|\,(\max(\mathrm{supp}(\mathbf{y}_2) \cap \kappa))^+$.

Case 2: $\max(\mathrm{supp}(\mathbf{y}_1) \cap \kappa) = \max(\mathrm{supp}(\mathbf{y}_2) \cap \kappa)$. Since $y_1 \neq y_2$ there is a $\mathbf{z} \in \mathrm{dom}(\mathbf{x})$ such that w.l.o.g. $\mathbf{z}^G \in \mathbf{y}_1^G$ but $\mathbf{z}^G \notin \mathbf{y}_2^G$. Let $G \ni (t, V) \geq (p(\mathbf{y}_i), U(\mathbf{y}_i))$ for $i = 1, 2$ such that $\mathrm{dom}_1(t) = \mathrm{dom}_1(p(\mathbf{y}_1)) \cup \mathrm{dom}_1(p(\mathbf{y}_2)) \cup \mathrm{supp}(\mathbf{z})$ and $(t, V) \Vdash \mathbf{z} \in \mathbf{y}_1 \wedge \mathbf{z} \notin \mathbf{y}_2$. We set $s = t|\,\kappa$, $W_i = V|\,(\mathrm{dom}_1(p(\mathbf{y}_i)) \cup \kappa)$ and $q_i = t|\,\mathrm{dom}_1(p(\mathbf{y}_i)) \in U(\mathbf{y}_i)$. If we set $a_i = \mathrm{dom}_1(p(\mathbf{y}_i) \cup \kappa)$ it follows that $(t|\,a_i, V|\,a_i) = (q_i \cup s, W_i)$. Since $\mathrm{supp}(\mathbf{y}_i, \mathbf{z}) \subseteq a_i$, we see that $(q_1 \cup s, W_1) \Vdash \mathbf{z} \in \mathbf{y}_1$ and $(q_2 \cup s, W_2) \Vdash \mathbf{z} \notin \mathbf{y}_2$. Therefore, $f(y_1)(\mathbf{z}, s) = 1$ and $f(y_2)(\mathbf{z}, s) = 0$.

Now $\mathcal{P}^{N_G}(x) \in M[G]$ holds by Replacement. Again by Replacement we get an $\alpha \in \mathrm{Reg}$ such that $\mathcal{P}^{N_G}(x) \subseteq N_{G_\alpha}$. Now being a subset of x is absolute for transitive models, and $N_{G_\alpha} \vDash \mathrm{ZF}$, so that $\mathcal{P}^{N_G}(x) = \mathcal{P}^{N_{G_\alpha}}(x) \in N_{G_\alpha} \subseteq N_G$ follows. $\square$

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