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Abstract

The goal of this thesis is to study the functional central limit theorems, especially the extension of Donsker's approximation of Brownian motion the so-called rough Donsker (rDonsker) theorem, which helps us approximate the fractional Brownian motion essential for further implementations of rough volatility models. Furthermore, based on the results those convergence theorems, the numerical implementation of rough Donsker volatility model is presented and its results are discussed .

This work is largely based on the paper *Functional central limit theorems for rough volatility* by Blanka Horvath, Antoine Jacquier, and Aitor Muguruza [1].

Contents

1. **Introduction**

Alongside the financial market development and introduction of derivatives of primary assets, a strong need for mathematical modeling of their prices emerged. Financial mathematics is a relatively new branch of mathematics dealing with modeling of these prices without introducing an arbitrage into the financial market. Intuitively, an arbitrage opportunity means investing in an asset which, with positive probability, yields a profit without any downside risk. Assuming the financial market is arbitrage free, all the investments are exposed to some kind of downside risk. By Financial Times lexicon the volatility is defined as the extent to which the price of a security or commodity, or the level of a market, interest rate or currency, changes over time. High volatility implies rapid and large upward and downward movements over a relatively short period of time; low volatility implies much smaller and less frequent changes in value. In other words, volatility gives us the idea about our investment risk by showing the range to which the price may change while keeping the direction of the change unrevealed. The log-prices of derivatives are usually modeled as continuous semi-martingales.For X_t being a log-price of the asset at the time t, the price-process is given by

$$\mathrm{d}X_t = \mu_t \mathrm{dt} + \sigma_t \mathrm{d}W_t,$$

where μ_t denotes the drift, σ_t volatility and W_t standard one-dimensional Brownian motion. A several models were introduced throughout the years such as Black-Scholes where the volatility function is either constant or a deterministic function of time, Dupire's local volatility model, see [2], the local volatility $\sigma(Y_t; t)$ is a deterministic function of the underlying price and time, chosen to match observed European option prices exactly. Such a model is by definition time-inhomogenus; its dynamics are highly unrealistic, typically generating future volatility surfaces completely unlike those we observe. On the other hand, in so-called stochastic volatility models, the volatility σ_t is modeled as a continuous Brownian semi-martingale. Notable amongst such stochastic volatility models are the Hull and White model [5], the Heston model [4], and the SABR model [3]. Whilst stochastic volatility dynamics are more realistic than local volatility dynamics, generated option prices are not consistent with observed European option prices [21]. From an analysis of the time series of realized variance using recent high-frequency data, see [21], previously showed that the logarithm of realized variance behaves essentially as a fractional Brownian motion with Hurst exponent H of order 0.1, at any reasonable timescale.

The resulting Rough Fractional Stochastic Volatility (RFSV) model is remarkably consistent with financial time series data [24]. Throughout the thesis, the exstention of Donsker theorem, the so called rDonsker theorem, is explored along with its applications general volatility models given by the following system

$$dX_t = -\tfrac{1}{2}V_t\ dt + \sqrt{V}dB_t, \qquad X_0 = 0,$$

$$V_t = \Phi(\mathcal{G}^\alpha Y)(t), \qquad V_0 > 0,$$

Also, the precise definition of $\mathcal{G}^\alpha Y$ is given and the approximation sequence for X_t derived. In following chapter the mathematical background is introduced. Then, in Chapter 3, the theorems and results of the paper, see[1], are presented. Finally, in Chapter 4, numerical implementation, two "R" codes, of rough Bergomi model is given and explained. In chapter 5, all the results of implementations are discussed and final conclusions are given.

2. Mathematical background

In order to properly explain the model and its implementation, it is necessary to introduce the following mathematical concepts.

Definition 2.1. For $\beta \in (0,1]$, the β-Hölder space $\mathcal{C}^\beta(\mathbb{I})$, equipped with the norm

$$\|f\|_\beta := |f|_\beta + \|f\|_\infty = \sup_{\substack{t,s\in\mathbb{I}\\ t\neq s}} \frac{|f(t)-f(s)|}{|t-s|^\beta} + \max_{t\in\mathbb{I}} |f(t)|,$$

is a non-separable Banach space [26, Chapter 3].

Definition 2.2. For any $\lambda \in (0,1)$, the left Riemann-Liouville fractional operator is defined on $\mathcal{C}^\lambda(\mathbb{I})$ as

$$(I^\alpha f)(t) := \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\mathrm{d}s, & \text{for } \alpha \in [0,1), \\ \left(\dfrac{\mathrm{d}}{\mathrm{d}t} I^{1+\alpha} f\right)(t) = \dfrac{1}{\Gamma(1+\alpha)}\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_0^t (t-s)^\alpha f(s)\mathrm{d}s, & \text{for } \alpha \in (-\lambda, 0). \end{cases} \tag{2.1}$$

Following the spirit of Riemann-Liouville fractional operators we introduce the class of Generalised Fractional Operators (GFO). For any $\alpha \in (-1,1)$, we introduce the space $\mathcal{L}^\alpha := \{u \mapsto u^\alpha L(u) : L \in \mathcal{C}_b^1(\mathbb{I})\}$, as well as the following subset of $\mathbb{R}^2$:

$$\mathfrak{R} := \Big\{(\alpha,\lambda) \in (-1,1)\times(0,1) \text{ such that } \alpha+\lambda \in (0,1)\Big\}.$$

Definition 2.3. For any $(\alpha,\lambda) \in \mathfrak{R}$, the GFO associated to $g \in \mathcal{L}^\alpha$ is defined on $\mathcal{C}^\lambda(\mathbb{I})$ as

$$(\mathcal{G}^\alpha f)(t) := \begin{cases} \displaystyle\int_0^t f(s)\frac{\mathrm{d}}{\mathrm{d}t} g(t-s)\mathrm{d}s, & \text{if } \alpha \in [0, 1-\lambda), \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_0^t f(s) g(t-s)\mathrm{d}s, & \text{if } \alpha \in (-\lambda, 0). \end{cases} \tag{2.2}$$

We shall further use the notation $G(t) := \int_0^t g(u)\mathrm{d}u$, for any $t \in \mathbb{I}$. The following kernels and operators are well-known examples of Generalised Fractional Operators:

$$\begin{array}{lll} \text{Riemann-Liouville:} & g(u) = u^\alpha, & \text{for } \alpha \in (-1,1); \\ \text{Gamma fractional:} & g(u) = u^\alpha \mathrm{e}^{\beta u}, & \text{for } \alpha \in (-1,1), \beta > 0; \\ \text{Power-law:} & g(u) = u^\alpha (1+u)^{\beta-\alpha}, & \text{for } \alpha \in (-1,1), \beta < -1. \end{array} \tag{2.3}$$

The following proposition generalises the classical mapping properties of Riemann-Liouville fractional operators first proved by Hardy and Littlewood [28], and will be of fundamental importance in the rest of our analysis.

Proposition 2.4. *For $(\alpha,\lambda)\in\mathfrak{R}$, the operator $\mathcal{G}^\alpha:\mathcal{C}^\lambda(\mathbb{I})\to\mathcal{G}^{\lambda+\alpha}(\mathbb{I})$ is continuous.*

Proof. Since $g\in\mathcal{L}^\alpha$, there exists $C>0$ such that $|g(u)|\le Cu^\alpha$; hence, for $t\in\mathbb{I}$,

$$\frac{d}{dt}\int_0^t |f(s)g(t-s)|\,ds \le C\frac{d}{dt}\int_0^t |f(s)(t-s)^\alpha|\,ds.$$

Therefore, for $f\in\mathcal{C}^\lambda(\mathbb{I})$, the inequalities involving the Riemann-Liouville fractional operator

$$(\mathcal{G}^\alpha f)(t)\le C(I^\alpha f)(t)\le C\|f\|_\lambda \tag{2.4}$$

hold for $\alpha\le 0$ and all $t\in\mathbb{I}$. Since Riemann-Liouville operators are continuous, continuity of the GFO follows directly from (2.4) along with linearity. To prove that $\mathcal{G}^\alpha$ belongs to $\mathcal{C}^{\lambda+\alpha}(\mathbb{I})$, we may invoke (2.4) and easily adapt Theorem 2.11. Similarly, when $\alpha\ge 0$, for any $u\in\mathbb{I}$, $g'(u)=u^\alpha L'(u)+u^{\alpha-1}L(u)\le C_1+C_2u^{\alpha-1}$, and the λ-Hölder continuity of f yields, for any $t\in\mathbb{I}$,

$$\int_0^t\frac{d}{dt}\,g(t-s)f(s)\,ds\le\frac{C_1}{\lambda+1}t^{\lambda+1}+C_2\int_0^t(t-s)^{\alpha-1}f(s)\,ds\le C_1+C_2\int_0^t(t-s)^{\alpha-1}f(s)\,ds.$$

Since the time horizon $\mathbb{I}$ is compact, the first constant does not affect continuity or mapping properties of the GFO. The second term is bounded by the Riemann-Liouville integral operator, hence continuity and mapping properties follow as before by straightforward modification of Theorem 2.9.

□

Definition 2.5. Let $(\Omega,\ F\ ,\ \mathbb{P})$ be a probability space. A fractional Brownian motion W^H with Hurst parameter $H\in(0,1]$, is an almost surely continuous, centered Gaussian process $(W_t^H)_{t\in\mathbb{R}}$ with

$$\mathrm{Cov}(W_t^H,W_s^H)=\tfrac{1}{2}(|t|^{2H}+|s|^{2H}-|t-s|^{2H}),\ \ t,s\in\mathbb{R}$$

Definition 2.6. A stationary fractional Ornstein-Uhlenbeck process (X_t) is defined as the stationary solution of the stochastic differential equation

$$dX_t=\nu\,dW_t^H-\alpha\,(X_t-m)dt,$$

where $m\in\mathbb{R}$ and ν and α are positive parameters, see [32].

For the usual Ornstein-Uhlenbeck processes, there exists an explicit form for the solution given by

$$X_t = \nu \int_{-\infty}^{t} e^{-\alpha(t-s)} dW_t^H + m. \tag{2.5}$$

Where the stochastic integral with respect to fBM is a pathwise Riemann-Stieltjes integral, see again [32].

2.1. **Rough fractional volatility models.** In this paper, an approximation scheme for the following system is developed generalising the concept of rough volatility introduced in [19, 20, 22] in the context of mathematical finance, where the X process represents the dynamics of the logarithm of a stock price process:

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\mathrm{d}B_t, \quad X_0 = 0, \\ V_t &= \Phi\left(\mathcal{G}^\alpha Y\right)(t), \qquad\qquad V_0 > 0, \end{aligned} \tag{2.6}$$

with $\alpha \in (-1, 1)$, and Y the (strong) solution to the stochastic differential equation

$$\mathrm{d}Y_t = b(Y_t)\mathrm{d}t + a(Y_t)\mathrm{d}W_t, \quad Y_0 \in \mathbb{R}, \tag{2.7}$$

The two Brownian motions B and W, defined on a common filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{I}}, \mathbb{P})$, are correlated by the parameter $\rho \in [-1, 1]$. The functional Φ is continuous on $\mathcal{C}(\mathbb{I})$, and for any $\varphi \in \mathcal{C}(\mathbb{I})$, $\Phi(\varphi)$ is continuously differentiable and integrable. This is enough to ensure that the first stochastic differential equation is well defined. It remains to formulate the precise definition for $\mathcal{G}^\alpha Y$ (Propsition 2.8) to fully specify the system (2.6) and clarify the existence of solutions. Existence and (strong) uniquess of a solution to the second SDE in (2.7) is guaranteed by the following standard assumption [18]:

Assumption 2.7. There exist $C_b, C_a > 0$ and an increasing function $\varrho : (0, \infty) \to (0, \infty)$ with $\lim_{\epsilon \downarrow 0} \int_\epsilon^1 \frac{\mathrm{d}x}{\varrho(x)} = \infty$. such that

$$|b(x) - b(y)| \le C_b|x - y| \qquad \text{and} \qquad |a(x) - a(y)| \le C_a\sqrt{\varrho(|x - y|)}.$$

Not only is the solution to (2.7) continuous, but $\frac{1}{2}$-Hölder continuous as a consequence of Kolmogorov-Čentsov's theorem [7]. Existence and precise meaning of the term $\mathcal{G}^\alpha Y$ is more delicate, and is treated further below.

Proposition 2.8. *The equality* $(\mathcal{G}^{\alpha}W)(t) = \int_0^t g(t-s)\mathrm{d}W_s$ *holds almost surely for all* $t \in \mathbb{I}$.

Proof. Since the paths of Brownian motion are 1/2-Hölder continuous, existence (and continuity) of $\mathcal{G}^{\alpha}W$ is guaranteed for all $\alpha \in (-1/2, 1/2)$. When $\alpha \in [0, 1/2)$, the kernel is smooth and square integrable, so that Itô's product rule yields (since $g(0) = 0$)

$$(\mathcal{G}^{\alpha}W)(t) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}t} g(t-s)W(s)\mathrm{d}s = \int_0^t g(t-s)\mathrm{d}W_s,$$

and the claim holds. For $\alpha \in (-1/2, 0)$, and any $\epsilon > 0$, introduce the operator

$$\left(\mathcal{G}_{\epsilon}^{1+\alpha} f\right)(t) := \int_0^{t-\epsilon} g(t-s)f(s)\mathrm{d}s, \qquad \forall t \in \mathbb{I},$$

which satisfies $\lim_{\epsilon \downarrow 0} \left(\mathcal{G}_{\epsilon}^{1+\alpha} f\right)(t) = \left(\mathcal{G}^{1+\alpha} f\right)(t)$ pointwise. Now, for any $t \in \mathbb{I}$, almost surely,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_{\epsilon}^{1+\alpha} W\right)(t) = g(\epsilon)W(t-\epsilon) + \int_0^{t-\epsilon} \frac{\mathrm{d}}{\mathrm{d}t} g(t-s)W(s)\mathrm{d}s = \int_0^{t-\epsilon} g(t-s)\mathrm{d}W_s. \tag{2.8}$$

Then, as ϵ tends to zero, the right-hand side of (2.8) tends to $\int_0^t g(t-s)\mathrm{d}W_s$, and furthermore, the convergence is uniform. On the other hand, the equalities

$$\begin{aligned}(\mathcal{G}^{1+\alpha}W)(t) - (\mathcal{G}^{1+\alpha}W)(0) &= \lim_{\epsilon \to 0} (\mathcal{G}_{\epsilon}^{1+\alpha}W)(t) - (\mathcal{G}_{\epsilon}^{1+\alpha}W)(0) = \lim_{\epsilon \to 0} \int_0^t \left(\tfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{G}_{\epsilon}^{1+\alpha}W\right)(s)\mathrm{d}s \\ &= \int_0^t \lim_{\epsilon \to 0} \left(\tfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{G}_{\epsilon}^{1+\alpha}W\right)(s)\mathrm{d}s = \int_0^t \left(\int_0^s g(s-u)\mathrm{d}W_u\right)\mathrm{d}s,\end{aligned}$$

hold since convergence is uniform on compacts, and the fundamental theorem of calculus concludes the proof.

□

Theorem 2.9. *For any* $f \in \mathcal{C}^{\lambda}(\mathbb{I})$, *with* $\lambda \in (0,1)$ *and* $\alpha \in (0,1)$, *the identity*

$$(I^{\alpha}f)(t) = \frac{f(0)}{\Gamma(1+\alpha)} t^{\alpha} + \psi(t),$$

holds for all $t \in \mathbb{I}$, *for some* $\psi \in \mathcal{C}^{\lambda+\alpha}(\mathbb{I})$ *satisfying* $|\psi(t)| \leq Ct^{\lambda+\alpha}$ *on* $\mathbb{I}$ *for some* $C > 0$.

Proof. We may easily represent

$$(I^{\alpha}f)(t) = \frac{f(0)}{\Gamma(\alpha)} \int_0^t \frac{\mathrm{d}u}{(t-u)^{1-\alpha}} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(u)-f(0)}{(t-u)^{1-\alpha}}\mathrm{d}u = \frac{f(0)}{\Gamma(1+\alpha)} t^{\alpha} + \psi(t)$$

with $\psi(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(u) - f(0)}{(t-u)^{1-\alpha}} \mathrm{d}u$. Since $f \in \mathcal{C}^\lambda(\mathbb{I})$, we obtain $|\psi(t)| \le \frac{|f|_\lambda}{\Gamma(\alpha)} \int_0^t \frac{u^\lambda}{(t-u)^{1-\alpha}} \mathrm{d}u$, and hence

$$|\psi(t)| \le \frac{\Gamma(2+\lambda)|f|_\lambda}{(1+\lambda)\Gamma(\alpha+\lambda+1)} t^{\alpha+\lambda},$$

which proves the estimate for $|\psi|$. Next, we prove that $\psi \in \mathcal{C}^{\lambda+\alpha}(\mathbb{I})$. For this, introduce $\phi(t) := f(t) - f(0)$ and consider $t, t+h \in \mathbb{I}$ with $h > 0$,

$$\begin{aligned}
\psi(t+h) - \psi(t) &= \frac{1}{\Gamma(\alpha)} \left(\int_{-h}^t \frac{\phi(t-u)}{(u+h)^{1-\alpha}} \mathrm{d}u - \int_0^t \frac{\phi(t-u)}{u^{1-\alpha}} \mathrm{d}u \right) \\
&= \frac{\phi(t)}{\Gamma(1+\alpha)} \left[(t+h)^\alpha - t^\alpha \right] + \frac{1}{\Gamma(\alpha)} \left(\int_{-h}^0 \frac{\phi(t-u) - \phi(t)}{(u+h)^{1-\alpha}} \mathrm{d}u \right) \\
&+ \frac{1}{\Gamma(\alpha)} \left(\int_0^t \left[(u+h)^{\alpha-1} - u^{\alpha-1} \right] \left[\phi(t-u) - \phi(t) \right] \mathrm{d}u \right) =: J_1 + J_2 + J_3.
\end{aligned}$$

We first consider J_1. If $h > t$, then

$$|J_1| \le \frac{|f|_\lambda}{\Gamma(1+\alpha)} t^\lambda \left[(t+h)^\alpha - t^\alpha \right] \le C h^{\lambda+\alpha}.$$

On the other hand, when $0 < h < t$, since $(1+u)^\alpha - 1 \le \alpha u$ for $u > 0$, then

$$|J_1| \le \frac{|f|_\lambda}{\Gamma(1+\alpha)} t^{\lambda+\alpha} \left| \left(1 + \frac{h}{t} \right)^\alpha - 1 \right| \le C h t^{\lambda+\alpha-1} \le C h^{\lambda+\alpha}.$$

For J_2, since $f \in \mathcal{C}^\lambda(\mathbb{I})$, we can write

$$|J_2| \le \frac{|f|_\lambda}{\Gamma(\alpha)} \int_{-h}^0 \frac{|u|^\lambda}{(u+h)^{1-\alpha}} \le C h^{\lambda+\alpha}.$$

Finally,

$$|J_3| \le \frac{|f|_\lambda}{\Gamma(\alpha)} \int_0^t u^\lambda [u^{\alpha-1} - (u+h)^{\alpha-1}] \mathrm{d}u = \frac{|f|_\lambda}{\Gamma(\alpha)} h^{\lambda+\alpha} \int_0^{t/h} u^\lambda [u^{\alpha-1} - (u+1)^{\alpha-1}] \mathrm{d}u.$$

Hence, if $t \le h$, then $|J_3| \le C h^{\lambda+\alpha}$. Likewise, if $t > h$ and $\lambda + \alpha < 1$, then $|J_3| \le C h^{\lambda+\alpha}$ since

$$\left| u^{\alpha-1} - (u+1)^{\alpha-1} \right| = u^{\alpha-1} \left[1 - \left(1 + \frac{1}{u} \right)^{\alpha-1} \right] \le C u^{\alpha-2}.$$

Thus ψ satisfies the $(\lambda+\alpha)$-Hölder condition and belongs to $\mathcal{C}^{\lambda+\alpha}(\mathbb{I})$. □

Corollary 2.10. *For any $\alpha, \lambda \in (0,1)$ such that $\lambda + \alpha \le 1$, I^α is a continuous operator from $\mathcal{C}^\lambda(\mathbb{I})$ to $\mathcal{C}^{\lambda+\alpha}(\mathbb{I})$.*

Proof. It is clear that I^α is a linear operator. Using the estimate in Theorem 2.9 we have

$$\|I^\alpha f\|_{\alpha+\lambda} \le \frac{f(0)}{\Gamma(1+\alpha)}\|(\cdot)^\alpha\|_{\lambda+\alpha} + \|\psi\|_{\lambda+\alpha} \le C_1\|f\|_\lambda\|(\cdot)^\alpha\|_{\lambda+\alpha} + C_2\|f\|_\lambda\|(\cdot)^{\alpha+\lambda}\|_{\lambda+\alpha} \le C\|f\|_\lambda,$$

since $|f|_\lambda \le \|f\|_\lambda$, $f(0) \le \|f\|_\lambda$. Therefore I^α is also bounded and hence continuous. □

Theorem 2.11. *For any* $0 < -\alpha < \lambda \le 1$, *let* $f \in \mathcal{C}^\lambda(\mathbb{I})$. *Then* $I^\alpha f$ *exists,* $I^{-\alpha}I^\alpha f = f$ *and, for all* $t \in \mathbb{I}$,

$$(I^\alpha f)(t) = -\frac{\alpha}{\Gamma(1+\alpha)}\int_0^t (t-u)^{\alpha-1}[f(t)-f(u)]\mathrm{d}u.$$

Proof. For $f \in \mathcal{C}^\lambda(\mathbb{I})$, define, for any $\varepsilon > 0$ and $t \in \mathbb{I}$,

$$(I_\varepsilon^{1+\alpha} f)(t) := \frac{1}{\Gamma(\alpha+1)}\int_0^{t-\varepsilon}(t-u)^\alpha f(u)\mathrm{d}u,$$

and note that $I_0^{1+\alpha} = I^{1+\alpha}$. Then, we have

$$\begin{aligned}\Gamma(1+\alpha)\left(\frac{\mathrm{d}}{\mathrm{d}t}I_\varepsilon^{1+\alpha}f\right)(t) &= \varepsilon^\alpha f(t-\varepsilon) + \alpha\int_0^{t-\varepsilon}(t-u)^{\alpha-1}f(u)\mathrm{d}u \\ &= -\alpha\int_0^{t-\varepsilon}(t-u)^{\alpha-1}(f(t)-f(u))\mathrm{d}u - \varepsilon^\alpha(f(t)-f(t-\varepsilon)). \qquad (2.9)\end{aligned}$$

where Hölder continuity implies that $f(t) - f(u) \le C(t-u)^\lambda$, so that the integral in (2.9) is well defined. Then, as ε tends to zero, the right-hand side of (2.9) tends uniformly to

$$\psi(t) = -\alpha\int_0^t (t-u)^{\alpha-1}(f(t)-f(u))\mathrm{d}u.$$

Now, for $t \in \mathbb{I}$,

$$\begin{aligned}(I^{1+\alpha}f)(t) - (I^{1+\alpha}f)(0) &= \lim_{\varepsilon\downarrow 0}\left\{(I_\varepsilon^{1+\alpha}f)(t) - (I_\varepsilon^{1+\alpha}f)(0)\right\} = \lim_{\varepsilon\downarrow 0}\int_0^t\left(\frac{\mathrm{d}}{\mathrm{d}u}I_\varepsilon^{1+\alpha}f\right)(u)\mathrm{d}u \\ &= \int_0^t \lim_{\varepsilon\downarrow 0}\left(\frac{\mathrm{d}}{\mathrm{d}u}I_\varepsilon^{1+\alpha}f\right)(u)\mathrm{d}u = \frac{1}{\Gamma(1+\alpha)}\int_0^t \psi(u)\mathrm{d}u,\end{aligned}$$

where the exchange of limit and integral holds since the convergence is uniform and the interval compact. Therefore, $\Gamma(\alpha+1)(I^{1+\alpha}f)$ is the integral of ψ and, by the Fundamental Theorem of Calculus,

$$\psi(t) = \Gamma(\alpha+1)\left(\frac{\mathrm{d}}{\mathrm{d}t}I^{1+\alpha}f\right)(t) = \Gamma(\alpha+1)(I^\alpha f)(t).$$

Therefore it exists and, similarly to Theorem 2.9, $\psi \in \mathcal{C}^{\lambda+\alpha}(\mathbb{I})$. Finally, since, for $0 < \beta < 1$, the equality

$$(I^\beta I^{1-\beta}f)(t) = (I^{1-\beta}I^\beta f)(t) = (I^1 f)(t) = \int_0^t f(u)\mathrm{d}u$$

holds for all $t \in \mathbb{I}$, we conclude that

$$\Big(\Gamma(1+\alpha)I^{1+\alpha}f - I^{-\alpha}\psi\Big)(t) = \int_0^t \Gamma(1+\alpha)f(u) - (I^{-\alpha}\psi)(u))(t-u)^\alpha \mathrm{d}u = 0,$$

and hence, by continuity of both f and $I^{-\alpha}\psi$, $f = I^{-\alpha}I^\alpha f$. □

This chapter is concluded with two definitions necessary to understand the algorithms in chapter 4.

Definition 2.12 (Discrete convolution)**.** For any $\mathrm{a}, \mathrm{b} \in \mathbb{R}^n$, the discrete convolution operator $* : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$(\mathrm{a} * \mathrm{b})_i := \sum_{m=0}^{i} \mathrm{a}_m \mathrm{b}_{i-m}, \quad i = 0, \ldots, n-1.$$

When simulating $\mathcal{G}^\alpha W$ on the uniform partition $\mathcal{T}$, the scheme reads

$$(\mathcal{G}^\alpha W)^j(t_i) = \sum_{k=1}^{i} g(t_i - t_{k-1})\xi_k = \sum_{k=1}^{i}(t_k)\xi_{j,k-i+1}, \qquad \text{for } i = 1, \ldots, n,$$

which has the form of the discrete convolution in Definition 2.12. Rewritten in matrix form,

$$\begin{pmatrix} g(t_1) & 0 & \cdots & 0 \\ g(t_2) & g(t_1) & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ g(t_n) & g(t_{n-1}) & \cdots & g(t_1) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix},$$

it is clear that this operator yields a complexity of order $\mathcal{O}(n^2)$, which can be improved drastically.

Definition 2.13. The Discrete Fourier Transform (DFT) of a sequence $\mathrm{c} := (c_0, c_1, ..., c_{n-1}) \in \mathbb{C}^n$ is given by

$$\widehat{f}(\mathrm{c})[j] := \sum_{k=0}^{n-1} c_k \exp\left(-\frac{2\mathtt{i}\pi jk}{n}\right), \qquad \text{for } j = 0, \ldots, n-1,$$

and the Inverse DFT of c is given by

$$f(\mathrm{c})[k] := \frac{1}{n}\sum_{j=0}^{n-1} c_j \exp\left(\frac{2\mathtt{i}\pi jk}{n}\right), \qquad \text{for } k = 0, \ldots, n-1.$$

In general, both transforms require a computational effort of order $\mathcal{O}(n^2)$, but the the Fast Fourier Transform (FFT) algorithm reduces the complexity of both transforms to $\mathcal{O}(n \log n)$. Thankfully, many numerical packages offer a direct implementation of the discrete convolution which simplifies and fastens the implementation. Although the FFT step is the heaviest computation on the simulation of rough volatility models, the actual time grid $\mathcal{T}$ is not specially large, i.e. $n < 1000$. Hence, it is not important to have the fastest possible FFT for very large n, it is much more important for the implementation to be fast on small time grids.

3. Main theorems and convergence results

We now move on to main results of the paper by B. Horvath, A. Jacquier and A. Muguruza [1] which further made possible and also inspired the implementation presented in this thesis.

Assumption 3.1. The family $(\xi_i)_{i\geq 1}$ forms an iid sequence of centered random variables with finite moments of all orders and $\mathbb{E}(\xi_1^2) = \sigma^2 > 0$.

Following Donsker [8] and Lamperti [16], we first define, for any $\omega \in \Omega$, $n \geq 1$, $t \in \mathbb{I}$, the approximating sequence for the driving Brownian motion B as

$$B_n(t) := \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \xi_k + \frac{nt - \lfloor nt \rfloor}{\sigma\sqrt{n}} \xi_{\lfloor nt \rfloor + 1}. \tag{3.1}$$

As will be explained later, a similar construction holds to approximate the process Y:

(3.2)

$$Y_n(t) := \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} b\left(Y_n^{k-1}\right) + \frac{nt - \lfloor nt \rfloor}{n} b\left(Y_n^{\lfloor nt \rfloor}\right) + \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} a\left(Y_n^{k-1}\right) \xi_k + \frac{nt - \lfloor nt \rfloor}{\sigma\sqrt{n}} a\left(Y_n^{\lfloor nt \rfloor}\right) \xi_{\lfloor nt \rfloor + 1},$$

where $Y_n^k := Y_n(t_k)$, where $t_k := \frac{k}{n}$, from which we naturally deduce an approximating scheme (up to the interpolating term which decays to zero by Chebyshev's inequality) for X as

$$X_n(t) \;\; := -\frac{1}{2n} \sum_{k=1}^{\lfloor nt \rfloor} \Phi\left(\mathcal{G}^\alpha Y_n\right)(t_k) + \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \sqrt{\Phi\left(\mathcal{G}^\alpha Y_n\right)(t_k)} \left(B_n^{k+1} - B_n^k\right) \tag{3.3}$$

All the approximations above, as well as all the convergence statements below should be understood pathwise, but we omit the ω dependence in the notations for clarity. The main result of the paper is a convergence statement about the approximating sequence $(X_n)_{n\geq 1}$. In usual weak convergence analysis [6], convergence is stated in the Skorohod space $(\mathcal{D}(\mathbb{I}), \|\cdot\|_{\mathcal{D}})$ of càdlàg processes equipped with the Skorohod topology. While Theorem 3.2 proves it, it further provides convergence in the stronger Hölder space $\left(\mathcal{C}^\lambda(\mathbb{I}), \|\cdot\|_\lambda\right)$ for $\lambda < \frac{1}{2}$, but with additional restrictions.

Theorem 3.2. *The sequence $(X_n)_{n\geq 1}$ converges weakly to X in $(\mathcal{D}(\mathbb{I}), \|\cdot\|_{\mathcal{D}})$. Furthermore, for $\lambda < \frac{1}{2}$ and $\alpha \in (-\frac{1}{2}, \frac{1}{2})$, convergence in the Hölder space $\left(\mathcal{C}^\lambda(\mathbb{I}), \|\cdot\|_\lambda\right)$ holds if either of the following two conditions holds:*

(i) all the ξ_i are distributed as $\mathcal{N}(0,1)$ and $\mathbb{E}[\mathrm{e}^{\Phi(\mathcal{G}^\alpha Y)}]$ is finite;

(ii) all the ξ_i are bounded almost surely and $\mathbb{E}[\mathrm{e}^{\Phi(\mathcal{G}^\alpha Y_n)}]$ is finite for each n.

In (ii), the moment condition on the sequence (Y_n) is difficult to check. However, it clearly holds as soon as the iid sequence (ζ_i), approximating the stochastic driver of Y, is bounded. The construction of the proof allows to extend the convergence to the case where Y is a d-dimensional diffusion without additional work. The proof of the theorem requires a certain number of steps: we start with the convergence of the approximation (Y_n) in some Hölder space, which we then translate, first into convergence of the stochastic integral in (2.6), then, by continuity of the mapping Φ into convergence of the sequence $(\Phi(\mathcal{G}^\alpha Y_n))$. All these ingredients are detailed down below. Once this is achieved, the proof of the theorem itself is relatively straightforward, as will be illustrated.

The standard convergence result for Brownian motion can be stated as follows:

Theorem 3.3. *For $\alpha < \frac{1}{2}$, the sequence (B_n) in* (3.1) *converges weakly to a Brownian motion in $(\mathcal{C}^\alpha(\mathbb{I}), \|\cdot\|_\alpha)$.*

Theorem 3.4 (Sufficient conditions for weak convergence in Hölder spaces)**.** *Let $Z \in \mathcal{C}^\lambda(\mathbb{I})$ and $(Z_n)_{n\geq 1}$ its corresponding approximating sequence in the sense that for any $t_1 \leq \ldots \leq t_k$ in $\mathbb{I}$, $(Z_n(t_1), \ldots, Z_n(t_k))$ converges in distribution to $(Z(t_1), \ldots, Z(t_k))$ as n tends to infinity. Assume further that the tightness criterion*

$$\mathbb{E}\left(|Z_n(t) - Z_n(s)|^\alpha\right) \leq C|t-s|^{1+\beta} \tag{3.4}$$

holds for all $n \geq 1$, $t, s \in \mathbb{I}$, and some $C, \alpha, \beta > 0$. Then $(Z^n)_{n\geq 1}$ converges weakly to Z in $\mathcal{C}^\lambda(\mathbb{I})$ for $\lambda < \frac{\beta}{\alpha}$.

As pointed out by Račkauskas and Suquet in [15], strictly speaking the convergence takes place in the Hölder space $C_0^\lambda(\mathbb{I})$ endowed with the norm $\|f\|_\lambda^0 := |f|_\lambda + |f(0)|$, for all functions that satisfy

$$\lim_{\delta\downarrow 0} \sup_{\substack{0<t-s<\delta \\ t,s\in\mathbb{I}}} \frac{|f(t)-f(s)|}{(t-s)^\alpha} = 0.$$

Then $\left(C_0^\lambda(\mathbb{I}), \|\cdot\|_\lambda^0\right)$ becomes a separable closed subspace of $\left(\mathcal{C}^\lambda(\mathbb{I}), \|\cdot\|_\lambda\right)$ (see [15, 10] for details), To conclude our review of weak convergence in Hölder spaces, the following theorem, due to Račkauskas and Suquet [15] provides necessary and sufficient conditions ensuring convergence in Hölder space:

Theorem 3.5 (Račkauskas-Suquet [15])**.** *Let* $\alpha \in (0, \frac{1}{2})$ *and* $p(\alpha) := \frac{1}{1-2\alpha}$. *The sequence* $(B_n)_{n\geq 1}$ *in* (3.1) *converges (pathwise) weakly to a Brownian motion in* $\mathcal{C}^\alpha(\mathbb{I})$ *if and only if* $\mathbb{E}(\xi_1) = 0$ *and* $\lim_{t\uparrow\infty} t^{p(\alpha)}\mathbb{P}(|\xi_1| \geq t) = 0$.

The first important step in our analysis is to extend Donsker-Lamperti's weak convergence from Brownian motion to the Itô diffusion Y in (2.7).

Theorem 3.6. *The sequence* $(Y_n)_{n\geq 1}$ *converges weakly to* Y *in* (2.7) *in* $(\mathcal{C}^\alpha(\mathbb{I}), \|\cdot\|_\alpha)$ *for all* $\alpha < \frac{1}{2}$.

Proof. Finite-dimensional convergence is a classical result by Kushner [12], so only tightness needs to be checked. Using $Y_n^i := Y_n\left(\frac{i}{n}\right)$ as above, and without loss of generality assume $Y_n^0 = 0$ and $b(Y_n^0) = 0$, so that

$$\mathbb{E}\left(\left|Y_n^1\right|^{2p}\right) = \mathbb{E}\left(\left|\frac{b\left(Y_n^0\right)}{n} + \frac{a\left(Y_n^0\right)}{\sigma\sqrt{n}}\xi_1\right|^{2p}\right) \leq \frac{C}{n^p}\mathbb{E}\left(|\xi_1|^{2p}\right).$$

Assumption 2.7 yields
$\mathbb{E}\left(|Y_n^2|^{2p}\right) = \left(\left|\frac{1}{n}\sum_{k=1}^2 b\left(Y_n^{k-1}\right) + \frac{1}{\sigma\sqrt{n}}\sum_{k=1}^2 a\left(Y_n^{k-1}\right)\xi_k\right|^{2p}\right)$
$\leq \left\{\mathbb{E}\left[|Y_n^1|\right] + \frac{1}{\sqrt{n}}\mathbb{E}\left[\left(\left|\frac{C_b Y_n^1}{\sqrt{n}}\right| + \frac{\left|b(Y_n^0)\right|}{\sqrt{n}} + \frac{C_a}{\sigma}\sqrt{\rho\left(|Y_n^1|\right)}\xi_2 + |a(Y_n^0)\xi_2|\right)\right]\right\}^{2p}$
$\leq \frac{C}{n^p}\mathbb{E}\left(\left(|\xi_1| + |\xi_2|\right)^{2p}\right)$.
By induction we find $\mathbb{E}\left(|Y_n^i - Y_0|^{2p}\right) \leq \frac{C}{n^p}\mathbb{E}\left[\left(\sum_{k=1}^i |\xi_i|\right)^{2p}\right]$, which implies the tightness criterion (3.4) for $p > 1$ for $\alpha = 2p$ and $\beta = p - 1$. □

We have set the ground to extend our results to processes that are not necessarily 1/2-Hölder continuous, Markovian nor semimartingales. More precisely, we are interested in α-Hölder continuous paths with $\alpha \in (0, 1)$, such as Riemann-Liouville fractional Brownian motion. A key tool is the Continuous Mapping Theorem which establishes the preservation of weak convergence under continuous operators.

Theorem 3.7 (Continuous Mapping Theorem [14]). *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be two normed spaces and assume that $g : \mathcal{X} \to \mathcal{Y}$ is a continuous operator. If the sequence of random variables $(Z_n)_{n\geq 1}$ converges weakly to Z in $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$, then $(g(Z_n))_{n\geq 1}$ also converges weakly to $g(Z)$ in $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$.*

Many authors have exploited the combination of Theorems 3.3 and 3.7 in order to prove weak convergence [27, Chapter IV]. This path avoids the lengthy computations of tightness and finite-dimensional convergence in classical proofs [6]. In fact, Hamadouche [10] already realised that Riemann-Liouville fractional operators are continuous, hence Theorem 3.7 holds under mapping by Hölder continuous functions. In contrast, the novelty of our approach is to consider, on the one hand the family of GFO applied to a Brownian motion, and on the other hand the extension of Brownian motion to Itô diffusions. In fact, minimal changes to the proof in Proposition 2.8 yield the following:

Corollary 3.8. *If Y is the solution to (2.7), then $(\mathcal{G}^\alpha Y)(t) = \int_0^t g(t-s)\mathrm{d}Y_s$ almost surely for all $t \in \mathbb{I}$.*

The analogue of Theorem 3.6 for Y follows by continuous mapping along with the fact that $\mathcal{G}^\alpha$ is a continuous operator from $\left(\mathcal{C}^\lambda(\mathbb{I}), \|\cdot\|_\alpha\right)$ to $\left(\mathcal{C}^{\lambda+\alpha}(\mathbb{I}), \|\cdot\|_\alpha\right)$ for all $\lambda \in (0,1)$ such that $(\alpha, \lambda) \in \mathfrak{R}$.

Theorem 3.9 (Generalised rough Donsker). *For (Y_n) in (3.2), Y its weak limit in $\left(\mathcal{C}^\lambda(\mathbb{I}), \|\cdot\|_\lambda\right)$ for $\lambda < \frac{1}{2}$,*

(3.5)
$$(\mathcal{G}^\alpha Y_n)(t) = \sum_{i=1}^{\lfloor nt \rfloor} n \left[G\left(t - \frac{i-1}{n}\right) - G\left(t - \frac{i}{n}\right)\right] \left(Y_n^i - Y_n^{i-1}\right) + nG\left(t - \frac{\lfloor nt \rfloor}{n}\right) \left(Y_n(t) - Y_n^{\lfloor nt \rfloor}\right)$$

converges weakly to $\mathcal{G}^\alpha Y$ in $\left(\mathcal{C}^{\alpha+\lambda}(\mathbb{I}), \|\cdot\|_{\alpha+\lambda}\right)$ for any $\alpha \in (-\frac{1}{2}, \frac{1}{2})$.

Proof. We apply directly the definition (2.2) of the GFO to the sequence (3.2), recalling that the latter is differentiable in time. For $\alpha > 0$, integration by parts yields, for any $n \geq 1$ and

$t \in \mathbb{I}$,

$$
\begin{aligned}
(\mathcal{G}^{\alpha} Y_n)(t) &= \int_0^t g'(t-s) Y_n(s) \mathrm{d}s = \int_0^t g(t-s) \frac{\mathrm{d}Y_n(s)}{\mathrm{d}s} \mathrm{d}s \\
&= \frac{1}{\sigma\sqrt{n}} \left[\sum_{i=1}^{\lfloor nt \rfloor} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} g(t-s)\, a\left(Y_n^{i-1}\right) \xi_i \mathrm{d}s + n \int_{\frac{\lfloor nt \rfloor}{n}}^{t} g(t-s)\, a\left(Y_n^{\lfloor nt \rfloor}\right) \xi_{\lfloor nt \rfloor + 1} \mathrm{d}s \right] \\
&+ \frac{1}{n} \left[n \sum_{i=1}^{\lfloor nt \rfloor} \int_{\frac{i-1}{n}}^{\frac{i}{n}} g(t-s)\, b\left(Y_n^{i-1}\right) \mathrm{d}s + n \int_{\frac{\lfloor nt \rfloor}{n}}^{t} g(t-s)\, b\left(Y_n^{\lfloor nt \rfloor}\right) \mathrm{d}s \right] \\
&= \sum_{i=1}^{\lfloor nt \rfloor} n \left[G\left(t - \frac{i-1}{n}\right) - G\left(t - \frac{i}{n}\right) \right] \left(Y_n^i - Y_n^{i-1}\right) + nG\left(t - \frac{\lfloor nt \rfloor}{n}\right) \left(Y_n(t) - Y_n^{\lfloor nt \rfloor}\right)
\end{aligned}
$$

since $G(0) = g(0) = 0$. When $\alpha < 0$, similar steps imply

$$
\begin{aligned}
(\mathcal{G}^{\alpha} Y_n)(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t g(t-s)\, Y_n(s) \mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t G(t-s) \frac{\mathrm{d}Y_n(s)}{\mathrm{d}s} \mathrm{d}s \\
&= \sum_{i=1}^{\lfloor nt \rfloor} n \left[G\left(t - \frac{i-1}{n}\right) - G\left(t - \frac{i}{n}\right) \right] \left(Y_n^i - Y_n^{i-1}\right) + nG\left(t - \frac{\lfloor nt \rfloor}{n}\right) \left(Y_n(t) - Y_n\left(\frac{\lfloor nt \rfloor}{n}\right)\right);
\end{aligned}
$$

when $\frac{\lfloor nt \rfloor}{n} = t$, $G(0) = 0$, and the expression is well defined. □

We may omit the interpolation term in Donker's linear interpolating sequence (3.1), and Lamperti's proof [13] still holds with the sequence $B_n(t) := \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i$, as the rightmost term in (3.1) converges to zero by Chebychev inequality. This statement also holds for the sequence (3.2), so that, $\mathcal{G}^{\alpha} Y_n$ in (3.5) reduces to

$$
(3.6) \qquad (\mathcal{G}^{\alpha} Y_n)(t) = \sum_{k=1}^{\lfloor nt \rfloor} g(t - t_{k-1}) \left(Y_n^k - Y_n^{k-1}\right) = \sum_{k=1}^{\lfloor nt \rfloor - 1} \left[g(t - t_{k-1}) - g(t - t_k) \right] Y_n^{k-1}
$$

which coincides with the usual left-point forward Euler approximation. For numerical purposes, (3.6) is much more efficient, since the integral G required in (3.5) is not necessarily available in closed form. The speed of convergence of the rDonsker scheme is not of order $\mathcal{O}\left(n^{-1/2}\right)$ as one might assume. In fact the Hurst parameter (in particular $\alpha \in (-\frac{1}{2}, 0)$) influences the speed of convergence adversely, as the following proposition shows.

Proposition 3.10. *The speed of convergence of the rDonsker scheme is of order* $\mathcal{O}\left(n^{-\alpha-1/2}\right)$ *when* $\alpha \in (-\frac{1}{2}, 0]$ *and* $\mathcal{O}\left(n^{-1/2}\right)$ *when* $\alpha \in (0, \frac{1}{2})$.

Proof. Let $\alpha \in (-\frac{1}{2}, 0]$. Since $g \in \mathcal{L}^\alpha$, the approximation (3.6) reads, for any $n \geq 1$,

$$(\mathcal{G}^\alpha Y_n)(t_i) = \frac{1}{n^{1/2-\alpha}\sigma} \sum_{k=1}^{i} (nt_i - (k+1)T)^\alpha L(t_i - t_{k-1}) \left(Y_n^k - Y_n^{k-1}\right) \sigma\sqrt{n}, \quad \text{for } i = 0, \ldots, n.$$

Here, $(nt_i - T(k-1))^\alpha \leq (t_i)^\alpha$ is bounded for any $n \geq 1$ so that the claim follows directly. When $\alpha \in (0, \frac{1}{2})$ we may rewrite (3.6) as

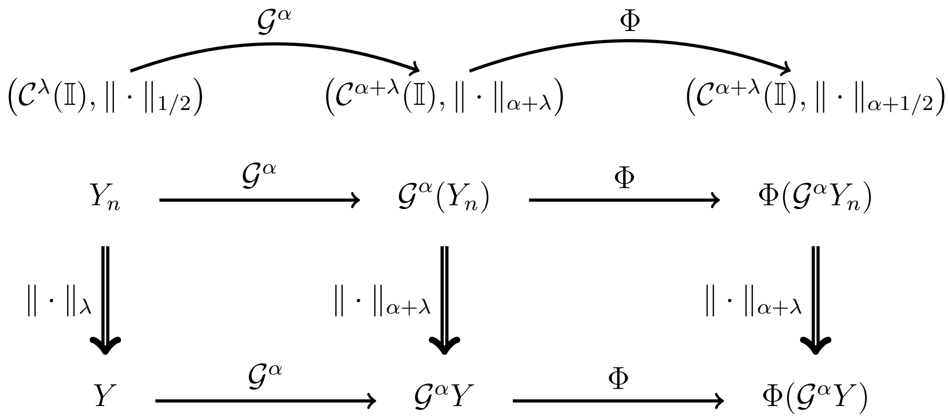
$$(\mathcal{G}^\alpha Y_n)(t_i) = \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{i} (t_i - t_{k-1})^\alpha L(t_i - t_{k-1}) \left(Y_n^k - Y_n^{k-1}\right) \sigma\sqrt{n}, \quad \text{for } i = 0, \ldots, n.$$

In this case, $(t_i - t_{k-1})^\alpha \leq (t_i)^\alpha$ is also bounded for any $n \geq 1$, and the proposition follows. □

So far, our results hold for a class of α-Hölder continuous functions. It is often necessary, at least for practical reasoning purposes, to constrain the volatility process $(V_t)_{t\in\mathbb{I}}$ to remain strictly positive at all times. The stochastic integral $\mathcal{G}^\alpha Y$ need not be so in general. However, a simple transformation (e.g. exponential) can easily overcome this fact. The remaining question is to know whether the α-Hölder continuity is preserved after this composition.

Proposition 3.11. *Let* $(Y_n)_{n\geq 1}$ *be the approximating sequence* (3.2). *Then* $(\Phi(\mathcal{G}^\alpha Y_n))$ *converges weakly to* $\Phi(\mathcal{G}^\alpha Y)$ *in* $\left(\mathcal{C}^{\alpha+1/2}(\mathbb{I}), \|\cdot\|_{\alpha+1/2}\right)$ *for any* $\alpha \in (-\frac{1}{2}, \frac{1}{2})$.

Proof. Drábek [9] found necessary and sufficient conditions ensuring that Hölder continuity is preserved under composition (which he calls Nemyckij operators). More precisely, he proved that the composition $f \circ g$ is continuous from $\left(\mathcal{C}^\lambda(\mathbb{I}), \|\cdot\|_\lambda\right)$ to $\left(\mathcal{C}^\lambda(\mathbb{I}, \|\cdot\|_\lambda\right)$ if and only if f is of class $\mathcal{C}^1$. The proof of the proposition then follows by applying the Continuous Mapping Theorem to Theorem 3.9 along with Drábek's continuity property. The following diagram summarises the steps, where $\lambda < 1/2$. The double arrows indicate weak convergence, and we indicate next to them the topology in which it takes place.



□

3.1. **Convergence of the (log-)stock process in the Hölder topology.** We extend here the convergence to the log-stock process maintaining the Hölder space framework. To start with, the Hölder regularity coefficient of an Itô integral with an integrand having λ-Hölder continuous paths is not at all obvious. The following proposition gives an answer to this question.

Proposition 3.12. *Let W be a standard Brownian motion, and Θ a càdlàg process on the same filtered probability space with finite moments up to order $2p$. Then $\Theta \bullet W \in \mathcal{C}^\lambda(\mathbb{I})$ for all $\lambda < \frac{1}{2}\left(1 - \frac{1}{p}\right)$.*

Proof. For this we will use Kolmogorov-Čentsov's continuity theorem [7].

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t \Theta(u)\mathrm{d}W_u - \int_0^s \Theta(u)\mathrm{d}W_u\right)^{2p}\right] &= \mathbb{E}\left[\left(\int_s^t \Theta(u)\mathrm{d}W_u\right)^{2p}\right] = \mathbb{E}\left[\left(\int_s^t \Theta(u)^2\mathrm{d}u\right)^{p}\right] \\ &\leq C(t-s)^{p-1}\left(\int_s^t \mathbb{E}\left[\Theta(u)^{2p}\right]\mathrm{d}u\right) \leq C(t-s)^p\end{aligned}$$

by Itô's isometry and Hölder's inequality along with the finite moments of Θ. Thus, by Kolmogorov's continuity criterion the stochastic integral $\Theta \bullet W$ has continuous paths with Hölder regularity$\frac{1-1/p}{2}$ for all $p \geq 1$. □

The finiteness of all moments might be too restrictive for some applications, and in fact this will be relaxed in Section 3.2 at the cost of switching to the Skorohod topology. Nevertheless, in the Hölder setting, once Proposition 3.12 applies, it suffices to prove continuity of the Itô map between the corresponding Hölder spaces.

Proposition 3.13. *The Itô map* $\Theta \mapsto \Theta \bullet W$ *is continuous from* $\mathcal{C}^{\lambda}(\mathbb{I})$ *to* $\mathcal{C}^{\upsilon}(\mathbb{I})$ *for all* $\lambda \in (0,1)$, $\upsilon < \frac{1}{2}$.

Proof. Let $f \in \mathcal{C}^{\lambda}(\mathbb{I})$ and $W \in \mathcal{C}^{\upsilon}(\mathbb{I})$. Since the Itô map is linear, it suffices to check boundedness.

$$\left\| \int_0^t f(s)\mathrm{d}W_s \right\|_{\Upsilon} \leq \left\| \int_0^t ||f||_{\lambda}\mathrm{d}W_s \right\|_{\upsilon} \leq ||f||_{\lambda} \, \|W_t\|_{\Upsilon} \leq ||f||_{\lambda} \left\| Ct^{1/2} \right\|_{\upsilon} \leq CT^{1/2}||f||_{\lambda},$$

where we have used the Hölder continuity of W, and the proposition follows. □

Finally, we present the main convergence result.

Theorem 3.14. *Let* $\Phi(\mathcal{G}^{\alpha}Y_n)$ *as in* (3.6) *with* $\xi_i \sim \mathcal{N}(0,1)$, *and weak limit* $\Phi(\mathcal{G}^{\alpha}Y)$ *in* $\left(\mathcal{C}^{\alpha+\lambda}(\mathbb{I}), \|\cdot\|_{\alpha+\lambda}\right)$, *for* $\lambda < \frac{1}{2}$ *and* $\alpha \in (-\frac{1}{2}, \frac{1}{2})$. *If* $\mathbb{E}[\mathrm{e}^{\Phi(\mathcal{G}^{\alpha}Y)}] < \infty$, *then the sequence defined by*

$$-\frac{1}{2n}\sum_{i=1}^{\lfloor nt \rfloor} \Phi\left(\mathcal{G}^{\alpha}Y_n\right)(t_i) + \frac{\rho}{\sqrt{n}}\sum_{i=1}^{\lfloor nt \rfloor} \sqrt{\Phi\left(\mathcal{G}^{\alpha}Y_n\right)(t_i)}\xi_i + \frac{\overline{\rho}}{\sqrt{n}}\sum_{i=1}^{\lfloor nt \rfloor} \sqrt{\Phi\left(\mathcal{G}^{\alpha}Y_n\right)(t_i)}\zeta_i$$

where (ζ_i) *is an iid family of* $\mathcal{N}(0,1)$ *random variables, converges weakly in* $\left(\mathcal{C}^{\lambda}(\mathbb{I}), \|\cdot\|_{\lambda}\right)$ *to*

$$-\frac{1}{2}\int_0^t \Phi\left(\mathcal{G}^{\alpha}Y\right)(s)\mathrm{d}s + \int_0^t \sqrt{\Phi\left(\mathcal{G}^{\alpha}Y\right)(s)}\left(\rho \mathrm{d}W_s + \overline{\rho}\mathrm{d}W_s^{\perp}\right).$$

Proof. The proof follows by repeatedly applying the continuous mapping theorem after Proposition 3.11. For the deterministic integral part one can easily prove that the integral mapping is continuous from $\mathcal{C}^{\alpha+\lambda}$ to $\mathcal{C}^{\lambda}$ using a similar argument to Proposition 3.13. Then we get

$$\int_0^t \Phi\left(\mathcal{G}^{\alpha}Y_n\right)(s)\,\mathrm{d}s = \sum_{i=1}^{\lfloor nt \rfloor}\int_{\frac{i}{n}}^{\frac{i+1}{n}} \Phi\left(\mathcal{G}^{\alpha}Y_n\right)\left(\frac{i}{n}\right)\mathrm{d}s = \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor}\Phi\left(\mathcal{G}^{\alpha}Y_n\right)\left(\frac{i}{n}\right).$$

For the stochastic integral by definition we have that $\Phi\left(\mathcal{G}^{\alpha}Y\right) \in L^1$ is well defined and the finiteness of all moments allows us to apply Proposition 3.12. Then using the continuity of the Itô map we obtain the following approximating sequence weakly convergent in $\mathcal{C}^{1/2}(\mathbb{I})$:

$$\int_0^t \sqrt{\Phi\left(\mathcal{G}^{\alpha}Y_n\right)(s)}\mathrm{d}Z_s = \sum_{i=1}^{\lfloor nt \rfloor}\int_{\frac{i}{n}}^{\frac{i+1}{n}} \sqrt{\Phi\left(\mathcal{G}^{\alpha}Y_n\right)(t_i)}\mathrm{d}Z_s = \sum_{i=1}^{\lfloor nt \rfloor}\sqrt{\Phi\left(\mathcal{G}^{\alpha}Y_n\right)(t_i)}\left(Z(t_{i+1}) - Z(t_i)\right).$$

Then, the problem reduces to being able to simulate the increments of Z exactly, taking into account that $\mathrm{corr}(Z,W) = \rho$ must also hold. Since the increments of Z are Gaussian we may easily construct this explicitly

$$Z(t_{i+1}) - Z(t_i) = \frac{1}{\sqrt{n}}\left(\rho\xi_i + \sqrt{1-\rho^2}\zeta_i\right)$$

where the independence of the iid $\mathcal{N}(0,1)$ sequences (ξ_i) and (ζ_i) is crucial for this to be exact. □

We used here the approximation (3.6), instead of (3.5), essentially for computational reasons. It is of course possible to use the latter, at the cost of increasing complexity of the approximating sequence due to the interpolating term involving double integrals, in general not available in closed form. Proposition 3.13 allows to maintain the Hölder space framework but only if the family (ξ_i) is restricted to be Gaussian, which is in any case sufficient for Monte-Carlo simulations. Nevertheless, the following proposition relaxes this condition.

Theorem 3.15. *Let the sequences $(\Phi(\mathcal{G}^\alpha Y_n), W_n)$ defined by 3.6 and 3.1 converge weakly to $(\mathcal{G}^\alpha Y, W))$ in the joint Hölder topology $\mathcal{C}^{\alpha+\lambda}\times\mathcal{C}^\lambda(\mathbb{I})$ for $\lambda < \frac{1}{2}$ and $\alpha \in (-\frac{1}{2}, \frac{1}{2})$. Assume further $\mathbb{E}[\mathrm{e}^{\Phi(\mathcal{G}^\alpha Y_n)}] < \infty$ and that the iid family (ξ_i) in Assumption 3.1 is bounded. Then the sequence of stochastic integrals $(\Phi(\mathcal{G}^\alpha Y_n) \bullet W_n)$ also converges to $\Phi(\mathcal{G}^\alpha Y) \bullet W$ in $\mathcal{C}^\lambda(\mathbb{I})$.*

Proof. We will make use of Theorem 3.4. Finite dimensional convergence follows from Jakubowski, Memin and Pagès [11], since the approximating sequence 3.1 with bounded random variables satisfies the Uniform Tightness (see [11] for details) criterion. Then it remains to prove tightness of the approximating sequence,

$$\begin{aligned}
\mathbb{E}\left[\left\{\sum_{j=ns}^{nt}\Phi\left(\mathcal{G}^\alpha Y_n\right)(t_j)\left(W_n(t_{j+1})-W_n(t_j)\right)\right\}^{2p}\right] &\le \frac{C}{n^{2p}}\mathbb{E}\left[\left(\sum_{j=ns}^{nt}\Phi\left(\mathcal{G}^\alpha Y_n\right)(t_j)^{2p}\right)^{2p}\right]\\
&\le \frac{C}{n^{2p}}\sum_{j=ns}^{nt}\mathbb{E}\left[\Phi\left(\mathcal{G}^\alpha Y_n\right)(t_j)^{2p}\right] \le \frac{C}{n^{2p}},
\end{aligned}$$

where we have made use of the boundedness of ξ, Jensen's inequality and the finiteness of all moments of $\Phi\left(\mathcal{G}^\alpha Y_n\right)$. The inequality then gives the desired convergence result in $\mathcal{C}^\lambda(\mathbb{I})$ □

As opposed to Theorem 3.14 (where the driving random variables are forced to be Gaussian), Theorem 3.15 allows to use any family of bounded random variables as approximating sequences of $W^\perp$ and any family random variables ensuring the moment condition $\mathbb{E}[\mathrm{e}^{\Phi(\mathcal{G}^\alpha Y_n)}] < \infty$. The gap between these two sets of conditions, that neither theorem covers, but this will be discussed in Section 3.2.

3.2. **Extending the weak convergence to the Skorohod space and proof of Theorem 3.2.** The Skorohod space of càdlàg processes equipped with the Skorohod topology has been widely used to prove weak convergence [6]. The Skorohod space of càdlàg processes equipped with the Skorohod norm, which we denote $(\mathcal{D}(\mathbb{I}), \|\cdot\|_{\mathcal{D}})$, markedly simplifies when we only consider continuous processes (as is the case of our framework with Hölder continuous processes). Billingsley [6, Chapter 3 Section 12] proved that the identity $(\mathcal{D}(\mathbb{I}) \cap \mathcal{C}(\mathbb{I}), \|\cdot\|_{\mathcal{D}}) \equiv (\mathcal{C}(\mathbb{I}), \|\cdot\|_\infty)$ always holds. This seemingly simple statement allows us to reduce proofs of weak convergence of continuous processes in the Skorohod topology to that in the supremum norm, usually much simpler. We start with the following straightforward observation:

Lemma 3.16. *The identity map is continuous from* $\left(\mathcal{C}^\lambda(\mathbb{I}), \|\cdot\|_\lambda\right)$ *to* $(\mathcal{D}(\mathbb{I}), \|\cdot\|_{\mathcal{D}})$ *for all* $\lambda \in (0,1)$.

Proof. Since the identity map is linear, it suffices to check that it is bounded. For this observe that $\|f\|_\lambda = |f|_\lambda + \sup_{t\in\mathbb{I}} |f(t)| = |f|_\lambda + \|f\|_\infty > \|f\|_\infty$, where $|f|_\lambda > 0$, which concludes the proof since the Skorohod norm in the space of continuous functions is equivalent to the supremum norm. □

Applying the Continuous Mapping Theorem twice, first with the Generalised fractional operator (Theorem 3.9), then with the identity map, yields the following result directly:

Theorem 3.17. *The sequence* $(\Phi(\mathcal{G}^\alpha Y_n))$ *converges weakly to* $\Phi\left(\mathcal{G}^\alpha Y\right)$ *in* $(\mathcal{D}(\mathbb{I}), ||\cdot||_{\mathcal{D}})$ *for any* $\alpha \in (-\frac{1}{2}, \frac{1}{2})$.

The final step in the proof of our main theorem, is to extend weak convergence to the log-stock price. For this, the following result on weak convergence of stochastic integrals $X \bullet Y := \int X \mathrm{d}Y$ due to Jakubowski, Memin and Pagès [11], and later generalised to SDEs by Kurtz and Protter [17] is the key ingredient.

Theorem 3.18. *Let* $(W_n)_{n\geq 1}$ *be as in* (3.1)*,* N *a càdlàg process on* $\mathbb{I}$*, and* $(N_n)_{n\geq 1}$ *an approximating sequence such that* (N_n, W_n) *converges weakly in* $(\mathcal{D}(\mathbb{I}^2), \|\cdot\|_{\mathcal{D}})$ *to* (N, W)*. Then, there exists a filtration* $\mathcal{H}$ *under which* W *is an* $\mathcal{H}$*-continuous martingale and* $(N_n, W_n, N_n \bullet W_n)_{n\geq 1}$ *converges weakly to* $(N, W, N \bullet W)$.

As noted in [17], the Skorohod topology in $\mathcal{D}(\mathbb{I}^2)$ is stronger than in $\mathcal{D}(\mathbb{I}) \times \mathcal{D}(\mathbb{I})$. In order to use this result, we first need to have the joint convergence of the two correlated driving Brownian motions W and Z. Let $(W_n)_{n\geq 1}$ and $(W_n^{\perp})_{n\geq 1}$ be two sequences as in (3.1), with weak limits W and $W^{\perp}$, and let $\overline{\rho} := \sqrt{1-\rho^2}$. Donsker's invariance implies that $(W_n, W_n^{\perp})_{n\geq 1}$ converges weakly to $(W, W^{\perp})$ in $(\mathcal{C}^{\alpha}(\mathbb{I}^2), \|\cdot\|_{\alpha})$, and hence by the Continuous Mapping Theorem with $f(x,y) := \left(x, \rho x + \sqrt{1-\rho^2}y\right)$, the sequence $(W_n, \rho W_n + \overline{\rho} W_n^{\perp})_{n\geq 1}$ converges weakly to $(W, \rho W + \overline{\rho} W^{\perp})$ in $(\mathcal{C}^{\alpha}(\mathbb{I}^2), \|\cdot\|_{\alpha})$ for all $\alpha < \frac{1}{2}$. Finally, the first term on the right-hand side of (3.3) converges weakly to $-\frac{1}{2}\int_0^T \Phi\left(\mathcal{G}^{\alpha} Y\right)(s)\mathrm{d}s$ by the Continuous Mapping Theorem, as the integral is a continuous operator from $(\mathcal{D}(\mathbb{I}), ||\cdot||_{\mathcal{D}})$ to itself. Since the couple (Y_n, W_n) converges weakly to (Y, W) in $(\mathcal{D}(\mathbb{I}^2), ||\cdot||_{\mathcal{D}})$, Theorem 3.18 implies that the second term on the right-hand side of (3.3) converges weakly to $\sqrt{\Phi(\mathcal{G}^{\alpha} Y)} \bullet W$, and Theorem 3.2 follows.

4. **Numerical implementation**

In this chapter, the core of this thesis is presented. The numerical implementation of the above mentioned results applied on rBergomi model is given trough two algorithms. Results as well as pros and cons are further commented. Both algorithms have been written based on instructions given in [1]. First, the rBergomi model is defined then the instructions for writing the algorithms are given with further comments on rBergomi model and finally the R-codes are introduced:

Definition 4.1. Rough Bergomi model introduced by Bayer, Friz and Gatheral [24], where

$$V_t = \xi_0(t)\mathcal{E}\left(2\nu C_H \int_0^t (t-s)^\alpha \mathrm{d}W_s\right),$$

with parameters $V_0, \xi_0(\cdot) > 0$, $\alpha \in (-\frac{1}{2}, \frac{1}{2})$, $\nu > 0$ given in definition of fractional Ornstein-Uhlenbeck process and $\mathcal{E}(\cdot)$ is the stochastic exponential. This corresponds exactly to (2.6) with $g(u) \equiv u^\alpha$, $Y = W$ and

$$\Phi(\varphi)(t) := \xi_0(t)\exp\left(2\nu C_H \varphi(t)\right)\exp\left\{-2\nu^2 C_H^2 \int_0^t (t-s)^{2\alpha}\mathrm{d}s\right\}.$$

Algorithm 4.2 (Simulation of rough volatility models)**.**

(1) Simulate two $\mathcal{N}(0,1)$ matrices $\{\xi_{j,i}\}_{\substack{j=1,\dots,M\\ i=1,\dots,n}}$ and $\{\zeta_{j,i}\}_{\substack{j=1,\dots,M\\ i=1,\dots,n}}$ with $\mathrm{corr}(\xi_{j,i}, \zeta_{j,i}) = \rho$;

(2) As can be seen above, the driver of volatility in rDonsker model is in fact Brownian motion, e.a. $Y = W$ following $\Delta Y_n^j(t_i) = \sqrt{T/n}\xi_{i,j}$

(3) Simulate M paths of the fractional driving process $((\mathcal{G}^\alpha Y_n)(t))_{t\in\mathcal{T}}$ using

$$(\mathcal{G}^\alpha Y_n)^j(t_i) := \sum_{k=1}^{i} g(t_{i-k+1})\Delta Y_n^j(t_k) = \sum_{k=1}^{i} g(t_k)\Delta Y_n^j(t_{i-k+1}), \quad i = 1,\dots,n \text{ and } j = 1,\dots,M.$$

The complexity of this step is in general of order $\mathcal{O}(n^2)$. However, this step is easily implemented using discrete convolution with complexity $\mathcal{O}(n\log n)$. With the vectors $\mathfrak{g} := (g(t_i))_{i=1,\dots,n}$ and $\Delta Y_n^j := (\Delta Y_n^j(t_i))_{i=1,\dots,n}$ for $j = 1,\dots,M$, we can write $(\mathcal{G}^\alpha Y_n)^j(\mathcal{T}) = \sqrt{\frac{T}{n}}(\mathfrak{g} * \Delta Y_n^j)$, for $j = 1,\dots,M$, where $*$ represents the discrete convolution operator.

(4) Use the forward Euler scheme to simulate the log-stock process, for all $i = 1, \dots, n$, $j = 1, \dots, M$, as

$$X^j(t_i) = X^j(t_{i-1}) - \frac{1}{2}\frac{T}{n}\sum_{k=1}^{i} \Phi\left(\mathcal{G}^\alpha Y_n\right)^j(t_{k-1}) + \sqrt{\frac{T}{n}}\sum_{k=1}^{i}\sqrt{\Phi\left(\mathcal{G}^\alpha Y_n\right)^j(t_{k-1})}\zeta_{j,k}.$$

As Bayer, Friz and Gatheral [24] and Bennedsen, Lunde and Pakkanen [29] pointed out, a major drawback in simulating rough volatility models is the very high variance of the estimators, so that a large number of simulations are needed to produce a decent price estimate. Nevertheless, the rDonsker scheme admits a very simple conditional expectation technique which reduces both memory requirements and variance while also admitting antithetic variates. This approach is best suited for calibrating European type options. We consider $\mathcal{F}_t^B = \sigma(B_s : s \le t)$ and $\mathcal{F}_t^W = \sigma(W_s : s \le t)$ the natural filtrations generated by the Brownian motions B and W. In particular the conditional variance process $V_t|\mathcal{F}_t^W$ is deterministic. As discussed by Romano and Touzi [31], and recently adapted to the rBergomi case by McCrickerd and Pakkanen [30], we can decompose the stock price process as

$$\mathrm{e}^{X_t} = \mathcal{E}\left(\rho\int_0^t \sqrt{\Phi\left(\mathcal{G}^\alpha Y\right)(s)}\mathrm{d}B_s\right)\mathcal{E}\left(\sqrt{1-\rho^2}\int_0^t \sqrt{\Phi\left(\mathcal{G}^\alpha Y\right)(s)}\mathrm{d}B_s^\perp\right) := \mathrm{e}^{X_t^1}\mathrm{e}^{X_t^2},$$

and notice that

$$X_t|(\mathcal{F}_t^W \wedge \mathcal{F}_0^B) \sim \mathcal{N}\left(\mathrm{e}^{X_t^1} - (1-\rho^2)\int_0^t \Phi\left(\mathcal{G}^\alpha Y\right)(s)\mathrm{d}s, (1-\rho^2)\int_0^t \Phi\left(\mathcal{G}^\alpha Y\right)(t)\mathrm{d}s\right).$$

Thus $\exp(X_t)$ becomes log-normal and the Black-Scholes closed-form formulae are valid here (European, Barrier options, maximum,...). The advantage of this approach is that the orthogonal Brownian motion $B^\perp$ is completely unnecessary for the simulation, hence the generation of random numbers is reduced to a half, yielding proportional memory saving. Not only this, but also this simple trick reduces the variance of the Monte-Carlo estimate, hence fewer simulations are needed to obtain the same precision. We present a simple algorithm to implement the rDonsker with conditional expectation and assuming that $Y = W$.

Algorithm 4.3 (Simulation of rough volatility models with Brownian drivers)**.** Consider the equidistant grid $\mathcal{T}$.

(1) Draw a random matrix $\{\xi_{j,i}\}_{\substack{j=1,\dots,M/2\\ i=1,\dots,n}}$ with unit variance, and create antithetic variates $\{-\xi_{j,i}\}_{\substack{j=1,\dots,M/2\\ i=1,\dots,n}}$;

(2) Simulate M paths of the fractional driving process $\mathcal{G}^\alpha W$ using discrete convolution:

$$(\mathcal{G}^\alpha W)^j(\mathcal{T}) = \sqrt{\frac{T}{n}}(\mathfrak{g} * \xi_j), \quad j = 1, \dots, M,$$

and store in memory $(1-\rho^2)\int_0^T (\mathcal{G}^\alpha W)^j(s)\mathrm{d}s \approx (1-\rho^2)\frac{T}{n}\sum_{k=0}^{n-1}(\mathcal{G}^\alpha W)^j(t_k) =: \Sigma^j$ for each $j = 1, \dots, M$;

(3) use the forward Euler scheme to simulate the log-stock process, for each $i = 1, \dots, n$, $j = 1, \dots, M$, as

$$X^j(t_i) = X^j(t_{i-1}) - \frac{\rho^2}{2}\frac{T}{n}\sum_{k=1}^{i}\Phi\left(\mathcal{G}^\alpha W\right)^j(t_{k-1}) + \rho\sqrt{\frac{T}{n}}\sum_{k=1}^{i}\sqrt{\Phi\left(\mathcal{G}^\alpha W\right)^j(t_{k-1})}\xi_{j,i};$$

(4) Finally, we have $X^j(T) \sim \mathcal{N}(X_T^j - \Sigma^j, \Sigma^j)$ for $j = 1, \dots, M$; we may compute any option using the Black-Scholes formula. For instance a Call option with strike K would be given by

$$C^j(K) = \exp(X_T^j)\mathcal{N}(d_1^j) - K\mathcal{N}(d_2^j) \text{ for } j = 1, \dots, M,$$

where

$$d_1^j := \frac{1}{\sqrt{\Sigma^j}}(X_T^j - \log(K) + \frac{1}{2}\Sigma^j) \text{ and } d_2^j = d_1^j - \sqrt{\Sigma^j}.$$

Thus, the output of the model would be $C(K) = \frac{1}{M}\sum_{k=1}^{M} C^j(K)$.

Remark 4.4. These algorithms have been used in order to create the R-codes which simulate the rBergomi model. Of course, they are only general guidelines for implementation of a larger group of rough volatility models. In order to write the codes each model has to be further studied.

Looking at the definition of the rough Bergomi model, one can observe that not all the information needed for implementation of this specific model is given above. The articles Pricing under rough volatility by Bayer, Friz and Gatheral [24] and Volatility is rough by Gatheral, Jaisson and Rosenbaum [22] provide the information about the missing components C_H, and $\xi_0(\cdot)$.

Let us start with C_H. As stated in [22], empirically, the increments of the log-volatility of various assets enjoy a scaling property with constant smoothness parameter and that their distribution is close to Gaussian. This naturally suggests the simple model:

$$\log \sigma_{t+\Delta} - \log \sigma_t = \nu \left(W^H_{t+\Delta} - W^H_t \right), \tag{4.1}$$

where W^H is a fractional Brownian motion with Hurst parameter equal to the measured smoothness of the volatility and ν is a positive constant. We may of course write (4.1) under the form

$$\sigma_t = \sigma \exp \left\{ \nu \, W^H_t \right\}, \tag{4.2}$$

where σ is another positive constant. Realized variance was found to be consistent with this simple model as the relationship was found to hold for all 21 equity indices in the Oxford-Man database, Bund futures, Crude Oil futures, and Gold futures according to [24]. Furthermore consider the Mandelbrot-Van Ness representation of fractional Brownian motion W^H in terms of Wiener integrals:

$$W^H_t = C_H \left\{ \int_{-\infty}^{t} \frac{dW_s}{(t-s)^\gamma} - \int_{-\infty}^{0} \frac{dW_s}{(-s)^\gamma} \right\}$$

where $\gamma = \frac{1}{2} - H$ and the choice

$$C_H = \frac{\Gamma(3/2 - H)}{\Gamma(H + 1/2)\, \Gamma(2 - 2H)}.$$

ensures that W^H satisfies the definition, i.e.

$$\mathrm{Cov}(W^H_t, W^H_s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$$

thus, we obtained the necessary C_H. Now, the only thing left unknown is $\xi_0(\cdot)$. In Bayer, Fritz and Gatheral[24] it is stated that

$$\xi_t(u) = \mathbb{E}[v_u \mid \mathcal{F}_t], \quad u \geq t$$

is a forward variance curve, where $v_t^2 = \sigma_t^2$ denotes instantaneous variance at time u > t. This means that in our case $\xi_0(\cdot)$ represents the evolution of variance expressed as the expected value of the variance of an asset based on the information available up until the time of our investment. Normally, the forward variance curve can be obtained from the Market but this is not the case here because one has to be actively involved in Market trading in order to obtain such information so we turn to an alternative way of constructing the forward variance curve. The idea is to obtain the plain vanilla call and put prices for as many different strikes as possible from the internet, then to compute the price of the log contract for available expiration dates and finally, extract the forward variance curve by differentiation. One can of course argue that this approach influences the precision of our simulations but no better alternative was found.

4.1. **Construction of forward variance curve.**

Definition 4.5. The log contract is the future type contract whose payoff is equal to the logarithm of the price of an asset at the time of expiration.

As shown in [21], we consider a contract whose payoff at time T is $\log(S_T/\text{F})$ where F represents the time-T forward price of the stock. Then

$$g^{"}(K) = -\frac{1}{S^2_{T\,|S_T=K}}$$

and it follows from the payoff relation

$$\mathbb{E}[g(S_T) \mid S_t] = g(F) + \int_0^F dK\widetilde{P}(K)g^{"}(K) + \int_0^F dK\widetilde{C}(K)g^{"}(K)$$

,with $\widetilde{P}$ and $\widetilde{C}$ being undiscounted put and call prices respectively and $k := \log(\frac{K}{F})$, that

$$\mathbb{E}\left[\log\left(\frac{S_T}{F}\right)\right] = -\int_{-\infty}^0 dkp(k) - \int_0^\infty dkc(k)$$

with

$$c(y) := \frac{\widetilde{C}(Fe^y)}{Fe^y} \quad ; \quad p(y) := \frac{\widetilde{P}(Fe^y)}{Fe^y}$$

representing option prices expressed in terms of percentage of the strike price. In settings of no interest rates or dividends $F = S_0$ hence

$$\log\left(\frac{S_T}{S_0}\right) = \int_0^T d\log(S_t) = \int_0^T \frac{dS_t}{S_t} - \int_0^T \frac{\sigma^2_{S_t}}{2}dt$$

One can immediately see that the second term on the right side is the half of total variance which we are trying to obtain. Taking the risk neutral expectation of this equation, we get

$$\mathbb{E}\left[\int_0^T \sigma^2_{S_t}dt\right] = -2\mathbb{E}\left[\log\left(\frac{S_T}{F}\right)\right] = 2\quad\left\{\int_{-\infty}^0 dkp(k) + \int_0^\infty dkc(k)\right\}$$

It is now explicitly shown that the fair value of total variance is given by the value of an infinite strip of European options in a completely model independent way so long as the underlying process is a diffusion. Another small problem here is the assumption that we can obtain prices of European call and put option for all strikes $K \in \mathbb{R}$. In this paper, the integral is obtained via upper and lower Darboux sums with mash size being equal to the difference of obtainable strikes for which we have both, European call and put, prices. In short, we use the arithmetic middle of upper and lower Darboux sum to calculate this integral with respect to strike and we do it for different maturity dates. After calculating this integral, differentiation with respect to time was calculated in order to obtain the forward variance curve. Finally we have all the ingredients necessary for our implementation and now we move on to the R codes.

4.2. **R codes.**

The asset chosen for testing the implementation was S&P 500 Index. Because of its importance, European call and put prices are available for a decent number of strikes which makes our forward variance curve more accurate and for maturities up to 13 weeks. The obtained European call and put prices for available strikes were sorted in excel tables and then imported into RStudio. The prices of call and put options for different strikes are taken as arithmetic middle of bid/ask spread. Next graphic shows how the data is presented on the webpage marketwatch.com used to obtain them.

https://www.marketwatch.com/investing/index/spx/options

Home News Viewer Markets Investing Personal Finance Retirement Economy Real Estate Entertainment Watchlist Alerts

SEPTEMBER, 2018 OPTIONS Hide

CALLS PUTS

Expires September 4, 2018

Symbol	Last	Change	Vol	Bid	Ask	Open Int.	Strike	Symbol	Last	Change	Vol	Bid	Ask	Open Int.
1,500	quote	0.08	-0.07	247.00	0.05	0.05	4,240							
quote	1,354	0.00	60.00	1,354	1,362		1,550	quote	0.12	-0.03	27.00	0.05	0.05	4,231
1,600	quote	0.05	-0.07	4.00	0.05	0.05	8,389							
quote	0.00	0.00	0.00	1,255	1,256		1,650	quote	0.05	-0.06	8.00	0.05	0.05	4,124
quote	0.00	0.00	0.00	1,205	1,206		1,700							
quote	0.00	0.00	0.00	1,104	1,106		1,800							
quote	0.00	0.00	0.00	1,054	1,056		1,850	quote	0.10	-0.05	1.00	0.10	0.05	63.00
quote	993.58	22.73	1.00	1,005	1,006	2.00	1,900							
1,950	quote	0.10	-0.15	7.00	0.05	0.05	16.00							
quote	870.60	0.00	1.00	904.70	906.20	1.00	2,000	quote	0.10	-0.15	5.00	0.05	0.05	132.00
2,050	quote	0.10	-0.20	10.00	0.05	0.05	115.00							
2,100	quote	0.05	-0.10	11.00	0.05	0.05	2,332							
quote	0.00	0.00	0.00	754.30	755.80		2,150	quote	0.10	-0.02	25.00	0.05	0.05	99.00
2,200	quote	0.05	-0.05	5.00	0.05	0.05	1,153							
quote	0.00	0.00	0.00	629.80	631.30		2,275							
2,300	quote	0.11	-0.04	2.00	0.05	0.05	143.00							
2,325	quote	0.05	-0.15	4.00	0.05	0.05	120.00							
quote	0.00	0.00	0.00	554.80	556.30		2,350	quote	0.05	-0.25	50.00	0.05	0.05	179.00
2,400	quote	0.04	0.01	1.00	0.05	0.05	107.00							
quote	0.00	0.00	0.00	494.40	495.90		2,410	quote	0.00	0.00	0.00	0.05	0.10	4.00
2,425	quote	0.10	-0.80	3.00	0.05	0.10	385.00							
quote	371.65	0.00	1.00	454.90	456.30	1.00	2,450							
quote	0.00	0.00	0.00	429.90	431.40		2,475	quote	0.05	-0.02	13.00	0.05	0.10	29.00
quote	0.00	0.00	0.00	404.90	406.40		2,500	quote	0.05	-0.05	312.00	0.05	0.10	6,043
quote	0.00	0.00	0.00	394.90	396.40		2,510	quote	0.40	-0.60	2.00	0.05	0.10	2.00
2,520	quote	0.21	-2.44	2.00	0.05	0.10	6.00							
quote	0.00	0.00	0.00	379.90	381.40		2,525							
2,530	quote	0.10	-1.20	1,531	0.05	0.10	1,283							
quote	0.00	0.00	0.00	349.50	351.00		2,555							
quote	0.00	0.00	0.00	344.50	346.00		2,560	quote	0.10	-0.50	75.00	0.05	0.10	4.00
quote	0.00	0.00	0.00	334.50	336.00		2,570							
quote	0.00	0.00	0.00	329.50	331.00		2,575	quote	0.65	-0.05	2.00	0.05	0.10	765.00
quote	0.00	0.00	0.00	315.00	316.40		2,590	quote	0.18	-0.37	115.00	0.05	0.15	120.00
quote	0.00	0.00	0.00	265.00	266.50		2,640							
quote	0.00	0.00	0.00	260.00	261.50		2,645	quote	0.60	-0.92	300.00	0.05	0.15	303.00
quote	0.00	0.00	0.00	255.00	256.50		2,650	quote	0.15	-0.10	120.00	0.05	0.15	3,829
quote	0.00	0.00	0.00	250.00	251.50		2,655	quote	0.15	-0.50	120.00	0.10	0.20	606.00
2,670	quote	0.15	-0.60	120.00	0.10	0.20	602.00							
quote	0.00	0.00	0.00	230.00	231.50		2,675							
quote	229.58	37.31	1.00	225.10	226.50	1.00	2,680	quote	0.25	0.05	10.00	0.10	0.20	530.00
quote	0.00	0.00	0.00	220.10	221.50		2,685	quote	0.20	-0.47	51.00	0.10	0.20	1,913
quote	0.00	0.00	0.00	215.10	216.60		2,690							
quote	0.00	0.00	0.00	210.10	211.60		2,695	quote	0.20	-0.25	200.00	0.10	0.20	3,673
2,700	quote	0.15	-0.20	1,320	0.15	0.25	4,140							
quote	0.00	0.00	0.00	199.80	201.30		2,705	quote	0.25	-0.10	33.00	0.10	0.20	1,564
2,710	quote	0.20	-0.60	23.00	0.15	0.25	892.00							
2,715	quote	0.35	0.10	115.00	0.15	0.25	2,831							
quote	139.23	35.93	2.00	185.10	186.60	2.00	2,720							
2,725	quote	0.20	-0.20	54.00	0.15	0.25	3,272							
quote	114.30	0.00	2.00	165.20	166.60	2.00	2,740							
quote	102.60	15.90	14.00	160.20	161.70	35.00	2,745	quote	0.25	-0.20	68.00	0.20	0.25	3,055
2,750	quote	0.25	-0.25	100.00	0.20	0.25	7,587							
quote	73.60	0.00	1.00	150.20	151.70	1.00	2,755	quote	0.45	0.15	17.00	0.20	0.30	160.00
quote	116.88	0.00	1.00	145.20	146.70	1.00	2,760	quote	0.25	-0.25	20.00	0.20	0.30	1,089
2,775	quote	0.45	-0.16	5.00	0.20	0.30	6,592							
2,785	quote	0.33	-0.41	29.00	0.25	0.35	168.00							
2,795	quote	0.68	0.28	78.00	0.25	0.40	1,106							
2,800	quote	0.35	-0.35	1,579	0.35	0.40	2,054							
quote	106.45	39.55	1.00	100.30	101.80	30.00	2,805							
2,810	quote	0.40	-0.25	5.00	0.35	0.45	1,493							
quote	90.60	8.20	10.00	89.90	91.40	85.00	2,815	quote	0.60	-0.10	300.00	0.35	0.45	619.00
quote	91.30	14.30	8.00	85.10	86.60	136.00	2,820	quote	0.45	-0.43	36.00	0.40	0.50	668.00
quote	86.30	11.65	6.00	80.20	81.70	357.00	2,825							
quote	71.50	-5.30	32.00	65.30	66.70	252.00	2,840	quote	0.60	-0.75	612.00	0.55	0.65	1,379
2,845	quote	0.80	-0.46	11.00	0.60	0.70	1,331							
quote	51.40	-14.25	67.00	55.40	56.90	826.00	2,850							
quote	49.84	3.58	2.00	50.50	52.00	139.00	2,855							
quote	35.56	-4.59	6.00	40.80	42.20	1,458	2,865							
quote	37.00	3.63	21.00	36.30	37.70	1,751	2,870	quote	1.35	-1.05	218.00	1.25	1.45	1,516
quote	26.88	1.33	18.00	24.70	26.00	1,973	2,880							
2,885	quote	2.60	-1.80	107.00	2.50	2.90	998.00							
quote	14.80	-0.81	117.00	18.20	19.00	575.00	2,890	quote	3.25	-2.09	190.00	3.30	3.80	1,546
quote	14.79	2.59	156.00	12.70	13.20	1,081	2,895							
quote	10.05	0.04	312.00	9.40	10.00	2,543	2,900							
							2,903	Current price as of 8/31/2018 10:45:10 AM						
quote	7.40	0.09	271.00	6.70	7.10	1,088	2,905	quote	7.80	-3.05	79.00	8.00	8.70	955.00
quote	4.90	-0.40	232.00	4.60	4.90	841.00	2,910	quote	11.05	-2.53	37.00	10.40	11.60	188.00
quote	2.25	0.00	352.00	2.00	2.15	1,970	2,920							
2,925	quote	19.75	2.95	10.00	22.00	23.50	27.00							
2,930	quote	25.16	-5.63	32.00	24.10	26.20	60.00							
quote	0.45	-0.30	121.00	0.65	0.80	2,342	2,935							
quote	0.25	-0.10	1,330	0.20	0.25	2,082	2,950							
quote	0.20	-0.10	47.00	0.05	0.10	67.00	2,985							
3,000	quote	0.00	0.00	0.00	94.00	95.50								
3,025	quote	0.00	0.00	0.00	118.60	120.00								
quote	0.05	-0.05	460.00	0.05	0.05	3,412	3,050	quote	0.00	0.00	0.00	143.50	145.00	
3,075	quote	0.00	0.00	0.00	168.50	170.00								
quote	0.05	-0.05	364.00	0.05	0.05	1,436	3,100	quote	0.00	0.00	0.00	193.50	195.00	
quote	0.05	0.02	6.00	0.05	0.05	52.00	3,150							
quote	0.05	0.00	1.00	0.00	0.05	1.00	3,200							
quote	0.00	0.00	0.00	0.00	0.05		3,250	quote	0.00	0.00	0.00	343.40	344.90	
quote	0.00	0.00	0.00	0.00	0.05		3,300							

FIGURE 1. An example of data for different strikes and single maturity date

4.2.1. ***Forward variance curve.***

```
library(stats)
library(pracma)
library("xlsx")
a <- 0
c <- 0
p <- 0
d <- 0
Ic <- 0
Ip <- 0
min <- 200
ind <- 0

i <- 1
fK <- c(length(table_12$X__1))
fcall <- c(length(table_12$X__1))
fput <- c(length(table_12$X__1))

for(l in 1:length(table_12$X__1)) {
    fK[l]<-table_12[l,8]$X__8
    fcall[l]<-table_12[l,9]$X__9
    fput[l]<-table_12[l,7]$X__7
}

LK <- log(fK,10)

for (i in 1:(length(fK)-1))
  {
    diff <- fcall[i]-fput[i]
    if(diff < 0) {diff <- (-diff)}
```

```
    if(diff < min)
      {
        min <- diff
        ind <- i
    }
}
for (j in 1:(ind-1))
  {
    a <- (LK[j+1]-LK[j])
    p <- ((fput[j+1]+fput[j])/2)
    d <- ((LK[j+1]+LK[j])/2)
    Ip <- Ip + (a*(p/d))
}
for (k in ind:(length(K)-1))
  {
    c <- ((fcall[k+1]+fcall[k])/2)
    a <- (LK[k+1]-LK[k])
    d <- ((LK[k+1]+LK[k])/2)
    Ic <- Ic + (a*(c/d))
}

#obtained values

fwdvc <- c(length(13))
fw <-c(0.01838119, 0.11818865, 0.21840105, 0.35303439, 0.45754196, 0.58420218,
       0.73187350, 0.91971890, 1.10343434, 1.21383215, 1.39843313, 1.58303411,
       1.79916553)
fwdvc[1] <- fw[1]

for(o in 2:13) {
  fwdvc[o] <- fw[o]-fw[o-1]
```

```
}

plot(fwdvc[1:13], xlab="time (weeks)",
     ylab="Expectation of sigma squared (variance)", main="plot", type="l")

points(0, 0, type = "l")
```

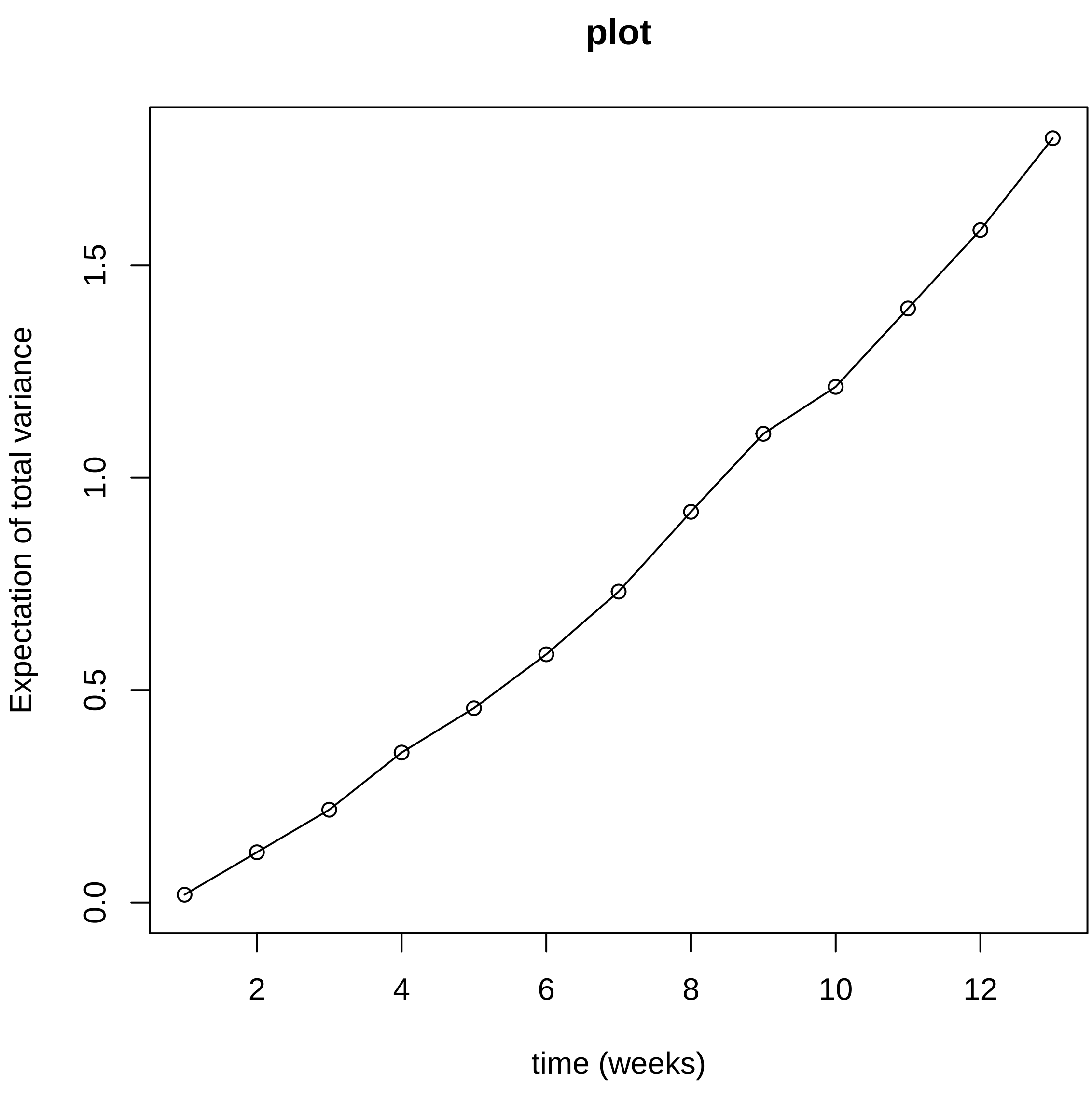


Figure 2. Total variance curve

Remark 4.6. One may notice that the obtained results were written down and saved in a vector at the end of the program, that is done in order to prevent the tedious import of excel tables in case of something going wrong. Furthermore, if we consider the plot of expected values of forward variance and take into consideration that the differentiation with respect to time was done for a time unit of one week, then the values of our forward variance curve, are actually the slopes of lines connecting the points of the plot.

4.2.2. ***Algorithm 4.2.***

```
library(stats)
library(pracma)

n <- 182
m <- 1000
t <- 0.07
alph <- 0.1
H <- 0.1
nu <- 2.8
acall <- 0
aput <- 0
rho <- 0.5
put <- 0
call <- 0
ksi <- 0
var <- c(length(13))
sp <- c(2760, 2755, 2760, 2760, 2770, 2745, 2760, 2755, 2750, 2760, 2765, 2765,
        2760)
```

```
start_time <- Sys.time()

#foward variance curve

fwdvc <- c(length(13))
fw <- c(0.01838119, 0.11818865, 0.21840105, 0.35303439, 0.45754196, 0.58420218,
        0.73187350, 0.91971890, 1.10343434, 1.21383215, 1.39843313, 1.58303411,
        1.79916553)
fwdvc[1] <- fw[1]
for(o in 2:13) {
  fwdvc[o] <- fw[o]-fw[o-1]
}


#simulating correlated matrices

matrica <- matrix(rnorm(n*m,mean=0, sd=1),m,n)
tmpMatrica <- matrix(rnorm(n*m, mean=0, sd=1),m,n)
matrica2 <- rho*matrica + sqrt(1-rho*rho)*tmpMatrica
matrica3 <- matrix(nrow=m, ncol=n)

#volatility simulation

sum(matrica*matrica2)/(m*n)
for (j in 1:m)
  {for (i in 1:n)
    {
     G <- 0
    for (k in 1:i)
    {
```

```
        G <- + ((((i-k+1)*(t/n))^alph)*sqrt(t/n)*matrica[j,k])
        }
      matrica3[j,i] <- G
      }
}

#simulating prices

matrica4 <- matrix(nrow=m, ncol=n)
CH <- sqrt((2*H*gamma(1.5-H))/(gamma(H+0.5)*gamma(2-2*H)))

for (j in 1:m)
 {  X <- log(2760,exp(1))

   for (i in 1:n)
    {
      S1 <- 0
      S2 <- 0
      if(i %% 14 == 1){ksi <- fwdvc[(i+13)/14]}
   for (k in 1:i)
       {
      integrand <- function(s) {(((k*t)/n)-s)^(2*alph)}
      Itp <- integrate(integrand, lower = 0, upper = ((k*t)/n))
      I <- Itp$value
      Y <- (-2)*(nu*nu)*(CH*CH)*I
      Z <- 2*nu*CH*matrica3[j,k]
      S1 <- + (0.5*t/n*ksi*exp(Z)*(exp(Y)))
      S2 <- + (sqrt(t/n*ksi*exp(Z)*(exp(Y)))*matrica2[j,k])
   }
     W <- (S2-S1)
     X <- X + W
```

```
    matrica4[j,i] <- exp(X)
    }
}

plot(matrica4[1,1:n], xlab="time", ylab="price", main="plot", type="l")
points(0,2760, type = "l")
lines(matrica4[33,1:n], col="darkgreen")
lines(matrica4[50,1:n], col="brown")
lines(matrica4[89,1:n], col="darkblue")

#pricing asian options with strike K

K <- 2760
avgp <- c(length(m))
for (j in 1:m) {
    avg <- 0
    for(i in 1:n) {
    avg <- avg + matrica4[j,i]
    }
     avgp[j] <- avg/n
}

for (j in 1:m) {
    put <- put + (max(0,K-avgp[j]))
    call <- call + (max(0,avgp[j]-K)) }
acall <- call/m
aput <- put/m
print(acall)
print(aput)

end_time <- Sys.time()
```

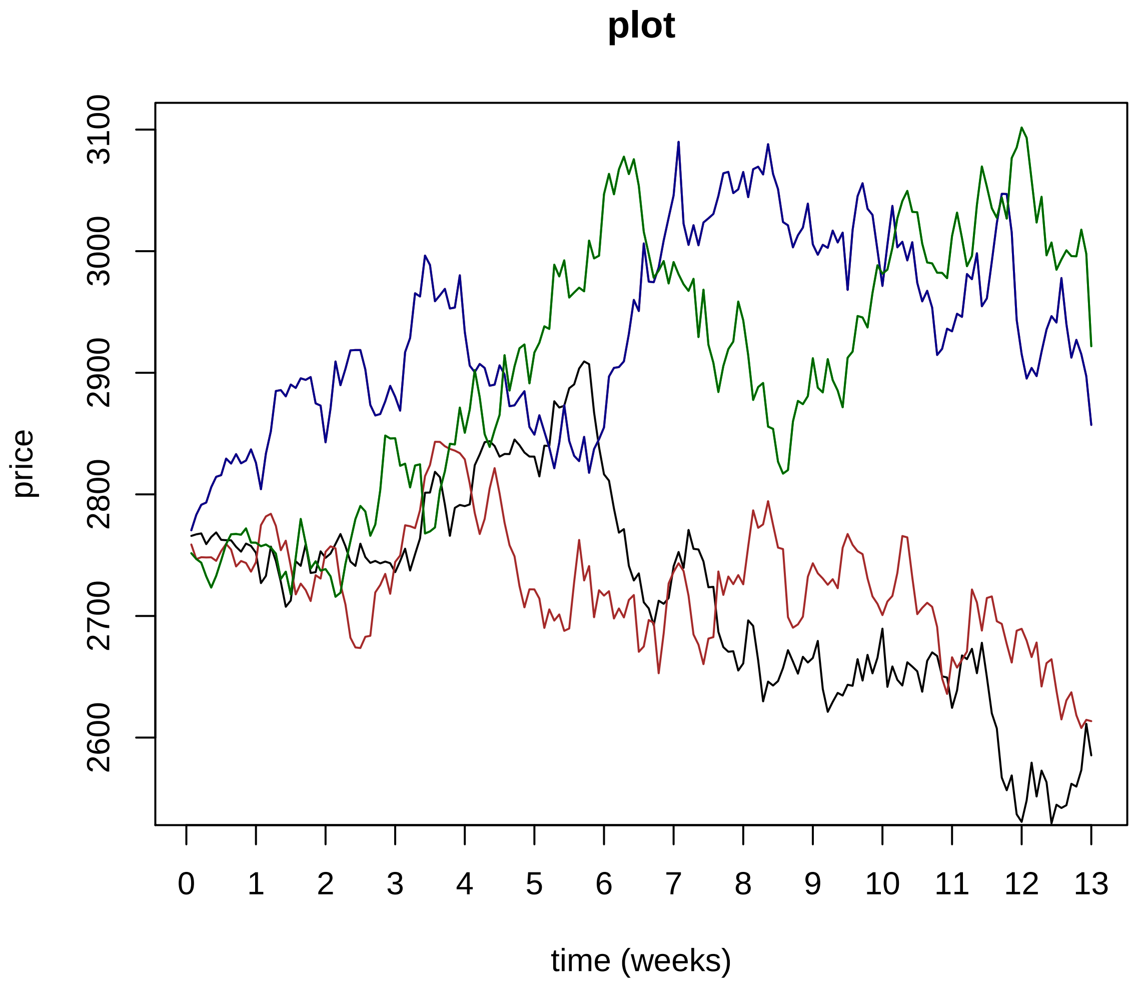


FIGURE 3. Path simulations of Algorithm 4.2

```
end_time − start_time
```

Remark 4.7. The vector sp contains the weekly forward prices of the asset during 13 weeks. Also, Sys.time() was introduced in order to measure the speed of both algorithms. Variables m and n denote the number of simulated paths and number of discretisation points during 13 weeks respectively.

4.2.3. ***Algorithm 4.3.***

```
library(stats)
library(pracma)

m <- 1000
n <- 182
alph <- 0.1
H <- 0.1
ksi <- 0
nu <- 2.8
t <- 0.07
call <- 0
put <- 0
acall <- 0
aput <- 0
var <- c(length(13))
sp <- c(2760, 2755, 2760, 2760, 2770, 2745, 2760, 2755, 2750, 2760, 2765, 2765,
        2760)
lg <- 0
dg <- 0

start_time <- Sys.time()

#foward variance curve

fwdvc <- c(length(13))
fw <- c(0.01838119, 0.11818865, 0.21840105, 0.35303439, 0.45754196, 0.58420218,
        0.73187350, 0.91971890, 1.10343434, 1.21383215, 1.39843313, 1.58303411,
        1.79916553)
fwdvc[1] <- fw[1]
```

```
for(o in 2:13) {
  fwdvc[o] <- fw[o]-fw[o-1]
}




#antithetic variates matrix


matrica <- matrix(rnorm(n*(m/2),mean=0, sd=1),(m/2),n)
matrica2 <- matrix(nrow=(m/2),ncol=n)
matrica2 <- (-1)*matrica
matrica3 <- rbind(matrica,matrica2)
matrica4 <- matrix(nrow = m, ncol = n)
g <- c(length(n))
h <- c(length(n))

for(j in 1:m)
  {
    for (i in 1:n)
      {
        g[i] <- ((i*t)/n)^alph
        h[i] <- matrica3[j,i]
      }
    x <- fft(g)
    y <- fft(h)
    z <- x*y
    matrica4[j,1:n] <- Re(sqrt((t)/n)*ifft(z))
  }

#simulating log-prices
```

```
matrica5 <- matrix(nrow=m, ncol=n)
CH <- sqrt((2*H*gamma(1.5-H))/(gamma(H+0.5)*gamma(2-2*H)))

for (j in 1:m)
  {  X <- log(2760,exp(1))
    for (i in 1:n)
     {
       S1 <- 0
       S2 <- 0
       if( i %% 14 == 1 ) {ksi <- fwdvc[(i+13)/14]}
     for (k in 1:i)
         {
           integrand <- function(s) {((((k*t)/n)-s)^(2*alph)}
           Itp<-integrate(integrand, lower = 0, upper = (k*t)/n)
           I<-Itp$value
           Y<-(-2)*(nu*nu)*(CH*CH)*I
           Z<-2*nu*CH*matrica4[j,k]
           S1 <- + (0.5*(t/n)*ksi*exp(Z)*(exp(Y)))
           S2 <- + (sqrt((t/n)*ksi*exp(Z)*(exp(Y)))*matrica3[j,k])
         }
       W <- (S2-S1)
       X <- X + W
       matrica5[j,i] <- exp(X)
     }
  }
plot(matrica5[3,1:n], xlab="time", ylab="price", main="plot", type="l")
points(0,2760, type = "l")
lines(matrica5[28,1:n], col="blue")
lines(matrica5[57,1:n], col="red")
lines(matrica5[99,1:n], col="purple")
```

```
#pricing asian options with strike K

K <- 2870
avgp <- c(length(m))
for (j in 1:m)
  {
    avg <- 0
    for(i in 1:n)
      {
        avg <- avg + matrica5[j,i]
      }
    avgp[j] <- avg/n
  }

for (j in 1:m) {
    call <- call + (max(0,avgp[j]-K))
    put <- put + (max(0,K-avgp[j]))
  }

acall <- call/m
aput <- put/m

print(acall)
print(aput)

end_time <- Sys.time()
end_time - start_time
```

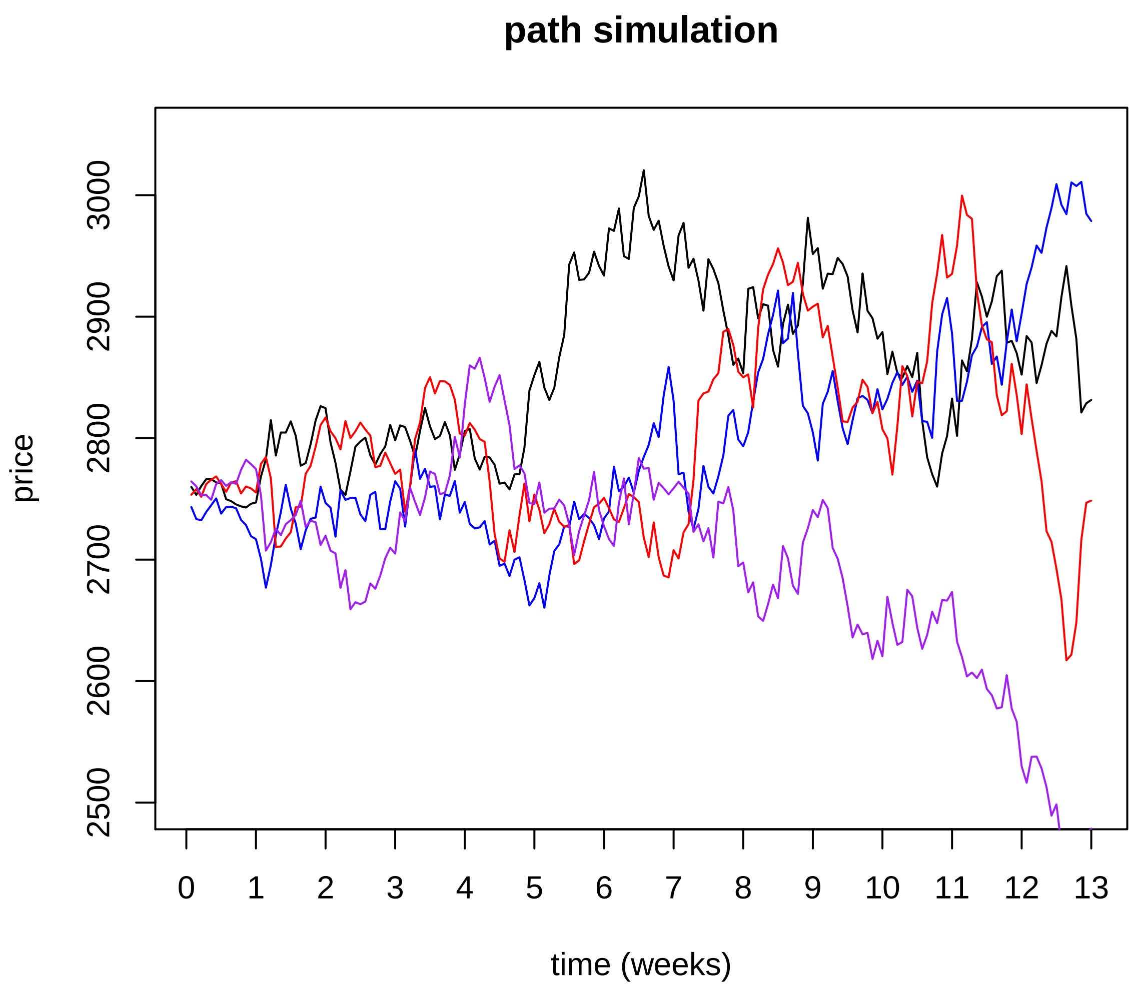


FIGURE 4. Path simulations of Algorithm 4.3

Remark 4.8. Looking at the sample paths of both algorithms, it is easily observed that their values differ more with time. That is consistent with the fact that total variance of the asset price is monotone increasing with respect to time. One can also notice that the paths of Algorithm 4.3 are less dispersed for a first few weeks than the paths of Algorithm 4.2. That is due to antithetic variates used in algorithm 4.3 which is a known method of reducing the variance of Monte Carlo simulations.

For the final part, the title of the thesis is justified as we move on to pricing of Asian options using the above introduced algorithms adapted to fit the rBergomi model.

Definition 4.9. Given an underlying asset S_t with exercise date T and strike price K, the payoff of the Asisan call option is given by

$$C := \left(\frac{1}{T}\int_0^T S_t dt - K\right)^+$$

whereas the payoff of the Asian put option is given by

$$C := \left(K - \frac{1}{T}\int_0^T S_t dt\right)^+$$

As we can see, Asian options are path dependent options on average price and thus less volatile than the European ones. Looking at the Asian call option price, it can be shown that its upper bound is given by the corresponding European call price using convexity arguments and Jensens inequality. To my knowledge, this is the first attempt of pricing Asian options in the settings of rough Bergomi model and no data on Asian option prices was found on the internet, meaning that this was the only bound one could turn to in order to see if the obtained prices could be realistic. Some may argue that Black-Scholes formula could have been used, but these results wouldnt be useful or helpful because their significance is questionable for a number of reasons one of them being that the model assumes constant volatility and does not take into consideration the volatility smile observed at the market.

A. put (Alg. 4.2)	A. put (Alg. 4.3)	Put	Strike	Call	A. call (Alg. 4.3)	A. call (Alg. 4.2)
18.54	17.82	38.30	2670	130.70	113.79	109.66
19.62	17.04	39.25	2675	126.72	107.20	106.21
27.77	22.90	42.45	2690	115.10	99.18	94.26
24.51	24.76	44.70	2700	107.35	89.72	95.28
34.52	29.25	48.45	2715	96.15	78.91	79.31
48.93	49.65	62.05	2760	64.95	54.01	56.42
56.13	54.87	63.80	2765	61.75	51.17	49.72
61.36	56.74	65.65	2770	58.60	49.25	45.51
97.72	95.48	93.55	2830	26.85	31.97	29.57
127.11	125.95	119.50	2870	13.25	12.48	11.93

FIGURE 5. Results of both algorithms for different strikes and 3 months maturity

In the end, we state a few observations that were made while creating the above introduced algorithms. In Algorithm 4.2, the correlation coefficient ρ between the price and the volatility drivers was assumed to be negative because one would expect the prices to drop as the volatility rises and vice versa. At the end, -0.1 value was chosen because the results best matched the results obtained using algorithm 4.3. Discretization was proved to be of great importance. As often found in discretization processes, the finer the subdivision of the given time interval is, the better the end result is. For instance, using five days a week or even daily time discretization, i.e. n=65 and n=91 respectively, the obtained results for Asian call option would sometimes break the upper bound. In the end twice a day time discretization was used in testings which gave the above presented results. An even better time discretization would probably yield better results in sense of accuracy but that requires a lot of computational power which, unfortunately, was not available at the time being so we leave that for some further research. It was also found that the obtained Asian options prices do not vary much for multiple Monte Carlo simulations if thousand paths are simulated. Of course, more paths would make this variance even smaller but the reason for not simulating more than one thousand paths is again the lack of computational power.

5. **Conclusion**

To conclude, rough volatility models have a great potential because they explain the volatility behavior in the most accurate way possible today. The paper written by Horvath, Jaquier and Muguruza[1] enables faster and easier implementation methods. Of course they have some downsides, the obvious one being lack of closed forms, as in rBergomi for example, which leaves us with Monte Carlo simulations as the only possible implementation method. The biggest problem of Monte Carlo is the need for a vast number of simulations which in turn sets the need for lot of computational power in order to achieve accuracy in a sensible amount of time. The accuracy of the above presented results cannot be properly discussed as this is the first simulation of that kind so no results were provided to make the comparison. We can only state that they can be improved via more refined subdivision of time interval and larger number of simulated paths. The questions about the optimal correlation coefficient ρ and constant $\nu > 0$ are left for some further research because with result comparison possibility. It is also worth mentioning that they probably arent unique in sense that they depend on the underlying asset.

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