Diplomarbeit

Convergence of the collocation schemes for systems of nonlinear ODEs with a time singularity in the right-hand side

ausgeführt am Institut für
Analysis und Scientific Computing
der Technischen Universität Wien

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Wien, August 2018

Abstract

This thesis deals with the convergence of the collocation method applied to solve systems of nonlinear ordinary differential equations with variable coefficient matrices and a time singularity in the right-hand side of the form

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad b(x(0),x(1)) = 0,$$

where $M : [0,1] \to \mathbb{R}^{n\times n}$ and $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous matrix-valued and vector-valued functions. Moreover, $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous nonlinear mapping, which is specified according to the spectrum of the matrix $M(0)$.

An experimental approach is taken by applying the collocation method and observing occurring convergence or superconvergence. The considered differential equations show different qualities in regard to the existence of a known solution, the smoothness of the inhomogenity, and the question if analytical conditions for the existence and uniqueness of solutions are satisfied.

The results of the numerical simulations are interpreted in light of the convergence theory for the linear case.

Keywords nonlinear systems of ordinary differential equations · time singularity · singular boundary value problems · collocation method · convergence · superconvergence

Vorwort

Bereits seit jungen Jahren hat mich die Mathematik fasziniert. Durch einen mehr oder minder glücklichen Zufall ist aus dieser Faszination ein Studium der Technischen Mathematik geworden. Ich habe schnell gemerkt, dass mir besonders die "greifbareren" Aspekte der wissenschaftlichen Mathematik Freude bereiten, allen voran mathematische Simulationen mittels Differentialgleichungen und die Numerik. Ich habe mich daher sehr gefreut, von Dr. Weinmüller das Angebot zu bekommen, unter ihrer Anleitung eine Diplomarbeit über die Schnittstelle dieser beiden Teildisziplinen zu verfassen.

An dieser Stelle möchte ich mich bei meiner Betreuerin Ewa Weinmüller herzlichst dafür bedanken, mir die Möglichkeit gegeben zu haben, diese Diplomarbeit zu verfassen, und dadurch auch einen Teil für die Forschung an forderster Front der Mathematik beizutragen. Vielen Dank für deine freundliche und kompetente Unterstützung während dem Schreiben der Arbeit.

Vielen Dank auch an meine Kollegen und Freunde aus dem Studium. Nach meinem Wechsel an die TU Wien konnte ich mich glücklich schätzen, schnell wunderbare Menschen wie euch kennen zu lernen, und kann ehrlich sagen: ohne euch hätte ich das nicht geschafft.

Großer Dank gilt meiner Familie, die mich in meinem Weg vorbehaltslos unterstüzt hat und mir die Möglichkeit gegeben hat, das Studium zu absolvieren.

Danke Manu (und Lotta), dass ihr beide mir auch während besonders stressigen Phasen geholfen habt, nicht die Nerven zu verlieren.

Felix Auer
Wien, August 2018

Contents

Chapter 1

Introduction

We are interested in the convergence properties of the polynomial collocation for boundary value problems for systems of nonlinear ordinary differential equations (ODEs) with a time singularity in the right hand side of the form

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad b(x(0),x(1)) = 0, \tag{1.1}$$

where $M : [0,1] \to \mathbb{R}^{n\times n}$ and $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous matrix-valued and vector-valued functions. Moreover, $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous nonlinear mapping, which is specified according to a spectrum of the matrix $M(0)$.
We want to take an experimental approach and apply the collocation method for a number of different ODEs of the from (1.1), and observe any occurring convergence or superconvergence. The considered differential equations will consist of different inital value problems (IVPs), terminal value problems (TVPs), and boundary value problems (BVPs), and show different qualities in regard to the existence of a known solution, the smoothness of the function f, and the question whether analytical conditions for the existence and uniqueness of solutions are satisfied.

The aim of this work is to provide a first overview of the convergence of the collocation method for a number of different types of nonlinear ODEs of the above form. While the existence and uniqueness of solutions are already discussed in [1], there are no published results about the convergence of the collocation schemes. The results of our experiments could help to further analyze the convergence properties of nonlinear ODEs with a time singularity in the right hand side, and lead to a more general theory about the convergence of the collocation methods. This theory would build upon the existence results in [1] in the same way as the results about convergence for the linear case in [3] build upon the existence results in [2].
Because of the superior convergence properties of polynomial collocation in the presence of a singularity compared to other high order methods, collocation was proposed in [5] to be used to compute numerical solutions for ODEs with a time singularity of the form (1.1). Among others, the MATLAB package `bvpsuite1.1` based on collocation has been implemented [6], and further refined in `bvpsuite2.0` [8].
In [3] the package `bvpsuite2.0` was used to illustrate the convergence theory for linear ODES with numerical examples.

This thesis is organized as follows:
In Chapter 2 the necessary notation and the collocation method are introduced. We give an overview of the most important results about the existence and the convergence of the collocation schemes for the linear case. A summary about the existence and solution for (1.1) is given as well. The lion's share of the thesis can be found in Chapter 3, where we take a look at four different types of ODEs of the form above. The results of the numerical experiments are shown, and the reasons for the occurrence or lack of superconvergence are discussed. A summary of the most important results can be found in Chapter 4. The appendix in Chapter 5 shows part of the MATLAB source code used for the simulations in Chapter 3.

Chapter 2

Preliminaries

2.1 Notation

The following notation is used throughout this paper.
Let $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the n-dimensional vector space of real-valued and complex-valued vectors, respectively. The maximum vector norm is denoted by

$$|x| := |(x_1, ..., x_n)^T| = \max_{1 \le i \le n} |x_i|$$

for $x \in \mathbb{R}^n$ or $x \in \mathbb{C}^n$.
Let $C_n[0,1]$ denote the space of continuous real n-dimensional vector-valued functions on $[0,1]$ with the maximum norm restricted to the interval $[0,\delta]$, $\delta > 0$ denoted by

$$||y||_\delta := \max_{t \in [0,\delta]} |y(t)|$$

and further

$$||y|| := ||y||_1 = \max_{t \in [0,1]} |y(t)|.$$

If possible, we omit subscripts and use $C[0,1] = C_n[0,1]$ if the dimensions are obvious.

2.2 Collocation Method

The following collocation method is applied to approximate the solution y to problems of the form

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t, y(t))}{t}, \quad B_0 y(0) + B_1 y(1) = \beta \tag{2.1}$$

given on $[0,1]$. We assume that the solution of the Boundary Value Problem (2.1) exists and is unique in $C[0,1]$.
We first discretize $[0,1]$ by introducing a mesh with step size h containing $N+1$ equidistant points

$$\Delta = \{0 = \tau_0 < \tau_1 < ... < \tau_N = 1, \tau_j = jh, j = 0, ..., N = 1/h\}.$$

Moreover we generate k (inner) collocation points in each subinterval $[\tau_i, \tau_{i+1}]$ by choosing a vector $\rho = (\rho_1, \rho_2, ..., \rho_k)$ with $\rho_j \in (0,1)$ and $\rho_i \neq \rho_j$ for $i \neq j$ and setting

$$\tau_{ij} := \tau_i + \rho_j h, \quad i = 0, ..., N-1, \quad j = 1, ..., k.$$

The idea of collocation is to approximate the components of y with a piecewise polynomial function. Let $\mathscr{P}_{k,h}$ denote the class of piecewise polynomial functions which are continuous in $[0,1]$ and reduce to a polynomial of degree less or equal than k in each subinterval $[\tau_j, \tau_{j+1}], j = 0, ..., N-1$. The analytical solution y of (2.1) is now approximated by a function $p \in \mathscr{P}_{k,h}$ such that

$$p'(\tau_{ij}) = \frac{M(\tau_{ij})}{\tau_{ij}} p(\tau_{ij}) + \frac{f(\tau_{ij}, y(\tau_{ij}))}{\tau_{ij}}, \quad i = 0, ..., N-1, \quad j = 1, ..., k$$

holds at the collocation points, and the boundary conditions

$$B_0 p(0) + B_1 p(1) = \beta$$

as well as the continuity relations

$$p_{i-1}(\tau_i) = p_i(\tau_i), \quad i = 1, ..., N-1$$

with $p(t) := p_i(t), t \in [\tau_i, \tau_{i+1}]$ are satisfied.[8]
We will use two different kinds of collocation points resulting in two variants of the vector ρ. The first one being equidistant collocation points, which yield

$$\rho = (\frac{1}{k+1}, \frac{2}{k+1}, ..., \frac{k}{k+1})$$

for k collocation points.
The second kind are so called Gaussian collocation points. For k collocation points the vector ρ consists of the zeros of the n-th *Legendre* polynomial

$$P_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \frac{(2n-2i)!}{(n-i)!\ (n-2i)!\ i!\ 2^n} x^{n-2i}$$

shifted from $(-1, -1)$ to $(0, 1)$.

The maximum global error of the collocation method with step size h, a resulting numerical solution y_h, an exact solution y, and $N+1$ mesh points $t_j, 0 \leq j \leq N$ is computed either in the mesh points by

$$||y_h - y||_\Delta := \max_{0 \leq j \leq N} |y_h(\tau_j) - y(\tau_j)|$$

or uniformely in t over 2048 equidistant points $t_j, 0 \leq j < 2048$, between 0 and 1, by

$$||y_h - y||_u := \max_{0 \leq j < 2048} |y_h(t_j) - y(t_j)|.$$

The order of convergence and the error constant c are estimated using two consecutive meshes with step sizes h and $h/2$. From the ansatz $||y_h - y|| \approx ch^p$ for $h \to 0$, we have

$$||y_h - y|| = ch^p, \quad ||y_{h/2} - y|| = c\left(\frac{h}{2}\right)^p \Rightarrow p = \ln\left(\frac{||y_h - y||}{||y_{h/2} - y||}\right)\frac{1}{\ln(2)}.$$

Having p, we calculate the error constant from $c = ||y_{h/2} - y|| / \left(\frac{h}{2}\right)^p$. [3]

2.2.1 Existence of a solution and convergence for the linear case

To help us interpret the results of the numerical experiments in this thesis we will give a short overview of the most important results about the existence of a solution and the convergence of the collocation method for linear ODEs of the form

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \quad B_0 y(0) + B_1 y(1) = \beta \tag{2.2}$$

as found in [2] and [3].

Theorem 2.1 *Let us assume that all eigenvalues of $M(0)$ have negative real parts. For any $f \in C[0,1]$ exists a unique solution $y \in C[0,1]$ of the initial value problem*

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \quad B_0 y(0) = \beta.$$

This solution satisfies the initial condition

$$M(0)y(0) = -f(0), \tag{2.3}$$

which is necessary and sufficient for y to be continuous on $[0,1]$. Moreover, if $f \in C^r[0,1]$, $M \in C^r[0,1]$, $M^{(r)}(0) = 0$, $r \geq 1$, then $y \in C^r[0,1]$.

Theorem 2.2 *Let $M(0)$ have only eigenvalues with negative real parts. Let us assume that $y \in C^{k+1}[0,1]$ is the unique solution of problem (2.2) with the condition (2.3) and $M \in C^1[0,1]$, $f \in C[0,1]$. Let the function $p \in \mathcal{P}_{k,h}$ be the unique solution of the collocation scheme*

$$p'(\tau_{ij}) - \frac{M(\tau_{ij})}{\tau_{ij}} p(\tau_{ij}) = \frac{f(\tau_{ij})}{\tau_{ij}}, \quad i = 0, ..., N-1, \quad j = 1, ..., k, \quad p(0) = y(0).$$

Then

$$||p - y|| \leq const \cdot h^k.$$

Theorem 2.3 *Let us assume that all eigenvalues of $M(0)$ have positive real parts and let the matrix $B_1 \in \mathbb{R}^{n \times n}$ in (2.2) be nonsingular, $\beta \in \mathbb{R}^n$, $f \in C[0,1]$ and $M \in C[0,1]$. Then there exists a solution $y \in C[0,1]$ of the terminal value problem*

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \quad B_1 y(t) = \beta. \tag{2.4}$$

Moreover, if $f \in C^r[0,1]$, $M \in C^r[0,1]$, $M^{(r)}(0) = 0$, $r \geq 1$ and the smallest positive real part σ_+ of the eigenvalues of $M(0)$ satisfies $\sigma_+ > r$ then $y \in C^r[0,1]$.

Theorem 2.4 *Let us assume that $M \in C^1[0,1]$ only has eigenvalues with positive real parts, $f \in C[0,1]$ and $y \in C^{k+1}[0,1]$ is the unique solution of (2.4). Let the function $p \in \mathcal{P}_{k,h}$ satisfy the collocation scheme*

$$p'(\tau_{ij}) - \frac{M(\tau_{ij})}{\tau_{ij}} p(\tau_{ij}) = \frac{f(\tau_{ij})}{\tau_{ij}}, \quad i = 0, ..., N-1, \quad j = 1, ..., k, \quad p(1) = y(1).$$

Then, provided that h is sufficiently small,

$$||p - y|| \leq const \cdot h^k.$$

Before we can generalize the results for IVPs and TVPS for general BVPs, we introduce the following notation in regards to the matrix $M(0)$.

- X_+ is the invariant subspace associated with the eigenvalues with $\sigma > 0$
- $X_0^{(e)}$ is the space spanned by the eigenvectors associated with zero eigenvalues
- X_- is the invariant subspace associated with the eigenvalues with $\sigma < 0$
- $X_0^{(h)}$ is the space spanned by the generalized eigenvectors associated with $\lambda = 0$
- S is the orthogonal projection onto X_+
- R is the orthogonal projection onto $X_0^{(e)}$
- $P := R + S$ is the projection onto $X_+ \oplus X_0^{(e)}$
- $Q := I - P$ is the projection onto $X_- \oplus X_0^{(h)}$
- Z is the orthogonal projection onto $X_0^{(e)} \oplus X_0^{(h)}$
- N is the orthogonal projection onto X_-
- H is the orthogonal projection onto $X_0^{(h)}$

All projections are constructed using the generalized eigenbasis of $M(0)$. We let $\tilde{R}, \tilde{P}$ denote the matrices consisting of the maximal set of linearly independent columns of the respective projections.

Theorem 2.5 *Consider BVP* (2.2)*, where the inhomogenity f is given in such a way that $f \in C[0,1]$ and Zf satisfies*

$$f(t) = \mathcal{O}(t^{\alpha} h(t))$$

for $t \to 0$, $\alpha > 0$ and $h \in C[0,\delta]$,$\delta > 0$. Let the coefficient matrix $M \in C[0,1]$ be such that its projections ZM satisfy

$$ZM(t) = ZM(0) + t^{\gamma} D(t), \quad \gamma > 0, \;\; D \in C[0,1], \;\; t \in [0,1].$$

Moreover, let $B_0, B_1 \in \mathbb{R}^{m \times n}$,$\beta \in \mathbb{R}^m$,$m = rankP$, and the $m \times m$ matrix $B_0\tilde{R} + B_1\tilde{P}$ be nonsingular. Then the BVP (2.2) *has a unique solution $y \in C[0,1]$. This solution satisfies two sets of inital conditions,*

$$Hy(0) = 0, \quad M(0)Ny(0) = -Nf(0),$$

which are necessary and sufficient for a solution of (2.2) *to be continuous on* $[0,1]$.

Remark 2.5.1 *The smoothness results $y \in C^r[0,1]$ follow by applying the smoothness results derived seperately for components of the solutions associated with eigenvalues with negative or positive real parts, respectively.*

The BVP (2.2) is well-posed if and only if the boundary conditions can be equivalently written as

$$Hy(0) = 0, \quad M(0)Ny(0) = -Nf(0), \quad Ry(0) = R\eta, \quad Sy(1) = S\eta \tag{2.5}$$

with $\eta \in \mathbb{R}^n$.

Theorem 2.6 *Let us assume that $y \in C^{k+2}[0,1]$ is the unique solution of the BVP* (2.2)*. There exists a unique solution $p \in \mathcal{P}_{k,h}$ of the collocation scheme* (2.2), (2.5)*, $f \in C^{k+1}[0,1]$, $M \in C^{k+2}[0,1]$, and real parts of nonnegative eigenvalues of $M(0)$ satisfy $\sigma > k+2$. Let $p \in \mathcal{P}_{k,h}$ satisfy the collocation scheme*

$$p'(\tau_{ij}) - \frac{M(\tau_{ij})}{\tau_{ij}} p(\tau_{ij}) = \frac{f(\tau_{ij})}{\tau_{ij}}, \quad i = 0, ..., N-1, \quad j = 1, ..., k,$$

$$Hp(0) = 0, \quad M(0)Np(0) = -Nf(0), \quad Rp(0) = R\eta, \quad Sp(1) = S\eta.$$

Then, provided that h is sufficiently small,

$$||p - y|| \leq const \cdot h^k.$$

2.2.2 Superconvergence

The use of collocation schemes to approximate solutions can lead to high accuracy at the used mesh points $\tau_0, ..., \tau_n$, when applied to ordinary differential equations. This effect is called *superconvergence*. The use of continuous piecewise polynomial functions for approximation leads to a uniform error order k. Depending on the type of collocation points (Gaussian, uniform, etc.), higher order at the mesh points is possible. If

$$\int_0^1 t^r \prod_{i=1}^{k} (t - \rho_i) \, \mathrm{d}t = 0, r = 0, ..., \nu < k, \tag{2.6}$$

the error estimates can be written as

$$|p(\tau_i) - y(\tau_i)| \leq ch^{k+\nu+1}, \quad j = 0, ..., N,$$

with $k+\nu+1 =: s_+$ being the superconvergence order in the context for nonstiff regular explicit ODEs.[4] [5]
We will now take a look at the order of superconvergence one can expect to achieve with the use of Gaussian and uniform collocation points for regular ODEs.

Gaussian Collocation Points

As mentioned above, for Gaussian collocation points the k points $\rho = (\rho_1, ..., \rho_k)$ consists of the zeros of the n-th *Legendre* polynomial

$$P_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \frac{(2n-2i)!}{(n-i)!\ (n-2i)!\ i!\ 2^n} x^{n-2i}$$

shifted from $(-1, -1)$ to $(0, 1)$. From the orthogonality properties of the n-th *Legendre* polynomial condition (2.6) follows for $\nu = k - 1$, leading to a superconvergence of order of $k_+ = 2k$, as mentioned in [7].

Equidistant Collocation Points

For k equidistant collocation points $\rho = (\frac{1}{k+1}, \frac{2}{k+1}, ..., \frac{k}{k+1})$ the order of superconvergence k_+ is dependent on the parity of k. Inspecting

$$w(t) := \prod_{i=1}^{k} (t - \rho_i)$$

in (2.6) we can immediately see that $w(t)$ is an even function (in regards to symmetry around 0.5) for even k and an odd function (around 0.5) for an odd number k.
For $r = 0$ equation (2.6) reduces to

$$\int_0^1 w(t)\,\mathrm{d}t,$$

which yields a value of 0 for an odd function $w(t)$ (in regards to symmetry around 0.5) and therefore for $w(t)$ generated by an odd number of equidistant collocation points. For an even function $w(t)$ we can not give a general statement about the value of the above integral. However, we can easily see that the functions $w(t)$ generated by an even number of collocation points will not be able to fulfill condition (2.6).
For $r > 0$ the symmetry, which allowed us to arrive at a value of 0 for the integral in (2.6) for the case of $r = 0$, is missing. It follows that 2.6 is neither satisfiable for even nor for odd functions $w(t)$.
This gives us a superconvergence order of $k_+ = k + 1$ for odd k and no superconvergence $k_+ = k$ for even k.

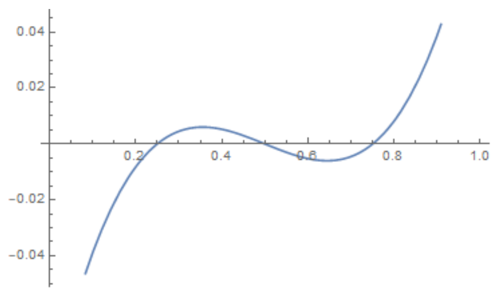

Figure 2.1: $w(t)$ for $k = 3$

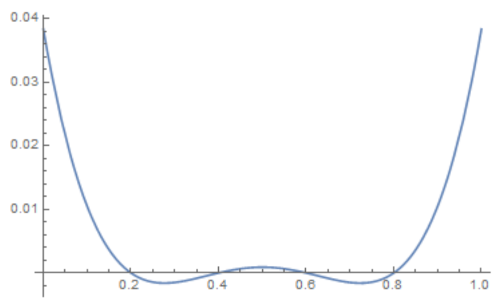

Figure 2.2: $w(t)$ for $k = 4$

2.3 Existence and Uniqueness of Solutions

In this section we will summarize the most important results from [1] about the existence and uniqueness of solutions for system (1.1)

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1].$$

We will assume that the eigenvalues λ_k of $M(0)$ satisfy

$$\lambda_k = \sigma_k + i\rho_k, \quad \lambda_k \neq 0, \quad k = 1, ..., m.$$

Let further

$$M(0) = EJE^{-1},\ t^{M(0)} = Et^J E^{-1},\ J = \text{diag}(J_1, ..., J_m), t^J = \text{diag}(t^{J_1}, ..., t^{J_m}),$$

where $E \in \mathbb{C}^{n\times n}$ is the matrix of generalized eigenvectors of $M(0)$ and J_k is the k-th *Jordan* box with dimension n_k, $k = 1, ..., m$. In addition we denote

$$\begin{aligned} g(t,x) &:= (M(t) - M(0))x + f(t,x), \quad t \in [0,1], x \in \mathbb{R}^n, \\ \beta &:= |E| \cdot |E^{-1}| \cdot \max_{1\le k\le m} \sum_{j=0}^{n_k-1} \frac{1}{|\sigma_k|^{j+1}}. \end{aligned} \tag{2.7}$$

Banach's Fixed Point Theorem allows us to deal with the difficulties caused by the singularity at $t = 0$. It provides both existence and uniqueness of a solution of problem (1.1), when subjected to certain initial, terminal or boundary conditions on a restricted interval $[0,\delta], 0 < \delta \le 1$. The form of initial, terminal or boundary conditions that specifies the unique continuous solution on $[0,\delta]$ depends on the spectral properties of the constant matrix $M(0)$. Depending on the spectrum of $M(0)$, we shall distinguish between IVPs, TVPs, and two-point BVPs for system (1.1).

Initial Value Problems

For the case of only negative real parts of eigenvalues of $M(0)$ it turns out that the unique continuous solution on $[0,\delta]$ is determined by the following structure of the initial condition:

$$M(0)x(0) + f(0,x(0)) = 0. \tag{2.8}$$

This condition follows from the requirement that the solution of (1.1) is continuous on the closed intervall $[0,1]$ including the singular point $t = 0$. Hence, we can interpret (2.8) as the necessary condition for (1.1) to be well-posed.

Theorem 2.7 *Assume that all eigenvalues λ_k of $M(0)$ have only negative real parts and let β be specified by* (2.7). *Let $a \in (0,1)$ and $L \in (0,1/\beta)$ exist such that f satisfies the Lipschitz condition*

$$|f(t,x) - f(t,y)| \le L|x-y|, \quad t \in [0,a],\ x,y \in \mathbb{R}^n. \tag{2.9}$$

Finally, assume that there exists $W > 0$; such that

$$|f(t,x)| \le W + \omega(|x|), \quad t \in [a,1],\ x \in \mathbb{R} \tag{2.10}$$

where $\omega \in C([0,\infty);(0,\infty))$ is nondecreasing, $\omega(s) > s$ for $s \in [0,\infty)$, and

$$\int_0^\infty \frac{\mathrm{d}s}{\omega(s)} = \infty.$$

Then, the IVP (1.1), (2.8) *has at least one solution on* $[0,1]$.

Corollary 2.7.1 *Let the Lipschitz condition* (2.9) *hold on* $[0,1]$, *which means that $a = 1$ in Theorem 2.7. Then, the IVP* (1.1),(2.8) *has a unique solution on* $[0,1]$.

Terminal Value Problems

We want to consider the case where all eigenvalues of $M(0)$ have positive real parts. In this situation every solution of 1.1 satisfies condition (2.8). Consequently, there is no uniquely solvable IVP and therefore, we have to study the system (1.1) subject to a terminal condition

$$x(1) = c, \quad c \in \mathbb{R}^n. \tag{2.11}$$

Theorem 2.8 *Assume that all eigenvalues λ_k of $M(0)$ have only positive real parts and let β be specified by* (2.7). *Let $a \in (0,1)$ and $L \in (0, 1/\beta)$ exist such that f satisfies the Lipschitz condition* (2.9). *Finally, assume that f satisfies* (2.10). *Then, for each $c \in \mathbb{R}^n$, the TVP* (1.1), (2.11) *has at least one solution on* $[0,1]$. *Moreover, this solution satisfies the initial condition* (2.8).

Corollary 2.8.1 *Let the Lipschitz condition* (2.9) *hold on* $[0,1]$, *which means that $a = 1$ in Theorem 2.8. Then, the IVP* (1.1),(2.11) *has a unique solution on* $[0,1]$. *Moreover, this solution satisfies the initial condition* (2.8).

Boundary Value Problems

Finally, we want to consider the case of a mixed spectrum of $M(0)$ without zero or purely imaginary eigenvalues. Based on the results for spectrums with only negative and only positive real parts of eigenvalues, system (1.1) equipped with the following boundary conditions is studied:

$$NM(0)x(0) + Nf(0, x(0)) = 0, \quad Px(1) = Pc, \; c \in \mathbb{R}^n, \tag{2.12}$$

where N and P are appropriately defined projection matrices.

Theorem 2.9 *Assume that all eigenvalues λ_k of $M(0)$ have nonzero real parts and let β be specified by* (2.7). *Let $a \in (0,1)$ and $L \in (0, 1/\beta)$ exist such that f satisfies the Lipschitz condition* (2.9). *Finally, assume that f satisfies* (2.10) *and*

$$Pf(t,x) = Pf(t, Px), \quad t \in [\delta, 1], x \in \mathbb{R}^n. \tag{2.13}$$

Then, for each $c \in \mathbb{R}^n$, the TVP (1.1), (2.12) *has at least one solution on* $[0,1]$. *Moreover, this solution satisfies the initial condition* (2.8).

Corollary 2.9.1 *Let the Lipschitz condition* (2.9) *hold on* $[0,1]$, *which means that $a = 1$ in Theorem 2.9. Then, the IVP* (1.1),(2.12) *has a unique solution on* $[0,1]$. *Moreover, this solution satisfies the initial condition* (2.8).

Remark 2.9.1 *Theorem* (2.9) *and Corollary* (2.9.1) *also hold if we replace assumption* (2.13) *by*

$$Nf(t,x) = Nf(t, Px), \quad t \in [\delta, 1], x \in \mathbb{R}^n. \tag{2.14}$$

Chapter 3

Numerical Simulations

In the following chapter we illustrate the convergence behavior of the collocation method for various types of ODEs of the form (1.1). These feature ODEs with and without a known solution, some of them not satisfying the Lipschitz condition (2.9) and the growth condition (2.10). We also take a look at the effect of low differentiability of the function f.
The convergence order is analyzed for Gaussian and equidistant points. The number of these points will range from 2 to 5.
For the examples without a known exact solutions we use a numerical solution as reference instead of an analytical solution. These numerical solutions are obtained by using 7 Gaussian collocation points and a mesh containing 2048 or twice as many points as the finest mesh we use for our collocation method.
All calculations are done using the MATLAB package `bvpsuite2.0`.

Since all coefficient matrices $M(t) \in \mathbb{R}^{n \times n}$ used in the following examples are diagonalizable, all Jordan boxes are of size $n_k = 1$ for $k = 1, ..., n$. Therefore, (2.7) reduces to choosing the reciprocal to the absolute value of the smallest real part σ_k of an eigenvalue λ_k of $M(0)$, $k \in 1, ..., n$:

$$\beta = \max_{1 \leq k \leq n} \frac{1}{|\sigma_k|}$$

3.1 Examples from [1]

The first class of ODEs we consider are the examples found in [1]. While all of these examples satisfy the analytical conditions for the existence of a solution, none of them have a solution known to us.

3.1.1 Example 1.1

We consider the following IVP (Example 1.1 from [1]) without a known solution:

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{1}{3}\frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad M(0)x(0) + \frac{1}{3}f(0,x(0)) = 0,$$

where

$$M(t) = \begin{pmatrix} -1 + (\exp(t) - 1) & \exp(-t) - 1 \\ \exp(2t) - 1 & -2 + (\exp(-2t) - 1) \end{pmatrix}, \quad M(0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad \beta = \max\left\{1, \frac{1}{2}\right\} = 1$$

and

$$f(t,x(t)) = p(t) + H(t)r(t,x)$$

with

$$p(t) = \begin{pmatrix} t^2/2 \\ -1 \end{pmatrix}, \quad r(t,x) = \begin{pmatrix} \sin(x_1 + x_2) \\ \cos(x_1 + x_2) \end{pmatrix}, \quad H(t) = \begin{pmatrix} t & 0 \\ \sin(t) & \cos(t) \end{pmatrix}.$$

Since $\max_{t\in[0,1]} |H(t)| = \sqrt{2}$ and $|r(t,x) - r(t,y)| \leq 2|x-y|$, for $t \in [0,1], x, y \in \mathbb{R}^2$, we conclude

$$\left|\frac{1}{3}f(t,x) - \frac{1}{3}f(t,y)\right| \leq \frac{2\sqrt{2}}{3}|x-y|, \quad t \in [0,1], \quad x, y \in \mathbb{R}^2.$$

Hence, $L = \frac{2\sqrt{2}}{3} \in (0, 1/\beta) = (0,1)$ and $\frac{1}{3}f$ satisfies the Lipschitz condition (2.9) on $[0,1]$, and by Corollary (2.7.1) Example 1.1 has a unique solution on $[0,1]$.

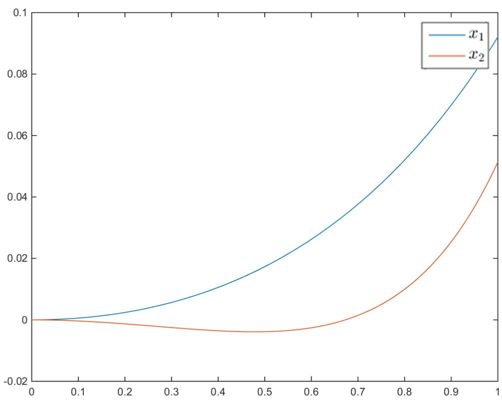

Figure 3.1: Solution of Example 1.1

As we can see in Table 3.1 to Table 3.4, for Gaussian points we observe a possible superconvergence order of $2k-1$. For uniformly spaced equidistant collocation points we observe an order of k for even k and an order of $k+1$ for odd k uniformly in t. For this example we use an error tolerance of 1e-15. One should note that the negative values for p are a result of an error that is smaller than the used error tolerance. Once this tolerance is satisfied a decrease of step size h does not yield any more improvements.

Table 3.1: Example 1.1: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	6.59e-04	9.59e-03	3.86	8.74e-03	3.47e-02	1.98	8.74e-03	3.47e-02	1.98
1/4	4.53e-05	1.11e-02	3.97	2.20e-03	3.51e-02	1.99	2.20e-03	3.51e-02	1.99
1/8	2.88e-06	3.14e-03	3.36	5.50e-04	3.52e-02	2.00	5.50e-04	3.52e-02	2.00
1/16	2.80e-07	1.25e-03	3.03	1.37e-04	3.52e-02	2.00	1.37e-04	3.52e-02	2.00
1/32	3.43e-08	1.17e-03	3.01	3.44e-05	3.52e-02	2.00	3.44e-05	3.52e-02	2.00
1/64	4.25e-09	1.14e-03	3.00	8.60e-06	3.52e-02	2.00	8.60e-06	3.52e-02	2.00
1/128	5.29e-10	1.12e-03	3.00	2.15e-06	3.52e-02	2.00	2.15e-06	3.52e-02	2.00
1/256	6.60e-11	1.11e-03	3.00	5.37e-07	3.52e-02	2.00	5.37e-07	3.52e-02	2.00
1/512	8.24e-12	1.11e-03	3.00	1.34e-07	3.52e-02	2.00	1.34e-07	3.52e-02	2.00
1/1024	1.03e-12	-	-	3.36e-08	-	-	3.36e-08	-	-

Table 3.2: Example 1.1: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	6.95e-06	2.97e-04	5.41	9.17e-04	1.27e-02	3.79	9.17e-04	1.27e-02	3.79
1/4	1.62e-07	2.39e-04	5.26	6.60e-05	1.56e-02	3.94	6.60e-05	1.56e-02	3.94
1/8	4.23e-09	1.95e-04	5.16	4.29e-06	1.70e-02	3.98	4.29e-06	1.70e-02	3.98
1/16	1.18e-10	1.61e-04	5.09	2.70e-07	1.75e-02	3.99	2.70e-07	1.75e-02	3.99
1/32	3.45e-12	1.39e-04	5.05	1.69e-08	1.77e-02	3.99	1.69e-08	1.77e-02	3.99
1/64	1.04e-13	2.79e-06	4.11	1.06e-09	1.78e-02	3.99	1.06e-09	1.78e-02	3.99
1/128	6.01e-15	5.53e-15	-0.01	6.63e-11	1.79e-02	4.00	6.63e-11	1.79e-02	4.00
1/256	6.08e-15	5.94e-15	-0.004	4.14e-12	2.05e-02	4.02	4.14e-12	2.05e-02	4.02
1/512	6.09e-15	6.16e-15	0.001	2.54e-13	2.62e-01	4.43	2.54e-13	8.39e-03	3.88
1/1024	6.09e-15	-	-	1.17e-14	-	-	1.72e-14	-	-

Table 3.3: Example 1.1: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	6.41e-08	7.58e-06	6.88	1.77e-04	2.65e-03	3.90	1.77e-04	2.65e-03	3.90
1/4	5.42e-10	3.23e-06	6.26	1.17e-05	2.89e-03	3.96	1.17e-05	2.89e-03	3.96
1/8	7.03e-12	1.75e-06	5.97	7.52e-07	3.03e-03	3.99	7.52e-07	3.03e-03	3.99
1/16	1.11e-13	1.05e-08	4.13	4.73e-08	3.08e-03	3.99	4.73e-08	3.08e-03	3.99
1/32	6.37e-15	8.88e-15	0.09	2.96e-09	3.10e-03	3.99	2.96e-09	3.10e-03	3.99
1/64	5.96e-15	6.26e-15	0.01	1.85e-10	3.11e-03	4.00	1.85e-10	3.11e-03	4.00
1/128	5.91e-15	5.77e-15	-0.005	1.15e-11	3.23e-03	4.00	1.15e-11	3.23e-03	4.00
1/256	5.93e-15	6.01e-15	0.002	7.18e-13	6.79e-03	4.14	7.18e-13	1.09e-03	3.81
1/512	5.92e-15	6.04e-15	0.003	4.07e-14	6.97e-05	3.40	5.11e-14	1.05e-18	-1.72
1/1024	5.91e-15	-	-	3.83e-15	-	-	1.69e-13	-	-

Table 3.4: Example 1.1: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.68e-09	9.71e-07	9.17	1.57e-05	8.19e-04	5.70	1.57e-05	8.19e-04	5.70
1/4	2.92e-12	1.25e-06	9.35	3.01e-07	1.09e-03	5.91	3.01e-07	1.09e-03	5.91
1/8	4.47e-15	2.09e-15	-0.3	4.98e-09	1.24e-03	5.97	4.98e-09	1.24e-03	5.97
1/16	5.76e-15	5.92e-15	0.01	7.91e-11	1.32e-03	5.99	7.91e-11	1.32e-03	5.99
1/32	5.72e-15	5.82e-15	0.004	1.23e-12	4.06e-03	6.32	1.23e-12	8.69e-04	5.87
1/64	5.70e-15	5.71e-15	0.0002	1.54e-14	9.97e-12	1.55	2.10e-14	1.45e-14	-0.08
1/128	5.70e-15	5.67e-15	-0.001	5.25e-15	3.96e-15	-0.05	2.23e-14	3.79e-12	1.05
1/256	5.70e-15	5.33e-15	-0.01	5.46e-15	5.17e-15	-0.01	1.07e-14	1.72e-13	0.5
1/512	5.75e-15	7.13e-15	0.03	5.50e-15	6.31e-15	0.02	7.59e-15	4.78e-17	-0.8
1/1024	5.61e-15	-	-	5.42e-15	-	-	1.33e-14	-	-

3.1.2 Example 1.2

We consider the following TVP (Example 2.1 from [1]) without a known solution:

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{1}{3}\frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad x(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where

$$M(t) = \begin{pmatrix} 1 + (\exp(t) - 1) & \exp(-t) - 1 \\ \exp(2t) - 1 & 2 + (\exp(-2t) - 1) \end{pmatrix}, \quad M(0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \beta = \max\left\{1, \frac{1}{2}\right\} = 1$$

and

$$f(t,x(t)) = p(t) + H(t)r(t,x)$$

with

$$p(t) = \begin{pmatrix} t^2/2 \\ -1 \end{pmatrix}, \quad r(t,x) = \begin{pmatrix} \sin(x_1 + x_2) \\ \cos(x_1 + x_2) \end{pmatrix}, \quad H(t) = \begin{pmatrix} t & 0 \\ \sin(t) & \cos(t) \end{pmatrix}.$$

Since Example 1.2 shares β as well as $p(t)$, $r(t,x)$, $H(t)$ and therefore $f(t,x)$ with Example 1.1, the existence of a unique solution on $[0,1]$ follows immediately with Corollary (2.8.1).

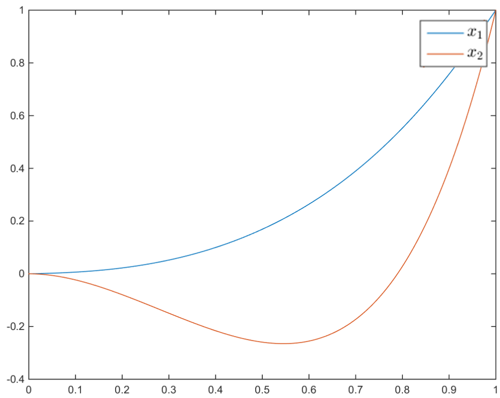

Figure 3.2: Solution of Example 1.2

As we can see in Tables 3.5 - 3.8, we observe a convergence order of 2, regardless if we use Gaussian or uniformly spaced collocation points. This is a consequence of the small positive eigenvalues of $M(0)$.
Again, an error tolerance of 1e-15 is used.

Table 3.5: Example 1.2: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	3.10e-02	2.26e-01	2.86	1.12e-01	5.03e-01	2.16	1.12e-01	4.23e-01	1.91
1/4	4.26e-03	2.42e-01	2.91	2.50e-02	3.32e-01	1.86	2.98e-02	5.05e-01	2.04
1/8	5.66e-04	3.96e-01	3.15	6.88e-03	4.42e-01	2.00	7.24e-03	4.92e-01	2.02
1/16	6.37e-05	1.90e+00	3.71	1.71e-03	4.40e-01	2.00	1.77e-03	4.78e-01	2.01
1/32	4.84e-06	1.26e+00	3.59	4.29e-04	4.37e-01	1.99	4.38e-04	4.71e-01	2.01
1/64	3.99e-07	3.76e-06	5.39	1.07e-04	4.39e-01	2.00	1.08e-04	4.58e-01	2.00
1/128	2.75e-07	5.80e-04	1.57	2.68e-05	4.39e-01	2.00	2.69e-05	4.50e-01	2.00
1/256	9.22e-08	2.33e-03	1.82	6.71e-06	4.39e-01	2.00	6.73e-06	4.45e-01	2.00
1/512	2.59e-08	4.24e-03	1.92	1.67e-06	4.39e-01	2.00	1.68e-06	4.45e-01	2.00
1/1024	6.83e-09	-	-	4.19e-07	-	-	4.19e-07	-	-

Table 3.6: Example 1.2: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	3.37e-04	1.39e-03	2.00	2.26e-03	5.73e-03	1.34	1.26e-02	1.20e-01	3.24
1/4	8.44e-05	9.63e-03	3.41	8.92e-04	1.50e-01	3.69	1.33e-03	2.06e-01	3.63
1/8	7.89e-06	3.32e-05	0.69	6.88e-05	3.22e-02	2.95	1.07e-04	1.90e-01	3.59
1/16	4.89e-06	6.69e-04	1.76	8.87e-06	1.06e-03	1.72	8.87e-06	1.06e-03	1.72
1/32	1.43e-06	1.27e-03	1.95	2.67e-06	2.29e-03	1.94	2.67e-06	2.29e-03	1.94
1/64	3.69e-07	1.41e-03	1.99	6.93e-07	2.78e-03	1.99	6.93e-07	2.78e-03	1.99
1/128	9.26e-08	1.57e-03	2.00	1.73e-07	2.89e-03	2.00	1.73e-07	2.89e-03	2.00
1/256	2.31e-08	1.57e-03	2.00	4.33e-08	2.91e-03	2.00	4.33e-08	2.91e-03	2.00
1/512	5.75e-09	1.66e-03	2.01	1.08e-08	2.97e-03	2.00	1.08e-08	2.97e-03	2.00
1/1024	1.42e-09	-	-	2.68e-09	-	-	2.68e-09	-	-

Table 3.7: Example 1.2: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	3.09e-04	7.14e-03	4.53	2.92e-03	2.93e-02	3.32	3.27e-03	3.69e-02	3.49
1/4	1.33e-05	4.22e-05	0.83	2.90e-04	7.55e-02	4.01	2.90e-04	7.55e-02	4.01
1/8	7.51e-06	3.96e-04	1.90	1.80e-05	8.60e-04	1.85	1.80e-05	8.60e-04	1.85
1/16	2.00e-06	5.22e-04	2.00	4.96e-06	1.29e-03	2.00	4.96e-06	1.29e-03	2.00
1/32	4.99e-07	5.30e-04	2.01	1.23e-06	1.32e-03	2.01	1.23e-06	1.32e-03	2.01
1/64	1.23e-07	5.21e-04	2.00	3.06e-07	1.29e-03	2.00	3.06e-07	1.29e-03	2.00
1/128	3.08e-08	5.18e-04	2.00	7.62e-08	1.27e-03	2.00	7.62e-08	1.27e-03	2.00
1/256	7.68e-09	5.31e-04	2.00	1.89e-08	1.28e-03	2.00	1.89e-08	1.28e-03	2.00
1/512	1.90e-09	6.16e-04	2.03	4.73e-09	1.35e-03	2.01	4.73e-09	1.35e-03	2.01
1/1024	4.65e-10	-	-	1.17e-09	-	-	1.17e-09	-	-

Table 3.8: Example 1.2: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.59e-05	5.12e-05	0.98	1.18e-03	3.51e-02	4.90	1.18e-03	3.51e-02	4.90
1/4	1.31e-05	1.95e-04	1.95	3.94e-05	5.34e-04	1.88	3.94e-05	5.34e-04	1.88
1/8	3.39e-06	2.27e-04	2.02	1.07e-05	7.23e-04	2.03	1.07e-05	7.23e-04	2.03
1/16	8.36e-07	2.23e-04	2.01	2.62e-06	7.11e-04	2.02	2.62e-06	7.11e-04	2.02
1/32	2.07e-07	2.18e-04	2.01	6.46e-07	6.87e-04	2.01	6.46e-07	6.87e-04	2.01
1/64	5.14e-08	2.15e-04	2.00	1.60e-07	6.72e-04	2.01	1.60e-07	6.72e-04	2.01
1/128	1.28e-08	2.17e-04	2.01	3.99e-08	6.68e-04	2.00	3.99e-08	6.68e-04	2.00
1/256	3.19e-09	2.33e-04	2.02	9.96e-09	6.80e-04	2.01	9.96e-09	6.80e-04	2.01
1/512	7.86e-10	3.37e-04	2.08	2.48e-09	7.59e-04	2.03	2.48e-09	7.59e-04	2.03
1/1024	1.86e-10	-	-	6.08e-10	-	-	6.08e-10	-	-

3.1.3 Example 1.3

Next, we consider the following BVP (Example 3.1 from [1]) without a known solution:

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{1}{3}\frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad -x_1(0) + \frac{1}{3}f_1(0,x(0)) = 0,\ x_2(1) = 1,$$

where

$$M(t) = \begin{pmatrix} -1 + (\exp(t) - 1) & \exp(-t) - 1 \\ \exp(2t) - 1 & 2 + (\exp(-2t) - 1) \end{pmatrix}, \quad M(0) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \beta = \max\left\{1, \frac{1}{2}\right\} = 1$$

and

$$f(t,x(t)) = p(t) + H(t)r(t,x)$$

with

$$p(t) = \begin{pmatrix} t^2/2 \\ -1 \end{pmatrix}, \quad r(t,x) = \begin{pmatrix} \sin(x_1 + x_2) \\ \cos(x_1 + x_2) \end{pmatrix}, \quad H(t) = \begin{pmatrix} \sin(t) & \cos(t) \\ t & 0 \end{pmatrix}.$$

Since $M(0)$ is diagonalizable it follows for the projections N and P that

$$N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and thus,

$$NM(0)x(0) = \begin{pmatrix} \lambda_1 x_1(0) \\ 0 \end{pmatrix}, \quad N\frac{1}{3}f(0,x(0)) = \begin{pmatrix} \frac{1}{3}f_1(0,x(0)) \\ 0 \end{pmatrix}, \quad Px(1) = \begin{pmatrix} 0 \\ x_2(1) \end{pmatrix}.$$

With this, the boundary conditions (2.12) for $c = (1,1)^T$ amount to the boundary conditions above:

$$\lambda_1 x_1(0) + \frac{1}{3}f_1(0,x(0)) = 0,\ x_2(0) = 1 \iff -x(0) + \frac{1}{3}f_1(0,x(0)) = 0,\ x_2(0) = 1$$

From Corollary (2.9.1) we conclude the existence of a unique solution to the BVP Example 1.3.

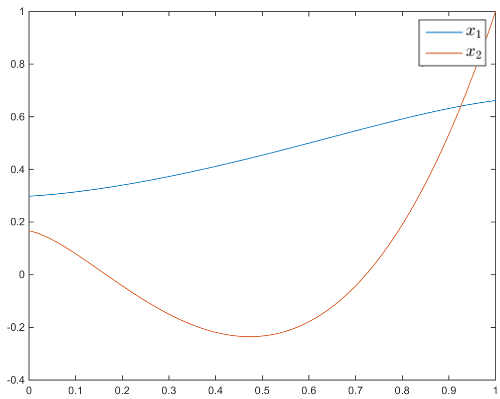

Figure 3.3: Solution of Example 1.3

We can see in Tables 3.9 - 3.12 that once again we can observe a convergence order of 2, regardless of the spacing of the collocation points.
For `bvpsuite2.0` to be able to terminate, we have to raise the error tolerance to 1e-13.

Table 3.9: Example 1.3: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	7.74e-02	2.89e-01	1.90	1.01e-01	3.70e-01	1.87	1.01e-01	3.70e-01	1.87
1/4	2.07e-02	4.01e-01	2.14	2.76e-02	5.35e-01	2.14	2.76e-02	5.35e-01	2.14
1/8	4.70e-03	4.88e-01	2.23	6.26e-03	6.50e-01	2.23	6.26e-03	6.50e-01	2.23
1/16	1.00e-03	4.77e-01	2.22	1.33e-03	6.33e-01	2.22	1.33e-03	6.33e-01	2.22
1/32	2.14e-04	3.98e-01	2.17	2.85e-04	5.29e-01	2.17	2.85e-04	5.29e-01	2.17
1/64	4.76e-05	3.15e-01	2.12	6.34e-05	4.19e-01	2.12	6.34e-05	4.19e-01	2.12
1/128	1.10e-05	2.54e-01	2.07	1.46e-05	3.38e-01	2.07	1.46e-05	3.38e-01	2.07
1/256	2.61e-06	2.16e-01	2.04	3.48e-06	2.88e-01	2.04	3.48e-06	2.88e-01	2.04
1/512	6.34e-07	1.95e-01	2.03	8.46e-07	2.59e-01	2.02	8.46e-07	2.59e-01	2.02
1/1024	1.56e-07	-	-	2.08e-07	-	-	2.08e-07	-	-

Table 3.10: Example 1.3: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.82e-02	9.21e-02	2.34	3.74e-02	1.92e-01	2.36	3.74e-02	1.92e-01	2.36
1/4	3.61e-03	9.78e-02	2.38	7.28e-03	2.10e-01	2.42	7.28e-03	2.10e-01	2.42
1/8	6.93e-04	7.39e-02	2.25	1.36e-03	1.54e-01	2.28	1.36e-03	1.54e-01	2.28
1/16	1.46e-04	5.24e-02	2.12	2.80e-04	1.05e-01	2.14	2.80e-04	1.05e-01	2.14
1/32	3.36e-05	4.16e-02	2.05	6.36e-05	8.08e-02	2.06	6.36e-05	8.08e-02	2.06
1/64	8.09e-06	3.67e-02	2.02	1.52e-05	7.00e-02	2.03	1.52e-05	7.00e-02	2.03
1/128	1.99e-06	3.45e-02	2.01	3.74e-06	6.52e-02	2.01	3.74e-06	6.52e-02	2.01
1/256	4.93e-07	3.37e-02	2.01	9.25e-07	6.32e-02	2.01	9.25e-07	6.32e-02	2.01
1/512	1.23e-07	3.49e-02	2.01	2.30e-07	6.37e-02	2.01	2.30e-07	6.37e-02	2.01
1/1024	3.04e-08	-	-	5.72e-08	-	-	5.72e-08	-	-

Table 3.11: Example 1.3: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	5.00e-03	2.79e-02	2.48	1.43e-02	8.55e-02	2.58	1.43e-02	8.55e-02	2.58
1/4	8.96e-04	2.02e-02	2.25	2.40e-03	5.97e-02	2.32	2.40e-03	5.97e-02	2.32
1/8	1.88e-04	1.48e-02	2.10	4.82e-04	4.03e-02	2.13	4.82e-04	4.03e-02	2.13
1/16	4.40e-05	1.26e-02	2.04	1.10e-04	3.26e-02	2.05	1.10e-04	3.26e-02	2.05
1/32	1.07e-05	1.17e-02	2.02	2.66e-05	2.95e-02	2.02	2.66e-05	2.95e-02	2.02
1/64	2.64e-06	1.12e-02	2.01	6.54e-06	2.81e-02	2.01	6.54e-06	2.81e-02	2.01
1/128	6.56e-07	1.11e-02	2.01	1.62e-06	2.74e-02	2.01	1.62e-06	2.74e-02	2.01
1/256	1.63e-07	1.13e-02	2.01	4.04e-07	2.74e-02	2.01	4.04e-07	2.74e-02	2.01
1/512	4.06e-08	1.31e-02	2.03	1.01e-07	2.89e-02	2.01	1.01e-07	2.89e-02	2.01
1/1024	9.91e-09	-	-	2.49e-08	-	-	2.49e-08	-	-

Table 3.12: Example 1.3: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.33e-03	6.33e-03	2.25	4.71e-03	2.41e-02	2.35	4.71e-03	2.41e-02	2.35
1/4	2.81e-04	5.11e-03	2.09	9.21e-04	1.77e-02	2.13	9.21e-04	1.77e-02	2.13
1/8	6.60e-05	4.56e-03	2.04	2.10e-04	1.50e-02	2.05	2.10e-04	1.50e-02	2.05
1/16	1.61e-05	4.31e-03	2.02	5.06e-05	1.38e-02	2.02	5.06e-05	1.38e-02	2.02
1/32	3.98e-06	4.19e-03	2.01	1.24e-05	1.32e-02	2.01	1.24e-05	1.32e-02	2.01
1/64	9.89e-07	4.14e-03	2.01	3.08e-06	1.29e-02	2.01	3.08e-06	1.29e-02	2.01
1/128	2.46e-07	4.17e-03	2.01	7.68e-07	1.28e-02	2.00	7.68e-07	1.28e-02	2.00
1/256	6.13e-08	4.49e-03	2.02	1.91e-07	1.31e-02	2.01	1.91e-07	1.31e-02	2.01
1/512	1.51e-08	6.49e-03	2.08	4.76e-08	1.46e-02	2.03	4.76e-08	1.46e-02	2.03
1/1024	3.58e-09	-	-	1.17e-08	-	-	1.17e-08	-	-

3.1.4 Example 1.4

In Example 1.2 we are not able to observe any kind of superconvergence or even a convergence order that changes with the number of used collocation points. The reason for this is the relatively small positive eigenvalues of $M(0)$, as described in Section 2.2.1. In this example we will choose larger values and observe the new convergence order.
Let

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{1}{3}\frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad x(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where

$$M(t) = \begin{pmatrix} 10 + (\exp(t) - 1) & \exp(-t) - 1 \\ \exp(2t) - 1 & 20 + (\exp(-2t) - 1) \end{pmatrix}, \; M(0) = \begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix}, \; \beta = \max\left\{\frac{1}{10}, \frac{1}{20}\right\} = \frac{1}{10}$$

and

$$f(t,x(t)) = p(t) + H(t)r(t,x)$$

with

$$p(t) = \begin{pmatrix} t^2/2 \\ -1 \end{pmatrix}, \quad r(t,x) = \begin{pmatrix} \sin(x_1 + x_2) \\ \cos(x_1 + x_2) \end{pmatrix}, \quad H(t) = \begin{pmatrix} t & 0 \\ \sin(t) & \cos(t) \end{pmatrix}.$$

There is no known solution for this TVP.
Just like in the previous examples we have

$$\left|\frac{1}{3}f(t,x) - \frac{1}{3}f(t,y)\right| \le \frac{2\sqrt{2}}{3}|x-y|, \quad t \in [0,1], \quad x,y \in \mathbb{R}^2.$$

Hence, $L = \frac{2\sqrt{2}}{3} \in (0, 1/\beta) = (0,10)$ and $\frac{1}{3}f$ satisfies the Lipschitz condition (2.9) on $[0,1]$ and by Corollary (2.7.1) Example 1.4 has a unique solution on $[0,1]$. The larger positive eigenvalues do not change the existence and uniqueness of a solution.

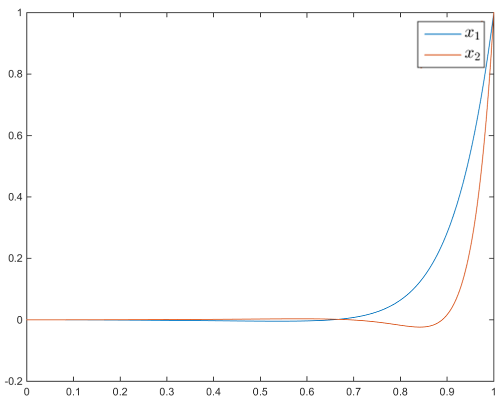

Figure 3.4: Solution of Example 1.4

As we can see in Table 3.13 to Table 3.16, we are now able to observe the same convergence order as in Example 1.1.
Gaussian collocation points result in a superconvergence of order $2k$ for k collocation points and equidistant collocation points yield order k for even k and a small superconvergence of order $k+1$ for odd k.
In line with Example 1.2, we use an error tolerance of 1e-15.

Table 3.13: Example 1.4: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	5.37e-01	1.57e+00	1.55	6.62e-01	1.40e+00	1.08	6.62e-01	1.40e+00	1.08
1/4	1.83e-01	7.56e+00	2.68	3.11e-01	2.88e+00	1.60	3.11e-01	2.88e+00	1.60
1/8	2.85e-02	6.00e+01	3.68	1.02e-01	6.49e+00	1.99	1.02e-01	5.86e+00	1.94
1/16	2.22e-03	1.90e+02	4.09	2.57e-02	1.16e+01	2.20	2.65e-02	9.33e+00	2.11
1/32	1.30e-04	1.48e+02	4.02	5.56e-03	6.54e+00	2.03	6.14e-03	7.92e+00	2.06
1/64	7.98e-06	1.32e+02	3.99	1.35e-03	5.66e+00	2.00	1.46e-03	7.70e+00	2.06
1/128	5.00e-07	1.35e+02	4.00	3.37e-04	5.60e+00	2.00	3.51e-04	6.68e+00	2.03
1/256	3.12e-08	1.34e+02	4.00	8.41e-05	5.51e+00	2.00	8.60e-05	6.16e+00	2.01
1/512	1.95e-09	1.34e+02	4.00	2.10e-05	5.52e+00	2.00	2.12e-05	6.16e+00	2.01
1/1024	1.21e-10	-	-	5.25e-06	-	-	5.25e-06	-	-

Table 3.14: Example 1.4: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.34e-01	1.83e+00	2.97	4.30e-01	1.42e+00	1.73	4.30e-01	1.42e+00	1.73
1/4	2.99e-02	1.74e+01	4.59	1.29e-01	6.08e+00	2.77	1.29e-01	6.08e+00	2.77
1/8	1.23e-03	1.89e+02	5.73	1.89e-02	4.89e+01	3.77	1.89e-02	2.12e+01	3.37
1/16	2.31e-05	4.21e+02	6.02	1.38e-03	1.12e+02	4.07	1.82e-03	3.28e+01	3.53
1/32	3.55e-07	3.90e+02	6.00	8.20e-05	9.12e+01	4.01	1.57e-04	6.83e+01	3.74
1/64	5.52e-09	3.81e+02	6.00	5.06e-06	8.64e+01	4.00	1.17e-05	1.10e+02	3.85
1/128	8.62e-11	3.34e+02	5.97	3.15e-07	8.51e+01	4.00	8.11e-07	1.52e+02	3.92
1/256	1.37e-12	3.90e-02	4.34	1.97e-08	8.47e+01	4.00	5.33e-08	1.86e+02	3.96
1/512	6.77e-14	6.77e-14	2.75e-05	1.23e-09	8.44e+01	3.99	3.42e-09	6.31e+02	4.15
1/1024	6.77e-14	-	-	7.70e-11	-	-	1.91e-10	-	-

Table 3.15: Example 1.4: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	7.85e-02	2.10e+00	4.74	2.63e-01	1.55e+00	2.55	2.63e-01	1.55e+00	2.55
1/4	2.93e-03	3.40e+01	6.74	4.48e-02	7.36e+00	3.68	4.48e-02	7.36e+00	3.68
1/8	2.73e-05	6.04e+02	8.13	3.49e-03	1.97e+01	4.15	3.49e-03	1.28e+01	3.94
1/16	9.72e-08	2.89e+02	7.86	1.96e-04	1.68e+01	4.09	2.26e-04	2.11e+01	4.12
1/32	4.16e-10	4.45e+02	7.99	1.14e-05	1.35e+01	4.03	1.29e-05	1.85e+01	4.08
1/64	1.63e-12	3.22e-04	4.59	7.01e-07	1.22e+01	4.00	7.61e-07	1.62e+01	4.05
1/128	6.77e-14	6.77e-14	1.12e-04	4.35e-08	1.17e+01	4.00	4.57e-08	1.45e+01	4.03
1/256	6.77e-14	6.77e-14	0	2.72e-09	1.17e+01	4.00	2.78e-09	1.35e+01	4.02
1/512	6.77e-14	6.77e-14	-1.72e-05	1.70e-10	1.18e+01	4.00	1.71e-10	1.30e+01	4.01
1/1024	6.77e-14	-	-	1.06e-11	-	-	1.06e-11	-	-

Table 3.16: Example 1.4: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.94e-02	2.15e+00	6.78e	1.44e-01	1.81e+00	3.64	1.44e-01	1.81e+00	3.64
1/4	1.76e-04	5.46e+01	9.11e	1.15e-02	2.35e+01	5.49	1.15e-02	1.50e+01	5.17
1/8	3.17e-07	2.18e+02	9.78e	2.54e-04	2.68e+01	5.56	3.18e-04	7.64e+00	4.85
1/16	3.60e-10	4.14e+00	8.35e	5.37e-06	1.98e+01	5.45	1.10e-05	4.57e+00	4.66
1/32	1.09e-12	1.25e-06	4.02e	1.22e-07	9.38e+01	5.90	4.34e-07	2.61e+01	5.16
1/64	6.75e-14	6.68e-14	-2.75e-03	2.05e-09	1.14e+02	5.94	1.20e-08	1.63e+02	5.60
1/128	6.77e-14	6.77e-14	1.23e-05	3.31e-11	1.14e+02	5.95	2.47e-10	4.81e+02	5.83
1/256	6.77e-14	6.77e-14	-6.18e-06	5.36e-13	8.34e-06	2.98	4.34e-12	8.96e+02	5.94
1/512	6.77e-14	6.77e-14	-1.16e-05	6.77e-14	6.77e-14	3.57e-05	7.05e-14	1.02e-13	5.96e-02
1/1024	6.77e-14	-	-	6.77e-14	-	-	6.77e-14	-	-

3.1.5 Example 1.5

We will now raise the positive eigenvalue of $M(t)$ in Example 1.3 and try to see if we can observe the same effects as in Example 1.4.
Let

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{1}{3}\frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad -x_1(0) + \frac{1}{3}f_1(0,x(0)) = 0, \; x_2(1) = 1,$$

where

$$M(t) = \begin{pmatrix} -1 + (\exp(t) - 1) & \exp(-t) - 1 \\ \exp(2t) - 1 & 20 + (\exp(-2t) - 1) \end{pmatrix}, \quad M(0) = \begin{pmatrix} -1 & 0 \\ 0 & 20 \end{pmatrix}, \quad \beta = \max\left\{1, \frac{1}{20}\right\} = 1$$

and

$$f(t,x(t)) = p(t) + H(t)r(t,x)$$

with

$$p(t) = \begin{pmatrix} t^2/2 \\ -1 \end{pmatrix}, \quad r(t,x) = \begin{pmatrix} \sin(x_1 + x_2) \\ \cos(x_1 + x_2) \end{pmatrix}, \quad H(t) = \begin{pmatrix} \sin(t) & \cos(t) \\ t & 0 \end{pmatrix}.$$

We can not provide a known analytical solution for this BVP.
Similar to Example 1.4 changing the magnitude of the positive eigenvalue λ_2 does not alter the existence or uniqueness of a solution for Example 1.5, which therefore follows from the existence and uniqueness for Example 1.3.

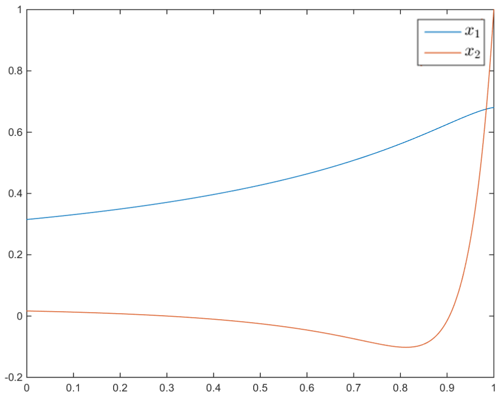

Figure 3.5: Solution of Example 1.5

Table 3.17 to 3.20 show that with a larger positive eigenvalue in $M(0)$ we are able to observe the same convergence orders as in Example 1.1 and Example 1.4: $2k$ for Gaussian collocation points, k and $k+1$ for even and odd k, respectively, for equidistant collocation points.
Once again for `bvpsuite2.0` to terminate, we have to change the tolerance from 1e-15 to 1e-13.

Table 3.17: Example 1.5: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	4.89e-01	1.62	1.73	6.03e-01	1.39	1.20	6.03e-01	1.39	1.20
1/4	1.48e-01	7.47	2.83	2.62e-01	2.78	1.70	2.62e-01	2.78	1.70
1/8	2.08e-02	4.92e+01	3.74	8.03e-02	5.53	2.04	8.03e-02	4.80	1.97
1/16	1.56e-03	1.31e+02	4.09	1.96e-02	8.53	2.19	2.05e-02	7.67	2.14
1/32	9.17e-05	1.04e+02	4.02	4.29e-03	5.26	2.05	4.67e-03	5.85	2.06
1/64	5.64e-06	9.57e+01	4.00	1.03e-03	4.19	2.00	1.12e-03	5.89	2.06
1/128	3.52e-07	9.42e+01	4.00	2.59e-04	4.31	2.00	2.69e-04	5.06	2.03
1/256	2.20e-08	9.46e+01	4.00	6.45e-05	4.25	2.00	6.59e-05	4.68	2.01
1/512	1.37e-09	9.47e+01	4.00	1.61e-05	4.23	2.00	1.63e-05	4.71	2.02
1/1024	8.58e-11	-	-	4.03e-06	-	-	4.03e-06	-	-

Table 3.18: Example 1.5: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.97e-01	1.82	3.20	3.76e-01	1.45	1.95	3.76e-01	1.45	1.95
1/4	2.15e-02	1.41e+01	4.68	9.73e-02	5.17	2.87	9.73e-02	5.17	2.87
1/8	8.37e-04	1.25e+02	5.73	1.33e-02	3.09e+01	3.73	1.33e-02	1.29e+01	3.30
1/16	1.58e-05	3.05e+02	6.05	1.01e-03	8.21e+01	4.08	1.35e-03	2.46e+01	3.54
1/32	2.38e-07	2.67e+02	6.01	5.97e-05	6.70e+01	4.02	1.16e-04	5.40e+01	3.76
1/64	3.68e-09	2.55e+02	6.00	3.68e-06	6.29e+01	4.00	8.56e-06	8.63e+01	3.88
1/128	5.74e-11	1.89e+02	5.94	2.29e-07	6.10e+01	4.00	5.82e-07	1.16e+02	3.94
1/256	9.35e-13	4.05e-03	4.00	1.44e-08	6.17e+01	4.00	3.80e-08	1.37e+02	3.97
1/512	5.84e-14	5.56e-13	0.36	8.97e-10	6.13e+01	4.00	2.43e-09	4.58e+02	4.16
1/1024	4.54e-14	-	-	5.61e-11	-	-	1.35e-10	-	-

Table 3.19: Example 1.5: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	5.88e-02	1.78	4.92	2.09e-01	1.43	2.78	2.09e-01	1.43	2.78
1/4	1.94e-03	2.02e+01	6.67	3.05e-02	4.76	3.64	3.05e-02	4.76	3.64
1/8	1.91e-05	2.08e+02	7.79	2.44e-03	1.27e+01	4.12	2.44e-03	8.58	3.93
1/16	8.61e-08	4.14e+02	8.04	1.41e-04	1.56e+01	4.19	1.60e-04	1.81e+01	4.20
1/32	3.27e-10	4.33e+02	8.05	7.71e-06	9.60	4.05	8.75e-06	1.31e+01	4.10
1/64	1.23e-12	9.00e-04	4.91	4.65e-07	8.23	4.01	5.10e-07	1.12e+01	4.07
1/128	4.10e-14	2.23e-14	-1.26e-01	2.88e-08	7.79	4.00	3.04e-08	9.88	4.04
1/256	4.48e-14	3.87e-14	-2.62e-02	1.80e-09	7.79	4.00	1.85e-09	9.17	4.03
1/512	4.56e-14	5.18e-14	0.02	1.12e-10	8.08	4.01	1.14e-10	8.97	4.02
1/1024	4.49e-14	-	-	6.99e-12	-	-	6.99e-12	-	-

Table 3.20: Example 1.5: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.32e-02	1.49	6.82	1.02e-01	1.38	3.76	1.02e-01	1.38	3.76
1/4	1.17e-04	2.16e+01	8.75	7.49e-03	7.56	4.99	7.49e-03	7.56	4.99
1/8	2.71e-07	1.99e+02	9.82	2.36e-04	5.23e+01	5.92	2.36e-04	1.75e+01	5.39
1/16	3.00e-10	1.96e+02	9.81	3.90e-06	7.68e+01	6.06	5.62e-06	3.73e+01	5.67
1/32	3.34e-13	7.90e-09	2.91	5.85e-08	6.51e+01	6.01	1.11e-07	7.35e+01	5.86
1/64	4.45e-14	4.76e-14	0.02	9.08e-10	6.12e+01	6.00	1.90e-09	1.03e+02	5.94
1/128	4.40e-14	4.49e-14	0.00	1.42e-11	2.15e+01	5.78	3.10e-11	1.33e+02	6.00
1/256	4.39e-14	3.91e-14	-2.08e-02	2.59e-13	2.42e-07	2.48	4.86e-13	6.93e-05	3.39
1/512	4.46e-14	4.88e-14	0.01	4.64e-14	9.16e-14	0.11	4.64e-14	9.16e-14	0.11
1/1024	4.41e-14	-	-	4.31e-14	-	-	4.31e-14	-	-

3.1.6 Example 1.6

We next consider Example 1.2 from [1]. Let

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{5}{2}\frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad M(0)x(0) + \frac{5}{2}f(0,x(0)) = 0,$$

where

$$M(t) = \begin{pmatrix} -1 + (\exp(t) - 1) & \exp(-t) - 1 \\ \exp(2t) - 1 & -2 + (\exp(-2t) - 1) \end{pmatrix}, \quad M(0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad \beta = 1$$

and

$$f(t,x(t)) = p(t) + H(t)r(t,x)$$

with

$$p(t) = \begin{pmatrix} t^2/2 \\ -1 \end{pmatrix}, \qquad H(t) = \begin{pmatrix} t & 0 \\ \sin(t) & \cos(t) \end{pmatrix}$$

and

$$r(t,x) = \alpha_1(t)\begin{pmatrix} \sin(x_1 + x_2) \\ \cos(x_1 + x_2) \end{pmatrix} + \alpha_2(t)\begin{pmatrix} x_1 \log(1 + |x_1 + x_2|) \\ x_2 \log(1 + |x_1 + x_2|) \end{pmatrix},$$

where

$$\alpha_1(t) = 0.5(t - 0.5)^2 \quad \alpha_2(t) = \begin{cases} 0, & t < 0.5, \\ -(t - 0.5)^2, & t \geq 0.5. \end{cases}$$

There is no known solution for this example.
We want to apply Theorem (2.7). Let $a = 0.5$. Then $\max_{t\in[0,0.5]} |H(t)| < \sqrt{2}$ and $\max_{t\in[0,0.5]} |\alpha_1(t)| = \frac{1}{8}$. Thus,

$$|r(t,x) - r(t,y)| \leq \frac{1}{4}|x - y|$$

for $t \in [0, 0.5], x, y \in \mathbb{R}^2$. Consequently, $\frac{5}{2}f$ satisfies the Lipschitz condition (2.9) on $[0, 0.5]$ with

$$L = \frac{5\sqrt{2}}{8} \in \left(0, \frac{1}{\beta}\right).$$

Furthermore, $\frac{5}{2}f$ satisfies the growth condition (2.10) on $[0.5, 1]$ with

$$w(s) = (1 + 2s)\ln(1 + 2s) + 1, \quad s \in [0, \infty).$$

From Theorem (2.7) it follows that Example 1.6 has a solution on $[0, 1]$.

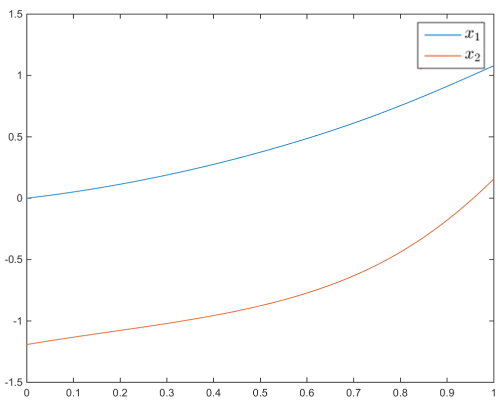

Figure 3.6: Solution of Example 1.6

As we can see in Table 3.21 to Table 3.24, we are not able to observe a clear superconvergence. Example 1.6 uses an error tolerance of 1e-11.

Table 3.21: Example 1.6: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	6.72e-03	8.01e-02	3.58	3.11e-02	1.70e-01	2.45	3.11e-02	1.51e-01	2.28
1/4	5.63e-04	4.98e-02	3.23	5.67e-03	9.77e-02	2.05	6.38e-03	1.09e-01	2.04
1/8	5.99e-05	1.34e-02	2.60	1.37e-03	8.72e-02	2.00	1.55e-03	1.42e-01	2.17
1/16	9.86e-06	6.50e-05	0.68	3.42e-04	9.14e-02	2.01	3.42e-04	9.14e-02	2.01
1/32	6.16e-06	4.49e-04	1.24	8.47e-05	1.11e-01	2.07	8.47e-05	1.03e-01	2.05
1/64	2.61e-06	3.03e-02	2.25	2.02e-05	6.43e-02	1.94	2.05e-05	7.18e-02	1.96
1/128	5.49e-07	5.23e-01	2.84	5.25e-06	9.41e-02	2.02	5.25e-06	9.15e-02	2.01
1/256	7.68e-08	3.73e-01	2.78	1.30e-06	9.03e-02	2.01	1.30e-06	9.35e-02	2.02
1/512	1.12e-08	8.18e-08	0.32	3.22e-07	7.28e-02	1.98	3.22e-07	7.28e-02	1.98
1/1024	8.99e-09	-	-	8.18e-08	-	-	8.18e-08	-	-

Table 3.22: Example 1.6: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	8.75e-04	2.47e-02	4.82	3.13e-03	1.42e-02	2.18	4.15e-03	2.49e-02	2.59
1/4	3.10e-05	9.48e-06	-8.54e-01	6.90e-04	4.79e-02	3.06	6.90e-04	4.79e-02	3.06
1/8	5.60e-05	4.74e-03	2.14	8.29e-05	4.78e-02	3.06	8.29e-05	4.96e-03	1.97
1/16	1.27e-05	1.42	4.19	9.96e-06	2.88e-05	0.38	2.12e-05	1.07e-03	1.42
1/32	6.98e-07	7.40e-07	0.02	7.63e-06	1.01e-03	1.41	7.95e-06	1.28e-03	1.47
1/64	6.90e-07	7.72e-03	2.24	2.87e-06	3.04e-03	1.67	2.87e-06	3.03e-03	1.67
1/128	1.46e-07	1.87e-01	2.90	9.00e-07	2.73e+01	3.55	9.01e-07	2.75e+01	3.55
1/256	1.96e-08	2.05e-07	0.42	7.68e-08	1.19e-05	0.91	7.68e-08	1.19e-05	0.91
1/512	1.46e-08	7.71e-03	2.11	4.09e-08	2.32e-04	1.39	4.09e-08	2.32e-04	1.39
1/1024	3.37e-09	-	-	1.56e-08	-	-	1.56e-08	-	-

Table 3.23: Example 1.6: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	6.72e-04	4.12e-03	2.62	1.67e-03	3.74e-03	1.16	1.76e-03	4.14e-03	1.24
1/4	1.10e-04	4.22e-04	0.97	7.46e-04	7.99e-02	3.37	7.46e-04	6.00e-02	3.17
1/8	5.59e-05	6.76e-03	2.31	7.21e-05	2.30e-02	2.77	8.32e-05	6.91e-03	2.13
1/16	1.13e-05	7.36	4.83	1.06e-05	2.56e-04	1.15	1.91e-05	7.49e-04	1.32
1/32	3.97e-07	2.75e-06	0.56	4.76e-06	3.82e-04	1.27	7.61e-06	6.43e-03	1.94
1/64	2.70e-07	1.40e-04	1.50	1.98e-06	1.18e-02	2.09	1.98e-06	2.54e-03	1.72
1/128	9.52e-08	6.63e-03	2.30	4.65e-07	1.38e-01	2.60	6.00e-07	1.06	2.97
1/256	1.94e-08	2.56e-03	2.13	7.68e-08	3.71e+01	3.61	7.68e-08	2.73	3.14
1/512	4.43e-09	6.06e-06	1.16	6.31e-09	1.10e-09	-2.80e-01	8.74e-09	2.88e-08	0.19
1/1024	1.99e-09	-	-	7.66e-09	-	-	7.66e-09	-	-

Table 3.24: Example 1.6: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	6.52e-05	2.74e-04	2.07	2.66e-03	1.25e-02	2.23	2.66e-03	1.25e-02	2.23
1/4	1.55e-05	5.87e-06	-7.01e-01	5.66e-04	2.31e-03	1.02	5.66e-04	2.31e-03	1.02
1/8	2.52e-05	6.31e-03	2.66	2.80e-04	3.19e-02	2.28	2.80e-04	3.19e-02	2.28
1/16	4.01e-06	2.76e-04	1.53	5.77e-05	2.20	3.80	5.77e-05	2.20	3.80
1/32	1.39e-06	5.96e-04	1.75	4.13e-06	1.43e-01	3.02	4.13e-06	1.43e-01	3.02
1/64	4.14e-07	9.91e+01	4.64	5.11e-07	4.70e-07	-2.01e-02	5.11e-07	4.70e-07	-2.01e-02
1/128	1.66e-08	5.21e-07	0.71	5.18e-07	3.28e-01	2.75	5.18e-07	3.28e-01	2.75
1/256	1.01e-08	2.61e+11	8.06	7.68e-08	5.99e-06	0.79	7.68e-08	5.99e-06	0.79
1/512	3.80e-11	6.89e-19	-2.86e+00	4.46e-08	4.89e-03	1.86	4.46e-08	4.89e-03	1.86
1/1024	2.76e-10	-	-	1.23e-08	-	-	1.23e-08	-	-

3.1.7 Example 1.7

Consider Example 2.2 from [1] without a known solution:

$$x'(t) = \frac{M(t)}{t} x(t) + \frac{5}{2}\frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad x(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where

$$M(t) = \begin{pmatrix} 1 + (\exp(t) - 1) & \exp(-t) - 1 \\ \exp(2t) - 1 & 2 + (\exp(-2t) - 1) \end{pmatrix}, \quad M(0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \beta = 1$$

and

$$f(t, x(t)) = p(t) + H(t) r(t, x)$$

with

$$p(t) = \begin{pmatrix} t^2/2 \\ -1 \end{pmatrix}, \qquad H(t) = \begin{pmatrix} \sin(t) & \cos(t) \\ t & 0 \end{pmatrix}$$

and

$$r(t,x) = \alpha_1(t) \begin{pmatrix} \sin(x_1 + x_2) \\ \cos(x_1 + x_2) \end{pmatrix} + \alpha_2(t) \begin{pmatrix} x_1 \log(1 + |x_1 + x_2|) \\ x_2 \log(1 + |x_1 + x_2|) \end{pmatrix},$$

where

$$\alpha_1(t) = 0.5(t - 0.5)^2 \quad \alpha_2(t) = \begin{cases} 0, & t < 0.5, \\ -(t-0.5)^2, & t \geq 0.5. \end{cases}$$

Since M and f remain unchanged from Example 1.6 it follows by Theorem (2.8) that a solution of the TVP Example 1.7 exists.

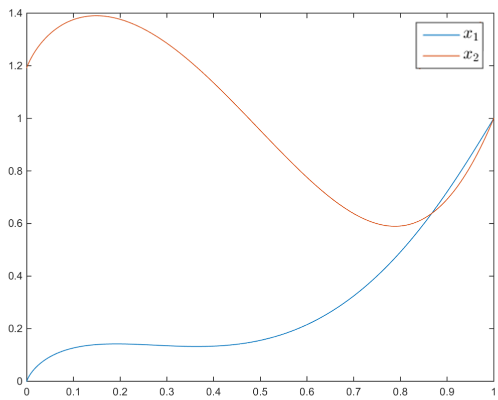

Figure 3.7: Solution of Example 1.7

As expected, we can not observe any kind of superconvergence due to the small positive eigenvalues. See Tables 3.25 to 3.28.
This example was calculated with an error tolerance of 1e-11.

Table 3.25: Example 1.7: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.38e-01	3.41e-01	1.30	1.83e-01	4.46e-01	1.29	1.83e-01	4.46e-01	1.29
1/4	5.61e-02	2.81e-01	1.16	7.50e-02	3.78e-01	1.17	7.50e-02	3.78e-01	1.17
1/8	2.51e-02	2.54e-01	1.11	3.34e-02	3.39e-01	1.11	3.34e-02	3.39e-01	1.11
1/16	1.16e-02	2.28e-01	1.08	1.54e-02	3.04e-01	1.07	1.54e-02	3.04e-01	1.07
1/32	5.50e-03	2.08e-01	1.05	7.32e-03	2.76e-01	1.05	7.32e-03	2.76e-01	1.05
1/64	2.66e-03	1.95e-01	1.03	3.54e-03	2.58e-01	1.03	3.54e-03	2.58e-01	1.03
1/128	1.30e-03	1.89e-01	1.03	1.73e-03	2.49e-01	1.02	1.73e-03	2.49e-01	1.02
1/256	6.38e-04	1.93e-01	1.03	8.53e-04	2.51e-01	1.02	8.53e-04	2.51e-01	1.02
1/512	3.12e-04	2.14e-01	1.05	4.19e-04	2.69e-01	1.04	4.19e-04	2.69e-01	1.04
1/1024	1.51e-04	-	-	2.04e-04	-	-	2.04e-04	-	-

Table 3.26: Example 1.7: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	5.28e-02	1.13e-01	1.10	8.60e-02	1.81e-01	1.07	8.60e-02	1.81e-01	1.07
1/4	2.47e-02	1.14e-01	1.11	4.09e-02	1.90e-01	1.11	4.09e-02	1.90e-01	1.11
1/8	1.15e-02	1.06e-01	1.07	1.90e-02	1.78e-01	1.08	1.90e-02	1.78e-01	1.08
1/16	5.46e-03	9.90e-02	1.04	9.00e-03	1.64e-01	1.05	9.00e-03	1.64e-01	1.05
1/32	2.65e-03	9.41e-02	1.03	4.35e-03	1.55e-01	1.03	4.35e-03	1.55e-01	1.03
1/64	1.30e-03	9.21e-02	1.03	2.13e-03	1.50e-01	1.02	2.13e-03	1.50e-01	1.02
1/128	6.37e-04	9.38e-02	1.03	1.05e-03	1.50e-01	1.02	1.05e-03	1.50e-01	1.02
1/256	3.12e-04	1.03e-01	1.05	5.17e-04	1.57e-01	1.03	5.17e-04	1.57e-01	1.03
1/512	1.51e-04	1.34e-01	1.09	2.53e-04	1.81e-01	1.05	2.53e-04	1.81e-01	1.05
1/1024	7.11e-05	-	-	1.22e-04	-	-	1.22e-04	-	-

Table 3.27: Example 1.7: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	3.06e-02	6.68e-02	1.13	6.09e-02	1.35e-01	1.15	6.09e-02	1.35e-01	1.15
1/4	1.40e-02	6.25e-02	1.08	2.75e-02	1.25e-01	1.09	2.75e-02	1.25e-01	1.09
1/8	6.61e-03	5.86e-02	1.05	1.29e-02	1.16e-01	1.06	1.29e-02	1.16e-01	1.06
1/16	3.19e-03	5.59e-02	1.03	6.20e-03	1.09e-01	1.04	6.20e-03	1.09e-01	1.04
1/32	1.56e-03	5.45e-02	1.03	3.02e-03	1.05e-01	1.02	3.02e-03	1.05e-01	1.02
1/64	7.67e-04	5.49e-02	1.03	1.49e-03	1.04e-01	1.02	1.49e-03	1.04e-01	1.02
1/128	3.77e-04	5.84e-02	1.04	7.32e-04	1.06e-01	1.02	7.32e-04	1.06e-01	1.02
1/256	1.83e-04	7.02e-02	1.07	3.60e-04	1.15e-01	1.04	3.60e-04	1.15e-01	1.04
1/512	8.71e-05	1.14e-01	1.15	1.75e-04	1.43e-01	1.07	1.75e-04	1.43e-01	1.07
1/1024	3.92e-05	-	-	8.31e-05	-	-	8.31e-05	-	-

Table 3.28: Example 1.7: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.92e-02	4.11e-02	1.10	4.39e-02	9.52e-02	1.12	4.39e-02	9.52e-02	1.12
1/4	8.98e-03	3.90e-02	1.06	2.02e-02	8.97e-02	1.07	2.02e-02	8.97e-02	1.07
1/8	4.31e-03	3.72e-02	1.04	9.60e-03	8.44e-02	1.05	9.60e-03	8.44e-02	1.05
1/16	2.10e-03	3.61e-02	1.03	4.65e-03	8.06e-02	1.03	4.65e-03	8.06e-02	1.03
1/32	1.03e-03	3.59e-02	1.02	2.28e-03	7.85e-02	1.02	2.28e-03	7.85e-02	1.02
1/64	5.06e-04	3.71e-02	1.03	1.12e-03	7.84e-02	1.02	1.12e-03	7.84e-02	1.02
1/128	2.47e-04	4.15e-02	1.06	5.53e-04	8.13e-02	1.03	5.53e-04	8.13e-02	1.03
1/256	1.19e-04	5.62e-02	1.11	2.71e-04	9.16e-02	1.05	2.71e-04	9.16e-02	1.05
1/512	5.51e-05	1.29e-01	1.24	1.31e-04	1.25e-01	1.10	1.31e-04	1.25e-01	1.10
1/1024	2.33e-05	-	-	6.11e-05	-	-	6.11e-05	-	-

3.1.8 Example 1.8

We finally consider Example 3.2 from [1]. Let

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{5}{2}\frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad -x_1(0) + \frac{5}{2}f_1(0,x(0)) = 0, \; x_2(1) = 1,$$

where

$$M(t) = \begin{pmatrix} -1 + (\exp(t) - 1) & \exp(-t) - 1 \\ \exp(2t) - 1 & 2 + (\exp(-2t) - 1) \end{pmatrix}, \quad M(0) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \beta = 1$$

and

$$f(t,x(t)) = p(t) + H(t)r(t,x)$$

with

$$p(t) = \begin{pmatrix} t^2/2 \\ -1 \end{pmatrix}, \qquad H(t) = \begin{pmatrix} t & 0 \\ \sin(t) & \cos(t) \end{pmatrix}$$

and

$$r(t,x) = \alpha_1(t)\begin{pmatrix} \sin(x_1) \\ \cos(x_1 + x_2) \end{pmatrix} + \alpha_2(t)\begin{pmatrix} x_1 \log(1 + |x_1 + x_2|) \\ x_2 \log(1 + |x_1 + x_2|) \end{pmatrix},$$

where

$$\alpha_1(t) = 0.5(t - 0.5)^2 \quad \alpha_2(t) = \begin{cases} 0, & t < 0.5, \\ -(t - 0.5)^2, & t \geq 0.5. \end{cases}$$

There is no known solution.
As before in Example 1.3, we can derive the correctly posed boundary conditions and see that (2.14) holds. With (2.9) and (2.9.1) the existence of a solution follows.

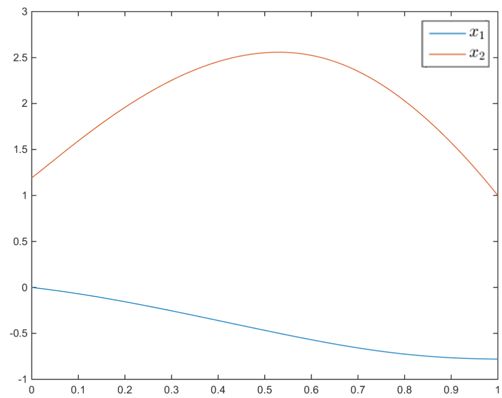

Figure 3.8: Solution of Example 1.8

In line with Example 1.2, Example 1.3, and Example 1.7, we are not able to observe any kind of superconvergence, as we can see Tables 3.29 to 3.32.
Again, we use an error tolerance of 1e-11.

Table 3.29: Example 1.8: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	3.91e-02	1.37e-01	1.81	8.64e-02	3.55e-01	2.04	9.32e-02	4.13e-01	2.15
1/4	1.11e-02	1.09e-01	1.64	2.10e-02	4.09e-01	2.14	2.10e-02	3.97e-01	2.12
1/8	3.56e-03	1.11e-01	1.65	4.77e-03	1.48e-01	1.65	4.84e-03	1.57e-01	1.67
1/16	1.13e-03	1.24e-01	1.69	1.52e-03	1.65e-01	1.69	1.52e-03	1.65e-01	1.69
1/32	3.51e-04	1.33e-01	1.71	4.69e-04	1.78e-01	1.71	4.69e-04	1.78e-01	1.71
1/64	1.07e-04	1.35e-01	1.72	1.43e-04	1.81e-01	1.72	1.43e-04	1.81e-01	1.72
1/128	3.25e-05	1.35e-01	1.72	4.35e-05	1.80e-01	1.72	4.35e-05	1.80e-01	1.72
1/256	9.91e-06	1.34e-01	1.72	1.32e-05	1.79e-01	1.72	1.32e-05	1.79e-01	1.72
1/512	3.02e-06	1.36e-01	1.72	4.03e-06	1.80e-01	1.72	4.03e-06	1.80e-01	1.72
1/1024	9.17e-07	-	-	1.23e-06	-	-	1.23e-06	-	-

Table 3.30: Example 1.8: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	8.22e-03	2.14e-02	1.38	1.34e-02	3.17e-02	1.24	1.34e-02	3.17e-02	1.24
1/4	3.15e-03	2.91e-02	1.60	5.66e-03	4.91e-02	1.56	5.66e-03	4.91e-02	1.56
1/8	1.04e-03	3.64e-02	1.71	1.92e-03	6.68e-02	1.71	1.92e-03	6.68e-02	1.71
1/16	3.18e-04	3.80e-02	1.73	5.88e-04	7.10e-02	1.73	5.88e-04	7.10e-02	1.73
1/32	9.60e-05	3.74e-02	1.72	1.77e-04	6.96e-02	1.72	1.77e-04	6.96e-02	1.72
1/64	2.91e-05	3.67e-02	1.72	5.37e-05	6.79e-02	1.72	5.37e-05	6.79e-02	1.72
1/128	8.86e-06	3.64e-02	1.71	1.63e-05	6.69e-02	1.71	1.63e-05	6.69e-02	1.71
1/256	2.70e-06	3.69e-02	1.72	4.98e-06	6.71e-02	1.71	4.98e-06	6.71e-02	1.71
1/512	8.21e-07	4.02e-02	1.73	1.52e-06	7.00e-02	1.72	1.52e-06	7.00e-02	1.72
1/1024	2.47e-07	-	-	4.60e-07	-	-	4.60e-07	-	-

Table 3.31: Example 1.8: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	4.06e-03	1.19e-02	1.55	1.60e-02	7.78e-02	2.28	1.60e-02	7.78e-02	2.28
1/4	1.39e-03	1.52e-02	1.73	3.29e-03	3.57e-02	1.72	3.29e-03	3.57e-02	1.72
1/8	4.20e-04	1.55e-02	1.73	9.97e-04	3.74e-02	1.74	9.97e-04	3.74e-02	1.74
1/16	1.26e-04	1.50e-02	1.72	2.98e-04	3.58e-02	1.73	2.98e-04	3.58e-02	1.73
1/32	3.82e-05	1.46e-02	1.72	8.99e-05	3.47e-02	1.72	8.99e-05	3.47e-02	1.72
1/64	1.16e-05	1.45e-02	1.71	2.73e-05	3.41e-02	1.71	2.73e-05	3.41e-02	1.71
1/128	3.54e-06	1.46e-02	1.72	8.33e-06	3.40e-02	1.71	8.33e-06	3.40e-02	1.71
1/256	1.08e-06	1.55e-02	1.73	2.54e-06	3.47e-02	1.72	2.54e-06	3.47e-02	1.72
1/512	3.26e-07	1.94e-02	1.76	7.72e-07	3.81e-02	1.73	7.72e-07	3.81e-02	1.73
1/1024	9.62e-08	-	-	2.32e-07	-	-	2.32e-07	-	-

Table 3.32: Example 1.8: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.24e-03	7.39e-03	1.72	6.43e-03	2.06e-02	1.68	6.43e-03	2.06e-02	1.68
1/4	6.81e-04	7.65e-03	1.74	2.01e-03	2.32e-02	1.76	2.01e-03	2.32e-02	1.76
1/8	2.03e-04	7.35e-03	1.73	5.93e-04	2.19e-02	1.74	5.93e-04	2.19e-02	1.74
1/16	6.14e-05	7.17e-03	1.72	1.78e-04	2.10e-02	1.72	1.78e-04	2.10e-02	1.72
1/32	1.87e-05	7.09e-03	1.71	5.40e-05	2.06e-02	1.72	5.40e-05	2.06e-02	1.72
1/64	5.69e-06	7.10e-03	1.71	1.64e-05	2.04e-02	1.71	1.64e-05	2.04e-02	1.71
1/128	1.73e-06	7.31e-03	1.72	5.01e-06	2.05e-02	1.71	5.01e-06	2.05e-02	1.71
1/256	5.26e-07	8.27e-03	1.74	1.53e-06	2.13e-02	1.72	1.53e-06	2.13e-02	1.72
1/512	1.57e-07	1.35e-02	1.82	4.63e-07	2.50e-02	1.75	4.63e-07	2.50e-02	1.75
1/1024	4.45e-08	-	-	1.38e-07	-	-	1.38e-07	-	-

3.1.9 Example 1.9

Similar to Example 1.4 we raise the eigenvalues from Example 1.7 and repeat the experiment. Let

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{5}{2}\frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad x(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where

$$M(t) = \begin{pmatrix} 10 + (\exp(t) - 1) & \exp(-t) - 1 \\ \exp(2t) - 1 & 20 + (\exp(-2t) - 1) \end{pmatrix}, \quad M(0) = \begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix}, \quad \beta = 10$$

and

$$f(t, x(t)) = p(t) + H(t)r(t,x)$$

with

$$p(t) = \begin{pmatrix} t^2/2 \\ -1 \end{pmatrix}, \qquad H(t) = \begin{pmatrix} \sin(t) & \cos(t) \\ t & 0 \end{pmatrix}$$

and

$$r(t,x) = \alpha_1(t) \begin{pmatrix} \sin(x_1 + x_2) \\ \cos(x_1 + x_2) \end{pmatrix} + \alpha_2(t) \begin{pmatrix} x_1 \log(1 + |x_1 + x_2|) \\ x_2 \log(1 + |x_1 + x_2|) \end{pmatrix},$$

where

$$\alpha_1(t) = 0.5(t - 0.5)^2 \quad \alpha_2(t) = \begin{cases} 0, & t < 0.5, \\ -(t-0.5)^2, & t \geq 0.5. \end{cases}$$

We do not have an analytical solution at hand.
As before, the existence of a solution immediately follows from the existence for Example 1.7.

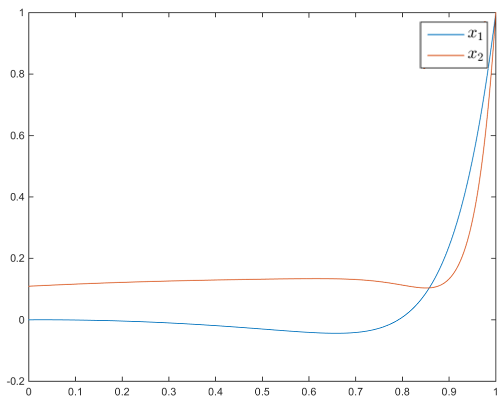

Figure 3.9: Solution of Example 1.9

In Tables 3.33 to 3.36 we are able to observe the same kind of superconvergence order for k collocation points as before: $2k$ for Gaussian points and k or $k+1$ for an even or odd number of uniform points respectively, although these results are not as clearly visible as in the examples above.
Once again, an error tolerance of 1e-11 was used for the computations.

Table 3.33: Example 1.9: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	4.92e-01	1.47	1.58	6.15e-01	1.33	1.11	6.15e-01	1.33	1.11
1/4	1.65e-01	7.19	2.72	2.85e-01	2.79	1.65	2.85e-01	2.79	1.65
1/8	2.50e-02	5.99e+01	3.74	9.10e-02	6.58	2.06	9.10e-02	5.74	1.99
1/16	1.87e-03	1.57e+02	4.09	2.18e-02	9.77	2.20	2.28e-02	9.08	2.16
1/32	1.10e-04	1.25e+02	4.02	4.75e-03	5.86	2.05	5.12e-03	6.21	2.05
1/64	6.77e-06	1.16e+02	4.01	1.14e-03	4.73	2.00	1.24e-03	6.45	2.06
1/128	4.22e-07	1.13e+02	4.00	2.85e-04	4.75	2.00	2.97e-04	5.64	2.03
1/256	2.64e-08	1.13e+02	4.00	7.12e-05	4.69	2.00	7.27e-05	5.19	2.02
1/512	1.65e-09	1.13e+02	4.00	1.78e-05	4.66	2.00	1.80e-05	5.21	2.02
1/1024	1.03e-10	-	-	4.44e-06	-	-	4.44e-06	-	-

Table 3.34: Example 1.9: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.11e-01	1.70	3.01	3.94e-01	1.33	1.75	3.94e-01	1.33	1.75
1/4	2.62e-02	1.76e+01	4.69	1.17e-01	6.39	2.89	1.17e-01	6.39	2.89
1/8	1.01e-03	2.02e+02	5.87	1.58e-02	5.56e+01	3.93	1.58e-02	2.54e+01	3.55
1/16	1.73e-05	3.05e+02	6.02	1.04e-03	8.25e+01	4.07	1.35e-03	2.94e+01	3.60
1/32	2.68e-07	2.82e+02	5.99	6.17e-05	6.83e+01	4.02	1.11e-04	5.50e+01	3.78
1/64	4.20e-09	2.82e+02	5.99	3.82e-06	6.21e+01	3.99	8.03e-06	8.18e+01	3.88
1/128	6.59e-11	2.55e+02	5.97	2.40e-07	6.46e+01	4.00	5.45e-07	1.08e+02	3.94
1/256	1.05e-12	4.54e-02	4.42	1.50e-08	6.44e+01	4.00	3.56e-08	1.28e+02	3.97
1/512	4.91e-14	4.29e-13	0.35	9.35e-10	6.41e+01	4.00	2.27e-09	4.25e+02	4.16
1/1024	3.86e-14	-	-	5.85e-11	-	-	1.27e-10	-	-

Table 3.35: Example 1.9: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	7.00e-02	2.03	4.86	2.41e-01	1.50	2.64	2.41e-01	1.50	2.64
1/4	2.41e-03	3.97e+01	7.00	3.87e-02	8.93	3.93	3.87e-02	8.93	3.93
1/8	1.88e-05	3.73e+02	8.08	2.55e-03	1.95e+01	4.30	2.55e-03	1.14e+01	4.04
1/16	6.95e-08	2.99e+02	8.00	1.29e-04	1.02e+01	4.07	1.55e-04	1.43e+01	4.13
1/32	2.71e-10	2.58e+02	7.96	7.71e-06	8.70	4.02	8.86e-06	1.35e+01	4.11
1/64	1.09e-12	8.00e-04	4.91	4.75e-07	8.15	4.01	5.13e-07	1.09e+01	4.06
1/128	3.64e-14	2.72e-14	-6.02e-02	2.96e-08	8.03	4.00	3.08e-08	9.64	4.03
1/256	3.79e-14	3.71e-14	-3.81e-03	1.84e-09	7.93	4.00	1.89e-09	9.07	4.02
1/512	3.80e-14	3.72e-14	-3.72e-03	1.15e-10	7.94	4.00	1.16e-10	8.62	4.01
1/1024	3.81e-14	-	-	7.20e-12	-	-	7.20e-12	-	-

Table 3.36: Example 1.9: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.69e-02	2.31	7.10	1.32e-01	1.98	3.91	1.32e-01	1.98	3.91
1/4	1.23e-04	1.27e+02	9.99	8.75e-03	4.41e+01	6.15	8.75e-03	1.15e+01	5.18
1/8	1.21e-07	4.10e+01	9.44	1.23e-04	2.62	4.79	2.41e-04	1.46e+01	5.29
1/16	1.74e-10	8.09e-01	8.03	4.45e-06	6.12e+01	5.93	6.16e-06	2.85	4.70
1/32	6.66e-13	1.16e-06	4.15	7.32e-08	7.18e+01	5.97	2.36e-07	7.24	4.97
1/64	3.76e-14	3.93e-14	0.01	1.16e-09	7.78e+01	5.99	7.51e-09	7.68e+01	5.54
1/128	3.73e-14	3.77e-14	0.00	1.83e-11	7.30e+01	5.98	1.61e-10	2.76e+02	5.81
1/256	3.73e-14	3.86e-14	0.01	2.90e-13	7.16e-06	3.07	2.88e-12	5.85e+02	5.94
1/512	3.71e-14	3.55e-14	-7.02e-03	3.45e-14	2.04e-14	-8.39e-02	4.69e-14	4.39e-13	0.36
1/1024	3.73e-14	-	-	3.65e-14	-	-	3.66e-14	-	-

3.1.10 Example 1.10

We want to repeat Example 1.8 for larger positive eigenvalues. Let

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{5}{2}\frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad -x_1(0) + \frac{1}{3}f_1(0,x(0)) = 0, \; x_2(1) = 1,$$

where

$$M(t) = \begin{pmatrix} -1 + (\exp(t) - 1) & \exp(-t) - 1 \\ \exp(2t) - 1 & 20 + (\exp(-2t) - 1) \end{pmatrix}, \quad M(0) = \begin{pmatrix} -1 & 0 \\ 0 & 20 \end{pmatrix}, \quad \beta = 1$$

and

$$f(t,x(t)) = p(t) + H(t)r(t,x)$$

with

$$p(t) = \begin{pmatrix} t^2/2 \\ -1 \end{pmatrix}, \qquad H(t) = \begin{pmatrix} t & 0 \\ \sin(t) & \cos(t) \end{pmatrix}$$

and

$$r(t,x) = \alpha_1(t)\begin{pmatrix} \sin(x_1) \\ \cos(x_1 + x_2) \end{pmatrix} + \alpha_2(t)\begin{pmatrix} x_1 \log(1 + |x_1 + x_2|) \\ x_2 \log(1 + |x_1 + x_2|) \end{pmatrix},$$

where

$$\alpha_1(t) = 0.5(t - 0.5)^2 \quad \alpha_2(t) = \begin{cases} 0, & t < 0.5, \\ -(t - 0.5)^2, & t \geq 0.5. \end{cases}$$

We do not have an analytical solution.
The existence of a solution follows from the existence for Example 1.8.

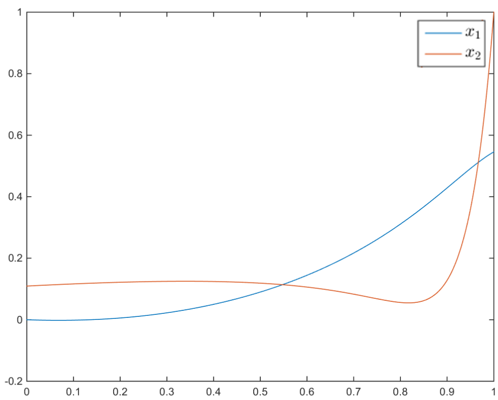

Figure 3.10: Solution of Example 1.10

As we can see in Tables 3.37 to 3.40, larger positive eigenvalue once again allow us to observe our usual superconvergence orders.
The used error tolerance was 1e-11.

Table 3.37: Example 1.10: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	4.21e-01	1.40	1.74	5.19e-01	1.20	1.21	5.19e-01	1.20	1.21
1/4	1.26e-01	6.85	2.88	2.24e-01	2.52	1.75	2.24e-01	2.52	1.75
1/8	1.71e-02	4.58e+01	3.80	6.69e-02	5.12	2.09	6.69e-02	4.29	2.00
1/16	1.23e-03	1.01e+02	4.08	1.58e-02	6.72	2.18	1.67e-02	6.66	2.16
1/32	7.26e-05	8.19e+01	4.02	3.47e-03	4.22	2.05	3.74e-03	4.64	2.06
1/64	4.47e-06	7.67e+01	4.01	8.38e-04	3.55	2.01	9.00e-04	4.59	2.05
1/128	2.78e-07	7.52e+01	4.00	2.08e-04	3.45	2.00	2.17e-04	4.12	2.03
1/256	1.74e-08	7.47e+01	4.00	5.20e-05	3.42	2.00	5.31e-05	3.78	2.01
1/512	1.09e-09	7.50e+01	4.00	1.30e-05	3.41	2.00	1.31e-05	3.79	2.02
1/1024	6.78e-11	-	-	3.25e-06	-	-	3.25e-06	-	-

Table 3.38: Example 1.10: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.70e-01	1.63	3.26	3.27e-01	1.31	2.01	3.27e-01	1.31	2.01
1/4	1.77e-02	1.34e+01	4.78	8.14e-02	5.02	2.97	8.14e-02	5.02	2.97
1/8	6.44e-04	1.16e+02	5.82	1.04e-02	2.94e+01	3.82	1.04e-02	1.27e+01	3.42
1/16	1.14e-05	2.15e+02	6.04	7.32e-04	5.84e+01	4.07	9.68e-04	1.94e+01	3.57
1/32	1.73e-07	1.93e+02	6.01	4.36e-05	4.86e+01	4.02	8.14e-05	3.95e+01	3.78
1/64	2.68e-09	1.83e+02	6.00	2.69e-06	4.60e+01	4.00	5.94e-06	6.09e+01	3.88
1/128	4.20e-11	1.25e+02	5.92	1.68e-07	4.53e+01	4.00	4.03e-07	8.08e+01	3.94
1/256	6.95e-13	1.71e-04	3.48	1.05e-08	4.50e+01	4.00	2.63e-08	9.54e+01	3.97
1/512	6.20e-14	1.68e-13	0.16	6.54e-10	4.46e+01	4.00	1.68e-09	3.17e+02	4.16
1/1024	5.55e-14	-	-	4.09e-11	-	-	9.36e-11	-	-

Table 3.39: Example 1.10: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	4.96e-02	1.63	5.04	1.80e-01	1.35	2.91	1.80e-01	1.35	2.91
1/4	1.50e-03	1.97e+01	6.84	2.40e-02	4.94	3.84	2.40e-02	4.94	3.84
1/8	1.31e-05	1.82e+02	7.91	1.67e-03	1.08e+01	4.22	1.67e-03	6.61	3.98
1/16	5.47e-08	2.42e+02	8.01	9.00e-05	9.20	4.16	1.06e-04	1.18e+01	4.19
1/32	2.12e-10	2.64e+02	8.04	5.04e-06	6.11	4.04	5.80e-06	9.09	4.12
1/64	8.09e-13	1.09e-05	3.95	3.06e-07	5.36	4.01	3.34e-07	7.44	4.07
1/128	5.25e-14	4.09e-14	-5.14e-02	1.90e-08	5.16	4.00	1.99e-08	6.40	4.04
1/256	5.44e-14	5.60e-14	0.01	1.18e-09	5.12	4.00	1.22e-09	6.00	4.03
1/512	5.42e-14	4.87e-14	-1.70e-02	7.39e-11	5.42	4.01	7.46e-11	6.00	4.03
1/1024	5.48e-14	-	-	4.58e-12	-	-	4.58e-12	-	-

Table 3.40: Example 1.10: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.06e-02	1.41	7.05	8.51e-02	1.38	4.02	8.51e-02	1.38	4.02
1/4	8.00e-05	2.15e+01	9.02	5.23e-03	8.25	5.31	5.23e-03	8.25	5.31
1/8	1.55e-07	1.37e+02	9.91	1.32e-04	2.62e+01	5.87	1.32e-04	7.00	5.23
1/16	1.61e-10	4.48e+01	9.50	2.25e-06	3.73e+01	6.00	3.50e-06	8.77	5.31
1/32	2.21e-13	2.67e-10	2.05	3.53e-08	3.78e+01	6.00	8.79e-08	2.43e+01	5.61
1/64	5.36e-14	5.52e-14	0.01	5.52e-10	3.70e+01	5.99	1.80e-09	5.59e+01	5.81
1/128	5.33e-14	5.30e-14	-1.13e-03	8.66e-12	6.36	5.63	3.21e-11	9.83e+01	5.93
1/256	5.34e-14	5.44e-14	0.00	1.75e-13	2.15e-09	1.70	5.29e-13	4.58e-05	3.30
1/512	5.32e-14	5.57e-14	0.01	5.38e-14	7.29e-14	0.05	5.38e-14	7.27e-14	0.05
1/1024	5.30e-14	-	-	5.20e-14	-	-	5.20e-14	-	-

3.1.11 Summary

For the IVP in Example 1.1 we can almost reach the superconvergence order of $2k$ for Gaussian collocation points mentioned in [7] and are also able to observe a small superconvergence order of $k+1$ for an odd number k of equidistant collocation points as described in Section 2.2.2.
Because of the small positive eigenvalues of the TVPs in Example 1.2 and Example 1.7 and the BVPs in Example 1.3 and Example 1.8, we are not able to observe any kind of superconvergence at all.
Choosing larger positive eigenvalues for $M(0)$ in these examples allows us to observe the same convergence orders as in Example 1.1, as shown in Example 1.4, Example 1.5, Example 1.9 and Example 1.10.
For Example 1.6 we are not able to see superconvergence.

3.2 Examples not satisfying the analytical conditions with a known small degree polynomial as solution

We want to examine examples with known explicit solutions, which do not satisfy the analytical assumptions we pose for the right-hand side.

We consider nonlinear boundary value problems of the form (1.1)

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad b(x(0),x(1)) = 0,$$

where the right-hand side does not satisfy either the Lipschitz condition (2.9) nor the growth condition (2.10), which are needed for the proof of existence and uniqueness of a solution. A solution, however, is known and given in the explicit form

$$x_1(t) = t, \quad x_2(t) = t^2.$$

The analytical solution is the same for all examples.

3.2.1 Example 2.1

We consider the following IVP :

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad M(0)x(0) + f(0,x(0)) = 0,$$

where

$$M(t) = \begin{pmatrix} -1 & t \\ -t^3 & -2 \end{pmatrix}, \quad M(0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad \beta = \max\left\{1, \frac{1}{2}\right\} = 1$$

and

$$f(t,x(t)) = \begin{pmatrix} 2\sqrt{tx_1} - x_2^{3/2} \\ 4x_1^2 + x_1^2 x_2 \end{pmatrix}.$$

The functions f_1 and f_2 obviously do not satisfy the conditions (2.9) and (2.10). The known solutions is

$$x_1(t) = t, \quad x_2(t) = t^2.$$

This example uses an error tolerance of 1e-15.
As we can see in Tables 3.41 to 3.45 for $k=2$ to $k=5$ we can not observe any clear superconvergence in the mesh points, and a convergence order of 2 for the uniform error, independent of the number of collocation points used. The reason for this may be the very low error achieved in the meshpoints; for $k=2$, $h=1/2$ and Gaussian points we already have an error as low as 5.66e-16, while using an error tolerance of 1e-15.

This effect is a possible consequence of the very tame solution. The functions $x_1(t) = t$ and $x_2(t) = t^2$ are both in $C^\infty[0,1]$ and can of course be perfectly approximated with a polynomial of second order, which in turn makes the collocation method very powerful and allows it to yield a very good approximation with very little effort.
We want to verify this theory by repeating the tests with $k=1$, and are now able to observe a superconvergence of $2k$ in the mesh points, both for Gaussian and equidistant mesh points.

Table 3.41: Example 2.1: Convergence of the collocation scheme, $k = 1$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	8.67e-02	3.57e-01	2.04	8.67e-02	3.57e-01	2.04	8.67e-02	3.57e-01	2.04
1/4	2.11e-02	3.42e-01	2.01	2.11e-02	3.42e-01	2.01	2.11e-02	3.42e-01	2.01
1/8	5.22e-03	3.36e-01	2.00	5.22e-03	3.36e-01	2.00	5.22e-03	3.36e-01	2.00
1/16	1.30e-03	3.34e-01	2.00	1.30e-03	3.34e-01	2.00	1.30e-03	3.34e-01	2.00
1/32	3.26e-04	3.34e-01	2.00	3.26e-04	3.34e-01	2.00	3.26e-04	3.34e-01	2.00
1/64	8.14e-05	3.33e-01	2.00	8.14e-05	3.33e-01	2.00	8.14e-05	3.33e-01	2.00
1/128	2.03e-05	3.33e-01	2.00	2.03e-05	3.33e-01	2.00	2.03e-05	3.33e-01	2.00
1/256	5.09e-06	3.33e-01	2.00	5.09e-06	3.33e-01	2.00	5.09e-06	3.33e-01	2.00
1/512	1.27e-06	3.33e-01	2.00	1.27e-06	3.33e-01	2.00	1.27e-06	3.33e-01	2.00
1/1024	3.18e-07	-	-	3.18e-07	-	-	3.18e-07	-	-

Table 3.42: Example 2.1: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	5.66e-16	7.00e-16	0.31	2.22e-16	-7.84e-17	2.63	5.56e-02	2.22e-01	2.00
1/4	4.58e-16	7.16e-16	0.36	0	-2.52e-19	-2.13e+00	1.39e-02	2.22e-01	2.00
1/8	3.51e-16	1.33e-15	0.68	1.57e-16	-1.36e-18	-7.96e-01	3.47e-03	2.22e-01	2.00
1/16	2.22e-16	1.29e-16	-1.56e-01	2.46e-16	-3.91e-16	0.30	8.68e-04	2.22e-01	2.00
1/32	2.48e-16	2.45e-16	-3.96e-04	2.22e-16	2.92e-17	-5.85e-01	2.17e-04	2.22e-01	2.00
1/64	2.48e-16	3.08e-14	1.16	3.33e-16	2.44e-13	1.59	5.43e-05	2.22e-01	2.00
1/128	1.11e-16	3.86e-19	-1.16e+00	1.11e-16	3.94e-19	-1.16e+00	1.36e-05	2.22e-01	2.00
1/256	2.48e-16	5.76e-16	0.16	2.48e-16	2.37e-18	-8.39e-01	3.39e-06	2.22e-01	2.00
1/512	2.22e-16	3.55e-18	-6.61e-01	4.44e-16	8.33e-14	0.84	8.48e-07	2.22e-01	2.00
1/1024	3.51e-16	-	-	2.48e-16	-	-	2.12e-07	-	-

Table 3.43: Example 2.1: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.40e-15	1.46e-15	0.06	0	-2.12e-16	-5.35e-01	9.38e-02	3.75e-01	2.00
1/4	1.35e-15	1.91e-15	0.25	4.44e-16	4.87e-45	-4.81e+01	2.34e-02	6.75e-04	-2.56e+00
1/8	1.13e-15	6.44e-16	-2.56e-01	1.32e-01	-4.68e+29	5.04e+01	1.38e-01	1.16e+05	6.56
1/16	1.35e-15	1.80e-14	0.95	0	-4.14e-21	-1.46e+00	1.46e-03	3.75e-01	2.00
1/32	7.02e-16	1.26e-13	1.50	2.48e-16	4.09e-16	0.16	3.66e-04	3.75e-01	2.00
1/64	2.48e-16	3.84e-18	-1.00e+00	2.22e-16	-5.07e-15	0.75	9.16e-05	3.75e-01	2.00
1/128	4.97e-16	1.77e-11	2.16	1.10e-16	-2.33e-20	-1.25e+00	2.29e-05	3.75e-01	2.00
1/256	1.11e-16	4.31e-19	-1.00e+00	3.14e-16	1.29e-12	1.50	5.72e-06	3.75e-01	2.00
1/512	2.22e-16	2.20e-16	-2.16e-04	1.11e-16	7.94e-20	-1.16e+00	1.43e-06	3.75e-01	2.00
1/1024	2.22e-16	-	-	2.48e-16	-	-	3.58e-07	-	-

Table 3.44: Example 2.1: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.05e-15	1.51e-15	0.59	6.15e-15	6.29e-15	0.03	1.20e-01	4.80e-01	2.00
1/4	7.02e-16	2.61e-16	-6.82e-01	6.01e-15	6.12e-15	0.01	3.00e-02	4.80e-01	2.00
1/8	1.13e-15	3.11e-15	0.49	5.95e-15	5.41e-15	-4.52e-02	7.50e-03	4.80e-01	2.00
1/16	8.08e-16	1.66e-15	0.26	6.14e-15	1.15e-14	0.23	1.88e-03	4.80e-01	2.00
1/32	6.75e-16	5.57e-12	2.60	5.25e-15	3.89e-15	-8.65e-02	4.69e-04	4.80e-01	2.00
1/64	1.11e-16	8.82e-19	-1.16e+00	5.58e-15	3.68e-14	0.45	1.17e-04	4.80e-01	2.00
1/128	2.48e-16	3.42e-18	-8.83e-01	4.07e-15	1.66e-11	1.71	2.93e-05	4.80e-01	2.00
1/256	4.58e-16	3.82e-15	0.38	1.24e-15	7.40e-11	1.98	7.32e-06	4.80e-01	2.00
1/512	3.51e-16	3.22e-17	-3.83e-01	3.14e-16	1.60e-13	1.00	1.83e-06	4.80e-01	2.00
1/1024	4.58e-16	-	-	1.57e-16	-	-	4.58e-07	-	-

Table 3.45: Example 2.1: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	4.53e-15	4.33e-15	-6.62e-02	7.40e-15	3.72e-28	-4.42e+01	1.39e-01	1.05e-01	-4.01e-01
1/4	4.74e-15	4.16e-15	-9.43e-02	1.47e-01	5.09e+25	4.41e+01	1.83e-01	8.18e+01	4.40
1/8	5.06e-15	4.27e-15	-8.19e-02	7.90e-15	8.91e-53	-4.20e+01	8.68e-03	1.08e-04	-2.11e+00
1/16	5.36e-15	9.37e-15	0.20	3.52e-02	9.25e+48	4.19e+01	3.74e-02	8.48e+05	6.11
1/32	4.66e-15	1.08e-14	0.24	8.75e-15	8.87e-15	0.00	5.43e-04	5.56e-01	2.00
1/64	3.94e-15	2.29e-14	0.42	8.73e-15	2.80e-14	0.28	1.36e-04	5.56e-01	2.00
1/128	2.94e-15	8.46e-12	1.64	7.19e-15	1.06e-11	1.50	3.39e-05	5.56e-01	2.00
1/256	9.42e-16	6.48e-14	0.76	2.54e-15	1.06e-11	1.50	8.48e-06	5.56e-01	2.00
1/512	5.55e-16	7.76e-13	1.16	8.95e-16	6.58e-14	0.69	2.12e-06	5.56e-01	2.00
1/1024	2.48e-16	-	-	5.55e-16	-	-	5.30e-07	-	-

3.2.2 Example 2.2

We consider the following TVP :

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad x(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where

$$M(t) = \begin{pmatrix} 1 & t \\ 3t^4 & 2 \end{pmatrix}, \quad M(0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \beta = \max\left\{1, \frac{1}{2}\right\} = 1$$

and

$$f(t,x(t)) = \begin{pmatrix} -x_1^3 \\ -3x_1x_2^2 \end{pmatrix}.$$

The functions f_1 and f_2 do not satisfy the conditions (2.9) and (2.10). The known solutions is

$$x_1(t) = t, \quad x_2(t) = t^2.$$

For $k = 2$ to $k = 5$ we can not produce any useful data. For $k = 1$ we observe a superconvergence order of 2. See Example 2.1 for further explanation.

Table 3.46: Example 2.2: Convergence of the collocation scheme, $k = 1$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	9.29e-02	4.84e-01	2.38	9.29e-02	4.84e-01	2.38	9.29e-02	3.08e-01	1.73
1/4	1.78e-02	3.37e-01	2.12	1.78e-02	3.37e-01	2.12	2.80e-02	4.08e-01	1.93
1/8	4.10e-03	2.71e-01	2.02	4.10e-03	2.71e-01	2.02	7.35e-03	4.51e-01	1.98
1/16	1.01e-03	2.62e-01	2.00	1.01e-03	2.62e-01	2.00	1.86e-03	4.73e-01	2.00
1/32	2.53e-04	2.58e-01	2.00	2.53e-04	2.58e-01	2.00	4.67e-04	4.75e-01	2.00
1/64	6.32e-05	2.59e-01	2.00	6.32e-05	2.59e-01	2.00	1.17e-04	4.78e-01	2.00
1/128	1.58e-05	2.59e-01	2.00	1.58e-05	2.59e-01	2.00	2.92e-05	4.78e-01	2.00
1/256	3.95e-06	2.59e-01	2.00	3.95e-06	2.59e-01	2.00	7.30e-06	4.78e-01	2.00
1/512	9.87e-07	2.59e-01	2.00	9.87e-07	2.59e-01	2.00	1.82e-06	4.78e-01	2.00
1/1024	2.47e-07	-	-	2.47e-07	-	-	4.56e-07	-	-

Table 3.47: Example 2.2: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.83e-16	6.45e-16	1.19	6.13e-16	1.91e-15	1.64	5.56e-02	2.22e-01	2.00
1/4	1.24e-16	3.88e-17	-8.39e-01	1.96e-16	1.54e-16	-1.77e-01	1.39e-02	2.22e-01	2.00
1/8	2.22e-16	4.49e-16	0.34	2.22e-16	6.58e-17	-5.85e-01	3.47e-03	2.22e-01	2.00
1/16	1.76e-16	1.12e-14	1.50	3.33e-16	6.12e-12	3.54	8.68e-04	2.22e-01	2.00
1/32	6.21e-17	1.40e-17	-4.29e-01	2.86e-17	1.86e-20	-2.12e+00	2.17e-04	2.22e-01	2.00
1/64	8.36e-17	8.10e-16	0.55	1.24e-16	7.94e-15	1.00	5.43e-05	2.22e-01	2.00
1/128	5.72e-17	1.14e-18	-8.06e-01	6.21e-17	8.27e-21	-1.84e+00	1.36e-05	2.22e-01	2.00
1/256	1.00e-16	1.70e-19	-1.15e+00	2.22e-16	3.89e-12	1.76	3.39e-06	2.22e-01	2.00
1/512	2.22e-16	3.78e-13	1.19	6.55e-17	5.26e-19	-7.73e-01	8.48e-07	2.22e-01	2.00
1/1024	9.72e-17	-	-	1.12e-16	-	-	2.12e-07	-	-

Table 3.48: Example 2.2: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.83e-16	1.44e-16	-9.72e-01	1.39e-16	8.41e-17	-7.22e-01	9.37e-02	3.75e-01	2.00
1/4	5.55e-16	1.11e-15	0.50	2.29e-16	3.89e-15	2.04	2.34e-02	3.75e-01	2.00
1/8	3.93e-16	1.90e-16	-3.48e-01	5.55e-17	6.94e-18	-1.00e+00	5.86e-03	3.75e-01	2.00
1/16	5.00e-16	5.12e-14	1.67	1.11e-16	1.78e-15	1.00	1.46e-03	3.75e-01	2.00
1/32	1.57e-16	8.54e-16	0.49	5.55e-17	1.73e-18	-1.00e+00	3.66e-04	3.75e-01	2.00
1/64	1.12e-16	6.00e-17	-1.50e-01	1.11e-16	8.88e-16	0.50	9.16e-05	3.75e-01	2.00
1/128	1.24e-16	2.57e-16	0.15	7.85e-17	6.94e-18	-5.00e-01	2.29e-05	3.75e-01	2.00
1/256	1.12e-16	1.19e-14	0.84	1.11e-16	4.07e-16	0.23	5.72e-06	3.75e-01	2.00
1/512	6.25e-17	3.52e-19	-8.30e-01	9.44e-17	4.96e-18	-4.72e-01	1.43e-06	3.75e-01	2.00
1/1024	1.11e-16	-	-	1.31e-16	-	-	3.58e-07	-	-

Table 3.49: Example 2.2: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	3.14e-16	2.96e-16	-8.50e-02	1.74e-15	1.48e-15	-2.32e-01	1.20e-01	4.80e-01	2.00
1/4	3.33e-16	4.13e-16	0.16	2.04e-15	2.08e-15	0.01	3.00e-02	4.80e-01	2.00
1/8	2.99e-16	6.29e-17	-7.50e-01	2.02e-15	2.48e-15	0.10	7.50e-03	4.80e-01	2.00
1/16	5.03e-16	1.62e-13	2.08	1.89e-15	9.40e-16	-2.52e-01	1.87e-03	4.80e-01	2.00
1/32	1.19e-16	2.95e-18	-1.07e+00	2.25e-15	2.10e-15	-1.95e-02	4.69e-04	4.80e-01	2.00
1/64	2.48e-16	3.10e-14	1.16	2.28e-15	5.30e-10	2.97	1.17e-04	4.80e-01	2.00
1/128	1.11e-16	1.11e-16	0	2.91e-16	3.05e-14	0.96	2.93e-05	4.80e-01	2.00
1/256	1.11e-16	1.11e-16	0	1.49e-16	1.66e-17	-3.97e-01	7.32e-06	4.80e-01	2.00
1/512	1.11e-16	4.54e-17	-1.43e-01	1.97e-16	1.99e-16	0.00	1.83e-06	4.80e-01	2.00
1/1024	1.23e-16	-	-	1.97e-16	-	-	4.58e-07	-	-

Table 3.50: Example 2.2: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.33e-15	1.56e-15	0.23	1.31e-14	2.71e-14	1.05	1.39e-01	5.56e-01	2.00
1/4	1.14e-15	6.24e-16	-4.35e-01	6.33e-15	1.37e-14	0.56	3.47e-02	5.56e-01	2.00
1/8	1.54e-15	1.52e-15	-6.54e-03	4.30e-15	9.15e-15	0.36	8.68e-03	5.56e-01	2.00
1/16	1.55e-15	1.08e-15	-1.30e-01	3.34e-15	4.46e-15	0.10	2.17e-03	5.56e-01	2.00
1/32	1.69e-15	4.35e-16	-3.92e-01	3.11e-15	1.03e-14	0.35	5.43e-04	5.56e-01	2.00
1/64	2.22e-15	8.45e-09	3.64	2.45e-15	1.65e-16	-6.48e-01	1.36e-04	5.56e-01	2.00
1/128	1.78e-16	1.71e-17	-4.82e-01	3.83e-15	6.39e-06	4.38	3.39e-05	5.56e-01	2.00
1/256	2.48e-16	2.95e-15	0.45	1.84e-16	3.89e-17	-2.81e-01	8.48e-06	5.56e-01	2.00
1/512	1.82e-16	6.94e-16	0.21	2.24e-16	1.06e-16	-1.20e-01	2.12e-06	5.56e-01	2.00
1/1024	1.57e-16	-	-	2.43e-16	-	-	5.30e-07	-	-

3.2.3 Example 2.3

We consider the following BVP :

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad -x_1(0) + f_1(0,x(0)) = 0,\ x_2(1) = 1,$$

where

$$M(t) = \begin{pmatrix} -1 & t \\ 3t^4 & 2 \end{pmatrix}, \quad M(0) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \beta = \max\left\{1, \frac{1}{2}\right\} = 1$$

and

$$f(t,x(t)) = \begin{pmatrix} 2\sqrt{tx_1} - x_2^{3/2} \\ -3x_1x_2^2 \end{pmatrix}.$$

The functions f_1 and f_2 do not satisfy the conditions (2.9) and (2.10). The known solutions is

$$x_1(t) = t, \quad x_2(t) = t^2.$$

To get `bvpsuite2.0` to terminate in a reasonable amount of time, we had to reduce the error tolerance from 1e-15 to 1e-6 for $k = 1$.
For $k = 2$ to $k = 5$ we can not produce any useful data. For $k = 1$ we observe a superconvergence order of 2. See Example 2.1 for further explanation.

Table 3.51: Example 2.3: Convergence of the collocation scheme, $k = 1$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	8.93e-02	4.52e-01	2.34	8.93e-02	4.52e-01	2.34	8.93e-02	3.03e-01	1.76
1/4	1.76e-02	3.40e-01	2.13	1.76e-02	3.40e-01	2.13	2.64e-02	3.63e-01	1.89
1/8	4.02e-03	2.74e-01	2.03	4.02e-03	2.74e-01	2.03	7.10e-03	4.40e-01	1.98
1/16	9.84e-04	2.57e-01	2.01	9.84e-04	2.57e-01	2.01	1.79e-03	4.50e-01	1.99
1/32	2.45e-04	2.52e-01	2.00	2.45e-04	2.52e-01	2.00	4.51e-04	4.60e-01	2.00
1/64	6.11e-05	2.51e-01	2.00	6.11e-05	2.51e-01	2.00	1.13e-04	4.61e-01	2.00
1/128	1.53e-05	2.50e-01	2.00	1.53e-05	2.50e-01	2.00	2.82e-05	4.62e-01	2.00
1/256	3.81e-06	2.50e-01	2.00	3.81e-06	2.50e-01	2.00	7.05e-06	4.63e-01	2.00
1/512	9.54e-07	2.50e-01	2.00	9.54e-07	2.50e-01	2.00	1.76e-06	4.66e-01	2.00
1/1024	2.38e-07	-	-	2.38e-07	-	-	4.40e-07	-	-

Table 3.52: Example 2.3: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.48e-16	3.08e-16	0.31	3.51e-16	1.11e-15	1.66	5.56e-02	2.22e-01	2.00
1/4	2.00e-16	6.35e-17	-8.29e-01	1.11e-16	4.44e-16	1.00	1.39e-02	2.22e-01	2.00
1/8	3.55e-16	3.69e-16	0.02	5.55e-17	4.97e-18	-1.16e+00	3.47e-03	2.22e-01	2.00
1/16	3.51e-16	2.25e-14	1.50	1.24e-16	1.94e-16	0.16	8.68e-04	2.22e-01	2.00
1/32	1.24e-16	1.20e-18	-1.34e+00	1.11e-16	1.08e-19	-2.00e+00	2.17e-04	2.22e-01	2.00
1/64	3.14e-16	3.27e-17	-5.44e-01	4.44e-16	8.99e-17	-3.84e-01	5.43e-05	2.22e-01	2.00
1/128	4.58e-16	3.32e-14	0.88	5.80e-16	4.78e-13	1.38	1.36e-05	2.22e-01	2.00
1/256	2.48e-16	4.74e-17	-2.98e-01	2.22e-16	1.74e-17	-4.59e-01	3.39e-06	2.22e-01	2.00
1/512	3.05e-16	1.28e-13	0.97	3.05e-16	2.95e-13	1.10	8.48e-07	2.22e-01	2.00
1/1024	1.56e-16	-	-	1.42e-16	-	-	2.12e-07	-	-

Table 3.53: Example 2.3: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	6.66e-16	6.66e-16	0.00	3.89e-16	2.72e-15	2.81	9.37e-02	3.75e-01	2.00
1/4	6.66e-16	3.75e-16	-4.15e-01	5.55e-17	2.78e-18	-2.16e+00	2.34e-02	3.75e-01	2.00
1/8	8.88e-16	2.08e-15	0.41	2.48e-16	2.48e-16	-1.46e-13	5.86e-03	3.75e-01	2.00
1/16	6.68e-16	7.87e-15	0.89	2.48e-16	1.55e-15	0.66	1.46e-03	3.75e-01	2.00
1/32	3.61e-16	1.10e-16	-3.43e-01	1.57e-16	4.91e-18	-1.00e+00	3.66e-04	3.75e-01	2.00
1/64	4.58e-16	4.58e-16	0	3.14e-16	7.95e-18	-8.84e-01	9.16e-05	3.75e-01	2.00
1/128	4.58e-16	9.27e-12	2.04	5.80e-16	4.27e-12	1.84	2.29e-05	3.75e-01	2.00
1/256	1.11e-16	1.52e-20	-1.60e+00	1.62e-16	7.31e-19	-9.75e-01	5.72e-06	3.75e-01	2.00
1/512	3.38e-16	5.38e-15	0.44	3.19e-16	3.71e-13	1.13	1.43e-06	3.75e-01	2.00
1/1024	2.48e-16	-	-	1.46e-16	-	-	3.58e-07	-	-

Table 3.54: Example 2.3: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	4.44e-16	4.44e-16	0.00	2.89e-15	2.68e-15	-1.07e-01	1.20e-01	4.80e-01	2.00
1/4	4.44e-16	1.92e-16	-6.05e-01	3.11e-15	3.34e-15	0.05	3.00e-02	4.80e-01	2.00
1/8	6.75e-16	1.55e-15	0.40	3.00e-15	3.00e-15	0	7.50e-03	4.80e-01	2.00
1/16	5.12e-16	9.24e-15	1.04	3.00e-15	1.96e-15	-1.53e-01	1.87e-03	4.80e-01	2.00
1/32	2.48e-16	1.78e-18	-1.42e+00	3.33e-15	5.08e-15	0.12	4.69e-04	4.80e-01	2.00
1/64	6.66e-16	3.88e-15	0.42	3.06e-15	1.14e-11	1.98	1.17e-04	4.80e-01	2.00
1/128	4.97e-16	2.91e-14	0.84	7.78e-16	3.74e-14	0.80	2.93e-05	4.80e-01	2.00
1/256	2.78e-16	9.07e-17	-2.02e-01	4.48e-16	3.12e-15	0.35	7.32e-06	4.80e-01	2.00
1/512	3.19e-16	2.54e-14	0.70	3.51e-16	1.59e-14	0.61	1.83e-06	4.80e-01	2.00
1/1024	1.96e-16	-	-	2.30e-16	-	-	4.58e-07	-	-

Table 3.55: Example 2.3: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.01e-15	1.73e-15	-2.14e-01	1.32e-14	2.08e-14	0.66	1.39e-01	5.56e-01	2.00
1/4	2.33e-15	2.12e-15	-6.71e-02	8.35e-15	2.04e-14	0.64	3.47e-02	5.56e-01	2.00
1/8	2.44e-15	2.81e-15	0.07	5.34e-15	7.47e-15	0.16	8.68e-03	5.56e-01	2.00
1/16	2.33e-15	1.37e-15	-1.93e-01	4.78e-15	6.35e-15	0.10	2.17e-03	5.56e-01	2.00
1/32	2.66e-15	4.59e-15	0.16	4.45e-15	1.46e-15	-3.20e-01	5.43e-04	5.56e-01	2.00
1/64	2.39e-15	1.25e-14	0.40	5.55e-15	1.32e-14	0.21	1.36e-04	5.56e-01	2.00
1/128	1.81e-15	7.17e-12	1.71	4.80e-15	3.52e-11	1.83	3.39e-05	5.56e-01	2.00
1/256	5.55e-16	8.47e-13	1.32	1.35e-15	1.56e-12	1.27	8.48e-06	5.56e-01	2.00
1/512	2.22e-16	2.23e-18	-7.38e-01	5.58e-16	4.35e-15	0.33	2.12e-06	5.56e-01	2.00
1/1024	3.70e-16	-	-	4.44e-16	-	-	5.30e-07	-	-

3.2.4 Example 2.4

We consider the following BVP :

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad x_2(1) = 1, \ -2x_2(0) + f_2(0,x(0)) = 0,$$

where

$$M(t) = \begin{pmatrix} 1 & t \\ -t^3 & -2 \end{pmatrix}, \ M(0) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \ \beta = \max\left\{1, \frac{1}{2}\right\} = 1$$

and

$$f(t,x(t)) = \begin{pmatrix} -x_1^3 \\ 4x_1^2 + x_1^2 x_2 \end{pmatrix}.$$

The functions f_1 and f_2 do not satisfy the conditions (2.9) and (2.10). The known solutions is

$$x_1(t) = t, \quad x_2(t) = t^2.$$

For $k = 2$ to $k = 5$ we can not produce any useful data. For $k = 1$ we observe a superconvergence order of 2. See Example 2.1 for further explanation.

Table 3.56: Example 2.4: Convergence of the collocation scheme, $k = 1$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	8.23e-02	3.26e-01	1.99	8.23e-02	3.26e-01	1.99	8.23e-02	3.26e-01	1.99
1/4	2.08e-02	3.31e-01	2.00	2.08e-02	3.31e-01	2.00	2.08e-02	3.31e-01	2.00
1/8	5.20e-03	3.33e-01	2.00	5.20e-03	3.33e-01	2.00	5.20e-03	3.33e-01	2.00
1/16	1.30e-03	3.33e-01	2.00	1.30e-03	3.33e-01	2.00	1.30e-03	3.33e-01	2.00
1/32	3.26e-04	3.33e-01	2.00	3.26e-04	3.33e-01	2.00	3.26e-04	3.33e-01	2.00
1/64	8.14e-05	3.33e-01	2.00	8.14e-05	3.33e-01	2.00	8.14e-05	3.33e-01	2.00
1/128	2.03e-05	3.33e-01	2.00	2.03e-05	3.33e-01	2.00	2.03e-05	3.33e-01	2.00
1/256	5.09e-06	3.33e-01	2.00	5.09e-06	3.33e-01	2.00	5.09e-06	3.33e-01	2.00
1/512	1.27e-06	3.33e-01	2.00	1.27e-06	3.33e-01	2.00	1.27e-06	3.33e-01	2.00
1/1024	3.18e-07	-	-	3.18e-07	-	-	3.18e-07	-	-

Table 3.57: Example 2.4: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.11e-16	1.11e-16	0	2.78e-17	6.94e-18	-2.00e+00	5.56e-02	2.22e-01	2.00
1/4	1.11e-16	2.22e-16	0.50	1.11e-16	1.11e-16	0	1.39e-02	2.22e-01	2.00
1/8	7.85e-17	3.47e-18	-1.50e+00	1.11e-16	1.11e-16	0	3.47e-03	2.22e-01	2.00
1/16	2.22e-16	1.42e-14	1.50	1.11e-16	1.11e-16	0	8.68e-04	2.22e-01	2.00
1/32	7.85e-17	4.66e-15	1.18	1.11e-16	1.11e-16	0	2.17e-04	2.22e-01	2.00
1/64	3.47e-17	5.05e-22	-2.68e+00	1.11e-16	1.11e-16	0	5.43e-05	2.22e-01	2.00
1/128	2.22e-16	2.94e-12	1.96	1.11e-16	8.67e-19	-1.00e+00	1.36e-05	2.22e-01	2.00
1/256	5.72e-17	1.11e-21	-1.96e+00	2.22e-16	8.66e-18	-5.85e-01	3.39e-06	2.22e-01	2.00
1/512	2.22e-16	2.22e-16	0	3.33e-16	2.50e-17	-4.15e-01	8.48e-07	2.22e-01	2.00
1/1024	2.22e-16	-	-	4.44e-16	-	-	2.12e-07	-	-

Table 3.58: Example 2.4: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	3.48e-16	4.42e-16	0.35	1.11e-16	8.88e-16	3.00	9.37e-02	3.75e-01	2.00
1/4	2.73e-16	1.66e-16	-3.61e-01	1.39e-17	3.47e-18	-1.00e+00	2.34e-02	3.75e-01	2.00
1/8	3.51e-16	1.49e-15	0.70	2.78e-17	4.34e-19	-2.00e+00	5.86e-03	3.75e-01	2.00
1/16	2.17e-16	9.08e-17	-3.14e-01	1.11e-16	1.11e-18	-1.66e+00	1.46e-03	3.75e-01	2.00
1/32	2.69e-16	7.09e-16	0.28	3.51e-16	1.11e-13	1.66	3.66e-04	3.75e-01	2.00
1/64	2.22e-16	1.14e-16	-1.61e-01	1.11e-16	1.39e-17	-5.00e-01	9.16e-05	3.75e-01	2.00
1/128	2.48e-16	3.91e-16	0.09	1.57e-16	1.04e-16	-8.50e-02	2.29e-05	3.75e-01	2.00
1/256	2.33e-16	5.67e-17	-2.55e-01	1.67e-16	1.67e-17	-4.15e-01	5.72e-06	3.75e-01	2.00
1/512	2.78e-16	1.14e-15	0.23	2.22e-16	1.14e-13	1.00	1.43e-06	3.75e-01	2.00
1/1024	2.37e-16	-	-	1.11e-16	-	-	3.58e-07	-	-

Table 3.59: Example 2.4: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.99e-16	2.01e-16	-5.71e-01	1.54e-15	1.87e-15	0.28	1.20e-01	4.80e-01	2.00
1/4	4.44e-16	1.78e-15	1.00	1.27e-15	9.68e-16	-1.94e-01	3.00e-02	4.80e-01	2.00
1/8	2.22e-16	1.27e-15	0.84	1.45e-15	1.53e-15	0.03	7.50e-03	4.80e-01	2.00
1/16	1.24e-16	1.94e-16	0.16	1.42e-15	1.12e-15	-8.69e-02	1.87e-03	4.80e-01	2.00
1/32	1.11e-16	3.47e-18	-1.00e+00	1.51e-15	7.03e-15	0.44	4.69e-04	4.80e-01	2.00
1/64	2.22e-16	9.58e-15	0.91	1.11e-15	2.82e-15	0.22	1.17e-04	4.80e-01	2.00
1/128	1.19e-16	3.02e-18	-7.56e-01	9.50e-16	6.76e-14	0.88	2.93e-05	4.80e-01	2.00
1/256	2.00e-16	8.77e-17	-1.49e-01	5.17e-16	1.36e-15	0.17	7.32e-06	4.80e-01	2.00
1/512	2.22e-16	2.22e-16	0	4.58e-16	5.63e-16	0.03	1.83e-06	4.80e-01	2.00
1/1024	2.22e-16	-	-	4.47e-16	-	-	4.58e-07	-	-

Table 3.60: Example 2.4: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.29e-15	1.67e-15	0.37	1.33e-14	3.63e-14	1.45	1.39e-01	5.56e-01	2.00
1/4	1.00e-15	8.36e-16	-1.30e-01	4.89e-15	2.28e-14	1.11	3.47e-02	5.56e-01	2.00
1/8	1.09e-15	6.62e-16	-2.42e-01	2.26e-15	4.50e-15	0.33	8.68e-03	5.56e-01	2.00
1/16	1.29e-15	2.13e-15	0.18	1.80e-15	6.84e-16	-3.49e-01	2.17e-03	5.56e-01	2.00
1/32	1.14e-15	7.87e-15	0.56	2.29e-15	1.32e-13	1.17	5.43e-04	5.56e-01	2.00
1/64	7.77e-16	2.23e-14	0.81	1.02e-15	1.73e-13	1.23	1.36e-04	5.56e-01	2.00
1/128	4.44e-16	1.54e-16	-2.18e-01	4.34e-16	6.81e-18	-8.56e-01	3.39e-05	5.56e-01	2.00
1/256	5.17e-16	2.17e-15	0.26	7.85e-16	2.25e-15	0.19	8.48e-06	5.56e-01	2.00
1/512	4.32e-16	8.79e-11	1.96	6.88e-16	3.56e-14	0.63	2.12e-06	5.56e-01	2.00
1/1024	1.11e-16	-	-	4.44e-16	-	-	5.30e-07	-	-

3.2.5 Summary

For $k \geq 2$ we are able to approximate the very smooth analytical solution $x_1(t) = t, x_2(t) = t^2$ exactly, up to the round-off errors, with a polynomial of degree k or higher, thus making it impossible to observe any kind of convergence in the mesh points. However, the error tolerance of 1e-15 used for the calculations in the mesh points in `bvpsuite2.0` introduces an error big enough to divert the numerical solution from the analytical solution on the rest of the interval $[0, 1]$, resulting in an error larger than the chosen tolerance. This effect yields a convergence order of 2 for the uniform error.
For $k = 1$ we are not able to approximate the solution for x_2 in such a way and are able to observe a superconvergence order of 2 for both Gaussian and equidistant mesh points. Due to missing data for $k \geq 2$, we are not able to identify whether this represents a superconvergence order of $2k$ or $k + 1$.
The missing conditions (2.9) and (2.10) did not have any consequences for these examples.

3.3 Examples satisfying the analytical conditions without a known solution

We will consider additional examples for nonlinear boundary value problems, where the analytical assumptions for the existence of a solution are satisfied.

3.3.1 Example 3.1

Consider the following IVP without a known solution:

$$x'(t) = \frac{M(t)}{t} x(t) + \frac{f(t, x(t))}{t}, \quad t \in (0, 1], \quad M(0)x(0) + f(0, x(0)) = 0,$$

where

$$M(t) = \begin{pmatrix} -1 + \sqrt{t} & -2\sqrt{t} \\ \sqrt{t} & -2 + 3t^{3/2} \end{pmatrix}, \quad M(0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad \beta = \max\left\{1, \frac{1}{2}\right\} = 1$$

and

$$f(t, x(t)) = \begin{pmatrix} \sqrt{t} - \frac{1}{3}\arctan(x_1 + x_2) + \alpha(t)\sqrt{|x_1|}\operatorname{sgn} x_1 \\ 1 + \frac{x_1}{2(1+x_1^2)} + \alpha(t)\sqrt{|x_2|}\operatorname{sgn} x_2 \end{pmatrix},$$

with

$$\alpha(t) = \begin{cases} 0, & t \in [0, 0.2], \\ (t - 0.2)^3, & t \in (0.2, 1]. \end{cases}$$

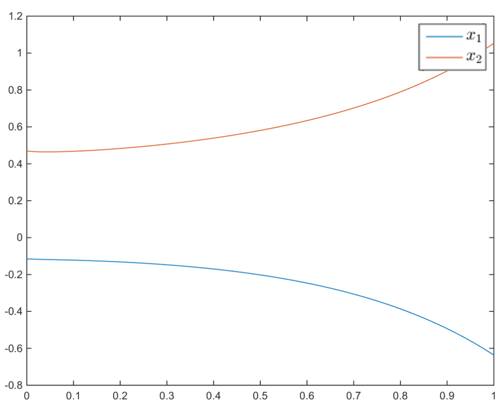

Figure 3.11: Solution of Example 3.1

As we can see in Tables 3.61 to 3.64, we are not able to observe any kind of superconvergence in the mesh points and are left with $p \approx 0.75$ for Gaussian and $p \approx 0.5$ for equidistant collocation points. It appears there is no clear value for p for the uniform error.

Table 3.61: Example 3.1: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	4.32e-03	1.64e-02	1.92	2.81e-02	9.73e-02	1.79	1.18e-01	3.44e-01	1.55
1/4	1.14e-03	8.04e-03	1.41	8.11e-03	1.03e-01	1.84	4.04e-02	4.56e-01	1.75
1/8	4.31e-04	5.32e-03	1.21	2.27e-03	1.47e-02	0.90	1.20e-02	5.79e-01	1.86
1/16	1.87e-04	3.49e-03	1.06	1.22e-03	5.70e-03	0.56	3.30e-03	3.36e-02	0.84
1/32	8.98e-05	2.29e-03	0.94	8.29e-04	5.18e-03	0.53	1.85e-03	1.17e-02	0.53
1/64	4.70e-05	1.51e-03	0.83	5.75e-04	4.87e-03	0.51	1.28e-03	1.09e-02	0.51
1/128	2.63e-05	9.88e-04	0.75	4.03e-04	4.69e-03	0.51	8.93e-04	1.05e-02	0.51
1/256	1.57e-05	6.62e-04	0.67	2.83e-04	4.60e-03	0.50	6.28e-04	2.39e-02	0.66
1/512	9.83e-06	4.67e-04	0.62	2.00e-04	4.55e-03	0.50	3.99e-04	4.49	1.50
1/1024	6.40e-06	-	-	1.41e-04	-	-	1.41e-04	-	-

Table 3.62: Example 3.1: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	9.13e-04	2.98e-03	1.71	5.71e-03	1.99e-02	1.81	1.86e-01	5.71e-01	1.62
1/4	2.80e-04	2.07e-03	1.44	1.63e-03	4.45e-03	0.72	6.03e-02	7.34e-01	1.80
1/8	1.03e-04	1.43e-03	1.27	9.89e-04	3.68e-03	0.63	1.73e-02	8.99e-01	1.90
1/16	4.28e-05	1.03e-03	1.15	6.38e-04	3.16e-03	0.58	4.63e-03	2.16e-01	1.39
1/32	1.94e-05	7.47e-04	1.05	4.28e-04	2.83e-03	0.55	1.77e-03	1.31e-02	0.58
1/64	9.32e-06	5.24e-04	0.97	2.93e-04	2.62e-03	0.53	1.19e-03	1.01e-02	0.51
1/128	4.76e-06	3.40e-04	0.88	2.03e-04	2.49e-03	0.52	8.32e-04	9.73e-03	0.51
1/256	2.59e-06	2.04e-04	0.79	1.42e-04	2.40e-03	0.51	5.86e-04	3.67e-02	0.75
1/512	1.50e-06	1.22e-04	0.71	9.99e-05	2.35e-03	0.51	3.49e-04	4.90e+02	2.27
1/1024	9.19e-07	-	-	7.04e-05	-	-	7.24e-05	-	-

Table 3.63: Example 3.1: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.13e-04	6.85e-04	1.69	1.28e-03	1.97e-03	0.62	2.11e-01	6.16e-01	1.54
1/4	6.62e-05	3.87e-04	1.27	8.34e-04	1.77e-03	0.54	7.25e-02	8.25e-01	1.75
1/8	2.74e-05	2.93e-04	1.14	5.72e-04	1.71e-03	0.53	2.15e-02	1.06	1.87
1/16	1.24e-05	2.35e-04	1.06	3.98e-04	1.64e-03	0.51	5.88e-03	2.19e-01	1.30
1/32	5.96e-06	1.88e-04	1.00	2.79e-04	1.60e-03	0.50	2.38e-03	2.52e-02	0.68
1/64	2.99e-06	1.44e-04	0.93	1.97e-04	1.58e-03	0.50	1.48e-03	2.42e-02	0.67
1/128	1.57e-06	1.04e-04	0.86	1.39e-04	1.57e-03	0.50	9.31e-04	3.64e-02	0.76
1/256	8.61e-07	7.23e-05	0.80	9.83e-05	1.57e-03	0.50	5.52e-04	6.06e-02	0.85
1/512	4.95e-07	5.49e-05	0.75	6.96e-05	1.57e-03	0.50	3.07e-04	4.32e+03	2.64
1/1024	2.93e-07	-	-	4.92e-05	-	-	4.92e-05	-	-

Table 3.64: Example 3.1: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	6.44e-05	2.03e-04	1.66	1.04e-03	1.68e-03	0.69	2.36e-01	6.76e-01	1.52
1/4	2.04e-05	9.41e-05	1.10	6.45e-04	1.56e-03	0.64	8.22e-02	9.21e-01	1.74
1/8	9.51e-06	8.16e-05	1.03	4.14e-04	1.39e-03	0.58	2.46e-02	1.19	1.87
1/16	4.65e-06	7.21e-05	0.99	2.77e-04	1.27e-03	0.55	6.73e-03	1.61e-01	1.14
1/32	2.34e-06	6.08e-05	0.94	1.90e-04	1.19e-03	0.53	3.04e-03	3.10e-02	0.67
1/64	1.22e-06	4.85e-05	0.89	1.31e-04	1.13e-03	0.52	1.91e-03	2.93e-02	0.66
1/128	6.60e-07	3.70e-05	0.83	9.17e-05	1.10e-03	0.51	1.22e-03	4.18e-02	0.73
1/256	3.72e-07	2.88e-05	0.78	6.44e-05	1.07e-03	0.51	7.33e-04	2.11e-01	1.02
1/512	2.16e-07	3.02e-05	0.79	4.53e-05	1.05e-03	0.50	3.61e-04	1.09e+06	3.50
1/1024	1.25e-07	-	-	3.19e-05	-	-	3.19e-05	-	-

3.3.2 Example 3.2

We consider the following IVP without a known solution:

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad M(0)x(0) + f(0,x(0)) = 0,$$

where

$$M(t) = \begin{pmatrix} -3+\sqrt{t} & -2\sqrt{t} \\ \sqrt{t} & -2+3t^{3/2} \end{pmatrix}, \quad M(0) = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}, \quad \beta = \max\left\{\frac{1}{3}, \frac{1}{2}\right\} = \frac{1}{2}$$

and

$$f(t,x(t)) = \begin{pmatrix} \sqrt{t} - \arctan(x_1+x_2) + \alpha(t)\sqrt{|x_1|}\operatorname{sgn} x_1 \\ 1 + \frac{x_1}{(1+x_1^2)} + \alpha(t)\sqrt{|x_2|}\operatorname{sgn} x_2 \end{pmatrix},$$

with

$$\alpha(t) = \begin{cases} 0, & t \in [0, 0.2], \\ (t-0.2)^3, & t \in (0.2, 1]. \end{cases}$$

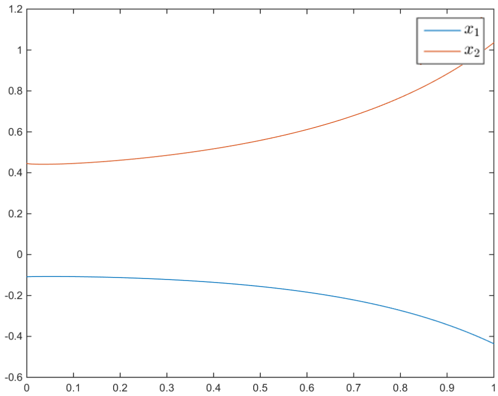

Figure 3.12: Solution of Example 3.2

As we can see in Tables 3.65 to 3.68, we achieve almost the same results as in Example 3.1, although for Gaussian collocation points we now have $p \approx 0.5$ as well.

Table 3.65: Example 3.2: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	3.56e-03	1.01e-02	1.50	2.19e-02	6.95e-02	1.67	9.74e-02	2.72e-01	1.48
1/4	1.26e-03	4.99e-03	0.99	6.87e-03	9.21e-02	1.87	3.50e-02	3.92e-01	1.74
1/8	6.33e-04	3.88e-03	0.87	1.88e-03	1.15e-02	0.87	1.04e-02	5.13e-01	1.87
1/16	3.46e-04	2.98e-03	0.78	1.03e-03	5.29e-03	0.59	2.85e-03	5.30e-02	1.05
1/32	2.02e-04	2.30e-03	0.70	6.81e-04	4.59e-03	0.55	1.37e-03	9.28e-03	0.55
1/64	1.24e-04	1.83e-03	0.65	4.65e-04	4.16e-03	0.53	9.37e-04	8.26e-03	0.52
1/128	7.93e-05	1.50e-03	0.61	3.23e-04	3.90e-03	0.51	6.52e-04	7.81e-03	0.51
1/256	5.21e-05	1.27e-03	0.58	2.26e-04	3.76e-03	0.51	4.57e-04	2.00e-02	0.68
1/512	3.49e-05	1.11e-03	0.55	1.59e-04	3.68e-03	0.50	2.85e-04	1.26	1.35
1/1024	2.38e-05	-	-	1.12e-04	-	-	1.12e-04	-	-

Table 3.66: Example 3.2: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	8.72e-04	2.47e-03	1.50	4.44e-03	1.47e-02	1.73	1.60e-01	4.89e-01	1.61
1/4	3.07e-04	1.34e-03	1.06	1.33e-03	3.50e-03	0.70	5.23e-02	6.21e-01	1.79
1/8	1.47e-04	9.51e-04	0.90	8.24e-04	2.93e-03	0.61	1.52e-02	7.67e-01	1.89
1/16	7.90e-05	6.92e-04	0.78	5.39e-04	2.57e-03	0.56	4.10e-03	2.88e-01	1.53
1/32	4.59e-05	5.27e-04	0.70	3.65e-04	2.33e-03	0.53	1.42e-03	1.56e-02	0.69
1/64	2.82e-05	4.17e-04	0.65	2.52e-04	2.18e-03	0.52	8.76e-04	7.66e-03	0.52
1/128	1.80e-05	3.42e-04	0.61	1.76e-04	2.09e-03	0.51	6.10e-04	7.28e-03	0.51
1/256	1.18e-05	2.90e-04	0.58	1.23e-04	2.04e-03	0.51	4.28e-04	2.91e-02	0.76
1/512	7.91e-06	2.54e-04	0.56	8.70e-05	2.01e-03	0.50	2.53e-04	8.66e+01	2.04
1/1024	5.38e-06	-	-	6.13e-05	-	-	6.13e-05	-	-

Table 3.67: Example 3.2: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.38e-04	4.72e-04	1.77	1.27e-03	2.11e-03	0.73	1.82e-01	5.23e-01	1.52
1/4	4.06e-05	2.63e-04	1.35	7.66e-04	1.79e-03	0.61	6.34e-02	7.12e-01	1.75
1/8	1.60e-05	1.40e-04	1.04	5.00e-04	1.64e-03	0.57	1.89e-02	9.21e-01	1.87
1/16	7.74e-06	7.53e-05	0.82	3.37e-04	1.50e-03	0.54	5.18e-03	3.04e-01	1.47
1/32	4.38e-06	4.56e-05	0.68	2.32e-04	1.41e-03	0.52	1.87e-03	2.48e-02	0.75
1/64	2.74e-06	3.26e-05	0.59	1.62e-04	1.35e-03	0.51	1.12e-03	2.13e-02	0.71
1/128	1.82e-06	2.65e-05	0.55	1.14e-04	1.32e-03	0.51	6.83e-04	2.62e-02	0.75
1/256	1.24e-06	2.35e-05	0.53	8.01e-05	1.30e-03	0.50	4.05e-04	4.58e-02	0.85
1/512	8.58e-07	2.21e-05	0.52	5.65e-05	1.29e-03	0.50	2.25e-04	1.26e+03	2.49
1/1024	5.98e-07	-	-	3.99e-05	-	-	3.99e-05	-	-

Table 3.68: Example 3.2: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.49e-05	9.63e-05	1.95	7.86e-04	1.19e-03	0.60	2.03e-01	5.75e-01	1.50
1/4	6.43e-06	6.11e-05	1.63	5.19e-04	1.20e-03	0.61	7.14e-02	7.83e-01	1.73
1/8	2.08e-06	2.72e-05	1.24	3.41e-04	1.09e-03	0.56	2.15e-02	1.03	1.86
1/16	8.85e-07	5.97e-06	0.69	2.32e-04	1.01e-03	0.53	5.94e-03	2.33e-01	1.32
1/32	5.49e-07	2.52e-06	0.44	1.60e-04	9.65e-04	0.52	2.38e-03	3.00e-02	0.73
1/64	4.04e-07	2.24e-06	0.41	1.12e-04	9.34e-04	0.51	1.43e-03	2.55e-02	0.69
1/128	3.04e-07	2.46e-06	0.43	7.86e-05	9.14e-04	0.51	8.85e-04	3.36e-02	0.75
1/256	2.26e-07	2.77e-06	0.45	5.54e-05	9.02e-04	0.50	5.27e-04	1.63e-01	1.03
1/512	1.65e-07	3.25e-06	0.48	3.91e-05	8.95e-04	0.50	2.57e-04	1.36e+05	3.22
1/1024	1.18e-07	-	-	2.76e-05	-	-	2.76e-05	-	-

3.3.3 Example 3.3

We consider the following TVP without a known solution:

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad x(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where

$$M(t) = \begin{pmatrix} 1+\sqrt{t} & -2\sqrt{t} \\ \sqrt{t} & 2+3t^{3/2} \end{pmatrix}, \quad M(0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \beta = \max\left\{1, \frac{1}{2}\right\} = 1$$

and

$$f(t,x(t)) = \begin{pmatrix} \sqrt{t} - \frac{1}{3}\arctan(x_1 + x_2) + \alpha(t)\sqrt{|x_1|}\text{sgn}x_1 \\ 1 + \frac{x_1}{2(1+x_1^2)} + \alpha(t)\sqrt{|x_2|}\text{sgn}x_2 \end{pmatrix},$$

with

$$\alpha(t) = \begin{cases} 0, & t \in [0, 0.2], \\ (t-0.2)^3, & t \in (0.2, 1]. \end{cases}$$

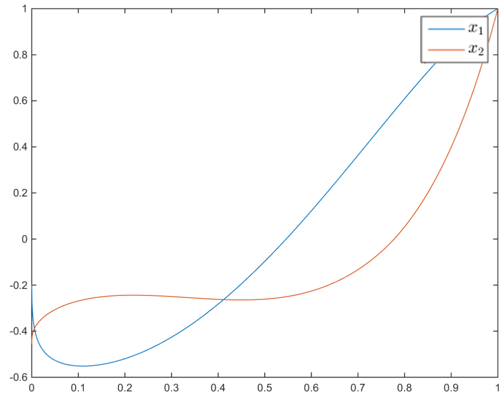

Figure 3.13: Solution of Example 3.3

As can be seen in Tables 3.69 to 3.72, we get $p \approx 0.5$, regardless of the type of collocation points and whether we observe the uniform error or the error in the mesh points.

Table 3.69: Example 3.3: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	3.63e-01	4.60e-01	0.34	4.02e-01	5.18e-01	0.37	4.02e-01	5.18e-01	0.37
1/4	2.87e-01	4.57e-01	0.34	3.12e-01	4.80e-01	0.31	3.12e-01	4.80e-01	0.31
1/8	2.27e-01	4.90e-01	0.37	2.51e-01	5.18e-01	0.35	2.51e-01	5.18e-01	0.35
1/16	1.76e-01	5.36e-01	0.40	1.98e-01	5.70e-01	0.38	1.98e-01	5.70e-01	0.38
1/32	1.33e-01	5.94e-01	0.43	1.52e-01	6.36e-01	0.41	1.52e-01	6.36e-01	0.41
1/64	9.87e-02	6.73e-01	0.46	1.14e-01	7.22e-01	0.44	1.14e-01	7.22e-01	0.44
1/128	7.16e-02	7.90e-01	0.49	8.36e-02	8.41e-01	0.48	8.36e-02	8.41e-01	0.48
1/256	5.08e-02	9.82e-01	0.53	6.01e-02	1.03	0.51	6.01e-02	1.03	0.51
1/512	3.51e-02	1.37	0.59	4.22e-02	1.36	0.56	4.22e-02	1.36	0.56
1/1024	2.34e-02	-	-	2.87e-02	-	-	2.87e-02	-	-

Table 3.70: Example 3.3: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.88e-01	3.61e-01	0.32	3.30e-01	3.98e-01	0.27	7.99e-01	2.06	1.37
1/4	2.30e-01	3.84e-01	0.37	2.74e-01	4.35e-01	0.33	3.09e-01	6.26e-01	0.51
1/8	1.78e-01	4.11e-01	0.40	2.17e-01	4.71e-01	0.37	2.17e-01	4.71e-01	0.37
1/16	1.35e-01	4.46e-01	0.43	1.68e-01	5.16e-01	0.41	1.68e-01	5.16e-01	0.41
1/32	1.00e-01	4.95e-01	0.46	1.27e-01	5.73e-01	0.44	1.27e-01	5.73e-01	0.44
1/64	7.26e-02	5.67e-01	0.49	9.36e-02	6.51e-01	0.47	9.36e-02	6.51e-01	0.47
1/128	5.16e-02	6.85e-01	0.53	6.78e-02	7.68e-01	0.50	6.78e-02	7.68e-01	0.50
1/256	3.56e-02	9.15e-01	0.59	4.79e-02	9.65e-01	0.54	4.79e-02	9.65e-01	0.54
1/512	2.37e-02	1.50	0.67	3.29e-02	1.37	0.60	3.29e-02	1.37	0.60
1/1024	1.50e-02	-	-	2.18e-02	-	-	2.18e-02	-	-

Table 3.71: Example 3.3: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.47e-01	3.17e-01	0.36	3.08e-01	3.83e-01	0.32	8.74e-01	2.09	1.26
1/4	1.92e-01	3.31e-01	0.39	2.47e-01	4.04e-01	0.35	3.66e-01	1.31	0.92
1/8	1.46e-01	3.53e-01	0.42	1.93e-01	4.33e-01	0.39	1.93e-01	4.33e-01	0.39
1/16	1.09e-01	3.83e-01	0.45	1.47e-01	4.72e-01	0.42	1.47e-01	4.72e-01	0.42
1/32	7.96e-02	4.27e-01	0.48	1.10e-01	5.24e-01	0.45	1.10e-01	5.24e-01	0.45
1/64	5.69e-02	4.98e-01	0.52	8.07e-02	5.98e-01	0.48	8.07e-02	5.98e-01	0.48
1/128	3.96e-02	6.27e-01	0.57	5.78e-02	7.15e-01	0.52	5.78e-02	7.15e-01	0.52
1/256	2.67e-02	9.24e-01	0.64	4.03e-02	9.29e-01	0.57	4.03e-02	9.29e-01	0.57
1/512	1.71e-02	1.93	0.76	2.73e-02	1.42	0.63	2.73e-02	1.42	0.63
1/1024	1.01e-02	-	-	1.76e-02	-	-	1.76e-02	-	-

Table 3.72: Example 3.3: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.14e-01	2.79e-01	0.38	2.85e-01	3.58e-01	0.33	9.05e-01	2.02	1.16
1/4	1.65e-01	2.91e-01	0.41	2.26e-01	3.77e-01	0.37	4.05e-01	2.17	1.21
1/8	1.24e-01	3.09e-01	0.44	1.75e-01	4.03e-01	0.40	1.75e-01	4.03e-01	0.40
1/16	9.11e-02	3.37e-01	0.47	1.33e-01	4.39e-01	0.43	1.33e-01	4.39e-01	0.43
1/32	6.57e-02	3.79e-01	0.51	9.83e-02	4.88e-01	0.46	9.83e-02	4.00e-01	0.40
1/64	4.63e-02	4.51e-01	0.55	7.14e-02	5.60e-01	0.50	7.43e-02	3.93e-01	0.40
1/128	3.17e-02	5.99e-01	0.61	5.06e-02	6.79e-01	0.54	5.63e-02	7.71e-01	0.54
1/256	2.08e-02	1.00	0.70	3.49e-02	9.14e-01	0.59	3.87e-02	2.30	0.74
1/512	1.28e-02	3.00	0.87	2.32e-02	1.52	0.67	2.32e-02	1.52	0.67
1/1024	7.00e-03	-	-	1.46e-02	-	-	1.46e-02	-	-

3.3.4 Example 3.4

We consider the following TVP without a known solution:

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad x(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where

$$M(t) = \begin{pmatrix} 3+\sqrt{t} & -2\sqrt{t} \\ \sqrt{t} & 2+3t^{3/2} \end{pmatrix}, \quad M(0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \quad \beta = \max\left\{\frac{1}{3}, \frac{1}{2}\right\} = \frac{1}{2}$$

and

$$f(t,x(t)) = \begin{pmatrix} \sqrt{t} - \arctan(x_1+x_2) + \alpha(t)\sqrt{|x_1|}\text{sgn}x_1 \\ 1 + \frac{x_1}{(1+x_1^2)} + \alpha(t)\sqrt{|x_2|}\text{sgn}x_2 \end{pmatrix},$$

with

$$\alpha(t) = \begin{cases} 0, & t \in [0, 0.2], \\ (t-0.2)^3, & t \in (0.2, 1]. \end{cases}$$

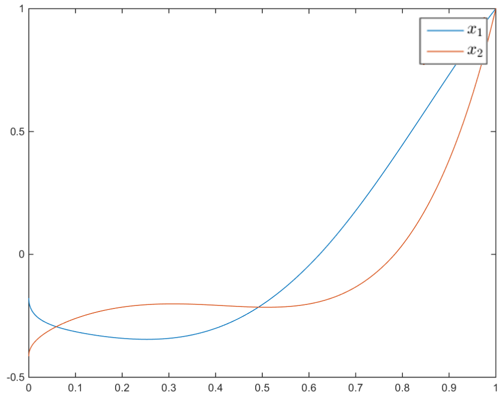

Figure 3.14: Solution of Example 3.4

Tables 3.73 to 3.76 yield almost the same results as for Example 3.3.

Table 3.73: Example 3.4: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.21e-01	1.55e-01	0.36	1.53e-01	2.27e-01	0.57	4.34e-01	1.11	1.36
1/4	9.40e-02	1.72e-01	0.44	1.03e-01	1.88e-01	0.43	1.69e-01	8.26e-01	1.14
1/8	6.95e-02	1.93e-01	0.49	7.68e-02	2.11e-01	0.49	7.68e-02	2.11e-01	0.49
1/16	4.94e-02	2.04e-01	0.51	5.48e-02	2.24e-01	0.51	5.48e-02	2.24e-01	0.51
1/32	3.47e-02	2.13e-01	0.52	3.86e-02	2.33e-01	0.52	3.86e-02	2.33e-01	0.52
1/64	2.41e-02	2.25e-01	0.54	2.69e-02	2.46e-01	0.53	2.69e-02	2.46e-01	0.53
1/128	1.66e-02	2.46e-01	0.56	1.86e-02	2.67e-01	0.55	1.86e-02	2.67e-01	0.55
1/256	1.13e-02	2.87e-01	0.58	1.27e-02	3.05e-01	0.57	1.27e-02	3.05e-01	0.57
1/512	7.56e-03	3.76e-01	0.63	8.54e-03	3.86e-01	0.61	8.54e-03	3.86e-01	0.61
1/1024	4.90e-03	-	-	5.59e-03	-	-	5.59e-03	-	-

Table 3.74: Example 3.4: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	9.59e-02	1.29e-01	0.43	1.14e-01	1.51e-01	0.41	8.57e-01	2.30	1.42
1/4	7.12e-02	1.42e-01	0.50	8.63e-02	1.70e-01	0.49	3.19e-01	3.20	1.66
1/8	5.03e-02	1.46e-01	0.51	6.14e-02	1.76e-01	0.51	1.01e-01	1.28	1.22
1/16	3.53e-02	1.51e-01	0.52	4.33e-02	1.81e-01	0.52	4.33e-02	1.81e-01	0.52
1/32	2.45e-02	1.58e-01	0.54	3.03e-02	1.88e-01	0.53	3.03e-02	1.88e-01	0.53
1/64	1.69e-02	1.70e-01	0.55	2.10e-02	2.00e-01	0.54	2.10e-02	2.00e-01	0.54
1/128	1.15e-02	1.94e-01	0.58	1.44e-02	2.22e-01	0.56	1.44e-02	2.22e-01	0.56
1/256	7.69e-03	2.45e-01	0.62	9.75e-03	2.66e-01	0.60	9.75e-03	2.66e-01	0.60
1/512	4.99e-03	3.83e-01	0.70	6.45e-03	3.68e-01	0.65	6.45e-03	3.68e-01	0.65
1/1024	3.08e-03	-	-	4.12e-03	-	-	4.12e-03	-	-

Table 3.75: Example 3.4: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	7.90e-02	1.12e-01	0.51	1.04e-01	1.48e-01	0.50	9.55e-01	2.40	1.33
1/4	5.56e-02	1.13e-01	0.51	7.35e-02	1.47e-01	0.50	3.80e-01	3.56	1.61
1/8	3.90e-02	1.15e-01	0.52	5.19e-02	1.50e-01	0.51	1.24e-01	4.88	1.77
1/16	2.72e-02	1.19e-01	0.53	3.65e-02	1.55e-01	0.52	3.65e-02	1.55e-01	0.52
1/32	1.88e-02	1.26e-01	0.55	2.54e-02	1.62e-01	0.53	2.54e-02	1.62e-01	0.53
1/64	1.28e-02	1.39e-01	0.57	1.75e-02	1.74e-01	0.55	1.75e-02	1.74e-01	0.55
1/128	8.63e-03	1.67e-01	0.61	1.20e-02	1.97e-01	0.58	1.20e-02	1.97e-01	0.58
1/256	5.66e-03	2.34e-01	0.67	8.02e-03	2.47e-01	0.62	8.02e-03	2.47e-01	0.62
1/512	3.55e-03	4.70e-01	0.78	5.22e-03	3.77e-01	0.69	5.22e-03	3.77e-01	0.69
1/1024	2.06e-03	-	-	3.25e-03	-	-	3.25e-03	-	-

Table 3.76: Example 3.4: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	6.45e-02	9.16e-02	0.51	9.05e-02	1.27e-01	0.49	1.00	2.38	1.25
1/4	4.54e-02	9.30e-02	0.52	6.44e-02	1.30e-01	0.51	4.23e-01	3.84	1.59
1/8	3.17e-02	9.51e-02	0.53	4.54e-02	1.32e-01	0.51	1.40e-01	4.05	1.62
1/16	2.20e-02	9.88e-02	0.54	3.17e-02	1.36e-01	0.53	4.57e-02	1.60e-01	0.45
1/32	1.51e-02	1.06e-01	0.56	2.20e-02	1.43e-01	0.54	3.34e-02	1.92e-01	0.50
1/64	1.02e-02	1.20e-01	0.59	1.52e-02	1.56e-01	0.56	2.36e-02	2.49e-01	0.57
1/128	6.79e-03	1.53e-01	0.64	1.03e-02	1.81e-01	0.59	1.59e-02	4.32e-01	0.68
1/256	4.36e-03	2.45e-01	0.73	6.82e-03	2.37e-01	0.64	9.92e-03	2.18	0.97
1/512	2.63e-03	7.06e-01	0.90	4.38e-03	4.02e-01	0.72	5.06e-03	1.70	0.93
1/1024	1.41e-03	-	-	2.65e-03	-	-	2.65e-03	-	-

3.3.5 Example 3.5

We consider the following BVP without a known solution:

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad -x_1(0) + f_1(0,x(0)) = 0, \ x_2(1) = 1,$$

where

$$M(t) = \begin{pmatrix} -1+\sqrt{t} & -2\sqrt{t} \\ \sqrt{t} & 2+3t^{3/2} \end{pmatrix}, \quad M(0) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \beta = \max\left\{1, \frac{1}{2}\right\} = 1$$

and

$$f(t,x(t)) = \begin{pmatrix} \sqrt{t} - \frac{1}{3}\arctan(x_1+x_2) + \alpha(t)\sqrt{|x_1|}\text{sgn}x_1 \\ 1 + \frac{x_1}{2(1+x_1^2)} + \alpha(t)\sqrt{|x_2|}\text{sgn}x_2 \end{pmatrix},$$

with

$$\alpha(t) = \begin{cases} 0, & t \in [0, 0.2], \\ (t-0.2)^3, & t \in (0.2, 1]. \end{cases}$$

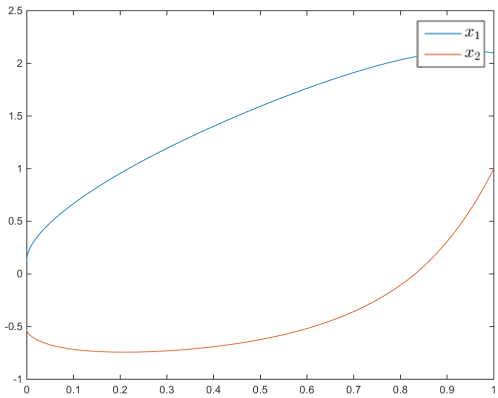

Figure 3.15: Solution of Example 3.5

Once more, we get $p \approx 0.5$ for all errors, as can be seen in Tables 3.77 to 3.80.

Table 3.77: Example 3.5: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	9.05e-02	1.43e-01	0.67	2.13e-01	6.23e-01	1.55	4.05e-01	9.57e-01	1.24
1/4	5.70e-02	1.74e-01	0.80	7.31e-02	2.70e-01	0.94	1.72e-01	7.19e-01	1.03
1/8	3.27e-02	1.50e-01	0.73	3.81e-02	1.43e-01	0.64	8.38e-02	2.38e-01	0.50
1/16	1.97e-02	1.13e-01	0.63	2.45e-02	9.61e-02	0.49	5.91e-02	2.38e-01	0.50
1/32	1.27e-02	9.17e-02	0.57	1.74e-02	9.75e-02	0.50	4.17e-02	2.39e-01	0.50
1/64	8.56e-03	8.40e-02	0.55	1.23e-02	9.91e-02	0.50	2.94e-02	2.48e-01	0.51
1/128	5.85e-03	8.61e-02	0.55	8.72e-03	9.97e-02	0.50	2.06e-02	2.48e-01	0.51
1/256	3.98e-03	9.79e-02	0.58	6.15e-03	9.97e-02	0.50	1.45e-02	3.51e-01	0.58
1/512	2.67e-03	1.28e-01	0.62	4.34e-03	9.90e-02	0.50	9.72e-03	9.48e+01	1.47
1/1024	1.74e-03	-	-	3.07e-03	-	-	3.50e-03	-	-

Table 3.78: Example 3.5: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	5.99e-02	1.08e-01	0.85	8.60e-02	1.61e-01	0.91	8.65e-01	2.21	1.35
1/4	3.32e-02	9.30e-02	0.74	4.59e-02	1.44e-01	0.82	3.38e-01	3.31	1.65
1/8	1.98e-02	7.37e-02	0.63	2.59e-02	1.06e-01	0.67	1.08e-01	8.19e-01	0.97
1/16	1.28e-02	6.16e-02	0.57	1.62e-02	8.11e-02	0.58	5.51e-02	2.25e-01	0.51
1/32	8.64e-03	5.74e-02	0.55	1.09e-02	7.16e-02	0.54	3.88e-02	2.24e-01	0.51
1/64	5.91e-03	5.87e-02	0.55	7.45e-03	7.08e-02	0.54	2.73e-02	2.21e-01	0.50
1/128	4.03e-03	6.57e-02	0.58	5.12e-03	7.64e-02	0.56	1.92e-02	2.20e-01	0.50
1/256	2.71e-03	8.30e-02	0.62	3.48e-03	9.08e-02	0.59	1.36e-02	6.06e-01	0.68
1/512	1.76e-03	1.31e-01	0.69	2.31e-03	1.26e-01	0.64	8.46e-03	2.78e+03	2.04
1/1024	1.09e-03	-	-	1.48e-03	-	-	2.06e-03	-	-

Table 3.79: Example 3.5: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	3.85e-02	6.59e-02	0.77	6.28e-02	1.17e-01	0.90	8.95e-01	2.01	1.17
1/4	2.25e-02	5.59e-02	0.66	3.38e-02	9.48e-02	0.74	3.98e-01	3.76	1.62
1/8	1.43e-02	4.76e-02	0.58	2.02e-02	7.19e-02	0.61	1.30e-01	6.28e-01	0.76
1/16	9.59e-03	4.38e-02	0.55	1.32e-02	6.09e-02	0.55	7.67e-02	3.94e-01	0.59
1/32	6.56e-03	4.38e-02	0.55	9.00e-03	5.80e-02	0.54	5.10e-02	3.82e-01	0.58
1/64	4.49e-03	4.74e-02	0.57	6.20e-03	6.01e-02	0.55	3.41e-02	4.39e-01	0.61
1/128	3.03e-03	5.65e-02	0.60	4.25e-03	6.74e-02	0.57	2.22e-02	7.82e-01	0.73
1/256	1.99e-03	7.97e-02	0.67	2.86e-03	8.42e-02	0.61	1.34e-02	1.61	0.86
1/512	1.26e-03	1.61e-01	0.78	1.87e-03	1.28e-01	0.68	7.35e-03	1.10e+05	2.65
1/1024	7.34e-04	-	-	1.17e-03	-	-	1.17e-03	-	-

Table 3.80: Example 3.5: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.79e-02	4.55e-02	0.70	4.91e-02	8.92e-02	0.86	9.68e-01	2.07	1.09
1/4	1.71e-02	3.95e-02	0.60	2.71e-02	6.94e-02	0.68	4.54e-01	3.97	1.56
1/8	1.13e-02	3.58e-02	0.56	1.69e-02	5.61e-02	0.58	1.54e-01	5.46e-01	0.61
1/16	7.69e-03	3.49e-02	0.55	1.13e-02	5.07e-02	0.54	1.01e-01	5.09e-01	0.58
1/32	5.27e-03	3.63e-02	0.56	7.79e-03	5.03e-02	0.54	6.71e-02	4.90e-01	0.57
1/64	3.58e-03	4.08e-02	0.59	5.36e-03	5.35e-02	0.55	4.51e-02	5.49e-01	0.60
1/128	2.39e-03	5.19e-02	0.63	3.66e-03	6.17e-02	0.58	2.97e-02	8.86e-01	0.70
1/256	1.54e-03	8.38e-02	0.72	2.44e-03	8.10e-02	0.63	1.83e-02	4.85	1.01
1/512	9.34e-04	2.44e-01	0.89	1.58e-03	1.37e-01	0.72	9.12e-03	5.73e+06	3.25
1/1024	5.03e-04	-	-	9.60e-04	-	-	9.60e-04	-	-

3.3.6 Example 3.6

We consider the following BVP without a known solution:

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad -3x_1(0) + f_1(0,x(0)) = 0,\ x_2(1) = 1,$$

where

$$M(t) = \begin{pmatrix} -3+\sqrt{t} & -2\sqrt{t} \\ \sqrt{t} & 2+3t^{3/2} \end{pmatrix}, \quad M(0) = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}, \quad \beta = \max\left\{\frac{1}{3}, \frac{1}{2}\right\} = \frac{1}{2}$$

and

$$f(t,x(t)) = \begin{pmatrix} \sqrt{t} - \arctan(x_1+x_2) + \alpha(t)\sqrt{|x_1|}\mathrm{sgn} x_1 \\ 1 + \frac{x_1}{(1+x_1^2)} + \alpha(t)\sqrt{|x_2|}\mathrm{sgn} x_2 \end{pmatrix},$$

with

$$\alpha(t) = \begin{cases} 0, & t \in [0, 0.2], \\ (t-0.2)^3, & t \in (0.2, 1]. \end{cases}$$

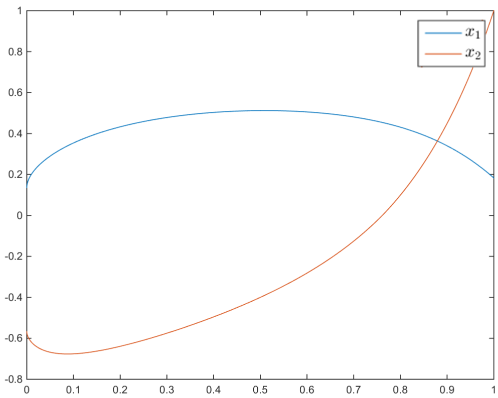

Figure 3.16: Solution of Example 3.6

Again, we observe values of $p \approx 0.5$ to $p \approx 0.75$. See Tables 3.81 to 3.84.

Table 3.81: Example 3.6: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.04e-01	1.65e-01	0.66	1.41e-01	2.56e-01	0.86	3.21e-01	7.93e-01	1.30
1/4	6.59e-02	1.86e-01	0.75	7.81e-02	2.28e-01	0.77	1.30e-01	1.06	1.51
1/8	3.92e-02	1.62e-01	0.68	4.57e-02	1.98e-01	0.71	4.57e-02	1.98e-01	0.71
1/16	2.45e-02	1.33e-01	0.61	2.80e-02	1.58e-01	0.62	2.80e-02	1.58e-01	0.62
1/32	1.60e-02	1.15e-01	0.57	1.82e-02	1.33e-01	0.57	1.82e-02	1.33e-01	0.57
1/64	1.08e-02	1.09e-01	0.56	1.22e-02	1.23e-01	0.56	1.22e-02	1.23e-01	0.56
1/128	7.34e-03	1.13e-01	0.56	8.32e-03	1.25e-01	0.56	8.32e-03	1.25e-01	0.56
1/256	4.97e-03	1.27e-01	0.58	5.65e-03	1.37e-01	0.58	5.65e-03	1.37e-01	0.58
1/512	3.31e-03	1.65e-01	0.63	3.79e-03	1.70e-01	0.61	3.79e-03	1.70e-01	0.61
1/1024	2.15e-03	-	-	2.49e-03	-	-	2.49e-03	-	-

Table 3.82: Example 3.6: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	6.85e-02	1.18e-01	0.79	9.63e-02	1.71e-01	0.83	6.50e-01	1.70	1.38
1/4	3.96e-02	1.03e-01	0.69	5.41e-02	1.52e-01	0.75	2.49e-01	2.53	1.67
1/8	2.46e-02	8.69e-02	0.61	3.23e-02	1.22e-01	0.64	7.81e-02	1.17	1.30
1/16	1.61e-02	7.73e-02	0.57	2.07e-02	1.03e-01	0.58	3.17e-02	2.25e-01	0.71
1/32	1.09e-02	7.42e-02	0.55	1.39e-02	9.45e-02	0.55	1.94e-02	2.08e-01	0.68
1/64	7.43e-03	7.64e-02	0.56	9.45e-03	9.40e-02	0.55	1.21e-02	1.31e-01	0.57
1/128	5.04e-03	8.52e-02	0.58	6.44e-03	1.01e-01	0.57	8.12e-03	9.35e-02	0.50
1/256	3.36e-03	1.07e-01	0.62	4.35e-03	1.18e-01	0.60	5.73e-03	2.39e-01	0.67
1/512	2.18e-03	1.66e-01	0.69	2.88e-03	1.61e-01	0.65	3.59e-03	1.47	0.96
1/1024	1.35e-03	-	-	1.84e-03	-	-	1.84e-03	-	-

Table 3.83: Example 3.6: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	4.56e-02	7.49e-02	0.72	7.17e-02	1.25e-01	0.80	7.12e-01	1.72	1.27
1/4	2.77e-02	6.58e-02	0.62	4.11e-02	1.06e-01	0.68	2.95e-01	2.78	1.62
1/8	1.80e-02	5.90e-02	0.57	2.56e-02	8.87e-02	0.60	9.61e-02	9.17e-01	1.09
1/16	1.21e-02	5.61e-02	0.55	1.69e-02	7.96e-02	0.56	4.53e-02	3.29e-01	0.72
1/32	8.26e-03	5.67e-02	0.56	1.15e-02	7.69e-02	0.55	2.76e-02	2.95e-01	0.68
1/64	5.62e-03	6.12e-02	0.57	7.84e-03	7.93e-02	0.56	1.72e-02	2.92e-01	0.68
1/128	3.77e-03	7.26e-02	0.61	5.33e-03	8.79e-02	0.58	1.07e-02	4.52e-01	0.77
1/256	2.47e-03	1.02e-01	0.67	3.57e-03	1.09e-01	0.62	6.28e-03	9.46e-01	0.90
1/512	1.55e-03	2.04e-01	0.78	2.33e-03	1.63e-01	0.68	3.35e-03	6.26	1.21
1/1024	9.04e-04	-	-	1.45e-03	-	-	1.45e-03	-	-

Table 3.84: Example 3.6: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	3.38e-02	5.34e-02	0.66	5.72e-02	9.71e-02	0.76	7.24e-01	1.60	1.14
1/4	2.14e-02	4.83e-02	0.59	3.37e-02	8.18e-02	0.64	3.28e-01	2.98	1.59
1/8	1.43e-02	4.54e-02	0.56	2.16e-02	7.16e-02	0.58	1.09e-01	6.90e-01	0.89
1/16	9.70e-03	4.48e-02	0.55	1.45e-02	6.68e-02	0.55	5.87e-02	4.17e-01	0.71
1/32	6.61e-03	4.68e-02	0.56	9.90e-03	6.65e-02	0.55	3.60e-02	3.68e-01	0.67
1/64	4.47e-03	5.25e-02	0.59	6.76e-03	7.02e-02	0.56	2.26e-02	3.58e-01	0.66
1/128	2.97e-03	6.62e-02	0.64	4.58e-03	8.01e-02	0.59	1.43e-02	5.17e-01	0.74
1/256	1.90e-03	1.06e-01	0.73	3.04e-03	1.04e-01	0.64	8.55e-03	2.76	1.04
1/512	1.15e-03	3.06e-01	0.90	1.96e-03	1.73e-01	0.72	4.15e-03	3.19e+02	1.80
1/1024	6.19e-04	-	-	1.19e-03	-	-	1.19e-03	-	-

3.3.7 Example 3.7

We consider the following BVP without a known solution:

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad x_1(1) = 1, \; -2x_2(0) + f_2(0, x(0)) = 0,$$

where

$$M(t) = \begin{pmatrix} -1+\sqrt{t} & -2\sqrt{t} \\ \sqrt{t} & 2+3t^{3/2} \end{pmatrix}, \quad M(0) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \beta = \max\left\{1, \frac{1}{2}\right\} = 1$$

and

$$f(t,x(t)) = \begin{pmatrix} \sqrt{t} - \frac{1}{3}\arctan(x_1+x_2) + \alpha(t)\sqrt{|x_1|}\mathrm{sgn}x_1 \\ 1 + \frac{x_1}{2(1+x_1^2)} + \alpha(t)\sqrt{|x_2|}\mathrm{sgn}x_2 \end{pmatrix},$$

with

$$\alpha(t) = \begin{cases} 0, & t \in [0, 0.2], \\ (t-0.2)^3, & t \in (0.2, 1]. \end{cases}$$

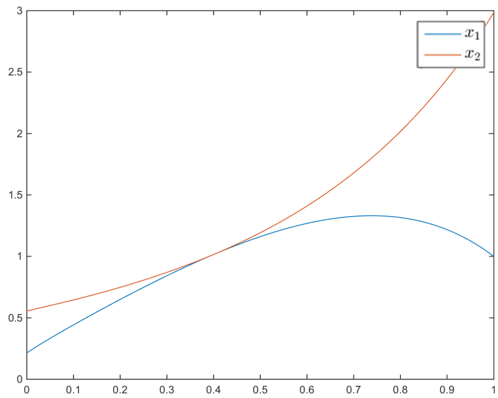

Figure 3.17: Solution of Example 3.7

Tables 3.85 to 3.88 show us the same results as for the last few examples.

Table 3.85: Example 3.7: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.28e-02	1.64e-01	3.69	1.46e-01	5.18e-01	1.83	4.11e-01	1.22	1.57
1/4	9.91e-04	7.95e-05	-1.82e+00	4.10e-02	6.08e-01	1.95	1.38e-01	1.64	1.79
1/8	3.50e-03	1.52e-03	-4.02e-01	1.07e-02	1.57e-01	1.29	4.01e-02	2.05	1.89
1/16	4.62e-03	4.56e-03	-4.86e-03	4.35e-03	2.94e-03	-1.41e-01	1.08e-02	2.78e-01	1.17
1/32	4.64e-03	8.66e-03	0.18	4.79e-03	6.94e-03	0.11	4.79e-03	6.94e-03	0.11
1/64	4.09e-03	1.38e-02	0.29	4.45e-03	1.23e-02	0.24	4.45e-03	1.23e-02	0.24
1/128	3.34e-03	2.05e-02	0.37	3.76e-03	1.92e-02	0.34	3.76e-03	1.92e-02	0.34
1/256	2.58e-03	3.05e-02	0.45	2.98e-03	2.89e-02	0.41	2.98e-03	2.89e-02	0.41
1/512	1.89e-03	4.86e-02	0.52	2.24e-03	4.49e-02	0.48	2.24e-03	4.49e-02	0.48
1/1024	1.32e-03	-	-	1.61e-03	-	-	1.61e-03	-	-

Table 3.86: Example 3.7: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	6.32e-04	1.17e-04	-2.43e+00	6.84e-03	4.08e-02	2.58	6.05e-01	1.87	1.63
1/4	3.41e-03	1.81e-03	-4.56e-01	1.15e-03	9.22e-05	-1.82e+00	1.96e-01	2.39	1.80
1/8	4.68e-03	4.61e-03	-7.64e-03	4.05e-03	2.28e-03	-2.75e-01	5.62e-02	2.91	1.90
1/16	4.71e-03	7.77e-03	0.18	4.90e-03	5.68e-03	0.05	1.51e-02	1.55	1.67
1/32	4.15e-03	1.15e-02	0.29	4.72e-03	9.87e-03	0.21	4.72e-03	9.87e-03	0.21
1/64	3.39e-03	1.61e-02	0.37	4.08e-03	1.50e-02	0.31	4.08e-03	1.50e-02	0.31
1/128	2.62e-03	2.26e-02	0.44	3.28e-03	2.18e-02	0.39	3.28e-03	2.18e-02	0.39
1/256	1.92e-03	3.41e-02	0.52	2.50e-03	3.20e-02	0.46	2.50e-03	3.20e-02	0.46
1/512	1.34e-03	6.17e-02	0.61	1.82e-03	5.12e-02	0.53	1.82e-03	5.12e-02	0.53
1/1024	8.77e-04	-	-	1.26e-03	-	-	1.26e-03	-	-

Table 3.87: Example 3.7: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.81e-03	1.73e-03	-6.96e-01	2.95e-03	2.84e-03	-5.68e-02	7.10e-01	2.10	1.56
1/4	4.55e-03	4.06e-03	-8.13e-02	3.07e-03	1.29e-03	-6.25e-01	2.40e-01	2.80	1.77
1/8	4.81e-03	6.47e-03	0.14	4.73e-03	4.08e-03	-7.17e-02	7.05e-02	3.53	1.88
1/16	4.36e-03	9.16e-03	0.27	4.97e-03	7.43e-03	0.14	1.91e-02	4.15	1.94
1/32	3.62e-03	1.24e-02	0.35	4.50e-03	1.14e-02	0.27	4.98e-03	2.09e-02	0.41
1/64	2.83e-03	1.67e-02	0.43	3.74e-03	1.63e-02	0.35	3.74e-03	1.63e-02	0.35
1/128	2.11e-03	2.36e-02	0.50	2.92e-03	2.29e-02	0.42	2.92e-03	2.29e-02	0.42
1/256	1.49e-03	3.81e-02	0.58	2.18e-03	3.37e-02	0.49	2.18e-03	3.37e-02	0.49
1/512	9.96e-04	8.61e-02	0.71	1.55e-03	5.68e-02	0.58	1.55e-03	5.68e-02	0.58
1/1024	6.07e-04	-	-	1.04e-03	-	-	1.04e-03	-	-

Table 3.88: Example 3.7: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	4.08e-03	3.44e-03	-2.46e-01	8.15e-04	1.66e-04	-2.30e+00	7.84e-01	2.27	1.54
1/4	4.84e-03	5.32e-03	0.07	4.01e-03	2.57e-03	-3.20e-01	2.70e-01	3.07	1.75
1/8	4.62e-03	7.34e-03	0.22	5.00e-03	5.37e-03	0.03	8.02e-02	3.94	1.87
1/16	3.96e-03	9.65e-03	0.32	4.88e-03	8.54e-03	0.20	2.19e-02	4.68	1.94
1/32	3.17e-03	1.26e-02	0.40	4.25e-03	1.22e-02	0.31	5.73e-03	7.38e-02	0.74
1/64	2.40e-03	1.68e-02	0.47	3.44e-03	1.69e-02	0.38	3.44e-03	1.69e-02	0.38
1/128	1.74e-03	2.44e-02	0.54	2.64e-03	2.35e-02	0.45	2.64e-03	2.35e-02	0.45
1/256	1.19e-03	4.42e-02	0.65	1.93e-03	3.52e-02	0.52	1.93e-03	3.52e-02	0.52
1/512	7.59e-04	1.41e-01	0.84	1.34e-03	6.36e-02	0.62	1.34e-03	6.36e-02	0.62
1/1024	4.25e-04	-	-	8.73e-04	-	-	8.73e-04	-	-

3.3.8 Example 3.8

We consider the following BVP without a known solution:

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad x_1(1) = 1, \; -2x_2(0) + f_2(0,x(0)) = 0,$$

where

$$M(t) = \begin{pmatrix} -3+\sqrt{t} & -2\sqrt{t} \\ \sqrt{t} & 2+3t^{3/2} \end{pmatrix}, \quad M(0) = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}, \quad \beta = \max\left\{\frac{1}{3}, \frac{1}{2}\right\} = \frac{1}{2}$$

and

$$f(t,x(t)) = \begin{pmatrix} \sqrt{t} - \arctan(x_1+x_2) + \alpha(t)\sqrt{|x_1|}\mathrm{sgn}x_1 \\ 1 + \frac{x_1}{(1+x_1^2)} + \alpha(t)\sqrt{|x_2|}\mathrm{sgn}x_2 \end{pmatrix},$$

with

$$\alpha(t) = \begin{cases} 0, & t \in [0, 0.2], \\ (t-0.2)^3, & t \in (0.2, 1]. \end{cases}$$

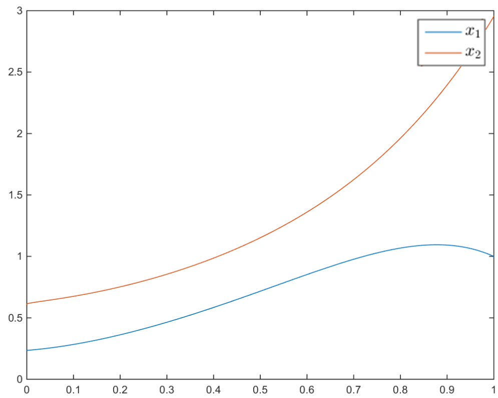

Figure 3.18: Solution of Example 3.8

As we can see in Tables 3.89 to 3.92, the values for p remain unchanged.

Table 3.89: Example 3.8: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.63e-02	1.91e-01	2.86	1.48e-01	5.37e-01	1.86	3.97e-01	1.13	1.50
1/4	3.64e-03	1.03e-01	2.41	4.10e-02	6.14e-01	1.95	1.40e-01	1.56	1.74
1/8	6.83e-04	1.01e-01	2.40	1.06e-02	6.60e-01	1.99	4.21e-02	2.04	1.87
1/16	1.29e-04	1.77e-05	-7.17e-01	2.68e-03	3.25e-02	0.90	1.15e-02	2.46	1.93
1/32	2.12e-04	1.99e-04	-1.88e-02	1.43e-03	7.78e-03	0.49	3.02e-03	1.58e-02	0.48
1/64	2.15e-04	7.18e-04	0.29	1.02e-03	8.01e-03	0.49	2.17e-03	1.66e-02	0.49
1/128	1.76e-04	1.34e-03	0.42	7.25e-04	8.16e-03	0.50	1.54e-03	1.72e-02	0.50
1/256	1.31e-04	2.09e-03	0.50	5.13e-04	8.24e-03	0.50	1.09e-03	3.81e-02	0.64
1/512	9.30e-05	3.29e-03	0.57	3.63e-04	8.27e-03	0.50	7.02e-04	6.03	1.45
1/1024	6.26e-05	-	-	2.56e-04	-	-	2.56e-04	-	-

Table 3.90: Example 3.8: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.69e-03	5.06e-03	1.58	1.33e-02	1.03e-01	2.95	5.69e-01	1.61	1.50
1/4	5.66e-04	8.49e-02	3.61	1.71e-03	2.95e-03	0.39	2.02e-01	2.25	1.74
1/8	4.62e-05	5.56e-07	-2.13e+00	1.30e-03	3.09e-03	0.41	6.03e-02	2.94	1.87
1/16	2.02e-04	1.76e-04	-4.81e-02	9.79e-04	3.52e-03	0.46	1.65e-02	3.52	1.93
1/32	2.08e-04	5.31e-04	0.27	7.11e-04	3.79e-03	0.48	4.32e-03	1.85e-01	1.08
1/64	1.73e-04	9.30e-04	0.40	5.09e-04	3.95e-03	0.49	2.04e-03	1.54e-02	0.49
1/128	1.31e-04	1.40e-03	0.49	3.61e-04	4.03e-03	0.50	1.45e-03	1.60e-02	0.50
1/256	9.31e-05	2.12e-03	0.56	2.56e-04	4.07e-03	0.50	1.03e-03	5.91e-02	0.73
1/512	6.30e-05	3.75e-03	0.65	1.81e-04	4.09e-03	0.50	6.21e-04	8.21e+02	2.26
1/1024	4.00e-05	-	-	1.28e-04	-	-	1.30e-04	-	-

Table 3.91: Example 3.8: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.12e-03	1.75e-02	3.97	4.30e-03	1.31e-02	1.61	6.49e-01	1.72	1.41
1/4	7.16e-05	1.26e-05	-1.25e+00	1.41e-03	2.78e-03	0.49	2.44e-01	2.57	1.70
1/8	1.71e-04	9.59e-05	-2.77e-01	1.01e-03	2.74e-03	0.48	7.52e-02	3.51	1.85
1/16	2.07e-04	3.55e-04	0.19	7.23e-04	2.85e-03	0.49	2.09e-02	4.33	1.92
1/32	1.81e-04	6.40e-04	0.37	5.13e-04	2.91e-03	0.50	5.50e-03	7.90e-01	1.43
1/64	1.40e-04	9.56e-04	0.46	3.63e-04	2.93e-03	0.50	2.04e-03	1.32e-02	0.45
1/128	1.02e-04	1.39e-03	0.54	2.56e-04	2.94e-03	0.50	1.49e-03	2.85e-02	0.61
1/256	7.01e-05	2.23e-03	0.62	1.81e-04	2.93e-03	0.50	9.78e-04	9.87e-02	0.83
1/512	4.55e-05	4.91e-03	0.75	1.28e-04	2.92e-03	0.50	5.50e-04	6.43e+03	2.61
1/1024	2.71e-05	-	-	9.01e-05	-	-	9.01e-05	-	-

Table 3.92: Example 3.8: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	3.00e-04	8.94e-04	1.58	1.10e-03	1.45e-03	0.40	7.04e-01	1.82	1.37
1/4	1.00e-04	2.58e-05	-9.79e-01	8.38e-04	1.46e-03	0.40	2.72e-01	2.77	1.67
1/8	1.98e-04	2.18e-04	0.05	6.36e-04	1.67e-03	0.46	8.52e-02	3.87	1.83
1/16	1.92e-04	4.39e-04	0.30	4.61e-04	1.78e-03	0.49	2.39e-02	4.86	1.92
1/32	1.56e-04	6.68e-04	0.42	3.29e-04	1.84e-03	0.50	6.33e-03	4.50e-01	1.23
1/64	1.16e-04	9.44e-04	0.50	2.33e-04	1.86e-03	0.50	2.70e-03	1.75e-02	0.45
1/128	8.22e-05	1.39e-03	0.58	1.65e-04	1.87e-03	0.50	1.97e-03	4.25e-02	0.63
1/256	5.49e-05	2.48e-03	0.69	1.17e-04	1.87e-03	0.50	1.27e-03	2.86e-01	0.98
1/512	3.41e-05	7.71e-03	0.87	8.23e-05	1.87e-03	0.50	6.47e-04	1.68e+06	3.47
1/1024	1.87e-05	-	-	5.82e-05	-	-	5.82e-05	-	-

3.3.9 Example 3.9

We want to raise the positive eigenvalues of Example 3.3 by one order of magnitude, to see if the low values for p are a consequence of the small positive eigenvalues, similar to Example 1.2.
We consider the following TVP without a known solution:

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad x(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where

$$M(t) = \begin{pmatrix} 10+\sqrt{t} & -2\sqrt{t} \\ \sqrt{t} & 20+3t^{3/2} \end{pmatrix}, \quad M(0) = \begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix}, \quad \beta = \max\left\{\frac{1}{10}, \frac{1}{20}\right\} = \frac{1}{10}$$

and

$$f(t,x(t)) = \begin{pmatrix} \sqrt{t} - \frac{1}{3}\arctan(x_1+x_2) + \alpha(t)\sqrt{|x_1|}\mathrm{sgn}x_1 \\ 1 + \frac{x_1}{2(1+x_1^2)} + \alpha(t)\sqrt{|x_2|}\mathrm{sgn}x_2 \end{pmatrix},$$

with

$$\alpha(t) = \begin{cases} 0, & t \in [0, 0.2], \\ (t-0.2)^3, & t \in (0.2, 1]. \end{cases}$$

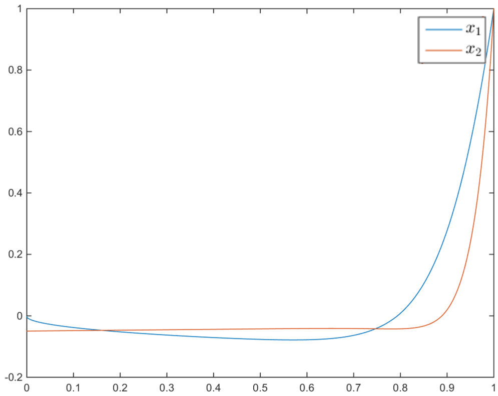

Figure 3.19: Solution of Example 3.9

As we can see in Tables 3.93 to 3.96, the values for p for the error in the mesh points stay small. However, it seems like p is now slightly larger for the equidistant error.

Table 3.93: Example 3.9: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	5.27e-01	1.48	1.49	6.57e-01	1.40	1.10	9.65e-01	1.48	0.61
1/4	1.88e-01	5.01	2.37	3.07e-01	2.81	1.60	6.31e-01	2.27	0.92
1/8	3.63e-02	4.55	2.32	1.02e-01	5.19	1.89	3.33e-01	4.66	1.27
1/16	7.27e-03	3.06e-02	0.52	2.74e-02	1.40e+01	2.25	1.38e-01	1.07e+01	1.57
1/32	5.07e-03	3.15e-02	0.53	5.76e-03	4.08e-02	0.56	4.65e-02	2.14e+01	1.77
1/64	3.52e-03	3.31e-02	0.54	3.89e-03	3.60e-02	0.53	1.36e-02	3.41e+01	1.88
1/128	2.42e-03	3.60e-02	0.56	2.69e-03	3.89e-02	0.55	3.70e-03	5.01e-01	1.01
1/256	1.65e-03	4.19e-02	0.58	1.83e-03	4.44e-02	0.57	1.83e-03	4.44e-02	0.57
1/512	1.10e-03	5.49e-02	0.63	1.23e-03	5.63e-02	0.61	1.23e-03	5.63e-02	0.61
1/1024	7.12e-04	-	-	8.06e-04	-	-	8.06e-04	-	-

Table 3.94: Example 3.9: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.34e-01	1.46	2.64	4.22e-01	1.31	1.63	1.75	2.48	0.51
1/4	3.75e-02	9.74e-01	2.35	1.36e-01	4.57	2.53	1.23	4.30	0.90
1/8	7.37e-03	2.17e-02	0.52	2.35e-02	1.32	1.94	6.57e-01	1.04e+01	1.33
1/16	5.15e-03	2.21e-02	0.53	6.15e-03	2.62e-02	0.52	2.62e-01	2.47e+01	1.64
1/32	3.57e-03	2.31e-02	0.54	4.28e-03	2.70e-02	0.53	8.40e-02	4.55e+01	1.82
1/64	2.46e-03	2.48e-02	0.56	2.96e-03	2.87e-02	0.55	2.39e-02	6.65e+01	1.91
1/128	1.67e-03	2.82e-02	0.58	2.03e-03	3.18e-02	0.57	6.36e-03	8.32e+01	1.95
1/256	1.12e-03	3.57e-02	0.62	1.37e-03	3.83e-02	0.60	1.64e-03	1.96e-01	0.86
1/512	7.26e-04	5.60e-02	0.70	9.03e-04	5.40e-02	0.66	9.03e-04	5.40e-02	0.66
1/1024	4.48e-04	-	-	5.73e-04	-	-	5.73e-04	-	-

Table 3.95: Example 3.9: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	8.39e-02	8.63e-01	3.36	2.64e-01	1.50	2.51	1.29	1.45	0.17
1/4	8.16e-03	1.67e-02	0.52	4.63e-02	1.85	2.66	1.14	2.93	0.68
1/8	5.71e-03	1.70e-02	0.52	7.33e-03	2.15e-02	0.52	7.13e-01	8.86	1.21
1/16	3.97e-03	1.75e-02	0.53	5.11e-03	2.20e-02	0.53	3.08e-01	2.47e+01	1.58
1/32	2.74e-03	1.84e-02	0.55	3.55e-03	2.29e-02	0.54	1.03e-01	5.02e+01	1.79
1/64	1.87e-03	2.03e-02	0.57	2.45e-03	2.47e-02	0.56	2.99e-02	7.80e+01	1.89
1/128	1.26e-03	2.43e-02	0.61	1.66e-03	2.81e-02	0.58	8.05e-03	1.01e+02	1.95
1/256	8.25e-04	3.42e-02	0.67	1.11e-03	3.56e-02	0.63	2.09e-03	1.05e+01	1.54
1/512	5.18e-04	6.88e-02	0.78	7.20e-04	5.61e-02	0.70	7.20e-04	5.61e-02	0.70
1/1024	3.01e-04	-	-	4.44e-04	-	-	4.44e-04	-	-

Table 3.96: Example 3.9: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	2.36e-02	8.37e-02	1.83	1.57e-01	2.39	3.93	1.58	2.02	0.36
1/4	6.66e-03	1.37e-02	0.52	1.04e-02	2.76e-02	0.71	1.23	3.14	0.67
1/8	4.64e-03	1.39e-02	0.53	6.34e-03	1.88e-02	0.52	7.71e-01	8.89	1.18
1/16	3.21e-03	1.45e-02	0.54	4.42e-03	1.92e-02	0.53	3.41e-01	2.52e+01	1.55
1/32	2.21e-03	1.55e-02	0.56	3.06e-03	2.02e-02	0.54	1.16e-01	5.34e+01	1.77
1/64	1.50e-03	1.76e-02	0.59	2.10e-03	2.20e-02	0.57	3.41e-02	8.58e+01	1.88
1/128	9.92e-04	2.23e-02	0.64	1.42e-03	2.57e-02	0.60	9.26e-03	1.14e+02	1.94
1/256	6.36e-04	3.58e-02	0.73	9.37e-04	3.44e-02	0.65	2.41e-03	5.34e+01	1.80
1/512	3.84e-04	1.03e-01	0.90	5.97e-04	6.14e-02	0.74	6.90e-04	2.62e-01	0.95
1/1024	2.06e-04	-	-	3.57e-04	-	-	3.57e-04	-	-

3.3.10 Example 3.10

We repeat the same procedure for Example 3.4. Consider the following TVP without a known solution:

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad x(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where

$$M(t) = \begin{pmatrix} 30+\sqrt{t} & -2\sqrt{t} \\ \sqrt{t} & 20+3t^{3/2} \end{pmatrix}, \quad M(0) = \begin{pmatrix} 30 & 0 \\ 0 & 20 \end{pmatrix}, \quad \beta = \max\left\{\frac{1}{30}, \frac{1}{20}\right\} = \frac{1}{20}$$

and

$$f(t,x(t)) = \begin{pmatrix} \sqrt{t} - \arctan(x_1+x_2) + \alpha(t)\sqrt{|x_1|}\mathrm{sgn}x_1 \\ 1 + \frac{x_1}{(1+x_1^2)} + \alpha(t)\sqrt{|x_2|}\mathrm{sgn}x_2 \end{pmatrix},$$

with

$$\alpha(t) = \begin{cases} 0, & t \in [0, 0.2], \\ (t-0.2)^3, & t \in (0.2, 1]. \end{cases}$$

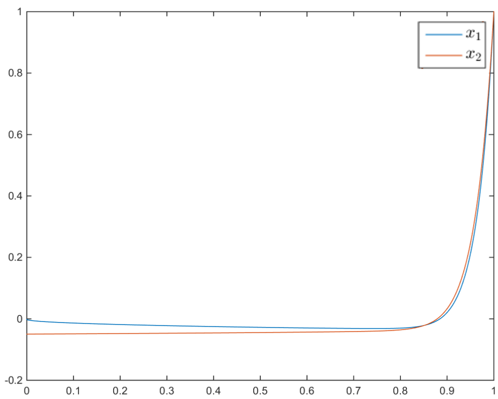

Figure 3.20: Solution of Example 3.10

Tables 3.97 to 3.100 yield the same results as in Example 3.9. We observe small p-values for the mesh point error and larger ones for the uniform error.

Table 3.97: Example 3.10: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	7.31e-01	1.77	1.27	8.62e-01	1.63	0.92	1.12	1.53	0.45
1/4	3.02e-01	5.98	2.15	4.57e-01	3.48	1.46	8.20e-01	2.42	0.78
1/8	6.78e-02	6.50e+01	3.30	1.66e-01	6.95	1.80	4.77e-01	5.31	1.16
1/16	6.88e-03	1.93	2.03	4.77e-02	1.87e+01	2.15	2.14e-01	1.35e+01	1.49
1/32	1.68e-03	1.04e-02	0.53	1.07e-02	1.56e+01	2.10	7.58e-02	3.00e+01	1.73
1/64	1.17e-03	1.09e-02	0.54	2.49e-03	1.20	1.48	2.29e-02	5.20e+01	1.86
1/128	8.03e-04	1.19e-02	0.56	8.91e-04	1.29e-02	0.55	6.33e-03	7.30e+01	1.93
1/256	5.46e-04	1.39e-02	0.58	6.09e-04	1.47e-02	0.57	1.66e-03	8.91e+01	1.96
1/512	3.65e-04	1.81e-02	0.63	4.09e-04	1.86e-02	0.61	4.26e-04	2.85e-02	0.67
1/1024	2.36e-04	-	-	2.67e-04	-	-	2.67e-04	-	-

Table 3.98: Example 3.10: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	3.71e-01	1.88	2.34	6.10e-01	1.67	1.45	1.92	2.35	0.29
1/4	7.31e-02	1.18e+01	3.67	2.23e-01	5.34	2.29	1.57	4.32	0.73
1/8	5.75e-03	2.25e-01	1.76	4.55e-02	3.14e+01	3.14	9.51e-01	1.17e+01	1.21
1/16	1.70e-03	7.29e-03	0.53	5.15e-03	9.23e-01	1.87	4.11e-01	3.18e+01	1.57
1/32	1.18e-03	7.60e-03	0.54	1.41e-03	8.90e-03	0.53	1.39e-01	6.59e+01	1.78
1/64	8.11e-04	8.17e-03	0.56	9.75e-04	9.44e-03	0.55	4.04e-02	1.04e+02	1.89
1/128	5.52e-04	9.30e-03	0.58	6.68e-04	1.05e-02	0.57	1.09e-02	1.36e+02	1.94
1/256	3.68e-04	1.18e-02	0.62	4.50e-04	1.26e-02	0.60	2.84e-03	1.59e+02	1.97
1/512	2.39e-04	1.85e-02	0.70	2.97e-04	1.79e-02	0.66	7.24e-04	1.32e+02	1.94
1/1024	1.47e-04	-	-	1.88e-04	-	-	1.88e-04	-	-

Table 3.99: Example 3.10: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.53e-01	2.04	3.73	4.10e-01	1.89	2.21	1.09	9.09e-01	-2.60e-01
1/4	1.15e-02	4.29e-01	2.61	8.88e-02	5.09	2.92	1.30	2.26	0.40
1/8	1.88e-03	5.60e-03	0.52	1.17e-02	3.96	2.80	9.90e-01	8.92	1.06
1/16	1.31e-03	5.76e-03	0.53	1.69e-03	7.26e-03	0.53	4.76e-01	3.02e+01	1.50
1/32	9.05e-04	6.08e-03	0.55	1.17e-03	7.56e-03	0.54	1.69e-01	7.06e+01	1.74
1/64	6.19e-04	6.71e-03	0.57	8.06e-04	8.13e-03	0.56	5.04e-02	1.20e+02	1.87
1/128	4.16e-04	8.03e-03	0.61	5.48e-04	9.27e-03	0.58	1.38e-02	1.64e+02	1.93
1/256	2.72e-04	1.13e-02	0.67	3.66e-04	1.18e-02	0.63	3.61e-03	1.97e+02	1.97
1/512	1.71e-04	2.28e-02	0.78	2.37e-04	1.86e-02	0.70	9.23e-04	2.18e+02	1.98
1/1024	9.93e-05	-	-	1.46e-04	-	-	2.34e-04	-	-

Table 3.100: Example 3.10: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	5.24e-02	1.25	4.58	2.60e-01	2.29	3.14	1.73	2.05	0.25
1/4	2.20e-03	4.52e-03	0.52	2.95e-02	3.03	3.34	1.45	2.71	0.45
1/8	1.53e-03	4.60e-03	0.53	2.91e-03	2.33e-02	1.00	1.06	8.97	1.03
1/16	1.06e-03	4.77e-03	0.54	1.45e-03	6.33e-03	0.53	5.23e-01	3.03e+01	1.46
1/32	7.29e-04	5.11e-03	0.56	1.01e-03	6.65e-03	0.54	1.89e-01	7.39e+01	1.72
1/64	4.94e-04	5.80e-03	0.59	6.90e-04	7.25e-03	0.57	5.74e-02	1.31e+02	1.86
1/128	3.28e-04	7.36e-03	0.64	4.66e-04	8.48e-03	0.60	1.58e-02	1.84e+02	1.93
1/256	2.10e-04	1.18e-02	0.73	3.08e-04	1.14e-02	0.65	4.16e-03	2.24e+02	1.96
1/512	1.27e-04	3.43e-02	0.90	1.96e-04	2.04e-02	0.74	1.07e-03	2.50e+02	1.98
1/1024	6.82e-05	-	-	1.17e-04	-	-	2.70e-04	-	-

3.3.11 Example 3.11

We now raise the eigenvalues in Example 3.10 by another order of magnitude and consider the TVP without a known solution

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad x(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where

$$M(t) = \begin{pmatrix} 300+\sqrt{t} & -2\sqrt{t} \\ \sqrt{t} & 200+3t^{3/2} \end{pmatrix}, \quad M(0) = \begin{pmatrix} 300 & 0 \\ 0 & 200 \end{pmatrix}, \quad \beta = \max\left\{\frac{1}{300}, \frac{1}{200}\right\} = \frac{1}{200}$$

and

$$f(t,x(t)) = \begin{pmatrix} \sqrt{t} - \arctan(x_1+x_2) + \alpha(t)\sqrt{|x_1|}\operatorname{sgn} x_1 \\ 1 + \frac{x_1}{(1+x_1^2)} + \alpha(t)\sqrt{|x_2|}\operatorname{sgn} x_2 \end{pmatrix},$$

with

$$\alpha(t) = \begin{cases} 0, & t \in [0,0.2], \\ (t-0.2)^3, & t \in (0.2,1]. \end{cases}$$

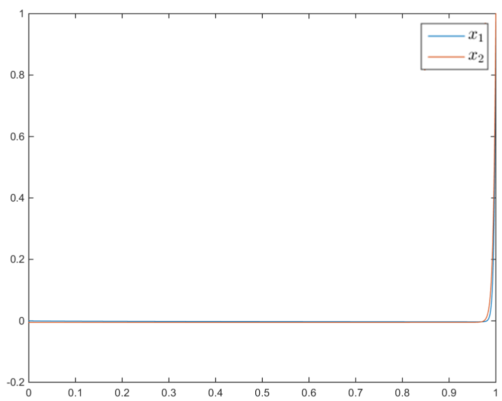

Figure 3.21: Solution of Example 3.11

As we can see in Tables 3.101 to 3.100, raising the eigenvalues to $\lambda_1 = 300$ and $\lambda_2 = 200$ does not yield another improvement in convergence of the uniform error.
We will therefore omit similar simulations for Examples 3.3.5 to 3.3.8.

Table 3.101: Example 3.11: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.32	1.45	0.14	1.34	1.45	0.11	1.38	1.43	0.05
1/4	1.20	1.76	0.28	1.25	1.67	0.21	1.33	1.54	0.10
1/8	9.85e-01	3.08	0.55	1.08	2.53	0.41	1.24	1.89	0.20
1/16	6.74e-01	1.20e+01	1.04	8.12e-01	6.56	0.75	1.07	3.08	0.38
1/32	3.28e-01	1.81e+02	1.82	4.82e-01	3.17e+01	1.21	8.25e-01	8.11	0.66
1/64	9.26e-02	1.06e+04	2.80	2.09e-01	1.66e+02	1.61	5.23e-01	3.56e+01	1.02
1/128	1.33e-02	6.68e+05	3.65	6.85e-02	1.00e+03	1.98	2.59e-01	2.00e+02	1.37
1/256	1.06e-03	7.23e+06	4.08	1.74e-02	2.63e+03	2.15	1.00e-01	8.99e+02	1.64
1/512	6.22e-05	9.81e-01	1.55	3.92e-03	1.32e+03	2.04	3.20e-02	2.57e+03	1.81
1/1024	2.13e-05	-	-	9.53e-04	-	-	9.14e-03	-	-

Table 3.102: Example 3.11: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.23	1.49	0.28	1.30	1.46	0.17	1.96	1.95	-6.72e-03
1/4	1.01	2.15	0.55	1.15	1.84	0.34	1.97	1.99	0.01
1/8	6.90e-01	6.01	1.04	9.11e-01	3.48	0.64	1.96	2.25	0.07
1/16	3.36e-01	6.17e+01	1.88	5.83e-01	1.41e+01	1.15	1.87	3.69	0.24
1/32	9.11e-02	4.58e+03	3.12	2.63e-01	1.64e+02	1.86	1.58	1.24e+01	0.59
1/64	1.05e-02	1.32e+06	4.49	7.25e-02	6.25e+03	2.73	1.05	8.07e+01	1.04
1/128	4.67e-04	5.59e+04	3.83	1.09e-02	4.34e+05	3.61	5.08e-01	5.62e+02	1.44
1/256	3.27e-05	1.05e-03	0.63	8.95e-04	5.55e+06	4.07	1.87e-01	2.39e+03	1.71
1/512	2.12e-05	1.65e-03	0.70	5.34e-05	1.86	1.68	5.73e-02	5.88e+03	1.85
1/1024	1.31e-05	-	-	1.67e-05	-	-	1.59e-02	-	-

Table 3.103: Example 3.11: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	1.11	1.53	0.46	1.25	1.48	0.24	1.25	1.48	0.24
1/4	8.07e-01	2.76	0.89	1.05	2.04	0.48	1.05	1.74	0.36
1/8	4.36e-01	1.33e+01	1.64	7.58e-01	4.89	0.90	8.19e-01	4.02e-01	-3.42e-01
1/16	1.40e-01	4.21e+02	2.89	4.07e-01	3.25e+01	1.58	1.04	5.38e-01	-2.37e-01
1/32	1.89e-02	1.47e+05	4.58	1.36e-01	7.18e+02	2.47	1.22	2.83	0.24
1/64	7.90e-04	7.41e+04	4.41	2.45e-02	2.30e+04	3.31	1.03	3.66e+01	0.86
1/128	3.70e-05	7.15e-04	0.61	2.48e-03	4.02e+05	3.90	5.71e-01	4.06e+02	1.35
1/256	2.43e-05	1.01e-03	0.67	1.67e-04	2.49e+03	2.98	2.23e-01	2.21e+03	1.66
1/512	1.52e-05	2.03e-03	0.78	2.11e-05	1.66e-03	0.70	7.08e-02	6.23e+03	1.82
1/1024	8.84e-06	-	-	1.30e-05	-	-	2.00e-02	-	-

Table 3.104: Example 3.11: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	9.84e-01	1.58	0.68	1.20	1.50	0.32	2.28	2.39	0.07
1/4	6.14e-01	3.65	1.29	9.59e-01	2.27	0.62	2.18	2.64	0.14
1/8	2.52e-01	3.24e+01	2.34	6.23e-01	7.14	1.17	1.98	3.15	0.22
1/16	4.98e-02	3.38e+03	4.01	2.76e-01	8.27e+01	2.06	1.70	3.19	0.23
1/32	3.09e-03	5.24e+06	6.13	6.64e-02	5.68e+03	3.28	1.45	5.36	0.38
1/64	4.40e-05	5.17e-04	0.59	6.85e-03	1.20e+06	4.56	1.12	3.75e+01	0.85
1/128	2.92e-05	6.56e-04	0.64	2.90e-04	4.23e+03	3.40	6.22e-01	3.75e+02	1.32
1/256	1.87e-05	1.05e-03	0.73	2.74e-05	1.01e-03	0.65	2.49e-01	2.15e+03	1.63
1/512	1.13e-05	3.06e-03	0.90	1.75e-05	1.82e-03	0.74	8.03e-02	6.47e+03	1.81
1/1024	6.07e-06	-	-	1.04e-05	-	-	2.29e-02	-	-

3.3.12 Summary

As expected, if f is not smooth enough, we observe a reduction of the convergence order for the collocation method. This effect is mostly independent of the size of the positive eigenvalues for TVPs and BVPs.

3.4 Additional Examples with a known solution not satisfying the analytical conditions

Similar to Section 3.2, we study additional examples with a known solution, which do not satisfy the analytical conditions for uniqueness and existence. To circumvent the effect of the very low error for polynomials of small degree we consider examples with a solution featuring the trigonometric functions $\sin(t)$ and $\cos(t)$.

3.4.1 Example 4.1

Consider the IVP

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t},\ t \in (0,1],$$
$$M(0)x(0) + f(0,x(0)) = 0,$$

where

$$M(t) = \begin{pmatrix} -1 & \sin^2(t) \\ -t^2 & -2 \end{pmatrix},\quad M(0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix},$$

$$f_1(t,x_1,x_2) = \sin\left(\sqrt{(tx_1)^2 + x_2^2}\right) + x_2\cos^2(t),$$
$$f_2(t,x_1,x_2) = 3\cos(t)\sqrt{(tx_1)^2 + x_2^2}.$$

Function f_1, f_2 do not satisfy the Lipschitz and the growth conditions.
The known solution is

$$x_1(t) = \sin(t), x_2(t)) = t\ \cos(t).$$

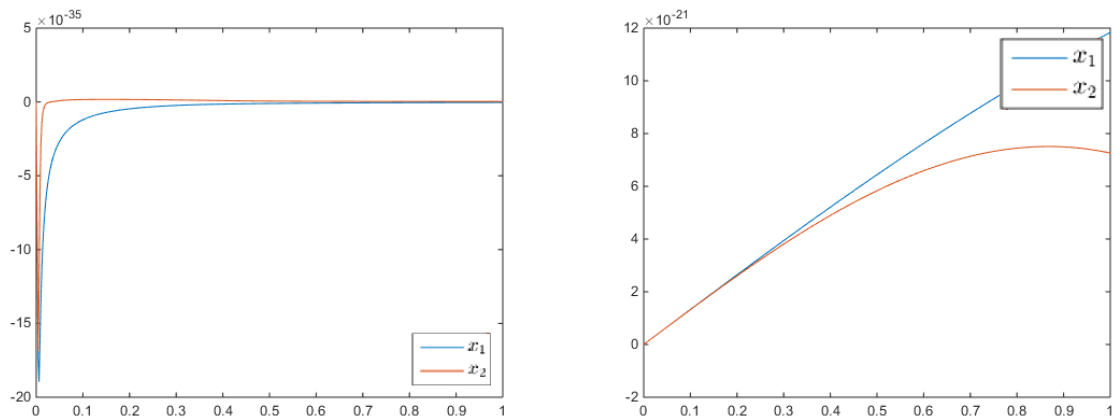

Figure 3.22: Solutions of Example 4.1

We can not test this example for convergence, since `bvpsuite2.0` is not able to deliver a unique solution. Depending on the chosen error tolerance and the initial profile for the Newton solver, we get two different types of solutions as pictured above.
The solution in the left picture is an approximation for the solution $x = 0$. The right picture depicts a solution of the form $x_1(t) = c\sin(t), x_2(t) = c\ t\ cos(t)$ with $c \ll 1$. Very small values for c yield an error

of the appropriate size, while showing a plot similar to the analytical solution, although with a much smaller order of magnitude.
We were not able to produce a solution with $c \approx 1$ as mentioned above.

3.4.2 Example 4.2

We consider the TVP

$$\begin{aligned} x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t},\ t \in (0,1], \\ x(1) = (\sin(1), \cos(1))^\top, \end{aligned}$$

where

$$M(t) = \begin{pmatrix} 1 & t^2\sin^2(t) \\ -t^2 & 2 + t\sin(t) \end{pmatrix},\ M(0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

$$\begin{aligned} f_1(t,x_1,x_2) &= x_1^2\cos(t) + x_2\cos^2(t) - x_1^2x_2, \\ f_2(t,x_1,x_2) &= -x_1x_2. \end{aligned}$$

Functions f_1, f_2 do not satisfy the Lipschitz and the growth conditions.
The known solution is

$$x_1(t) = t\sin(t), x_2(t)) = t^2\cos(t).$$

For this example we are able to observe a superconvergence of $k+2$ for an even number of Gaussian mesh points and $k+1$ for an odd number of Gaussian mesh points. For equidistant mesh points we once again observe a superconvergence of k for even k and $k+1$ for odd k in the mesh points. The uniform error does not show superconvergence at all, but a fixed order of 2.

Table 3.105: Example 4.2: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	4.83e-03	7.47e-02	3.95	1.66e-02	8.16e-02	2.30	7.52e-02	2.91e-01	1.95
1/4	3.12e-04	7.85e-02	3.99	3.37e-03	5.86e-02	2.06	1.94e-02	3.21e-01	2.02
1/8	1.97e-05	7.99e-02	3.99	8.09e-04	5.18e-02	2.00	4.78e-03	3.21e-01	2.02
1/16	1.24e-06	8.01e-02	4.00	2.02e-04	5.23e-02	2.00	1.18e-03	2.94e-01	1.99
1/32	7.75e-08	8.12e-02	4.00	5.04e-05	5.15e-02	2.00	2.96e-04	2.74e-01	1.97
1/64	4.84e-09	8.12e-02	4.00	1.26e-05	5.17e-02	2.00	7.54e-05	2.91e-01	1.99
1/128	3.03e-10	8.12e-02	4.00	3.15e-06	5.16e-02	2.00	1.90e-05	3.01e-01	1.99
1/256	1.89e-11	8.12e-02	4.00	7.87e-07	5.16e-02	2.00	4.78e-06	3.07e-01	2.00
1/512	1.18e-12	8.12e-02	4.00	1.97e-07	5.16e-02	2.00	1.20e-06	3.11e-01	2.00
1/1024	7.39e-14	-	-	4.92e-08	-	-	3.00e-07	-	-

Table 3.106: Example 4.2: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	6.76e-04	1.03e-02	3.93	1.36e-03	2.08e-02	3.93	1.07e-01	3.67e-01	1.77
1/4	4.45e-05	1.11e-02	3.98	8.93e-05	2.24e-02	3.98	3.13e-02	4.70e-01	1.95
1/8	2.81e-06	1.14e-02	4.00	5.64e-06	2.29e-02	4.00	8.09e-03	5.21e-01	2.00
1/16	1.76e-07	1.15e-02	4.00	3.53e-07	2.31e-02	4.00	2.02e-03	5.19e-01	2.00
1/32	1.10e-08	1.16e-02	4.00	2.21e-08	2.32e-02	4.00	5.03e-04	4.77e-01	1.98
1/64	6.90e-10	1.16e-02	4.00	1.38e-09	2.32e-02	4.00	1.28e-04	5.00e-01	1.99
1/128	4.31e-11	1.16e-02	4.00	8.63e-11	2.32e-02	4.00	3.22e-05	5.14e-01	1.99
1/256	2.70e-12	1.16e-02	4.00	5.40e-12	2.32e-02	4.00	8.08e-06	5.22e-01	2.00
1/512	1.68e-13	1.16e-02	4.00	3.37e-13	2.32e-02	4.00	2.02e-06	5.26e-01	2.00
1/1024	1.05e-14	-	-	2.11e-14	-	-	5.07e-07	-	-

Table 3.107: Example 4.2: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	8.96e-06	5.64e-04	5.98	8.03e-05	1.60e-03	4.31	1.43e-01	5.07e-01	1.82
1/4	1.42e-07	5.78e-04	5.99	4.04e-06	1.11e-03	4.05	4.05e-02	6.17e-01	1.96
1/8	2.23e-09	5.83e-04	6.00	2.43e-07	1.02e-03	4.01	1.04e-02	6.71e-01	2.00
1/16	3.49e-11	5.85e-04	6.00	1.51e-08	9.93e-04	4.00	2.59e-03	6.91e-01	2.01
1/32	5.46e-13	5.86e-04	6.00	9.43e-10	9.93e-04	4.00	6.41e-04	5.99e-01	1.97
1/64	8.53e-15	2.85e-06	4.72	5.89e-11	9.86e-04	4.00	1.63e-04	6.33e-01	1.99
1/128	3.24e-16	1.53e-18	-1.10e+00	3.68e-12	9.73e-04	4.00	4.12e-05	6.54e-01	1.99
1/256	6.96e-16	1.03e-14	0.49	2.30e-13	6.50e-04	3.92	1.03e-05	6.65e-01	2.00
1/512	4.97e-16	1.23e-16	-2.23e-01	1.52e-14	1.50e-05	3.32	2.59e-06	6.72e-01	2.00
1/1024	5.80e-16	-	-	1.52e-15	-	-	6.48e-07	-	-

Table 3.108: Example 4.2: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	7.95e-07	4.89e-05	5.94	4.04e-06	2.49e-04	5.95	1.69e-01	6.07e-01	1.84
1/4	1.29e-08	5.19e-05	5.99	6.54e-08	2.63e-04	5.99	4.71e-02	7.22e-01	1.97
1/8	2.04e-10	5.30e-05	6.00	1.03e-09	2.69e-04	6.00	1.20e-02	7.79e-01	2.00
1/16	3.19e-12	5.34e-05	6.00	1.61e-11	2.70e-04	6.00	3.00e-03	8.17e-01	2.02
1/32	4.99e-14	2.55e-05	5.79	2.52e-13	2.71e-04	6.00	7.39e-04	6.85e-01	1.97
1/64	9.04e-16	1.87e-15	0.17	3.95e-15	1.67e-12	1.45	1.89e-04	7.28e-01	1.99
1/128	8.01e-16	3.72e-17	-6.33e-01	1.44e-15	2.19e-15	0.09	4.76e-05	7.54e-01	1.99
1/256	1.24e-15	3.75e-14	0.61	1.36e-15	1.12e-15	-3.43e-02	1.20e-05	7.69e-01	2.00
1/512	8.11e-16	3.59e-15	0.24	1.39e-15	1.73e-15	0.04	3.00e-06	7.77e-01	2.00
1/1024	6.87e-16	-	-	1.35e-15	-	-	7.50e-07	-	-

3.4.3 Example 4.3

Consider the BVP

$$
\begin{aligned}
x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t, x(t))}{t},\ t \in (0, 1],\\
-x_1(0) + f_1(0, x_1(0), x_2(0)) = 0\\
x_2(1) = \cos(1),
\end{aligned}
$$

where

$$
M(t) = \begin{pmatrix} -1 & \sin^2(t) \\ -t^2 & 2 \end{pmatrix},\ M(0) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix},
$$

$$
\begin{aligned}
f_1(t, x_1, x_2) &= \sin\left(\sqrt{(tx_1)^2 + x_2^2}\right) + x_2 \cos^2(t),\\
f_2(t, x_1, x_2) &= -\left(x_1^2 + \cos^2(t)\right) x_2.
\end{aligned}
$$

Function f_1, f_2 do not satisfy the Lipschitz and the growth conditions.
The known solution is

$$
x_1(t) = \sin(t), x_2(t)) = t\cos(t).
$$

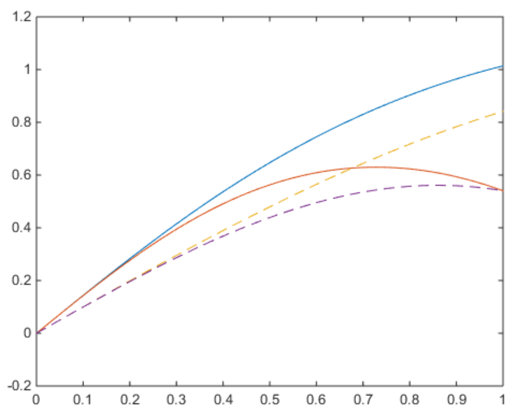

Figure 3.23: Numerical solution of Example 4.3, when no initial profile for the Newton solver is supplied.

If we do not supply any initial profile for the Newton solver, we are not able to get a solution from `bvpsuite2.0` that is close to the analytical solution, independent of chosen error tolerance or number of collocation points. The picture above shows a numerical solution recieved from `bvpsuite2.0` (solid line) and the analytical solution (dashed line).
The simulations yield an error of 2.1e-01, regardless of the used convergence points and whether we observe the uniform error or the error in the mesh points.

Supplying the Newton solver with the known analytical solution as the initial profile gives us a numerical solution that is almost exactly the same and yields the following error tables:

Table 3.109: Example 4.3: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	3.57e-03	2.98e-02	3.06	4.04e-02	1.40e-01	1.79	4.04e-02	1.39e-01	1.78
1/4	4.28e-04	2.69e-02	2.99	1.17e-02	1.70e-01	1.93	1.18e-02	1.69e-01	1.92
1/8	5.41e-05	2.75e-02	3.00	3.07e-03	1.89e-01	1.98	3.11e-03	1.97e-01	2.00
1/16	6.78e-06	2.77e-02	3.00	7.76e-04	1.98e-01	2.00	7.79e-04	1.99e-01	2.00
1/32	8.48e-07	2.77e-02	3.00	1.94e-04	1.98e-01	2.00	1.95e-04	2.00e-01	2.00
1/64	1.06e-07	2.78e-02	3.00	4.86e-05	1.99e-01	2.00	4.86e-05	2.00e-01	2.00
1/128	1.32e-08	2.78e-02	3.00	1.21e-05	1.99e-01	2.00	1.22e-05	2.00e-01	2.00
1/256	1.66e-09	2.78e-02	3.00	3.04e-06	1.99e-01	2.00	3.04e-06	1.99e-01	2.00
1/512	2.07e-10	2.78e-02	3.00	7.59e-07	1.99e-01	2.00	7.59e-07	1.99e-01	2.00
1/1024	2.59e-11	-	-	1.90e-07	-	-	1.90e-07	-	-

Table 3.110: Example 4.3: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	3.19e-05	1.01e-03	4.98	7.90e-04	1.37e-02	4.12	9.62e-04	1.60e-02	4.06
1/4	1.01e-06	1.03e-03	4.99	4.55e-05	1.20e-02	4.02	5.79e-05	1.56e-02	4.04
1/8	3.17e-08	1.04e-03	5.00	2.80e-06	1.16e-02	4.01	3.53e-06	1.46e-02	4.00
1/16	9.93e-10	1.04e-03	5.00	1.74e-07	1.15e-02	4.00	2.20e-07	1.46e-02	4.00
1/32	3.10e-11	1.04e-03	5.00	1.09e-08	1.14e-02	4.00	1.37e-08	1.44e-02	4.00
1/64	9.70e-13	1.04e-03	5.00	6.79e-10	1.14e-02	4.00	8.56e-10	1.44e-02	4.00
1/128	3.03e-14	3.82e-04	4.79	4.25e-11	1.14e-02	4.00	5.35e-11	1.44e-02	4.00
1/256	1.09e-15	6.05e-13	1.14	2.65e-12	1.14e-02	4.00	3.34e-12	1.42e-02	4.00
1/512	4.97e-16	2.57e-26	-3.80e+00	1.66e-13	2.29e-03	3.74	2.09e-13	4.28e-03	3.81
1/1024	6.90e-15	-	-	1.24e-14	-	-	1.49e-14	-	-

Table 3.111: Example 4.3: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	8.09e-06	2.53e-04	4.97	2.04e-04	3.41e-03	4.06	2.04e-04	3.37e-03	4.05
1/4	2.58e-07	2.62e-04	4.99	1.22e-05	3.15e-03	4.00	1.24e-05	3.21e-03	4.01
1/8	8.11e-09	2.65e-04	5.00	7.62e-07	3.18e-03	4.01	7.69e-07	3.25e-03	4.02
1/16	2.54e-10	2.66e-04	5.00	4.73e-08	3.10e-03	4.00	4.75e-08	3.15e-03	4.00
1/32	7.93e-12	2.66e-04	5.00	2.96e-09	3.11e-03	4.00	2.96e-09	3.13e-03	4.00
1/64	2.48e-13	2.66e-04	5.00	1.85e-10	3.10e-03	4.00	1.85e-10	3.11e-03	4.00
1/128	7.75e-15	6.64e-11	1.87	1.15e-11	3.12e-03	4.00	1.15e-11	3.13e-03	4.00
1/256	2.12e-15	5.41e-11	1.83	7.20e-13	3.63e-03	4.03	7.21e-13	3.60e-03	4.03
1/512	5.98e-16	1.35e-25	-3.56e+00	4.41e-14	2.66	5.09	4.42e-14	2.69	5.09
1/1024	7.05e-15	-	-	1.30e-15	-	-	1.30e-15	-	-

Table 3.112: Example 4.3: Convergence of the collocation scheme, $k = 5$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_\Delta$	c	p	$\|\|Y_h - Y\|\|_u$	c	p
1/2	4.77e-08	6.07e-06	6.99	1.14e-06	7.76e-05	6.08	1.40e-06	9.10e-05	6.03
1/4	3.75e-10	6.13e-06	7.00	1.69e-08	7.21e-05	6.03	2.14e-08	9.11e-05	6.03
1/8	2.94e-12	6.15e-06	7.00	2.58e-10	6.89e-05	6.01	3.28e-10	8.84e-05	6.01
1/16	2.30e-14	3.24e-08	5.11	4.01e-12	6.53e-05	5.99	5.08e-12	8.48e-05	6.00
1/32	6.66e-16	2.93e-16	-2.37e-01	6.32e-14	1.98e-06	4.98	7.94e-14	3.18e-06	5.05
1/64	7.85e-16	9.52e-17	-5.07e-01	2.00e-15	1.40e-14	0.47	2.40e-15	2.82e-14	0.59
1/128	1.12e-15	1.29e-15	0.03	1.45e-15	3.49e-17	-7.68e-01	1.59e-15	5.19e-17	-7.05e-01
1/256	1.09e-15	5.64e-18	-9.50e-01	2.47e-15	1.31e-14	0.30	2.59e-15	1.33e-14	0.30
1/512	2.11e-15	5.34e-19	-1.33e+00	2.00e-15	5.61e-20	-1.68e+00	2.11e-15	8.57e-20	-1.62e+00
1/1024	5.30e-15	-	-	6.41e-15	-	-	6.50e-15	-	-

3.4.4 Example 4.4

We consider the BVP

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \ t \in (0,1],$$
$$x_1(1) = \sin(1)$$
$$-2x_2(0) + f_2(0, x_1(0), x_2(0)) = 0$$

where

$$M(t) = \begin{pmatrix} 1 & t^2\sin^2(t) \\ -t^2 & -2 \end{pmatrix}, \ M(0) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix},$$

$$f_1(t, x_1, x_2) = x_1^2\cos(t) + x_2\cos^2(t) - x_1^2 x_2,$$
$$f_2(t, x_1, x_2) = 4\cos(t)\sqrt{(tx_1)^2 + x_2^2}.$$

Function f_1, f_2 do not satisfy the Lipschitz and the growth conditions.
The known solution is

$$x_1(t) = t\sin(t), x_2(t)) = t^2\cos(t).$$

If we do not supply an initial profile to the Newton solver `bvpsuite2.0` yields no solution and stops with the following error:

```
Error using snls (line 50)
Jacobian at initial point contains Inf or NaN values. lsqnonlin cannot continue.
```

Supplying the Newton solver with the analytical solution disturbed by a small constant to avoid the above error the program does not terminate for $k > 2$. For $k = 2$ we get

Table 3.113: Example 4.4: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\lVert Y_h - Y\rVert_\Delta$	c	p	$\lVert Y_h - Y\rVert_\Delta$	c	p	$\lVert Y_h - Y\rVert_u$	c	p
1/2	3.77e-02	7.37e-02	0.97	3.68e-02	8.85e-02	1.27	3.68e-02	8.22e-02	1.16
1/4	1.93e-02	7.23e-02	0.95	1.53e-02	4.87e-02	0.84	1.65e-02	5.87e-02	0.92
1/8	9.98e-03	7.17e-02	0.95	8.58e-03	5.38e-02	0.88	8.74e-03	5.71e-02	0.90
1/16	5.17e-03	7.56e-02	0.97	4.65e-03	6.21e-02	0.93	4.67e-03	6.31e-02	0.94
1/32	2.64e-03	2.28e-06	-2.04e+00	2.43e-03	1.44e-06	-2.14e+00	2.44e-03	1.45e-06	-2.14e+00
1/64	1.08e-02	9.24e-01	1.07	1.08e-02	9.17e-01	1.07	1.08e-02	9.18e-01	1.07
1/128	5.17e-03	7.83e-01	1.03	5.13e-03	7.78e-01	1.03	5.13e-03	7.78e-01	1.03
1/256	2.52e-03	7.12e-01	1.02	2.50e-03	7.07e-01	1.02	2.50e-03	7.07e-01	1.02
1/512	1.25e-03	6.74e-01	1.01	1.24e-03	6.69e-01	1.01	1.24e-03	6.69e-01	1.01
1/1024	6.20e-04	-	-	6.15e-04	-	-	6.15e-04	-	-

3.4.5 Summary

In contrast to Section 3.2 conditions (2.9) and (2.10) are not satisfied, leading to severe consequences for the numerical solutions received from `bvpsuite2.0`. The quality of these solutions is very much dependent on the initial profile supplied to the Newton solver. While using the analytical solution as input sometimes yields good results, in general, we can not expect to arrive at a numerical solution that is close to the analytical one. Sometimes, this type of problem does not allow `bvpsuite2.0` to produce a solution at all, even with the analytical solution as the initial profile.

Chapter 4

Conclusion

The convergence rate of the collocation methods for ODEs of the form

$$x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t, x(t))}{t}, \quad t \in (0, 1], \quad b(x(0), x(1)) = 0$$

depends on various features.

For ODEs satisfying the analytical conditions (2.9) and (2.10), and with relatively tame functions f, as seen in Section 3.1, the value of any positive eigenvalues of $M(0)$ is an important factor for the superconvergence.
For the IVP in Example 1.1 we can almost reach the superconvergence order of $2k$ for Gaussian collocation points mentioned in [7] and are also able to observe a small superconvergence order of $k + 1$ for odd k.
For the TVPs with small positive eigenvalues in Example 1.2 and Example 1.7 and the BVPs in Example 1.3 and Example 1.8 we are not able to observe any kind of superconvergence at all.
Choosing larger positive eigenvalues for $M(0)$ in these examples allows us to observe the same convergence orders as in Example 1.1, which is shown in Example 1.4, Example 1.5, Example 1.9, and Example 1.10.

For the ODEs not satisfying the analytical conditions for existence and uniqueness, but with a low order polynomial as a known solution in Section 3.2, the polynomial based approach of the collocation method yields very good results, and a small error in the mesh points. For $k \geq 2$ this error is always smaller than the chosen error tolerance of `bvpsuite2.0`. Outside of the mesh points however, the error remains larger, and does not fall below the error tolerance.
For $k = 1$ we are able to observe a superconvergence order of 2. We are not able to identify whether this represents an order of $2k$ or $k + 1$ due to missing data from $k > 1$.

In case that the function f is not appropriately smooth, we observe order reductions, see Section 3.3. This is also the case if the ODEs satisfy the conditions for the existence and uniqueness of a solution. This reduction is largely independent of other features of the equations.

In Section 3.4, we have found out that the collocation method in general is not able to yield a unique solution for ODEs which have a known solution but do not satisfy the analytical conditions for uniqueness and existence. For our examples, with solutions featuring the trigonometric functions sin and cos, `bvpsuite2.0`'s ability to reproduce this solution relies very much on the initial profile supplied to the Newton solver, and how it resembles the analytical solution. Once a good initial profile has been found, we are able to observe superconvergence.
Generally, we require a good starting solution profile to achieve convergence of the related Newton procedure. However, even with a good starting profile, the Newton procedure can converge to a wrong solution of the problem, or not at all.

Chapter 5

Appendix

5.1 Matlab Code

5.1.1 Examples

A variation of the following code is used to model the examples in this thesis for `bvpsuite2.0`. To save space, we only present the full code for Example 1.1 as a representative for an example where the initial profile of the Newton solver does not affect the generated solution, and the code for Example 4.3, where the supplied inital profile is crucial for the numerical solution.

Since all examples use the same settings for `bvpsuite2.0`, except for the error tolerance and the number of collocation points when calculating the reference solution, only one settings file is presented. When needed, the values for `'absTolSolver'` and `'relTolSolver'` are adjusted. The values for `'collPoints'` and `'collMethod'` are only used for the reference solutions and are therefore set to `7` and `'gauss'`, respectively. These attributes are set programmatically to the appropriate values during runtime for the other calculations.

To save space, the first lines in `'jacobian'` and `'dBV'` will be omitted. For further details on how to model problems with `bvpsuite2.0` see [8]. The code for the examples is based on code by Stefan Wurm written for the simulations in [1].

5.1.2 Testing

To quickly generate and print error tables the following Matlab code is used.

The file `test.m` defines a function `test(problem,settings,refSettings)`, which expects the problem, the settings used during the calculation of the solutions for different collocation points, and the settings used to calculate the reference solution, and prints the content of the error table to a file named after the provided problem. These contents are then to be copied and pasted into the `.tex` file. The function also generates a `.png` file of the plot of the reference solution. For examples with a known analytical solution the file is modified to use this solution as a reference.

The function in `printTab.m` is used to format the calculated error table, i.e. print the numbers in different formats according to their value and add separators between them.

In addition, a modified version of the function `bvpsuite2.0` called `bvpsuite2plus` is used, which takes the stepsize, collocation method and number of collocation points as additional parameters. Compare the signatures of the two functions below:

```
bvpsuite2(problem,settings,initProfile,valuesAt)
bvpsuite2plus(problem,settings,h,collMethod,collPoints,initProfile,valuesAt)
```

example1-1.m

```
function [ret] = example1-1(request,z,za,zb,zc,t,p,lambda)

mu = 1/3;
u = -[1;2];
M = @(t) [exp(t)-1, exp(-t)-1; exp(2*t)-1, exp(-2*t)-1];
p = @(t) [t^2/2; -1];
H = @(t) [t, 0; sin(t), cos(t)];
r = @(t,x) [sin(sum(x)); cos(sum(x))];
drdx = @(t,x) [cos(sum(x)); -sin(sum(x))];

J = diag(u);
alpha = norm(inv(J),1);
M = @(t) J+M(t);
f = @(t,x) p(t)+H(t)*r(t,x);
Df = @(t,x) 1/t*(M(t) + mu*H(t) * [drdx(t,x), drdx(t,x)]);

switch request
    case 'n'
        ret = 2;
    case 'orders'
        ret = [1, 1];
    case 'problem'
        ret = z(:,2) - (M(t)*z(:,1) + mu*f(t,z(:,1)))/t;
    case 'jacobian'
         ret(1, 1, 2) = 1;
         ret(2, 2, 2) = 1;
         Dfx = Df(t,z(:,1));
         ret(1:2, 1:2, 1) = -Dfx;
    case 'interval'
        ret = [ 0,1 ];
    case 'linear'
        ret = 0;
    case 'parameters'
        ret = 0;
    case 'c'
        ret = [];
    case 'BV'
        ftmp = f(0,za(:,1));
        M0tmp = M(0);
        ret = za(:,1) + 1/3*M0tmp\ftmp;
    case 'dBV'
        ret(1, 1:2, 1:2, 1) = eye(2) + 1/3*M(0)\[(H(0)*drdx(0,za(:,1))),...
            (H(0)*drdx(0,za(:,1)))];
    case 'dP'
        ret = [];
    case 'dP_BV'
        ret = [];
    case 'initProfile'
        ret.initialMesh = [];
        ret.initialValues = [];
    case 'EVP'
        ret = 0;
    case 'dLambda'
        ret = 0;
    otherwise
        ret = 0;
end
```

example4-3.m

```
function [ret] = example4-3(request,z,za,zb,zc,t,p,lambda)
u = [-1;2];
M = @(t) [0, sin(t)^2; -t^2, 0];

J = diag(u);
M = @(t) J+M(t);
f = @(t,x) [sin(sqrt((t*x(1))^2 + x(2)^2)) + x(2)*cos(t)^2;
    -(x(1)^2 + cos(t)^2) * x(2)];
Df = @(t,x) 1/t*(M(t) + ...
    [t^2*x(1) * cos(sqrt(t^2*x(1)^2+x(2)^2))/sqrt(t^2*x(1)^2+x(2)^2), ...
    x(2) * cos(sqrt(t^2*x(1)^2 + x(2)^2))/sqrt(t^2*x(1)^2+x(2)^2)+cos(t)^2;
    -2*x(1)*x(2), -cos(t)^2-x(1)^2]);

switch request
    case 'n'
        ret = 2;
    case 'orders'
        ret = [1, 1];
    case 'problem'
        ret = z(:,2) - (M(t)*z(:,1) + f(t,z(:,1)))/t;
    case 'jacobian'
         ret(1, 1, 2) = 1;
         ret(2, 2, 2) = 1;
         Dfx = Df(t,z(:,1));
         ret(1:2, 1:2, 1) = -Dfx;
    case 'interval'
        ret = [ 0,1 ];
    case 'linear'
        ret = 0;
    case 'parameters'
        ret = 0;
    case 'c'
        ret = [];
    case 'BV'
        ret=[-za(1,1);zb(2,1)-cos(1)]+[1;0].*f(0,za(:,1));
    case 'dBV'
        ret(1,1,1:2,1 )= [0; -2*za(1,1)*za(2,1)];
        ret(1,1,1,1)=ret(1,1,1,1)-1;
        ret(2,2,2,1)=1;
    case 'dP'
        ret = [];
    case 'dP_BV'
        ret = [];
    case 'initProfile'
        x1 = linspace(0,1,100);
        ret.initialMesh = x1;
        ret.initialValues = [sin(x1);x1.*cos(x1)];
    case 'EVP'
        ret = 0;
    case 'dLambda'
        ret = 0;
    otherwise
        ret = 0;
end
```

ref_settings.m

```
function [ret] = ref_settings(request)

switch request
    case 'mesh'
        ret = 0:1/2^11:1;
    case 'collMethod'
        ret = 'gauss';
    case 'collPoints'
        ret = 7;
    case 'meshAdaptation'
        ret = 0;
    case 'errorEstimate'
        ret = 0;
    case 'absTolSolver'
        ret = 1e-15;
    case 'relTolSolver'
        ret = 1e-15;
    case 'absTolMeshAdaptation'
        ret = 1e-15;
    case 'relTolMeshAdaptation'
        ret = 1e-15;
    case 'minInitialMesh'
        ret = 50;
    case 'finemesh'
        ret = 0;
    case 'allowTRM'
        ret = 1;
    case 'maxFunEvalsTRM'
        ret = 90000000;
    case 'maxIterationsTRM'
        ret = 90000000;
    case 'lambdaMin'
        ret = 0.001;
    case 'maxAdaptations'
        ret = 18;
    case 'switchToFFNFactor'
        ret = 0.5;
    case 'updateJacFactor'
        ret = 0.5;
    case 'K'
        ret = 200;
end
end
```

test.m

```
function test(problem,settings,refSettings)
fid = fopen(problem,'wt');
[refx,refz,refs] = bvpsuite2(problem,refSettings); %calculate and plot
plot(refx,refz);                                   %reference solution
l = legend('$x_1$','$x_2$');
set(l,'Interpreter','latex');
print(problem,'-dpng'); %save reference plot as .png
for collPoints = 2:5
    error1 = [];
    p1 = [];
    c1 = [];
    for i = 1:10
        step = (2^11)/(2^i);
        [x,z,s] = bvpsuite2plus(problem,settings,2^i,'gauss',collPoints);
        refvals = refz(:,1:step:end);
        errorvec = z-refvals;
        error1(end+1) = max(sqrt(sum(errorvec.^2,1)));
    end
    for i = 1:9
        p1(end+1) = log(error1(i)/error1(i+1))/log(2);
        c1(end+1) = error1(i+1)/((1/2^(i+1)).^p1(end));
    end
    p1(end+1) = 0; %add 0s as last elements to p and c vectors
    c1(end+1) = 0;

    error2 = [];
    p2 = [];
    c2 = [];
    for i = 1:10
        step = (2^11)/(2^i);
        [x,z,s] = bvpsuite2plus(problem,settings,2^i,'uniform',collPoints);
        refvals = refz(:,1:step:end);
        errorvec = z-refvals;
        error2(end+1) = max(sqrt(sum(errorvec.^2,1)));
    end
    for i = 1:9
        p2(end+1) = log(error2(i)/error2(i+1))/log(2);
        c2(end+1) = error2(i+1)/((1/2^(i+1)).^p2(end));
    end
    p2(end+1) = 0;
    c2(end+1) = 0;

    error3 = [];
    p3 = [];
    c3 = [];
    for i = 1:10
        [x,z,s] = bvpsuite2plus(problem,settings,2^i,'uniform',collPoints);
        z = coeffToValues(s.coeff,s.x1,feval_problem(problem,'orders'), ...
            1/(collPoints+1)*(1:collPoints),refx);
        errorvec = z-refz;
        error3(end+1) = max(sqrt(sum(errorvec.^2,1)));
    end
    for i = 1:9
        p3(end+1) = log(error3(i)/error3(i+1))/log(2);
        c3(end+1) = error3(i+1)/((1/2^(i+1)).^p3(end));
    end
    p3(end+1) = 0;
    c3(end+1) = 0;
```

```
    A = [error1;c1;p1;error2;c2;p2;error3;c3;p3]';
    fprintf(fid, 'k = ');
    fprintf(fid,num2str(collPoints));
    fprintf(fid, '\n');
    fprintf(fid, printTab(A));
    fprintf(fid, '----------------\n');
end
fclose(fid);
```

printTab.m

```
function output = printTab(A)
[rows, columns] = size(A);
output = '';
for r = 1:rows
   output = [output,'1/',my2str(2^r,-1), '&   ',my2str(A(r,1),1), '  &  '];
   for c = 2:columns-1
       output = [output,my2str(A(r,c),c), '  &  '];
   end
   output = [output, my2str(A(r,columns),columns), '  \\\\ \n'];
end
end

function formatted = my2str(numb,c)
max_len = 10;
if(c == -1)
    formatted = num2str(numb);
    spacing_arg = ['%-', num2str(max_len),'s'];
    formatted = sprintf(spacing_arg, formatted);
    return;
end
if(numb == 0)
    formatted = '-';
    spacing_arg = ['%-', num2str(max_len),'s'];
    formatted = sprintf(spacing_arg, formatted);
    return;
end
if(numb < 1 && (mod(c,3)>0))
    formatted = num2str(numb,'%.2e');
    spacing_arg = ['%-', num2str(max_len),'s'];
    formatted = sprintf(spacing_arg, formatted);
    return;
end
if(numb > 0 && numb <= 10)
    formatted = num2str(numb,'%.2f');
    spacing_arg = ['%-', num2str(max_len),'s'];
    formatted = sprintf(spacing_arg, formatted);
    return;

end
formatted = num2str(numb,'%.2e');
spacing_arg = ['%-', num2str(max_len),'s'];
formatted = sprintf(spacing_arg, formatted);
return;
end
```

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