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D I P L O M A R B E I T

Theoretical approach to the inverse-J-shaped relationship of fertility and economic development

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Contents

Abstract

In many common macroeconomic growth models, population increases at a fixed, exogenously given rate or the relationship between fertility and economic development is assumed to be negative. This thesis considers the recently discovered fertility rebound in countries with high levels of development and two overlapping generations models with endogenous fertility are discussed. Assuming two different types of labor and allowing for both occupation and fertility choice, Kimura and Yasui (2007) show that capital accumulation increases the income and the education level of an economy. Furthermore, fertility decreases with output per capita at low and medium levels of development, and stagnates at high levels of output per capita and education. Day (2015) extends the model of Kimura and Yasui (2007). She introduces purchased child rearing inputs and a production function for child rearing. This leads to an inverse J-shaped association between fertility and economic development. Moreover, it allows for different population growth rates across countries at the same stage of economic development.

1 Motivation

The development of the population structure and population size over time is an essential factor for many recent and upcoming political decisions, as well as for economic growth all over the world. A particularly big challenge in most developed countries is population ageing due to low fertility rates and a high life expectancy. Figure 1.1 illustrates the increase of the ratio of the population aged 65 years or older to the one aged 15–64, also referred to as working–age population, over time. From 1960 to 2015 this ratio rose by 7 percentage points in the United States of America and by 14 percentage points in the European Union. Among the political consequences are increasing governmental expenditures on the social systems, due to pension and health costs for old–aged citizens. Those lead to either higher taxes, increasing debt levels or the need for sizeable reforms. Higher taxes result in less disposable income for citizens. This has a negative effect on economic growth, because taxpayers have less money to consume or invest. Moreover, it can be argued that with a smaller share of working–age citizens, there is a change in the composition and amount of human capital. Human capital is important for innovations and ideas, which stimulate productivity (Prskawetz and Lindh, 2006). Additionally, population developments in general play a key role in economic growth theory, as they influence human capital and all per capita variables through the denominator.
Thus, it is of special interest for governments and economic players to understand the mechanics behind changes of the population's magnitude and structure. While these changes depend on all three demographic components fertility, mortality and migration, in this thesis, I focus on the substantial role of fertility rates on population developments. Consequently, the fertility rate is considered as endogenous and it is examined how the economic status and the development stage of a country, as possible important factors, influence fertility. Appendix A provides a few definitions concerning population and economic analysis.

For the second half of the twentieth century, there is empirical evidence verifying a negative relationship between economic development and fertility. For instance Myrskylä et al. (2009) identified a negative relationship between fertility and human development in 1970, using a data set of 107 different countries. While the global gross domestic product per capita increased from 446 \$ to 5460 \$ between 1950 and 2000 (Worldbank, 2016c), during the same period of time,

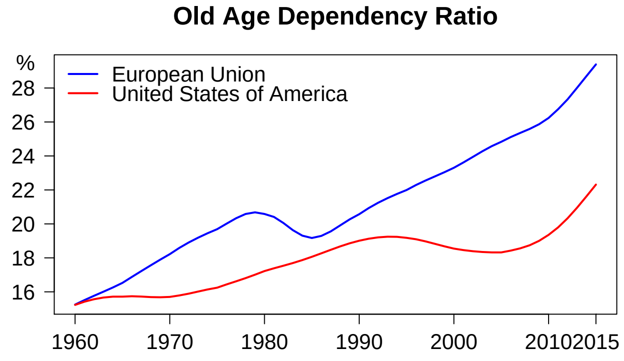


Figure 1.1: Old age dependency ratio for the EU and the US
Source of data: Worldbank (2016a), own illustration.

the global total fertility rate decreased from 4.9 births per woman to 2.7 (Worldbank, 2016b). Besides the contraceptive revolution, the reversal of intergenerational transfers and the decrease in children's mortality, there are two major arguments, why higher income per capita went along with lower fertility. Firstly, the quality–quantity trade–off: Parents invest more in the education of their children, in order to give them better chances in their working–age life. In exchange, they have a smaller number of offspring, due to the higher total expenditures per child. Indeed, Hanushek (1992) finds that children within larger families perform worse in educational life than offspring with less siblings. The second main argument for lower fertility at higher stages of economic development are rising opportunity costs of children due to increasing real wages and rising average working hours of women: With higher income, a household is able to afford a higher standard of living. This standard suffers during child rearing years, because of the wage loss during that period and as a consequence, parents have fewer children.

The observed association between fertility and economic status went along with the demographic transition theory (see Kirk (1996) and Lee (2003)). The theory states that a country's fertility rate is high and stable before the demographic transition and at low levels of income. In developed countries, this time before the transition coincided with the pre–industrial age and up to the end of the nineteenth century. Afterwards, during the demographic transition phase, fertility decreases along with economic development. This phase could be observed in the twentieth century in developed countries. Finally, the demographic transition theory suggests that birth rates settle at a low level with small variations after the demographic transition and at very high levels of income per capita.

Remarkably, just a few years ago, Myrskylä et al. (2009) made the discovery that the well-established negative correlation between fertility and economic development might be positive again for the highest developed countries, suggesting an inverse J-shaped relationship between fertility and economic development. To retrace their arguments, the paper of Myrskylä et al. (2009) is presented, using one of their illustrations. The authors attempted to prove that at a certain level of the human development index (HDI), somewhere between 0.85–0.9, the negative relation between the total fertility rate (TFR) and HDI changes into a positive relationship among the highest developed countries. To graphically observe that effect at high HDI–levels even better, they applied the following transformation on HDI:

$$\text{HDI}^* = -\ln(1-\text{HDI})$$

From the properties of the logarithm, the difference between high HDI–values is larger than the difference between low HDI–values for the transformed index HDI*.
Moreover, they also made a transformation on the TFR that reflects long–term population growth rates corresponding to the TFR–level:

$$\text{TFR}^* = \frac{log(0.4886 \cdot \text{TFR})}{31}$$

Here, 0.4886 denotes the female fraction of births and 31 is a rough measure for the mean age of a mother at childbirth in countries with a high HDI. In Figure 1.2, the transformed variables HDI* and TFR* are used instead of the original ones.
For the year 1975, Myrskylä et al. (2009) worked with the data of 107 and for the year 2005 with the data of 140 different countries, respectively. As can be seen in Figure 1.2, in 1975 no country with an HDI over 0.9 existed and therefore the blue curve just describes the negative relationship. However, the new positive correlation over an HDI–level of 0.9 can be detected for the year 2005. Moreover, the Spearman's rank correlation is significantly negative for HDI–values below 0.85 and significantly positive for HDI–values above 0.9.

Due to these observations, it is assumed that the point at which a fertility rebound occurs on the basis of the dataset, is located at an HDI–level somewhere between 0.85 and 0.9. Myrskylä et al. (2009) carried out different regressions based on the maximum likelihood method to calculate this turning point at a value of approximately 0.86. Moreover, they found a negative effect of HDI–level increase on fertility on the left–hand side of the calculated turning point and, vice versa, a positive effect on the right–hand side of the turning point.

Still, one cannot say which parts of the human development index are most important for the fer-

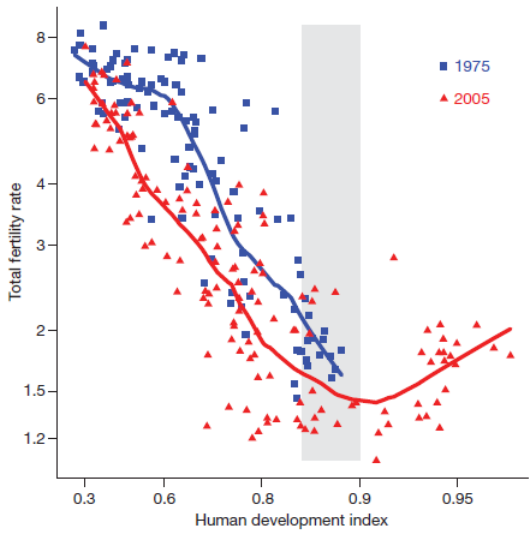


Figure 1.2: Relationship between HDI and TFR in 1975 and 2005
(Myrskylä et al., 2009, page 741, Figure 1)

tility rebound, as income, schooling years and life expectancy contribute equally to the final HDI level. Moreover, the three factors are partially correlated, as e.g. a higher education could lead to more human capital, more productivity and therefore higher income. In order to identify if it is economic growth that drives the fertility rebound, Luci and Thevenon (2010) took a dataset of 30 OECD countries from 1960–2007 with their respective total fertility rates and GDP per capita levels, that were measured in purchasing power parity constant 2005 US Dollars. In Figure 1.3, the panel data for the 1960s until the 2000s, for the 26 OECD countries, where data for the whole period was available, is illustrated. It leads to the assumption of an inverse J-shaped relationship again, but here between TFR and GDP per capita in comparison to TFR and level of HDI in Myrskylä et al. (2009). Similar as before, the more recent data points (here those of the 2000s) are responsible for the seemingly positive relationship after a certain amount of income.

To support the stated relations, Luci and Thevenon (2010) made a sizeable panel data regression study. In Table 1.1, the three pooled OLS models with the corresponding regressors and endogenous variables that were compared in a first step, are illustrated. Here, GDPpc stands for gross domestic product per capita. While all of the used regressors have significant coefficients, the quadratic model is better than the linear or the exponential model considering the adjusted R squared measure with 0.459 in comparison to 0.359 and 0.2. As the coefficient of $\ln(\text{GDPpc})^2$ is positive, they showed a convex relationship between TFR and ln(GDPpc) with this dataset is most realistic.

Figure 1.3: TFR and GDP per capita PPP in 26 OECD countries 1960–2007
(Luci and Thevenon, 2010, page 15, Figure 3)

In a next step, Luci and Thevenon (2010) examine the best quadratic panel data model comparing the pooled OLS method with the instrumental variables (IV), fixed effects (FE), random effects (RE), between effects (BE) and first difference estimator model. Here, the FE model has the highest adjusted R squared measure and hence they use the estimates of this model to determine a turning point in the relationship between TFR and GDP per capita. As can be observed in Figure 1.4, where a curve corresponding to the FE model coefficients is drawn, this turning point is located at a logarithm of GDP per capita PPP level of approximately 10.39 US Dollars, which reconverts into 32,600 US Dollars, and a total fertility rate of 1.51 births per woman, which lies 0.59 births per woman under the 2.1–rate, a population would need to maintain its current level.

Furthermore, in 2006, the fertility rate in France was much higher than in Germany, while the two countries had approximately the same GDP per capita PPP level. And while English speaking and Scandinavian countries are located above the FE estimation curve, others like Eastern European countries Poland and Hungary and Southern European countries Italy, Spain and Greece are below the estimation curve. Those observations suggest that different compositions of GDP could be a reason why countries are located at different sides of the curve.

Consequently, Luci and Thevenon (2010) examine how different contributors to GDP influence fertility. They find that while increasing average working hours of women are negative for fertil-

	linear model	exponential model	quadratic model
Endogenous variable:	***TFR***	***lnTFR***	***TFR***
Type of regression:	Pooled OLS	Pooled OLS	Pooled OLS
Regressors:			
GDPpc		-0.0000166***	
		(-16.21)	
lnGDPpc	-1.013***		-15.63***
	(-24.24)		(-14.91)
lnGDPpc²			0.760***
			(13.95)
constant	11.87***	0.943***	81.92***
	(28.98)	(43.24)	(16.27)
N	1050	1050	1050
nb. of countries:	30	30	30
time period:	1960-2007	1960-2007	1960-2007
R²:	0.359	0.200	0.460
R² adj.:	0.359	0.200	0.459

t statistics in parentheses, * p<0.05, ** p<0.01, *** p<0.001

Table 1.1: Testing for the best model considering adjusted R squared measure (Luci and Thevenon, 2010, page 20, Table 2)

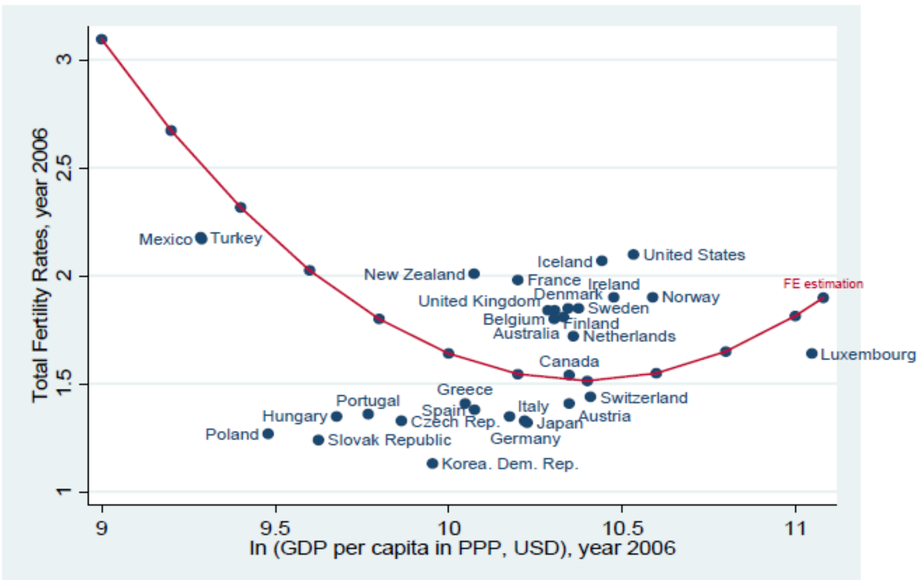


Figure 1.4: Convex relationship from FE model and 2006 scatter plot of 30 OECD countries (Luci and Thevenon, 2010, page 21, Figure 4)

ity growth, there is a significantly positive relationship between female labor force participation and fertility. Therefore, the possibility of reconciling work and family is considered to be a very important factor for the allowance of higher fertility rates. This could also explain, why certain countries are below or above the estimation curve in Figure 1.4.

To conclude their paper, Luci and Thevenon (2010) find that the driving factors of the fertility rebound are GDP and female labor force participation. But as stated before, education is correlated with the latter two. In a study comprising 17 countries, England et al. (2012) identify a positive relationship of women's education and women's employment. In another paper, considering Norway and women cohorts born between 1940–1964, the effects of education levels on fertility are compared over time. Kravdal and Rindfuss (2008) find that within early cohorts of women with a medium total level of education, higher educated women had a negative effect on total fertility. But in younger cohorts this negative effect diminished with a high total level of education, suggesting for a convex relationship between education and fertility.

All of the presented studies mainly rely on the TFR to measure fertility. However, Bongaarts (2001) found a significant difference between desired family size and the TFR in the 1990s. Besides unwanted fertility, gender preferences or replacement of deceased children, he found that the rising age of mothers at childbearing is the main reason for the gap between desired family size and period fertility rates. When mothers decide to get their offspring later in life, the TFR decreases although the number of children, these women intend to have, remains the same. Therefore, there is a short-term decrease in period fertility, but one cannot conclude that these lower fertility rates will last in the long-term. Similarly, TFR levels increase again, when the postponement of childbearing is completed. Consequently, Bongaarts and Feeney (1998) suggested the tempo-adjusted TFR as a new measure of period fertility rates. It adjusts for birth postponement through changes of the mean age of a mother at the birth of their first, second and third child. Using the tempo-adjusted measure, Sobotka (2004) and Goldstein et al. (2009) argue that lowest–low fertility around 2000 in Europe only occurred due to the birth postponements of mothers, and that completed fertility in the corresponding cohorts will actually be at higher levels compared to observed TFR values.

Thus, it is also necessary to ask whether the recent rebound of fertility is only due to the end of birth postponements or if there is also a quantum effect such that completed fertility levels increased as well. In a paper on demography and fertility trends in Europe since 1990, Sobotka (2004) points out that with tempo-adjusted measures the slope of the increase in fertility is less steep than with TFR levels in most European countries. Moreover, completed cohort fertility and desired family size levels are relatively stable in comparison to period fertility levels. The latter ones are also influenced by economic insecurity, unemployment and career involvement of women. Furthermore, regional differences in Europe are visible, as the North and West have higher rates than the rest of Europe, which Sobotka (2004) partly attributes to higher gender equality and policies that support both childcare expansions and family subsidies.

In another study, Bongaarts and Sobotka (2012) argue that the recent rise in period fertility in Europe is in major parts due to demographic explanations. The authors find that tempo and also parity distortions of the TFR lead to the recent fertility trends, which may not persist in the long–term. The parity of a woman in her reproductive age is the amount of children she already gave birth to. In the TFR and tempo-adjusted TFR measurements, the births at a certain time t at birth order i and of women aged a are related to all women aged a at time t. Therefore, for a certain birth order i, also women are considered if they already have $\geq i$ children and hence, parity distortions within these methods can occur. To control for parity and tempo effects, Bongaarts and Sobotka (2012) present a tempo- and parity-adusted TFR, which they abbreviated as TFRp*.[1] In Table 1.2, the absolute TFR increases from the time of the lowest level of TFR since 1990 until the time of the highest TFR level since 1990 are given for nine European countries. It can be observed that with the tempo-adjusted TFR suggested by Bongaarts and Feeney (1998), which is abbreviated as TFR* in Table 1.2, parts of the increases can be captured, while with the TFRp* levels almost all of the increases can be explained by diminishing distortions from tempo and parity composition effects. This study would suggest that in most European countries, the recent rebound in TFR levels is just due to the end of birth postponement and not due to quantum effects.

Nevertheless, Myrskylä et al. (2011) find that also in one of the latest cohorts, where women have completed their reproductive ages, namely the one of 1970, the inverse J-shaped association between fertility and development is still evident. In Figure 1.5, the overall fertility of the 1970 cohort for 29 developed countries is plotted against the average human development index levels of the countries between 1995 and 2005, when the 1970 cohorts were in their prime childbearing years, or aged 25-35. The curve of a quadratic fit has its minimum between HDI levels of 0.83 and 0.84. Similar results can be found for the cohorts of 1960 and 1965. Furthermore, Myrskylä et al. (2011) came to the conclusion that fertility at older reproductive ages and gender equality account for the positive fertility-development link, using a dataset of 16 developed countries and corresponding tempo-adjusted, period fertility rates.

[1] The mathematical formulas for calculating the TFR, the tempo-adjusted TFR and the parity- and tempo-adjusted TFR can be found in Appendix 1 of their paper (Bongaarts and Sobotka, 2012, pages 113-115).

Country	Period	Absolute TFR increase	Percent TFR increase due to diminishing distortions	
			TFR*	TFRp*
Bulgaria	1997–2008	0.36	38	90
Czech Republic	1999–2008	0.37	56	100
Estonia	1998–2006	0.26	3	57
Finland	1998–2007	0.14	13	82
Netherlands	1996–2003	0.22	24	85
Russia	1999–2007	0.25	41	71
Slovenia	2003–2008	0.32	28	71
Spain	1998–2007	0.24	93	100
Sweden	1999–2006	0.35	14	69

Table 1.2: Percentage of TFR increase attributable to diminishing tempo and parity effects (Bongaarts and Sobotka, 2012, page 109, Table 3)

The consequences of the discussed papers are versatile. The facts from Myrskylä et al. (2009) and Luci and Thevenon (2010) suggest changes in projections of population developments in developed countries due to the recently observed fertility rebounds, although tempo- and parity-distortions might have accounted for both lowest–low fertility (Goldstein et al. (2009) and Sobotka (2004)) and parts of the recent increase (Sobotka (2013) and Bongaarts and Sobotka (2012)). Still, Myrskylä et al. (2011) also found an inverse J-shaped fertility–development relationship for the 1970 cohort, leading to an expectation of a slowly increase in long-term fertility within the highest developed countries. In the context of higher fertility rates, an alleviation of expenditures on social systems might also be expected. The size of a potential fertility rebound probably depends on gender equality (Myrskylä et al. (2011)) and the possibility to combine work and family, suggesting corresponding governmental policies, which support both child-care extensions and subsidies for families (Sobotka (2013)).

Another consequence from the presented discussions so far might be a necessary adjustment of endogenous fertility in growth theory. In the early attempts to implement endogenous fertility in economic growth models, the empirical facts of a negative income–fertility link were used. To get a short overview, I introduce existing fertility theories and how endogenous fertility can be implemented within the framework of household optimization in chapter 2, based on Jones et al. (2011). In chapter 3, an overlapping generations (OLG) model from Kimura and Yasui (2007) with two skill groups and both fertility and occupation choice is presented, which leads to constant fertility at a high levels of education and income. Afterwards, in chapter 4, an extension of the model of Kimura and Yasui (2007) by Day (2015) is examined. There, the author already put to work the recently detected positive consequences of high education and

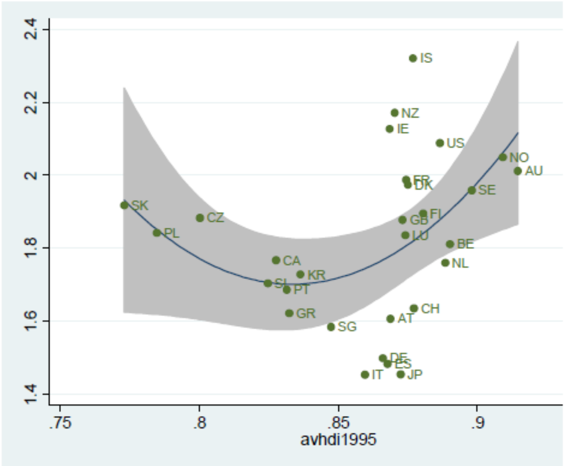


Figure 1.5: Relationship between 1970 cohort fertility and average HDI for the years 1995–2005
(Myrskylä et al., 2011, page 18, Figure 4)

GDP per capita on fertility rates through the implementation of a production function for child rearing, and children's goods and services expenditures. For both models, I extend the qualitative analysis by a quantitative numerical simulation. Finally, a discussion of the theoretical results concludes the thesis in chapter 5.

2 Fertility theories

In models based on neoclassical economics, on the production side of the economy firms maximize their profit, while on the consumption side households optimize their utility. Within such a theoretical approach, e.g. different pension systems, tax systems, or family policies can be evaluated a priori, where the latter ones are of special interest in this thesis.

The fertility theories, as a part of the macroeconomic theory, incorporate the empirically observed relationship between fertility and economic development into economic growth models. Specifically, the relationship is implemented through the allowance of optimal fertility choice of households in the utility maximization.
I consider corresponding static utility optimization problems, based on Jones et al. (2011). In their work, the authors present an overview on how specific constraints and different assumptions on the form of the utility function influence the relationship between fertility and wages. In section 2.1, their general household maximization problem is studied. It introduces a general utility function as an objective function. Moreover, concepts that explain why fertility rates in developed countries have declined in the twentieth century are implemented through budget, time, and children's quality constraints. In section 2.2, the maximization problem is also solved, with different assumptions on the utility function and on the constraints.

2.1 General model

The general model, based on Jones et al. (2011), includes agents that try to maximize their utility over their life time. Utility can be gained by consumption c, leisure ℓ, the number of children n and the quality of children q. The latter two deserve special attention as individuals have a fertility choice in comparison to models with a fixed population growth. Additionally, it is assumed that each variable contributes additively to the utility function of an agent with declining marginal products to obtain the following form:

$$U(c,n,\ell,q) = u_c(c) + u_n(n) + u_q(q) + u_\ell(\ell)$$
$$u_i'(i) > 0 \quad u_i''(i) < 0 \quad i = c,n,q,\ell$$

Each agent can decide how she wants to spend her available life time, which is set to 1 without loss of generality. An individual can work, denoted by l_w, enjoy leisure ℓ or use time for child rearing zn. Here, z is the parameter that determines the fixed amount of time per child that an agent has to stay at home during child rearing. This leads to the time constraint:

$$l_w + zn + \ell = 1$$

Moreover, each agent can also choose if she wants to spend her wage for consumption goods and services c or for investments in the children's quality sn. Here, s stands for the sum of a parent's educational investments per child, e.g. tuition fees or financial support through the pre–working life of their offspring. Additionally, a parent has to pay the fixed costs x for every child she gets. x stands for the standard required child rearing goods and services costs.[1] The following budget constraint is obtained:

$$c + (x+s)n = wl_w$$

Here, w denotes the wage per working unit and the left–hand side of the equation represents the expenditures, while the right–hand side represents the income of an agent. The investments for the education of a child s convert into the obtained quality of a child with a quality function $q = f(s)$, where more investments lead to more quality: $f'(s) > 0 \quad \forall s$. Consequently, the corresponding maximization problem of households is given by:

$$\max_{c,n,s,l_w} \quad U(c,n,q,\ell) = \max_{c,n,s,l_w} \; u_c(c) + u_n(n) + u_q(q) + u_\ell(\ell) \tag{2.1}$$
$$s.t. \quad l_w + zn + \ell = 1 \tag{2.2}$$
$$c + (x+s)n = wl_w \tag{2.3}$$
$$q = f(s) \tag{2.4}$$

It can be observed that in this general model of household optimization with fertility choice, the decision of having children and investing in their quality influences the objective function as well as all of the constraints. Setting up the problem this way, the main arguments of possible

[1] The notation of time per child z and expenditures per child x differs from the one in Jones et al. (2011) in order to be consistent throughout the thesis.

fertility decline due to higher income, as mentioned in the introduction, are reflected in this set–up.

Firstly, it considers the opportunity costs of time. On the one hand, a higher wage raises the right–hand side of (2.3) and leads to more possibilities to consume, therefore also the possibility to afford more children. This is called the income effect. On the other hand, it also causes higher opportunity costs of time, which are reflected in constraint (2.2). The relative price of a child increases with higher wages due to the increased wage loss an agent has if she spends time at home raising children. Therefore, an agent spends a higher share of her wage for consumption goods and a lower share for child rearing goods, in comparison to the lower wage scenario. This is well–known as substitution effect. Now, the overall effect of higher income on the fertility decision depends on which of the effects is higher considering the agent's utility and hence depends on the form of the utility function.
Secondly, the quantity–quality trade–off of children is also reflected in the problem. If the wage w increases in (2.3), parents will increase their investments in the education of their offspring per child s if the obtained higher quality $q = f(s)$ gains more utility in the utility function U than having more children n. This implies that the derivative of $f(s)$ has to be high enough. Again, it can be observed that the form of the utility function (2.1) determines the influence of higher income on the fertility decision. Furthermore, the type of the quality function $f(s)$ is crucial for the form of the optimal results.

In the main parts of this thesis, chapters 3 and 4, simplified problems without the mentioned quality–quantity trade–off are considered. Therefore, the general problem for different functional forms of f is not presented here, but I concentrate on the analysis of the special case without children's quality. The following sections describe the corresponding solutions for two different types of utility functions.

2.2 Solution of the general problem without the quality aspect of children

In the case of individuals not gaining utility by investing in the quality of their children, problem (2.1)-(2.4) can be reduced. The quality part of the utility function is set to zero ($u_q(q) = 0$) and the optimal choice of investments in the children's education is zero as well ($s = 0$). Moreover, it is assumed that agents cannot choose the amount of leisure ℓ they want to spend and therefore $\ell = 0$ and $u_\ell(\ell) = 0$. Consequently, the allocated time spent working l_w can be formulated as

$l_w = 1 - zn$. The reduced form of the problem is given by:

$$\max_{c,n} \quad u_c(c) + u_n(n) \tag{2.5}$$

$$s.t. \quad c + xn = w(1 - zn) \tag{2.6}$$

Constraint (2.6) reflects both the limited time and the limited disposable income of an agent. I distinguish between logarithmic and general isoelastic utilities in solving the problem.

2.2.1 Logarithmic utilities

Firstly, logarithmic utilities $u_i(i) = \alpha_i log(i)$ for $i = c$ and $i = n$, are assumed, where the coefficients α_i weight the preferences of an individual. The corresponding non–linear optimization problem is given by:

$$\max_{c,n} \quad \alpha_c log(c) + \alpha_n log(n) \tag{2.7}$$

$$s.t. \quad c + xn = w(1 - zn) \tag{2.8}$$

The optimal fertility choice n^* can be calculated with the Lagrangian method, resulting in:

$$n^* = \frac{\alpha_n w}{(\alpha_c + \alpha_n)(x + wz)} \tag{2.9}$$

It can be observed that the optimal solution is increasing in the preference of having children α_n. To examine the influence of income on fertility choice, the first derivation of n^* with respect to the wage w is considered:

$$\begin{aligned} \frac{\partial n^*}{\partial w} &= \frac{\alpha_n(\alpha_c + \alpha_n)(x + zw) - \alpha_n w(\alpha_c + \alpha_n)z}{[(\alpha_c + \alpha_n)(x + zw)]^2} \\ &= \frac{\alpha_n(\alpha_c + \alpha_n)x}{[(\alpha_c + \alpha_n)(x + zw)]^2} > 0 \quad (\text{for } x > 0) \end{aligned} \tag{2.10}$$

The derivation (2.10) is positive, as long as the good costs per child x are greater than 0. That means, the income effect of a wage increase is greater than the substitution effect and income has a positive overall effect on fertility. This relationship contradicts the negative link between fertility and income in most developed countries during the twentieth century if it is assumed that all agents are identical, and hence are paid the same wages. Nevertheless, a similar approach is used in chapter 4 to explain the recently found inverse J-shaped relation by additionally allowing for two different types of a worker's skill level and consequently two different types of payment. If x would be set to zero in model (2.7), there would be no influence of an agent's wage on the fertility choice. Such an approach is applied in the OLG model of Kimura and Yasui (2007) in chapter 3.

2.2.2 Isoelastic utility with $x = 0$

As a second special case, a similar problem to (2.7), is considered. The difference in this setup is that the expenditures on child rearing goods and services x are set to zero. Therefore the costs of children just consist of the opportunity costs of time per child z. Furthermore, general isoelastic utility functions $u_i(i) = \alpha_i \frac{i^{1-\sigma}-1}{1-\sigma}$ for $i = c$ and $i = n$ are assumed, to obtain the maximization problem:

$$\max_{c,n} \quad \alpha_c \frac{c^{1-\sigma}-1}{1-\sigma} + \alpha_n \frac{n^{1-\sigma}-1}{1-\sigma} \tag{2.11}$$

$$s.t. \quad c = w(1-zn) \tag{2.12}$$

The Lagrangian method leads to the following optimal fertility choice n^*:

$$n^* = \frac{1}{\left(\frac{\alpha_c z}{\alpha_n}\right)^{\frac{1}{\sigma}} w^{\frac{(1-\sigma)}{\sigma}} + z}$$

It can be seen that the value of σ is decisive for the effect of income on fertility. For $\sigma > 1$ and therefore a low elasticity of substitution $\frac{1}{\sigma}$, fertility increases in w similarly to the case with logarithmic utilities and good costs of children greater than 0. Vice versa, for $\sigma < 1$ and therefore a high elasticity of substitution, fertility decreases with the wage. Note that in this scenario only the opportunity costs of time are responsible for the negative relationship.

The two special cases show that certain assumptions on the utility function can be crucial for the results. The second case with $\sigma < 1$ would explain the negative relationship between fertility and income in most developed countries in the twentieth century, but fails to explain the recently observed inverse J-shaped relationship.
A further discussion of the general problem (2.1)-(2.4) with certain assumptions on utilities, and the solution of the problem with the allowance of a quality aspect of children can be found in Jones et al. (2011). However, the models in the following chapters implement a possibility for parents to invest in their own education instead of investing in the quality of their children. Therefore, this short overview of the early fertility theory is sufficient for the purposes of the thesis.

3 Occupational choice, Educational Attainment, and Fertility by Kimura/Yasui

In this chapter, the growth model of Kimura and Yasui (2007) is presented. It incorporates the interplay between education and output per capita by distinguishing between two skill groups, the possibility for parents to invest in their own education and also allows for endogenous fertility. Furthermore, it serves as a basis for the model of Day (2015) in chapter 4, where the empirically observed fertility rebound for high development levels will be implemented. Detailed calculation steps used for the results in this chapter are elaborated in Appendix B.

An overlapping generations model, where individuals live for three periods, is assumed. In the first period, as a child, agents receive goods and services from their parents and do not have to work. In the second period, individuals are in their working age. In comparison to other models, parents can invest in their own education instead of investments in the education of their children. Therefore, an agent can decide to become skilled. This requires time and hence a loss of income during the additional education phase. Otherwise she remains unskilled, with the disadvantage of a lower income per working unit. Moreover, in the second time span, individuals get children, and thus decide the number of their offspring, and save money for their old age. In the third and last period of her life, a person is retired and consumes the savings of the previous period. The number of working–age people at time t is defined by N_t and each agent is endowed with one unit of time during each period. Generation t spends its working–age life time at time period t. Correspondingly, generation $t+1$ is in old age and generation $t-1$ in childhood years in that period.

3.1 Firms

The mathematical implementation of the model starts with the production side of an economy, where production technology of firms is given by the function:

$$F(K_t, L_t^s, L_t^u) = A\left[(K_t)^\alpha (L_t^s)^{1-\alpha} + bL_t^u\right] = Y_t \tag{3.1}$$
$$\alpha \in (0,1), \quad A > 0, \quad b > 0$$

Here A stands for the productivity level, hence a constant level of productivity over time is assumed. K_t is the total capital in the economy at time t, L_t^s denotes the overall labor supply of skilled working–age individuals and L_t^u the overall labor supply of the unskilled working–age individuals. Furthermore, α stands for the capital's elasticity of production and b is a weight factor for the contribution of unskilled labor. Note that capital complements labor of skilled workers, while the contribution of unskilled labor is additive.

As mentioned above, agents cannot supply work if they need the time for education or child rearing. Hence, the involved overall labor supplies L_t^s and L_t^u at time t can be rewritten in the following way:

$$L_t^s = (1 - \tau - n_t^s z)\varphi_t N_t \tag{3.2}$$
$$L_t^u = (1 - n_t^u z)(1 - \varphi_t)N_t \tag{3.3}$$

Here, φ_t is the ratio of the skilled working–age population to the total working–age population at time t. The amount of time that an individual needs to become skilled, is denoted by τ and z defines the fixed time per child that is required for child rearing. Finally, n_t^s stands for the amount of children every skilled agent defines as optimal at time t, while n_t^u denotes the children of an unskilled parent.
In (3.2) and (3.3), the factors $(1 - \tau - n_t^s z)$ and $(1 - n_t^u z)$ describe the labor supply of an individual. $\varphi_t N_t$ and $(1 - \varphi_t)N_t$ reflect the population size of skilled and unskilled agents, respectively.

Additionally, the market is assumed to be perfectly competitive and the price of the aggregate good is normalized to 1. Therefore, the production factors of firms from production function (3.1) are paid their marginal products (w_t^s and w_t^u stand for the wage of skilled and unskilled workers, respectively):

$$w_t^s = A(1-\alpha)\left(\frac{k_t}{h_t\varphi_t}\right)^{\alpha} \tag{3.4}$$

$$w_t^u = Ab \tag{3.5}$$

$$R_{t+1} = 1 + r_{t+1} = A\alpha\left(\frac{k_t}{h_t\varphi_t}\right)^{\alpha-1} \tag{3.6}$$

Here, $k_t = \frac{K_t}{N_t}$ is the capital per capita at time t and $h_t = \frac{L_t^s}{\varphi_t N_t}$ represents the ratio of the skilled labor supply to the potential labor supply from the entire population of skilled workers. R_{t+1} denotes the gross capital investment return at time $t+1$ that an agent receives for her savings of time t. The equation $r_{t+1} = R_{t+1} - 1$ corresponds to full depreciation of 1 over one generation and hence, r_{t+1} stands for the net interest rate.
Moreover, it can be observed that capital intensity k_t has a positive direct effect on the wage of skilled workers and a negative direct effect on the net interest rate of savings. Vice versa, the ratio of overall labor supply of skilled workers to the total population $h_t\varphi_t$ has a negative direct effect on the wage of skilled workers and a positive one on the net interest rate of savings. With constant productivity level A, the wage of unskilled workers remains constant over time as well and depends on the weight factor of unskilled labor in the production function, b.

3.2 Households

Agents maximize their utility over their life time. Within the model design, an individual can gain utility by the number of her children n_t and old age consumption d_{t+1}. Similar to the approach of the objective function (2.7) in chapter 2, logarithmic utilities are assumed:

$$u_t(n_t, d_{t+1}) = \gamma \ln(n_t) + (1-\gamma)\ln(d_{t+1}), \quad \gamma \in (0,1) \tag{3.7}$$

Old–age consumption is given by the savings in working–age times the gross interest rate R_{t+1}:

$$d_{t+1} = R_{t+1}s_t \tag{3.8}$$

The budget constraint of an agent depends on the skill group that she is part of. The working time over an agent's life times the wage of the skill group, that she is part of, equals the savings for old age, as in this setting individuals cannot gain utility by consuming during working age:

$$s_t^s = w_t^s(1-\tau-zn_t^s) \tag{3.9}$$
$$s_t^u = w_t^u(1-zn_t^u) \tag{3.10}$$

This leads to the following household optimization problem for skilled workers:

$$\max_{n_t,s_t} \quad \gamma\ln(n_t)+(1-\gamma)\ln(R_{t+1}s_t) \tag{3.11}$$
$$s.t. \quad s_t = w_t(1-\tau-zn_t) \tag{3.12}$$

Exchanging the budget constraints (3.10) and (3.12) yields the corresponding problem for unskilled workers. Appendix B.1 provides the calculation steps that lead to the optimal allocations.

The optimal number of children and savings for skilled workers are given by:

$$n_t^s = \frac{\gamma(1-\tau)}{z} \tag{3.13}$$
$$s_t^s = w_t^s(1-\tau)(1-\gamma) \tag{3.14}$$

The optimal number of children and savings for unskilled workers are given by:

$$n_t^u = \frac{\gamma}{z} \tag{3.15}$$
$$s_t^u = w_t^u(1-\gamma) \tag{3.16}$$

It can be seen that the numbers of children n_t^i are independent of the wages w_t^i for both skill groups.[1] The fraction of working–age life time per child z has a negative influence, while the preference for children γ has a positive influence on the optimal number of offspring. Skilled agents get fewer children than unskilled workers as τ is greater than 0. Furthermore, the savings depend on the wages of the agents w_t^i and their preference on consumption in old age $1-\gamma$.

It is assumed that in the optimum parents are indifferent between becoming skilled and remaining unskilled. Therefore, the values of the corresponding utility functions have to be exactly the same. By equalizing the utilities (3.11) for skilled and unskilled workers and afterwards, inserting optimal solutions for the number of children ((3.13) and (3.15)) and for the savings ((3.14)

[1] For the rest of the thesis, the superscript i can be substituted by either s for skilled workers or by u for unskilled workers in equations. If there is an economic variable with superscript i in the text, the corresponding economic variables of both, skilled and unskilled agents have to be considered.

and (3.16)), the following no–arbitrage condition is obtained:

$$\frac{w_t^u}{w_t^s} = (1-\tau)^{\frac{1}{1-\gamma}} \tag{3.17}$$

With this condition, the wage of skilled workers is higher than the wage of unskilled workers. Furthermore, skilled individuals have to gain more utility from old age consumption and hence save more money in working–age life in order to obtain the same overall utility as unskilled agents, who get more children.

3.3 Dynamic System

Inserting wages (3.4) and (3.5), and the optimal number of children (3.13) into the no–arbitrage condition (3.17) leads to an expression of the fraction of skilled workers φ_t as a function of k_t:

$$\varphi(k_t) := \varphi_t = \frac{(1-\tau)^{\frac{1}{\alpha(1-\gamma)}-1}}{1-\gamma}\left(\frac{1-\alpha}{b}\right)^{\frac{1}{\alpha}} k_t \equiv \theta k_t \tag{3.18}$$

The variable $\bar{k}$ is defined as $\bar{k} := \theta^{-1}$ to divide the result into two scenarios:

$$\varphi(k_t) = \begin{cases} 1 & \text{if } k_t \geq \bar{k} \\ \theta k_t & \text{if } k_t < \bar{k} \end{cases} \tag{3.19}$$

Therefore, the ratio of skilled workers to the total working population strictly increases with capital per capita for $k_t < \bar{k}$ and is constant at a value of 1 for $k_t \geq \bar{k}$. That means that all workers decide to become skilled, when the capital per capita level in the economy is high enough. Moreover, φ_t decreases in τ, as can be seen in (3.18). Therefore, more agents decide to remain unskilled if the education time of skilled workers increases.

Recalling the output in the economy (3.1) and the overall labor supplies (3.2) and (3.3), output per capita can be expressed as[2]:

$$\begin{aligned} y_t &= A\left[k_t^{\alpha}\left(\frac{L_t^s}{N_t}\right)^{1-\alpha} + b\frac{L_t^u}{N_t}\right] \\ &= A\left[k_t^{\alpha}\left((1-\tau-zn_t^s)\varphi_t\right)^{1-\alpha} + b(1-zn_t^u)(1-\varphi_t)\right] \end{aligned}$$

[2]The resulting condition for a positive relation between capital per capita and outcome per capita is missing in the paper of Kimura and Yasui (2007), and hence elaborated in the following calculation.

Inserting the number of children (3.13) and (3.15), as well as the fraction of skilled workers (3.19) for $k_t < \bar{k}$ into the previous equation, yields an expression of output per capita as a function of k_t for the case that $k_t < \bar{k}$:

$$\begin{aligned} y_t &= A\left[k_t^\alpha\left((1-\tau-z\frac{(1-\tau)\gamma}{z})\theta k_t\right)^{1-\alpha}+b(1-z\frac{\gamma}{z})(1-\theta k_t)\right] \\ &= A\left[k_t(1-\tau)^{1-\alpha}(1-\gamma)^{1-\alpha}\theta^{1-\alpha}+b(1-\gamma)(1-\theta k_t)\right] \end{aligned} \quad (3.20)$$

The derivation of y_t with respect to the capital per capita level k_t is positive for $k_t < \bar{k}$ if it holds that:

$$\begin{aligned} \frac{\partial y_t}{\partial k_t}(k_t) = A\left[(1-\tau)^{1-\alpha}(1-\gamma)^{1-\alpha}\theta^{1-\alpha} - b(1-\gamma)\theta\right] > 0 \\ \Leftrightarrow \frac{(1-\tau)^{1-\alpha}}{(1-\gamma)^\alpha\theta^\alpha} > b \end{aligned} \quad (3.21)$$

It is assumed that (3.21) is fulfilled. For the case that $k_t \geq \bar{k}$, there is just skilled labor and $\theta k_t = 1$. Using result (3.20), output per capita is obtained as:

$$y_t = A\left[k_t^\alpha(1-\tau)^{1-\alpha}(1-\gamma)^{1-\alpha}\right]$$

Hence, y_t also increases in k_t for this scenario and under assumption (3.21), the output per capita increases with the capital per capita level for all k_t. Moreover, capital accumulation leads to an increase in the fraction of skilled workers and therefore, education increases alongside economic development.

In a next step, the fertility rate in the economy f_t and its relation to capital per capita is considered.[3] f_t is defined as the average number of children per citizen. The children born at time t consist of all children of skilled workers $n_t^s\varphi_t N_t$ and all children of unskilled individuals $n_t^u(1-\varphi_t)N_t$. Using the optimal solutions for n_t^s (3.13), n_t^u (3.15) and φ_t (3.19), the fertility rate can be expressed as a function of capital per capita:

$$\begin{aligned} f_t := f(k_t) &= \varphi_t n_t^s + (1-\varphi)n_t^u \\ f(k_t) &= \begin{cases} (1-\tau\theta k_t)\frac{\gamma}{z} & \text{if } k_t < \bar{k} \\ (1-\tau)\frac{\gamma}{z} & \text{if } k_t \geq \bar{k} \end{cases} \end{aligned} \quad (3.22)$$

[3]The fertility rate is denoted as f_t instead of m_t, as in the original paper, in order to be consistent throughout the thesis.

Both the time an agent requires to become skilled τ and the time she requires to raise children z have a negative effect on fertility, while the preference for children γ has a positive effect on the average number of children in the economy. Fertility decreases linearly in k_t if the economy comprises both skilled and unskilled workers and remains constant at a low level if the economy only comprises skilled agents. Note that the working–age population at time $t+1$ is equal to the children of the working–age population at time t, given by:

$$N_{t+1} = f_t N_t \tag{3.23}$$

Finally, it is studied how capital evolves over time and which possible equilibria exist, i.e. levels of capital per capita stock with $k_{t+1} = k_t$. The fraction of skilled workers, fertility and output per capita were derived as functions of capital per capita above. Therefore, together with the information on the development of capital per capita over time, it can be analyzed how these important variables of an economy develop over time for a certain starting level of capital per capita k_1.

Capital accumulates according to the following equation:

$$K_{t+1} - K_t = I_t - \omega K_t$$

Here, I_t denotes the investments of the economy at time t. It is assumed that the savings of the working–age population are used for these investments ($I_t = s_t N_t$). ω stands for the depreciation rate and, as full depreciation was assumed, the capital accumulation equation from above can be reduced to:

$$K_{t+1} = s_t N_t \tag{3.24}$$

Here, s_t are the total savings per citizen and can be expressed as:

$$s_t = \varphi_t s_t^s + (1 - \varphi_t) s_t^u \tag{3.25}$$

Capital per capita at time $t+1$ is calculated by recalling the evolutions of the working-age population (3.23) and capital (3.24), as well as the equation for the average savings per agent (3.25):

$$k_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{s_t N_t}{f_t N_t} = \frac{\varphi_t s_t^s + (1 - \varphi_t) s_t^u}{f_t} \tag{3.26}$$

Nevertheless, to find equilibria of the economy, where the level of capital per capita stock does not change over time, i.e. $k_{t+1} = k_t$, a function k_{t+1} that depends on the capital per capita in the former period t is required.

Consequently, the optimal choices for savings (3.14) and (3.16), the optimal fertility rate (3.22), as well as the fraction of skilled workers (3.19) are inserted into equation (3.26). The result is divided into two scenarios again:

$$k_{t+1} = \begin{cases} \frac{\theta k_t w_t^s (1-\tau)(1-\gamma) + (1-\theta k_t) w_t^u (1-\gamma) N_t}{(1-\tau\theta k_t)\frac{\gamma}{z} N_t} & \text{if } k_t < \bar{k} \\ \frac{w_t^s (1-\tau)(1-\gamma)}{\frac{\gamma(1-\tau)}{z}} & \text{if } k_t \geq \bar{k} \end{cases} \tag{3.27}$$

Inserting the wages (3.4) and (3.5), the expression $\frac{L_t^s}{\varphi_t N_t}$ for h_t and the optimal number of children (3.13) into (3.27), yields a function for k_{t+1} in k_t, which is defined as $\phi(k_t)$:

$$k_{t+1}(k_t) =: \phi(k_t) = \begin{cases} Az\frac{1-\gamma}{\gamma}\frac{1}{1-\tau\theta k_t}\left[\frac{(1-\alpha)(1-\tau)^{1-\alpha}\theta^{1-\alpha}k_t}{(1-\gamma)^\alpha} + (1-\theta k_t)b\right] & \text{if } k_t < \bar{k} \\ Az\frac{1-\gamma}{\gamma}\frac{(1-\alpha)k_t^\alpha}{(1-\tau)^\alpha(1-\gamma)^\alpha} & \text{if } k_t \geq \bar{k} \end{cases} \tag{3.28}$$

The corresponding first and second derivations of $\phi(k_t)$ are given by:

$$\phi'(k_t) = \begin{cases} Az\frac{1-\gamma}{\gamma}\frac{\theta b}{(1-\tau\theta k_t)^2}\left[(1-\tau)^{-\frac{\gamma}{1-\gamma}} - (1-\tau)\right] > 0 & \text{if } k_t < \bar{k} \\ Az\frac{1-\gamma}{\gamma}\frac{\alpha(1-\alpha)k_t^{\alpha-1}}{(1-\tau)^\alpha(1-\gamma)^\alpha} > 0 & \text{if } k_t \geq \bar{k} \end{cases} \tag{3.29}$$

$$\phi''(k_t) = \begin{cases} Az\frac{1-\gamma}{\gamma}\frac{2\tau\theta^2 b}{(1-\tau\theta k_t)^3}\left[(1-\tau)^{-\frac{\gamma}{1-\gamma}} - (1-\tau)\right] > 0 & \text{if } k_t < \bar{k} \\ -Az\frac{1-\gamma}{\gamma}\frac{\alpha(1-\alpha)^2 k_t^{\alpha-2}}{(1-\tau)^\alpha(1-\gamma)^\alpha} < 0 & \text{if } k_t \geq \bar{k} \end{cases} \tag{3.30}$$

Therefore, ϕ is increasing for all k_t, convex for $k_t < \bar{k}$ and concave for $k_t \geq \bar{k}$.
Moreover, the equation of motion fulfills the properties:

$$\phi(0) = Az\frac{(1-\gamma)}{\gamma}b > 0 \tag{3.31}$$

$$\lim_{k_t \to \infty} \phi'(k_t) = 0 \tag{3.32}$$

Considering the convexity properties together with (3.31) and (3.32), it follows that the model has at least one steady state with $\phi(k_t) = k_t$ up to a maximum of three equilibria. The illustra-

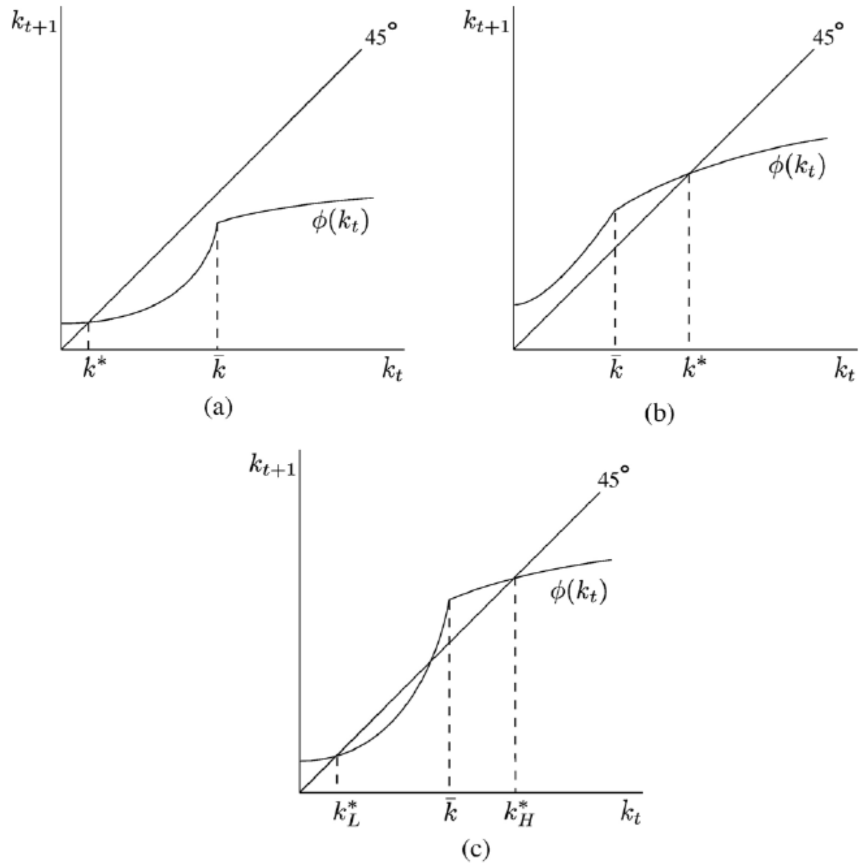
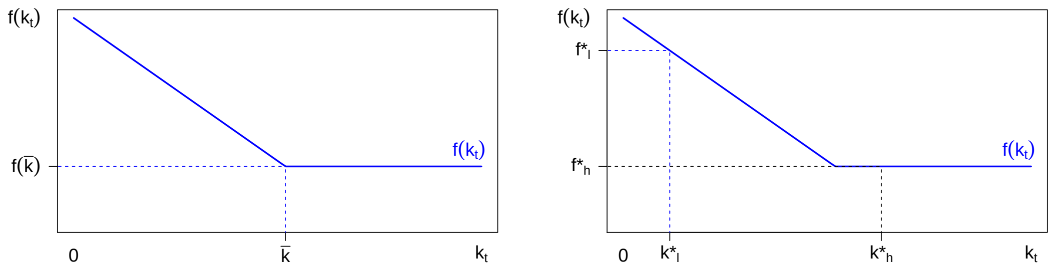


Figure 3.1: Equation of motion with three possible scenarios for steady states
(Kimura and Yasui, 2007, page 233, Figure 1)



(a) Fertility at $\bar{k}$ (b) Fertility in stable steady states

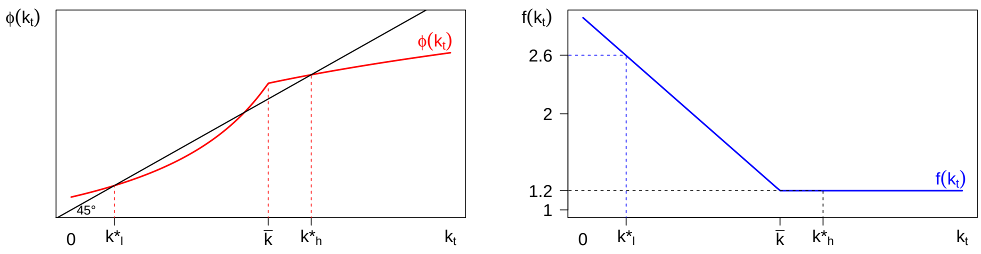
Figure 3.2: Own illustrations concerning the fertility function of set *KY*

tions in Figure 3.1 show the three possible cases. In panel (a), just one steady state k^* exists. From all starting levels of capital per capita, the economy would converge to a low outcome per capita and high fertility scenario in such a setting. In panel (b), there is one high capital per capita equilibrium. From a low starting level of k, i.e. $k < \bar{k}$, where the economy comprises both skilled and unskilled workers, capital accumulation increases the fraction of skilled workers, wages and the outcome per capita in the economy. Fertility decreases with a higher level of income and education until a level of capital per capita higher than $\bar{k}$ is reached, where the economy just consists of skilled agents. Then, the fertility rate remains stable at a relatively low value. Finally, panel (c) illustrates one low, k_l^*, and one high, k_h^*, stable equilibrium. For starting levels of capital per capita higher than the medium, unstable equilibrium level, k_m^*, the economy behaves similar to case (b), while for lower levels than k_m^* the economy is in a poverty trap and converges to a level with low outcome per capita and high fertility like in scenario (a).

The association between capital per capita and fertility is illustrated in Figure 3.2. In the left panel (a), the change of the slope of the fertility function from negative and constant to 0 can be observed at $\bar{k}$. In the right panel (b) of Figure 3.2, the possible equilibria are illustrated. The high steady state f_l^* of fertility corresponds to a low stable equilibrium of capital per capita, k_l^*, while the low level fertility f_h^* corresponds to the high level k_h^*.

3.4 Parameter analysis

In this section I present a quantitative analysis of the model for various parameter constellations. At first, I consider a parameter scenario from the original paper (Kimura and Yasui, 2007, foot-



(a) Equation of motion $\phi(k_t)$ (b) Fertility function $f(k_t)$

Figure 3.3: Own illustrations of functions ϕ and f for the parameter set KY

note on page 233), that leads to three steady states, as in Figure 3.1(c). It is denoted as parameter set KY, summarized in Table 3.1. The production elasticity of capital α is set to 0.33, unskilled labor contributes with the weight factor b of 0.1, and firms produce with a total productivity level A of 4.5. Furthermore, an agent requires 60 percent of his working–age life time to become skilled and the preference on having children γ is 0.6 in comparison to the preference on old–age consumption $(1-\gamma)$ of 0.4. Finally, a parent requires 20 percent of her life time in working age for child rearing ($z = 0.2$). Hence, within this setting a skilled individual cannot have more than 2 children.

α	b	τ	A	z	γ
0.33	0.1	0.6	4.5	0.2	0.6

Table 3.1: Parameter set KY

Note that condition (3.21) is fulfilled here, as 0.59 > 0.1, and therefore, output per capita y_t increases in k_t. The corresponding equation of motion $\phi(k_t)$ with stable steady states k_l^* and k_h^* is plotted in Figure 3.3, as well as the corresponding fertility function. The value of the fertility function is approximately 2.61 for k_l^* and 1.2 for k_h^*.

Transforming equation (3.23), leads to:

$$f_t N_t = N_{t+1} \tag{3.33}$$

$$\Leftrightarrow f_t N_t - N_t = N_{t+1} - N_t$$

$$\Leftrightarrow f_t - 1 = \frac{N_{t+1} - N_t}{N_t} \tag{3.34}$$

Hence, $f_t - 1$ describes the working–age population growth rate in the economy. Consequently, the working–age population growth rate is 1.61 if the economy is in the steady state k_l^*. Within k_h^*, the working-age population is still increasing per generation, but at a much lower growth rate of 0.2.

The equilibria required for the figures and tables in this chapter can be calculated with the results elaborated in Appendix B.6. The level of the instable steady state k_m^* in parameter set KY is approximately 0.46. Next, I consider two initial levels of capital per capita stock, one below and one above k_m^*, respectively, to examine, how macroeconomic variables evolve over time to the corresponding steady states. Figure 3.4 illustrates the case with $k_1 = 0.48$. Capital per capita increases over time, and hence the fraction of skilled workers increases until all agents decide to become skilled, i.e. $\varphi = 1$. The increase of k_t leads to a decrease in fertility until a level greater than $\bar{k}$ is reached. Then, fertility remains at the relatively low level. The wage of skilled workers increases from the moment on, when the economy just comprises skilled workers. Output per capita y_t increases alongside k_t.

Figure 3.5 displays the development of the fraction of skilled workers, fertility and wages of skilled workers for the case of an initial level $k_1 = 0.44$. For that case, the economy converges towards the low steady state of capital per capita. Fertility increases, the fraction of skilled workers decreases and the wage of skilled workers remains constant over time.

Finally, I examine how ceteris paribus changes to the parameter set KY influence the economy with its steady states. The results are given in Table 3.2. With the relatively high technology level $A = 6$, the poverty trap can be avoided, as there just exists one stable high equilibrium with a fertility level of 1.2. Vice versa, a low technology level $A = 3.5$ leads to a poverty trap with low capital per capita and a high fertility level of 2.75, which is even higher than the fertility level of the low equilibrium in the original setting of 2.61. Furthermore, the technology level A does not influence the lowest level of k, where all workers become skilled, $\bar{k}$ and the fertility level at this stage.

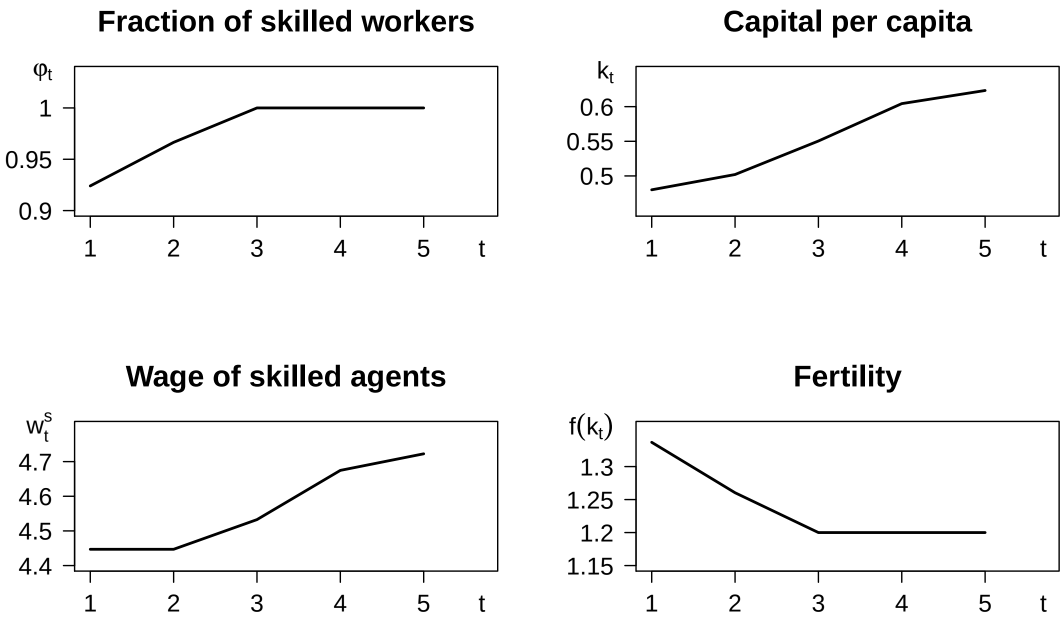


Figure 3.4: Own plot of economic variables over time for the initial level $k_1 = 0.48$

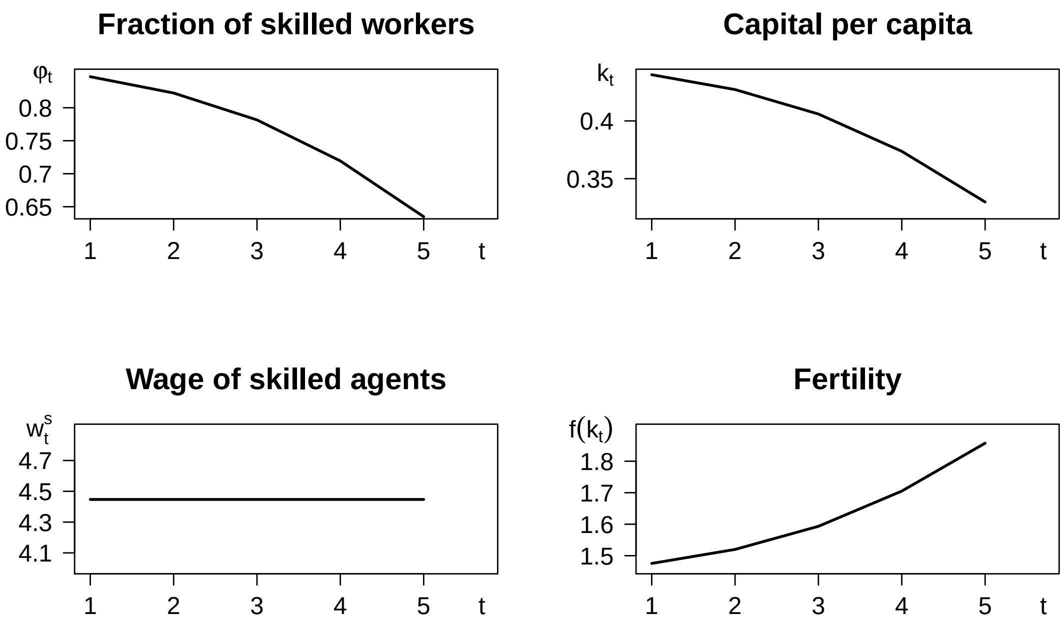


Figure 3.5: Own plot of economic variables over time for the initial level $k_1 = 0.44$

	$\bar{k}$	k_l^*	k_h^*	f($\bar{k}$)	f(k_l^*)	f(k_h^*)
set *KY*	0.52	0.11	0.63	1.2	2.61	1.2
A = 6	0.52		0.97	1.2		1.2
A = 3.5	0.52	0.07		1.2	2.75	
$\gamma = 0.4$	0.07		1.7	0.8		0.8
$\tau = 0.3$	0.01		0.48	2.1		2.1

Table 3.2: Sensitivity analysis of parameter set *KY*

The preference on children, γ, of 0.6 in the original setting KY is relatively high. With a ceteris paribus change to $\gamma = 0.4$, $\bar{k}$ changes to 0.07 instead of 0.52, which means that all agents decide to become skilled at a lower level of capital per capita. There is just one steady state k_h^*, where the economy just comprises skilled workers. Moreover, it is much higher than in the original setting (1.7 instead of 0.63), which means the outcome per capita in that equilibrium is higher, while fertility is lower (0.8 instead of 1.2). The reason of the lower fertility is straightforward as both, the number of children of skilled workers n_t^s, (3.13), and the number of children of unskilled agents n_t^u, (3.15), are positively associated with γ. The explanation for the lower value of $\bar{k}$ is that with a higher preference on old age consumption $1-\gamma$ more agents want to become skilled in order to save more money for old age.

The time required to become skilled τ was also set rather high at 0.6 in the parameter specification KY. This means that skilled citizens can just work 40 percent of their working–age life time. Therefore, I set τ at a lower level of 0.3 ceteris paribus. Similarly to the case with lower preferences on children, that results in just one high steady state k_h^*, but this time the fertility in the equilibrium is higher (2.1 instead of 1.2), while capital per capita is lower than in the original setting (0.48 instead of 0.63). The explanation for the higher average number of children in this scenario is that with less required effort of becoming skilled, agents can spend more time rearing offspring.

4 Skill Composition, Fertility, and Economic Growth by Creina Day

The following chapter is based on the paper of Day (2015). Her overlapping generations model is an extension of the model of Kimura and Yasui (2007). She additionally allows for purchased child rearing inputs and country differences in the possibility of reconciling family and work through a production function for child rearing. Eventually, this leads to an inverse J-shaped relationship between fertility and income per capita as it was suggested in the motivation of this thesis. Moreover, countries can subsidize child rearing goods and services with the revenues of a tax on old–age consumption. Consuming in working age gains additional utility for agents. With a similar mechanism as in chapter 3, the fraction of skilled workers and therefore, education, increases with the level of capital per capita stock. The corresponding model and a sensitivity analysis for various parameter settings of the model are presented.

An overlapping generations model is assumed, where individuals live for three periods. In the first period, as a child, agents receive goods and services from their parents. In the second period, individuals can decide to invest in their own education and hence, can become skilled. This requires time τ and therefore a loss of income during the additional education phase. Otherwise, they are employed as unskilled workers. Moreover, in the second time span of their life, citizens get children, consume goods and services and save money for their old age. In the third and last period of their life, agents are retired and consume the savings of the previous period. Again, at time t, generation t spends its working–age life time.

4.1 Firms

The production side of the economy remains almost unchanged in comparison to the model of Kimura and Yasui (2007). The notation from section 3.1 is used, except for the variable denoting the required time per child in child rearing, which is endogenous, time-dependent and skill–dependent in this model extension and therefore denoted as $\hat{z}_t^i$ instead of z. Therefore, the formulas concerning the labor supplies of the economy (3.2) and (3.3) are substituted by:

$$L_t^s = (1 - \tau - n_t^s \hat{z}_t^s)\varphi_t N_t \tag{4.1}$$

$$L_t^u = (1 - n_t^u \hat{z}_t^u)(1 - \varphi_t)N_t \tag{4.2}$$

The production technology and the corresponding wages remain the same:

$$F(K_t, L_t^s, L_t^u) = A\left[(K_t)^\alpha (L_t^s)^{1-\alpha} + bL_t^u\right] = Y_t, \quad \alpha \in (0,1), A > 0, b > 0 \tag{4.3}$$

$$w_t^s = A(1-\alpha)\left(\frac{k_t}{h_t\varphi_t}\right)^\alpha \tag{4.4}$$

$$w_t^u = Ab \tag{4.5}$$

$$R_{t+1} = 1 + r_{t+1} = A\alpha\left(\frac{k_t}{h_t\varphi_t}\right)^{\alpha-1} \tag{4.6}$$

Capital complements skilled labor, while unskilled labor contributes additively to the final output level of firms, depending on the weight factor of unskilled labor b. A higher productivity level A raises the output. Moreover, the direct effect of capital per capita k_t on the wage of skilled workers is positive, while it is negative for the gross interest rate R_{t+1}. Opposite direct effects of the fraction of skilled workers φ_t on wage w_t^s and interest rate R_{t+1} can be observed as well. The wage of unskilled workers just depends on the technology level A and the weight factor of unskilled labor in the production process b.

4.2 Households

The consumption side of the economy depends on the individuals' utilities as introduced in chapter 2. In comparison to the household optimization in section 3.2, purchased child rearing inputs are introduced similarly as in the presented problems (2.1) and (2.7). There, the parameters for goods and services expenditures per child, x, and for the time per child that parents are required to stay at home and cannot work during child rearing, z, were exogenous. However, these parameters could vary in different countries and over time.

For example, if better day care possibilities for children of very young ages are available in one country, a parent can put her child in day care and is not required to stay at home for the same period of time, as a parent of another country, where those possibilities are not available. In that scenario, the optimal value of goods and services inputs per child would be higher in the first country due to the higher services expenditures on day care, while the amount of time needed for

child rearing would be higher in the second country. In the first country, this would also lead to a higher female labor force participation. Recalling the arguments of Luci and Thevenon (2010), especially different shares of female labor force participation could be the reason for different levels of countries' fertility rates.
Obviously, the labor market and child rearing conditions can also change within a country. Consequently, to allow for different country specifications and a change of the mentioned parameters over time, Day (2015) introduced a production function for child rearing, where the values of good expenditures and time costs per child are endogenously calculated.

With the number of children at time t, n_t, fixed, parents optimize the time spent on childrearing, z_t and the utilized amount of child goods and services units, x_t. A Cobb-Douglas production function for child rearing is assumed with production elasticity of time denoted by a:

$$n_t = z_t^a x_t^{1-a}, \quad a \in (0,1) \tag{4.7}$$

With that specification, a one percent increase in time z_t would lead to an a percent increase in children at time t. Similarly, a one percent increase in goods units x_t would lead to a $(1-a)$ percent increase in n_t. Hence, different values of the exogenous parameter a stand for different efficiencies of the production factors in child rearing across countries.

The price of the goods unit x_t is exogenously given by p_x. Additionally, the government subsidizes each of those units by βp_x ($\beta \in (0,1)$ is the share of the price) through a tax on old–age consumption T_t. The corresponding governmental budget constraint can be formulated as:

$$\beta p_x(\varphi_t n_t^s + (1-\varphi_t)n_t^u)N_t = (1+r_t)(\varphi_t s_{t-1}^s + (1-\varphi_t)s_{t-1}^u)N_{t-1}T_t \tag{4.8}$$

In the whole thesis, the superscript s stands for skilled, while u denotes unskilled. s_t^s and s_t^u are the amounts of money agents save during their working–age life. As an individual receives interest on the savings and has to pay the tax in old age, the amount of money for consumption of a skilled old–aged citizen at time t is given by $(1+r_t)s_{t-1}^s(1-T_t)$ and the corresponding money for consumption in old age of an unskilled agent by $(1+r_t)s_{t-1}^u(1-T_t)$. Therefore, the right–hand side of equation (4.8) describes the revenues gained by the tax at time t. The left–hand side describes the subsidies on children's goods at time t.

With the subsidized price of a goods unit, overall costs of offspring for skilled and unskilled parents C_t^i can be expressed as:

$$C_t^i = w_t^i z_t^i + x_t^i p_x(1-\beta), \quad \beta \in (0,1) \tag{4.9}$$

In equation (4.9), the summand $w_t^i z_t^i$ reflects the opportunity costs of time, as agents are assumed to have no income during child rearing. A parent loses the wage w_t^i for the time she misses work z_t^i. The expenditures for child rearing goods are given by the amount of units x_t^i times the subsidized price $p_x(1-\beta)$.
Subsequently, to find the optimal choice of time and goods intensity in parenting, each agent solves the problem:

$$\text{argmin}_{x_t^i, z_t^i} \quad w_t^i z_t^i + x_t^i p_x(1-\beta) \tag{4.10}$$

$$s.t. \quad n_t = z_t^a x_t^{1-a} \Leftrightarrow z_t = \left(\frac{n_t}{x_t^{1-a}}\right)^{\frac{1}{a}} \tag{4.11}$$

A comprehensive derivation of the following optimal solutions derived by the Lagrangian method can be found in Appendix C.1. The optimal share of time and share of goods, respectively, are obtained as:

$$x_t^i = \left(\frac{w_t^i(1-a)}{p_x(1-\beta)a}\right)^a n_t^i =: \hat{x}_t{}^i(w_t^i, \beta) n_t^i \tag{4.12}$$

$$z_t^i = \left(\frac{p_x(1-\beta)a}{w_t^i(1-a)}\right)^{1-a} n_t^i =: \hat{z}_t{}^i(w_t^i, \beta) n_t^i \tag{4.13}$$

It can be observed that the fixed parameters x and z from the models (2.1) and (2.7) have changed into the endogenous variables $\hat{x}_t{}^i$ and $\hat{z}_t{}^i$ that depend on w_t^i and β in this setup.
The endogenous share of time per child invested in child rearing at time t, $\hat{z}_t{}^i$, decreases with the wage of an agent w_t^i, while the share of units of goods per child at time t, $\hat{x}_t{}^i$, increases with income. The reason is that opportunity costs of time increase with higher wages, while it is easier to afford purchased child rearing goods and services with more available money.
If parents can afford more child rearing inputs due to a lower price after subsidies $p_x(1-\beta)$, they also use more units x_t^i and spend less time at home z_t^i than in the case of more expensive goods and services.

A formula for the overall price per child, defined as $p_t^i(w_t^i, p_x, \beta) := C_t^i/n_t^i$, is obtained by inserting the optimal choices x_t^i and z_t^i into the objective function (4.10) and dividing the function by the number of offspring n_t^i:

$$p(w_t^i, p_x, \beta) = (w_t^i)^a \left(p_x(1-\beta)\right)^{1-a} \cdot B \tag{4.14}$$

In the formula, the auxiliary variable B is defined as $B := a^{-a}(1-a)^{-(1-a)}$.

The overall costs per child increase both in wage and child rearing goods price, where the latter one suggests that a higher share of subsidy β could result in higher fertility rates.

Now, in a second step, the calculated overall price per child (4.14) is used to optimize a parent's utility. The utility is composed of the number of children n_t alongside consumption in working–age life c_t and consumption in retirement d_{t+1}. The utility of each of those arguments is described by a log–utility function and contributes additively to the overall utility at time t, defined as u_t:

$$u_t(c_t, n_t, d_{t+1}) = \lambda \ln(c_t) + \gamma \ln(n_t) + \delta \ln(d_{t+1}), \quad \lambda + \gamma + \delta = 1, \quad \lambda, \gamma, \delta > 0 \tag{4.15}$$

γ, λ and δ stand for the preferences of an agent and weigh the different arguments of u_t. The higher the value of γ is in comparison to the other parameters, the more agents prefer children over consumption in working age and retirement age and vice versa. It is assumed that all agents have the same preferences and therefore the same weight factors γ, λ and δ. The assumption that the parameters sum up to 1 is made without loss of generality.
To obtain the optimal allocations, two constraints on the maximization have to be considered. Firstly, old–age consumption is defined as:

$$d_{t+1} = s_t(1 + r_{t+1})(1 - T_{t+1}) \tag{4.16}$$

The variable s_t denotes the savings during working age and T_{t+1} is the tax on old–age consumption, as mentioned in the governmental budget constraint (4.8). As a simplifying assumption in this model design, there is no inheritance from generation to generation and consequently all of the savings are being consumed.
Secondly, the budget constraint comprises the fact of limited time and money and depends on the skill level of a citizen:

$$s_t^s + p(w_t^s, \beta, p_x)n_t^s + c_t^s = (1-\tau)w_t^s \tag{4.17}$$

$$s_t^u + p(w_t^u, \beta, p_x)n_t^u + c_t^u = w_t^u \tag{4.18}$$

The right–hand side reflects the income of the agents during working–age life. During the education phase a skilled individual cannot work, therefore the time spent working is denoted by $(1-\tau)$. The left–hand side comprises the expenditures of an agent. She pays the overall price $p(w_t^i, \beta, p_x)n_t^i$ for her children that was calculated via problem (4.10), (4.11). Furthermore, the individual can divide the remaining income into consumption during working–age life and savings, for consumption during her retirement.

Inserting the first constraint (4.16) into the objective function (4.15) leads to the following utility optimization problem for skilled workers:

$$\text{argmax}_{c_t^s, s_t^s, n_t^s} \quad \gamma \ln(c_t) + \lambda \ln(n_t) + \delta \ln[s_t(1+r_{t+1})(1-T_{t+1})] \tag{4.19}$$

$$s.t. \quad (1-\tau)w_t^s = s_t^s + p(w_t^s, \beta, p_x)n_t^s + c_t^s \tag{4.20}$$

The corresponding problem for unskilled individuals can be formulated by substituting the superscript s by u and using constraint (4.18) instead of (4.17).

Note that within a skill class, the treated problem resembles model (2.7) of chapter 2. The differences are that agents save for the future and spend time to become skilled in this framework. Furthermore, the fixed parameters x and z in (2.8) have become the endogenous terms $\hat{x}_t p_x(1-\beta)$ and $\hat{z}_t$ in this setting, as $p(w_t, \beta, p_x) = \hat{z}_t n_t + \hat{x}_t p_x(1-\beta)n_t$. Due to the resemblance, one might expect that the number of children of a skilled and unskilled worker, respectively, increases with her wage, as a positive relationship between fertility and income was obtained in model (2.7).

The optimization problem (4.19) is solved with the Lagrangian method after inserting equation (4.14) for the overall price per child $p(w_t^s, \beta, p_x)$ into constraint (4.20). The corresponding calculation steps that lead to the following results are stated in Appendix C.2.

The optimal allocations of skilled agents are given by:

$$n_t^s = \frac{(1-\tau)\gamma}{B}\left[\frac{w_t^s}{p_x(1-\beta)}\right]^{1-a} \tag{4.21}$$

$$c_t^s = \lambda(1-\tau)w_t^s \tag{4.22}$$

$$s_t^s = (1-\gamma-\lambda)(1-\tau)w_t^s \tag{4.23}$$

Similarly, the corresponding optimal choices for unskilled workers are obtained as:

$$n_t^u = \frac{\gamma}{B}\left[\frac{w_t^u}{p_x(1-\beta)}\right]^{1-a} \tag{4.24}$$

$$c_t^u = \lambda w_t^u \tag{4.25}$$

$$s_t^u = (1-\gamma-\lambda)w_t^u \tag{4.26}$$

Indeed, within a skill group, the number of children n_t^i increases with the wage w_t^i. This is an important difference in comparison to the model of Kimura and Yasui (2007), where the numbers of children were independent of the wages (see (3.13) and (3.15)). The reason for the different results is the introduction of purchased child rearing inputs x_t^i in this model extension. Similar relationships between fertility and wages were obtained in the introduction of fertility theories in chapter 2. Specifically, in subsection 2.2.1, fertility and income had a positive relationship for the case that children's goods and services expenditures are greater than 0, i.e. $x > 0$, and were independent for the case that those costs are zero, i.e. $x = 0$.
Considering equations (4.22), (4.23), (4.25) and (4.26), income has a positive effect on both consumption and savings. Moreover, agents have more children if the price of child rearing goods and services after subsidies $p_x(1-\beta)$ decreases. Consumption, savings and fertility increase the higher they are valued in the utility function (i.e. the higher their weight factor), while they all decrease for skilled workers if the education time of becoming skilled τ increases.

Citizens are assumed to have identical preferences. Hence, if both types of skill levels exist, a necessary condition is that the overall utility of an unskilled agent has the same total value as the overall utility of a skilled worker. Equating these utilities and inserting the optimal choices of fertility and consumption leads to the following no–arbitrage condition, as can be reviewed in Appendix C.3:

$$(1-\tau)^{\frac{1}{1-a\gamma}} = \frac{w_t^u}{w_t^s} \tag{4.27}$$

The left–hand side of the equation is smaller than 1. Consequently, the wage per working unit for skilled workers is larger than the wage for unskilled agents. The more the education time to become skilled increases (the higher τ), the higher is the wage gap between skilled and unskilled individuals. An increase in the preference for children γ has the opposite effect and hence, reduces the gap.

Multiplying equation (4.27) by w_t^s leads to the following expression for the wage w_t^u:

$$w_t^u = (1-\tau)^{\frac{1}{1-a\gamma}} w_t^s \tag{4.28}$$

Using (4.28), the following inequality proves that the optimal amount of consumption is higher for skilled agents:

$$c_t^u = \lambda w_t^u = \lambda (1-\tau)^{\frac{1}{1-a\gamma}} w_t^s < \lambda(1-\tau) w_t^s = c_t^s \tag{4.29}$$

Similarly, it can be observed that unskilled agents save less money than skilled individuals, i.e. $s_t^u < s_t^s$.
Hence, in order to gain the same total utility as skilled workers, despite consuming less in both, working age and old age, unskilled individuals have more children:

$$n_t^u = \frac{\gamma}{B}\left[\frac{w_t^u}{p_x(1-\beta)}\right]^{1-a}$$

Inserting (4.28), leads to:

$$\begin{aligned} n_t^u &= \frac{\gamma}{B}\left[\frac{(1-\tau)^{\frac{1}{1-a\gamma}} w_t^s}{p_x(1-\beta)}\right]^{1-a} = \frac{\gamma(1-\tau)^{\frac{1-a}{1-a\gamma}}}{B}\left[\frac{w_t^s}{p_x(1-\beta)}\right]^{1-a} \\ &> \frac{\gamma(1-\tau)}{B}\left[\frac{w_t^s}{p_x(1-\beta)}\right]^{1-a} = n_t^s \end{aligned} \tag{4.30}$$

4.3 Dynamic system

Next, the development of capital per capita over time and the existence of steady states in this model design are considered for the general case without parameter specifications. Expressing the fraction of skilled workers and the average number of children as functions of capital per capita, the developments of these variables over time can be analyzed as well.

The optimal amount of time in child rearing (4.13) and the optimal number of children n_t^s lead to the following auxiliary expression of the term $(1-\tau-\hat{z}_t^s n_t^s)$:

$$(1-\tau-\hat{z}_t^s n_t^s) = (1-a\gamma)(1-\tau) \tag{4.31}$$

Substituting wages (4.4) and (4.5) in the no–arbitrage condition (4.27) and inserting for $h_t = \frac{L_t^s}{\varphi_t N_t}$

and for L_t^s (4.1) yields:

$$(1-\tau)^{\frac{1}{1-a\gamma}} = \frac{Ab}{A(1-\alpha)\left(\frac{k_t}{\varphi_t(1-\tau-\hat{z}_t^s n_t^s)}\right)^{\alpha}} \tag{4.32}$$

Finally, inserting expression (4.31) into (4.32) and solving for the fraction of skilled workers leads to the fraction of skilled workers in the economy at time t[1]:

$$\varphi(k_t) := \varphi_t = \frac{(1-\tau)^{\frac{1}{\alpha(1-a\gamma)}-1}}{1-a\gamma}\left(\frac{1-\alpha}{b}\right)^{\frac{1}{a}} k_t \equiv \theta k_t \tag{4.33}$$

The variable $\bar{k}$ is defined as $\bar{k} := \theta^{-1}$ to divide the result into two cases:

$$\varphi(k_t) = \begin{cases} 1, & \text{if } k_t \geq \bar{k} \\ \theta k_t, & \text{if } k_t < \bar{k} \end{cases} \tag{4.34}$$

The ratio of skilled workers to the total working population strictly increases with capital per capita for $k_t < \bar{k}$ and is constant at a value of 1 for $k_t \geq \bar{k}$. It means that all workers decide to become skilled, when the capital per capita level in the economy is high enough. Moreover, φ_t decreases in τ, as can be seen in (4.33). Therefore, more agents remain unskilled if the education time of skilled workers increases. Note that those properties concerning the fraction of skilled workers did not change in this extension as compared to the OLG model of Kimura and Yasui (2007) and that the mechanism behind those properties is the same.

The explanation for the increase of the fraction of skilled workers in capital per capita k_t, for the case that both types of skill levels exist, can be observed in the no–arbitrage condition formula (4.32). The left–hand side of the equation, as well as the wage for unskilled workers Ab, are constant. Consequently, the wage for skilled workers has to be constant as well over time. It is given by:

$$w_t^s = A(1-\alpha)\left(\frac{k_t}{\varphi_t(1-\tau-\hat{z}_t^s n_t^s)}\right)^{\alpha} \tag{4.35}$$

With equation (4.31), the only terms that can change over time in formula (4.35) are k_t and φ_t. Moreover, formula (4.35) shows that capital per capita k_t has a positive effect on the wage w_t^s, while the fraction of skilled workers φ_t has a negative effect on the wage w_t^s. Consequently, an increase in k_t leads to an increase in the fraction φ_t. This way, the wage of skilled workers

[1] A comprehensive calculation of $\varphi(k_t)$ can be found in Appendix C.4.

remains constant for the case that both types of skill levels exist and the no–arbitrage condition is fulfilled.

In a next step of analyzing the dynamics of the model an equation of motion for the capital per capita is calculated that will help to understand the dynamics of various other economic variables and the existence of steady states in the model.

Exactly as in section 3.3 of chapter 3, and with the same arguments, capital per capita evolves according to:

$$k_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{s_t N_t}{f_t N_t} = \frac{\varphi_t s_t^s + (1-\varphi_t) s_t^u}{\varphi_t n_t^s + (1-\varphi_t) n_t^u} \tag{4.36}$$

Here, $f_t := \frac{N_{t+1}}{N_t}$ stands for the average number of children per parent in the economy.

As a next step, the numbers of children (4.21) and (4.24), the optimal choices for savings (4.23) and (4.26), as well as the fraction of skilled workers (4.34) are inserted into (4.36). The resulting expression is divided into the scenario, where all workers are skilled ($k_t \geq \bar{k}$) and into the scenario, where also a fraction of unskilled workers exists ($k_t < \bar{k}$):

$$k_{t+1} = B\frac{1-\gamma-\lambda}{\gamma} \begin{cases} (w_t^s)^a [p_x(1-\beta)]^{1-a} & \text{if } k_t \geq \bar{k} \\ \dfrac{\theta k_t w_t^s (1-\tau) + (1-\theta k_t) w_t^u}{\left[\theta k_t \left(\frac{w_t^s}{p_x(1-\beta)}\right)^{1-a} (1-\tau) + (1-\theta k_t)\left(\frac{w_t^u}{p_x(1-\beta)}\right)^{1-a}\right]} & \text{if } k_t < \bar{k} \end{cases} \tag{4.37}$$

Using $h_t = \frac{L_t^s}{\varphi_t N_t}$, and equations (4.1) and (4.31), the wage of a skilled worker from formula (4.4) can be rewritten as:

$$w_t^s = A(1-\alpha) \begin{cases} [k_t/(1-a\gamma)(1-\tau)]^{\alpha} & \text{if } k_t \geq \bar{k} \\ [(1-a\gamma)(1-\tau)\theta]^{-\alpha} & \text{if } k_t < \bar{k} \end{cases} \tag{4.38}$$

Finally, an expression for the capital per capita at time $t+1$, k_{t+1}, depending on the value of the former period k_t is obtained by inserting for the wage of skilled workers (4.38) and for $\theta^{-\alpha}$

(4.33) into (4.37). It is defined as the function $\phi(k_t)$:

$$k_{t+1}(k_t) =: \phi(k_t) = A^a \tilde{B} \frac{1-\gamma-\lambda}{\gamma} \begin{cases} \left[\frac{(1-\alpha)}{(1-a\gamma)^{\alpha}(1-\tau)^{\alpha}}\right]^a k_t^{a\alpha} & \text{if } k_t \geq \bar{k} \\ b^a \frac{1+\theta\left\{(1-\tau)^{\frac{-a\gamma}{(1-a\gamma)}}-1\right\}k_t}{1-\theta\left\{1-(1-\tau)^{\frac{a(1-\gamma)}{(1-a\gamma)}}\right\}k_t} & \text{if } k_t < \bar{k} \end{cases} \tag{4.39}$$

Here, $\tilde{B}$ is an auxiliary variable for $B[px(1-\beta)]^{(1-a)}$. A comprehensive, stepwise calculation of $\phi(k_t)$ can be reviewed in Appendix C.5.

Now, the properties of ϕ provide information on the existence of steady states in the economy, i.e. constant levels of capital per capita stock over time ($k_{t+1} = k_t$). The first derivative of ϕ is positive for all k_t, while the second derivative is positive for $k_t < \bar{k}$, and negative for $k_t \geq \bar{k}$.[2] Hence, ϕ is increasing and convex for $k_t < \bar{k}$ and increasing and concave for $k_t > \bar{k}$.
Moreover, it holds that:

$$\phi(0) = (Ab)^a \tilde{B} \frac{(1-\gamma)}{\gamma} > 0 \quad \text{and} \quad \lim_{k_t \to \infty} \phi'(k_t) = 0. \tag{4.40}$$

It follows that $\phi(k_t) = k_{t+1}(k_t)$ has at least one point of intersection with the 45°–line, which guarantees one steady state. Together with the other characteristics of ϕ, there are up to three equilibria in the economy if parameters are appropriately selected. I will consider different parameter sets and corresponding steady state scenarios in the next section. Moreover, time paths of k_t, the wage of skilled workers w_t^s and consumption of skilled workers c_t^s for a specific initial level k_1 are provided. The possible steady state scenarios of this model extension match the scenarios of chapter 3, reflected in Figure 3.1.

The main incentive for this alternative overlapping generations model was to implement the recently observed inverse J-shaped relationship between economic development and fertility in a theoretical growth model. An expression of the total fertility rate f_t as a function k_t is calculated to examine if this model design matches the empirical facts. The corresponding function $f(k_t)$ reflects the relationship between fertility and economic development if and only if outcome per capita y_t increases in k_t. Consequently, I elaborate an inequality that ensures the positive link between per capita outcome and capital per capita. The resulting inequality is missing in the

[2] The first and second derivative of ϕ are given in Appendix C.7.

paper of (Day, 2015). Dividing total outcome Y_t, (4.3), by N_t yields:

$$y_t = A\left[k_t^{\alpha}\left(\frac{L_t^s}{N_t}\right)^{1-\alpha} + b\frac{L_t^u}{N_t}\right]$$

Inserting skilled and unskilled labor supplies (4.1) and (4.2), leads to:

$$y_t = A\left[k_t^{\alpha}\left((1-\tau-n_t^s\hat{z}_t^s)\varphi_t\right)^{1-\alpha} + b(1-n_t^u\hat{z}_t^u)(1-\varphi_t)\right]$$

Using (4.31) and substituting (4.24) for n_t^u, as well as (4.13) for $\hat{z}_t^u$, helps to evaluate the influence of k_t on y_t:

$$y_t = A\left[k_t^{\alpha}\left((1-\tau)(1-\gamma)\theta k_t\right)^{1-\alpha} + b(1-\frac{\gamma}{B}\left[\frac{w_t^u}{p_x(1-\beta)}\right]^{1-a}\left[\frac{p_x(1-\beta)a}{w_t^u(1-a)}\right]^{1-a}\hat{z}_t^u)(1-\theta k_t)\right]$$
$$y_t = A\left[\left((1-\tau)(1-\gamma)\theta\right)^{1-\alpha}k_t + b(1-a\gamma)(1-\theta k_t)\right]$$

The first derivative of y_t with respect to k_t is greater than 0 if and only if the following condition is fulfilled:

$$\frac{\partial y_t}{\partial k_t}(k_t) = \left((1-\tau)(1-\gamma)\theta\right)^{1-\alpha} - b(1-a\gamma)\theta > 0$$
$$\Leftrightarrow \frac{\left((1-\tau)(1-\gamma)\right)^{1-\alpha}}{(1-a\gamma)\theta^{\alpha}} > b \tag{4.41}$$

It is assumed that condition (4.41) is met. Hence, outcome per capita increases with capital per capita and the following function $f(k_t)$ captures the association between fertility and economic development as well. The calculation steps to obtain the new fertility function are presented in Appendix C.6. At first, the optimal choices of children (4.21) and (4.24) and the formula for the fraction of skilled workers (4.34) are inserted into $f_t = \varphi_t n_t^s + (1-\varphi_t)n_t^u$. After that, the wage formulas (4.5) and (4.38) are substituted for w_t^s and w_t^u as well as the expression for θ, (4.33). Finally, the function, which is once again divided into two scenarios, is obtained:

$$f(k_t) := f_t = \begin{cases} \frac{\gamma}{B}\left[\frac{Ab}{p_x(1-\beta)}\right]^{1-a}\left[1-\theta\left(1-(1-\tau)^{\frac{a(1-\gamma)}{(1-a\gamma)}}\right)k_t\right] & \text{if } k_t < \bar{k} \\ \frac{\gamma(1-\tau)^{1-\alpha(1-a)}}{B}\left[\frac{A(1-\alpha)}{p_x(1-\beta)(1-a\gamma)^{\alpha}}\right]^{1-a}k_t^{\alpha(1-a)} & \text{if } k_t \geq \bar{k} \end{cases} \tag{4.42}$$

The derivatives can be reviewed in Appendix C.7. They fulfill the following properties:

$$f'(k_t) < 0, \quad f''(k_t) = 0 \quad \text{if } k_t < \bar{k} \tag{4.43}$$

$$f'(k_t) > 0, \quad f''(k_t) < 0 \quad \text{if } k_t \geq \bar{k} \tag{4.44}$$

Fertility is decreasing in capital per capita as long as the economy comprises both skilled and unskilled workers and is increasing without unskilled agents. Therefore, a fertility rebound at stage $\bar{k}$, where every individual decides to become skilled can be observed within the model. Furthermore, it is linear for $k_t < \bar{k}$ and concave for $k_t \geq \bar{k}$. The properties of f_t can be explained by taking a look at the following form of the fertility function:

$$f(k_t) = \varphi(k_t)n_t^s(k_t) + (1 - \varphi(k_t))n_t^u(k_t)$$

For $k_t < \bar{k}$, both types of skill levels exist and wages w_t^s and w_t^u are constant, therefore independent of the level of capital per capita. Hence, the optimal numbers of children n_t^u and n_t^s are constant for that case as well. With inequality (4.30), skilled agents get fewer children than unskilled agents. Therefore, the total fertility rate decreases in k_t for $k_t < \bar{k}$, as the fraction of skilled workers (4.33) increases. For the case that $k_t \geq \bar{k}$, the fraction of skilled workers is 1 and the total fertility rate is therefore given as $f_t = n_t^s$. The wage of skilled workers increases in k_t (see (4.38)) and as the amount of children n_t^s increases with the wage w_t^s (see (4.21)), there is indeed a fertility rebound for a capital per capita level higher than $\bar{k}$.
Note that the dependence of the number of children n_t^s on the wage w_t^s is also the reason for the difference in fertility behavior in comparison to the model of Kimura and Yasui (2007), where fertility was constant for a high level of capital per capita.

The level of the turning point $\bar{k}$ depends among other parameters on the production elasticity of time a in the production function for child rearing (4.7):

$$\frac{\partial \bar{k}}{\partial a} = -\frac{\gamma}{(1-a\gamma)}\left[1 + \frac{\ln(1-\tau)}{\alpha(1-a\gamma)}\right]\bar{k} > 0$$

$$\Leftrightarrow \quad \ln(1-\tau) < -\alpha(1-a\gamma) \tag{4.45}$$

It can be observed that the level of $\bar{k}$ increases in a as long as inequality (4.45) is fulfilled. $\frac{\partial \bar{k}}{\partial a} > 0$ means that in countries, where the elasticity of children's goods and services $(1-a)$ is higher, e.g. due to inexpensive day care possibilities and therefore a lower elasticity a, the fertility rebound corresponds to a lower level of capital per capita than in countries with a higher production elasticity of time a.

4.4 Parameter analysis

In the following, I present an own, numerical analysis of the model. Different steady state scenarios are studied as well as the dynamics of capital per capita, wages, consumption and fertility. The equilibria used for this analysis have been calculated with the formulas from Appendix C.8.

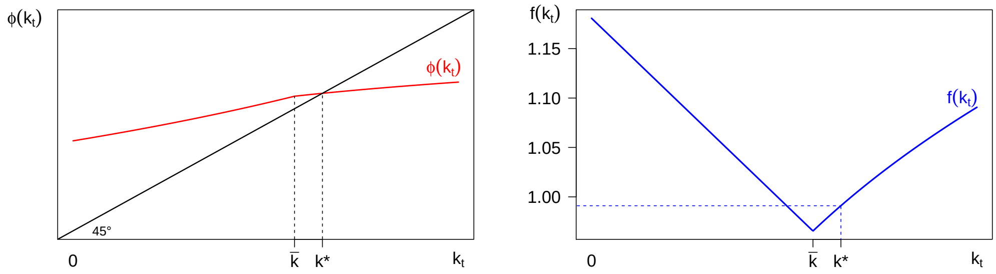
Table 4.1 shows a particular parameter set, denoted as $P1$. The production elasticity of time in the production function for child rearing a is set to 0.45, hence time is a little less efficient than purchased goods and services inputs in parenting. Moreover, the price of child rearing goods and services p_x coincides with the price of the aggregate consumption good, that is 1. However, the government subsidizes child rearing inputs with 40 percent ($\beta = 0.4$), resulting in a price per unit $p_x(1-\beta)$ of 0.6. Similarly to parameter set KY in section 3.4, the desire to have children is weighted slightly higher than the preference in consumption, i.e. $\gamma > \lambda$. In $P1$, the weight factor of consumption in working age is 0.3, while the weight factor of children is 0.5. While with parameter set KY, an agent required 60 percent of her working–age life time to become skilled, here the education phase lasts half of the working–age life, i.e. $\tau = 0.5$. In the production technology of firms, a one percent increase in capital corresponds to a 0.4 percent increase in total output, i.e. $\alpha = 0.4$. Finally, the total productivity A and the weight factor of unskilled labor supply b are set to 20 and 0.5, respectively, in order to allow for a shape of the fertility function that corresponds to the observed relationship between fertility and economic growth from chapter 1.

a	α	b	τ	A	p_x	β	γ	λ
0.45	0.4	0.5	0.5	20	1	0.4	0.5	0.3

Table 4.1: Own parameter set $P1$ for the model of Day (2015)

Assuming specification $P1$ holds, conditions (4.41) and (4.45) are fulfilled and therefore $\bar{k}$ increases in a, and outcome per capita y_t increases with capital per capita k_t.

A plot of the corresponding equation of motion (4.39) is drawn in panel (a) of Figure 4.1. It can be observed that there is just one point of intersection between $\phi(k_t)$ and the 45°–line, and therefore one equilibrium with $k_{t+1} = k_t$. This high equilibrium level k^* is calculated as approximately 2.59. $\bar{k}$ has the approximate value of 2.3. From every starting level of capital per capita stock k_1, the economy eventually converges towards the steady state k^*. In panel (b) of Figure 4.1 fertility is drawn as a function of k_t, using (4.42). The inverse J-shape in the plot is indeed



(a) Equation of motion $\phi(k_t)$ (b) Fertility function $f(k_t)$

Figure 4.1: Functions ϕ and f for the parameter set $P1$

very similar to the corresponding Figures 1.2 and 1.4 from the introduction. The average fertility rates between 0.95 and 1.2 are lower than the fertility steady state levels of 1.2 and 2.6 in chapter 3. The fertility rate in the steady state k^* is 0.99. This would mean that the working–age population nearly remains the same over each time period in the steady state. Recalling that the desired number of children is relatively stable at around 2 per family over time, as was discussed in chapter 1, this number for $f(k^*)$ seems reasonable.

I apply $\phi(k_t)$ and other functions of economic variables in k_t from the sections 4.2 and 4.3 to obtain time paths of certain economic variables from an initial level of capital per capita k_1 of 1. At level $k_1 = 1$, the economy comprises both types of skill levels and the fraction of skilled workers is below 0.5. As can be observed in Figure 4.2, capital per capita increases over time which leads to a simultaneous increase in outcome per capita. Furthermore, the economic development leads to a higher fraction of skilled workers of over 0.8 and therefore a higher level of education in the second generation, i.e. $t = 2$, until all workers decide to become skilled from the third generation onwards, i.e. $t \geq 3$. Moreover, together with the illustrations in Figure 4.3, it can be seen that the wage of skilled workers and hence also the consumption of skilled workers remain constant, as long as the economy comprises both types of skill groups, and increase in capital per capita afterwards.
The average consumption per citizen increases with every generation. The reason is that skilled agents consume more than unskilled agents (see inequality (4.29)). As the fraction of skilled workers increases, average consumption increases as well. From the third generation on, the economy just consists of skilled individuals and average consumption increases due to the higher disposable income over time.

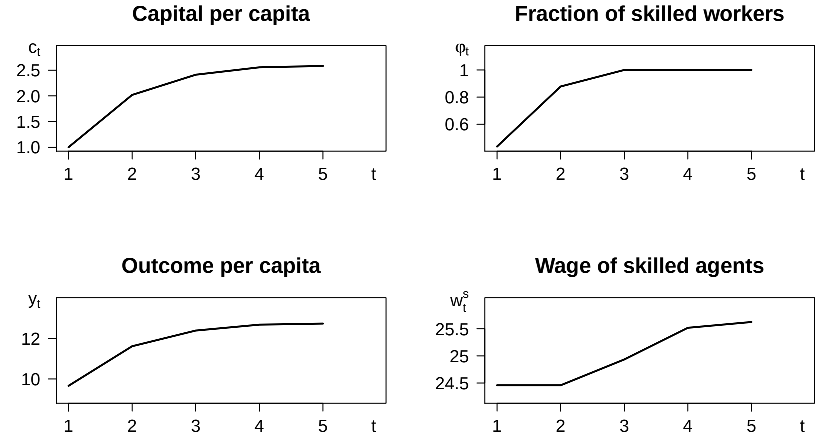


Figure 4.2: Development of economic variables over time for set $P1$ and $k_1 = 1$

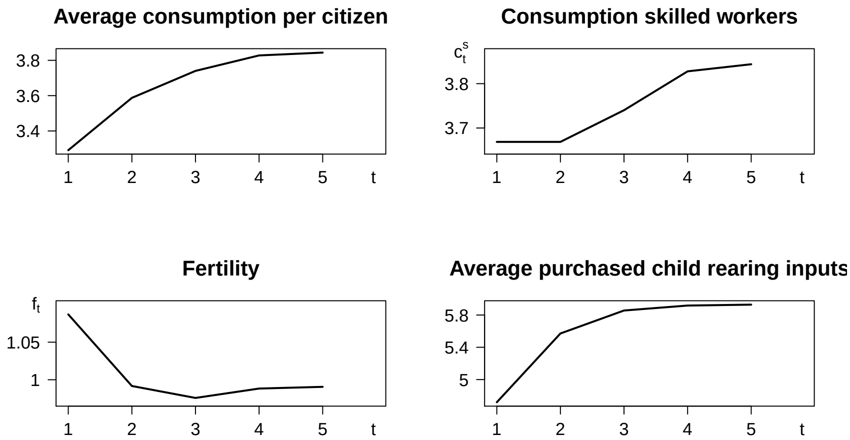
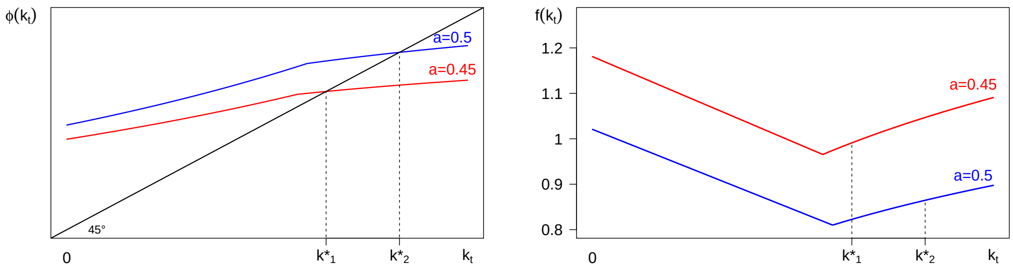


Figure 4.3: Development of economic variables over time for set $P1$ and $k_1 = 1$

As skilled workers get fewer children than unskilled individuals within the model, fertility decreases with the fraction of skilled workers between the first and third generation. After that, there is a small fertility rebound, when the economy just consists of skilled citizens and the fertility converges to the steady state level of approximately 0.99.
Therefore, if the assumptions of parameter set $P1$ hold, the model reflects an empirical fact from chapter 1: Until a certain level of education and economic output is reached and hence until a certain level of human development is attained, fertility decreases with output and education. Afterwards, in this model reflected through a level of capital per capita stock that is greater than $\bar{k}$, fertility increases again.

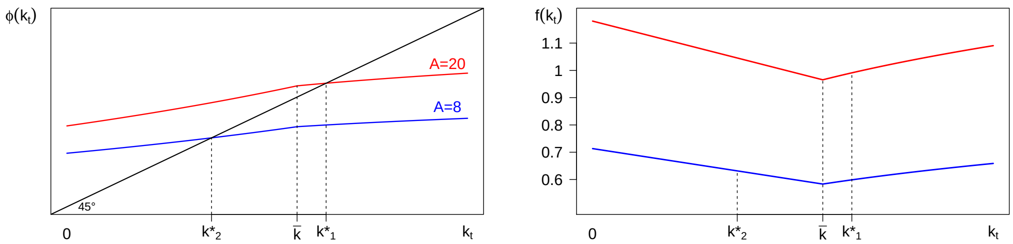
Finally, the plot in the bottom right of Figure 4.3 illustrates the development of the average units of purchased child rearing inputs over time. Recalling formula (4.12) for the optimal number of units x_t^i, the number increases with the wage w_t^i. As skilled workers get higher wages than unskilled workers (see equation (4.27)), they also use more units of purchased inputs for child rearing. Moreover, the wage of skilled workers increases in k_t, for the case that $k_t \geq \bar{k}$. Together with the development of the fraction of skilled workers over time, this explains why x_t^i increases with every generation.

Parameter set $P1$ is used for a sensitivity analysis on the elasticity of time a in the production function for child rearing. Changing a ceteris paribus from 0.45 to the higher elasticity 0.5, condition (4.45) is fulfilled and the fertility rebound takes place at a higher level of capital per capita stock in the new scenario. In Figure 4.4 the results of the parameter sets with $a = 0.45$ and $a = 0.5$, respectively, are compared. The levels of $\bar{k}$ for the different scenarios correspond to the values of k_t, where the slopes of the fertility functions change from negative to positive. For the higher rate $a = 0.5$ both, the initial level and the slope of ϕ are higher than for $a = 0.45$, resulting in a higher steady state k_2^*. Nevertheless, the growth rate of the working–age population is lower for all k_t in the new setting and it also holds that $f(k_2^*) - 1 < f(k_1^*) - 1$. These results can be explained by the higher relative price per child for a higher production elasticity of time a (see (4.14)): Due to the relative higher price, agents have less children. Instead, they purchase additional goods and services, and save more of their income, respectively.
The different fertility rates in Figure 4.4 might explain the difference in fertility rates of countries with similar income levels. In this context, considering Figure 1.4 from chapter 1, one might expect that Germany had a higher production elasticity of time than France in 2006, as both countries had a similar level of GDP per capita, while TFR levels were higher in France than in Germany.



(a) Equation of motion $\phi(k_t)$ (b) Fertility function $f(k_t)$

Figure 4.4: Comparison between different levels of production elasticity of time a



(a) Equation of motion $\phi(k_t)$ (b) Fertility function $f(k_t)$

Figure 4.5: Comparison between different levels of total productivity A

A similar approach leads to an analysis of different total productivity levels. Parameter set $P1$ is used once again, but with a ceteris paribus change of A to a value of 20 instead of 8. As can be observed in Figure 4.5, both ϕ and f have higher function values for the case of a higher total productivity. The reason is that a higher A leads to higher income for agents and with more disposable money, individuals are able to both, save more, which leads to capital accumulation, and afford more offspring. The total productivity level A does not influence the level of $\bar{k}$ (see (4.33)), hence the level is the same for both scenarios. It can be seen that the new steady state k_2^* is located below $\bar{k}$, i.e. the economy comprises both skilled and unskilled workers in the new stable equilibrium k_2^*.
The model introduced in this chapter suggests that with a high level of production technology, countries can avoid a poverty trap.

In general, there are either one or three steady states depending on the parameter specification.

Under certain circumstances, three equilibria are possible, similar as with parameter set KY in chapter 3. A corresponding parameter set denoted as $P2$ with illustrations on $\phi(k)$ and the fertility function can be found in Appendix C.9, as well as a short discussion. Nevertheless, both, the resulting values of fertility and the slope of the fertility function for the case that $k_t \geq \bar{k}$ are unrealistically low. Therefore, the results induced by parameter set $P2$ do not match the empirical facts discussed in chapter 1.

4.5 Robustness test

In this final part of chapter 4, I discuss the assumptions on the production and utility function of Day (2015). Therefore, in a first step, the production technology (4.3) is exchanged with a Cobb-Douglas production function. Here, ε stands for the production elasticity of skilled labor:

$$F(K_t, L_t^s, L_t^u) := AK_t^{\alpha}(L_t^s)^{\varepsilon}(L_t^u)^{(1-\alpha-\varepsilon)}, \quad \alpha, \varepsilon, \alpha+\varepsilon \in (0,1) \tag{4.46}$$

Again, a perfect competitive market is assumed and the price of the aggregate good is set to 1. Therefore, the three production factors are rewarded with the corresponding marginal products of $F(K_t, L_t^s, L_t^u)$:

$$w_t^s = A\varepsilon K_t^{\alpha}(L_t^s)^{\varepsilon-1}(L_t^u)^{1-\alpha-\varepsilon} = \frac{\partial F}{\partial L_t^s}(K_t, L_t^s, L_t^u) \tag{4.47}$$

$$w_t^u = A(1-\alpha-\varepsilon)K_t^{\alpha}(L_t^s)^{\varepsilon}(L_t^u)^{-\alpha-\varepsilon} = \frac{\partial F}{\partial L_t^u}(K_t, L_t^s, L_t^u) \tag{4.48}$$

$$R_{t+1} = 1 + r_{t+1} = A\alpha K_t^{\alpha-1}(L_t^s)^{\varepsilon}(L_t^u)^{1-\alpha-\varepsilon} = \frac{\partial F}{\partial K_t}(K_t, L_t^s, L_t^u)$$

In comparison to the original model, the wage of unskilled labor increases with capital K_t. It is assumed that the utility function and the production function for child rearing remain the same as in section 4.2. As the type of the production technology of firms does not influence decisions in the utility optimization of households, all the results from section 4.2 still hold in the new model. That includes the optimal time used for child rearing $\hat{z}_t^i n_t^i$, the optimal choice of children n_t^i, the optimal savings s_t^i, the optimal amount of consumption during the working–age life c_t^i, and the no–arbitrage condition, which is given by:

$$(1-\tau)^{\frac{1}{1-\alpha\gamma}} = \frac{w_t^u}{w_t^s} \tag{4.49}$$

Consequently, using the auxiliary equation (4.31), the overall labor supplies in the economy can be expressed as:

$$L_t^s = (1-\tau-\hat{z}_t^s n_t^s)\varphi_t N_t = (1-a\gamma)(1-\tau)\varphi_t N_t$$
$$L_t^u = (1-\hat{z}_t^u n_t^u)(1-\varphi_t)N_t = (1-a\gamma)(1-\varphi_t)N_t$$

Inserting the wages corresponding to the new production function (4.47) and (4.48) and afterwards, these overall labor supplies into the no–arbitrage condition (4.49), leads to an expression for the fraction of skilled workers at time t, φ_t:

$$\begin{aligned}
(1-\tau)^{\frac{1}{1-a\gamma}} &= \frac{A(1-\alpha-\varepsilon)K_t^{\alpha}(L_t^s)^{\varepsilon}(L_t^u)^{-\alpha-\varepsilon}}{A\varepsilon K_t^{\alpha}(L_t^s)^{\varepsilon-1}(L_t^u)^{(1-\alpha-\varepsilon)}} && (4.50)\\
\Leftrightarrow (1-\tau)^{\frac{1}{1-a\gamma}} &= \frac{1-\alpha-\varepsilon}{\varepsilon}\frac{L_t^s}{L_t^u}\\
\Leftrightarrow (1-\tau)^{\frac{1}{1-a\gamma}} &= \frac{1-\alpha-\varepsilon}{\varepsilon}\frac{(1-a\gamma)(1-\tau)\varphi_t N_t}{(1-a\gamma)(1-\varphi_t)N_t}\\
\Leftrightarrow (1-\varphi_t)(1-\tau)^{\frac{1}{1-a\gamma}-1} &= \frac{1-\alpha-\varepsilon}{\varepsilon}\varphi_t\\
\Leftrightarrow (1-\tau)^{\frac{a\gamma}{1-a\gamma}} &= \frac{1-\alpha-\varepsilon}{\varepsilon}\varphi_t + (1-\tau)^{\frac{a\gamma}{1-a\gamma}}\varphi_t\\
\Leftrightarrow \varphi_t =: \varphi &= \frac{(1-\tau)^{\frac{a\gamma}{1-a\gamma}}}{\frac{1-\alpha-\varepsilon}{\varepsilon} + (1-\tau)^{\frac{a\gamma}{1-a\gamma}}} < 1 && (4.51)
\end{aligned}$$

Note that with the Cobb-Douglas production function, the ratio is constant over time in comparison to the original model, where it was directly proportional to the level of capital per capita stock k_t. The reason is that capital increases the wage of unskilled workers with the same factor K_t^{α} as it increases the wage of skilled workers. Consequently, K_t^{α} can be canceled in the no–arbitrage condition (4.50) and the fraction of skilled workers becomes independent of capital K_t. Moreover, the economy consists of both skill groups for all generations, as $\varphi < 1$. In the original model of Day (2015), the fertility rate fell for $k_t < \bar{k}$ due to the decrease of unskilled workers in k_t, who preferred more children over more consumption. Now, there is a constant ratio of skilled and unskilled workers and the wages w_t^i also increase in k_t for the case that both type of workers exist. Consequently, there is a positive relationship between fertility and capital per capita for all k_t, as within skill groups the relationship between fertility and income w_t^i is positive (see (4.21) and (4.24)). This contradicts the findings of chapter 1.

Similarly, the original model was tested for the case of general isoelastic utilities instead of logarithmic utilities, while leaving the other functions unchanged. It is not presented in detail in this thesis as it did not lead to explicit solutions of consumption, the number of children or the fraction of skilled workers.

To summarize the robustness check, the type of the production function and the form of the utility function are crucial for the results and mechanisms in the framework.

5 Discussion

In this final part of the thesis, a summary of the theoretical models and corresponding parameter analyses from the previous chapters is given and the model outcomes are linked to empirical results from chapter 1. Moreover, I consider possible model extensions for further research.

In chapter 3, the paper "*Occupational choice, Educational Attainment, and Fertility*" by Masako Kimura and Daishin Yasui (2007) was presented. The authors set up an overlapping generations model with a separable production technology in the skill levels of the labor force, where capital complements skilled labor, and unskilled labor contributes additively to the output level of the economy. Agents have identical utilities, which consist of consumption in working age, consumption in old age and the number of children they get over their life time. In an optimum that comprises both skill groups, agents are indifferent between becoming skilled and remaining unskilled. Skilled workers receive a higher wage and consume more in comparison to unskilled agents. In order to gain the same utility as skilled individuals despite the lower consumption, unskilled workers have a greater number of offspring. The direct effect of an increase in the level of capital per capita stock on skilled labor is positive as capital complements skilled labor. Hence, more people decide to become skilled with economic development, i.e. with increasing levels of capital per capita, and the fraction of skilled workers increases simultaneously. This leads to a higher average education level. Moreover, capital accumulation increases the outcome per capita level. The average fertility rate decreases for a higher fraction of skilled workers, as educated agents get fewer children.
Since outcome per capita and education are two of the three components of the human devolopment index, the model manages to reflect the empirical findings from the introduction that fertility and human development are negatively connected for low and medium levels of education and outcome per capita. At a certain level of capital per capita all agents decide to become skilled which results in a constantly low, average fertility rate. This reflects the findings of the Demographic Transition Theory, which predicts a stable, low fertility at high development stages and also corresponds with Bongaarts and Sobotka (2012), who suppose that the recently observed increase in total fertility rates within European countries is mainly due to parity- and tempo-effects and will not endure in the long-term, when fertility rates should even out at de-

sired fertility levels of families.
Additionally, a quantitative numerical analysis was carried out, where the parameter specification *KY* as introduced in the paper of Kimura and Yasui (2007) led to one unstable and two stable steady states. At the high, stable equilibrium all agents were skilled and fertility was relatively low, while in the low, stable steady state, the economy is in a poverty trap with a low level of capital per capita and high fertility. Furthermore, it was demonstrated that a high level of technology can prevent the poverty trap in an economy and that a shorter education time, which is required to work in skilled labor, leads to higher fertility levels as agents have more time to raise children. Finally, a low preference in children results in a high preference in savings, and therefore, higher capital accumulation and a high level of capital per capita in the steady state of the economy.

In chapter 4, the work of Creina Day (2015), "*Skill Composition, Fertility, and Economic Growth*", was considered. The corresponding overlapping generations model is an extension of the model in chapter 3. Parents can additionally purchase child rearing inputs, which allows them to spend more time working. Again, capital accumulation ensures economic growth and growth in the fraction of skilled workers. At low and medium levels of capital per capita, development and fertility are negatively related, as well. But in the model of Day (2015), due to the possibility of purchased child rearing inputs, the optimal number of children is positively associated with the income of an agent. Hence, at high levels of capital per capita, where all citizens decide to become skilled, capital accumulation leads to more offspring. Consequently, this model accomplished to reflect the inverse J-shaped relationship between fertility and human development that was found for both, period (Myrskylä et al., 2009) and cohort (Myrskylä et al., 2011) fertility rates.
Again, an own parameter analysis was elaborated where a certain parameter set led to an economy with one high and stable steady state and realistic fertility rates around replacement fertility. From a low level of capital per capita stock, capital accumulation increases outcome per capita, average consumption per citizen and purchased child rearing inputs per offspring until the economy reaches its steady state level. Furthermore, capital accumulation results in a fertility decrease until the moment, when all workers decide to become skilled. Afterwards fertility converges at a small positive rate towards the equilibrium. Similar to the model of Kimura and Yasui (2007), production technology increases the wage of all citizens and therefore fertility, consumption and savings. To allow for country differences, different specifications of the production elasticity of purchased child rearing inputs $(1-a)$ can be assumed. A low value of a and therefore a high value $(1-a)$ suggests that a country has cheap day care possibilities or policies that facilitate to combine child rearing and work. The parameter analysis in chapter 4 points out

that a lower a ensures higher fertility rates while the outcome per capita level in the steady state is lower than in the case of a higher production elasticity of time in the production function for child rearing.

Concerning further theoretical approaches on population developments, there are various possible extensions of the presented models. As tempo–effects accounted for the lowest-low fertility levels and to a certain degree for the fertility rebound, it could be interesting to implement the postponement of childbearing within growth models through a positive association between economic development and the average age of mothers at childbirth. Within the presented models, the technology level A and the time required for child rearing τ were exogenously given. As an endogenous approach for the latter one could implement an endogenous wage of skilled workers given by $w_t^s = \tau^{\mu}$, similar as in (Tertilt and Jones, 2008, pages 58–60), where μ describes the rate of return on human capital accumulation τ.

Fertility developments were the main focus in the thesis, but also life expectancy and migration contribute to population changes. Chen (2010) extends the model of Kimura and Yasui (2007) by allowing for an agent's probability of survival into old age. The author finds that an increase in the survival probability increases the fraction of skilled workers and decreases the fertility level.

The implementation of migration into the models could also be an interesting extension, particularly for European countries that experienced a large flow of refugees in recent years. A corresponding comment is made in the preface of a UN expert paper: "*Finally, the author argues that the impact of migration, both in terms of the fertility of migrants in the receiving country and the impact of the migration of women of reproductive age in the country of origin, is sufficiently significant that the analysis of fertility and migration trends should be integrated.*" (Sobotka, 2013, page iii)

As concluding remarks, the future empirical research on population developments will help to understand the reasons of the recently observed fertility rebound of TFR levels in developed countries even better. It remains to be seen if completed cohort fertility of 1975 and 1980 cohorts confirms the inverse J-shaped relationship between fertility and human development or if a stable, relatively low period fertility can be expected in the long term for wealthy countries, when tempo effects vanish. For both cases, the presented models form an excellent basis for further theoretical approaches in the future.

Appendices

A Definitions concerning population and economic analysis

Human development index
The human development index (HDI) is a measure of a country's development stage. Besides economic development, which is measured by the gross national income (GNI), the HDI also takes into account education, through mean and expected years of schooling, and the life expectancy of a country's population. For all of the three components indices between 0 and 1 are used. The HDI is defined as the product of the income, education and life expectancy index.

Crude birth rate
"A vital statistics summary rate based on the number of live births occurring in a population during a given period of time, usually a calendar year, i.e., the number of live births occurring among the population of a given geographical area during a given year, per 1,000 mid-year total population of the given geographical area during the same year" (UNdata, 2016)

Total fertility rate
"A basic indicator of the level of fertility, calculated by summing age-specific birth rates over all reproductive ages. It may be interpreted as the expected number of children a women who survives to the end of the reproductive age span will have during her life time if she experiences the given age-specific rates" (UNdata, 2016)

The total fertility rate is mainly used over the crude birth rate, because it controls for population structure and is therefore a better measure for comparisons across countries.

GDP per capita PPP 2011 international dollars
As income compared by ordinary currency exchange rates does not always reflect the living standards of citizens across countries appropriately, one can also consider the measure of purchasing power parity (PPP) in international dollars. GDP per capita levels, given in a certain currency, are converted into PPP 2011 international dollars by using an exchange rate based on

comparable consumption baskets of 2011.

B Mathematical Appendix of chapter 3

B.1 Household optimization

The maximization problem for skilled workers is given by (see (3.11)):

$$\begin{aligned} \max_{n_t^s, s_t^s} \quad & \gamma \ln(n_t^s) + (1-\gamma)\ln(R_{t+1}s_t^s) \\ s.t. \quad & s_t^s = w_t^s(1-\tau - zn_t) \end{aligned} \tag{B.1}$$

By inserting the budget constraint (B.1) into the objective function (3.11), the problem can reformulated as:

$$\max_{n_t^s} u_t^s = \gamma \ln(n_t^s) + (1-\gamma)\ln(R_{t+1}w_t^s(1-\tau - zn_t^s)) \tag{B.2}$$

From the first order conditions, one obtains:

$$\begin{aligned} \frac{\gamma}{n_t^s} &= (1-\gamma)\frac{R_{t+1}w_t^s z}{R_{t+1}w_t^s(1-\tau-zn_t^s)} \\ \Leftrightarrow \gamma(1-\tau-zn_t^s) &= (1-\gamma)zn_t^s \\ \Leftrightarrow n_t^s &= \frac{\gamma(1-\tau)}{z} > 0 \end{aligned}$$

As n_t^s is greater than 0, it is a feasible solution and also maximizes problem (B.2) due to the concavity of the objective function. Inserting the optimal number of children n_t^s into the formula for savings of skilled workers s_t^s (B.1), leads to:

$$\begin{aligned} s_t^s &= w_t^s(1-\tau - z\frac{\gamma(1-\tau)}{z} \\ s_t^s &= w_t^s(1-\tau-\gamma(1-\tau)) \\ s_t^s &= w_t^s(1-\tau)(1-\gamma) \end{aligned}$$

The difference in the maximization problem of unskilled workers in comparison to the one of skilled agents only consists of the term $(1-\tau)$ and the superscript u instead of s in w_t^u, s_t^u and n_t^u. Therefore, the optimal results for unskilled workers are directly obtained from the solutions above:

$$n_t^u = \frac{\gamma}{z}$$
$$s_t^u = w_t^u(1-\gamma)$$

B.2 No–arbitrage condition and fraction of skilled workers

At first, I apply a monotonic transformation on the utilities of skilled and unskilled agents (3.11), using the exponential function. Then, I recall the equations for the optimal numbers of children (3.13) and (3.15) and optimal savings (3.14) and (3.16):

$$exp(u_t^s) = (n_t^s)^\gamma \cdot (R_{t+1}s_t^s)^{1-\gamma} = (\frac{\gamma(1-\tau)}{z})^\gamma \cdot (R_{t+1}w_t^s(1-\tau)(1-\gamma))^{1-\gamma} \tag{B.3}$$

$$exp(u_t^u) = (n_t^u)^\gamma \cdot (R_{t+1}s_t^u)^{1-\gamma} = (\frac{\gamma}{z})^\gamma \cdot (R_{t+1}w_t^u(1-\gamma))^{1-\gamma} \tag{B.4}$$

Now, to ensure that agents are indifferent between becoming skilled and remaining unskilled, the corresponding exponential utilities (B.3) and (B.4) are equated, to obtain the no–arbitrage condition:

$$\left(\frac{\gamma(1-\tau)}{z}\right)^\gamma \cdot (R_{t+1}w_t^s(1-\tau)(1-\gamma))^{(1-\gamma)} \stackrel{!}{=} \left(\frac{\gamma}{z}\right)^\gamma \cdot (R_{t+1}w_t^u(1-\gamma))^{(1-\gamma)}$$
$$\Leftrightarrow (1-\tau)^\gamma \cdot (w_t^s(1-\tau))^{(1-\gamma)} = (w_t^u)^{(1-\gamma)}$$
$$\Leftrightarrow (1-\tau)^{\frac{1}{1-\gamma}} = \frac{w_t^u}{w_t^s} \tag{B.5}$$

Inserting wages (3.4) and (3.5), and afterwards, the optimal number of children (3.13) into this condition, leads to a function for the fraction of skilled workers φ_t depending on k_t:

$$
\begin{aligned}
(1-\tau)^{\frac{1}{1-\gamma}} &= \frac{w_t^u}{w_t^s} \\
(1-\tau)^{\frac{1}{1-\gamma}} &= \frac{Ab}{A(1-\alpha)\left[\frac{k_t}{(1-\tau-zn_t^s)\varphi_t}\right]^{\alpha}} \\
(1-\tau)^{\frac{1}{1-\gamma}} &= \frac{b}{(1-\alpha)\left[\frac{k_t}{(1-\tau-z\frac{\gamma(1-\tau)}{z})\varphi_t}\right]^{\alpha}} \\
(1-\tau)^{\frac{1}{1-\gamma}}\left[\frac{k_t}{(1-\tau)(1-\gamma)\varphi_t}\right]^{\alpha} &= \frac{b}{(1-\alpha)} \\
(1-\tau)^{\frac{1}{1-\gamma}-\alpha}\left[\frac{k_t}{(1-\gamma)}\right]^{\alpha}\frac{(1-\alpha)}{b} &= \varphi_t^{\alpha} \\
\varphi_t &= \frac{(1-\tau)^{\frac{1}{\alpha(1-\gamma)}-1}}{(1-\gamma)}\left(\frac{(1-\alpha)}{b}\right)^{\frac{1}{\alpha}} k_t \equiv \theta k_t
\end{aligned}
$$

B.3 Fertility

The fertility $f(k_t)$ equals the optimal number of children n_t^s for the case that all workers are skilled ($k_t \geq \bar{k}$).

For the case $k_t < \bar{k}$, the fertility function $f(k_t)$ is obtained by inserting the fraction of skilled workers (3.18) and the optimal numbers of children (3.13) and (3.15):

$$
\begin{aligned}
f(k_t) &= \varphi_t n_t^s + (1-\varphi_t)n_t^u \\
&= \theta k_t(1-\tau)\frac{\gamma}{z} + (1-\theta k_t)\frac{\gamma}{z} \\
&= (\theta k_t - \tau\theta k_t + 1 - \theta k_t)\frac{\gamma}{z} \\
&= (1-\tau\theta k_t)\frac{\gamma}{z}
\end{aligned}
$$

B.4 Equation of motion

To start the calculation for capital per capita at time $t+1$ as a function of capital per capita in period t, I recall formula (3.26):

$$
k_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{\varphi_t s_t^s + (1-\varphi_t)s_t^u}{f_t}
$$

Now, I distinguish between levels of capital per capita stock above and below $\bar{k}$ again. For $k_t < \bar{k}$, the following term can be obtained by using the equations for the fraction of skilled workers (3.19), the fertility function (3.22) and the optimal savings (3.14) and (3.16):

$$k_{t+1} = \frac{\theta k_t w_t^s (1-\tau)(1-\gamma) + (1-\theta k_t) w_t^u (1-\gamma)}{(1-\tau\theta k_t)\frac{\gamma}{z}}$$

Substituting the wages w_t^s and w_t^u using equations (3.4) and (3.5), and afterwards, replacing h_t with $\frac{L_t^s}{\varphi_t N_t}$, as well as the optimal number of children n_t^s with formula (3.13), leads to:

$$\begin{aligned}
k_{t+1} &= \frac{z(1-\gamma)}{\gamma} \frac{\theta k_t A(1-\alpha)\left[\frac{k_t}{h_t \varphi_t}\right]^{\alpha}(1-\tau) + (1-\theta k_t)Ab}{(1-\tau\theta k_t)} \\
&= \frac{z(1-\gamma)}{\gamma} \frac{\theta k_t A(1-\alpha)\left[\frac{k_t}{(1-\tau-zn_t^s)\varphi_t}\right]^{\alpha}(1-\tau) + (1-\theta k_t)Ab}{(1-\tau\theta k_t)} \\
&= \frac{z(1-\gamma)}{\gamma} \frac{\theta k_t A(1-\alpha)\left[\frac{k_t}{(1-\tau)(1-\gamma)\theta k_t}\right]^{\alpha}(1-\tau) + (1-\theta k_t)Ab}{(1-\tau\theta k_t)} \\
&= Az\frac{(1-\gamma)}{\gamma} \frac{(1-\alpha)\theta^{1-\alpha}(1-\tau)^{1-\alpha}(1-\gamma)^{-\alpha} + (1-\theta k_t)b}{(1-\tau\theta k_t)}
\end{aligned} \tag{B.6}$$

For the scenario that $k_t \geq \bar{k}$, the fraction of skilled workers φ_t is 1. Therefore, the expression for k_{t+1} reduces to:

$$k_{t+1} = \frac{s_t^s}{n_t^s}$$

Inserting the formulas for savings (3.14), children (3.13) and wages (3.4) of skilled workers, results in:

$$\begin{aligned}
k_{t+1} &= \frac{w_t^s(1-\tau)(1-\gamma)}{\frac{\gamma(1-\tau)}{z}} \\
&= \frac{A(1-\alpha)\left[\frac{k_t}{(1-\tau-zn_t^s)}\right]^{\alpha}(1-\tau)(1-\gamma)}{\frac{\gamma(1-\tau)}{z}} \\
&= \frac{A(1-\alpha)\left[\frac{k_t}{(1-\tau)(1-\gamma)}\right]^{\alpha}(1-\tau)(1-\gamma)}{\frac{\gamma(1-\tau)}{z}} \\
&= Az\frac{(1-\gamma)}{\gamma} \frac{(1-\alpha)k_t^{\alpha}}{(1-\tau)^{\alpha}(1-\gamma)^{\alpha}}
\end{aligned} \tag{B.7}$$

B.5 Derivatives of $\phi(k)$

With definition (3.18), $\theta^{1-\alpha}$ can be expressed as:

$$\theta^{1-\alpha} = \frac{(1-\tau)^{(1-\alpha)\left[\frac{1}{\alpha(1-\gamma)}-1\right]}}{(1-\gamma)^{1-\alpha}} \left(\frac{1-\alpha}{b}\right)^{\frac{1-\alpha}{\alpha}} \tag{B.8}$$

For a level of capital per capita stock $k < \bar{k}$, equation (B.8) inserted into formula (3.28), leads to:

$$\phi(k) = Az\frac{1-\gamma}{\gamma}\frac{b}{1-\tau\theta k} + Az\frac{1-\gamma}{\gamma}\frac{k}{1-\tau\theta k}\left[\frac{(1-\alpha)^{\frac{1}{\alpha}}(1-\tau)^{\frac{1-\alpha}{\alpha(1-\gamma)}}}{(1-\gamma)b^{\frac{1-\alpha}{\alpha}}} - \theta b\right]$$

Therefore, the first derivative of $\phi(k)$ for $k < \bar{k}$ is given by:

$$\phi'(k) = Az\frac{1-\gamma}{\gamma}\frac{b\tau\theta}{(1-\tau\theta k)^2} + Az\frac{1-\gamma}{\gamma}\frac{(1-\tau\theta k)+\tau\theta k}{(1-\tau\theta k)^2}\left[\frac{(1-\alpha)^{\frac{1}{\alpha}}(1-\tau)^{\frac{1-\alpha}{\alpha(1-\gamma)}}}{(1-\gamma)b^{\frac{1-\alpha}{\alpha}}} - \theta b\right]$$

Factorizing and afterwards inserting equation (3.18) for θ into the term $(1-\gamma)b^{1+\frac{1-\alpha}{\alpha}}\theta$ leads to the final result:

$$\begin{aligned}
\phi'(k) &= Az\frac{1-\gamma}{\gamma}\frac{\theta b}{(1-\tau\theta k)^2}\left[\frac{(1-\alpha)^{\frac{1}{\alpha}}(1-\tau)^{\frac{1-\alpha}{\alpha(1-\gamma)}}}{(1-\gamma)b^{1+\frac{1-\alpha}{\alpha}}\theta} - (1-\tau)\right] \\
&= Az\frac{1-\gamma}{\gamma}\frac{\theta b}{(1-\tau\theta k)^2}\left[\frac{(1-\alpha)^{\frac{1}{\alpha}}(1-\tau)^{\frac{1-\alpha}{\alpha(1-\gamma)}}}{(1-\gamma)b^{1+\frac{1-\alpha}{\alpha}}\frac{(1-\alpha)^{\frac{1}{\alpha}}}{b^{\frac{1}{\alpha}}}\frac{(1-\tau)^{\frac{1}{\alpha(1-\gamma)}-1}}{1-\gamma}} - (1-\tau)\right] \\
&= Az\frac{1-\gamma}{\gamma}\frac{\theta b}{(1-\tau\theta k)^2}\left[(1-\tau)^{\frac{-\gamma}{1-\gamma}} - (1-\tau)\right]
\end{aligned} \tag{B.9}$$

As $(1-\tau) < 1$ and therefore, $(1-\tau)^{\frac{-\gamma}{(1-\gamma)}} > 1$, the factor enclosed in square brackets is positive and $\phi'(k) > 0$ for $k < \bar{k}$. The first derivative for $k \geq \bar{k}$, as well as the second derivative of $\phi(k)$ and their corresponding signs follow directly from the terms in equations (3.28) (for $k \geq \bar{k}$) and (B.9).

B.6 Equilibria

For the case that $k < \bar{k}$, with (B.6), k_{t+1} is given by:

$$k_{t+1} = Az\frac{1-\gamma}{\gamma}\frac{1}{1-\theta\tau k_t}\left[\frac{(1-\alpha)(1-\tau)^{1-\alpha}\theta^{1-\alpha}k_t}{(1-\gamma)^{\alpha}} + (1-\theta k_t)b\right]$$

Equilibrium levels of capital per capita stock k^* have to fulfill $k_{t+1} = k_t = k^*$:

$$\begin{aligned} k - \theta\tau k^2 &= Az\frac{1-\gamma}{\gamma}\left[\frac{(1-\alpha)(1-\tau)^{1-\alpha}\theta^{1-\alpha}k_t}{(1-\gamma)^{\alpha}} + (1-\theta k_t)b\right] \\ \Leftrightarrow k^2 &+ \left(Az\frac{1-\gamma}{\gamma}\left[\frac{(1-\alpha)(1-\tau)^{1-\alpha}\theta^{1-\alpha}}{(1-\gamma)^{\alpha}} - \theta b\right] - 1\right)k + Az\frac{1-\gamma}{\gamma}b = 0 \end{aligned} \tag{B.10}$$

Now, the two possible equilibria k_l^* and k_m^* for the case $k < \bar{k}$, can be calculated by solving the quadratic equation (B.10).

For the case that $k \geq \bar{k}$, with (B.7), k_{t+1} is given by:

$$k_{t+1} = Az\frac{1-\gamma}{\gamma}\frac{(1-\alpha)k_t^{\alpha}}{(1-\tau)^{\alpha}(1-\gamma)^{\alpha}}$$

Again, a possible steady state k^* has to fulfill $k_{t+1} = k_t$:

$$\begin{aligned} k^{1-\alpha} &= Az\frac{1-\gamma}{\gamma}\frac{(1-\alpha)}{(1-\tau)^{\alpha}(1-\gamma)^{\alpha}} \\ k_h^* &= \left[Az\frac{1-\gamma}{\gamma}\frac{(1-\alpha)}{(1-\tau)^{\alpha}(1-\gamma)^{\alpha}}\right]^{\frac{1}{1-\alpha}} \end{aligned}$$

C Mathematical Appendix of chapter 4 and discussion of parameter set $P2$

C.1 Optimal choice of time and goods in child rearing

To solve the optimization problem (4.10), I define the Lagrangian function $L(x,z,\lambda)$ and find the arguments that fulfill the first order conditions. Note that during the calculation, the sub– and superscripts of the variables are omitted, as they are not required:

$$L(x,z,\lambda) = wz + xp_x(1-\beta) + \lambda(n - z^a x^{1-a})$$

The corresponding first order conditions are given by:

$$\frac{\partial L}{\partial z} = w - \lambda a \left(\frac{x}{z}\right)^{1-a} \stackrel{!}{=} 0 \tag{C.1}$$

$$\frac{\partial L}{\partial x} = p_x(1-\beta) - \lambda(1-a)\left(\frac{z}{x}\right)^{a} \stackrel{!}{=} 0 \tag{C.2}$$

Using (C.1) and (C.2), λ equates to the following expressions:

$$\lambda = \frac{w}{a}\left(\frac{z}{x}\right)^{1-a} \tag{C.3}$$

$$\lambda = \frac{p_x(1-\beta)}{1-a}\left(\frac{x}{z}\right)^{a} \tag{C.4}$$

Equating (C.3) and (C.4), results in:

$$\frac{w}{a}\left(\frac{z}{x}\right)^{1-a} = \frac{p_x(1-\beta)}{1-a}\left(\frac{x}{z}\right)^{a}$$

$$\Longleftrightarrow \quad \frac{w}{a}\frac{z}{x} = \frac{p_x(1-\beta)}{1-a}$$

Now, (4.11) is inserted for z, to obtain the following optimal number of goods and services units:

$$\frac{w}{a}\frac{n^{\frac{1}{a}}}{x^{\frac{1-a}{a}} \cdot x} = \frac{p(1-\beta)}{1-a}$$

$$\Longleftrightarrow \quad x = \left[\left(\frac{w}{a}\right)\left(\frac{1-a}{p(1-\beta)}\right)\right]^{a} n$$

In the more elaborate form, where sub- and superscripts are used, one obtains:

$$x_t^i = \left(\frac{w_t^i(1-a)}{p_x(1-\beta)a}\right)^a n_t^i =: \hat{x}_t^i(w_t^i,\beta)n_t^i \tag{C.5}$$

To obtain the optimal share of time in child rearing, x_t^i is inserted into equation (4.11):

$$z_t^i = \frac{n_t^{\frac{1}{a}}}{\left(\frac{w_t^i(1-a)}{p_x(1-\beta)a}\right)^{1-a} n_t^{\frac{1-a}{a}}}$$

$$\iff \quad z_t^i = \left(\frac{p_x(1-\beta)a}{w_t^i(1-a)}\right)^{1-a} n_t^i =: \hat{z}_t^i(w_t^i,\beta)n_t^i \tag{C.6}$$

Those arguments are indeed those of a global minimum. The reason is differentiability and convexity of the Lagrangian function $L(x,z,\lambda^*)$, which is sufficient for a global minimum.

Finally, to obtain the overall price per child, the results for x_t^i (C.5) and z_t^i (C.6) are inserted into the objective function (4.10) and then the overall price for children per agent is divided by n_t^i:

$$\frac{C_t^i}{n_t^i} =: p = (w_t^i)^{1-1+a}\left(\frac{p_x(1-\beta)a}{1-a}\right)^{1-a} + (p_x(1-\beta))^{1-a}(w_t^i)^a\left(\frac{1-a}{a}\right)^a =$$
$$= (w_t^i)^a(p_x(1-\beta))^{1-a}\cdot\left[\frac{a^{1-a}}{(1-a)^{1-a}} + \left(\frac{1-a}{a}\right)^a\right]$$

The last factor can be simplified and is defined as the auxiliary variable B:

$$B := \frac{a^{1-a+a}+(1-a)^{a+1-a}}{(1-a)^{1-a}a^a} = \frac{a+1-a}{(1-a)^{1-a}a^a}$$
$$= a^{-a}(1-a)^{-(1-a)}$$

Hence, the price per child for an working-age individual can be expressed as:

$$p(w_t^i,p_x,\beta) = (w_t^i)^a(p_x(1-\beta))^{1-a}\cdot B \tag{C.7}$$

It directly follows that $\hat{z}_t^i = \frac{\partial p(\cdot)}{\partial w_t^i}$ and that $\hat{x}_t^i = \frac{\partial p(\cdot)}{\partial p_x(1-\beta)}$.

C.2 Optimization of an agent's utility function

The overall price per child (4.14) is used to reformulate the utility maximization problem of skilled agents (4.19):

$$\begin{aligned} \text{argmax}_{c_t^s, s_t^s, n_t^s} \quad & \gamma \ln(c_t) + \lambda \ln(n_t) + \delta \ln\left[s_t(1+r_{t+1})(1-T_{t+1})\right] \\ s.t. \quad & (1-\tau)w_t^s = s_t^s + \left[w^a\left(p_x(1-\beta)\right)^{1-a} B\right] n_t^s + c_t^s \end{aligned} \tag{C.8}$$

I apply the Lagrangian method. Sub– and superscripts of the variables are omitted during the calculations. The corresponding Lagrangian function $L(c,s,n,\Lambda)$ is given as:

$$\begin{aligned} L(c,s,n,\Lambda) = {} & \gamma \ln(c) + \lambda \ln(n) + \delta \ln(s(1+r)(1-T)) \\ & + \Lambda\left((1-\tau)w - (s + \left[w^a\left(p_x(1-\beta)\right)^{1-a} B\right] n + c)\right) \end{aligned}$$

The first order conditions are obtained as:

$$\frac{\partial L}{\partial c} = \frac{\lambda}{c} - \Lambda \stackrel{!}{=} 0 \quad \Leftrightarrow \Lambda = \frac{\lambda}{c} \tag{C.9}$$

$$\frac{\partial L}{\partial n} = \frac{\gamma}{n} - \Lambda w^a\left(p_x(1-\beta)\right)^{1-a} B \stackrel{!}{=} 0 \tag{C.10}$$

$$\frac{\partial L}{\partial s} = \frac{\delta}{s} - \Lambda \stackrel{!}{=} 0 \tag{C.11}$$

Equating (C.9) and (C.11), leads to:

$$\frac{\delta}{s} = \frac{\lambda}{c} \Leftrightarrow c = \frac{\lambda}{\delta} s \tag{C.12}$$

Similarly, equating (C.10) and (C.11), leads to:

$$\begin{aligned} \frac{\delta}{s} &= \frac{\gamma}{n w^a\left(p_x(1-\beta)\right)^{1-a} B} \\ s &= \frac{\delta}{\gamma} n w^a\left(p_x(1-\beta)\right)^{1-a} B \end{aligned} \tag{C.13}$$

Now, the formulas for c and s, (C.12) and (C.13), respectively, are inserted into the budget constraint (C.8):

$$(1-\tau)w = s + \left[w^a\left(p_x(1-\beta)\right)^{1-a}B\right]n + c$$
$$(1-\tau)w = \left(\frac{\delta}{\gamma} + \frac{\lambda}{\delta}\frac{\delta}{\gamma} + 1\right)\left[w^a\left(p_x(1-\beta)\right)^{1-a}B\right]n$$

The use of the equation $\lambda + \gamma + \delta = 1$ leads to the optimal solution for n:

$$(1-\tau)w = \frac{1}{\gamma}\left[w^a\left(p_x(1-\beta)\right)^{1-a}B\right]n$$
$$n = \frac{(1-\tau)\gamma}{B}\left[\frac{w}{p_x(1-\beta)}\right]^{1-a}$$

As the next step, the result for n is inserted in (C.13) to obtain the optimal savings rate for old age:

$$s = \frac{\delta}{\gamma}\frac{(1-\tau)\gamma}{B}\left[\frac{w}{p_x(1-\beta)}\right]^{1-a}w^a\left(p_x(1-\beta)\right)^{1-a}B$$
$$\Leftrightarrow s = \delta(1-\tau)w = (1-\gamma-\lambda)(1-\tau)w$$

Optimal consumption in working age is derived by using (C.12):

$$\Leftrightarrow c = \frac{\lambda}{\delta}\delta(1-\tau)w = \lambda(1-\tau)w$$

The constraint of the problem is linear in n, c and s and the objective function is a linear combination of logarithmic functions, therefore concave. Hence $L(c,s,n,\Lambda)$ is concave as well and the obtained arguments are those of a global maximum.
The solutions for the unskilled agents directly follow from the problem above, as one deals with the same terms and factors, except for the omission of the time spent for becoming skilled τ, and the different superscript u instead of s.

C.3 Derivation of the No–arbitrage condition

Firstly, note that the utility function (4.15) can be rewritten as: $\ln\left[(n_t^i)^\gamma \cdot (c_t^i)^\lambda \cdot (d_{t+1}^i)^\delta\right]$
As the utility functions of skilled and unskilled workers have the same value if and only if the

arguments of the logarithms have the same value, it is sufficient to equate those arguments:

$$(n_t^s)^\gamma \cdot (c_t^s)^\lambda \cdot (d_{t+1}^s)^\delta \stackrel{!}{=} (n_t^u)^\gamma \cdot (c_t^u)^\lambda \cdot (d_{t+1}^u)^\delta$$

Inserting the optimal solutions of c_t^s, n_t^s and d_{t+1}^s from the equations (4.22), (4.21) and (4.23) and the corresponding solutions of unskilled agents leads to:

$$\begin{aligned}&\left(\frac{(1-\tau)\gamma}{B}\right)^\gamma \left[\frac{w_t^s}{p_x(1-\beta)}\right]^{\gamma-a\gamma} \cdot (1-\tau)^\lambda \lambda^\lambda (w_t^s)^\lambda \cdot (1-\tau)^\delta (1-\gamma-\lambda)^\delta (w_t^s)^\delta \left[(1+r_{t+1})(1-T_{t+1})\right]^\delta \\ &= \left(\frac{\gamma}{B}\right)^\gamma \left[\frac{w_t^u}{p_x(1-\beta)}\right]^{\gamma-a\gamma} \cdot \lambda^\lambda (w_t^u)^\lambda \cdot (1-\gamma-\lambda)^\delta (w_t^u)^\delta \left[(1+r_{t+1})(1-T_{t+1})\right]^\delta\end{aligned}$$

Now, the equation is reduced by the same factors on both sides:

$$(1-\tau)^{\gamma+\lambda+\delta} \cdot (w_t^s)^{\gamma-a\gamma+\lambda+\delta} = (w_t^u)^{\gamma-a\gamma+\lambda+\delta}$$

Finally, by using $\gamma+\lambda+\delta = 1$, the no–arbitrage condition can be formulated as:

$$(1-\tau) = \left(\frac{w_t^u}{w_t^s}\right)^{1-a\gamma}$$

C.4 Derivation of a formula for the fraction of skilled workers

The following auxiliary calculation is required for the final results. By inserting (4.21) and (C.6) for n_t^s and $\hat{z}$, using $B = a^{(-a)}(1-a)^{-(1-a)}$ and reducing the term, I find that:

$$\begin{aligned}(1-\tau-\hat{z}n_t^s) &= 1-\tau-\left(\frac{p_x(1-\beta)a}{w_t^s(1-a)}\right)^{1-a} \frac{(1-\tau)\gamma}{B}\left(\frac{w_t^s}{p_x(1-\beta)}\right)^{1-a} \\ &= (1-\tau)-(1-\tau)a\gamma = (1-a\gamma)(1-\tau) \qquad \text{(C.14)}\end{aligned}$$

To express the fraction of skilled workers as a function of capital per capita, at first, I recall the no–arbitrage condition (4.27):

$$(1-\tau)^{\frac{1}{1-a\gamma}} = \frac{w_t^u}{w_t^s}$$

The wage formulas (4.4) and (4.5) are applied. Afterwards, recalling that the labor supply of skilled individuals is given by $L_t^s = (1-\tau-\hat{z}n_t^s)\varphi_t N_t$, and hence $h_t = (\frac{L_t^s}{\varphi_t N_t}) = (1-\tau-\hat{z}n_t^s)$, leads to:

$$(1-\tau)^{\frac{1}{1-a\gamma}} = \frac{Ab}{A(1-\alpha)\left(\frac{k_t}{\varphi_t(1-\tau-\hat{z}n_t^s)}\right)^{\alpha}}$$

Raising the equation to the power of $\frac{1}{\alpha}$ and rearranging yields:

$$(1-\tau)^{\frac{1}{\alpha(1-a\gamma)}} = \frac{\varphi_t(1-\tau-\hat{z}n_t^s)}{k_t}\left(\frac{b}{1-\alpha}\right)^{\frac{1}{\alpha}}$$

$$\Leftrightarrow \varphi_t = (1-\tau)^{\frac{1}{\alpha(1-a\gamma)}}\left(\frac{1-\alpha}{b}\right)^{\frac{1}{\alpha}} k_t \Big/ (1-\tau-\hat{z}n_t^s)$$

Finally, by using (C.14) to replace the factor $(1-\tau-\hat{z}n_t^s)$, the function for the fraction of skilled workers can be expressed as:

$$\varphi(k_t) := \varphi_t = \frac{(1-\tau)^{\frac{1}{\alpha(1-a\gamma)}-1}}{(1-a\gamma)}\left(\frac{1-\alpha}{b}\right)^{\frac{1}{\alpha}} k_t$$

C.5 Derivation of the equation of motion

At first, an auxiliary calculation is applied again to express the wage of skilled workers in a different form.
With (4.4), the wage of skilled workers can be written as:

$$w_t^s = A(1-\alpha)\left(\frac{k_t}{h_t\varphi_t}\right)^{\alpha}$$

Inserting for h_t, it follows that:

$$w_t^s = A(1-\alpha)\left(\frac{k_t}{L_t^s/N_t}\right)^{\alpha}$$

Inserting for L_t^s with (4.1) leads to:

$$w_t^s = A(1-\alpha)\left(\frac{k_t}{(1-\tau-\hat{z}n_t^s)\varphi_t}\right)^{\alpha}$$

Finally, with (C.14), the new expression of the wage is given by:

$$w_t^s = A(1-\alpha)\left(\frac{k_t}{(1-a\gamma)(1-\tau)}\right)^{\alpha} \tag{C.15}$$

This new form of the wage will be applied in the following calculation steps. To express k_{t+1} as a function of k_t, I recall equation (4.36):

$$k_{t+1} = \frac{\varphi_t s_t^s + (1-\varphi_t)s_t^u}{\varphi_t n_t^s + (1-\varphi_t)n_t^u}$$

As the next step, the optimal choices for savings and children (formulas (4.23), (4.26), (4.21) and (4.24)) are inserted:

$$k_{t+1} = \frac{\varphi_t(1-\gamma-\lambda)(1-\tau)w_t^s + (1-\varphi_t)(1-\gamma-\lambda)w_t^u}{\varphi_t\frac{(1-\tau)\gamma}{B}\left[\frac{w_t^s}{p_x(1-\beta)}\right]^{1-a} + (1-\varphi_t)\frac{\gamma}{B}\left[\frac{w_t^u}{p_x(1-\beta)}\right]^{1-a}} \tag{C.16}$$

Two cases are considered separately.

Equation of motion for $k_t \geq \bar{k}$

When k_t is greater or equal to $\bar{k}$, the fraction of skilled workers φ_t is 1. Hence, equation (C.16) reduces to:

$$k_{t+1} = \frac{B(1-\gamma-\lambda)}{\gamma}(w_t^s)^a\left[p_x(1-\beta)\right]^{1-a}$$

Using (C.15), leads to:

$$\begin{aligned}
k_{t+1} &= \frac{B(1-\gamma-\lambda)}{\gamma}A^a(1-\alpha)^a\frac{k_t^{a\alpha}}{[(1-\tau)(1-a\gamma)]^{a\alpha}}\left[p_x(1-\beta)\right]^{1-a} \\
&= A^a\tilde{B}\frac{(1-\gamma-\lambda)}{\gamma}\left[\frac{(1-\alpha)}{(1-\tau)^{\alpha}(1-a\gamma)^{\alpha}}\right]^a k_t^{a\alpha}
\end{aligned}$$

$\tilde{B}$ is an auxiliary variable for $B\left[p_x(1-\beta)\right]^{1-a}$.

Equation of motion for $k_t < \bar{k}$

For low levels of capital per capita, the equation of motion (C.16) develops into the following form using (C.15) for the wage w_t^s and (4.5) for w_t^u:

$$\phi(k_t) = A^a B[p_x(1-\beta)]^{(1-a)} \frac{(1-\gamma-\lambda)}{\gamma} \left[\frac{(1-\tau)\theta k_t(1-\alpha)[(1-a\gamma)(1-\tau)\theta]^{-\alpha} + (1-\theta k_t)b}{\theta k_t(1-\tau)(1-\alpha)^{1-a}[(1-a\gamma)(1-\tau)\theta]^{-\alpha(1-a)} + (1-\theta k_t)b^{1-a}} \right]$$

$$= A^a \tilde{B} \frac{(1-\gamma-\lambda)}{\gamma} \left[\frac{b + \left\{ \frac{\theta^{1-\alpha}(1-\tau)^{1-\alpha}(1-\alpha)}{(1-a\gamma)^{\alpha}} - b\theta \right\} k_t}{b^{1-a} + \left\{ \frac{\theta^{1-\alpha(1-a)}(1-\tau)^{1-\alpha(1-a)}(1-\alpha)^{1-a}}{(1-a\gamma)^{\alpha(1-a)}} - b^{1-a}\theta \right\} k_t} \right]$$

Inserting for $\theta^{-\alpha}$ from the formula for the fraction of skilled workers (4.33) equates to:

$$= A^a \tilde{B} \frac{(1-\gamma-\lambda)}{\gamma} \frac{b + \left[\frac{(1-\tau)^{-\alpha(\frac{1}{\alpha(1-a\gamma)}-1)}}{(1-a\gamma)^{-\alpha}} \frac{b\theta(1-\tau)^{1-\alpha}(1-\alpha)}{(1-\alpha)(1-a\gamma)^{\alpha}} - b\theta \right] k_t}{b^{1-a} + \left[\frac{(1-\tau)^{-\alpha(1-a)(\frac{1}{\alpha(1-a\gamma)}-1)}}{(1-a\gamma)^{-\alpha(1-a)}} \left[\frac{b}{1-\alpha}\right]^{1-a} - \frac{(1-\tau)^{1-\alpha(1-a)}(1-\alpha)^{1-a}}{(1-a\gamma)^{\alpha(1-a)}} b^{1-a}\theta \right] k_t}$$

$$= A^a \tilde{B} \frac{(1-\gamma-\lambda)}{\gamma} b^a \frac{1 + \theta \left[(1-\tau)^{-\alpha(\frac{1}{\alpha(1-a\gamma)}-1)+1-\alpha} - 1 \right] k_t}{1 + \theta \left[(1-\tau)^{-\alpha(1-a)(\frac{1}{\alpha(1-a\gamma)}-1)+1-\alpha(1-a)} \right] k_t}$$

$$= A^a \tilde{B} \frac{(1-\gamma-\lambda)}{\gamma} b^a \frac{1 + \theta \left[(1-\tau)^{\frac{-a\gamma}{1-a\gamma}} - 1 \right] k_t}{1 + \theta \left[(1-\tau)^{\frac{a(1-\gamma)}{1-a\gamma}} \right] k_t}$$

C.6 Derivation of the fertility function $f(k_t)$

The average fertility in a country at time t, $f(k_t)$, can be expressed as:

$$f(k_t) = \varphi_t n_t^s + (1-\varphi_t) n_t^u$$

Inserting the optimal choices of children (4.21) and (4.24), equates to:

$$= \varphi_t \frac{(1-\tau)\gamma}{B} \left[\frac{w_t^s}{p_x(1-\beta)} \right]^{1-a} + (1-\varphi_t) \frac{\gamma}{B} \left[\frac{w_t^u}{p_x(1-\beta)} \right]^{1-a}$$

Two cases of capital per capita levels are treated separately.

Fertility function for $k_t \geq \bar{k}$

Here the fraction of skilled workers φ_t is 1. Therefore, the second summand from above disappears and using equation (4.38) for the wage w_t^s, it can be obtained that:

$$
\begin{aligned}
f(k_t) &= \frac{\gamma(1-\tau)}{B}\left[\frac{w_t^s}{p_x(1-\beta)}\right]^{1-a} \\
&= \frac{\gamma(1-\tau)}{B}\left[\frac{A(1-\alpha)k_t^{\alpha}}{(1-a\gamma)^{\alpha}(1-\tau)^{\alpha}p_x(1-\beta)}\right]^{1-a} \\
&= \frac{\gamma(1-\tau)^{1-\alpha(1-a)}}{B}\left[\frac{A(1-\alpha)}{(1-a\gamma)^{\alpha}p_x(1-\beta)}\right]^{1-a} k_t^{\alpha(1-a)}
\end{aligned}
$$

Fertility function for $k_t < \bar{k}$

Similarly, I fill in for the wages, for the case, where agents hold a low level of capital per capita:

$$
\begin{aligned}
f(k_t) &= \theta k_t \frac{(1-\tau)\gamma}{B}\left[\frac{w_t^s}{p_x(1-\beta)}\right]^{1-a} + (1-\theta k_t)\frac{\gamma}{B}\left[\frac{w_t^u}{p_x(1-\beta)}\right]^{1-a} \\
&= \theta k_t \frac{(1-\tau)\gamma}{B}\left[\frac{A(1-\alpha)[(1-a\gamma)(1-\tau)\theta]^{-\alpha}}{p_x(1-\beta)}\right]^{1-a} + (1-\theta k_t)\frac{\gamma}{B}\left[\frac{Ab}{p_x(1-\beta)}\right]^{1-a} \\
&= \frac{\gamma}{B}\left[\frac{A}{p_x(1-\beta)}\right]^{1-a}\left[(1-\tau)\theta k_t(1-\alpha)^{1-a}(1-a\gamma)^{-\alpha(1-a)}(1-\tau)^{-\alpha(1-a)}\theta^{-\alpha(1-a)} + (1-\theta k_t)b^{1-a}\right]
\end{aligned}
$$

$\theta^{-\alpha(1-a)}$ can be substituted using equation (4.33), which leads to:

$$
\begin{aligned}
f(k_t) &= \frac{\gamma}{B}\left[\frac{A}{p_x(1-\beta)}\right]^{1-a} \cdot \\
&\left[(1-\tau)^{1-\alpha(1-a)}\theta k_t(1-\alpha)^{1-a}(1-a\gamma)^{-\alpha(1-a)}\frac{(1-\tau)^{-\alpha(1-a)\left[\frac{1}{\alpha(1-a\gamma)}-1\right]}}{(1-a\gamma)^{-\alpha(1-a)}}\frac{(1-\alpha)^{\frac{-\alpha(1-a)}{\alpha}}}{b^{\frac{-\alpha(1-a)}{\alpha}}} + (1-\theta k_t)b\right. \\
&= \frac{\gamma}{B}\left[\frac{Ab}{p_x(1-\beta)}\right]^{1-a}\left[\theta k_t(1-\tau)^{1-\alpha(1-a)-\alpha(1-a)\left[\frac{1}{\alpha(1-a\gamma)}-1\right]} + (1-\theta k_t)\right] \\
&= \frac{\gamma}{B}\left[\frac{Ab}{p_x(1-\beta)}\right]^{1-a}\left[1-\theta\left(1-(1-\tau)^{\frac{a(1-\gamma)}{1-a\gamma}}\right)k_t\right]
\end{aligned}
$$

C.7 First and second derivatives of the equation of motion and fertility function

$$\phi'(k_t) = A^a\tilde{B}\frac{1-\gamma-\lambda}{\gamma}\begin{cases} a\alpha\left[\frac{(1-\alpha)}{(1-a\gamma)^{\alpha}(1-\tau)^{\alpha}}\right]^a k_t^{a\alpha-1} > 0 & \text{if } k_t \geq \bar{k} \\ \frac{b^a\theta\left[(1-\tau)^{\frac{-a\gamma}{(1-a\gamma)}}-(1-\tau)^{\frac{a(1-\gamma)}{(1-a\gamma)}}\right]}{\left[1-\theta\left\{1-(1-\tau)^{\frac{a(1-\gamma)}{(1-a\gamma)}}\right\}k_t\right]^2} > 0 & \text{if } k_t < \bar{k} \end{cases}$$

$$\phi''(k_t) = A^a\tilde{B}\frac{1-\gamma-\lambda}{\gamma}\begin{cases} (a\alpha-1)a\alpha\left[\frac{(1-\alpha)}{(1-a\gamma)^{\alpha}(1-\tau)^{\alpha}}\right]^a k_t^{a\alpha-2} < 0 & \text{if } k_t \geq \bar{k} \\ \frac{b^a 2\theta^2\left\{1-(1-\tau)^{\frac{a(1-\gamma)}{(1-a\gamma)}}\right\}\left[(1-\tau)^{\frac{-a\gamma}{(1-a\gamma)}}-(1-\tau)^{\frac{a(1-\gamma)}{(1-a\gamma)}}\right]}{\left[1-\theta\left\{1-(1-\tau)^{\frac{a(1-\gamma)}{(1-a\gamma)}}\right\}k_t\right]^3} > 0 & \text{if } k_t < \bar{k} \end{cases}$$

$$f'(k_t) = \begin{cases} -\frac{\gamma}{B}\left[\frac{Ab}{p_x(1-\beta)}\right]^{1-a}\theta\left(1-(1-\tau)^{\frac{a(1-\gamma)}{(1-a\gamma)}}\right) < 0 & \text{if } k_t < \bar{k} \\ \alpha(1-a)\frac{f(k_t)}{k_t} > 0 & \text{if } k_t \geq \bar{k} \end{cases}$$

$$f''(k_t) = \begin{cases} 0 & \text{if } k_t < \bar{k} \\ (\alpha(1-a)-1)\frac{f'(k_t)}{k_t} < 0 & \text{if } k_t \geq \bar{k} \end{cases}$$

C.8 Equilibria

An auxiliary variable $\tilde{X}$ is defined as:

$$\tilde{X} = A^a\tilde{B}\frac{1-\gamma-\lambda}{\gamma}b^a$$

To calculate the possible equilibria k_i^* for the scenario, where $k < \bar{k}$, I have to set $k_{t+1} = k_t$ in the equation of motion (4.39).

$$
\begin{aligned}
\tilde{X}\frac{1+\theta\left\{(1-\tau)^{\frac{-a\gamma}{(1-a\gamma)}}-1\right\}k}{1-\theta\left\{1-(1-\tau)^{\frac{a(1-\gamma)}{(1-a\gamma)}}\right\}k} &\overset{!}{=} k \\
\tilde{X}\left(1+\theta\left\{(1-\tau)^{\frac{-a\gamma}{(1-a\gamma)}}-1\right\}k\right) &= k\left(1-\theta\left\{1-(1-\tau)^{\frac{a(1-\gamma)}{(1-a\gamma)}}\right\}k\right) \\
\tilde{X}+\tilde{X}\theta\left((1-\tau)^{\frac{-a\gamma}{(1-a\gamma)}}-1\right)k &= k+\theta(1-\tau)^{\frac{a(1-\gamma)}{(1-a\gamma)}}k^2-\theta k^2 \\
\theta\left(1-(1-\tau)^{\frac{a(1-\gamma)}{(1-a\gamma)}}\right)k^2+\left[\tilde{X}\theta\left((1-\tau)^{\frac{-a\gamma}{1-a\gamma}}-1\right)-1\right]k+\tilde{X} &= 0
\end{aligned}
$$

Now, the possible equilibria k_m^* and k_l^* can be calculated by solving the quadratic equation.

To find a formula for a possible equilibrium k_h^* in the scenario, where $k \geq \bar{k}$, I have to set $k_{t+1} = k_t$ in the equation of motion (4.39) again:

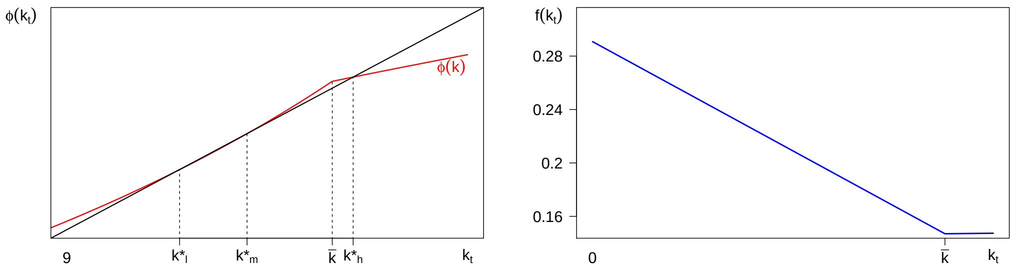
$$
\begin{aligned}
A^a\tilde{B}\frac{1-\gamma-\lambda}{\gamma}\left[\frac{1-\alpha}{(1-a\gamma)^\alpha(1-\tau)^\alpha}\right]^a k^{a\alpha} &\overset{!}{=} k \\
k^{1-a\alpha} &= A^a\tilde{B}\frac{1-\gamma-\lambda}{\gamma}\left[\frac{1-\alpha}{(1-a\gamma)^\alpha(1-\tau)^\alpha}\right]^a \\
k_h^* &= \left[A^a\tilde{B}\frac{1-\gamma-\lambda}{\gamma}\right]^{\frac{1}{1-a\alpha}}\left[\frac{1-\alpha}{(1-a\gamma)^\alpha(1-\tau)^\alpha}\right]^{\frac{a}{1-a\alpha}}
\end{aligned}
$$

C.9 Discussion of parameter set *P*2

As can be seen in Table C.1, in the parameter set defined as *P*2, I assume a very high elasticity of time (a = 0.95), a long time required to become skilled (τ = 0.52) and a medium total productivity level of 15.2. Moreover, the preference on consumption is higher than in parameter set *P*1 (λ = 0.6 > 0.3) as well as the weight factor of unskilled labor (b = 0.9 > 0.5). Moreover, it is assumed that the price of children's goods is lower than the price of the aggregate good with $p_x = 0.8$.

a	α	b	τ	A	p_x	β	γ	λ
0.95	0.4	0.9	0.52	15.2	0.8	0.4	0.3	0.6

Table C.1: Own parameter set *P*2 for the model of Day (2015)



(a) Equation of motion $\phi(k_t)$ (b) Fertility function $f(k_t)$

Figure C.1: Functions ϕ and f for the parameter set P2

With parameter set $P2$, the model comprises three equilibria. In Figure C.1, the corresponding equation of motion and fertility function are plotted. For $k_t < k_m^*$ the economy is in a poverty trap, because it converges towards the lowest equilibrium k_l^* for an arbitrary level of capital per capita lower than the unstable equilibrium k_m^*. In this equilibrium both skilled and unskilled workers exist. For $k_t > k_m^*$ the economy converges towards the highest equilibrium k_h^*. This high steady state k_h^* has similar properties as the steady state of parameter set $P1$, as it just comprises skilled workers.

It can be observed that the values of f_t are very low, namely between 0.15 and 0.28. Moreover, the slope of the fertility function is low for $k > \bar{k}$, as can be observed in Figure C.1(b). Therefore this model specification contradicts the inverse J-shape introduced in chapter 1 of the thesis and setting $P1$ remains the most realistic one concerning fertility.

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