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D I P L O M A R B E I T

On meromorphic Lévy processes and option pricing

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Zusammenfassung

Meromorphe Lévy-Prozesse bilden eine Familie von Lévy-Prozessen mit absolut monotoner Lévy-Dichte. Die Wiener-Hopf-Zerlegung eines solchen meromorphen Lévy-Prozesses sowie ihre Dichten auf $(0, \infty)$ und $(-\infty, 0)$ lassen sich explizit als Reihen darstellen. In der Finanzmathematik kann bei einem exponentiellen Lévy-Marktmodell diese Darstellung bei der Bepreisung von exotischen Optionen, die vom Maximum oder Minimum des Preisprozesses abhängig sind, verwendet werden. Insbesondere lassen sich die Laplace-Transformierten bezüglich der Restlaufzeit der arbitragefreien Preise von Barrier- und Lookback-Optionen explizit darstellen. In dieser Diplomarbeit werden die zentralen Erkenntnisse für meromorphe Lévy-Prozesse und deren Wiener-Hopf-Zerlegung sowie der Zusammenhang zur Theorie der Pick-Funktionen ausführlich im ersten Kapitel besprochen. Im zweiten Abschnitt werden Laplace-Transformierte arbitragefreier Optionspreise in einem meromorphen Lévy-Marktmodell hergeleitet und im abschließenden dritten Teil mittels numerischer Laplace-Inversion für einige Beispiele berechnet. Der zugehörige R Code findet sich ebenfalls im dritten Kapitel.

Abstract

Meromorphic Lévy processes form a class of Lévy processes with a completely monotone Lévy density. An analytic expression for the Wiener-Hopf factors of meromorphic Lévy processes and their densities on $(0, \infty)$ und $(-\infty, 0)$ can be derived with tools from complex analysis. In the field of financial mathematics these identities can be used to price exotic options with payoffs depending on the maximum or minimum stock process in an exponential Lévy market model. In particular, explicit formulas for the Laplace transforms with respect to time of arbitrage-free prices of lookback and barrier options can be derived. This thesis provides an in-depth discussion of the fundamental results on meromorphic Lévy processes and their Wiener-Hopf factors in the first part. In the second part we obtain formulas for the Laplace transforms of arbitrage-free option prices using the results on the Wiener-Hopf factorization. Subsequently we compute some explicit examples of exotic option prices by numeric Laplace inversion. The corresponding R code can be found in the final chapter.

Contents

Chapter 1

Meromorphic Lévy processes

1.1 Definition

The class of meromorphic Lévy processes has first been introduced by Kuznetsov (2010b) and later received its name by Kuznetsov *et al.* (2012). It can be regarded as a straight generalisation of double-exponential jump-diffusions (DEJD) and a hyper-exponential jump-diffusions (HEJD) introduced by Kou (2002) and Levendorskiĭ (2004), respectively. These classes of processes contain all Lévy process that have a completely monotone Lévy density of the form

$$f_\nu(x) = 1_{\{x>0\}} \sum_{n=1}^{N} a_n p_n e^{-p_n x} + 1_{\{x<0\}} \sum_{n=1}^{N} \hat{a}_n \hat{p}_n e^{\hat{p}_n x}, \quad x \in \mathbb{R}$$

with positive parameters $p_n, \hat{p}_n, a_n$ and $\hat{a}_n$ for all $n = 1, \ldots, N$. In the DEJD case $N = 1$ and in the HEJD case $N \in \mathbb{N}$. Thus we have a Lévy jump diffusion process with N summands of double exponentially distributed compound Poisson jumps. The rate parameters of the exponential distributions are given by p_n and $\hat{p}_n$ while the expectations of the Poisson distributions are given by a_n and $\hat{a}_n$.

Kuznetsov (2010b) generalized this family of Lévy processes by allowing for an infinite number of summands of double exponentially distributed compound Poisson jumps. However some restrictions on the parameters have to be imposed to make it possible to pass to the limit $N \to \infty$. In Proposition 1.1 we study the case when the rate parameters $(p_n)_{n\in\mathbb{N}}$ and $(\hat{p}_n)_{n\in\mathbb{N}}$ tend to infinity. We call a Lévy process $(L_t)_{t\in\mathbb{R}^+}$ with a Lévy density $f_\nu(x)$ as in Proposition 1.1 *meromorphic*. In Proposition 1.2 we show how the parameters $p_n, \hat{p}_n, a_n$ and $\hat{a}_n$ determine the behavior of a meromorphic Lévy process.

Proposition 1.1. *For four real-valued positive sequences $(a_n)_{n\in\mathbb{N}}$, $(\hat{a}_n)_{n\in\mathbb{N}}$, $(p_n)_{n\in\mathbb{N}}$ and $(\hat{p}_n)_{n\in\mathbb{N}}$ with $p_n \nearrow \infty$ and $\hat{p}_n \nearrow \infty$ as $n \to \infty$ let*

$$f_\nu(x) := 1_{\{x>0\}} \sum_{n\in\mathbb{N}} a_n p_n e^{-p_n x} + 1_{\{x<0\}} \sum_{n\in\mathbb{N}} \hat{a}_n \hat{p}_n e^{\hat{p}_n x}, \quad x \in \mathbb{R}. \tag{1.1}$$

Then $f_\nu(x)$ is a Lévy density iff

$$\sum_{n\in\mathbb{N}} \left(\frac{a_n}{p_n^2} + \frac{\hat{a}_n}{\hat{p}_n^2} \right) < \infty. \tag{1.2}$$

Proof. SUFFICIENCY. First we show that $f_\nu(x)$ is a density function if (1.2) holds. As non-negativity and measurability of $f_\nu(x)$ is obvious it remains to show that $f_\nu(x) < \infty$ for all $x \in \mathbb{R}$. For some $x > 0$ we have

$$f_\nu(x) = \sum_{n\in A(x)} a_n p_n e^{-p_n x} + \sum_{n\in A(x)^{\mathrm{C}}} a_n p_n e^{-p_n x}$$

with $A(x) := \{n \in \mathbb{N} : e^{-p_n x} > p_n^{-3}\}$. The first sum is finite and the second convergences due to condition (1.2). The same argument holds for $x < 0$. To show that $f_\nu(x)$ is a Lévy density we have to check whether

$$\int_{\mathbb{R}} (1 \wedge x^2) f_\nu(x) dx < \infty.$$

By the monotone convergence theorem and $(1 \wedge x^2) \leq x^2$ we find

$$\int_{\mathbb{R}^+} (1 \wedge x^2) f_\nu(x) dx \leq \int_{\mathbb{R}^+} x^2 f_\nu(x) dx = \sum_{n\in\mathbb{N}} a_n p_n \int_{\mathbb{R}^+} x^2 e^{-p_n x} dx.$$

Changing the variable of integration $x \mapsto y = p_n x$ yields in

$$\int_{\mathbb{R}^+} (1 \wedge x^2) f_\nu(x) dx \leq \sum_{n\in\mathbb{N}} \frac{a_n}{p_n^2} \int_{\mathbb{R}^+} y^2 e^{-y} dy = 2 \sum_{n\in\mathbb{N}} \frac{a_n}{\rho_n^2} < \infty.$$

Proceeding similarly for the negative half line concludes the proof.

NECESSITY. Since $f_\nu(x)$ is a Lévy density it has to satisfy

$$\int_{\{|x|\leq 1\}} x^2 f_\nu(x) dx < \infty.$$

By the monotone convergence theorem and the change of the variable of integration $x \mapsto y = p_n x$ we find

$$\int_{\{x\leq 1\}} x^2 \nu(dx) = \sum_{n\in\mathbb{N}} a_n p_n \int_{\{x\leq 1\}} x^2 e^{-p_n x} dx = \sum_{n\in\mathbb{N}} \frac{a_n}{p_n^2} \int_{\{y\leq p_n\}} y^2 e^{-y} dy.$$

As the sequence $(p_n)_{n\in\mathbb{N}}$ is monotonically increasing $\{x \leq p_1\} \subseteq \{x \leq p_n\}$ for all $n \in \mathbb{N}$. Thus

$$\infty > \int_{\{x\leq 1\}} x^2 \nu(dx) > \sum_{n\in\mathbb{N}} \frac{a_n}{p_n^2} \int_{\{x\leq p_1\}} y^2 e^{-y} dx = c \sum_{n\in\mathbb{N}} \frac{a_n}{p_n^2}$$

with $c := (p_1^2 + 2p_1 + 2)e^{-p_1} > 0$. $\square$

Proposition 1.2. *A meromorphic Lévy process is*

- *of finite activity iff*
$$\sum_{n\in\mathbb{N}} (a_n + \hat{a}_n) < \infty,$$
- *of finite variation iff*
$$\sigma = 0, \quad \sum_{n\in\mathbb{N}} \left(\frac{a_n}{p_n} + \frac{\hat{a}_n}{p_n}\right) < \infty.$$

Proof. A Lévy process is of finite activity if the Lévy measure ν satisfies $\nu(\mathbb{R}) < \infty$. By Fubini's Theorem (all terms are non-negative) we find

$$\nu(\mathbb{R}) = \int_{\mathbb{R}} f_\nu(x)dx = \sum_{n\in\mathbb{N}} a_n p_n \int_{\mathbb{R}+} e^{-p_n x}dx + \sum_{n\in\mathbb{N}} \hat{a}_n \hat{p}_n \int_{\mathbb{R}-} e^{\hat{p}_n x}dx = \sum_{n\in\mathbb{N}} (a_n + \hat{a}_n).$$

Second, a Lévy process is of finite variation if $\sigma = 0$ and the Lévy measure ν satisfies $\int_{[-1,1]} |x|\nu(dx) < \infty$. By Fubini's theorem and the change of the variable of integration $x \mapsto y = p_n x$ we find the identity for the positive half line:

$$\int_{\{0<x\le 1\}} x\nu(dx) = \sum_{n\in\mathbb{N}} a_n p_n \int_{\{x\le 1\}} xe^{-p_n x}dx = \sum_{n\in\mathbb{N}} \frac{a_n}{p_n} \int_{\{y\le p_n\}} ye^{-y}dy.$$

This yields in the double inequality

$$c_1 \sum_{n\in\mathbb{N}} \frac{a_n}{p_n} < \int_{\{0<x\le 1\}} x\nu(dx) < c_2 \sum_{n\in\mathbb{N}} \frac{a_n}{p_n}$$

with the values c_1 and c_2 given by

$$c_1 = \int_{\{y\le p_1\}} ye^{-y}dy = 1-(p_1+1)e^{-p1} \quad c_2 = \int_{\mathbb{R}+} ye^{-y}dy = 1$$

since the sequence $(p_n)_{n\in\mathbb{N}}$ is monotonically increasing. Proceeding similarly for the negative half line concludes the proof. $\square$

1.2 Laplace exponent

The exponential decay of the Lévy density of meromorphic Lévy processes allows us to work with the Laplace exponent $\varphi_L(z)$. It is defined via the equation

$$\mathbb{E}\left(e^{zL_t}\right) = e^{t\varphi_L(z)} \quad \text{for} \quad z \in \mathbb{C} : \mathbb{E}\left(\left|e^{zL_t}\right|\right) < \infty.$$

For heavy-tailed Lévy processes this is satisfied if $\mathrm{Re}(z) = 0$ only. In contrast, for meromorphic Lévy processes $\varphi_L(z)$ is analytic on a vertical strip in $\mathbb{C}$ and can be continued meromorphically to the whole complex plane. We will denote the meromorphic continuation by $\varphi_L(z)$ too. This result has first been stated in Kuznetsov (2010b), Proposition 1. We give a proof in all details in Theorem 1.1. Moreover we derive a Taylor series expansion for the Laplace exponent at 0 in Theorem 1.2 revealing the cumulants of a meromorphic Lévy process.

Theorem 1.1. *The Laplace exponent $\varphi_L(z)$ of a meromorphic Lévy process is an analytic function on the vertical strip $C := \{z \in \mathbb{C} : -\hat{p}_1 < \mathrm{Re}(z) < p_1\}$. It has representation*

$$\varphi_L(z) = \frac{\sigma^2 z^2}{2} + \mu z + z^2 \sum_{n\in\mathbb{N}} \frac{a_n}{p_n(p_n - z)} + z^2 \sum_{n\in\mathbb{N}} \frac{\hat{a}_n}{\hat{p}_n(\hat{p}_n + z)}, \quad z \in C \tag{1.3}$$

with $\mu \geq 0, \sigma \geq 0$ and the right-hand side being meromorphic in $\mathbb{C}$.

Proof. The proof is organized in three steps: In the first step we show that

$$f(z) := \frac{\sigma^2 z^2}{2} + \mu z + z^2 \sum_{n\in\mathbb{N}} \frac{a_n}{p_n(p_n - z)} + z^2 \sum_{n\in\mathbb{N}} \frac{\hat{a}_n}{\hat{p}_n(\hat{p}_n + z)}$$

is meromorphic in $\mathbb{C}$ with poles $(p_n)_{n\in\mathbb{N}}$ and $(-\hat{p}_n)_{n\in\mathbb{N}}$, where we will denote the set of poles with $\mathcal{P}$. In the second step we proof that $\varphi_L(z)$ exists and is finite for $z \in C$ with representation

$$\varphi_L(z) = \frac{\sigma^2 z^2}{2} + \mu z + \int_{\mathbb{R}} (e^{zx} - 1 - zx)\, f_\nu(x) dx, \quad z \in C$$

for some $\mu \geq 0, \sigma \geq 0$. Finally we show that $\varphi_L(z) = f(z)$ for $z \in C$.

1ST STEP. For an arbitrary fixed compact set K in $\mathbb{C}\backslash\mathcal{P}$ we find $R \in \mathbb{R}^+$ such that $|z| < R$ for all $z \in K$ as K is bounded. So we may consider

$$\sum_{n\in\mathbb{N}} \frac{a_n}{p_n(p_n - z)} = \sum_{\substack{n\in\mathbb{N} \\ |p_n|\leq 2R}} \frac{a_n}{p_n(p_n - z)} + \sum_{\substack{n\in\mathbb{N} \\ |p_n|> 2R}} \frac{a_n}{p_n(p_n - z)}.$$

Since $p_n \nearrow \infty$ the first sum is bounded on K. Moreover for $|z| < R$ and $p_n > 2R$ we have the inequality

$$|z| < R < \frac{p_n}{2} = p_n - \frac{p_n}{2}.$$

Rearranging the inequality and using the reverse triangle inequality yields in

$$\frac{p_n}{2} < p_n - |z| = |p_n - |z|| \leq |p_n - z|.$$

So we have

$$\left|\frac{a_n}{p_n(p_n - z)}\right| = \frac{a_n}{p_n|p_n - z|} < 2\frac{a_n}{p_n^2} =: M_n.$$

Since $M_n \in \ell_1(\mathbb{N})$ by condition (1.2) the second sum convergences uniformly on K by the Weierstrass M-test. The same argument holds for the second series of $f(z)$.

2ND STEP. According to the exponential moments theorem for Lévy processes (see Sato (1999), Theorem 25.17) it is sufficient to show

$$\int_{\{|x|>1\}} e^{cx} f_\nu(x)dx < \infty, \quad \text{for} \quad c \in (-\hat{p}_1, p_1).$$

By the monotone convergence theorem we obtain

$$\int_{\{x>1\}} e^{cx} f_\nu(x)dx = \sum_{n\in\mathbb{N}} a_n p_n \int_{\{x>1\}} e^{-(p_n-c)x}dx.$$

For $c \in (-\hat{p}_1, p_1)$ we have $\epsilon_n := p_n - c \geq 0$. Thus

$$\int_{\{x>1\}} e^{cx} f_\nu(x)dx = \sum_{n\in\mathbb{N}} a_n p_n \int_{\{x>1\}} e^{-\epsilon_n x}dx = \sum_{n\in\mathbb{N}} \frac{a_n p_n}{\epsilon_n} e^{-\epsilon_n}.$$

As $\epsilon_n \nearrow \infty$ we find

$$\int_{\{x>1\}} e^{cx} f_\nu(x)dx < \frac{e^c}{\epsilon_1} \left(\sum_{n\in\mathbb{N}} a_n p_n e^{-p_n} \right) = \frac{e^c}{\epsilon_1} f_\nu(1) < \infty.$$

The same argument holds for the integral over $\{x < -1\}$.

3RD STEP. We already know that $(e^{zx} - 1 - zx)f_\nu(x)$ is integrable as $\varphi_L(z)$ is finite for $z \in C$. Hence we can apply Lebesgue's dominated convergence theorem to find

$$\int_{\mathbb{R}} (e^{zx} - 1 - zx)\, f_\nu(x)dx = \sum_{n\in\mathbb{N}} (e^{zx} - 1 - zx) \int_{\mathbb{R}} a_n p_n e^{-p_n x}dx.$$

We split up the integral into three terms. Standard calculus results for $z \in C$ in

$$\int_{\mathbb{R}+} e^{zx} e^{-p_n x}dx = \frac{1}{p_n - z}, \quad \int_{\mathbb{R}+} e^{-p_n x}dx = \frac{1}{p_n} \quad \text{and} \quad \int_{\mathbb{R}+} zxe^{-p_n x}dx = \frac{z}{p_n^2}$$

$$\frac{1}{p_n - z} - \frac{1}{p_n} - \frac{z}{p_n^2} = \frac{z^2}{p_n^2(p_n - z)}.$$

Proceeding similarly for the negative half line and plugging into the representation of $\varphi_L(z)$ gives the result. □

Theorem 1.2. *The Laplace exponent $\varphi_L(z)$ of a meromorphic Lévy process L has the Taylor series expansion*

$$\varphi_n(z) = \sum_{n\in\mathbb{N}} c_n z^n \quad \textit{for} \quad |z| < (p_1 \wedge \hat{p}_1)$$

with coefficients

$$c_1 = \mu, \quad c_2 = \sigma^2 + \sum_{k\in\mathbb{N}} \left(\frac{a_k}{p_k^2} + \frac{\hat{a}_k}{\hat{p}_k^2} \right) \quad \textit{and} \quad c_n = \sum_{k\in\mathbb{N}} \left(\frac{a_k}{p_k^n} + \frac{\hat{a}_k}{(-\hat{p}_k)^n} \right) \quad \textit{if } n > 2.$$

The n^{th} cumulant $\kappa_n(L_1)$ of L_1 is given via $\kappa_n(L_1) = c_n$.

Proof. By Theorem 1.1 $\varphi_L(z)$ is analytic for $|z| < (p_1 \wedge \hat{p}_1)$. The coefficients for a Taylor series expansion at $z = 0$ are given by $c_n = \varphi_L^{(n)}(0)/n!$. To compute $\varphi_L^{(n)}(0)$ we introduce the functions

$$f_n(z) := \frac{a_n z^2}{p_n(p_n - z)}, \quad g_n(z) := \frac{\hat{a}_n z^2}{\hat{p}_n(\hat{p}_n + z)}$$

for $n \in \mathbb{N}$. Standard calculus results in

$$f_n'(z) = \frac{a_n z(2p_n - z)}{p_n(p_n - z)^2}, \quad g_n'(z) = \frac{\hat{a}_n z(2\hat{p}_n - z)}{\hat{p}_n(\hat{p}_n - z)^2}$$

$$f_n^{(2)}(z) = \frac{2\, a_n p_n}{(p_n - z)^3}, \quad g_n^{(2)}(z) = \frac{2\, \hat{a}_n \hat{p}_n}{(\hat{p}_n + z)^3}$$

$$f_n^{(m)}(z) = \frac{m!\, a_n p_n}{(p_n - z)^{m+1}}, \quad g_n^{(m)}(z) = (-1)^m \frac{m!\, \hat{a}_n \hat{p}_n}{(\hat{p}_n + z)^{m+1}}$$

Interchanging summation and differentiation gives the result. This is justified since $f_n(z)$ and $g_n(z)$ are analytic in z for all $n \in \mathbb{N}$ and since $\sum_{n\in\mathbb{N}} |f_n(z)|$ and $\sum_{n\in\mathbb{N}} |g_n(z)|$ are locally bounded on $|z| < (p_1 \wedge \hat{p}_1)$ by Theorem 1.1. (see e.g. Conclusion C3 in the Theorem of Mattner (2001)) □

1.3 Wiener-Hopf factorization

In this section we study the extrema of a meromorphic Lévy process $\underline{L}_t := \inf_{s\in[0,t]} L_s$ and $\overline{L}_t := \sup_{s\in[0,t]} L_s$ by using the Wiener-Hopf factorization. Therefore we introduce an exponential random variable $\mathbf{e}_q$ with rate parameter $q > 0$ which is independent of the process $(L_t)_{t\in\mathbb{R}^+}$. Moreover we denote the moment generating functions for the infimum und supremum process at the random time $\mathbf{e}_q$ by $\psi_{\underline{L}_{\mathbf{e}_q}}(z) := \mathbb{E}(e^{z\underline{L}_{\mathbf{e}_q}})$ and $\psi_{\overline{L}_{\mathbf{e}_q}}(z) := \mathbb{E}(e^{z\overline{L}_{\mathbf{e}_q}})$ which we will call the Wiener-Hopf factors. Then the Wiener-Hopf factorization states:

1. $\overline{L}_{\mathbf{e}_q}$ and $L_{\mathbf{e}_q} - \overline{L}_{\mathbf{e}_q}$ are independent,
2. $L_{\mathbf{e}_q} - \overline{L}_{\mathbf{e}_q}$ and $\underline{L}_{\mathbf{e}_q}$ are equal in distribution,
3. $\psi_{\overline{L}_{\mathbf{e}_q}}(z)$ and $\psi_{\underline{L}_{\mathbf{e}_q}}(-z)$ is the only pair of moment generating functions in the class of infinitely divisible distributions supported on $\mathbb{R}^+$ with zero drift satisfying
$$\frac{q}{q - \varphi_L(z)} = \psi_{\overline{L}_{\mathbf{e}_q}}(z)\psi_{\underline{L}_{\mathbf{e}_q}}(z) \quad \text{for} \quad \mathrm{Re}(z) = 0.$$

For a detailed treatment see Sato (1999), Chapter 9, Section 45 and Kyprianou (2006), Chapter 6. For general Lévy processes only integral expressions for the Wiener-Hopf factors are available. In this section we reproduce the results of Kuznetsov (2010b) who derived analytic identities for the Wiener-Hopf factors of meromorphic Lévy processes. His results are based on the theory of Pick functions[1], i.e. analytic functions on

[1] Georg Alexander Pick (1859, Vienna - 1942, KZ Theresienstadt)

$\mathbb{C}^+ \to \mathbb{C}^+$ where $\mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ denotes the open upper half complex plane. Rogers (1983) first used this class of functions in a study of the Wiener-Hopf factorization. Before beginning, we review some results on Pick functions which we will need in the sequel. Details can be found in Donoghue (1974), chapter 2. We start with the canonical representation: Any Pick function $f(z)$ can be uniquely represented in the form

$$f(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \mu(dt), \quad \forall z \in \mathbb{C}^+$$

with μ being a Borel measure on $\mathbb{R}$ satisfying $\int (t^2+1)^{-1}\mu(dt) < \infty$, $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$. Conversely any function of this form is Pick. Pick functions naturally can be extended on the open lower half complex plane via reflection, i.e. $f(\overline{z}) := \overline{f(z)}$ for $z \in \mathbb{C}^+$. We are particularly interested in Pick functions that can be extended to meromorphic functions on $\mathbb{C}$ satisfying the reflection property $f(\overline{z}) = \overline{f(z)}$. We will call such meromorphic functions *real* as the reflection property is equivalent to $f(z) \in \mathbb{R}$ for $z \in \mathbb{R}$ except for poles (Schwarz reflection principle). A key result is the following: Analytical continuation of a Pick function by reflection across a real open interval I is possible iff the corresponding measure of the canonical representation satisfies $\mu(I) = 0$. Then the canonical representation is still true for $z \in I$. In particular a real meromorphic function $f(z)$ on $\mathbb{C}$ is Pick for $z \in \mathbb{C}^+$ iff it admits the representation

$$f(z) = a + bz + \frac{c_0}{z} + \sum_{n=\omega_1}^{\omega_2} c_n \left(\frac{1}{a_n - z} - \frac{1}{a_n} \right), \quad \forall z \in \mathbb{C} \tag{1.4}$$

with $a \in \mathbb{R}^+$, $b \in \mathbb{R}$, $c_n \geq 0$, $\omega_1 \leq \omega_2 \in \overline{\mathbb{Z}}$ and $\sum_{n=\omega_1}^{\omega_2} c_n a_n^{-2} < \infty$ (see also Ahiezer and Krein (1962), Article I, Chapter 2, Theorem 8). A standard example for a real meromorphic function satisfying (1.4) is $\tan(z)$. In Theorem 1.3 we establish a different representation of such real meromorphic function via interlacing zeros and poles following Levin (1996), Chapter 27.2. Here two real-valued sequences $(a_n)_{n\in\mathbb{Z}}$ and $(b_n)_{n\in\mathbb{Z}}$ are called *interlacing* if $b_n < a_n < b_{n+1}, \quad \forall n \in \mathbb{Z}$. Then we will use this two representations to obtain analytic identities for the Wiener-Hopf factors of meromorphic Lévy process following Kuznetsov (2010b), Theorem 1.

Theorem 1.3. *A real meromorphic function $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ with poles accumulating at ∞ and $-\infty$ is Pick for $z \in \mathbb{C}^+$ iff it admits the representation*

$$f(z) = c\frac{z-a_0}{z-b_0} \prod_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left(1 - \frac{z}{a_n}\right)\left(1 - \frac{z}{b_n}\right)^{-1} \tag{1.5}$$

with $c > 0$ and some interlacing zeros $(a_n)_{n\in\mathbb{Z}}$ and poles $(b_n)_{n\in\mathbb{Z}}$ satisfying

$$a_{-1} < 0 < a_1 \quad \textit{and} \quad b_{-1} < 0 < b_1.$$

Proof. SUFFICIENCY. First we proof that $f(z)$ with representation (1.5) is a meromorphic function with poles $(b_n)_{n\in\mathbb{Z}}$. It is sufficient to show that the infinite product for $n \in \mathbb{N}$ converges uniformly on every compact set of $\mathbb{C}\backslash\{b_n\}_{n\in\mathbb{N}}$ due to the symmetry of the definition. Hence we have to proof that

$$\sum_{n\in\mathbb{N}} \left(1 - \left(1 - \frac{z}{a_n}\right)\left(1 - \frac{z}{b_n}\right)^{-1}\right) = z \sum_{n\in\mathbb{N}} \left(\frac{1}{b_k} - \frac{1}{a_k}\right)\left(1 - \frac{z}{b_n}\right)^{-1}$$

converges uniformly on every compact set of $\mathbb{C}\backslash\{b_n\}_{n\in\mathbb{N}}$. For an arbitrary compact set K we find $\epsilon_K > 0$ and $R_K > 0$ such that

$$|b_k - z| > \epsilon_K, \quad \text{and} \quad |z| < R_K, \quad \forall z \in K$$

since K is closed and bounded. Together with the the triangle inequality this yields in

$$\left|\left(1 - \frac{z}{b_n}\right)^{-1}\right| = \frac{|b_n|}{|b_n - z|} < \frac{|b_n - z| + |z|}{|b_n - z|} < 1 + \frac{R_K}{\epsilon_K} =: c_K.$$

Thus be the interlacing property $a_n^{-1} < b_{n+1}^{-1}$ and therefore

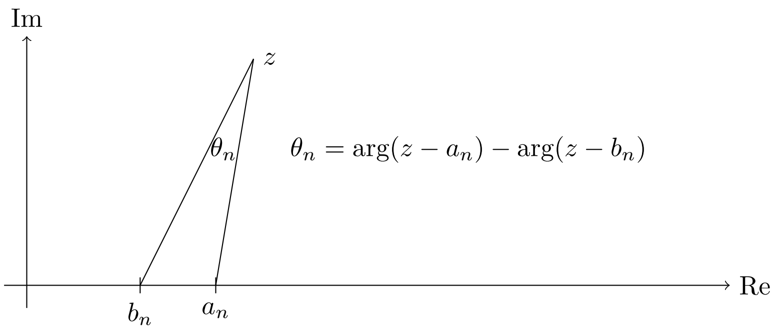
$$\sum_{n\in\mathbb{N}} \left|\left(\frac{1}{b_k} - \frac{1}{a_k}\right)\left(1 - \frac{z}{b_n}\right)^{-1}\right| < c_k \sum_{n\in\mathbb{N}} \left(\frac{1}{b_k} - \frac{1}{a_k}\right) < c_k \sum_{n\in\mathbb{N}} \left(\frac{1}{b_n} - \frac{1}{b_{n+1}}\right) = \frac{c_k}{b_1}.$$

By the Weierstrass M-test we have analyticity on $\mathbb{C}\backslash\{b_n\}_{n\in\mathbb{N}}$.

As it is obvious that $f(z)$ is real meromorphic it remains to show that $\arg(f(z)) \in (0,\pi)$ for $\arg(z) \in (0,\pi)$. The exponentiation identity gives

$$\arg\left(\left(1 - \frac{z}{a_n}\right)\left(1 - \frac{z}{b_n}\right)^{-1}\right) = \arg\left(\frac{b_n}{a_n}\frac{z - a_n}{z - b_n}\right) = \arg(z - a_n) - \arg(z - b_n).$$

Geometrically $\theta_n := \arg(z-a_n) - \arg(z-b_n)$ is the angle at which the segment $[b_n, a_n]$ is seen from z in the complex plane:



The function $\arg(z)$ is strictly monotonically decreasing in $\mathrm{Re}(z)$ as

$$\arg(z) = \pi - \tan^{-1}\left(\mathrm{Im}(z)\mathrm{Re}(z)^{-1}\right).$$

Hence by the interlacing property $\arg(z-b_n) > \arg(z-a_{n+1})$. Thus for $\arg(z) \in (0,\pi)$

$$\begin{aligned}\arg(f(z)) &= \sum_{n\in\mathbb{Z}} \arg(z-a_n) - \arg(z-b_n) < \sum_{n\in\mathbb{Z}} \arg(z-a_n) - \arg(z-a_{n+1}) \\ &= \lim_{n\to\infty} (\arg(z-a_{-n}) + \arg(z-a_n)) = \pi\end{aligned}$$

Since $\arg(z-a_n) - \arg(z-b_n) > 0$ we have $\arg(f(z)) > 0$.

Necessity. Let $f(z)$ be an arbitrary real meromorphic Pick function. The structure of the poles and zeros of $f(z)$ can be easily deduced from Cauchy's argument principle (see e.g. Greene & Krantz (2006), Section 5.1): For some contour γ with domain $[t_1,t_2]$ enclosing poles P_γ and zeros Z_γ we have

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{z\in Z_\gamma \cup P_\gamma} \mathrm{ind}_\gamma(z)\, \mathrm{ord}_z\, f = \frac{1}{2\pi} \Delta_\gamma \arg(f)$$

where $\Delta_\gamma \arg(f) = f(\gamma(t_2)) - f(\gamma(t_1))$ denotes the change of argument, $\mathrm{ind}_\gamma(z)$ the winding number of γ and $\mathrm{ord}_z\, f$ the order of the pole or zero z. To proof the interlacing property we consider for $a \in \mathbb{R}$ and $r \in \mathbb{R}^+$ the simple contours

$$\gamma(a,r) := \gamma_1(a,r) + \gamma_2(a,r) \quad \text{with}$$

$$\gamma_1(a,r) := a + re^{i\theta},\ \theta \in [0,\pi] \quad \text{and} \quad \gamma_2(a,r) := a + re^{i\theta},\ \theta \in [\pi, 2\pi].$$

Since $\mathrm{img}(\gamma_1(a,r)) \in \mathbb{C}^+ \cup \mathbb{R}$ we have

$$\arg(f(z)) \in [0,\pi] \quad \text{for} \quad z \in \mathrm{img}(\gamma_1(a,r))$$

as $f(z)$ is a real Pick function. Similarly we find

$$\arg(f(z)) \in [-\pi, 0] \quad \text{for} \quad z \in \mathrm{img}(\gamma_2(a,r)).$$

Thus

$$|\Delta_\gamma \arg(f)| \le 2\pi.$$

So by Cauchy's argument principle all poles and zeros are simple and differ on any open interval at most by one. We introduce

$$\tilde{f}(z) := \frac{z-a_0}{z-b_0} \prod_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left(1 - \frac{z}{a_n}\right)\left(1 - \frac{z}{b_n}\right)^{-1}$$

with $(a_n)_{n\in\mathbb{Z}}$ being the zeros of $f(z)$ and $(b_n)_{n\in\mathbb{Z}}$ its poles. Then $f(z)/\tilde{f}(z)$ is entire with no zeros on $\mathbb{C}$ satisfying

$$\left| \arg\left(\frac{f(z)}{\tilde{f}(z)}\right) \right| = |\arg(f(z)) - \arg(\tilde{f}(z))| \le 2\pi$$

as both $f(z)$ and $\tilde{f}(z)$ are real Pick functions. Thus $u(z) := \log(f(z)/\tilde{f}(z))$ is entire with $|\mathrm{Im}(u(z))| \le 2\pi$. So by Picard's Little Theorem $u(z)$ is constant and $f(z)/\tilde{f}(z) = c > 0$. $\square$

Theorem 1.4. *For a meromorphic Lévy process and $q > 0$ it holds that:*

(i) *There exist two increasing real-valued sequences $(z_n)_{n\in\mathbb{N}}$ and $(\hat{z}_n)_{n\in\mathbb{N}}$ such that*

$$\{z \in \mathbb{C} : \varphi_L(z) = q\} = \bigcup_{n\in\mathbb{N}} \{z_n, -\hat{z}_n\} \quad \text{and} \quad z_n < p_n,\ \hat{z}_n < \hat{p}_n \quad \forall n \in \mathbb{N}.$$

(ii) *The Wiener-Hopf factors $\psi_{\overline{L}_{e_q}}(z)$ and $\psi_{\underline{L}_{e_q}}(z)$ are analytic on $\{z \in \mathbb{C} : \mathrm{Re}(z) < z_1\}$ and $\{z \in \mathbb{C} : \mathrm{Re}(z) > -\hat{z}_1\}$, respectively and take on the form*

$$\psi_{\overline{L}_{\mathbf{e}_q}}(z) = \prod_{n\in\mathbb{N}} \left(1 - \frac{z}{p_n}\right)\left(1 - \frac{z}{z_n}\right)^{-1}, \quad \mathrm{Re}(z) < z_1,$$

$$\psi_{\underline{L}_{\mathbf{e}_q}}(z) = \prod_{n\in\mathbb{N}} \left(1 + \frac{z}{\hat{p}_n}\right)\left(1 + \frac{z}{\hat{z}_n}\right)^{-1}, \quad \mathrm{Re}(z) > -\hat{z}_1.$$

Proof. In the first step of the proof we show that $\varphi_L(z)$ can be written in the form

$$\frac{q}{q - \varphi_L(z)} = \prod_{n\in\mathbb{N}} \left(1 - \frac{z}{p_n}\right)\left(1 - \frac{z}{z_n}\right)^{-1}\left(1 + \frac{z}{\hat{p}_n}\right)\left(1 + \frac{z}{\hat{z}_n}\right)^{-1} \tag{1.6}$$

for $z \in \mathbb{C}$ with $(z_n)_{n\in\mathbb{N}}$ and $(\hat{z}_n)_{n\in\mathbb{N}}$ satisfying the proposed properties by using our results on real Pick functions. In the second step of the proof we show that

$$\psi_{\overline{L}_{\mathbf{e}_q}}(z) = \prod_{n\in\mathbb{N}} \left(1 - \frac{z}{a_n}\right)\left(1 - \frac{z}{b_n}\right)^{-1} \quad \text{and} \quad \psi_{\underline{L}_{\mathbf{e}_q}}(z) = \prod_{n\in\mathbb{N}} \left(1 + \frac{z}{\hat{p}_n}\right)\left(1 + \frac{z}{\hat{z}_n}\right)^{-1}$$

for $\mathrm{Re}(z) = 0$ via the uniqueness property of the Wiener-Hopf factorization. Then we extend both identities to $\{z \in \mathbb{C} : \mathrm{Re}(z) < z_1\}$ and $\{z \in \mathbb{C} : \mathrm{Re}(z) > -\hat{z}_1\}$, respectively, in the final step.

1ST STEP. Using the representation of $\varphi_L(z)$ in (1.3) we obtain for $z \in \mathbb{C}$

$$\frac{\varphi_L(z) - q}{z} = \frac{\sigma^2 z}{2} + \mu - \frac{q}{z} + \sum_{n\in\mathbb{N}} a_n \left(\frac{1}{p_n - z} - \frac{1}{p_n}\right) + \sum_{n\in\mathbb{N}} \hat{a}_n \left(\frac{1}{\hat{p}_n - z} - \frac{1}{\hat{p}_n}\right).$$

Thus $(\varphi_L(z) - q)/z$ takes on the form (1.4) and is therefore a real meromorphic Pick function with poles $(p_n)_{n\in\mathbb{N}}$, $(-\hat{p}_n)_{n\in\mathbb{N}}$ and 0. Applying Theorem 1.3 we find for $z \in \mathbb{C}$

$$\frac{\varphi_L(z) - q}{z} = \frac{c(z - z_1)}{z} \prod_{n\in\mathbb{N}} \left(1 - \frac{z}{z_{n+1}}\right)\left(1 - \frac{z}{p_n}\right)^{-1}\left(1 + \frac{z}{\hat{z}_n}\right)\left(1 + \frac{z}{\hat{p}_n}\right)^{-1}.$$

The sequences $(z_n)_{n\in\mathbb{N}}$ and $(\hat{z}_n)_{n\in\mathbb{N}}$ satisfy (i) due to the interlacing condition of Theorem 1.3. Algebraic manipulation moreover gives for all $z \in \mathbb{C}$

$$\frac{q}{q - \varphi_L(z)} = \frac{qz_1}{c} \prod_{n\in\mathbb{N}} \left(1 - \frac{z}{p_n}\right)\left(1 - \frac{z}{z_n}\right)^{-1}\left(1 + \frac{z}{\hat{p}_n}\right)\left(1 + \frac{z}{\hat{z}_n}\right)^{-1}.$$

Since $\varphi_L(z) = 0$ we have $c = qz_1$ giving (1.6).

2ND STEP. First we introduce the meromorphic functions

$$m_1(z) := \prod_{n\in\mathbb{N}} \left(1-\frac{z}{p_n}\right)\left(1-\frac{z}{z_n}\right)^{-1} \quad \text{and} \quad m_2(z) := \prod_{n\in\mathbb{N}} \left(1+\frac{z}{\hat{p}_n}\right)\left(1+\frac{z}{\hat{z}_n}\right)^{-1}$$

for $z \in \mathbb{C}$ such that $q/(\varphi_L(z) - q) = m_1(z)m_2(z)$. We recall the Lévy-Khintchine representation of an exponential distributed random variable with rate parameter $\lambda \in \mathbb{R}^+$ (see Sato (1999), Example 8.10) for $\mathrm{Re}(z) = 0$:

$$\frac{\lambda}{\lambda - z} = \left(1-\frac{z}{\lambda}\right)^{-1} = \exp\left(\int_{\mathbb{R}^+} (e^{zx}-1)\,\frac{e^{-\lambda x}}{x}dx\right).$$

Using this identity for $m_1(z)$ we find for $\mathrm{Re}(z) = 0$

$$\begin{aligned} m_1(z) &= \prod_{n\in\mathbb{N}} \left(\exp \int_{\mathbb{R}} (e^{zx}-1)\,\frac{e^{-z_n x}-e^{-p_n x}}{x}dx\right) \\ &= \exp\left(\lim_{N\to\infty}\int_{\mathbb{R}^+} \frac{e^{zx}-1}{x}\sum_{n=1}^{N}\left(e^{-z_n x}-e^{-p_n x}\right)dx\right). \end{aligned}$$

Since $|e^{zx}-1| < |zx|$ and the by interlacing property we have

$$\left|\frac{e^{zx}-1}{x}\sum_{n=1}^{N}\left(e^{-z_n x}-e^{-p_n x}\right)\right| < |z|e^{-z_1 x}.$$

By using the continuity of the exponential function and Lebesgue's dominated convergence theorem we obtain

$$f_1(z) = \exp\left(\int_{\mathbb{R}} (e^{zx}-1)\,\nu(dx)\right) \quad \text{with} \quad \nu(dx) = \sum_{n\in\mathbb{N}} \frac{e^{-z_n x}-e^{-p_n x}}{x}dx.$$

Moreover $\nu(dx)$ is a Lévy measure as it is dominated by a Lévy measure of an exponential distributed random variable with rate parameter z_1. Thus $m_1(z)$ is the characteristic function of a non-negative, infinite divisible random variables with zero drift. One finds similarly that $m_2(z)$ is the characteristic function of a non-positive, infinite divisible random variables with zero drift. Hence we can apply the uniqueness property of the Wiener-Hopf factorization.

3RD STEP. First we want to establish for $\mathrm{Re}(z) < z_1$

$$\psi_{\overline{L}_{\mathbf{e}_q}}(z) = \exp\left(\int_{\mathbb{R}} (e^{zx}-1)\,\nu(dx)\right) \quad \text{with} \quad \nu(dx) = \sum_{n\in\mathbb{N}} \frac{e^{-z_n x}-e^{-p_n x}}{x}dx.$$

According to the exponential moments theorem for Lévy processes (see Sato (1999), Theorem 25.17) it is sufficient to show

$$\int_{x>1} e^{cx}\nu(dx) < \infty \quad \text{for} \quad c \in (-\infty, z_1).$$

This is a simple consequence of $\nu(dx)$ being dominated by a Lévy measure of an exponential distributed random variable with rate parameter z_1. As $\psi_{\overline{L}_{\mathbf{e}_q}}(z)$ is an one-side Laplace transform it is analytic on $\{z \in \mathbb{C} : \mathrm{Re}(z) < z_1\}$. Thus the assertion follows from the identity theorem of analytic functions. $\square$

1.4 Distributions of the Wiener-Hopf factors

The distributions of the Wiener-Hopf factors for completely monotone Lévy densities have already been precisely characterized by Rogers (1983). In particular a Lévy process has a completely monotone Lévy measure iff its Wiener-Hopf factors are in the *class ME* (Mixtures of Exponential Distributions). This class of infinitely divisible distributions on the half line consists of all probability distributions such that

$$\mu(dx) = c\delta_0 + 1_{(0,\infty)} m(x)dx$$

with $c \in [0,1)$ and $m(x)$ being completely monotone on $(0,\infty)$. It coincides with the class of mixtures of all exponential distributions and δ_0. For details see Sato (1999), Chapter 51 and Bondesson (1981). In the case of meromorphic Lévy processes it turns out that we have a countable mixture of exponential distributions:

Theorem 1.5. *For a meromorphic Lévy processes the distributions of the Wiener-Hopf factors are absolutely continuous on* $(0,\infty)$ *and* $(-\infty,0)$*, respectively. The corresponding densities are given by*

$$\overline{v}^q(x) = \sum_{n\in\mathbb{N}} b_n z_n e^{-z_n x} \quad \textit{for} \quad x \in (0,\infty),$$

$$\underline{v}^q(x) = \sum_{n\in\mathbb{N}} \hat{b}_n \hat{z}_n e^{\hat{z}_n x} \quad \textit{for} \quad x \in (-\infty,0).$$

with mixing weights $(b_n)_{n\in\mathbb{N}}$ *and* $(\hat{b}_n)_{n\in\mathbb{N}}$ *given via*

$$b_n = \prod_{k\in\mathbb{N}}\left(1-\frac{z_n}{p_k}\right)\prod_{k\in\mathbb{N}\setminus\{n\}}\left(1-\frac{z_n}{z_k}\right)^{-1} \quad \textit{and} \quad \hat{b}_n = \prod_{k\in\mathbb{N}}\left(1-\frac{\hat{z}_n}{\hat{p}_k}\right)\prod_{k\in\mathbb{N}\setminus\{n\}}\left(1-\frac{\hat{z}_n}{\hat{z}_k}\right)^{-1}.$$

Proof. For the Wiener-Hopf factor $\psi_{\overline{L}_{\mathbf{e}_q}}(z)$ we find that by Ferreiro-Castilla & Utzet (2012), Theorem 4.1. that it is absolutely continuous on $(0,\infty)$ with density

$$\overline{v}^q(x) = -\sum_{n\in\mathbb{N}} \frac{h(z_n)}{g'(z_n)} e^{-z_n x} \quad \text{for} \quad x \in (0,\infty),$$

where $h(z) = \prod_{k\in\mathbb{N}}(1-\frac{z}{p_n})$ and $g(z) = \prod_{k\in\mathbb{N}}(1-\frac{z}{z_n})$. As $h(z)$ is analytic in z_n and $f(z)$ has a zero in z_n we have

$$\overline{v}^q(x) = -\sum_{n\in\mathbb{N}} \mathrm{Res}_{z_n}\left(\frac{h}{g}\right) e^{-z_n x} \quad \text{for} \quad x \in (0,\infty).$$

By standard residue calculus we obtain the result. □

Theorem 1.6. *For a meromorphic Lévy process the distributions of the Wiener-Hopf factors have a probability mass at 0:*

$$\mathbb{P}\left(\overline{L}_{\mathbf{e}_q} = 0\right) = \prod_{n\in\mathbb{N}} \frac{z_n}{p_n} \quad \text{and} \quad \mathbb{P}\left(\underline{L}_{\mathbf{e}_q} = 0\right) = \prod_{n\in\mathbb{N}} \frac{\hat{z}_n}{\hat{p}_n}.$$

If the paths of a meromorphic Lévy processes are a.s. of unbounded variation, i.e.

$$\sigma \neq 0 \quad \text{or} \quad \sum_{n\in\mathbb{N}} \left(\frac{a_n}{p_n} + \frac{\hat{a}_n}{\hat{p}_n}\right) = \infty.$$

the distributions of the Wiener-Hopf factors are free of atoms.

Proof. For $z \in (-\infty, 0]$ we have by Theorem 1.4 and the definition of the moment generating function

$$\psi_{\overline{L}_{\mathbf{e}_q}}(z) = \mathbb{P}(\overline{L}_{\mathbf{e}_q} = 0) + \int_{(0,\infty)} e^{zt}\overline{v}^q(t)dt.$$

Thus by Lebesgue's dominated convergence theorem we can recover $\mathbb{P}\left(\overline{L}_{\mathbf{e}_q} = 0\right)$ via

$$\mathbb{P}(\overline{L}_{\mathbf{e}_q} = 0) = \lim_{z\to\infty} \psi_{\overline{L}_{\mathbf{e}_q}}(-z) = \lim_{z\to\infty} \left(\prod_{n\in\mathbb{N}} \left(1 + \frac{z}{p_n}\right)\left(1 + \frac{z}{z_n}\right)^{-1}\right)$$

By analyticity $\lim_{N\to\infty} \prod_{n=1}^{N} \left(1 + \frac{z}{p_n}\right)\left(1 + \frac{z}{z_n}\right)^{-1}$ converges uniformly in N for all $x \in (-\infty, 0]$. So we can interchange the limits and obtain the first result. For the second result we start with the observation that $\inf\{t \in \mathbb{R}^+ : L_t > 0\} = \inf\{t \in \mathbb{R}^+ : \overline{L}_t > 0\} = 0$ a.s. iff $\mathbb{P}(\overline{L}_t > 0) = 1$ for all $t \in \mathbb{R}^+$. So we have that $\mathbb{P}\left(\overline{L}_{\mathbf{e}_q} = 0\right) = 0$ iff 0 is regular for $(0, \infty)$. This holds true for any Lévy processes of unbounded variation (see Bertoin (1996), chapter VI, section 3). Therefore the last statement of the Theorem follows by Proposition 1.2. □

1.5 Beta class

A ten-parameter family of meromorphic Lévy processes called Beta class was introduced by Kuznetsov (2010a). In this section we reproduce the main properties of this family, in particular Theorem 9 of Kuznetsov (2010a). The Lévy density of a Lévy processes in the Beta class is of the form

$$f_\nu(x) = 1_{\{x>0\}} c \frac{e^{-\alpha\beta x}}{(1 - e^{-\beta x})^\gamma} + 1_{\{x<0\}} \hat{c} \frac{e^{\hat{\alpha}\hat{\beta} x}}{(1 - e^{\hat{\beta} x})^{\hat{\gamma}}}, \quad x \in \mathbb{R}$$

for $\alpha, \hat{\alpha}, \beta, \hat{\beta}, c, \hat{c} > 0$ and $\gamma, \hat{\gamma} \in (0, 3)\backslash\{1, 2\}$. In Theorem 1.7 we show that a Lévy process with such a Lévy density is meromorphic. Moreover we will see that a Lévy process of the Beta class allows for a closed form expression for the Laplace exponent

$\varphi_L(z)$ involving the Beta function instead of an infinite series representation as in Theorem 1.1. The Beta class is a rich family of Lévy processes as it contains processes of finite and infinite activity as well as processes of finite and infinite variation (see Theorem 1.8). In addition to Kuznetsov (2010a) we give a representation of the cumulants of a Beta class Lévy process by using complete Bell polynomials. For the definition and main properties of Bell polynomials see Comtet (1974), Chapter III, Section 3.3.

Theorem 1.7. *A meromorphic Lévy process with coefficients for $n \in \mathbb{N}$*

$$a_n := \frac{c}{p_n B(n,\gamma)(n+\gamma-1)} \quad \textit{and} \quad p_n := \beta(\alpha+n-1),$$

$$\hat{a}_n := \frac{\hat{c}}{\hat{p}_n B(n,\hat{\gamma})(n+\hat{\gamma}-1)} \quad \textit{and} \quad \hat{p}_n := \hat{\beta}(\hat{\alpha}+n-1)$$

where $\alpha, \hat{\alpha}, \beta, \hat{\beta}, c, \hat{c} > 0$ is well-defined if $\gamma, \hat{\gamma} \in (0,3)\backslash\{1,2\}$. Then it holds that:

(i) *The Lévy density $f_\nu(x)$ admits the representation*

$$f_\nu(x) = 1_{\{x>0\}} c \frac{e^{-\alpha\beta x}}{(1-e^{-\beta x})^{\gamma}} + 1_{\{x<0\}} \hat{c} \frac{e^{\hat{\alpha}\hat{\beta} x}}{(1-e^{\hat{\beta} x})^{\hat{\gamma}}}, \quad x \in \mathbb{R}.$$

(ii) *The Lévy exponent $\varphi_L(z)$ admits the representation*

$$\begin{aligned}\varphi_L(z) =& \frac{1}{2}\sigma^2 z + \mu z + \frac{c}{\beta}\left(B\left(\alpha - \frac{z}{\beta}, 1-\gamma\right) - B(\alpha, 1-\gamma)\right) \\ &+ \frac{\hat{c}}{\hat{\beta}}\left(B\left(\hat{\alpha} + \frac{z}{\hat{\beta}}, 1-\hat{\gamma}\right) - B(\hat{\alpha}, 1-\hat{\gamma})\right)\end{aligned}$$

for some $\sigma > 0$ and $\mu \in \mathbb{R}$.

Proof. First we show that $\sum_{n\in\mathbb{N}} \left(\frac{a_n}{p_n^2} + \frac{a_n}{p_n^2}\right) < \infty$ (see Proposition 1.1). Therefore we use the asymptotic equivalence $B(a,b) \sim \Gamma(b)\, a^{-b}$ for $a \to \infty$ to find for $n \to \infty$

$$\frac{a_n}{p_n^2} \sim \frac{(n+1)^{\gamma}}{(n+\gamma)(n+\alpha)^3} \frac{c}{\Gamma(\gamma)\beta^3} \sim n^{\gamma-4} \frac{c}{\Gamma(\gamma)\beta^3}.$$

Thus by the Limit comparison test we have a meromorphic Lévy process if $\gamma \in (0,3)\backslash\{1,2\}$ and with similar reasoning if $\hat{\gamma} \in (0,3)\backslash\{1,2\}$. For $x > 0$ we have per definition (1.1) and the relation of the binomial coefficient and the Beta function

$$\binom{a-1}{b-1} = \frac{1}{aB(b, a-b+1)}$$

the series representation of the Lévy density:

$$f_\nu(x) = ce^{-\beta\alpha x} \sum_{n\in\mathbb{N}} \binom{n+\gamma-2}{n-1} (e^{-\beta x})^{n-1} = ce^{-\beta\alpha x} \sum_{n=0}^{\infty} \binom{n+\gamma-1}{n} (e^{-\beta x})^n, \quad x > 0.$$

As the Taylor series at $z = 0$ of the function $(1-z)^{1-y}$ (Binomial series) is given via

$$\frac{1}{(1-z)^{y-1}} = \sum_{n=0}^{\infty} \binom{n+y}{n} z^n$$

we arrive at (i). Using the same arguments we can verify (i) in the case of $x < 0$ too.

Second, for $\gamma \in (0,1)$ and $\mathrm{Re}(z) \in (-\hat{\alpha}\hat{\beta}, \alpha\beta)$ we consider the integrals

$$\int_{\mathbb{R}^+} e^{zx} \frac{e^{-\alpha\beta x}}{(1-e^{-\beta x})^{\gamma}} dx = \frac{1}{\beta} \int_{[0,1]} u^{\frac{z}{-\beta}+\alpha} (1-u)^{-\gamma} du$$

$$\int_{\mathbb{R}^+} \frac{e^{-\alpha\beta x}}{(1-e^{-\beta x})^{\gamma}} dx = \frac{1}{\beta} \int_{[0,1]} u^{\alpha} (1-u)^{-\gamma} du$$

where we performed a change of variables $u \mapsto e^{-\beta x}$. Then the integral representation of the Beta function

$$B(a,b) = \int_{[0,1]} t^{a-1}(1-t)^{b-1}\, dt, \quad \mathrm{Re}(a) > 0,\, \mathrm{Re}(b) > 0$$

yields in

$$\int_{\mathbb{R}^+} (e^{zx} - 1 - zx) \frac{e^{-\alpha\beta x}}{(1-e^{-\beta x})^{\gamma}} dx = \frac{1}{\beta} \left(B\left(\alpha - \frac{z}{\beta}, 1-\gamma\right) - B(\alpha, 1-\gamma) \right) + \tilde{\mu} z$$

for some $\tilde{\mu} := \int_{\mathbb{R}^+} \frac{xe^{-\alpha\beta x}}{(1-e^{-\beta x})^{\gamma}} dx \in \mathbb{R}$. To extend this result on $\gamma \in (0,3)\backslash\{1,2\}$ we use analytic continuation. The right-hand-side extends to an analytic function on $\mathrm{Re}(\gamma) \in (0,3)\backslash\{1,2\}$. To conclude the proof we show that the left-hand-side is analytic on $\Gamma := \mathrm{Re}(\gamma) \in (0,3)$. As the integrand

$$f(\gamma, x) := (e^{zx} - 1 - zx) \frac{e^{-\alpha\beta x}}{(1-e^{-\beta x})^{\gamma}}$$

$f(\gamma, x)$ is analytic on Γ it is sufficient to show that for any compact set $K \subset \Gamma$ we find an integrable function $g_K(x)$ on $\mathbb{R}^+$ such that $|f(\gamma, x)| < g_K(x)$ for all $\gamma \in \Gamma$ and $x \in \mathbb{R}^+$ (see e.g. Conclusion C3 in the Theorem of Mattner (2001)). With $\gamma_K := \arg\max_{\gamma \in K} \mathrm{Re}(\gamma)$ we can choose $g_K(x) := f(\gamma_K, x)$ as

$$|(1-e^{-\beta x})^{-\gamma}| = (1-e^{-\beta x})^{-\mathrm{Re}(\gamma)}$$

and $g_K(x)$ is integrable as $\gamma_K \in (0,3)$. $\square$

Theorem 1.8. *A meromorphic Lévy process L of the Beta class is*

- *of finite activity iff $\gamma < 1$ and $\hat{\gamma} < 1$,*
- *of finite variation iff $\sigma = 0, \gamma < 2$ and $\hat{\gamma} < 2$.*

The n^{th} cumulant $\kappa_n(L_1)$ of L_1 is given via

$$\kappa_1(L_1) = \mu + \frac{c}{\beta^2} B(\alpha, 1-\gamma)\psi_0 - \frac{\hat{c}}{\hat{\beta}^2} B(\hat{\alpha}, 1-\hat{\gamma})\hat{\psi}_0,$$

$$\kappa_2(L_1) = \sigma^2 + \frac{c}{\beta^3} B(\alpha, 1-\gamma)(\psi_0^2 + \psi_1) + \frac{\hat{c}}{\hat{\beta}^3} B(\hat{\alpha}, 1-\hat{\gamma})(\hat{\psi}_0^2 + \hat{\psi}_1),$$

$$\kappa_n(L_1) = \frac{c}{\beta^{n+1}} B(\alpha, 1-\gamma) B_n(\psi_0, \ldots, \psi_{n-1}) + \frac{(-1)^n \hat{c}}{\hat{\beta}^{n+1}} B(\hat{\alpha}, 1-\hat{\gamma}) B_n(\hat{\psi}_0, \ldots, \hat{\psi}_{n-1})$$

for $n \geq 3$ where $B_n(x_1, \ldots, x_n)$ denotes the n^{th} complete Bell polynomial and

$$\psi_n := \psi^{(n)}(\alpha) - \psi^{(n)}(\alpha + 1 - \gamma),$$

$$\hat{\psi}_n := \psi^{(n)}(\alpha) - \psi^{(n)}(\hat{\alpha} + 1 - \hat{\gamma})$$

with $\psi^{(n)}(x)$ being the polygamma function of order n.

Proof. According to Proposition 1.2 a meromorphic Lévy process is of finite activity iff $\sum_{n\in\mathbb{N}}(a_n + \hat{a}_n) < \infty$ and of finite variation iff $\sum_{n\in\mathbb{N}}\left(\frac{a_n}{p_n} + \frac{\hat{a}_n}{\hat{p}_n}\right) < \infty$ and $\sigma = 0$. By using the asymptotic equivalence $B(a,b) \sim \Gamma(b)\, a^{-b}$ for $a \to \infty$ we find similarly to the proof of Theorem 1.7 for $n \to \infty$

$$a_n \sim \frac{n^\gamma}{n^2}\frac{c\Gamma(\gamma)}{\beta}, \quad \frac{a_n}{p_n} \sim \frac{n^\gamma}{n^3}\frac{c\Gamma(\gamma)}{\beta}.$$

Thus the first two results follow by the Limit comparison test.

The n^{th} cumulant is given by $\kappa_n(L_1) = \varphi_L^{(n)}(0)/n!$. To derive the n^{th} derivative of the Laplace exponent we apply Faà di Bruno's formula on $B(\alpha - \frac{x}{\beta}, 1-\gamma) = f(g(x))$ where $f(x) := \exp(x)$ and $g(x) := \log(B(\alpha - \frac{x}{\beta}, 1-\gamma)$. This yields

$$\frac{d^n}{dx^n} B(\alpha - \frac{x}{\beta}, 1-\gamma) = \sum_{k=1}^{n} B(\alpha - \frac{x}{\beta}, 1-\gamma) B_{n,k}\left(\frac{d}{dx}g(x), \ldots, \frac{d^{n-k+1}}{dx^{n-k+1}}g(x)\right)$$

with the Bell polynomials $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$. As $\log(B(a,b)) = \log(\Gamma(a)) + \log(\Gamma(b)) - \log(\Gamma(a+b))$ we have that

$$\frac{d^n}{dx^n} g(x) = \frac{1}{(-\beta)^n}\left(\varphi^{(n-1)}(\alpha - \frac{x}{\beta}) - \varphi^{(n-1)}(\alpha - \frac{x}{\beta} + 1 - \gamma)\right)$$

by definition of the polygamma function of order n. Then standard calculus gives the result. □

Chapter 2

Option pricing with meromorphic Lévy processes

2.1 Laplace transforms of arbitrage-free option prices

There is a vast literature concerning path-depended option pricing in a hyper-exponential jump diffusion model (HEJD model) and its special case the double-exponential jump diffusion model (DEJD model). The Laplace transforms with respect to time of arbitrage-free prices of many exotic options have a rather simple explicit form in these models. Formulas for lookback and barrier options were derived by Lipton (2002), Kou and Wang (2003, 2004) and Sepp (2004, 2005).

In many papers these results have been used to obtain arbitrage-free prices for more general market models: Carr & Crosby (2010) extended the framework to HEJD models with stochastic volatility and jump intensity. Moreover several papers used HEJD model prices to approximate prices in more general Lévy markets (see e.g. Asmussen *et al.* (2007), Jeannin & Pistorius (2010) and Crosby *et al.* (2010)).

In this chapter we study a *meromorphic model* with a bond yielding a constant riskless rate of return $r > 0$ and stock evolving as $S_t := S_0 \exp(L_t)$ with $S_0 > 0$ where $(L_t)_{t\in\mathbb{R}^+}$ is a meromorphic Lévy process under the risk-neutral measure $\mathbb{Q}$ chosen by the market. We will give analytic identities for vanilla, lookback and barrier options in a meromorphic model as one main contribution of the thesis. This extends the framework of exotic option pricing in the HEJD model to a much richer subclass of exponential Lévy market models.

Throughout this chapter we require that $\mathbb{E}(S_t) < \infty$. In the second part of the proof of Theorem 1.1 we have already seen that this equivalent to assume $p_1 > 1$. Moreover we assume that $\mathbb{Q}(\overline{L}_t = 0) = \mathbb{Q}(\underline{L}_t = 0) = 0$ (see Theorem 1.6 for a sufficient condition).

This assumption is made just for convenience. It will be clear from the proofs of the results within this chapter how it may be removed (see also Jeannin & Pistorius (2010) which give formulas in the HEJD model without this assumption).

By the equivalent martingale measure requirement (EMM condition) it must hold that

$$e^{-rt}\mathbb{E}_{\mathbb{Q}}(S_t) = S_0 \iff \varphi(1) = r$$

(see Cont & Tankov (2003), section 8.4.1.). For a meromorphic Lévy market this is equivalent to:

$$\mu = r - \frac{\sigma^2}{2} - \sum_{n\in\mathbb{N}} \left(\frac{a_n}{p_n(p_n-1)} + \frac{\hat{a}_n}{\hat{p}_n(\hat{p}_n+1)} \right).$$

by Theorem 1.1. We consider path-dependent options with a payoff $g(L_t, \overline{L}_t, \underline{L}_t)$ that depends on at most of two of the random variables L_t, $\overline{L}_t$ and $\underline{L}_t$. Most important examples are options with barrier and/or lookback features. Then the arbitrage-free price of the option is given by

$$G_t := e^{-rt}\mathbb{E}_{\mathbb{Q}}\left(g(L_t, \overline{L}_t, \underline{L}_t)\right).$$

The Laplace transform of G_t can be interpreted as expectation, i.e. for an independent, exponentially distributed random variable $\mathbf{e}_{u+r}$ with rate parameter $(u+r)$ we have for $u \in \mathbb{R}^+$

$$\mathscr{L}(G_t)(u) := \int_0^\infty e^{-ut} G_t dt = \frac{1}{u+r}\mathbb{E}_{\mathbb{Q}}\left(g(L_{\mathbf{e}_{u+r}}, \overline{L}_{\mathbf{e}_{u+r}}, \underline{L}_{\mathbf{e}_{u+r}})\right).$$

With the help of the Wiener-Hopf factorization we can find the distribution of $(L_{\mathbf{e}_q}, \overline{L}_{\mathbf{e}_q})$ and $(L_{\mathbf{e}_q}, \underline{L}_{\mathbf{e}_q})$ and the marginal distribution of $L_{\mathbf{e}_q}$. Note that these representations of the density of $L_{\mathbf{e}_q}$ have already been established by Kuznetsov *et al.* (2012), Theorem 2.

Proposition 2.1. *For a meromorphic Lévy process satisfying* $\mathbb{Q}(\overline{L}_t = 0) = \mathbb{Q}(\underline{L}_t = 0) = 0$ *the joint distributions of* $(L_{\mathbf{e}_q}, \overline{L}_{\mathbf{e}_q})$ *and* $(L_{\mathbf{e}_q}, \underline{L}_{\mathbf{e}_q})$ *admit densities* $\overline{v}^q(x,y)$ *on* $\mathbb{R}\times(0,\infty)$ *and* $\underline{v}^q(x,y)$ *on* $\mathbb{R}\times(-\infty,0)$. *They have representations:*

$$\overline{v}^q(x,y) = \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} 1_{(-\infty,y)}(x)\, b_n\hat{b}_m z_n\hat{z}_m\, \frac{e^{\hat{z}_m x}}{e^{y(z_n+\hat{z}_m)}},$$

$$\underline{v}^q(x,y) = \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} 1_{(y,\infty)}(x)\, b_n\hat{b}_m z_n\hat{z}_m\, \frac{e^{y(z_n+\hat{z}_m)}}{e^{z_n x}}.$$

Moreover the density $v^q(x)$ *of the marginal distribution of* $L_{\mathbf{e}_q}$ *has representations*

$$\begin{aligned} v^q(x) &= \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} \frac{b_n\hat{b}_m z_n\hat{z}_m}{z_n+\hat{z}_m}\, \frac{e^{\min(0,x)(z_n+\hat{z}_m)}}{e^{xz_n}} \qquad (2.1) \\ &= q\left(1_{\{x>0\}}\sum_{n\in\mathbb{N}}\frac{e^{-z_n x}}{\varphi'(z_n)} - 1_{\{x<0\}}\sum_{n\in\mathbb{N}}\frac{e^{\hat{z}_n x}}{\varphi'(-\hat{z}_n)}\right). \end{aligned}$$

Proof. As $\underline{L}_{\mathbf{e}_q}$ and $\overline{L}_{\mathbf{e}_q}$ are independent by the Wiener-Hopf factorization (see Theorem 1.4) we obtain the joint density of $(\overline{L}_{\mathbf{e}_q}, \underline{L}_{\mathbf{e}_q})$ as

$$\overline{v}^q(x)\underline{v}^q(y) = \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} 1_{(0,\infty)}(x)1_{(-\infty,0)}(y)\, b_n\hat{b}_m z_n\hat{z}_m\, \frac{e^{y\hat{z}_m}}{e^{xz_n}}$$

(see Theorem 1.5) since $\mathbb{Q}(\overline{L}_t = 0) = \mathbb{Q}(\underline{L}_t = 0) = 0$ by assumption. As $L_{\mathbf{e}_q} \overset{\mathrm{d}}{=} \overline{L}_{\mathbf{e}_q} + \underline{L}_{\mathbf{e}_q}$ by the Wiener-Hopf factorization we find $\overline{v}^q(x,y)$ and $\underline{v}^q(x,y)$ by substitutions $(x,y) \mapsto (x+y, x)$ and $(x,y) \mapsto (x+y, y)$. Then the marginal distribution of $L_{\mathbf{e}_q}$ has density

$$v^q(x) = \int_{-\infty}^{\min(0,x)} \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} b_n\hat{b}_m z_n\hat{z}_m\, e^{-z_n x} e^{y(z_n+\hat{z}_m)} dy$$

Interchanging the order of integration and summation is justified by Fubini's theorem as all terms are non-negative. This results in the first representation. Using this representation of $v^q(x)$ we can derive the moment generation function of $L_{\mathbf{e}_q}$ as $\int_{\mathbb{R}} e^{zx}v^q(x)$ for the vertical strip $-\hat{z}_1 < \mathrm{Re}(z) < z_1$. On the other hand we know that the moment generation function of $L_{\mathbf{e}_q}$ is given by $q(q-\varphi(z))^{-1}$. Thus we arrive at the equation

$$\begin{aligned} \frac{q}{q-\varphi(z)} &= \int_{-\infty}^{0} \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} \frac{b_n\hat{b}_m z_n\hat{z}_m}{z_n+\hat{z}_m}\, e^{x(z+\hat{z}_m)} + \int_0^{\infty} \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} \frac{b_n\hat{b}_m z_n\hat{z}_m}{z_n+\hat{z}_m} e^{x(z-z_n)} \\ &= \sum_{n\in\mathbb{N}} \left(\sum_{m\in\mathbb{N}} \frac{b_n\hat{b}_m z_n\hat{z}_m}{z_n+\hat{z}_m} \right) \frac{1}{z_n - z} + \sum_{m\in\mathbb{N}} \left(\sum_{n\in\mathbb{N}} \frac{b_n\hat{b}_m z_n\hat{z}_m}{z_n+\hat{z}_m} \right) \frac{1}{\hat{z}_m + z} \end{aligned} \tag{2.2}$$

for $-\hat{z_1} < \mathrm{Re}(z) < z_1$. Both sides have a meromorphic continuation to $\mathbb{C}$ with poles at $(z_n)_{n\in\mathbb{N}}$ and $(-\hat{z}_n)_{n\in\mathbb{N}}$. Comparing the residues at the poles we find:

$$\frac{q}{-\varphi'(z_n)} = -\left(\sum_{m\in\mathbb{N}} \frac{b_n\hat{b}_m z_n\hat{z}_m}{z_n+\hat{z}_m} \right) \quad \text{and} \quad \frac{q}{-\varphi'(-\hat{z}_n)} = \left(\sum_{n\in\mathbb{N}} \frac{b_n\hat{b}_m z_n\hat{z}_m}{z_n+\hat{z}_m} \right).$$

Inserting these expressions in the first representation gives the second representation of $v^q(x)$. □

2.2 Simple options

Simple options only depend on the terminal value of the stock, e.g. these options have payoff functions of the form $g(L_t)$. Most important examples are put and call options with arbitrage-free prices

$$P_t := e^{-rt}\mathbb{E}_{\mathbb{Q}}\left((K - S_t)_+\right) \quad \text{and} \quad C_t := e^{-rt}\mathbb{E}_{\mathbb{Q}}\left((S_t - K)_+\right)$$

for some strike $K > 0$. With the density of $L_{\mathbf{e}_{u+r}}$ given in Theorem 2.1 we can derive the Laplace transforms of C_t and P_t. Formulas for the Laplace transforms in the DEJD model can be found in Sepp (2004).

Theorem 2.1. *In a meromorphic model the Laplace transforms with respect to time of the arbitrage-free prices of call and put options have representations*

$$\mathscr{L}(C_t)(u) = \sum_{n\in\mathbb{N}} 1_{\{k\geq 0\}} \frac{Ke^{-z_n k}}{z_n(z_n-1)\varphi'(z_n)} - 1_{\{k<0\}}\left(\frac{Ke^{\hat{z}_n k}}{\hat{z}_n(\hat{z}_n+1)\varphi'(-\hat{z}_n)} - \frac{S_0}{u} + \frac{K}{u+r}\right),$$

$$\mathscr{L}(P_t)(u) = \mathscr{L}(C_t)(u) + \frac{K}{r+u} - \frac{S_0}{u}$$

with $k := \log(K) - \log(S_0)$.

Proof. The Laplace transform of the price of call options can be displayed as

$$\mathscr{L}(C_t)(u) = S_0\mathscr{A}(1) - K\mathscr{A}(0) \quad \text{with} \quad \mathscr{A}(a) := \frac{\mathbb{E}(e^{aL_{\mathbf{e}_{u+r}}} 1_{\{L_{\mathbf{e}_{u+r}}>k\}})}{u+r}. \tag{2.3}$$

We consider two cases:
1ST CASE. The call out of the money or at the money, i.e $k \geq 0$: Using the density of $L_{\mathbf{e}_{u+r}}$ given in Theorem 2.1 we find for $a \in [0, z_1)$:

$$\mathscr{A}(a) = \sum_{n\in\mathbb{N}} \int_k^\infty \frac{e^{x(a-z_n)}}{\varphi'(z_n)} dx = \sum_{n\in\mathbb{N}} \frac{e^{k(a-z_n)}}{(z_n - a)\varphi'(z_n)} \tag{2.4}$$

As all involved terms are non-negative interchanging the order of integration and summation is justified by Fubini's theorem. Next we show that (2.4) holds for $a \in \{0, 1\}$, i.e. $z_1 > 1$. By the interlacing condition of Theorem 1.4 we have $z_1 \in (0, p_1)$ where $p_1 > 1$ by assumption. Note that $\varphi(z)$ is convex on $(-\hat{p}_1, p_1) \subset [0, 1]$ as

$$\varphi''(z) = \sigma^2 + 2\sum_{n\in\mathbb{N}} \left(\frac{a_n p_n}{(p_n - z)^3} + \frac{\hat{a}_n \hat{p}_n}{(\hat{p}_n + z)^3}\right) > 0 \quad \text{for } z \in (-\hat{p}_1, p_1).$$

Since $\varphi(0) = 0$ and $\varphi(1) = r$ by the EMM condition we have $\varphi(z) < r$ for $z \in [0, 1]$. Consequently $z_1 > 1$ since $\varphi(z_1) = r + u > 0$ by definition. Inserting (2.4) in (2.3) concludes the first case.

2ND CASE. The call is in the money, i.e $k < 0$: Using the density of $L_{\mathbf{e}_{u+r}}$ given in Theorem 2.1 we find with Fubini's theorem for $a \in \mathbb{R}^+$:

$$\begin{aligned}\mathscr{A}(a) &= \sum_{n\in\mathbb{N}} \int_0^\infty \frac{e^{x(a-z_n)}}{\varphi'(z_n)} dx - \sum_{n\in\mathbb{N}} \int_k^0 \frac{e^{x(a+\hat{z}_n)}}{\varphi'(-\hat{z}_n)} dx \\ &= \sum_{n\in\mathbb{N}} \frac{1}{(z_n - a)\varphi'(z_n)} - \sum_{n\in\mathbb{N}} \frac{1 - e^{k(a+\hat{z}_n)}}{(a+\hat{z}_n)\varphi'(-\hat{z}_n)}\end{aligned} \tag{2.5}$$

Inserting (2.5) in (2.3) and some little algebraic manipulation yields in

$$\begin{aligned}\mathscr{L}(C_t)(u) = &S_0 \sum_{n\in\mathbb{N}} \left(\frac{1}{\varphi'(z_n)(z_n-1)} - \frac{1}{\varphi'(-\hat{z}_n)(\hat{z}_n+1)} \right) \\ &- K \sum_{n\in\mathbb{N}} \left(\frac{1}{\varphi'(z_n)z_n} - \frac{1}{\varphi'(-\hat{z}_n)\hat{z}_n} \right) \\ &- K \sum_{n\in\mathbb{N}} \frac{e^{k\hat{z}_n}}{\varphi'(-\hat{z}_n)} \left(\frac{1}{\hat{z}_n(1+\hat{z}_n)} \right).\end{aligned} \tag{2.6}$$

To conclude the second part of the proof it remains to show that

$$S_0 = \sum_{n\in\mathbb{N}} \left(\frac{1}{\varphi'(-\hat{z}_n)\hat{z}_n} - \frac{1}{\varphi'(z_n)z_n} \right) = \frac{S_0}{u}, \tag{2.7}$$

$$K \sum_{n\in\mathbb{N}} \left(\frac{1}{\varphi'(z_n)(z_n-1)} - \frac{1}{\varphi'(-\hat{z}_n)(\hat{z}_n+1)} \right) = \frac{K}{r+u}. \tag{2.8}$$

By the the EMM condition we have that $\mathscr{L}(e^{-rt}\mathbb{E}_{\mathbb{Q}}(S_t)) = \mathscr{L}(S_0) = \frac{S_0}{u}$. On the other hand we can use the density of $L_{\mathbf{e}_{u+r}}$ given in Theorem 2.1 to find

$$\mathscr{L}\left(e^{-rt}\mathbb{E}_{\mathbb{Q}}\left(S_t\right)\right)(u) = S_0\mathbb{E}_{\mathbb{Q}}(e^{L_{\mathbf{e}_{u+r}}}) = S_0 \sum_{n\in\mathbb{N}} \left(\frac{1}{\varphi'(z_n)(z_n-1)} - \frac{1}{\varphi'(-\hat{z}_n)(\hat{z}_n+1)} \right).$$

Combining this two results gives (2.7). As $v^{u+r}(x)$ is a probability density we find $\int_{\mathbb{R}} v^{u+r}(x)dx = 1$. Computing the integral reveals that

$$\int_{\mathbb{R}} v^{u+r}(x)dx = (u+r) \sum_{n\in\mathbb{N}} \left(\frac{1}{\varphi'(z_n)z_n} - \frac{1}{\varphi'(-\hat{z}_n)\hat{z}_n} \right).$$

Thus we obtain (2.8). Finally the Laplace transform of P_t is the given by the put-call parity, i.e.

$$P_t - C_t = e^{-rt}\mathbb{E}_{\mathbb{Q}}\left(K - S_t\right) = e^{-rt}K - S_0.$$

$\square$

2.3 Barrier options

One of the most popular options with a payoff depending on the maximum or minimum stock process are barrier options. There exist two types of barrier options: Knock-out barrier options have the same payoff as corresponding simple options, but the additional feature that the contract gets worthless if the price of the asset crosses a barrier before maturity. Knock-in barrier options in contrast are worthless unless the price of the asset crosses a barrier before maturity.

We study in this section the standard examples of a down-and-in put, an up-and-in call, a down-and-out call and an up-and-out put with strike $K \in \mathbb{R}^+$ and barrier $C \in \mathbb{R}^+$. Their arbitrage-free prices are given by:

$$\begin{aligned}
\mathrm{P}_t^{\mathrm{DI}} &:= e^{-rt}\mathbb{E}_{\mathbb{Q}}\left(1_{\{\underline{S}_t<C\}}(K-S_t)_+\right) \quad &&\text{for} \quad \min(S_0,K)>C,\\
\mathrm{C}_t^{\mathrm{UI}} &:= e^{-rt}\mathbb{E}_{\mathbb{Q}}\left(1_{\{\overline{S}_t>C\}}(S_t-K)_+\right) \quad &&\text{for} \quad \max(S_0,K)<C,\\
\mathrm{C}_t^{\mathrm{DO}} &:= e^{-rt}\mathbb{E}_{\mathbb{Q}}\left(1_{\{\underline{S}_t>C\}}(S_t-K)_+\right) \quad &&\text{for} \quad \min(S_0,K)>C,\\
\mathrm{P}_t^{\mathrm{UO}} &:= e^{-rt}\mathbb{E}_{\mathbb{Q}}\left(1_{\{\overline{S}_t<C\}}(K-S_t)_+\right) \quad &&\text{for} \quad \max(S_0,K)<C
\end{aligned}$$

where $\overline{S}_t := S_0 \exp\left(\overline{L}_t\right)$ and $\underline{S}_t := S_0 \exp\left(\underline{L}_t\right)$.

The option prices for the excluded areas of (S_0, K, C) can be easily traced back to call and put options (Theorem 2.1) and thus are not considered in this section.

In the Black Scholes framework analytic formulas for the arbitrage-free prices have been first derived in Rubinstein & Reiner (1991) and Rich (1994). For the DEJD and HELD model formulas for the Laplace transforms of the prices with respect to time can be found in Kou & Wang (2004), Sepp (2005) and Jeannin & Pistorius (2010). Results Lévy processes for in general are given be Kudryavtsev & Levendorskiĭ (2011).

With the help of Theorem 1.5 and Proposition 2.1 we can derive the Laplace transforms of these arbitrage-free prices in the meromorphic model in the same way as Jeannin & Pistorius (2010), Proposition 4.2 for the HEJD model:

Theorem 2.2. *In a meromorphic model the Laplace transforms with respect to time of the arbitrage-free prices of barrier call and put options have representations*

$$\begin{aligned}
\mathscr{L}(P_t^{\mathrm{DI}})(u) =& \frac{K}{u+r}\sum_{m\in\mathbb{N}}\hat{b}_m e^{\hat{z}_m c}\left(\sum_{n\in\mathbb{N}}\frac{\hat{z}_m b_n e^{z_n(c-k)}}{(z_n+\hat{z}_m)(z_n-1)}+1\right)\\
&-\frac{C}{u+r}\left(1-\sum_{n\in\mathbb{N}}\frac{b_n}{1-z_n}\right)\sum_{m\in\mathbb{N}}\frac{\hat{z}_m\hat{b}_m}{1+\hat{z}_m}e^{\hat{z}_m c},
\end{aligned}$$

$$\begin{aligned}\mathscr{L}(C_t^{\mathrm{UI}})(u) =& \frac{K}{u+r}\sum_{n\in\mathbb{N}} b_n e^{-z_n c}\left(\sum_{m\in\mathbb{N}} \frac{z_n \hat{b}_m e^{\hat{z}_m(k-c)}}{(z_n+\hat{z}_m)(\hat{z}_m+1)} - 1\right)\\ &+ \frac{C}{u+r}\left(1-\sum_{m\in\mathbb{N}}\frac{\hat{b}_m}{1+\hat{z}_m}\right)\sum_{n\in\mathbb{N}}\frac{z_n b_n}{z_n-1}e^{-z_n c},\end{aligned}$$

$$\mathscr{L}(\mathrm{C}_t^{\mathrm{DO}})(u) = \mathscr{L}(\mathrm{C}_t)(u) + \frac{K}{u+r}\sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}\frac{b_n\hat{b}_m\hat{z}_m}{z_n-1}e^{z_n(c-k)+\hat{z}_m c},$$

$$\mathscr{L}(\mathrm{P}_t^{\mathrm{UO}})(u) = \mathscr{L}(\mathrm{P}_t)(u) - \frac{K}{u+r}\sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}\frac{b_n\hat{b}_m z_n}{\hat{z}_m+1}e^{\hat{z}_m(k-c)-z_n c}$$

where $k:=\log(K)-\log(S_0)$ and $c:=\log(C)-\log(S_0)$. The sequences $(z_n)_{n\in\mathbb{N}}$ and $(\hat{z}_n)_{n\in\mathbb{N}}$ are defined in Theorem 1.4 with $q=u+r$, the sequences $(b_n)_{n\in\mathbb{N}}$ and $(\hat{b}_n)_{n\in\mathbb{N}}$ are defined in Theorem 1.5.

Proof. The Laplace transforms of the prices of a down-and-in put and an up-and-in call can be displayed as

$$\mathscr{L}(P_t^{\mathrm{DI}})(u) = \frac{K\mathscr{B}(0)-S_0\mathscr{B}(1)}{u+r} \quad \text{and} \quad \mathscr{L}(C_t^{\mathrm{UI}})(u) = \frac{S_0\mathscr{C}(1)-K\mathscr{C}(0)}{u+r} \tag{2.9}$$

where for $a\in\{0,1\}$ we have

$$\begin{aligned}\mathscr{B}(a) &:= \mathbb{E}_{\mathbb{Q}}\left(e^{aL_{\mathbf{e}_{u+r}}}1_{\{L_{\mathbf{e}_{u+r}}<k\}}1_{\{\underline{L}_{\mathbf{e}_{u+r}}<c\}}\right) \quad \text{with} \quad c<0,\ k>c,\\ \mathscr{C}(a) &:= \mathbb{E}_{\mathbb{Q}}\left(e^{aL_{\mathbf{e}_{u+r}}}1_{\{L_{\mathbf{e}_{u+r}}>k\}}1_{\{\overline{L}_{\mathbf{e}_{u+r}}>c\}}\right) \quad \text{with} \quad c>0,\ k<c.\end{aligned}$$

Using the joint density of $(L_{\mathbf{e}_{u+r}}, \underline{L}_{\mathbf{e}_{u+r}})$ given in Theorem 2.1 we find for $a\in\mathbb{R}^+$ with Fubini's Theorem:

$$\begin{aligned}\mathscr{B}(a) &= \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} z_n\hat{z}_m b_n\hat{b}_m\int_{-\infty}^{c}\int_{y}^{k} e^{(a-z_n)x}e^{(z_n+\hat{z}_m)y}dxdy\\ &= \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} z_n\hat{z}_m b_n\hat{b}_m\int_{-\infty}^{c}\frac{e^{(z_n+\hat{z}_m)y}}{a-z_n}\left(e^{(a-z_n)k}-e^{(a-z_n)y}\right)dy\\ &= \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}\frac{z_n\hat{z}_m b_n\hat{b}_m}{a-z_n}\left(\frac{e^{(a-z_n)k+(z_n+\hat{z}_m)c}}{z_n+\hat{z}_m}-\frac{e^{(a+\hat{z}_m)c}}{a+\hat{z}_m}\right).\end{aligned}$$

Similarly we arrive with the density of $(L_{\mathbf{e}_{u+r}}, \overline{L}_{\mathbf{e}_{u+r}})$ for $a\in[0,z_1)$ at:

$$\begin{aligned}\mathscr{C}(a) &= \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} z_n\hat{z}_m b_n\hat{b}_m\int_{c}^{\infty}\int_{k}^{y} e^{(a+\hat{z}_m)x}e^{-(z_n+\hat{z}_m)y}dxdy\\ &= \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} z_n\hat{z}_m b_n\hat{b}_m\int_{c}^{\infty}\frac{e^{-(z_n+\hat{z}_m)y}}{a+\hat{z}_m}\left(e^{(a+\hat{z}_m)y}-e^{(a+\hat{z}_m)k}\right)dy\\ &= \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}\frac{z_n\hat{z}_m b_n\hat{b}_m}{a+\hat{z}_m}\left(\frac{e^{(a-z_n)c}}{z_n-a}-\frac{e^{(a+\hat{z}_m)k-(z_n+\hat{z}_m)c}}{z_n+\hat{z}_m}\right).\end{aligned}$$

In the proof of Theorem 2.1 we have already verified that $z_1 > 1$. Inserting this representations of $\mathscr{B}(a)$ and $\mathscr{C}(a)$ in (2.9) and a little algebraic manipulation concludes the first part of the proof. Note that we used the identities

$$\sum_{n\in\mathbb{N}} \frac{z_n b_n}{z_n - 1} = 1 + \sum_{n\in\mathbb{N}} \frac{b_n}{z_n - 1} \quad \text{and} \quad \sum_{n\in\mathbb{N}} \frac{\hat{z}_n \hat{b}_n}{1+\hat{z}_n} = 1 - \sum_{n\in\mathbb{N}} \frac{\hat{b}_n}{1+\hat{z}_n}$$

to simplify the expressions. These identities are based on the property $\sum_{n\in\mathbb{N}} b_n = \sum_{n\in\mathbb{N}} \hat{b}_n = 1$ which can be verified easily with Theorem 1.5:

$$\sum_{n\in\mathbb{N}} b_n = \int_0^\infty \overline{v}^{u+r}(x)dx = 1 \quad \text{and} \quad \sum_{n\in\mathbb{N}} \hat{b}_n = \int_{-\infty}^0 \underline{v}^{u+r}(x)dx = 1.$$

Second, the Laplace transforms of the prices of a down-and-out call and an up-and-out put can be displayed as

$$\mathscr{L}(\mathrm{C}_t^{\mathrm{DO}})(u) = \mathscr{L}(\mathrm{C}_t)(u) - \frac{S_0\mathscr{D}(1) - K\mathscr{D}(0)}{u+r},$$

$$\mathscr{L}(\mathrm{P}_t^{\mathrm{UO}})(u) = \mathscr{L}(\mathrm{P}_t)(u) - \frac{K\mathscr{E}(0) - S_0\mathscr{E}(1)}{u+r} \tag{2.10}$$

where for $a \in \{0,1\}$ we have

$$\mathscr{D}(a) := \mathbb{E}\left(e^{aL_{\mathbf{e}_{u+r}}} 1_{\{L_{\mathbf{e}_{u+r}}>k\}} 1_{\{\underline{L}_{\mathbf{e}_{u+r}}<c\}}\right) \quad \text{with} \quad c<0,\ k>c,$$

$$\mathscr{E}(a) := \mathbb{E}\left(e^{aL_{\mathbf{e}_{u+r}}} 1_{\{L_{\mathbf{e}_{u+r}}<k\}} 1_{\{\overline{L}_{\mathbf{e}_{u+r}}>c\}}\right) \quad \text{with} \quad c>0,\ k<c.$$

Using the densities of $(L_{\mathbf{e}_{u+r}}, \underline{L}_{\mathbf{e}_{u+r}})$ and $(L_{\mathbf{e}_{u+r}}, \overline{L}_{\mathbf{e}_{u+r}})$ given in Theorem 2.1 we find for $a < z_1$ and $a \in \mathbb{R}$, respectively with Fubini's Theorem:

$$\begin{aligned}
\mathscr{D}(a) &= \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} z_n\hat{z}_m b_n\hat{b}_m \int_{-\infty}^{c}\int_{k}^{\infty} e^{(a-z_n)x}e^{(z_n+\hat{z}_m)y}dxdy\\
&= \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} \frac{z_n\hat{z}_m b_n\hat{b}_m}{(z_n-a)(z_n+\hat{z}_m)} e^{(a-z_n)k+(z_n+\hat{z}_m)c},\\
\mathscr{E}(a) &= \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} z_n\hat{z}_m b_n\hat{b}_m \int_{c}^{\infty}\int_{-\infty}^{k} e^{(a+\hat{z}_m)x}e^{-(z_n+\hat{z}_m)y}dxdy\\
&= \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} \frac{z_n\hat{z}_m b_n\hat{b}_m}{(a+\hat{z}_m)(z_n+\hat{z}_m)} e^{k(a+\hat{z}_m)-c(z_n+\hat{z}_m)}.
\end{aligned}$$

Inserting this representations of $\mathscr{C}(a)$ and $\mathscr{D}(a)$ in (2.10) gives the result. $\square$

2.4 Lookback options

Another popular example of path-dependent options with a payoff depending on the maximum or minimum asset price over the life of the option are lookback options. Standard or floating strike lookback put and call options enable the holder to sell a the lowest (highest) prices observed during a period. Their arbitrage-free prices are:

$$\begin{aligned}\mathrm{LC}_t &:= e^{-rt}\mathbb{E}_{\mathbb{Q}}\left(S_t - \underline{S}_t\right),\\ \mathrm{LP}_t &:= e^{-rt}\mathbb{E}_{\mathbb{Q}}\left(\overline{S}_t - S_t\right).\end{aligned}$$

where $\overline{S}_t := \overline{S}_0 \exp\left(\overline{L}_t\right)$ and $\underline{S}_t := \underline{S}_0 \exp\left(\underline{L}_t\right)$ with $\overline{S}_0 \in [S_0, \infty)$ and $\underline{S}_0 \in (0, S_0]$. Note that we do not assume $\overline{S}_0 = S_0 = \underline{S}_0$ as after the date of issue the already observed minimum (maximum) of the asset may differ from its current value. So we can price these options during their whole life time.

In the Black Scholes framework analytic formulas for the arbitrage-free prices have been first derived in Goldman *et al.* (1979) and Conze & Viswanathan (1992). In the DEJD and HEJD model formulas for the Laplace transforms with respect to time of the prices are given by Kou & Wang (2004) and Sepp (2005). A representation in terms of the Wiener-Hopf factors of Laplace transforms of the arbitrage-free prices of floating strike put and call lookback options in an arbitrary Lévy market model can be found in Nguyen-Ngoc & Yor (2001) and Kudryavtsev & Levendorskiĭ (2011).

With the help of Theorem 1.5 we can derive the Laplace transforms of these arbitrage-free prices in the meromorphic model similarly as Kou & Wang (2004), Theorem 4.1 for the DEJD model:

Theorem 2.3. *In a meromorphic model the Laplace transforms with respect to time of the arbitrage-free prices of standard lookback put and call options have representations*

$$\begin{aligned}\mathscr{L}(\mathrm{LP}_t)(u) &= \frac{\overline{S}_0}{u+r}\left(\sum_{n\in\mathbb{N}} \frac{b_n}{z_n - 1} e^{-\overline{m} z_n} + 1\right) - \frac{S_0}{u},\\ \mathscr{L}(\mathrm{LC}_t)(u) &= \frac{\underline{S}_0}{u+r}\left(\sum_{n\in\mathbb{N}} \frac{\hat{b}_n}{1+\hat{z}_n} e^{\underline{m} z_n} + 1\right) - \frac{S_0}{u}\end{aligned}$$

where $\overline{m} := \log(\overline{S}_0) - \log(S_0)$ and $\underline{m} := \log(\underline{S}_0) - \log(S_0)$. The sequences $(z_n)_{n\in\mathbb{N}}$ and $(\hat{z}_n)_{n\in\mathbb{N}}$ are defined in Theorem 1.4 with $q = u + r$, the sequences $(b_n)_{n\in\mathbb{N}}$ and $(\hat{b}_n)_{n\in\mathbb{N}}$ are defined in Theorem 1.5.

Proof. Due to the EMM condition we have $\mathscr{L}(e^{-rt}\mathbb{E}_{\mathbb{Q}}(S_t))(u) = \frac{S_0}{u}$. Thus the Laplace transforms of the arbitrage-free prices of standard lookback put and call options can be displayed as

$$\mathscr{L}(\mathrm{LP}_t^{\mathrm{fix}})(u) = \frac{S_0}{u+r}\left(\mathscr{F}(1) - \mathscr{F}(0)\right) + \frac{\overline{S}_0}{u+r} - \frac{S_0}{u},$$

$$\mathscr{L}(\mathrm{LC}_t^{\mathrm{fix}})(u) = \frac{S_0}{u} - \left(\frac{S_0}{u+r}\left(\mathscr{G}(1) - \mathscr{G}(0)\right) + \frac{\underline{S}_0}{u+r}\right),$$

where for $a \in \{0,1\}$ we have

$$\mathscr{F}(a) := \mathbb{E}\left(e^{a\overline{L}_{\mathbf{e}_{u+r}}} 1_{\{\overline{L}_{\mathbf{e}_{u+r}} > \overline{m}\}}\right) \quad \text{and} \quad \mathscr{G}(a) := \mathbb{E}\left(e^{a\underline{L}_{\mathbf{e}_{u+r}}} 1_{\{\underline{L}_{\mathbf{e}_{u+r}} < \underline{m}\}}\right).$$

Note that for $\overline{S}_0 = S_0 = \underline{S}_0 \iff \overline{m} = \underline{m} = 0$ we can identify $\mathscr{F}(a)$ and $\mathscr{G}(a)$ as the Wiener-Hopf factors. Using the densities of $\underline{L}_{\mathbf{e}_{u+r}}$ and $\overline{L}_{\mathbf{e}_{u+r}}$ given in Theorem 1.5 we find with Fubini's Theorem for $a < z_1$ and $a \in \mathbb{R}^+$, respectively:

$$\mathscr{F}(a) = \sum_{n\in\mathbb{N}} z_n b_n \int_{\overline{m}}^{\infty} e^{(a-z_n)x} dx = \sum_{n\in\mathbb{N}} \frac{z_n b_n}{z_n - a} e^{(a-z_n)\overline{m}},$$

$$\mathscr{G}(a) = \sum_{n\in\mathbb{N}} \hat{z}_n \hat{b}_n \int_{-\infty}^{\underline{m}} e^{(a+\hat{z}_n)x} dx = \sum_{n\in\mathbb{N}} \frac{\hat{z}_n \hat{b}_n}{a + \hat{z}_n} e^{(a+\hat{z}_n)\underline{m}}.$$

Combining these results gives the two desired representations. □

Chapter 3

Numerical results

3.1 Numerical Laplace inversion

As the option prices in the meromorphic Lévy market model are deduced in terms of Laplace transforms which we cannot invert analytically we need a numerical inversion algorithm. A survey of Laplace inversion algorithms can be found in Cohen (2010). For our purpose we shall use the Gaver-Stehfest algorithm (see Stehfest (1970)). It has been used for the DEJD model by Kou (2002) and Sepp (2004) as well for the Beta class in the case $\gamma = \hat{\gamma} = 1$ by Schoutens & Van Damme (2011). The algorithm aims to approximate $f(x)$ on $x > 0$ by a sequence of functions with $n \in \mathbb{N}$ given by

$$f_n(x) = \frac{\log(2)}{x} \sum_{k=1}^{2n} w_k(n) \mathscr{L}(f) \left(\frac{k \log(2)}{x} \right) \tag{3.1}$$

where $\mathscr{L}(f)$ denotes the Laplace transform of $f(x)$, i.e.

$$\mathscr{L}(f)(z) := \int_0^\infty e^{-zx} f(x) dx.$$

The weights $(w_k)_{k=1}^{2n}$ only depend on $n \in \mathbb{N}$ and are given by

$$w_k(n) := \frac{(-1)^{n+k}}{n!} \sum_{j=\lfloor \frac{k+1}{2} \rfloor}^{\min(k,n)} j^{n+1} \binom{n}{j} \binom{2j}{j} \binom{j}{k-j}. \tag{3.2}$$

Kuznetsov (2013) lists four desirable properties of the Gaver-Stehfest algorithm:

- The algorithm only requires values of $\mathscr{L}(f)(z)$ for $\mathrm{Im}(z) = 0$.
- The weights $(w_k)_{k=1}^{2n}$ can be computed easily.
- The approximations $f_n(x)$ are linear in values of $\mathscr{L}(f)(z)$.

- For a constant function $f(x) = c$ we also have exact approximations for $n > 1$, i.e. $f_n(x) = c$.

As the weights $(w_k)_{k=1}^{2n}$ are growing rather fast with alternating signs a high-precision arithmetic is needed for the implementation of the Gaver-Stehfest algorithm. For a detailed study of the convergence of the algorithm see Kuznetsov (2013) too.

3.2 Numerical examples of option prices

For all our numerical examples of meromorphic models we will use an asset driven by a Lévy process L of the Beta class with infinite variation (see Theorem 1.8) satisfying $\mathbb{E}(\exp(L_t)) < \infty$, i.e $\min(\alpha\beta, \hat{\alpha}\hat{\beta}) > 1$ (see Theorem 1.7 and 1.1) under the equivalent martingale measure. We consider two different types of models with three different parameter sets. For the first set of three Lévy models we choose:

$$\textbf{Model 1:}\quad (\sigma, c, \alpha, \beta, \gamma, \hat{c}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}) = (\sigma, 1, 5, 5, \gamma, 1, 3, 3, \hat{\gamma}):$$

$$\textit{Set 1: } (\sigma, \gamma, \hat{\gamma}) = (0.23, 0.5, 0.5),$$

$$\textit{Set 2: } (\sigma, \gamma, \hat{\gamma}) = (0.22, 1.5, 1.5),$$

$$\textit{Set 3: } (\sigma, \gamma, \hat{\gamma}) = (0.001, 2.5, 2.5).$$

Thus according to Theorem 1.8 we have finite activity of the jump component in Set 1 and paths of bounded variation for the jump part in Set 1 and 2. For all three sets the return distribution of the asset agrees up to the second digit on a volatility of 0.24. We find a negative skew of -0.08 (Set 1), -0.13 (Set 2) and -0.31 (Set 3). Further we have an excess kurtosis of 0.15 (Set 1), 0.20 (Set 2) and 0.34 for Set 3. For the second set of Lévy models we choose:

$$\textbf{Model 2:}\quad (\sigma, c, \alpha, \beta, \gamma, \hat{c}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}) = (\sigma, 1, 1, 5, \gamma, 1, 1, 4, \hat{\gamma})$$

$$\textit{Set 1: } (\sigma, \gamma, \hat{\gamma}) = (0.24, 0.5, 0.5),$$

$$\textit{Set 2: } (\sigma, \gamma, \hat{\gamma}) = (0.21, 1.5, 1.5),$$

$$\textit{Set 3: } (\sigma, \gamma, \hat{\gamma}) = (0.02, 2.5, 2.5)$$

Similarly we find finite activity of the jump component in Set 1 and paths of bounded variation for the jump part in Set 1 and 2. For all three sets the return distribution of the asset agrees up to the second digit on a higher volatility of 0.33. We find a higher negative skew of -0.40 (Set 1), -0.44 (Set 2) and -0.49 (Set 3) and an excess kurtosis of 1.35, 1.41 and 1.46.

We compare our six meromorphic models to DEJD models and Black Scholes models. For the Black Scholes model we choose the volatility of the asset returns to agree with the volatility of the returns in the meromorphic model. In the DEJD model the Brownian part of the Lévy processes, volatility and skew of the asset returns and as well the degree of the exponential decay of the Lévy measure agrees with the meromorphic model. Thus we have in the DEJD model a Lévy density for $x \in \mathbb{R}$ of the form

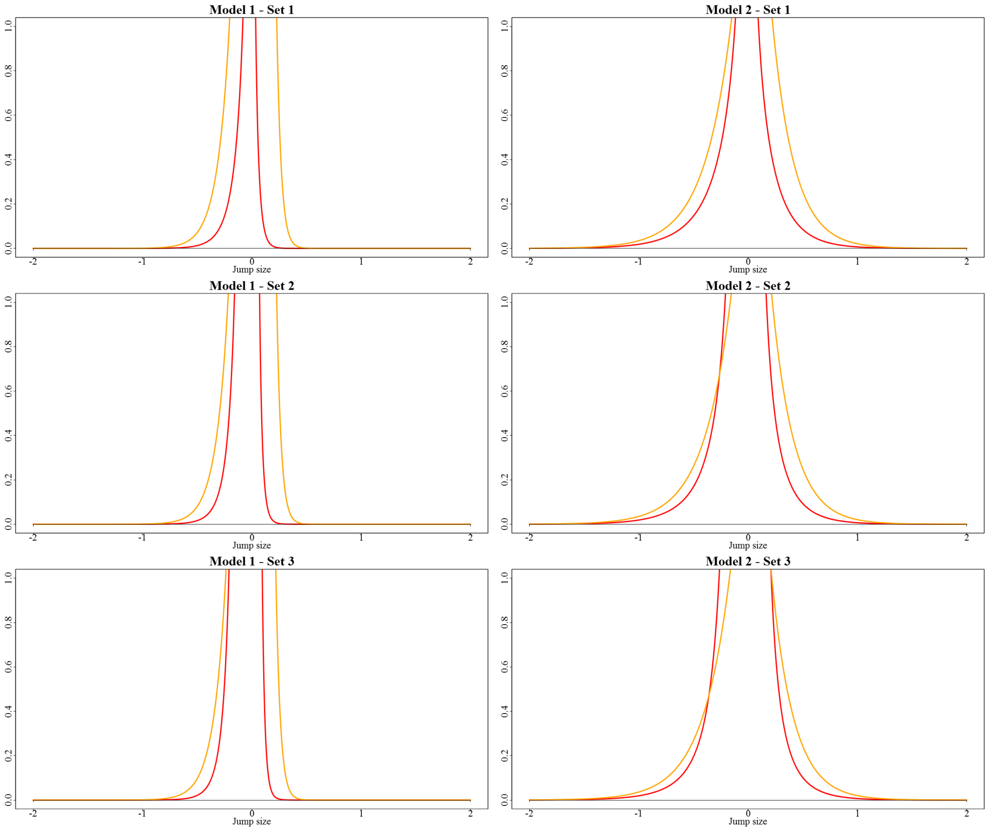
$$f_\nu(x) = 1_{\{x>0\}} a e^{-px} + 1_{\{x<0\}} \hat{a} e^{px} \quad \text{with} \quad p = \alpha\beta,\, \hat{p} = \hat{\alpha}\hat{\beta}. \tag{3.3}$$

The intensities a and $\hat{a}$ are uniquely determined be the second and third moment of the asset returns in the meromorphic model, i.e $(a, \hat{a})$ satisfy (see Theorem 1.2)

$$\begin{pmatrix} \frac{2}{(\alpha\beta)^2} & \frac{2}{(\hat{\alpha}\hat{\beta})^2} \\ \frac{6}{(\alpha\beta)^3} & -\frac{6}{(\hat{\alpha}\hat{\beta})^3} \end{pmatrix} \begin{pmatrix} a \\ \hat{a} \end{pmatrix} = \begin{pmatrix} \kappa_2(L_1) \\ \kappa_3(L_1) \end{pmatrix} \tag{3.4}$$

where $\kappa_n(L_1)$ denotes the n^{th} cumulant of the meromorphic Lévy process (see Theorem 1.8 for an analytic representation).

In Figure 3.1 we see that in several cases the Lévy densities of the associated DEJD models differ severely from the corresponding densities in meromorphic models. Table 3.1 reveals that both the DEJD and the Black Scholes model fails to reproduce the arbitrage-free prices in the meromorphic model for several different types of options. We conclude that for markets where an assets jump behavior cannot be describe sufficiently by a DEJD process option prices will be quite different in a probably more suitable meromorphic model.



(a) Levy densities for Model 1 (b) Levy densities for Model 2

Figure 3.1: Lévy densities in the six different settings for the meromorphic model (red) and the associated DEJD model (orange).

Put: $S_0 = 100, K = 90$			
$\gamma = \hat{\gamma}$	Meromorphic	DEJD	Black Scholes
Set 1	8.33	6.21	3.41
Set 2	3.50	6.05	3.51
Set 3	3.41	3.16	3.37
Up-and-in call: $S_0 = 100, K = 100, C = 120$			
$\gamma = \hat{\gamma}$	Meromorphic	DEJD	Black Scholes
Set 1	11.08	14.89	11.13
Set 2	11.16	14.66	11.28
Set 3	10.59	9.90	11.06
Lookback put: $S_0 = 100, \overline{S}_0 = 110$			
$\gamma = \hat{\gamma}$	Meromorphic	DEJD	Black Scholes
Set 1	14.47	19.09	14.88
Set 2	14.44	18.68	15.06
Set 3	13.40	12.40	14.79

Put: $S_0 = 100, K = 90$			
$\gamma = \hat{\gamma}$	Meromorphic	Double Exponential	Brownian motion
Set 1	5.63	7.87	6.28
Set 2	5.57	7.15	6.27
Set 3	5.57	4.60	6.30
Up-and-in call: $S_0 = 100, K = 100, C = 120$			
$\gamma = \hat{\gamma}$	Meromorphic	Double Exponential	Brownian motion
Set 1	13.87	16.76	15.04
Set 2	13.66	15.76	15.03
Set 3	13.24	10.78	15.06
Lookback put: $S_0 = 100, \overline{S}_0 = 110$			
$\gamma = \hat{\gamma}$	Meromorphic	Double Exponential	Brownian motion
Set 1	16.89	19.70	20.19
Set 2	16.30	18.17	20.17
Set 3	15.09	12.56	20.22

Table 3.1: Arbitrage-free prices of three different options in a meromorphic model (Model 1 and Model 2) and the corresponding DEJD and Black Scholes model. For computing the Gaver-Stehfest algorithm with $n = 7$ has been used.

3.3 R Code

All computational work has been done in R. The code requires the package `numDeriv`. To compute option prices in the Black Scholes model the packages `fOptions` and `fExoticOptions` may be used (see Wuertz (2013b) and Wuertz (2013a) for details).

The parameters of the Lévy process in the Beta class are defined globally as `a1` $= \alpha$, `a2` $= \hat{\alpha}$, `b1` $= \beta$, `b2` $= \hat{\beta}$, `c1` $= c$, `c2` $= \hat{c}$, `l1` $= \gamma$, `l2` $= \hat{\gamma}$ and `sigma` $= \sigma$. Similarly we have for the double exponential Lévy jump diffusion process `a1d` $= a$, `a2d` $= \hat{a}$, `p` $= p$ and `p2` $= \hat{p}$. The parametrization of the DEJD model is done according to (3.3) and (3.4). For computing the second and third cumulant `k2` and `k3` of the Beta class Lévy process see Theorem 1.8:

```
Beta<-function(a,b) gamma(a)*gamma(b)/gamma(a+b)
psi1<-function(n) psigamma(a1,n)-psigamma(a1+1-l1,n)
psi2<-function(n) psigamma(a2,n)-psigamma(a2+1-l2,n)

k2<-sigma^2+c1/b1^3*Beta(a1,1-l1)*(psi1(0)^2+psi1(1))
 +c2/b2^3*Beta(a2,1-l2)*(psi2(0)^2+psi2(1))
k3<-c1/b1^4*Beta(a1,1-l1)*(psi1(0)^3+3*psi1(0)*psi1(1)+psi1(2))
 +c2/b2^4*Beta(a2,1-l2)*(psi2(0)^3+3*psi2(0)*psi2(1)+psi2(2))

p1<-a1*b1
p2<-a2*b2
a1d<-solve(matrix(c(2/p1^2,6/p1^3,2/p2^2,-6/p2^3),nrow=2),c(k2,k3))[1]
a2d<-solve(matrix(c(2/p1^2,6/p1^3,2/p2^2,-6/p2^3),nrow=2),c(k2,k3))[2]
```

Next we define the Laplace exponents `lap.exp` and `lapd.exp` of the Beta class and DEJD Lévy process (see Theorem 1.1 and 1.7) with the EMM condition being satisfied. The interest rate is defined globally as `r` $= r$.

```
lap<-function(z,a) sigma^2*z^2/2+a*z+c1/b1*(Beta(a1-z/b1,1-l1)
 -Beta(a1,1-l1))+c2/b2*(Beta(a2+z/b2,1-l2)-Beta(a2,1-l2))
lap.exp<-function(z) lap(z,r-lap(1,0))

lapd<-function(z,a) sigma^2*z^2/2+a*z+z^2*a1d/(p1*(p1-z))
 +z^2*a2d/(p2*(p2+z))
lapd.exp<-function(z) lapd(z,r-lap(1,0))
```

To compute the zeros of $\varphi_L(z) - u$ (see Theorem 1.4) we use the function `unitroot`. As `unitroot` requires closed intervals we replace e.g. the open interval

$$(\alpha\beta, (\alpha+1)\beta) \ni z_2$$

by the closed interval

$$[\beta\alpha + 10^{-6}, \beta(\alpha+1) - 10^{-6}].$$

For the meromorphic model we compute the first `n=75` positive roots `z(u)` and as well the first 75 negative roots `-zh(u)`. In the DEJD model we have two positive roots `zd(u)` and two negative roots `-zhd(u)`.

```
eps1<-10^(-10)
eps2<-10^6

zh<-function(u){
z<-numeric(n)
foo<-function(z) lap.exp(z)-u
z[1]<- -uniroot(foo, c(-b2*a2+eps1,-eps1),tol = 1e-10)$root
for(i in 2:n) z[i]<- -uniroot(foo,c(-(b2*(a2+i-1))+eps1,
 -(b2*(a2+i-2))-eps1),tol = 1e-10 )$root
z}

z<-function(u){
z<-numeric(2)
foo<-function(z) lap.exp(z)-u
z[1]<- uniroot(foo, c(eps1,a1*b1-eps1),tol = 1e-10)$root
for(i in 2:n) z[i]<- uniroot(foo,c(b1*(a1+i-2)+eps1,
 b1*(a1+i-1)-eps1),tol = 1e-10)$root
z}

zhd<-function(u){
z<-numeric(2)
foo2<-function(z){lapd.exp(z)-u}
z[1]<- -uniroot(foo2,c(-p2+eps1,-eps1),tol = 1e-10)$root
z[2]<- -uniroot(foo2,c(-p2-eps2,-p2-eps1),tol = 1e-10)$root
z}

zd<-function(u){
z<-numeric(2)
foo2<-function(z){lapd.exp(z)-u}
z[1]<- uniroot(foo2,c(eps1,p1-eps1),tol = 1e-10)$root
z[2]<- uniroot(foo2,c(p1+eps1,p1+eps2),tol = 1e-10)$root
z}
```

Next we compute the mixing weights of the densities of the Wiener-Hopf factors (see Theorem 1.5). In the Beta class case we have for the positive factor the 75 mixing weights `b(u)` and the for the negative factor 75 mixing weights `bh(u)`. For the DEJD model we obtain similarly in each case two mixing weights `bd(u)` and `bhd(u)`.

```
b<-function(u){
s<-numeric(n)
t<-numeric(n-1)
b<-numeric(n)
z<-z(u+r)
for(j in 1:n){
for(i in 1:n) s[i]<- 1 - z[j]/(b1*(a1+i-1))
if(j==1) for(i in 2:n) t[i-1]<- 1-z[j]/z[i]
else{
if(j==n) for(i in 1:(n-1)) t[i]<- 1-z[j]/z[i]
else{
 for(i in 1:(j-1)) t[i]<- 1-z[j]/z[i]
 for(i in (j+1):n) t[i-1]<- 1-z[j]/z[i]}}
b[j]<-prod(s)/prod(t)}
b}

bh<-function(u){
s<-numeric(n)
t<-numeric(n-1)
b<-numeric(n)
zh<-zh(u+r)
for(j in 1:n){
for(i in 1:n) s[i]<- 1-zh[j]/(b2*(a2+i-1))
if(j==1) for(i in 2:n) t[i-1]<- 1-zh[j]/zh[i]
else{
if(j==n) for(i in 1:(n-1)) t[i]<- 1-zh[j]/zh[i]
else{
 for(i in 1:(j-1)) t[i]<- 1-zh[j]/zh[i]
 for(i in (j+1):n) t[i-1]<- 1-zh[j]/zh[i]}}
b[j]<-prod(s)/prod(t)}
b}

bd<- function(u){
b<-numeric(2)
b[1]<- (1-zd(u+r)[1]/p1)/(1-zd(u+r)[1]/zd(u+r)[2])
b[2]<- (1-zd(u+r)[2]/p1)/(1-zd(u+r)[2]/zd(u+r)[1])
b}
```

```
bhd<- function(u){
b<-numeric(2)
b[1]<- (1-zhd(u+r)[1]/p2)/(1-zhd(u+r)[1]/zhd(u+r)[2])
b[2]<- (1-zhd(u+r)[2]/p2)/(1-zhd(u+r)[2]/zhd(u+r)[1])
b}
```

To implement the Gaver-Stehfest algorithm we first compute the weights `Q` as a function of `n` $= n$ as in (3.2). Then we can implement the algorithm as a function (`stehfest`) straightforward according to (3.1):

```
Q<-function(j,n){
q<-0
for (k in (floor((j+1)/2)):(min(j,n))) {
 q<-q+k^n*factorial(2*k)/(factorial(n-k)*factorial(k)*
    factorial(k-1)*factorial(j-k)*factorial(2*k-j))}
q<-(-1)^(n+j)*q
q}

stehfest<-function(func,u,n){
s<-numeric(2*n)
q<-numeric(2*n)
for (j in 1:(2*n)) {s[j]<-func(log(2)*j/u)
                    q[j]<-Q(j,n)}
crossprod(q,s)*log(2)/u }
```

Finally we implement the formulas for a call and put option (see Theorem 2.1: in the meromorphic model `call` and `put`, in the DEJD model `calld` and `putd`), for an up-and-in call (see Theorem 2.2: `uicall` and `uicalld`) and a lookback put (see Theorem 2.3: `floatput` and `floatputd`). Note that the strike K =`K`, the barrier C =`C`, the current value S_0 =`S` and the current maximum $\overline{S}_0$ =`M` needs to be defined globally.

```
call<-function(u){
k<-log(K/S)
z<-z(u+r)
zh<-zh(u+r)
if(k>0) sum(K*exp(-z*k)/(grad(lap.exp,z)*z*(z-1)))
else -sum(K*exp(zh*k)/(grad(lap.exp,-zh)*zh*(zh+1)))+S/u-K/(u+r)}

put<-function(u) call(u)+K/(u+r)-S/u
```

```
calld<-function(u){
k<-log(K/S)
z<-zd(u+r)
zh<-zhd(u+r)
if(k>0) sum(K*exp(-z*k)/(grad(foo,z)*z*(z-1)))
else -sum(K*exp(zh*k)/(grad(foo,-zh)*zh*(zh+1)))+S/u-K/(u+r)}

putd<-function(u) calld(u)+K/(u+r)-S/u

uicall<-function(u){
k<-log(K/S)
c<-log(C/S)
z<-z(u+r)
zh<-zh(u+r)
b<-b(u)
bh<-bh(u)
w<-numeric(n)
for (i in 1:n)  w[i]<- sum(z[i]*bh*exp(zh*(k-c))/((zh+z[i])*(zh+1)))-1
 K/(u+r)*sum(b*exp(-z*c)*w)+C/(u+r)*(1-sum(bh/(1+zh)))*
 sum(z*b*exp(-z*c)/(z-1))}

uicalld<-function(u){
k<-log(K/S)
c<-log(C/S)
z<-zd(u+r)
zh<-zhd(u+r)
b<-bd(u)
bh<-bhd(u)
w<-numeric(2)
for (i in 1:2) w[i]<- sum(z[i]*bh*exp(zh*(k-c))/((zh+z[i])*(zh+1)))-1
 K/(u+r)*sum(b*exp(-z*c)*w)+C/(u+r)*(1-sum(bh/(1+zh)))*
 sum(z*b*exp(-z*c)/(z-1))}

floatput<-function(u){
m<-log(M/S)
z<-z(u+r)
b<-b(u)
M/(u+r)*(1+sum(b/(z-1)*exp(-m*z)))-S/u}

floatputd<-function(u){
m<-log(M/S)
z<-zd(u+r)
b<-bd(u)
M/(u+r)*(1+sum(b/(z-1)*exp(-m*z)))-S/u}
```

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