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DIPLOMARBEIT

Flat Space Holographic Renormalization

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ABSTRACT

Einstein gravity on three-dimensional flat space is holographically renormalized by supplementing the bulk action with one half of the Gibbons–Hawking–York boundary term. One-point functions for the vacuum and flat space cosmologies are derived.

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CONTENTS

1 INTRODUCTION

1.1 HOLOGRAPHIC PRINCIPLE

The holographic principle arose from considerations on the number of degrees of freedom in a region of space. In classical field theories, as well as in their quantum mechanical analogs, fields can in principle take on distinct values at each point in space. The degrees of freedom in any finite region are said to be infinite. When considering quantum-gravity we are led to introducing a small distance cutoff, which makes the number of degrees of freedom finite and proportional to the volume of space under consideration. That this may still be a vast overestimation can be concluded [1] from observations on Bekenstein and Hawking's formula relating entropy of a black hole and area of its horizon,

$$S = \frac{A}{4G} \,. \tag{1.1}$$

When considering a region of space with an entropy higher than that of a corresponding black hole, the second law of thermodynamics can be violated by adding more mass to the system therefore creating a black hole. It follows that (1.1) gives an upper bound for the entropy inside a closed region of space with boundary area A. This leads to the idea that it might be possible to describe all phenomena within a region of space by a set of degrees of freedom on the surface bounding the region [2].

1.2 GAUGE/GRAVITY DUALITY

The *gauge/gravity duality* is a possible realization of the holographic principle. It is the general statement that there is a one-to-one correspondence between a quantum theory of gravity on a particular space on one hand, and a gauge theory on the boundary of space on the other hand. Its first and most famous realization is the proposed equivalence of a string theory on a space of constant negative curvature (anti-de Sitter space) and a supersymmetric Yang-Mills theory on its boundary [3]. Since the theory on the boundary has conformal symmetry the duality is often referred to as *AdS/CFT correspondence*, short for anti-de Sitter/conformal field theory correspondence.

The precise relation between the correlation functions in field theory and the supergravity or string theory action is [4, 5]

$$Z_S[\phi^{(0)}] = \left\langle \exp \int_{\partial M} \mathcal{O}\phi^{(0)} \right\rangle_{FT} \,, \tag{1.2}$$

where the right hand side is the generating functional of the field theory with $\phi^{(0)}$ acting as a source on the operator $\mathcal{O}$. The bulk partition function Z_S is calculated by integrating over all field configurations ϕ satisfying the boundary condition $\phi^{(0)}$,

$$Z_S[\phi^{(0)}] = \int_{\phi^{(0)}} \mathcal{D}\phi \exp(-\Gamma_S[\phi]) \ . \tag{1.3}$$

The fields $\phi^{(0)}$ act as sources for operators of the field theory. In the classical supergravity approximation one simply has

$$Z_S[\phi^{(0)}] = \exp(-\Gamma_S[\phi^{(0)}]) \ , \tag{1.4}$$

where $\Gamma_S[\phi^{(0)}]$ is the classical supergravity action evaluated on solution of the equations of motion satisfying boundary condition $\phi^{(0)}$.

1.3 VARIATIONAL PRINCIPLE

The *principle of stationary action* is ubiquitous in physics. It is a variational principle that can be applied to a mechanical system to obtain the equations of motion. [1] It is used to study classical mechanics as well as relativistic particles and fields in space. In quantum mechanics it appears in the semi-classical approximation to the path integral as in (1.4). This quantum mechanical connection makes it relevant to the gauge/gravity duality.

A well defined principle of stationary action (cf. section 7.1.2 in [6]) consists of an action functional Γ and functionals B_i serving as boundary conditions such that:

(i) Evaluating $\delta\Gamma = 0$ under the conditions $\delta B_i = 0$ yields just the equations of motion without additional constraints.

(ii) Those equations of motion have a unique solution consistent with given values of the B_i.

For example the action of a point particle in its usual form,

$$\Gamma = \int_{t_A}^{t_B} \frac{\dot{x}(t)^2}{2} \, dt \ , \tag{1.5}$$

with variation

$$\delta\Gamma = \dot{x}(t)\,\delta x(t)\Big|_{t_A}^{t_B} - \int_{t_A}^{t_B} \ddot{x}(t)\,\delta x(t)\,dt \ , \tag{1.6}$$

leads to a well defined variational principle when the endpoints are kept fixed:

$$B_1 = x(t_A) \qquad B_2 = x(t_B) \tag{1.7}$$

1 The expressions “variational principle” and “principle of stationary action” will be used interchangeably.

A modified action with an additional boundary term,

$$\Gamma = \int_{t_A}^{t_B} \frac{\dot{x}(t)^2}{2}\, dt + x(t_A)\dot{x}(t_A) \tag{1.8a}$$

$$\delta\Gamma = x(t_A)\,\delta\dot{x}(t_A) + \dot{x}(t_B)\,\delta x(t_B) - \int_{t_A}^{t_B} \ddot{x}(t)\,\delta x(t)\, dt\,, \tag{1.8b}$$

gives a well defined variational principle for different boundary conditions:

$$B_1 = \dot{x}(t_A) \qquad B_2 = x(t_B) \tag{1.9}$$

In Einstein gravity a similar boundary term introduced by Gibbons, Hawking, and York [7, 8] is needed for Dirichlet boundary conditions, when considering a compact manifold with boundary M:

$$\Gamma = \frac{1}{16\pi G}\int_M R\,\boldsymbol{\epsilon} + \frac{1}{8\pi G}\int_{\partial M} K\,\tilde{\boldsymbol{\epsilon}} \tag{1.10}$$

The Gibbons–Hawking–York boundary term makes it possible to obtain a well defined variational principle when the metric is kept fixed on the boundary ∂M.

1.4 HOLOGRAPHIC RENORMALIZATION

Until now we have only considered compact domains of integration in the action. In holography we want to study the Universe as a whole so it is important to consider spaces with infinite extent. In classical field theory, a well defined variational principle can easily be maintained by requiring the fields to fall off at spatial infinity rapidly enough not to give boundary contributions. In general relativity the situation is not as easily resolved: The boundary conditions affect the global structure of space-time. They are ingredients to a theory of gravity and can not simply be changed to make the variational principle well defined.

To remove any divergences occurring in the action and to obtain a well defined variational principle, a local counterterm must be added [4]. The purpose of this work is to holographically renormalize Einstein gravity on three-dimensional flat space in order to obtain a well defined variational principle and to calculate one-point functions of the corresponding boundary theory.

In chapter 2 the variational principle in General Relativity is summarized. This information is used to review in chapter 3 the holographic renormalization of Einstein spaces with negative curvature in Lorentzian signature (anti-de Sitter space), as well as Euclidean signature (hyperbolic space). One-point functions are calculated. The same methods are applied to flat space with Lorentzian signature in chapter 4, and Euclidean signature in chapter 5.

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VARIATIONAL PRINCIPLE IN GENERAL RELATIVITY

This chapter gives a short review of the variational principle in general relativity. It is shown that general relativity can be formulated in terms of a principle of stationary action. Boundary terms that do not usually appear in introductory texts are discussed. The equations in this chapter are valid in any dimension unless stated otherwise.

The action usually considered is the Einstein–Hilbert action supplemented with a Gibbons–Hawking–York boundary term [7, 8]. For a pseudo-Riemannian manifold M with boundary ∂M and its metric metric g_{ab} this is

$$\Gamma = \frac{1}{16\pi G}\int_M (R - 2\Lambda)\,\boldsymbol{\epsilon} + \frac{1}{8\pi G}\int_{\partial M} K\,\tilde{\boldsymbol{\epsilon}}\,, \tag{2.1}$$

where $\boldsymbol{\epsilon}$ is the natural volume element and $\tilde{\boldsymbol{\epsilon}}$ the volume element induced on the boundary. R is the scalar curvature, K the trace of the extrinsic curvature of the boundary, and Λ is the cosmological constant. The variation of the action,

$$\begin{aligned}\delta\Gamma = &-\frac{1}{16\pi G}\int_M \left(G^{ab} + \Lambda g^{ab}\right)\delta g_{ab}\,\boldsymbol{\epsilon} \\ &+ \frac{1}{16\pi G}\int_{\partial M}\left(K\gamma^{ij} - K^{ij}\right)\delta\gamma_{ij}\,\tilde{\boldsymbol{\epsilon}}\,,\end{aligned} \tag{2.2}$$

vanishes for solutions satisfying the vacuum Einstein equations when keeping the boundary metric fixed. See appendix A for the conventions used.

The action (2.1) is not necessarily well suited for non compact spaces. When considering such manifolds, a boundary at infinity has to be included. The integrals makes no sense in this case unless some kind of limiting procedure is involved: A radial coordinate r is introduced and bulk integrals are restrict to a region $r < r_c$. Integrals at the boundary are evaluated at an $r = r_c$ cut-off hypersurface. The limit $r_c \to \infty$ can then be taken.

On a space- or time-like hypersurface, we can define the unit vector n^a normal to the hypersurface. In the current chapter, whenever there occurs a $\pm$ or $\mp$ sign, it refers to the sign of the norm $n^a n_a = \pm 1$. The induced metric is then $\gamma_{ab} = g_{ab} \mp n_a n_b$. Assuming our boundary ∂M is nowhere null we can apply these notions to the space- and time-like components of ∂M separately and cover the whole boundary.

Let us consider a more general action with arbitrary coefficients α and β, and a manifold M with corners (c.f. appendix C),

$$\Gamma_{\alpha,\beta} = \frac{1}{16\pi G}\int_M (R - 2\Lambda)\,\boldsymbol{\epsilon} + \frac{1}{8\pi G}\int_{\partial M} (\alpha K + \beta)\,\tilde{\boldsymbol{\epsilon}}\,. \tag{2.3}$$

We obtain from (B.3), (B.6), (B.9f), (B.14) and (B.15)

$$\begin{aligned}
\delta\Gamma_{\alpha,\beta} &= \frac{1}{16\pi G}\int_M \left(\left(\frac{R}{2}-\Lambda\right) g^{ab}\delta g_{ab} - R^{ab}\delta g_{ab} + \nabla_a v^a\right)\epsilon \\
&\quad + \frac{1}{16\pi G}\int_{\partial M}\left((\alpha K+\beta)\gamma^{ab}\delta g_{ab} + 2\alpha\delta K\right)\tilde{\epsilon} \qquad (2.4a)\\
&= -\frac{1}{16\pi G}\int_M \left(R^{ab} - \frac{1}{2}Rg^{ab} + \Lambda g^{ab}\right)\delta g_{ab}\,\epsilon \\
&\quad + \int_{\partial M}\left(n_a v^a + (\alpha K+\beta)\gamma^{ab}\delta g_{ab} + 2\alpha\delta K\right)\tilde{\epsilon} \qquad (2.4b)\\
&= -\frac{1}{16\pi G}\int_M \left(R^{ab} - \frac{1}{2}Rg^{ab} + \Lambda g^{ab}\right)\delta g_{ab}\,\epsilon \\
&\quad + \frac{1}{16\pi G}\int_{\partial M}\left((\alpha K+\beta)\gamma^{ab} - K^{ab}\right)\delta g_{ab}\tilde{\epsilon} \\
&\quad + \frac{1-\alpha}{16\pi G}\int_{\partial M}\left(\pm K n^a n^b \delta g_{ab} - \gamma^{ab}n^c\nabla_c\delta g_{ab}\right)\tilde{\epsilon} \\
&\quad + \frac{1-2\alpha}{16\pi G}\int_{\partial M}\tilde{\nabla}_a\left(\gamma^{ab}n^c\delta g_{bc}\right)\tilde{\epsilon}\,. \qquad (2.4c)
\end{aligned}$$

This can be rewritten by applying Stokes theorem to the divergence on the boundary to obtain:

$$\begin{aligned}
\delta\Gamma_{\alpha,\beta} &= -\frac{1}{16\pi G}\int_M \left(G^{ab} + \Lambda g^{ab}\right)\delta g_{ab}\,\epsilon \qquad (2.5a)\\
&\quad + \frac{1}{16\pi G}\int_{\partial M}\left((\alpha K+\beta)\gamma^{ij} - K^{ij}\right)\delta\gamma_{ij}\,\tilde{\epsilon} \qquad (2.5b)\\
&\quad + \frac{1-\alpha}{16\pi G}\int_{\partial M}\left(\pm K n^a n^b\,\delta g_{ab} - \gamma^{ab}n^c\,\nabla_c\delta g_{ab}\right)\tilde{\epsilon} \qquad (2.5c)\\
&\quad + \frac{1-2\alpha}{16\pi G}\int_{\partial^2 M}\tilde{n}^a n^b\,\delta g_{ab}\,\tilde{\tilde{\epsilon}}\,, \qquad (2.5d)
\end{aligned}$$

where $\tilde{n}^a$ is the normal vector of $\partial^2 M$ and $\tilde{\tilde{\epsilon}}$ is the induced volume element on it. From (2.5) we see that $\alpha = 1$ is required to obtain a well defined variational principle when keeping the metric on the boundary fixed. With $\beta = 0$ we obtain the action from before (2.1), whose variation including corner terms is

$$\begin{aligned}
\delta\Gamma_{\mathrm{GHY}} &= -\frac{1}{16\pi G}\int_M \left(G^{ab} + \Lambda g^{ab}\right)\delta g_{ab}\,\epsilon \\
&\quad + \frac{1}{16\pi G}\int_{\partial M}\left(K\gamma^{ij} - K^{ij}\right)\delta\gamma_{ij}\,\tilde{\epsilon} \qquad (2.6)\\
&\quad - \frac{1}{16\pi G}\int_{\partial^2 M}\tilde{n}^a n^b\delta g_{ab}\tilde{\tilde{\epsilon}}\,.
\end{aligned}$$

With one half the usual Gibbons-Hawking-York term

$$\Gamma_{\frac{1}{2}} = \frac{1}{16\pi G}\int_M (R-2\Lambda)\,\epsilon + \frac{1}{16\pi G}\int_{\partial M} K\,\tilde{\epsilon} \qquad (2.7)$$

the corner term of the variation vanishes and we have

$$\delta\Gamma_{\frac{1}{2}} = -\frac{1}{16\pi G}\int_M \left(G^{ab} + \Lambda g^{ab}\right)\delta g_{ab}\,\boldsymbol{\epsilon} \tag{2.8a}$$
$$-\frac{1}{16\pi G}\int_{\partial M}\left(K^{ab} - \frac{1}{2}K g^{ab}\right)\delta g_{ab}\,\tilde{\boldsymbol{\epsilon}} \tag{2.8b}$$
$$-\frac{1}{32\pi G}\int_{\partial M}\gamma^{ab} n^c\,\nabla_c \delta g_{ab}\tilde{\boldsymbol{\epsilon}}\,. \tag{2.8c}$$

One more boundary term will be of importance,

$$B = \int_{\partial M}\tilde{R}^k\,\tilde{\boldsymbol{\epsilon}}\,, \tag{2.9}$$

where $\tilde{R}$ is the scalar curvature of the boundary ∂M. Its variation is

$$\delta B = \int_{\partial M}\left(\frac{1}{2}\tilde{R}^k\gamma^{ab}\delta\gamma_{ab} + k\tilde{R}^{k-1}\left(\tilde{\nabla}_a\tilde{v}^a - \tilde{R}^{ab}\delta\gamma_{ab}\right)\right)\tilde{\boldsymbol{\epsilon}}\,, \tag{2.10}$$

with the vector $\tilde{v}^a$ defined as

$$\tilde{v}^a \equiv \gamma^{ab}\tilde{\nabla}^c\delta\gamma_{bc} - \gamma^{bc}\tilde{\nabla}^a\delta\gamma_{bc}\,. \tag{2.11}$$

For three-dimensional M this can be rewritten using $\tilde{R}_{ab} = \frac{1}{2}\tilde{R}\gamma_{ab}$ as

$$\begin{aligned}\delta B = \frac{1-k}{2}&\int_{\partial M}\left(\tilde{R}^k\gamma^{ab}\delta\gamma_{ab} + 2k\tilde{R}^{k-2}\tilde{v}^a\tilde{\nabla}_a\tilde{R}\right)\tilde{\boldsymbol{\epsilon}}\\ &+ k\int_{\partial^2 M}\tilde{n}_a\tilde{v}^a\tilde{R}^{k-1}\,\tilde{\tilde{\boldsymbol{\epsilon}}}\,.\end{aligned} \tag{2.12}$$

These are the basic building blocks for constructing a well defined variational principle with covariant boundary terms.

3 EINSTEIN SPACE WITH NEGATIVE CURVATURE

Before turning to three-dimensional flat space, the variational principle in AdS_3 and three-dimensional hyperbolic space is revisited.

In section 3.1 the Brown–Henneaux boundary conditions are recalled. In section 3.2 the variational principle is reviewed. In section 3.3 one-point functions are reconsidered.

3.1 BOUNDARY CONDITIONS

Asymptotically AdS_3 metrics satisfying Brown–Henneaux boundary conditions are defined by having the form [9]

$$\begin{aligned} g_{rr} &= \ell^2/r^2 + h_{rr}\,\ell^4/r^4 + O(1/r^5) & g_{rt} &= O(1/r^3) \\ g_{tt} &= \sigma\, r^2/\ell^2 + h_{tt} + O(1/r) & g_{r\varphi} &= O(1/r^3) \\ g_{\varphi\varphi} &= r^2 + h_{\varphi\varphi}\,\ell^2 + O(1/r) & g_{t\varphi} &= h_{t\varphi}\,\ell + O(1/r)\ , \end{aligned} \tag{3.1}$$

where ℓ is the AdS radius and σ equals 1 or -1 for hyperbolic, resp. AdS space. For convenience all h_{ab} are scaled with appropriate factors of ℓ so that they are dimensionless quantities. The set of variations preserving these boundary is called normalizable. It is given by

$$\begin{aligned} \delta g_{rr} &= \delta h_{rr}\,\ell^4/r^4 + O(1/r^5) & \delta g_{rt} &= O(1/r^3) \\ \delta g_{tt} &= \delta h_{tt} + O(1/r) & \delta g_{r\varphi} &= O(1/r^3) \\ \delta g_{\varphi\varphi} &= \delta h_{\varphi\varphi}\,\ell^2 + O(1/r) & \delta g_{t\varphi} &= O(1)\ . \end{aligned} \tag{3.2}$$

All other variations are called non-normalizable and are used to calculate one-point functions in section 3.3. The functions in (3.1) and (3.2) depend on t and φ.

With these definitions the following relations for constant r hypersurfaces are obtained:

$$n_a = \frac{\ell}{r}\,\delta^r_a + O(1/r^3) \tag{3.3a}$$

$$\sqrt{\sigma\gamma} = \frac{r^2}{\ell} + O(1) \tag{3.3b}$$

$$K = \frac{2}{\ell} - \frac{\ell}{r^2}\,(h_{rr} + \sigma h_{tt} + h_{\varphi\varphi}) + O(1/r^3) \tag{3.3c}$$

$$\gamma^{ab}\delta\gamma_{ab} = \frac{\ell^2}{r^2}\,(\delta h_{\varphi\varphi} + \sigma\delta h_{tt}) + O(1/r^3) \tag{3.3d}$$

$$K^{ab}\delta\gamma_{ab} = \frac{\ell}{r^2}\,(\delta h_{\varphi\varphi} + \sigma\delta h_{tt}) + O(1/r^3) \tag{3.3e}$$

$$n^a n^b \delta g_{ab} = \frac{\ell^2}{r^2}\delta h_{rr} + O(1/r^3) \tag{3.3f}$$

$$\gamma^{ab} n^c \nabla_c \delta g_{ab} = -\frac{2\ell}{r^2}\,(\delta h_{\varphi\varphi} + \sigma\delta h_{tt}) + O(1/r^3) \tag{3.3g}$$

3.2 VARIATIONAL PRINCIPLE

The action (2.3) with cosmological constant $\Lambda = -\frac{1}{\ell^2}$ is considered. The coefficient β is rescaled to be dimensionless and an overall minus sign is added to account for the Euclidean signature of the metric in hyperbolic space.

$$\Gamma_{\alpha,\beta} = -\sigma\frac{1}{16\pi G}\int_M d^3x\,\sqrt{\sigma g}\left(R + \frac{2}{\ell^2}\right) - \sigma\frac{1}{8\pi G}\int_{\partial M} d^2x\,\sqrt{\sigma\gamma}\left(\alpha K + \frac{\beta}{\ell}\right) \tag{3.4}$$

Accordingly, the variation of this action is:

$$\begin{aligned}\delta\Gamma_{\alpha,\beta} = {} & \sigma\frac{1}{16\pi G}\int_M d^3x\,\sqrt{\sigma g}\left(G^{ab} - \frac{1}{\ell^2}g^{ab}\right)\delta g_{ab} \\ & - \sigma\frac{1}{16\pi G}\int_{\partial M} d^2x\,\sqrt{\sigma\gamma}\left[\left(\alpha K + \frac{\beta}{\ell}\right)\gamma^{ab} - K^{ab}\right]\delta\gamma_{ab} \\ & - \sigma\frac{1-\alpha}{16\pi G}\int_{\partial M} d^2x\,\sqrt{\sigma\gamma}\left(K n^a n^b\,\delta g_{ab} - \gamma^{ab} n^c\,\nabla_c\delta g_{ab}\right)\end{aligned} \tag{3.5}$$

Evaluated on fields obeying the equations of motion, the bulk term vanishes. Inserting the relations (3.3), this becomes:

$$\delta\Gamma|_{\text{EOM}} = \frac{1}{16\pi G}\int_{\partial M} d^2x\,\big(2\sigma(\alpha-1)\delta h_{rr} - (\beta+1)(\delta h_{tt} + \sigma\delta h_{\varphi\varphi})\big) + O(1/r) \tag{3.6}$$

Imposing no further restrictions on the boundary metric, for the first variation to vanish we need to set

$$\alpha = 1 \qquad \beta = -1\,. \tag{3.7}$$

For these values a well known action [5] for asymptotically AdS_3 is recovered,

$$\Gamma = -\sigma\frac{1}{16\pi G}\int_M d^3x\,\sqrt{\sigma g}\left(R+\frac{2}{\ell^2}\right) - \sigma\frac{1}{8\pi G}\int_{\partial M} d^2x\,\sqrt{\sigma\gamma}\left(K-\frac{1}{\ell}\right) . \tag{3.8}$$

3.3 ONE-POINT FUNCTIONS

One-point functions of operators of the field theory are derived by functionally differentiating the renormalized on shell action (3.8) with respect to the corresponding source [10, 11]. This is expressed by the relation

$$\delta\Gamma|_{\text{EOM}} = \int_{\partial M} d^2x\,\sqrt{|\hat{\gamma}|}\,\langle\mathcal{O}\rangle\,\delta\varphi^{(0)} , \tag{3.9}$$

where $\hat{\gamma}_{ij} = \lim_{r\to\infty}\gamma_{ij}/r^2$ is the induced metric on the conformal boundary. To compute this expression, the same asymptotic metric as in (3.1) is considered, but non-normalizable metric fluctuations to accommodate for sources are allowed,

$$\begin{aligned} \delta g_{tt} &= \delta h^{(0)}_{tt}\, r^2/\ell^2 + O(1) \\ \delta g_{t\varphi} &= \delta h^{(0)}_{t\varphi}\, r^2/\ell + O(r) \\ \delta g_{\varphi\varphi} &= \delta h^{(0)}_{\varphi\varphi}\, r^2 + O(1) . \end{aligned} \tag{3.10}$$

In this way the identities (3.3) are generalized. The equations that differ from (3.3) are

$$\begin{aligned} \gamma^{ab}\delta\gamma_{ab} = \delta h^{(0)}_{\varphi\varphi} + \sigma\delta h^{(0)}_{tt} + \frac{\ell^2}{r^2}\Big(&\delta h_{\varphi\varphi} + \sigma\delta h_{tt} \\ &- h_{\varphi\varphi}\delta h^{(0)}_{\varphi\varphi} - 2\sigma h_{t\varphi}\delta h^{(0)}_{t\varphi} - h_{tt}\delta h^{(0)}_{tt}\Big) + O(1/r^3) \end{aligned} \tag{3.11a}$$

$$\begin{aligned} K^{ab}\delta\gamma_{ab} = \frac{1}{\ell}\left(\delta h^{(0)}_{\varphi\varphi} + \sigma\delta h^{(0)}_{tt}\right) + \frac{\ell}{2r^2}\Big(&2\delta h_{\varphi\varphi} + 2\sigma\delta h_{tt} \\ &- (4h_{\varphi\varphi} + h_{rr})\delta h^{(0)}_{\varphi\varphi} - 4(2\sigma h_{t\varphi} + h_{tt})\delta h^{(0)}_{t\varphi} \\ &+ \sigma h_{rr}\delta h^{(0)}_{tt}\Big) + O(1/r^3) . \end{aligned} \tag{3.11b}$$

The variation of the action with the cutoff removed ($r \to \infty$) becomes

$$\begin{aligned} \delta\Gamma|_{\text{EOM}} = \frac{1}{32\pi G}\int_{\partial M} d^2x\,\Big(&(h_{rr} + 2h_{\varphi\varphi})\,\delta h^{(0)}_{tt} - 4h_{t\varphi}\,\delta h^{(0)}_{t\varphi} \\ &+ (\sigma h_{rr} + 2h_{tt})\,\delta h^{(0)}_{\varphi\varphi}\Big) . \end{aligned} \tag{3.12}$$

The non-normalizable modes of the metric act as sources for the stress tensor, so the analog of equation (3.9) for Einstein gravity is

$$\delta\Gamma|_{\text{EOM}} = -\sigma\frac{1}{2}\int_{\partial M} d^2x\,\sqrt{|\hat{\gamma}|}\,\left\langle T^{ij}\right\rangle\delta\hat{\gamma}_{ij} . \tag{3.13}$$

Rewritten using the expansion (3.10) this gives

$$\begin{aligned}\delta\Gamma|_{\mathrm{EOM}} &= -\sigma\int_{\partial M} d^2x \left(\frac{1}{2\ell^3}\left\langle T^{tt}\right\rangle \delta h^{(0)}_{tt} + \frac{1}{\ell^2}\left\langle T^{t\varphi}\right\rangle \delta h^{(0)}_{t\varphi}\right. \\ &\qquad\left. + \frac{1}{2\ell}\left\langle T^{\varphi\varphi}\right\rangle \delta h^{(0)}_{\varphi\varphi}\right) \qquad (3.14a) \\ &= -\int_{\partial M} d^2x \left(\sigma\frac{\ell}{2}\left\langle T_{tt}\right\rangle \delta h^{(0)}_{tt} + \left\langle T_{t\varphi}\right\rangle \delta h^{(0)}_{t\varphi}\right. \\ &\qquad\left. + \sigma\frac{1}{2\ell}\left\langle T_{\varphi\varphi}\right\rangle \delta h^{(0)}_{\varphi\varphi}\right), \qquad (3.14b)\end{aligned}$$

where the indices of $\left\langle T^{ij}\right\rangle$ are lowered with $\hat{\gamma}_{ij}$. With the standard definitions of mass and angular momentum [12] it is possible to write (3.14) in a form that can be easily compared to (3.12):

$$\delta\Gamma|_{\mathrm{EOM}} = \int_{\partial M} d^2x \left(\frac{M}{4\pi}\left(\delta h^{(0)}_{tt} - \sigma\delta h^{(0)}_{\varphi\varphi}\right) + \frac{J}{2\pi\ell}\delta h^{(0)}_{t\varphi}\right). \qquad (3.15)$$

In the following subsections the result (3.12) is evaluated for BTZ black holes, and globally AdS space. The expressions are compared with the definition (3.15).

3.3.1 *BTZ Black Holes*

The metric of a BTZ black hole [13] can be written as

$$\begin{aligned}ds^2 &= \sigma\frac{(r^2 - r_+^2)(r^2 + \sigma r_-^2)}{\ell^2 r^2} dt^2 + \frac{\ell^2 r^2}{(r^2 - r_+^2)(r^2 + \sigma r_-^2)} dr^2 \\ &\quad + r^2\left(d\varphi - \frac{r_+ r_-}{l r^2} dt\right)^2 .\end{aligned} \qquad (3.16)$$

Non-trivial functions h_{ab} are accordingly

$$h_{rr} = -\sigma h_{tt} = \frac{r_+^2 - \sigma r_-^2}{\ell^2} \qquad h_{t\varphi} = -\frac{r_+ r_-}{\ell^2} . \qquad (3.17)$$

Evaluating (3.12) gives

$$\delta\Gamma|_{\mathrm{EOM}} = \frac{1}{32\pi G\ell^2}\int_{\partial M} d^2x \left((r_+^2 - \sigma r_-^2)(\delta h^{(0)}_{tt} - \sigma\delta h^{(0)}_{\varphi\varphi}) + 4 r_+ r_- \delta h^{(0)}_{t\varphi}\right), \qquad (3.18)$$

and the standard quantities for mass and angular momentum are recovered by comparison with (3.15),

$$M_{\mathrm{BTZ}} = \frac{r_+^2 - \sigma r_-^2}{8G\ell^2} \qquad J_{\mathrm{BTZ}} = \frac{r_+ r_-}{4G\ell} . \qquad (3.19)$$

3.3.2 *Anti-de Sitter Spaces*

Globally anti-de Sitter spaces (or hyperbolic spaces) can be given in the form

$$ds^2 = \sigma\left(1 + \frac{r^2}{\ell^2}\right) dt^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\varphi^2 , \qquad (3.20)$$

so that non-trivial h_{ab} are

$$h_{rr} = -\sigma h_{tt} = -1 \ . \tag{3.21}$$

Evaluating (3.12) gives

$$\delta\Gamma|_{EOM} = \frac{1}{32\pi G}\int_{\partial M} d^2x \left(\sigma\delta h^{(0)}_{\varphi\varphi} - \delta h^{(0)}_{tt}\right) \ , \tag{3.22}$$

and as before the standard quantities are recovered,

$$M_{AdS} = -\frac{1}{8G} \qquad J_{AdS} = 0 \ . \tag{3.23}$$

4

LORENTZIAN FLAT SPACE

The methods used in chapter 3 are now employed to study flat space. This is done in a formulation that is only applicable to Lorentzian metric signature. Flat space with Euclidean signature is studied in the next chapter.

In section 4.1 the boundary conditions for flat space in Eddington–Finkelstein gauge are recalled. In section 4.2 the variational principle is reviewed.

4.1 BOUNDARY CONDITIONS

Consistent boundary conditions for flat space are given by [14]

$$\begin{aligned} g_{rr} &= h_{rr}/r^2 + O(1/r^3) \\ g_{uu} &= h_{uu} + h^{(1)}_{uu}/r + O(1/r^2) \\ g_{\varphi\varphi} &= r^2 + (h_2(\varphi) + u h_3(\varphi))\, r + h_{\varphi\varphi} + O(1/r) \\ g_{ru} &= -1 + h_{ru}/r + O(1/r^2) \\ g_{r\varphi} &= h_1(\varphi) + O(1/r) \\ g_{u\varphi} &= h_{u\varphi} + O(1/r)\,. \end{aligned} \tag{4.1}$$

These are looser boundary conditions than the ones by Barnich and Compère [15]. The corresponding variations are

$$\begin{aligned} \delta g_{rr} &= O(1/r^2) & \delta g_{ru} &= O(1/r) \\ \delta g_{uu} &= \delta h_{uu} + \delta h^{(1)}_{uu}/r + O(1/r^2) & \delta g_{r\varphi} &= O(1) \\ \delta g_{\varphi\varphi} &= (\delta h_2(\varphi) + u\delta h_3(\varphi))\, r + O(1) & \delta g_{u\varphi} &= O(1)\,. \end{aligned} \tag{4.2}$$

All functions in (4.1) and (4.2) depend on u and φ if not stated otherwise. Evaluating the equations of motion ($R_{ab} - \frac{1}{2} R g_{ab} = 0$) using the metric (4.1) gives to leading order

$$\partial_u h_{uu} = 0 \tag{4.3a}$$

$$\partial_u h_{rr} = -2h_{ru} \tag{4.3b}$$

$$\partial_u h_{r\varphi} = u\partial_\varphi h_{uu} + \partial_\varphi h_{ru} - 2h_{u\varphi} + h_4(\varphi)\,. \tag{4.3c}$$

Equation (4.3a) will be user to simplify expressions in what follows. Using definitions (4.1) and (4.2) the following relations are obtained:

$$n_a = \frac{1}{\sqrt{-h_{uu}}}\,\delta^r_a + O(1/r) \tag{4.4a}$$

$$\sqrt{\sigma\gamma} = r\,\sqrt{-h_{uu}} + O(1) \tag{4.4b}$$

$$K = -\frac{\partial_u h^{(1)}_{uu} - h_3 h_{uu} + 2h_{uu}\partial_u h_{ru} - 2h^2_{uu}}{2r\,(-h_{uu})^{3/2}} + O(1/r^2) \tag{4.4c}$$

$$\gamma^{ab}\delta\gamma_{ab} = \frac{\delta h_{uu}}{h_{uu}} + O(1/r) \tag{4.4d}$$

$$K^{ab}\delta\gamma_{ab} = \frac{\delta h_{uu}\left(\partial_u h^{(1)}_{uu} + 2h_{uu}\partial_u h_{ru}\right)}{2r(-h_{uu})^{5/2}} + O(1/r^2) \tag{4.4e}$$

$$n^a n^b \delta g_{ab} = -\frac{\delta h_{uu}}{h_{uu}} + O(1/r) \tag{4.4f}$$

$$\gamma^{ab}n^c\nabla_c\delta g_{ab} = \frac{\partial_u\delta h_{uu}}{(-h_{uu})^{3/2}} - \frac{2\partial_u h_{ru}\,\delta h_{uu} + \partial_u\delta h^{(1)}_{uu}}{r(-h_{uu})^{3/2}} + \frac{3h^{(1)}_{uu}\,\partial_u\delta h_{uu} - 2h^2_{uu}\,\delta h_3}{2r(-h_{uu})^{5/2}} + O(1/r^2) \tag{4.4g}$$

4.2 VARIATIONAL PRINCIPLE

The action (2.3) with vanishing cosmological is considered. The coefficient β is set to zero. The boundary term corresponding to β is treated afterwards as a special case of the boundary term (2.9).

$$\Gamma_\alpha = \frac{1}{16\pi G}\int_M d^3x\,\sqrt{g}\,R + \frac{1}{8\pi G}\int_{\partial M} d^2x\,\sqrt{\gamma}\,\alpha K \tag{4.5}$$

The variation of (4.5) is

$$\begin{aligned}\delta\Gamma_\alpha = &-\frac{1}{16\pi G}\int_M d^3x\,\sqrt{g}\,G^{ab}\delta g_{ab}\\ &+\frac{1}{16\pi G}\int_{\partial M} d^2x\,\sqrt{\gamma}\left(\alpha K\gamma^{ab} - K^{ab}\right)\delta\gamma_{ab}\\ &+\frac{1-\alpha}{16\pi G}\int_{\partial M} d^2x\,\sqrt{\gamma}\left(Kn^a n^b\delta g_{ab} - \gamma^{ab}n^c\nabla_c\delta g_{ab}\right)\,.\end{aligned} \tag{4.6}$$

After inserting the relations (3.3) and removing some terms by integration along the u coordinate this becomes:

$$\delta\Gamma|_{EOM} = \frac{1}{16\pi G}\int_{\partial M} d^2x\left[\left(1 - 2\alpha - \frac{\alpha h_3}{2h_{uu}}\right)\delta h_{uu} + (1-\alpha)\,\delta h_3\right] + O(1/r) \tag{4.7}$$

Since this is in general nonzero for any fixed α, a well defined variational principle can not be obtained with the boundary term in (4.5).

With the aim of removing leading order contributions in (4.7), additional boundary terms in the form of

$$\Gamma_{R^k} = \frac{1}{8\pi G} \int_{\partial M} d^2x \sqrt{\gamma}\, \tilde{R}^k \,, \tag{4.8}$$

are considered. The variation of (4.8) for any $k \in \mathbb{R}$ is:

$$\delta\Gamma_{R^k} = \frac{1}{16\pi G} \int_{\partial M} d^2x \sqrt{\gamma}\,(1-k)\left(\tilde{R}^k \gamma^{ij}\delta\gamma_{ij} + 2k\tilde{R}^{k-2}\,\tilde{v}^a \tilde{\nabla}_a \tilde{R}\right) \tag{4.9}$$

Using the relations

$$\begin{aligned} \tilde{R} &= \frac{h_3^2 + 4\partial_u \partial_\varphi h_{u\varphi} - 2\partial_u^2 h_{\varphi\varphi} - 2\partial_\varphi^2 h_{uu}}{2r^2 h_{uu}} \\ &\quad + \frac{h_3 \partial_u h_{uu}^{(1)} + (\partial_\varphi h_{uu})^2}{2r^2 h_{uu}^2} + O(1/r^3) && \text{(4.10a)} \\ \tilde{v}^a \tilde{\nabla}_a \tilde{R} &= \left(h_3 \partial_u^2 h_{uu}^{(1)} + 4h_{uu}\partial_u^2 \partial_\varphi h_{u\varphi} - 2h_{uu}\partial_u^3 h_{\varphi\varphi}\right) \\ &\quad \times \frac{h_3\, \delta h_{uu} - 2h_{uu}\, \delta h_3}{4r^3 h_{uu}^4} + O(1/r^4)\,, && \text{(4.10b)} \end{aligned}$$

the orders of r of the expressions in (4.9) are:

$$\sqrt{\gamma}\, \tilde{R}^k \gamma^{ij}\delta\gamma_{ij} = O(r^{1-2k}) \tag{4.11a}$$

$$\sqrt{\gamma}\, \tilde{R}^{k-2}\tilde{v}^a \tilde{\nabla}_a \tilde{R} = O(r^{2-2k}) \tag{4.11b}$$

It can be concluded that the variation of (4.8) is of the following order:

$$\delta\Gamma_{R^k} = \begin{cases} O(r) & k = 0 \\ 0 & k = 1 \\ O(r^{2-2k}) & \text{otherwise} \end{cases} \tag{4.12}$$

There is no k that results in a contribution independent of r. Consequently the addition of the term (4.8) to the action can not cancel (4.7).

Using these simple boundary terms did not result in a well defined variational principle in Eddington–Finkelstein gauge. In the next section a similar procedure is applied – with greater success – to flat space with Euclidean signature.

5

EUCLIDEAN FLAT SPACE

A well defined variational principle for flat space with Euclidean signature is formulated in this chapter. In section 3.1 a specific set of flat space boundary conditions is obtained. In section 3.2 the variational principle is reviewed. In section 3.3 one-point functions are calculated.

5.1 BOUNDARY CONDITIONS

Since there are no null vectors in Euclidean signature, the flat space boundary conditions in Eddington–Finkelstein gauge (4.1) and (4.2) have to be translated into a more suitable gauge. To transform (4.1) into diagonal gauge, the coordinate u is replaced by the time coordinate t,

$$u = t + K(r, \varphi) \; . \tag{5.1}$$

From the equations of motion (4.3) we have

$$\partial_u h_{uu} = 0 \; , \tag{5.2}$$

and with the choice

$$K(r, \varphi) = \frac{r}{h_{uu}(\varphi)} + K_0(\varphi) \; , \tag{5.3}$$

the $dr\,dt$ term of the metric vanishes. If we furthermore restrict the analysis to zero mode solutions $\partial_\varphi h_{uu} = 0$ and upon converting t to Euclidean time τ the form of the metric becomes

$$\begin{aligned} g_{rr} &= h_{rr}(\varphi) + h_{rr}^{(1)}/r + O(1/r^2) & g_{r\tau} &= h_{r\tau}(\varphi)/r + O(1/r^2) \\ g_{\tau\tau} &= h_{\tau\tau}(\varphi) + h_{\tau\tau}^{(1)}/r + O(1/r^2) & g_{r\varphi} &= h_{r\varphi} + h_{r\varphi}^{(1)}/r + O(1/r^2) \\ g_{\varphi\varphi} &= r^2 + h_{\varphi\varphi}\, r + O(1) & g_{\tau\varphi} &= h_{\tau\varphi} + O(1/r) \; , \end{aligned} \tag{5.4}$$

with $h_{rr}h_{\tau\tau} = 1$. The following set of variations preserving the metric (5.4) are considered:

$$\begin{aligned} \delta g_{rr} &= \delta h_{rr}(\varphi) + O(1/r) & \delta g_{r\tau} &= \delta h_{r\tau}(\varphi)/r + O(1/r^2) \\ \delta g_{\tau\tau} &= \delta h_{\tau\tau}(\varphi) + O(1/r) & \delta g_{r\varphi} &= O(1) \\ \delta g_{\varphi\varphi} &= O(r) & \delta g_{\tau\varphi} &= O(1) \end{aligned} \tag{5.5}$$

All functions in (5.4) and (5.5) depend on τ and φ if not stated otherwise. The following relations are obtained:

$$n_a = \sqrt{h_{rr}}\,\delta^r_a + O(1/r) \tag{5.6a}$$

$$\sqrt{\gamma} = r\sqrt{h_{\tau\tau}} + \frac{h_{\tau\tau}h_{\varphi\varphi} + h^{(1)}_{\tau\tau}}{2\sqrt{h_{\tau\tau}}} + O(1/r) \tag{5.6b}$$

$$K = \frac{1}{r\sqrt{h_{rr}}} + O(1/r^2) \tag{5.6c}$$

$$\gamma^{ab}\delta\gamma_{ab} = \frac{\delta h_{\tau\tau}}{h_{\tau\tau}} + O(1/r) \tag{5.6d}$$

$$K^{ab}\delta\gamma_{ab} = O(1/r^2) \tag{5.6e}$$

$$n^a n^b \delta g_{ab} = -\frac{\delta h_{\tau\tau}}{h_{\tau\tau}} + O(1/r) \tag{5.6f}$$

$$\gamma^{ab}n^c\nabla_c\delta g_{ab} = O(1/r^2) \tag{5.6g}$$

5.2 VARIATIONAL PRINCIPLE

Up to a changed sign to account for Euclidean signature, the same action (4.5) as in chapter 4 is considered here.

$$\Gamma_\alpha = -\frac{1}{16\pi G}\int_M d^3x\,\sqrt{g}\,R - \frac{1}{8\pi G}\int_{\partial M} d^2x\,\sqrt{\gamma}\,\alpha K \tag{5.7}$$

The variation of the full action yields:

$$\begin{aligned}\delta\Gamma_\alpha = &\frac{1}{16\pi G}\int_M d^3x\,\sqrt{g}\,G^{ab}\delta g_{ab}\\ &- \frac{1}{16\pi G}\int_{\partial M} d^2x\,\sqrt{\gamma}\left(\alpha K\gamma^{ab} - K^{ab}\right)\delta\gamma_{ab}\\ &- \frac{1-\alpha}{16\pi G}\int_{\partial M} d^2x\,\sqrt{\gamma}\left(K n^a n^b \delta g_{ab} - \gamma^{ab}n^c\nabla_c\delta g_{ab}\right)\end{aligned} \tag{5.8}$$

Inserting (5.6) into the variation gives

$$\delta\Gamma|_{EOM} = \frac{1}{16\pi G}\int_{\partial M} d^2x\,\frac{1-2\alpha}{\sqrt{h_{rr}h_{\tau\tau}}}\delta h_{\tau\tau} + O(1/r)\,, \tag{5.9}$$

which vanishes for the choice

$$\alpha = 1/2\,. \tag{5.10}$$

A well defined variational principle is obtained with this value of α. The resulting boundary term is one half the usual Gibbons–Hawking–York boundary term.

5.3 ONE-POINT FUNCTIONS

One-point functions are computed in analogy with chapter 3. The set of metric fluctuations that include sources is

$$\begin{aligned} \delta g_{rr} &= \delta h_{rr} + O(1/r) & \delta g_{r\tau} &= O(1/r) \\ \delta g_{\tau\tau} &= \delta h_{\tau\tau} + O(1/r) & \delta g_{r\varphi} &= O(1) \\ \delta g_{\varphi\varphi} &= \delta h^{(0)}_{\varphi\varphi}\, r^2 + O(r) & \delta g_{\tau\varphi} &= \delta h^{(0)}_{\tau\varphi}\, r^2 + O(1)\,. \end{aligned} \tag{5.11}$$

The relations generalizing (5.6) are:

$$\gamma^{ab}\delta\gamma_{ab} = \delta h^{(0)}_{\varphi\varphi} + \frac{\delta h_{\tau\tau} - 2h_{\tau\varphi}\,\delta h^{(0)}_{\tau\varphi}}{h_{\tau\tau}} + O(1/r) \tag{5.12a}$$

$$\begin{aligned} K^{ab}\delta\gamma_{ab} = &-\frac{\partial_\tau h_{r\varphi}\,\delta h^{(0)}_{\tau\varphi}}{\sqrt{h_{rr}}h_{\tau\tau}} + \frac{\delta h^{(0)}_{\varphi\varphi}}{r\sqrt{h_{rr}}} + \Bigg(\frac{h^{(1)}_{rr}\partial_\tau h_{r\varphi}}{2rh^{3/2}_{rr}h_{\tau\tau}} \\ &+ \frac{\partial_\tau(h_{\varphi\varphi}h_{r\varphi}) - 2h_{\tau\varphi} - \partial_\varphi h_{r\tau} - \partial_\tau h^{(1)}_{r\varphi}}{r\sqrt{h_{rr}}h_{\tau\tau}} \\ &+ \frac{h^{(1)}_{\tau\tau}\partial_\tau h_{r\varphi} + h_{r\tau}\partial_\varphi h_{\tau\tau}}{r\sqrt{h_{rr}}h^2_{\tau\tau}}\Bigg)\delta h^{(0)}_{\tau\varphi} + O(1/r^2) \end{aligned} \tag{5.12b}$$

$$n^a n^b \delta g_{ab} = -\frac{\delta h_{\tau\tau}}{h_{\tau\tau}} + O(1/r) \tag{5.12c}$$

$$\begin{aligned} \gamma^{ab}n^c\nabla_c\delta g_{ab} = &\frac{h_{r\varphi}\partial_\tau h^{(1)}_{rr} + h_{r\tau}\partial_\varphi h_{rr} + 2h_{r\tau}h_{rr}\partial_\tau h_{\tau\varphi}}{rh^{3/2}_{rr}h_{\tau\tau}}\,\delta h^{(0)}_{\tau\varphi} \\ &+ \frac{h_{r\tau}}{r\sqrt{h_{rr}}h^2_{\tau\tau}}\left(2h_{\tau\varphi}\,\partial_\tau\delta h^{(0)}_{\tau\varphi} - \partial_\tau\delta h_{\tau\tau}\right) \\ &- \frac{h_{r\tau}\,\partial_\tau\delta h^{(0)}_{\varphi\varphi}}{r\sqrt{h_{rr}}h_{\tau\tau}} + O(1/r^2) \end{aligned} \tag{5.12d}$$

Inserting these equations into the variation of the action and removing some terms by integration along the τ coordinate gives:

$$\begin{aligned} \delta\Gamma|_{\mathrm{EOM}} = &\frac{1}{32\pi G}\int_{\partial M} d^2x\,\frac{1}{\sqrt{h_{rr}h_{\tau\tau}}}\Bigg[h_{\tau\tau}\,\delta h^{(0)}_{\varphi\varphi} \\ &+ (2\,r\,h_{r\varphi} + O(1))\,\partial_\tau\delta h^{(0)}_{\tau\varphi} - \frac{\delta(h_{\tau\tau}h_{rr})}{h_{rr}} \\ &+ \Bigg(h_{r\tau}\partial_\varphi \ln\!\big(h_{rr}h^2_{\tau\tau}\big) - 2h_{\tau\varphi} - 2\partial_\varphi h_{r\tau} \\ &+ \frac{h^{(1)}_{\tau\tau}\partial_\tau h_{r\varphi}}{h_{\tau\tau}} + h_{r\varphi}\partial_\tau h_{\varphi\varphi}\Bigg)\delta h^{(0)}_{\tau\varphi}\Bigg] \end{aligned} \tag{5.13}$$

This expression diverges for nonzero $\partial_\tau\delta h^{(0)}_{\tau\varphi}$, so we require

$$\partial_\tau\delta h^{(0)}_{\tau\varphi} = 0\,. \tag{5.14}$$

Restricting the variations (5.11) by $\delta(h_{\tau\tau}h_{rr}) = 0$ we finally obtain

$$\begin{aligned}\delta\Gamma|_{\text{EOM}} = \frac{1}{32\pi G}\int_{\partial M} d^2x\,\frac{1}{\sqrt{h_{rr}h_{\tau\tau}}}\Big[h_{\tau\tau}\,\delta h^{(0)}_{\varphi\varphi} \\ + \Big(h_{r\tau}\partial_\varphi \ln\big(h_{rr}h^2_{\tau\tau}\big) - 2h_{\tau\varphi} - 2\partial_\varphi h_{r\tau} \\ + \frac{h^{(1)}_{\tau\tau}\partial_\tau h_{r\varphi}}{h_{\tau\tau}} + h_{r\varphi}\partial_\tau h_{\varphi\varphi}\Big)\delta h^{(0)}_{\tau\varphi}\Big] \,.\end{aligned} \tag{5.15}$$

When considering only stationary axisymmetric solutions this simplifies to

$$\delta\Gamma|_{\text{EOM}} = \frac{1}{32\pi G}\int_{\partial M} d^2x\,\frac{1}{\sqrt{h_{rr}h_{\tau\tau}}}\left(h_{\tau\tau}\,\delta h^{(0)}_{\varphi\varphi} - 2h_{\tau\varphi}\delta h^{(0)}_{\tau\varphi}\right) \,. \tag{5.16}$$

With the definitions

$$M = \frac{h_{\tau\tau}}{8G} \qquad J_{\text{BTZ}} = \frac{h_{\tau\varphi}}{4G} \,, \tag{5.17}$$

and using $h_{rr}h_{\tau\tau} = 1$, an expression analogous to (3.15) can be written down:

$$\delta\Gamma|_{\text{EOM}} = \int_{\partial M} d^2x\left(\frac{M}{4\pi}\,\delta h^{(0)}_{\varphi\varphi} - \frac{J}{4\pi}\,\delta h^{(0)}_{\tau\varphi}\right) \tag{5.18}$$

This has a close resemblance to the corresponding equation in the AdS case (3.15). The mass M and angular momentum J are calculated for flat space and flat space cosmologies in what follows.

5.3.1 *Flat Space*

Starting with the flat metric in cylindrical coordinates

$$ds^2 = dr^2 + d\tau^2 + r^2 d\varphi^2 \,, \tag{5.19}$$

leads to the response

$$\delta\Gamma|_{\text{EOM}} = \frac{1}{32\pi G}\int_{\partial M} d^2x\,\delta h^{(0)}_{\varphi\varphi} \,. \tag{5.20}$$

The resulting expressions

$$M_{\text{flat}} = \frac{1}{8G} \qquad J_{\text{flat}} = 0 \,, \tag{5.21}$$

coincide precisely with mass and angular momentum of flat space [15, 14].

5.3.2 *Flat Space Cosmologies*

Similarly the metric of flat space cosmologies is given by [16]

$$ds^2 = \frac{dr^2}{r_+^2\left(1 - \frac{r_0^2}{r^2}\right)} + r_+^2\left(1 - \frac{r_0^2}{r^2}\right)d\tau^2 + r^2\left(d\varphi - \frac{r_+ r_0}{r^2}d\tau\right)^2 . \quad (5.22)$$

With the response

$$\delta\Gamma|_{\text{EOM}} = \frac{1}{32\pi G}\int_{\partial M} d^2x \left(r_+^2 h_{\varphi\varphi}^{(0)} + 2r_+ r_0 \delta h_{\tau\varphi}^{(0)}\right) , \quad (5.23)$$

the expressions

$$M_{\text{FSC}} = \frac{r_+^2}{8G} \qquad J_{\text{FSC}} = -\frac{r_+ r_0}{4G} , \quad (5.24)$$

are obtained. They are again in agreement with the values for flat space cosmologies [15, 14].

6 CONCLUSION

The variational principle in general relativity was reviewed and boundary terms that supplement the Einstein–Hilbert action were studied. For anti-de Sitter space and its Euclidean counterpart – hyperbolic space – a well defined variational principle was recovered using the usual Gibbons–Hawking–York boundary term and a constant counterterm. Using this counterterm, one-point functions were calculated according to the gauge/gravity duality and the usual terms for mass and angular momentum were recovered. The same methods were then used to study the variational principle in flat space, where a well defined variational principle was obtained for zero mode solutions in Euclidean signature. It was shown that the bulk action has to be supplemented by *one half* of the Gibbons–Hawking–York boundary term. The one-point functions for zero mode solutions were derived using this action.

Several generalizations and extensions to the present work can be thought of. First, it is interesting to generalize the discussion to describe non-zero mode solutions in Euclidean signature. It is also an open issue to calculate two- and three-point functions for flat space. Zero-point functions can be found in a related work [17]. Moreover, extension of the analysis to different theories would be of interest. Examples are the zero cosmological constant cases of topologically massive gravity [18, 19] in general and flat space chiral gravity [20, 14] in particular. So far all work on three-dimensional flat space holography is restricted to boundary conditions adapted to null infinity. An important extension to this description is inclusion of spatial infinity to describe all components of the asymptotic boundary.

A

CONVENTIONS

The same sign conventions as in [21, 22, 23] are used throughout the text. In particular, the metric signature consists of mostly pluses, and the Riemann and Ricci tensors are defined as

$$R_{abc}{}^{d} = \partial_b \Gamma^{d}{}_{ac} - \partial_a \Gamma^{d}{}_{bc} - \Gamma^{d}{}_{ae} \Gamma^{e}{}_{bc} + \Gamma^{d}{}_{be} \Gamma^{e}{}_{ac} \tag{A.1}$$

$$R_{ab} = R_{acb}{}^{c} \; . \tag{A.2}$$

Letters from the beginning of the alphabet ($a, b, c, \dots$) and letters from the middle of the alphabet ($i, j, k, \dots$) are used as vector indices in the bulk M and on the boundary ∂M, respectively.

The natural volume element on the bulk is $\boldsymbol{\epsilon}$. The induced metric on the boundary is γ_{ij}. Symbols with a tilde refer to quantities on the boundary: $\tilde{\boldsymbol{\epsilon}}$ and $\tilde{R}$ denote the natural volume element and the scalar curvature on the boundary, respectively. The derivative $\tilde{\nabla}_a$ is the unique covariant derivative compatible with γ_{ab},

$$\tilde{\nabla}_a \gamma_{bc} = 0 \; . \tag{A.3}$$

For a tensor Π^{ab}, symmetrization is written as

$$\Pi^{(ab)} = \frac{1}{2}(\Pi^{ab} + \Pi^{ba}) \; . \tag{A.4}$$

The symbol $\top$ denotes projection to the tangent space of the boundary,

$$\left(\Pi^{ab}\right)^{\top} = \gamma^{a}_{c} \gamma^{b}_{d} \Pi^{cd} \; . \tag{A.5}$$

B VARIATIONS

B.1 VARIATIONS IN THE BULK

In this section we derive variations of expressions that occur in the integral over M. Expressions that can be defined on the boundary ∂M are considered in appendix B.2. We recall the expressions for the derivatives of the inverse and determinant of an invertible matrix A

$$\frac{\partial A^{-1}}{\partial x} = -A^{-1}\frac{\partial A}{\partial x}A^{-1} , \tag{B.1a}$$

$$\frac{\partial \det(A)}{\partial x} = \det(A)\ \mathrm{tr}\left(A^{-1}\frac{\partial A}{\partial x}\right) . \tag{B.1b}$$

These formulas yield the variations of the metric g_{ab}, its inverse g^{ab} and its determinant g

$$\delta g^{ab} = -g^{ac}g^{bd}\,\delta g_{cd} , \tag{B.2a}$$

$$\delta g = g g^{ab}\,\delta g_{ab} , \tag{B.2b}$$

$$\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}\,g^{ab}\,\delta g_{ab} . \tag{B.2c}$$

The natural volume element $\boldsymbol{\epsilon}$ can be written as $\sqrt{|g|}\boldsymbol{e}$, where $\boldsymbol{e}$ is a total antisymmetric tensor whose nonzero components are 1 or -1. This leads to the coordinate independent expression

$$\delta\boldsymbol{\epsilon} = \frac{1}{2}g^{ab}\,\delta g_{ab}\,\boldsymbol{\epsilon} . \tag{B.3}$$

The Levi-Civita connection is expressed in a coordinate system via Christoffel symbols $\Gamma^a{}_{bc}$. From simple comparison of the two sides it can be verified that

$$\delta\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}\left(\nabla_b\delta g_{cd} + \nabla_c\delta g_{bd} - \nabla_d\delta g_{bc}\right) . \tag{B.4}$$

Similarly we can check that the Riemann tensor $R_{abc}{}^d$ has the variation

$$\delta R_{abc}{}^d = \nabla_b\delta\Gamma^d{}_{ac} - \nabla_a\delta\Gamma^d{}_{bc} , \tag{B.5}$$

which leads to the variation of the scalar curvature

$$\delta R = \nabla_a v^a - R^{ab}\,\delta g_{ab} , \tag{B.6a}$$

$$v^a \equiv g^{ab}\,\nabla^c\delta g_{bc} - g^{bc}\,\nabla^a\delta g_{bc} . \tag{B.6b}$$

B.2 VARIATIONS ON THE BOUNDARY

On a space- or time-like hypersurface, we can define the vector n^a normal to all vectors lying in the hypersurface. We require it to be normalized $n^a n_a = \pm 1$. The induced metric is then

$$\gamma_{ab} = g_{ab} \mp n_a n_b \ . \tag{B.7}$$

We apply these notions to space- and time-like components of the boundary ∂M.

Since n_a is a covector, the condition for it to be normal to a vector v^a does not depend on the metric. This means that the direction of n_a does not depend on the metric and we have $\delta n_a \propto n_a$. From the normalization condition we conclude that:

$$\delta n_a = \pm \frac{1}{2} n_a n^b n^c \, \delta g_{bc} \tag{B.8a}$$

$$\delta n^a = \left(\mp \frac{1}{2} n^a n^b - \gamma^{ab} \right) n^c \, \delta g_{bc} \tag{B.8b}$$

We obtain the following relations in a straightforward manner:

$$\delta \gamma_{ab} = \left(\delta_a^c \delta_b^d - n_a n_b n^c n^d \right) \delta g_{cd} \tag{B.9a}$$

$$\delta \gamma_a^b = \pm n_a \gamma^{bc} n^d \, \delta g_{cd} \tag{B.9b}$$

$$\delta \gamma^{ab} = -\gamma^{ac} \gamma^{bd} \, \delta g_{cd} \tag{B.9c}$$

$$\delta \gamma = \gamma \gamma^{ij} \, \delta \gamma_{ij} \tag{B.9d}$$

$$\delta \sqrt{|\gamma|} = \frac{1}{2} \sqrt{|\gamma|} \, \gamma^{ij} \, \delta \gamma_{ij} \tag{B.9e}$$

$$(\delta \tilde{\epsilon})^\top = \frac{1}{2} \gamma^{ij} \, \delta \gamma_{ij} \tilde{\epsilon} \tag{B.9f}$$

The extrinsic curvature is defined as

$$K_{ab} = \gamma_a^c \nabla_c n_b \ , \tag{B.10}$$

which implies that $K_{ab} = K_{ab}^\top$. Its variation is

$$\delta K_{ab} = \delta \left(\gamma_a^c \gamma_b^d \nabla_c n_d \right) \tag{B.11a}$$

$$\begin{aligned} &= \delta \gamma_a^c \gamma_b^d \nabla_c n_d + \delta \gamma_b^d K_{ad} \\ &\quad + \gamma_a^c \gamma_b^d \nabla_c \delta n_d - \gamma_a^c \gamma_b^d \delta \Gamma^e{}_{cd} \, n_e \end{aligned} \tag{B.11b}$$

$$\begin{aligned} &= \pm n_a K_b^c n^d \delta g_{cd} \pm n_b K_a^c n^d \delta g_{cd} \\ &\quad \pm \frac{1}{2} \gamma_a^c \gamma_b^d \nabla_c \left(n_d n^e n^f \delta g_{ef} \right) \\ &\quad - \frac{1}{2} \gamma_a^c \gamma_b^d n^e \left(\nabla_c \delta g_{de} + \nabla_d \delta g_{ce} - \nabla_e \delta g_{cd} \right) \end{aligned} \tag{B.11c}$$

$$\begin{aligned} &= \pm \left(2 n_{(a} K_{b)}^c n^d + \frac{1}{2} K_{ab} n^c n^d \right) \delta g_{cd} \\ &\quad - \frac{1}{2} \gamma_a^c \gamma_b^d n^e \left(\nabla_c \delta g_{de} + \nabla_d \delta g_{ce} - \nabla_e \delta g_{cd} \right) \ . \end{aligned} \tag{B.11d}$$

To rewrite derivatives of δg_{ab} in the bulk as boundary terms the following relation comes in handy

$$\begin{aligned}\gamma^{ab}n^c\,\nabla_a\delta g_{bc} &= \gamma^{ab}\nabla_a\left(n^c\delta g_{bc}\right) - \gamma^{ab}\nabla_a n^c\delta g_{bc} && \text{(B.12a)}\\ &= \gamma^{ab}\nabla_a\left(n^c\left(\gamma^d_b \pm n^d n_b\right)\delta g_{cd}\right)\\ &\quad - \gamma^{ab}\nabla_a n^c\delta g_{bc} && \text{(B.12b)}\\ &= \tilde{\nabla}_a\left(n^c\gamma^{ad}\delta g_{cd}\right) \pm \gamma^{ab}\nabla_a n_b n^c n^d\delta g_{cd}\\ &\quad - \gamma^{ab}\nabla_a n^c\delta g_{bc} && \text{(B.12c)}\\ &= \tilde{\nabla}_a\left(\gamma^{ab}n^c\,\delta g_{bc}\right) + \left(\pm K n^a n^b - K^{ab}\right)\delta g_{ab}\,, && \text{(B.12d)}\end{aligned}$$

where $\tilde{\nabla}_a$ is the covariant derivative on the boundary compatible with γ_{ab},

$$\tilde{\nabla}_a\gamma_{bc} = 0\,. \tag{B.13}$$

Using (B.12) the variation of the trace of the extrinsic curvature can be written as

$$\begin{aligned}\delta K &= \delta\left(g^{ab}K_{ab}\right) = g^{ab}\delta K_{ab} - K^{ab}\delta g_{ab} && \text{(B.14a)}\\ &= \pm\frac{1}{2}K n^a n^b\delta g_{ab} - \gamma^{ab}n^c\nabla_a\delta g_{bc}\\ &\quad + \frac{1}{2}\gamma^{ab}n^c\nabla_c\delta g_{ab} - K^{ab}\delta g_{ab} && \text{(B.14b)}\\ &= \mp\frac{1}{2}K n^a n^b\delta g_{ab} - \tilde{\nabla}_a\left(\gamma^{ab}n^c\delta g_{bc}\right) + \frac{1}{2}\gamma^{ab}n^c\nabla_c\delta g_{ab}\,. && \text{(B.14c)}\end{aligned}$$

Similarly the divergence term appearing in (B.6) contracted with n_a is

$$\begin{aligned}n_a v^a &= g^{ab}n^c\nabla_a\delta g_{bc} - g^{ab}n^c\nabla_c\delta g_{ab} && \text{(B.15a)}\\ &= n^a\gamma^{bc}\nabla_c\delta g_{ab} - \gamma^{ab}n^c\nabla_c\delta g_{ab} && \text{(B.15b)}\\ &= \left(\pm K n^a n^b - K^{ab}\right)\delta g_{ab}\\ &\quad + \tilde{\nabla}_a\left(\gamma^{ab}n^c\delta g_{bc}\right) - \gamma^{ab}n^c\nabla_c\delta g_{ab}\,. && \text{(B.15c)}\end{aligned}$$

B.3 VARIATIONS OF NON COVARIANT BOUNDARY TERMS

In this section the variations of non covariant boundary terms are calculated.

(I) For any Π^{ab} with $\Pi^{ab} = \left(\Pi^{ab}\right)^{\top}$

$$\begin{aligned}
\delta\left(\Pi^{ab}\nabla_a n_b\right) &= \delta\Pi^{ab}\nabla_a n_b + \Pi^{ab}\nabla_a \delta n_b - \Pi^{ab}\delta\Gamma^c{}_{ab}\, n_c && \text{(B.16a)}\\
&= \delta\Pi^{ab}\nabla_a n_b \pm \frac{1}{2}\Pi^{ab}\nabla_a\left(n_b\delta g_{nn}\right)\\
&\quad - \Pi^{(ab)}n^c\nabla_a\delta g_{bc} + \frac{1}{2}\Pi^{ab}n^c\nabla_c\delta g_{ab} && \text{(B.16b)}\\
&= \delta\Pi^{ab}\nabla_a n_b \pm \frac{1}{2}\Pi^{ab}K_{ab}\delta g_{nn}\\
&\quad - \gamma^a_d\nabla_a\left(\Pi^{(db)}n^c\delta g_{bc}\right)\\
&\quad + \gamma^a_d\nabla_a\left(\Pi^{(db)}n^c\right)\delta g_{bc} + \frac{1}{2}\Pi^{ab}n^c\nabla_c\delta g_{ab} && \text{(B.16c)}\\
&= \left(K^c_a\Pi^{(ad)} + \gamma^a_b\nabla_a\Pi^{(bc)}n^d \pm \frac{1}{2}\Pi^{ab}K_{ab}n^c n^d\right)\delta g_{cd}\\
&\quad + \delta\Pi^{ab}\nabla_a n_b + \frac{1}{2}\Pi^{ab}n^c\nabla_c\delta g_{ab}\\
&\quad - \tilde{\nabla}_a\left(\Pi^{(ab)}n^c\delta g_{bc}\right)\ , && \text{(B.16d)}
\end{aligned}$$

where $\delta g_{nn} \equiv n^a n^b \delta g_{ab}$.

(II) For any v^a with $v^a = \left(v^a\right)^{\top}$

$$\begin{aligned}
\delta\left(n^a v^b\nabla_a n_b\right) &= \delta n^a v^b\nabla_a n_b + n^a\delta v^b\nabla_a n_b\\
&\quad + n^a v^b\nabla_a\delta n_b - n^a v^b\delta\Gamma^c{}_{ab}\, n_c && \text{(B.17a)}\\
&= \left(\mp\frac{1}{2}n^a n^c - \gamma^{ac}\right)n^d v^b\nabla_a n_b\delta g_{cd}\\
&\quad + n^a\delta v^b\nabla_a n_b \pm \frac{1}{2}n^a v^b\nabla_a\left(n_b\delta g_{nn}\right)\\
&\quad - \frac{1}{2}n^a v^b n^c\nabla_b\delta g_{ac} && \text{(B.17b)}\\
&= -K^c_b n^d v^b\delta g_{cd} + n^a\delta v^b\nabla_a n_b \pm\\
&\quad - \frac{1}{2}v^a\nabla_a\delta g_{nn} + \frac{1}{2}v^a\nabla_a\left(n^b n^c\right)\delta g_{bc} && \text{(B.17c)}\\
&= n^a\delta v^b\nabla_a n_b - \frac{1}{2}v^a\nabla_a\delta g_{nn} && \text{(B.17d)}\\
&= n^a\delta v^b\nabla_a n_b + \frac{1}{2}\tilde{\nabla}_a v^a\delta g_{nn}\\
&\quad - \frac{1}{2}\tilde{\nabla}_a\left(v^a\delta g_{nn}\right)\ . && \text{(B.17e)}
\end{aligned}$$

(III) For $k^a = (k^a)^\top$, $p^a = (p^a)^\top$, $k^a p_a = 0$, $p^2 =$ fixed

$$\begin{aligned}
\delta\left(k^a k^b \nabla_a p_b\right) &= \delta\left(k^a k^b\right)\nabla_a p_b + k^a k_b \nabla_a \delta p^b \\
&\quad + k^a k_b \delta\Gamma^b{}_{ac} p^c + k^a k^b \nabla_a p^c \delta g_{bc} && \text{(B.18a)} \\
&= \delta\left(k^a k^b\right)\nabla_a p_b - k^a k_b \nabla_a \left(\frac{1}{2p^2} p^b p^c p^d \delta g_{cd}\right) \\
&\quad + \frac{1}{2} k^a k^b p^c \nabla_c \delta g_{ab} + k^a k^b \nabla_a p^c \delta g_{bc} && \text{(B.18b)} \\
&= \delta\left(k^a k^b\right)\nabla_a p_b - \frac{1}{2} k^a k^b \nabla_a p_b \frac{p^c p^d}{p^2} \delta g_{cd} \\
&\quad + \frac{1}{2}\tilde{\nabla}_a\left(p^a k^b k^c \delta g_{bc}\right) - \frac{1}{2}\gamma^d_a \nabla_d \left(p^a k^b k^c\right)\delta g_{bc} \\
&\quad + k^a k^b \nabla_a p^c \delta g_{bc} && \text{(B.18c)}
\end{aligned}$$

C

MANIFOLDS WITH CORNERS

When dealing with general relativity it is sometimes useful to consider manifolds like cylinders and cubes that are topological manifolds but are not smooth manifolds with boundary because they have "corners". This is a short review of manifolds with corners closely following [24]. We use the notion of boundary for a manifold with corners as given in [25], which is inequivalent to the one usually given in the context of manifolds with boundary.

Let $\bar{\mathbb{R}}^n_+$ denote the subset of $\mathbb{R}^n$ where all of the coordinates are nonnegative:

$$\bar{\mathbb{R}}^n_+ = \left\{ \left(x^1, \dots, x^n\right) \in \mathbb{R}^n : x^1 \geqslant 0, \dots, x^n \geqslant 0 \right\} . \qquad \text{(C.1)}$$

Suppose M is a topological n-manifold with boundary. A chart with corners for M is a pair (U, φ) were $U \subseteq M$ is open and φ is a homeomorphism from U to an open subset $\hat{U} \subseteq \bar{\mathbb{R}}^n_+$. Two charts with corners (U, φ), (V, ψ) are smoothly compatible if the composite map $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is smooth. A smooth structure with corners on a topological manifold with boundary is a maximal collection of smoothly compatible interior charts and charts with corners whose domains cover M.

Definition 1 (Manifold with corners)**.** *A topological manifold with boundary together with a smooth structure with corners is called a smooth manifold with corners.*

Smooth maps, partitions of unity, tangent vectors, covectors, tensors, differential forms, orientations, and integrals of differential forms can be defined on smooth manifolds with corners in exactly the same way as for smooth manifolds and smooth manifolds with boundary, using smooth charts with corners in place of smooth boundary charts. To summarize, with ascending generality:

- A smooth manifold is locally isomorphic to $\mathbb{R}^n$.
- A smooth manifold with boundary is locally isomorphic to a half-space $\mathbb{H}^n$.
- A smooth manifold with corners is locally isomorphic to $\bar{\mathbb{R}}^n_+$.

We now turn to the definition of the boundary of a manifold with corners. Let $U \subseteq \bar{\mathbb{R}}^n_+$ be open. For each $u = (u_1, \dots, u_n)$ in U, define the *depth* $\mathrm{depth}_U u$ of u in U to be the number of $u_1, \dots, u_n$ which are zero. Let M be an n-manifold with corners. For $x \in M$, choose a chart (U, φ) on the manifold M with $\varphi(u) = x$ for $u \in U$, and define the *depth* $\text{depth}_M x$ of x in M by $\text{depth}_M x = \text{depth}_U u$. This is independent of the choice of (U, φ). For each $k = 0, \ldots, n$, define the *depth k stratum* of M to be

$$S^k(M) = \{x \in M : \text{depth}_M x = k\} . \tag{C.2}$$

Let M be a manifold with corners, and $x \in M$. A local boundary component β of M at x is a local choice of connected component of $S^1(M)$ near x. That is, for each sufficiently small open neighborhood V of x in M, β gives a choice of connected component W of $V \cap S^1(M)$ with $x \in \overline{W}$, and any two such choices V, W and V', W' must be compatible in the sense that $x \in \overline{(W \cap W')}$. There are exactly $\text{depth}_M x$ distinct local boundary components β of M at x for each $x \in M$.

Definition 2 (Boundary of a manifold with corners)**.** *Let X be a manifold with corners. The boundary is defined as the set*

$$\partial M = \{(x, \beta) : x \in M, \beta \textit{ is a local boundary component for } M \textit{ at } x\} . \tag{C.3}$$

Given this definition ∂M naturally has the structure of an $(n-1)$-manifold with corners and we can iterate the boundary construction to obtain $\partial M, \partial^2 M, \ldots, \partial^n M$, with $\partial^k M$ an $(n-k)$-manifold with corners.

When viewing a manifold with corners M as a topological manifold with boundary we inherit a different notion of boundary that we denote by $\tilde{\partial} M$. That ∂M and $\tilde{\partial} M$ are in general not equal can be seen from:

- ∂M is a manifold with corners whereas $\tilde{\partial} M$ may not be one.
- $\tilde{\partial}^2 M = \emptyset$ always holds, whereas $\partial^2 M \neq \emptyset$ in general.

These differences make ∂M the preferred notion of boundary when working with stokes theorem, since then it holds without modification:

$$\int_M d\omega = \int_{\partial M} \omega . \tag{C.4}$$

Example (The cube as a manifold with corners)**.** *Let M be the three-dimensional cube which is a 3-manifold with corners. The depth 1 stratum $S^1(M)$ consists of six disconnected regions and is given by the faces, excluding edges and vertices. A point in the interior has no local boundary component. A point on a face that does not lie on an edge has exactly one boundary component, the interior of the corresponding face. A point on an edge that is not a vertex has two boundary components and each vertex has three boundary components. The boundary ∂M consists of six disjoint squares, $\partial^2 M$ consists of 24 disjoint lines, and $\partial^3 M$ consists of 48 points.*

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