

DIPLOMARBEIT

Null canonical formulation and integrability of cylindrical gravitational waves

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Abstract:

The gravitational field in four dimensional spacetime may be described using free initial data on a pair of intersecting null hypersurfaces swept out by the future null normal geodesics to their two dimensional intersection surface.

A Poisson bracket on such initial data was calculated in [32][33]. The expressions obtained are tractable but still rather intricate, and it is not at all obvious how this bracket might be quantized. A change of variables that simplifies the bracket would thus be desirable.

The bracket does have the feature (reflecting causality) that it is non-zero only between data lying on the same generating null geodesic, and that it only depends on the data on this generator. That is, the data on each generator forms an essentially autonomous Poisson algebra. The limited role of the two transverse dimensions suggests that the Poisson algebra would remain substantially the same in a symmetry reduced model in which the transverse dimensions have been eliminated.

Here this expectation is confirmed in the context of cylindrically symmetric gravitational waves. Specifically, the Poisson algebra of the metric variables in free null initial data for cylindrically symmetric gravitational waves is obtained, and it is found to be essentially identical to the bracket on the metric sector of the initial data found in [32][33]. Then, using the integrability of the dynamics of cylindrically symmetric gravitational waves an explicit transformation from metric data on a null hypersurface to so called “monodromy data”, a one parameter family of unimodular matrices, is obtained. The

Poisson brackets of the monodromy data are then obtained from that of the null data. These have been obtained earlier via another route in [24] in a slightly more restricted context. They are quite simple, and what is more, a unique preferred quantization is known [24]. It is also demonstrated that the transformation to monodromy data is invertible.

Aside from these original results extensive background material is presented, including a review of the Geroch group of symmetries in cylindrically symmetrical gravity.

The original results presented here are joint work with Michael Reisenberger.

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Introduction

Canonical formulations of general relativity most commonly use initial data on spacelike hypersurfaces to parametrise the space of solutions of Einstein's equations. In spite of the advantage of comforting familiarity and similarity to successful approaches in other field theories, the complexity of the resulting structure—in particular the non-linearity of the constraints on such initial data—led to difficulties when trying to pass from the Poisson structure to an algebra of quantum operators. Some, but not all, of these difficulties have been overcome with the discovery of loop quantum gravity [34][37].

An alternative approach is to formulate canonical general relativity using initial data on piecewise null hypersurfaces. One advantage of null initial data is that it is easy to solve all the constraints explicitly and identify free initial data (see discussion and references in [32][33]). In [32][33], Reisenberger calculated the Poisson bracket on free initial data on a certain class of piecewise null hypersurfaces. More precisely he found a Poisson bracket on the free data which reproduces Peierls' form of the Poisson bracket on a “sufficiently rich family of sufficiently nice” observables of the spacetime geometry [32][33]. The initial data surface used is the union of a pair of intersecting null hypersurfaces swept out by the future null normal geodesics (called the generators) emerging from the two dimensional intersection surface. The data consist of some quantities specified only on the intersection surface, and the main datum, which describes the induced metric on the null hypersurfaces. The generators are both tangent and normal to the hypersurfaces they sweep out. Therefore, in a suitable chart, the induced 3-metric consists of a 2-metric of signature $(+,+)$ and zeros. The determinant of this 2-metric can be used as a coordinate along the generators and the main part of the initial data becomes the unimodular 2-metric, the 2-metric divided by the square root of its determinant.

The Poisson bracket between these metric data obtained in [32][33] is elegant, but nevertheless somewhat intricate, and it is by no means obvious how it might be quantized. Can it be simplified by a change of variables, or be put into a form that has already been studied? In the present work steps are taken in this direction.

The Poisson brackets vanish between data at points that cannot be connected by a causal curve, a fact that reflects causality — one cannot act on fields outside one's lightcone. On a null hypersurface this means that it vanishes unless both points lie on the same generator. Furthermore, the bracket only depends on data on that generator. The Poisson algebra thus essentially decomposes into autonomous Poisson algebras on each generator. (This is not quite correct because the brackets are distributions over the coordinates transverse to the generators, so there is no Poisson algebra literally on only one generator, one always has to work with a congruence of generators.)

This observation gave birth to the idea of the present work: If the transverse dimensions have such a limited role then perhaps in a symmetry reduced model in which the

dependence of the fields on the transverse coordinates has been eliminated the Poisson brackets would be essentially the same, but could be studied more easily.

This turns out to be true. In the present work we study cylindrically symmetric gravitational waves, that is vacuum gravitational fields that are invariant under rotation about a symmetry axis and under translations along the symmetry axis. More technically, they have two commuting, hypersurface orthogonal Killing fields, a feature which is captured in the little acronym EG2CHSKF (Einstein gravity with two commuting, hypersurface orthogonal Killing fields). One of the Killing fields vanishes on the symmetry axis, and we will assume that the metric is regular on the axis. Cylindrically symmetric fields can be described as fields on a two dimensional spacetime, obtained from the full spacetime by factoring the Killing orbits. Initial data can be set on a null hypersurface consisting of the Killing orbit of a null geodesic emerging from a point on the symmetry axis at a given instant of time. In the two dimensional quotient spacetime this hypersurface is represented by a null curve emerging from the timelike worldline of the symmetry axis. All the generators have been collapsed to one.

By restricting the Einstein-Hilbert action functional to cylindrically symmetric fields an action is obtained that correctly describes the dynamics of these symmetric fields, and defines Poisson brackets. We evaluate the Poisson bracket on the main (metric) free null initial datum in the symmetry reduced theory by a procedure similar to that employed in [32][33], and obtain a bracket that is virtually identical to that found in [32][33] for the full theory. Specifically, the bracket on the lone generator in the symmetry reduced theory is just the bracket of the full theory stripped from the transverse delta function.

Cylindrically symmetric gravitational waves and EG2CHSKF more generally, has been studied intensively in the past because the class of solutions is large enough to include many interesting situations and yet the field equations are solvable. If one of the Killing fields is timelike, the solutions are the stationary and axially symmetric gravitational fields. If both are spacelike and one Killing field vanishes on a spacelike curve then they correspond to colliding gravitational plane waves or Gowdy universes. Finally, if both Killing fields are spacelike and one vanishes on a timelike curve, then the solutions are the cylindrical gravitational waves [18]. For a related subject, the Hamiltonian treatment of plane waves in terms of evolution along a null direction see [4].

The field equations can be solved basically because cylindrically symmetric gravitational waves (or indeed any class of spacetimes described EG2CHSKF) form an integrable system. Roughly speaking, this means that there exist sufficiently many Poisson commuting conserved quantities to form a complete set of canonical momenta. This is associated with the existence of a very large symmetry group, called the Geroch group, on the space of solutions. Indeed this group is transitive — it can be used to map any solution to any other [9][17].

Integrability makes the theory unusually manageable and also makes available several useful tools from the general theory of integrable systems [2]. Using precisely such tools Korotkin and Samtleben (KS) were able to construct in [24] a quantization of cylindrically symmetric gravitational waves that are asymptotically flat in a suitable sense. To arrive at this result they calculate the Poisson bracket on spacelike initial data. Then they define two one complex parameter families of conserved quantities $T_{\pm}(w)$ in terms of these data and compute their Poisson brackets. They note that the resulting Poisson algebra is a modification of a semi-classical Yangian double, a Poisson algebra which has

a well known quantization. Incorporating the modification at the quantum level they are able to define a $*$ -algebra that quantizes the Poisson algebra of the $T_{\pm}(w)$. Finally, they argue that the $T_{\pm}(w)$ constitute complete data to describe the field, as follows: A so called monodromy matrix $\mathcal{M}(w)$ may be constructed from the $T_{\pm}(w)$. For real w the monodromy matrix $\mathcal{M}(w)$ is closely related to the metric on the symmetry axis at an instant of time determined by w — \mathcal{M} is its Kramer-Neugebauer transform, which corresponds to a metric not regular on the axis. Korotkin and Samtleben argue that the monodromy matrix \mathcal{M} determines the field on all of spacetime.

The monodromy matrix is a very interesting object. Classically KS find that it has a simple, closed and elegant Poisson algebra, and the quantization of the $T_{\pm}(w)$ determines a similarly closed and elegant quantization of this Poisson algebra. But what makes it especially interesting for null canonical gravity is that it may be defined without reference to what happens at infinity. Although it is defined in [21] in terms of the asymptotic quantities $T_{\pm}(w)$, this is not necessary. In particular, suppose \mathcal{N} is a compact null line segment in the symmetry reduced spacetime that meets the worldline of the symmetry axis. (This worldline forms a boundary of the symmetry reduced spacetime which consists of only the radial and time dimensions of the original spacetime.) Then data on \mathcal{N} determines the metric and its Kramer-Neugebauer transform on the symmetry axis at times lying within the domain of dependence of \mathcal{N} . It thus determines $\mathcal{M}(w)$ on an interval \mathcal{W} of real values of w . This is important because in general for gravitational fields, without Killing vectors there usually are no smooth and infinitely extended null hypersurfaces, because the generators inevitably form caustics, and so it is convenient to formulate the canonical theory in terms of data on compact, truncated null hypersurfaces.

Here an explicit expression for $\mathcal{M}(w)$ in terms of initial data on \mathcal{N} is found, and it is proved that the null initial data may in turn be recovered from $\mathcal{M}(w)$ on the interval \mathcal{W} . Thus what has been found is a (non-local, non-linear) change of variables: $\mathcal{M}(w)$ on \mathcal{W} can be used to describe the gravitational field in the domain of dependence of \mathcal{N} . The Poisson bracket of $\mathcal{M}(w)$ is then calculated from the Poisson brackets on the data on \mathcal{N} . As expected it coincides with the bracket found in [24], but our derivation is somewhat more widely applicable; since it involves only fields in the domain of dependence of \mathcal{N} it is not necessary to assume asymptotic flatness or anything else about the behaviour at infinity.

Finally, as a further check, the matrices $T_{u/d}(w)$, the analogues of $T_{\pm}(w)$ in [24], are defined in terms of null initial data in the asymptotically flat context, and their Poisson bracket is calculated from that on the null initial data. Again the result of [24] is recovered.

This work is divided into two main parts. The first part, encompassing chapters 1 to 4, is devoted to reviewing the existing theory of cylindrically symmetrical gravitational waves. In chapter 1, starting with Einstein's equations, we derive an action for general relativity. In Chapter 2, the independence of the fields of two of the coordinates is used to perform a Kaluza-Klein reduction and introduce two interrelated formulations of the theory as a non-linear sigma model. Chapter 3 covers the basics of symplectic geometry, integrability, the role of initial data and symmetries. Chapter 4 introduces the auxiliary linear system which will be the basis of all our work. In Chapter 4 it is used to construct an infinite dimensional algebra of symmetries of the theory — the algebra of generators of the Geroch group. Another part of the chapter is dedicated to the analytic properties of solutions to the linear system.

In the second part, consisting of chapter 5, we calculate the symplectic 2-form on null initial data and derive a Poisson bracket. This expression is shown to be analogous to the bracket obtained in [32]. A change of variables to the monodromy matrix is proposed and the inverse of this transformation is shown to exist. We then show that our bracket yields the same Poisson algebras for the conserved quantities and the monodromy matrix found in [24]. This provides a cross-check and indicates that the expressions for the bracket on null initial data in [33] and in this work are correct.

The original results presented in this thesis are the result of a collaboration with Michael Reisenberger. I am very grateful for all his feedback, comments, explanations and contributions. Also, I thank Herbert Balasin from the Technische Universität Wien for making possible my stay in Uruguay and initiating our collaboration.

Part I

The state of the art of Einstein gravity with
two commuting, hypersurface orthogonal, spacelike
Killing fields

Chapter 1

The Action of General Relativity

In this section we derive the Einstein-Hilbert-Hawking action for general relativity including the boundary term starting from Einstein's equations for vacuum. This action will be the starting point for section 2.1, where we will use two coordinate independences induced by two commuting Killing fields in order to reduce the action integral and obtain an effectively 2-dimensional theory.

Einstein's equations in vacuum are

$$G_{\mu\nu} = 0 \quad \text{with} \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \quad \text{being the Einstein tensor.} \quad (1.1)$$

$R_{\mu\nu}$ and R are the Ricci tensor and scalar respectively.

What we are looking for, is a functional $S[g]$ of the metric tensor such that

$$\delta_g S[g] = \int_D \delta_g [\varepsilon_g \mathcal{L}(g)] = c \int_D \varepsilon_g G^{\mu\nu} \delta g_{\mu\nu} \quad \text{with} \quad c = \text{const.} \quad (1.2)$$

because then the requirement that the action be stationary under a variation $\delta g_{\mu\nu}$ will yield the field equations (1.1). $D \subset M$ is some domain in spacetime M such that ∂D is spacelike, ε_g is the volume element associated with the metric g (see also appendix E). The variation $\delta g_{\mu\nu}$ is required to vanish on the boundary ∂M of M , hence also the derivative in any direction along the boundary, but not necessarily the derivative in the direction normal to the boundary. This last fact will produce the ‘‘Hawking’’-boundary term.

Let us pause for a moment to look at the variation from the viewpoint of differential geometry. We may view δ_g as a derivative operator acting on functionals on the space \mathcal{C} of field configurations, which in this case of a field theory is an infinite-dimensional manifold, a function space or ‘‘function manifold’’. ‘‘Functions’’ on \mathcal{C} are called functionals and taking derivatives of them is called functional derivation. The application of a tangent vector field such as δ_g on a functional $F[g]$ on \mathcal{C} at an element $g_0 \in \mathcal{C}$ may be realized by first evaluating the functional along the integral curve $g(\lambda)$ through $g_0 = g(\lambda)|_{\lambda=0}$ of the vector field δ_g with parameter λ and then taking the ordinary derivative with respect to the parameter λ of $F[g(\lambda)]$ at the value $\lambda = 0$.

$$(\delta_g F)[g_0] = \left. \frac{d}{d\lambda} \right|_{\lambda=0} F[g(\lambda)], \quad g(0) = g_0.$$

When dealing with the relation of the equations of motion to the action functional the exact form of δ_g is not so important. Only some general characteristics of δ_g such as the vanishing on the boundary of some domain are fixed. In these situations it is sometimes useful to imagine a curve $g(\lambda)$ in \mathcal{C} having the desired properties and then applying $\left. \frac{d}{d\lambda} \right|_{\lambda=\lambda_0}$.

What about the object $\delta g(x)_{\mu\nu}$? For a fixed point x and a fixed pair of numbers (μ, ν) with $0 \leq \mu, \nu \leq 3$, $g_{\mu\nu}(x)$ can also be regarded as a functional on \mathcal{C} - it maps an element g of \mathcal{C} to its value of the component (μ, ν) at the point x . For different points x and values of (μ, ν) we get a family of functionals, parametrized by x, μ and ν . $\delta g(x)_{\mu\nu}$ is just the result of applying the tangent vector δ to the functional $g(x)_{\mu\nu}$. The set of functionals

$$\{g(x)_{\mu\nu} \mid x \in M, 0 \leq \mu, \nu \leq 3\}$$

may be viewed as a basis set of functionals. Any other functional F may be expressed as a function of elements of this set.

Let us continue with (1.2). We start the computation trying to invert the chain rule in order to find the appropriate $\mathcal{L}(g)$. What we certainly have to know, is the variation of the volume element

$$\delta \varepsilon_g = \delta \sqrt{\det g_{\mu\nu}} dx^0 \wedge \dots \wedge dx^3 = \frac{\delta \det g_{\mu\nu}}{2\sqrt{\det g_{\mu\nu}}} dx^0 \wedge \dots \wedge dx^3.$$

$g_{\mu\nu}$ is a symmetric matrix and can thus be brought to diagonal form by a similarity transformation $g = A \cdot \text{diag}(\lambda_1, \dots, \lambda_4) \cdot A^{-1}$, λ_i being the eigenvalues. The determinant of a matrix doesn't change under similarity transformations and so is equal to the product of the eigenvalues. We get

$$\begin{aligned} \delta \det g &= \delta \exp \ln \det g = \delta \exp \ln \prod_i \lambda_i = \delta \exp \sum_i \ln \lambda_i = \\ &= \sum_i \frac{\delta \lambda_i}{\lambda_i} \det g = g^{\mu\nu} \delta g_{\mu\nu} \det g \\ \Rightarrow \delta \varepsilon_g &= \frac{1}{2} \varepsilon_g g^{\mu\nu} \delta g_{\mu\nu} . \end{aligned}$$

The variation δ is defined on the metric with two covariant indices. The variation of the inverse, contravariant metric can be computed in the following way:

$$\delta \delta_\nu^\mu = 0 = \delta(g^{\mu\rho} g_{\rho\nu}) = \delta g^{\mu\rho} g_{\rho\nu} + g^{\mu\rho} \delta g_{\rho\nu} \Rightarrow \delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma} . \quad (1.3)$$

Considering (1.1) and (1.3) we obtain

$$\delta(\varepsilon_g R) = \varepsilon_g \left(\frac{1}{2} g^{\mu\nu} R - R_{\sigma\rho} g^{\sigma\mu} g^{\rho\nu} \right) \delta g_{\mu\nu} + \varepsilon_g g^{\mu\nu} \delta R_{\mu\nu} = -\varepsilon_g G^{\mu\nu} \delta g_{\mu\nu} + \varepsilon_g g^{\mu\nu} \delta R_{\mu\nu} .$$

The variation of $\varepsilon_g R$ gives us the desired Einstein tensor, but also one term we don't want, so we may consider taking the Ricci scalar $R(g)$ as one part of the Lagrangian density \mathcal{L} and adding another term compensating the variation of the Ricci tensor. To

this end we will further differentiate $\delta R_{\mu\nu}$. As explained before, we imagine a curve $g(\lambda)$ giving us a λ -dependent family of Ricci tensors $R(\lambda)_{\mu\nu}$. In this picture

$$\delta R_{\mu\nu} = \frac{d}{d\lambda} \Big|_{\lambda=0} \left[\nabla_\rho C(\lambda)^\rho_{\mu\nu} - \nabla_\nu C(\lambda)^\rho_{\mu\rho} + C(\lambda)^\rho_{\rho\sigma} C(\lambda)^\sigma_{\mu\nu} - C(\lambda)^\rho_{\nu\sigma} C(\lambda)^\sigma_{\mu\rho} \right],$$

where we have expressed the covariant derivative ∇_λ associated to $g(\lambda)$ using the covariant derivative ∇ associated to $g(0)$ and the corresponding difference tensor

$$C(\lambda)^\mu_{\nu\rho} = \frac{1}{2} g(\lambda)^{\mu\sigma} [\nabla_\nu g(\lambda)_{\sigma\rho} + \nabla_\rho g(\lambda)_{\sigma\nu} - \nabla_\sigma g(\lambda)_{\rho\nu}]. \quad (1.4)$$

At $\lambda = 0$ they vanish because the covariant derivative ∇ annihilates $g(0)$. Consequently the derivatives of the terms quadratic in $C^\mu_{\rho\sigma}$ will vanish because the one factor not differentiated will always be 0 when evaluated at $\lambda = 0$. A similar thing happens when $\frac{d}{d\lambda}$ acts on the first factor in (1.4). The terms ∇g evaluated at $\lambda = 0$ all vanish. In δC we are left with terms of the form $\nabla \delta g$ yielding

$$\begin{aligned} g^{\mu\nu} \delta R_{\mu\nu} &= \frac{1}{2} g^{\mu\nu} g^{\rho\sigma} \nabla_\rho (\nabla_\mu \delta g_{\sigma\nu} + \nabla_\nu \delta g_{\mu\sigma} - \nabla_\sigma \delta g_{\mu\nu}) - \\ &\quad - \frac{1}{2} g^{\mu\nu} g^{\rho\sigma} \nabla_\nu (\nabla_\mu \delta g_{\rho\sigma} + \nabla_\rho \delta g_{\sigma\mu} - \nabla_\sigma \delta g_{\rho\mu}) = \\ &= \nabla^\rho \nabla^\mu \delta g_{\rho\mu} - \nabla^2 \delta g^\mu_\mu = \nabla_\rho (\nabla^\mu \delta g^\rho_\mu - \nabla^\rho \delta g^\mu_\mu). \end{aligned}$$

This divergence gives us a boundary term, that is we have

$$\delta \int_D \varepsilon_g R = - \int_D \varepsilon_g \delta g_{\mu\nu} G^{\mu\nu} + \int_{\partial D} \varepsilon_h n_\rho (\nabla^\mu \delta g^\rho_\mu - \nabla^\rho \delta g^\mu_\mu), \quad (1.5)$$

where h is the induced metric on the boundary, ε_h the corresponding volume form and n_μ the unit normal to the boundary implying $g_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu$. In the following we assume that on the boundary in $\delta g = \delta h - \delta n n - n \delta n$ the terms δn and δh are each 0.

The integrand in the boundary term of (1.5) simplifies as follows:

$$\begin{aligned} &n^\rho (h^{\mu\nu} - n^\mu n^\nu) (\nabla_\nu \delta g_{\rho\mu} - \nabla_\rho \delta g_{\nu\mu}) = \\ &= n^\rho h^{\mu\nu} (\nabla_\nu \delta g_{\rho\mu} - \nabla_\rho \delta g_{\nu\mu}) = -n^\rho h^{\mu\nu} \nabla_\rho \delta g_{\nu\mu} \end{aligned} \quad (1.6)$$

since $\delta g = 0$ on the boundary and therefore also the derivative $h^{\mu\nu} \nabla_\nu \delta g_{\rho\sigma}$ along it.

The boundary is a 3-dimensional submanifold of spacetime. A natural measure for how much it is ‘‘curved’’ in the surrounding spacetime is the exterior curvature

$$K_{\mu\nu} = h_\mu^\rho \nabla_\rho n_\nu.$$

It measures the lack of parallel transport of the normal vector n_μ as one moves along the boundary.

The variation of the curvature scalar is

$$\delta K = \delta (h^{\mu\nu} \nabla_\mu n_\nu) = -h^{\mu\nu} \delta C^\rho_{\mu\nu} n_\rho = \frac{1}{2} h^{\mu\nu} n_\rho \nabla^\rho \delta g_{\mu\nu}.$$

The variation of ε_h also vanishes

$$\delta \varepsilon_h = \frac{1}{2} \varepsilon_h h^{\mu\nu} \delta h_{\mu\nu}.$$

Taking a look at (1.6) and (1.5), we finally arrive at the complete action functional

$$S = \frac{\kappa}{2} \int_D \varepsilon_g R + \kappa \int_{\partial D} \varepsilon_h K. \quad (1.7)$$

The factor $\frac{\kappa}{2}$ is convention. We keep it to allow comparison of our results e.g. with the ones obtained in [32], [33] and [24] at different stages.

Chapter 2

The action of vacuum Einstein gravity with two commuting, hypersurface orthogonal, spacelike Killing fields (EG2CHSKF)

In this chapter we restrict our attention to the class of spacetimes, which respect the symmetries corresponding to two commuting, hypersurface orthogonal, spacelike Killing fields. Using the method of Kaluza-Klein dimensional reduction we adapt the quantities in the action (1.7) to these symmetries. The resulting reduced action will be defined on the subspace of solutions to Einstein's equations, which respect our symmetries. In order not to leave this subspace when considering variations of the fields, variations will also have to respect these symmetries.

2.1 Dimensional reduction

2.1.1 General procedure

Kaluza and Klein developed a technique, which in the situation of independence of the metric of a coordinate x^{n+1} or equivalently existence of a Killing field $\frac{\partial}{\partial x^{n+1}}$, effectively reduces the dimension of the problem by interpreting the $(n+1)$ -dimensional vielbein

$$\check{g}_{\mu\nu} = \check{\eta}_{\check{a}\check{b}} \check{e}_{\mu}^{\check{a}} \check{e}_{\nu}^{\check{b}} \quad \text{with} \quad \check{e}_{\mu}^{\check{a}} \in GL(n+1) \quad , \quad \eta = \text{diag}(-1, 1, \dots, 1) \quad (2.1)$$

as a compound of an n -dimensional vielbein e_{μ}^a , two scalar fields ψ and ϕ , and a 1-form A_{μ} . The factorization (2.1) is locally invariant under

$$\check{e}_{\mu}^{\check{a}} \mapsto L^{\check{a}}_{\check{b}} \check{e}_{\mu}^{\check{b}}, \quad L \in SO(1, n+1), \quad (2.2)$$

because $SO(1, n+1)$ is the isometry group of $\check{\eta}$. We deal with spacelike Killing vectors. Thus, by exploiting the freedom of local Lorentz transformations and thereby partially fixing the corresponding gauge, we can achieve that \check{e}_{μ}^1 is the timelike basis element, $\check{e}_{\mu}^1, \dots, \check{e}_{\mu}^n$ are perpendicular to the spacelike Killing vector $\frac{\partial}{\partial x^{n+1}}$ while \check{e}_{μ}^{n+1} has a component in the direction of the Killing vector. The $(n+1)$ -dimensional vielbein can then

be related to the n -dimensional vielbein e_μ^a and the fields ψ, ϕ, A by

$$\begin{pmatrix} \check{e}_1^1 & \dots & \check{e}_1^n & \check{e}_1^{n+1} \\ \vdots & \ddots & \vdots & \vdots \\ \check{e}_n^1 & \dots & \check{e}_n^n & \check{e}_n^{n+1} \\ 0 & \dots & 0 & \check{e}_{n+1}^{n+1} \end{pmatrix} = \begin{pmatrix} \psi e_1^1 & \dots & \psi e_1^n & \phi A_1 \\ \vdots & \ddots & \vdots & \vdots \\ \psi e_n^1 & \dots & \psi e_n^n & \phi A_n \\ 0 & \dots & 0 & \phi \end{pmatrix}. \quad (2.3)$$

Note that this equation determines ϕ completely to $\phi = \check{e}_{n+1}^{n+1}$, while it determines ψ and e_μ^a only up to multiplication by a scalar factor and its inverse respectively

$$\psi e_\mu^a = \psi \varsigma \varsigma^{-1} e_\mu^a = (\psi \varsigma) (\varsigma^{-1} e_\mu^a) = \psi' e'_\mu{}^a. \quad (2.4)$$

Here and in the following we use the following index convention: Latin indices from the beginning of the alphabet (a, b, c, \dots) will range from 1 to n , if they carry a $\check{}$ ($\check{a}, \check{b}, \check{c}, \dots$) they will range from 1 to $n+1$. In general quantities carrying a $\check{}$ will be $(n+1)$ -dimensional while those without a $\check{}$ will be n -dimensional. Accordingly, spacetime indices μ, ν, \dots on $(n+1)$ -dimensional (n -dimensional) quantities will range from 1 to $n+1$ (1 to n).

The question now arises if the fields ψ, ϕ and A_μ really transform as scalar fields and a one form respectively. We consider an infinitesimal coordinate transformation generated by a tangent vector field $\check{\xi}^\mu(x)$ independent of x^{n+1} . The infinitesimal change of the vielbein is given by the Lie derivative along $\check{\xi}$

$$\delta \check{e}_\mu^{\check{a}} = L_{\check{\xi}} \check{e}_\mu^{\check{a}} = \check{\xi}^\nu \partial_\nu \check{e}_\mu^{\check{a}} + \partial_\mu \check{\xi}^\nu \check{e}_\nu^{\check{a}}.$$

Denoting the first n components of $\check{\xi}^\mu$ by ξ^μ and using (2.3), we have

$$\delta \check{e} = \begin{pmatrix} \delta \psi e_\mu^a + \psi \delta e_\mu^a & \delta \phi A_\mu + \phi \delta A_\mu \\ 0 & \delta \phi \end{pmatrix},$$

$$L_{\check{\xi}} \check{e} = \begin{pmatrix} \xi^\nu \partial_\nu \psi e_\mu^a + \psi L_\xi e_\mu^a & \xi^\nu \partial_\nu \phi A_\mu + \phi (L_\xi A_\mu + \partial_\mu \xi^{n+1}) \\ 0 & \xi^\nu \partial_\nu \phi \end{pmatrix}.$$

Comparing the two expressions, we see that ψ and ϕ really transform as scalars and A_μ transforms like a $U(1)$ gauge potential in electrodynamics, $\partial_\mu \xi^{n+1}$ plays the role of the $\mathfrak{u}(1)$ -valued one-form, the difference between two equivalent gauge potentials. Below we will see that only the field strength tensor $F = dA$, which is independent of this gauge, will appear in the action integral.

For the metric $\check{g}_{\mu\nu} = \check{\eta}_{\check{a}\check{b}} \check{e}_\mu^{\check{a}} \check{e}_\nu^{\check{b}}$ we get

$$\check{g}_{\mu\nu} = \begin{pmatrix} \psi^2 g_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\nu & \phi^2 \end{pmatrix}, \quad \check{g}^{\mu\nu} = \begin{pmatrix} \psi^{-2} g^{\mu\nu} & -\psi^{-2} A^\mu \\ -\psi^{-2} A^\nu & \phi^{-2} + \psi^{-2} A^2 \end{pmatrix}. \quad (2.5)$$

The task now is to reexpress the quantities built out of the $(n+1)$ -dimensional vielbein appearing in the action integral in terms of the n -dimensional metric and the new fields

A_μ, ϕ, ψ . We will apply the Kaluza-Klein reduction twice, once for each of our commuting Killing fields, ending up with a new, symmetry reduced model on a 2-dimensional Lorentzian spacetime. In the sequel we show the necessary calculations to reduce the action from $n+1$ to n dimensions as indicated in (2.3). The main quantity we have to reexpress is the Ricci scalar. The others will behave quite simply. An introduction to the vielbein formalism we will use is provided in appendix A.

According to (2.3)

$$\check{e}^a = \psi e^a, \quad \check{e}^{n+1} = \phi A + \phi dx^{n+1}. \quad (2.6)$$

The corresponding covielbein is

$$\check{E}_a = \psi^{-1} E_a - A_a \psi^{-1} \partial_{n+1}, \quad \check{E}^{n+1} = \phi^{-1} \partial_{n+1}. \quad (2.7)$$

We will frequently use the notation

$$\partial_a := E_a^\mu \partial_\mu \quad \text{and more generally} \quad v_a := E_a^\mu v_\mu \quad (2.8)$$

for a covector v_μ . Following (A.3), we compute

$$d\check{e}^a = \psi de^a + \partial_b \psi e^b \wedge e^a = \psi de^a + \psi^{-2} \partial_b \psi \check{e}^b \wedge \check{e}^a, \quad (2.9)$$

$$d\check{e}^{n+1} = \partial_a \phi e^a \wedge A + \frac{1}{2} \phi F_{ab} e^a \wedge e^b + \partial_a \phi e^a \wedge dx^{n+1} = \quad (2.10)$$

$$= \psi^{-1} \phi^{-1} \partial_a \phi \check{e}^a \wedge \check{e}^{n+1} + \frac{1}{2} \psi^{-2} \phi F_{ab} \check{e}^a \wedge \check{e}^b. \quad (2.11)$$

$$(2.12)$$

Since

$$d\check{e}^{n+1} = \check{e}^a \wedge \check{\omega}^{n+1}_a,$$

we can read off

$$\check{\omega}^{n+1}_a = \psi^{-1} \phi^{-1} \partial_a \phi \check{e}^{n+1} + \frac{1}{2} \psi^{-2} \phi F_{ab} \check{e}^b, \\ \check{\omega}^a_{n+1} = -\psi^{-1} \phi^{-1} \partial^a \phi \check{e}_{n+1} - \frac{1}{2} \psi^{-2} \phi F^a_b \check{e}^b.$$

This implies that

$$d\check{e}^a = \check{e}^b \wedge \check{\omega}^a_b + \check{e}^{n+1} \wedge \check{\omega}^a_{n+1} = \check{e}^b \wedge \check{\omega}^a_b - \frac{1}{2} \psi^{-2} \phi F^a_b \check{e}^{n+1} \wedge \check{e}^b.$$

But by (2.9)

$$d\check{e}^a = \check{e}^b \wedge \omega^a_b + \psi^{-2} \partial_b \psi \check{e}^b \wedge \check{e}^a. \quad (2.13)$$

Thus because of the antisymmetry condition (A.5), we have

$$\check{\omega}^a_b = \omega^a_b + \partial_b \psi \psi^{-2} \check{e}^a - \partial^a \psi \psi^{-2} \check{e}_b - \frac{1}{2} \psi^{-2} \phi F^a_b \check{e}^{n+1}.$$

Now that $\check{\omega}$ has been found we may begin to evaluate the Riemann tensor by computing

$$\begin{aligned} d\check{\omega}_b^a &= d\omega_b^a - \psi^{-4}\partial_c\psi\partial_b\psi\check{e}^c \wedge \check{e}^a + \psi^{-4}\partial^a\psi\partial_c\psi\check{e}^c \wedge \check{e}_b + \\ &\quad + \psi^{-3}\partial_c\partial_b\psi\check{e}^c \wedge \check{e}^a - \psi^{-3}\partial_c\partial^a\psi\check{e}^c \wedge \check{e}_b + \psi^{-1}\partial_b\psi de^a - \psi^{-1}\partial^a\psi de_b - \\ &\quad - \frac{1}{4}\psi^{-4}\phi^2 F_b^a F_{cd}\check{e}^c \wedge \check{e}^d + \check{e}^{n+1} \wedge \check{e}^c \quad (\dots)_{bc}^a, \\ d\check{\omega}_{n+1}^a &= (\psi^{-3}\phi^{-1}\partial_b\psi\partial^a\phi - \psi^{-2}\phi^{-1}\partial_b\partial^a\phi)\check{e}^b \wedge \check{e}_{n+1} + \check{e}^d \wedge \check{e}^f \quad (\dots)_{df}^a. \end{aligned}$$

The Riemann tensor is

$$\begin{aligned} \check{R}_b^a &= d\check{\omega}_b^a + \check{\omega}_c^a \wedge \check{\omega}_b^c + \check{\omega}_{n+1}^a \wedge \check{\omega}_{n+1}^b = \\ &= d\omega_b^a - \psi^{-4}\partial_c\psi(\partial_b\psi\check{e}^c \wedge \check{e}^a - \partial^a\psi\check{e}^c \wedge \check{e}_b) + \psi^{-3}(\partial_c\partial_b\psi\check{e}^c \wedge \check{e}^a - \\ &\quad - \partial_c\partial^a\psi\check{e}^c \wedge \check{e}_b) + \psi^{-2}(\partial_b\psi\check{e}^c \wedge \omega_c^a - \partial^a\psi\check{e}^c \wedge \omega_{bc}) - \frac{1}{4}\psi^{-4}\phi^2 F_b^a F_{cd}\check{e}^c \wedge \check{e}^d + \\ &\quad + \omega_c^a \wedge \omega_b^c + \psi^{-2}\omega_c^a \wedge (\check{e}^c\partial_b\psi - \check{e}_b\partial^c\psi) + \psi^{-2}(\partial_c\psi\check{e}^a - \partial^a\psi\check{e}_c) \wedge \omega_b^c + \\ &\quad + \psi^{-4}(\partial_c\psi\check{e}^a - \partial^a\psi\check{e}_c) \wedge (\partial_b\psi\check{e}^c - \partial^c\psi\check{e}_b) - \frac{1}{4}\phi^2\psi^{-4}F_d^a F_{bc}\check{e}^d \wedge \check{e}^c + \\ &\quad + \check{e}^{n+1} \wedge (\dots)_b^a, \end{aligned}$$

$$\begin{aligned} \check{R}_{n+1}^a &= d\check{\omega}_{n+1}^a + \check{\omega}_b^a \wedge \check{\omega}_{n+1}^b = \\ &= (\psi^{-3}\phi^{-1}\partial_b\psi\partial^a\phi - \psi^{-2}\phi^{-1}\partial_b\partial^a\phi)\check{e}^b \wedge \check{e}_{n+1} - \psi^{-1}\phi^{-1}\partial^b\phi\omega_b^a \wedge \check{e}^{n+1} - \\ &\quad - \psi^{-3}\phi^{-1}\partial_b\psi\partial^b\phi\check{e}^a \wedge \check{e}^{n+1} + \psi^{-3}\phi^{-1}\partial^a\psi\partial^b\phi\check{e}_b \wedge \check{e}^{n+1} + \\ &\quad + \frac{1}{4}\psi^{-4}\phi^2 F_b^a F_c^b \check{e}^{n+1} \wedge \check{e}^c + \check{e}^b \wedge \check{e}^c (\dots)_{bc}^a. \end{aligned}$$

The Ricci tensor is

$$\begin{aligned} \check{R}_b &= \check{E}_a \lrcorner \check{R}_b^a + \check{E}_{n+1} \lrcorner \check{R}_{n+1}^b = \\ &= \psi^{-1}E_a \lrcorner d\omega_b^a - \psi^{-4}\partial_c\psi(\partial_b\psi(1-n)\check{e}^c - \partial^c\psi\check{e}_b + \partial_b\psi\check{e}^c) + \\ &\quad + \psi^{-3}\partial_c\partial_b\psi\check{e}^c(1-n) - \psi^{-3}\partial_a\partial^a\psi\check{e}_b + \psi^{-3}\partial_c\partial_b\psi\check{e}^c - \psi^{-3}\partial_b\psi(E_a \lrcorner \omega_c^a)\check{e}^c - \\ &\quad - \psi^{-2}\partial^a\psi\omega_{ba} + \psi^{-3}\partial^a\psi(E_a \lrcorner \omega_{bc})\check{e}^c + \psi^{-1}E_a \lrcorner (\omega_c^a \wedge \omega_b^c) + \\ &\quad + (E_a \lrcorner \omega_c^a)\psi^{-3}(\check{e}^c\partial_b\psi - \check{e}_b\partial^c\psi) + \omega_{bc}\psi^{-2}\partial^c\psi + n\partial_c\psi\psi^{-2}\omega_b^c - \\ &\quad - \partial_c\psi\psi^{-2}\omega_b^c - (E_a \lrcorner \omega_b^c)\psi^{-3}(\partial_c\psi\check{e}^a - \partial^a\psi\check{e}_c) - \\ &\quad - \frac{1}{2}\psi^{-4}\phi^2 F_b^a F_{ad}\check{e}^d + \psi^{-4}\partial_c\psi(n-1)(\partial_b\psi\check{e}^c - \partial^c\psi\check{e}_b) - \\ &\quad - \psi^{-4}(\partial_c\psi\check{e}^a - \partial^a\psi\check{e}_c)(\partial_b\psi\delta_c^a - \partial^c\psi\eta_{ab}) + \\ &\quad + \frac{1}{4}\phi^2\psi^{-4}F_d^a F_{ba}\check{e}^d + \psi^{-3}\phi^{-1}\partial_c\psi\partial_b\phi\check{e}^c - \psi^{-2}\phi^{-1}\partial_c\partial_b\phi\check{e}^c - \\ &\quad - \psi^{-1}\phi^{-1}\partial_c\phi\omega_b^c - \\ &\quad - \psi^{-3}\phi^{-1}\partial_c\psi\partial^c\phi\check{e}_b - \frac{1}{4}\psi^{-4}\phi^2 F_{bc}F_d^c\check{e}^d + \partial_b\psi\psi^{-3}\phi^{-1}\partial^c\phi\check{e}_c = \\ &= \psi^{-1}E_a \lrcorner d\omega_b^a + \psi^{-4}\partial_c\psi\partial_b\psi 2(n-2)\check{e}^c - (n-2)\psi^{-4}\partial_c\psi\partial^c\psi\check{e}_b + \\ &\quad + \psi^{-3}\partial_c\partial_b\psi\check{e}^c(2-n) - \psi^{-3}\partial_a\partial^a\psi\check{e}_b + \psi^{-1}E_a \lrcorner (\omega_c^a \wedge \omega_b^c) - \\ &\quad - (E_a \lrcorner \omega_c^a)\psi^{-3}\check{e}_b\partial^c\psi + (n-1)\psi^{-2}\partial_c\psi\omega_b^c - (E_a \lrcorner \omega_b^c)\psi^{-3}\partial_c\psi\check{e}^a + \\ &\quad + \psi^{-4}\partial_c\psi\partial^c\psi\check{e}_b + \psi^{-3}\phi^{-1}\partial_c\psi\partial_b\phi\check{e}^c - \psi^{-2}\phi^{-1}\partial_c\partial_b\phi\check{e}^c - \psi^{-1}\phi^{-1}\partial_c\phi\omega_b^c - \\ &\quad - \psi^{-3}\phi^{-1}\partial_c\psi\partial^c\phi\check{e}_b + \psi^{-3}\phi^{-1}\partial_b\psi\partial_c\phi\check{e}^c - \frac{1}{2}\psi^{-4}\phi^2 F_b^a F_{ad}\check{e}^d, \\ \check{R}_{n+1} &= \check{E}_a \lrcorner \check{R}_{n+1}^a = \\ &= (2-n)\psi^{-3}\phi^{-1}\partial_a\psi\partial^a\phi\check{e}^{n+1} - \psi^{-2}\phi^{-1}\partial_c\partial^c\phi\check{e}^{n+1} - \\ &\quad - \psi^{-2}\phi^{-1}\partial^b\phi(E_a \lrcorner \omega_b^a)\check{e}^{n+1} - \frac{1}{4}\psi^{-4}\phi^2 F_b^a F_a^b \check{e}^{n+1}. \end{aligned}$$

Finally, the Ricci scalar is

$$\begin{aligned}
\check{R} &= \check{E}_b \lrcorner \check{R}^b + \check{E}_{n+1} \lrcorner \check{R}^{n+1} = \\
&= \psi^{-2} R - \psi^{-4} \partial_b \psi \partial^b \psi ((n-2)^2 - n) + 2(1-n) \psi^{-3} \partial_c \partial^c \psi + \\
&\quad + 2(1-n) \psi^{-3} \partial_c \psi (E_a \lrcorner \omega^{ac}) - 2\psi^{-2} \phi^{-1} \partial_b \partial^b \phi - \\
&\quad - 2\psi^{-2} \phi^{-1} \partial_c \phi (E_a \lrcorner \omega^{ac}) + \\
&\quad + 2(2-n) \psi^{-3} \phi^{-1} \partial_c \psi \partial^c \phi - \frac{1}{4} \psi^{-4} \phi^2 F^{ab} F_{ab}.
\end{aligned}$$

Considering (A.2), we have for an arbitrary covector v_μ

$$\begin{aligned}
\nabla_a v_\mu &= E_a^\nu \nabla_\nu (v_b e_\mu^b) = E_a^\nu (\partial_\nu v_b e_\mu^b - v_b \omega_{\nu c}^b e_\mu^c) = \\
&= (\partial_a v_b - (E_a \lrcorner \omega^c_b) v_c) e_\mu^b = (\nabla_a v_b) e_\mu^b.
\end{aligned}$$

The volume element simply transforms as

$$\varepsilon_{\check{g}} = \sqrt{-\det \check{g}} dx^1 \wedge \dots \wedge dx^{n+1} = \det \check{e} dx^1 \wedge \dots \wedge dx^{n+1} = \psi^n \phi \varepsilon_g \wedge dx^{n+1},$$

eventually yielding

$$\begin{aligned}
\frac{\kappa}{2} \int_D \varepsilon_{\check{g}} \check{R} &= \frac{\kappa}{2} \int_D \varepsilon_g \wedge dx^{n+1} \psi^n \phi \left\{ \psi^{-2} R - \psi^{-4} \partial_b \psi \partial^b \psi ((n-2)^2 - n) + \right. \\
&\quad \left. + 2(1-n) \psi^{-3} \nabla_c \nabla^c \psi - 2\psi^{-2} \phi^{-1} \nabla_b \nabla^b \phi + 2(2-n) \psi^{-3} \phi^{-1} \partial_c \psi \partial^c \phi - \frac{1}{4} \psi^{-4} \phi^2 F^{ab} F_{ab} \right\}. \quad (2.14)
\end{aligned}$$

Note that in F_{ab} we do not have to substitute partial derivatives by covariant derivatives in order to get a covariant quantity because $F_{ab} = E_b \lrcorner (E_a \lrcorner dA)$ and the exterior derivative d is independent of the connection used.

Let us now look at the behaviour of K under the reduction:

For the induced metric \check{h} on the boundary of D we have $\check{h}^{\mu\nu} = \check{g}^{\mu\nu} + \check{n}^\mu \check{n}^\nu$ with $\check{g}^{\mu\nu} \check{n}_\mu \check{n}_\nu = -1$. It follows that $\check{n}^\mu \check{\nabla}_\nu \check{n}_\mu = \frac{1}{2} \check{\nabla}_\nu (\check{n}^\mu \check{n}_\mu) = \frac{1}{2} \check{\nabla}_\nu (-1) = 0$ and hence

$$\check{K} = \check{h}^{\mu\nu} \check{\nabla}_\mu \check{n}_\nu = \check{g}^{\mu\nu} \check{\nabla}_\mu \check{n}_\nu = \check{g}^{\check{a}\check{b}} \check{\nabla}_{\check{a}} \check{n}_{\check{b}} = \check{\eta}^{\check{a}\check{b}} (\check{\partial}_{\check{a}} \check{n}_{\check{b}} + \check{E}_{\check{a}} \lrcorner \check{\omega}_{\check{b}}^{\check{c}} \check{n}_{\check{c}}).$$

We assume that \check{n} is orthogonal to the dimension being reduced, that is $\check{n}_{n+1} = \frac{\partial}{\partial x^{n+1}} \lrcorner \check{n} = 0$. Note that

$$\check{\eta}^{ab} = \eta^{ab}, \quad \partial_{n+1} \check{n}_{\check{b}} = 0, \quad \check{\partial}_a = \check{E}_a^\mu \partial_\mu = e_\mu^b \check{E}_a^\mu \partial_b = \psi^{-1} \partial_a, \quad \check{n}_a = n_a. \quad (2.15)$$

The last equation comes from the fact that if $\check{n} = \check{n}_a \check{e}^a$ has norm -1 w.r.t. \check{g} then $n = n_a e^a = \check{n}_a e^a$ has norm -1 w.r.t. g :

$$-1 = \check{g}^{\mu\nu} \check{n}_\mu \check{n}_\nu = \eta^{ab} \check{E}_a^\mu \check{E}_b^\nu \check{e}_\mu^c \check{e}_\nu^d \check{n}_c \check{n}_d = \eta^{ab} E_a^\mu E_b^\nu e_\mu^c e_\nu^d \check{n}_c \check{n}_d = g^{\mu\nu} n_\mu n_\nu.$$

Now let us reduce

$$\begin{aligned}
\check{K} &= \psi^{-1}\eta^{ab}\partial_a n_b - \check{E}_a \lrcorner (\omega^{ba} + \psi^{-2}\partial^a\psi\check{e}^b - \psi^{-2}\partial^b\psi\check{e}^a)n_b + \\
&\quad + \check{E}_{n+1} \lrcorner (\psi^{-1}\phi^{-1}\partial^b\phi\check{e}^{n+1})n_b \\
&= \psi^{-1}\partial^a n_a - \psi^{-1}E_a \lrcorner \omega^{ba}n_b - \psi^{-2}\partial^a\psi n_a + n\psi^{-2}\partial^a\psi n_a + \\
&\quad + \psi^{-1}\phi^{-1}\partial^a\phi n_a = \\
&= \psi^{-1}\nabla^a n_a + (n-1)\psi^{-2}n_a\partial^a\psi + \psi^{-1}\phi^{-1}n_a\partial^a\phi,
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
\kappa \int_{\partial D} \varepsilon_{\check{h}} \check{K} &= \\
= \kappa \int_{\partial D} \varepsilon_h \wedge dx^{n+1} \left\{ \psi^{n-2}\phi\nabla^a n_a + (n-1)\psi^{n-3}\phi n_a\partial^a\psi + \psi^{n-2}n_a\partial^a\phi \right\}.
\end{aligned} \tag{2.17}$$

If we integrate the third and the fourth term in the right hand side of (2.14) partially, the boundary terms exactly cancel the second and third term of (2.17). We are thus left with

$$\begin{aligned}
\frac{\kappa}{2} \int_D \varepsilon_{\check{g}} \check{R} + \kappa \int_{\partial D} \varepsilon_{\check{h}} \check{K} &= \\
= \frac{\kappa}{2} \int_D \varepsilon_g \wedge dx^{n+1} \left\{ \psi^{n-2}\phi R - \psi^{n-4}\phi(\partial\psi)^2((n-2)^2 - n) - \right. \\
&\quad - 2(1-n)\partial_a(\psi^{n-3}\phi)\partial^a\psi + 2\partial_a\psi^{n-2}\partial^a\phi + 2(2-n)\psi^{n-3}\partial_c\psi\partial^c\phi - \\
&\quad \left. - \frac{1}{4}\psi^{n-4}\phi^3 F^{ab}F_{ab} \right\} + \kappa \int_{\partial D} \varepsilon_h \wedge dx^{n+1}\psi^{n-2}\phi K = \\
= \frac{\kappa}{2} \int_D \varepsilon_g \wedge dx^{n+1} \left\{ \psi^{n-2}\phi R + \psi^{n-4}\phi(\partial\psi)^2(n^2 - 3n + 2) + \right. \\
&\quad \left. + 2(n-1)\psi^{n-3}\partial_a\phi\partial^a\psi - \frac{1}{4}\psi^{n-4}\phi^3 F^{ab}F_{ab} \right\} + \kappa \int_{\partial D} \varepsilon_h \wedge dx^{n+1}\psi^{n-2}\phi K.
\end{aligned} \tag{2.18}$$

$$\tag{2.19}$$

We are going to reduce the dimension twice, so in the second step we will have to reduce terms involving inner products in the action, in particular $F_{\check{a}\check{b}}F^{\check{a}\check{b}}$ and $\partial_{\check{a}}s\partial^{\check{a}}t$, s and t being scalars. These can be expanded in terms of lower dimensional fields via identities much like those of (2.15):

$$\begin{aligned}
F_{\check{a}\check{b}}F_{\check{c}\check{d}}\eta^{\check{a}\check{c}}\eta^{\check{b}\check{d}} &= \psi^{-4}F_{ab}F_{cd}\eta^{ac}\eta^{bd} + 2\psi^{-2}\phi^{-2}F_{an+1}F_{bn+1}\eta^{ab} \\
\text{and } \partial_{\check{a}}s\partial_{\check{b}}t\eta^{\check{a}\check{b}} &= \psi^{-2}\partial_a s\partial_b t\eta^{ab}.
\end{aligned} \tag{2.20}$$

We have now reduced all the quantities we need. The coordinate x^{n+1} can simply be integrated over its domain because the integrand is independent of it. This integration gives a constant factor, which we will drop, since it cannot influence the equations of motion.

2.1.2 Applying the reduction process twice

In our situation of Einstein gravity with two commuting, hypersurface orthogonal, space-like Killing fields the coordinates and the vierbein may be chosen so that the vierbein is independent of two of the coordinates (see e.g. chapter 7.1. of [39]):

Commutativity of the two Killing fields, let's call them ξ and χ , implies by Frobenius's theorem that coordinates x^μ can be found such that

$$\xi = \frac{\partial}{\partial x^2} \quad \text{and} \quad \chi = \frac{\partial}{\partial x^3}. \quad (2.21)$$

The Killing fields are assumed to be spacelike, which is why we called these coordinates x^2 and x^3 , and none of them x^0 , which we want to reserve for a timelike coordinate. Using these coordinates, Killing's equation takes the simple form

$$0 = L_\xi g_{\mu\nu} = \frac{\partial}{\partial x^2} g_{\mu\nu}, \quad 0 = L_\chi g_{\mu\nu} = \frac{\partial}{\partial x^3} g_{\mu\nu}, \quad (2.22)$$

which tells us that the metric is independent of x^2 and x^3 . We are in a situation where we may apply the Kaluza Klein reduction technique twice to reduce first from 4 to 3 and then from 3 to 2 dimensions. Each time we will choose the functions ψ and ϕ in a convenient way.

In the first step ($n = 3$) we choose $\psi^{-1} = \phi$ and call ϕ " $\Delta^{1/2}$ ", and we name the one-form B_μ , that is

$${}^{(4)}e_{\check{\mu}}^{\check{a}} = \begin{pmatrix} \Delta^{-1/2} {}^{(3)}e_{\mu}^a & \Delta^{1/2} B_{\mu} \\ 0 & \Delta^{1/2} \end{pmatrix}.$$

We get

$${}^{(3)}S = \frac{\kappa}{2} \int_{(3)D} \varepsilon_{(3)g} \left\{ {}^{(3)}R - \frac{1}{2} \Delta^{-2} (\partial \Delta)^2 - \frac{1}{4} \Delta^2 F(B)^2 \right\} - \kappa \int_{\partial(3)D} \varepsilon_{(3)h} {}^{(3)}K, \quad (2.23)$$

where we have denoted the induced metric and exterior curvature on the 2-dimensional boundary $\partial(3)D$ of $(3)D$ by $(3)h$ and $(3)K$ respectively.

Now we invoke the hypersurface orthogonality of the Killing fields. This implies that the coordinates x^2 and x^3 may be chosen to be constant on hypersurfaces orthogonal to the Killing fields, and thus ∂_{x^0} and ∂_{x^1} are orthogonal to ∂_{x^2} and ∂_{x^3} . In other words, the metric is block diagonal, having two 2×2 blocks. The vierbein may then also be chosen to be block diagonal in the same way, with the consequence that $B_0 = 0 = B_1$ and hence $F_{01} = 0$.

Remark: The requirement of hypersurface orthogonality may also be derived from two alternative conditions:

- One of the Killing fields vanishes at one point in spacetime.
- $R_{\mu\nu} = 0$.

See again [39] for details.

In the second step ($n = 2$) we call ψ " λ " and ϕ " ρ ", and the Kaluza-Klein one-form will be denoted by A_μ . Thus

$${}^{(3)}e_{\check{\mu}}^{\check{a}} = \begin{pmatrix} \lambda {}^{(2)}e_{\mu}^a & \rho A_{\mu} \\ 0 & \rho \end{pmatrix}.$$

For the treatment of the terms in (2.23) involving the fields Δ and B_μ let's take a look at (2.20) and remember $F_{01} = 0 = F_{10}$. This together with (2.19) now tells us that

$${}^{(2)}S = \frac{\kappa}{2} \int_{(2)D} \varepsilon_{(2)g\rho} \left\{ {}^{(2)}R + 2\lambda^{-1} \partial_a \lambda \rho^{-1} \partial^a \rho - \frac{1}{4} \lambda^{-2} \rho^2 F(A)^2 - \frac{1}{2} \Delta^{-2} (\partial \Delta)^2 + \right. \\ \left. - \frac{1}{2} \rho^{-2} \Delta^2 (\partial B_2)^2 \right\} + \kappa \int_{\partial(2)D} \varepsilon_{(2)h\rho} \rho K. \quad (2.24)$$

Again for hypersurface orthogonality A must be zero. The 4-dimensional vielbein now has the form

$${}^{(4)}e_{\check{\mu}}^{\check{a}} = \begin{pmatrix} \Delta^{-1/2} \lambda {}^{(2)}e_{\nu}^a & 0 & 0 \\ 0 & \Delta^{-1/2} \rho & \Delta^{1/2} B_2 \\ 0 & 0 & \Delta^{1/2} \end{pmatrix} \quad (2.25)$$

and the metric is

$$g_{\mu\nu} = \begin{pmatrix} \Delta^{-1} \lambda^2 {}^{(2)}g_{\sigma\tau} & 0 & 0 \\ 0 & \rho^2 \Delta^{-1} + \Delta (B_2)^2 & \Delta B_2 \\ 0 & \Delta B_2 & \Delta \end{pmatrix}. \quad (2.26)$$

The field Δ is the squared norm of the Killing vector $\frac{\partial}{\partial x^3}$, ΔB_2 is the inner product of the two Killing vectors, and $\rho dx^2 dx^3$ is the area element on the Killing orbits. Note that λ is not determined by the metric, only the combination $\lambda^2 {}^{(2)}g$ is. Later, in subsection 2.3.3, we will see that one of the field equations requires ρ to be a harmonic function with respect to $\lambda^2 {}^{(2)}g$ (or ${}^{(2)}g$, it makes no difference). Thus if ρ and it's harmonic conjugate are taken as coordinates on the two dimensional quotient manifold then $\lambda^2 {}^{(2)}g_{\mu\nu}$ becomes proportional to $\text{diag}(-1, 1)$. Setting ${}^{(2)}g_{\mu\nu}$ equal to $\text{diag}(-1, 1)$ then gives an unambiguous value to λ , determined by the 4-metric.

This parametrization (2.26) of the metric is well adapted to the cylindrically symmetric context if we take x^2 to be the azimuthal angle θ , and x^3 the axial coordinate z . We will demand of our cylindrically symmetric spacetime that it contains the symmetry axis $\rho = 0$, and that the metric is regular at the axis in a certain coordinate system X, Y, Z, T about the axis constructed from the fields.

The three spatial coordinates X, Y, Z are constructed from ρ, θ, z just like Cartesian coordinates are constructed from cylindrical coordinates in Euclidean space: $X = \rho \cos \theta$, $Y = \rho \sin \theta$, and $Z = z$. (The fact that $\rho d\theta dz$ is the area element on the Killing orbits makes ρ a curved spacetime analog of the cylindrical radius.) The time coordinate T is then chosen so that the equal T hypersurfaces are orthogonal to the curves of constant X, Y, Z . That this is possible is a consequence of the field equations, which, as already mentioned, require ρ to be harmonic on the quotient spacetime. The requirement is fulfilled if and only if T is the harmonic conjugate $\tilde{\rho}$ of ρ , or a function of it only.

The components of the metric in these coordinates are easily worked out¹ and they are regular at the axis iff Δ, B_2, λ , and ${}^{(2)}g$ are regular functions of x^μ and, as $\rho \rightarrow 0$

1

$$ds^2 = -\Delta^{-1} \lambda^2 dT^2 + \Delta^{-1} \lambda^2 (\cos^2 \theta dX^2 + \sin^2 \theta dY^2 + 2 \cos \theta \sin \theta dX dY) \\ + (\Delta^{-1} + \Delta \frac{B_2^2}{\rho^2}) (\sin^2 \theta dX^2 + \cos^2 \theta dY^2 - 2 \cos \theta \sin \theta dX dY) \\ + \Delta dz^2 + 2\Delta \frac{B_2}{\rho} (\cos \theta dY dZ - \sin \theta dX dZ).$$

1. Δ has a non-zero limit,
2. $B_2/\rho \rightarrow 0$,
3. $\lambda \rightarrow 1$ (if ${}^{(2)}g$ is set to $\text{diag}(-1, 1)$ in the coordinates $\rho, \tilde{\rho}$).

Condition 2. is equivalent to requiring that that B_2 and its gradient dB_2 on the quotient spacetime vanish at $\rho = 0$.

2.2 The Kramer-Neugebauer transformation

EG2CHSKF possesses a remarkable discrete symmetry, the symmetry under the so called *Kramer-Neugebauer transformation*. Because of this symmetry the solutions may be obtained as stationary points of two distinct actions, an action S and another action \tilde{S} obtained from S by replacing the fields by their Kramer-Neugebauer transforms. The two actions thus have exactly the same form in their respective field variables, but these are defined differently in terms of the 4-metric.

Here the symmetry will be demonstrated, basically following [9], by obtaining the two actions. We begin with \tilde{S} which is identical to the action ${}^{(2)}S$ given in (2.24), but will be expressed in terms of different variables. Taking into account that the hypothesis of hypersurface orthogonality implies that $F(A) = 0$, the action ${}^{(2)}S$ becomes

$${}^{(2)}S = \frac{\kappa}{2} \int_{(2)D} \varepsilon_{(2)g} \rho \left\{ {}^{(2)}R - \frac{1}{2} \Delta^{-2} (\partial \Delta)^2 - \frac{1}{2} \rho^{-2} \Delta^2 (\partial B_2)^2 + 2\lambda^{-1} \partial_a \lambda \rho^{-1} \partial^a \rho \right\} + \kappa \int_{\partial(2)D} \varepsilon_{(2)h} \rho K, \quad (2.27)$$

Suppose we rewrite the two terms $-\frac{1}{2} \Delta^{-2} (\partial \Delta)^2 + 2\lambda^{-1} \partial_a \lambda \rho^{-1} \partial^a \rho$, in terms of $\tilde{\Delta} = \rho/\Delta$. The result is

$$-\frac{1}{2} \tilde{\Delta}^{-2} (\partial \tilde{\Delta})^2 + \tilde{\Delta}^{-1} \partial_a \tilde{\Delta} \rho^{-1} \partial^a \rho - \frac{1}{2} \rho^{-2} (\partial \rho)^2 + 2\lambda^{-1} \partial_a \lambda \rho^{-1} \partial^a \rho.$$

It is clear that the second and third terms can be absorbed in the fourth by a suitable redefinition of λ , yielding an expression of the same form as we started with: With $\tilde{\lambda} = \tilde{\Delta}^{1/2} \rho^{-1/4} \lambda = \Delta^{-1/2} \rho^{1/4} \lambda$ one obtains $-\frac{1}{2} \tilde{\Delta}^{-2} (\partial \tilde{\Delta})^2 + 2\tilde{\lambda}^{-1} \partial_a \tilde{\lambda} \rho^{-1} \partial^a \rho$. The action can thus be rewritten as

$$\tilde{S} = \frac{\kappa}{2} \int_{(2)D} \varepsilon_{(2)g} \rho \left\{ {}^{(2)}R - \frac{1}{2} \tilde{\Delta}^{-2} (\partial \tilde{\Delta})^2 - \frac{1}{2} \tilde{\Delta}^{-2} (\partial \tilde{B})^2 + 2\tilde{\lambda}^{-1} \partial_a \tilde{\lambda} \rho^{-1} \partial^a \rho \right\} + \kappa \int_{\partial(2)D} \varepsilon_{(2)h} \rho K. \quad (2.28)$$

Here B_2 has been named \tilde{B} , and the action has been named \tilde{S} , because it involves the \sim fields. It is precisely the action ${}^{(2)}S$ though.

Now we find another action equivalent to ${}^{(2)}S$. Define the field B “conjugate” to \tilde{B} by the differential equation

$$\partial_a B = \rho^{-1} \Delta^2 \partial^b \tilde{B} \varepsilon_{ab}, \quad (2.29)$$

or equivalently

$$\Delta^{-1} \partial_a B = \tilde{\Delta}^{-1} \partial^b \tilde{B} \varepsilon_{ab}. \quad (2.30)$$

This equation has solutions because by virtue of the equation of motion for \tilde{B} , easily obtained from (2.28),

$$\nabla_a(\rho\tilde{\Delta}^{-2}\partial^a\tilde{B}) = 0.$$

Equation (2.29) defines B up to the addition of a constant. We will leave this constant undetermined, as the results obtained are valid for any value.

Note that $\Delta^{-2}(\partial B)^2 = \tilde{\Delta}^{-2}\partial_b\tilde{B}\partial_d\tilde{B}\varepsilon^{ab}\varepsilon^{cd}\eta_{ac} = -\tilde{\Delta}^{-2}(\partial\tilde{B})^2$ so ${}^{(2)}S$ written in terms of B , Δ and λ takes almost the same form as the same action written in terms of \tilde{B} , $\tilde{\Delta}$, and $\tilde{\lambda}$:

$$\begin{aligned} {}^{(2)}S = & \frac{\kappa}{2} \int_{(2)D} \varepsilon_{(2)g} \rho \left\{ {}^{(2)}R - \frac{1}{2}\Delta^{-2}(\partial\Delta)^2 + \frac{1}{2}\Delta^{-2}(\partial B)^2 + \right. \\ & \left. + 2\lambda^{-1}\partial_a\lambda\rho^{-1}\partial^a\rho \right\} + \kappa \int_{\partial(2)D} \varepsilon_{(2)h} \rho K. \end{aligned} \quad (2.31)$$

It differs in form only in the sign in front of the B term.

Now consider variations of ${}^{(2)}S$. If only B is varied then

$$\delta {}^{(2)}S = -\frac{\kappa}{2} \int_{(2)D} \varepsilon_{(2)g} \nabla_a(\rho\Delta^{-2}\partial^a B)\delta B + \frac{\kappa}{2} \int_{\partial(2)D} \varepsilon_{(2)h} n_a \rho \Delta^{-2}\partial^a B \delta B, \quad (2.32)$$

where n is the normal 1-form to the boundary. At solutions ${}^{(2)}S$ is stationary with respect to variations that leave the original fields in the expression (2.27) invariant on the boundary. In particular, $\tilde{B} = B_2$ must be invariant on the boundary. This does not imply that B is invariant on the boundary. Thus under the variations in question the boundary term in δB does not in general vanish.

But suppose we add ($\frac{\kappa}{2}$ times)

$$B\varepsilon^{ab}\nabla_a\partial_b\tilde{B} = 0,$$

to the Lagrangean density. It is identically zero, so it does not change the action at all. This term may be expressed as

$$\nabla_a(B\varepsilon^{ab}\partial_b\tilde{B}) - \partial_a B\varepsilon^{ab}\partial_b\tilde{B}$$

or, using (2.29),

$$\nabla_a(\rho\Delta^{-2}B\partial^a B) - \rho\Delta^{-2}\partial_a B\partial^a B.$$

Therefore

$${}^{(2)}S = S + \frac{\kappa}{2} \int_{\partial(2)D} \varepsilon_{(2)h} n_a \rho \Delta^{-2} B \partial^a B,$$

with

$$\begin{aligned} S = & \frac{\kappa}{2} \int_{(2)D} \varepsilon_{(2)g} \rho \left\{ {}^{(2)}R - \frac{1}{2}\Delta^{-2}(\partial\Delta)^2 - \frac{1}{2}\Delta^{-2}(\partial B)^2 + \right. \\ & \left. + 2\tilde{\lambda}^{-1}\partial_a\lambda\rho^{-1}\partial^a\rho \right\} + \kappa \int_{\partial(2)D} \varepsilon_{(2)h} \rho K. \end{aligned} \quad (2.33)$$

Notice that S is of precisely the same form as \tilde{S} , even the B term now has the same sign. The variation of S under a variation B is minus the variation ${}^{(2)}S$ given in (2.32). The variation of the boundary term is

$$\frac{\kappa}{2} \int_{\partial(2)D} \varepsilon_{(2)h} n_a \rho \Delta^{-2} (\delta B \partial^a B + B \partial^a \delta B) = \frac{\kappa}{2} \int_{\partial(2)D} \varepsilon_{(2)h} \left\{ n_a \rho \Delta^{-2} \delta B \partial^a B + n_a B \varepsilon^{ab} \partial_b \delta \tilde{B} \right\}.$$

When $\delta\tilde{B} = 0$ on the boundary then also $n_a\varepsilon^{ab}\partial_b\delta\tilde{B} = 0$ since this is a tangential derivative along the boundary of $\delta\tilde{B}$. Thus, when \tilde{B} is fixed on the boundary the variation of the boundary term precisely cancels the boundary term in the variation of S . It follows that $(^2)S = \tilde{S}$ is stationary under variations of B that fix \tilde{B} on the boundary iff $\nabla_a(\rho\Delta^{-2}\delta^a B) = 0$, or equivalently iff S is stationary under variations B that fix B on the boundary.

There are thus two actions for the system, S and \tilde{S} , where both \tilde{S} and the corresponding boundary conditions on the variations are obtained from the action S and its associated boundary conditions by the substitution

$$\begin{aligned}\lambda \mapsto \tilde{\lambda} &= \lambda\rho^{1/4}\Delta^{-1/2}, \\ \Delta \mapsto \tilde{\Delta} &= \frac{\rho}{\Delta}, \\ B \mapsto \tilde{B}, &\quad \text{with} \quad \tilde{\Delta}^{-1}\partial_a\tilde{B} = \Delta^{-1}\partial^b B\varepsilon_{ab}.\end{aligned}\tag{2.34}$$

The space of solutions is therefore invariant under this transformation. This is the Kramer-Neugebauer (KN) transformation. Note that it is its own inverse.

The KN transformation will play an important role in chapter 4 when we derive the Geroch group of symmetries of cylindrically symmetric gravitational waves. By itself it is not, however, a symmetry of the space of such waves, because it does not preserve the regularity of the metric on the symmetry axis: From the discussion after equation (2.26) we know that regularity requires that Δ have a non-zero limit on the axis $\rho = 0$. But in the KN transformed spacetime Δ is replaced by $\tilde{\Delta} = \rho/\Delta$ which tends to 0 if Δ had a non-zero limit. The KN transform of a metric regular on the axis is not regular there.

2.3 Formulation of EG2CHSKF as a model on a symmetric space

2.3.1 The action

EG2CHSKF can be reformulated as a σ -model; it can be described by a field taking values in the symmetric space $G/H = SL(2)/SO(2)$ and some further fields. Equivalently, it can be described in terms of a field taking values in the group $G = SL(2)$, and additional fields, with an $H = SO(2)$ gauge invariance. There are two ways to do this, using $SL(2)$ valued fields \mathcal{V} and $\tilde{\mathcal{V}}$ related by the Kramer-Neugebauer transformation.

We will begin with $\tilde{\mathcal{V}}$ defined to be the lower right 2×2 block of the vierbein (2.25) divided by the square root of its determinant:

$$\tilde{\mathcal{V}} := \begin{pmatrix} (\frac{\rho}{\Delta})^{1/2} & (\frac{\Delta}{\rho})^{1/2}B_2 \\ 0 & (\frac{\Delta}{\rho})^{1/2} \end{pmatrix} = \begin{pmatrix} \tilde{\Delta}^{1/2} & \tilde{\Delta}^{-1/2}\tilde{B} \\ 0 & \tilde{\Delta}^{-1/2} \end{pmatrix}.\tag{2.35}$$

$\tilde{\mathcal{V}}$ is a matrix of determinant 1, and thus an element of $SL(2)$. It is upper triangular, but this is only a convenient choice. General relativity formulated in terms of a vierbein has an $SO(3,1)$ gauge invariance, corresponding to a local Lorentz transformation of the vierbein. Since the internal index is the second index in the vierbein (2.25) local Lorentz transformations are realized by left multiplication by the corresponding $SO(3,1)$ matrix. In the process of dimensional reduction this gauge freedom was fixed, and so we ended

up with a definite vierbein (2.25), however it is useful to restore an $SO(2)$ subgroup of the local Lorentz gauge freedom, the subgroup that rotates in the tangent plane to the Killing orbits. It acts as

$$\tilde{\mathcal{V}} \mapsto \tilde{\mathcal{V}}h, \quad (2.36)$$

with $h \in SO(2)$. With this additional gauge freedom $\tilde{\mathcal{V}}$ can be any $SL(2)$ element. Put another way, any $SL(2)$ matrix can be put into upper triangular form by right multiplication with a suitable $SO(2)$ element h . Then $\tilde{\Delta}$ and \tilde{B} can be read off the triangular matrix.

There is a unique representative of the form (2.35) in each equivalence class of $SL(2)$ matrices under the gauge action (2.36) of $SO(2)$: It is easy to check that there are precisely two upper triangular matrices in each equivalence class, differing only by overall sign. If one also requires that the 11 matrix element be positive, as is the case in (2.35), then the matrix becomes unique. (This matrix element cannot vanish since the determinant is 1.) Thus the fields $\tilde{\Delta}$ and \tilde{B} are uniquely defined by the gauge equivalence class of $\tilde{\mathcal{V}}$, and of course vice versa. The gauge equivalence classes, which are the elements of $SL(2)/SO(2)$, thus represent the fields.

The KN dual of $\tilde{\mathcal{V}}$,

$$\mathcal{V} := \begin{pmatrix} \Delta^{1/2} & \Delta^{-1/2}B \\ 0 & \Delta^{-1/2} \end{pmatrix}, \quad (2.37)$$

and its $SO(2)$ equivalence class represent the fields Δ and B in the same manner.

In the context of cylindrically symmetric gravitational waves the description in terms of \mathcal{V} has an important advantage over that in terms of $\tilde{\mathcal{V}}$: The 4-metric on the axis is regular iff \mathcal{V} and λ are regular on the axis. Regularity of the 4-metric requires regularity of Δ , λ and B_2 , and moreover that B_2 and its gradient vanishes on the axis $\rho = 0$. (This is discussed after (2.26).) The conditions on $B_2 = \tilde{B}$ can be satisfied by setting $B_2 = 0$ at a point on the axis iff B and Δ are regular at the axis, as can be seen immediately from (2.29). This demonstrates the claim.

For this reason we will usually use the field \mathcal{V} . Let us express S in terms \mathcal{V} (and ρ , λ , and the metric ${}^{(2)}g$ on the symmetry reduced spacetime). We need

$$J_\mu := \mathcal{V}^{-1}\partial_\mu\mathcal{V} = \frac{1}{2} \begin{pmatrix} \Delta^{-1}\partial_\mu\Delta & 2\Delta^{-1}\partial_\mu B \\ 0 & -\Delta^{-1}\partial_\mu\Delta \end{pmatrix}, \quad (2.38)$$

and its symmetric part is

$$P_\mu := \frac{1}{2} \begin{pmatrix} \Delta^{-1}\partial_\mu\Delta & \Delta^{-1}\partial_\mu B \\ \Delta^{-1}\partial_\mu B & -\Delta^{-1}\partial_\mu\Delta \end{pmatrix}.$$

We denote its antisymmetric part by Q_μ . Note that

$$\text{Tr}(P^2) = \frac{1}{2} \left(\Delta^{-2}(\partial\Delta)^2 + \Delta^{-2}(\partial B)^2 \right) = \frac{1}{2} \left((\rho^{-1}\partial\rho - \Delta^{-1}\partial\Delta)^2 + \Delta^2\rho^{-2}(\partial B_2)^2 \right) \quad (2.39)$$

so (2.24) can be cast in the form

$$S = \frac{\kappa}{2} \int_{(2)D} \varepsilon_{(2)g} \rho \left\{ (2)R - \text{Tr}(P^2) + 2\lambda^{-1} \partial_a \lambda \rho^{-1} \partial^a \rho \right\} + \kappa \int_{\partial(2)D} \varepsilon_{(2)h} \rho K. \quad (2.40)$$

Obviously

$$\tilde{S} = \frac{\kappa}{2} \int_{(2)D} \varepsilon_{(2)g} \rho \left\{ (2)R - \text{Tr}(\tilde{P}^2) + 2\tilde{\lambda}^{-1} \partial_a \tilde{\lambda} \rho^{-1} \partial^a \rho \right\} + \kappa \int_{\partial(2)D} \varepsilon_{(2)h} \rho K. \quad (2.41)$$

By construction these actions ought to depend only on the gauge equivalence class of \mathcal{V} or $\tilde{\mathcal{V}}$. Let us verify this directly. Under the gauge action (2.36)

$$P = \frac{1}{2}(J + J^T) \mapsto \frac{1}{2} \left[h^{-1} \mathcal{V}^{-1} d(\mathcal{V}h) + d(h^T \mathcal{V}^T) \mathcal{V}^{-1T} h^{-1T} \right] = \quad (2.42)$$

$$= \frac{1}{2} h^{-1} (J + J^T) h + \frac{1}{2} (h^{-1} dh + dh^{-1} h) = h^{-1} P h, \quad (2.43)$$

since $SO(2)$ elements satisfy $h^{-1} = h^T$. It follows that $\text{Tr}(P^2)$ is unaffected by the $SO(2)$ action.

2.3.2 Facts about symmetric space models

It is useful to understand the σ model structure found in the last subsection from a wider perspective. Here we present some facts about symmetric spaces and theories modelled on them, as well as the notation that we will use to describe both. For details on Lie-groups and Lie-algebras see [38][29][5].

The definition of a globally symmetric space has a priori nothing to do with the quotient of a Lie group by a compact subgroup. But as shown in chapter 4 of [19] the two concepts are closely related and basically equivalent. We are confronted with the quotient of the non-compact Lie group $G = SL(2, \mathbb{R})$ by the compact subgroup $H = SO(2, \mathbb{R})$ and by the above equivalence, this quotient may be called a symmetric space as e.g. in [20]. We denote the Lie algebra of G by \mathfrak{g} , that of H by \mathfrak{h} and the orthogonal² complement of \mathfrak{h} in \mathfrak{g} by \mathfrak{k}

$$\mathfrak{h} \oplus \mathfrak{k} = \mathfrak{g}, \quad (2.44)$$

\oplus denoting the direct sum of vector spaces. The projections onto the subspaces \mathfrak{k} and \mathfrak{h} will be denoted by $\big|_{\mathfrak{k}}$ and $\big|_{\mathfrak{h}}$ respectively or sometimes simply by subscripts \mathfrak{k} and \mathfrak{h} . What about the various commutators of elements of the subspaces? Since H is a subgroup, \mathfrak{h} is a Lie subalgebra and so $[\mathfrak{h}, \mathfrak{h}] \in \mathfrak{h}$.

As explained in [19], a maximal compact subgroup H of a semisimple Lie Group G can be characterised as the fixed points of an involutive³ automorphism η on G called Cartan involution,

$$H = \{h \in G \mid \eta(h) = h\}$$

It should always be clear from the context whether we talk about $\eta = \text{diag}(-1, 1 \dots 1)$ or this involution η and this double use of the symbol η sticks best with the conventions of our most important references.

²Orthogonal with respect to the Cartan-Killing form given by the trace of the adjoint representation.

³Involutive means $\eta^2 = \text{id}$.

Since η is an automorphism $\eta(id) = id$, id being the identity element of the group. Consequently the differential of η at id maps \mathfrak{g} (linearly) to \mathfrak{g} . We use the same symbol η for this differential map on \mathfrak{g} . It is an automorphism of \mathfrak{g} since

$$\begin{aligned} e^{t\eta(A)+t\eta(B)+\frac{t^2}{2}\eta([A,B])+\mathcal{O}(t^3)} &= \eta\left(e^{tA+tB+\frac{t^2}{2}[A,B]+\mathcal{O}(t^3)}\right) = \eta(e^{tA}e^{tB}) = \\ &= \eta(e^{tA})\eta(e^{tB}) = e^{t\eta(A)}e^{t\eta(B)} = e^{t\eta(A)+t\eta(B)+\frac{t^2}{2}[\eta(A),\eta(B)]+\mathcal{O}(t^3)}. \end{aligned}$$

Differentiating with respect to t twice and evaluating at $t = 0$, we get that

$$\eta([A, B]) = [\eta(A), \eta(B)].$$

$\eta^2 = id$ and so η can only have the two eigenvalues $+1$ and -1 ⁴. The eigenspace for the eigenvalue $+1$ (-1) is \mathfrak{h} (\mathfrak{k}). For the various commutators we have

$$\begin{aligned} \eta([\mathfrak{h}, \mathfrak{h}]) &= [\eta(\mathfrak{h}), \eta(\mathfrak{h})] = [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \\ \eta([\mathfrak{h}, \mathfrak{k}]) &= [\eta(\mathfrak{h}), \eta(\mathfrak{k})] = -[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}, \\ \eta([\mathfrak{k}, \mathfrak{k}]) &= [\eta(\mathfrak{k}), \eta(\mathfrak{k})] = [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}. \end{aligned} \tag{2.45}$$

Given a Lie group valued field \mathcal{V} one may define the Lie algebra valued one-form $J = \mathcal{V}^{-1}d\mathcal{V}$. This is a flat connection because the curvature vanishes:

$$d \wedge J + J \wedge J = -\mathcal{V}^{-1}d\mathcal{V}\mathcal{V}^{-1} \wedge d\mathcal{V} + \mathcal{V}^{-1}d\mathcal{V} \wedge \mathcal{V}^{-1}d\mathcal{V} = 0. \tag{2.46}$$

It can be decomposed according to (2.44)

$$J_\mu = P_\mu + Q_\mu \quad \text{with} \quad P_\mu \in \mathfrak{k}, \quad Q_\mu \in \mathfrak{h}.$$

Using (2.45) the \mathfrak{k} and \mathfrak{h} components of the (vanishing) curvature may be expressed in terms of Q and P , yielding the equations

$$\partial_\mu P_\nu - \partial_\nu P_\mu + [Q_\mu, P_\nu] - [Q_\nu, P_\mu] = 0, \tag{2.47}$$

$$\partial_\mu Q_\nu - \partial_\nu Q_\mu + [Q_\mu, Q_\nu] + [P_\mu, P_\nu] = 0. \tag{2.48}$$

As mentioned above we want to consider fields taking values in the quotient space G/H or equivalently we want a theory which is (gauge) invariant under the right multiplication of the G -valued field $\mathcal{V}(x)$ by an H -valued field h . How does J transform under $\mathcal{V}(x) \mapsto \mathcal{V}(x)h(x)$?

$$\mathcal{V}^{-1}d\mathcal{V} \mapsto h^{-1}\mathcal{V}^{-1}(d\mathcal{V}h + \mathcal{V}dh) = h^{-1}\mathcal{V}^{-1}d\mathcal{V}h + h^{-1}dh = h^{-1}Jh + h^{-1}dh.$$

Note that since η is an automorphism

$$\eta(h^{-1}Jh) = \eta(h^{-1})\eta(J)\eta(h) = h^{-1}\eta(J)h$$

⁴If it had only $+1$ or -1 and not both it would be the identity map or minus the identity map respectively.

so $h^{-1}Ph \in \mathfrak{k}$ and $h^{-1}Qh \in \mathfrak{h}$. Clearly $h^{-1}dh \in \mathfrak{h}$, so

$$P \mapsto h^{-1}Ph, \quad (2.49)$$

$$Q \mapsto h^{-1}Qh + h^{-1}dh.$$

We see that P transforms in the adjoint representation of H , while Q transforms as an H connection. We can use this connection to construct a covariant derivative

$$D_\mu = \partial_\mu + ad_{Q_\mu} \quad (2.50)$$

on fields in the adjoint representation. This allows us to rewrite (2.47) as

$$D_\mu P_\nu - D_\nu P_\mu = 0.$$

2.3.3 Equations of motion

We will now derive the equations of motion from the action (2.40)

$$S = \frac{\kappa}{2} \int_D \varepsilon_g \rho \left\{ R - Tr(P^2) + 2\lambda^{-1} \partial_a \lambda \rho^{-1} \partial^a \rho \right\} + \kappa \int_{\partial^{(2)}D} \varepsilon_h \rho K$$

for \mathcal{V} , ρ , λ and the metric g (from now on we drop the superscript ⁽²⁾). We are working with an adapted action and fields restricted to satisfy the symmetries invoked by the two Killing fields. Variations of these fields will therefore also belong to a restricted class satisfying the symmetries. When deriving the equations of motion by considering corresponding variations of the action, we might miss some equations, which would only appear if more general variations were considered. A comparison of our equations with the ones derived in [39] directly from Einstein's equations without the detour of a reduced action shows that this is not the case — our set of equations is complete.

We start with a variation $\delta_{\mathcal{V}}$ such that $0 = \delta_{\mathcal{V}}\lambda = \delta_{\mathcal{V}}\rho = \delta_{\mathcal{V}}g$, but $\delta_{\mathcal{V}}\mathcal{V}$ arbitrary, except on the boundary where we require it to vanish. This gives

$$\begin{aligned} 0 = \delta_{\mathcal{V}}S &= -\kappa \int_D \varepsilon_g \rho Tr \{ P_\mu \delta_{\mathcal{V}} P_\nu \} g^{\mu\nu} = \\ &= -\kappa \int_D \varepsilon_g \rho Tr \{ P_\mu (-\mathcal{V}^{-1} \delta \mathcal{V} \mathcal{V}^{-1} \partial_\nu \mathcal{V} + \mathcal{V}^{-1} \partial_\nu \delta \mathcal{V}) \} g^{\mu\nu} = \\ &= -\kappa \int_D \varepsilon_g Tr \left\{ -\rho J_\nu P_\mu \mathcal{V}^{-1} \delta \mathcal{V} + \rho P_\mu J_\nu \mathcal{V}^{-1} \delta \mathcal{V} - \nabla_\nu (\rho P_\mu) \mathcal{V}^{-1} \delta \mathcal{V} \right\} g^{\mu\nu} = \\ &= \kappa \int_D \varepsilon_g Tr \left\{ \left(\nabla_\mu (\rho P^\mu) + [J_\mu, \rho P^\mu] \right) \mathcal{V}^{-1} \delta \mathcal{V} \right\}. \end{aligned}$$

As explained in chapter 1 and section 2.3.2, for fixed x , $\delta \mathcal{V}(x)$ is a tangent vector in $\mathcal{T}_{\mathcal{V}(x)}G$ and $\mathcal{V}^{-1}(x)\delta \mathcal{V}(x)$ is its pullback (by the left group action) to $\mathcal{T}_{id}G = \mathfrak{g}$, the Lie algebra. Letting x vary $\mathcal{V}^{-1}\delta \mathcal{V}$ therefore is a \mathfrak{g} -valued field. According to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}$ we can consider variations for which $\mathcal{V}^{-1}\delta \mathcal{V} \in \mathfrak{k}$ or $\mathcal{V}^{-1}\delta \mathcal{V} \in \mathfrak{h}$. Using the orthogonality of the inner product Tr and (2.45) we get from the first case (with (2.50))

$$\boxed{0 = (\nabla_\mu + ad_{Q_\mu})(\rho P^\mu) = D_\mu(\rho P^\mu)} \quad (2.51)$$

and from the second case just $[P_\mu, P^\mu] = 0$ which is an identity. Stationarity with respect to variations of λ requires

$$0 = \delta_\lambda S = \kappa \int_D \varepsilon_g \partial_\mu \left(\frac{\delta \lambda}{\lambda} \right) \partial^\mu \rho = -\kappa \int_D \varepsilon_g \frac{\delta \lambda}{\lambda} \nabla^2 \rho.$$

Thus

$$\boxed{\nabla^2 \rho = 0.} \quad (2.52)$$

Now we vary w.r.t. the metric $g_{\mu\nu}$. As explained in chapter 1

$$\delta_g \left(\frac{\kappa}{2} \int_D \varepsilon_g R + \kappa \int_{\partial M} \varepsilon_h K \right) = -\frac{\kappa}{2} \int_D \varepsilon_g \rho G^{\mu\nu} \delta g_{\mu\nu}$$

and so

$$\begin{aligned} \delta_g S = \frac{\kappa}{2} \int_D \varepsilon_g \rho \delta g_{\mu\nu} \left\{ -G^{\mu\nu} + \rho^{-1} \nabla^\mu \nabla^\nu \rho - \rho^{-1} g^{\mu\nu} \nabla^2 \rho + \text{Tr}(P^\mu P^\nu) - \right. \\ \left. - 2\lambda^{-1} \partial^\mu \lambda \rho^{-1} \partial^\nu \rho - \frac{1}{2} g^{\mu\nu} \text{Tr}(P^2) + g^{\mu\nu} \lambda^{-1} \partial_\sigma \lambda \rho^{-1} \partial^\sigma \rho \right\}. \end{aligned} \quad (2.53)$$

Therefore

$$\begin{aligned} G^{\mu\nu} = \text{Tr}(P^\mu P^\nu) - 2\lambda^{-1} \partial^{(\mu} \lambda \rho^{-1} \partial^{\nu)} \rho - \frac{1}{2} g^{\mu\nu} \text{Tr}(P^2) + g^{\mu\nu} \lambda^{-1} \partial_\sigma \lambda \rho^{-1} \partial^\sigma \rho + \\ + \rho^{-1} \nabla^\mu \nabla^\nu \rho - \rho^{-1} g^{\mu\nu} \nabla^2 \rho. \end{aligned} \quad (2.54)$$

Finally, varying ρ yields

$$0 = \delta_\rho S = \frac{\kappa}{2} \int_D \varepsilon_g \left\{ R - \text{Tr}(P^2) - 2\nabla_\mu (\lambda^{-1} \partial^\mu \lambda) \right\} \delta \rho + \kappa \int_{\partial D} \varepsilon_h K \delta \rho, \quad (2.55)$$

so stationarity with respect to variations of ρ vanishing on the boundary requires

$$\boxed{0 = 2\nabla^2 \ln \lambda - R + \text{Tr}(P^2)} \quad (2.56)$$

This last equation can be simplified using the freedom to make a local rescaling $\psi = \lambda \mapsto \Omega \lambda$ and a compensating rescaling of the zweibein $e_\mu^a \mapsto \Omega^{-1} e_\mu^a$ in the decomposition (2.25) or, equivalently, a rescaling $g \mapsto \Omega^{-2} g$ of the 2-metric. Since any 2-metric is conformally flat this allows us to make g flat, and thus remove the Ricci scalar term in (2.56):

In two dimensions the rescaling of the metric has the effect $R \mapsto \Omega^{-2}(R - 2\nabla^2 \ln \Omega)$ on the Ricci scalar [39], so if Ω is chosen to satisfy $R = 2\nabla^2 \ln \Omega$ then the Ricci scalar of the rescaled metric vanishes. (The disappearance of the R term is compensated by the change in the λ term $2\nabla^2 \ln \lambda \mapsto 2\nabla^2 \ln(\Omega \lambda) = 2\nabla^2 \ln \lambda + 2\nabla^2 \ln \Omega$.) Furthermore, in two dimensions the Riemann tensor is proportional to the Ricci scalar: $R^a{}_{b\mu\nu} = R e_{[\mu}^a e_{\nu]} b$, so the rescaled metric is flat.

All this can be seen quite easily in the null coordinates defined in appendix B. In these coordinates $g_{\mu\nu} dx^\mu dx^\nu = 2g_{+-} dx^+ dx^-$. That is, $g = \Omega^2 \tilde{\eta}$ with $\Omega^2 = -2g_{+-}$. Rescaling by the conformal factor Ω reduces the metric to the Minkowski metric. Letting $\sigma := \ln \lambda$ the field equation (2.56) becomes

$$\partial_+ \partial_- \sigma = -\frac{1}{2} \text{Tr}(P_+ P_-). \quad (2.57)$$

The other field equations also take simple forms in null coordinates and with the metric flat: The $+-$ and $-+$ components of (2.54) become the identity $0 = 0$, the $\pm\pm$ components are

$$\partial_{\pm}\sigma\partial_{\pm}\rho = \frac{1}{2}\rho\text{Tr}(P_{\pm}P_{\pm}) + \frac{1}{2}\partial_{\pm}\partial_{\pm}\rho. \quad (2.58)$$

Equation (2.51) takes the form

$$D_{+}(\rho P_{-}) + D_{-}(\rho P_{+}) = 0, \quad (2.59)$$

and equation (2.52) becomes

$$\partial_{+}\partial_{-}\rho = 0. \quad (2.60)$$

This equation can be solved immediately. Its general solution is

$$\rho(x) = \frac{1}{2}\left(\rho^{+}(x^{+}) - \rho^{-}(x^{-})\right). \quad (2.61)$$

The factor $\frac{1}{2}$ as well as the minus sign is convention. We shall also define

$$\tilde{\rho} := \frac{1}{2}\left(\rho^{+}(x^{+}) + \rho^{-}(x^{-})\right). \quad (2.62)$$

ρ is a harmonic function and $\tilde{\rho}$ is clearly also harmonic, it is the harmonic conjugate to ρ , defined (up to a constant) by $\nabla_{\mu}\tilde{\rho} = \varepsilon_{\mu}^{\nu}\nabla_{\nu}\rho$: In null coordinates this equation becomes

$$\partial_{\pm}\tilde{\rho} = \pm\partial_{\pm}\rho,$$

which is solved by (2.62).

More can be said about ρ and $\tilde{\rho}$. We are considering cylindrically symmetric gravitational fields, so there is a symmetry axis where the rotation Killing field has magnitude zero, and increases linearly in distance away from this axis to lowest order. This axis has a worldline in the dimensionally reduced spacetime, a timelike curve on which $\rho = 0$, but $d\rho \neq 0$. See the discussion at the end of subsection 2.1.2.

The conjugate function $\tilde{\rho}$ must increase (or decrease) monotonically along this curve. If it did not there would be a point on the worldline at which the derivative of $\tilde{\rho}$ along the tangent t to the worldline vanishes: $0 = t \lrcorner d\tilde{\rho} = t^{\mu}\varepsilon_{\mu}^{\nu}\nabla_{\nu}\rho$. But since t is timelike $t^{\mu}\varepsilon_{\mu}^{\nu}$ is linearly independent of t , so this could only be the case if $d\rho = 0$ at this point, which we require not to be the case. $\tilde{\rho}$ can therefore be used as a time coordinate.

Equation (2.57) is actually not an independent equation, but rather a consequence of the other equations. In fact only one of the two equations (2.58) is needed. To prove this claim note first the identity $[D_{\mp}P_{\pm}]_{\mathfrak{k}} = [D_{\pm}P_{\mp}]_{\mathfrak{k}}$:

$$[\partial_{\mp}P_{\pm}]_{\mathfrak{k}} = [\partial_{\mp}(\mathcal{V}^{-1}\partial_{\pm}\mathcal{V})]_{\mathfrak{k}} = [-J_{\mp}J_{\pm} + \mathcal{V}^{-1}\partial_{\mp}\partial_{\pm}\mathcal{V}]_{\mathfrak{k}},$$

so

$$[\partial_{\mp}P_{\pm} - \partial_{\pm}P_{\mp}]_{\mathfrak{k}} = [J_{\pm}, J_{\mp}]_{\mathfrak{k}} = [Q_{\pm}, P_{\mp}] - [Q_{\mp}, P_{\pm}],$$

which establishes the identity.

Now we take the derivative ∂_{-} of the $++$ component of (2.58). The derivative of the left side is $\partial_{-}(\partial_{+}\sigma\partial_{+}\rho) = \partial_{-}\partial_{+}\sigma\partial_{+}\rho$ by (2.60). That of the right side is

$$\begin{aligned} & \frac{1}{2}\text{Tr}\left\{\partial_{-}(\rho P_{+})P_{+} + \rho P_{+}\partial_{-}P_{+}\right\} = \\ & = \frac{1}{2}\text{Tr}\left\{D_{-}(\rho P_{+})P_{+} + \rho P_{+}D_{-}P_{+} - 2\rho[Q_{-}, P_{+}]P_{+}\right\}. \end{aligned}$$

But the commutator terms have vanishing trace, so, invoking the recently demonstrated identity, the preceding expression is seen to equal

$$\begin{aligned}
& \frac{1}{2}Tr\{D_-(\rho P_+)P_+ + \rho P_+D_+P_-\} = \\
& = \frac{1}{2}Tr\{P_+[D_-(\rho P_+) + D_+(\rho P_-)]\} - \frac{1}{2}\partial_+\rho Tr\{P_+P_-\} = \\
& = -\frac{1}{2}\partial_+\rho Tr\{P_+P_-\}
\end{aligned}$$

by (2.59).

To establish (2.57) it suffices to show that $\partial_+\rho$ is non-zero at every point. But if $\partial_+\rho = 0$ at a point p of the reduced spacetime then $\partial_+\rho = 0$ also at the point on the axis with the same x^+ , that is where the future directed null geodesic through p cuts the axis worldline. But ∂_+ is not parallel to the axis worldline, since it is null instead of timelike, so this would imply that $d\rho = 0$ there, contrary to our hypothesis. The claim that (2.57) is not an independent field equation is thus established.

But this is not all. The function $\rho^+(x^+)$ and the $++$ component of (2.58) provide an expression for $\partial_+\sigma$ which can be integrated along constant x^- curves to determine σ up to a function of x^- . Similarly $\rho^-(x^-)$ and the $-$ component determines σ up to a function of x^+ . Our calculation allows us to show that these determinations are consistent, and thus that both equations (2.58) may be solved.

Exchanging $+$ and $-$ in the calculation we see that (when (2.59) and (2.60) hold) the ∂_+ derivative of the $--$ component of (2.58) is equivalent to (2.57). Thus, if (2.59), (2.60), and the $++$ component of (2.58) hold on all of the 2-d spacetime, and the $-$ component of (2.58) holds on a constant x^+ curve, then all the field equations hold everywhere.

This implies that given a solution of (2.59), the equations (2.58) can be integrated as follows: The value of σ is fixed at one point. Then the $--$ component of (2.58) is integrated along the constant x^+ curve through this point. Finally the $++$ component is integrated along all the constant x^- curves through the constant x^+ curve. This determines σ and, according to our argument, assures that all the field equations hold. Solving the field equations thus requires only solving (2.59).

Chapter 3

Integrability, symmetries and conserved quantities

In this chapter we try develop an understanding of integrable classical systems and its connection with symmetries and conserved quantities. For finite dimensional systems there is a well developed theory on integrable systems which allows to exploit the symmetries and conserved quantities to solve the equations of motion and/or to lay the foundations for a possible quantization. In the infinite dimensional cases much more effort has to be made to develop a general theory of integrable systems — even the definition of integrability cannot simply be adopted. Still, the basic mechanism and ideas seem to be the same as for finite dimensional systems. The symmetry algebra is related to a subalgebra of the Poisson algebra, namely to the algebra of conserved charges, which may be a convenient algebra to quantize. In this chapter we stay at the purely classical level and give the above mentioned connection between integrability, symmetries and conserved quantities in the finite dimensional context. From time to time we will hint at a possible generalization to field theories.

3.1 Basics of symplectic geometry

Symplectic geometry is the framework for Hamiltonian mechanics. An understanding of the tools that it provides us with is therefore essential. In this section we give a brief review of symplectic manifolds, Hamiltonian vector fields, and Poisson brackets and introduce our sign conventions.

A (finite dimensional) *symplectic manifold* is a pair (M, ω) consisting of a differentiable manifold M and a closed, nondegenerate¹ 2-form ω . In any coordinate system we can compute

$$\det \omega = \det(-\omega^T) = (-1)^n \det \omega^T = (-1)^n \det \omega \neq 0,$$

with $n = \dim M$. It follows that M has to be even dimensional.

The most prominent example for a symplectic manifold is the cotangent bundle \mathcal{T}^*Q of some manifold Q . Choosing coordinates q^i on Q we can coordinatize the cotangent

¹by nondegenerate we mean that at every point $x \in M$ ω viewed as a linear map from $\mathcal{T}_x M$ to $\mathcal{T}_x^* M$ has kernel $\{0\}$.

spaces by $p_i(\alpha) := \frac{\partial}{\partial q^i} \lrcorner \alpha$, $\alpha \in \mathcal{T}^*Q$. As a symplectic 2-form ω we take

$$\omega = dq^i \wedge dp_i = -d(p_i dq^i) \in \mathcal{T}^{(0,2)}(\mathcal{T}^*Q).$$

Since $p_i dq^i$ transforms covariantly under coordinate changes on Q this is well defined. Locally on *any* symplectic manifold M it is possible to find coordinates (q^i, p^i) (a “symplectic chart”) such that

$$\omega = dq^i \wedge dp_i.$$

This is the content of Darboux’s theorem (see e.g. [40]). The symplectic form in this coordinate system is called the canonical 2-form. It can also be written as $\omega = d\theta_c$ with $\theta_c := q_i dp^i$ the canonical 1-form. This construction is only local, still it shows that locally every symplectic manifold is diffeomorphic to a subset U of a cotangent bundle. Similarly we can only locally conclude from $d\omega = 0$ that $\exists \theta$, a symplectic potential, such that $\omega = d\theta$. Note that the canonical 1-form is only one particular symplectic potential.

At a fixed point $x \in M$ we can view ω as a linear map from $\mathcal{T}_x M$ to $\mathcal{T}_x^* M$, $X \mapsto \omega_x(X, \cdot)$. Since it is nondegenerate its inverse exists — we call it Π_x , which can consequently be viewed as bilinear antisymmetric map from $\mathcal{T}_x^* M \otimes \mathcal{T}_x^* M$ to \mathbb{R} , i.e.

$$\Pi(\omega(X, \cdot), \cdot) = id. \tag{3.1}$$

This can be done at every point to define it on the whole of M .

A *locally Hamiltonian vector field* X on M is defined by

$$L_X \omega = 0. \tag{3.2}$$

For forms the Lie derivative can be expressed by the exterior derivative d and the dual pairing \lrcorner of a vector with a form

$$L_X \omega = X \lrcorner (d\omega) + d(X \lrcorner \omega) = d(X \lrcorner \omega)$$

since $d\omega = 0$. The condition for a vector field X to be locally Hamiltonian is thus

$$d(X \lrcorner \omega) = 0.$$

From this it follows that again *locally* $X \lrcorner \omega = \omega(X, \cdot) = df$ for some function f . If $X \lrcorner \omega = df$ holds globally we say that $X = X_f$ is globally Hamiltonian and generated by f . In this case we can use the inverse Π to write $X_f = \Pi(df, \cdot)$.

Now let us consider a one parameter family of transformations ρ_λ of M depending differentiably on the parameter λ . We say that it is a *canonical transformation* if the associated pull-back ρ_λ^* of ω satisfies

$$\rho_\lambda^* \omega = \omega.$$

The infinitesimal transformation is induced by the vector field $\partial_\lambda \rho_\lambda|_{\lambda=0}$ and the infinitesimal condition of invariance of ω is then

$$L_{\partial_\lambda \rho_\lambda|_{\lambda=0}} \omega = 0. \tag{3.3}$$

This is just the condition for this vector field to be locally Hamiltonian. Hamiltonian vector fields thus correspond to generators of canonical transformations.

For two functions f and g we define their *Poisson bracket* by

$$\{f, g\} := \omega(X_f, X_g) = X_g \lrcorner df = \Pi(dg, df) = -\Pi(df, dg). \quad (3.4)$$

Clearly this bracket is antisymmetric.

In a symplectic chart where $\omega = dq^i \wedge dp_i$ we have

$$\begin{aligned} X_f \lrcorner \omega &= \left((X_f)_q^i \frac{\partial}{\partial q^i} + (X_f)_i^p \frac{\partial}{\partial p_i} \right) \lrcorner \omega = -(X_f)_i^p dq^i + (X_f)_q^i dp_i = \frac{\partial f}{\partial p_i} dp_i + \frac{\partial f}{\partial q^i} dq^i \\ &\Rightarrow X_f = -\frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} + \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} \\ \{f, g\} &= X_g \lrcorner df = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}. \end{aligned} \quad (3.5)$$

This also shows that the Poisson bracket is a bi-derivation and satisfies the Leibniz rule.

Let X and Y be two locally Hamiltonian vector fields. Since $L_{[X, Y]} = [L_X, L_Y]$, $[X, Y]$ is also locally Hamiltonian. It is even globally Hamiltonian

$$[X, Y] \lrcorner \omega = L_X(Y \lrcorner \omega) - Y \lrcorner L_X \omega = \quad (3.6)$$

$$= X \lrcorner d(Y \lrcorner \omega) + d(X \lrcorner (Y \lrcorner \omega)) = -d\omega(X, Y). \quad (3.7)$$

For two globally Hamiltonian vector fields X_f and X_g generated by f and g respectively we get

$$[X_f, X_g] = -X_{\{f, g\}}. \quad (3.8)$$

The Poisson bracket defined in this way satisfies the Jacobi relation:

Using the general formula² for an $(n-1)$ -form α and n vector fields X_1, \dots, X_n (see [26])

$$d\alpha(X_1, \dots, X_n) = X_{[1}(\alpha(X_2, \dots, X_n)) - \frac{n-1}{2}\alpha([X_{[1}, X_2], X_3, \dots, X_n]) \quad (3.9)$$

we get, using the closedness of ω , that

$$\begin{aligned} 0 &= X_{[f_1}(\omega(X_{f_2}, X_{f_3})) - \omega([X_{[f_1}, X_{f_2}], X_{f_3}]) = \\ &= Cyc \ 2X_{f_1}(\omega(X_{f_2}, X_{f_3})) - 2\omega([X_{f_1}, X_{f_2}], X_{f_3}) = \\ &= -Cyc \ 2\{f_1, \{f_2, f_3\}\} + 2\{f_3, \{f_1, f_2\}\} = \\ &= -Cyc \ 4\{f_1, \{f_2, f_3\}\}, \end{aligned}$$

the Jacobi relation.

(3.4) shows that the object $\{\cdot, g\}$ for some function $g \in \mathcal{F}(M)$ is a tangent vector field. Take another tangent vector field δ , then because of (3.1)

$$\omega(\{\cdot, g\}, \delta) = \delta g \quad (3.10)$$

²This formula can actually be used to define the exterior derivative, also for infinite dimensional manifolds.

at least if ω is nondegenerate. If it is degenerate, the relation remains valid for non-degeneracy vectors δ .

Now suppose we have a dynamical system with n degrees of freedom described by a Hamiltonian $H(q^i, p_i)$ of the position and the momentum coordinates. If we construct a symplectic manifold with the canonical 2-form ω where the q^i and p_i form a symplectic chart, the differential equations for the flow of the associated Hamiltonian vector field X_H (check (3.5)),

$$\begin{aligned}\dot{q}^i &= X_H(q^i) = \{q^i, H\} = \frac{\partial H}{\partial p_i}, \\ \dot{p}^i &= X_H(p^i) = \{p^i, H\} = -\frac{\partial H}{\partial q^i},\end{aligned}$$

are identical to Hamilton's equations of motion (where the role of time is now played by the parameter of the flow). Picking a point (q_0, p_0) on M and considering the integral curve passing through this point corresponds to giving initial data for a "trajectory" of whatever dynamical system is considered. In general, if the dynamics of a system are given by a Hamiltonian, the evolution of an observable $A \in \mathcal{F}(M)$ is given by

$$\dot{A} = \{A, H\} \quad .$$

In particular if $\{A, H\} = 0$ then A is constant along trajectories.

Since for any function $f \in \mathcal{F}(M)$ $\{, f\}$ is a tangent vector, namely the Hamiltonian vector field generated by f , any function in this way induces a canonical transformation by (3.3).

We will now introduce the concepts of generating functions of canonical transformations and polarizations. These will help us understand the geometry of the Hamilton Jacobi equation and the motivation for the definition of (Liouville) integrability given below.

A *submanifold* N of a symplectic manifold M is said to be *Lagrangian* if at any point $x \in N$ and for all $Y \in \mathcal{T}_x M$ we have

$$\omega_x(X, Y) = 0 \quad \forall X \in \mathcal{T}_x N, Y \in \mathcal{T}_x M \quad \Rightarrow \quad Y \in \mathcal{T}_x N. \quad (3.11)$$

More generally, we can define the complement S^\perp of a subspace S of $\mathcal{T}_x M$ at some point x by

$$S^\perp := \{Y \in \mathcal{T}_x M \mid \omega_x(X, Y) = 0 \quad \forall X \in S\}.$$

(3.11) then takes the form

$$(\mathcal{T}_x N)^\perp = \mathcal{T}_x N \quad \forall x \in N.$$

Also generally from the nondegeneracy of ω we have that

$$\dim S^\perp = \dim \mathcal{T}_x M - \dim S$$

and thus the dimension of a Lagrangian submanifold is $\frac{1}{2} \dim M$.

A (*real*) *polarization* of a symplectic manifold M is a foliation of M by Lagrangian submanifolds. An example is the "vertical polarization", the sets of points where $q^i = \text{const.}$, of a cotangent bundle. It can be shown [40] that if P is a real polarization

of a symplectic manifold (M, ω) then one can find a symplectic chart (q^i, p_i) in some neighbourhood of any $x \in M$ such that the leaves of P coincide locally with the level surfaces of the q^i . In other words a polarization can be used to locally endow a symplectic manifold with the structure of a cotangent bundle³.

Now let Q be a manifold, then the graph Λ of a 1-form α ,

$$\Lambda = \{(q, p) \mid p = \alpha_q\},$$

is a submanifold of \mathcal{T}^*Q . It is of dimension $\frac{1}{2} \dim \mathcal{T}^*Q$. If additionally $\omega|_{\Lambda} = 0$, we know that Λ is Lagrangian (taking ω to be the canonical 2-form on \mathcal{T}^*Q). α can be interpreted as a map from Q to \mathcal{T}^*Q , $\alpha: q \mapsto (q, \alpha_q)$. Therefore $\omega|_{\Lambda} = 0$ iff $\alpha^*\omega = 0$. We have

$$\alpha^*\omega = \alpha^*d\Theta = d(\alpha^*\Theta) = d\alpha$$

and therefore Λ is Lagrangian iff α is closed. Furthermore, if this is the case, then locally on Q we find a function S such that $\alpha = dS$. S is then called the generating function of Λ , $dS = \frac{\partial S}{\partial q^i} dq^i = \alpha = p_i dq^i$ and Λ can (locally) directly be given by

$$\Lambda = \{(q^i, p_i) \mid p_i = \frac{\partial S}{\partial q^i}\}. \quad (3.12)$$

It is clear that if we have not only one such generating function, but a family of functions depending differentiably on n parameters, which we shall call q^i , then this family generates a (local) polarization if the leaves fill out at least an open subset of \mathcal{T}^*M . Since in (3.12) the q^i don't depend on S and consequently not on the q^i , only the p_i -part of the points on Λ can change when varying q^i . We should be able to move them in n different directions in order to really fill out an open subset of \mathcal{T}^*M . Infinitesimally these variations are given by the vectors $\frac{\partial p_i}{\partial q^j} = \frac{\partial^2 S}{\partial q^i \partial q^j}$. If they are linearly independent, i.e. if

$$\det \frac{\partial^2 S}{\partial q^i \partial q^j} \neq 0,$$

we can be sure that we indeed get a polarization. Furthermore, we can use q^i and q^i as local coordinates on \mathcal{T}^*M . In these coordinates the canonical 1-form and 2-form are given by

$$\Theta = -p_i dq^i = -\frac{\partial S}{\partial q^i} dq^i \quad \text{and} \quad \omega = -\frac{\partial^2 S}{\partial q^i \partial q^j} dq^i dq^j.$$

If we define

$$p'_i := -\frac{\partial S}{\partial q^i}, \quad (3.13)$$

then we get

$$\begin{aligned} \omega = dq^i \wedge dp_i &= \frac{\partial}{\partial q^j} \frac{\partial S}{\partial q^i} dq^i \wedge dq^j = \frac{\partial}{\partial q^i} \frac{\partial S}{\partial q^j} dq^i \wedge dq^j = \\ &= -\frac{\partial}{\partial q^i} p'^j dq^i \wedge dq^j = dq^i \wedge dp^i. \end{aligned}$$

We have constructed a canonical transformation to coordinates (q', p') adapted to the polarization generated by S , adapted in the sense that the leaves $q'^i = \text{const}$ are the leaves of the polarization.

³From Darboux's theorem we already know that this is possible, but a polarization can be used to construct such coordinates.

3.2 Integrability

Integrability

A dynamical system with n degrees of freedom described by a Hamiltonian H is said to be (Liouville) integrable iff there exist n independent functions F_i in involution $\{F_i, F_j\} = 0$, which are also invariant under the flow generated by H , $\{F_i, H\} = 0$. [2]

The term independent means that at any point the gradients of the functions are linearly independent or more geometrically that their level surfaces are mutually nowhere tangent to each other. The invariance under the flow of X_H means constancy along solution curves and thus the F_i are conserved quantities. Note that it is an additional requirement that these quantities Poisson commute. Furthermore by (3.4) we have

$$0 = \{F_i, F_j\} = -\Pi(dF_i, dF_j). \quad (3.14)$$

Π is nondegenerate and therefore we can view the $\Pi(\cdot, dF_j)$ as n linearly independent vectors. Looking for a vector v satisfying

$$\Pi(v, dF_j) = 0$$

amounts to requiring v to lie in the intersection of the kernels of n linearly independent linear functionals. In an m -dimensional space the intersection of these kernels is $(m - n)$ -dimensional. In (3.14) we are even looking for n linearly independent such vectors. Thus the intersection of the kernels must be at least n -dimensional and we get that $m - n \geq n$ or $n \leq \frac{m}{2}$. Therefore, on a $2n$ -dimensional symplectic manifold there cannot exist more than n such functions, in particular H has to be a function of the F_i .

We motivate the above definition of (Liouville) integrability for finite dimensional systems by the following (see again [2])

Liouville theorem

The equations of motion for a Liouville integrable system are obtained by performing an integral.

In the previous section we showed how a real polarization could be used to perform a canonical transformation. Since we have the n independent functions F_i we could try to use them as one half of the new coordinates. As they are conserved in time we would then have

$$\dot{F}_i = 0 \quad (3.15)$$

and, since H is a function of the F_i only, for the evolution of the corresponding momenta p_F^i (the other half of the symplectic chart, not yet defined) we would get

$$\dot{p}_F^i = \left\{ \left\{ p_F^i, H \right\}, H \right\} = - \left\{ \frac{\partial H}{\partial F_i}, H \right\} = 0. \quad (3.16)$$

The solutions of these equations are straight forward.

So let's us try to do such a transformation:

For the submanifolds of the foliation we take the

$$\Lambda_{\{f_j\}} := \{(q^i, p_i) \mid F_j(q^i, p_i) = f_j \ \forall j\},$$

with $\{f_j\}$ being constant and in the image of F_j . Assuming that the relations $F_i(q^j, p_j) = f_j$ can be solved for the p_i , $\Lambda_{\{f_j\}}$ is given by the graph of the one form

$$\alpha = p_i(f_j, q^j) dq^i.$$

As shown in the previous section, these submanifolds are Lagrangian iff $d\alpha = 0$ or equivalently iff $\omega|_{\Lambda_{\{f_j\}}} = 0$. Now the vector fields $\{\cdot, F_i\}$ are linearly independent and tangent to $\Lambda_{\{f_j\}}$ and thus span the tangent space. They vanishing of ω on these vector fields

$$\omega(\{\cdot, F_i\}, \{\cdot, F_j\}) = \{\cdot, F_i\} \lrcorner dF_j = \{F_j, F_i\} = 0$$

thus implies $\omega|_{\Lambda_{\{f_j\}}} = 0$.

We can define the generating function S of the polarization via the path independent integral

$$S(f_i, q^i) = \int_{q_0}^q \alpha$$

and the momenta p_F^i by

$$p_F^i = \frac{\partial S}{\partial f_i}.$$

The transformation is canonical and the equations of motion are (3.15) and (3.16). The only trivial step is the computation of the integral above.

Having developed the tools of generating functions of polarizations, we can understand the Hamilton-Jacobi-equation

$$H\left(q^i, \frac{\partial S}{\partial q^i}\right) = \text{const.} \quad (3.17)$$

from a geometrical point of view. It is a partial, nonlinear differential equation for the function $S(q^i)$. One looks for a generating function of a Lagrangian submanifold on which H is constant, i.e. which is tangent to X_H . If one has a solution S one knows explicitly a submanifold on which the motion takes place and it is no more necessary to consider the whole of M . The equations of motion are no longer $2n$ equations for $2n$ variables, but n equations for n variables:

$$\dot{q}^i = \frac{\partial H}{\partial p^i} \Big|_{p_i = \frac{\partial S}{\partial q^i}}.$$

Having an n -parameter family of such solutions S , as explained above, allows to introduce new coordinates and trivialize the equations of motion by a canonical transformation.

3.3 3 more symplectic manifolds

The cotangent bundle of a manifold is the most prominent and easiest to handle example of a symplectic manifold. We also mentioned that locally any symplectic manifold has the structure of a cotangent bundle. Still it is not always useful to exploit this fact and actually perform the Legendre transformation, as it is called in classical mechanics. Especially when one deals with a presymplectic manifold, a pair (M, ω) consisting of

manifold M and a closed, possibly degenerate 2-form ω , this transformation can become tedious, although it has been well analysed [12].

In most physical theories an action, from which the equations of motion can be derived, is known. In this section we briefly show how in these cases the tools of symplectic geometry can be applied to dictate the dynamics, basically following [40]. We give two examples of (pre)symplectic manifolds for finite dimensional systems and then generalize one of them to describe field theories.

3.3.1 The tangent space

Consider a dynamical system with a finite number of degrees of freedom $q^i \in Q$, the dynamics of which can be described by Hamilton's principle:

There exists a Lagrangian function $L(q^i, v^i)$, $(q^i, v^i) \in \mathcal{T}Q$, and the corresponding action

$$I_{01} = \int_{t_0}^{t_1} L(q^i(t), v^i = \dot{q}^i(t)) dt \quad (3.18)$$

being a functional⁴ on the space of all possible curves $q^i(t)$ with $(q^i(t), \dot{q}^i(t)) \in \mathcal{T}Q$. The principle states that the physical trajectories are the points (in that space of curves) which are stationary under variations vanishing on the endpoints $q(t_0)$ and $q(t_1)$. This gives us the famous Lagrange's equations

$$\left. \frac{d}{dt} \left(\frac{\partial L}{\partial v^i} \right) \right|_{v^i = \dot{q}^i} - \left. \frac{\partial L}{\partial q^i} \right|_{v^i = \dot{q}^i} = 0. \quad (3.19)$$

Now we are looking for a 2-form ω_L and a function H such that the integral curves of X_H , the Hamiltonian vector field of H defined by

$$X_H \lrcorner \omega_L = dH, \quad (3.20)$$

are the solutions of (3.19). In the previous section we had the Hamiltonian of classical mechanics as a function on \mathcal{T}^*Q as the generating function of X_H . Remembering the definition of the Hamiltonian function in classical mechanics

$$H(q^j, p_j) = v^i(q^j, p_j) p_i - L(q^i, v^i(q^j, p_j)) \quad \text{with} \quad p_i(q^j, v^j) = \frac{\partial L}{\partial v^i}$$

as a function of $(q^i, p_i) \in \mathcal{T}^*Q$, it is only natural to consider the same function on $\mathcal{T}Q$

$$H(q^j, v^j) = v^i \frac{\partial L}{\partial v^i} - L(q^j, v^j). \quad (3.21)$$

Again on \mathcal{T}^*Q we had

$$\omega = -d(p_i dq^i),$$

so we attempt

$$\omega_L = -d \left(\frac{\partial L}{\partial v^i} dq^i \right) = \frac{\partial^2 L}{\partial q^j \partial v^i} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial v^i \partial v^j} dq^i \wedge dv^j. \quad (3.22)$$

⁴Originally $L(q^i, v^i)$ is a function on $\mathcal{T}Q$. In many situations we will use L on different spaces, such as the space of curves or solutions $q^i(t)$, where we have $L = L(q^i = q^i(t), v^i = \dot{q}^i(t))$. In those situations the evaluation of L in this different manner will not always be explicitly mentioned. e.g. $\frac{\partial L}{\partial \dot{q}^i(t)} = \frac{\partial L}{\partial v^i} \Big|_{v^i = \dot{q}^i(t)}$

Let us check if this yields what we want:

If $(q^i, v^i) = (q(t), \dot{q}^i(t))$ is an integral curve of X_H and a solution to (3.19), then

$$X_H = \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial v^i}$$

and so

$$X_H \lrcorner \omega_L = 2 \frac{\partial^2 L}{\partial q^j \partial v^i} \dot{q}^i dq^j + \frac{\partial^2 L}{\partial v^i \partial v^j} (\dot{q}^i dv^j - \ddot{q}^j dq^i). \quad (3.23)$$

On the other hand

$$dH = v^i \frac{\partial^2 L}{\partial v^i \partial q^j} dq^j + v^i \frac{\partial^2 L}{\partial v^i \partial v^j} dv^j - \frac{\partial L}{\partial q^i} dq^i. \quad (3.24)$$

Considering again solution curves $(q^i, v^i) = (q(t), \dot{q}^i(t))$, using the expressions (3.23) and (3.24), (3.20) becomes

$$-\dot{q}^i \frac{\partial^2 L}{\partial v^j \partial q^i} dq^j - \frac{\partial^2 L}{\partial v^i \partial v^j} \ddot{q}^j dq^i = -\frac{\partial L}{\partial q^i} dq^i,$$

or

$$\left(\frac{d}{dt} \left(\frac{\partial L}{\partial v^i} \Big|_{v^i=\dot{q}^i} \right) - \frac{\partial L}{\partial q^i} \right) dq^i = 0,$$

which is identically satisfied because $q^i(t)$ is a solution of (3.19).

The trajectories of a dynamical system described by an action integral (3.18) are the integral curves of the Hamiltonian vector field on $M = \mathcal{T}Q$ generated by the Hamiltonian (3.21) and 2-form (3.22).

Note that we haven't made use of the nondegeneracy of ω_L . In fact ω_L is only nondegenerate if

$$\det \left(\frac{\partial^2 L}{\partial v^i \partial v^j} \right) \neq 0.$$

A degenerate ω_L has the following consequences⁵

- 1) The equations of motion and the initial data do not determine a solution uniquely. Instead one can consider equivalence classes of solutions to the same initial data. This is generally referred to as gauge freedom. Observables must then be functions independent of the representative of the equivalence class.
- 2) The Legendre transformation from $\mathcal{T}Q$ to \mathcal{T}^*Q is singular. The treatment of such singular cases was examined by P. Dirac [12] and generally leads to constraints.
- 3) The definition of the Poisson bracket as the inverse of ω_L does no more make sense. Only an inverse on gauge invariant functions can be defined (which on the other hand should be sufficient).

EG2CHSKF formulated as a nonlinear σ -model on a symmetric space can be described by a presymplectic manifold, as we will see later.

⁵A manifold equipped with a degenerate 2-form ω is called presymplectic.

3.3.2 The space of solutions

The action functional of a dynamical system is a functional on the space of all possible curves connecting an initial state with a final state. We now consider a similar but of course not equivalent and much smaller space M , the space of curves $q^i(t)$, which are actually solutions to the equations of motion. The values of $q^i(t_0)$ and $\dot{q}^i(t_0)$ at some specific value t_0 can be used as coordinates on that space if ω_L is nondegenerate⁶. We will now construct a closed 2-form on this space and thereby give it the structure of a (pre)symplectic manifold. The definition of a Hamiltonian vector field whose integral curves are solutions will be straightforward. Although of course this space is also finite dimensional if the system has a finite number of degrees of freedom conceptually it is a space of functions. We are therefore looking for a Hamiltonian functional and tangent vectors regarded as operators on $\mathcal{F}(M)$ are represented by functional derivatives.

As Hamiltonian we take, similar to before,

$$H(q(t)) = \left(v^i \frac{\partial L}{\partial v^i} - L(q^j, v^j) \right) \Bigg|_{\substack{q^i=q^i(t_0) \\ v^i=\dot{q}^i(t_0)}}.$$

The equations of motions can again be interpreted as equations for a curve in M , since the curve $q^i(t, s) = q^i(t + s)$ parametrized by s solves

$$\frac{d}{ds} \left(\frac{\partial L}{\partial v^i} \right) \Bigg|_{v^i=\frac{d}{ds}q^i} - \frac{\partial L}{\partial q^i} \Bigg|_{v^i=\frac{d}{ds}q^i} = 0 \quad (3.25)$$

and may be considered a solution passing through $q^i(t)$ for fixed t .

We proceed analogously as before. If $q^i(t, s)$ is a solution of (3.25) and an integral curve of X_H , then

$$X_H = \int ds' \dot{q}^i(s') \frac{\delta}{\delta q^i(s')} . \quad (3.26)$$

More generally, the components of a tangent vector to the space of solutions can be characterised in the following way:

We consider a curve of solutions $q^i(t; \lambda)$ parametrized by λ . The tangent vector $u^i(t)$ to this curve at $\lambda = \lambda_0$ is then given by

$$u^i(t) = \frac{dq^i(t; \lambda)}{d\lambda} \Bigg|_{\lambda=\lambda_0} .$$

For any λ , $q^i(t; \lambda)$ must satisfy (3.25). Differentiating (3.25) with respect to λ , we get a relation for $u^i(t)$

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial^2 L}{\partial v^i \partial v^j} \Bigg|_{v^i, j = \dot{q}^i, j(t; \lambda_0)} \dot{u}^j(t) + \frac{\partial^2 L}{\partial v^i \partial q^j} \Bigg|_{v^i = \dot{q}^i(t; \lambda_0)} u^j(t) \right) - \\ - \frac{\partial^2 L}{\partial q^i \partial v^j} \Bigg|_{v^j = \dot{q}^j(t; \lambda_0)} \dot{u}^j(t) - \frac{\partial^2 L}{\partial q^i \partial q^j} \Bigg|_{\lambda=\lambda_0} u^j(t) = 0. \end{aligned}$$

⁶If it is degenerate, they can still be used to label the gauge equivalence classes.

The components of tangent vectors are solutions of the linearised equations of motion.

A natural 2-form ω on M can be obtained in the following way. The action integral I_{01} (3.18) is primarily a functional on the space of all curves connecting two points $q(t_0)$ with $q(t_1)$. But of course it can be considered on the smaller space of solutions. We can then compute its derivative along a tangent vector U

$$\begin{aligned}
U \lrcorner dI_{01} &= \\
&= \int ds u^i(s) \frac{\delta I_{01}}{\delta q^i(s)} = \int ds \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i(t)} u^i(s) \delta(t-s) + \frac{\partial L}{\partial \dot{q}^i(t)} \frac{d}{dt} \delta(t-s) u^i(s) \right) dt = \\
&= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i(t)} u^i(t) + \frac{\partial L}{\partial \dot{q}^i(t)} \dot{u}^i(t) \right) dt = \\
&= \left(\frac{\partial L}{\partial \dot{q}^i(t)} u^i \right) \Big|_{t=t_0}^{t=t_1} + \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i(t)} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i(t)} \right) \right) u^i dt = \left(\frac{\partial L}{\partial \dot{q}^i(t)} u^i \right) \Big|_{t=t_0}^{t=t_1}
\end{aligned}$$

since $q^i(t)$ satisfies (3.19). Defining for each t a 1-form Θ_t by

$$U \lrcorner \Theta_t = \frac{\partial L}{\partial \dot{q}^i(t)} u^i(t),$$

we get

$$dI_{01} = \Theta_{t_1} - \Theta_{t_0}.$$

Thus

$$\omega := -d\Theta_{t_1} = -d(\Theta_{t_0} + dI_{01}) = -d\Theta_{t_0} = -d\Theta_t$$

is independent of t and can be well defined at every point of M (a point of M is an entire solution $q^i(t)$ and ω has to be independent of t in order to be well defined). The minus sign is convention and will assure that the analogue of (3.20) holds.

Now for any tangent vector U and X_H as in (3.26) we get, using (3.9),

$$\begin{aligned}
\omega(X_H, U) &= -X_H \lrcorner d(U \lrcorner \Theta_{t_0}) + U \lrcorner d(X_H \lrcorner \Theta_{t_0}) + [X_H, U] \lrcorner \Theta_{t_0} = \\
&= -X_H \lrcorner d \left(\frac{\partial L}{\partial \dot{q}^i(t_0)} u^i(t_0) \right) + U \lrcorner d \left(\frac{\partial L}{\partial \dot{q}^i(t_0)} \dot{q}^i(t_0) \right) - \\
&\quad - X_H \lrcorner du^i(t_0) \frac{\partial L}{\partial \dot{q}^j(t_0)} + U \lrcorner d\dot{q}^i(t_0) \frac{\partial L}{\partial \dot{q}^j(t_0)} = \\
&= - \int ds \left(\frac{\partial^2 L}{\partial \dot{q}^i(t_0) \partial q^j(t_0)} \delta(t_0-s) + \frac{\partial^2 L}{\partial \dot{q}^i(t_0) \partial \dot{q}^j(t_0)} \frac{d}{dt_0} \delta(t_0-s) \right) u^i(t_0) \dot{q}^j(s) + \\
&+ \int ds \left(\frac{\partial^2 L}{\partial \dot{q}^i(t_0) \partial q^j(t_0)} \delta(t_0-s) u^i(t_0) + \frac{\partial^2 L}{\partial \dot{q}^i(t_0) \partial \dot{q}^j(t_0)} \frac{d}{dt_0} \delta(t_0-s) u^i(t_0) \right) \dot{q}^i(s) u^j(s) = \\
&= 2 \frac{\partial^2 L}{\partial \dot{q}^i(t_0) \partial q^j(t_0)} u^j(t_0) \dot{q}^i(t_0) - \frac{\partial^2 L}{\partial \dot{q}^i(t_0) \partial \dot{q}^j(t_0)} (u^i(t_0) \ddot{q}^j(t_0) - \dot{q}^i(t_0) \dot{u}^j(t_0)).
\end{aligned}$$

On the other hand

$$U \lrcorner dH = \dot{q}^i(t_0) \frac{\partial^2 L}{\partial \dot{q}^i(t_0) \partial q^j(t_0)} u^j(t_0) + \dot{q}^i(t_0) \dot{u}^j(t_0) \frac{\partial^2 L}{\partial \dot{q}^i(t_0) \partial \dot{q}^j(t_0)} - \frac{\partial L}{\partial q^j(t_0)} u^j(t_0).$$

As before

$$\omega(X_H, U) - U \lrcorner dH = \left(-\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i(t)} \right) + \frac{\partial L}{\partial q^i(t)} \right) u^i(t) = 0. \quad (3.27)$$

We see that although conceptually this (pre)symplectic manifold is quite different to the tangent bundle $\mathcal{T}Q$ covered in the last section, formally there are many similarities and again we have found a Hamiltonian H and a symplectic form ω such that H generates the dynamical flow via the Poisson structure.

3.3.3 The space of solutions for fields

The preceding example allows for an easy generalization to fields, which is eventually what we need. In this case also the space \mathcal{S} of solutions is generally an infinite dimensional manifold. For the theory of infinite dimensional manifolds see e.g. [26].

We consider maps from an n -dimensional base manifold Q equipped with a metric g to some other space F , possibly a vector space or a group,

$$\phi^\alpha : Q \ni x \mapsto \phi^\alpha(x) \in F,$$

α labelling the degrees of freedom in F . These maps form the space \mathcal{C} of field configurations. Equivalently we can consider the fields to be sections of the fibre bundle $E \xrightarrow{\pi} Q$ with typical fibre F , which we denote by Φ^α

$$\Phi^\alpha : Q \ni x \mapsto (x, \phi^\alpha(x)) \in E.$$

The metric g may also be the dynamical field itself, as in general relativity.

The dynamics shall be determined by an action of the form

$$I_D = \int_D \mathcal{L}(\phi^\alpha, \nabla_\mu \phi^\alpha, x^\nu) \varepsilon_g, \quad (3.28)$$

where $D \subset Q$, the Lagrangian density \mathcal{L} being a function on the first jet bundle $J^1(E)$. In the case of general relativity, the volume element contains the dynamical field, the metric g . In a coordinate system, it can be written as

$$\varepsilon_g = \sqrt{|g|} dx^0 \wedge \dots \wedge dx^3 = \sqrt{|g|} \varepsilon.$$

In this case we agree to include the square root of the determinant into \mathcal{L} . As before I_D can be considered on the space of all field configurations \mathcal{C} or on the smaller space of solutions \mathcal{S} . The field equations are derived by setting

$$\delta I_D = 0$$

for variations δ vanishing on the boundary of D . This yields

$$\int_D \varepsilon \left[\left(\frac{\partial \mathcal{L}}{\partial \phi^\alpha} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi^\alpha} \right) \delta \phi^\alpha + \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi^\alpha} \delta \phi^\alpha \right) \right] = 0. \quad (3.29)$$

Here we used that $[\delta, \nabla_\mu] = 0$. This can be understood as follows: What we do is we choose a field configuration $\phi^\alpha(x)$ and then consider arbitrary variations $\delta \phi^\alpha$ of the values

$\phi^\alpha(x)$ at every point x , subject to the condition that $\delta\phi^\alpha|_{\partial D} = 0$. Viewing the field as the section $\Phi^\alpha(x)$ of the fibre bundle $E \xrightarrow{\pi} Q$, the variations at a point x happen inside of the fibre $\pi^{-1}(x)$. On the contrary taking the gradient ∇_μ means considering variations of this section along the base manifold Q . We think of $[\delta, \nabla_\mu]\phi^\alpha$ as the infinitesimal version of the change of ϕ^α along the curve

$$(x, \lambda = 0) \rightarrow (x + \epsilon, 0) \rightarrow (x + \epsilon, \lambda) \rightarrow (x, \lambda) \rightarrow (x, 0),$$

where $x \in Q$ and λ parametrises the integral curve of δ . On E this gives the curve

$$\phi^\alpha(x; \lambda = 0) \rightarrow \phi^\alpha(x + \epsilon; 0) \rightarrow \phi^\alpha(x + \epsilon; \lambda) \rightarrow \phi^\alpha(x; \lambda) \rightarrow \phi^\alpha(x; 0),$$

which is a closed curve and therefore $[\delta, \nabla_\mu]\phi^\alpha = 0$.

In (3.29) the total divergence gives a boundary term, which vanishes since $\delta\phi^\alpha|_{\partial D} = 0$. On the other hand, inside of D $\delta\phi^\alpha$ is arbitrary and therefore

$$\frac{\partial \mathcal{L}}{\partial \phi^\alpha} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi^\alpha} = 0. \quad (3.30)$$

Now we consider two $(n-1)$ -dimensional hypersurfaces Σ and Σ' such that first $\Sigma \cap \Sigma' = \partial\Sigma = \partial\Sigma'$ and such that secondly both can be used to pose an initial value problem for some region in spacetime. We call two hypersurfaces with this property equivalent. Furthermore let us assume that at every point $p \in \Sigma$ or $\in \Sigma'$ there exists a vector transverse to Σ or Σ' , which is future directed. The region in between Σ and Σ' we call $D_{\Sigma\Sigma'}$. Let further U be a tangent vector to the space of solutions with components u^α . We again compute

$$\begin{aligned} U \lrcorner dI_{D_{\Sigma\Sigma'}} &= \int_{D_{\Sigma\Sigma'}} \left(\frac{\partial \mathcal{L}}{\partial \phi^\alpha} u^\alpha + \frac{\partial \mathcal{L}}{\partial (\nabla_\mu \phi^\alpha)} \nabla_\mu u^\alpha \right) \varepsilon = \int_{\partial D_{\Sigma\Sigma'}} \frac{\partial \mathcal{L}}{\partial (\nabla \phi^\alpha)} u^\alpha \lrcorner \varepsilon = \\ &= \int_{\Sigma'} \frac{\partial \mathcal{L}}{\partial (\nabla \phi^\alpha)} u^\alpha \lrcorner \varepsilon + \int_{\Sigma} \frac{\partial \mathcal{L}}{\partial (\nabla \phi^\alpha)} u^\alpha \lrcorner \varepsilon = \pm U \lrcorner \Theta_{\Sigma'} \pm U \lrcorner \Theta_\Sigma. \end{aligned}$$

The signs have to be chosen in the following way. First Stokes theorem asks us to orient the boundary positively w.r.t. an outward pointing transverse vector (see appendix E for explanations). So the signs of the integrals over Σ and Σ' depend on which one lies to the future of the other one. Secondly we define Θ_Σ , analogously to the previous example, by

$$U \lrcorner \Theta_\Sigma := \int_{\Sigma} \frac{\partial \mathcal{L}}{\partial (\nabla \phi^\alpha)} u^\alpha \lrcorner \varepsilon. \quad (3.31)$$

Here, in the definition of Θ we define the orientation to be positive w.r.t. a future directed vector.

Again, for such equivalent initial value surfaces Σ and Σ' , the 2-form

$$\omega = -d\Theta_\Sigma$$

will not depend on the choice of surface Σ .

If we wanted to continue analogous to the previous example, we would have to find again a Hamiltonian H and then show that $X_H \lrcorner \omega = dH$. This is what is done in the canonical

treatment of field theories. It requires a choice of “time”, along which the evolution with H takes place. But actually we don’t really need such a Hamiltonian. We got the field equations directly from the action (3.28) via Hamilton’s principle and we also already have a closed 2-form ω , which can be used to calculate Poisson brackets (with the above restrictions coming from a possible degeneracy of ω).

3.4 Symmetries of the e.o.m.

In this section we define symmetries of the equations of motion and study their connection to conserved quantities. Since symmetries are very important in EG2CHSKF a detailed treatment seems reasonable.

Everything will be done for the case of a field theory since it is more general.

A *symmetry* of the equations of motion is a vector field δ_s on \mathcal{C} , the flow of which leaves invariant its subspace \mathcal{S} of solutions. It maps solutions to solutions. Let

$$\begin{aligned}\sigma : \mathbb{R} \times \mathcal{C} &\rightarrow \mathcal{C} \\ (\lambda, \phi^\alpha) &\mapsto \sigma(\lambda, \phi^\alpha)\end{aligned}$$

be such a flow. Then at a solution $\phi^\alpha \in \mathcal{S}$, $\sigma(\lambda, \phi^\alpha) = \phi^\alpha(\lambda)$ has to satisfy (3.30) for every value of λ . Differentiating at $\lambda = 0$ we get the linearised field equation

$$\frac{\partial^2 \mathcal{L}}{\partial \phi^\alpha \partial \phi^\beta} \delta_s \phi^\beta + \frac{\partial^2 \mathcal{L}}{\partial \phi^\alpha \partial \nabla_\mu \phi^\beta} \delta_s \nabla_\mu \phi^\beta - \nabla_\mu \left(\frac{\partial^2 \mathcal{L}}{\partial \nabla_\mu \phi^\alpha \partial \phi^\beta} \delta_s \phi^\beta + \frac{\partial^2 \mathcal{L}}{\partial \nabla_\mu \phi^\alpha \partial \nabla_\nu \phi^\beta} \nabla_\nu \delta_s \phi^\beta \right).$$

We allow $\delta_s \phi^\alpha(x)$ to depend on the value of $\phi^\alpha(x')$ where x' is infinitesimally close to x , that is on the value of $\nabla_\mu \phi^\alpha(x)$. Since the equations of motion can be derived from the Lagrangian it is natural to look for a condition for $\delta_s I$ such that δ_s be a symmetry if this condition holds. If $\phi^\alpha(x; \lambda)$ is the flow of δ_s with parameter λ , then for infinitesimal λ we can write

$$I[\phi^\alpha(x; \lambda)] = I[\phi^\alpha(x; \lambda = 0)] + \lambda \delta_s I. \quad (3.32)$$

If $\phi^\alpha(x; \lambda)$ is a solution for every λ if $\phi^\alpha(x; 0)$ is a solution (δ_s is a symmetry), then infinitesimally we must have

$$0 = \delta I[\phi^\alpha(x; \lambda)] = \lambda \int \varepsilon \delta \delta_s \mathcal{L} = \lambda \int \varepsilon \left(\frac{\partial \delta_s \mathcal{L}}{\partial \phi^\alpha} - \nabla_\mu \frac{\partial \delta_s \mathcal{L}}{\partial \nabla_\mu \phi^\alpha} \right) \delta \phi^\alpha$$

for arbitrary variations δ vanishing on the boundary. $\delta_s \mathcal{L}$ must satisfy the Euler-Lagrange equations.

We will now proof the following Lemma: The Euler-Lagrange equations hold for a quantity $A = A(\phi^\alpha(x), \nabla_\mu \phi^\alpha(x), x)$ iff A is locally a total divergence $A = \nabla_\mu a^\mu$ for some vector a^μ .

Proof: We follow [6]. “ \Rightarrow ” Suppose that ϕ^α takes values in a vector space. Then for $\lambda \in \mathbb{R}$

$$A(\lambda \phi^\alpha(x), \lambda \nabla_\mu \phi^\alpha(x), x)$$

is well defined. We now calculate

$$\begin{aligned} \frac{d}{d\lambda} A(\lambda\phi^\alpha(x), \lambda\nabla_\mu\phi^\alpha(x), x) &= \frac{\partial A}{\partial\phi^\alpha}\phi^\alpha + \frac{\partial A}{\partial\nabla_\mu\phi^\alpha}\nabla_\mu\phi^\alpha = \nabla_\mu \left(\frac{\partial A}{\partial\nabla_\mu\phi^\alpha}\phi^\alpha \right) = \nabla_\mu a_I^\mu \\ A(\phi^\alpha, \nabla_\mu\phi^\alpha, x) - A(0, 0, x) &= \int_0^1 d\lambda \frac{d}{d\lambda} A(\lambda\phi^\alpha(x), \lambda\nabla_\mu\phi^\alpha(x), x). \end{aligned}$$

$A(0, 0, x)$ is a function of x only, independent of ϕ . We locally choose a coordinate system x^μ and define

$$a_{II}^1 = \int_{x_0^1}^{x^1} A(0, 0, x') dx'^1 \quad \text{and} \quad 0 = a_{II}^i \text{ for } i \neq 1$$

Then locally $A(0, 0, x) = \nabla_\mu a_{II}^\mu$. Hence locally

$$A(\phi^\alpha, \nabla_\mu\phi^\alpha, x) = \nabla_\mu (a_I^\mu + a_{II}^\mu) = \nabla_\mu a^\mu.$$

“ \Leftarrow ” Suppose that A is locally a divergence. Then we split the region D in (3.28) into regions U_i such that $A = \cup_i U_i$ and such that in every U_i $A = \nabla_\mu a_i^\mu$. Then for every U_i and all variations δ vanishing on ∂U_i

$$\begin{aligned} \delta \int_{U_i} \varepsilon_g A &= \int_{U_i} \varepsilon_g \nabla_\mu \delta a_i^\mu = \int_{\partial U_i} (\varepsilon_g)_\mu \delta a_i^\mu = \\ &= \int_{\partial U_i} (\varepsilon_g)_\mu \left[\left(\frac{\partial a_i^\mu}{\partial\phi^\alpha} - \nabla_\nu \frac{\partial a_i^\mu}{\partial\nabla_\nu\phi^\alpha} \right) \delta\phi^\alpha + \nabla_\nu \left(\frac{\partial a_i^\mu}{\partial\nabla_\nu\phi^\alpha} \delta\phi^\alpha \right) \right] = \int_{\partial U_i} (\varepsilon_g)_\mu \frac{\partial A}{\partial\nabla_\mu\phi^\alpha} \delta\phi^\alpha. \end{aligned}$$

On the other hand

$$\begin{aligned} \delta \int_{U_i} \varepsilon A &= \int_{U_i} \varepsilon \left(\frac{\partial A}{\partial\phi^\alpha} - \nabla_\mu \left(\frac{\partial A}{\partial\nabla_\mu\phi^\alpha} \right) \right) \delta\phi^\alpha + \int_{\partial U_i} (\varepsilon)_\mu \frac{\partial A}{\partial\nabla_\mu\phi^\alpha} \delta\phi^\alpha \\ &\Rightarrow \frac{\partial A}{\partial\phi^\alpha} - \nabla_\mu \frac{\partial A}{\partial\nabla_\mu\phi^\alpha} = 0. \quad \blacktriangle \end{aligned}$$

Since neither \mathcal{L} nor $\delta_s\phi^\alpha$ depends on higher derivatives of ϕ^α and therefore also $\delta_s\mathcal{L}$ does not, we can apply the result just obtained to conclude that

$$\delta_s \text{ is a symmetry} \Leftrightarrow \text{locally } \exists a^\mu \text{ such that } \delta_s\mathcal{L} = \nabla_\mu a^\mu$$

Remark: We assumed that ϕ^α takes values in a vector space. In EG2CHSKF we deal with group- $(SL(2))$ -valued fields $\mathcal{V}(x)$ ($\tilde{\mathcal{V}}(x)$). Nevertheless, the conclusion remains valid because in (2.41) the Lagrangian depends on $\mathcal{V}(x)$ ($\tilde{\mathcal{V}}(x)$) only via $P_\mu(x)$ ($\tilde{P}_\mu(x)$), which is Lie algebra-valued and therefore an element of a vector space. Actually one could even use the J_1 (\tilde{J}_1) instead of the \mathcal{V} ($\tilde{\mathcal{V}}$) as configuration variables as it is done e.g. in [24].

3.5 Noether's theorem

In this section we treat Noether's theorem, which in many cases enables one to easily calculate conserved quantities from symmetries of the equations of motion.

For a symmetry δ_s defined on \mathcal{C} we had that $\delta_s \mathcal{L} = \nabla_\mu a^\mu$ in some domain U . For any $D_{\Sigma\Sigma'} \subset U$ being a region bounded by two Cauchy surfaces Σ and Σ' we have

$$\delta_s I_{D_{\Sigma\Sigma'}} = \pm \int_\Sigma (\varepsilon)_\mu a^\mu \pm \int_{\Sigma'} (\varepsilon)_\mu a^\mu,$$

but also on \mathcal{S}

$$\delta_s I_{D_{\Sigma\Sigma'}} = \pm \int_\Sigma (\varepsilon)_\mu \frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi^\alpha} \delta_s \phi^\alpha \pm \int_{\Sigma'} (\varepsilon)_\mu a^\mu.$$

Varying Σ and Σ' such that their boundaries are kept fix and such that $D_{\Sigma\Sigma'} \subset U$ stays valid, but otherwise arbitrarily, we conclude that on \mathcal{S}

$$\int_\Sigma (\varepsilon)_\mu \left(a^\mu - \frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi^\alpha} \delta_s \phi^\alpha \right) = \text{const.}, \quad (3.33)$$

where “const.” means independent of Σ . This is actually necessary if (3.33) with Σ “arbitrary” (in the sense defined above) wants to define a functional on \mathcal{S} . A functional on \mathcal{S} must map an entire point, a solution, to one value. If we define the functional to be the integral over an “arbitrary” Cauchy surface, it has to be independent of the choice of Cauchy surface. A field theory can be given a dynamical interpretation if we foliate (a subset of) spacetime into Cauchy surfaces and consider the evolution of the field in time as the change of the field from one surface to another. In this picture (3.33) can be called a conserved quantity (conserved in time).

Rewriting (3.33) in the form of an integral over $D_{\Sigma\Sigma'}$, we have

$$\int_{D_{\Sigma\Sigma'}} \varepsilon \nabla_\mu \left(a^\mu - \frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi^\alpha} \delta_s \phi^\alpha \right) = 0.$$

Making $D_{\Sigma\Sigma'}$ arbitrarily small, we conclude

$$\nabla_\mu \left(a^\mu - \frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi^\alpha} \delta_s \phi^\alpha \right) = 0, \quad (3.34)$$

which is a continuity equation. We therefore call

$$a^\mu - \frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi^\alpha} \delta_s \phi^\alpha$$

a conserved current. Clearly linear combinations of conserved currents are again conserved.

3.6 Symmetry groups and algebras

It is intuitively clear that finite symmetry transformations form a group, continuous symmetry transformations form a Lie group and the infinitesimal symmetry transformations δ_s treated above form a Lie algebra. Suppose that we have

$$\delta_1 \mathcal{L} = \nabla_\mu a_1^\mu \text{ in } U_1 \subset Q, \quad \delta_2 \mathcal{L} = \nabla_\mu a_2^\mu \text{ in } U_2 \subset Q.$$

Then $[\delta_1, \delta_2]$ is also a symmetry since

$$\begin{aligned}
[\delta_1, \delta_2]\mathcal{L} &= \nabla_\mu(\delta_1 a_2^\mu - \delta_2 a_1^\mu) \text{ in } U_1 \cap U_2 \text{ and again on } \mathcal{S} \\
[\delta_1, \delta_2]\mathcal{L} &= \frac{\partial^2 \mathcal{L}}{\partial \phi^\alpha \partial \phi^\beta} \delta_2 \phi^\alpha \delta_1 \phi^\beta + \frac{\partial^2 \mathcal{L}}{\partial \nabla_\mu \phi^\alpha \partial \phi^\beta} \nabla_\mu \delta_2 \phi^\alpha \delta_1 \phi^\beta + \frac{\partial^2 \mathcal{L}}{\partial \phi^\alpha \partial \nabla_\nu \phi^\beta} \delta_2 \phi^\alpha \nabla_\nu \delta_1 \phi^\beta + \\
&+ \frac{\partial^2 \mathcal{L}}{\partial \nabla_\mu \phi^\alpha \partial \nabla_\nu \phi^\beta} \nabla_\mu \delta_2 \phi^\alpha \nabla_\nu \delta_1 \phi^\beta + \frac{\partial \mathcal{L}}{\partial \phi^\alpha} \delta_1 \delta_2 \phi^\alpha + \frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi^\alpha} \nabla_\mu \delta_1 \delta_2 \phi^\alpha - (\delta_1 \leftrightarrow \delta_2) = \\
&= \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi^\alpha} [\delta_1, \delta_2] \phi^\alpha \right)
\end{aligned}$$

and

$$\int_\Sigma (\varepsilon_g)_\mu \left(\delta_1 a_2^\mu - \delta_2 a_1^\mu - \frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi^\alpha} [\delta_1, \delta_2] \phi^\alpha \right)$$

is a constant of motion.

We might not know the entire symmetry group and algebra from the beginning. But having found two symmetries δ_1 and δ_2 , we know that $[\delta_1, \delta_2]$ is also a symmetry. If $\{[\delta_1, \delta_2], \delta_1, \delta_2\}$ are linearly independent the commutator constitutes a new symmetry. Higher commutators might give even more symmetry generators. We will later see how in EG2CHSKF these repeated commutators of initially only 6 generators will give us an infinite dimensional symmetry algebra.

3.7 Symplectic, Poissonian and Lie Poisson actions of symmetry groups and algebras

The notion of a symmetry of the equations of motion does not yet specify how it acts on the symplectic structure of the manifold under consideration. In this section we present two distinct ways, which cover many examples.

Let G be a Lie group acting on a symplectic manifold M by symmetry transformations on points and on functions by

$$\begin{aligned}
s_G: G \times M &\rightarrow M & G \times \mathcal{F}(M) &\rightarrow \mathcal{F}(M) \\
(g, x) &\mapsto s(g, x) =: gx & (g, f(\cdot)) &\mapsto f \circ s(g^{-1}, \cdot) =: gf(\cdot)
\end{aligned}$$

and let \mathcal{G} be its Lie algebra acting on functions by

$$s_{\mathcal{G}}: \mathcal{G} \times \mathcal{F}(M) \rightarrow \mathcal{F}(M) \quad (X, f(\cdot)) \mapsto \left. \frac{d}{dt} \right|_{t=0} f(e^{-tX} \cdot) =: X \cdot f(\cdot).$$

An action of G is said to be *symplectic* if

$$\{f_1(gx), gf_2(gx)\} = \{f_1, f_2\}(gx) \tag{3.35}$$

or in terms of \mathcal{G} if

$$\{X \cdot f_1(x), f_2(x)\} + \{f_1(x), X \cdot f_2(x)\} = X \cdot \{f_1(x), f_2(x)\}. \tag{3.36}$$

One essentially demands that the symplectic form not change under the flows generated by G . A sufficient condition for (3.36) to be satisfied is that locally there are functions h such that the action of \mathcal{G} on functions f can be realized by the locally Hamiltonian vector fields X_h . Then

$$X \cdot f = \{f, h\}$$

and (3.36) is satisfied by virtue of the Jacobi relation of the Poisson bracket. This condition is also necessary (for the proof see [2], p. 526).

A symplectic action of G on M is said to be *Poissonian*⁷ if it satisfies (3.36) and additionally \mathcal{G} acts by globally Hamiltonian vector fields $X \lrcorner \omega = dH_X$, the H_X depend linearly on X and

$$H_{[X,Y]} = \{H_X, H_Y\}.$$

In this case one can define a *moment* μ

$$\mu : M \rightarrow \mathcal{G}^*$$

such that for $X \in \mathcal{G}$

$$\mu(x) \cdot X = H_X.$$

The action of X can then simply be written as

$$X \cdot f = X \cdot \{f, \mu\}.$$

The μ are conserved quantities (see [1], p. 277).

Another natural (and more general) condition for the behaviour of the symplectic 2-form under symmetry transformations can be given if the Lie group itself is equipped with a Poisson structure. For example, if the manifold M and the Lie Group G are in fact identical, one has a Poisson structure on both spaces. To analyse this case we note that, functions on the Cartesian product of two sets X and Y can be written as

$$f(x, y) = \sum_i f_i^{(X)}(x) f_i^{(Y)}(y) \quad x \in X, y \in Y, f_i^{(X)} \in \mathcal{F}(X), f_i^{(Y)} \in \mathcal{F}(Y),$$

where the sum is in general infinite and requires some topology on $\mathcal{F}(X) \times \mathcal{F}(Y)$. If X and Y are Poisson manifolds the product Poisson structure on $X \times Y$ is

$$\begin{aligned} & \{f(x, y), g(x, y)\}_{X \times Y} := \\ & = \sum_{i,j} \left\{ f_i^{(X)}(x), g_j^{(X)}(x) \right\}_X f_i^{(Y)}(y) g_j^{(Y)}(y) + \sum_{i,j} f_i^{(X)}(x) g_j^{(X)}(x) \left\{ f_i^{(Y)}(y), g_j^{(Y)}(y) \right\}_Y. \end{aligned}$$

This allows to define the notion of a *Poisson Lie group*. A function f on a Lie group evaluated at a point gg' , a product, can be viewed as a function on $G \times G$. Now suppose G is equipped with a Poisson structure. We call G a *Poisson Lie group* [2] if the Poisson structure is compatible with the multiplication of the group in the sense that

$$\{f_1(gg'), f_2(gg')\}_{G \times G} = \{f_1, f_2\}_G(gg'). \quad (3.37)$$

⁷[40] for example uses the term *Hamiltonian*

Similarly, suppose we are given a manifold M and the action of a Poisson Lie Group G . We can view the expression $f(gx)$ as a function on $G \times M$. Since now G and M are equipped with a Poisson structure, compatibility of the respective Poisson structures with the action of the group reads

$$\{f_1(gx), f_2(gx)\}_G + \{f_1(gx), f_2(gx)\}_M = \{f_1(gx), f_2(gx)\}_{G \times M} = \{f_1, f_2\}_M(gx). \quad (3.38)$$

If this condition is satisfied, we say that the action is a *Lie Poisson action*. Note that it reduces to (3.35) if the Poisson bracket on G is trivial (vanishes).

While symplectic actions are always generated locally Hamiltonian vector fields, this is not true for Lie Poisson actions. Instead, one can construct so-called non-Abelian Hamiltonians. To this end consider a Poisson Lie group G . Then on \mathcal{G}^* one can introduce a Lie algebra structure with the following idea (chapter 14.5 of [2]):

Choose a basis E_a of \mathcal{G} . Then the vector fields ∇_a^E defined by

$$\nabla_a^E f(g) = \left. \frac{d}{dt} \right|_{t=0} f(e^{tE_a} g)$$

form a basis of $\mathcal{T}_g G$ at any point g . Since a Poisson bracket is a biderivation we can define $\eta^{ab}(g)$ by

$$\{f_1(g), f_2(g)\}_G = \eta^{ab}(g) \nabla_a^E f_1(g) \nabla_b^E f_2(g).$$

Then $\eta(g) := \eta^{ab}(g) E_a \otimes E_b \in \mathcal{G} \otimes \mathcal{G}$ for every g . It is just the Poisson bivector Π introduced in (3.1) in a specific basis given by the E_a . Consequently, $d\eta(e) \in \mathcal{G}^* \otimes \mathcal{G} \otimes \mathcal{G}$ and we can use $d\eta(e)$ to define a multiplication on \mathcal{G}^* . Denoting by $\{E^a\}$ the basis dual to $\{E_a\}$, then

$$[E^a, E^b]_{\mathcal{G}^*} := (d\eta(e))_c^{ab} E^c$$

defines a Lie algebra structure (Antisymmetry is clear, the Jacobi relation holds because it holds for $\{, \}_G$, (3.37) ensures that for $f \in \mathcal{F}(G)$, $d_e f \in \mathcal{G}^*$ one can actually define $[d_e f_1, d_e f_2]_{\mathcal{G}^*} = d_e \{f_1, f_2\}$ because $\eta(e) = 0$).

This allows us to formulate the infinitesimal version of (3.38) in a compact way. We put $g = e^{-tX}$ and differentiate w.r.t. t at $t = 0$

$$- \{X \cdot f_1(x), f_2\}_M - \{f_1(x), X \cdot f_2\}_M - X \lrcorner [d_e f_1, d_e f_2]_{\mathcal{G}^*} = -X \cdot \{f_1(x), f_2(x)\}. \quad (3.39)$$

The Lie algebra structure on \mathcal{G}^* represents the Poisson structure on G . If \mathcal{G}^* is abelian, that is if the Poisson structure on G is trivial, we get a symplectic action. We cannot expect that a Lie Poisson action is generated by local Hamiltonians. But luckily there is a generalization:

Introduce Darboux coordinates (q^i, p_i) . The action of $X \in \mathcal{G}$ is a vector field on M . We can expand it as $X = X^{q^i} \frac{\partial}{\partial q^i} + X^{p_i} \frac{\partial}{\partial p_i}$. Define the 1-form $\Omega_X = X^{q^i} dp^i - X^{p_i} dq^i$ and $\Omega = E^a \Omega_{E_a}$. Then (3.39) is equivalent to

$$d\Omega + [\Omega, \Omega]_{\mathcal{G}^*} = 0$$

Hence, locally $\Omega = \Gamma^{-1} d\Gamma$ and

$$X \cdot f = X \lrcorner (\Gamma^{-1} \{f, \Gamma\}) \quad (3.40)$$

The symmetry group of EG2CHSKF will act via a Lie Poisson action as we will see.

Chapter 4

The Symmetry Algebra of EG2CHSKF

In this section we derive the subalgebra of the full symmetry algebra of EG2CHSKF, which acts on the field \mathcal{V} . The central extension of the group corresponding to this subalgebra is what is known as the Geroch group [9]. The central element can be shown to act on the conformal factor λ . In [20] the Geroch group is further extended and transformations of the dilaton field ρ are introduced.

4.1 The $SL(2) \times SL(2)$ Symmetry

In section 2.3 we saw that there were two ways to formulate EG2CHSKF on a symmetric space, (2.40) and (2.41). As explained in section 2.3.3 the metric on the two dimensional symmetry reduced spacetime can always be chosen to be flat, so the Ricci scalar vanishes. Furthermore, the boundary term is irrelevant for the upcoming discussions. It does not influence the symmetries of the matter sector. (2.40) and (2.41) then become

$$\frac{\kappa}{2} \int \varepsilon_g \rho \left\{ -Tr(P^2) + 2\lambda^{-1} \partial_a \lambda \rho^{-1} \partial^a \rho \right\} \quad \text{and} \quad \frac{\kappa}{2} \int \varepsilon_g \rho \left\{ -Tr(\tilde{P}^2) + 2\tilde{\lambda}^{-1} \partial_a \tilde{\lambda} \rho^{-1} \partial^a \rho \right\}. \quad (4.1)$$

One sees directly from (2.49) and by invariance of the trace under similarity transformations that the first action integral is invariant under transformations

$$\mathcal{V}(x) \mapsto g\mathcal{V}(x)h(x) \quad g \in G = SL(2), h(x) \in H = SO(2). \quad (4.2)$$

The right multiplication by the H group valued field h is a remnant of the internal Lorentz transformations of the vierbein. (To reduce the vierbein to the form (2.25) the internal Lorentz transformation freedom was partly fixed, leaving only internal boosts of the zweibein of the reduced spacetime, and the right action of h on \mathcal{V} .) On the other hand the left multiplication by the constant G element g changes the 4-metric. It maps a solution to another solution with a distinct geometry. These will be the important, physical, transformations that generate the Geroch group of symmetries. The h transformation will be used to maintain \mathcal{V} always in upper triangular form, with $\mathcal{V}_{11} > 0$. This can always be done and determines h completely.

The same transformation of $\tilde{\mathcal{V}}$,

$$\tilde{\mathcal{V}}(x) \mapsto \tilde{g}\tilde{\mathcal{V}}(x)\tilde{h}(x) \quad \tilde{g} \in G = SL(2), \tilde{h}(x) \in H = SO(2) \quad (4.3)$$

leaves the second action in (4.1) invariant, and thus also maps solutions to solutions. Here \tilde{h} does not have an obvious interpretation in terms of internal local Lorentz transformations, but the Kramer-Neugebauer transformation maps triangular \mathcal{V} with $V_{11} > 0$ to triangular $\tilde{\mathcal{V}}$ with $\tilde{V}_{11} > 0$ (indeed we have only defined $\tilde{\mathcal{V}}$ in this triangular frame) so, in the same manner as in (4.2), \tilde{h} is completely determined by the requirement that the transformation preserves the triangularity of $\tilde{\mathcal{V}}$ and the positivity of its 11 component. Note that although (4.3) has the same form as (4.2), $\tilde{\mathcal{V}}$ is not \mathcal{V} . The transformations are in fact generally distinct. We have therefore two $SL(2)$ symmetries, that is, transformations that map solutions to solutions. But as we shall see, this is much more than it seems at first, because applying several g and \tilde{g} transformations in succession one obtains new symmetries, and ultimately an infinite dimensional and transitive group of symmetries, the Geroch group.

The generators of the transformations (4.2) read

$$\delta\mathcal{V} = \delta g\mathcal{V} + \mathcal{V}\delta h(\mathcal{V}, \delta g). \quad (4.4)$$

The corresponding conserved Noether current is

$$j_{\delta}^{\mu} = 2\rho \operatorname{Tr}[P^{\mu}\mathcal{V}^{-1}(\delta g\mathcal{V} + V\delta h)]. \quad (4.5)$$

The generators of the \sim transformations (4.3), and the corresponding Noether current are of course analogous.

As a basis of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ we use the set of the elements

$$\delta g_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \delta g_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \delta g_3 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \quad (4.6)$$

The $\mathfrak{h} = \mathfrak{so}(2)$ transformations are generated by

$$\delta h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \delta g_2 + \delta g_3. \quad (4.7)$$

Let us calculate how these symmetry transformations act on (\mathcal{V}, λ) , i.e. on the fields Δ and B contained in \mathcal{V} (2.37) and the conformal factor λ .

$$\begin{aligned} \delta\mathcal{V} &= \begin{pmatrix} \frac{1}{2}\Delta^{-1/2}\delta\Delta & -\frac{1}{2}\Delta^{-3/2}\delta\Delta B + \Delta^{-1/2}\delta B \\ 0 & -\frac{1}{2}\Delta^{-3/2}\delta\Delta \end{pmatrix} \\ \delta g_1\mathcal{V} &= \begin{pmatrix} -\Delta^{1/2} & -\Delta^{-1/2}B \\ 0 & \Delta^{-1/2} \end{pmatrix} \Rightarrow \delta_1\Delta = -2\Delta, \quad \delta_1B = -2B \\ \delta g_2\mathcal{V} &= \begin{pmatrix} 0 & \Delta^{-1/2} \\ 0 & 0 \end{pmatrix} \Rightarrow \delta_2\Delta = 0, \quad \delta_2B = 1 \\ \delta g_3\mathcal{V} + \mathcal{V}\delta h(\delta g_3, \mathcal{V}) &= \begin{pmatrix} \Delta^{1/2}B & -\Delta^{3/2} \\ 0 & -\Delta^{-1/2}B \end{pmatrix} \Rightarrow \delta_3\Delta = 2\Delta B, \quad \delta_3B = B^2 - \Delta^2 \end{aligned}$$

The conformal factor λ is invariant under $\delta_1, \delta_2, \delta_3$. The corresponding generators of the \sim transformations $\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3$ act in precisely the same way on $(\tilde{\mathcal{V}}, \tilde{\lambda})$, i.e. on the fields $\tilde{B}, \tilde{\Delta}$ and $\tilde{\lambda}$. The currents (4.5) corresponding to these generators are easily calculated from

$$P^{\mu}\mathcal{V}^{-1} = \frac{1}{2} \begin{pmatrix} \Delta^{-3/2}\partial^{\mu}\Delta & -\Delta^{-3/2}B\partial^{\mu}\Delta + \Delta^{-1/2}\partial^{\mu}B \\ \Delta^{-3/2}\partial^{\mu}B & -\Delta^{-3/2}B\partial^{\mu}B - \Delta^{-1/2}\partial^{\mu}\Delta \end{pmatrix}.$$

They are

$$\begin{aligned}
j_{\delta_1}^\mu &= -2\rho\Delta^{-1}\partial^\mu\Delta + \Delta^{-2}B\partial^\mu B, \\
j_{\delta_2}^\mu &= \partial^\mu B\rho\Delta^{-2}, \\
j_{\delta_3}^\mu &= 2\rho\Delta^{-1}\partial^\mu\Delta B + \partial^\mu B\rho\Delta^{-2}(B^2 - \Delta^2).
\end{aligned} \tag{4.8}$$

The currents corresponding to $\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3$ are given by the same expressions in terms of $\tilde{\Delta}$ and \tilde{B} , which in turn can be expressed in terms of Δ and B using the Kramer-Neugebauer transformation (2.34). The result is

$$\begin{aligned}
j_{\tilde{\delta}_1}^\mu &= 2\rho(\Delta^{-1}\partial^\mu\Delta - \rho^{-1}\partial^\mu\rho - \Delta^2\rho^{-2}\tilde{B}\partial^\mu\tilde{B}), \\
j_{\tilde{\delta}_2}^\mu &= \partial^\mu\tilde{B}\Delta^2\rho^{-1}, \\
j_{\tilde{\delta}_3}^\mu &= -2\rho\tilde{B}(\Delta^{-1}\partial^\mu\Delta - \rho^{-1}\partial^\mu\rho) + \partial^\mu\tilde{B}\Delta^2\rho^{-1}\left(\tilde{B}^2 - \frac{\rho^2}{\Delta^2}\right).
\end{aligned} \tag{4.9}$$

The action of the generators $\tilde{\delta}_i$ on $(\tilde{\mathcal{V}}, \tilde{\lambda})$ can be transferred to an action on (\mathcal{V}, λ) using the Kramer-Neugebauer transformation:

$$\frac{\tilde{\delta}\Delta}{\Delta} = -\frac{\tilde{\delta}\tilde{\Delta}}{\tilde{\Delta}} \tag{4.10}$$

$$\frac{\tilde{\delta}\lambda}{\lambda} = -\frac{1}{2}\frac{\tilde{\delta}\tilde{\Delta}}{\tilde{\Delta}} \tag{4.11}$$

$$\varepsilon^{\mu\nu}\partial_\nu\tilde{\delta}B = \rho\left(\tilde{\Delta}^{-2}\partial^\mu\tilde{\delta}\tilde{B} - 2\tilde{\Delta}^{-3}\tilde{\delta}\tilde{\Delta}\partial^\mu\tilde{B}\right). \tag{4.12}$$

Note that the last equation does not determine $\tilde{\delta}B$ in terms of $\tilde{\delta}\tilde{B}$ and $\tilde{\delta}\tilde{\Delta}$ uniquely, but only up to the addition of a constant. This reflects a freedom in the choice of basis in the symmetry algebra: Consider δ_2 . It leaves invariant Δ and λ , while $\delta_2 B = 1$. Constant multiples of δ_2 yield a constant δB . Thus, the fact that the Kramer-Neugebauer transformation defines the action of $\tilde{\delta}$ on B only up to constants amounts to the freedom to add multiples of δ_2 to $\tilde{\delta}$ thereby shifting $\tilde{\delta}B$ by a constant. In other words, we may choose the constants in $\delta\tilde{B}$ and $\tilde{\delta}B$ in a convenient way without altering the space of symmetries as a whole.

At this point, we have six symmetry generators $\delta_1, \delta_2, \delta_3, \tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3$ acting on both (\mathcal{V}, λ) and $(\tilde{\mathcal{V}}, \tilde{\lambda})$. We will now investigate their linear dependence. Explicitly, for the action of $\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3$ on (\mathcal{V}, λ) from (4.10) - (4.12) we get

$$\begin{aligned}
\tilde{\delta}_1: \quad & \tilde{\delta}_1\Delta = 2\Delta, & \tilde{\delta}_1\lambda = \lambda, & \varepsilon^{\mu\nu}\partial_\nu\tilde{\delta}_1B = 2\rho^{-1}\Delta^2\partial^\mu\tilde{B} = 2\varepsilon^{\mu\nu}\partial_\nu B, \\
\tilde{\delta}_2: \quad & \tilde{\delta}_2\Delta = 0, & \tilde{\delta}_2\lambda = 0, & \varepsilon^{\mu\nu}\partial_\nu\tilde{\delta}_2B = 0, \\
\tilde{\delta}_3: \quad & \tilde{\delta}_3\Delta = -2\Delta\tilde{B}, & \tilde{\delta}_3\lambda = -\tilde{B}\lambda, & \varepsilon^{\mu\nu}\partial_\nu\tilde{\delta}_3B = -2\rho(\rho^{-1}\partial^\mu\rho - \Delta^{-1}\partial^\mu\Delta + \Delta^2\rho^{-2}\tilde{B}\partial^\mu\tilde{B}).
\end{aligned} \tag{4.13}$$

The equations on the right are always well defined because taking the divergence ∂_μ gives $\varepsilon^{\mu\nu}\partial_\mu\partial_\nu\tilde{\delta}B = 0$ on the left sides whereas the right sides are always linear combinations of the conserved currents (4.9) and their divergences are thus also 0. For easy comparison

we again state the action of $\delta_1, \delta_2, \delta_3$ on (\mathcal{V}, λ)

$$\begin{aligned} \delta_1 : \quad & \delta_1 \Delta = -2\Delta, & \delta_1 \lambda &= 0, & \delta_1 B &= -2B, \\ \delta_2 : \quad & \delta_2 \Delta = 0, & \delta_2 \lambda &= 0, & \delta_2 B &= 1, \\ \delta_3 : \quad & \delta_3 \Delta = 2\Delta B, & \delta_3 \lambda &= 0, & \delta_3 B &= B^2 - \Delta^2. \end{aligned} \quad (4.14)$$

We see that δ_2 is a constant multiple of $\tilde{\delta}_2$. Apart from the action on λ , δ_1 up to constant multiples of δ_2 equals $-\tilde{\delta}_2$. But their actions on the conformal factor λ are clearly not opposite. Taking a look at the equation of motion (2.58) for λ , we see that in the case of δ_1 , the rescaling $\delta\lambda = \lambda$ amounts to a shift in the integration constant of $\sigma = \ln \lambda$, $\delta\sigma = 1$. $\tilde{\delta}_3$ is also linearly independent. Its action on Δ , λ and B can certainly not be expressed as linear combinations of the actions of the other generators. $\tilde{\delta}_3$ acting on (\mathcal{V}, λ) is an independent symmetry and calling it δ_4 it may be added to the $\mathfrak{sl}(2, \mathbb{R})$ algebra generated by $\delta_1, \delta_2, \delta_3$. Analogously, the action of δ_3 can be transferred to act on $(\mathcal{V}, \tilde{\lambda})$ giving rise to a new symmetry $\tilde{\delta}_4$.

In contrast to $\delta_i B$ ($i = 1, 2, 3$) the expression for $\delta_4 B$ is not an algebraic combination of the fields Δ , ρ , B . It has to be computed from a differential equation. We could though simply define a new field ϕ , a potential [9], up to the addition of a constant by

$$\varepsilon^{\mu\nu} \partial_\nu \phi := \varepsilon^{\mu\nu} \partial_\nu \delta_4 B = 2\rho(\rho^{-1} \partial^\mu \rho - \Delta^{-1} \partial^\mu \Delta + \Delta^2 \rho^{-1} \tilde{B} \partial^\mu \tilde{B}). \quad (4.15)$$

In a first step, forgetting about the definition of ϕ and considering it an independent field makes $\delta_4 B$ resemble the $\delta_i B$ ($i = 1, 2, 3$) in the sense that it is an algebraic expression in the fields. In a second step imposing the equation (4.15) recovers the original situation. (As always we also get a ϕ).

In section 3.6 we showed that the commutator of any two symmetries is again a symmetry. In the beginning we had only the action of \mathfrak{g} , which was spanned by $\delta_1, \delta_2, \delta_3$. Having now also δ_4 at our disposal we don't know yet, if we won't get even more symmetries by building commutators such as $[\delta_i, \delta_4]$ and again transferring the actions between \mathcal{V} and $\tilde{\mathcal{V}}$. In principle we could do this and would see that the process never stops. We would get more and more symmetries and, insisting on algebraic relations of the form $\delta_i B = f(\phi_j)$, more and more potentials ϕ_j . Trying to calculate this is of little use because it is first very tedious, second not instructive and third the algebra is infinite dimensional, hence we would never come to an end. As explained in the next section there is a more practical way to implement and expose this infinite dimensional symmetry algebra.

4.2 The Linear System

In the previous section we saw that the action of the symmetries δ_i , $i > 3$ on the original fields Δ , B are defined via differential equations, but can be turned into algebraic transformations if extra fields, the potentials, are introduced, e.g. (4.15). Furthermore, commutators of symmetries such as $[\delta_1, \delta_4]$ are also symmetries, and in these commutators expressions like $\delta_1 \phi$ appear — the potentials are also acted upon by the δ_i . It turns out that the original fields and the additional potentials can be organized into matrices which are the coefficients of a G valued power series

$$\hat{\mathcal{V}}(\gamma) = \mathcal{V} + \gamma \hat{\mathcal{V}}_1 + \gamma^2 \hat{\mathcal{V}}_2 + \dots \quad (4.16)$$

in a complex parameter γ , called the spectral parameter, such that the full infinite dimensional symmetry algebra \mathfrak{g}^∞ acts in a more or less manageable way on $\hat{\mathcal{V}}(\gamma)$.

Here we will not define $\hat{\mathcal{V}}$ via a direct specification of the coefficients in the power series, but rather from another point of view, namely as the central object in a complete reformulation of cylindrically symmetric gravity which will be essential to the remainder of the present work. The action of the symmetry algebra \mathfrak{g}^∞ on $\hat{\mathcal{V}}$ will then be developed in section 4.6.

In the reformulation of cylindrically symmetric gravity $\hat{\mathcal{V}}$ will be determined by a linear system of field equations

$$\hat{\mathcal{V}}^{-1}\partial_\mu\hat{\mathcal{V}} = \hat{J}_\mu, \quad (4.17)$$

which has a solution if and only if $\mathcal{V} = \hat{\mathcal{V}}|_{\gamma=0}$ satisfies its (non-linear) field equation (2.51), $D_\mu(\rho P^\mu)$. That is, (2.51) is the integrability condition of the linear system. This is achieved by choosing \hat{J}_μ to be a suitable function of $J_\mu = \mathcal{V}^{-1}\partial_\mu\mathcal{V}$ and of γ .

Let us find this function. As an ansatz we adopt some reasonable restrictions on the form of \hat{J}_μ . Firstly we would like $\hat{\mathcal{V}}$ to transform in the same way under $H = SO(2)$ transformations as V does, and this implies that that the projections of \hat{J} onto \mathfrak{h} and \mathfrak{k} must transform as Q and P do, according to (2.49). This is the case if \hat{J}_μ is the sum of Q_μ and a linear combination of the components of P :

$$\hat{J}_\mu = Q_\mu + A_\mu{}^\nu(\gamma)P_\nu. \quad (4.18)$$

Furthermore, we require $\hat{\mathcal{V}}$ to be a spacetime scalar like \mathcal{V} , so \hat{J} must be a 1-form, like Q and P . Admitting that A depends on the metric, and spacetime area form, ε , this still requires that A be invariant under Lorentz transformations. That is, $A = \Lambda A \Lambda^{-1}$, with

$$\Lambda = \left[\begin{array}{cc} 1 & \beta \\ \beta & 1 \end{array} \right],$$

in an orthonormal spacetime basis. Equivalently

$$0 = [A, \Lambda] = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \right] = \beta \begin{pmatrix} b-c & a-d \\ d-a & c-b \end{pmatrix} \Rightarrow b=c, a=d.$$

Note that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_\mu^\nu = \varepsilon_{\mu\rho}\eta^{\rho\nu}.$$

Thus

$$\hat{J}_\mu = Q_\mu + a(\gamma)P_\mu + b(\gamma)\varepsilon_{\mu\nu}P^\nu. \quad (4.19)$$

The linear system (4.17) indicates that \hat{J} is a flat $SL(2)$ connection. It is precisely the connection obtained from the zero connection by rotating the gauge by $\hat{\mathcal{V}}(x)$ at each point x . The linear system can also be regarded as the statement that $\hat{\mathcal{V}}$ is constant with respect to the $SL(2)$ covariant derivative with connection \hat{J} : $d\hat{\mathcal{V}} - \hat{\mathcal{V}}\hat{J} = 0$, or equivalently $d\hat{\mathcal{V}}^{-1} + \hat{J}\hat{\mathcal{V}}^{-1} = 0$. Thus $\hat{\mathcal{V}}(x)$ is the parallel transport of $\hat{\mathcal{V}}$ at a reference point along any path to x . The necessary and sufficient condition for parallel transport to be path independent is that the curvature of the connection vanishes. In other words

$$\partial_\nu\hat{J}_\mu - \partial_\mu\hat{J}_\nu + [\hat{J}_\nu, \hat{J}_\mu] = 0. \quad (4.20)$$

Since the spacetime is two dimensional it is equivalent to require that the contraction of the curvature with $\varepsilon^{\mu\nu}$ vanishes. Substituting the form (4.19) into this contraction one obtains

$$\begin{aligned} & \varepsilon^{\mu\nu} \left(\partial_\mu \hat{J}_\nu - \partial_\nu \hat{J}_\mu + [\hat{J}_\mu, \hat{J}_\nu] \right) = \\ & = 2\varepsilon^{\mu\nu} \left\{ \partial_\mu Q_\nu + \frac{1}{2} [Q_\mu, Q_\nu] + \partial_\mu a P_\nu + a(\partial_\mu P_\nu + [Q_\mu, P_\nu]) + \right. \\ & \left. + \partial_\mu b \varepsilon_{\nu\rho} P^\rho + b \varepsilon_{\nu\rho} \left(\partial_\mu P^\rho + [Q_\mu, P^\rho] \right) + \frac{1}{2} [P_\rho, P_\sigma] \left(\delta_\mu^\rho \delta_\nu^\sigma a^2 + b^2 \varepsilon_\mu^\rho \varepsilon_\nu^\sigma + 2ab \delta_\mu^\rho \varepsilon_\nu^\sigma \right) \right\}. \end{aligned} \quad (4.21)$$

But the curvature of $J = Q + P := \mathcal{V}^{-1} d\mathcal{V}$ is always zero, which implies (2.48), $\partial_\mu Q_\nu - \partial_\nu \hat{Q}_\mu + [Q_\mu, Q_\nu] + [P_\mu, P_\nu] = 0$, and (2.47), $\partial_\mu P_\nu - \partial_\nu \hat{P}_\mu + [Q_\mu, P_\nu] + [P_\mu, Q_\nu] = 0$. Substituting these relations into (4.21) one obtains

$$\begin{aligned} & 2\varepsilon^{\mu\nu} \partial_\mu a P_\nu + 2\partial_\mu b P^\mu + 2b \left(\partial_\mu P^\mu + [Q_\mu, P^\mu] \right) + [P_\rho, P_\sigma] \left(\varepsilon^{\rho\sigma} (a^2 - 1 - b^2) + 2ab \eta^{\rho\sigma} \right) = \\ & = 2(\varepsilon_{\nu\mu} \partial^\nu a + \partial_\mu b - b \rho^{-1} \partial_\mu \rho) P^\mu + 2b \rho^{-1} (\rho D_\mu P^\mu + \partial_\mu \rho P^\mu) + [P_\mu, P_\nu] \varepsilon^{\mu\nu} (a^2 - 1 - b^2) = \\ & = 2(\partial_\mu b - \varepsilon_\mu^\nu \partial_\nu a - b \rho^{-1} \partial_\mu \rho) P^\mu + [P_\mu, P_\nu] \varepsilon^{\mu\nu} (a^2 - b^2 - 1) + 2b \rho^{-1} D_\mu (\rho P^\mu). \end{aligned} \quad (4.22)$$

The third term vanishes for \mathcal{V} satisfying the e.o.m. The first term is in \mathfrak{k} , the second term in \mathfrak{h} . Hence, their scalar coefficient functions of a and b have to vanish separately:

$$\Rightarrow \partial_\mu b - \varepsilon_\mu^\nu \partial_\nu a = b \rho^{-1} \partial_\mu \rho, \quad (4.23)$$

$$\Rightarrow a^2 - b^2 = 1. \quad (4.24)$$

The algebraic equation can be satisfied by setting

$$a(\gamma) = \frac{1 + \gamma^2}{1 - \gamma^2}, \quad b(\gamma) = -\frac{2\gamma}{1 - \gamma^2}.$$

This defines the parameter γ . It is by no means the only good way to parametrize the one dimensional set of solutions to (4.24). (Indeed we shall make extensive use of another parameter $u = a + b$). Different parametrizations lead to equivalent formulations of the theory. The γ parameter is the one used in [9, 20, 30, 24].¹ Note, a and b are only defined for $\gamma \neq \pm 1$. Thus at present, the curvature can only be required to vanish for $\gamma \neq \pm 1$. In section 4.4 we analyse more carefully what happens at these values.

In terms of γ the linear system becomes

$$\hat{\mathcal{V}}^{-1} \partial_\mu \hat{\mathcal{V}} = Q_\mu + \frac{1 + \gamma^2}{1 - \gamma^2} P_\mu - \frac{2\gamma}{1 - \gamma^2} \varepsilon_{\mu\nu} P^\nu. \quad (4.25)$$

This is seen to agree with the form of the system used in [30, 24] once the fact that their metric has the opposite sign is taken into account.

Substituting the γ parametrization into the differential equation (4.23) one obtains

$$\frac{2(1 + \gamma^2)}{(1 - \gamma^2)^2} \partial_\mu \gamma + \varepsilon_\mu^\nu \partial_\nu \gamma \frac{4\gamma}{(1 - \gamma^2)^2} = \frac{2\gamma}{1 - \gamma^2} \rho^{-1} \partial_\mu \rho. \quad (4.26)$$

¹The first two of these references call it t instead of γ .

The only constant solution is $\gamma = 0$, and with this solution $\hat{J} = J$, and the linear system is integrable identically even when \mathcal{V} does not satisfy the field equations. To obtain a linear system that implies the field equation of \mathcal{V} we must admit a spacetime dependent spectral parameter γ . The x dependence of γ is the essential difference between cylindrically symmetric gravity and the principal chiral model (see e.g. [35]). Multiplying (4.26) by $\frac{1+\gamma^2}{2\gamma}$ yields

$$\frac{(1+\gamma^2)^2}{(1-\gamma^2)^2} \gamma^{-1} \partial_\mu \gamma + 2\varepsilon_\mu^\nu \partial_\nu \gamma \frac{1+\gamma^2}{(1-\gamma^2)^2} = \frac{1+\gamma^2}{1-\gamma^2} \rho^{-1} \partial_\mu \rho.$$

On the other hand, multiplying (4.26) by $-\varepsilon_\rho^\mu$ and renaming indices gives

$$-2\varepsilon_\mu^\nu \partial_\nu \gamma \frac{1+\gamma^2}{(1-\gamma^2)^2} - \frac{4\gamma^2}{(1-\gamma^2)^2} \gamma^{-1} \partial_\mu \gamma = -\frac{2\gamma}{1-\gamma^2} \rho^{-1} \varepsilon_\mu^\nu \partial_\nu \rho.$$

Addition of the two previous relations yields

$$\gamma^{-1} \partial_\mu \gamma = \frac{1+\gamma^2}{1-\gamma^2} \rho^{-1} \partial_\mu \rho - \frac{2\gamma}{1-\gamma^2} \varepsilon_\mu^\nu \rho^{-1} \partial_\nu \rho \quad (4.27)$$

(an equation that is remarkably similar to (4.25)). This equation can be solved in closed form. Multiplying it by $\rho^{\frac{1-\gamma^2}{2\gamma}}$ it becomes

$$-\rho \frac{1}{2} \partial_\mu \left(\gamma + \frac{1}{\gamma} \right) = \frac{1}{2} \left(\gamma + \frac{1}{\gamma} \right) \partial_\mu \rho - \varepsilon_\mu^\nu \partial_\nu \rho.$$

Taking into account that $\partial_\mu \tilde{\rho} = \varepsilon_\mu^\nu \partial_\nu \rho$ this is equivalent to

$$0 = \partial_\mu \left(\frac{1}{2} \rho \left(\gamma + \frac{1}{\gamma} \right) - \tilde{\rho} \right). \quad (4.28)$$

Thus

$$\frac{1}{2} \rho \left(\gamma + \frac{1}{\gamma} \right) - \tilde{\rho} = w, \quad (4.29)$$

a complex constant. w is called the constant spectral parameter and will play a central role in the sequel. The solution of (4.29) is

$$\begin{aligned} \gamma(x, w) &= \frac{1}{\rho} \left(w + \tilde{\rho} - \sqrt{(w + \tilde{\rho})^2 - \rho^2} \right) = \\ &= \frac{\sqrt{w + \tilde{\rho} + \rho} - \sqrt{w + \tilde{\rho} - \rho}}{\sqrt{w + \tilde{\rho} + \rho} + \sqrt{w + \tilde{\rho} - \rho}} = \frac{\sqrt{w + \rho^+} - \sqrt{w + \rho^-}}{\sqrt{w + \rho^+} + \sqrt{w + \rho^-}}. \end{aligned} \quad (4.30)$$

To obtain all solutions for γ both branches of the square root must be admitted. However, in the second line $a^+ := \sqrt{w + \rho^+}$ and $a^- := \sqrt{w + \rho^-}$ must take the same values in the denominator and the numerator.

We have now achieved our aim of formulating a linear system which is integrable (for all values of the constant spectral parameter w) iff \mathcal{V} satisfies its field equation, for $\partial_\mu \hat{J}_\nu - \partial_\nu \hat{J}_\mu + [\hat{J}_\mu, \hat{J}_\nu] = \varepsilon_{\mu\nu} \frac{2\gamma}{1-\gamma^2} \rho^{-1} D_\sigma(\rho P^\sigma)$. This formulation of the field equation is certainly rather roundabout, but it offers great advantages which will become clearer in the sequel. Reformulation of the field equations or equations of motion in terms of a linear system is one of the central tools in the theory of integrable systems. (See e.g. [2]).

The linear system for $\hat{\mathcal{V}}$ takes an especially simple form in null coordinates (B.1). In null coordinates the only non-zero components of $\varepsilon_{\hat{\mu}}^{\hat{\nu}}$ are $\varepsilon_{\pm}^{\pm} = \pm 1$, and therefore

$$\hat{\mathcal{V}}^{-1} \partial_{\pm} \hat{\mathcal{V}} = \hat{J}_{\pm} = Q_{\pm} + \frac{1 \mp \gamma}{1 \pm \gamma} P_{\pm} \quad (4.31)$$

$$= Q_{\pm} + u^{\pm 1} P_{\pm}, \quad (4.32)$$

where

$$u(x, w) := \frac{1 - \gamma}{1 + \gamma} = \frac{\sqrt{w + \rho^-}}{\sqrt{w + \rho^+}} = \frac{a^-}{a^+}. \quad (4.33)$$

(u is the parameter $a + b$ mentioned earlier, and we could have obtained its dependence on ρ^{\pm} and w directly from (4.23), without passing through γ .) The differential equation for γ takes the form

$$\gamma^{-1} \partial_{\pm} \gamma = u^{\pm 1} \rho^{-1} \partial_{\pm} \rho. \quad (4.34)$$

The solution to the linear system (4.25) is a path ordered exponential

$$\hat{\mathcal{V}}(x, w) = \hat{\mathcal{V}}(x_0, w) \mathcal{P} e^{\int_{x_0}^x \hat{J}}, \quad (4.35)$$

where $\hat{\mathcal{V}}(x_0, w)$ is the value of $\hat{\mathcal{V}}$ at an arbitrarily chosen reference point x_0 in spacetime. $\hat{\mathcal{V}}(x_0, w)$ is not restricted by (4.25), $\hat{\mathcal{V}}$ is thus defined by the linear system only up to a left multiplication by a w dependent group element $S(w) \in G$.

Originally the idea was that $\hat{\mathcal{V}}$ be a power series from which all the physical fields and potentials can be recovered. In particular \mathcal{V} was to be the order zero term in the series. With a little care we can ensure that this is the case. Recall that $\gamma(x) = 0$ is a possible γ field (corresponding to $w = \infty$). When $\gamma = 0$ the connection \hat{J} reduces to just J , so

$$\hat{\mathcal{V}}(x, w) = \hat{\mathcal{V}}(x_0, w) \mathcal{V}(x_0)^{-1} \mathcal{V}(x). \quad (4.36)$$

Thus $\mathcal{V}(x) = \hat{\mathcal{V}}(x)$ when $\gamma = 0$, or equivalently $w = \infty$, provided $\hat{\mathcal{V}}(x_0, w = \infty) = \mathcal{V}(x_0)$. We will in fact choose $\hat{\mathcal{V}} = \mathcal{V}$ for all w at a point on the axis $\rho = 0$. With this choice (and with many others) $\hat{\mathcal{V}}$ turns out to be analytic in γ in a neighbourhood of 0 for a large and reasonable class of solutions \mathcal{V} , so $\hat{\mathcal{V}}$ is indeed a power series in γ . It will also often be convenient to put $\hat{\mathcal{V}}$ in upper triangular form, by means of a local H transformation $\hat{\mathcal{V}}(x, w) \rightarrow \hat{\mathcal{V}}(x, w) h(x)$ with $h(x) \in H$.

Following [9] we denote these conditions, analyticity in γ about 0 and triangularity of $\mathcal{V} = \hat{\mathcal{V}}(\gamma = 0)$, as the *triangular gauge*.²

4.3 The analytical properties of $\gamma(x, w)$

4.3.1 The complex square root and its Riemann surface

The variable spectral parameter γ contains square roots of the constant spectral parameter w . Here we elaborate some details about the complex square root function and the Riemann surface associated to it.

²In Chapter 5 the H transformations are shown to be true gauge transformations. It less clear in what sense the analyticity condition is a gauge condition. Nevertheless we will keep using this already established terminology.

Consider the equation

$$v^2 = z. \quad (4.37)$$

For fixed $z \neq 0$ this equation for v has two solutions differing by a sign. If we want to define the square root such that

$$\sqrt{z^2} = z, \quad (4.38)$$

then it is not uniquely defined, but a multi-valued function. Setting $z = r e^{i\phi}$ and $v = s e^{i\psi}$ in (4.37) then

$$(s e^{i\psi})^2 = s^2 e^{i2\psi} = r e^{i\phi} \Leftrightarrow s^2 = r, \quad 2\psi = \phi + 2n\pi \Leftrightarrow s = \sqrt{r}, \quad \psi = \frac{\phi}{2} + n\pi \quad n \in \mathbb{Z},$$

with \sqrt{r} being the real (positive) square root of r . Therefore

$$\sqrt{z} = \sqrt{r e^{i\phi}} = \pm \sqrt{r} e^{i\phi/2} \quad (4.39)$$

When writing a complex number in its polar form we always have the freedom of adding $2n\pi$, $n \in \mathbb{Z}$, to the argument ϕ

$$z = r e^{i\phi} = r e^{i(\phi+2n\pi)}.$$

If we agree that we always take the positive sign in $s = \pm\sqrt{r}$, then the sign in (4.39) corresponds to our choice of n when writing z in its polar form. If n is even (odd), the sign is $+$ ($-$). This already suggests that we define the square root function not as a bi-valued function on the complex plane, but as a single-valued function on two copies of the complex plane where

$$z_1 = r e^{i\phi} \quad \text{and} \quad z_2 = r e^{i(\phi+2\pi)}$$

are no longer coinciding points, but merely equivalent points on different copies of the complex plane.

By analysing continuity properties of the square root (4.39) on the complex plane we will see how two copies of the complex plane appropriately cut and glued together will give a single-valued continuous function on what is called a Riemann surface.

First note that we can construct two single-valued functions each mapping \mathbb{C} to \mathbb{C} by requiring that the argument of a complex number be in a certain interval

$$z = r e^{i\phi} \quad \phi \in [a, a + 2\pi[, \quad a \in \mathbb{R} \quad (4.40)$$

(for example $[-\pi, \pi[$). Then (4.39) defines two single-valued functions differing by a sign. Now we consider a small loop

$$\begin{aligned} c: [0, 1] \subset \mathbb{R} &\rightarrow \mathbb{C} \\ t &\mapsto \epsilon e^{i(\phi+2t\pi)} = \epsilon e^{i(\phi+2t\pi+2n\pi)} \end{aligned} \quad (4.41)$$

around 0, choose an interval for the argument³ and evaluate the functions (4.39) on this curve. Denote by t_0 the value of t where

$$\lim_{\varepsilon \rightarrow 0} \left\{ \phi + 2\pi \left[t_0 - \varepsilon + n(t_0 - \varepsilon, a, \phi) \right] \right\} = a + 2\pi,$$

³In this step n in (4.41) becomes a function of t , a and ϕ

which is the parameter value of the curve where according to our prescribed interval we have to shift the argument:

$$\phi + 2\pi(t + n(t, a, \phi)) \in [a, a + 2\pi[\quad \forall t$$

$$\lim_{\epsilon \rightarrow 0} \left\{ \phi + 2\pi[t_0 + \epsilon + n(t_0 + \epsilon, a, \phi)] - \phi + 2\pi[t_0 - \epsilon + n(t_0 - \epsilon, a, \phi)] \right\} = -2\pi.$$

The square root functions are discontinuous at $c(t_0)$. Since the argument jumps by -2π , we have that

$$\lim_{\epsilon \rightarrow 0} \sqrt{c(t_0 + \epsilon)} = -\lim_{\epsilon \rightarrow 0} c(t_0 - \epsilon). \quad (4.42)$$

The functions change its sign.

We call a point with the property that $\sqrt{}$ cannot be defined continuously on an arbitrarily small loop around it a branch point. The “point at infinity” is also a branch point. This can be seen as follows. We substitute $y = \frac{1}{z}$. Then “ $z = \infty$ ” corresponds to $y = 0$ and

$$\sqrt{z} = \frac{1}{\sqrt{y}}.$$

We consider again a loop (4.41) around $y = 0$. Then again at t_0 the argument jumps and the functions change its sign.

It is clear that the square root functions will be discontinuous along any loop around 0 or ∞ - the number of points of discontinuity on a particular loop will depend on how often the argument of the curve jumps between the boundaries of the interval (4.40). The choice of a corresponds to a straight line in the complex plane connecting the two branch points where the square root functions are discontinuous. We could even construct a curved line by making a a continuous function of r in (4.40) or, even more generally, choose any line connecting the two branch points and make the square root functions discontinuous along this line. This line is called the branch cut.

By (4.42) $\sqrt{}$ changes by a factor of -1 when crossing the branch cut. The two square root functions (4.39) also differ by factor of -1 . So we consider two copies of the complex plane $\mathbb{C}_{1,2}$. We choose an a , i.e. we define a branch cut for $\mathbb{C}_{1,2}$. On \mathbb{C}_1 (\mathbb{C}_2) we define (4.39) with the positive (the negative) sign. Now we agree that every time we cross the branch cut on \mathbb{C}_1 we jump to the corresponding point behind the branch cut on \mathbb{C}_2 and vice versa. Then we have one continuous single-valued function defined on two copies of the complex plane cut along a branch cut and glued together alternately. This surface is called the Riemann surface of the complex square root. For a fixed complex number z the choice of sign of \sqrt{z} amounts to the choice on which of the two sheets of the Riemann surface we choose this number to be. Pictures of this and Riemann surfaces of other functions can be found in abundance on the internet.

4.3.2 γ and its Riemann surface

From (4.30) we see that $\gamma(x, w)$ contains a square root. We will thus need a Riemann surface to continuously define γ on. In the previous section we showed that the Riemann surface of the function \sqrt{z} consists of two copies of the complex plane, cut along a line connecting the two branch points 0 and ∞ and glued together such that every loop around

0 (or ∞) encircles this point an even number of times. We look for branch points of γ . In the expression for $\gamma(x, w)$, w is mapped to the value

$$(w + \tilde{\rho})^2 - \rho^2 = (w + \tilde{\rho} + \rho)(w + \tilde{\rho} - \rho)$$

before taking the square root. This expression is 0 at $w = -\tilde{\rho} \pm \rho$. An arbitrarily small loop

$$\begin{aligned} c: [0, 1] \subset \mathbb{R} &\rightarrow \mathbb{C} \\ t &\mapsto -\tilde{\rho} \pm \rho + \epsilon e^{i(\phi+2t\pi)} = -\tilde{\rho} \pm \rho + \epsilon e^{i(\phi+2t\pi+2n\pi)} \end{aligned} \quad (4.43)$$

in the w plane is mapped to

$$(\pm 2\rho + \epsilon e^{i(\phi+2\pi(n+t))}) \epsilon e^{i(\phi+2\pi(n+t))} = \pm 2\rho \epsilon e^{i(\phi+2\pi(n+t))} + \epsilon^2 e^{i(2\phi+4\pi(n+t))}$$

before the square root is taken. Since ϵ is arbitrarily small, this is a loop encircling 0 once in the $(w + \tilde{\rho})^2 - \rho^2 = (w + \tilde{\rho} + \rho)(w + \tilde{\rho} - \rho)$ -plane. The points $w = -\tilde{\rho} \pm \rho$ are branch points of γ . What about the point at infinity? Again we set $w = \frac{1}{w'}$, $\tilde{\rho} \pm \rho = \pm \frac{1}{2\rho'^{\pm}}$. Then

$$(w + \tilde{\rho} + \rho)(w + \tilde{\rho} - \rho) = \left(\frac{1}{w'} + \frac{1}{2\rho'^+} \right) \left(\frac{1}{w'} - \frac{1}{2\rho'^-} \right) = \left(1 + \frac{w'}{2\rho'^+} \right) \left(1 - \frac{w'}{2\rho'^-} \right) \frac{1}{w'^2}.$$

We consider an arbitrarily small loop around 0, $w' = \epsilon e^{i(\phi+2t\pi+2n\pi)}$. Then we may write

$$\begin{aligned} \left(1 + \frac{\epsilon e^{i(\phi+2t\pi+2n\pi)}}{2\rho'^+} \right) &= r_1(t) e^{i\psi_1(t)}, \\ \left(1 - \frac{\epsilon e^{i(\phi+2t\pi+2n\pi)}}{2\rho'^-} \right) &= r_2(t) e^{i\psi_2(t)}, \\ \frac{1}{(\epsilon e^{i(\phi+2t\pi+2n\pi)})^2} &= \frac{1}{\epsilon^2} e^{-i(2\phi+4t\pi+2n\pi)}, \\ \left(1 + \frac{w'}{2\rho'^+} \right) \left(1 - \frac{w'}{2\rho'^-} \right) \frac{1}{w'^2} &= r_1(t)r_2(t) \frac{1}{\epsilon^2} e^{i(\psi_1(t)+\psi_2(t)-2\phi-4t\pi-4n\pi)}. \end{aligned}$$

Since ϵ is arbitrarily small there will surely exist a value of a such that $r_1(t)$, $r_2(t)$, $\psi_1(t)$, $\psi_2(t)$ vary continuously, arbitrarily little and have the same values for $t = 0$ and $t = 1$. $e^{i(-2\phi-4t\pi-4n\pi)}$ describes 2 complete circles. Hence when doing a single loop around “ ∞ ” in the w -plane we do a double loop in the $(w + \tilde{\rho})^2 - \rho^2$ -plane. We can define γ on this loop continuously and therefore ∞ is not a branch point.

As a branch cut connecting our two branch points $w = -\tilde{\rho} \pm \rho$ we may choose the segment of the real line between $w = -\tilde{\rho} \pm \rho$. Then the Riemann surface of γ consists of two copies of the complex plane cut and glued together along the ρ - and $\tilde{\rho}$ -dependent branch cut. Using the figure used in [24], but with a different notation, we label the two sheets of the Riemann surface by W_+ and W_- . If we need to distinguish between their upper and lower half planes, we write $W_{+u}, W_{+d}, W_{-u}, W_{-d}$ (see figure 4.1).

4.3.3 Properties of γ

Finally we work out some properties of γ also mentioned in the appendix of [24]. We had

$$\gamma(x; w) = \frac{1}{\rho} \left(w + \tilde{\rho} - \sqrt{(w + \tilde{\rho})^2 - \rho^2} \right) = \frac{\sqrt{w + \rho + \tilde{\rho}} - \sqrt{w + \tilde{\rho} - \rho}}{\sqrt{w + \rho + \tilde{\rho}} + \sqrt{w + \tilde{\rho} - \rho}}. \quad (4.44)$$

This relation can be inverted for $\gamma \neq 0$ to express w as a function of γ :

$$w = \frac{\rho}{2} \left(\gamma + \frac{1}{\gamma} \right) - \tilde{\rho}. \quad (4.45)$$

For now let $0 < \rho < \infty$ and $\tilde{\rho}$ be finite.

1) γ is obtained from $\frac{1}{\gamma}$ by changing the sign of the square root in (4.44).

We set $y := \frac{w + \tilde{\rho}}{\rho}$. Then

$$\gamma = y - \sqrt{y^2 - 1}$$

and so

$$\frac{1}{\gamma} = \frac{1}{y - \sqrt{y^2 - 1}} = \frac{y + \sqrt{y^2 - 1}}{y^2 - (y^2 - 1)} = y + \sqrt{y^2 - 1}.$$

Since exchanging the sign of the square root means exchanging the two sheets, this means that for a complex number w corresponding to two different points on the two sheets of the Riemann surface the value of γ at one of the points is the reciprocal of the value of γ at the other point. This is also mirrored in (4.45). γ and $\frac{1}{\gamma}$ are mapped to the same number w (again corresponding to two points on the two sheets of the Riemann surface).

2) $w \in \mathbb{R}$ ($y \in \mathbb{R}$) implies that either $\gamma \in \mathbb{R} \setminus \{0\}$ or $|\gamma| = 1$:

$$\begin{aligned} w \in \mathbb{R} &\Leftrightarrow \gamma + \frac{1}{\gamma} \in \mathbb{R} \Leftrightarrow 0 = \gamma + \frac{1}{\gamma} - \bar{\gamma} - \frac{1}{\bar{\gamma}} = 2i \operatorname{Im}(\gamma) - \frac{2i \operatorname{Im}(\gamma)}{|\gamma|^2} = 2i \operatorname{Im}(\gamma) \left(1 - \frac{1}{|\gamma|^2} \right) \Leftrightarrow \\ &\Leftrightarrow (\gamma \in \mathbb{R} \setminus \{0\} \vee |\gamma| = 1) \end{aligned}$$

3) $\gamma \in \mathbb{R} \setminus \{0\} \Rightarrow (y \in \mathbb{R} \wedge |y| \geq 1)$:

From 2) we already know that $\gamma \in \mathbb{R} \setminus \{0\} \Rightarrow y \in \mathbb{R}$.

Consider the two polynomials $\gamma^2 \pm 2\gamma + 1$ with $\gamma \in \mathbb{R}$. Each has only one zero at $\mp\gamma = 1$. These are also the global minima. Hence

$$\gamma^2 \pm 2\gamma + 1 \geq 0 \Rightarrow \gamma^2 + 1 \geq \mp 2\gamma \Rightarrow \left| \gamma + \frac{1}{\gamma} \right| \geq 2 \Rightarrow |y| \geq 1 \text{ for } \gamma \in \mathbb{R} \setminus \{0\}.$$

4) $|\gamma| = 1 \Rightarrow (y \in \mathbb{R} \wedge |y| \leq 1)$

Again from 2) we know that $|\gamma| = 1 \Rightarrow y \in \mathbb{R}$. Furthermore if $|\gamma| = 1$ we can write $\gamma = e^{i\phi}$. Then $\gamma + \frac{1}{\gamma} = 2 \operatorname{Re} \gamma = 2 \cos(\phi)$ and so

$$|y| = |\cos(\phi)| \leq 1$$

For the specific value⁴ $y = \sqrt[{\mathbb{R}}]{2}$, $\gamma(y) = \sqrt[{\mathbb{R}}]{2} - \sqrt{1} = \sqrt[{\mathbb{R}}]{2} \mp 1$. We now specify that if y denotes the point with value $\sqrt[{\mathbb{R}}]{2}$ on $W_+ = W_{+u} \cup W_{+d}$ we choose the value $\sqrt[{\mathbb{R}}]{2} - 1$ while for the corresponding point on $W_- = W_{-u} \cup W_{-d}$ we choose $\sqrt[{\mathbb{R}}]{2} + 1$. By continuity real points $y \in W_+$ with $y > 1$ get mapped to points $\gamma \in \mathbb{R}$ with $0 \leq \gamma < 1$ and the corresponding points on W_- get mapped to points $\gamma \in \mathbb{R}$ with $\gamma > 1$. What about the real negative points in the γ -plane. Since

$$2y = \gamma + \frac{1}{\gamma},$$

a negative and real γ implies a negative and real w . Now we think of a path in W_+ connecting a positive real point to a negative real point without crossing the branch cut. This path in the γ -plane will then not cross the unit circle and hence when arriving at a point $y \in W_+$, real and $y < -1$, in the γ -plane we arrive somewhere on the segment of the negative real axis inside the unit circle. So real $y \in W_+$ with $y < -1$ are mapped to real negative γ with $0 \geq \gamma > -1$. By the same arguments real $y \in W_-$ with $y < -1$ are mapped to real negative γ with $\gamma < -1$.

Now for a fixed real number y_0 (corresponding to two points) with $|y_0| > 1$ we consider the curve $y(t) = y_0 + it$. At $t = 0$ the tangent to the curve $\gamma(y(t))$ is

$$\left. \frac{d}{dt} \right|_{t=0} \gamma(y(t)) = i - \frac{2y_0i}{2\sqrt{y_0^2 - 1}} = i \left(1 - \frac{y_0}{\sqrt{y_0^2 - 1}} \right), \quad (4.46)$$

which vanishes only if $y_0 = \infty$. Hence crossing the real line without the branch cut in W_+ or W_- means crossing (and not only touching) the real line in the γ plane.

Finally, if we set $y_0 = \sqrt[{\mathbb{R}}]{2}$, $y \in W_+$, we have

$$\left. \frac{d}{dt} \right|_{t=0} \gamma(y(t)) = i \left(1 - \frac{\sqrt[{\mathbb{R}}]{2}}{\sqrt{1}} \right) = i(1 - \sqrt[{\mathbb{R}}]{2}),$$

which is purely imaginary and “negative” because we fixed that the sign of the square root for real points $y > 1$, $y \in W_+$ is positive. So moving from W_{+d} to W_{+u} in the γ -plane corresponds to moving from Γ_{+u} to Γ_{+d} (see figure 4.1). If we then cross the branch cut to get to W_{-d} in the γ -plane, we cross the unit circle⁵ and enter the region Γ_{-d} . Finally, if we then move to W_{-u} in the γ -plane, we arrive at Γ_{-u} , which completes this analysis. Under the map γ

$$W_{+u} \rightarrow \Gamma_{+d}, \quad W_{+d} \rightarrow \Gamma_{+u}, \quad W_{-u} \rightarrow \Gamma_{-u}, \quad W_{-d} \rightarrow \Gamma_{-d}. \quad (4.47)$$

5) The point $\gamma = 0$ is reached in the limit $w \rightarrow \infty$ on W_+
The point $\gamma = \infty$ is reached in the limit $w \rightarrow \infty$ on W_-

$$\lim_{w \rightarrow \infty} \gamma = \lim_{y \rightarrow \infty} \frac{1 - \sqrt{1 - y^{-2}}}{y^{-1}}$$

⁴ $\sqrt[{\mathbb{R}}]{}$ denotes the positive real square root function.

⁵(4.46) can also be used for $|y_0| < 1$ to show that indeed we cross the unit circle and do not only touch it

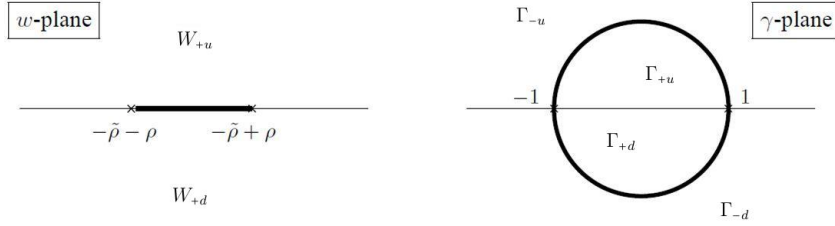


Figure 4.1: On the properties of γ

Depending on the sign of the square root in the limit, this is an expression of the form $\frac{0}{0}$ or it is ∞ . In the former case we continue

$$\lim_{w \rightarrow \infty} \gamma = \lim_{y \rightarrow \infty} \frac{-\frac{1}{2} \frac{2y^{-3}}{\sqrt{1-y^{-2}}}}{-y^{-2}} = \frac{y^{-1}}{\sqrt{1-y^{-2}}} = 0.$$

Taking into account (4.47), we know which case belongs to which sheet.

6) $\gamma(x, w = -\tilde{\rho} \pm \rho) = \pm 1$ is obvious.

7) $\lim_{\rho \rightarrow 0} \gamma = 0$ or ∞ depending on the sheet

$\rho \rightarrow 0$ means that $y \rightarrow \infty$. From 5) we know what happens in this case.

9) $\lim_{\rho \rightarrow \infty} \gamma = \pm i$ $\rho \rightarrow \infty$ corresponds to $y \rightarrow 0$ and so $\gamma \rightarrow -\sqrt{-1} = \pm i$ again depending on the sheet.

4.4 The analytical properties of $\hat{\mathcal{V}}(x, w)$

In this section we investigate the analytical properties of $\hat{\mathcal{V}}(x, w)$. It is obtained from $\mathcal{V}(x)$ by solving the linear system (4.25) for some initial condition. Since the $\hat{J}(x, w)$ appearing in this linear system is singular along the lines $\rho^\pm = -w$, a detailed treatment of the behaviour of $\hat{\mathcal{V}}(x, w)$ near these singular points seems necessary.

Throughout this section, we assume that $J(x) = \mathcal{V}^{-1} \partial \mathcal{V}$ is continuous on the closed subset \mathcal{D}

$$\mathcal{D} := \{(x^-, x^+) \mid x_0^+ \leq x^+ \leq x_1^+, x_0^- \leq x^- \leq x_1^-, x^+ \geq x^-\} \quad (4.48)$$

and that furthermore the mixed partial derivatives $\partial_+ J_-$ and $\partial_- J_+$ exist and are continuous. For real w , we define the line segments ℓ_w^\pm to be the intersections of \mathcal{D} with the lines $x^\pm = -w$. If $x_0^+ < -w < x_1^+$, these line segments separate \mathcal{D} into regions \mathcal{D}_I , \mathcal{D}_{II} and \mathcal{D}_{III} illustrated in figure 4.2. We define, that these regions contain those points of their boundary, which are also points of the boundary of \mathcal{D} , but which do not lie on ℓ^\pm .

4.4.1 Continuity with respect to spacetime-coordinates

We start with the easy part, namely the properties of the solution $\hat{\mathcal{V}}$ to the linear system in the domains where the connection \hat{J} is non-singular.

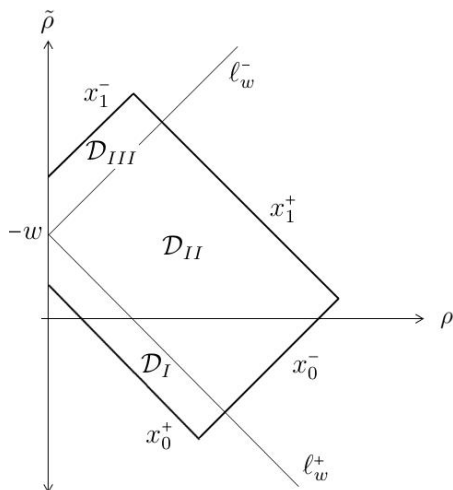


Figure 4.2: The domain \mathcal{D} where J is assumed to be continuous.

Proposition 4.4.1. *If the imaginary part of w is non-zero or if w is real, but such that ℓ_w^\pm are empty, then the solution of the linear system with initial condition $\hat{\mathcal{V}}(x_0, w) = \mathbb{1}$ is given by the path ordered exponential*

$$\hat{\mathcal{V}}(x, w) = \mathcal{P}e^{\int_{x_0}^x \hat{J}}, \quad (4.49)$$

where x_0 as well as the entire path connecting x_0 to x must lie in \mathcal{D} . This solution is continuous in x , analytic in w and independent of the path chosen.

Proof. For such w , the connection \hat{J} is non-singular in all of \mathcal{D} . In particular, along a specific curve it is continuous with respect to the parameter of the curve and analytic in w . By theorem (10.3) in chapter II of [27], the solution is therefore analytic in w . Since the curvature (4.22) of the connection \hat{J} is identically zero for those w considered in the proposition, the solution is independent of the path. By the properties of the path ordered exponential (see appendix D) it is clear that (4.49) solves the linear system and satisfies $\hat{\mathcal{V}}(y, w) = \mathbb{1}$. \square

If w is real, the lines ℓ_w^\pm may or may not separate \mathcal{D} into two or three regions, depending on the value of w . In any case, we denote these regions by \mathcal{D}_i , where i may take values I, II, III . $\hat{J}(x, w)$ is then non-singular in these regions \mathcal{D}_i . An analogue of the above proposition holds with \mathcal{D} replaced by \mathcal{D}_i and w real.

In the following, for real w with $x_0^+ < -w < x_1^+$, step by step we define a function on the lines ℓ_w^\pm , which are the limiting values of the solutions of the linear system in the regions \mathcal{D}_i . Note that although the linear system in the form (4.25) and the curvature (4.22) is singular on the lines ℓ_w^\pm , parts of these singularities may be removed by considering the \pm -components (4.31). In that expression, \hat{J}_- is singular only on ℓ_w^- and \hat{J}_+ is singular only on ℓ_w^+ . We will see, that the function we define on ℓ_w^\pm is differentiable along these lines and satisfies the corresponding components of (4.31).

For $x_0^+ < -w < x_1^+$, let $\mathcal{I} \subset \mathbb{R}$ by any real, closed interval such that points $(x^-, -w)$ with $x^- \in \mathcal{I}$ lie on ℓ^+ and not on the axis $\rho = 0$ (this means that $\mathcal{I} \ni x^- < -w$). We start with

the region D_I . Choose a point $x_i = (x_i^-, x_i^+)$ in D_I , such that the points $x_m := (x^-, x_i^+)$, $x^- \in \mathcal{I}$, all lie in D_I . We proof the following propositions

Lemma 4.4.1. *The improper integral*

$$\int_{x_i^+}^{-w} \hat{J}_+(x^-, z^+, w) dz^+ = \int_{x_i^+}^{-w} \left\{ Q_+(x^-, z^+, w) + \frac{\sqrt{w+x^-}}{\sqrt{w+z^+}} P_+(x^-, z^+, w) \right\} dz^+ \quad (4.50)$$

and its derivative with respect to x^- exist for all $x^- \in \mathcal{I}$. The derivative is given by the improper integral over the derivative of the integrand.

Proof. We introduce the sequence of functions $(a_n(x^-))_{n \in \mathbb{N}}$ of $x^- \in \mathcal{I}$ given by

$$a_n(x^-) = \int_{x_i^+}^{-w-\frac{\chi}{n}} \left\{ Q_+ + \frac{\sqrt{w+x^-}}{\sqrt{w+z^+}} P_+ \right\} dz^+, \quad (4.51)$$

where $\chi > 0$ is such that $-w - \chi > x_i^+$. For two integers $m < n$

$$\begin{aligned} \|a_n - a_m\| &\leq \int_{-w-\frac{\chi}{m}}^{-w-\frac{\chi}{n}} \left\{ \|Q_+\| + \frac{\sqrt{|w+x^-|}}{\sqrt{|w+z^+|}} \|P_+\| \right\} dz^+ \leq \\ &\leq |\chi/m - \chi/n| \max_{z^+ \in [x_i^+, -w]} \|Q_+\| + \sqrt{|w+x^-|} \max_{z^+ \in [x_i^+, -w]} \|P_+\| 2 \int_{\sqrt{\chi/m}}^{\sqrt{\chi/n}} d(|\sqrt{w+z^+}|) = \\ &= |\chi/m - \chi/n| \max_{z^+ \in [x_i^+, -w]} \|Q_+\| + \sqrt{|w+x^-|} \max_{z^+ \in [x_i^+, -w]} \|P_+\| 2 \left| \sqrt{\chi/n} - \sqrt{\chi/m} \right|, \end{aligned}$$

which by the assumed regularity of J_μ becomes arbitrarily small for sufficiently high m, n . The sequence converges pointwise, that is the improper integral exists for every $x^- \in \mathcal{I}$.

Now to the statement about its derivative: By a standard theorem on sequences of functions (see e.g. [16]) we need to prove that the sequence of derivatives, $(\partial_{x^-}(a_n))_{n \in \mathbb{N}}$, converges uniformly on \mathcal{I} . Then

$$\partial_{x^-}(\lim_{n \rightarrow \infty} a_n) = \lim_{n \rightarrow \infty} (\partial_{x^-} a_n).$$

The sequence of derivatives is

$$\partial_{x^-} a_n(x^-) = \int_{x_i^+}^{-w-\frac{\chi}{n}} \left\{ \partial_{x^-} Q_+ + \partial_{x^-} \left(\frac{\sqrt{w+x^-}}{\sqrt{w+z^+}} P_+(x^-, z^+) \right) \right\} dz^+.$$

For $m < n$ we get

$$\begin{aligned} \|\partial_{x^-} a_n(x^-) - \partial_{x^-} a_m(x^-)\| &\leq |\chi/m - \chi/n| \max_{z^+ \in [x_i^+, -w]} \|\partial_{x^-} Q_+\| + \\ &+ \left\| \frac{1}{2\sqrt{w+x^-}} \int_{-w-\chi/m}^{-w-\chi/n} \frac{P_+(x^-, z^+)}{\sqrt{w+z^+}} dz^+ + \int_{-w-\chi/m}^{-w-\chi/n} \frac{\sqrt{w+x^-}}{\sqrt{w+z^+}} \partial_{x^-} P_+(x^-, z^+) dz^+ \right\| \leq \\ &\leq |\chi/m - \chi/n| \max_{z^+ \in [x_i^+, -w]} \|\partial_{x^-} Q_+\| + \frac{|\sqrt{\chi/n} - \sqrt{\chi/m}|}{|\sqrt{w+x^-}|} \max_{z^+ \in [x_i^+, -w]} \|P_+\| + \\ &+ 2|\sqrt{w+x^-}| |\sqrt{\chi/n} - \sqrt{\chi/m}| \max_{z^+ \in [x_i^+, -w]} \|\partial_{x^-} P_+\|, \end{aligned} \quad (4.53)$$

which again becomes arbitrarily small for sufficiently high m, n . This proves pointwise convergence: For every $x^- \in \mathcal{I}$, the improper integral

$$\int_{x_i^+}^{-w} \partial_{x^-} \hat{J}_+(x^-, z^+) dz^+$$

exists. Now let $\alpha < \beta$ be the endpoints of \mathcal{I} , $\mathcal{I} = [\alpha, \beta]$. Then we furthermore have

$$\begin{aligned} \sup_{x^- \in \mathcal{I}} \|\partial_{x^-} a_n(x^-) - \lim_{m \rightarrow \infty} [\partial_{x^-} a_m(x^-)]\| &= \sup_{x^- \in \mathcal{I}} \left\| \int_{-w-\frac{\chi}{n}}^{-w} \partial_{x^-} \hat{J}_+(x^-, z^+) dz^+ \right\| \leq \\ &\leq \frac{|\chi|}{n} \max_{\substack{x^- \in \mathcal{I} \\ z^+ \in [-w-1/n, -w]}} \|\partial_{x^-} Q_+\| + \frac{|\sqrt{|w|} - \sqrt{|w + \chi/n|}|}{\sqrt{|w + \beta|}} \max_{\substack{x^- \in \mathcal{I} \\ z^+ \in [-w-\chi/n, -w]}} \|P_+\| + \\ &+ 2\sqrt{|w + \alpha|} \left| \sqrt{w} - \sqrt{|w + \chi/n|} \right| \max_{\substack{x^- \in \mathcal{I} \\ z^+ \in [-w-\chi/n, -w]}} \|\partial_{x^-} P_+\|, \end{aligned}$$

which goes to zero as n goes to infinity. Hence, $(\partial_{x^-} a_n)_{n \in \mathbb{N}}$ converges uniformly on \mathcal{I} and the proposition is proved. \square

Let w, \mathcal{I}, x_i and χ be as above.

Lemma 4.4.2. *The multiple integral*

$$\int_{x_i^+}^{-w} dz_1^+ \int_{x_i^+}^{z_1^+} dz_2^+ \dots \int_{x_i^+}^{z_{k-1}^+} dz_k^+ \hat{J}_+(x^-, z_k^+) \dots \hat{J}_+(x^-, z_1^+) \quad (4.54)$$

and its derivative with respect to x^- exist for all $x^- \in \mathcal{I}$ and $k \in \mathbb{N}$ with $k \geq 2$. The derivative is given by taking the multiple, improper integral over the derivative of the integrand.

Proof. Throughout this proof, we write simply

$$\hat{J}_i := \hat{J}_+(x^-, z_i^+), \quad \hat{J} = \hat{J}_+(x^-, z^+).$$

We consider the sequence $(a_n(x^-))_{n \in \mathbb{N}}$ with

$$a_n(x^-) = \frac{1}{k!} \int_{x_i^+}^{-w-\chi/n} dz_1^+ \int_{x_i^+}^{-w-\chi/n} dz_2^+ \dots \int_{x_i^+}^{-w-\chi/n} dz_k^+ \mathcal{P}[\hat{J}_k \dots \hat{J}_1],$$

where the path ordering operator \mathcal{P} orders the factors of the product such, that factors with lower values of z^+ are to left of factors with higher values of z^+ (see also appendix D). By the properties of the norm $\|\cdot\|$ we have

$$\|\mathcal{P}[AB]\| \leq \|A\| \|B\|$$

and thus for $m < n$

$$\|a_n(x^-)\| \leq \frac{1}{k!} \left(\int_{-w-\chi/n}^{-w-\chi/n} dz^+ \|\hat{J}\| \right)^k \leq \frac{A_{(mn)}^k(x^-)}{k!}, \quad (4.55)$$

with

$$A_{(mn)}(x^-) := \left| \frac{\chi}{m} - \frac{\chi}{n} \right| \max_{z^+ \in [x_i^+, -w]} \|Q_+\| + \sqrt{|w+x^-|} \max_{z^+ \in [x_i^+, -w]} \|P_+\| 2 \left| \sqrt{\chi/n} - \sqrt{\chi/m} \right|. \quad (4.56)$$

$A_{(mn)}(x^-)$ becomes arbitrarily small for sufficiently high m, n . The sequence converges pointwise.

We now consider the sequence of derivatives. The path ordered product can be expressed as a sum of normal products multiplied by Θ -distributions⁶. Thus ∂_{x^-} commutes with the path ordering operator \mathcal{P} and for $m < n$ we have

$$\begin{aligned} & \|\partial_{x^-} a_n(x^-) - \partial_{x^-} a_m(x^-)\| \leq \\ & \leq \frac{1}{k!} \left\| \int_{-w-\chi/m}^{-w-\chi/n} dz_1^+ \int_{-w-\chi/m}^{-w-\chi/n} dz_2^+ \dots \int_{-w-\chi/m}^{-w-\chi/n} dz_k^+ \mathcal{P}[\partial_{x^-} \{\hat{J}_k \dots \hat{J}_1\}] \right\| \leq \quad (4.57) \\ & \leq \frac{1}{(k-1)!} \left(\int_{-w-\chi/m}^{-w-\chi/n} dz^+ \|\hat{J}\| \right)^{k-1} \int_{-w-\chi/m}^{-w-\chi/n} \|\partial_{x^-} \hat{J}_1\| dz_1^+ \leq \frac{A_{(mn)}^{k-1}(x^-) B_{(mn)}(x^-)}{(k-1)!}, \end{aligned}$$

where

$$\begin{aligned} B_{(mn)}(x^-) := & \left| \frac{\chi}{m} - \frac{\chi}{n} \right| \max_{z^+ \in [x_i^+, -w]} \|\partial_{x^-} Q_+\| + \frac{\left| \sqrt{\chi/n} - \sqrt{\chi/m} \right|}{\sqrt{|w+x^-|}} \max_{z^+ \in [x_i^+, -w]} \|P_+\| + \\ & + 2\sqrt{|w+x^-|} \left| \sqrt{\chi/n} - \sqrt{\chi/m} \right| \max_{z^+ \in [x_i^+, -w]} \|\partial_{x^-} P_+\|. \quad (4.58) \end{aligned}$$

Once again, the right hand side of (4.57) becomes arbitrarily small for m, n high enough, which implies pointwise convergence.

To prove uniform convergence of the sequence of derivatives, we note that

$$\begin{aligned} & \sup_{x^- \in \mathcal{I}} \|\lim_{m \rightarrow \infty} [\partial_{x^-} a_m(x^-)] - \partial_{x^-} a_n(x^-)\| = \\ & = \sup_{x^- \in \mathcal{I}} \left\| \int_{-w-\chi/n}^{-w} dz_1^+ \int_{x_i^+}^{z_1^+} dz_2^+ \dots \int_{x_i^+}^{z_{k-1}^+} dz_k^+ \partial_{x^-} [\hat{J}_k \dots \hat{J}_1] \right\| \leq \\ & \leq \frac{\sup_{x^- \in \mathcal{I}}}{(k-1)!} \int_{-w-\chi/n}^{-w} dz_1^+ \|\hat{J}_1\| \int_{x_i^+}^{z_1^+} dz_2^+ \dots \int_{x_i^+}^{z_1^+} dz_k^+ \|\mathcal{P}[\partial_{x^-} \{\hat{J}_k \dots \hat{J}_2\}]\| + \\ & \quad + \frac{\sup_{x^- \in \mathcal{I}}}{(k-1)!} \int_{-w-\chi/n}^{-w} dz_1^+ \|\partial_{x^-} \hat{J}_1\| \int_{x_i^+}^{z_1^+} dz_2^+ \dots \int_{x_i^+}^{z_1^+} dz_k^+ \|\mathcal{P}[\hat{J}_k \dots \hat{J}_2]\| \leq \\ & \leq \frac{\sup_{x^- \in \mathcal{I}}}{(k-2)!} \int_{-w-\chi/n}^{-w} dz_1^+ \|\hat{J}_1\| \sup_{x^- \in \mathcal{I}} \left(\int_{x_i^+}^{z_1^+} dz_2 \|\hat{J}_2\| \right)^{k-2} \int_{x_i^+}^{z_1^+} dz_3^+ \|\partial_{x^-} \hat{J}_3\| + \\ & \quad + \frac{\sup_{x^- \in \mathcal{I}}}{(k-1)!} \int_{-w-\chi/n}^{-w} dz_1^+ \|\partial_{x^-} \hat{J}_1\| \sup_{x^- \in \mathcal{I}} \left(\int_{x_i^+}^{z_1^+} dz_2 \|\hat{J}_2\| \right)^{k-1} \leq \\ & \leq \frac{1}{(k-2)!} \sup_{x^- \in \mathcal{I}} \left(\int_{-w-\chi/n}^{-w} dz_1^+ \|\hat{J}_1\| \right) \sup_{x^- \in \mathcal{I}} A^{k-2}(x^-) \sup_{x^- \in \mathcal{I}} B(x^-) + \\ & \quad + \frac{1}{(k-1)!} \sup_{x^- \in \mathcal{I}} \left(\int_{-w-\chi/n}^{-w} dz_1^+ \|\partial_{x^-} \hat{J}_1\| \right) \sup_{x^- \in \mathcal{I}} A^{k-1}(x^-), \end{aligned}$$

⁶unambiguously except on sets of measure zero

where

$$A(x^-) := \int_{x_i^+}^{-w} \|\hat{J}\| dz, \quad B(x^-) := \int_{x_i^+}^{-w} \|\partial_{x^-} \hat{J}\| dz. \quad (4.59)$$

From the previous proposition and its proof we know that the integrals over dz_1 from $-w - \chi/n$ to $-w$ will go to zero for $n \rightarrow \infty$ while the other factors remain finite. Thus the sequence of derivatives converges uniformly on \mathcal{I} and again by the standard theorem on the derivative of the limit of a sequence [16], the proposition is proved. \square

We arrive at the proposition, which allows us to define the continuous extension to ℓ^+ of a solution to the linear system.

Proposition 4.4.2. *Define*

$$\hat{\mathcal{V}}_{\ell_w^+}(x^-) = \mathcal{P}e^{\int_{x_i^-}^{x^-} \hat{J}_-(z^-, x_i^+) dz^-} \mathcal{P}e^{\int_{x_i^+}^{-w} \hat{J}_+(x^-, z^+) dz^+}, \quad (4.60)$$

where

$$\mathcal{P}e^{\int_{x_i^+}^{-w} \hat{J}_+(x^-, z^+) dz^+} = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k!} \lim_{m \rightarrow \infty} \int_{x_i^+}^{-w-\chi/m} dz_1 \dots \int_{x_i^+}^{-w-\chi/m} dz_k \mathcal{P}[\hat{J}_1 \dots \hat{J}_k]. \quad (4.61)$$

Then

1. $\hat{\mathcal{V}}_{\ell_w^+}$ exists (is finite),
2. the two limits in its definition (4.61) may be interchanged,
3. its x^- -derivative exists for all $x^- \in \mathcal{I}$ and
4. it satisfies the x^- -component of the linear system (4.31).

Proof. 1. The first factor in (4.60) exists by the results in appendix D. For the second factor, we consider the sequence of partial sums

$$\sum_{k=0}^n \frac{1}{k!} \lim_{m \rightarrow \infty} \int_{x_i^+}^{-w-1/m} dz_1 \dots \int_{x_i^+}^{-w-1/m} dz_k \mathcal{P}[\hat{J}_1 \dots \hat{J}_k],$$

where $\hat{J}_k := \hat{J}_+(x^-, z_k^+)$ as before. We know from the previous propositions and proofs that the limit $m \rightarrow \infty$ may be taken — the multiple integrals exist. Furthermore, for $m < n$

$$\begin{aligned} & \left\| \sum_{k=m}^n \frac{1}{k!} \int_{x_i^+}^{-w} dz_1 \dots \int_{x_i^+}^{-w} dz_k \mathcal{P}[\hat{J}_1 \dots \hat{J}_k] \right\| \leq \\ & \leq \sum_{k=m}^{\infty} \frac{1}{k!} \int_{x_i^+}^{-w} dz_1 \dots \int_{x_i^+}^{-w} dz_k \|\mathcal{P}[\hat{J}_1 \dots \hat{J}_k]\| \leq \\ & \leq \sum_{k=m}^{\infty} \frac{1}{k!} A^k(x^-) = e^{A(x^-)} - \sum_{k=0}^{m-1} \frac{1}{k!} A^k(x^-), \end{aligned} \quad (4.62)$$

where $A(x^-)$ is given by (4.59). Since (4.62) is arbitrarily small for sufficiently high m, n , the path ordered exponential as an infinite series converges pointwise for all $x^- \in \mathcal{I}$.

2. We write the infinite sum as a Lebesgue-integral: Consider the natural numbers as a measurable space by taking the set of all subsets as sigma-algebra Σ . On this measurable space we define the counting measure $\mu_{\mathbb{N}}$ by

$$\begin{aligned} \mu_{\mathbb{N}}(X) &= |X|, & \text{if } X \text{ is finite, where } |X| \text{ is the number of elements of } X \text{ and} \\ \mu_{\mathbb{N}}(X) &= \infty, & \text{if } X \text{ is infinite.} \end{aligned}$$

Now consider

$$f_m(k) = \frac{1}{k!} \int_{x_i^+}^{-w-\eta/m} dz_1 \dots \int_{x_i^+}^{-w-\eta/m} dz_k \mathcal{P}[\hat{J}_1 \dots \hat{J}_k]$$

a sequence, indexed by m , of functions of the natural numbers (k acts as the variable). Then we may write

$$\sum_{k=0}^{\infty} \frac{1}{k!} \int_{x_i^+}^{-w-\chi/m} dz_1 \dots \int_{x_i^+}^{-w-\chi/m} dz_k \mathcal{P}[\hat{J}_1 \dots \hat{J}_k] = \int_{\mathbb{N}} f_m d\mu_{\mathbb{N}}.$$

We have

$$\|f_m(k)\| \leq \frac{1}{k!} A^k(x^-) =: g(k) \quad (4.63)$$

for all m . $g(k)$ is integrable over the natural numbers and thus by the Lebesgue dominated convergence theorem⁷ we have

$$\lim_{m \rightarrow \infty} \int_{\mathbb{N}} f_m d\mu_{\mathbb{N}} = \int_{\mathbb{N}} \lim_{m \rightarrow \infty} f_m d\mu_{\mathbb{N}},$$

or equivalently

$$\lim_{m \rightarrow \infty} \sum_{k=0}^{\infty} f_m(k) = \sum_{k=0}^{\infty} \lim_{m \rightarrow \infty} f_m(k).$$

There is one more subtlety: the theorem is stated for real valued functions and $f_m(k)$ is an $SL(2, \mathbb{C})$ -valued function. But $f_m(k)$ in (4.63) may be replaced by the real or imaginary part of any of its components, which are real-valued and so the theorem may applied separately to them.

3. Concerning its x^- -derivative, once again we consider the sequence of derivatives of the partial sums. By the previous propositions we have for $m < n$

$$\begin{aligned} & \left\| \sum_{k=m}^n \frac{1}{k!} \partial_{x^-} \int_{x_i^+}^{-w} dz_1 \dots \int_{x_i^+}^{-w} dz_k \mathcal{P}[\hat{J}_1 \dots \hat{J}_k] \right\| \leq \\ & \leq \sum_{k=m}^n \frac{1}{(k-1)!} A^{k-1}(x^-) B(x^-) \leq \\ & \leq \left\{ e^{A(x^-)} - \sum_{k=0}^{m-2} \frac{1}{k!} A^k(x^-) \right\} B(x^-), \end{aligned}$$

⁷see theorem 5.6 in [3]

which becomes arbitrarily small for sufficiently high m, n . The limit $n \rightarrow \infty$ of the sequence of derivatives exists and it converges uniformly for $x \in \mathcal{I}$ because

$$\begin{aligned} \sup_{x^- \in \mathcal{I}} \left\| \sum_{k=n+1}^{\infty} \frac{1}{k!} \partial_{x^-} \int_{x_i^+}^{-w} dz_1 \dots \int_{x_i^+}^{-w} dz_k \mathcal{P}[\hat{J}_1 \dots \hat{J}_k] \right\| &\leq \\ &\leq \left\{ e^{\sup_{x^- \in \mathcal{I}} A(x^-)} - \sum_{k=0}^n \frac{\sup_{x^- \in \mathcal{I}} A^k(x^-)}{k!} \right\} \sup_{x^- \in \mathcal{I}} B(x^-) \end{aligned}$$

becomes arbitrarily small if n is sufficiently high.

4. Together with the previous lemmas, we have all the necessary permissions to calculate

$$\begin{aligned} \partial_{x^-} \mathcal{P} e^{\int_{x_i^+}^{-w} \hat{J}_+(x^-, z^+) dz^+} &= \partial_{x^-} \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k!} \lim_{m \rightarrow \infty} \int_{x_i^+}^{-w-1/m} dz_1 \dots \int_{x_i^+}^{-w-1/m} dz_k \mathcal{P}[\hat{J}_1 \dots \hat{J}_k] = \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k!} \partial_{x^-} \lim_{m \rightarrow \infty} \int_{x_i^+}^{-w-1/m} dz_1 \dots \int_{x_i^+}^{-w-1/m} dz_k \mathcal{P}[\hat{J}_1 \dots \hat{J}_k] = \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k!} \lim_{m \rightarrow \infty} \partial_{x^-} \int_{x_i^+}^{-w-1/m} dz_1 \dots \int_{x_i^+}^{-w-1/m} dz_k \mathcal{P}[\hat{J}_1 \dots \hat{J}_k]. \end{aligned} \quad (4.64)$$

Applying again the dominated convergence theorem to the sequence of functions $\partial_{x^-} f_m(k)$ with

$$\begin{aligned} \|\partial_{x^-} f_m(k)\| &= \left\| \frac{1}{k!} \int_{x_i^+}^{-w-\eta/m} dz_1 \dots \int_{x_i^+}^{-w-\eta/m} dz_k \partial_{x^-} \mathcal{P}[\hat{J}_1 \dots \hat{J}_k] \right\| \leq \\ &\leq \frac{1}{(k-1)!} A^{k-1}(x^-) B(x^-) =: g'(x^-) \end{aligned}$$

allows to exchange the two limits in the last line of (4.64) and we get

$$\partial_{x^-} \mathcal{P} e^{\int_{x_i^+}^{-w} \hat{J}_+(x^-, z^+) dz^+} = \lim_{m \rightarrow \infty} \sum_{k=0}^{\infty} \frac{1}{k!} \partial_{x^-} \int_{x_i^+}^{-w-1/m} dz_1 \dots \int_{x_i^+}^{-w-1/m} dz_k \mathcal{P}[\hat{J}_1 \dots \hat{J}_k]. \quad (4.65)$$

Similar to the calculations in the appendix D, (4.65) is

$$\begin{aligned} \partial_{x^-} \mathcal{P} e^{\int_{x_i^+}^{-w} \hat{J}_+(x^-, z^+) dz^+} &= \\ &= \lim_{m \rightarrow \infty} \int_{x_i^+}^{-w-1/m} \mathcal{P} e^{\int_{x_i^+}^{z_1^+} \hat{J}_+(x^-, z_1^+) dz_1^+} \partial_{x^-} \hat{J}_+(x^-, z^+) \mathcal{P} e^{\int_{z_1^+}^{-w-1/m} \hat{J}_+(x^-, z_2^+) dz_2^+} dz^+. \end{aligned} \quad (4.66)$$

The integrand is evaluated on points off ℓ^+ only, where the zero-curvature equation (4.22) is valid and tells us that

$$\partial_{x^-} \hat{J}_+(x^-, z^+) = \partial_{z^+} \hat{J}_-(x^-, z^+) + [\hat{J}_+(x^-, z^+), \hat{J}_-(x^-, z^+)].$$

Also, since the path ordered exponential solves the linear system (4.25) we have

$$\partial_{z^+} \mathcal{P} e^{\int_{x_i^+}^{z_1^+} \hat{J}_+(x^-, z_1^+) dz_1^+} = \mathcal{P} e^{\int_{x_i^+}^{z_1^+} \hat{J}_+(x^-, z_1^+) dz_1^+} \hat{J}_+(x^-, z^+)$$

and

$$\partial_{z^+} \mathcal{P} e^{\int_{z^+}^{-w-1/m} \hat{J}_+(x^-, z_2^+) dz_2^+} = -\hat{J}_+(x^-, z^+) \mathcal{P} e^{\int_{z^+}^{-w-1/m} \hat{J}_+(x^-, z_2^+) dz_2^+}.$$

Thus

$$\begin{aligned} \partial_{x^-} \mathcal{P} e^{\int_{x_i^+}^{-w} \hat{J}_+(x^-, z^+) dz^+} &= \\ &= \lim_{m \rightarrow \infty} \int_{x_i^+}^{-w-1/m} \partial_{z^+} \left\{ \mathcal{P} e^{\int_{x_i^+}^{z_1^+} \hat{J}_+(x^-, z_1^+) dz_1^+} \hat{J}_-(x^-, z^+) \mathcal{P} e^{\int_{z^+}^{-w-1/m} \hat{J}_+(x^-, z_2^+) dz_2^+} dz^+ \right\} = \\ &= \mathcal{P} e^{\int_{x_i^+}^{-w} \hat{J}_+(x^-, z_1^+) dz_1^+} \hat{J}_-(x^-, -w) - \hat{J}_-(x^-, x_i^+) \mathcal{P} e^{\int_{x_i^+}^{-w} \hat{J}_+(x^-, z_2^+) dz_2^+}. \end{aligned}$$

Finally, since

$$\partial_{x^-} \mathcal{P} e^{\int_{x_i^+}^{x^-} \hat{J}_-(z^-, x_i^+) dz^-} = \mathcal{P} e^{\int_{x_i^+}^{x^-} \hat{J}_-(z^-, x_i^+) dz^-} \hat{J}_-(x^-, x_i^+)$$

the x^- -derivative of $\hat{\mathcal{V}}_{\ell_w^+}(x^-)$, (4.60), is

$$\partial_{x^-} \hat{\mathcal{V}}_{\ell_w^+}(x^-) = \hat{\mathcal{V}}_{\ell_w^+}(x^-) \hat{J}_-(x^-, -w) \quad (4.67)$$

and thus indeed it satisfies the limit of the x^- -component of the linear system on ℓ^+ . □

At this point we introduce the transport matrix

$$T(y, x; w) := \mathcal{P} e^{\int_y^x \hat{J}(w)} \quad (4.68)$$

for two points y and x inside of \mathcal{D}_I . It is the solution $\hat{\mathcal{V}}$ of the linear system in \mathcal{D}_I , which equals $\mathbb{1}$ at the point y . By the zero curvature equation (4.22) the transport matrix is independent of the path chosen. ‘‘Transport’’ because if we know $\hat{\mathcal{V}}(y, w)$, then $\hat{\mathcal{V}}(x, w)$ is given by

$$\hat{\mathcal{V}}(x, w) = \hat{\mathcal{V}}(y, w) T(y, x; w).$$

Let w still be real and $x_0^+ < -w < x_1^+$. The transport matrix may be extended to ℓ^+ . Let $y \in \mathcal{D}_I$ and $x = (x^-, -w) \in \ell_w^+$ and define

$$T(y, x; w) = T(y, x_i; w) \mathcal{P} e^{\int_{x_i^+}^{x^-} \hat{J}_-(z^-, x_i^+) dz^-} \mathcal{P} e^{\int_{x_i^+}^{-w} \hat{J}_+(x^-, z^+) dz^+}, \quad (4.69)$$

where, as before, x_i must be such that (x^-, x_i^+) lies inside of \mathcal{D}_I . Note that at this point, we have defined that at least the last part of the transport of $\hat{\mathcal{V}}$ to points on ℓ_w^+ must be along a line of constant x^- .

It is pretty clear, that the above procedure can be applied to extend a solution of the linear system inside of \mathcal{D}_{II} to ℓ_w^+ and ℓ_w^- and a solution in \mathcal{D}_{III} to ℓ_w^- . The path ordered exponential along lines with constant x^+ or x^- from a point in \mathcal{D}_i to one of the line segments ℓ_w^\pm exists and on these lines satisfies the corresponding components of the linear system (4.31). If y is in \mathcal{D}_{II} and x on ℓ_w^+ we define

$$T(y, x; w) = T(y, x_i; w) \mathcal{P} e^{\int_{x_i^+}^{x^-} \hat{J}_-(z^-, x_i^+) dz^-} \mathcal{P} e^{\int_{x_i^+}^{-w} \hat{J}_+(x^-, z^+) dz^+},$$

where now $x_i \in \mathcal{D}_{II}$ must be such that (x^-, x_i^+) lies inside of \mathcal{D}_{II} .

Analogously, for $y, x_i \in \mathcal{D}_{II}$ and $x = (-w, x^+)$ on ℓ_w^- , we define

$$T(y, x; w) = T(y, x_i; w) \mathcal{P}e^{\int_{x_i^+}^{x^+} \hat{J}_+(x_i^-, z^+) dz^+} \mathcal{P}e^{\int_{x_i^-}^{-w} \hat{J}_-(z^-, x^+) dz^-}.$$

For transport across ℓ^\pm , we set

$$T(y, x; w) = T(y, z; w)T(z, x; w),$$

where $z \in \ell_w^+$ if $y \in \mathcal{D}_I$ and $x \in \mathcal{D}_{II}$ and $z \in \ell_w^-$ if $y \in \mathcal{D}_{II}$ and $x \in \mathcal{D}_{III}$.

As mentioned above, inside of \mathcal{D}_i , the transport matrices between two points are path independent and continuous, that is, for fixed y , $T(y, x; w)$ is a continuous function of x . The continuous extension of the transport matrices to points on ℓ_w^\pm was defined along paths, which near ℓ_w^\pm have constant x^\mp . We now show, that this extension is indeed not only continuous along these paths, but along any path.

Proposition 4.4.3. *For real w with $x_0^+ < -w < x_1^-$, let y be a point on ℓ_w^+ and D_r be a disk with radius r centred at y . Then for every $\epsilon > 0$ there is a value $r > 0$ such that*

$$\|T(y, x; w) - \mathbb{1}\| < \epsilon \quad (4.70)$$

for all points x in D_r .

Proof. We distinguish two cases.

1. First, let $x \in D_r$ lie in \mathcal{D}_I or \mathcal{D}_{II} , not on ℓ_w^+ . Introduce the point $\tilde{x} = (y^-, x^+)$. Then we have

$$T(y, x; w) = \mathcal{P}e^{\int_{-w}^{x^+} \hat{J}_+(y^-, z^+) dz^+} \mathcal{P}e^{\int_{y^-}^{x^-} \hat{J}_-(z^-, x^+) dz^-}$$

and

$$\|T(y, x; w) - \mathbb{1}\| \leq e^{|\int_{-w}^{x^+} \|\hat{J}_+(y^-, z^+)\| dz^+|} e^{|\int_{y^-}^{x^-} \|\hat{J}_-(z^-, x^+)\| dz^-|} - 1. \quad (4.71)$$

Since J_μ is assumed to be non-singular throughout \mathcal{D} , we define

$$Q := \max_{\mathcal{D}, \mu \in \{0,1\}} \|Q_\mu\|, \quad P := \max_{\mathcal{D}, \mu \in \{0,1\}} \|P_\mu\|.$$

For the exponents we get

$$\begin{aligned} \left| \int_{-w}^{x^+} \|\hat{J}_+(y^-, z^+)\| dz^+ \right| &\leq |w + x^+|Q + 2|\sqrt{w + y^-}| \left| \int_0^{\sqrt{|w+x^+|}} d(\sqrt{|w+z^+|}) \right| P \leq \\ &\leq \sqrt{r} \left\{ Q + 2|\sqrt{w + y^-}|P \right\}, \end{aligned} \quad (4.72)$$

where we assumed $r < 1$ such that $\sqrt{r} > r$. Furthermore

$$\begin{aligned} \left| \int_{y^-}^{x^-} \|\hat{J}_-(z^-, x^+)\| dz^- \right| &\leq |y^- - x^-|Q + P \left| \int_{y^-}^{x^-} \frac{\sqrt{w+x^+}}{\sqrt{w+z^-}} dz^- \right| P \leq \\ &\leq \sqrt{r} \left\{ Q + 2|\sqrt{w+x^+}|(\sqrt{w+x^-} - \sqrt{w+y^-})|P \right\}. \end{aligned} \quad (4.73)$$

Hence,

$$\|T(y, x; w) - \mathbb{1}\| \leq e^{\sqrt{r}\{Q+2|\sqrt{w+y^-}|P\}} e^{r\left\{Q+P\frac{\sqrt{r}}{\sqrt{w+y^-+r}}\right\}} - 1, \quad (4.74)$$

which, by choosing r sufficiently small, will be smaller than ϵ .

2. Now let $x \in \ell_w^+$. By the construction of the extension of $\hat{\mathcal{V}}$ to ℓ^+ , $\hat{\mathcal{V}}$ at a point $x = (x^-, -w)$ on ℓ^+ is obtained by integrating the linear system to a point $(x^-, -w - \epsilon)$ and then taking the limit $\epsilon \rightarrow 0$. We have proofed above that $\hat{\mathcal{V}}$ defined in this way satisfies the x^- -component of the linear system. On the other hand, for two points $(x^-, -w)$ and $(y^-, -w)$ on ℓ^+ , consider

$$\hat{\mathcal{V}}(x^-, -w, w) := \hat{\mathcal{V}}(y^-, -w, w) \mathcal{P} e^{\int_{y^-}^{x^-} \hat{J}_-(z^-, -w, w) dz^-}.$$

This matrix also satisfies

$$\partial_{x^-} \hat{\mathcal{V}}(x^-, -w, w) = \hat{\mathcal{V}}(x^-, -w, w) \hat{J}(x^-, -w, w)$$

and for $y = x$ is equal to $\hat{\mathcal{V}}$. Hence the two are equal. For the transport matrices, this means that for two points x and y on ℓ_w^+ , we can transport $\hat{\mathcal{V}}$ directly along ℓ_w^+ using the path ordered exponential and get the same result as we get when transporting as in (4.69). For $x, y \in \ell_w^+ \cap D_r$ we thus get

$$T(y, x; w) = \mathcal{P} e^{\int_{y^-}^{x^-} \hat{J}_-(z^-, -w) dz^-} = \mathcal{P} e^{\int_{y^-}^{x^-} Q_-(z^-, -w) dz^-},$$

because $\hat{J}_-(z^-, -w) = Q_-(z^-, -w) dz^-$ and

$$\|T(y, x; w) - \mathbb{1}\| \leq e^{|\int_{y^-}^{x^-} Q_-(z^-, -w) dz^-|} - 1 \leq e^{rQ} - 1, \quad (4.75)$$

which will also be arbitrarily small for r sufficiently small and the proposition is proofed. □

An analogous proposition holds for ℓ_w^- . Thus, also for real w , we can construct a solution $\hat{\mathcal{V}}(x, w)$, continuous in x for $x \in \mathcal{D} \setminus (\ell_w^+ \cap \ell_w^-)$, which satisfies the (components of the) linear system everywhere, where it is valid. If we set

$$\hat{\mathcal{V}}(x, w) = T(y, x; w),$$

it satisfies the initial condition $\hat{\mathcal{V}}(x, w) = \mathbb{1}$. A different initial condition would be for example to set $\hat{\mathcal{V}}(y, w) = \mathcal{V}(y)$ by

$$\hat{\mathcal{V}}(x, w) = \mathcal{V}(y) T(y, x; w).$$

In section 5 we will use such an initial condition to define $\hat{\mathcal{V}}_0$ to be the solution, which equals \mathcal{V} on a point on the axis $\rho = 0$.

Our last task is an analysis of the behaviour of $\hat{\mathcal{V}}$ near the point $y_{ax} \in (\ell_w^+ \cap \ell_w^-)$ on the axis⁸. First, we note that at all other points on the axis, by the results of section 4.3.3, we have $\gamma = 0$ or $\gamma = \infty$, depending on the sheet and thus

$$\hat{J}_\mu(\rho = 0, \tilde{\rho}, w) = J_\mu(\rho = 0, \tilde{\rho}) \quad \text{or} \quad \hat{J}_\mu(\rho = 0, \tilde{\rho}, w) = -J_\mu^T(\rho = 0, \tilde{\rho}) \quad \text{for } \tilde{\rho} \neq -w. \quad (4.76)$$

⁸if there is such a point

Thus along the axis $\rho = 0$, the connection is not singular, but simply has a discontinuity at y_{ax} . For the transport matrices, this means that for points x on the axis approaching y_{ax} ,

$$\lim_{x \rightarrow y_{ax}} T(y, x; w),$$

certainly exists. Similar as before, we define the transport matrix to the point y_{ax} to use a path, which at least in its last part is along the axis $\rho = 0$. Then continuity can be proofed as before. One can make the radius of a half-disc D_r of radius r around y_{ax} so small, that for every point x in D_r the norm $\|T(x, y_{ax}; w) - \mathbb{1}\| < \epsilon$. The upper bounds for the integrals (4.73) and (4.72) are valid in all of \mathcal{D} .

Thus finally, we have proofed that for every w , there exists a matrix $\hat{\mathcal{V}}(x, w)$, which is continuous on all of \mathcal{D} and satisfies the linear system or one of its components everywhere, where they can be defined.

4.4.2 Continuity with respect to the spectral parameter

Above we kept w fixed and investigated the properties of a solution $\hat{\mathcal{V}}$ of the linear system with respect to spacetime points x , in particular near the the points where the connection \hat{J} is singular. Now we keep x fixed and study continuity and analyticity properties with respect to w . $\hat{\mathcal{V}}(x)$ is obtained from $\hat{\mathcal{V}}(x_0)$ by integration of the linear system along a suitable path connecting the two points. It turns out that if $\hat{\mathcal{V}}(x_0) = \mathcal{V}$ for all w on the first sheet W_+ of the Riemann surface (i.e. $|\gamma| \leq 1$), then for fixed x (and x_0) $\hat{\mathcal{V}}(x, w)$ is analytic on that entire sheet of the Riemann surface excluding the branch cut, the segment of the real axis corresponding to $|\gamma| = 1$, where it is still has continuous limits.

The domain of analyticity

We have already mentioned in proposition 4.4.1 that, separately, in the upper and lower half plane of any of the sheets of the Riemann surface without the real line the solution $\hat{\mathcal{V}}$ is analytic in w , if the constants of integration do not introduce singularities. This follows from theorem (10.3) in chapter II of [27] because for such w the connection \hat{J} is analytic in w . For real w , the theorem has to be applied with more care.

Fix a point x_0 , where the initial condition for $\hat{\mathcal{V}}$ is posed for all $|\gamma| \leq 1$, and a point x , and consider a smooth curve $x(t)$ such that $x(0) = x_0$ and $x(1) = x$. The curve defines a pull-back of \hat{J}_μ to the real interval $[0, 1]$ by

$$\hat{J}(t, w) = \dot{x}^\mu(t) \hat{J}_\mu(x(t), w).$$

A solution $\hat{\mathcal{V}}(t, w)$ along this curve must satisfy

$$\begin{aligned} \partial_t \hat{\mathcal{V}}(t, w) &= \hat{\mathcal{V}}(t, w) \dot{x}^\mu(t) \hat{J}_\mu(x(t), w) = \\ &= \hat{\mathcal{V}}(t, w) \dot{x}^\mu(t) \left(Q_\mu(t) + \frac{1 + \gamma^2(t, w)}{1 - \gamma^2(t, w)} P_\mu - \frac{2\gamma(t, w)}{1 - \gamma^2(t, w)} \varepsilon_{\mu\nu} P^\nu(t) \right). \end{aligned}$$

The right hand side depends analytically on $\hat{\mathcal{V}}$, continuously on t and analytically on w as long as w does not lie on the branch points corresponding to t , i.e. as long as $x^\pm(t) \neq -w$. Denote by \mathfrak{D} the subset of the product of the real interval $[0, 1]$ with the sheet W_+ ,

where \hat{J} is regular, i.e. those pairs $(t, w), w \in W_+$ for which $x^\pm(t) \neq -w$. Theorem (10.3) in chapter II of [27] now tell us that the solution $\hat{\mathcal{V}}$ will depend analytically on w for those points w , for which $(0, w) \in \mathfrak{D}$ and that this solution can be extended to the value $t = 1$ if

$$\{(t, w) \mid t \in [0, 1]\} \subset \mathfrak{D}.$$

In other words, for fixed x (and x_0), the solution $\hat{\mathcal{V}}(x, w)$ is analytic in w for those points w , for which a curve connecting x and x_0 and avoiding the singularities at the branch points can be found.

Of course, for a fixed curve the possibilities for avoiding the singularities are very limited. As t goes from 0 to 1, the movements of the branch points on the real axis of W_+ cover at least the intervals $[-x_0^+, -x^+]$ and $[-x_0^-, -x^-]$. The only thing one can do is to avoid detours enlarging those intervals. A straight line segment is optimal.

Thus the solution $\hat{\mathcal{V}}(x, w)$ is analytic on W_+ without the segment of the real axis corresponding to the intervals $[-x_0^+, -x^+]$ and $[-x_0^-, -x^-]$.

The domain of continuity

Above we have seen that $\hat{\mathcal{V}}(x, w)$ is analytic for $w \in W_+$ except at some parts of the real axis. We now give a sketch of the proof that $\hat{\mathcal{V}}$ can be extended continuously to these segments from W_{+u} and W_{+d} . A more detailed proof can be found in various parts in [18].

First, to get a solution $\hat{\mathcal{V}}(x, w)$ one can always choose a curve in \mathcal{D} , which consists of two null pieces, i.e. with $x^+ = \text{const.}$ and $x^- = \text{const.}$, respectively, on the curve. If we use the initial condition $\hat{\mathcal{V}}(x_0, w) = \mathcal{V}$ then $\hat{\mathcal{V}}(x, w)$ can be written as the product

$$\hat{\mathcal{V}}(x, w) = \mathcal{V}(x_0)T(x_0, z; w)T(z, x; w). \quad (4.77)$$

Denote by $T_\pm(y, \tilde{y}; w)$ the transport matrix along that piece of the curve, for which $x^\mp = \text{const.}$, that is $y^\mp = \tilde{y}^\mp$. Then $T_\pm(y, \tilde{y}; w)$ satisfies the differential equation and initial condition

$$\partial_\pm T = T\hat{J}_\pm, \quad T(y, y; w) = \mathbb{1},$$

except, if w is real and $-w \in [y^\pm, \tilde{y}^\pm]$, at the point $x^\pm = -w$, where it is continuous in \tilde{y} (see section 4.4.1). This is equivalent to the integral equation

$$T_\pm(y, \tilde{y}; w) = \mathbb{1} + \int_{y^\pm}^{\tilde{y}^\pm} dz^\pm T_\pm(y, z; w)\hat{J}_\pm(z, w). \quad (4.78)$$

We now give a useful lemma and the sketch of its proof.

Lemma 4.4.3. *Let $g(x, y, w)$ be a continuous complex valued function defined for arbitrary x and y , and for all $w \in W_{+u/d}$. Define*

$$f(x, y, w) := \int_x^y \frac{1}{\sqrt{w+z}} g(x, z, w) dz. \quad (4.79)$$

Then f exists and is continuous in w for $W \in W_{+u/d}$.

Proof. First take $w \neq \infty$.

From lemma 4.4.1 we know that the integral exists, even if $-w \in [x, y]$ and the integrand is thus singular at $z = -w$. We now proof continuity with respect to w . By the multi-variable Weierstrass approximation theorem [36], g being a continuous function can be represented as the limit of a uniformly convergent sequence (g_n) of polynomials in z and w for $z \in [x, y]$ and w in a compact rectangular subspace, $\mathcal{R} \subset W_{+u/d}$, of the Riemann surface. Analogously, define the sequence of integrals

$$f_n(x, y, w) := \int_x^y \frac{1}{\sqrt{w+z}} g_n(x, z, w) dz. \quad (4.80)$$

Then by the uniform convergence of the sequence of polynomials, for every $\epsilon > 0$ there is an $N(\epsilon)$ such that for all $n > N$ and points $(x, z, w) \in \{x\} \times [x, y] \times \mathcal{R}$ we have

$$|g - g_n| < \epsilon.$$

Therefore, for all $n > N$ and $w \in \mathcal{R}$

$$\begin{aligned} |f(x, y, w) - f_n(x, y, w)| &\leq \text{sign}(x - y) \int_x^y \frac{1}{|\sqrt{w+z}|} |g(x, z, w) - g_n(x, z, w)| dz \leq \\ &\leq \epsilon \text{sign}(x - y) \int_x^y \frac{1}{|\sqrt{w+z}|} dz \leq \epsilon D, \end{aligned}$$

with D some positive constant, which is finite due to the integrability of the inverse power of the square root. Thus, on \mathcal{R} f_n converges uniformly to f . Now, f_n can be written as

$$\begin{aligned} f_n(x, y, w) &= \int_x^y \frac{1}{\sqrt{w+z}} [g_n(x, z, w) - g_n(x, w, w)] dz + 2[\sqrt{w+y} - \sqrt{w-x}] g_n(x, w, w) = \\ &= \int_x^y \sqrt{w+z} \frac{g_n(x, z, w) - g_n(x, w, w)}{w+z} dz + 2[\sqrt{w+y} - \sqrt{w-x}] g_n(x, w, w). \end{aligned}$$

The quotient in the second line is a polynomial because the division can be carried out without remainder:

$$z^n - w^n = (z^{n-1} - wz^{n-2} + w^2z^{n-3} - \dots + (-1)^n w^{n-2}z - w^{n-1})(z + w).$$

Thus, the singularity can be removed, f_n is a sum of terms continuous in w for $w \in \mathcal{R}$ and since a uniformly convergent sequence of continuous functions is continuous, f is continuous.

Now, for continuity at $w = \infty$, introduce $\tilde{w} = w^{-1}$. Then

$$f_n(x, y, \tilde{w}) = \int_x^y \frac{\sqrt{\tilde{w}}}{\sqrt{1+z\tilde{w}}} g_n(x, z, \tilde{w}) dz.$$

Clearly, the integrand is continuous at $\tilde{w} = 0$. Therefore also $f_n(x, y, \tilde{w})$ is continuous at $\tilde{w} = 0$ and $f_n(x, y, w)$ is continuous at $w = \infty$. \square

We are now in a position to proof

Proposition 4.4.4. *The solution of the integral equation (4.78) is continuous in w for $w \in W_{+u/d}$.*

Proof. One tries to iterate the solution by a sequence defined recursively

$$T_{\pm}^{(0)}(y, \tilde{y}; w) = \mathbb{1}, \quad T_{\pm}^{(n+1)}(y, \tilde{y}; w) = \mathbb{1} + \int_{y^{\pm}}^{\tilde{y}^{\pm}} dz^{\pm} T_{\pm}^{(n)}(y, z; w) \hat{J}_{\pm}(z, w). \quad (4.81)$$

If the integrand is regular, that is if $-w \notin [y^{\pm}, \tilde{y}^{\pm}]$ the limit of the sequence is just the path ordered exponential, see appendix D. If $-w \in [y^{\pm}, \tilde{y}^{\pm}]$ the integration domain can be split into two intervals $[y^{\pm}, -w]$ and $[-w, \tilde{y}^{\pm}]$. $T^{(n)}$ is then just the product of two factors, where each factor is of the form of the n -th partial sum in (4.61). In proposition 4.4.2 we showed that the sequence of partial sums converges uniformly. Therefore also the sequence of products of two partial sums of that kind converges uniformly and thus the sequence (4.81) converges uniformly.

Furthermore, since $T^{(0)} = \mathbb{1}$ is continuous, repeated application of the above lemma 4.4.3 yields that every element $T^{(n)}$ is continuous in w . And since $T^{(n)}$ converges uniformly, also the limit $\lim_{n \rightarrow \infty} T^{(n)}$ is continuous.

Clearly, $\lim_{n \rightarrow \infty} T^{(n)}$ satisfies the integral equation (4.78). \square

Therefore, all the factors in (4.77) are continuous in w for $w \in W_{+u/d}$. Together with the analyticity properties we have showed: $\hat{\mathcal{V}}(x, w)$ is analytic on W_+ without the real line segments corresponding to the intervals $[-x_0^+, -x^+]$ and $[-x_0^-, -x^-]$ where it has continuous limits. Now consider any point p on these line segments, which does not lie on the branch cut, and an open disc D_r centred at p with radius r so small that the branch points do not lie in D_r . Denote by s the intersection of the real line with D_r . Now take an arbitrary triangle $\Delta \subset D_r$ and consider the contour integral

$$\int_{\partial\Delta} \hat{\mathcal{V}}(x, w) dw. \quad (4.82)$$

If Δ is such that s does not split Δ into two halves, then $\hat{\mathcal{V}}$ is holomorphic in Δ and the integral vanishes. If s splits Δ into two triangles Δ_1 and Δ_2 , the integral can be written as

$$\int_{\partial\Delta_1} \hat{\mathcal{V}}(x, w) dw + \int_{\partial\Delta_2} \hat{\mathcal{V}}(x, w) dw,$$

because in this expression the integrals over s contained in $\partial\Delta_1$ and $\partial\Delta_2$ cancel. But these two integrals also vanish because $\hat{\mathcal{V}}$ is holomorphic inside of Δ_1 and Δ_2 . Hence, (4.82) vanishes for all triangles $\Delta \in \mathcal{D}_r$ and by Morera's theorem [16], $\hat{\mathcal{V}}$ is analytic in D_r . Since there is such a D_r for any point not on the branch cut, $\hat{\mathcal{V}}$ is analytic on the entire real line except on the branch cut, and thus it is analytic on W_+ except on the branch cut corresponding to $|\gamma| = 1$.

4.4.3 A Hölder condition for $\hat{\mathcal{V}}$

We state without proof that for real w the transport matrices $T(x, y(w); w)$ with $y(w) = (-w, y^+)$, $x = (x^-, y^+)$ or $y(w) = (y^-, -w)$, $x = (y^-, x^+)$ satisfy a Hölder condition of index $1/2$ in w on any closed subinterval of the segments of the real axis corresponding to $[-x_0^-, -x_1^-]$ or $[-x_0^+, -x_1^+]$, that is

$$\|T(x, y(w); w) - T(x, y(v); v)\| \leq C|w - v|^{1/2}, \quad (4.83)$$

C being a constant.

This is theorem 4F.5 in [18], which is also where the proof can be found.

4.5 The extended Kramer-Neugebauer transformation

We now proceed with the revelation of the Geroch group. In section 2.3 we introduced the Kramer-Neugebauer transformation (2.34), which enabled us to transform \mathcal{V} to $\tilde{\mathcal{V}}$ and vice versa. Following [9] we will now look for an extended version K^∞ of this transformation that transforms $\hat{\mathcal{V}}$ to $\hat{\tilde{\mathcal{V}}}$ and vice versa.

We shall work in triangular gauge. Recall that $\hat{\mathcal{V}}$ is said to be in triangular gauge iff it is regular in γ at $\gamma = 0$, i.e. at $W_+ \ni w = \infty$, and $\mathcal{V} = \hat{\mathcal{V}}|_{\gamma=0}$ is upper triangular. The form

$$\begin{pmatrix} a + \mathcal{O}(\gamma) & b + \mathcal{O}(\gamma) \\ c\gamma + \mathcal{O}(\gamma^2) & d + \mathcal{O}(\gamma) \end{pmatrix} \quad (4.84)$$

for $\hat{\mathcal{V}}$ assures that in the limit $W_+ \ni w \rightarrow \infty$ ($\gamma \rightarrow 0$), $\hat{\mathcal{V}}$ is upper triangular. We will now compute the components of $\hat{\mathcal{V}}(x, \gamma)$ and $\hat{\tilde{\mathcal{V}}}(x, \gamma)$ to the order indicated in (4.84) to get an idea what K^∞ might look like.

The coefficient of P_\pm on the right hand side of the linear system in null coordinates, (4.31), is

$$\frac{1 \mp \gamma}{1 \pm \gamma} = 1 \mp 2\gamma + \mathcal{O}(\gamma^2),$$

so the right hand side is

$$\hat{J}_\pm = J_\pm \mp 2\gamma P_\pm + \mathcal{O}(\gamma^2).$$

The left hand side may be obtained to the desired order by expanding $\hat{\mathcal{V}}$ in γ :

$$\hat{\mathcal{V}} = \mathcal{V} + \gamma \hat{\mathcal{V}}_1 + \mathcal{O}(\gamma^2) \quad \Rightarrow \quad \hat{\mathcal{V}}^{-1} = \mathcal{V}^{-1} - \gamma \mathcal{V}^{-1} \hat{\mathcal{V}}_1 \mathcal{V}^{-1} + \mathcal{O}(\gamma^2).$$

The linear system (4.31) to the order γ^2 thus becomes

$$\begin{aligned} \hat{\mathcal{V}}^{-1} \partial_\pm \hat{\mathcal{V}} &= (\mathcal{V}^{-1} - \gamma \mathcal{V}^{-1} \hat{\mathcal{V}}_1 \mathcal{V}^{-1}) (\partial_\pm \mathcal{V} + \partial_\pm \gamma \hat{\mathcal{V}}_1 + \gamma \partial_\pm \hat{\mathcal{V}}_1) + \mathcal{O}(\gamma^2) = \\ &= J_\pm + \gamma \rho^{-1} \partial_\pm \rho \mathcal{V}^{-1} \hat{\mathcal{V}}_1 + \gamma \mathcal{V}^{-1} \partial_\pm \hat{\mathcal{V}}_1 - \gamma \mathcal{V}^{-1} \hat{\mathcal{V}}_1 J_\pm + \mathcal{O}(\gamma^2), \end{aligned}$$

where we have used

$$\gamma^{-1} \partial_\pm \gamma = \frac{1 \mp \gamma}{1 \pm \gamma} \rho^{-1} \partial_\pm \rho, \quad (4.85)$$

which is a consequence of (4.27). This implies that

$$\partial_\pm \gamma = \gamma \rho^{-1} \partial_\pm \rho + \mathcal{O}(\gamma^2).$$

Hence, the first order term of (4.31) is

$$\mp 2P_\pm = \rho^{-1} \partial_\pm \rho \mathcal{V}^{-1} \hat{\mathcal{V}}_1 + \mathcal{V}^{-1} \partial_\pm \hat{\mathcal{V}}_1 - \mathcal{V}^{-1} \hat{\mathcal{V}}_1 J_\pm,$$

or equivalently

$$\mp 2\rho \mathcal{V} P_\pm \mathcal{V}^{-1} = \partial_\pm (\rho \hat{\mathcal{V}}_1 \mathcal{V}^{-1}). \quad (4.86)$$

If we want to compute $\hat{\mathcal{V}}$ to the order given in (4.84), then we are only interested in the 21-component, c , of $\hat{\mathcal{V}}_1$. By the triangularity of \mathcal{V}^{-1} the 21-component of (4.86) only

contains c and no other components of $\hat{\mathcal{V}}_1$. Using the definition of \mathcal{V} and the explicit expression for P given in section 2.3, we get

$$2\mathcal{V}P_{\pm}\mathcal{V}^{-1} = \begin{pmatrix} * & * \\ \Delta^{-2}\partial_{\pm}B & * \end{pmatrix}.$$

The 21-component of (4.86) becomes

$$\mp\rho\Delta^{-2}\partial_{\pm}B = \partial_{\pm}(\rho\Delta^{-1/2}c)$$

or, using (2.30) and (B.4),

$$-\partial_{\pm}\tilde{B} = \partial_{\pm}(\rho\Delta^{-1/2}c) \quad \Rightarrow \quad c = -\Delta^{1/2}\rho^{-1}\tilde{B},$$

where we absorbed the integration constant into \tilde{B} . Remember, \tilde{B} is determined by (2.29) only up to constant shifts. Therefore,

$$\hat{\mathcal{V}} = \begin{pmatrix} \Delta^{1/2} + \mathcal{O}(\gamma) & \Delta^{-1/2}B + \mathcal{O}(\gamma) \\ -\rho^{-1}\Delta^{1/2}\tilde{B}\gamma + \mathcal{O}(\gamma^2) & \Delta^{-1/2} + \mathcal{O}(\gamma) \end{pmatrix}. \quad (4.87)$$

The expression for $\hat{\tilde{\mathcal{V}}}$ can be obtained using the Kramer-Neugebauer transformation (2.34)

$$\hat{\tilde{\mathcal{V}}} = \begin{pmatrix} \tilde{\Delta}^{1/2} + \mathcal{O}(\gamma) & \tilde{\Delta}^{-1/2}\tilde{B} + \mathcal{O}(\gamma) \\ -\rho^{-1}\tilde{\Delta}^{1/2}\tilde{B}\gamma + \mathcal{O}(\gamma^2) & \tilde{\Delta}^{-1/2} + \mathcal{O}(\gamma) \end{pmatrix}.$$

Now we introduce the matrix

$$K(x) := \begin{pmatrix} 0 & -ix^{-1/2} \\ ix^{1/2} & 0 \end{pmatrix}$$

and compute

$$\begin{aligned} K\left(\frac{\gamma}{\rho}\right)\hat{\mathcal{V}}K(\gamma) &= \\ &= \begin{pmatrix} 0 & -i\gamma^{-1/2}\rho^{1/2} \\ i\gamma^{1/2}\rho^{-1/2} & 0 \end{pmatrix} \begin{pmatrix} \Delta^{1/2} + \mathcal{O}(\gamma) & \Delta^{-1/2}B + \mathcal{O}(\gamma) \\ -\rho^{-1}\Delta^{1/2}\tilde{B}\gamma + \mathcal{O}(\gamma^2) & \Delta^{-1/2} + \mathcal{O}(\gamma) \end{pmatrix} \begin{pmatrix} 0 & -i\gamma^{-1/2} \\ i\gamma^{1/2} & 0 \end{pmatrix} = \\ &= \begin{pmatrix} \sqrt{\frac{\rho}{\Delta}} + \mathcal{O}(\gamma) & \sqrt{\frac{\Delta}{\rho}}\tilde{B} + \mathcal{O}(\gamma) \\ -\frac{\gamma}{\sqrt{\rho\Delta}}B + \mathcal{O}(\gamma^2) & \sqrt{\frac{\Delta}{\rho}} + \mathcal{O}(\gamma) \end{pmatrix} = \begin{pmatrix} \tilde{\Delta}^{1/2} + \mathcal{O}(\gamma) & \tilde{\Delta}^{-1/2}\tilde{B} + \mathcal{O}(\gamma) \\ -\rho^{-1}\tilde{\Delta}^{1/2}\tilde{B}\gamma + \mathcal{O}(\gamma^2) & \tilde{\Delta}^{-1/2} + \mathcal{O}(\gamma) \end{pmatrix} = \hat{\tilde{\mathcal{V}}}. \end{aligned} \quad (4.88)$$

To the order given in (4.84) $\hat{\mathcal{V}}$ gets mapped to $\hat{\tilde{\mathcal{V}}}$.

From (4.29) we see that

$$s := \frac{1}{2w} = \frac{\gamma}{\rho(\gamma^2 + 1) - 2\gamma\tilde{\rho}} = \frac{\gamma}{\rho} + \mathcal{O}(\gamma^2). \quad (4.89)$$

We may thus suspect that the transformation

$$K^{\infty} : \hat{\mathcal{V}} \mapsto K(s)\hat{\tilde{\mathcal{V}}}K(\gamma) \quad (4.90)$$

maps $\hat{\mathcal{V}}$ corresponding to a solution \mathcal{V} to $\hat{\tilde{\mathcal{V}}}$ corresponding to the solution $\tilde{\mathcal{V}}$ to all orders of γ . We verify this:

$$\begin{aligned} [K(s)\hat{\mathcal{V}}K(\gamma)]^{-1} \partial_{\pm} [K(s)\hat{\mathcal{V}}K(\gamma)] &= K(\gamma)\hat{\mathcal{V}}^{-1} \partial_{\pm} [\hat{\mathcal{V}}K(\gamma)] = \\ &= K(\gamma)\hat{J}_{\pm}K(\gamma) + K(\gamma)\partial_{\pm}K(\gamma). \end{aligned} \quad (4.91)$$

We wish to show that this is equal to \hat{J}_{\pm} . From (2.38) we see that

$$\hat{J}_{\pm} = \frac{1}{2} \begin{pmatrix} u^{\pm 1} \Delta^{-1} \partial_{\pm} \Delta & \Delta^{-1} \partial_{\pm} B(u^{\pm 1} + 1) \\ \Delta^{-1} \partial_{\pm} B(u^{\pm 1} - 1) & -u^{\pm 1} \Delta^{-1} \partial_{\pm} \Delta \end{pmatrix}.$$

Thus

$$\begin{aligned} K(\gamma)\hat{J}_{\pm}K(\gamma) &= \\ &= \begin{pmatrix} 0 & -i\gamma^{-1/2} \\ i\gamma^{1/2} & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} u^{\pm 1} \Delta^{-1} \partial_{\pm} \Delta & \Delta^{-1} \partial_{\pm} B(u^{\pm 1} + 1) \\ \Delta^{-1} \partial_{\pm} B(u^{\pm 1} - 1) & -u^{\pm 1} \Delta^{-1} \partial_{\pm} \Delta \end{pmatrix} \begin{pmatrix} 0 & -i\gamma^{-1/2} \\ i\gamma^{1/2} & 0 \end{pmatrix} = \\ &= \frac{1}{2} \begin{pmatrix} -u^{\pm 1} \Delta^{-1} \partial_{\pm} \Delta & -\gamma^{-1} \Delta^{-1} \partial_{\pm} B(u^{\pm 1} - 1) \\ -\gamma \Delta^{-1} \partial_{\pm} B(u^{\pm 1} + 1) & u^{\pm 1} \Delta^{-1} \partial_{\pm} \Delta \end{pmatrix}. \end{aligned}$$

Furthermore

$$\partial_{\pm} K(\gamma) = \begin{pmatrix} 0 & \frac{1}{2} i \gamma^{-1/2} \frac{1 \mp \gamma}{1 \pm \gamma} \rho^{-1} \partial_{\pm} \rho \\ \frac{1}{2} i \gamma^{1/2} \frac{1 \mp \gamma}{1 \pm \gamma} \rho^{-1} \partial_{\pm} \rho & 0 \end{pmatrix},$$

so the second term is

$$K(\gamma)\partial_{\pm}K(\gamma) = \frac{1}{2} \begin{pmatrix} u^{\pm 1} \rho^{-1} \partial_{\pm} \rho & 0 \\ 0 & -u^{\pm 1} \rho^{-1} \partial_{\pm} \rho \end{pmatrix}$$

and (4.91) becomes

$$\begin{aligned} [K(s)\hat{\mathcal{V}}K(\gamma)]^{-1} \partial_{\pm} [K(s)\hat{\mathcal{V}}K(\gamma)] &= \\ &= \frac{1}{2} \begin{pmatrix} u^{\pm 1} (\rho^{-1} \partial_{\pm} \rho - \Delta^{-1} \partial_{\pm} \Delta) & -\Delta^{-1} \partial_{\pm} B(u^{\pm 1} - 1) \gamma^{-1} \\ -\Delta^{-1} \partial_{\pm} B(u^{\pm 1} + 1) \gamma & u^{\pm 1} (\Delta^{-1} \partial_{\pm} \Delta - \rho^{-1} \partial_{\pm} \rho) \end{pmatrix}. \end{aligned}$$

Using $\tilde{\Delta} = \rho/\Delta$ and $\tilde{\Delta}^{-1} \partial_{\pm} \tilde{B} = \pm \Delta^{-1} \partial_{\pm} B$ to write this in terms of $\tilde{\Delta}$ and \tilde{B} , and the identities

$$\begin{aligned} (u^{\pm 1} - 1) \gamma^{-1} &= \frac{\mp 2}{1 \pm \gamma} = \mp (u^{\pm 1} + 1) \\ (u^{\pm 1} + 1) \gamma &= \frac{2\gamma}{1 \pm \gamma} = \mp (u^{\pm 1} - 1), \end{aligned}$$

we see that indeed

$$[K(s)\hat{\mathcal{V}}K(\gamma)]^{-1} \partial_{\pm} [K(s)\hat{\mathcal{V}}K(\gamma)] = \hat{J}_{\pm}. \quad (4.92)$$

By (4.88), if $\hat{\mathcal{V}}$ is in triangular gauge, then $K(s)\hat{\mathcal{V}}K(\gamma)$ will also be in triangular gauge. K^{∞} maps a triangular $\hat{\mathcal{V}}$ to a triangular $\hat{\tilde{\mathcal{V}}}$ and thereby a solution \mathcal{V} of the equations of motions to the corresponding solution $\tilde{\mathcal{V}}$. As will be made clear in the next section, the big advantage over the original Kramer-Neugebauer transform is that K^{∞} is linear. The price we had to pay was that we had to introduce the w -dependent quantities $\hat{\mathcal{V}}$.

4.6 Using K^∞ to generate the symmetry algebra

In section 4.1 we encountered the \mathfrak{g} symmetry (4.4)

$$\delta\mathcal{V} = \mathcal{V} + \delta g\mathcal{V} + \mathcal{V}\delta h(\mathcal{V}, \delta g) \quad \delta g \in \mathfrak{g}, \delta h \in \mathfrak{h}.$$

The action of \mathfrak{g} on the left is combined with an \mathfrak{h} -action on the right, which maintains the γ triangular form. If $\hat{\mathcal{V}}$ is in γ triangular form, with $\mathcal{V} = \hat{\mathcal{V}}|_{\gamma=0}$, then this transformation of \mathcal{V} implies a transformation of $\hat{\mathcal{V}}$ of exactly the same form:

$$\delta\hat{\mathcal{V}} = \delta g\hat{\mathcal{V}} + \hat{\mathcal{V}}\delta h(\mathcal{V}, \delta g) \quad (4.93)$$

The w -independence of δg and δh assure regularity of $\hat{\mathcal{V}}$ at $\gamma = 0$.

Another type of \mathfrak{g} symmetry transformation, $\tilde{\delta}$, acts according to an expression of the form (4.93) on $\hat{\mathcal{V}}$ and $\tilde{\mathcal{V}}$.

We can use K^∞ to obtain the action of $\tilde{\delta}$ on $\hat{\mathcal{V}}$:

$$\begin{aligned} \tilde{\delta}\hat{\mathcal{V}} &= \tilde{\delta}(K(s)\hat{\mathcal{V}}K(\gamma)) = K(s)\tilde{\delta}\hat{\mathcal{V}}K(\gamma) = \\ &= K(s)(\tilde{\delta}\hat{g}\hat{\mathcal{V}} + \hat{\mathcal{V}}\tilde{\delta}\tilde{h})K(\gamma) = K(s)\tilde{\delta}\hat{g}K(s)\hat{\mathcal{V}} + \hat{\mathcal{V}}K(\gamma)\tilde{\delta}\tilde{h}K(\gamma) = \\ &= \tilde{\delta}g(s)\hat{\mathcal{V}} + \hat{\mathcal{V}}\tilde{\delta}h(\gamma), \end{aligned}$$

with

$$\begin{aligned} \tilde{\delta}g(s) &= K(s)\delta gK(s) \\ \tilde{\delta}h(\gamma) &= K(\gamma)\delta hK(\gamma). \end{aligned} \quad (4.94)$$

The action of $\tilde{\delta}$ on $\hat{\mathcal{V}}$ is similar in form to (4.93). It is the sum of a left and a right action, and the generator of the left action, $\tilde{\delta}g(s)$, is an element of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ since $Tr(\tilde{\delta}g(s)) = Tr(K^2(s)\delta g) = Tr(\delta g) = 0$. There are however differences. $\tilde{\delta}g(s)$ is not constant, it depends on $s = (2w)^{-1}$, and the generator of the right action, $\tilde{\delta}h(\gamma)$, need not even lie in \mathfrak{h} . $\tilde{\delta}$ does preserve the γ triangularity of $\hat{\mathcal{V}}$, because it preserves that of $\tilde{\mathcal{V}}$, and $K^\infty : \tilde{\mathcal{V}} \mapsto \hat{\mathcal{V}}$ also preserves γ triangular form.

Let us work out $\tilde{\delta}g(s)$ and $\tilde{\delta}h(\gamma)$ explicitly. First let's consider the transformation

$$\tilde{\delta}\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{\delta}\tilde{h} = 0$$

The $\hat{\mathcal{V}}$ transformation differs only by a sign: $\tilde{\delta}g(s) = -\delta g$, $\tilde{\delta}h(\gamma) = 0$. We combine the action of the other two generators. Thus

$$\tilde{\delta}\tilde{g} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \quad \tilde{\delta}\tilde{h} = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix},$$

with $c = \frac{b\tilde{\mathcal{V}}_{11}}{\tilde{\mathcal{V}}_{22}} = \frac{b\rho\mathcal{V}_{22}}{\mathcal{V}_{11}}$ so that triangularity is maintained.⁹ We get

$$\tilde{\delta}g(s) = \begin{pmatrix} 0 & -bs^{-1} \\ -as & 0 \end{pmatrix}, \quad \tilde{\delta}h(\gamma) = \begin{pmatrix} 0 & c\gamma^{-1} \\ -c\gamma & 0 \end{pmatrix}. \quad (4.95)$$

⁹Note that since \mathcal{V} is triangular and $\det(\mathcal{V}) = \mathcal{V}_{11}\mathcal{V}_{22} = 1$, so both \mathcal{V}_{11} and \mathcal{V}_{22} are non-zero. The same can be said of $\tilde{\mathcal{V}}$.

Note that $\delta h(\gamma)$ does not lie in \mathfrak{h} , because it is not antisymmetric (unless $\gamma = \pm 1$). It does, however, lie in $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, since it is trace free, and it has a generalized antisymmetry $\eta^\infty(\delta h(\gamma)) = \delta h(\gamma)$, where

$$\eta^\infty(a(\gamma)) := \eta(a) \left(\frac{1}{\gamma} \right) = -a^T \left(\frac{1}{\gamma} \right) \quad \forall a \in \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}). \quad (4.96)$$

The γ independent \mathfrak{h} elements δh acting from the right in (4.93) of course also have these properties, since \mathfrak{h} is the subalgebra of \mathfrak{g} characterized by $\eta(a) := -a^T = a$. The map η^∞ is called the *extended involution*, being an extension to \mathfrak{g} valued functions of γ of the involution η on \mathfrak{g} . The analogous extension to G valued functions of γ is

$$\eta^\infty(g(\gamma)) = \eta(g) \left(\frac{1}{\gamma} \right) = (g^T)^{-1} \left(\frac{1}{\gamma} \right).$$

It will be important in the following.

Just as in section 4.1 we can take commutators of the transferred symmetries (4.94) and the original ones (4.93). Let us examine these. The action on $\hat{\mathcal{V}}$ of the commutator of two symmetries, δ_1 and δ_2 , is

$$\begin{aligned} [\delta_1, \delta_2] \hat{\mathcal{V}} &= \delta_1 g \delta_2 g \hat{\mathcal{V}} + \delta_1 g \hat{\mathcal{V}} \delta_2 h + \delta_2 g \hat{\mathcal{V}} \delta_1 h + \hat{\mathcal{V}} \delta_2 h \delta_1 h + \hat{\mathcal{V}} \delta_1 (\delta_2 h) - (1 \leftrightarrow 2) \\ &= [\delta_1 g, \delta_2 g] \hat{\mathcal{V}} + \hat{\mathcal{V}} \{ [\delta_2 h, \delta_1 h] + \delta_1 (\delta_2 h) - \delta_2 (\delta_1 h) \}, \end{aligned} \quad (4.97)$$

where $\delta_1(\delta_2 h) = \frac{\partial \delta_2 h}{\partial \mathcal{V}_{ij}} \delta_1 \mathcal{V}_{ij}$ is the variation of $\delta_2 h$ under δ_1 due to \mathcal{V} dependence of $\delta_2 h$. Thus the action of a commutator takes the same basic form as (4.93), consisting of a (possibly w dependent) left \mathfrak{g} action which is simply the commutator of the \mathfrak{g} actions of δ_1 and δ_2 , and a right action generated by the matrix

$$\delta_{[1,2]} h := [\delta_2 h, \delta_1 h] + \delta_1 (\delta_2 h) - \delta_2 (\delta_1 h). \quad (4.98)$$

It is easy to see from this last formula that commutators preserve certain properties: If both δh_1 and δh_2 take values in \mathfrak{g} , then so does $\delta_{[1,2]} h$. If both are Laurent polynomials in γ , (and of course also functions of \mathcal{V}) then so is $\delta_{[1,2]} h$. If both are invariant under η^∞ , then $\delta_{[1,2]} h$ is also.

For now we concentrate only on the left \mathfrak{g} action. Later we will analyse the right action and its task to maintain the γ triangular form. We have at our disposal the following five generators

$$t_3^{(0)} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t_+^{(0)} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t_-^{(0)} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$t_-^{(1)} := s t_-^{(0)}, \quad t_+^{(-1)} := s^{-1} t_+^{(0)}.$$

Using the commutation relations

$$[t_3^{(0)}, t_\pm^{(0)}] = \pm 2 t_\pm^{(0)}, \quad [t_+^{(0)}, t_-^{(0)}] = t_3^{(0)},$$

we can construct

$$t_3^{(1)} = [t_+^{(0)}, t_-^{(1)}] = s t_3^{(0)}, \quad t_3^{(-1)} = [t_+^{(-1)}, t_-^{(0)}] = s^{-1} t_3^{(0)}.$$

Taking further commutators one obtains the infinite collection of symmetry generators

$$\begin{aligned} t_{\pm}^{(n)} &= \left[ad_{\pm \frac{1}{2} t_3^{(1)}} \right]^n t_{\pm}^{(0)}, & t_{\pm}^{(-n)} &= \left[ad_{\pm \frac{1}{2} t_3^{(-1)}} \right]^n t_{\pm}^{(0)}, \\ t_3^{(\pm n)} &= [t_+^{(\pm n)}, t_-^{(0)}], & & n \text{ any positive integer.} \end{aligned}$$

In all cases

$$t_i^{(\pm n)} := s^{\pm n} t_i^{(0)} \quad i \in \{+, -, 3\}.$$

(The generators $t_-^{(1)}$ and $t_+^{(-1)}$ have of course been defined twice, but clearly the definitions are consistent). The commutation relations between these generators are

$$[t_i^{(m)}, t_j^{(n)}] = f_{ij}{}^k t_k^{(m+n)} \quad m, n \in \mathbb{Z},$$

$f_{ij}{}^k$ being the structure constants of \mathfrak{g} .

The $t_i^{(\pm n)}$ generate the algebra of \mathfrak{g} valued Laurent polynomials in s , that is, the loop algebra of \mathfrak{g} [9],[21], which we shall denote \mathfrak{g}^∞ (or \mathfrak{g}_x^∞ if we refer to a realization by Laurent polynomials in the variable x).

We admit as symmetries only transformations that preserve the γ triangularity of $\hat{\mathcal{V}}$. In the transformations (4.93) this was ensured by a suitable right \mathfrak{h} action. Also the transformations (4.95) contain a suitable right (not necessarily \mathfrak{h}) action so that they too preserve γ triangularity. Since the symmetries associated with the generators $t_i^{(\pm n)}$ are commutators of the symmetries (4.93) and (4.95) they also preserve γ triangularity. The matrices $\delta h(\mathcal{V}, \gamma)$ acting from the right in each transformation can be calculated from (4.98). Thus there certainly is a whole loop algebra worth of symmetries. However calculating the right actions this way is quite tedious. Moreover, there are several ways to obtain the same \mathfrak{g}^∞ elements as commutators, but it is not immediately obvious that all lead, via (4.98), to the same right action. In other words, it is *a priori* conceivable that several different symmetry transformations correspond to the same left \mathfrak{g}^∞ action. This is in fact not the case. The left action defines a unique right action, and the symmetry group generated by the commutators is precisely \mathfrak{g}^∞ . In the following a more direct way of constructing γ triangularity restoring right actions will be presented, and their uniqueness demonstrated.

Recall that γ triangularity requires that $\delta \hat{\mathcal{V}}$ be regular at γ , and that $\mathcal{V} = \hat{\mathcal{V}}|_{\gamma=0}$ is upper triangular. We already have right actions that restore γ triangularity for each of the five left action generators $t_i^{(0)}$, $t_-^{(1)}$ and $t_+^{(-1)}$. Let us examine the cases $t_-^{(1)}$ and $t_+^{(-1)}$ explicitly. For

$$\delta g(s) = \begin{pmatrix} 0 & -bs^{-1} \\ -as & 0 \end{pmatrix} \quad (4.99)$$

the generator of the right action is

$$\delta h(\mathcal{V}, \gamma) = \begin{pmatrix} 0 & c\gamma^{-1} \\ -c\gamma & 0 \end{pmatrix}, \quad (4.100)$$

with

$$c = \frac{b\tilde{\mathcal{V}}_{11}}{\tilde{\mathcal{V}}_{22}} = \frac{b\rho\mathcal{V}_{22}}{\mathcal{V}_{11}}.$$

This $\delta h(\gamma)$ was obtained in section 4.1 from the requirement that the transformation preserve the triangularity of $\tilde{\mathcal{V}}$. Let us see how in the present context of a transformation of $\hat{\mathcal{V}}$ the right action by $\delta h(\gamma)$ assures the γ triangularity of $\hat{\mathcal{V}}$.

First, c does not depend on a , which means that for $\delta g(s) = t_-^{(1)}$ $\delta h(\gamma) = 0$. γ Triangularity is preserved because by (4.89) no poles at $\gamma = 0$ are introduced and $\hat{\mathcal{V}}|_{\gamma=0}$ is left unchanged. (In fact this makes $t_-^{(1)}$ a trivial symmetry generator in the sense that it does not affect the physical field $\mathcal{V} = \hat{\mathcal{V}}|_{\gamma=0}$.)

In the case $\delta g(s) = t_+^{(-1)}$, we have $\delta h(\gamma) = \frac{\rho \mathcal{V}_{22}}{\mathcal{V}_{11}}(\gamma t_-^{(0)} - \gamma^{-1} t_+^{(0)})$. Since

$$s^{-1} = 2w = \rho \left(\gamma + \frac{1}{\gamma} \right) - 2\tilde{\rho}, \quad (4.101)$$

$\delta g(s)$ introduces a pole at $\gamma = 0$. But this pole is cancelled by the $\delta h(\gamma)$ term: Using the Taylor expansion $\hat{\mathcal{V}} = \mathcal{V} + \gamma \hat{\mathcal{V}}_1 + \mathcal{O}(\gamma^2)$ we find that

$$\begin{aligned} \delta \hat{\mathcal{V}} &= t_+^{(-1)} \hat{\mathcal{V}} + \hat{\mathcal{V}} \frac{\rho \mathcal{V}_{22}}{\mathcal{V}_{11}} (\gamma t_-^{(0)} - \gamma^{-1} t_+^{(0)}) = \left(\frac{\rho}{\gamma} - 2\tilde{\rho} \right) \begin{pmatrix} 0 & \mathcal{V}_{22} \\ 0 & 0 \end{pmatrix} + \rho \begin{pmatrix} (\hat{\mathcal{V}}_1)_{21} & (\hat{\mathcal{V}}_1)_{22} \\ 0 & 0 \end{pmatrix} + \\ &+ \frac{\rho \mathcal{V}_{22}}{\mathcal{V}_{11}} \begin{pmatrix} 0 & -\mathcal{V}_{11} \gamma^{-1} \\ 0 & 0 \end{pmatrix} + \frac{\rho \mathcal{V}_{22}}{\mathcal{V}_{11}} \begin{pmatrix} 0 & -(\hat{\mathcal{V}}_1)_{11} \\ 0 & -(\hat{\mathcal{V}}_1)_{21} \end{pmatrix} + \mathcal{O}(\gamma) = \\ &= \begin{pmatrix} \rho(\hat{\mathcal{V}}_1)_{21} & -2\tilde{\rho} \mathcal{V}_{22} + \rho(\hat{\mathcal{V}}_1)_{22} - \frac{\rho \mathcal{V}_{22}}{\mathcal{V}_{11}} (\hat{\mathcal{V}}_1)_{11} \\ 0 & -\frac{\rho \mathcal{V}_{22}}{\mathcal{V}_{11}} (\hat{\mathcal{V}}_1)_{21} \end{pmatrix} + \mathcal{O}(\gamma). \end{aligned}$$

The right action by $\delta h(\gamma)$ indeed cancels the γ^{-1} -term introduced by $\delta g(s)$.

The example of $t_-^{(1)}$ makes it clear that more generally for the generators $t_i^{(n)}$ with $n > 0$ no compensating $\delta h(\gamma)$ is needed and the physical fields are not transformed at all ($s = \mathcal{O}(\gamma)$). On the other hand, the $t_i^{(-n)}$ with $n > 0$ introduce poles at $\gamma = 0$, which have to be cancelled by suitable $\delta h(\gamma)$ s. Suppose

$$\delta g(s) = s^{-k} \delta g, \quad k > 0, \quad \delta g(s) \in \mathfrak{g}.$$

s^{-k} may be written as a Laurent polynomial in γ with coefficients s_n (see (4.101))

$$s^{-k} = \left(\rho \left(\gamma + \frac{1}{\gamma} \right) - 2\tilde{\rho} \right)^k = \sum_{n=-k}^k \gamma^n s_n.$$

We also write $\delta h(\gamma)$ in the form of a Laurent polynomial

$$\delta h(\gamma) = \sum_{n=-k}^l \gamma^n \delta h_n, \quad (4.102)$$

where the lowest order $-k$ is so that the poles of order k coming from $\delta g(s)$ may be cancelled, but no higher order poles are introduced. The highest order l is not specified yet. Finally, we write the Taylor expansion of $\hat{\mathcal{V}}$ as before:

$$\hat{\mathcal{V}} = \sum_{n=0}^{\infty} \gamma^n \hat{\mathcal{V}}_n \quad \hat{\mathcal{V}}_0 = \mathcal{V}.$$

With this notation

$$\begin{aligned}\delta\hat{\mathcal{V}} &= \delta g(s)\hat{\mathcal{V}} + \hat{\mathcal{V}}\delta h(\gamma) = \\ &= \sum_{n=-k}^k \gamma^n s_n \delta g \sum_{m=0}^{\infty} \gamma^m \hat{\mathcal{V}}_m + \sum_{m=0}^{\infty} \gamma^m \hat{\mathcal{V}}_m \sum_{n=-k}^l \gamma^n \delta h_n.\end{aligned}$$

Requiring that the coefficients of negative powers of γ vanish determines the δh_n for negative n :

$$\begin{aligned}\gamma^{-k} : & \quad 0 = s_{-k} \delta g \mathcal{V} + \mathcal{V} \delta h_{-k} \\ & \Rightarrow \delta h_{-k} = -s_{-k} \mathcal{V}^{-1} \delta g \mathcal{V} \\ \gamma^{-k+1} : & \quad 0 = s_{-k+1} \delta g \mathcal{V} + s_{-k} \delta g \hat{\mathcal{V}}_1 + \hat{\mathcal{V}}_1 \delta h_{-k} + \mathcal{V} \delta h_{-k+1} \\ & \Rightarrow \delta h_{-k+1} = -\mathcal{V}^{-1} (s_{-k+1} \delta g \mathcal{V} + s_{-k} \delta g \hat{\mathcal{V}}_1 + \hat{\mathcal{V}}_1 \delta h_{-k}) \\ & \quad \vdots \\ \gamma^{-k+i} : & \quad 0 = \sum_{m=0}^i s_{-k+i-m} \delta g \hat{\mathcal{V}}_m + \sum_{m=0}^i \hat{\mathcal{V}}_m \delta h_{-k+i-m} \tag{4.103} \\ & \Rightarrow \delta h_{-k+i} = -\mathcal{V}^{-1} \left(\sum_{m=0}^i s_{-k+i-m} \delta g \hat{\mathcal{V}}_m + \sum_{m=1}^i \hat{\mathcal{V}}_m \delta h_{-k+i-m} \right) \\ & \quad \vdots \\ \gamma^{-1} : & \quad 0 = \sum_{m=0}^{k-1} s_{-1-m} \delta g \hat{\mathcal{V}}_m + \sum_{m=0}^{k-1} \hat{\mathcal{V}}_m \delta h_{-1-m} \\ & \Rightarrow \delta h_{-1} = -\mathcal{V}^{-1} \left(\sum_{m=0}^{k-1} s_{-1-m} \delta g \hat{\mathcal{V}}_m + \sum_{m=1}^{k-1} \hat{\mathcal{V}}_m \delta h_{-1-m} \right).\end{aligned}$$

In this way the δh_n , $-k \leq n \leq -1$ can be used to cancel the singularities. The next coefficient, δh_0 , may be used to preserve the triangularity of \mathcal{V} .

$$\delta \mathcal{V}_{21} = \left(\sum_{m=0}^k s_{-m} \delta g \hat{\mathcal{V}}_m + \sum_{m=0}^k \hat{\mathcal{V}}_m \delta h_{-m} \right)_{21}$$

This determines the 21-component of δh_0 .

$$\begin{aligned}\mathcal{V}_{22}(\delta h_0)_{21} &= (\mathcal{V} \delta h_0)_{21} = \left(\sum_{m=0}^k s_{-m} \delta g \hat{\mathcal{V}}_m + \sum_{m=1}^k \hat{\mathcal{V}}_m \delta h_{-m} \right)_{21} \\ (\delta h_0)_{21} &= \frac{1}{\mathcal{V}_{22}} \left(\sum_{m=0}^k s_{-m} \delta g \hat{\mathcal{V}}_m + \sum_{m=1}^k \hat{\mathcal{V}}_m \delta h_{-m} \right)_{21}.\end{aligned}$$

Now let us demonstrate the uniqueness of the right actions: We have noted earlier that the matrices $\delta h(\mathcal{V}, \gamma)$ corresponding to the five fundamental left action generators $t_i^{(0)}$, $t_-^{(1)}$ and $t_+^{(-1)}$ are \mathfrak{g} valued Laurent polynomials in γ and are invariant under the extended involution η^∞ , they satisfy $\eta^\infty(\delta h) = \delta h$. We have also noted that these properties are preserved by commutators: if the right action generators of two transformations have these properties then so does the generator of the right action of the commutator. Since our algebra of symmetries is the span of the symmetries generated by the five fundamental

generators and their commutators, it follows that all right actions are generated by η^∞ invariant \mathfrak{g} valued Laurent polynomials in γ .

This implies the uniqueness of the right action, for as we have seen, the preservation of γ triangularity determines all the coefficients of negative powers of γ in δh and also the component $(\delta h_0)_{21}$ of the γ independent term. The invariance $\delta h(\gamma) = \eta^\infty(\delta h)(\gamma) = \eta(\delta h(\frac{1}{\gamma}))$ determines all the remaining coefficients in (4.102):

$$\delta h_n = \eta(\delta h_{-n}) \quad \text{for } n > 0,$$

$$\delta h_0 = \eta(\delta h_0) \Rightarrow \delta h_0 \in \mathfrak{so}(2) \Rightarrow (\delta h_0)_{12} = -(\delta h_0)_{21}, (\delta h_0)_{11} = 0 = (\delta h_0)_{22}.$$

Let us summarize what we have found: The algebra of (infinitesimal) symmetries we have found is isomorphic to the infinite dimensional loop group \mathfrak{g}^∞ . The symmetries act in a relatively simple way on the field $\hat{\mathcal{V}}$, related to the physical field \mathcal{V} by

$$\hat{\mathcal{V}}^{-1} \partial_\pm \hat{\mathcal{V}} = Q_\pm + \frac{1 \mp \gamma}{1 \pm \gamma} P_\pm, \quad Q_\pm = (\mathcal{V}^{-1} \partial_\pm \mathcal{V})|_{\mathfrak{h}}, \quad P_\pm = (\mathcal{V}^{-1} \partial_\pm \mathcal{V})|_{\mathfrak{k}}.$$

The variations of $\hat{\mathcal{V}}$ under the symmetry transformations take the form

$$\hat{\mathcal{V}} \rightarrow \hat{\mathcal{V}}' = \hat{\mathcal{V}} + \delta \hat{\mathcal{V}}, \tag{4.104}$$

$$\delta \hat{\mathcal{V}} = \delta g(s) \hat{\mathcal{V}} + \hat{\mathcal{V}} \delta h(\gamma, x), \quad \delta g \in \mathfrak{g}_s^\infty, \quad \delta h(x) \in \mathfrak{h}_\gamma^\infty. \tag{4.105}$$

Here

$$\mathfrak{h}_\gamma^\infty := \{h \in \mathfrak{g}_\gamma^\infty \mid \eta^\infty(h(\gamma)) = h(\gamma)\}.$$

is the subalgebra of $\mathfrak{g}_\gamma^\infty$ which is invariant under η^∞ . Note that δh depends on x not only via γ , but also via its dependence on $\hat{\mathcal{V}}$ (see (4.103)). δh maintains the γ triangular gauge. Therefore we may extract the variation of the physical field \mathcal{V}

$$\delta \mathcal{V} = \delta \hat{\mathcal{V}}(\gamma = 0). \tag{4.106}$$

The physical field \mathcal{V} does not determine $\hat{\mathcal{V}}$ uniquely, but only up to a transformation

$$\hat{\mathcal{V}} \rightarrow S(w) \hat{\mathcal{V}}, \tag{4.107}$$

with $S(w)$ being an $SL(2)$ valued function such that $S(w = \infty) = \mathbb{1}$. This freedom introduces no ambiguity in the definition of the symmetry algebra, since the transformations (4.107) are generated by a subalgebra of the symmetry algebra.

4.7 Example: $\delta g(s) = s^{-1} \delta g$

In this section we explicitly calculate $\delta h(\gamma)$ and $\delta \hat{J}$ for the case $\delta g(s) = s^{-1} \delta g$. We concentrate on the γ -dependent terms of $\delta h(\gamma)$ — the way how δh_0 maintains triangularity is obvious and so we set it to zero for now. This computation gives an impression of how the form (4.31) is maintained.

$$s^{-1} = \rho \left(\gamma + \frac{1}{\gamma} \right) + 2\tilde{\rho} \Rightarrow s_1 = \rho = s_{-1}, \quad s_0 = 2\tilde{\rho}$$

$$\delta h_{-1} = -\rho \mathcal{V}^{-1} \delta g \mathcal{V} =: -\rho \delta g'$$

$$\delta h_1 = \eta(\delta h_{-1}) = -\rho \eta(\delta g')$$

$$\begin{aligned} \delta \hat{J}_\pm &= [\hat{J}_\pm, -\rho \delta g' \gamma^{-1} - \rho \eta(\delta g') \gamma] - \partial_\pm \rho (\delta g' \gamma^{-1} + \eta(\delta g') \gamma) + \\ &\quad + [J_\pm, \rho \delta g' \gamma^{-1}] + [\eta(J_\pm), \rho \eta(\delta g') \gamma] + u^{\pm 1} \partial_\pm \rho (\delta g' \gamma^{-1} - \eta(\delta g') \gamma) \\ &\quad - \gamma^{-1} \hat{J}_\pm + \gamma^{-1} J_\pm = \gamma^{-1} (1 - u^{\pm 1}) P_\pm = \frac{\pm 2}{1 \pm \gamma} P_\pm \\ &\quad - \gamma \hat{J}_\pm + \gamma \eta(J_\pm) = \gamma (-1 - u^{\pm 1} P_\pm) = \frac{-2\gamma}{1 \pm \gamma} P_\pm \\ \delta \hat{J}_\pm &= \left[P_\pm, \frac{\pm 2\rho}{1 \pm \gamma} \delta g' - \frac{2\gamma\rho}{1 \pm \gamma} \eta(\delta g') \right] + \partial_\pm \rho \left(\frac{\mp 2\delta g'}{1 \pm \gamma} - \frac{2\gamma\eta(\delta g')}{1 \pm \gamma} \right) = \\ &= \pm 4\rho [P_\pm, \delta g'|_{\mathfrak{k}}] \pm 4\rho u^{\pm 1} [P_\pm, \delta g'|_{\mathfrak{h}}] \mp 4\partial_\pm \rho (u^{\pm 1} \delta g'|_{\mathfrak{k}} + \delta g'|_{\mathfrak{h}}) = \\ &= \pm 4\rho [P_\pm, \delta g']|_{\mathfrak{h}} \pm 4\rho u^{\pm 1} [P_\pm, \delta g']|_{\mathfrak{k}} \mp 4\partial_\pm \rho (u^{\pm 1} \delta g'|_{\mathfrak{k}} + \delta g'|_{\mathfrak{h}}) = \\ &= \pm 4\rho [P_\pm, \delta g' \mp 4\partial_\pm \rho \delta g']|_{\mathfrak{h}} + u^{\pm 1} (\pm 4\rho [P_\pm, \delta g'] \mp 4\partial_\pm \rho \delta g')|_{\mathfrak{k}} \end{aligned}$$

Including a γ -independent δh_0 would simply add a term

$$[Q_\pm + u^{\pm 1} P_\pm, \delta h_0] + \partial_\pm \delta h_0 = ([Q_\pm, \delta h_0] + \partial_\pm \delta h_0) + u^{\pm 1} [P_\pm, \delta h_0]$$

4.8 $\mathcal{M}(w)$ and its symmetry transformation behaviour

We conclude this chapter by introducing the function

$$\mathcal{M}(x, w) := \hat{\mathcal{V}} \eta^\infty (\hat{\mathcal{V}}^{-1}), \quad (4.108)$$

which in the literature is often called the monodromy matrix associated with $\hat{\mathcal{V}}$ [30],[9] (see [2] for a more general treatment of the role of $\mathcal{M}(w)$ in integrable systems). By continuity of $\hat{\mathcal{V}}$ on the closed γ -unit disc, $\mathcal{M}(w)$ is well defined at least for pairs (x, w) , such that w lies on the branch cut dependent on x .

Because of the invariance of \hat{J} under η^∞ , we have

$$\partial \mathcal{M}(x, w) = \hat{\mathcal{V}}(x, w) \hat{J}(x, w) \eta^\infty (\hat{\mathcal{V}}^{-1}(x, w)) - \hat{\mathcal{V}}(x, w) \hat{J}(x, w) \eta^\infty (\hat{\mathcal{V}}^{-1}(x, w)) = 0, \quad (4.109)$$

so $\mathcal{M}(x, w) = \mathcal{M}(w)$ is constant w.r.t. x . Under a transformation (4.105) $\mathcal{M}(w)$ transforms as

$$\delta \mathcal{M}(w) = \delta g(s) \mathcal{M}(w) - \mathcal{M}(w) \delta g(s). \quad (4.110)$$

This action is quite simple. It involves only the algebra \mathfrak{g}_s^∞ . This fact is quite valuable because we saw earlier that the computation of $\delta h(\gamma)$ can be very tedious if $\delta g(s)$ contains high negative powers s^{-k} .

4.9 Finite transformations

Since the infinitesimal action (4.105) of the Geroch group is quite intricate due to its nonlinearity, exponentiation of this action seems to be out of reach. In contrast, the transformation (4.110) of $\mathcal{M}(w)$ is linear. It can be exponentiated to

$$\mathcal{M}_g(w) = g(s)\mathcal{M}(w)g^{-1}(s), \quad (4.111)$$

with $g(s)$ being members of the Geroch group. Viewing elements of \mathfrak{g}_s^∞ as functions from the complex plane into the algebra \mathfrak{g} , pointwise exponentiation will lead to maps from the complex plane into the complexified group G .

From an solution $\hat{\mathcal{V}}$ one can use (4.108) to get $\mathcal{M}(w)$ and then use the simple transformation behaviour (4.111) to get the finitely transformed $\mathcal{M}_g(w)$. In order to get back to $\hat{\mathcal{V}}_g$ one has to factorise $\mathcal{M}_g(w)$ into two functions, one analytic around $\gamma = 0$, the other one analytic around $\gamma = \infty$. This factorisation is called Birkhoff [14] or (Riemann-)Hilbert factorization [28]. It is discussed in section 5.5.

Part II

Null canonical formulation

Chapter 5

Null initial data for EG2CHSKF versus null canonical gravity

In this chapter we investigate the possibility of profiting from the machinery and ideas developed for EG2CHSKF, an integrable system, even for the far more general theory, canonical gravity with no Killing fields using a null initial data surface. In section 5.1 we review the relevant results of [32] and explain how an aspect of the full theory of gravity formulated with null initial data resembles the situation where two Killing fields are present. We show how the role of the complex fields μ and $\bar{\mu}$ used in [32] to parametrise the degenerate 3-metric on a null initial data surface is quite similar to the role of the field \mathcal{V} of EG2CHSKF in view of the Poisson structure of μ and $\bar{\mu}$. In section 5.2 we compute the symplectic 2-form for the \mathcal{V} -sector of EG2CHSKF, in 5.3 we derive a Poisson bracket of the field \mathcal{V} on a null initial data surface and in section 5.4 we fix the gauge of \mathcal{V} by parametrising it by complex fields $\mu, \bar{\mu}$. This allows to derive a bracket of μ and $\bar{\mu}$ from the bracket of \mathcal{V} and compare it with the bracket obtained in [32]. It turns out that the expression is completely analogous. In a next step, in section 5.5, we investigate in detail how the monodromy matrix $\mathcal{M}(w)$ can be used as initial data instead of \mathcal{V} . In [24] it is shown, using an infinitely extended spacelike initial data surface, that the monodromy matrix forms a quite simple Poisson algebra and even a quantization of this algebra is proposed. In section 5.7 we derive this Poisson algebra and in section 5.8 also the Yangian algebra of conserved charges found in [24] using our Poisson structure on a null initial data surface. This provides a check that our bracket of null initial data is correct.

5.1 Motivation

The initial data hypersurface $\tilde{\mathcal{N}}$ defined in [32] consists of two null hypersurface branches $\tilde{\mathcal{N}}_L$ and $\tilde{\mathcal{N}}_R$, which are joined on a spacelike 2-surface S_0 . Each branch is swept out by a congruence of null geodesics emerging normally from S_0 . The geodesic congruences and thus $\tilde{\mathcal{N}}_L$ and $\tilde{\mathcal{N}}_R$ generically have to be truncated at spacelike discs S_L and S_R before caustics form. As a consequence, $\tilde{\mathcal{N}}$ in general supports initial data only for a bounded region of spacetime.

Denote by $\tilde{\mathcal{N}}_A$ one of the branches $\tilde{\mathcal{N}}_L$ and $\tilde{\mathcal{N}}_R$. The tangent vectors n_A to the geodesic congruences sweeping out $\tilde{\mathcal{N}}_A$ are null by definition and thus not only tangent, but also

orthogonal to $\tilde{\mathcal{N}}_A$. Hence the 3-metric on $\tilde{\mathcal{N}}_A$ must be degenerate. In a basis, where n_A forms the first basis vector, the 3-metric is thus of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & h_{11} & h_{12} \\ 0 & h_{21} & h_{22} \end{pmatrix}. \quad (5.1)$$

The 2-metric h_{ij} measures the distance between points on neighbouring generators and by the degeneracy with respect to n_A , this distance does not depend on the separation of the points along the generators, but only on the separation of the corresponding generators themselves. By the Raychaudhuri focussing equation [39], if the expansion rate of the geodesic congruence is negative (positive) in one direction of the generators at one point, then it must remain negative (positive). Thus, the cross sectional area may be used as a coordinate r along the generators. To be precise, the definition of r introduced in [32] is

$$r(p) = \sqrt{\rho(p)/\bar{\rho}(\gamma)},$$

where

$$\rho = \sqrt{\det(h_{ij})} \quad (5.2)$$

and $\bar{\rho}(\gamma)$ is the value ρ has at the point on S_A where the generator, on which p lies, meets S_A . The requirement that $\tilde{\mathcal{N}}_A$ be truncated before neighbouring generators form caustics may now be satisfied by demanding that ρ be greater or equal to zero. To label the different generators two more coordinates, \tilde{y}^1 and \tilde{y}^2 , are used. Since the information about the determinant of the 2-metric h_{ij} is now contained entirely in the coordinate r , the remaining freedom of h_{ij} is the unimodular metric

$$e_{ij} = h_{ij}/\rho. \quad (5.3)$$

It may be conveniently parametrised by the complex field μ as

$$e_{ij} = \frac{1}{1-|\mu|^2} \begin{pmatrix} |1+\mu|^2 & \frac{\mu-\bar{\mu}}{i} \\ \frac{\mu-\bar{\mu}}{i} & |1-\mu|^2 \end{pmatrix}. \quad (5.4)$$

For points x and y lying on the same generator of the same branch $\tilde{\mathcal{N}}_A$ of $\tilde{\mathcal{N}}$ the bracket of the field μ and its complex conjugate $\bar{\mu}$ is then calculated to be

$$\{\mu(x), \bar{\mu}(y)\} = 2\pi G \operatorname{sign}(x, y) \left[\frac{1-|\mu|^2}{\sqrt{\rho}} \right] \Big|_x \left[\frac{1-|\mu|^2}{\sqrt{\rho}} \right] \Big|_y e^{\int_x^y \frac{\bar{\mu}d\mu - \mu d\bar{\mu}}{1-|\mu|^2}}, \quad (5.5)$$

where $\operatorname{sign}(x, y)$ is 1 (-1) if y is further (closer) from S_0 along the generator than x . If x and y lie on different generators the bracket vanishes. Also $\{\mu(x), \mu(y)\} = 0 = \{\bar{\mu}(x), \bar{\mu}(y)\}$.

We complete this review by noting that the three coordinates r , \tilde{y}^1 and \tilde{y}^2 on $\tilde{\mathcal{N}}_A$ may be supplemented by a fourth one. At every point on $\tilde{\mathcal{N}}_A$, there exists a transverse normal null geodesic. The above three coordinates may be convected along these geodesics and the parameter u of these geodesics, normalized such that $\partial_r \cdot \partial_u = -1$ and $u = 0$ on $\tilde{\mathcal{N}}_A$,

may be used as a fourth coordinate for a neighbourhood of $\tilde{\mathcal{N}}_A$. In these coordinates the full 4-metric $\tilde{g}_{\mu\nu}$ on $\tilde{\mathcal{N}}_A$ has the form

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & h_{11} & h_{12} \\ 0 & 0 & h_{21} & h_{22} \end{pmatrix}_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \rho e_{11} & \rho e_{12} \\ 0 & 0 & \rho e_{21} & \rho e_{22} \end{pmatrix}_{\mu\nu}. \quad (5.6)$$

We compare this expression with (2.26),

$$g_{\mu\nu} = \begin{pmatrix} \Delta^{-1}\lambda^2 {}^{(2)}g_{00} & \Delta^{-1}\lambda^2 {}^{(2)}g_{01} & 0 & 0 \\ \Delta^{-1}\lambda^2 {}^{(2)}g_{10} & \Delta^{-1}\lambda^2 {}^{(2)}g_{11} & 0 & 0 \\ 0 & 0 & \rho^2\Delta^{-1} + \Delta(B_3)^2 & \Delta B_3 \\ 0 & 0 & \Delta B_3 & \Delta \end{pmatrix}_{\mu\nu}, \quad (5.7)$$

the 4-metric of EG2CHSKF in coordinates x^0, x^1, y^1, y^2 , where ∂_{y^1} and ∂_{y^2} are the two Killing fields: Both expressions are block-diagonal. In both cases the determinant of the lower right 2×2 -block is ρ^2 . In the former case, the unimodular part of this block is e_{ij} , in the latter, in view of the definition (2.35) of $\tilde{\mathcal{V}}$, it is $\tilde{\mathcal{V}}\tilde{\mathcal{V}}^T$. The upper left 2×2 -block of (5.6) seems simpler than the corresponding block in (5.7). But this is only a matter of choice of coordinates. The 2-metric ${}^{(2)}g$ can be brought to conformally flat form

$${}^{(2)}g = -2\lambda' dx^+ dx^- \quad (5.8)$$

at least locally in null coordinates x^\pm . The 4-metric $\tilde{g}_{\mu\nu}$ is of the form (5.6) only on $\tilde{\mathcal{N}}_A$, i.e. on the surface where $u = 0$. For EG2CHSKF analogous hypersurfaces \mathcal{N}_A would be the surfaces of constant x^+ (\mathcal{N}_L) or x^- (\mathcal{N}_R). Consider the surface $x^+ = x_0^+$. We may rescale

$$x^+ = \tilde{x}^+ \frac{\Delta(x_0^+)}{\lambda'(x_0^+)\lambda^2(x_0^+)}.$$

Then, using \tilde{x}^+ instead of x^+ , the 4-metric on the surface $x^+ = x_0^+$ is of the form

$$g_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \rho^2\Delta^{-1} + \Delta(B_3)^2 & \Delta B_3 \\ 0 & 0 & \Delta B_3 & \Delta \end{pmatrix}_{\mu\nu}, \quad (5.9)$$

just like g' . The last difference of the two expressions, which at the same time is the most important difference, is that all the quantities in $\tilde{g}_{\mu\nu}$ depend on \tilde{y}^1 and \tilde{y}^2 , while in $g_{\mu\nu}$ they are all independent of y^1 and y^2 . But, as far as the Poisson structure of the lower right 2×2 -block is concerned, i.e. the bracket of μ and $\bar{\mu}$ on the one side and the bracket of the components of the matrix $\tilde{\mathcal{V}}$ on the other side, the special form of (5.5) suggests that this difference not influence the Poisson structure. Whether the bracket vanishes for points on different generators of the hypersurface $\tilde{\mathcal{N}}_A$ and is nonzero only for points on the same generator or whether the bracket is the same for all pairs of points, which have the same x^+ (x^-) coordinate, but possibly different y^1 and y^2 coordinates, might not matter.

In the next sections, this question is answered. We derive a Poisson bracket for \mathcal{V} , the Kramer-Neugebauer transform of $\tilde{\mathcal{V}}$, parametrise \mathcal{V} by a complex field μ , similar to (5.4), and deduce a bracket of this field μ with its complex conjugate $\bar{\mu}$, which is completely analogous to (5.5). Since $\tilde{\mathcal{V}}$ has a completely analogous Lagrangian density, all the results on the Poisson structure are equally valid for $\tilde{\mathcal{V}}$. The advantage of \mathcal{V} over $\tilde{\mathcal{V}}$ is, that it is regular at the axis.

5.2 The symplectic potential Θ and symplectic 2-form ω

As mentioned in section 2.3.3 the dynamics of the $G = SL(2, \mathbb{R})$ -valued field \mathcal{V} is the most difficult part of the equations of motion. A solution for ρ is any function of the form

$$\rho = \frac{1}{2}\rho^+(x^+) + \frac{1}{2}\rho^-(x^-)$$

and λ can be obtained by integration once a solution for \mathcal{V} is known. We therefore concentrate on the part

$$I_{\mathcal{V}} = -\frac{\kappa}{2} \int_D \varepsilon_g \rho \text{Tr}(P^2) \quad (5.10)$$

of the action (2.40). It is the only one containing \mathcal{V} . The factor $\frac{\kappa}{2}$ is a normalization, which doesn't affect the e.o.m.. We keep this factor to allow easy comparison of our results with those of [32] and [24]. Proceeding as explained in section 3.3.3 we compute $\delta I_{\mathcal{V}}$ on the space of solutions \mathcal{S} for a vector δ with $\delta\rho = 0 = \delta g$. (This restriction on δ will be assumed throughout this chapter.)

$$\begin{aligned} \delta I_{\mathcal{V}} &= -\kappa \int_{D_{\Sigma\Sigma'}} \varepsilon_g \rho \text{Tr} \left\{ P^\mu (-\mathcal{V}^{-1} \delta \mathcal{V} J_\mu + \mathcal{V}^{-1} \nabla_\mu \delta \mathcal{V}) \right\} = \\ &= \kappa \int_{D_{\Sigma\Sigma'}} \varepsilon_g \text{Tr} \left\{ (\nabla_\mu (\rho P^\mu) + [Q_\mu, \rho P^\mu]) \mathcal{V}^{-1} \delta \mathcal{V} \right\} - \kappa \int_{\partial D_{\Sigma\Sigma'}} (\varepsilon_g)_\mu \rho \text{Tr} \left\{ P^\mu \mathcal{V}^{-1} \delta \mathcal{V} \right\}. \end{aligned}$$

The dot in $(\varepsilon_g)_\mu$ refers to the uncontracted index. Therefore

$$\delta \lrcorner \Theta_\Sigma = -\kappa \int_\Sigma (\varepsilon_g)_\mu \rho \text{Tr} \left\{ P^\mu \mathcal{V}^{-1} \delta \mathcal{V} \right\} = -\kappa \int_\Sigma (\varepsilon_g)_\mu \rho \text{Tr} \left\{ J^\mu (\mathcal{V}^{-1} \delta \mathcal{V}) \Big|_{\mathfrak{t}} \right\}. \quad (5.11)$$

For two vectors δ_1, δ_2 the symplectic 2-form $\omega = -\text{d}\Theta$ is then

$$\begin{aligned} \omega(\delta_1, \delta_2) &= -\delta_1(\delta_2 \lrcorner \Theta) + \delta_2(\delta_1 \lrcorner \Theta) + [\delta_1, \delta_2] \lrcorner \Theta = \\ &= -\kappa \int_\Sigma (\varepsilon_g)_\mu \rho \text{Tr} \left\{ \delta_2 P^\mu \mathcal{V}^{-1} \delta_1 \mathcal{V} - P^\mu \mathcal{V}^{-1} \delta_2 \mathcal{V} \mathcal{V}^{-1} \delta_1 \mathcal{V} \right\} - (1 \leftrightarrow 2) \\ &= -\kappa \int_\Sigma (\varepsilon_g)_\mu \rho \text{Tr} \left\{ \delta_2 P^\mu \mathcal{V}^{-1} \delta_1 \mathcal{V} + P^\mu [\mathcal{V}^{-1} \delta_1 \mathcal{V}, \mathcal{V}^{-1} \delta_2 \mathcal{V}] - \delta_1 P^\mu \mathcal{V}^{-1} \delta_2 \mathcal{V} \right\}. \end{aligned}$$

But

$$\begin{aligned} &\text{Tr} \left\{ P^\mu [\mathcal{V}^{-1} \delta_1 \mathcal{V}, \mathcal{V}^{-1} \delta_2 \mathcal{V}] \right\} = \\ &= \text{Tr} \left\{ P^\mu [(\mathcal{V}^{-1} \delta_1 \mathcal{V}) \Big|_{\mathfrak{t}}, (\mathcal{V}^{-1} \delta_2 \mathcal{V}) \Big|_{\mathfrak{h}}] + P^\mu [(\mathcal{V}^{-1} \delta_1 \mathcal{V}) \Big|_{\mathfrak{h}}, (\mathcal{V}^{-1} \delta_2 \mathcal{V}) \Big|_{\mathfrak{t}}] \right\} = \\ &= \text{Tr} \left\{ (\mathcal{V}^{-1} \delta_1 \mathcal{V}) \Big|_{\mathfrak{t}} [(\mathcal{V}^{-1} \delta_2 \mathcal{V}) \Big|_{\mathfrak{h}}, P^\mu] \right\} - (1 \leftrightarrow 2) \end{aligned}$$

and so

$$\omega(\delta_1, \delta_2) = -\kappa \int_{\Sigma} (\varepsilon_g)_{\mu, \rho} \text{Tr} \left\{ (\delta_2 P^\mu + [(\mathcal{V}^{-1} \delta_2 \mathcal{V})|_{\mathfrak{h}}, P^\mu]) \mathcal{V}^{-1} \delta_1 \mathcal{V} \right\} - (1 \leftrightarrow 2).$$

We are now in the position to examine the degeneracy of ω . When formulated in terms of the $SL(2)$ group elements \mathcal{V} the theory possesses a local $H = SO(2)$ symmetry (2.49). This is a gauge symmetry: The corresponding infinitesimal transformations are degeneracy vectors of the symplectic 2-form. If δ is an infinitesimal H transformation, i.e. $\delta \mathcal{V} = \mathcal{V} \delta h$ with $\delta h \in \mathfrak{h} = \mathfrak{so}(2)$, then $\mathcal{V}^{-1} \delta \mathcal{V} \in \mathfrak{h}$, $(\mathcal{V}^{-1} \delta \mathcal{V})|_{\mathfrak{h}} = \mathcal{V}^{-1} \delta \mathcal{V}$ and $(\mathcal{V}^{-1} \delta \mathcal{V})|_{\mathfrak{k}} = 0$. Recall that $\mathfrak{k} = \mathfrak{g} \setminus \mathfrak{h} = \mathfrak{sl}(2, \mathbb{R}) \setminus \mathfrak{so}(2)$ is identified with the orthogonal complement of \mathfrak{h} in \mathfrak{g} . We introduce the two distributions in the tangent bundle of the space of solutions \mathcal{S}

$$\begin{aligned} \mathfrak{H} &:= \{ \delta \mid \mathcal{V}^{-1} \delta \mathcal{V} \in \mathfrak{h} \} \subset \mathcal{TS}, \\ \mathfrak{K} &:= \{ \delta \mid \mathcal{V}^{-1} \delta \mathcal{V} \in \mathfrak{k} \} \subset \mathcal{TS}. \end{aligned}$$

An arbitrary δ can be decomposed according to

$$\mathcal{V}^{-1} \delta \mathcal{V} = (\mathcal{V}^{-1} \delta \mathcal{V})|_{\mathfrak{k}} + (\mathcal{V}^{-1} \delta \mathcal{V})|_{\mathfrak{h}}$$

into an $\mathfrak{so}(2)$ transformation and an orthogonal remainder. We denote the orthogonal remainder of δ by Δ :

$$\mathcal{V}^{-1} \Delta \mathcal{V} := (\mathcal{V}^{-1} \delta \mathcal{V})|_{\mathfrak{k}} = \mathcal{V}^{-1} \delta \mathcal{V} - (\mathcal{V}^{-1} \delta \mathcal{V})|_{\mathfrak{h}}. \quad (5.12)$$

Obviously

$$\begin{aligned} \Delta &= \delta \quad \text{if } \delta \in \mathfrak{K}, \\ \Delta &= 0 \quad \text{if } \delta \in \mathfrak{H}. \end{aligned}$$

How does this decomposition translate to variations of P_μ ?

$$\begin{aligned} \Delta P_\mu &= \Delta (J_\mu|_{\mathfrak{k}}) = (\Delta J_\mu)|_{\mathfrak{k}} = (-\mathcal{V}^{-1} \Delta \mathcal{V} J_\mu + \mathcal{V}^{-1} \nabla_\mu \Delta \mathcal{V})|_{\mathfrak{k}} = \\ &= \left(-\mathcal{V}^{-1} \delta \mathcal{V} J_\mu + (\mathcal{V}^{-1} \delta \mathcal{V})|_{\mathfrak{h}} J_\mu + \mathcal{V}^{-1} \nabla_\mu (\delta \mathcal{V} - \mathcal{V} (\mathcal{V}^{-1} \delta \mathcal{V})|_{\mathfrak{h}}) \right)|_{\mathfrak{k}} = \\ &= \left(\delta J_\mu + [(\mathcal{V}^{-1} \delta \mathcal{V})|_{\mathfrak{h}}, J_\mu] - \nabla_\mu (\mathcal{V}^{-1} \delta \mathcal{V})|_{\mathfrak{h}} \right)|_{\mathfrak{k}} = \delta P_\mu + [(\mathcal{V}^{-1} \delta \mathcal{V})|_{\mathfrak{h}}, P_\mu] \in \mathfrak{k}. \end{aligned}$$

With this definition we get

$$\omega(\delta_1, \delta_2) = \kappa \int_{\Sigma} (\varepsilon_g)_{\mu, \rho} \text{Tr} \left\{ \Delta_2 P^\mu \mathcal{V}^{-1} \Delta_1 \mathcal{V} \right\} - (1 \leftrightarrow 2). \quad (5.13)$$

If $\delta_1 \in \mathfrak{H}$, then $\mathcal{V}^{-1} \delta_1 \mathcal{V} \in \mathfrak{h}$, $\Delta_1 = 0$ and $\omega(\delta_1, \delta_2)$ vanishes for all δ_2 . The same is of course true for δ_2 . Thus vectors in \mathfrak{H} are indeed degeneracy vectors.

As mentioned in 2.3.3 the 2-dimensional metric locally can be brought to conformally flat form in coordinates (x^0, x^1) and by absorbing the conformal factor in λ we can actually make it flat. In these coordinates (5.13) takes the form

$$\omega(\delta_1, \delta_2) = -\kappa \int_{\Sigma} dx^\nu \epsilon_{\mu\nu\rho} \text{Tr} [\Delta_2 P^\mu \mathcal{V}^{-1} \Delta_1 \mathcal{V}] - (1 \leftrightarrow 2). \quad (5.14)$$

5.3 Poisson bracket on a truncated light cone

For the null initial data surface \mathcal{N} we choose the set

$$\mathcal{N} = \mathcal{N}_R \cup \mathcal{N}_L = \{(x_0^-, x_0^+ + s) \mid s \in [0, a]\} \cup \{(x_0^- + t, x_0^+) \mid t \in [0, b]\}, \quad (5.15)$$

with $a, b > 0$. In [32] it is explained that this is a Cauchy surface for a diamond shaped region in the future of \mathcal{N} , such as \mathcal{D} considered section 4.4. As initial data the \mathcal{V} on \mathcal{N} suffice. No derivative is needed to coordinatize \mathcal{S} up to gauge equivalence. We have to orient \mathcal{N} coherently and choose to orient it positively with respect to the future, as explained in appendix E. On the reduced spacetime we define the chart (x^0, x^1) to be positively oriented. Then also (x^-, x^+) is positively oriented (see appendix B for definitions). So $\varepsilon_g = \frac{1}{2}dx^- \wedge dx^+$. On \mathcal{N}_L , ∂_{x^+} is future directed, $\partial_{x^+} \lrcorner (dx^- \wedge dx^+) = -dx^-$. So $-x^-$ is a future oriented chart on \mathcal{N}_L . On \mathcal{N}_R , ∂_{x^-} is future directed, $\partial_{x^-} \lrcorner (dx^- \wedge dx^+) = dx^+$. So x^+ is a future oriented chart on \mathcal{N}_R . Hence, when evaluating integrals over \mathcal{N} we have to integrate from smaller to greater x^+ , but from greater to smaller x^- . (5.14) further turns into

$$\begin{aligned} \omega(\delta_1, \delta_2) &= \\ &= -\frac{\kappa}{2} \int_0^a dx^+ \epsilon_{+-} \rho \text{Tr}[\Delta_2 P^- \mathcal{V}^{-1} \Delta_1 \mathcal{V}] - \frac{\kappa}{2} \int_b^0 dx^- \epsilon_{+-} \rho \text{Tr}[\Delta_2 P^+ \mathcal{V}^{-1} \Delta_1 \mathcal{V}] - (1 \leftrightarrow 2) = \\ &= \kappa \int_0^a dx^+ \rho \text{Tr}[\Delta_2 P_+ \mathcal{V}^{-1} \Delta_1 \mathcal{V}] + \kappa \int_0^b dx^- \rho \text{Tr}[\Delta_2 P_- \mathcal{V}^{-1} \Delta_1 \mathcal{V}] - (1 \leftrightarrow 2). \end{aligned}$$

Expressing ΔP_μ in terms of $\Delta \mathcal{V}$

$$\Delta P_\mu = (-\mathcal{V}^{-1} \Delta \mathcal{V} J_\mu + \mathcal{V}^{-1} \partial_\mu \Delta \mathcal{V})|_{\mathfrak{f}} = (\partial_\mu (\mathcal{V}^{-1} \Delta \mathcal{V}) + [J_\mu, \mathcal{V}^{-1} \Delta \mathcal{V}])|_{\mathfrak{f}} = D_\mu (\mathcal{V}^{-1} \Delta \mathcal{V}),$$

the first integral becomes

$$\begin{aligned} &\int_0^a dx^+ \rho \text{Tr} \{D_+ (\mathcal{V}^{-1} \Delta_1 \mathcal{V}) \mathcal{V}^{-1} \Delta_2 \mathcal{V}\} - \int_0^a dx^+ \rho \text{Tr} \{(-\mathcal{V}^{-1} \Delta_2 \mathcal{V} J_+ + \mathcal{V}^{-1} \partial_+ \Delta_2 \mathcal{V}) \mathcal{V}^{-1} \Delta_1 \mathcal{V}\} = \\ &= \int_0^a dx^+ \rho \text{Tr} \{D_+ (\mathcal{V}^{-1} \Delta_1 \mathcal{V}) \mathcal{V}^{-1} \Delta_2 \mathcal{V} + \mathcal{V}^{-1} \Delta_2 \mathcal{V} J_+ \mathcal{V}^{-1} \Delta_1 \mathcal{V} - J_+ \mathcal{V}^{-1} \Delta_2 \mathcal{V} \mathcal{V}^{-1} \Delta_1 \mathcal{V} + \\ &\quad + \mathcal{V}^{-1} \Delta_2 \mathcal{V} \partial_+ (\mathcal{V}^{-1} \Delta_1 \mathcal{V}) + \rho^{-1} \partial_+ \rho \mathcal{V}^{-1} \Delta_2 \mathcal{V} \mathcal{V}^{-1} \Delta_1 \mathcal{V}\} - [\rho \text{Tr} \{\mathcal{V}^{-1} \Delta_2 \mathcal{V} \mathcal{V}^{-1} \Delta_1 \mathcal{V}\}] \Big|_0^a = \\ &= \int_0^a dx^+ 2\sqrt{\rho} \text{Tr} \{D_+ (\sqrt{\rho} \mathcal{V}^{-1} \Delta_1 \mathcal{V}) \mathcal{V}^{-1} \Delta_2 \mathcal{V}\} - [\rho \text{Tr} \{\mathcal{V}^{-1} \Delta_2 \mathcal{V} \mathcal{V}^{-1} \Delta_1 \mathcal{V}\}] \Big|_0^a. \end{aligned}$$

With the second integral we do the exact same thing, but with the roles of δ_1 and δ_2 exchanged, giving us an additional minus sign by the antisymmetry of the integral w.r.t. $\delta_1 \leftrightarrow \delta_2$. We get

$$-2 \int_0^a dx^- \sqrt{\rho} \text{Tr} \{D_- (\sqrt{\rho} \mathcal{V}^{-1} \Delta_2 \mathcal{V}) \mathcal{V}^{-1} \Delta_1 \mathcal{V}\} + [\rho \text{Tr} \{\mathcal{V}^{-1} \Delta_2 \mathcal{V} \mathcal{V}^{-1} \Delta_1 \mathcal{V}\}] \Big|_0^a.$$

Adding the two cancels the boundary term at 0 (the point (x_0^-, x_0^+) where Nl and \mathcal{N}_R meet). We are left with

$$\begin{aligned} \omega(\delta_1, \delta_2) &= 2\kappa \int_0^a dx^+ \sqrt{\rho} \text{Tr} \{D_+ (\sqrt{\rho} \mathcal{V}^{-1} \Delta_1 \mathcal{V}) \mathcal{V}^{-1} \Delta_2 \mathcal{V}\} - \\ &\quad - 2\kappa \int_0^a dx^- \sqrt{\rho} \text{Tr} \{D_- (\sqrt{\rho} \mathcal{V}^{-1} \Delta_2 \mathcal{V}) \mathcal{V}^{-1} \Delta_1 \mathcal{V}\} - \\ &\quad - [\rho \text{Tr} \{\mathcal{V}^{-1} \Delta_2 \mathcal{V} \mathcal{V}^{-1} \Delta_1 \mathcal{V}\}] \Big|_{(x_0^-, x_0^+)}^{(x_0^-, x_0^+ + a)}. \end{aligned}$$

Following [32], we will now use (3.10) to calculate the Poisson bracket on \mathcal{N} as a generalized inverse of ω

$$\omega(\{\cdot, \phi\}, \delta) = \delta\phi \quad (5.16)$$

for ϕ a gauge invariant function on phase space. By the term generalized we mean the following: The ultimate goal of a Poisson bracket should be the computation of the bracket of observables, which carry the physically relevant content of the theory. In [32] it is suggested that one take as observables functions on phase space that are gauge invariant and such that the support of the functional gradient

$$\frac{\delta\phi}{\delta\mathcal{V}}$$

is compact and disjoint from the boundary. Then in an expression such as

$$\{\phi, \psi\} = \int_{\mathcal{N}} dx \int_{\mathcal{N}} dy \frac{\delta\phi}{\delta\mathcal{V}(x)_{ij}} \frac{\delta\psi}{\delta\mathcal{V}(y)_{kl}} \{\mathcal{V}(x)_{ij}, \mathcal{V}(y)_{kl}\}$$

the value of the Poisson bracket on the boundary $\partial\mathcal{N}$ will not contribute. We therefore drop the boundary terms at a ($(x_0^- + b, x_0^+)$ and $(x_0^-, x_0^+ + a)$). Although in the sequel we will calculate brackets between functions, which are not 0 on $\partial\mathcal{N}$, particularly the \mathcal{V} , at the end of the day this should not make a difference. Furthermore, two points $x \in \mathcal{N}_R \setminus \{(x_0^+, x_0^-)\}$ and $y \in \mathcal{N}_L \setminus \{(x_0^+, x_0^-)\}$ cannot be causally connected. Therefore we set $\{\mathcal{V}(x), \mathcal{V}(y)\} = 0$ for such points (see [32] for a more rigorous justification of this step). The two integrals separately define the Poisson bracket on their domain. For $\frac{\delta\phi}{\delta\mathcal{V}}$ having support only in $\mathcal{N}_R \setminus \{(x_0^+, x_0^-)\}$ and δ and thus Δ vanishing on the boundary $\partial\mathcal{N}_R$, we thus have

$$\delta\phi = \omega(\{\cdot, \phi\}, \delta) = 2\kappa \int_0^a dx^+ \sqrt{\rho} Tr [D_+(\sqrt{\rho}\mathcal{V}^{-1} \{\mathcal{V}, \phi\})\mathcal{V}^{-1}\Delta\mathcal{V}]. \quad (5.17)$$

We are interested in the Poisson bracket of \mathcal{V} with \mathcal{V} in the following sense: The matrix \mathcal{V} contains the original fields Δ (not the variation, but the field introduced in section 2.1) and B . The Poisson bracket between any two elements $\mathcal{V}_{ij}(x)$ and $\mathcal{V}_{kl}(y)$ at points x and y will certainly be sufficient to determine the bracket between the original fields. In index notation the expression

$$\{\mathcal{V}_{ij}(x), \mathcal{V}_{kl}(y)\}$$

is an element of the tensor space $\mathcal{T}_x G \otimes \mathcal{T}_y G$ and when pulled back to $\mathcal{T}_e G = \mathfrak{g}$

$$\mathcal{V}_{ij}^{-1}(x)\mathcal{V}_{lm}^{-1}(y) \{\mathcal{V}_{jk}(x), \mathcal{V}_{mn}(y)\}$$

is an element of $\mathcal{T}_e G \otimes \mathcal{T}_e G = \mathfrak{g} \otimes \mathfrak{g}$. We introduce the notation

$$\overset{i}{A}$$

for a \mathfrak{g} -valued field A living in the i -th copy of \mathfrak{g} in the tensor product of sufficiently many copies of \mathfrak{g} . Usually 2 copies will be sufficient. Then

$$(\overset{1}{A})_{ijkl} = (\overset{1}{A} \otimes \mathbb{1})_{ijkl} = A_{ij}\delta_{kl} \quad \text{and} \quad (\overset{2}{A})_{ijkl} = (\mathbb{1} \otimes A)_{ijkl} = \delta_{ij}A_{kl}.$$

Analogously

$$Tr^i$$

denotes the trace taken only in the i -th copy.

We have the expression (5.17) for $\delta\phi$, but by the chain rule we also have

$$\delta\phi = \int_0^a dy^+ \frac{\delta\phi}{\delta\mathcal{V}_{ij}(y)} \delta\mathcal{V}_{ij}(y) = \int_0^a dy^+ \frac{\delta\phi}{\delta\mathcal{V}_{ij}(y)} \mathcal{V}_{ik}(y) \mathcal{V}_{kl}^{-1}(y) \delta\mathcal{V}_{lj}(y). \quad (5.18)$$

Since ϕ is an observable this expression must vanish for all gauge transformations $\delta \in \mathfrak{K}$ by (5.17). The gauge generators are degeneracy vectors of the symplectic form by definition and hence

$$\frac{\delta\phi}{\delta\mathcal{V}_{ij}} \mathcal{V}_{ik} \in \mathfrak{k}^*.$$

δ can be substituted by Δ in (5.18).

Subtracting (5.17) from (5.18) and using the chain rule once more in $\{\mathcal{V}, \phi\}$ in (5.17), we have

$$\begin{aligned} 0 &= \int_0^a dy^+ \frac{\delta\phi}{\delta\mathcal{V}(y)} \mathcal{V}(y) \lrcorner \\ &\lrcorner \left(2\kappa \int_0^a dx^+ \sqrt{\rho(x)} Tr^1 \left[D_{x^+} \left(\sqrt{\rho(x)} \mathcal{V}^{-1}(x) \mathcal{V}^{-1}(y) \{ \mathcal{V}^1(x), \mathcal{V}^2(y) \} \right) \mathcal{V}^{-1}(x) \Delta \mathcal{V}^1(x) \right] - \right. \\ &\quad \left. - \mathcal{V}^{-1}(y) \Delta \mathcal{V}^2(y) \right). \end{aligned} \quad (5.19)$$

Here \lrcorner denotes the dual pairing between \mathfrak{k}^* and \mathfrak{k} . Now, $\frac{\delta\phi}{\delta\mathcal{V}(y)} \mathcal{V}(y)$ can be any element of \mathfrak{k}^* and so

$$\begin{aligned} 2\kappa \int_0^a dx^+ \sqrt{\rho(x)} Tr^1 \left[D_{x^+} \left(\sqrt{\rho(x)} \mathcal{V}^{-1}(x) \mathcal{V}^{-1}(y) \{ \mathcal{V}^1(x), \mathcal{V}^2(y) \} \right) \right]_{\mathfrak{k}} \mathcal{V}^{-1}(x) \Delta \mathcal{V}^1(x) &= \\ &= \mathcal{V}^{-1}(y) \Delta \mathcal{V}^2(y). \end{aligned} \quad (5.20)$$

We see that only the $\mathfrak{K} \otimes \mathfrak{K}$ -part of the Poisson bracket¹ is specified by this equation.

We have to choose the bracket such that

$$D_{\pm} \left(\sqrt{\rho} \mathcal{V}^{-1}(x) \mathcal{V}^{-1}(y) \{ \mathcal{V}^1(x), \mathcal{V}^2(y) \} \right) \Big|_{\mathfrak{k} \otimes \mathfrak{k}}^{1 \otimes 2}$$

1) transfers $\mathcal{V}^{-1} \Delta_2 \mathcal{V}^1$ to the second copy of the tensor product via the Trace operation, 2) cancels the factor $2\kappa\sqrt{\rho}$ and 3) eats up the integral and evaluates the remaining $\mathcal{V}^{-1} \Delta \mathcal{V}$ at y . This can be achieved by

$$D_{\pm} \left(\sqrt{\rho} \mathcal{V}^{-1}(x) \mathcal{V}^{-1}(y) \{ \mathcal{V}^1(x), \mathcal{V}^2(y) \} \right) \Big|_{\mathfrak{k} \otimes \mathfrak{k}}^{1 \otimes 2} = \frac{\delta(x-y)}{2\kappa\sqrt{\rho}} \Omega_{\mathfrak{k}}, \quad (5.21)$$

where $\Omega_{\mathfrak{k}}$ is an element of $\mathfrak{g} \otimes \mathfrak{g}$ which takes care of 1) above. What exactly is $\Omega_{\mathfrak{k}}$? Note that the trace in the fundamental 2×2 -representation of $\mathfrak{gl}(2, R)$, which enters the integral (5.20), defines a symmetric, invariant², and non-degenerate inner product $\langle a, b \rangle = Tr[ab]$

¹Or equivalently the $\mathfrak{k} \otimes \mathfrak{k}$ -part of $\mathcal{V}^{-1}(x) \mathcal{V}^{-1}(y) \{ \mathcal{V}^1(x), \mathcal{V}^2(y) \}$

²invariant under the adjoint action in the sense that $\langle [x, y], z \rangle + \langle y, [x, z] \rangle = 0$.

on \mathfrak{g} . So

$$\langle \Omega, A \rangle = A \quad \forall A \in \mathfrak{g} \quad (5.22)$$

defines Ω uniquely. In a basis $\{e_i\}$ of \mathfrak{g} the form $\langle \cdot, \cdot \rangle$ has components $\kappa_{ij} = \langle e_i, e_j \rangle$. The set $\{e^i\}$ defined by $e^i \kappa_{ij} = e_j$ is basis for \mathfrak{g}^* because of the assumed non-degeneracy of $\langle \cdot, \cdot \rangle$. Then

$$\langle e^i \otimes e^j, e_j \rangle = e^i \quad \forall j$$

and so $e^i \otimes e^j$ fulfils all the requirements on Ω and so is equal to Ω since they define it uniquely.

We now work out another property of Ω , which will be of use later on. We assumed invariance of $\langle \cdot, \cdot \rangle$. This implies

$$\begin{aligned} - \langle [B, \Omega], A \rangle &= \langle \Omega, [B, A] \rangle = [B, A] = [B, \langle \Omega, A \rangle] = \langle [B, \Omega], A \rangle \quad \forall A, B \in \mathfrak{g} \\ \Rightarrow \langle [B + B, \Omega], A \rangle &= 0 \quad \forall A, B \in \mathfrak{g} \quad \Rightarrow [B + B, \Omega] = 0 \quad \forall B \in \mathfrak{g} \end{aligned}$$

and for a group element in the vicinity of the identity

$$\frac{d}{dt} (e^{tB} e^{tB} \Omega e^{-tB} e^{-tB}) = e^{tB} e^{tB} [B + B, \Omega] e^{-tB} e^{-tB} = 0$$

and

$$\begin{aligned} (e^{tA} e^{tA} \Omega e^{-tA} e^{-tA}) \Big|_{t=0} &= \Omega \\ \Rightarrow e^{tA} e^{tA} \Omega e^{-tA} e^{-tA} &= \Omega. \end{aligned} \quad (5.23)$$

Ω is invariant under the adjoint action of G .

Ω , defined in this way, is almost what we need. ‘‘Almost’’ because since the left hand side of (5.21) is in $\mathfrak{k} \otimes \mathfrak{k}$, the right hand side also has to be in $\mathfrak{k} \otimes \mathfrak{k}$, which is why we use only the part of Ω which is in $\mathfrak{k} \otimes \mathfrak{k}$, namely $\Omega_{\mathfrak{k} \otimes \mathfrak{k}} := \Omega \Big|_{\mathfrak{k} \otimes \mathfrak{k}}$. Note that $\Omega_{\mathfrak{k}}$ is not invariant under the adjoint action of the whole of G because the projection onto \mathfrak{k} does not commute with it. It is though invariant under the action of $H \subset G$, which preserves the subspaces. Finally, we mention that if we use $\kappa = \langle \cdot, \cdot \rangle$ as a metric on \mathfrak{g} , then Ω is the inverse metric. Just like in differential geometry using abstract indices a, b, c, \dots (5.22) would be

$$\kappa_{ab} A^b \Omega^{ac} = A^c \quad \forall A^b \quad \Rightarrow \quad \kappa_{ab} \Omega^{ac} = \delta_b^c.$$

As mentioned above only the $\mathfrak{k} \otimes \mathfrak{k}$ -part of $\mathcal{V}^{-1} \mathcal{V}^{-1} \{\mathcal{V}, \mathcal{V}\}$ is specified by (5.17). We denote this part by $A_{\mathfrak{k} \otimes \mathfrak{k}}$ ³ and the other parts by

$$\mathcal{V}^{-1} \mathcal{V}^{-1} \{\mathcal{V}, \mathcal{V}\} = A_{\mathfrak{g}\mathfrak{g}}^{12} = A_{\mathfrak{k}\mathfrak{k}}^{12} + A_{\mathfrak{k}\mathfrak{h}}^{12} + A_{\mathfrak{h}\mathfrak{k}}^{12} + A_{\mathfrak{h}\mathfrak{h}}^{12}. \quad (5.24)$$

Let us now solve equation (5.21) for $A_{\mathfrak{k}\mathfrak{k}}$. Suppose we choose the gauge $Q_+ = 0$ and thus $D_+ = \partial_+$. Then

$$A_{\mathfrak{k}\mathfrak{k}}(x, y) = \frac{\text{sign}(x - y)}{4\kappa \sqrt{\rho(x)} \sqrt{\rho(y)}} \Omega_{\mathfrak{k}} + \frac{1}{4\kappa \sqrt{\rho(x)} \sqrt{\rho(y)}} C,$$

³We hope not to confuse by omitting dependences on spacetime coordinates and numbers indicating in which copies of \mathfrak{g} an object lies in situations where this should be clear.

where $C \in \mathfrak{k} \otimes \mathfrak{k}$ is independent of x and y , would do the job. We chose the $\text{sign}(x - y)$ -function as antiderivative of the $\delta(x - y)$ because it is antisymmetric. Since the left hand side of the equation and the first term on the right hand side are antisymmetric w.r.t. the simultaneous interchange of x and y and spaces 1 and 2 the last term on the right hand side must be also. C is thus independent of y as well as x , and antisymmetric with respect to interchange of $\overset{1}{\mathfrak{k}}$ and $\overset{2}{\mathfrak{k}}$. Since \mathfrak{k} is 2-dimensional this implies

$$C = c(e_1 \otimes e_2 - e_2 \otimes e_1) =: c \varepsilon_{\mathfrak{k}},$$

where c is a constant. The value of c is the only free parameter in the solution (5.21). $\varepsilon_{\mathfrak{k}}$ is “the volume element on \mathfrak{k} ”. In terms of an arbitrary basis $\{t_1, t_2\}$ we could write

$$\varepsilon_{\mathfrak{k}} = \sqrt{|\det(g)|} (t_1 \otimes t_2 - t_2 \otimes t_1),$$

where g is the component matrix of the metric on \mathfrak{k} w.r.t. the t_i . Since the adjoint action of an element of H preserves \mathfrak{k} and the metric we have

$$\overset{1}{h} \overset{2}{h} \varepsilon_{\mathfrak{k}} \overset{1}{h}^{-1} \overset{2}{h}^{-1} = \varepsilon_{\mathfrak{k}} \quad h \in H,$$

just as it is the case for $\Omega_{\mathfrak{k}}$.

(5.24) is the result of integrating the differential equation (5.21) for the gauge $Q_+ = 0$. In an arbitrary gauge, where Q_+ in general does not vanish, the covariant derivative D_+ in the differential equation (5.21) contains an additional, adjoint action of the Lie algebra element $Q_+(x)$. For the integral we thus need the integrated version of this action, which is the adjoint action of the path ordered exponential, a Lie group element and its inverse acting from the right and left respectively,

$$\overset{1}{\mathcal{P}e} \int_{x_0}^x Q_+ \quad \text{and} \quad \overset{1}{\mathcal{P}e} \int_{x_0}^{x_0+} Q_+$$

where $\mathcal{P}e$ denotes the path ordered exponential. Definitions and conventions are given in appendix D. In general such a path ordered exponential depends on the path and is related to the holonomy of the connection used. In our case, modulo reparametrisation, which of course does not change the integral, there is only one path since the light cone in our case is only one dimensional. Therefore this expression really is a function of x (and the initial point x_0) only. In order to keep antisymmetry we also include the factors

$$\overset{2}{\mathcal{P}e} \int_{x_0}^y Q_+ \quad \text{and} \quad \overset{2}{\mathcal{P}e} \int_{y_0}^{x_0+} Q_+$$

arriving at

$$A_{\mathfrak{k}\mathfrak{k}}(x, y) = \frac{1}{4\kappa\sqrt{\rho(x)}\sqrt{\rho(y)}} \overset{1}{\mathcal{P}e} \int_{x_0}^{x_0+} Q_+ \overset{2}{\mathcal{P}e} \int_{y_0}^{x_0+} Q_+ \left(\text{sign}(x - y)\Omega_{\mathfrak{k}} + C \right) \overset{2}{\mathcal{P}e} \int_{x_0}^y Q_+ \overset{1}{\mathcal{P}e} \int_{x_0}^x Q_+.$$

This indeed satisfies (5.21).

$$\begin{aligned} \partial_{x^+} \left(\sqrt{\rho(x)} A_{\mathfrak{k}\mathfrak{k}}(x, y) \right) &= \frac{\delta(x - y)}{2\kappa\sqrt{\rho(y)}} \overset{1}{\mathcal{P}e} \int_{x_0}^{x_0+} Q_+ \overset{2}{\mathcal{P}e} \int_{y_0}^{x_0+} Q_+ \Omega_{\mathfrak{k}} \overset{2}{\mathcal{P}e} \int_{x_0}^y Q_+ \overset{1}{\mathcal{P}e} \int_{x_0}^x Q_+ \\ &- \left[Q_+(x), \frac{1}{2\kappa\sqrt{\rho(y)}} \overset{1}{\mathcal{P}e} \int_{x_0}^{x_0+} Q_+ \overset{2}{\mathcal{P}e} \int_{y_0}^{x_0+} Q_+ \left(\text{sign}(x - y)\Omega_{\mathfrak{k}} + C \right) \overset{2}{\mathcal{P}e} \int_{x_0}^y Q_+ \overset{1}{\mathcal{P}e} \int_{x_0}^x Q_+ \right]. \end{aligned}$$

In the first term on the right hand side the $\delta(x - y)$ allows us to evaluate both the $\mathcal{P}e^1$ and the $\mathcal{P}e^2$ factor at x . Because of the invariance of Ω and $\Omega_{\mathfrak{t}}$ (5.23), this term is just the right hand side of (5.21). The commutator is $[Q_+, \sqrt{\rho}A_{kk}]$, just the term which makes a covariant derivative D_+ out of the partial derivative ∂_+ . We get

$$\begin{aligned} & \mathcal{V}^{-1}(x)\mathcal{V}^{-1}(y)\{\mathcal{V}(x), \mathcal{V}(y)\} = \\ & = \frac{1}{4\kappa\sqrt{\rho(x)\rho(y)}}\mathcal{P}e^1 \int_{x_0}^{x_0-xQ_+} \mathcal{P}e^2 \int_{y_0}^{y_0+Q_+} \left(\text{sign}(x-y)\Omega_{\mathfrak{t}} + C\right)\mathcal{P}e^2 \int_{x_0}^{y_0+Q_+} \mathcal{P}e^1 \int_{x_0}^{x_0+Q_+}. \end{aligned} \quad (5.25)$$

As mentioned above, $\Omega_{\mathfrak{t}}$ and $\varepsilon_{\mathfrak{t}}$ are invariant under the simultaneous adjoint action of H on both spaces. This fact allows us to write

$$\begin{aligned} & \{\mathcal{V}(x), \mathcal{V}(y)\} = \\ & = \frac{1}{4\kappa\sqrt{\rho(x)\rho(y)}}\mathcal{V}(x)\mathcal{V}(y)\mathcal{P}e^1 \int_x^{y_0+Q_+} \left(\text{sign}(x-y)\Omega_{\mathfrak{t}} + c\varepsilon_{\mathfrak{t}}\right)\mathcal{P}e^1 \int_y^{x_0+Q_+} \end{aligned} \quad (5.26)$$

$$\{\mathcal{V}(x), \mathcal{V}(y)\} = 0 \quad \text{for } x, y \text{ on different branches of } \Sigma, \quad (5.27)$$

where we set $C = c\varepsilon_{\mathfrak{t}}$. (5.26) shows the independence of x_0 . We see that as opposed to Poisson brackets on spacelike Cauchy surfaces, our bracket $\{\mathcal{V}(x), \mathcal{V}(y)\}$ is no longer 0 for $x \neq y$. This result reflects the fact that points on the same generator of the light cone are causally connected (see also [32]).

5.4 Gauge fixing and relation to the bracket of the variables $\mu, \bar{\mu}$ in [32]

In this section we fix the gauge corresponding to the local $\mathfrak{h} = \mathfrak{so}(2)$ -symmetry of \mathcal{V} . A computation of the Dirac bracket on the constraint surface defined by the gauge fixing condition will allow us to fix the parts of A_{gg} not determined by the symplectic 2-form. In the gauge we choose, the entries of \mathcal{V} are especially simple functions of the variables μ and $\bar{\mu}$ used in [32] to parametrise the unimodular metric on the initial data surface used there. Note that our initial data surface (a part of the light cone) is a special form of the surface used in [32] when two commuting, spacelike, hypersurface orthogonal Killing fields are present.

We recall that in the Hamiltonian treatment⁴ of a theory with possibly singular Lagrangian one is typically left with the cotangent bundle and canonical 2-form as a symplectic manifold, first class constraints corresponding to gauge symmetries and possibly second class constraints (see [12]). One may then introduce a gauge fixing condition of the type

$$C(q) = 0 \quad (5.28)$$

⁴We use the terms Hamiltonian and Lagrangian formulation to distinguish between models using the cotangent and the tangent bundle respectively as symplectic manifolds.

q being the dynamical fields. If such a condition is to fix the gauge under consideration uniquely, then the surface defined by (5.28) must intersect each gauge orbit exactly once. This gauge fixing condition is another constraint which, together with the previously first class constraint generating the gauge transformation in question, forms a pair of second class constraints. One first class constraint is replaced by two second class constraints. As explained by Dirac, second class constraints are generically a problem when one wants to use the canonical Poisson structure to quantize the system. This is because by definition the Poisson bracket of a second class constraints $\phi_2(q)$ with at least one other constraint $\phi_o(q)$ does not vanish weakly (that is does not vanish when the constraints are imposed). For example one might have

$$\{\phi_2, \phi_o\} = c$$

c being a non-zero constant. In a quantization one would want to turn ϕ_2 and ϕ_o into operators $\hat{\phi}_2$ and $\hat{\phi}_o$ obeying

$$\hat{\phi}_2\psi = 0 \quad \hat{\phi}_o\psi = 0 \quad [\hat{\phi}_2, \hat{\phi}_o]\psi = i\hbar c\psi,$$

ψ being the wave function, which is not possible ($[\hat{\phi}_2, \hat{\phi}_o]\psi = \hat{\phi}_2\hat{\phi}_o\psi - \hat{\phi}_o\hat{\phi}_2\psi = 0 \neq c\hbar\psi$ if $\hat{\phi}_2\psi = 0$ and $\hat{\phi}_o\psi = 0$). Dirac proposed to solve this problem by introducing a new bracket, which satisfies

$$\{q, \phi_2^i\}_* = 0 \tag{5.29}$$

for all the second class constraints ϕ_2^i . This bracket, called the Dirac bracket, is the inverse of the symplectic 2-form pulled back to the gauge fixed constraint surface. One may say that it is permissible to impose the second class constraints already before computing the bracket. Variations generated by phase space functions via the Dirac bracket do not take the system out of the constraint surface. If the second class constraints arose from gauge fixing then it is the bracket on the gauge fixed constraint surface.

Although we do not use a Hamiltonian formulation EG2CHSKF, we are in a similar situation. We have an \mathfrak{h} gauge symmetry, the freedom to locally rotate the zweibein. (3.10) can be used to determine the Poisson bracket by the symplectic 2-form only for gauge invariant functions, while there remains some freedom, $A_{\mathfrak{t}\mathfrak{h}}, A_{\mathfrak{h}\mathfrak{t}}$ and $A_{\mathfrak{h}\mathfrak{h}}$, in the bracket of functions not gauge invariant. In the Hamiltonian treatment this corresponds to the fact that one may actually change the Poisson bracket from the canonical Poisson bracket to the Dirac bracket without altering the dynamics, the essential content of the theory. The condition (5.29) may be imposed on any Poisson manifold, not only on a cotangent bundle with canonical symplectic and Poisson structure, in order to consistently fix the gauge. This is what we will do now. We will impose a gauge condition and use (5.29) in order to fix $A_{\mathfrak{g}\mathfrak{g}}$ completely.

5.4.1 A different group structure

Up to now the fields \mathcal{V} were $SL(2, \mathbb{R})$ -valued ($SL(2, \mathbb{R}) \subset SL(2, \mathbb{C})$). In this subsection we will introduce a different subgroup of $SL(2, \mathbb{C})$, which is convenient for the purpose of this section. First we remember how we obtained these group valued fields. We used the two Killing vectors to perform two Kaluza-Klein reductions to obtain an effectively two dimensional problem. This resulted in a two dimensional metric in the remaining dimensions, which we could choose to be flat. The remaining fields were ρ , λ and \mathcal{V} or

dually ρ , $\tilde{\lambda}$ and $\tilde{\mathcal{V}}$. $\tilde{\mathcal{V}}$ was simply the lower right 2×2 block in (2.25) divided by its determinant. Although \mathcal{V} ($\tilde{\mathcal{V}}$) was introduced in a triangular gauge, we later spoke of \mathcal{V} ($\tilde{\mathcal{V}}$) simply as an $SL(2, \mathbb{R})$ -valued field. The triangular gauge is in no way mandatory. Thinking of $\tilde{\mathcal{V}}$ as a unimodular zweibein, we may regard it simply as a collection of the components of two one-forms in our preferred coordinate system, given by the parameters of the integral curves of the Killing fields. We denote the two coordinates given by the Killing vectors $\{\Theta^1, \Theta^2\}$. From this point of view the two indices of \mathcal{V} are quite different in their nature. One is an external one-form index and one is an internal index. Still \mathcal{V} is a real $SL(2, \mathbb{R})$ -matrix. If we call the unimodular metric e (as it is called in [32]), then we may write

$$e_{\sigma\nu} = \tilde{\mathcal{V}}_{\sigma}{}^i \delta_{ij} (\tilde{\mathcal{V}}^T)^j{}_{\nu}. \quad (5.30)$$

The same can be said about the Kramer-Neugebauer transform \mathcal{V} with the difference that it does not correspond to a unimodular metric regular on the axis. From now on we state everything in terms of \mathcal{V} . But all the results, in particular those of this section, which do not require regularity at the axis, are also valid for both \mathcal{V} and $\tilde{\mathcal{V}}$.

Let's now try to find a relation between this $SL(2, \mathbb{R})$ -valued field \mathcal{V} and the fields μ and $\bar{\mu}$ used in [32]. There the metric is factorised in the following way:

$$z := \Theta^1 + i\Theta^2 \quad \alpha := \frac{dz + \mu d\bar{z}}{\sqrt{1 - \mu\bar{\mu}}} =: v^1 + iv^2,$$

v^1 and v^2 being real one-forms. μ and $\bar{\mu}$ are related to the unimodular metric by

$$e = \frac{1}{2}(\alpha\bar{\alpha} + \bar{\alpha}\alpha) = \frac{1}{2}\left((v^1 + iv^2)(v^1 - iv^2) + (v^1 - iv^2)(v^1 + iv^2)\right) = v^1v^1 + v^2v^2$$

or in components

$$e_{\sigma\nu} = v_{\sigma}{}^i \delta_{ij} (v^T)^j{}_{\nu}.$$

We see that w.r.t. the Θ^1, Θ^2 -coordinate system the component matrix $v_{\sigma}{}^i$ defined in this way is used in the exact same way as $\mathcal{V}_{\sigma}{}^i$. It is also real and has unit determinant. In terms of μ and $\bar{\mu}$

$$\alpha = \frac{(1 + \mu)d\Theta^1 + i(1 - \mu)d\Theta^2}{\sqrt{1 - \mu\bar{\mu}}},$$

so

$$v^1 = \text{Re}(\alpha) = \text{Re}\left(\frac{dz + \mu d\bar{z}}{\sqrt{1 - \mu\bar{\mu}}}\right)$$

$$v^2 = \text{Im}(\alpha) = \frac{(1 + \mu)d\Theta^1 + i(1 - \mu)d\Theta^2}{\sqrt{1 - \mu\bar{\mu}}},$$

or in components

$$v_{\nu}{}^i = \frac{1}{\sqrt{1 - \mu\bar{\mu}}} \begin{pmatrix} \text{Re}(1 + \mu) & \text{Im}(\mu) \\ \text{Im}(\mu) & \text{Re}(1 - \mu) \end{pmatrix}_{\nu}{}^i. \quad (5.31)$$

If we interpret v as \mathcal{V} , then it is in the symmetric gauge. We could now proceed as follows: Formulate a condition for the symmetric gauge (actually an x -dependent family of conditions), compute the Dirac bracket using $\{\mathcal{V}(x), C(z)\} = 0$ and relate the bracket of the \mathcal{V} to the bracket of the μ and $\bar{\mu}$ by using (5.31) and the chain rule. But a slightly

different approach is instructive and somewhat easier to handle:

Instead of using as \mathcal{V}_μ^i the components w.r.t. the real coordinates Θ^1, Θ^2 we introduce new, complex conjugate coordinates

$$\begin{aligned} Z^1 := z &:= \frac{1}{\sqrt{2}}(\Theta^1 + i\Theta^2), & Z^2 := \bar{z} &:= \frac{1}{\sqrt{2}}(\Theta^1 - i\Theta^2), \\ \Theta^1 &= \frac{1}{\sqrt{2}}(Z^1 + Z^2), & \Theta^2 &= \frac{-i}{\sqrt{2}}(Z^1 - Z^2). \end{aligned}$$

The Jacobians for this transformation are

$$\begin{aligned} u_\nu^\sigma &:= \frac{\partial Z^\sigma}{\partial \Theta^\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}_\nu^\sigma, \\ (u^{-1})_\sigma^\nu &= \frac{\partial \Theta^\nu}{\partial Z^\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}_\sigma^\nu. \end{aligned}$$

This transformation acts on the one-form index of \mathcal{V} . We now do a second transformation acting on the internal index. For this second transformation we use the same matrix u defined above

$$\mathcal{V}_\sigma^i \mapsto \mathcal{V}_\sigma^j u_j^i.$$

The combination of these transformations yields

$$\mathcal{V}_\sigma^i \mapsto (u^{-1})_\sigma^\nu \mathcal{V}_\nu^j u_j^i$$

or explicitly

$$\begin{aligned} \frac{1}{\sqrt{1-\mu\bar{\mu}}} \begin{pmatrix} \operatorname{Re}(1+\mu) & \operatorname{Im}(\mu) \\ \operatorname{Im}(\mu) & \operatorname{Re}(1-\mu) \end{pmatrix} &\mapsto \frac{1}{2\sqrt{1-\mu\bar{\mu}}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} \operatorname{Re}(1+\mu) & \operatorname{Im}(\mu) \\ \operatorname{Im}(\mu) & \operatorname{Re}(1-\mu) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \\ &= \frac{1}{2\sqrt{1-\mu\bar{\mu}}} \begin{pmatrix} 1+\bar{\mu} & -i(1-\bar{\mu}) \\ 1+\mu & i(1-\mu) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \frac{1}{\sqrt{1-\mu\bar{\mu}}} \begin{pmatrix} 1 & \bar{\mu} \\ \mu & 1 \end{pmatrix}. \end{aligned}$$

We see that this transformed \mathcal{V} contains the fields μ and $\bar{\mu}$ in a simple way. Again, from now on, when talking about \mathcal{V} we mean this transformed \mathcal{V}

$$\mathcal{V}_\nu^i = \frac{1}{\sqrt{1-\mu\bar{\mu}}} \begin{pmatrix} 1 & \bar{\mu} \\ \mu & 1 \end{pmatrix}_\nu^i. \quad (5.32)$$

What about the group structure? Before the transformation \mathcal{V} was a real matrix of determinant 1, that is, an element of $SL(2, \mathbb{R})$. The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ consists of the real traceless matrices and is thus spanned by the basis

$$e_1 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Alternatively we may say that $\mathfrak{sl}(2, \mathbb{R})$ is spanned by the Pauli matrices themselves, but the coefficients of σ_x and σ_z have to be real while the coefficients of σ_y have to be purely imaginary.

Our new \mathcal{V} matrices will also be members of a group because they are related to the old \mathcal{V} by an adjoint action of the matrix u , which does not affect the group property. The new \mathcal{V} will certainly also have unit determinant and, by their relation to $SL(2, \mathbb{R})$, can be characterized as the subgroup of $SL(2, \mathbb{C})$, we call it \tilde{G} , of matrices \mathcal{V} for which

$$\bar{\mathcal{V}} = \overline{u^{-1}Au} = \overline{u^{-1}A\bar{u}} = \overline{u^{-1}u}\mathcal{V}u^{-1}\bar{u} = \sigma_x\mathcal{V}\sigma_x, \quad A \in SL(2, \mathbb{R}).$$

The Lie algebra $\tilde{\mathfrak{g}}$ of \tilde{G} can be characterized as follows

$$\mathcal{V}^{-1}\delta\mathcal{V} = u^{-1}A^{-1}uu^{-1}\delta Au = u^{-1}A^{-1}\delta Au, \quad A \in SL(2, \mathbb{R}).$$

The adjoint action of u translates directly from the group level to the Lie algebra level

$$\tilde{\mathfrak{g}} = u^{-1}\mathfrak{sl}(2, \mathbb{R})u.$$

For the Pauli matrices we have

$$u^{-1}\sigma_x u = \sigma_y, \quad u^{-1}\sigma_y u = \sigma_z, \quad u^{-1}\sigma_z u = \sigma_x. \quad (5.33)$$

$\tilde{\mathfrak{g}}$ is also spanned by $\sigma_x, \sigma_y, \sigma_z$, but now the coefficients of σ_x and σ_y have to be real while those of σ_z have to be imaginary.

The maximal compact subgroup $H = SO(2, \mathbb{R})$ of $G = SL(2, \mathbb{R})$ gets mapped to \tilde{H} and is generated by $u^{-1}i\sigma_y u = i\sigma_z$.

$$\tilde{H} = \{e^{it\sigma_z}, t \in \mathbb{R}\} = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, t \in \mathbb{R} \right\}$$

Finally note that the metric on the Lie algebra is not altered (the trace is invariant under the adjoint action).

This completes our analysis of the new group structure.

5.4.2 Derivation of $\{\mu(x), \bar{\mu}(y)\}$ from A_{eff}

We will now formulate a gauge fixing condition of the form $C(\mathcal{V}) = 0$ which restricts $\mathcal{V} \in \tilde{G}$ to the form (5.32). We found out that \tilde{G} can be characterized as the group of $SL(2, \mathbb{C})$ -matrices, which satisfy

$$\bar{\mathcal{V}} = \sigma_x\mathcal{V}\sigma_x.$$

Solving this relation, we find for the components

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} s & r \\ q & p \end{pmatrix} = \begin{pmatrix} \bar{p} & \bar{q} \\ \bar{r} & \bar{s} \end{pmatrix} \Rightarrow \bar{p} = s, \bar{q} = r.$$

If we define

$$\mu' = \frac{r}{\sqrt{1+r\bar{r}}}$$

then

$$r = \frac{\mu'}{\sqrt{1-\mu'\bar{\mu}'}}$$

and the condition for unit determinant takes the form

$$p\bar{p} - \frac{\mu'\bar{\mu}'}{1 - \mu'\bar{\mu}'} = 1$$

$$p\bar{p} = \frac{1}{1 - \mu'\bar{\mu}'}$$

Therefore \mathcal{V} is of the form

$$\frac{1}{\sqrt{1 - \mu'\bar{\mu}'}} \begin{pmatrix} e^{-it} & \bar{\mu}' \\ \mu' & e^{it} \end{pmatrix}, \quad t \in \mathbb{R}. \quad (5.34)$$

This can indeed be reduced to the form (5.32) via the gauge transformation $\mathcal{V} \rightarrow \mathcal{V}h$, with $h = e^{it\sigma_z} \in \tilde{H}$ and $\mu = e^{it}\mu'$.

If we impose the condition $p = s$ then the matrices (5.34) are precisely those of the form (5.32) (with $\mu = \mu'$). Our gauge condition will therefore be

$$C(\mathcal{V}) = Tr(\sigma_z \mathcal{V}) = Tr \begin{pmatrix} p & q \\ -r & -s \end{pmatrix} = p - s = 0. \quad (5.35)$$

Let us now impose (5.29).

$$0 = \mathcal{V}^{-1} \{ \mathcal{V}, C(\mathcal{V}) \}_* = \mathcal{V}^{-1} \{ \mathcal{V}, Tr \left[\sigma_z^2 \mathcal{V} \right] \}_* = Tr \left[\sigma_z^2 \mathcal{V} A_{\text{gg}}^{12} \right], \quad (5.36)$$

where A_{gg}^{12} is now defined using the Dirac bracket. The Dirac bracket may be obtained directly from this equation, sidestepping the calculation of the Poisson brackets of the constraints. Since $\mathcal{V}^{-1} \Delta \mathcal{V} = \mathcal{V}^{-1} \delta \mathcal{V}|_{\tilde{\mathfrak{h}}}$ is gauge invariant, A_{ff} calculated with the Dirac bracket and with the original Poisson bracket is the same. The equation (5.36) then determines the remaining components of A_{gg} , A_{fh} , A_{hf} and A_{hh} , in terms of A_{ff} . For A_{ff} we use (5.26) with c still undetermined. We introduce a useful parametrisation⁵

$$A_{\text{gg}}^{12} = A_{\text{ff}}^{12} + \sigma_z^2 \beta(x, y) + \gamma(x, y) \sigma_z^2 + \xi(x, y) \sigma_z^1 \sigma_z^2.$$

Antisymmetry implies

$$\gamma(x, y) = -\beta(y, x).$$

Using (5.33) we may transform A_{ff} , which we computed in the $SL(2, \mathbb{R})$ structure, to our new \tilde{G} structure.

$$\Omega_{\text{f}} = \frac{1}{2}(\sigma_x \sigma_x + \sigma_y \sigma_y) = \frac{1}{4}[(\sigma_x - i\sigma_y)(\sigma_x + i\sigma_y) + (\sigma_x + i\sigma_y)(\sigma_x - i\sigma_y)] = (\sigma_- \sigma_+ + \sigma_+ \sigma_-),$$

where $\sigma_+ = \frac{1}{2}(\sigma_x + i\sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\sigma_- = \frac{1}{2}(\sigma_x - i\sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

$$\varepsilon_{\text{f}} = \frac{1}{2}(\sigma_x \sigma_y - \sigma_y \sigma_x) = \frac{i}{4}[(\sigma_x + i\sigma_y)(\sigma_x - i\sigma_y) - (\sigma_x - i\sigma_y)(\sigma_x + i\sigma_y)] = i(\sigma_+ \sigma_- - \sigma_- \sigma_+)$$

⁵In our new group structure $\tilde{\mathfrak{h}}$ is spanned by σ_z .

$$\begin{aligned}
Q &= (\mathcal{V}^{-1}d\mathcal{V})|_{\tilde{\mathfrak{h}}} \\
\mathcal{V}^{-1}d\mathcal{V} &= \frac{1}{\sqrt{1-\mu\bar{\mu}}} \begin{pmatrix} 1 & -\bar{\mu} \\ -\mu & 1 \end{pmatrix} d \left[\frac{1}{\sqrt{1-\mu\bar{\mu}}} \begin{pmatrix} 1 & \bar{\mu} \\ \mu & 1 \end{pmatrix} \right] = \\
&= \frac{1}{2} \frac{d[\mu\bar{\mu}]}{1-\mu\bar{\mu}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{1-\mu\bar{\mu}} \begin{pmatrix} -\bar{\mu}d\mu & d\bar{\mu} \\ d\mu & -\mu d\bar{\mu} \end{pmatrix} = \\
&= \frac{1}{1-\mu\bar{\mu}} \begin{pmatrix} \frac{1}{2}[\mu d\bar{\mu} - \bar{\mu}d\mu] & d\bar{\mu} \\ d\mu & -\frac{1}{2}[\mu d\bar{\mu} - \bar{\mu}d\mu] \end{pmatrix} \tag{5.37}
\end{aligned}$$

$$Q = \frac{1}{2} \frac{\mu d\bar{\mu} - \bar{\mu}d\mu}{1-\mu\bar{\mu}} \sigma_z \tag{5.38}$$

$$\mathcal{P}e^{\int_x^y Q} = e^{\frac{1}{2}\alpha\sigma_z} \quad \alpha := \int_x^y \frac{\mu d\bar{\mu} - \bar{\mu}d\mu}{1-\mu\bar{\mu}}$$

$$\mathcal{P}e^{\int_x^y Q} \sigma_{\pm} \mathcal{P}e^{\int_y^x Q} = e^{\frac{1}{2}\alpha\sigma_z} \sigma_{\pm} e^{-\frac{1}{2}\alpha\sigma_z} = e^{\pm\frac{1}{2}\alpha} \sigma_{\pm} e^{\pm\frac{1}{2}\alpha} = e^{\pm\alpha} \sigma_{\pm}$$

$A_{\mathfrak{tt}}$ in our new group structure is thus

$$A_{\mathfrak{tt}}(x, y) = \frac{1}{4\kappa\sqrt{\rho(x)\rho(y)}} \left((\text{sign}(x-y) + ic)e^{-\alpha(x,y)}\sigma_+\sigma_- + (\text{sign}(x-y) - ic)e^{\alpha(x,y)}\sigma_-\sigma_+ \right)$$

With this knowledge we continue with (5.36). Projecting onto $\tilde{\mathfrak{k}}$ in the first space gives

$$0 = \overset{2}{T}r \left[\overset{2}{\sigma}_z \overset{2}{\mathcal{V}}(z) \overset{2}{\sigma}_z \right] \overset{1}{\gamma}(x, z) + \overset{2}{T}r \left[\overset{2}{\sigma}_z \overset{2}{\mathcal{V}}(z) A_{\mathfrak{tt}}^2(x, z) \right]$$

$$\sigma_z \mathcal{V} = \frac{1}{\sqrt{1-\mu\bar{\mu}}} \begin{pmatrix} 1 & \bar{\mu} \\ -\mu & -1 \end{pmatrix}$$

Therefore

$$\begin{aligned}
0 &= \frac{2\gamma(x, y)}{\sqrt{1-\mu(y)\bar{\mu}(y)}} + \frac{1}{\sqrt{1-\mu(y)\bar{\mu}(y)}} \frac{1}{4\kappa\sqrt{\rho(x)\rho(y)}} \times \\
&\quad \times \left(-\mu(y)(\text{sign}(x-y) - ic)e^{-\alpha(x,y)}\sigma_- + \bar{\mu}(y)(\text{sign}(x-y) + ic)e^{\alpha(x,y)}\sigma_+ \right)
\end{aligned}$$

$$\gamma(x, y) = \frac{1}{8\kappa\sqrt{\rho(x)\rho(y)}} \left(\mu(y)(\text{sign}(x-y) - ic)e^{-\alpha(x,y)}\sigma_- - \bar{\mu}(y)(\text{sign}(x-y) + ic)e^{\alpha(x,y)}\sigma_+ \right)$$

$$\beta(x, y) = -\gamma(y, x) =$$

$$= \frac{1}{8\kappa\sqrt{\rho(x)\rho(y)}} \left(\mu(x)(\text{sign}(x-y) + ic)e^{\alpha(x,y)}\sigma_- - \bar{\mu}(x)(\text{sign}(x-y) - ic)e^{-\alpha(x,y)}\sigma_+ \right),$$

while projection of (5.36) onto $\tilde{\mathfrak{h}}$ gives

$$0 = \overset{2}{T}r \left(\overset{2}{\sigma}_z \overset{2}{\mathcal{V}}(z) \overset{2}{\sigma}_z \right) \overset{1}{\sigma}_z \xi(x, z) + \overset{2}{T}r \left(\overset{2}{\sigma}_z \overset{2}{\mathcal{V}}(z) \overset{2}{\beta}(x, z) \right) \overset{1}{\sigma}_z$$

$$\begin{aligned}
0 &= \frac{2\xi(x, y)}{\sqrt{1-\mu(y)\bar{\mu}(y)}} + \frac{1}{\sqrt{1-\mu(y)\bar{\mu}(y)}} \frac{1}{8\kappa\sqrt{\rho(x)\rho(y)}} \left(\mu(y)\bar{\mu}(x)(\text{sign}(x-y) - ic)e^{-\alpha(x,y)} + \right. \\
&\quad \left. + \bar{\mu}(y)\mu(x)(\text{sign}(x-y) + ic)e^{\alpha(x,y)} \right)
\end{aligned}$$

$$\xi(x, y) = \frac{-1}{16\kappa\sqrt{\rho(x)\rho(y)}} \left(\mu(y)\bar{\mu}(x) \left(\text{sign}(x-y) - ic \right) e^{-\alpha(x,y)} + \bar{\mu}(y)\mu(x) \left(\text{sign}(x-y) + ic \right) e^{\alpha(x,y)} \right).$$

The final result can be written in a factorized form

$$A_{\text{gg}}(x, y) = \frac{1}{4\kappa\sqrt{\rho(x)\rho(y)}} \left[\left(\text{sign}(x-y) - ic \right) e^{-\alpha(x,y)} \left(\sigma_- - \frac{\bar{\mu}(x)}{2}\sigma_z \right) \otimes \left(\sigma_+ + \frac{\mu(y)}{2}\sigma_z \right) + \left(\text{sign}(x-y) + ic \right) e^{\alpha(x,y)} \left(\sigma_+ + \frac{\mu(x)}{2}\sigma_z \right) \otimes \left(\sigma_- - \frac{\bar{\mu}(y)}{2}\sigma_z \right) \right]. \quad (5.39)$$

To derive $\{\mu(x), \bar{\mu}(y)\}$ we note that for any variation δ , just like in (5.37) above, we have

$$\begin{aligned} \mathcal{V}^{-1}\delta\mathcal{V} &= \frac{1}{1-\mu\bar{\mu}} \begin{pmatrix} \frac{1}{2}(\mu\delta\bar{\mu} - \bar{\mu}\delta\mu) & \delta\bar{\mu} \\ \delta\mu & -\frac{1}{2}(\mu\delta\bar{\mu} - \bar{\mu}\delta\mu) \end{pmatrix} = \\ &= \frac{1}{1-\mu\bar{\mu}} \left(\delta\mu\sigma_- + \delta\bar{\mu}\sigma_+ + \frac{1}{2}(\mu\delta\bar{\mu} - \bar{\mu}\delta\mu)\sigma_z \right) = \\ &= \frac{1}{1-\mu\bar{\mu}} \left(\delta\mu \left[\sigma_- - \frac{\bar{\mu}}{2}\sigma_z \right] + \delta\bar{\mu} \left[\sigma_+ + \frac{\mu}{2}\sigma_z \right] \right) \end{aligned}$$

and thus

$$\begin{aligned} \mathcal{V}^{-1}\mathcal{V}^{-1}\{\mathcal{V}^1, \mathcal{V}^2\} &= \frac{1}{1-\mu(x)\bar{\mu}(x)} \frac{1}{1-\mu(y)\bar{\mu}(y)} \times \\ &\times \left[\{\mu(x), \mu(y)\} \left(\sigma_- - \frac{\bar{\mu}(x)}{2}\sigma_z \right) \left(\sigma_- - \frac{\bar{\mu}(y)}{2}\sigma_z \right) + \right. \\ &+ \{\bar{\mu}(x), \mu(y)\} \left(\sigma_+ + \frac{\mu(x)}{2}\sigma_z \right) \left(\sigma_- - \frac{\bar{\mu}(y)}{2}\sigma_z \right) + \\ &+ \{\mu(x), \bar{\mu}(y)\} \left(\sigma_- - \frac{\bar{\mu}(x)}{2}\sigma_z \right) \left(\sigma_+ + \frac{\mu(y)}{2}\sigma_z \right) + \\ &\left. + \{\bar{\mu}(x), \bar{\mu}(y)\} \left(\sigma_+ + \frac{\mu(x)}{2}\sigma_z \right) \left(\sigma_+ + \frac{\mu(y)}{2}\sigma_z \right) \right]. \end{aligned}$$

Equating this with (5.39) yields

$$\begin{aligned} \{\mu(x), \mu(y)\} &= 0 \\ \{\bar{\mu}(x), \bar{\mu}(y)\} &= 0 \\ \{\bar{\mu}(x), \mu(y)\} &= \frac{1}{4\kappa} \left(\frac{1-\mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_x \left(\frac{1-\mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_y (\text{sign}(x-y) + ic) e^{\alpha(x,y)} \\ \{\mu(x), \bar{\mu}(y)\} &= \frac{1}{4\kappa} \left(\frac{1-\mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_x \left(\frac{1-\mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_y (\text{sign}(x-y) - ic) e^{-\alpha(x,y)} \end{aligned} \quad (5.40)$$

in accordance with [32] for $\kappa = (8\pi G)^{-1}$. Note that

$$\overline{\{\bar{\mu}(x), \mu(y)\}} = \{\mu(x), \bar{\mu}(y)\} \quad (5.41)$$

only for c real.

5.4.3 The Jacobi relation for brackets of the μ and $\bar{\mu}$ fields

We define the Jacobi sum $J(x, y, z)$ to be

$$\begin{aligned} J(x, y, z) &= \{ \{ \mu(x), \bar{\mu}(y) \}, \mu(z) \} + \{ \{ \bar{\mu}(y), \mu(z) \}, \mu(x) \} + \{ \{ \mu(z), \mu(x) \}, \bar{\mu}(y) \} = \\ &= \{ \{ \mu(x), \bar{\mu}(y) \}, \mu(z) \} - \{ \{ \mu(z), \bar{\mu}(y) \}, \mu(x) \}. \end{aligned} \quad (5.42)$$

The Jacobi relation holds if

$$J(x, y, z) = 0.$$

For an arbitrary variation δ we find

$$\begin{aligned} \delta \{ \mu(x), \bar{\mu}(y) \} &= \frac{1}{4\kappa} (\text{sign}(x-y) - ic) e^{-\alpha(x,y)} \times \\ &\times \left\{ \left(\frac{-\delta\mu\bar{\mu} - \mu\delta\bar{\mu}}{\sqrt{\rho}} \right) \Big|_x \left(\frac{1-\mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_y + \left(\frac{1-\mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_x \left(\frac{-\delta\mu\bar{\mu} - \mu\delta\bar{\mu}}{\sqrt{\rho}} \right) \Big|_y - \right. \\ &\left. - \left(\frac{1-\mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_x \left(\frac{1-\mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_y \left(\int_x^y \frac{\delta\mu d\bar{\mu} + \mu d\delta\bar{\mu} - \delta\bar{\mu} d\mu - \bar{\mu} d\delta\mu}{1-\mu\bar{\mu}} + \int_x^y \frac{\mu d\bar{\mu} - \bar{\mu} d\mu}{(1-\mu\bar{\mu})^2} \delta(\mu\bar{\mu}) \right) \right\}. \end{aligned}$$

We intend to substitute $\{ \cdot, \mu(z) \}$ for δ so we set $\delta\mu = 0$ and keep only terms with $\delta\bar{\mu}$. The two integral terms above become

$$\begin{aligned} \int_x^y \frac{(\mu d\delta\bar{\mu} - \delta\bar{\mu} d\mu)(1-\mu\bar{\mu}) + (\mu d\bar{\mu} - \bar{\mu} d\mu)\mu\delta\bar{\mu}}{(1-\mu\bar{\mu})^2} &= \int_x^y \frac{\mu d\delta\bar{\mu}}{1-\mu\bar{\mu}} + \int_x^y \frac{-\delta\bar{\mu} d\mu + \mu^2 \delta\bar{\mu} d\bar{\mu}}{(1-\mu\bar{\mu})^2} = \\ &= \left[\frac{\mu\delta\bar{\mu}}{1-\mu\bar{\mu}} \right]_x^y + \int_x^y \frac{-\delta\bar{\mu}(1-\mu\bar{\mu})d\mu - \mu\delta\bar{\mu}d(\mu\bar{\mu}) - \delta\bar{\mu}d\mu + \mu^2\delta\bar{\mu}d\bar{\mu}}{(1-\mu\bar{\mu})^2} = \\ &= \left[\frac{\mu\delta\bar{\mu}}{1-\mu\bar{\mu}} \right]_x^y - \int_x^y \frac{2d\mu}{(1-\mu\bar{\mu})^2} \delta\bar{\mu} \end{aligned}$$

$$\begin{aligned} \delta \{ \mu(x), \bar{\mu}(y) \} &= \frac{1}{4\kappa} \left(\frac{1-\mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_x \left(\frac{1-\mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_y (\text{sign}(x-y) - ic) e^{-\alpha(x,y)} \times \\ &\times \left\{ \left(\frac{-2\mu\delta\bar{\mu}}{1-\mu\bar{\mu}} \right) \Big|_y + \int_x^y \frac{2d\mu}{(1-\mu\bar{\mu})^2} \delta\bar{\mu} \right\} \end{aligned}$$

$$\begin{aligned} \{ \{ \mu(x), \bar{\mu}(y) \}, \mu(z) \} &= \frac{1}{8\kappa^2} (\text{sign}(x-y) - ic) e^{-\alpha(x,y)} \left(\frac{1-\mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_x \left(\frac{1-\mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_y \left(\frac{1-\mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_z \times \\ &\times \left\{ \frac{-\mu(y)}{\sqrt{\rho(y)}} (\text{sign}(y-z) + ic) e^{\alpha(y,z)} + \int_x^y \left(\frac{\partial_s \mu}{\sqrt{\rho}(1-\mu\bar{\mu})} \right) \Big|_s (\text{sign}(s-z) + ic) e^{\alpha(s,z)} ds \right\} = \\ &= \frac{1}{8\kappa^2} \left(\frac{1-\mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_x \left(\frac{1-\mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_y \left(\frac{1-\mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_z \times \\ &\times \left\{ \frac{\mu(y)}{\sqrt{\rho(y)}} (\text{sign}(x-y) - ic) (\text{sign}(z-y) - ic) e^{\alpha(y,z)} e^{\alpha(y,x)} + \right. \\ &\left. + \int_x^y \left(\frac{\partial_s \mu}{\sqrt{\rho}(1-\mu\bar{\mu})} \right) \Big|_s (\text{sign}(x-y) - ic) (\text{sign}(s-z) + ic) e^{\alpha(s,z)} e^{\alpha(y,x)} ds \right\}. \end{aligned}$$

To get the complete Jacobi sum (5.42) we have to antisymmetrize w.r.t. x and z . Since the first term is symmetric w.r.t. x and z we only have to worry about the integral term.

$$\begin{aligned}
J(x, y, z) &= \frac{1}{8\kappa^2} \left(\frac{1 - \mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_x \left(\frac{1 - \mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_y \left(\frac{1 - \mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_z e^{\alpha(y,z)} e^{\alpha(y,x)} \times \\
&\times \left\{ \int_x^y \left(\frac{\partial_s \mu}{\sqrt{\rho}(1 - \mu\bar{\mu})} \right) \Big|_s (\text{sign}(x - y) - ic)(\text{sign}(s - z) + ic) e^{\alpha(s,y)} ds - \right. \\
&\left. - \int_z^y \left(\frac{\partial_s \mu}{\sqrt{\rho}(1 - \mu\bar{\mu})} \right) \Big|_s (\text{sign}(z - y) - ic)(\text{sign}(s - x) + ic) e^{\alpha(s,y)} ds \right\} \quad (5.43)
\end{aligned}$$

We introduce the function

$$f(s) := \left(\frac{\partial_s \mu}{\sqrt{\rho}(1 - \mu\bar{\mu})} \right) \Big|_s e^{\alpha(s,y)}$$

and note that for an arbitrary locally integrable function g

$$\begin{aligned}
\int_a^b g(s) \text{sign}(s - t) ds &= \left(\int_a^t + \int_t^b \right) g(s) \text{sign}(s - t) ds = \\
&= \int_a^t g(s) \text{sign}(a - t) ds + \int_t^b g(s) \text{sign}(b - t) ds.
\end{aligned}$$

Denoting by \mathcal{P} the factor

$$\mathcal{P} = \frac{1}{8\kappa^2} \left(\frac{1 - \mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_x \left(\frac{1 - \mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_y \left(\frac{1 - \mu\bar{\mu}}{\sqrt{\rho}} \right) \Big|_z e^{\alpha(y,z)} e^{\alpha(y,x)},$$

which is nonzero if ρ is finite at all points x, y, z , we continue with (5.43)

$$\begin{aligned}
\mathcal{P}^{-1} J(x, y, z) &= \\
&= ic \int_x^y f \text{sign}(x - y) ds + c^2 \int_x^y f ds + \int_x^z f (\text{sign}(x - y) - ic) \text{sign}(x - z) ds + \\
&+ \int_z^y f (\text{sign}(x - y) - ic) \text{sign}(y - z) ds - ic \int_z^y f \text{sign}(z - y) ds - c^2 \int_z^y f ds - \\
&- \int_z^x f (\text{sign}(z - y) - ic) \text{sign}(z - x) ds - \int_x^y f (\text{sign}(z - y) - ic) \text{sign}(y - x) ds = \\
&= \int_x^z f [c^2 - \text{sign}(z - x) \text{sign}(x - y) - \text{sign}(x - y) \text{sign}(y - z) - \text{sign}(y - z) \text{sign}(z - x)] ds.
\end{aligned}$$

If all points are distinct, define

$$M := \max(x, y, z) \quad m := \min(x, y, z) \quad n \in \{x, y, z\} \setminus \{M, m\},$$

then the second factor in the integrand of the Jacobi sum is

$$\begin{aligned}
&c^2 - \text{sign}(z - x) \text{sign}(x - y) - \text{sign}(x - y) \text{sign}(y - z) - \text{sign}(y - z) \text{sign}(z - x) = \\
&= c^2 - \text{sign}(M - m) \text{sign}(m - n) - \text{sign}(m - n) \text{sign}(n - M) - \text{sign}(n - M) \text{sign}(M - m) = \\
&= c^2 + 1 - 1 + 1 = c^2 + 1.
\end{aligned}$$

The vanishing of the Jacobi sum $J(x, y, z) = 0$ implies

$$J(x, y, z) = \mathcal{P} \int_x^z f(c^2 + 1) = 0. \quad (5.44)$$

Since in general $\mathcal{P} \neq 0$ and $f \neq 0$, this implies

$$c = \pm i.$$

We see that if we want the Jacobi relation to hold, then (5.41) does not hold or vice versa. This issue is addressed in [33].

Remark: The bracket $\{\mu(x), \bar{\mu}(y)\}$ is a bidistribution. It is characterized by the action on test functions via smearing by integration. As long as it does not have singular support at $x = y$, the value we assign the bracket at $x = y$ is irrelevant. The same is true for the Jacobi sum. It is a function of x, y, z , i.e. it does not have singular support at any triple of points (x, y, z) . Hence, it is sufficient to assume that x, y, z are all distinct when investigating under which condition the Jacobi relation holds.

5.5 The transformation of the field \mathcal{V} to the monodromy matrix $\mathcal{M}(w)$

In the previous sections, we have discovered that indeed the Poisson bracket of the field μ and its complex conjugate $\bar{\mu}$ computed via the bracket of the field \mathcal{V} in the context of EG2CHSKF is analogous to the bracket of the field μ and $\bar{\mu}$ found in [32]. While we have seen that the \mathcal{V} -bracket as well as the μ -bracket are quite complicated expressions, in [24] it is shown that the monodromy matrix $\mathcal{M}(w)$ already introduced in 4.8 satisfies a quite simple Poisson algebra and even a quantization of this algebra is proposed. Thus it is desirable to use the monodromy matrix $\mathcal{M}(w)$ as initial data instead of \mathcal{V} or μ . To this end in this section we will

- find a transformation, which expresses the monodromy matrix $\mathcal{M}(w)$ in terms of initial data \mathcal{V} on a truncated light cone \mathcal{N} .
- show that this transformation is invertible, that \mathcal{V} can be recovered from $\mathcal{M}(w)$.

In section 5.7 we will then use our \mathcal{V} -bracket on \mathcal{N} to derive the Poisson algebra of the monodromy matrix. As expected the algebra of [24] is obtained. Note however that unlike the derivation presented in [24] our calculation does not involve points infinitely far from the axis, it involves only the fields in a bounded region, so it is valid in a somewhat wider range of situations. In a further calculation we do assume that the field equations hold on a space that extends infinitely far from the axis, and also that the fields obey asymptotic conditions, to rederive the semi-classical Yangian algebra of conserved charges found in [24] from the perspective of our null canonical theory.

As was already done before, throughout this section we adopt ρ as a spacelike coordinate and $\tilde{\rho}$ as a timelike coordinate, that is $x^1 = \rho$, $x^0 = \tilde{\rho}$.

5.5.1 Getting to know the monodromy matrix $\mathcal{M}(w)$

Recall that at a given spacetime point x the function $\gamma(w)$ is two valued, γ and $1/\gamma$ correspond to the same value of w . γ defines a Riemann surface which is just the Riemann sphere. We call it the γ plane. Via the projection

$$\gamma \mapsto w(\gamma) = \frac{1}{2}\rho \left(\gamma + \frac{1}{\gamma} \right) - \tilde{\rho} \quad (5.45)$$

the γ plane covers the complex w plane twice.⁶ This gives us a second representation of the γ -plane: two copies of the w -plane, or “sheets”, W_+ and W_- , joined together on a branch cut. Specifically, the first sheet W_+ , the image of the unit disc $|\gamma| \leq 1$, is joined to the second sheet W_- , the image of $|\gamma| \geq 1$, along the segment $[-\rho^+, -\rho^-] = [-\tilde{\rho} - \rho, -\tilde{\rho} + \rho]$ of the real w axis, the image of the circle $|\gamma| = 1$. See figure 4.1 and subsections 4.3.2 and 4.3.3.

Let us define two single valued inverses to (5.45), γ_+ that maps $w \in W_+$ to the unit disc in the γ -plane, and γ_- that maps $w \in W_-$ to the complement of the disc. More precisely, if w is off the real axis $|\gamma_+(w)| < 1$ and $|\gamma_-(w)| > 1$, while if w lies on the branch cut $|\gamma_{\pm}| = 1$ and $\text{Im } \gamma_+(w) \geq 0$ and $\text{Im } \gamma_-(w) \leq 0$.

The linear system (4.25) thus admits two versions, one in which $\gamma = \gamma_+$, and one in which $\gamma = \gamma_- = 1/\gamma_+$ ($\gamma(x, w)$ being a solution to the differential equation (4.27) must be continuous in x). Let $\hat{\mathcal{V}}_+(x, w)$ and $\hat{\mathcal{V}}_-(x, w)$ be solutions to + and - versions of the linear system respectively. The two versions of the linear system are gathered together in the single linear system (4.25) parametrized by the value of γ at some reference point in spacetime. The + system is then the restriction of the system to the first sheet of the Riemann surface of γ , and the - system the restriction to the second sheet. The functions $\hat{\mathcal{V}}_{\pm}(x, \gamma) := \hat{\mathcal{V}}_{\pm}(x, w(\gamma))$ are solutions to this combined system restricted to their respective sheets, or equivalently to γ in the ranges of γ_+ and γ_- respectively.

Because the connection $\hat{J}_{\pm} = Q_{\pm} + \frac{1 \mp \gamma}{1 \pm \gamma} P_{\pm}$ is invariant under the extended involution η^{∞}

$$\hat{J}(\gamma) = \eta^{\infty}(\hat{J}(\gamma)) = \eta(\hat{J}) \left(\frac{1}{\gamma} \right),$$

there is a simple relation between the linear system on the two branches:

Recall that η^{∞} is defined on the loop group G_{γ}^{∞} by (4.96). At a given value of w

$$\hat{\mathcal{V}}_-^{-1} d\hat{\mathcal{V}}_- = \hat{J}(\gamma_-) = \eta(\hat{J}(1/\gamma_-)) = \eta(\hat{J}(\gamma_+)) = \eta(\hat{\mathcal{V}}_+)^{-1} d\eta(\hat{\mathcal{V}}_+).$$

Thus if $\hat{\mathcal{V}}_+(x, w)$ is a solution of the linear system on the first branch, then $\eta(\hat{\mathcal{V}}_+(x, w))$ is a solution on the second sheet. Note that in terms of γ this solution on the second sheet can be written as $\eta(\hat{\mathcal{V}}_+(x, 1/\gamma)) = \eta^{\infty}(\hat{\mathcal{V}}_+)(\gamma)$. Of course this is not the only solution to the linear system on the second branch: $C(w)\eta(\hat{\mathcal{V}}_+(w, t))$ is a solution for any function C of w , but independent of spacetime position.

Taken together the pair of solutions $\hat{\mathcal{V}}_+(x, \gamma)$ and $\eta^{\infty}(\hat{\mathcal{V}}_+)(x, \gamma)$ do not in general form a continuous function on the γ plane. There is a discontinuity at the unit circle. How are the limiting values there related? First, these limiting values exist by the results of 4.4. More, $\hat{\mathcal{V}}_+$ extends to a continuous function on the whole closed unit disc $|\gamma| \leq 1$, which

⁶The w -plane is actually also a Riemann sphere since it includes ∞ .

satisfies the linear system everywhere, except at $\gamma = \pm 1$ where the one component of the linear system is singular. Then

$$\hat{\mathcal{V}}_+(\gamma) = \mathcal{M}_+(\gamma)\eta^\infty(\hat{\mathcal{V}}_+(\gamma)), \quad (5.46)$$

where

$$\mathcal{M}_+ = \hat{\mathcal{V}}_+(\gamma)[\eta^\infty(\hat{\mathcal{V}}_+(\gamma))]^{-1} = \hat{\mathcal{V}}_+(\gamma)[\eta(\hat{\mathcal{V}}_+)(1/\gamma)]^{-1} = \hat{\mathcal{V}}_+(\gamma)\hat{\mathcal{V}}_+^T(1/\gamma) \quad (5.47)$$

is nothing but the monodromy matrix calculated from $\hat{\mathcal{V}}_+$.

Since $\hat{\mathcal{V}}_+$ is only defined on the unit disc, \mathcal{M}_+ is only defined on the unit circle $|\gamma| = 1$, or equivalently on the branch cuts of the w planes W_\pm , corresponding to a spacetime position dependent interval of real values of w . The unit circle corresponds to two copies of this interval, one corresponding to the upper semicircle, with $\text{Im } \gamma \geq 0$, and one to the lower semicircle.

Now recall that although built out of fields depending on both w and x , the monodromy matrix actually depends only on w . Consider the point $\gamma_+(w)$ corresponding to a value of w on the branch cut but not at either branch point, so $\gamma_+(w) \neq \pm 1$ and the linear system (4.25) is well defined. The spacetime gradient of the monodromy matrix at constant w is then

$$d\mathcal{M}_+ = \hat{\mathcal{V}}_+\hat{J}(\eta^\infty(\hat{\mathcal{V}}_+))^{-1} - \hat{\mathcal{V}}_+\eta^\infty(\hat{J})(\eta^\infty(\hat{\mathcal{V}}_+))^{-1},$$

which vanishes because $\hat{J} = \eta^\infty(\hat{J})$. That, is $\mathcal{M}(x, \gamma_+(x, w))$ depends only on w .

The values of the monodromy matrix on the upper semi-circle at a given spacetime point x reveals $\mathcal{M}_+(w)$ for w in the branch cut corresponding to x , i.e. the open interval $] -x^+, -x^-[$. The monodromy matrices at all spacetime points define a single universal function $\mathcal{M}_+(w)$ for all real w , for the real axis can be covered with open intervals each corresponding to the branch cut at some spacetime point, and should w lie in the branch cut of two spacetime points then $\mathcal{M}_+(w)$ at both points must be the same. It is also clear that $\mathcal{M}_+(w)$ must be continuous, since $\mathcal{M}_+(\gamma)$ is continuous along the circle. For this reason $\mathcal{M}(\gamma_+(w)) = \mathcal{M}_+(w)$ also at the endpoints $\gamma_+(w) = \pm 1$: By continuity in γ $\mathcal{M}_+(\gamma = \pm 1)$ is the limiting value of $\mathcal{M}_+(\gamma)$ as $\gamma \rightarrow \pm 1$, by continuity in w this is the limiting value of $\mathcal{M}_+(w)$ as w approaches the corresponding branch point.

The same argument can be repeated using the lower semi-circle, that is, using $\gamma = \gamma_-(w) = 1/\gamma_+(w)$. The monodromy matrix $\hat{\mathcal{V}}_+(\gamma)(\eta^\infty[\hat{\mathcal{V}}_+(\gamma)])^{-1}$ is again given by a universal function of w , \mathcal{M}'_+ . Since

$$\mathcal{M}'_+(w) = \hat{\mathcal{V}}_+(\gamma_-(w))\hat{\mathcal{V}}_+^T(1/\gamma_-(w)) = \hat{\mathcal{V}}_+(1/\gamma_+(w))\hat{\mathcal{V}}_+^T(\gamma_+(w))$$

it is just the transpose of $\mathcal{M}_+(w)$. Furthermore, at branch points, where $\gamma_-(w) = \pm 1$, $\gamma_+ = 1/\gamma_- = \gamma_-$. Thus $\mathcal{M}'_+(w) = \mathcal{M}_+(w)$ at such points. But any w can be put at a branch point by a suitable choice of spacetime point ($x^+ = -w$ for instance), so $\mathcal{M}'_+(w) = \mathcal{M}_+(w) =: \mathcal{M}(w)$ for all real w and $\mathcal{M}(w)$ is a symmetric matrix. In sum, at any x and any γ on the unit circle, the monodromy matrix for the solution $\hat{\mathcal{V}}_+$ is $\mathcal{M}(w(\gamma))$ where \mathcal{M} is a symmetric matrix depending continuously on w and nothing else.

$\hat{\mathcal{V}}_0$ is defined by the condition $\hat{\mathcal{V}}_{0+}(x, w) = \mathcal{V}(x)$ for all $w \in \mathcal{W}_+$ at a point x on the axis $\rho = 0$. (This implies that $\hat{\mathcal{V}}_{0+}(x, w) = \mathcal{V}(x)$ on all of the axis, see section 5.5.2) In section 4.4 it is demonstrated that $\hat{\mathcal{V}}_0$ is a solution to the linear system (4.25) for all $|\gamma| \leq 1$ except

at the singular values $\gamma = \pm 1$, where it still satisfies the non-singular component of the linear system. Moreover, it is continuous in γ throughout the closed disc in the topology of the disc. (That is, it is continuous on any continuous curve contained entirely in the closed disc.) Thus the hypothesis of our analysis apply, and $\hat{\mathcal{V}}_0$ is continuous in γ (in the topology of the γ plane) at all points of the closed unit disc.

Because of continuity at the unit circle we may write the monodromy matrix associated to $\hat{\mathcal{V}}_0$ as simply

$$\mathcal{M} = \hat{\mathcal{V}}_0(\gamma)[\eta^\infty(\hat{\mathcal{V}}_0(\gamma))]^{-1} = \hat{\mathcal{V}}_0(\gamma)\hat{\mathcal{V}}_0^T(1/\gamma). \quad (5.48)$$

Note that it may be defined on a larger region of the γ plane than just the unit circle, if $\hat{\mathcal{V}}_0(\gamma)$ can be defined on a region larger than the unit disc. (On the other hand note that $\hat{\mathcal{V}}_0$ need not be analytic, or even continuous, on the whole γ plane. That depends on $\mathcal{M}(w)$. $\eta^\infty(\hat{\mathcal{V}}_+)$ is analytic outside the unit disc, but $\hat{\mathcal{V}}_0$ is equal to $\mathcal{M}\eta^\infty(\hat{\mathcal{V}}_+)$ in this region.)

5.5.2 The monodromy matrix on the axis

The monodromy matrix $\mathcal{M}(w)$ of the solution $\hat{\mathcal{V}}_0$ has a simple physical interpretation for real w . It is the Kramer-Neugebauer transform of the unimodular metric on the axis $\rho = 0$ at $\tilde{\rho} = -w$.

Let us demonstrate this. Recall that $\hat{\mathcal{V}}_0$ is determined in the first branch, $\gamma = \gamma_+(w)$, by the linear system (4.25) and the requirement $\hat{\mathcal{V}}_0(x_0, w) = \mathcal{V}(x_0)$ for all w at some reference point x_0 on the $\rho = 0$ axis. First let us show that this means that $\hat{\mathcal{V}}_0(x_0, w) = \mathcal{V}(x_0)$ on the whole $\rho = 0$ axis.

For $\rho \rightarrow 0$ we have

$$\lim_{\rho \rightarrow 0} \gamma_+(\tilde{\rho}, \rho, w) = 0 \quad \Rightarrow \quad \lim_{\rho \rightarrow 0} \hat{J}(\tilde{\rho}, \rho, w) = J(\tilde{\rho}, \rho = 0),$$

unless $\tilde{\rho} = -w$. In this latter case the limit of γ_+ is undefined. On the axis $\rho = 0$, the connection \hat{J} of $\hat{\mathcal{V}}$ is the same as the connection $J = \mathcal{V}^{-1}d\mathcal{V}$ for \mathcal{V} . Hence, if we set $\hat{\mathcal{V}}_0 = \mathcal{V}$ at one point x_0 on the axis, then $\hat{\mathcal{V}}_0 = \mathcal{V}$ on the whole axis if w is not real, and at all points on the axis that are not separated from x_0 by $\tilde{\rho} = -w$ if w is real. At points on the other side of $\tilde{\rho} = -w$, for real w , $\hat{\mathcal{V}}_0(\tilde{\rho}, w) = F(w)\mathcal{V}(\tilde{\rho})$ for some matrix F depending only on w . But in subsection 4.4.1 it was demonstrated that $\hat{\mathcal{V}}_0$ is continuous at $\tilde{\rho} = -w$, so in fact $F = \mathbb{1}$ and $\hat{\mathcal{V}}_0 = \mathcal{V}$ along the whole axis. (This is of course also what is necessary in order that $\hat{\mathcal{V}}_0$ be continuous in w at the real w axis.) Note that $\hat{\mathcal{V}}_0$ at $\tilde{\rho} = -w$ on the axis has really been *defined* to be the limiting value of $\hat{\mathcal{V}}_0$ as this point is approached from elsewhere. What has been established in subsection 4.4.1 is that this can be done consistently, so the resulting function is continuous.

Now let us return to \mathcal{M} . At the point y_{ax} on the axis at $\tilde{\rho} = -w$, where we would like to evaluate

$$\mathcal{M}(w) = \hat{\mathcal{V}}_0(x, \gamma_+(w))\hat{\mathcal{V}}_0^T(x, 1/\gamma_+(w)),$$

we cannot, because $\gamma_+(w)$ is not unambiguously defined. However we can define it everywhere else, and then express the result in terms of $\hat{\mathcal{V}}_0$ at this singular point. Recall that $\mathcal{M}(w)$ is independent of position at any x where the linear system holds. For w real $\mathcal{M}(w)$ is thus constant at least everywhere off the lines $x^+ = -w$ and $x^- = -w$ where the

linear system becomes singular. But at these lines (but off the axis) $\mathcal{M}(w)$ is continuous, so in fact it is constant on all spacetime except at y_{ax} . To see this note that in the region $x^+ \geq -w$, $x^- \leq -w$, $x \neq y_{ax}$ real w correspond to γ on the unit circle, so

$$\hat{\mathcal{V}}_0(x, 1/\gamma_+(w)) = \hat{\mathcal{V}}_0(x, \bar{\gamma}_+(w)) = \tilde{\mathcal{V}}_0(x, \gamma_+(w)).$$

The continuity in spacetime of $\hat{\mathcal{V}}_0(x, 1/\gamma_+(w))$ at $x^+ = -w$ and $x^- = -w$ thus assures that $\mathcal{M}(w)$ is continuous there with its value in the region between these lines. Because $\hat{\mathcal{V}}_0$ is continuous in γ at $|\gamma| = 1$ the monodromy matrix $\mathcal{M}(w) = \hat{\mathcal{V}}_0(x, \gamma_+(w))\hat{\mathcal{V}}_0^T(x, 1/\gamma_+(w))$ approaches $\hat{\mathcal{V}}_0(x, \pm 1)\hat{\mathcal{V}}_0^T(x, \pm 1)$, the expression for $\mathcal{M}(w)$ on the lines.

Now consider the expression for $\mathcal{M}(w)$ on $x^- = -w$,

$$\mathcal{M}(w) = \hat{\mathcal{V}}_0(x, \gamma = 1)\hat{\mathcal{V}}_0^T(x, \gamma = 1) = \hat{\mathcal{V}}_0(x, w)\hat{\mathcal{V}}_0^T(x, w).$$

As the axis $\rho = 0$ is approached the left side is constant, while the right side approaches $\mathcal{V}(y_{ax})\mathcal{V}^T(y_{ax})$. Thus

$$\mathcal{M}(w) = \mathcal{V}(y_{ax})\mathcal{V}^T(y_{ax}). \quad (5.49)$$

Recall that $\rho^{1/2}\mathcal{V}$ is the Kramer-Neugebauer transform of the zweibein on the Killing orbits in the Killing coordinates x^2, x^3 defined in (2.21). See (2.37). $\rho\tilde{\mathcal{V}}\tilde{\mathcal{V}}^T$ is the metric in these coordinates. This metric of course vanishes on the axis, but this is to be expected, since the Killing coordinates together with ρ form essentially cylindrical coordinates. The Killing coordinates are not necessarily the cylindrical coordinates constructed from Riemann normal coordinates about the axis, but a rather linear combinations of these.

5.5.3 The definition of the monodromy matrix used in [24]

In [24] $\mathcal{M}(w)$ is defined in a slightly different manner than we have defined it. Since it is not so straightforward that the two definitions are equivalent, here we explain how they are related.

We start with the definition (5.48)

$$\mathcal{M}(w) = \hat{\mathcal{V}}(x, w)\eta^\infty(\hat{\mathcal{V}}^{-1}(x, w)).$$

Since η^∞ exchanges the two sheets of the Riemann surface, we have

$$\eta^\infty(\hat{\mathcal{V}}(x, w)) = \eta(\hat{\mathcal{V}}(x, \eta^\infty(w))),$$

and thus

$$\mathcal{M}(w) = \hat{\mathcal{V}}(x, w)\eta(\hat{\mathcal{V}}^{-1}(x, \eta^\infty(w))). \quad (5.50)$$

The monodromy matrix is independent of x . Provided that the domain of definition of $\hat{\mathcal{V}}$ extends to infinity, we may set the ρ -coordinate to infinity. Then the branch cut covers the entire real axes on W_+ and W_- . Then, for real w and $\epsilon \in \mathbb{R}$, we have

$$\lim_{\epsilon \rightarrow 0} (w - i\epsilon) = \lim_{\epsilon \rightarrow 0} \eta^\infty(w + i\epsilon) \quad \forall w.$$

We may thus write (5.50) as

$$\begin{aligned} \mathcal{M}(w) &= \lim_{\epsilon \rightarrow 0} \left[\hat{\mathcal{V}}(\tilde{\rho}, \rho = \infty, w + i\epsilon)\eta(\hat{\mathcal{V}}^{-1}(\tilde{\rho}, \rho = \infty, \eta^\infty(w + i\epsilon))) \right] = \\ &= \lim_{\epsilon \rightarrow 0} \left[\hat{\mathcal{V}}(\tilde{\rho}, \rho = \infty, w + i\epsilon)\eta(\hat{\mathcal{V}}^{-1}(\tilde{\rho}, \rho = \infty, w - i\epsilon)) \right]. \end{aligned} \quad (5.51)$$

Above we have shown that $\mathcal{M}(w)$ is the same on both W_+ and W_- . We may thus restrict w to lie in W_+ in (5.51). Furthermore, in [24], they define⁷

$$T_{u/d}(\tilde{\rho}, w) = \mathcal{V}(\tilde{\rho}, 0) \hat{\mathcal{V}}^{-1}(\tilde{\rho}, 0, w) \lim_{\rho \rightarrow \infty} \hat{\mathcal{V}}(\tilde{\rho}, \rho, w), \quad w \in W_{+u/d}. \quad (5.52)$$

In section 5.8, we will see that they are conserved in time for solutions \mathcal{V} constant at spatial infinity. Using our definition of $\hat{\mathcal{V}}_0$, $\hat{\mathcal{V}}_0 = \mathcal{V}$ on the axis, we simply have

$$T_{u/d}(w) = \lim_{\rho \rightarrow \infty} \hat{\mathcal{V}}_0(\tilde{\rho}, \rho, w), \quad w \in W_{+u/d}, \quad (5.53)$$

and so (5.51) may be written as

$$\mathcal{M}(w) = \lim_{\epsilon \rightarrow 0} (T_u(w + i\epsilon) \eta(T_d^{-1}(w - i\epsilon))), \quad (5.54)$$

in accordance with [24].

5.5.4 Relating the monodromy matrix to null initial data \mathcal{V}

If we want to replace null initial data \mathcal{V} on a truncated light cone by the monodromy matrix with its nice Poisson algebra, we must relate these data. The expression (5.47) uses $\hat{\mathcal{V}}$ evaluated at γ and γ^{-1} with $|\gamma| = 1$. On the Riemann surface, these points are the two points on the two branch cuts corresponding to the same value w . In section 5.7 when we calculate the Poisson algebra of the monodromy matrix, it will be necessary to evaluate both factors at the same point w . Looking for such an expression, we remember that at the branch points $\gamma = \pm 1 = \gamma^{-1}$. Since $\mathcal{M}(w)$ is x -independent, for given real w , we may choose x such that w corresponds to one of the branch points. That is, $w = -x^+$ or $w = -x^-$. Then we get

$$\mathcal{M}(w) = \hat{\mathcal{V}}_+(x^-, x^+ = -w, w) \hat{\mathcal{V}}_+^T(x^-, x^+ = -w, w), \quad (5.55)$$

or

$$\mathcal{M}(w) = \hat{\mathcal{V}}_+(x^- = -w, x^+, w) \hat{\mathcal{V}}_+^T(x^- = -w, x^+, w),$$

respectively. For fixed w and $x^+ = -w$, the relation

$$\mathcal{M}(w) = \hat{\mathcal{V}}(x^-, x^+ = -w, w) \hat{\mathcal{V}}^T(x^-, x^+ = -w, w)$$

indeed is valid for arbitrary x^- -coordinate. This is because

$$\begin{aligned} u^{-1}(x^-, x^+, w) &= \frac{\sqrt{w + x^+}}{\sqrt{w + x^-}} \Rightarrow u^{-1}(x^-, x^+ = -w, w) = 0 \Rightarrow \\ &\Rightarrow \hat{J}_-(x^+ = -w, w) = Q_- + u^{-1}(x^-, x^+ = -w, w) P_- = Q_-. \end{aligned}$$

Along the line $x^+ = -w$, the connection \hat{J}_- only has a gauge component, Q_- . If x and y are points on this line with different x^- -coordinate, for $\hat{\mathcal{V}}$ at these points we have

$$\hat{\mathcal{V}}(x, w) = \hat{\mathcal{V}}(y, w) \mathcal{P} e^{\int_y^x Q_-}$$

⁷They use the indices + and - to distinguish between $w \in W_{+u}$ and $w \in W_{+d}$. We use the indices u and d .

and thus

$$\begin{aligned}\hat{\mathcal{V}}(x, w)\hat{\mathcal{V}}^T(x, w) &= \hat{\mathcal{V}}(y, w)\mathcal{P}e^{\int_y^x Q_-} \left(\mathcal{P}e^{\int_y^x Q_-}\right)^T \hat{\mathcal{V}}^T(y, w) = \\ &= \hat{\mathcal{V}}(y, w)\mathcal{P}e^{\int_y^x Q_-}\mathcal{P}e^{-\int_y^x Q_-}\hat{\mathcal{V}}^T(y, w) = \hat{\mathcal{V}}(y, w)\hat{\mathcal{V}}^T(y, w).\end{aligned}$$

Analogously on the line $x^- = -w$.

The relation (5.55) is a big step in the right direction, it relates $\mathcal{M}(w)$ with $\hat{\mathcal{V}}(w)$ evaluated at a point where $x^+ = -w$ or $x^- = -w$. Again, w is taken to be real, as for the rest of this chapter. In figure 5.1 for two values of w , the corresponding lines $x^\pm = -w$ are illustrated.

Now suppose we know \mathcal{V} and thus, after integration of the linear system (4.25), $\hat{\mathcal{V}}$ on the null initial data surface

$$\mathcal{N} = \mathcal{N}_R \cup \mathcal{N}_L = \{(x_0^-, x_0^+ + s) \mid s \in [0, a]\} \cup \{(x_0^- + t, x_0^+) \mid t \in [0, b]\}, \quad (5.56)$$

with a, b real and $b \leq 2x_0^1$, introduced already in section 5.3, being an initial data surface for the region \mathcal{D} (see section 4.4). Then we get $\mathcal{M}(w)$ for those w , for which there is an x^+ or x^- on \mathcal{N} such that $x^\pm = -w$. These values are

$$\begin{aligned}x^+ &\in [x_0^+, x_0^+ + a] && \text{on } \mathcal{N}_R, \\ x^- &\in [x_0^-, x_0^+ + b] && \text{on } \mathcal{N}_L\end{aligned}$$

and so we get $\mathcal{M}(w)$ for the subset $\mathcal{W} = \mathcal{W}_L \cup \mathcal{W}_R$ consisting of the two intervals

$$\mathcal{W}_R = [-x_0^+ - a, -x_0^+], \quad (5.57)$$

$$\mathcal{W}_L = [-x_0^- - b, -x_0^-]. \quad (5.58)$$

For $b < 2x_0$, these intervals are disjoint as can be seen in figure 5.1, where they are depicted as parts of the $x^0 = \tilde{\rho}$ axis.

We select a special form of initial data surface \mathcal{N} , which allows for the recovery of \mathcal{V} in the domain of dependence of \mathcal{N} from $\mathcal{M}(w)$ on \mathcal{W} in a convenient way. This choice is not mandatory and there are other more general choices (see [18]). We define the initial data surface \mathcal{N} to consist of only one branch \mathcal{N}_R touching the axis

$$\mathcal{N} = \{(x_0^-, x_0^+ + s) \mid s \in [0, a]\}, \quad (5.59)$$

with $x_0^- = x_0^+$. Also, we use x_0 as our reference point for $\hat{\mathcal{V}}$ (and thus call it $\hat{\mathcal{V}}_0$). That is we define $\hat{\mathcal{V}}_0(x_0, w) = \mathcal{V}(x_0)$. With this choice, we can compute $\mathcal{M}(w)$ for all real w with

$$w \in \mathcal{W} = [-x_0^+ - a, x_0^+].$$

5.5.5 Preliminaries for the inverse transformation

With our choice of \mathcal{N} and x_0 , and for $w \in \mathcal{W}$ the monodromy matrix is

$$\mathcal{M}(w) = \hat{\mathcal{V}}_0(x, \lambda)\hat{\mathcal{V}}_0^T(x, \lambda^{-1}). \quad (5.60)$$

Here and from now on, λ (and later also σ) will always denote a point on the unit circle $|\lambda| = 1$, while γ can be any point in the complex plane. Consequently, equations involving

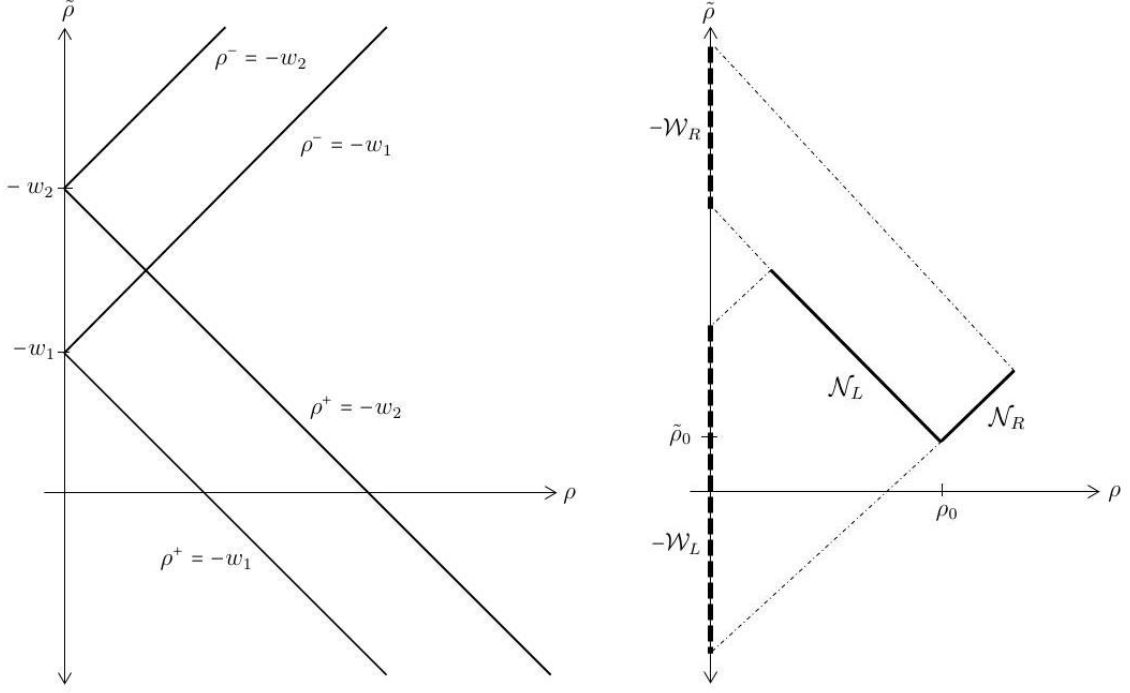


Figure 5.1: Left: The lines $\rho^\pm = -w$ for two values of w . Right: The intervals \mathcal{W}_L and \mathcal{W}_R of real values w , for which $\mathcal{M}(w)$ can be constructed from $\hat{\mathcal{V}}$ on \mathcal{N} .

λ are only valid on the unit circle, while those involving γ are valid inside, outside of the unit circle or both.

$\mathcal{M}(w)$ corresponds to the product of the limiting values $\hat{\mathcal{V}}_0(x, \lambda)$ and $\hat{\mathcal{V}}_0^T(x, \lambda^{-1})$ of the function $\hat{\mathcal{V}}(x, \gamma)$ holomorphic inside the unit circle and having continuous limit on the unit circle. Cauchy's theorem tells us [22] that a holomorphic function, continuous on the boundary, is determined by the values it has on the boundary. Thus the task of the inverse transformation for a given $\mathcal{M}(w)$ with $w \in \mathcal{W}$ is to find a function $\hat{\mathcal{U}}(x, \gamma)$ holomorphic in γ inside the unit circle and such that its boundary value on the unit circle satisfies

$$\mathcal{M}(w(x, \lambda)) = \hat{\mathcal{U}}(x, \lambda) \hat{\mathcal{U}}^T(x, \lambda^{-1}) \quad (5.61)$$

and such that $\hat{\mathcal{U}}(x, \gamma = 0) = \mathcal{U}(x)$, where $\mathcal{U}(x)$ solves the equations of motion.

In order to be able to perform the inverse transformation, x must be such, that the interval $[-x^+, -x^-]$ is a subset of \mathcal{W} . Because then we know $\mathcal{M}(w)$ on the entire branch cut and thus $\mathcal{M}(x, \lambda)$ on the entire unit circle. The equation (5.61) as a relation of the boundary value of $\hat{\mathcal{U}}$ is then given on the entire unit circle. Those points are exactly the points in the domain of dependence \mathcal{D} of \mathcal{N}

$$\mathcal{D} = \{(x^-, x^+) \mid x_0^- \leq x^- \leq x^+ \leq x_0^+ + a\}.$$

For all these points, the negative interval corresponding to the branch cut can be illustrated by extending the coordinate lines $x^\pm = \text{const.}$ through x to the axis. The segment of the axis between the points of intersection with these coordinate lines corresponds to the branch cut.

Before we continue, let us collect some properties $\mathcal{M}(w)$ has. First, it should be a symmetric $SL(2)$ matrix defined for $w \in \mathcal{W}$. Second, with our choice of initial data surface, for $w \in \mathcal{W}$ $\hat{J}(w)$ is real along the entire line of integration from the point x_0 to the point $(x_0^-, -w)$. Thus we require $\mathcal{M}(w)$ to be real. Third, by (4.83) $\hat{\mathcal{V}}(x^-, -w, w)$, being a product of \mathcal{V} and the transport matrix of the form considered there, satisfies a Hölder condition of index $1/2$. In chapter 1 of [28] it is shown that product of two such matrices will also satisfy the Hölder condition of the same index. Thus, we require $\mathcal{M}(w)$ to satisfy a Hölder condition. We require one additional property, namely that $\mathcal{M}(w)$ be differentiable (at least twice) in w . Then of course $\mathcal{M}(w(x, \gamma))$ is differentiable in x .

Summarizing, we say that $\mathcal{M}(w)$ is *admissible* if it is *symmetric, real, $SL(2)$ -valued, defined on \mathcal{W} differentiable in w* and satisfies a *Hölder condition of index $1/2$* .

5.5.6 The inverse transformation

We now investigate if an admissible matrix $\mathcal{M}(w)$ can be factorized such that we can recover a function $\hat{\mathcal{U}}(x, \gamma)$ with the same properties of $\hat{\mathcal{V}}(x, \gamma)$. For now we shall assume that there is a factorization of $\mathcal{M}(w)$ in the following sense. The proof will be given in the next subsection.

Proposition 5.5.1. *The Hilbert factorization problem*

Let $\mathcal{M}(w(x, \lambda))$ be admissible. Then there exist two matrices $\mathcal{Z}_-(x, \gamma)$ and $\mathcal{Z}_+(x, \gamma)$ holomorphic in γ outside and inside the unit circle respectively, having continuous limits thereon, and satisfying

$$\mathcal{M}(w(x, \lambda)) = \mathcal{Z}_+(x, \lambda)\mathcal{Z}_-(x, \lambda) \quad (5.62)$$

on the unit circle. This factorization is unique if we prescribe a value for $\mathcal{Z}_-(x, \gamma = \infty)$. Furthermore, for the same $\mathcal{M}(w)$ a different value for $\mathcal{Z}_-(x, \gamma = \infty)$ yields the solutions

$$\mathcal{Z}'_+(x, \gamma) = \mathcal{Z}_+(x, \gamma)Z^{-1}(x), \quad \mathcal{Z}'_-(x, \gamma) = Z(x)\mathcal{Z}_-(x, \gamma). \quad (5.63)$$

Now, $\mathcal{M}(w(x, \lambda))$ has a special form. Its dependence on λ via w implies invariance under the replacement of λ by λ^{-1} . And it is symmetric. Therefore,

$$\mathcal{M}(w(x, \lambda)) = \mathcal{M}^T(w(x, \lambda)) = \mathcal{M}^T(w(x, \lambda^{-1})).$$

Hence, (5.62) can also be written as

$$\mathcal{M}(w(x, \lambda)) = \mathcal{Z}_-^T(x, \lambda^{-1})\mathcal{Z}_+^T(x, \lambda^{-1}). \quad (5.64)$$

Both, $(\mathcal{Z}_+(x, \gamma), \mathcal{Z}_-(x, \gamma))$ and $(\mathcal{Z}_-^T(x, \gamma^{-1}), \mathcal{Z}_+^T(x, \gamma^{-1}))$ are factorizations of \mathcal{M} . By uniqueness of the solution, they thus have to satisfy relations of the form (5.63):

$$\mathcal{Z}_+(x, \gamma) = \mathcal{Z}_-^T(x, \gamma^{-1})Z^{-1}(x), \quad \mathcal{Z}_-(x, \gamma) = Z(x)\mathcal{Z}_+^T(x, \gamma^{-1}). \quad (5.65)$$

Expressing Z in both of these equations and restricting to values on the unit circle yields

$$\mathcal{Z}_+^{-1}(x, \lambda)\mathcal{Z}_-^T(x, \lambda^{-1}) = Z(x) = \mathcal{Z}_-(x, \lambda)\mathcal{Z}_+^{T-1}(x, \lambda^{-1}) = \mathcal{Z}_-(x, \lambda^{-1})\mathcal{Z}_+^{T-1}(x, \lambda), \quad (5.66)$$

where in the last step we used that λ and λ^{-1} could be exchanged by the independence of $Z(x)$ of γ . Since the left side is the transpose of the right side, $Z(x)$ is a symmetric matrix.

The second equation in (5.65) can now be used to express the original factorization (5.62) in the form

$$\mathcal{M}(w(x, \lambda)) = \mathcal{Z}_+(x, \lambda)Z(x)\mathcal{Z}_+^T(x, \lambda^{-1}). \quad (5.67)$$

The special form of $\mathcal{M}(w(x, \lambda))$ tells us that there is one function, $\mathcal{Z}_+(x, \lambda)$, which determines the holomorphicity properties of both factors in the Hilbert problem. We now interpret (5.67) as a Hilbert problem, the two factors being as indicated by the brackets,

$$\mathcal{M}(w(x, \lambda)) = [\mathcal{Z}_+(x, \lambda)Z(x)][\mathcal{Z}_+^T(x, \lambda^{-1})] \quad (5.68)$$

and require the function whose boundary values are given by the second factor to be equal to the identity matrix at $\gamma = \infty$, equivalently $\mathcal{Z}_+(x, 0) = \mathbb{1}$. Then the first factor evaluated at $\gamma = 0$ gives the symmetric matrix $Z(x)$. This product may be factorized into $Z = VV^T$ uniquely up to right multiplication of $V(x)$ by $SO(2)$ elements $h(x)$. Now we define

$$\hat{\mathcal{U}}(x, \gamma) = \mathcal{Z}_+(x, \gamma)V(x), \quad (5.69)$$

and (5.68) takes the form

$$\mathcal{M}(w(x, \gamma)) = \hat{\mathcal{U}}(x, \lambda)\hat{\mathcal{U}}^T(x, \lambda^{-1}) \quad (5.70)$$

where $\hat{\mathcal{U}}(x, \lambda)$ are the boundary values of a function $\hat{\mathcal{U}}(x, \gamma)$ holomorphic inside the unit circle, just like (5.61). In the next section we show that such a solution for admissible $\mathcal{M}(w)$ exists. In section 5.5.8, we will show that $\hat{\mathcal{U}}^{-1}\partial\hat{\mathcal{U}}$ indeed has the form

$$\hat{\mathcal{U}}^{-1}\partial_{\pm}\hat{\mathcal{U}} = (\mathcal{U}^{-1}\partial_{\pm}\mathcal{U})_{\mathfrak{h}} + u^{\pm 1}(\mathcal{U}^{-1}\partial_{\pm}\mathcal{U})_{\mathfrak{k}},$$

where $\mathcal{U} = \hat{\mathcal{U}}(\gamma = 0)$ and thus that \mathcal{U} satisfies the equations of motion.

Note also, that suppose we are given a $\hat{\mathcal{V}}$ constructed from a solution \mathcal{V} corresponding to some initial data on \mathcal{N} and build the monodromy matrix \mathcal{M} . Factorising this \mathcal{M} according to the Hilbert problem we recover $\hat{\mathcal{V}}$ up to right multiplication of an $SO(2)$ element, corresponding to the local Lorentz gauge freedom in the planes spanned by the Killing fields, which may be fixed requiring \mathcal{V} to be in triangular gauge. That we indeed recover $\hat{\mathcal{V}}$ and not some other solution is due to the uniqueness of the factorisation for prescribed behaviour at $\gamma = \infty$.

5.5.7 The Hilbert problem

We now prove that for admissible $\mathcal{M}(w)$, the Hilbert factorization problem as stated in 5.5.6 has a unique solution for a prescribed value of \mathcal{Z}_- at $\gamma = \infty$.

For now, we look for vectorial solutions, that is two vectors $\Phi_+(\gamma)$ and $\Phi_-(\gamma)$ holomorphic inside and outside of the unit circle with limits on the unit circle such that

$$\Phi_+(\lambda) = \mathcal{M}(\lambda)\Phi_-(\lambda), \quad (5.71)$$

Φ_- such that $\Phi_-(\infty) = v$, where v is some finite vector. Again, λ always lies on the unit circle \mathfrak{C} . The integral over \mathfrak{C} is to be taken in such a direction that the interior \mathfrak{C}^+ of the

circle is to left when λ increases, while the exterior \mathfrak{C}^- is to the right. Note that once we have found $\Phi_-(\lambda)$, $\Phi_-(\gamma)$ and $\Phi_+(\gamma)$ are given by

$$\begin{aligned}\Phi_-(\gamma) &= -\frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\Phi_-(\lambda)d\lambda}{\lambda-\gamma} + v, & \gamma \in \mathfrak{C}^-, \\ \Phi_+(\gamma) &= \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\mathcal{M}(\lambda)\Phi_-(\lambda)d\lambda}{\lambda-\gamma}, & \gamma \in \mathfrak{C}^+.\end{aligned}\tag{5.72}$$

Following chapter 18 of [28] we now derive a singular Fredholm integral equation of the second kind equivalent to (5.71). First, we state the necessary and sufficient condition for the vector $\Phi_+(\lambda)$ to be the boundary value of a vector $\Phi(\gamma)$ holomorphic in \mathfrak{C}^+ is

$$0 = -\frac{1}{2}\Phi_+(\sigma) + \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\phi_+(\lambda)d\lambda}{\lambda-\sigma}.\tag{5.73}$$

Similarly, the necessary and sufficient condition for the vector $\Phi_-(\lambda)$ to be the boundary value of vector $\Phi(\gamma)$ holomorphic in \mathfrak{C}^- and behaving at infinity as

$$\Phi(\gamma) = v + \mathcal{O}(\gamma^{-1})$$

is

$$0 = \frac{1}{2}\Phi_-(\sigma) + \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\Phi_-(\lambda)d\lambda}{\lambda-\gamma} - v.\tag{5.74}$$

Now we bring into the play the Hilbert problem. We require the boundary values Φ_- and Φ_+ to be related by (5.71). Inserting this into equation (5.73) gives us two equations for the boundary value $\phi_-(\lambda)$

$$\frac{1}{2}\Phi_-(\sigma) + \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\Phi_-(\lambda)d\lambda}{\lambda-\gamma} = v,\tag{5.75}$$

$$-\frac{1}{2}\mathcal{M}(\sigma)\Phi_-(\sigma) + \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\mathcal{M}(\lambda)\Phi_-(\lambda)d\lambda}{\lambda-\sigma} = 0.\tag{5.76}$$

These are just the Plemelj formulae applied to (5.72). Multiplying the second of these equations from the left by $\mathcal{M}^{-1}(\sigma)$ (the inverse exists because $\det \mathcal{M} = 1$) and subtracting it from the first equation gives

$$\Phi_-(\sigma) - \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\mathcal{M}^{-1}(\sigma)\mathcal{M}(\lambda) - \mathbb{1}}{\lambda-\sigma} \Phi_-(\lambda)d\lambda = v.\tag{5.77}$$

This is a Fredholm equation of the second kind. It is well posed because \mathcal{M} is required to satisfy a Hölder condition, which guarantees the existence of the integral.

The first question is now, are there (unique) solutions to the Fredholm equation (5.77)? The second question is, supposing that a solution to (5.77) exists, does it automatically satisfy both equations (5.75) and (5.76), from which it was derived, and thereby the homogeneous Hilbert problem? The second question has been analysed in chapter 18 of [28] and the answer is positive. Thus, we know that every solution of (5.77) corresponds to a solution of (5.71) behaving at infinity as $v + \mathcal{O}(\gamma^{-1})$.

We turn to the first question. Define a Fredholm operator \mathbf{K} acting on continuous functions $\phi(\lambda)$ on \mathfrak{C} by

$$\mathbf{K}\phi = \phi - \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\mathcal{M}^{-1}(\sigma)\mathcal{M}(\lambda) - \mathbb{1}}{\lambda-\sigma} \phi(\lambda)d\lambda.\tag{5.78}$$

It is a linear operator on the space of continuous functions satisfying a Hölder condition. Now, the space of Hölder continuous functions on a closed interval form a Banach space with respect to the maximum norm [15]. The functions on the circle can be represented as the subset of functions on a closed real interval, whose values agree at the endpoints. This subset is clearly closed, because the limit of a sequence of functions whose values at the endpoints agree, converging with respect to the maximum norm, will also have this property. Thus, the space of Hölder functions on the circle also form a Banach space.

For a Fredholm operator \mathbf{K} , one has the simple formula [28]

$$\dim \ker \mathbf{K} = \text{codim range } \mathbf{K}.$$

Since \mathbf{K} in our case is injective, the kernel has dimension zero, and thus the codimension of the image is also zero. Hence, \mathbf{K} is also surjective and thus invertible.

Our Fredholm equation (5.77) can then be compactly written as

$$\mathbf{K}\Phi_- = v.$$

In general, there is a theorem on Fredholm operators [28], called the Fredholm alternative, which basically states, that a Fredholm operator is the sum of an invertible operator and a compact operator. A quite vague but illustrative picture is: The compact operator can destroy the invertibility of the invertible operator in only a finite number of dimensions. The compact part of \mathbf{K} is the integral operator while the invertible operator is simply the identity in (5.78). The precise statement is that the kernel of a Fredholm operator is finite dimensional.

In chapter 18 of [28], a fundamental system of solutions to the homogeneous Fredholm equation

$$\mathbf{K}\phi = 0,$$

deduced from (5.77) by setting $v = 0$, is constructed. In the general case, the fundamental system for a homogeneous Fredholm equation in n dimensions is a set of n vectors $\{\chi_1, \dots, \chi_n\}$, which have the degrees $-\varsigma_1, \dots, -\varsigma_n$ at infinity, meaning that χ_j behaves as $\mathcal{O}(\gamma^{-\varsigma_j})$ at infinity and whose boundary values on \mathfrak{C} satisfy the integral equation. The general solution to the Hilbert problem is then given by linear combinations of these χ_j , where the coefficients are polynomials. Now, the sum ς of the degrees ς_j can be computed from \mathcal{M} by the formula [28]

$$\varsigma_1 + \dots + \varsigma_n = \varsigma = \frac{1}{2\pi} [\arg \det \mathcal{M}(\lambda)]_{\mathfrak{C}}, \quad (5.79)$$

where the subscript \mathfrak{C} denotes the increment when moving along the circle once. But in our case $\det \mathcal{M}(\lambda) = 1$ and thus $\varsigma = 0$. If this sum is to vanish, then either one of the summands, say ς_1 , has to be negative, or they are all zero. But functions with degree lesser or equal than zero do not vanish at infinity and thus they cannot be solutions of the homogeneous equation $\mathbf{K}\chi_1 = 0$. Thus there cannot be any such solutions. The kernel of \mathbf{K} is empty and so \mathbf{K} is injective. But it is also surjective because for The solution Φ to the Hilbert problem is uniquely determined by the inhomogeneous equation (5.77) and we have

$$\Phi = \mathbf{K}^{-1}v.$$

We remember that v determines the behaviour of Φ at infinity.

Up to now, we have only considered vectorial solutions to the Hilbert problem. If v_1, v_2 are linearly independent vectors, then clearly the corresponding solutions Φ_1, Φ_2 are also linearly independent. If they weren't, their difference would lie in the kernel, which is empty. Hence, the matrix \mathcal{Z}_- with columns Φ_1, Φ_2 uniquely solves the matrix form of Hilbert problem (5.62) for a given matrix with columns v_1, v_2 determining the behaviour at infinity. For the solution of (5.68) we need in particular $\mathcal{Z}_-(x, \gamma = \infty) = \mathbb{1}$. $\hat{\mathcal{U}}$ is then recovered as explained in the previous section.

The operator \mathbf{K} is invertible at least on the class of Hölder continuous vectors. Consider a family parametrised by x of vector valued functions $f(x, \lambda)$ on \mathfrak{C} . If f has any of the properties continuity, differentiability in x , Hölder, then since we we required $\mathcal{M}(w)$ to have them, by theorems 6C.2 and 6C.3 of [18] the integral term of $\mathbf{K}f$ and thus $\mathbf{K}f$ as a whole also has these properties. Thus \mathbf{K} maps an element with the desired properties to an element with the same properties.

It seems very likely that the inverse \mathbf{K}^{-1} also maps vectors v with these properties to vectors f having them, but we have not succeeded to prove it. We assume it to be true. A proof that in the case of analytic or smooth \mathcal{M} , the solutions will also be analytic or smooth, is given in [31]. Note that actually, if $\mathcal{M}(w)$ is analytic, $\hat{\mathcal{U}}$ holomorphic in the disc, can be holomorphically extended to a disc \mathfrak{C}_ϵ^+ of radius $1 + \epsilon$, because $\mathcal{M}(w)$ may be extended to an annular region containing the unit circle. One simply sets

$$\begin{aligned} \hat{\mathcal{U}}, & \quad \text{for } \gamma \in \mathfrak{C}^+ \\ \mathcal{M}\eta^\infty(\hat{\mathcal{U}}), & \quad \text{for } \gamma \in \mathfrak{C}_\epsilon^+ \setminus \mathfrak{C}^+. \end{aligned}$$

By the very definition of \mathcal{M} , (5.47), on \mathfrak{C} , the values agree, the function is thus continuous across \mathfrak{C} and therefore analytic in all of \mathfrak{C}_ϵ^+ , in particular on \mathfrak{C} .

5.5.8 Properties of $\hat{\mathcal{U}}$

We have at our disposal a function $\hat{\mathcal{U}}(x, \gamma)$, holomorphic in \mathfrak{C}^+ , continuous on \mathfrak{C} , and differentiable in x and along \mathfrak{C} . $\hat{\mathcal{U}}(x, \gamma^{-1})$ has the same properties except that it is holomorphic in \mathfrak{C}^- . The boundary values of these functions satisfy

$$\mathcal{M}(w(x, \lambda)) = \hat{\mathcal{U}}(x, \lambda)\hat{\mathcal{U}}^T(x, \lambda^{-1}) \quad \text{on } \mathfrak{C}.$$

Now, the spacetime differential operator with w held fix may be also be written as

$$\partial_\pm|_w = \partial_\pm|_\gamma + \frac{\gamma}{2\rho} \frac{1 \mp \gamma}{1 \pm \gamma} \partial_\gamma \quad (5.80)$$

and has simple poles at $\gamma = \pm 1$. We first consider only the +-component. Define the operator

$$D := (1 + \gamma)\partial_\pm|_w = (1 + \gamma)\partial_\pm|_\gamma + \frac{\gamma}{2\rho}(1 - \gamma)\partial_\gamma. \quad (5.81)$$

D is analytic in the finite complex plane. Note, that D may also act on functions defined only on \mathfrak{C} , because when varying x , γ on the unit circle moves along the unit circle and

does not leave it. $\hat{U}(x, \gamma)$ is defined by its boundary values $\hat{U}(x, \lambda)$ on \mathfrak{C} by

$$\hat{U}(x, \gamma) = \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\hat{U}(x, \lambda) d\lambda}{\lambda - \gamma}, \quad \gamma \in \mathfrak{C}^+, \quad (5.82)$$

and, by Cauchy's theorem, for $\gamma \in \mathfrak{C}^-$

$$\hat{U}(x, \gamma) = \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\hat{U}(x, \lambda) d\lambda}{\lambda - \gamma} = 0, \quad \gamma \in \mathfrak{C}^-. \quad (5.83)$$

$\partial_+|_{\gamma} \hat{U}(x, \gamma)$ is given by

$$\partial_+|_{\gamma} \hat{U}(x, \gamma) = \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\partial_+ \hat{U}(x, \lambda) d\lambda}{\lambda - \gamma} \quad \text{for } \gamma \in \mathfrak{C}^+,$$

which exists and is thus also holomorphic in \mathfrak{C}^+ . Therefore, $D\hat{U}$ will also be holomorphic in \mathfrak{C}^+ .

We define the function

$$\Xi(x, \gamma) := D\hat{U}(x, \gamma),$$

holomorphic in \mathfrak{C}^+ and vanishing on \mathfrak{C}^- , because by (5.83) \hat{U} vanishes identically on \mathfrak{C}^- . On the other hand, we can apply the differential operator D to the boundary values $\hat{U}(x, \lambda)$. This defines a continuous function on \mathfrak{C} . Now consider the integral

$$\psi(\gamma) = \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{D\hat{U}(x, \lambda) d\lambda}{\lambda - \gamma}, \quad \gamma \in \mathfrak{C}^+ \cup \mathfrak{C}^-. \quad (5.84)$$

$\psi(x, \gamma)$ is a holomorphic function of γ in \mathfrak{C}^+ and \mathfrak{C}^- . We now compute the difference $\psi - \Xi$. For ψ , we get

$$\psi(x, \gamma) = \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{(1 + \lambda) \partial_+ \hat{U}(x, \lambda) d\lambda}{\lambda - \gamma} + \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\partial_{\lambda} \hat{U}(x, \lambda)}{\lambda - \gamma} \frac{\lambda}{2\rho} (1 - \lambda) = I_1 + I_2.$$

In the second integral, I_2 , we may integrate partially to get

$$I_2 = \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\hat{V}(x, \lambda)}{(\lambda - \gamma)^2} \frac{1}{2\rho} (\lambda^2 + \gamma - 2\lambda\gamma) d\lambda.$$

Thus ψ is

$$\psi(x, \gamma) = \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\partial_+ \hat{U}(x, \lambda)}{\lambda - \gamma} (1 + \lambda) d\lambda + \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\hat{V}(x, \lambda)}{(\lambda - \gamma)^2} \frac{1}{2\rho} (\lambda^2 + \gamma - 2\lambda\gamma) d\lambda. \quad (5.85)$$

On the other hand, $\Xi(\gamma)$ is

$$\Xi(\gamma) = \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\partial_+ \hat{U}(x, \lambda)}{\lambda - \gamma} (1 + \gamma) d\lambda + \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\hat{V}(x, \lambda)}{(\lambda - \gamma)^2} \frac{1}{2\rho} (\gamma - \gamma^2).$$

Then the difference $\psi - \Xi$ is

$$\psi(x, \gamma) - \Xi(x, \gamma) = \frac{1}{2\pi i} \int_{\mathfrak{C}} \partial_+ \hat{U}(x, \lambda) d\lambda + \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\hat{V}(x, \lambda)}{2\rho} = 0, \quad (5.86)$$

because both integrands are the boundary values of functions holomorphic in \mathfrak{C}^+ . So, since Ξ and ψ are equal and Ξ vanishes in \mathfrak{C}^- , so does ψ

$$\psi(x, \gamma) = \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{D\hat{\mathcal{U}}(x, \lambda) d\lambda}{\lambda - \gamma} = 0, \quad \gamma \in \mathfrak{C}^-.$$

But this is the necessary and sufficient condition for $D\hat{\mathcal{U}}(x, \lambda)$ to be the boundary value of $D\hat{\mathcal{U}}(x, \gamma)$ as γ approaches \mathfrak{C} from inside (see [28]). Consequently $\Phi(\lambda) = \hat{\mathcal{U}}^{-1}(\lambda) D\hat{\mathcal{U}}(\lambda)$ is the boundary value of $\Phi(\gamma) = \hat{\mathcal{U}}^{-1}(\gamma) D\hat{\mathcal{U}}(\gamma)$, analytic in \mathfrak{C}^+ , and, save at $\lambda = -1$,

$$\hat{I}'(\lambda) := \hat{\mathcal{U}}^{-1}(\lambda) \partial_+ \big|_w \hat{\mathcal{U}}(\lambda) = \frac{1}{1 + \lambda} \Phi(\lambda) \quad (5.87)$$

is the boundary value of

$$\hat{I}'(\gamma) = \hat{\mathcal{U}}^{-1}(\gamma) \partial_+ \big|_w \hat{\mathcal{U}}(\gamma) = \frac{1}{1 + \gamma} \Phi(\gamma). \quad (5.88)$$

It follows that away from $\lambda = -1$ the function $\eta^\infty(\hat{I}')(\lambda)$ is the boundary value of $\eta^\infty(\hat{I}')(\gamma)$, which is analytic in \mathfrak{C}^- .

Now, \hat{I}' and $\eta^\infty(\hat{I}')$ are in fact the same on the unit circle: The monodromy data \mathcal{M} are given as a function of w only. Thus by construction $\mathcal{M}(x, \lambda) = \mathcal{M}(w(x, \lambda))$ satisfies $\partial_+ \big|_w \mathcal{M}(w) = 0$. But

$$\partial_+ \mathcal{M} \big|_w = \partial_+ \hat{\mathcal{U}}(\lambda) \big|_w \hat{\mathcal{U}}^T(\lambda^{-1}) + \hat{\mathcal{U}}(\lambda) \partial_+ \big|_w \hat{\mathcal{U}}^T(\lambda^{-1}) \big|_w = \hat{\mathcal{U}}(\lambda) \{ \hat{I}'(\lambda) - \eta^\infty(\hat{I}')(\lambda) \} \hat{\mathcal{U}}^T(\lambda^{-1}), \quad (5.89)$$

so

$$\hat{I}'(\lambda) = \eta^\infty(\hat{I}')(\lambda). \quad (5.90)$$

The two functions $\hat{I}'(\gamma)$ and $\eta^\infty(\hat{I}')(\gamma)$ together define a single function $\hat{I}(\gamma)$ on the whole Riemann sphere except $\gamma = -1$ defined by

$$\begin{aligned} \hat{I}(\gamma) &= \hat{I}'(\gamma), & \text{for } \gamma \in \mathfrak{C}^+ \cup \mathfrak{C} \setminus \{-1\}, \\ \hat{I}(\gamma) &= \eta^\infty(\hat{I}')(\gamma), & \text{for } \gamma \in \mathfrak{C}^- \cup \mathfrak{C} \setminus \{-1\}. \end{aligned}$$

\hat{I} is analytic in $\mathfrak{C}^+ \cup \mathfrak{C}^-$ and continuous on \mathfrak{C} except the point $\gamma = -1$. By Morera's theorem, \hat{I} is therefore analytic in all of $\mathfrak{C} \setminus \{-1\}$.

It is perhaps surprising, but these facts suffice to determine the dependence of \hat{I} on γ almost completely. Consider the function $(1 + \gamma)\hat{I}$. It is analytic throughout $\mathfrak{C} \setminus \{-1, \infty\}$, and, it is continuous at $\gamma = -1$ because it coincides with $\Phi(\gamma)$ in \mathfrak{C}^+ and this function tends to a finite boundary value $b := \Phi(-1)$ at $\gamma = -1$. (Note however, that it also coincides with $(1 + \gamma)\eta^\infty(\hat{I}') = \gamma\eta^\infty(\Phi)$ in \mathfrak{C}^- , and this function has the boundary value $-\eta(\Phi)(-1) = -\eta(b)$. Thus we also know that $b = -\eta(b)$, in other words $b \in \mathfrak{k}$.)

Consider again \hat{I} . When γ equals neither -1 nor ∞

$$\hat{I} = \frac{\Phi}{1 + \gamma} = \frac{\Phi - b}{1 + \gamma} + \frac{b}{1 + \gamma}. \quad (5.91)$$

Rearranging we have

$$\hat{I} - \frac{b}{1 + \gamma} = \frac{\Phi - b}{1 + \gamma}. \quad (5.92)$$

The left side is analytic on the whole Riemann sphere, save perhaps at $\gamma = -1$. The right side is analytic on the whole Riemann sphere except perhaps at $\gamma = \infty$, because the numerator vanishes at the zero of the denominator. The equality tells us that both sides are analytic on the entire Riemann sphere, and are thus independent of γ :

$$\hat{I} = a + \frac{b}{1 + \gamma}. \quad (5.93)$$

where a and b depend only on spacetime position x , but not γ .

Let us return to the condition (5.90). First, it is convenient to rewrite \hat{I} as

$$\hat{I} = A + \frac{1 - \gamma}{1 + \gamma} B. \quad (5.94)$$

where $A = a + b/2$ and $B = b/2$. (5.90) requires that at the unit circle

$$A + \frac{1 - \lambda}{1 + \lambda} B = \hat{I} = \eta^\infty(\hat{I}) = \eta(A) - \frac{1 - \lambda}{1 + \lambda} \eta(B). \quad (5.95)$$

Since this must hold for all λ on the circle (except $\lambda = -1$) it implies that

$$A = \eta(A), \quad B = -\eta(B).$$

The first equation states that $A \in \mathfrak{h}$, the second that $B \in \mathfrak{k}$ (which we already knew).

This suffices to show that the field $\mathcal{U}(x)$ obtained by evaluating $\hat{\mathcal{U}}(x, \gamma)$ at $\gamma = 0$ satisfies the field equations. At $\gamma = 0$

$$\hat{I} = \hat{\mathcal{U}}^{-1} \partial_+ \hat{\mathcal{U}}|_w = \hat{\mathcal{U}}^{-1} \partial_+ \hat{\mathcal{U}}|_{\gamma=0} = \mathcal{U}^{-1} \partial_+ \mathcal{U}. \quad (5.96)$$

because at $\gamma = 0$ differentiating at constant w is the same as differentiating at constant γ , since there

$$\partial_+ \gamma|_w = \frac{\gamma}{2\rho} \frac{1 - \gamma}{1 + \gamma} = 0.$$

On the other hand $\hat{I} = A + B$ at $\gamma = 0$. Thus

$$A = (\mathcal{U}^{-1} \partial_+ \mathcal{U})|_{\mathfrak{h}}, \quad B = (\mathcal{U}^{-1} \partial_+ \mathcal{U})|_{\mathfrak{k}}. \quad (5.97)$$

A completely analogous calculation for the x^- derivative yields

$$\hat{\mathcal{U}}^{-1} \partial_- \hat{\mathcal{U}}|_w = (\mathcal{U}^{-1} \partial_- \mathcal{U})|_{\mathfrak{h}} + \frac{1 + \gamma}{1 - \gamma} (\mathcal{U}^{-1} \partial_- \mathcal{U})|_{\mathfrak{k}}$$

The field $\hat{\mathcal{U}}$ therefore satisfies the linear system

$$\partial_\pm \hat{\mathcal{U}}|_w = \hat{\mathcal{U}} \left[A_\pm + \frac{1 \mp \gamma}{1 \pm \gamma} B_\pm \right]$$

at all $\gamma \in \mathfrak{C}^+$, with $A_\pm = (\mathcal{U}^{-1} \partial_\pm \mathcal{U})|_{\mathfrak{h}}$ and $B_\pm = (\mathcal{U}^{-1} \partial_\pm \mathcal{U})|_{\mathfrak{k}}$. Of course it then must also satisfy the integrability conditions of this linear system, which are precisely the field equations applied to U . See (4.22).

5.6 Poisson Bracket of $\hat{\mathcal{V}}_0(x, w)$

In sections 5.3 and 5.4 we calculated the Poisson brackets of the \mathcal{V} on a null initial data surface and compared it to [32] and [33]. In section 5.5 we have discussed the transformation from \mathcal{V} to the monodromy matrix $\mathcal{M}(w)$ and its inverse. We saw that the transformation is invertible, if we use a hypersurface \mathcal{N} touching the axis $\rho = 0$, which consists of only one branch, \mathcal{N}_R . In this section, we use our brackets of the \mathcal{V} on the initial data surface \mathcal{N} to compute the bracket of the $\hat{\mathcal{V}}_0$ for $w \in W_+$, which will be the starting point for the Poisson algebra of the monodromy matrix $\mathcal{M}(w)$ in section 5.7 and, following the ideas of [24], the Poisson algebra of conserved charges in section 5.8.

5.6.1 Notation and strategy

$\hat{\mathcal{V}}_0(y, w)$ is related to \mathcal{V} by

$$\hat{\mathcal{V}}_0(y, w) = \mathcal{V}(x_0)T(x_0, y; w) = \mathcal{V}(x_0)\mathcal{P}e^{\int_{x_0}^y \hat{J}(w)}, \quad (5.98)$$

where x_0 is a point on the axis and we introduced the transport matrix

$$T(x, y; w) := \mathcal{P}e^{\int_x^y \hat{J}(w)} = \hat{\mathcal{V}}^{-1}(x, w)\hat{\mathcal{V}}(y, w). \quad (5.99)$$

Let $y = (x_0^-, y^+)$ be a point⁸ on \mathcal{N} and fix $x_0 = (x_0^-, x_0^+)$ to be the point where \mathcal{N} meets the axis $\rho = 0$. Then $\hat{\mathcal{V}}_0(y, w)$ may be expressed entirely by $\mathcal{V}(z)$ with $z \in \mathcal{N}$ by

$$\hat{\mathcal{V}}_0(x_0^-, y^+, w) = \mathcal{V}(x_0)T(x_0, y; w) = \mathcal{V}(x_0^-, x_0^+)\mathcal{P}e^{\int_{x_0^+}^{y^+} \hat{J}_+(x_0^-, z^+, w)dz^+}. \quad (5.100)$$

We shift the coordinate x^+ such that $x_0^+ = 0$. Since in this section we exclusively deal with points on \mathcal{N} with the same x^- -coordinate, from now on we omit the x^- -dependence. Furthermore, instead of x^+, y^+, z^+, \dots we simply write x, y, z , and we do not write the subscript $+$ on one-forms to indicate that we refer to the $+$ -component. Furthermore, obvious w -dependences will be omitted. In this simplified notation, we have

$$\hat{\mathcal{V}}_0(y) = \mathcal{V}(0)T(0, y) = \mathcal{V}(0)\mathcal{P}e^{\int_0^y \hat{J}(z)dz}. \quad (5.101)$$

The Poisson bracket of the $\hat{\mathcal{V}}_0$ is related to the bracket of the \mathcal{V} by the chain rule

$$\begin{aligned} & \hat{\mathcal{V}}^{-1}(y_1)\hat{\mathcal{V}}^{-1}(y_2)\{\hat{\mathcal{V}}_0(y_1, w_1), \hat{\mathcal{V}}_0(y_2, w_2)\} = \\ & \int_0^\infty dx_3 \int_0^\infty dx_4 \left(\hat{\mathcal{V}}^{-1}(y_1) \overset{1 \otimes 3^*}{\frac{\delta \hat{\mathcal{V}}_0(y_1)}{\delta \mathcal{V}(x_3)}} \mathcal{V}(x_3) \right) \left(\hat{\mathcal{V}}^{-1}(y_2) \overset{2 \otimes 4^*}{\frac{\delta \hat{\mathcal{V}}_0(y_2)}{\delta \mathcal{V}(x_4)}} \mathcal{V}(x_4) \right) \underset{3 \ 4}{\lrcorner} \underset{3 \ 4}{\lrcorner} \\ & \underset{3 \ 4}{\lrcorner} \mathcal{V}^{-1}(x_3)\mathcal{V}^{-1}(x_4)\{\mathcal{V}(x_3), \mathcal{V}(x_4)\}, \end{aligned} \quad (5.102)$$

where the superscript $i \otimes j^*$ over $\hat{\mathcal{V}}_0^{-1}(y_i)(\delta \hat{\mathcal{V}}_0(y_i)/\delta \mathcal{V}(x_j))\mathcal{V}(x_j)$ means that it lies in the tensor product of the i -th copy of \mathfrak{g} with the j -th copy of the dual \mathfrak{g}^* of \mathfrak{g} (the notation is analogous to (5.19)). In a first step we calculate $\hat{\mathcal{V}}_0^{-1}(y)\delta \hat{\mathcal{V}}_0(y)$ for a generic variation δ in terms of $\delta \mathcal{V}$. From that we then extract the functional gradient and calculate the Poisson bracket (5.102).

⁸The x^- -coordinate of all points on \mathcal{N} is x_0^- .

5.6.2 The variation of $\hat{\mathcal{V}}_0$

From (5.101) we get

$$\delta\hat{\mathcal{V}}_0(y) = \delta\mathcal{V}(0)T(0, y) + \mathcal{V}(0) \int_0^y T(0, z)\delta\hat{J}(z)T(z, y)dz. \quad (5.103)$$

We express the variation $\delta\hat{J}$ by $\delta\mathcal{V}$.

$$\delta\hat{J} = \delta Q + u\delta P = \delta J|_{\mathfrak{h}} + u\delta J|_{\mathfrak{k}}$$

and

$$\begin{aligned} \delta J &= \delta(\mathcal{V}^{-1}\partial\mathcal{V}) = -\mathcal{V}^{-1}\delta\mathcal{V}\mathcal{V}^{-1}\partial\mathcal{V} + \mathcal{V}^{-1}\delta\partial\mathcal{V} = \\ &= -\mathcal{V}^{-1}\delta\mathcal{V}\mathcal{V}^{-1}\partial\mathcal{V} + \mathcal{V}^{-1}\partial\mathcal{V}\mathcal{V}^{-1}\delta\mathcal{V} + \partial(\mathcal{V}^{-1}\delta\mathcal{V}) = \\ &= \partial(\mathcal{V}^{-1}\delta\mathcal{V}) + [J, \mathcal{V}^{-1}\delta\mathcal{V}] = D(\mathcal{V}^{-1}\delta\mathcal{V}) + [P, \mathcal{V}^{-1}\delta\mathcal{V}]. \end{aligned}$$

Therefore

$$\delta\hat{J} = D(\mathcal{V}^{-1}\delta\mathcal{V})|_{\mathfrak{h}} + [P, (\mathcal{V}^{-1}\delta\mathcal{V})|_{\mathfrak{k}}] + uD(\mathcal{V}^{-1}\delta\mathcal{V})|_{\mathfrak{k}} + u[P, (\mathcal{V}^{-1}\delta\mathcal{V})|_{\mathfrak{h}}]. \quad (5.104)$$

At this point we introduce the action of “hatting” and “unhatting” of \mathfrak{g} -valued elements. Motivated by the relation between $J = Q + P$ and $\hat{J} = Q + uP$, for a \mathfrak{g} -valued field X we define

$$\hat{X}(w) := X_{\mathfrak{h}} + u(w)X_{\mathfrak{k}}, \quad (5.105)$$

where here and from now we simply use subscripts \mathfrak{h} and \mathfrak{k} instead of $|_{\mathfrak{h}}$ and $|_{\mathfrak{k}}$ to denote projections onto \mathfrak{h} and \mathfrak{k} respectively. The “unhatting”-operation is denoted by

$$\underset{\sim}{X}(w) := X_{\mathfrak{h}} + \frac{1}{u(w)}X_{\mathfrak{k}} =: X_{\mathfrak{h}} + X_{\mathfrak{k}}. \quad (5.106)$$

Later we will also need an extension of the “unhatting”-operation to the tensor product of the Lie algebra. For a $\mathfrak{g} \otimes \mathfrak{g}$ -valued field, we specify the factor, on which the unhatting operation acts, e.g.

$$A_{\mathfrak{g}\mathfrak{g}}(x_1, x_2, w_2) = A_{\mathfrak{g}\mathfrak{h}}(x_1, x_2) + \frac{1}{u(x_2, w_2)}A_{\mathfrak{g}\mathfrak{k}}(x_1, x_2),$$

if it only acts on the second factor of the spaces in the tensor product, or

$$\begin{aligned} A_{\mathfrak{g}\mathfrak{g}}(x_1, x_2, w_1, w_2) &= \\ &= A_{\mathfrak{h}\mathfrak{h}}(x_1, x_2, w_1, w_2) + A_{\mathfrak{k}\mathfrak{h}}(x_1, x_2, w_1, w_2) + A_{\mathfrak{h}\mathfrak{k}}(x_1, x_2, w_1, w_2) + A_{\mathfrak{k}\mathfrak{k}}(x_1, x_2, w_1, w_2) = \\ &= A_{\mathfrak{h}\mathfrak{h}}(x_1, x_2) + \frac{1}{u(x_1, w_1)}A_{\mathfrak{k}\mathfrak{h}}(x_1, x_2) + \frac{1}{u(x_2, w_2)}A_{\mathfrak{h}\mathfrak{k}}(x_1, x_2) + \\ &\quad + \frac{1}{u(x_1, w_1)u(x_2, w_2)}A_{\mathfrak{k}\mathfrak{k}}(x_1, x_2) \end{aligned}$$

if it acts on both factors.

In this notation we may write (5.104) as

$$\begin{aligned}
\delta \hat{J} &= D(\mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{h}} + [uP, u^{-1}(\mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{k}}] + uD[u(u^{-1} \mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{k}}] + [uP, (\mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{h}}] = \\
&= D(\mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{h}} + [\hat{P}, (\mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{k}}] + uD[u(\mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{k}}] + [\hat{P}, (\mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{h}}] = \\
&= D(\mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{h}} + [\hat{P}, (\mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{g}}] + uD[u(\mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{k}}].
\end{aligned} \tag{5.107}$$

The last term may be modified. We have

$$uD(uX_{\mathfrak{k}}) - DX_{\mathfrak{k}} = uDX_{\mathfrak{k}} - D\left(\frac{1}{u}X_{\mathfrak{k}}\right) = \left(u - \frac{1}{u}\right)DX_{\mathfrak{k}} - \partial\left(\frac{1}{u}\right)X_{\mathfrak{k}}.$$

On \mathcal{N} , $x^- = \rho^- = \text{const.}$. Defining $w_0 := w + \rho^-$ and using (4.33) and (2.61) on \mathcal{N} we have

$$\begin{aligned}
u &= \frac{\sqrt{w_0}}{\sqrt{w_0 + 2\rho}}, & u - \frac{1}{u} &= \frac{\sqrt{w_0}}{\sqrt{w_0 + 2\rho}} - \frac{\sqrt{w_0 + 2\rho}}{\sqrt{w_0}} = \frac{-2\rho}{\sqrt{w_0(w_0 + 2\rho)}}, \\
\partial \frac{1}{u} &= \frac{\partial \rho}{\sqrt{w_0(w_0 + 2\rho)}}, & -\partial \frac{1}{u} &= \left(u - \frac{1}{u}\right) \frac{1}{2} \rho^{-1} \partial \rho = \left(u - \frac{1}{u}\right) \rho^{-1/2} \partial \rho^{1/2} \\
&\Rightarrow uD(uX_{\mathfrak{k}}) &= \frac{\left(u - \frac{1}{u}\right)}{\sqrt{\rho}} D(\sqrt{\rho}X_{\mathfrak{k}}) + DX_{\mathfrak{k}}.
\end{aligned} \tag{5.108}$$

Hence, (5.107) is

$$\delta \hat{J} = [D + ad_{\hat{P}}](\mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{g}} + \frac{\left(u - \frac{1}{u}\right)}{\sqrt{\rho}} D[\sqrt{\rho}(\mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{k}}]. \tag{5.109}$$

For the variation of $\hat{\mathcal{V}}_0$ we thus have

$$\begin{aligned}
\delta \hat{\mathcal{V}}_0(y) &= \delta \mathcal{V}(0)T(0, y) + \mathcal{V}(0) \int_0^y T(0, z) \left\{ [D + ad_{\hat{P}}](\mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{g}} \right\} T(z, y) dz + \\
&+ \mathcal{V}(0) \int_0^y T(0, z) \left\{ \frac{\left(u - \frac{1}{u}\right)}{\sqrt{\rho}} D[\sqrt{\rho}(\mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{k}}] \right\} T(z, y) dz.
\end{aligned} \tag{5.110}$$

Now, for any \mathfrak{g} -valued field Y we have

$$\begin{aligned}
\partial_z [T(0, z)Y(z)T(z, y)] &= T(0, z) [\hat{J}(z)Y(z) - Y(z)\hat{J}(z) + \partial_z Y] T(z, y) = \\
&= T(0, z) \{ [D + ad_{\hat{P}}]Y(z) \} T(z, y).
\end{aligned} \tag{5.111}$$

Therefore, the second term in (5.110) may be integrated and we get

$$\begin{aligned}
\delta \hat{\mathcal{V}}_0(y) &= \delta \mathcal{V}(0)T(0, y) + \mathcal{V}(0) \left[T(0, z)(\mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{g}}(z)T(z, y) \right]_0^y + \\
&+ \mathcal{V}(0) \int_0^y T(0, z) \left\{ \frac{\left(u - \frac{1}{u}\right)}{\sqrt{\rho}} D[\sqrt{\rho}(\mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{k}}] \right\} T(z, y) dz.
\end{aligned} \tag{5.112}$$

On the axis, at $z = 0$, we have for $w \in W_+$ that $u(0) = 1$, and thus $X_{\mathfrak{g}}(0) = X_{\mathfrak{g}}(0)$. So the first term above cancels with the second term evaluated at $z = 0$. Multiplying from the left with $\hat{\mathcal{V}}_0^{-1}(y)$, we are finally left with

$$\hat{\mathcal{V}}_0^{-1} \delta \hat{\mathcal{V}}_0(y) = (\mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{g}}(y) + \int_0^y T(y, z) \left\{ \frac{\left(u - \frac{1}{u}\right)}{\sqrt{\rho}} D[\sqrt{\rho}(\mathcal{V}^{-1} \delta \mathcal{V})_{\mathfrak{k}}] \right\} T(z, y) dz. \tag{5.113}$$

5.6.3 Computation of the Poisson bracket

Following the idea presented in subsection 5.6.1, for the Poisson bracket of the $\hat{\mathcal{V}}_0$ we get

$$\begin{aligned}
\hat{\mathcal{V}}_0^{-1}(y_1)\hat{\mathcal{V}}_0^{-1}(y_2)\{\hat{\mathcal{V}}_0(y_1),\hat{\mathcal{V}}_0(y_2)\} &= A_{\mathfrak{gg}}^{12}(y_1,y_2)+ \\
&+ \int_0^{y_1} \int_0^{y_2} \left[\frac{\binom{1}{u-\frac{1}{u}}}{\sqrt{\rho}} \right]_{z_1} \left[\frac{\binom{2}{u-\frac{1}{u}}}{\sqrt{\rho}} \right]_{z_2} \overset{1}{T}(y_1,z_1)\overset{2}{T}(y_2,z_2) \times \\
&\times \overset{1}{D}_{z_1}\overset{2}{D}_{z_2}[\sqrt{\rho(z_1)\rho(z_2)}A_{\mathfrak{tt}}^{12}(z_1,z_2)]\overset{1}{T}(z_1,y_1)\overset{2}{T}(z_2,y_2)dz_2dz_1+ \\
&+ \int_0^{y_1} \left[\frac{\binom{1}{u-\frac{1}{u}}}{\sqrt{\rho}} \right]_{z_1} \overset{1}{T}(y_1,z_1)\overset{1}{D}_{z_1}[\sqrt{\rho(z_1)}A_{\mathfrak{tg}}^{12}(z_1,y_2)]\overset{1}{T}(z_1,y_1)dz_1+ \\
&+ \int_0^{y_2} \left[\frac{\binom{2}{u-\frac{1}{u}}}{\sqrt{\rho}} \right]_{z_2} \overset{2}{T}(y_2,z_2)\overset{2}{D}_{z_2}[\sqrt{\rho(z_2)}A_{\mathfrak{tg}}^{12}(y_1,z_2)]\overset{2}{T}(z_2,y_2)dz_2,
\end{aligned} \tag{5.114}$$

where it is understood that $\hat{\mathcal{V}}_0, u, T$ depend on $w_i, x_i, i \in \{1, 2\}$, and that $\hat{\mathcal{V}}_0, T$ lie in the i -th copy of the tensor product. In the last two terms we may use (5.21),

$$\overset{1}{D}[\sqrt{\rho(z_1)}A_{\mathfrak{tt}}^{12}(z_1,z_2)] = \frac{\delta(z_1-z_2)}{2\kappa\sqrt{\rho}}\Omega_{\mathfrak{t}}, \quad \overset{2}{D}[\sqrt{\rho(z_2)}A_{\mathfrak{tt}}^{12}(z_1,z_2)] = -\frac{\delta(z_1-z_2)}{2\kappa\sqrt{\rho}}\Omega_{\mathfrak{t}}, \tag{5.115}$$

to solve those parts of the integrals, which contain $A_{\mathfrak{tt}}$. The term in the second and third line requires more work to be done. Let us give the integrand without the transport matrices T a name

$$K = \frac{\binom{1}{u-\frac{1}{u}}\binom{2}{u-\frac{1}{u}}}{\sqrt{\rho_1\rho_2}}\overset{1}{D}\overset{2}{D}(\sqrt{\rho_1\rho_2}A_{\mathfrak{tt}}).$$

First we can again use (5.115). This gives

$$\begin{aligned}
K &= -\frac{\binom{1}{u-\frac{1}{u}}\binom{2}{u-\frac{1}{u}}}{\sqrt{\rho_1\rho_2}}\overset{1}{D}\left(\frac{\delta(z_1-z_2)}{2\kappa}\Omega_{\mathfrak{t}}\right) = \\
&= -\overset{1}{D}\left(\frac{\binom{1}{u-\frac{1}{u}}\binom{2}{u-\frac{1}{u}}}{\sqrt{\rho_1\rho_2}}\frac{\delta(z_1-z_2)}{2\kappa}\Omega_{\mathfrak{t}}\right) + \overset{1}{\partial}\left(\frac{\binom{1}{u-\frac{1}{u}}}{\sqrt{\rho_1}}\right)\frac{\binom{2}{u-\frac{1}{u}}}{\sqrt{\rho_2}}\frac{\delta(z_1-z_2)}{2\kappa}\Omega_{\mathfrak{t}} = \\
&= -(\overset{1}{D} + ad_{\hat{p}})(\dots) + \overset{1}{\partial}\left(\frac{\binom{1}{u-\frac{1}{u}}}{\sqrt{\rho_1}}\right)\frac{\binom{2}{u-\frac{1}{u}}}{\sqrt{\rho_2}}\frac{\delta(z_1-z_2)}{2\kappa}\Omega_{\mathfrak{t}} + \frac{\binom{1}{u-\frac{1}{u}}\binom{2}{u-\frac{1}{u}}}{\sqrt{\rho_1\rho_2}}\frac{\delta(z_1-z_2)}{2\kappa}\overset{1}{ad}_{\hat{p}}\Omega_{\mathfrak{t}}.
\end{aligned}$$

By (5.111), if the first term in the last line is again sandwiched with the transport matrices, it is a total derivative of a term containing a $\delta(z_1-z_2)$ by construction and the two integrals can be solved — one, because it is a total derivative, the other one using the $\delta(z_1-z_2)$. The rest of the last line is not a total derivative. Two things that are obviously

missing for it to also be a total derivative, are an $ad_Q \Omega_{\mathfrak{k}}$ -term and a $\partial_{z_1} \delta(z_1 - z_2)$ -term. Remembering the invariance of $\Omega_{\mathfrak{k}}$ under the adjoint action of H , we note that both terms could be “created” in a form symmetric in 1 and 2 because

$$\delta(z_1 - z_2)(ad_Q^1 + ad_Q^2)\Omega_{\mathfrak{k}} = 0 \quad \text{and} \quad (\overset{1}{\partial} + \overset{2}{\partial})\delta(z_1 - z_2) = 0. \quad (5.116)$$

But because of the antisymmetry of the Poisson bracket, and thus $A_{\mathfrak{k}\mathfrak{k}}$, K is antisymmetric w.r.t. 1 and 2. Is it possible to express the antisymmetric K as a product of a scalar antisymmetric term and a symmetric term containing $\Omega_{\mathfrak{k}}$? Then we might try to use the relations (5.116) to create the missing terms⁹. First we try an ansatz of the form

$$\delta(z_1 - z_2)[\overset{1}{\hat{P}} - \overset{2}{\hat{P}}, \Omega_{\mathfrak{k}}] = \delta(z_1 - z_2)a(z_1, z_2)[\overset{1}{\hat{P}} + \overset{2}{\hat{P}}, \Omega_{\mathfrak{k}}] \quad \text{with } a \text{ an antisymmetric scalar.} \quad (5.117)$$

Note again the intention of the ansatz: Like K , the left hand side is antisymmetric with respect to the exchange of the two factors of the tensor product and symmetric with respect to z_1 and z_2 . On the right hand side, symmetry and antisymmetry is reversed. But (5.117) does not have a solution for a . By using only the vector space structure of \mathfrak{k} , we will not succeed. So let’s try to use the algebra structure to find an additional term, which could be used in the ansatz (5.117). When $z_1 = z_2$ we have

$$[\overset{1}{\hat{P}}, \Omega_{\mathfrak{k}}] = \sum_{i,j \in \mathfrak{k}} P^i [e_i^1, e_j^1] e^j = \sum_{\substack{i,j \in \mathfrak{k} \\ k \in \mathfrak{h}}} P^i f_{ijk} e^k e^j = - \sum_{\substack{i,j \in \mathfrak{k} \\ k \in \mathfrak{h}}} P^i f_{ikj} e^k e^j = - \sum_{\substack{i,j \in \mathfrak{k} \\ k \in \mathfrak{h}}} P^i [e_i^2, e_k^2] e^k = -[\overset{2}{\hat{P}}, \Omega_{\mathfrak{h}}],$$

Thus

$$-\frac{\overset{2}{u}}{\overset{1}{u}}[\overset{1}{\hat{P}}, \Omega_{\mathfrak{k}}] = [\overset{2}{\hat{P}}, \Omega_{\mathfrak{h}}], \quad -\frac{\overset{1}{u}}{\overset{2}{u}}[\overset{2}{\hat{P}}, \Omega_{\mathfrak{k}}] = [\overset{1}{\hat{P}}, \Omega_{\mathfrak{h}}].$$

We enhance the ansatz (5.117)

$$\begin{aligned} \delta(z_1 - z_2)[\overset{1}{\hat{P}} - \overset{2}{\hat{P}}, \Omega_{\mathfrak{k}}] &= \delta(z_1 - z_2)a(z_1, z_2)[\overset{1}{\hat{P}} + \overset{2}{\hat{P}}, \Omega_{\mathfrak{k}}] + \delta(z_1 - z_2)b(z_1, z_2)[\overset{1}{\hat{P}} + \overset{2}{\hat{P}}, \Omega_{\mathfrak{h}}] = \\ &= \delta(z_1 - z_2)a(z_1, z_2)[\overset{1}{\hat{P}} + \overset{2}{\hat{P}}, \Omega_{\mathfrak{k}}] - \delta(z_1 - z_2)b(z_1, z_2) \left[\frac{\overset{2}{u}}{\overset{1}{u}} \overset{1}{\hat{P}} + \frac{\overset{1}{u}}{\overset{2}{u}} \overset{2}{\hat{P}}, \Omega_{\mathfrak{k}} \right], \end{aligned} \quad (5.118)$$

with a, b antisymmetric scalars. This time a solution exists. For $z_1 = z_2$

$$\begin{aligned} \Rightarrow \quad 1 &= a - b \frac{\overset{2}{u}}{\overset{1}{u}}, & -1 &= a - b \frac{\overset{1}{u}}{\overset{2}{u}}, \\ b &= \frac{2}{\frac{\overset{1}{u}}{\overset{2}{u}} - \frac{\overset{2}{u}}{\overset{1}{u}}}, & a &= \frac{\frac{\overset{1}{u}}{\overset{2}{u}} + \frac{\overset{2}{u}}{\overset{1}{u}}}{\frac{\overset{1}{u}}{\overset{2}{u}} - \frac{\overset{2}{u}}{\overset{1}{u}}}. \end{aligned}$$

⁹Of course this still would not guarantee that we can integrate, but these terms are surely necessary.

Although we have set $z_1 = z_2$, this definition of a, b may also be used for $z_1 \neq z_2$ by letting $\overset{1}{u}$ depend on z_1 and $\overset{2}{u}$ on z_2 . Substitution according to (5.118) yields

$$\begin{aligned} \overset{12}{K}(z_1, z_2; w_1, w_2) &= \frac{1}{2} \left(\overset{12}{K}(z_1, z_2; w_1, w_2) - \overset{21}{K}(z_2, z_1; w_2, w_1) \right) = \\ &= \frac{-\overset{1}{D} - \overset{1}{ad}_{\hat{P}} + \overset{2}{D} + \overset{2}{ad}_{\hat{P}}}{2} \left(\frac{\left(\overset{1}{u} - \frac{1}{\overset{1}{u}}\right) \left(\overset{2}{u} - \frac{1}{\overset{2}{u}}\right) \delta(z_1 - z_2)}{\sqrt{\rho_1 \rho_2}} \frac{\Omega_{\mathfrak{t}}}{2\kappa} \right) + \\ &+ \overset{1}{\partial} \left(\frac{\overset{1}{u} - \frac{1}{\overset{1}{u}}}{\sqrt{\rho_1}} \right) \frac{\left(\overset{2}{u} - \frac{1}{\overset{2}{u}}\right) \delta(z_1 - z_2)}{\sqrt{\rho_2}} \frac{\Omega_{\mathfrak{t}}}{4\kappa} - \overset{2}{\partial} \left(\frac{\overset{2}{u} - \frac{1}{\overset{2}{u}}}{\sqrt{\rho_2}} \right) \frac{\left(\overset{1}{u} - \frac{1}{\overset{1}{u}}\right) \delta(z_1 - z_2)}{\sqrt{\rho_1}} \frac{\Omega_{\mathfrak{t}}}{4\kappa} + \\ &+ \frac{\left(\overset{1}{u} - \frac{1}{\overset{1}{u}}\right) \left(\overset{2}{u} - \frac{1}{\overset{2}{u}}\right) \delta(z_1 - z_2)}{\sqrt{\rho_1 \rho_2} \left(\frac{\overset{1}{u}}{\overset{2}{u}} - \frac{\overset{2}{u}}{\overset{1}{u}}\right)} \frac{\delta(z_1 - z_2)}{4\kappa} \left[\left(\frac{\overset{1}{u}}{\overset{2}{u}} + \frac{\overset{2}{u}}{\overset{1}{u}} \right) [\overset{1}{\hat{P}} + \overset{2}{\hat{P}}, \Omega_{\mathfrak{t}}] + 2[\overset{1}{\hat{P}} + \overset{2}{\hat{P}}, \Omega_{\mathfrak{h}}] \right]. \end{aligned}$$

What we want, in order to be able to perform the integral $\int \int TTKT^{-1}T^{-1}$, is that the last two lines be of the form

$$\begin{aligned} &(\overset{1}{\partial}h + \overset{2}{\partial}h + h \overset{1}{ad}_{\hat{P}} + h \overset{2}{ad}_{\hat{P}}) \frac{\delta(z_1 - z_2)}{4\kappa} \Omega_{\mathfrak{t}} + \\ &+ (\overset{1}{\partial}f + \overset{2}{\partial}f + f \overset{1}{ad}_{\hat{P}} + f \overset{2}{ad}_{\hat{P}}) \frac{\delta(z_1 - z_2)}{4\kappa} \Omega_{\mathfrak{h}}. \end{aligned}$$

Then, according to (5.116), we may simply add the terms $(\overset{1}{\partial} + \overset{2}{\partial})\delta(z_1 - z_2)$ and $\overset{1}{ad}_Q + \overset{2}{ad}_Q$ and cast them into the form

$$\left(\overset{1}{D} + \overset{1}{ad}_{\hat{P}} + \overset{2}{D} + \overset{2}{ad}_{\hat{P}} \right) \left(\frac{\delta(z_1 - z_2)}{4\kappa} [h\Omega_{\mathfrak{t}} + f\Omega_{\mathfrak{h}}] \right). \quad (5.119)$$

Since K does not contain a term of the form $\delta(z_1 - z_2)(\overset{1}{\partial} + \overset{2}{\partial})f\Omega_{\mathfrak{h}}$, we need to have

$$\delta(z_1 - z_2)(\overset{1}{\partial} + \overset{2}{\partial})f = 0.$$

For $z_1 = z_2$ and using the explicit form of u we get

$$\begin{aligned} f &= \frac{(\overset{1}{u}^2 - 1)(\overset{2}{u}^2 - 1)}{\rho(\overset{1}{u}^2 - \overset{2}{u}^2)} & u^2 - 1 &= \frac{w + \rho^-}{w + \rho^+} - 1 = \frac{-2\rho}{w + \rho^+} \\ f &= \frac{4\rho^2}{(w_1 + \rho^+)(w_2 + \rho^+)} = \frac{4\rho^2}{(w_1 + \rho^+)(w_2 + \rho^+)} = \frac{2}{w_1 - w_2}. \end{aligned}$$

The factor f is indeed independent of z_1 and z_2 and thus the term containing $\Omega_{\mathfrak{h}}$ has the right form to be integrated twice. The last task is now to verify whether the derivative of the scalar factor of the $\Omega_{\mathfrak{t}}$ -term matches the given term, concretely if for $z_1 = z_2$

$$\partial \left(\frac{\overset{1}{u} - \frac{1}{\overset{1}{u}}}{\sqrt{\rho_1}} \right) \frac{\left(\overset{2}{u} - \frac{1}{\overset{2}{u}}\right)}{\sqrt{\rho_2}} - \partial \left(\frac{\overset{2}{u} - \frac{1}{\overset{2}{u}}}{\sqrt{\rho_2}} \right) \frac{\left(\overset{1}{u} - \frac{1}{\overset{1}{u}}\right)}{\sqrt{\rho_1}} \stackrel{?}{=} \frac{2}{w_1 - w_2} \left((\overset{1}{\partial} + \overset{2}{\partial}) \left(\frac{\overset{1}{u}}{\overset{2}{u}} + \frac{\overset{2}{u}}{\overset{1}{u}} \right) \right) \Big|_{z_1=z_2}. \quad (5.120)$$

By (5.108) we have

$$\partial \left(\frac{u - \frac{1}{u}}{\sqrt{\rho}} \right) = \frac{\partial u}{\sqrt{\rho}} - \frac{1}{\sqrt{\rho}} \partial \frac{1}{u} - \frac{1}{2} \rho^{-3/2} \partial \rho \left(u - \frac{1}{u} \right) = \frac{\partial u}{\sqrt{\rho}}$$

and so the left hand side of (5.120) is

$$\left(\frac{\partial u^1}{\frac{1}{u} - \frac{1}{\frac{1}{u}}} - \frac{\partial u^2}{\frac{2}{u} - \frac{1}{\frac{2}{u}}} \right) \frac{2}{w_1 - w_2} \left(\frac{\frac{1}{u}}{\frac{2}{u}} - \frac{\frac{2}{u}}{\frac{1}{u}} \right).$$

Using

$$\left((\partial^1 + \partial^2) \left(\frac{\frac{1}{u}}{\frac{2}{u}} + \frac{\frac{2}{u}}{\frac{1}{u}} \right) \right) \Big|_{z_1=z_2} = \partial u^1 \left(\frac{\frac{1}{u}}{\frac{2}{u}} - \frac{\frac{2}{u}}{\frac{1}{u}} \right) + \partial u^2 \left(\frac{\frac{1}{u}}{\frac{1}{u}} - \frac{\frac{2}{u}}{\frac{2}{u}} \right) = \left(\frac{\frac{1}{u}}{\frac{2}{u}} - \frac{\frac{2}{u}}{\frac{1}{u}} \right) \left(\frac{\partial u^1}{\frac{1}{u}} - \frac{\partial u^2}{\frac{2}{u}} \right),$$

the right hand side of (5.120) is

$$\frac{2}{w_1 - w_2} \left(\frac{\frac{1}{u}}{\frac{2}{u}} - \frac{\frac{2}{u}}{\frac{1}{u}} \right) \left(\frac{\partial u^1}{\frac{1}{u}} - \frac{\partial u^2}{\frac{2}{u}} \right).$$

But

$$\begin{aligned} \left(\frac{\partial u^1}{\frac{1}{u} - \frac{1}{\frac{1}{u}}} - \frac{\partial u^2}{\frac{2}{u} - \frac{1}{\frac{2}{u}}} \right) &= \frac{1}{2} \partial \ln(u^2 - 1) - \frac{1}{2} \partial \ln(u^2 - 1) = \frac{1}{2} \partial \ln(u^2 (1 - u^{-2})) - \frac{1}{2} \partial \ln(u^2 (1 - u^{-2})) = \\ &= \frac{1}{2} \partial \ln(u^2) + \frac{1}{2} \partial \ln \left(-\frac{2\rho}{w_1 + \rho^-} \right) - \frac{1}{2} \partial \ln(u^2) - \frac{1}{2} \partial \ln \left(-\frac{2\rho}{w_2 + \rho^-} \right) = \\ &= \frac{1}{2} \partial \ln(u^2) - \frac{1}{2} \partial \ln(u^2) = \frac{\partial u^1}{\frac{1}{u}} - \frac{\partial u^2}{\frac{2}{u}} \end{aligned}$$

and hence the left hand side of (5.120) is equal to its right hand side. Therefore

$$\begin{aligned} K &= \frac{(-D - ad_{\hat{P}} + \overset{2}{D} + \overset{2}{ad}_{\hat{P}})}{w_1 - w_2} \left(\left(\frac{\frac{1}{u}}{\frac{2}{u}} - \frac{\frac{2}{u}}{\frac{1}{u}} \right) \frac{\delta(z_1 - z_2)}{2\kappa} \Omega_{\mathfrak{k}} \right) + \\ &+ \frac{(\overset{1}{D} + \overset{1}{ad}_{\hat{P}} + \overset{2}{D} + \overset{2}{ad}_{\hat{P}})}{w_1 - w_2} \left(\left(\frac{\frac{1}{u}}{\frac{2}{u}} + \frac{\frac{2}{u}}{\frac{1}{u}} \right) \frac{\delta(z_1 - z_2)}{2\kappa} \Omega_{\mathfrak{k}} \right) + \\ &+ \frac{(\overset{1}{D} + \overset{1}{ad}_{\hat{P}} + \overset{2}{D} + \overset{2}{ad}_{\hat{P}})}{w_1 - w_2} \frac{\delta(z_1 - z_2)}{\kappa} \Omega_{\mathfrak{h}} = \\ &= \kappa^{-1} \frac{\overset{1}{D} + \overset{1}{ad}_{\hat{P}}}{w_1 - w_2} \left[\delta(z_1 - z_2) \left(\frac{\frac{2}{u}}{\frac{1}{u}} \Omega_{\mathfrak{k}} + \Omega_{\mathfrak{h}} \right) \right] + \kappa^{-1} \frac{(\overset{2}{D} + \overset{2}{ad}_{\hat{P}})}{w_1 - w_2} \left[\delta(z_1 - z_2) \left(\frac{\frac{1}{u}}{\frac{2}{u}} \Omega_{\mathfrak{k}} + \Omega_{\mathfrak{h}} \right) \right]. \end{aligned}$$

Eventually, the integrand of the second term in (5.114) may also be written entirely as a sum of total derivatives of terms containing δ -distributions

$$\begin{aligned}
& \left[\frac{\left(\frac{1}{u} - \frac{1}{\frac{1}{u}} \right)}{\sqrt{\rho}} \right]_{z_1} \left[\frac{\left(\frac{2}{u} - \frac{1}{\frac{1}{u}} \right)}{\sqrt{\rho}} \right]_{z_2} T^1(y_1, z_1) T^2(y_2, z_2) \times \\
& \quad \times \overset{1}{D}_{z_1} \overset{2}{D}_{z_2} [\sqrt{\rho(z_1)\rho(z_2)} A_{\text{eff}}^{12}(z_1, z_2)] T^1(z_1, y_1) T^2(z_2, y_2) = \quad (5.121) \\
& = \kappa^{-1} \frac{1}{w_1 - w_2} \left\{ \overset{1}{\partial}_{z_1} \left[T^1(y_1, z_1) T^2(y_2, z_2) \delta(z_1 - z_2) \left(\frac{2}{u} \Omega_{\text{t}} + \Omega_{\text{b}} \right) T^1(z_1, y_1) T^2(z_2, y_2) \right] + \right. \\
& \quad \left. + \overset{2}{\partial}_{z_2} \left[T^1(y_1, z_1) T^2(y_2, z_2) \delta(z_1 - z_2) \left(\frac{1}{u} \Omega_{\text{t}} + \Omega_{\text{b}} \right) T^1(z_1, y_1) T^2(z_2, y_2) \right] \right\}.
\end{aligned}$$

This second term is of the form

$$\int_0^{y_1} \int_0^{y_2} \{ \partial_{z_1} [a(z_1, z_2) \delta(z_1 - z_2)] + \partial_{z_2} [b(z_1, z_2) \delta(z_1 - z_2)] \} dz_2 dz_1, \quad (5.122)$$

where a and b are $\mathfrak{g} \otimes \mathfrak{g}$ -valued functions depending on z_1 and z_2 . Noting that

$$a(z_1, z_2) \delta(z_1 - z_2) = a(z_1, z_1) \delta(z_1 - z_2) \quad \text{and} \quad \partial_{z_1} \delta(z_1 - z_2) = -\partial_{z_2} \delta(z_1 - z_2),$$

we have

$$\partial_{z_2} [b(z_1, z_2) \delta(z_1 - z_2)] = -b(z_1, z_1) \partial_{z_1} \delta(z_1 - z_2)$$

and the integrand may be written as

$$\partial_{z_1} [a \delta(z_1 - z_2)] + \partial_{z_2} [b \delta(z_1 - z_2)] = \partial_{z_1} \{ [a - b] \delta(z_1 - z_2) \} + \partial_{z_1} b(z_1, z_1) \delta(z_1 - z_2). \quad (5.123)$$

The integral (5.122) becomes

$$\begin{aligned}
& \int_0^{y_1} \int_0^{y_2} \{ \partial_{z_1} b(z_1, z_1) \delta(z_1 - z_2) + \partial_{z_1} [(a - b) \delta(z_1 - z_2)] \} dz_2 dz_1 = \\
& = \int_0^{\min(y_1, y_2)} \partial_z \partial_z b(z, z) dz + \int_0^{y_2} \{ (a - b)(y_1) \delta(z_2 - y_1) - (a - b)(0) \delta(z_2) \} dz_2.
\end{aligned}$$

Since $u(0, w) = \pm 1$ for all w , $a(0) = b(0)$, the last term vanishes and we get

$$b(\min(y_1, y_2)) - b(0) + \Theta(0, y_1, y_2) (a - b)(y_1) = \begin{cases} a(y_1) - b(0) & \text{if } y_1 < y_2, \\ b(y_2) - b(0) & \text{if } y_1 > y_2, \end{cases} \quad (5.124)$$

where

$$\Theta(0, y_1, y_2) = \begin{cases} 1 & \text{if } 0 < y_1 < y_2, \\ 0 & \text{otherwise.} \end{cases} \quad (5.125)$$

Finally, plugging everything into (5.114) we get

$$\begin{aligned}
& \hat{\mathcal{V}}_0^{-1}(y_1, w_1) \hat{\mathcal{V}}_0^{-2}(y_2, w_2) \{ \hat{\mathcal{V}}_0^1(y_1, w_1), \hat{\mathcal{V}}_0^2(y_2, w_2) \} = A_{\underline{\text{gg}}}(y_1, y_2) - \\
& - \frac{\left(\frac{2}{u}(y_1) - \frac{1}{\frac{2}{u}(y_1)} \right)}{2\kappa \frac{1}{u}(y_1) \rho(y_1)} \Theta(0, y_1, y_2) \overset{2}{T}(y_2, y_1) \Omega_{\mathfrak{k}} \overset{2}{T}(y_1, y_2) + \\
& + \frac{\left(\frac{1}{u}(y_2) - \frac{1}{\frac{1}{u}(y_2)} \right)}{2\kappa \frac{2}{u}(y_2) \rho(y_2)} \Theta(0, y_2, y_1) \overset{1}{T}(y_1, y_2) \Omega_{\mathfrak{k}} \overset{1}{T}(y_2, y_1) + \\
& + \kappa^{-1} \overset{2}{T}(y_2, y_1) \left(\frac{\frac{2}{u}(y_1) \Omega_{\mathfrak{k}} + \Omega_{\mathfrak{h}}}{w_1 - w_2} \right) \Theta(0, y_1, y_2) \overset{2}{T}(y_1, y_2) - \\
& + \kappa^{-1} \overset{1}{T}(y_1, y_2) \left(\frac{\frac{1}{u}(y_2) \Omega_{\mathfrak{k}} + \Omega_{\mathfrak{h}}}{w_1 - w_2} \right) \Theta(0, y_2, y_1) \overset{1}{T}(y_2, y_1) + \\
& \mp \kappa^{-1} \overset{1}{T}(y_1, 0) \overset{2}{T}(y_2, 0) \left(\frac{\Omega_{\mathfrak{k}} + \Omega_{\mathfrak{h}}}{w_1 - w_2} \right) \overset{1}{T}(0, y_1) \overset{2}{T}(0, y_2) + \\
& + \int_0^{y_2} dz_2 \overset{1}{T} \overset{2}{T}(0, z_2) \left(\frac{\left(\frac{2}{u} - \frac{1}{\frac{2}{u}} \right)}{\sqrt{\rho_2}} \overset{2}{D}(\sqrt{\rho_2} A_{\mathfrak{h}\mathfrak{k}}(y_1, z_2)) \right) \overset{2}{T}(z_2, y_2) + \\
& + \int_0^{y_1} dz_1 \overset{1}{T}(0, z_1) \overset{2}{T} \left(\frac{\left(\frac{1}{u} - \frac{1}{\frac{1}{u}} \right)}{\sqrt{\rho_1}} \overset{1}{D}(\sqrt{\rho_1} A_{\mathfrak{h}\mathfrak{k}}(z_1, y_2)) \right) \overset{1}{T}(z_1, y_1) \Big\}, \tag{5.126}
\end{aligned}$$

where the sign in the sixth line is $-$ if w_1 and w_2 lie on the same sheet of the Riemann surface and $+$ otherwise.

5.7 The Poisson algebra of the monodromy matrix $\mathcal{M}(w)$

From section 5.5.4 we remember that $\mathcal{M}(w)$ can be related to $\hat{\mathcal{V}}_0$ on \mathcal{N} by

$$\mathcal{M}(w) = \hat{\mathcal{V}}_0(x_0^-, x^+ = -w, w) \hat{\mathcal{V}}_0^T(x_0^-, x^+ = -w, w), \tag{5.127}$$

where x_0^- is the x^- -coordinate on \mathcal{N} . For the Poisson bracket, we get

$$\begin{aligned}
\{ \overset{1}{\mathcal{M}}(v), \overset{2}{\mathcal{M}}(w) \} &= \{ \hat{\mathcal{V}}_0^1(v), \hat{\mathcal{V}}_0^2(w) \} \hat{\mathcal{V}}_0^T(v) \hat{\mathcal{V}}_0^T(w) + \hat{\mathcal{V}}_0^1(v) \{ \hat{\mathcal{V}}_0^T(v), \hat{\mathcal{V}}_0^2(w) \} \hat{\mathcal{V}}_0^T(w) + \\
& + \hat{\mathcal{V}}_0^2(w) \{ \hat{\mathcal{V}}_0^1(v), \hat{\mathcal{V}}_0^T(w) \} \hat{\mathcal{V}}_0^T(w) + \hat{\mathcal{V}}_0^1(v) \hat{\mathcal{V}}_0^2(w) \{ \hat{\mathcal{V}}_0^T(v), \hat{\mathcal{V}}_0^T(w) \}.
\end{aligned}$$

Since $k + k^T = 2k$ for $k \in \mathfrak{k}$ and $h + h^T = 0$ for $h \in \mathfrak{h}$, we may write

$$\{ \overset{1}{\mathcal{M}}(v), \overset{2}{\mathcal{M}}(w) \} = 4 \hat{\mathcal{V}}_0^1(v) \hat{\mathcal{V}}_0^2(w) \overset{12}{C}_{\mathfrak{h}\mathfrak{k}} \hat{\mathcal{V}}_0^T(v) \hat{\mathcal{V}}_0^T(w),$$

with

$${}^{12}C = \hat{\mathcal{V}}_0^{-1} \hat{\mathcal{V}}_0^{-1} \{ \hat{\mathcal{V}}_0^1, \hat{\mathcal{V}}_0^2 \}, \quad {}^{12}C_{\text{eff}} = \hat{\mathcal{V}}_0^{-1} \hat{\mathcal{V}}_0^{-1} \{ \hat{\mathcal{V}}_0^1, \hat{\mathcal{V}}_0^2 \} \Big|_{\mathfrak{t} \otimes \mathfrak{t}}^{1 \otimes 2}.$$

The points y_1 and y_2 , where we have to evaluate C_{eff} , are

$$y_1 = (x^- = x_0^-, x^+ = -v) \quad y_2 = (x^- = x_0^-, x^+ = -w)$$

Hence

$$\begin{aligned} \hat{u}^1(y_1) &= \frac{\sqrt{v - x_0^-}}{\sqrt{v - v}} = \infty & \hat{u}^1(y_2) &= \frac{\sqrt{v - x_0^-}}{\sqrt{v - w}} \\ \hat{u}^2(y_2) &= \frac{\sqrt{w - x_0^-}}{\sqrt{w - w}} = \infty & \hat{u}^2(y_1) &= \frac{\sqrt{w - x_0^-}}{\sqrt{w - v}} \end{aligned}$$

and so

$$\begin{aligned} {}^{12}C_{\text{eff}} &= \left[\hat{\mathcal{V}}_0^{-1}(y_1, v) \hat{\mathcal{V}}_0^{-1}(y_2, w) \{ \hat{\mathcal{V}}_0^1(y_1, v), \hat{\mathcal{V}}_0^2(y_2, w) \} \right] \Big|_{\mathfrak{t} \otimes \mathfrak{t}}^{1 \otimes 2} = \\ &= \left[A_{\text{hh}}(y_1, y_2) - \kappa^{-1} \hat{T}(y_1, 0) \hat{T}(y_2, 0) \frac{\Omega_{\mathfrak{g}}}{v - w} \hat{T}(0, y_1) \hat{T}(0, y_2) + \right. \\ &\quad + \int_0^{y_2} dz_2 \hat{T}(y_2, z_2) \left(\frac{\left(\hat{u}^2 - \frac{1}{\hat{u}^2} \right)}{\sqrt{\rho_2}} \hat{D}(\sqrt{\rho_2} A_{\text{ht}}(y_1, z_2)) \right) \hat{T}(z_2, y_2) + \\ &\quad \left. + \int_0^{y_1} dz_1 \hat{T}(y_1, z_1) \left(\frac{\left(\hat{u}^1 - \frac{1}{\hat{u}^1} \right)}{\sqrt{\rho_1}} \hat{D}(\sqrt{\rho_1} A_{\text{th}}(z_1, y_2)) \right) \hat{T}(z_1, y_1) \right] \Big|_{\mathfrak{t} \otimes \mathfrak{t}}^{1 \otimes 2} = \\ &= \left[-\kappa^{-1} \hat{T}(y_1, 0) \hat{T}(y_2, 0) \frac{\Omega_{\mathfrak{g}}}{v - w} \hat{T}(0, y_1) \hat{T}(0, y_2) \right] \Big|_{\mathfrak{t} \otimes \mathfrak{t}}^{1 \otimes 2} = \\ &= \left[-\kappa^{-1} \hat{T}(y_1, 0) \hat{T}(y_2, 0) \mathcal{V}^{-1}(0) \mathcal{V}^{-1}(0) \frac{\Omega_{\mathfrak{g}}}{v - w} \mathcal{V}(0) \mathcal{V}(0) \hat{T}(0, y_1) \hat{T}(0, y_2) \right] \Big|_{\mathfrak{t} \otimes \mathfrak{t}}^{1 \otimes 2} = \\ &= \left[-\kappa^{-1} \hat{\mathcal{V}}_0^{1-1} \hat{\mathcal{V}}_0^{2-1}(y_1) \hat{\mathcal{V}}_0^{2-1}(y_2) \frac{\Omega_{\mathfrak{g}}}{v - w} \hat{\mathcal{V}}_0^1(y_1) \hat{\mathcal{V}}_0^2(y_2) \right] \Big|_{\mathfrak{t} \otimes \mathfrak{t}}^{1 \otimes 2} = \\ &= -\frac{1}{4\kappa} \left(\hat{\mathcal{V}}_0^{-1} \hat{\mathcal{V}}_0^{-1} \frac{\Omega_{\mathfrak{g}}}{v - w} \hat{\mathcal{V}}_0^1 \hat{\mathcal{V}}_0^2 - \hat{\mathcal{V}}_0^{-1} \hat{\mathcal{V}}_0^{-1} \frac{\Omega_{\mathfrak{g}}^\eta}{v - w} \hat{\mathcal{V}}_0^1 (\hat{\mathcal{V}}_0^{-1})^T - \right. \\ &\quad \left. - \hat{\mathcal{V}}_0^T \hat{\mathcal{V}}_0^{-1} \frac{\Omega_{\mathfrak{g}}^\eta}{v - w} (\hat{\mathcal{V}}_0^{-1})^T \hat{\mathcal{V}}_0^2 + \hat{\mathcal{V}}_0^T \hat{\mathcal{V}}_0^{-1} \frac{\Omega_{\mathfrak{g}}}{v - w} (\hat{\mathcal{V}}_0^{-1})^T (\hat{\mathcal{V}}_0^{-1})^T \right). \end{aligned}$$

Finally, we get the remarkably simple result

$$\begin{aligned} \kappa \{ \hat{M}^1(v), \hat{M}^2(w) \} &= -\frac{\Omega_{\mathfrak{g}}}{v - w} \hat{M}^1(v) \hat{M}^2(w) - \hat{M}^1(v) \hat{M}^2(w) \frac{\Omega_{\mathfrak{g}}}{v - w} + \\ &\quad + \hat{M}^1(v) \frac{\Omega_{\mathfrak{g}}^\eta}{v - w} \hat{M}^2(w) + \hat{M}^2(w) \frac{\Omega_{\mathfrak{g}}^\eta}{v - w} \hat{M}^1(v), \end{aligned} \quad (5.128)$$

for $\kappa = -1$, in accordance with [24]. Note that in contrast to the derivation of this algebra in (5.135), we did not have to extend \mathcal{N} to infinity. It is a local construction as opposed to the calculation in [24].

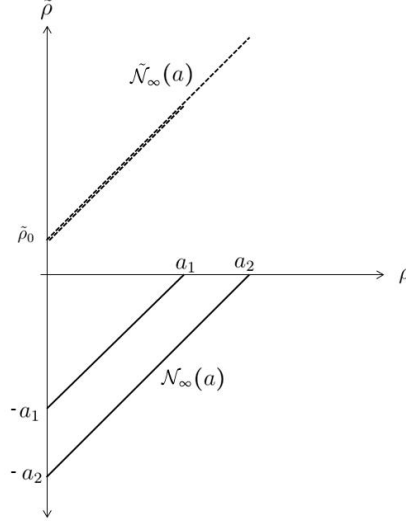


Figure 5.2: The two choices for the a -dependent initial data surfaces for two values a_1, a_2

5.8 The Yangian algebra of conserved charges

5.8.1 Extending \mathcal{N} to infinity

Following [24] we wish to extend our initial data surface,

$$\mathcal{N} = \{(x_0^-, x_0^+ + s) \mid s \in [0, a]\}, \quad x_0^- = x_0^+, \quad (5.129)$$

“to infinity” and evaluate the algebra of the $\hat{\mathcal{V}}_0(y)$ with the point y at infinity $\rho = \infty$. Two substantially different choices of an infinitely extended surface arise. First we could simply take the limit $a \rightarrow \infty$ and keep $x_0^0 = \frac{1}{2}(x_0^+ + x_0^-)$ finite (for example 0). This would correspond to a light ray starting on the x^0 axis (at 0) and extending towards null infinity. Another choice would be to keep the time coordinate of the future boundary of the surface finite, i.e. we make x_0^0 a -dependent: $x_0^0 = -a$. Then the line starts at $(x^1 = 0, x^0 = -a)$ and ends at $(x^1 = a, x^0 = 0)$. In the limit $a \rightarrow \infty$ this line will thus end at spatial infinity. These two choices $\tilde{\mathcal{N}}_\infty$ and \mathcal{N}_∞ are illustrated in 5.2 for two values of a .

As explained in [24] the quantity

$$\hat{\mathcal{V}}_1(\tilde{\rho}, \rho, w) := \mathcal{V}(\tilde{\rho}, 0) \hat{\mathcal{V}}^{-1}(\tilde{\rho}, 0, w) \hat{\mathcal{V}}(\tilde{\rho}, \rho, w) \quad \text{with } w \in W_+ \quad (5.130)$$

has time-dependence (x^0 -dependence)

$$\begin{aligned} \partial_t \hat{\mathcal{V}}_1(x, w) &= \mathcal{V}(x^0, 0) J_t(x^0, 0) \hat{\mathcal{V}}^{-1}(x^0, 0, w) \hat{\mathcal{V}}(x, w) - \\ &\quad - \mathcal{V}(x^0, 0) \hat{J}_t(x^0, 0, w) \hat{\mathcal{V}}^{-1}(x^0, 0, w) \hat{\mathcal{V}}(x, w) + \\ &\quad + \mathcal{V}(x^0, 0) \hat{\mathcal{V}}^{-1}(x^0, 0, w) \hat{\mathcal{V}}(x, w) \hat{J}_t(x, w) = \\ &= \mathcal{V}(x^0, 0) \hat{\mathcal{V}}^{-1}(x^0, 0, w) \hat{\mathcal{V}}(x, w) \hat{J}_t(x, w), \end{aligned}$$

where in the last equality we used that

$$\lim_{\rho \rightarrow 0} \gamma(x, w) = 0$$

for $w \in W_+$ and so

$$\lim_{\rho \rightarrow 0} \hat{J}(x, w) = J(x^0, 0). \quad (5.131)$$

For solutions constant at spatial infinity, i.e.

$$\lim_{\rho \rightarrow \infty} Q(x) = 0 = \lim_{\rho \rightarrow \infty} P(x) = \lim_{\rho \rightarrow \infty} J(x) = \lim_{\rho \rightarrow \infty} \hat{J}(x, w),$$

the quantities

$$T_{u/d}(w) := \lim_{\rho \rightarrow \infty} \hat{\mathcal{V}}_1(x, w), \quad (5.132)$$

already encountered in section 5.5.3, are independent of x^0 (time independent). Note that if $\rho = x^1$ goes to infinity then the branch cut blows up and covers the whole real axis. Therefore, γ and consequently $\lim_{x^1 \rightarrow \infty} \hat{\mathcal{V}}_1(x, w)$ become discontinuous along the entire real axis. The index u or d in (5.132) refers to w in W_{+u} or W_{+d} .

It turned out that these conserved quantities $T_{u/d}(w)$ form a twisted Yangian algebra with respect to the canonical Poisson structure and generate the action of the Geroch group [24][23].

We want to make an analogous construction using our Poisson structure on a light ray. We define a time-dependent family $\mathcal{N}(t)$ by: $\mathcal{N}(0) = N$ and $\mathcal{N}(t)$ is \mathcal{N} , translated a distance t in the x^0 -direction. We denote by $x_i(t)$ the point where $\mathcal{N}(t)$ meets the axis and by $x_f(t)$ the endpoint of $\mathcal{N}(t)$. On $\mathcal{N}(t)$ we define

$$\begin{aligned} \hat{\mathcal{V}}_0(x_f(t), w) &= \mathcal{V}(x_i(t)) \hat{\mathcal{V}}^{-1}(x_i(t), w) \hat{\mathcal{V}}(x_f(t), w) = \\ &= \mathcal{V}(x_i(t)) \mathcal{P} e^{\int_{x_i(t)}^{x_f(t)} \hat{J}_+(z, w) dz^+} = \mathcal{V}(x_i(t)) T(x_i(t), x_f(t), w). \end{aligned}$$

Analogously as before, $\hat{\mathcal{V}}_0(x_f(t), w)$ has time dependence

$$\partial_t \hat{\mathcal{V}}_0(x, w) = \mathcal{V}(x_i(t)) \hat{\mathcal{V}}^{-1}(x_i(t), w) \hat{\mathcal{V}}(x_f(t), w) \hat{J}_t(x_f, w).$$

If we extend \mathcal{N} and thereby $\mathcal{N}(t)$ to infinity according to $\mathcal{N}_\infty(a)$ and not $\tilde{\mathcal{N}}_\infty(a)$, then the time dependence of $\hat{\mathcal{V}}_0$ will be governed by \hat{J}_t at spatial infinity, which we assume to vanish, as above, and not by \hat{J}_t at null infinity. For $\hat{\mathcal{V}}_0$ on $\mathcal{N}_\infty(a)$, in the limit of $a \rightarrow \infty$ we get the conserved quantities corresponding to $\hat{\mathcal{V}}_1$ from [24] and therefore the Poisson bracket of the $\hat{\mathcal{V}}_0$ on $\mathcal{N}_\infty(a)$ in the limit $a \rightarrow \infty$ should form the same Yangian algebra.

5.8.2 Taking the limit

We want to extend \mathcal{N} to infinity and evaluate the bracket (5.126) of the $\hat{\mathcal{V}}(y_i, w_i)$, where y_1 and y_2 are ($\tilde{\rho} = 0, \rho = \infty$). We set $y_i = (\rho = a, \tilde{\rho} = 0)$ and take the limit $a \rightarrow \infty$. Although one might expect that when taking these limits the result depends on whether we first let $\rho(y_1) \rightarrow \infty$ and then $\rho(y_2) \rightarrow \infty$ or vice versa, this is not the case. The result is the same in any case.

We note that

$$\lim_{a \rightarrow \infty} u(y_i(a)) = \lim_{a \rightarrow \infty} \frac{\sqrt{w - a/2}}{\sqrt{w + a/2}} = \sqrt{-1}. \quad (5.133)$$

Assuming that $\lim_{\rho(y) \rightarrow \infty} A_{\text{gg}}(x, y) = 0$, the first line of (5.126) vanishes.

By (5.133) and as $\rho(y(a)) \rightarrow \infty$ the factors in the second and third line in (5.126) go to 0.

In the fourth and sixth line we have

$$\lim_{a \rightarrow \infty} \frac{\overset{j}{u}(y_i(a))}{\overset{k}{u}(y_i(a))} = \lim_{a \rightarrow \infty} \left(\frac{\sqrt{w_j - a/2} \sqrt{w_k + a/2}}{\sqrt{w_j + a/2} \sqrt{w_k - a/2}} \right) = \sqrt{-1} \frac{1}{\sqrt{-1}} = \pm 1,$$

where the sign is + if both w_k and w_j are in W_{+u} or in W_{+d} and - if not.

In analogy to [24] we define

$$T_{u/d}(w) := \lim_{a \rightarrow \infty} \hat{\mathcal{V}}_0(x_f(a), w), \quad (5.134)$$

where the u or d index means $w \in W_{+u/d}$.

Then the Poisson bracket of $T_{s_1}(w_1)$ and $T_{s_2}(w_2)$, where s_i is u or d , is

$$\begin{aligned} \{T_{s_1}^1(w_1), T_{s_2}^2(w_2)\} &= \\ &= \kappa^{-1} \overset{1}{\mathcal{V}} \overset{2}{\mathcal{V}} \left\{ T^1(0, \infty) T^2(0, \infty) \left(\frac{\pi[s_1 s_2] \Omega_{\mathfrak{t}} + \Omega_{\mathfrak{h}}}{w_1 - w_2} \right) - \left(\frac{\Omega_{\mathfrak{t}} + \Omega_{\mathfrak{h}}}{w_1 - w_2} \right) T^1(0, \infty) T^2(0, \infty) \right\} = \\ &= \kappa^{-1} T_{s_1}(w_1) T_{s_2}(w_2) \frac{\pi[s_1 s_2] \Omega_{\mathfrak{t}} + \Omega_{\mathfrak{h}}}{w_1 - w_2} - \kappa^{-1} \overset{1}{\mathcal{V}} \overset{2}{\mathcal{V}} \overset{1}{\mathcal{V}}^{-1} \overset{2}{\mathcal{V}}^{-1} \frac{\Omega_{\mathfrak{g}}}{w_1 - w_2} \overset{1}{\mathcal{V}} \overset{2}{\mathcal{V}} T^1(0, \infty) T^2(0, \infty) = \\ &= \kappa^{-1} T_{s_1}(w_1) T_{s_2}(w_2) \frac{\pi[s_1 s_2] \Omega_{\mathfrak{t}} + \Omega_{\mathfrak{h}}}{w_1 - w_2} - \kappa^{-1} \frac{\Omega_{\mathfrak{g}}}{w_1 - w_2} T_{s_1}(w_1) T_{s_2}(w_2). \end{aligned}$$

Here, $\pi[s_1 s_2]$ is 1 if both s_1 and s_2 are u or d , and -1 if they are opposite.

In accordance with [24] for $\kappa = -1$, we have

$$\begin{aligned} \kappa \left\{ T_{u/d}^1(w_1), T_{u/d}^2(w_2) \right\} &= \left[T_{u/d}^1(w_1) T_{u/d}^2(w_2), \frac{\Omega_{\mathfrak{g}}}{w_1 - w_2} \right] \\ \kappa \left\{ T_{u/d}^1(w_1), T_{d/u}^2(w_2) \right\} &= T_{u/d}^1(w_1) T_{d/u}^2(w_2) \frac{\Omega_{\mathfrak{g}}^{\eta}}{w_1 - w_2} - \frac{\Omega_{\mathfrak{g}}}{w_1 - w_2} T_{u/d}^1(w_1) T_{d/u}^2(w_2), \end{aligned} \quad (5.135)$$

where $\Omega_{\mathfrak{g}}^{\eta} := -\Omega_{\mathfrak{t}} + \Omega_{\mathfrak{h}}$.

5.8.3 Bracket of the $T_{u/d}$ and $\mathcal{M}(w)$ and the Lie-Poisson action of the Geroch group

In section 5.7 we mentioned that the calculation of $\{\mathcal{M}(v), \mathcal{M}(w)\}$ did not require an extension of \mathcal{N} to infinity, while the computation of $\{T_{s_1}(v), T_{s_2}(w)\}$ did require this extension. In this section we are interested in $\{T_{u/d}(v), \mathcal{M}(w)\}$, which with our definition (5.55) of $\mathcal{M}(w)$ is not as straightforward as in [24] because we didn't define $\mathcal{M}(w)$ directly via the $T_{u/d}(w)$. In order to compute a Poisson bracket of $T_{u/d}(w)$ with anything else we will have to extend \mathcal{N} to infinity by definition (5.134). Remembering the relation (5.55) of $\mathcal{M}(w)$ to the field $\hat{\mathcal{V}}_0$, we note that there is no obstacle in taking $\hat{\mathcal{V}}_0$ on $\mathcal{N}_{\infty}(a)$ and the limit $a \rightarrow \infty$. In this limit the coordinate x^- on $\mathcal{N}_{\infty}(a)$ will also simply go to (minus) infinity.

Let us again be more precise. We want to calculate

$$\begin{aligned} \{T_{u/d}^1(v), \mathcal{M}(w)\} &= \{T_{u/d}^1(v), \hat{\mathcal{V}}_0^2(x^-, x^+ = -w) \hat{\mathcal{V}}_0^T(x^-, x^+ = -w)\} = \\ &= \lim_{a \rightarrow \infty} \{\hat{\mathcal{V}}_0^1(y_1(a), v), \hat{\mathcal{V}}_0^2(y_2(a), w) \hat{\mathcal{V}}_0^T(y_2(a), w)\}, \end{aligned}$$

with

$$y_1(a) = (x^- = -a, x^+ = a) \quad y_2(a) = (x^- = -a, x^+ = -w).$$

In the limit $a \rightarrow \infty$ this means

$$\begin{aligned} \hat{u}(y_1) &= \frac{\sqrt{v-a}}{\sqrt{v+a}} \rightarrow \sqrt{-1} & \hat{u}(y_2) &= \frac{\sqrt{v-a}}{\sqrt{v-w}} \rightarrow \infty \\ \hat{u}(y_2) &= \frac{\sqrt{w-a}}{\sqrt{w-w}} = \infty & \hat{u}(y_1) &= \frac{\sqrt{w-a}}{\sqrt{w-a}} \rightarrow \sqrt{-1} \\ \rho(y_1) &\rightarrow \infty & \rho(y_2) &\rightarrow \infty \end{aligned} \tag{5.136}$$

and of course $\Theta(0, y_1, y_2) \rightarrow 0$. Note also that $\hat{u}(y_2) = \infty$ already for finite a if we understand the limit $\rho^+(y_2) \rightarrow -w$ as already taken.

Similar to the previous section we find

$$\begin{aligned} \{\hat{\mathcal{V}}_0^1, \hat{\mathcal{V}}_0^2 \hat{\mathcal{V}}_0^T\} &= \{\hat{\mathcal{V}}_0^1, \hat{\mathcal{V}}_0^2\} \hat{\mathcal{V}}_0^T + \hat{\mathcal{V}}_0^1 \{\hat{\mathcal{V}}_0^2, \hat{\mathcal{V}}_0^T\} = \\ &= \hat{\mathcal{V}}_0^1 \hat{\mathcal{V}}_0^2 C_{\mathfrak{gg}}^{12} \hat{\mathcal{V}}_0^T + \hat{\mathcal{V}}_0^1 \hat{\mathcal{V}}_0^2 C_{\mathfrak{gg}}^{12T} \hat{\mathcal{V}}_0^T = \\ &= 2 \hat{\mathcal{V}}_0^1 \hat{\mathcal{V}}_0^2 C_{\mathfrak{gf}}^{12} \hat{\mathcal{V}}_0^T. \end{aligned}$$

We take a look at (5.126) and think about taking the limit. The first line will vanish because of the factors of $\frac{1}{\sqrt{\rho}}$ in $A_{\mathfrak{gg}}$. The second line will vanish because of (5.136). The third line vanishes already for finite a because $\hat{u}(y_2) = \infty$. The fourth line will vanish because $\Theta(0, y_1, y_2) \rightarrow 0$. In the sixth line $\frac{\hat{u}(y_2)}{\hat{u}(y_1)} = 0$ again already at finite a . The $\Omega_{\mathfrak{h}}$ factor will not survive the projection $\big|_{\mathfrak{k}}^2$. In the seventh line we again assume that $\hat{D}(\sqrt{\rho_2} A_{\mathfrak{h}\mathfrak{k}}(y_1, z_2))$ contains a factor of $\frac{\delta(y_1 - z_2)}{\sqrt{\rho_1}}$. Then evaluation of the integrand at $z_2 = y_1$ and the limit $a \rightarrow \infty$ will kill the term because $\rho_i \rightarrow \infty$ and $\hat{u}(y_1) \rightarrow \sqrt{-1}$. The eight line already vanishes when projecting on \mathfrak{k} . So again we are left with the fifth line

only.

$$\begin{aligned}
\{T_{\pm}^1(v), M^2(w)\} &= \lim_{a \rightarrow \infty} \{\hat{\mathcal{V}}_0^1, \hat{\mathcal{V}}_0^2 \hat{\mathcal{V}}_0^T\} = 2 \lim_{a \rightarrow \infty} \hat{\mathcal{V}}_0^1 \hat{\mathcal{V}}_0^2 C_{\mathfrak{g}\mathfrak{t}}^{12} \hat{\mathcal{V}}_0^T = \\
&= - \lim_{a \rightarrow \infty} \kappa^{-1} 2 \hat{\mathcal{V}}_0^1 \hat{\mathcal{V}}_0^2 (T^1(y_1, 0) T^2(y_2, 0) \frac{\Omega_{\mathfrak{g}}}{v-w} T^1(y_1) T^2(y_2)) \Big|_{\mathfrak{t}}^2 \hat{\mathcal{V}}_0^T = \\
&= - \lim_{a \rightarrow \infty} \kappa^{-1} 2 \hat{\mathcal{V}}_0^1 \hat{\mathcal{V}}_0^2 (T^1(y_1, 0) T^2(y_2, 0) \mathcal{V}^{-1}(0) \mathcal{V}^{-1}(0) \frac{\Omega_{\mathfrak{g}}}{v-w} \mathcal{V}(0) \mathcal{V}(0) T^1(y_1) T^2(y_2)) \Big|_{\mathfrak{t}}^2 \hat{\mathcal{V}}_0^T = \\
&= - \lim_{a \rightarrow \infty} \kappa^{-1} 2 \hat{\mathcal{V}}_0^1 \hat{\mathcal{V}}_0^2 (\hat{\mathcal{V}}_0^{-1} \hat{\mathcal{V}}_0^{-1} \frac{\Omega_{\mathfrak{g}}}{v-w} \hat{\mathcal{V}}_0^1 \hat{\mathcal{V}}_0^2) \Big|_{\mathfrak{t}}^2 \hat{\mathcal{V}}_0^T = \\
&= - \kappa^{-1} \frac{\Omega_{\mathfrak{g}}}{v-w} M^2(w) T_{u/d}^1(v) + \kappa^{-1} M^2(w) \frac{\Omega_{\mathfrak{g}}^{\eta}}{v-w} T_{u/d}^1(v) = \\
&= - \kappa^{-1} \left(\frac{\Omega_{\mathfrak{g}}}{v-w} M^2(w) - M^2(w) \frac{\Omega_{\mathfrak{g}}^{\eta}}{v-w} \right) T_{u/d}^1(v) \tag{5.137}
\end{aligned}$$

Again for $\kappa = -1$ in accordance with [24].

One may easily construct the Lie Poisson action if $\mathcal{M}(w)$ is analytic on the entire real line. Then one can extend it to a strip σ on W_+ , which contains the real line. Denote by $\delta g(w)$ a \mathfrak{g} -valued function holomorphic in σ and define on $\mathcal{M}(w)$ the action

$$\delta \mathcal{M}(w) = \int_{\partial \sigma} Tr [T^{-1}(v) \delta g(v) \{T(v), \mathcal{M}(w)\} dv], \tag{5.138}$$

where one has to substitute $T_{u/d}$ for T when integrating over the part of $\partial \sigma$, which lies in $W_{+u/d}$. Due to the Poisson bracket (5.137), (5.138) becomes

$$\delta \mathcal{M}(w) = \delta g(w) \mathcal{M}(w) - \mathcal{M}(w) \delta g(w). \tag{5.139}$$

Thus, $\delta \mathcal{M}$ defined by (5.138) corresponds exactly to infinitesimal transformations of the Geroch algebra (4.110). (5.138) has exactly the form (3.40) of a Lie Poisson action, where the trace and integral play the role of the inner product between the loop algebra and its dual [2] and $T_{u/d}$ are the conserved, non-Abelian Hamiltonians.

Appendix A

Tetrad formalism

Let (M, g) be an n -dimensional Lorentzian manifold. At any point x we can choose a basis $\{E^a, a = 1, \dots, n\}$ of the tangent space $\mathcal{T}_x M$ such that $g^{\mu\nu} = \eta^{ab} E_a^\mu E_b^\nu$ with $\eta = \text{diag}(-1, 1, \dots, 1)$. Since the metric varies differentiably from point to point, we get a set of n differentiable vector fields $B = \{E^a(x), a = 1, \dots, n\}$ (we will generally omit the x -dependence) called a tetrad.

The covariant derivative ∇ maps $\mathcal{T}_x M$ to $\mathcal{T}_x^{(1,1)} M$ so we can expand

$$\nabla_\mu E_a^\nu = \omega_{\mu a}^b E_b^\nu, \quad (\text{A.1})$$

ω being an $(n \times n)$ -matrix of 1-forms. Introduce the basis $B^* = \{e_a, a = 1, \dots, n\}$, the cotetrad, of $\mathcal{T}^* M$ dual to B with $E_\mu^a e_b^\mu = \delta_b^a$. We have

$$\begin{aligned} 0 = \nabla_\mu \delta_\sigma^\nu &= \nabla_\mu (E_a^\nu e_\sigma^a) = \nabla_\mu E_a^\nu e_\sigma^a + E_a^\nu \nabla_\mu e_\sigma^a = \omega_{\mu a}^b E_b^\nu e_\sigma^a + E_b^\nu \nabla_\mu e_\sigma^b \\ &\Rightarrow \nabla_\mu e_\sigma^b = -\omega_{\mu a}^b e_\sigma^a. \end{aligned} \quad (\text{A.2})$$

Similarly for forms and dropping the spacetime indices,

$$de^a = e^b \wedge \omega_b^a. \quad (\text{A.3})$$

Furthermore we get an easy formula for the Riemann tensor:

$$\begin{aligned} (ddE_a^\sigma)_{\mu\nu} &= [\nabla_\mu, \nabla_\nu] E_a^\sigma = R^\sigma_{\rho\mu\nu} E_a^\rho = R^b_{a\mu\nu} E_b^\sigma \\ ddE_a &= d(\omega^b_a E_b) = d\omega^b_a E_b - \omega^b_a \wedge \omega^c_b E_c = (d\omega^b_a + \omega^b_c \wedge \omega^c_a) E_b \\ &\Rightarrow R^b_a = d\omega^b_a + \omega^b_c \wedge \omega^c_a. \end{aligned} \quad (\text{A.4})$$

(A.3) can be used to read off the connection 1-forms ω_b^a , but only modulo terms proportional to e_b (with respect to the indices used in (A.3)). In order to determine ω_b^a uniquely one must take into account that

$$\begin{aligned} \nabla(g^{\nu\sigma}) = 0 &= \nabla(E_a^\nu E_b^\sigma \eta^{ab}) = \omega^c_a E_c^\nu E_b^\sigma \eta^{ab} + E_a^\nu \omega^c_b E_c^\sigma \eta^{ab} = (\omega^{cb} + \omega^{bc}) E_c^\nu E_b^\sigma \\ &\Rightarrow \omega^{ab} = -\omega^{ba}. \end{aligned} \quad (\text{A.5})$$

(A.4) then allows to more or less rapidly compute the Riemann tensor, Ricci tensor and Ricci scalar, e.g.

$$R = E_b \lrcorner (E_a \lrcorner R^{ab}) = E_b^\nu E_a^\mu R_{\mu\nu}^{ab}, \quad (\text{A.6})$$

where \lrcorner denotes the dual pairing between a vector and a form “index”.

Appendix B

Null coordiantes

Here we collect the definitions and conventions concerning light cone coordinates. We adapt those used in [30] and [24]. Given a flat 2-dimensional manifold M with metric $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1)_{\mu\nu}$ in coordinates (x^0, x^1) we define

$$x^\pm := x^0 \pm x^1. \quad (\text{B.1})$$

The differentials and derivatives are

$$dx^\mp = dx^0 \mp dx^1, \quad \partial_\mp = \frac{1}{2}(\partial_0 \mp \partial_1). \quad (\text{B.2})$$

We take x^- to be “the first” coordinate because then the transformation is orientation preserving:

$$\begin{aligned} x^0 &= \frac{1}{2}(x^+ + x^-) & x^1 &= \frac{1}{2}(x^+ - x^-) \\ \frac{\partial(x^0, x^1)}{\partial(x^-, x^+)} &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\ \tilde{\varepsilon}_{\tilde{\mu}\tilde{\nu}} &= \left(\frac{\partial(x^0, x^1)^T}{\partial(x^-, x^+)} \right)_{\tilde{\mu}}^\rho \varepsilon_{\rho\sigma} \left(\frac{\partial(x^0, x^1)}{\partial(x^-, x^+)} \right)_{\tilde{\nu}}^\sigma = \frac{1}{4} \left(\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right)_{\tilde{\mu}\tilde{\nu}} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{\tilde{\mu}\tilde{\nu}} \\ \mu, \nu &\in \{0, 1\} & \tilde{\mu}, \tilde{\nu} &\in \{-, +\} \quad i.e. \quad \varepsilon_{-+} = \frac{1}{2}. \end{aligned}$$

For the metric in null coordinates we get

$$\tilde{\eta}_{\tilde{\mu}\tilde{\nu}} = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \eta_{\pm\mp} = -\frac{1}{2} \quad (\text{B.3})$$

and

$$\tilde{\varepsilon}_{\tilde{\mu}}^{\tilde{\nu}} = \tilde{\varepsilon}_{\tilde{\mu}\tilde{\rho}} \tilde{\eta}^{\tilde{\rho}\tilde{\nu}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_{\tilde{\mu}}^{\tilde{\nu}}, \quad \tilde{\varepsilon}_{\tilde{\nu}}^{\tilde{\mu}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\tilde{\nu}}^{\tilde{\mu}}. \quad (\text{B.4})$$

Appendix C

An orthonormal basis for $\mathfrak{sl}(2, \mathbb{R})$

Sometimes it is convenient to have an orthonormal basis for $\mathfrak{sl}(2, \mathbb{R})$. Such a basis is for example

$$e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (\text{C.1})$$

e_1 and e_2 with real coefficients span $\mathfrak{k} \subset \mathfrak{sl}(2, \mathbb{R})$ and e_3 with purely imaginary coefficients spans $\mathfrak{h} \subset \mathfrak{sl}(2, \mathbb{R})$. The components of the metric on $\mathfrak{sl}(2, \mathbb{R})$ given by the trace are, as already indicated,

$$\text{Tr}(e_\alpha e_\beta) = \delta_{\alpha\beta}, \quad \alpha, \beta \in \{1, 2, 3\}.$$

The structure coefficients are

$$[e_1, e_2] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i^2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sqrt{2}i e_3,$$

$$[e_3, e_1] = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \sqrt{2}i e_2$$

$$[e_2, e_3] = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \sqrt{2}i e_1,$$

or

$$[e_\alpha, e_\beta] = f_{\alpha\beta}{}^\gamma e_\gamma = \sqrt{2}i \epsilon_{\alpha\beta\gamma} e_\gamma.$$

Appendix D

The path ordered exponential $\mathcal{P}e$

Since we make extensive use of the path ordered exponential of a Lie algebra-valued one-form A as a solution of a differential equation of the form

$$\frac{d}{dt}g(t) = g(t)A(t), \quad g(0) = g_0, \quad (\text{D.1})$$

for a Lie group-valued function $g(t)$, in this section we show that it is well defined and how $\mathcal{P}e^{\int A}$ can be interpreted as the limit $n \rightarrow \infty$ of n factors of the form $\exp(A_i)$ with \exp being the exponential map. We will not explain the basics of Lie groups and algebras. They are covered very well in [29].

Let $M_{\mathbb{C}}^n$ the set of complex $n \times n$ matrices. On $M_{\mathbb{C}}^n$ we define the map

$$\begin{aligned} \langle \cdot, \cdot \rangle : M_{\mathbb{C}}^n \times M_{\mathbb{C}}^n &\rightarrow \mathbb{C}, \\ (A, B) &\mapsto \langle A, B \rangle := \text{Tr}(A^\dagger B) = \sum_{i,j} (A^*)_{ij} B_{ij}. \end{aligned}$$

Clearly

$$\langle A, B \rangle = \langle B, A \rangle, \quad \langle aA, B \rangle = a\langle A, B \rangle, \quad \langle A + C, B \rangle = \langle A, B \rangle + \langle C, B \rangle,$$

and since

$$\langle A, A \rangle = \sum_{i,j} (A^*)_{ij} A_{ij} = \sum_{i,j} |A_{ij}|^2$$

is a sum of positive real numbers, it is always positive and only zero if all the A_{ij} are zero. Thus, $\langle \cdot, \cdot \rangle$ satisfies all the properties of an inner product on $M_{\mathbb{C}}^n$. The norm induced by this inner product is

$$\|A\| := \sqrt{\langle A, A \rangle} = \sqrt{\text{Tr}(A^\dagger A)} = \sqrt{\sum_{i,j} |A_{ij}|^2}.$$

In particular, the norm satisfies

$$\begin{aligned} \|AB\|^2 &= \sum_{i,j} \left| \sum_k A_{ik} B_{kj} \right|^2 \leq \sum_{i,j} \sum_k |A_{ik}|^2 \sum_l |B_{lj}|^2 = \|A\|^2 \|B\|^2, \\ \|A + B\| &\leq \|A\| + \|B\| \end{aligned} \quad (\text{D.2})$$

Note also that if $\|A\| < \infty$ then all the entries A_{ij} of the matrix A are finite.

Now let G be a Lie-group of $n \times n$ matrices and let \mathfrak{g} be its Lie algebra. Since elements of G and \mathfrak{g} are matrices they are a subset of $M_{\mathbb{C}}^n$. Let $A(t)$ be a \mathfrak{g} -valued, bounded function on the interval $[0, 1]$. In $M_{\mathbb{C}}^n$, we construct the sequence of functions $(a_i(t))_{i \in \mathbb{N}}$

$$\begin{aligned} a_1(t) &= \mathbb{1} + \int_0^t A(t_1) dt_1 \\ a_i(t) &= \mathbb{1} + \int_0^t A(t_1) dt_1 + \dots + \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{i-1}} dt_i A(t_i) \dots A(t_1), \end{aligned}$$

where the composition $A(t_2)A(t_1)$ is simply the matrix multiplication. We define

$$A := \max_{t \in [0,1]} \|A(t)\|.$$

Hence, for fixed t two integers $i < j$, using the properties (D.2) of the norm we have

$$\begin{aligned} \|a_i(t) - a_j(t)\| &\leq A^i \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{i-1}} dt_i + \dots + A^j \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{j-1}} dt_j \leq \\ &\leq \sum_{k=i}^j \frac{A^k}{k!} \leq e^A - \sum_{k=0}^{i-1} \frac{A^k}{k!}. \end{aligned}$$

If i (and j) are sufficiently high, the right hand side is arbitrarily small. Hence the sequence is a Cauchy sequence and converges. The limit of $(a_i)_{i \in \mathbb{N}}$ exists as a matrix in $M_{\mathbb{C}}^n$ for every $t \in [0, 1]$ and thus the sequence of functions $(a_i(t))_{i \in \mathbb{N}}$ converges pointwise to a function $a(t) = \lim_{i \rightarrow \infty} (a_i(t))$. To emphasize the resemblance of the limit $a(t)$ to the exponential function and the appearance of integrals of $A(t)$, we give the limit the symbol

$$\mathcal{P}e^{\int_0^t A(t') dt'},$$

the path-ordered exponential,

$$\begin{aligned} \mathcal{P}e^{\int_0^t A(t') dt'} &:= \lim_{n \rightarrow \infty} a_n = \\ &= \mathbb{1} + \int_0^t A(t_1) dt_1 + \dots + \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{i-1}} dt_i A(t_i) \dots A(t_1) + \dots = \quad (D.3) \\ &= \mathbb{1} + \int_0^t \mathcal{P}[A(t_1)] dt_1 + \dots + \frac{1}{i!} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{i-1}} dt_i \mathcal{P}[A(t_i) \dots A(t_1)] + \dots, \end{aligned}$$

where we have introduced the path ordering operator \mathcal{P} . It orders the factors $A(t_i) \dots A(t_1)$ according to the values of their respective parameter. Factors with lower t -value are moved to the left of factors with higher t -value. A path ordered product may always be expressed as a sum of normal products where each factor is multiplied by appropriate combinations of Θ -distributions. For example, for two factors we have

$$\mathcal{P}[A(t_1)B(t_2)] = \Theta(t_2 - t_1)A(t_1)B(t_2) + \Theta(t_1 - t_2)B(t_2)A(t_1).$$

The fact, that this is undefined on the set $t_2 = t_1$ of measure zero does not matter under the integral sign¹.

¹Actually, if $A = B$ as it is in (D.3), even for $t_2 = t_1$ it may be assigned the value $A^2 = B^2$ unambiguously.

Every element of the sequence $(a_i(t))_{(i \in \mathbb{N})}$ is by its definition via integrals differentiable. We now prove that the sequence $(a'_i(t))_{(i \in \mathbb{N})}$ converges uniformly to a function $a'(t)$. Then according to a standard theorem on the differentiability of power series (see e.g. [16], Proposition 3.3), $a(t)$ is differentiable and its derivative is $a'(t)$, the limit $i \rightarrow \infty$ of $a'_i(t)$ — in other words the process of taking the limit and differentiation may be exchanged.

We have

$$a'_i(t) = a_i(t)A(t).$$

Similar as before, for $i < j$,

$$\|a'_i(t) - a'_j(t)\| \leq \|a_i(t) - a_j(t)\| \|A(t)\| \leq \left(e^A - \sum_{k=0}^{i-1} \frac{A^k}{k!} \right) A.$$

Thus

$$\lim_{i \rightarrow \infty} \|a'_i(t)\| \leq Ae^A,$$

the sequence $(a'_i(t))_{(i \in \mathbb{N})}$ converges pointwise to a function $a'(t)$. Now let $\epsilon > 0$. Then

$$\begin{aligned} \|a'(t) - a'_n(t)\| &= \left\| \sum_{i=n+1}^{\infty} \frac{1}{i!} \int_0^t dt_1 \dots \int_0^t dt_i \mathcal{P}[A(t_1) \dots A(t_i)] A(t) \right\| \leq \\ &\leq \sum_{i=n+1}^{\infty} \frac{1}{i!} \left\| \int_0^t dt_1 \dots \int_0^t dt_i \mathcal{P}[A(t_1) \dots A(t_i)] A(t) \right\| \leq \\ &\leq \sum_{i=n+1}^{\infty} \frac{1}{i!} \left(\int_0^t dt_1 \|A(t_1)\| \right)^i \|A(t)\| \leq \sum_{i=n+1}^{\infty} \frac{A^i}{i!} A = \\ &= \left(e^A - \sum_{i=0}^n \frac{A^i}{i!} \right) A. \end{aligned}$$

For n large enough, the right hand side will be arbitrarily small, in particular smaller than ϵ for all $t \in [0, 1]$, which proves uniform convergence. We have established

$$\frac{d}{dt} a(t) = \frac{d}{dt} \left[\lim_{i \rightarrow \infty} a_i(t) \right] = \lim_{i \rightarrow \infty} a'_i(t) = a'(t)$$

or using our symbol $\mathcal{P}e$

$$\frac{d}{dt} \mathcal{P}e^{\int_0^t A(t') dt'} = \mathcal{P}e^{\int_0^t A(t') dt'} A(t).$$

Looking back at the differential equation (D.1), its solution is thus

$$g(t) = g_0 \mathcal{P}e^{\int_0^t A(t') dt'}$$

because

$$\frac{d}{dt} g(t) = g(t)A(t), \quad g(0) = g_0.$$

If we use a different parameter \tilde{t} , then $g(\tilde{t})$ satisfies

$$\frac{d}{d\tilde{t}} g(t(\tilde{t})) = \frac{dt}{d\tilde{t}} \frac{d}{dt} g(t) = g(t(\tilde{t})) \frac{dt}{d\tilde{t}} A(t(\tilde{t})) =: g(\tilde{t}) \tilde{A}(\tilde{t}).$$

We will now show a different construction of $\mathcal{P}e^{\int_0^t A(t')dt'}$, which relates the path ordered exponential to the “normal” exponential mapping the Lie algebra to the group.

For a fixed value of $t \neq 0$ we can reparametrize such that $\tilde{t}(t) = 1$. This simplifies the following calculations. So from now on we assume that we are interested in the solution of (D.1) at the value $t = 1$.

We divide the interval $[0, 1]$ into n intervals of length $\frac{1}{n}$. For the i -th interval $[\frac{i-1}{n}, \frac{i}{n}]$ we fix an element $A_i = A\left(\frac{i-1/2}{n}\right) \in \mathfrak{g}$. The discrete version of (D.1) with initial condition $g(0) = g_0$ are then n differential equations with n boundary conditions

$$t \in \left[0, \frac{1}{n}\right]: \quad \frac{d}{dt}g(t) = g(t)A_1 \quad g(0) = g_0$$

$$t \in \left[\frac{i-1}{n}, \frac{i}{n}\right]: \quad \frac{d}{dt}g(t) = g(t)A_i \quad \lim_{\epsilon \rightarrow 0} g\left(\frac{i-1}{n} + \epsilon\right) = \lim_{\epsilon \rightarrow 0} g\left(\frac{i-1}{n} - \epsilon\right) \quad i \in \{2, \dots, n\}.$$

These equations are just the equations for integral curves of the left invariant vector fields, which are equal to A_i at $\mathcal{T}_e G$. The solution of the n -th equation for $t = 1$ is

$$g(1) = g_0 e^{\frac{1}{n}A_1} \dots e^{\frac{1}{n}A_n}. \quad (\text{D.4})$$

If G is a matrix group, then

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Let $\bar{b}_k = (b_i)_{1 \leq i \leq k}$ be a sequence of length $k \in \mathbb{N}$ with elements $b_i \in \{1, \dots, n\}$ such that $b_i \leq b_{i+1}$. By \mathfrak{B}_k we denote the set of all such sequences. On \mathfrak{B}_k we define the function

$$f(\bar{b}_k) = \prod_{i=1}^k \left(\sum_{j=i}^k \delta_{b_i b_j} \right)^{-1},$$

with δ_{mn} being the Kronecker delta. For example

$$f((1, 1, 4, 8, 8, 8)) = \frac{1}{3!2!}.$$

This allows to rewrite (D.4)

$$\begin{aligned} g(1) &= g_0 \prod_{l=1}^n \left(\sum_{k=0}^{\infty} \frac{\left(\frac{1}{n}A_l\right)^k}{k!} \right) = g_0 \left(\mathbb{1} + \sum_{l=1}^n \frac{1}{n}A_l + \sum_{\substack{l,m=1 \\ l < m}}^i \frac{1}{n^2}A_l A_m + \sum_{l=1}^n \frac{1}{2n^2}A_l^2 + \right. \\ &\quad \left. + \sum_{\substack{l,m,n=1 \\ l < m < n}} \frac{1}{n^3}A_l A_m A_n + \sum_{\substack{l,m=1 \\ l < m}}^n \frac{1}{2n^3}A_l^2 A_m + \sum_l \frac{1}{3!n^3}A_l^3 + \dots \right) = \\ &= g_0 \left(\mathbb{1} + \sum_{k=1}^{\infty} \sum_{\mathfrak{B}_k} f(\bar{b}_k) \frac{1}{n^k} A_{b_1} \dots A_{b_k} \right). \end{aligned} \quad (\text{D.5})$$

Now we define the step function $A_n(t)$ by

$$A_n(t) = A_i \quad t \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$$

and evaluate

$$\begin{aligned}
& \int_0^1 dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{k-1}} dt_k A_n(t_k) \dots A_n(t_1) = \\
& = \sum_{i_1=1}^n \int_{\frac{i_1-1}{n}}^{\frac{i_1}{n}} dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{k-1}} dt_k A_n(t_k) \dots A_n(t_2) A_{i_1} = \\
& = \sum_{i_1=1}^n \int_{\frac{i_1-1}{n}}^{\frac{i_1}{n}} dt_1 \sum_{i_2=1}^{i_1-1} \int_{\frac{i_2-1}{n}}^{\frac{i_2}{n}} dt_2 \int_0^{t_2} dt_3 \dots \int_0^{t_{k-1}} dt_k A_n(t_k) \dots A_n(t_3) A_{i_2} A_{i_1} + \\
& + \sum_{i_1=1}^n \int_{\frac{i_1-1}{n}}^{\frac{i_1}{n}} dt_1 \int_{\frac{i_1-1}{n}}^{t_1} dt_2 \int_0^{t_2} dt_3 \dots \int_0^{t_{k-1}} dt_k A_n(t_k) \dots A_n(t_3) A_{i_1} A_{i_1} = \\
& = \sum_{i_1=1}^n \frac{1}{n} \sum_{i_2=1}^{i_1-1} \int_{\frac{i_2-1}{n}}^{\frac{i_2}{n}} dt_2 \sum_{i_3=1}^{i_2-1} \int_{\frac{i_3-1}{n}}^{\frac{i_3}{n}} dt_3 \int_0^{t_3} dt_4 \dots \int_0^{t_{k-1}} dt_k A_n(t_k) \dots A_n(t_4) A_{i_3} A_{i_2} A_{i_1} + \\
& \sum_{i_1=1}^n \frac{1}{n} \sum_{i_2=1}^{i_1-1} \int_{\frac{i_2-1}{n}}^{\frac{i_2}{n}} dt_2 \int_{\frac{i_2-1}{n}}^{t_2} dt_3 \int_0^{t_3} dt_4 \dots \int_0^{t_{k-1}} dt_k A_n(t_k) \dots A_n(t_4) A_{i_2} A_{i_2} A_{i_1} + \\
& + \sum_{i_1=1}^n \int_{\frac{i_1-1}{n}}^{\frac{i_1}{n}} dt_1 \int_{\frac{i_1-1}{n}}^{t_1} dt_2 \sum_{i_3=1}^{i_2-1} \int_{\frac{i_3-1}{n}}^{\frac{i_3}{n}} dt_3 \int_0^{t_3} dt_4 \dots \int_0^{t_{k-1}} dt_k A_n(t_k) \dots A_n(t_4) A_{i_3} A_{i_1} A_{i_1} + \\
& + \sum_{i_1=1}^n \int_{\frac{i_1-1}{n}}^{\frac{i_1}{n}} dt_1 \int_{\frac{i_1-1}{n}}^{t_1} dt_2 \int_{\frac{i_1-1}{n}}^{t_2} dt_3 \int_0^{t_3} dt_4 \dots \int_0^{t_{k-1}} dt_k A_n(t_k) \dots A_n(t_4) A_{i_1} A_{i_1} A_{i_1} = \\
& = \sum_{\mathfrak{B}_k} f(\bar{b}_k) \frac{1}{n^k} A_{b_1} \dots A_{b_k}.
\end{aligned}$$

So for every $n \in \mathbb{N}$ we can write

$$\begin{aligned}
g(1) &= g_0 \left(\mathbb{1} + \int_0^1 dt_1 A_n(t_1) + \int_0^1 dt_1 \int_0^{t_1} dt_2 A_n(t_2) A_n(t_1) + \dots + \right. \\
& \left. + \dots + \int_0^1 dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{k-1}} dt_k A_n(t_k) A_n(t_{k-1}) \dots A_n(t_1) + \dots \right).
\end{aligned}$$

Now if $n \rightarrow \infty$, $A_n(t)$ will be equal to $A(t)$ and $g(t)$ will be the solution of (D.1). We may thus view

$$\mathcal{P}e^{\int_0^1 A(t') dt'} = \lim_{n \rightarrow \infty} e^{\frac{1}{n} A_1} \dots e^{\frac{1}{n} A_n}$$

as the limit $n \rightarrow \infty$ of the product of n group elements.

Suppose we are dealing with the Lie group $SL(n)$, the group of $n \times n$ matrices with unit determinant, with Lie algebra $\mathfrak{sl}(n)$. The determinant is a continuous function on $M_{\mathbb{C}}^n$ and so for a convergent sequence $(a_i)_{i \in \mathbb{N}}$ with $a_i \in M_{\mathbb{C}}^n$

$$\lim_{n \rightarrow \infty} \det(a_n) = \det\left(\lim_{n \rightarrow \infty} a_n\right).$$

Hence, if $A(t) \in \mathfrak{sl}(n)$, then $A_i \in \mathfrak{sl}(n)$ and $e^{\frac{1}{n} A_i} \in SL(n)$. So

$$\det \mathcal{P}e^{\int_0^t A(t') dt'} = \det \left(\lim_{n \rightarrow \infty} e^{\frac{1}{n} A_1} \dots e^{\frac{1}{n} A_{i-1}} e^{\tilde{t} A_i} \right) = \lim_{n \rightarrow \infty} \det \left(e^{\frac{1}{n} A_1} \right) \dots \det \left(e^{\frac{1}{n} A_{i-1}} \right) \det \left(e^{\tilde{t} A_i} \right) = 1$$

$$\Rightarrow \mathcal{P}e^{\int_0^t A(t')dt'} \in SL(n),$$

again showing that the path ordered exponential is a group element.

The second construction makes clear that for $0 < t_0 < t$ we have

$$\mathcal{P}e^{\int_0^t A(t')dt'} = \mathcal{P}e^{\int_0^{t_0} A(t_2)dt_2} \mathcal{P}e^{\int_{t_0}^t A(t_1)dt_1}.$$

The path ordered exponential is a functional of $A(t)$. The form (D.3) allows us to easily compute the functional gradient. We define the functional gradient $\frac{\delta F}{\delta A}$ of a functional F of Lie algebra valued functions such that for any Lie algebra valued function B we have,

$$\langle \frac{\delta F}{\delta A}, B \rangle = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} F[A(t) + \epsilon B(t)],$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing on the \mathfrak{g} -valued function space. Since these \mathfrak{g} -valued functions do not commute as matrices, the ordering is important. We denote by $\delta_{\mathfrak{g}} \in \mathcal{T}\mathfrak{g} \otimes \mathcal{T}^*\mathfrak{g}$ the identity map on \mathfrak{g} .

$$\begin{aligned} & \frac{\delta}{\delta A(t')} \mathcal{P}e^{\int_0^t A(t'')dt''} = \\ & = \int_0^t dt_1 \delta(t' - t_1) \delta_{\mathfrak{g}} + \int_0^t dt_1 \int_0^{t_1} dt_2 \left(\delta(t' - t_2) \delta_{\mathfrak{g}} A(t_1) + A(t_2) \delta(t' - t_1) \delta_{\mathfrak{g}} \right) + \dots + \\ & + \int_0^t dt_1 \dots \int_0^{t_{i-1}} dt_i \left(A(t_i) \dots A(t_2) \delta(t' - t_1) \delta_{\mathfrak{g}} + \dots + \delta_{\mathfrak{g}} \delta(t' - t_i) A(t_{i-1}) \dots A(t_1) \right) + \dots = \\ & = \Theta(t - t') \left\{ \delta_{\mathfrak{g}} + \delta_{\mathfrak{g}} \int_{t'}^t dt_1 A(t_1) + \int_0^{t'} dt_2 A(t_2) \delta_{\mathfrak{g}} + \right. \\ & \left. + \int_0^{t'} dt_2 \dots \int_0^{t_{i-1}} dt_i A(t_i) \dots A(t_2) \delta_{\mathfrak{g}} + \int_{t'}^t dt_1 \int_{t'}^{t_1} dt_2 \dots \int_{t'}^{t_{i-2}} dt_{i-1} \delta_{\mathfrak{g}} A(t_{i-1}) \dots A(t_1) \right\} = \\ & = \mathcal{P}e^{\int_0^{t'} A(t''')dt'''} \delta_{\mathfrak{g}} \mathcal{P}e^{\int_{t'}^t A(t'')dt''} \Theta(t - t') \end{aligned} \quad (\text{D.6})$$

A differential equation of the form (D.1) can result from considering a more general equation for Lie group-valued functions on a manifold (e.g. on spacetime) with points x

$$\partial_{\mu} g(x) = g(x) A_{\mu}(x) \quad (\text{D.7})$$

along a specific curve $x(t)$. Note that now A_{μ} is not simply a Lie algebra valued function, but a Lie algebra valued one-form. For a generic curve $x(t)$ the integrands in the path ordered exponential are

$$A(t)dt = A_{\mu}(x(t)) \frac{dx^{\mu}(t)}{dt} dt.$$

We can now ask whether the solution $g(x)$ is the same when evaluating the path ordered exponential along two different curves starting and ending at the same points. To investigate this we take the functional derivative of the path ordered exponential with respect

to the curve itself. Similar to (D.6) we get

$$\begin{aligned}
& \frac{\delta}{\delta x^\rho(t)} \mathcal{P}e^{\int_0^1 A(t') dt'} = \\
& = \int_0^1 dt_1 \frac{\delta}{\delta x^\rho(t)} \left\{ A_\mu(x(t_1)) \frac{dx^\mu(t_1)}{dt_1} \right\} + \\
& + \int_0^1 dt_1 \int_0^{t_1} dt_2 \left\{ \frac{\delta}{\delta x^\rho(t)} \left(A_\mu(x(t_2)) \frac{dx^\mu(t_2)}{dt_2} \right) A(t_1) + \right. \\
& + A(t_2) \frac{\delta}{\delta x^\rho(t)} \left(A_\mu(x(t_1)) \frac{dx^\mu(t_1)}{dt_1} \right) \left. \right\} + \\
& + \int_0^1 dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{i-1}} dt_i \left\{ \frac{\delta}{\delta x^\rho(t)} \left(A_\mu(x(t_i)) \frac{dx^\mu(t_i)}{dt_i} \right) A(t_{i-1}) \dots A(t_1) + \right. \\
& + \dots + A(t_i) \dots A(t_2) \frac{\delta}{\delta x^\rho(t)} \left(A_\mu(x(t_1)) \frac{dx^\mu(t_1)}{dt_1} \right) \left. \right\} + \dots .
\end{aligned}$$

Using

$$\frac{\delta}{\delta x^\rho(t)} \left(A_\mu(x(t')) \frac{dx^\mu(t')}{dt'} \right) = \partial_\rho A_\mu \frac{dx^\mu(t')}{dt'} \delta(t-t') + A_\rho(x(t')) \frac{d}{dt'} \delta(t-t'),$$

we get

$$\begin{aligned}
& \frac{\delta}{\delta x^\rho(t)} \mathcal{P}e^{\int_0^1 A(t') dt'} = \\
& = [\partial_\rho A_\mu(x(t)) - \partial_\mu A_\rho(x(t))] \dot{x}^\mu + \partial_\rho A_\mu(x(t)) \dot{x}^\mu \int_t^1 dt_1 A(t_1) + \\
& + A_\rho(x(t)) A(t) - \int_t^1 dt_1 \partial_\mu A_\rho(t) \dot{x}^\mu A(t_1) + \\
& + \int_0^t dt_2 A(t_2) \partial_\rho A_\mu(x(t)) \dot{x}^\mu - A(t) A_\rho(x(t)) - \int_0^t dt_2 A(t_2) \partial_\mu A_\rho(x(t)) \dot{x}^\mu + \\
& + \int_t^1 dt_1 \dots \int_t^{t_{i-2}} dt_{i-1} \partial_\rho A_\mu(x(t)) \dot{x}^\mu A(t_{i-1}) \dots A(t_1) + \\
& + \int_t^1 dt_1 \dots \int_t^{t_{i-3}} dt_{i-2} A_\rho(x(t)) A(t) A(t_{i-2}) \dots A(t_1) + \dots \\
& + \int_0^t dt_2 \dots \int_0^{t_{i-1}} dt_i A(t_i) \dots A(t_2) \partial_\rho A_\mu(x(t)) \dot{x}^\mu - \\
& - \int_0^t dt_3 \dots \int_0^{t_{i-1}} dt_i A(t_i) \dots A(t_3) A(t) A_\rho(x(t)) - \\
& - \int_0^t dt_2 \dots \int_0^{t_{i-1}} dt_i A(t_i) \dots A(t_2) \partial_\mu A_\rho(x(t)) \dot{x}^\mu + \dots = \\
& = \mathcal{P}e^{\int_0^t A(t') dt'} \{ \partial_\mu A_\rho(x(t)) - \partial_\rho A_\mu(x(t)) + [A_\rho(x(t)), A_\mu(x(t))] \} \dot{x}^\mu \mathcal{P}e^{\int_t^1 A(t'') dt''} .
\end{aligned}$$

We see that the quantity

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

measures the path independence of the path ordered exponential. It is called the curvature of A .

Probably the most elegant interpretation of the path ordered exponential and the differential equation (D.7) is in terms of parallel transport on fibre bundles. A detailed introduction to fibre bundles can be found in chapters 9 and 10 of [29] - we only give a brief overview.

A differentiable manifold E is called a fibre bundle if E “locally looks like” the product of a base manifold M and a manifold F called the typical fibre. That is over every point $x \in M$ one has a copy of F . The most prominent example of a fibre bundle is the tangent bundle of a differentiable manifold. Every point has its own copy of a vector space. If F is a Lie group G , then E is called a principal bundle. A section of E is an assignment of an element of F to every $x \in M$, sections of fibre bundles are F -valued functions on M - sections of principal bundles are G -valued functions on M . Just as in the case of the tangent bundle of a manifold there is no intrinsic way to compare values of sections at different points $x, y \in M$ because the fibres at x and y are merely two copies of the same manifold. One needs to define the notion of a parallel transport on the bundle. We are interested in the situation of a principal bundle because we mainly deal with group-valued fields. Let $A_\mu(x)$ be a Lie algebra valued one-form on the base space M . For a point $x_0 \in M$ and an element g_0 in the fibre (a copy of the group) at x_0 we define the parallel transport $g(t) \in G$ of g_0 along some curve $c(t) \in M$ with $c(0) = x_0$ by the differential equation

$$\frac{dg(t)}{dt} = g(t)A_\mu(x(t))\frac{dx^\mu(t)}{dt}.$$

This equation together with $g(0) = g_0$ has a unique solution given by

$$g(t) = g_0 \mathcal{P}e^{\int_0^t A(t')dt'}.$$

The path ordered exponential of A executes the parallel transport of fibre element along some curve in the base manifold M . We therefore call A a connection. Similarly the equation

$$\partial_\mu g(x) = g(x)A_\mu(x)$$

defines parallel transport along the coordinate lines. We may also write this as

$$D_\mu g(x) = 0,$$

with D_μ the covariant derivative defined in an obvious way. As mentioned above the solution of (D.7) in general depends on the curve chosen via the quantity

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = D_\mu A_\nu - D_\nu A_\mu$$

called the curvature of the connection A . If $F \neq 0$, parallel transport along two curves, which have the same starting- and endpoints, in general gives different results. Equivalently, when parallel-transporting a fibre element with the connection A along a loop will not return the same element. Only if the curvature is zero, that is if we deal with a flat connection, will the solution of (D.7) be path independent and we may say that

$$g(x) = g_0 \mathcal{P}e^{\int_0^t A_\mu \frac{dc^\mu(t')}{dt'} dt'}$$

is *the* solution of (D.7) where c^μ is a curve such that $c(t) = x$ and $c(0)$ is the point where the solution is to be equal to g_0 .

Appendix E

Integration and orientation

Most of the content of this chapter can be found in the appendix of [39].

Suppose that we are given an n -dimensional (pseudo-)Riemannian manifold (M, g) and a smooth nowhere vanishing n -form field ε on M . Then ε provides an orientation, that is a notion of right handed and left handed. If in a coordinate system $\{x^i\}$ the function $f(x)$ in

$$\varepsilon = f(x)dx^1 \wedge \dots \wedge dx^n$$

is positive (negative), then the coordinate system is right handed (left handed). Any n -form field $\varepsilon' = g\varepsilon$ with $g > 0$ everywhere defines the same orientation while it defines the opposite orientation if $g < 0$ everywhere. Consequently given an n -form field ε we could make a specific coordinate system right handed by using either ε or $-\varepsilon$. This latter approach can be used if we want to use the metric g to define a volume element by

$$\varepsilon_g^{\mu_1 \dots \mu_n} (\varepsilon_g)_{\mu_1 \dots \mu_n} = (-1)^s n! \quad (\text{E.1})$$

s being the number of minuses in the signature of g . Note that this definition specifies the volume element only up to sign. We can choose a coordinate system $\{x^i\}$ and make it right handed by choosing the sign of ε_g appropriately. Equivalently we can define ε_g by

$$\varepsilon_g := \sqrt{|g|} dx^1 \wedge \dots \wedge dx^n,$$

which is in accordance with (E.1). Furthermore, we say that a nowhere vanishing smooth n -form field ε is positive (negative) if in a right handed coordinate system $\{x^i\}$ we can write

$$\varepsilon = f dx^1 \wedge \dots \wedge dx^n,$$

with f greater (smaller) than 0.

Now let Σ be an $(n - 1)$ -dimensional submanifold of n and let v be a vector field transverse to Σ that is $v \in \mathcal{T}_p M \setminus \mathcal{T}_p \Sigma$, $p \in \Sigma$. Then we say that $\{y^1, \dots, y^{n-1}\}$ is a positively oriented chart on Σ oriented in the sense of v iff

$$v \lrcorner \varepsilon = f dy^1 \wedge \dots \wedge dy^{n-1}, \quad (\text{E.2})$$

with $f > 0$. Also if t is a function on M constant on Σ and the y^i extend smoothly off Σ so that $\{t, y^i\}$ forms a chart of a neighbourhood of Σ then Σ is oriented in the sense of

∂_t iff $dt \wedge dy^1 \wedge \dots \wedge dy^{n-1}$ is a positive n -form on M and $dy^1 \wedge \dots \wedge dy^{n-1}$ is a positive form on Σ or both are negative.

In Stokes theorem

$$\int_M d\alpha = \int_{\partial M} \alpha \quad (\text{E.3})$$

for example the orientation of the boundary has to be taken positive w.r.t. an everywhere outward pointing vector field. In the situation of a submanifold Σ of codimension 1, which is everywhere spacelike or lightlike, one might want to orient Σ positively w.r.t. to a future directed vector field. Following [39], we define a future directed vector in $\mathcal{T}_p M$ to be an element of the future half of the light cone at $p \in M$. Since in this case $\mathcal{T}_p M \setminus \mathcal{T}_p \Sigma$ is 1-dimensional, all its future directed elements differ by a factor of a strictly positive function. Therefore, the requirement of orientation towards the future of a codimension 1 surface, which is everywhere spacelike or lightlike, via (E.2) fixes the orientation of this surface uniquely. This is what we want to do in section 5.3. In computations care has to be taken of whether a coordinate system is left handed or right handed. It has an effect on the sign of integrals over forms. For example the integral of an $(n-1)$ -form α over a subset U of Σ admitting a single chart is defined using a right handed coordinate system $\{y^i\}$ by

$$\alpha = \alpha(y) dy^1 \wedge \dots \wedge dy^n$$

$$\int_U \alpha = \int_{y_a^1}^{y_b^1} \dots \int_{y_a^{n-1}}^{y_b^{n-1}} \alpha(y) dy^1 \dots dy^n,$$

where $y_b^i > y_a^i$ denote the boundaries of the subset U of Σ under consideration. In a left handed coordinate system an additional minus would arise or equivalently one could change the integration direction of an odd number of coordinates.

In the case that α in (E.3) is

$$\alpha = v \lrcorner \varepsilon_g \quad v \in \mathcal{T}M,$$

then

$$d\alpha = (\nabla_\mu v^\mu) \varepsilon_g$$

and so by Stokes's theorem

$$\int_N (\nabla_\mu v^\mu) \varepsilon_g = \int_N d(v \lrcorner \varepsilon_g) = \int_{\partial N} v \lrcorner \varepsilon_g$$

for $N \subset M$ (see [39]). Note that the restriction of the $(n-1)$ -form $v \lrcorner \varepsilon_g$ to $\mathcal{T}\partial N$ is understood. If ∂N is nowhere null, then the induced metric, h , on ∂N is non-degenerate and we may write

$$g_{\mu\nu} = h_{\mu\nu} \pm n_\mu n_\nu, \quad (\text{E.4})$$

where sign is + or - if ∂N is timelike or spacelike respectively. For the volume element we may then write

$$\varepsilon_g = n \wedge \varepsilon_h, \quad (\text{E.5})$$

where n is the outward pointing unit normal to ∂N and ε_h is the volume element on ∂N of the form (E.1), but constructed with h and coordinates on ∂N . In this case

$$(v \lrcorner \varepsilon_g)|_{\partial N} = v^\mu n_\mu \varepsilon_h$$

and we may write

$$\int_N \varepsilon_g(\nabla_\mu v^\mu) = \int_{\partial N} \varepsilon_h(n_\mu v^\mu).$$

If ∂N is null at least at one point, then the induced metric is degenerate (see e.g. [39],[32]) and so neither (E.4) nor (E.5) can be valid as stated for the timelike or spacelike case. Stokes's theorem of course still applies and so

$$\int_N (\nabla_\mu v^\mu) \varepsilon_g = \int_{\partial N} v \lrcorner \varepsilon_g = \int_{\partial N} (\varepsilon_g)_\mu v^\mu,$$

where

$$(\varepsilon_g)_\mu v^\mu := (v \lrcorner \varepsilon_g)|_{\partial N}$$

remains valid.

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