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DISSERTATION

A Generalization of Panjer's Recursion for Dependent Claim Numbers and an Approximation of Poisson Mixture Models

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Kurzfassung der Dissertation

Diese Dissertation befasst sich mit zwei Verallgemeinerungen des CreditRisk$^+$ Modells und Anwendungen der Panjer Rekursion. Im ersten Teil behandeln wir eine Verallgemeinerung des kollektiven Risikomodells und der Panjer Rekursion. Das Modell, das wir betrachten, besteht aus mehreren Geschäftsbereichen mit abhängigen Ausfallzahlen. Es ist eine Summe von kollektiven Risikomodellen. Eine Annahme des Modells ist, dass die Verteilungen der Schadenzahlen Poisson-Mischverteilungen sind. Eine Mischverteilung spiegelt einen Ausfallgrund im CreditRisk$^+$ Modell wider, das mathematisch gesehen ebenfalls ein kollektives Risikomodell ist.

In unserer Dissertation sind die Ausfallgründe mit bestimmten Abhängigkeitsstrukturen versehen und es wird bewiesen, dass die Panjer Rekursion ebenfalls anwendbar ist, indem wir eine geeignete äquivalente Darstellung der Anzahl der Schäden finden. Diese Abhängigkeitsstrukturen sind von stochastischer und linearer Natur. Solche stochastisch linearen Abhängigkeitsszenarien sind dazu geeignet, auch negative Korrelation zwischen den Schadenzahlen zu erzeugen. Bei der Mischung der Schadenzahlen mit gemeinsamen Verteilungen bleibt ebenfalls die Anwendbarkeit der Panjer Rekursion erhalten.

Des Weiteren beweisen wir, dass, wenn die Verteilungen der Schadenhöhen voneinander linear und stochastisch abhängen, die zusammengesetzte Verteilung mit Panjers Rekursion berechnet werden kann. Es ist auch möglich, eine multivariate Variante der Panjer Rekursion und von de Prils Rekursion zu beweisen, wenn die Schadenhöhen nicht unabhängig und identisch verteilt, sondern austauschbar sind. Wir formulieren diese Ergebnisse in entsprechenden Algorithmen.

Unter Benutzung dieser Resultate berechnen wir Risikobeiträge der Risiken aufgrund eines Ausfallgrundes mit solchen Abhängigkeitsstrukturen. Dazu nutzen wir das Riskomaß des bedingten Expected Shortfalls, das es auch zulässt, Risikobereitstellungen in einem Multiperiodenmodell zu betrachten. Beispiele zeigen, dass unterschiedliche Arten von Korrelationen unterschiedliche Verteilungen liefern.

Im zweiten Teil befassen wir uns mit dem kollektiven Risikomodell, wobei die Anzahl der Schäden eine bestimmte Poisson-Mischverteilung hat. Da es ein allgemeiner Ansatz ist, eine Gammaverteilung als Mischverteilung zu wählen, verallgemeinern wir diesen Misch-Ansatz zu verallgemeinerten Gammafaltungen. Auf diese Weise verallgemeinern wir auch das CreditRisk$^+$ Modell. Man definiert eine verallgemeinerte Gammafaltung als den schwachen Grenzwert einer Folge endlich vieler Faltungen von Gammaverteilungen.

Bereits bekannte Rekursionen für solch eine zusammengesetzte Poisson-Mischverteilung erfordern in jedem Schritt die Auswertung eines Integrals. Dies kann einen hohen Rechenaufwand und auch einen numerischen Fehler verursachen. Wir umgehen die Berechnung dieses Integrals in jedem Schritt, indem wir beweisen, dass eine geeignete Folge von Zufallsvariablen existiert, die gegen die Poisson-Mischverteilung konvergiert, was die Anwendung der Panjer Rekursion erlaubt. Infolgedessen geben wir eine Fehlerabschätzung für die Approximation dieser Folge bezüglich des Totalvariationsabstandes an. Unter Benutzung dieser Fehlerabschätzung einer geeigneten Approximation stellen wir einen Algorithmus für die Berechnung der Verteilung des Gesamtverlustes vor.

Bis jetzt ist dieses Modell nur im eindimensionalen Fall betrachtet worden. Es ist allerdings auch interessant, mehrere Geschäftsbereiche zu betrachten, was ein multivariates Modell impliziert. Dies geschieht mithilfe multivariater verallgemeinerter Gammafaltungen. Wir beweisen ein alternatives Ergebnis zur Abgeschlossenheit der multivariaten verallgemeinerten Gammafaltungen, die durch endliche viele Faltungen von multivariaten Gammaverteilungen approximiert werden können. Wir geben ebenfalls eine obere Schranke für die Fehlerabschätzung bezüglich des Totalvariationsabstandes wie im eindimensionalen Fall an. Es ist auch möglich, eine Darstellung zu finden, die eine Anwenung der Panjer Rekursion gestattet, um die Verteilung der entsprechenden Zufallssumme zu berechnen. Wir schließen mit Beispielen, die die Vorteile unseres Algorithmus' im Gegensatz zur schnellen Fourier Transformation und einer verbesserten Variante der schnellen Fourier Transformation hervorheben.

Abstract

This thesis examines two generalizations of the CreditRisk$^+$ model and applications of Panjer's recursion. In the first part we discuss a generalization of the collective risk model and of Panjer's recursion. The model we consider consists of several business lines with dependent claim numbers. It is a sum of collective risk models. The distributions of the claim numbers are assumed to be Poisson mixture distributions. A mixing distribution reflects a default cause in the CreditRisk$^+$ model which, mathematically seen, is also a collective risk model.

In our contribution we let the default causes have certain dependence structures and prove that Panjer's recursion is also applicable by finding an appropriate equivalent representation of the claim numbers. These dependence structures are of a stochastic linear nature. Such stochastically linear dependence scenarios are also capable of producing negative correlations between the default causes. Compounding the default causes by common distributions also keeps Panjer's recursion applicable.

In addition, we prove that if claim size distributions depend on each other linearly and stochastically, the compound distribution can be evaluated by Panjer's recursion. It is also possible to prove a multivariate variant of both Panjer's and de Pril's recursion if the claim sizes are not i.i.d. but exchangeable. We put all this into corresponding algorithms.

Using these results we compute risk contributions of the risks due to a default cause with such dependence structures. For this purpose we use the risk measure conditional expected shortfall that also allows us to consider risk allocations in a multiperiod model. Indeed, examples show that different types of correlation provide different distributions.

In the second part we discuss the collective risk model where the number of claims has a certain Poisson mixture distribution. Since it is a common approach to choose a gamma distribution as a mixing distribution, we generalize this mixture approach to generalized gamma convolutions. In doing so, we also generalize the CreditRisk$^+$ model. A generalized gamma convolution is defined as the weak limit of a sequence of finitely many convolutions of gamma distributions.

Already known recursions for such a compound Poisson mixture distribution require the evaluation of an integral in each step. This may cause high computational effort and also a numerical error. We circumvent the computation of this integral in each step by proving that an appropriate sequence of random variables exists converging to the Poisson mixture distribution, which allows for the application of Panjer's recursion. Consequently, we give an error estimate with respect to the total variation distance for the approximation by this sequence. Using this error bound of a proper approximation we present an algorithm for the calculation of the distribution of the total loss.

So far, this model has been discussed in the univariate case. However, it is also interesting to consider several lines of business, which indicates the usefulness of a multivariate model. This is realized with the help of multivariate generalized gamma convolutions. We prove an

alternative closure result that shows how to approximate a multivariate generalized gamma convolution by finite multivariate gamma convolutions. We also give an upper bound for an error estimate with respect to the total variation distance, as in the univariate case. It is possible to find a representation allowing for an application of Panjer's recursion in order to evaluate the distribution of the corresponding random sum, too. We conclude with examples that pinpoint the advantages of our algorithm in contrast to fast Fourier transform and an improved variant of fast Fourier transform.

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Contents

List of Figures

List of Tables

Part I

Generalized Panjer Recursion for Dependent Claim Numbers and Applications to Credit Risk Aggregation

Chapter 1

Introduction

The aggregation of risks in a portfolio is an important component in both insurance mathematics and risk management. It is important for an insurance company to estimate future claims of the insured persons, and banks have an increasing need to estimate the volume of credit defaults that becomes more and more important with the introduction of the Basel II and Basel III accords. The Basel II accord, cf. [4], also regulates operational risk, which is defined as the risk of losses resulting from inadequate or failed internal processes, people and systems, or from external events. In risk management there are several risk models. The CreditRisk$^+$ model (introduced by [10]) is one such with a lot of practical features. It does not require many assumptions and it enables the computation of the loss distribution recursively and exactly, i.e., it is not necessary to apply Monte Carlo methods that introduce a stochastic error. Gundlach and Lehrbass [27] offer a wide overview over CreditRisk$^+$. However, the shortcoming of CreditRisk$^+$ is that it does not allow for dependent default causes. Bürgisser et al. [9] treat this problem and introduce dependence between the default causes by sector correlation. Several approaches have been published that aim at tackling this problem: Reiß [53] models dependent default causes by incorporating market risk by using geometric Brownian motions. But he waives a solution in closed form for this feature. Deshpande and Iyer [15] also consider a dependence model for default causes in the usual CreditRisk$^+$ model that comprises linear combinations of risk factors and is a generalization of Giese [23]. They also consider VaR contributions. Instead, we consider a more general linear dependence structure in an extension of the CreditRisk$^+$ model that allows us to model negative correlation and we can also comprise a stochastic component in the linear dependence. Moreover, we treat expected shortfall contributions and present exact formulas for the risk contributions for gamma and τ-tempered α-stable distributions. Gordy [25] proposes a different approach to the evaluation of the CreditRisk$^+$ model, he proposes a saddle point approximation. For practical applications Vandendorpe et al. [69] propose a parameterization of the characteristics of a portfolio which could be also useful for a parameterization of our model.

Insurance companies too need to satisfy Solvency II requirements, and the model we consider is capable of estimating and quantifying risks and can thus be used to determine the minimal capital requirements, which is the first pillar of Solvency II. There is also an increasing demand to reflect dependencies between the risks. As can be found in [48, p. 15], it is interesting to note that in the definition of the target capital the expected shortfall is used, a risk measure that we are also going to use in our model for risk contributions.

In the (extended) CreditRisk$^+$ model as in the collective risk model it is necessary to calculate the distribution of a random sum. In risk management, e.g. the CreditRisk$^+$ model, the collective risk model is diversified into several risk clusters or lines of business and thus random sums where each claim number is driven by a default cause. Thus we consider a sum of several collective risk models. Since the claim numbers are modelled by Poisson mixture distributions, it is meaningful to speak of default cause intensities as the default causes are modelled by the mixing distributions. Hence we consider the random sum

$$S = \sum_{i=1}^{m} \sum_{h=1}^{N_i} X_{i,h}, \tag{1.1}$$

where $\{X_{i,h}\}_{h\in\mathbb{N}}$ are independent sequences of independent and identically distributed random variables for each $i \in \{1, \dots, m\}$ and $N_1, \dots, N_m$ are $\mathbb{N}_0$-valued random variables independent of $\{(X_{1,h}, \dots, X_{m,h})\}_{h\in\mathbb{N}}$. Using such a random sum, we are able to consider several business lines of an insurance company or several default cause intensities in the extended CreditRisk$^+$ model. Our model can be also used to estimate operational risk.

To put it more precisely, in former frameworks such as [21] the authors consider m default cause intensities $\Lambda_1, \dots, \Lambda_m$ where each claim number N_i for $i \in \{1, \dots, m\}$ depends on a default cause intensity Λ_i. These default cause intensities are independent, and the Poisson mixture distributions need to be in a Panjer(a, b, k) class. Hence, due to the independence, Panjer's recursion is applicable to these random sums. The distribution of a random variable N, denoted by $\{q_n\}_{n\in\mathbb{N}_0}$, belongs to a Panjer$(a, b, k)$ class with $a, b \in \mathbb{R}$ and $k \in \mathbb{N}_0$ if $q_0 = q_1 = \cdots = q_{k-1} = 0$ and

$$q_n = \Big(a + \frac{b}{n}\Big) q_{n-1} \qquad \text{for all } n \in \mathbb{N} \text{ with } n \geq k+1. \tag{1.2}$$

If $f_n = \mathbb{P}[X_1 = n]$ and $af_0 \neq 1$, then according to [21, Theorem 4.1] the distribution $p_n = \mathbb{P}[X_1 + \cdots + X_N = n]$ is given by the recursion

$$p_n = \frac{1}{1 - af_0} \bigg(\mathbb{P}[X_1 + \cdots + X_k = n]\, q_k + \sum_{j=1}^{n} \Big(a + \frac{bj}{n}\Big) f_j p_{n-j} \bigg) \tag{1.3}$$

for all $n \in \mathbb{N}$ with initial condition

$$p_0 = \begin{cases} q_0 & \text{if } f_0 = 0, \\ \sum_{k\geq 0} q_k f_0^k & \text{otherwise,} \end{cases} \tag{1.4}$$

which is the probability-generating function of the distribution of N at f_0. The recursive evaluation of such compound distributions was introduced by Panjer [50] and extended by Willmot and Panjer [77]. The distributions belonging to a Panjer(a, b, k) class were identified by Sundt and Jewell [63], Willmot [75], and Hess, Liewald, and Schmidt [30]. Sundt and Vernic [64] contributed exhaustively to this topic amassing many interesting results. There are many other contributions on the generalization of Panjer's recursion. Hipp [33] provides a speedy version for Panjer's recursion for certain types of severity distributions. Willmot [74] also considers a similar framework; besides mixed compound Poisson distributions he also considers linear combinations of random variables as mixing distributions of Poisson distributions. However, these mixture distributions only provide positive correlation.

The need for dependent claim numbers has already been mentioned. This issue gains increasing importance. Recent years have brought some economic examples of dependent claims. In 2009 the German car company Opel was having severe economic problems that also threatened suppliers and jobs with economic defaults. This is a typical example of a positive correlation between defaults in a market: a default from Opel would have meant the loss of their main customer to the suppliers. An example for a negative correlation might be the collapse of the former Swissair airline in 2001. The collapse of an airline does not decrease the general demand for flights so other airlines could profit from this collapse and carry former Swissair passengers, thus perhaps lowering their own default probability.

Besides this demand for dependent claim numbers, it is also of interest to generalize the extended CreditRisk$^+$ model and the collective risk model, respectively, from another point of view, from univariate dimensions to multivariate dimensions. In this case, it is possible to consider two kinds of generalizations. Either there are multivariate counting distributions and univariate severity distributions, or there are univariate counting distributions and multivariate severity distributions. Hesselager [31] and Vernic [71] started to consider recursions for the evaluation of bivariate compound distributions. In this case, the counting distributions are univariate and the severity distributions are bivariate. Sundt [61] and Eisele [18] considered recursions for multivariate compound distributions of this kind. Recursions for both types of generalizations can be found in Sundt and Vernic [64]. Considering multivariate claim sizes has the advantage that we can take time horizons into account, which is also an important factor of such a model. This becomes obvious, especially in the case of insurance companies where the occurrence of a claim usually causes several future payments, and it is very often not clear at the notification of claim how these payments might develop. This is also evident in e.g. health insurances during the recovery process of a patient. Naturally, this also applies to banks if a credit default occurs; the bank may expect recovery payments, but the size of these is not known at default.

This part of this thesis is organized as follows: in Chapter 2 we present some useful results on compound and mixture distributions and the probability-generating functions of the distributions of Poisson, negative multinomial, and logarithmic random vectors with different mixture random vectors and random variables.

In Chapter 3 we introduce a general dependence structure of the default cause intensities between risk factors and prove alternative representations of the claim numbers with the same distribution allowing for the application of Panjer's recursion. We generalize the framework given in [21] that introduces an extended version of the Credit Risk$^+$ model. Our major result here is a linear dependence scenario with a stochastic component, i.e., the linear structure is chosen stochastically. This scenario allows for both positive and negative correlation. A variant of Panjer's recursion is applicable. An interesting special case is a linear dependence structure with stochastic entries which makes negative correlation clearer, hence this general dependence scenario offers several modes of dependence. As an application we embed Giese's framework [23] of a dependence structure into our model and provide the application of Panjer's recursion.

In Chapter 4 we consider a different dependence structure: we mix the default cause intensities with common mixture distributions by choosing a parameter of the distribution stochastically. If we have the same choice for several default cause intensities, this also generates dependence. Then we prove that an alternative representation with the same distribution exists that also allows for an application of Panjer's recursion. In doing so, we also admit the default cause intensities to consist of a stochastic and linear dependence. Thus we observe both discrete and continuous dependence structures. This framework does not only work for default cause intensities with a gamma distribution as it is already known from the extended CreditRisk$^+$ model, but it also works for so-called τ-tempered α-stable distributions. This is a very flexible family of distributions and also allows for modelling of heavy tails.

In Chapter 5 we prove a generalization of Panjer's recursion for multivariate dependent claim size distributions and for claim numbers that are linked by more general relations than in Equation (1.2). We therefore relax the assumption of independent and identically distributed random variables and replace it with the assumption of exchangeable claim sizes. Then the application of Panjer's recursion requires a further step of computation, i.e., the evaluation of an expected value that becomes necessary by the tower property. Additionally, we present algorithms that show how to use Panjer's recursion for our alternative representations of claim numbers. We also generalize de Pril's recursion in the same manner. By adopting the approach for the construction of stochastically and linearly dependent claim numbers we also construct a dependence scenario for claim sizes. This is also a model in which a variant of Panjer's recursion is applicable.

Chapter 6 is a contribution to capital allocation of the portfolio loss S and also presents an algorithm for the evaluation of these contributions. Such dependence scenarios as introduced above raise the question of their contribution to the risk, i.e., how the total portfolio risk can be diversified to the risk coming from a single business line or default cause intensity. We consider such a risk capital allocation with respect to the risk measure expected shortfall. Because it is not only of interest to consider such a risk allocation for one time period, we also introduce such risk measures given a σ-algebra, see also [34]. Conditioning on a certain filtration of σ-algebras allows us to consider a multivariate framework, i.e., a model with several time periods.

Finally, in Chapter 7 we conclude this part of this thesis with some interesting examples that show that the choice of correlation has an impact on the distribution of the total portfolio.

Chapter 2

Properties of Compound and Mixture Distributions

In this chapter we provide some general representations of distributions when compounded or mixed that we will need in the following chapters. Additionally we introduce a dependence scenario between default cause intensities and prove a corresponding formula for the probability-generating function of the distributions of Poisson mixture vectors.

Remark 2.1. It is a very important result for us that a probability-generating function or a Laplace transform uniquely determines a distribution, and we will have frequent recourse to this result. A reference can be found in [42, Theorem 5.3]. The result is proven for characteristic functions but also carries over to probability-generating functions and Laplace transforms.

First, let us sketch the notation of a compound distribution.

Definition 2.2. Let N be an $\mathbb{N}_0$-valued random variable and let $\{X_h\}_{h\in\mathbb{N}}$ be a sequence independent of N, consisting of i.i.d. random variables such that X_1 has the distribution F. Define the random sum $S = \sum_{h=1}^{N} X_h$. Then

(a) if N has a Poisson distribution with parameter μ, then S has a compound Poisson distribution, and we write $S \sim \mathrm{CPoi}(\mu, F)$,

(b) if N has a negative binomial distribution with parameters $\alpha > 0$ and $p \in [0, 1)$, then S has a compound negative binomial distribution, and we wite $S \sim \mathrm{CNegBin}(\alpha, p, F)$,

(c) if N has a logarithmic distribution with parameter $q \in (0, 1)$, then S has a compound logarithmic distribution, and we write $S \sim \mathrm{CLog}(q, F)$.

We start with a lemma about an equality in distribution between compound distributions and Poisson mixture distributions.

Lemma 2.3. *Let N be an $\mathbb{N}_0$-valued random variable and Λ a non-negative random variable such that*

$$\mathcal{L}(N\,|\,\Lambda) \overset{a.s.}{=} \mathrm{Poisson}(\Lambda)\,. \tag{2.4}$$

Let $\{B_h\}_{h\in\mathbb{N}}$ be a sequence independent of (N, Λ), consisting of i.i.d. random vectors such that $B_1 \sim$ Multinomial$(1; p_1, \ldots, p_m)$. Let further $\{(X_{1,h}, \ldots, X_{m,h})\}_{h\in\mathbb{N}}$ be a sequence independent of (N, Λ) and $\{B_h\}_{h\in\mathbb{N}}$, consisting of i.i.d. $\mathbb{R}^m$-valued random vectors. Define

$S_i = \sum_{h=1}^{N} B_{i,h} X_{i,h}$ *for* $i \in \{1, \dots, m\}$ *with the notation* $B_h = (B_{1,h}, \dots, B_{m,h})$. *Then the random sum* $S = (S_1, \dots, S_m)$ *satisfies for each component*

$$\mathcal{L}(S_i \,|\, \Lambda) \overset{a.s.}{=} \mathrm{CPoi}(p_i\Lambda, \mathcal{L}(X_{i,1}))\,, \qquad i \in \{1, \dots, m\},$$

and the components $S_1, \dots, S_m$ *are conditionally independent given* Λ. *Additionally*

$$\mathcal{L}(S_1 + \cdots + S_m \,|\, \Lambda) \overset{a.s.}{=} \mathrm{CPoi}\biggl(\Lambda, \sum_{i=1}^{m} p_i\, \mathcal{L}(X_{i,1})\biggr) \tag{2.5}$$

holds.

Proof. Apply Remark 2.1. The probability-generating function of the conditional distribution of S given Λ gives by the independence of (N, Λ) of $\{B_h\}_{h\in\mathbb{N}}$ and $\{(X_{1,h}, \dots, X_{m,h})\}_{h\in\mathbb{N}}$ and since $\{B_h\}_{h\in\mathbb{N}}$ and $\{(X_{1,h}, \dots, X_{m,h})\}_{h\in\mathbb{N}}$ are sequences of i.i.d. random variables, respectively,

$$G_{S\,|\,\Lambda}(z) \overset{\text{a.s.}}{=} \mathbb{E}\biggl[\prod_{i=1}^{m} z_i^{\sum_{h=1}^{N} B_{i,h}X_{i,h}} \,\bigg|\, \Lambda\biggr] \overset{\text{a.s.}}{=} \mathbb{E}\biggl[\biggl(\mathbb{E}\biggl[\prod_{i=1}^{m} z_i^{B_{i,1}X_{i,1}}\biggr]\biggr)^{N} \,\bigg|\, \Lambda\biggr], \qquad z \in [0,1]^m.$$

Using that $\{B_h\}_{h\in\mathbb{N}}$ have a multinomial distribution and are independent of $\{(X_{1,h}, \dots, X_{m,h})\}_{h\in\mathbb{N}}$ and applying the law of total probability with the notation $G_{X_{i,1}}(z) = \mathbb{E}[z^{X_{i,1}}]$ provides

$$G_{S\,|\,\Lambda}(z) \overset{\text{a.s.}}{=} \mathbb{E}\biggl[\biggl(\sum_{i=1}^{m} p_i G_{X_{i,1}}(z_i)\biggr)^{N} \,\bigg|\, \Lambda\biggr].$$

Finally, using Equation (2.4) and $p_1 + \cdots + p_m = 1$, we observe

$$G_{S\,|\,\Lambda}(z) \overset{\text{a.s.}}{=} \exp\biggl(-\Lambda\biggl(1 - \sum_{i=1}^{m} p_i G_{X_{i,1}}(z_i)\biggr)\biggr) \overset{\text{a.s.}}{=} \prod_{i=1}^{m} \exp\bigl(-\Lambda p_i(1 - G_{X_{i,1}}(z_i))\bigr).$$

Then the marginal probability-generating function of S given Λ is with a slight abuse of notation $z_i = (1, \dots, 1, z_i, 1, \dots, 1)$

$$G_{S\,|\,\Lambda}(z_i) \overset{\text{a.s.}}{=} \exp\bigl(-\Lambda p_i(1 - G_{X_{i,1}}(z_i))\bigr),$$

which is the probability-generating function of the conditional distribution of a random variable Y given Λ, which has a compound Poisson distribution with Poisson parameter $p_i\Lambda$ and severity distribution $\mathcal{L}(X_{i,1})$ for $i \in \{1, \dots, m\}$. Equation (2.5) follows by [49, Proposition 3.3.4]. q.e.d.

It is also interesting to consider compound distributions as in Lemma 2.3 with negative binomial distributions instead. For this purpose we give a definition of the negative multinomial distribution (cf. [59, Section 2]), which is a multivariate variant of the negative binomial distribution.

Definition 2.6. A random vector $N = (N_1, \dots, N_m)$ has an m-dimensional negative multinomial distribution with parameters $\alpha > 0$ and $p_1, \dots, p_m \geq 0$ where $p_1 + \cdots + p_m < 1$ if

$$\mathbb{P}[N_1 = n_1, \dots, N_m = n_m] = \frac{\Gamma\bigl(\alpha + \sum_{i=1}^{m} n_i\bigr)}{\Gamma(\alpha)\prod_{i=1}^{m} n_i!} p_0^{\alpha} \prod_{i=1}^{m} p_i^{n_i}$$

for all $n_1, \dots, n_m \in \mathbb{N}_0$ where $p_0 := 1 - p_1 - \cdots - p_m$. We denote the distribution of N by $\mathrm{NegMult}(\alpha; p_1, \dots, p_m)$.

According to [78, Equation (2.1)] the corresponding probability-generating function is

$$[0,1]^m \ni (z_1,\ldots,z_m) \mapsto \left(\frac{1}{p_0} - \sum_{i=1}^m \frac{p_i}{p_0} z_i\right)^{-\alpha}.$$

In case $m=1$ we obtain $\mathbb{P}[N=n] = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)n!}(1-p)^\alpha p^n$ and

$$G_N(z) = \left(\frac{1-pz}{1-p}\right)^{-\alpha}, \quad z \le \left|\frac{1}{p}\right|,$$

which is a $\mathrm{NegBin}(\alpha,p)$ distribution. For further reading on the negative multinomial distribution the reader is referred to [59], [78], or [39, Chapter 36]. This negative multinomial distribution can be also seen as a generalization in the sense of a generalization of the geometric distribution, see also [47].

In the course of this thesis we will often use a result that is related to the negative binomial distribution and provides an alternative representation. We set it here:

Remark 2.7. Let $r>0$ and $p\in(0,1)$. According to [58, p. 95/96] and with an adjustment to our notation of the negative binomial distribution

$$\mathrm{NegBin}(r,1-p) = \mathrm{CPoi}(r\ln(1/p), \mathrm{Log}(1-p))$$

holds. Note that this transformation and shift of the convolution has already been considered by Ammeter [1].

The following lemma will be useful for further applications.

Lemma 2.8. *Let $\alpha,\beta>0$ and $\lambda_1,\ldots,\lambda_m\ge 0$. Let T be a strictly positive random variable and let Λ be a random variable such that $\mathcal{L}(\Lambda\,|\,T) \overset{a.s.}{=} \mathrm{Gamma}(\alpha T,\beta)$. Let $N=(N_1,\ldots,N_m)$ be a random vector with conditionally independent components given Λ, T such that*

$$\mathcal{L}(N_i\,|\,\Lambda,T) \overset{a.s.}{=} \mathrm{Poisson}(\lambda_i\Lambda)\,, \qquad i\in\{1,\ldots,m\}.$$

Furthermore, consider an m-dimensional random vector M given T such that $\mathcal{L}(M\,|\,T) \overset{a.s.}{=}$ $\mathrm{NegMult}(\alpha T; p_1,\ldots,p_m)$ with $p_i = \frac{\lambda_i}{\beta+\sum_{d=1}^m \lambda_d}$ for $i\in\{1,\ldots,m\}$. Then M and N have the same distribution.

Proof. For the proof we use Remark 2.1 and compute the probability-generating functions of the conditional distributions of N and M given T, respectively. We start with the random vector N given T for $z\in[0,1]^m$. Conditioning on (Λ,T) and using the conditional independence of $N_1,\ldots,N_m$ given (Λ,T) and the conditional distribution of N given Λ yields

$$G_{N|T}(z) \overset{\text{a.s.}}{=} \mathbb{E}\left[\prod_{i=1}^m z_i^{N_i}\,\middle|\,T\right] \overset{\text{a.s.}}{=} \mathbb{E}\left[\prod_{i=1}^m \mathbb{E}\big[z_i^{N_i}\,\big|\,\Lambda,T\big]\,\middle|\,T\right] \overset{\text{a.s.}}{=} \mathbb{E}\left[\prod_{i=1}^m \mathrm{e}^{-\lambda_i\Lambda(1-z_i)}\,\middle|\,T\right].$$

Using the conditional distribution of Λ given T, we observe

$$G_{N|T}(z) \overset{\text{a.s.}}{=} \left(\frac{\beta}{\beta+\sum_{i=1}^m \lambda_i(1-z_i)}\right)^{\alpha T}.$$

Consider now the conditional distribution of the random vector M given T. Using the conditional distribution of M given T with $z \in [0,1]^m$ yields

$$G_{M|T}(z) \overset{\text{a.s.}}{=} \mathbb{E}\Big[\prod_{i=1}^m z_i^{M_i}\Big|T\Big] \overset{\text{a.s.}}{=} \Big(\frac{1}{p_0} - \sum_{i=1}^m \frac{p_i}{p_0} z_i\Big)^{-\alpha T} \overset{\text{a.s.}}{=} \Big(\frac{\beta + \sum_{d=1}^m \lambda_d}{\beta} - \frac{\sum_{i=1}^m \lambda_i z_i}{\beta}\Big)^{-\alpha T}$$

by insertion of $p_i = \frac{\lambda_i}{\beta + \sum_{d=1}^m \lambda_d}$ for $i \in \{1,\dots,m\}$ and $p_0 = \frac{\beta}{\beta+\sum_{d=1}^m \lambda_d}$. q.e.d.

A special case of this lemma has also been treated in [59, Section 3.b]; in this case the random variable T is deterministic.

Remark 2.9. The negative multinomial distribution given in Definition 2.6 can also be obtained as a certain mixture distribution. If we choose in Lemma 2.3 the random variables $\Lambda \sim \text{Gamma}(\alpha,\beta)$ and $X_{i,1} \equiv 1$ for each $i \in \{1,\dots,m\}$, then the probability-generating function of the conditional distribution of S given Λ in the corresponding proof is for $z \in [0,1]^m$

$$\mathbb{E}\Big[\exp\Big(-\Lambda\Big(1 - \sum_{i=1}^m p_i z_i\Big)\Big)\Big] = \Big(\frac{\beta}{\beta + 1 - \sum_{i=1}^m p_i z_i}\Big)^{\alpha} = \Big(\frac{\beta + 1 - \sum_{i=1}^m p_i z_i}{\beta}\Big)^{-\alpha},$$

hence $\mathcal{L}(S) = \text{NegMult}\big(\alpha; \frac{p_1}{\beta+1},\dots,\frac{p_m}{\beta+1}\big)$ with $p_0 = \frac{\beta}{\beta+1}$.

Remark 2.10. The components of the negative multinomial distribution are not independent since the corresponding correlation coefficient is not zero,

$$\text{corr}(N_i, N_j) = \sqrt{\frac{p_i p_j p_0^2}{(1+p_i p_0)(1+p_j p_0)}}, \qquad i,j \in \{1,\dots,m\}, \quad i \neq j,$$

as can be found in [39, Equation (36.28)]. As [59, p. 411] also states, the correlation coefficient can only become zero if either p_i or p_j are zero, but then the distribution is degenerate. Yet, as can be also found in [39, Chapter 36.3], the marginal distributions of the negative multinomial distribution are negative binomial distributions. By Lemma 2.3 we see that for $i \in \{1,\dots,m\}$ the marginal distribution of S with degenerate severity $X_{i,1} \equiv 1$ is $\mathcal{L}(S_i\,|\,\Lambda) \overset{\text{a.s.}}{=} \text{Poisson}(p_i\Lambda)$. Due to Remark 2.9 this is also the marginal distribution of a negative multinomial distribution if $\Lambda \sim \text{Gamma}(\alpha,\beta)$ and also clarifies the type of dependence between the marginal distributions. Applying Lemma 2.8 with $m = 1$ in that framework and $T \equiv 1$, we observe that

$$\mathcal{L}(S_i) = \text{NegBin}\Big(\alpha, \frac{p_i}{\beta + p_i}\Big), \qquad i \in \{1,\dots,m\}.$$

In this thesis we will often use the Laplace transform of a multinomial distribution. We give it here:

Remark 2.11. Let $n \in \mathbb{N}$ and $0 \le p_1,\dots,p_m \le 1$. Let X be a random vector such that $X \sim \text{Multinomial}(n; p_1,\dots,p_m)$. Then the Laplace transform of the distribution of X satisfies

$$\mathbb{E}\big[e^{-\langle t,X\rangle}\big] = \Big(\sum_{i=1}^m p_i\, e^{-t_i}\Big)^n, \qquad t = (t_1,\dots,t_m) \in \mathbb{R}^m,$$

cf. e.g. [39, Equation (35.17)].

We now present a result on a compound distribution with a negative binomial distribution as a counting distribution.

Corollary 2.12. *Let $\alpha > 0$ and $q \in [0, 1)$ and let T be a strictly positive random variable and let N be a random variable such that $\mathcal{L}(N|T) \stackrel{a.s.}{=} \mathrm{NegBin}(\alpha T, q)$ and $\{B_h\}_{h\in\mathbb{N}}$ be a sequence independent of (N, T), consisting of i.i.d. m-dimensional random vectors such that $B_1 \sim \mathrm{Multinomial}(1; \tilde{p}_1, \dots, \tilde{p}_m)$. Consider the random vector $S = \sum_{h=1}^{N} B_h$. Then the random sum S satisfies $\mathcal{L}(S|T) \stackrel{a.s.}{=} \mathrm{NegMult}(\alpha T; q\tilde{p}_1, \dots, q\tilde{p}_m)$.*

Proof. We apply Remark 2.1 and consider the probability-generating function of the conditional distribution of S given T. For $z \in (0, 1]^m$ with $B_h = (B_{1,h}, \dots, B_{m,h})$ it is given as follows

$$G_{S|T}(z) \stackrel{\text{a.s.}}{=} \mathbb{E}\bigg[\prod_{i=1}^{m} z_i^{S_i}\bigg|T\bigg] \stackrel{\text{a.s.}}{=} \mathbb{E}\bigg[\prod_{i=1}^{m} z_i^{\sum_{h=1}^{N} B_{i,h}}\bigg|T\bigg] \stackrel{\text{a.s.}}{=} \mathbb{E}\bigg[\exp\bigg(\sum_{h=1}^{N}\sum_{i=1}^{m} B_{i,h}\ln z_i\bigg)\bigg|T\bigg].$$

Using the independence of (N, T) of $\{B_h\}_{h\in\mathbb{N}}$ and that $\{B_h\}_{h\in\mathbb{N}}$ are i.i.d. and that $\{B_h\}_{h\in\mathbb{N}}$ have a multinomial distribution and appling Remark 2.11 yields

$$G_{S|T}(z) \stackrel{\text{a.s.}}{=} \mathbb{E}\bigg[\bigg(\mathbb{E}\bigg[\exp\bigg(\sum_{i=1}^{m} B_{i,1}\ln z_i\bigg)\bigg]\bigg)^N\bigg|T\bigg] \stackrel{\text{a.s.}}{=} \mathbb{E}\bigg[\bigg(\sum_{i=1}^{m}\tilde{p}_i z_i\bigg)^N\bigg|T\bigg].$$

Since N has a negative binomial distribution given T, we observe

$$G_{S|T}(z) \stackrel{\text{a.s.}}{=} \bigg(\frac{1}{1-q} - \frac{q}{1-q}\sum_{i=1}^{m}\tilde{p}_i z_i\bigg)^{-\alpha T}.$$

It is easy to see that according to Definition 2.6 $p_0 = 1 - q$ and $\frac{q}{1-q}\sum_{i=1}^{m}\tilde{p}_i z_i = \sum_{i=1}^{m}\frac{p_i}{p_0} z_i$ implies $p_i = q\tilde{p}_i$ for $i \in \{1, \dots, m\}$, which completes the proof. q.e.d.

A further result also exists on mixtures of the negative multinomial distribution. It is a generalization of a result that can be found in [59, Section 3.d]. Let us first introduce a generalization of the logarithmic distribution to multivariate dimensions taken from [59, Section 3.d].

Definition 2.13. Let $p_1, \dots, p_m \geq 0$ with $0 < p_1 + \dots + p_m < 1$. A random vector $L = (L_1, \dots, L_m)$ has a multivariate logarithmic distribution if for $(l_1, \dots, l_m) \in \mathbb{N}_0^m \setminus \{0\}$

$$\mathbb{P}[L_1 = l_1, \dots, L_m = l_m] = \frac{-1}{\ln p_0}(l-1)!\prod_{i=1}^{m}\Big(\frac{p_i^{l_i}}{l_i!}\Big),$$

where $l = \sum_{i=1}^{m} l_i$ and $p_0 := 1 - p_1 - \dots - p_m$. We denote the distribution of L by $\mathrm{MultLog}(p_1, \dots, p_m)$.

According to [59, Equation (3.2)] its probability-generating function is given by

$$G_L(z) = \frac{\ln(1 - \sum_{i=1}^{m} p_i z_i)}{\ln p_0}, \qquad z \in [0, 1]^m. \tag{2.14}$$

There is the following analogy to Corollary 2.12.

Example 2.15. Let $L \sim \mathrm{Log}(q)$ with $q \in (0,1)$ and let $\{B_h = (B_{1,h}, \ldots, B_{m,h})\}_{h\in\mathbb{N}}$ be a sequence independent of L, consisting of i.i.d. random vectors with a multinomial distribution, i.e. $B_1 \sim \mathrm{Multinomial}(1; p_1, \ldots, p_m)$. Define $S = \sum_{h=1}^{L} B_h$. Thus we also write $S \sim \mathrm{CLog}(q, \mathrm{Multinomial}(1; p_1, \ldots, p_m))$. Applying Remark 2.11, the probability-generating function of the distribution of S for $z \in [0,1]^m$ is given as follows:

$$\begin{aligned} G_S(z) &= \mathbb{E}\Big[\prod_{i=1}^{m} z_i^{\sum_{h=1}^{L} B_{i,h}}\Big] = \mathbb{E}\Big[\Big(\mathbb{E}\Big[\prod_{i=1}^{m} z_i^{B_{i,1}}\Big]\Big)^{L}\Big] = \mathbb{E}\Big[\Big(\sum_{i=1}^{m} p_i z_i\Big)^{L}\Big] \\ &= \frac{\ln\big(1 - q\sum_{i=1}^{m} p_i z_i\big)}{\ln(1-q)}, \end{aligned}$$

hence $S \sim \mathrm{MultLog}(qp_1, \ldots, qp_m)$.

Then the aforementioned generalization is the following.

Lemma 2.16. *Let $\lambda > 0$, $p_0 \in (0,1)$ and $p_1, \ldots, p_m \geq 0$ with $p_0 + \cdots + p_m = 1$. Let T be a strictly positive random variable and let N be a random variable such that*

$$\mathcal{L}(N\,|\,T) \overset{a.s.}{=} \mathrm{Poisson}(\lambda T)\,. \tag{2.17}$$

Let $\{L_h\}_{h\in\mathbb{N}}$ be a sequence independent of (N,T), consisting of i.i.d. m-dimensional random vectors with $L_1 \sim \mathrm{MultLog}(p_1, \ldots, p_m)$. Define the random vector $S = \sum_{h=1}^{N} L_h$. Then

$$\mathcal{L}(S\,|\,T) \overset{a.s.}{=} \mathrm{NegMult}\Big(\frac{-\lambda T}{\ln p_0}; p_1, \ldots, p_m\Big).$$

Proof. Using Remark 2.1 we prove the claim by considering the probability-generating function of the conditional distribution of S given T for $z \in (0,1]^m$ with $L_h = (L_{1,h}, \ldots, L_{m,h})$

$$G_{S|T}(z) \overset{\text{a.s.}}{=} \mathbb{E}\Big[\prod_{i=1}^{m} z_i^{\sum_{h=1}^{N} L_{i,h}}\,\Big|\,T\Big] \overset{\text{a.s.}}{=} \mathbb{E}\Big[\exp\Big(\sum_{h=1}^{N}\sum_{i=1}^{m} L_{i,h}\ln z_i\Big)\,\Big|\,T\Big].$$

The sequence $\{L_h\}_{h\in\mathbb{N}}$ is i.i.d. and independent of (N,T). Hence, using also Equation (2.14), we observe

$$G_{S|T}(z) \overset{\text{a.s.}}{=} \mathbb{E}\Big[\Big(\mathbb{E}\Big[\exp\Big(\sum_{i=1}^{m} L_{i,1}\ln z_i\Big)\Big]\Big)^{N}\,\Big|\,T\Big] \overset{\text{a.s.}}{=} \mathbb{E}\Big[\Big(\frac{\ln(1-\sum_{i=1}^{m} p_i z_i)}{\ln p_0}\Big)^{N}\,\Big|\,T\Big].$$

Using Equation (2.17) provides

$$\begin{aligned} G_{S|T}(z) &\overset{\text{a.s.}}{=} \exp\Big(-\lambda T\Big(1 - \frac{\ln(1-\sum_{i=1}^{m} p_i z_i)}{\ln p_0}\Big)\Big) \\ &\overset{\text{a.s.}}{=} \exp\Big(\frac{-\lambda T}{\ln p_0}\Big(\ln p_0 - \ln\Big(1 - \sum_{i=1}^{m} p_i z_i\Big)\Big)\Big). \end{aligned}$$

A final simplification yields

$$G_{S|T}(z) \overset{\text{a.s.}}{=} \Big(\frac{1-\sum_{i=1}^{m} p_i z_i}{p_0}\Big)^{\frac{\lambda T}{\ln p_0}},$$

which completes the proof. q.e.d.

We now provide the probability-generating function of the distribution of a Poisson random vector that is mixed with a random vector whose components are non-negative. For simplicity we now introduce an assumption on the dependence scenario.

Assumption 2.18. *Let $\mathcal{J} \neq \varnothing$ be an arbitrary finite set. Let $A_j \in [0,\infty)^{m\times(n+1)}$ for $j \in \mathcal{J}$ and let J be a random variable with values in $\mathcal{J}$. Define the random matrix $A_J = \sum_{j\in\mathcal{J}} 1_{\{J=j\}} A_j$. Let $(R_1,\dots,R_n)$ be non-negative random variables and let R_0 be a non-negative constant. Define $(\Lambda_1,\dots,\Lambda_m)^\top = A_J(R_0,\dots,R_n)^\top$ and let $A_j = (a^j_{i,l})_{1\le i\le m, 0\le l\le n}$ for $j \in \mathcal{J}$.*

Lemma 2.19. *Let Assumption 2.18 be satisfied. Let $N_1,\dots,N_m$ be random variables conditionally independent given $J, R_1,\dots,R_n$ such that*

$$\mathcal{L}(N_i \,|\, J, R_1,\dots,R_n) \overset{a.s.}{=} \mathcal{L}(N_i \,|\, J, \Lambda_i) \overset{a.s.}{=} \operatorname{Poisson}(\lambda_{i,J}\Lambda_i)\,, \qquad i \in \{1,\dots,m\}, \tag{2.20}$$

where $\lambda_{1,j},\dots,\lambda_{m,j} \ge 0$ for $j \in \mathcal{J}$. Then the Laplace transform of the distribution of the random vector $(N,R) = (N_1,\dots,N_m,R_1,\dots,R_n)$ satisfies for $(y,z) \in [0,\infty)^{m+n}$

$$L_{(N,R)}(y,z) = \mathbb{E}\Bigg[\mathrm{e}^{-\langle z,R\rangle} \prod_{l=0}^{n} \exp\Bigg(-\sum_{i=1}^{m} \lambda_{i,J} a^J_{i,l} R_l (1-\mathrm{e}^{-y_i})\Bigg)\Bigg].$$

Remark 2.21. The random variable J can be interpreted as selecting a dependence scenario for $(\Lambda_1,\dots,\Lambda_m)$. Accordingly, the random variables $R_1,\dots,R_n$ can be interpreted as risk factors.

Proof of Lemma 2.19. We calculate the Laplace transform $L_{(N,R)}$ for $(y,z) \in [0,\infty)^{m+n}$. Conditioning on $J, R_1,\dots,R_n$ and using the conditional independence of $N_1,\dots,N_m$ given $J, R_1,\dots,R_n$ and that $R_1,\dots,R_n$ are measurable given $J, R_1,\dots,R_n$ and Equation (2.20) yields

$$\begin{aligned} L_{(N,R)}(y,z) &= \mathbb{E}\big[\mathrm{e}^{-\langle y,N\rangle}\,\mathrm{e}^{-\langle z,R\rangle}\big] = \mathbb{E}\Bigg[\mathrm{e}^{-\langle z,R\rangle} \prod_{i=1}^{m} \mathbb{E}\big[\mathrm{e}^{-y_i N_i} \,\big|\, J, \Lambda_i\big]\Bigg] \\ &= \mathbb{E}\Bigg[\mathrm{e}^{-\langle z,R\rangle} \prod_{i=1}^{m} \exp\big(-\lambda_{i,J}\Lambda_i(1-\mathrm{e}^{-y_i})\big)\Bigg]. \end{aligned}$$

Using $(\Lambda_1,\dots,\Lambda_m)^\top = A_J(R_0,\dots,R_n)^\top$ and the definition of A_J yields

$$L_{(N,R)}(y,z) = \mathbb{E}\Bigg[\mathrm{e}^{-\langle z,R\rangle} \exp\Bigg(-\sum_{i=1}^{m} \lambda_{i,J} \sum_{l=0}^{n} a^J_{i,l} R_l (1-\mathrm{e}^{-y_i})\Bigg)\Bigg],$$

and provides the result for the Laplace transform $L_{(N,R)}$. q.e.d.

There are several examples for this general setting that might make it more clear and allow for different representations. We consider the following model: A Poisson random vector is mixed with a random vector whose components are stochastically and linearly dependent. This also means that the choice for every component of the mixing random vector may include a stochastic number of random variables for each default cause intensity.

Example 2.22. Let $\mathcal{J} = \{0,1\}^{n+1}$ and J be an $(n+1)$-dimensional random vector such that $J = (J_0, \ldots, J_n)$ is $\mathcal{J}$-valued. Further let $(A)_{i,j} = a_{i,j}J_j$ where $a_{i,j} \geq 0$ with $i \in \{1, \ldots, m\}$ and $j \in \{0, \ldots, n\}$, and let $R_1, \ldots, R_n$ be n non-negative random variables which are independent of J, and let R_0 be a non-negative constant. Define $(\Lambda_1, \ldots, \Lambda_m)^\top = A(R_0, \ldots, R_n)^\top$.

Remark 2.23. The model in Example 2.22 may be embedded into Assumption 2.18. For each $j = (j_0, \ldots, j_n) \in \{0,1\}^{n+1}$ define $(A_j)_{i,l} = a^j_{i,l} j_l$ for $i \in \{1, \ldots, m\}$ and $l \in \{0, \ldots, n\}$. Then let $A_j = \sum_{j \in \{0,1\}^{n+1}} 1_{\{J=j\}} A_j$.

The set $\mathcal{J}$ allows for a very flexible choice of dependence scenarios. The random variables in $\mathcal{J}$ may be also matrix-valued. For the default cause intensities $\Lambda_1, \ldots, \Lambda_m$ each component is chosen stochastically. Hence we present a Poisson random vector that is mixed with a random vector, each component of which chosen stochastically. Therefore we make the following assumption:

Example 2.24. Let $\mathcal{J} = \{0,1\}^{m \times (n+1)}$ and let $J_1, \ldots, J_m$ be m $(n+1)$-dimensional random vectors such that $J_i = (J_{i,0}, \ldots, J_{i,n})$ is $\{0,1\}^{n+1}$-valued for each $i \in \{1, \ldots, m\}$. Let further $(A)_{i,j} = a_{i,j}J_{i,j}$ where $a_{i,j} \geq 0$ with $i \in \{1, \ldots, m\}$ and $j \in \{0, \ldots, n\}$, and let $R_1, \ldots, R_n$ be n non-negative random variables which are independent of $J_1, \ldots, J_m$, and let R_0 be a non-negative constant. Define $(\Lambda_1, \ldots, \Lambda_m)^\top = A(R_0, \ldots, R_n)^\top$.

Remark 2.25. This model can be also embedded into Assumption 2.18. Define $J = (J_1, \ldots, J_m)^\top$ with $J_i \in \{0,1\}^{n+1}$ for $i \in \{1, \ldots, m\}$. For each $j \in \{0,1\}^{m \times (n+1)}$ with $(j)_{i,l} = j_{i,l}$ for $i \in \{1, \ldots, m\}$ and $l \in \{0, \ldots, n\}$ define $(A_j)_{i,l} = a^j_{i,l} j_{i,l}$ and thus $A_J = \sum_{j \in \{0,1\}^{m \times (n+1)}} 1_{\{J=j\}} A_j$.

Chapter 3

Construction of Dependent Claim Numbers by Linear Combinations

In this chapter we present one of our major results. The starting point is the following. In an extension of the CreditRisk$^+$ model to be found in [21] the authors consider a random claim number $N = N_1 + \cdots + N_m$, where for each $i \in \{1, \ldots, m\}$ the claim number N_i has a Poisson mixture distribution and the mixing random variable is a gamma-distributed default cause intensity Λ_i. The distribution of the random claim numbers is assumed to satisfy

$$\mathcal{L}(N_i \,|\, \Lambda_1, \ldots, \Lambda_m) \overset{\text{a.s.}}{=} \mathcal{L}(N_i \,|\, \Lambda_i) \overset{\text{a.s.}}{=} \text{Poisson}(\lambda_i \Lambda_i)\,, \quad i \in \{1, \ldots, m\},$$

where $\lambda_i \geq 0$. The default cause intensities are assumed to be independent.

In contrast, we develop an extension of this model by admitting dependence between the claim numbers. We only stipulate that the claim numbers are conditionally independent given $\Lambda_1, \ldots, \Lambda_m$. Dependent default cause intensities provide the dependence structure. The default cause intensities consist of several non-negative risk factors. We construct dependence structures by linear combinations of the risk factors. These dependence structures are chosen stochastically. In this setting we can also relax the assumption of gamma-distributed risk factors and consider τ-tempered α-stable distributions, too.

3.1 Alternative Representation

We consider a scenario that can be described as a stochastic choice of a linear dependence structure between default cause intensities. If we introduce generally dependent default cause intensities, the independence between the random sums $S_i = \sum_{h=1}^{N_i} X_{i,h}$ for $i \in \{1, \ldots, m\}$ is lost, hence a convolution is not possible either. While the independence is also lost in our scenario, it is nevertheless possible to find an adequate representation of the claim numbers with the same distribution such that a variant of Panjer's recursion can be applied, as will be seen later. These default cause intensities may be also considered as a multivariate gamma distribution. There are several approaches to construct multivariate gamma distributions, and one of them includes a linear dependence between the marginal distributions, cf. [44, Chapter 48.3.4].

Based on Lemma 2.19, we derive a theorem that provides us with a general structure of a stochastically chosen linear dependence scenario:

Theorem 3.1. *Let Assumption 2.18 be satisfied. Let $N = (N_1, \dots, N_m)$ be a random vector with conditionally independent components given $J, R_1, \dots, R_n$ such that*

$$\mathcal{L}(N_i \,|\, J, R_1, \dots, R_n) \overset{a.s.}{=} \mathcal{L}(N_i \,|\, J, \Lambda_i) \overset{a.s.}{=} \mathrm{Poisson}(\lambda_{i,J}\Lambda_i)\,, \qquad i \in \{1, \dots, m\},$$

where $\lambda_{i,j} \geq 0$ for each $j \in \mathcal{J}$. On the other hand, consider $(n+1)|\mathcal{J}|$ independent sequences of i.i.d. random vectors $\{B_{j,l,h}\}_{h\in\mathbb{N}}$, such that

$$B_{j,l,1} \sim \mathrm{Multinomial}(1; p_{1,j,l}, \dots, p_{m,j,l})\,, \tag{3.2}$$

where $p_{i,j,l} \in [0,1]$ with $\sum_{i=1}^m p_{i,j,l} = 1$ satisfies the condition $p_{i,j,l}\sum_{d=1}^m \lambda_{d,j}a_{d,l}^j = \lambda_{i,j}a_{i,l}^j$ for each $j \in \mathcal{J}$ and $l \in \{0, \dots, n\}$ and $i \in \{1, \dots, m\}$. Let $\{Q_{j,l}\}_{j\in\mathcal{J}, l\in\{0,\dots,n\}}$ be random variables conditionally independent given $J, R_1, \dots, R_n$ which satisfy

$$\mathcal{L}(Q_{j,l} \,|\, J, R_0, \dots, R_n) \overset{a.s.}{=} \mathcal{L}(Q_{j,l} \,|\, R_l) \overset{a.s.}{=} \mathrm{Poisson}\Big(\sum_{i=1}^m \lambda_{i,j}a_{i,l}^j R_l\Big). \tag{3.3}$$

Assume further that $(Q_{j,0}, \dots, Q_{j,n})$ and the sequence $\{B_{j,l,h}\}_{h\in\mathbb{N}}$ are independent for each $j \in \mathcal{J}$ and $l \in \{0, \dots, n\}$. Finally, define the $\mathbb{N}_0^m$-valued random vector M by

$$M = \sum_{j\in\mathcal{J}} 1_{\{J=j\}} \sum_{l=0}^n \sum_{h=1}^{Q_{j,l}} B_{j,l,h}.$$

Then $(M, R_1, \dots, R_n)$ and $(N, R_1, \dots, R_n)$ have the same distribution.

Proof. For the proof we apply Remark 2.1. Let $R = (R_1, \dots, R_n)$. First, an application of Lemma 2.19 shows immediately that the Laplace transform of the distribution of $(N, R) = (N, R_1, \dots, R_n)$ satisfies for $(y, z) \in [0, \infty)^{m+n}$

$$L_{(N,R)}(y,z) = \mathbb{E}\Big[\mathrm{e}^{-\langle z,R\rangle}\prod_{l=0}^n \exp\Big(-\sum_{i=1}^m \lambda_{i,J}a_{i,l}^J R_l(1-\mathrm{e}^{-y_i})\Big)\Big].$$

Now consider the Laplace transform of the distribution of the random vector $(M, R) = (M_1, \dots, M_m, R_1, \dots, R_n)$. By partitioning J, we obtain for all $(y, z) \in [0, \infty)^{m+n}$

$$\begin{aligned}
L_{(M,R)}(y,z) &= \mathbb{E}\big[\mathrm{e}^{-\langle y,M\rangle}\,\mathrm{e}^{-\langle z,R\rangle}\big] \\
&= \mathbb{E}\Big[\exp\Big(-\Big\langle y, \sum_{j\in\mathcal{J}} 1_{\{J=j\}} \sum_{l=0}^n \sum_{h=1}^{Q_{j,l}} B_{j,l,h}\Big\rangle\Big)\,\mathrm{e}^{-\langle z,R\rangle}\Big] \\
&= \sum_{j\in\mathcal{J}} \mathbb{E}\Big[1_{\{J=j\}}\,\mathrm{e}^{-\langle z,R\rangle}\exp\Big(-\sum_{l=0}^n \sum_{h=1}^{Q_{j,l}} \langle y, B_{j,l,h}\rangle\Big)\Big].
\end{aligned}$$

Using that the sequences $\{B_{j,l,h}\}_{h\in\mathbb{N}}$ are i.i.d. and independent of $(Q_{j,0}, \dots, Q_{j,n})$ for $j \in \mathcal{J}$ and $l \in \{0, \dots, n\}$ and have the distribution given in Equation (3.2) and thus applying Remark 2.11, we observe

$$\begin{aligned}
L_{(M,R)}(y,z) &= \sum_{j\in\mathcal{J}} \mathbb{E}\Big[1_{\{J=j\}}\,\mathrm{e}^{-\langle z,R\rangle}\prod_{l=0}^n\Big(\mathbb{E}\Big[\exp\Big(-\langle y, B_{j,l,1}\rangle\Big)\Big]\Big)^{Q_{j,l}}\Big] \\
&= \sum_{j\in\mathcal{J}} \mathbb{E}\Big[1_{\{J=j\}}\,\mathrm{e}^{-\langle z,R\rangle}\prod_{l=0}^n\Big(\sum_{i=1}^m p_{i,j,l}\,\mathrm{e}^{-y_i}\Big)^{Q_{j,l}}\Big].
\end{aligned}$$

By conditioning on $(J, R_0, \dots, R_n)$, using the conditional independence of the random variables $Q_{j,0}, \dots, Q_{j,n}$ given $J, R_1, \dots, R_n$ for $j \in \mathcal{J}$ and Equation (3.3) and using that $R_1, \dots, R_n$ are measurable given $J, R_1, \dots, R_n$ provides

$$\begin{aligned} L_{(M,R)}(y,z) &= \sum_{j\in\mathcal{J}} \mathbb{E}\bigg[1_{\{J=j\}}\,\mathrm{e}^{-\langle z,R\rangle} \prod_{l=0}^{n} \mathbb{E}\bigg[\bigg(\sum_{i=1}^{m} p_{i,j,l}\,\mathrm{e}^{-y_i}\bigg)^{Q_{j,l}}\,\bigg|\, R_l\bigg]\bigg] \\ &= \sum_{j\in\mathcal{J}} \mathbb{E}\bigg[1_{\{J=j\}}\,\mathrm{e}^{-\langle z,R\rangle} \prod_{l=0}^{n} \exp\bigg(-\bigg(\sum_{i=1}^{m} \lambda_{i,j} a_{i,l}^{j} R_l\bigg)\bigg(1-\sum_{i=1}^{m} p_{i,j,l}\,\mathrm{e}^{-y_i}\bigg)\bigg)\bigg]. \end{aligned}$$

Noting $p_{i,j,l} \sum_{d=1}^{m} \lambda_{d,j} a_{d,l}^{j} = \lambda_{i,j} a_{i,l}^{j}$ yields

$$L_{(M,R)}(y,z) = \sum_{j\in\mathcal{J}} \mathbb{E}\bigg[1_{\{J=j\}}\,\mathrm{e}^{-\langle z,R\rangle} \prod_{l=0}^{n} \exp\bigg(-\sum_{i=1}^{m} \lambda_{i,j} a_{i,l}^{j} R_l (1-\mathrm{e}^{-y_i})\bigg)\bigg],$$

which completes the proof. q.e.d.

Remark 3.4. The random variables M_i for $i \in \{1, \dots, m\}$ do not in general have a distribution in a Panjer(a, b, k) class because the distributions of the random variables $Q_{j,l}$ for $j \in \mathcal{J}$ and $l \in \{0, \dots, n\}$ are not generally in a Panjer(a, b, k) class.

Remark 3.5. In the lemma the case $\sum_{d=1}^{m} \lambda_{d,j} a_{d,l}^{j} = 0$ for $j \in \mathcal{J}$ and $l \in \{0, \dots, n\}$ has not been considered separately. However, if $\sum_{d=1}^{m} \lambda_{d,j} a_{d,l}^{j} = 0$ holds for any j, l, then it is not immediately clear how to choose the probability distribution of $B_{j,l,1}$. Yet, we observe that in this case $\mathcal{L}(Q_{j,l}\,|\,R_l) \overset{\text{a.s.}}{=} \text{Poisson}(0) = \delta_0$. Hence for each $i \in \{1, \dots, m\}$ the corresponding sum in $M_i = \sum_{j\in\mathcal{J}} 1_{\{J=j\}} \sum_{l=0}^{n} \sum_{h=1}^{Q_{j,l}} B_{i,j,l,h}$ does not have summands for the corresponding j and l and this also holds for the random variables $N_1, \dots, N_m$. Thus the distribution of $B_{j,l,1}$ may be chosen arbitrarily.

Remark 3.6. The parameters $\lambda_{1,J}, \dots, \lambda_{m,J}$ might be considered as a redundant notation. Indeed, in order to apply this framework as an extension of the CreditRisk$^+$ model, they represent the default probability of the obligors in the CreditRisk$^+$ model, respectively. For this reason they should be retained.

Remark 3.7. There are several well-known distributions that could be chosen for J. It is possible to choose a multinomial distribution Multinomial$(1; p_1, \dots, p_n)$. Further, it is possible to choose a multivariate hypergeometric distribution (cf. [39, Chapter 39]) such that $J \sim \text{MultHyperGeom}(h; 1, \dots, 1)$ with $h \in \{1, \dots, n\}$ and J is n-dimensional.

Remark 3.8. Of course, there are also other methods to construct dependence between default cause intensities: Shevchenko and Luo [46] propose a bivariate model to model risks by a t-copula. They further use simulation and calibration methods. Yet, in such a case a recursion is not possible.

Remark 3.9. If the risk factors $R_1, \dots, R_n$ have a τ-tempered α-stable distribution, then it is possible to rewrite the corresponding Poisson mixture distribution as a compound Poisson distribution, cf. [21, Lemma 5.10], where the severity distribution is an extended negative binomial distribution. Furthermore, if the risk factors $R_1, \dots, R_n$ are infinitely divisible, then it is possible to rewrite the Poisson mixture distribution as a compound Poisson distribution, cf. [64, Corollary 4.1]. If the corresponding severity distribution is in a Panjer(a, b, k) class, then an extension of Panjer's recursion is also applicable.

Remark 3.10. Using this stochastically linear dependence, we are able to construct both positive and negative correlation. Recall that $\Lambda_i = \sum_{l=0}^{n} a_{i,l}^J R_l$ for $i \in \{1, \ldots, m\}$. For the sake of simplicity we first compute some preparing values we need.

If the risk factors $R_1, \ldots, R_n$ are independent, then we observe for $i, k \in \{1, \ldots, m\}$ with $i \neq k$

$$\mathbb{E}[\Lambda_i \,|\, J] \overset{\text{a.s.}}{=} \sum_{l=0}^{n} a_{i,l}^J \; \mathbb{E}[R_l]$$

and

$$\operatorname{Var}(\Lambda_i \,|\, J) \overset{\text{a.s.}}{=} \sum_{l=1}^{n} \big(a_{i,l}^J\big)^2 \operatorname{Var}(R_l)$$

and

$$\operatorname{cov}(\Lambda_i, \Lambda_k | J) \overset{\text{a.s.}}{=} \sum_{l,p=1}^{n} a_{i,l}^J a_{k,p}^J \operatorname{cov}(R_l, R_p) \overset{\text{a.s.}}{=} \sum_{l=1}^{n} a_{i,l}^J a_{k,l}^J \operatorname{Var}(R_l)\,.$$

Thus the variance of Λ_i is computed as follows

$$\begin{aligned}\operatorname{Var}(\Lambda_i) &= \mathbb{E}[\operatorname{Var}(\Lambda_i | J)] \,+\, \operatorname{Var}(\mathbb{E}[\Lambda_i \,|\, J]) \\ &= \sum_{l=1}^{n} \mathbb{E}\big[\big(a_{i,l}^J\big)^2\big] \; \operatorname{Var}(R_l) + \sum_{l=0}^{n} \operatorname{Var}\big(a_{i,l}^J\big) \; (\mathbb{E}[R_l])^2.\end{aligned}$$

Accordingly we compute the covariance between Λ_i and Λ_k for $i, k \in \{1, \ldots, m\}$ with $i \neq k$

$$\begin{aligned}\operatorname{cov}(\Lambda_i, \Lambda_k) &= \mathbb{E}[\,\operatorname{cov}(\Lambda_i, \Lambda_k | J)\,] \,+\, \operatorname{cov}(\,\mathbb{E}[\Lambda_i \,|\, J]\,,\, \mathbb{E}[\Lambda_k \,|\, J]\,) \\ &= \sum_{l=1}^{n} \mathbb{E}\big[a_{i,l}^J a_{k,l}^J\big] \; \operatorname{Var}(R_l) + \sum_{l,p=0}^{n} \operatorname{cov}\big(a_{i,l}^J, a_{k,p}^J\big) \; \mathbb{E}[R_l] \; \mathbb{E}[R_p]\,. \end{aligned} \tag{3.11}$$

Hence the correlation for $i \neq k$ is

$$\operatorname{corr}(\Lambda_i, \Lambda_k) = \frac{\sum_{l=1}^{n} \mathbb{E}\big[a_{i,l}^J a_{k,l}^J\big] \; \operatorname{Var}(R_l) + \sum_{l,p=0}^{n} \operatorname{cov}\big(a_{i,l}^J, a_{k,p}^J\big) \; \mathbb{E}[R_l] \; \mathbb{E}[R_p]}{\big(\operatorname{Var}(\Lambda_i)\big)^{1/2} \big(\operatorname{Var}(\Lambda_k)\big)^{1/2}}.$$

Due to the Poisson mixture distribution of the claim numbers $N_1, \ldots, N_m$ this also clarifies the correlation between the claim numbers. Thus the variance of N_i for $i \in \{1, \ldots, m\}$ is

$$\operatorname{Var}(N_i) = \mathbb{E}[\operatorname{Var}(N_i \,|\, J, \Lambda_i)] \,+\, \operatorname{Var}(\mathbb{E}[N_i \,|\, J, \Lambda_i]) \,=\, \mathbb{E}[\lambda_{i,J} \Lambda_i] \,+\, \operatorname{Var}(\lambda_{i,J} \Lambda_i)\,, \tag{3.12}$$

and the covariance between N_i and N_k for $i, k \in \{1, \ldots, m\}$ with $i \neq k$ is

$$\begin{aligned}\operatorname{cov}(N_i, N_k) &= \mathbb{E}[\,\operatorname{cov}(N_i, N_k \,|\, J, R_1, \ldots, R_n)\,] \,+\, \operatorname{cov}(\,\mathbb{E}[N_i \,|\, J, \Lambda_i]\,,\, \mathbb{E}[N_k \,|\, J, \Lambda_k]\,) \\ &= \operatorname{cov}(\lambda_{i,J} \Lambda_i, \lambda_{k,J} \Lambda_k) \end{aligned} \tag{3.13}$$

since $\mathbb{E}[N_i N_k \,|\, J, R_1, \ldots, R_n] - \mathbb{E}[N_i \,|\, J, \Lambda_i]\, \mathbb{E}[N_k \,|\, J, \Lambda_k] = 0$ due to the conditional independence of the $N_1, \ldots, N_m$ given $J, R_1, \ldots, R_n$ and the conditional distribution of $N_1, \ldots, N_m$.

It might not be clear from this remark how this dependence structure may also provide negative dependence. We give the following example.

Example 3.14. Let Assumption 2.18 be satisfied. Let R_1 and R_2 be two independent non-negative and non-degenerate risk factors. Then let $\mathcal{J} = \{0, 1\}$ and $n = 2$. A random variable J on $\mathcal{J}$ independent of R_1 and R_2 can thus take the values $j_1 = 1$ and $j_2 = 0$, and we define the following matrices:

$$A_{j_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad A_{j_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the default cause intensities are given by $\Lambda_1 = 1_{\{J=1\}} R_1$ and $\Lambda_2 = 1_{\{J=0\}} R_2$. Thus the covariance is

$$\begin{aligned} \operatorname{cov}(\Lambda_1, \Lambda_2) &= \mathbb{E}\big[1_{\{J=1\}} R_1 1_{\{J=0\}} R_2\big] - \mathbb{E}\big[1_{\{J=1\}} R_1\big]\, \mathbb{E}\big[1_{\{J=0\}} R_2\big] \\ &= -\, \mathbb{E}\big[1_{\{J=1\}} R_1\big]\, \mathbb{E}\big[1_{\{J=0\}} R_2\big] \\ &= -\, \mathbb{E}[R_1]\, \mathbb{P}[J=1]\, \mathbb{E}[R_2]\, \mathbb{P}[J=0] \end{aligned}$$

because $1_{\{J=1\}} 1_{\{J=0\}} = 0$ and J, R_1, R_2 are independent. Therefore, with this anti-thetic choice we obtain negative correlation between the default cause intensities since the expected values are positive. Consequently, the equivalent representations are with $B_{j,l,h} = (B_{1,j,l,h}, B_{2,j,l,h})$

$$M_1 = 1_{\{J=1\}} \sum_{h=1}^{Q_{1,1}} B_{1,1,1,h} \quad \text{and} \quad M_2 = 1_{\{J=0\}} \sum_{h=1}^{Q_{0,2}} B_{2,0,2,h}$$

because Poisson(0) and Ber(0) are δ_0-degenerate distributions. For a random sum S this means

$$S = \sum_{i=1}^{2} \sum_{h=1}^{N_i} X_{i,h} \stackrel{\mathrm{d}}{=} \sum_{i=1}^{2} \sum_{h=1}^{M_i} X_{i,h} = 1_{\{J=1\}} \sum_{h=1}^{Q_{1,1}} B_{1,1,1,h} X_{1,h} + 1_{\{J=0\}} \sum_{h=1}^{Q_{0,2}} B_{2,0,2,h} X_{2,h}.$$

If we assume R_l to be gamma-distributed for $l = 1, 2$, we can apply a simplified version of Algorithm 5.9.

3.2 Examples

3.2.1 Tempered Stable Distribution

We consider another and more general case as usual in the CreditRisk$^+$ model that has already been considered in [21]. We assume the risk factors to have a τ-tempered α-stable distribution instead of a gamma distribution. Stable distributions are denoted by $S_\alpha(\sigma, \beta, \mu)$ with $\alpha \in (0, 2]$, $\sigma > 0$, $\beta \in [-1, 1]$ and $\mu \in \mathbb{R}$ (cf. e.g. [55, p. 9]). If $\alpha \in (0, 1)$, $\beta = 1$, and $\mu = 0$, then the support of $S_\alpha(\sigma, \beta, \mu)$ is on the non-negative real line (cf. [55, p. 15]). We can generalize such a distribution by a change of measure of $Y \sim S_\alpha(\sigma, 1, 0)$ as in [21, Section 5.3] by introducing additional parameters $\tau \geq 0$ and $m \in \mathbb{N}_0$ and obtain the following distribution function

$$F_{\alpha,\sigma,\tau,m}(y) = \frac{\mathbb{E}\big[Y^{-m}\, \mathrm{e}^{-\tau Y}\, 1_{\{Y \leq y\}}\big]}{\mathbb{E}[Y^{-m}\, \mathrm{e}^{-\tau Y}]}, \qquad y \in \mathbb{R}.$$

In case $m = 0$ this is a τ-tempered α-stable distribution (cf. [54, Theorem 4.1]). If the risk factors $R_1, \ldots, R_n$ in scenarios as in Assumption 2.18 have such a distribution, then the respective distribution of the random sum S can be evaluated according to Algorithm 5.9 and Panjer's recursion replaced by [21, Algorithm 5.12].

3.2.2 Dependence on Common Risk Factors with a Multivariate Gamma Distribution

We give an example for our framework and embed Giese's model in [23, Section 10.1] of constructing a dependence structure. We present how his approach fits into our framework and thus find an alternative representation for a recursive calculation of the distribution. As in the independent case, Giese chooses the risk factors to be gamma-distributed. In order to construct a dependence between the m default cause intensities $\Lambda_1, \dots, \Lambda_m$, he then takes $m+1$ independent risk factors $R_1, \dots, R_{m+1}$ with $R_l \sim \mathrm{Gamma}(\alpha_l, 1)$ for $l \in \{1, \dots, m+1\}$. Giese supposes that $\sigma_{ii} = \frac{1}{\alpha_i + \alpha_{m+1}}$ for $i \in \{1, \dots, m\}$ is the variance of the risk factors. He thereby obtains a linear dependence structure with

$$\Lambda_i = \sigma_{ii}(R_i + R_{m+1}), \qquad i \in \{1, \dots, m\}.$$

It can be seen without much effort that for the distribution

$$\mathcal{L}(\Lambda_i) = \mathrm{Gamma}(\alpha_i + \alpha_{m+1}, 1/\sigma_{ii})$$

holds. Note that this approach of constructing a multivariate gamma distribution with dependent marginals had previously been introduced by Cheriyan and Ramabhadran, cf. [44, Chapter 48.3.1]. Using this approach we obtain only positive correlation, cf. also Remark 3.10 with $|\mathcal{J}| = 1$.

A generalization of the above dependence structure exists; in order to stay within the framework of the CreditRisk$^+$ model, we let the default cause intensities Λ_i for $i \in \{1, \dots, m\}$ depend on idiosyncratic risk factors

$$I_i \sim \mathrm{Gamma}\big(\alpha_i^I, \beta\big)\,, \qquad i \in \{1, \dots, m\},$$

and on common risk factors

$$C_j \sim \mathrm{Gamma}\big(\alpha_j^C, \beta\big)\,, \qquad j \in \{1, \dots, n-m\}.$$

These risk factors $I_1, \dots, I_m, C_1, \dots, C_{n-m}$ are assumed to be independent. Let R_0 be a non-negative constant.

As before for $i \in \{1, \dots, m\}$ we let the random claim numbers N_i have a mixture distribution with conditionally independent components given $\Lambda_1, \dots, \Lambda_m$ such that

$$\mathcal{L}(N_i \,|\, \Lambda_1, \dots, \Lambda_m) \overset{\text{a.s.}}{=} \mathcal{L}(N_i \,|\, \Lambda_i) \overset{\text{a.s.}}{=} \mathrm{Poisson}(\lambda_i \Lambda_i)\,, \qquad i \in \{1, \dots, m\},$$

where $\lambda_i \geq 0$. We let $A \in [0, \infty)^{m \times (n+1)}$ and obtain a linear dependence structure by setting

$$(\Lambda_1, \dots, \Lambda_m)^\top = A(R_0, I_1, \dots, I_m, C_1, \dots, C_{n-m})^\top, \tag{3.15}$$

where

$$A = \begin{cases} a_{i,0} = 0 & \text{for } i \in \{1, \dots, m\}, \\ \delta_{i,j} & \text{for } j \in \{1, \dots, m\}, \\ a_{i,j} & \text{for } j \in \{m+1, \dots, n\}. \end{cases}$$

Hence the default cause intensities Λ_i are linearly dependent for $i \in \{1, \dots, m\}$. Note that, given the risk factors $I_1, \dots, I_m, C_1, \dots, C_{n-m}$, the default cause intensities $\Lambda_1, \dots, \Lambda_m$ are conditionally independent.

This structure preserves the type of the distribution. An application of Theorem 3.1 with $|\mathcal{J}| = 1$ provides the equality in distribution of the random vectors M and N where

$$M = \sum_{l=1}^{n} \sum_{h=1}^{Q_l} B_{l,h},$$

and

$$Q_l \sim \begin{cases} \text{NegBin}\Big(\alpha_l^I, \frac{\sum_{i=1}^m \lambda_i \delta_{i,l}}{\beta+\sum_{i=1}^m \lambda_i \delta_{i,l}}\Big) & \text{for } l \in \{1,\dots,m\}, \\ \text{NegBin}\Big(\alpha_l^C, \frac{\sum_{i=1}^m \lambda_i a_{i,l}}{\beta+\sum_{i=1}^m \lambda_i a_{i,l}}\Big) & \text{for } l \in \{m+1,\dots,n\}, \end{cases}$$

are independent for $l \in \{1,\dots,n\}$.

Thus we can apply Panjer's recursion for the random sum

$$S = \sum_{i=1}^{m} \sum_{h=1}^{N_i} X_{i,h} \overset{\text{d}}{=} \sum_{l=1}^{n} \sum_{h=1}^{Q_l} \sum_{i=1}^{m} B_{i,l,h} X_{i,l,h},$$

where $\{X_{i,l,h}\}_{h\in\mathbb{N}}$ and $\{X_n\}_{n\in\mathbb{N}}$ are sequences of i.i.d. random variables that are equally distributed for $i \in \{1,\dots,m\}$, respectively. The distribution can be evaluated applying Algorithm 5.9.

Remark 3.16. If the common risk factors have a τ-tempered α-stable distribution, i.e. $C_l \sim F_{\alpha_l,\sigma_l,\tau_l,0}$ for $l \in \{1,\dots,n-m\}$, then, due to the additivity property (cf. [54, Corollary 2.12]) the sum of such risk factors again has a τ-tempered α-stable distribution. Then the distribution of the random variables Q_l for $l \in \{m+1,\dots,n\}$ in Equation (3.3) is a Poisson distribution mixed over a τ-tempered α-stable distribution and hence not generally in a Panjer(a,b,k) class, but the application of [21, Lemma 5.10] provides a means of converting this distribution into a random sum with distributions in a Panjer(a,b,k) class, namely

$$Q_l \overset{\text{d}}{=} \sum_{h=1}^{L_l} K_{l,h}, \qquad l \in \{m+1,\dots,n\}$$

where $L_l \sim \text{Poisson}(\delta_l)$ and $\{K_{l,h}\}_{h\in\mathbb{N}}$ is a sequence independent of L_l, consisting of i.i.d. random variables such that $K_{l,1} \sim \text{ExtNegBin}(-\alpha_l, 1, p_l)$, where the parameters are $\delta_l = \gamma_{\alpha_l,\sigma_l}((\sum_{i=1}^m \lambda_i a_{i,l} + \tau_l)^{\alpha_l} - \tau_l^{\alpha_l})$ and

$$p_l = \frac{\tau_l}{\sum_{i=1}^m \lambda_i a_{i,l} + \tau_l} \qquad \text{and} \qquad \gamma_{\alpha_l,\sigma_l} = \frac{\sigma_l^{\alpha_l}}{\cos(\alpha_l \pi/2)}.$$

Then Panjer's recursion can be applied using Theorem 5.1. The convolution of these compound Poisson distributions (the idiosyncratic risk factors also provide a compound Poisson distribution) may be circumvented by an application of [49, Proposition 3.3.4] which proves that a convolution of compound Poisson distributions is again a compound Poisson distribution. Hence it is only necessary to apply Panjer's recursion once.

3.3 A Generalization of the Dependence by Approximation

There is also another interesting application for mixtures with generalized gamma convolutions (cf. [8]). Under certain conditions we can generalize the stochastic linear dependence to stochastic polynomial dependence. We therefore cite a result in [7, Theorem 3]:

Theorem 3.17. *Let $X_1, \dots, X_n$ be independent random variables with gamma distribution. Then $X_1^{q_1} \cdots X_n^{q_n}$ is a generalized gamma convolution provided that $|q_j| \geq 1$ for $j \in \{1, \dots, n\}$.*

A generalized gamma convolution is explained in Definition 9.10. Thorin [66] shows how to extract the parameter a and Thorin measure U from a given distribution function, see also Definition 9.10.

Hence we obtain a corollary of Theorem 3.1:

Corollary 3.18. *Let Assumption 2.18 be satisfied. Define $R_l = X^l$ for $l \in \{0, \dots, n\}$ where $X \sim \mathrm{Gamma}(\alpha, \beta)$. Let $N = (N_1, \dots, N_m)$ be a random vector with conditionally independent components given $J, R_1, \dots, R_n$ such that*

$$\mathcal{L}(N_i \,|\, J, R_1, \dots, R_n) \stackrel{a.s.}{=} \mathcal{L}(N_i \,|\, J, \Lambda_i) \stackrel{a.s.}{=} \mathrm{Poisson}(\lambda_{i,J}\Lambda_i)\,, \qquad i \in \{1, \dots, m\},$$

where $\lambda_{i,j} \geq 0$ for each $j \in \mathcal{J}$. Define $(n+1)|\mathcal{J}|$ independent sequences of i.i.d. random vectors $\{B_{j,l,h}\}_{h \in \mathbb{N}}$ such that $B_{j,l,1} \sim \mathrm{Multinomial}(1; p_{1,j,l}, \dots, p_{m,j,l})$ for each $j \in \mathcal{J}$ and $l \in \{0, \dots, n\}$ where $p_{i,j,l} \in [0,1]$ with $\sum_{i=1}^m p_{i,j,l} = 1$ satisfies $p_{i,j,l} \sum_{d=1}^m \lambda_{d,j} a_{d,l}^j = \lambda_{i,j} a_{i,l}^j$. Further, let $M = (M_1, \dots, M_m)$ be an $\mathbb{N}_0^m$-valued random vector such that

$$M = \sum_{j \in \mathcal{J}} 1_{\{J=j\}} \sum_{l=0}^{n} \sum_{h=1}^{Q_{j,l}} B_{j,l,h},$$

where for each $j \in \mathcal{J}$ and $l \in \{0, \dots, n\}$

$$\mathcal{L}(Q_{j,l} \,|\, J, R_0, \dots, R_n) \stackrel{a.s.}{=} \mathcal{L}(Q_{j,l} \,|\, R_l) \stackrel{a.s.}{=} \mathrm{Poisson}\left(\sum_{i=1}^m \lambda_{i,j} a_{i,l}^j R_l \right)$$

holds. Assume further that for each $j \in \mathcal{J}$ and $l \in \{0, \dots, n\}$ the random variables $(Q_{j,0}, \dots, Q_{j,n})$ and $\{B_{j,l,h}\}_{h \in \mathbb{N}, l \in \{0, \dots, n\}}$ are independent. Then M and N have the same distribution, and there exist parameters $\alpha_{i,j,l}^{(k)} > 0$ and $\beta_{i,j,l}^{(k)} > 0$ for $k \in \mathbb{N}$ such that the random variables $P_{j,l}^{(k)} = \sum_{i=1}^k R_{i,j,l}^{(k)}$ with

$$R_{i,j,l}^{(k)} \sim \mathrm{NegBin}\left(\alpha_{i,j,l}^{(k)}, \frac{\sum_{d=1}^m \lambda_d a_{d,l}^j}{\beta_{i,j,l}^{(k)} + \sum_{d=1}^m \lambda_d a_{d,l}^j} \right)$$

converge weakly to $Q_{j,l}$ as $k \to \infty$ for $j \in \mathcal{J}$ and $l \in \{0, \dots, n\}$. For each $P_{j,l}^{(n)}$ Panjer's recursion can be applied.

Proof. Apply Theorem 3.1 to obtain the stochastic equality between M and N. The existence of the parameters $\alpha_{i,j,l}^{(k)}$ and $\beta_{i,j,l}^{(k)}$ for $k \in \mathbb{N}$ is proven in Proposition 9.28, and the representation of $R_{,i,j,l}^{(k)}$ is given by Lemma 9.32. Apply Lemma 9.32 to provide the stochastic convergence to each $Q_{j,l}$ for $j \in \mathcal{J}$ and $l \in \{0, \dots, n\}$. This also shows the applicability of Panjer's recursion, cf. also [21, Section 5.5], where the n-fold convolution is replaced by a convex combination and thus a single application of Panjer's recursion suffices. q.e.d.

Chapter 4

Dependent Claim Numbers by Continuous Mixtures

In Chapter 3 we constructed dependence between the default cause intensities by stochastically linear combinations of risk factors, i.e., we compounded them. However, in the literature the term compounding is also used to describe mixing random variables. Now we construct dependence by continuous mixture distributions. We choose one parameter of the distributions of the default cause intensities stochastically, and if it is equal for several default cause intensities, we obtain dependence. As in Chapter 3 the independence between the random sums $S_i = \sum_{h=1}^{N_i} X_{i,h}$ for $i \in \{1, \ldots, m\}$ is lost and there is also a need for an alternative to convolutions. Hence we establish an alternative representation of the claim numbers with equal distribution if the default cause intensities have either a gamma distribution or a τ-tempered α-stable distribution.

4.1 One Level of Compounding

In this section we mix risk factors with random variables that do not themselves have a mixture distribution. We consider a rather general result on mixture distributions to construct dependent claim numbers. As in the previous chapter we let the default cause intensities be stochastically and linearly dependent. Further, we specify a certain structure of the distribution of the risk factors that is a mixture distribution.

Lemma 4.1. *Let Assumption 2.18 be satisfied. Let $T_1, \ldots, T_n$ be strictly positive random variables. Let $\alpha_l, \beta_l > 0$ for all $l \in \{1, \ldots, n\}$ and $\lambda_{i,j} \geq 0$ for all $i \in \{1, \ldots, m\}$ and $j \in \mathcal{J}$. Then we define two $\mathbb{N}_0^m$-valued random vectors N and M as follows:*

(a) Let $R_1, \ldots, R_n$ be random variables with gamma distributions, conditionally independent given $J, T_1, \ldots, T_n$ and let their shape parameters be randomized, i.e. for each $l \in \{1, \ldots, n\}$

$$\mathcal{L}(R_l \,|\, J, T_1, \ldots, T_n) \overset{a.s.}{=} \mathcal{L}(R_l \,|\, J, T_l) \overset{a.s.}{=} \text{Gamma}(\alpha_l T_l, \beta_l)\,. \tag{4.2}$$

Further, let $N = (N_1, \ldots, N_m)$ be a random vector with conditionally independent components given $J, R_1, \ldots, R_n, T_1, \ldots, T_n$ such that for each $i \in \{1, \ldots, m\}$

$$\mathcal{L}(N_i \,|\, J, R_1, \ldots, R_n, T_1, \ldots, T_n) \overset{a.s.}{=} \mathcal{L}(N_i \,|\, J, \Lambda_i) \overset{a.s.}{=} \text{Poisson}(\lambda_{i,J} \Lambda_i) \tag{4.3}$$

with $(\Lambda_1, \ldots, \Lambda_m)$ as in Assumption 2.18.

(b) *For each $j \in \mathcal{J}$ consider random variables $P_{j,1}, \ldots, P_{j,n}$, which are conditionally independent given $J, T_1, \ldots, T_n$, such that for each $l \in \{1, \ldots, n\}$*

$$\mathcal{L}(P_{j,l}\,|\,J, T_1, \ldots, T_n) \overset{a.s.}{=} \mathcal{L}(P_{j,l}\,|\,T_l) \overset{a.s.}{=} \mathrm{Poisson}(-\alpha_l \ln(1 - q_{j,l})T_l) \tag{4.4}$$

with

$$q_{j,l} = \frac{\sum_{i=1}^m \lambda_{i,j} a_{i,l}^j}{\beta_l + \sum_{i=1}^m \lambda_{i,j} a_{i,l}^j},$$

and a random variable $P_{j,0}$ independent of $J, T_1, \ldots, T_n$ such that

$$\mathcal{L}(P_{j,0}) = \mathrm{Poisson}\biggl(\sum_{i=1}^m \lambda_{i,j} a_{i,0}^j R_0\biggr). \tag{4.5}$$

For each $j \in \mathcal{J}$ and $l \in \{0, \ldots, n\}$ let further $\{Y_{j,l,h}\}_{h\in\mathbb{N}}$ be $(n+1)|\mathcal{J}|$ independent sequences independent of $(P_{j,0}, \ldots, P_{j,n})$, consisting of i.i.d. random variables such that for $l \in \{1, \ldots, n\}$

$$Y_{j,l,1} \sim \begin{cases} \mathrm{Log}(q_{j,l}) & \text{if } q_{j,l} > 0, \\ \delta_1 & \text{if } q_{j,l} = 0, \end{cases} \tag{4.6}$$

and $Y_{j,0,1} \sim \delta_1$. For each $l \in \{0, \ldots, n\}$ and $j \in \mathcal{J}$ let $\{B_{j,l,h,k}\}_{h,k\in\mathbb{N}}$ be $(n+1)|\mathcal{J}|$ independent sequences independent of $\{(Y_{j,0,h}, \ldots, Y_{j,n,h})\}_{h\in\mathbb{N}}$ and $(P_{j,0}, \ldots, P_{j,n})$, consisting of i.i.d. random vectors such that

$$B_{j,l,1,1} \sim \mathrm{Multinomial}(1; p_{1,j,l}, \ldots, p_{m,j,l})\,, \tag{4.7}$$

where $p_{i,j,l} \in [0,1]$ with $\sum_{i=1}^m p_{i,j,l} = 1$ satisfies $p_{i,j,l} \sum_{d=1}^m \lambda_{d,j} a_{d,l}^j = \lambda_{i,j} a_{i,l}^j$. Define the random vector M by

$$M = \sum_{j\in\mathcal{J}} 1_{\{J=j\}} \sum_{l=0}^n \sum_{h=1}^{P_{j,l}} \sum_{k=1}^{Y_{j,l,h}} B_{j,l,h,k}. \tag{4.8}$$

Then $(M, T_1, \ldots, T_n)$ and $(N, T_1, \ldots, T_n)$ have the same distribution.

Proof. For the proof we apply Remark 2.1. Let $T = (T_1, \ldots, T_n)$. First, we compute the Laplace transform of the distribution of $(N,T) = (N, T_1, \ldots, T_n)$ for $(y,z) \in [0,\infty)^{m+n}$. Since $N_1, \ldots, N_m$ are conditionally independent given $J, R_1, \ldots, R_n, T_1, \ldots, T_n$ and since $T_1, \ldots, T_n$ are measurable given $J, R_1, \ldots, R_n, T_1, \ldots, T_n$ and because of Equation (4.3), we have

$$\begin{aligned} L_{(N,T)}(y,z) &= \mathbb{E}\bigl[\mathrm{e}^{-\langle y,N\rangle}\, \mathrm{e}^{-\langle z,T\rangle}\bigr] = \mathbb{E}\biggl[\mathrm{e}^{-\langle z,T\rangle} \prod_{i=1}^m \mathbb{E}\bigl[\mathrm{e}^{-y_i N_i}\,\big|J, \Lambda_i\bigr]\biggr] \\ &= \mathbb{E}\biggl[\mathrm{e}^{-\langle z,T\rangle} \prod_{i=1}^m \exp\bigl(-\lambda_{i,J}\Lambda_i(1 - \mathrm{e}^{-y_i})\bigr)\biggr]. \end{aligned}$$

Using $(\Lambda_1, \ldots, \Lambda_m)^\top = A_J(R_0, \ldots, R_n)^\top$ and the definition of A_J gives

$$\begin{aligned} L_{(N,T)}(y,z) &= \mathbb{E}\biggl[\mathrm{e}^{-\langle z,T\rangle} \prod_{i=1}^m \exp\biggl(-\lambda_{i,J} \sum_{l=0}^n a_{i,l}^J R_l (1 - \mathrm{e}^{-y_i})\biggr)\biggr] \\ &= \mathbb{E}\biggl[\mathrm{e}^{-\langle z,T\rangle} \prod_{l=0}^n \exp\biggl(-\sum_{i=1}^m \lambda_{i,J} a_{i,l}^J R_l (1 - \mathrm{e}^{-y_i})\biggr)\biggr]. \end{aligned}$$

Conditioning on $J, T_1, \ldots, T_n$, using the conditional independence of $R_1, \ldots, R_n$ given $J, T_1, \ldots, T_n$, using that $T_1, \ldots, T_n$ are measurable given $J, T_1, \ldots, T_n$, and using Equation (4.2) yields

$$\begin{aligned} L_{(N,T)}(y,z) = \ & \mathbb{E}\Bigg[\mathrm{e}^{-\langle z,T\rangle} \exp\Bigg(-R_0 \sum_{i=1}^m \lambda_{i,J} a_{i,0}^J (1-\mathrm{e}^{-y_i})\Bigg) \\ & \times \prod_{l=1}^n \mathbb{E}\Bigg[\exp\Bigg(-R_l \sum_{i=1}^m \lambda_{i,J} a_{i,l}^J (1-\mathrm{e}^{-y_i})\Bigg) \Bigg| J, T_l\Bigg]\Bigg] \\ = \ & \mathbb{E}\Bigg[\mathrm{e}^{-\langle z,T\rangle} \exp\Bigg(-R_0 \sum_{i=1}^m \lambda_{i,J} a_{i,0}^J (1-\mathrm{e}^{-y_i})\Bigg) \\ & \times \prod_{l=1}^n \Bigg(\frac{\beta_l}{\beta_l + \sum_{i=1}^m \lambda_{i,J} a_{i,l}^J (1-\mathrm{e}^{-y_i})}\Bigg)^{\alpha_l T_l}\Bigg]. \end{aligned}$$

Now consider the Laplace transform of the distribution of the random vector $(M, T) = (M_1, \ldots, M_m, T_1, \ldots, T_n)$. Using the definition of M, we obtain for $(y, z) \in [0, \infty)^{m+n}$

$$L_{(M,T)}(y,z) = \mathbb{E}\Bigg[\exp\Bigg(-\Bigg\langle y, \sum_{j\in\mathcal{J}} 1_{\{J=j\}} \sum_{l=0}^n \sum_{h=1}^{P_{j,l}} \sum_{k=1}^{Y_{j,l,h}} B_{j,l,h,k}\Bigg\rangle\Bigg) \mathrm{e}^{-\langle z,T\rangle}\Bigg].$$

Partitioning J yields

$$L_{(M,T)}(y,z) = \sum_{j\in\mathcal{J}} \mathbb{E}\Bigg[1_{\{J=j\}} \mathrm{e}^{-\langle z,T\rangle} \exp\Bigg(-\sum_{l=0}^n \sum_{h=1}^{P_{j,l}} \sum_{k=1}^{Y_{j,l,h}} \langle y, B_{j,l,h,k}\rangle\Bigg)\Bigg].$$

Using that $\{Y_{j,l,h}\}_{h\in\mathbb{N}}$ and $\{B_{j,l,h,k}\}_{h,k\in\mathbb{N}}$ are i.i.d. and independent of $P_{j,l}$ for $j \in \mathcal{J}$ and $l \in \{0, \ldots, n\}$ and that $\{Y_{j,l,h}\}_{h\in\mathbb{N}}$ are independent of $\{B_{j,l,h,k}\}_{h,k\in\mathbb{N}}$ gives

$$\begin{aligned} L_{(M,T)}(y,z) &= \sum_{j\in\mathcal{J}} \mathbb{E}\Bigg[1_{\{J=j\}} \mathrm{e}^{-\langle z,T\rangle} \prod_{l=0}^n \Bigg(\mathbb{E}\Bigg[\exp\Bigg(-\sum_{k=1}^{Y_{j,l,1}} \langle y, B_{j,l,1,k}\rangle\Bigg)\Bigg]\Bigg)^{P_{j,l}}\Bigg] \\ &= \sum_{j\in\mathcal{J}} \mathbb{E}\Bigg[1_{\{J=j\}} \mathrm{e}^{-\langle z,T\rangle} \prod_{l=0}^n \Big(\mathbb{E}\Big[\big(\mathbb{E}\big[\exp\big(-\langle y, B_{j,l,1,1}\rangle\big)\big]\big)^{Y_{j,l,1}}\Big]\Big)^{P_{j,l}}\Bigg]. \end{aligned}$$

By Equation (4.7) and by Remark 2.11 we obtain

$$\begin{aligned} L_{(M,T)}(y,z) &= \sum_{j\in\mathcal{J}} \mathbb{E}\Bigg[1_{\{J=j\}} \mathrm{e}^{-\langle z,T\rangle} \prod_{l=0}^n \Bigg(\mathbb{E}\Bigg[\Bigg(\sum_{i=1}^m p_{i,j,l}\, \mathrm{e}^{-y_i}\Bigg)^{Y_{j,l,1}}\Bigg]\Bigg)^{P_{j,l}}\Bigg] \\ &= \sum_{j\in\mathcal{J}} \mathbb{E}\Bigg[1_{\{J=j\}} \mathrm{e}^{-\langle z,T\rangle} \prod_{l=0}^n \Bigg(G_{Y_{j,l,1}}\Bigg(\sum_{i=1}^m p_{i,j,l}\, \mathrm{e}^{-y_i}\Bigg)\Bigg)^{P_{j,l}}\Bigg], \end{aligned}$$

where $G_{Y_{j,l,1}}(z)$ denotes the probability-generating function of the distribution of $Y_{j,l,1}$. Conditioning on $J, T_1, \ldots, T_n$, using that $T_1, \ldots, T_n$ are measurable given $J, T_1, \ldots, T_n$ and using the conditional independence of $P_{j,1}, \ldots, P_{j,n}$ given $J, T_1, \ldots, T_n$ and the independence of $P_{j,0}$ of $J, T_1, \ldots, T_n$ for each $j \in \mathcal{J}$ and using Equations (4.4) and (4.5) provides

$$L_{(M,T)}(y,z) = \sum_{j \in \mathcal{J}} \mathbb{E}\Bigg[1_{\{J=j\}} \,\mathrm{e}^{-\langle z,T\rangle} \exp\Bigg(-R_0 \sum_{i=1}^m \lambda_{i,j} a_{i,0}^j \Big(1 - G_{Y_{j,0,1}}\Big(\sum_{i=1}^m p_{i,j,0}\,\mathrm{e}^{-y_i}\Big)\Big)\Bigg) \times \prod_{l=1}^n \exp\Bigg(\alpha_l \ln(1-q_{j,l}) T_l \Big(1 - G_{Y_{j,l,1}}\Big(\sum_{i=1}^m p_{i,j,l}\,\mathrm{e}^{-y_i}\Big)\Big)\Bigg)\Bigg].$$

In case $q_{j,l} = 0$ for some j, l, it follows that $\ln(1 - q_{j,l}) = 0$, and we have

$$\exp\Bigg(\alpha_l \ln(1-q_{j,l}) T_l \Big(1 - G_{Y_{j,l,1}}\Big(\sum_{i=1}^m p_{i,j,l}\,\mathrm{e}^{-y_i}\Big)\Big)\Bigg) = 1.$$

If $q_{j,l} > 0$, then apply Equation (4.6) and note that $q_{j,l} \sum_{i=1}^m p_{i,j,l}\,\mathrm{e}^{-y_i} = \frac{\sum_{i=1}^m \lambda_{i,j} a_{i,l}^j \,\mathrm{e}^{-y_i}}{\beta_l + \sum_{i=1}^m \lambda_{i,j} a_{i,l}^j}$, hence a simplification yields

$$\begin{aligned}\exp\Bigg(\alpha_l \ln(1-q_{j,l}) T_l \Bigg(1 - \frac{\ln\big(1 - q_{j,l}\sum_{i=1}^m p_{i,j,l}\,\mathrm{e}^{-y_i}\big)}{\ln(1-q_{j,l})}\Bigg)\Bigg) \\ = \exp\Bigg(\alpha_l T_l \Bigg(\ln(1-q_{j,l}) - \ln\Bigg(1 - \frac{\sum_{i=1}^m \lambda_{i,j} a_{i,l}^j \,\mathrm{e}^{-y_i}}{\beta_l + \sum_{i=1}^m \lambda_{i,j} a_{i,l}^j}\Bigg)\Bigg)\Bigg),\end{aligned}$$

hence, since $q_{j,l} = 0$ implies $\sum_{i=1}^m \lambda_{i,j} a_{i,l}^j = 0$ and since $G_{Y_{j,0,1}}(z) = z$,

$$L_{(M,T)}(y,z) = \sum_{j \in \mathcal{J}} \mathbb{E}\Bigg[1_{\{J=j\}} \,\mathrm{e}^{-\langle z,T\rangle} \exp\Bigg(-R_0 \sum_{i=1}^m \lambda_{i,j} a_{i,0}^j (1 - \mathrm{e}^{-y_i})\Bigg) \times \prod_{l=1}^n \Bigg(\frac{\beta_l}{\beta_l + \sum_{i=1}^m \lambda_{i,j} a_{i,l}^j (1 - \mathrm{e}^{-y_i})}\Bigg)^{\alpha_l T_l}\Bigg],$$

which completes the proof. q.e.d.

Remark 4.9. The representation of M in Equation (4.8) could be also written as a random vector such that

$$M = \sum_{j \in \mathcal{J}} 1_{\{J=j\}} \sum_{l=0}^n \sum_{h=1}^{P_{j,l}} S_{j,l,h},$$

where $\{S_{j,l,h}\}_{h\in\mathbb{N}}$ are sequences of i.i.d. random vectors for each $j \in \mathcal{J}$ and $l \in \{0, \ldots, n\}$ such that with the notation $B_{j,l,h,k} = (B_{1,j,l,h,k}, \ldots, B_{m,j,l,h,k})$

$$S_{j,l,h} = (S_{1,j,l,h}, \ldots, S_{m,j,l,h}) = \Bigg(\sum_{k=1}^{Y_{j,l,h}} B_{1,j,l,h,k}, \ldots, \sum_{k=1}^{Y_{j,l,h}} B_{m,j,l,h,k}\Bigg).$$

If $q_{j,l} > 0$, then Example 2.15 shows that $S_{j,l,1} \sim \mathrm{MultLog}\big(r_{1,j,l}, \ldots, r_{m,j,l}\big)$ with

$$r_{i,j,l} = q_{j,l} p_{i,j,l} = \frac{\lambda_{i,j} a_{i,l}^j}{\beta_l + \sum_{d=1}^m \lambda_{d,j} a_{d,l}^j}, \qquad i \in \{1, \ldots, m\}.$$

If $q_{j,l} = 0$, then $Y_{j,l,1} \sim \delta_1$, and it holds for the probability-generating function of the distribution of $S_{j,l,1}$ with $z \in [0,1]^m$

$$G_{S_{j,l,1}}(z) = \mathbb{E}\Bigg[\bigg(\mathbb{E}\Big[\prod_{i=1}^{m} z_i^{B_{i,j,l,1}}\Big]\bigg)^{Y_{j,l,1}}\Bigg] = \mathbb{E}\Bigg[\bigg(\sum_{i=1}^{m} p_{i,j,l} z_i\bigg)^{Y_{j,l,1}}\Bigg] = \sum_{i=1}^{m} p_{i,j,l} z_i,$$

hence $S_{j,l,1} \sim \text{Multinomial}(1; p_{1,j,l}, \dots, p_{m,j,l})$. The marginal distributions of $S_{j,l,1}$ can be obtained by computing $G_{S_{j,l,1}}(1, \dots, 1, z_i, 1, \dots, 1)$ for $i \in \{1, \dots, m\}$, and we see that the marginal distributions have a compound logarithmic distribution, cf. Example 2.15.

Remark 4.10. It should be pointed out that the random variables $T_1, \dots, T_n$ need not be independent. This is crucial for the construction of dependence. For instance we could let $\tilde{T}_1, \dots, \tilde{T}_p$ be independent random variables and $\tilde{T}_0$ a non-negative constant. Define another random matrix $B = \sum_{k \in \mathcal{K}} 1_{\{K=k\}} B_k$ with $\mathcal{K} \neq \emptyset$ an arbitrary finite set and $B_k \in [0,\infty)^{n \times (p+1)}$ for $k \in \mathcal{K}$ and K a $\mathcal{K}$-valued random variable. Then let $(T_1, \dots, T_n)^\top = B(\tilde{T}_0, \dots, \tilde{T}_p)^\top$.

Remark 4.11. This approach also allows a construction with both positive and negative correlations. Assume that $\mathbb{E}[T_l] < \infty$ for each $l \in \{1, \dots, n\}$. Then the covariance between two default cause intensities Λ_i and Λ_k for $i, k \in \{1, \dots, m\}$ with $i \neq k$ is given by Equation (3.11)

$$\operatorname{cov}(\Lambda_i, \Lambda_k) = \sum_{l,p=1}^{n} \mathbb{E}\big[a_{i,l}^J a_{k,l}^J\big] \operatorname{cov}(R_l, R_p) + \sum_{l,p=0}^{n} \operatorname{cov}\big(a_{i,l}^J, a_{k,p}^J\big) \mathbb{E}[R_l] \mathbb{E}[R_p]$$

because $R_1, \dots, R_n$ are not independent. We have $\mathbb{E}[R_l] = \mathbb{E}[\mathbb{E}[R_l | T_l]] = \mathbb{E}\big[\frac{\alpha_l T_l}{\beta_l}\big]$ for $l \in \{1, \dots, n\}$. Further for $l, p \in \{1, \dots, n\}$

$$\begin{aligned} \operatorname{cov}(R_l, R_p) &= \mathbb{E}[\operatorname{cov}(R_l, R_p | T_1, \dots, T_n)] + \operatorname{cov}(\mathbb{E}[R_l | T_l], \mathbb{E}[R_p | T_p]) \\ &= \operatorname{cov}(\mathbb{E}[R_l | T_l], \mathbb{E}[R_p | T_p]) \end{aligned}$$

holds because $\operatorname{cov}(R_l, R_p | T_1, \dots, T_n) = 0$ by the conditional independence of R_l, R_p given $T_1, \dots, T_n$. Since $\mathbb{E}[R_l | T_l] \overset{\text{a.s.}}{=} \frac{\alpha_l T_l}{\beta_l}$ for $l \in \{1, \dots, n\}$, we have

$$\operatorname{cov}(\mathbb{E}[R_l | T_l], \mathbb{E}[R_p | T_p]) = \operatorname{cov}\Big(\frac{\alpha_l}{\beta_l} T_l, \frac{\alpha_p}{\beta_p} T_p\Big) = \frac{\alpha_l \alpha_p}{\beta_l \beta_p} \operatorname{cov}(T_l, T_p).$$

In order to obtain negative correlation we take an antithetic choice on another level as in Remark 3.10. Consider the following situation: Let $|\mathcal{J}| = 1$ and $A \in [0,\infty)^{2 \times 2}$ the identity matrix such that $\Lambda_i = R_i$ for $i = 1, 2$. Let $T \sim \text{Beta}(a, b)$ with $a, b > 0$. Define

$$T_1 = T \qquad \text{and} \qquad T_2 = 1 - T. \tag{4.12}$$

Since the mixing random variable needs to be strictly positive, this is an appropriate choice because the beta distribution meets this requirement in our definition. Let $\mathcal{L}(R_1 | T_1) \overset{\text{a.s.}}{=} \text{Gamma}(\alpha_1 T_1, \beta_1)$ and $\mathcal{L}(R_2 | T_2) \overset{\text{a.s.}}{=} \text{Gamma}(\alpha_2 T_2, \beta_2)$. Hence the covariance between Λ_1 and Λ_2 can be computed as follows

$$\operatorname{cov}(R_1, R_2) = \frac{\alpha_1 \alpha_2}{\beta_1 \beta_2} \operatorname{cov}(T_1, T_2),$$

and

$$\begin{aligned}\operatorname{cov}(T_1,T_2) &= \mathbb{E}[T(1-T)] \,-\, \mathbb{E}[T]\,\mathbb{E}[1-T]\\ &= \mathbb{E}[T] \,-\, \mathbb{E}\big[T^2\big] \,-\, \mathbb{E}[T] + \mathbb{E}[T]^2\\ &= -\operatorname{Var}(T)\,.\end{aligned}$$

Remark 4.13. We can use the representation of M in Lemma 4.1 for the computation of the distribution of the random sum S if $T_1,\dots,T_n$ all have a beta distribution. Then, using the notation $B_{j,l,h,k} = (B_{1,j,l,h,k},\dots,B_{m,j,l,h,k})$, we have

$$S = \sum_{i=1}^{m}\sum_{h=1}^{N_i} X_{i,h} \overset{\mathrm{d}}{=} \sum_{j\in\mathcal{J}} 1_{\{J=1\}} \sum_{l=0}^{n}\sum_{h=1}^{P_{j,l}}\sum_{k=1}^{Y_{j,l,h}}\sum_{i=1}^{m} B_{i,j,l,h,k}X_{i,l,h,k},$$

where for each $l\in\{0,\dots,n\}$ and $i\in\{1,\dots,m\}$ the sequences $\{X_{i,l,h,k}\}_{h,k\in\mathbb{N}}$ are independent and identical copies of $\{X_{i,h}\}_{h\in\mathbb{N}}$. The distribution of $P_{j,l}$ for $l\in\{1,\dots,n\}$ and $j\in\mathcal{J}$ is a Poisson-beta distribution, also known as a general Waring distribution. Hesselager [32, Theorem 1, Example 3] provides a recursive algorithm for a compound distribution with such a counting distribution. Finally an n-fold convolution becomes necessary. In this case $T_1,\dots,T_n$ should be independent. Applying Remark 4.10 it is possible to realize a dependence scenario as in Equation (4.12) with the following parameters

$$(T_1,T_2)^\top = \begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}(1,T)^\top.$$

For our next result it is necessary to introduce the notation of the extended negative binomial distribution, cf. also [40, p. 227]:

Definition 4.14. Let $k\in\mathbb{N}$, $0<p\le 1$, and $\alpha\in(-k,-k+1)$. A random variable N has an extended negative binomial distribution if

$$\mathbb{P}[N=n] = 0 \qquad \text{for } n\in\{0,1,\dots,k-1\}$$

and

$$\mathbb{P}[N=n] = \frac{\binom{n+\alpha-1}{n}p^n}{(1-p)^{-\alpha}-\sum_{j=0}^{k-1}\binom{j+\alpha-1}{j}p^j} \qquad \text{for } n\in\mathbb{N} \text{ with } n\ge k,$$

where the generalized binomial coefficient is given by

$$\binom{n+\alpha-1}{n} = \frac{\Gamma(n+\alpha)}{n!\,\Gamma(\alpha)} = (-1)^n\binom{-\alpha}{n}.$$

We denote the distribution of N by $\mathrm{ExtNegBin}(\alpha,k,p)$.

Then the probability-generating function of this distribution for the important case $k=1$ is given by

$$G_N(z) = \frac{1-(1-pz)^{-\alpha}}{1-(1-p)^{-\alpha}} \qquad \text{for } |z|\le\frac{1}{p},$$

cf. e.g. [21, Equation (2.3)] with an adjustment to our notation.

Thus we can formulate a similar but different result to Lemma 4.1. Here we consider τ-tempered α-stable distributions as distributions for the default cause intensities.

Lemma 4.15. *Let Assumption 2.18 be satisfied. Let $T_1, \dots, T_n$ be non-negative random variables and T_0 a non-negative constant and $\sigma_l > 0$, $\tau_l \geq 0$ and $\alpha_l \in (0,1)$ for each $l \in \{1, \dots, n\}$ and $\lambda_{i,j} \geq 0$ for each $i \in \{1, \dots, m\}$ and $j \in \mathcal{J}$. Then we define the two $\mathbb{N}_0^m$-valued random vectors N and M as follows:*

(a) Let $R_0^\star$ be a non-negative constant and let $R_1^\star, \dots, R_n^\star$ be random variables with τ-tempered α-stable distributions, conditionally independent given $J, T_1, \dots, T_n$ and let their parameters be random, i.e. for each $l \in \{1, \dots, n\}$

$$\mathcal{L}(R_l^\star \,|\, J, T_1, \dots, T_n) \overset{a.s.}{=} \mathcal{L}(R_l^\star \,|\, T_l) \overset{a.s.}{=} F_{\alpha_l, \sigma_l, \tau_l T_l, 0}. \tag{4.16}$$

Let $R_l = R_l^\star T_l$ for $l \in \{0, \dots, n\}$. Let further $N = (N_1, \dots, N_m)$ be a random vector with conditionally independent components given $J, R_1^\star, \dots, R_n^\star, T_1, \dots, T_n$ such that for $i \in \{1, \dots, m\}$

$$\mathcal{L}(N_i \,|\, J, R_1^\star, \dots, R_n^\star, T_1, \dots, T_n) \overset{a.s.}{=} \mathcal{L}(N_i \,|\, J, \Lambda_i) \overset{a.s.}{=} \mathrm{Poisson}(\lambda_{i,J} \Lambda_i) \tag{4.17}$$

with $(\Lambda_1, \dots, \Lambda_m)$ as in Assumption 2.18.

(b) For each $j \in \mathcal{J}$ consider random variables $P_{j,1}, \dots, P_{j,n}$ which are conditionally independent given $J, T_1, \dots, T_n$ such that for each $l \in \{1, \dots, n\}$ and $j \in \mathcal{J}$

$$\mathcal{L}(P_{j,l} \,|\, J, T_1, \dots, T_n) \overset{a.s.}{=} \mathcal{L}(P_{j,l} \,|\, J, T_l) \overset{a.s.}{=} \mathrm{Poisson}\big(\delta_{j,l} T_l^{\alpha_l}\big)\,, \tag{4.18}$$

where $\delta_{j,l} = \gamma_{\alpha_l,\sigma_l}((\sum_{i=1}^m \lambda_{i,j} a_{i,l}^j + \tau_l)^{\alpha_l} - \tau_l^{\alpha_l})$ with $\gamma_{\alpha_l,\sigma_l} = \frac{\sigma_l^{\alpha_l}}{\cos(\alpha_l \pi/2)}$, and for each $j \in \mathcal{J}$ a random variable $P_{j,0}$ which is independent of $J, T_1, \dots, T_n$ such that

$$\mathcal{L}(P_{j,0}) = \mathrm{Poisson}\bigg(\sum_{i=1}^m \lambda_{i,j} a_{i,0}^j R_0^\star T_0\bigg). \tag{4.19}$$

For each $j \in \mathcal{J}$ and $l \in \{0, \dots, n\}$ let further $\{Y_{j,l,h}\}_{h \in \mathbb{N}}$ be $(n+1)|\mathcal{J}|$ independent sequences independent of $(P_{j,0}, \dots, P_{j,n})$, consisting of i.i.d. random variables such that

$$Y_{j,l,1} \sim \begin{cases} \mathrm{ExtNegBin}(-\alpha_l, 1, q_{j,l}) & \text{if } q_{j,l} > 0, \\ \delta_0 & \text{if } q_{j,l} = 0, \end{cases} \tag{4.20}$$

for $l \in \{1, \dots, n\}$ and where

$$q_{j,l} = \frac{\sum_{i=1}^m \lambda_{i,j} a_{i,l}^j}{\sum_{i=1}^m \lambda_{i,j} a_{i,l}^j + \tau_l},$$

and $Y_{j,0,1} \sim \delta_1$, and we use the convention $0/0 := 0$. For each $j \in \mathcal{J}$ and $l \in \{0, \dots, n\}$ let $\{B_{j,l,h,k}\}_{h,k \in \mathbb{N}}$ be $(n+1)|\mathcal{J}|$ independent sequences independent of $\{(Y_{j,0,h}, \dots, Y_{j,n,h})\}_{h \in \mathbb{N}}$ and $(P_{j,0}, \dots, P_{j,n})$ for $j \in \mathcal{J}$, consisting of i.i.d. random vectors such that

$$B_{j,l,1,1} \sim \mathrm{Multinomial}(1; p_{1,j,l}, \dots, p_{m,j,l})\,, \tag{4.21}$$

where $p_{i,j,l} \in [0,1]$ with $\sum_{i=1}^m p_{i,j,l} = 1$ satisfies $p_{i,j,l} \sum_{d=1}^m \lambda_{d,j} a_{d,l}^j = \lambda_{i,j} a_{i,l}^j$. Define the random vector M by

$$M = \sum_{j \in \mathcal{J}} 1_{\{J=j\}} \sum_{l=0}^n \sum_{h=1}^{P_{j,l}} \sum_{k=1}^{Y_{j,l,h}} B_{j,l,h,k}.$$

Then $(M, T_1, \ldots, T_n)$ and $(N, T_1, \ldots, T_n)$ have the same distribution.

Proof. For the proof we apply Remark 2.1. Let $T = (T_1, \ldots, T_n)$. First, we compute the Laplace transform of the distribution of $(N, T) = (N, T_1, \ldots, T_n)$ for $(y, z) \in [0, \infty)^{m+n}$. Since $N_1, \ldots, N_m$ are conditionally independent given $J, R_1^\star, \ldots, R_n^\star, T_1, \ldots, T_n$ and since $T_1, \ldots, T_n$ are measurable given $J, R_1^\star, \ldots, R_n^\star, T_1, \ldots, T_n$ and because of Equation (4.17), we obtain

$$\begin{aligned} L_{(N,T)}(y,z) &= \mathbb{E}\big[\mathrm{e}^{-\langle y,N\rangle}\,\mathrm{e}^{-\langle z,T\rangle}\big] = \mathbb{E}\Bigg[\mathrm{e}^{-\langle z,T\rangle}\prod_{i=1}^{m}\mathbb{E}\big[\mathrm{e}^{-y_iN_i}\,\big|\,J,\Lambda_i\big]\Bigg] \\ &= \mathbb{E}\Bigg[\mathrm{e}^{-\langle z,T\rangle}\prod_{i=1}^{m}\exp\big(-\lambda_{i,J}\Lambda_i(1-\mathrm{e}^{-y_i})\big)\Bigg]. \end{aligned}$$

Using $(\Lambda_1, \ldots, \Lambda_m)^\top = A_J(R_0, \ldots, R_n)^\top$ and the definition of A_J and $R_0, \ldots, R_n$ yields

$$\begin{aligned} L_{(N,T)}(y,z) &= \mathbb{E}\Bigg[\mathrm{e}^{-\langle z,T\rangle}\prod_{i=1}^{m}\exp\Bigg(-\lambda_{i,J}\sum_{l=0}^{n}a_{i,l}^{J}R_l^\star T_l(1-\mathrm{e}^{-y_i})\Bigg)\Bigg] \\ &= \mathbb{E}\Bigg[\mathrm{e}^{-\langle z,T\rangle}\prod_{l=0}^{n}\exp\Bigg(-\sum_{i=1}^{m}\lambda_{i,J}a_{i,l}^{J}R_l^\star T_l(1-\mathrm{e}^{-y_i})\Bigg)\Bigg]. \end{aligned}$$

Conditioning on $J, T_1, \ldots, T_n$, using the conditional independence of $R_1^\star, \ldots, R_n^\star$ given $J, T_1, \ldots, T_n$ and that $T_1, \ldots, T_n$ are measurable given $J, T_1, \ldots, T_n$ and Equation (4.16) (and hence [21, Equation (5.25)]) provides

$$\begin{aligned} L_{(N,T)}(y,z) = {} & \mathbb{E}\Bigg[\mathrm{e}^{-\langle z,T\rangle}\exp\Bigg(-\sum_{i=1}^{m}\lambda_{i,J}a_{i,0}^{J}R_0^\star T_0(1-\mathrm{e}^{-y_i})\Bigg) \\ & \times\prod_{l=1}^{n}\mathbb{E}\Bigg[\exp\Bigg(-R_l^\star T_l\sum_{i=1}^{m}\lambda_{i,J}a_{i,l}^{J}(1-\mathrm{e}^{-y_i})\Bigg)\Bigg|\,J,T_l\Bigg]\Bigg] \\ = {} & \mathbb{E}\Bigg[\mathrm{e}^{-\langle z,T\rangle}\exp\Bigg(-\sum_{i=1}^{m}\lambda_{i,J}a_{i,0}^{J}R_0^\star T_0(1-\mathrm{e}^{-y_i})\Bigg) \\ & \times\prod_{l=1}^{n}\exp\Bigg(-\gamma_{\alpha_l,\sigma_l}\Bigg(T_l^{\alpha_l}\Bigg(\sum_{i=1}^{m}\lambda_{i,J}a_{i,l}^{J}(1-\mathrm{e}^{-y_i})+\tau_l\Bigg)^{\alpha_l}-(T_l\tau_l)^{\alpha_l}\Bigg)\Bigg)\Bigg]. \end{aligned}$$

Now consider the Laplace transform of the distribution of the random vector $(M, T) = (M_1, \ldots, M_m, T_1, \ldots, T_n)$. Using the definition of M for $(y, z) \in [0, \infty)^{m+n}$ we observe

$$L_{(M,T)}(y,z) = \mathbb{E}\Bigg[\exp\Bigg(-\Bigg\langle y,\sum_{j\in\mathcal{J}}1_{\{J=j\}}\sum_{l=0}^{n}\sum_{h=1}^{P_{j,l}}\sum_{k=1}^{Y_{j,l,h}}B_{j,l,h,k}\Bigg\rangle\Bigg)\,\mathrm{e}^{-\langle z,T\rangle}\Bigg].$$

Partitioning J yields

$$L_{(M,T)}(y,z) = \sum_{j\in\mathcal{J}}\mathbb{E}\Bigg[1_{\{J=j\}}\,\mathrm{e}^{-\langle z,T\rangle}\exp\Bigg(-\sum_{l=0}^{n}\sum_{h=1}^{P_{j,l}}\sum_{k=1}^{Y_{j,l,h}}\langle y,B_{j,l,h,k}\rangle\Bigg)\Bigg].$$

Using that $\{Y_{j,l,h}\}_{h\in\mathbb{N}}$ and $\{B_{j,l,h,k}\}_{h,k\in\mathbb{N}}$ are i.i.d. and independent of $(P_{j,0},\ldots,P_{j,n})$ for $j\in\mathcal{J}$ and $l\in\{0,\ldots,n\}$ and that $\{Y_{j,l,h}\}_{h\in\mathbb{N}}$ are independent of $\{B_{j,l,h,k}\}_{h,k\in\mathbb{N}}$, we observe

$$\begin{aligned} L_{(M,T)}(y,z) &= \sum_{j\in\mathcal{J}} \mathbb{E}\Big[1_{\{J=j\}}\,\mathrm{e}^{-\langle z,T\rangle}\prod_{l=0}^{n}\Big(\mathbb{E}\Big[\exp\Big(-\sum_{k=1}^{Y_{j,l,1}}\langle y,B_{j,l,1,k}\rangle\Big)\Big]\Big)^{P_{j,l}}\Big] \\ &= \sum_{j\in\mathcal{J}} \mathbb{E}\Big[1_{\{J=j\}}\,\mathrm{e}^{-\langle z,T\rangle}\prod_{l=0}^{n}\Big(\mathbb{E}\Big[\big(\mathbb{E}\big[\exp(-\langle y,B_{j,l,1,1}\rangle)\big]\big)^{Y_{j,l,1}}\Big]\Big)^{P_{j,l}}\Big]. \end{aligned}$$

By Equation (4.21) and due to Remark 2.11 we obtain

$$L_{(M,T)}(y,z) = \sum_{j\in\mathcal{J}} \mathbb{E}\Big[1_{\{J=j\}}\,\mathrm{e}^{-\langle z,T\rangle}\prod_{l=0}^{n}\Big(\mathbb{E}\Big[\Big(\sum_{i=1}^{m}p_{i,j,l}\,\mathrm{e}^{-y_i}\Big)^{Y_{j,l,1}}\Big]\Big)^{P_{j,l}}\Big].$$

Conditioning on $J,T_1,\ldots,T_n$ and using the conditional independence of $P_{j,1},\ldots,P_{j,n}$ given $J,T_1,\ldots,T_n$ and the independence of $P_{j,0}$ of $J,T_1,\ldots,T_n$ for each $j\in\mathcal{J}$ and using Equations (4.18) and (4.19) yields

$$\begin{aligned} L_{(M,T)}(y,z) = \sum_{j\in\mathcal{J}} \mathbb{E}\Big[&1_{\{J=j\}}\,\mathrm{e}^{-\langle z,T\rangle}\exp\Big(-\sum_{i=1}^{m}\lambda_{i,j}a_{i,0}^{j}R_0^{\star}T_0(1-\mathrm{e}^{-y_i})\Big) \\ &\times\prod_{l=1}^{n}\exp\Big(-\delta_{j,l}T_l^{\alpha_l}\Big(1-G_{Y_{j,l,1}}\Big(\sum_{i=1}^{m}p_{i,j,l}\,\mathrm{e}^{-y_i}\Big)\Big)\Big)\Big], \end{aligned}$$

where $G_{Y_{j,l,1}}(z)$ denotes the probability-generating function of the distribution of $Y_{j,l,1}$. If $q_{j,l}=0$ for some j,l, then by Equation (4.20) $G_{Y_{j,l,1}}(z)=1$ and hence

$$\exp\Big(-\delta_{j,l}T_l^{\alpha_l}\Big(1-G_{Y_{j,l,1}}\Big(\sum_{i=1}^{m}p_{i,j,l}\,\mathrm{e}^{-y_i}\Big)\Big)\Big)=1.$$

If $q_{j,l}>0$, then by Equation (4.20) and by $q_{j,l}\sum_{i=1}^{m}p_{i,j,l}\,\mathrm{e}^{-y_i}=\frac{\sum_{i=1}^{m}\lambda_{i,j}a_{i,l}^{j}\,\mathrm{e}^{-y_i}}{\sum_{i=1}^{m}\lambda_{i,j}a_{i,l}^{j}+\tau_l}$, we have

$$\begin{aligned} &\exp\Bigg(-\delta_{j,l}T_l^{\alpha_l}\Bigg(\frac{1-\Big(\frac{\tau_l}{\sum_{i=1}^{m}\lambda_{i,j}a_{i,l}^{j}+\tau_l}\Big)^{\alpha_l}-1+\Big(1-\frac{\sum_{i=1}^{m}\lambda_{i,j}a_{i,l}^{j}\,\mathrm{e}^{-y_i}}{\sum_{i=1}^{m}\lambda_{i,j}a_{i,l}^{j}+\tau_l}\Big)^{\alpha_l}}{1-\Big(\frac{\tau_l}{\sum_{i=1}^{m}\lambda_{i,j}a_{i,l}^{j}+\tau_l}\Big)^{\alpha_l}}\Bigg)\Bigg) \\ &= \exp\Bigg(-\gamma_{\alpha_l,\sigma_l}T_l^{\alpha_l}\Bigg(\Big(\tau_l+\sum_{i=1}^{m}\lambda_{i,j}a_{i,l}^{j}\Big)^{\alpha_l}-\tau_l^{\alpha_l}\Bigg)\frac{-\tau_l^{\alpha_l}+(\tau_l+\sum_{i=1}^{m}\lambda_{i,j}a_{i,l}^{j}(1-\mathrm{e}^{-y_i}))^{\alpha_l}}{(\sum_{i=1}^{m}\lambda_{i,j}a_{i,l}^{j}+\tau_l)^{\alpha_l}-\tau_l^{\alpha_l}}\Bigg), \end{aligned}$$

hence, since $q_{j,l}=0$ implies $\sum_{i=1}^{m}\lambda_{i,j}a_{i,l}^{j}=0$,

$$\begin{aligned} L_{(M,T)}(y,z) = \sum_{j\in\mathcal{J}} \mathbb{E}\Big[&1_{\{J=j\}}\,\mathrm{e}^{-\langle z,T\rangle}\exp\Big(-\sum_{i=1}^{m}\lambda_{i,j}a_{i,0}^{j}R_0^{\star}T_0(1-\mathrm{e}^{-y_i})\Big) \\ &\times\prod_{l=0}^{n}\exp\Big(-\gamma_{\alpha_l,\sigma_l}T_l^{\alpha_l}\Big(\Big(\tau_l+\sum_{i=1}^{m}\lambda_{i,j}a_{i,l}^{j}(1-\mathrm{e}^{-y_i})\Big)^{\alpha_l}-\tau_l^{\alpha_l}\Big)\Big)\Big]. \end{aligned}$$

q.e.d.

Unfortunately, due to the parameter $\alpha_l \in (0,1)$ for $l \in \{1,\dots,n\}$, it is not possible to choose, as before, the random variables T_l with a gamma distribution or a τ-tempered α-stable distribution. By [60, Example VI.12.8] $T_l^{\alpha_l}$ is not even infinitely divisible if T_l is gamma-distributed. It is only known that powers of τ-tempered α-stable distributed random variables $T_l^{\alpha_l}$ (then the distribution of T_l is by [60, Proposition VI.5.7] and [60, Proposition VI.5.26] a generalized gamma convolution) are infinitely divisible if $\alpha_l > 1$, cf. [60, Theorem VI.5.18] and [60, Proposition VI.5.19(i)]. If $\alpha_l \in (0,1)$, then $T_l^{\alpha_l}$ is only infinitely divisible if the corresponding characteristic function has no zeros (cf. [45, Theorem 8.4.1]). Hence we need to assume that $T_l^{\alpha_l}$ has a gamma distribution or a τ-tempered α-stable distribution.

Remark 4.22. In Lemma 4.15 it is not possible to find a representation for the distribution $\mathrm{ExtNegBin}(\alpha,k,p)$ with $k > 1$ since the extended negative binomial distribution is a truncated distribution and hence takes values greater than zero for $n \in \mathbb{N}$ with $n \geq k$. But N_i has a Poisson mixture distribution for $i \in \{1,\dots,m\}$ and can take the value 1. The same holds for the extended logarithmic distribution.

4.2 Several Levels of Compounding

In this section we also construct dependence between default cause intensities by choosing a parameter of their distributions commonly and stochastically. But now we engage in an iterative mixture of distributions and the respective compound distributions that have the same law. We present two basic results with a general alternative representation. The first lemma is based on the fact that a Poisson-gamma mixture distribution is a negative binomial distribution.

Lemma 4.23. *Let M and N be $\mathbb{R}^m$-valued random vectors and let $\lambda > 0$. Let T be a real-valued random variable and assume that the characteristic function of the conditional distribution of N given T satisfies*

$$\varphi_{N|T}(s) \overset{a.s.}{=} \exp\bigl(-\lambda T(1-\varphi_M(s))\bigr), \qquad s \in \mathbb{R}^m, \tag{4.24}$$

where $\varphi_M(s)$ denotes the characteristic function of the distribution of M. Given $\alpha > 0$ and $p \in (0,1)$, assume that $T \sim \mathrm{Gamma}\bigl(\alpha, \lambda\frac{1-p}{p}\bigr)$ and $L \sim \mathrm{NegBin}(\alpha,p)$. Define

$$M' = \sum_{l=1}^{L} \tilde{M}_l,$$

where $\{\tilde{M}_l\}_{l\in\mathbb{N}}$ is a sequence independent of L, consisting of i.i.d. $\mathbb{R}^m$-valued random vectors with $\mathcal{L}\bigl(\tilde{M}_1\bigr) = \mathcal{L}(M)$. Then M' and N are equal in distribution.

Proof. For the proof we apply Remark 2.1. Consider first the Laplace transform of the distribution of the random vector $N = (N_1,\dots,N_m)$. Condition on T and use Equation (4.24) with $s \in \mathbb{R}^m$

$$\varphi_N(s) = \mathbb{E}\Bigl[\prod_{i=1}^{m} \mathrm{e}^{\mathrm{i}\, s_i N_i}\Bigr] = \mathbb{E}\Bigl[\mathbb{E}\Bigl[\prod_{i=1}^{m} \mathrm{e}^{\mathrm{i}\, s_i N_i}\Bigm| T\Bigr]\Bigr] = \mathbb{E}\bigl[\exp\bigl(-\lambda T(1-\varphi_M(s))\bigr)\bigr]\,.$$

By the distribution of T and canceling λ/p we observe

$$\varphi_N(s) = \Bigl(\frac{\lambda\frac{1-p}{p}}{\lambda\frac{1-p}{p} + \lambda(1-\varphi_M(s))}\Bigr)^{\alpha} = \Bigl(\frac{1-p}{1-p\varphi_M(s)}\Bigr)^{\alpha}.$$

The characteristic function of the distribution of M', where we use the notation $\tilde{M}_l = (\tilde{M}_{1,l}, \dots, \tilde{M}_{m,l})$, is computed as follows for $s \in \mathbb{R}^m$

$$\varphi_{M'}(s) = \mathbb{E}\Big[\prod_{i=1}^{m} \mathrm{e}^{\mathrm{i}\, s_i \sum_{l=1}^{L} \tilde{M}_{i,l}}\Big] = \mathbb{E}\Big[\exp\Big(\sum_{l=1}^{L}\sum_{i=1}^{m} \tilde{M}_{i,l}\,\mathrm{i}\, s_i\Big)\Big].$$

The sequence $\{\tilde{M}_l\}_{l\in\mathbb{N}}$ is independent of L and i.i.d., and by the distribution of L we have

$$\varphi_{M'}(s) = \mathbb{E}\Big[\Big(\mathbb{E}\Big[\exp\Big(\sum_{i=1}^{m} \tilde{M}_{i,1}\,\mathrm{i}\, s_i\Big)\Big]\Big)^{L}\Big] = \Big(\frac{1-p}{1-p\varphi_M(s)}\Big)^{\alpha},$$

which completes the proof. q.e.d.

A further iteration is given in the next lemma:

Lemma 4.25. *Let M and N be $\mathbb{R}^m$-valued random vectors and let $\lambda > 0$. Let T be a real-valued random variable and let T' be a strictly positive random variable. Assume that the characteristic function of the conditional distribution of N given (T, T') satisfies*

$$\varphi_{N\,|\,(T,T')}(s) \overset{a.s.}{=} \exp\big(-\lambda T(1-\varphi_M(s))\big), \qquad s\in\mathbb{R}^m, \tag{4.26}$$

where $\varphi_M(s)$ denotes the characteristic function of the distribution of M. Let $q \in (0,1)$. Assume that the conditional distribution of T given T' satisfies

$$\mathcal{L}\big(T\,\big|\,T'\big) \overset{a.s.}{=} \mathrm{Gamma}\Big(\frac{-T'}{\ln(1-q)}, \lambda\frac{1-q}{q}\Big). \tag{4.27}$$

Let $L \sim \mathrm{Log}(q)$. Let $\{\tilde{M}_l\}_{l\in\mathbb{N}}$ be a sequence independent of L, consisting of $\mathbb{R}^m$-valued i.i.d. random vectors with $\mathcal{L}\big(\tilde{M}_1\big) = \mathcal{L}(M)$. Define

$$M' = \sum_{l=1}^{L} \tilde{M}_l.$$

Then

$$\varphi_{N\,|\,T'}(s) \overset{a.s.}{=} \exp\big(-T'(1-\varphi_{M'}(s))\big), \qquad s\in\mathbb{R}^m.$$

Proof. Let us first consider the characteristic function of the distribution of the random vector M' with $s \in \mathbb{R}^m$

$$\varphi_{M'}(s) = \mathbb{E}\Big[\prod_{i=1}^{m} \mathrm{e}^{\mathrm{i}\, s_i \sum_{l=1}^{L} \tilde{M}_{i,l}}\Big] = \mathbb{E}\Big[\exp\Big(\sum_{l=1}^{L}\sum_{i=1}^{m} \tilde{M}_{i,l}\,\mathrm{i}\, s_i\Big)\Big].$$

The sequence $\{\tilde{M}_l\}_{l\in\mathbb{N}}$ is i.i.d. and independent of L, hence by the distribution of L we obtain

$$\varphi_{M'}(s) = \mathbb{E}\Big[\Big(\mathbb{E}\Big[\exp\Big(\sum_{i=1}^{m} \tilde{M}_{i,1}\,\mathrm{i}\, s_i\Big)\Big]\Big)^{L}\Big] = \frac{\ln(1-q\varphi_M(s))}{\ln(1-q)}. \tag{4.28}$$

The characteristic function of the conditional distribution of $N = (N_1, \dots, N_m)$ given T' for $s \in \mathbb{R}^m$ is computed as follows: Condition on (T, T') and use Equation (4.26)

$$\begin{aligned}\varphi_{N\,|\,T'}(s) &\overset{\text{a.s.}}{=} \mathbb{E}\Big[\prod_{i=1}^{m} \mathrm{e}^{\mathrm{i}\, s_i N_i}\,\Big|\,T'\Big] \overset{\text{a.s.}}{=} \mathbb{E}\Big[\mathbb{E}\Big[\prod_{i=1}^{m} \mathrm{e}^{\mathrm{i}\, s_i N_i}\,\Big|\,T,T'\Big]\,\Big|\,T'\Big]\\ &\overset{\text{a.s.}}{=} \mathbb{E}\big[\exp\big(-\lambda T(1-\varphi_M(s))\big)\,\big|\,T'\big].\end{aligned}$$

Using Equation (4.27) and canceling λ/q

$$\varphi_{N|T'}(s) \overset{\text{a.s.}}{=} \left(\frac{\lambda\frac{1-q}{q}}{\lambda\frac{1-q}{q}+\lambda(1-\varphi_M(s))}\right)^{\frac{-T'}{\ln(1-q)}} \overset{\text{a.s.}}{=} \left(\frac{1-q}{1-q\varphi_M(s)}\right)^{\frac{-T'}{\ln(1-q)}}.$$

Rewriting this as an exponential term and inserting Equation (4.28)

$$\begin{aligned}\varphi_{N|T'}(s) &\overset{\text{a.s.}}{=} \exp\Big(\frac{-T'}{\ln(1-q)}\ln\Big(\frac{1-q}{1-q\varphi_M(s)}\Big)\Big)\\ &\overset{\text{a.s.}}{=} \exp\Big(-T'\Big(\frac{\ln(1-q)-\ln(1-q\varphi_M(s))}{\ln(1-q)}\Big)\Big) = \exp\big(-T'(1-\varphi_{M'}(s))\big),\end{aligned}$$

which completes the proof. q.e.d.

The characteristic function of the distribution of M opens a wide variety of distributions from which to choose. We give a few examples:

Example 4.29. (a) It is possible to consider several lines of business and thus let M be multivariate, e.g. $M \sim \text{Multinomial}(n; p_1, \ldots, p_m)$ with $n \in \mathbb{N}$.

(b) If $M \equiv 1$, the characteristic function $\varphi_{N|T}(s)$ in Equations (4.24) and (4.26) turns out to be the characteristic function of a Poisson mixture distribution a.s.

Now we consider a special case with iterated continuous mixtures. We apply the results of the Lemmata 4.23 and 4.25. Lemma 4.23 describes the transformation in the first step of iteration, and Lemma 4.25 indicates how to continue this iteration. This provides us with iterated compound sums for random claim numbers, which allows for the application of Panjer's recursion.

For reasons of completeness and since it is not readily clear how the transformation with M in N works, we give the complete proof of an application of these lemmata.

Corollary 4.30. *Fix $k, m \in \mathbb{N}$ and $\alpha > 0$ and $p, p_1, \ldots, p_m, q_2, \ldots, q_k \in (0,1)$. Let $l_1, \ldots, l_m \in (0,1)$ and $l_1 = 1$ if and only if $m = 1$ with $\sum_{i=1}^m l_i = 1$. Then we define two $\mathbb{N}_0^m$-valued random vectors N and M as follows:*

(a) Define $M = (M_1, \ldots, M_m)$ by

$$M_i = \sum_{j_1=1}^{M^1}\sum_{j_2=1}^{M^k_{j_1}}\cdots\sum_{j_k=1}^{M^2_{j_1,\ldots,j_{k-1}}} B_{i,j_1,\ldots,j_k}K_{i,j_1,\ldots,j_k}, \qquad i \in \{1, \ldots, m\},$$

where

$$M^1 \sim \text{NegBin}(\alpha, p) \tag{4.31}$$

and for each $j \in \{2, \ldots, k\}$ let $\{M^j_{j_1,\ldots,j_{k-j+1}}\}_{j_1,\ldots,j_{k-j+1}\in\mathbb{N}}$ be independent collections of i.i.d. random variables such that

$$M^j_{1,\ldots,1} \sim \text{Log}(q_j)\,, \qquad j \in \{2, \ldots, k\},$$

and let $\{(B_{1,j_1,\ldots,j_k}, \ldots, B_{m,j_1,\ldots,j_k})\}_{j_1,\ldots,j_k\in\mathbb{N}}$ be a collection of i.i.d. random vectors such that

$$B_{1,\ldots,1} \sim \text{Multinomial}(1; l_1, \ldots, l_m)\,,$$

and let $\{(K_{1,j_1,\dots,j_k},\dots,K_{m,j_1,\dots,j_k})\}_{j_1,\dots,j_k\in\mathbb{N}}$ *be a collection of i.i.d. random vectors with*

$$K_{i,1,\dots,1} \sim \operatorname{Log}(p_i)\,, \qquad i\in\{1,\dots,m\}.$$

Let all these random variables be independent of each other.

(b) Further, let the components of $N=(N_1,\dots,N_m)$ *be conditionally independent given* $\Lambda_1,\dots,\Lambda_m,T_1$ *such that for each* $i\in\{1,\dots,m\}$

$$\mathcal{L}(N_i\,|\,\Lambda_1,\dots,\Lambda_m,T_1) \overset{a.s.}{=} \mathcal{L}(N_i\,|\,\Lambda_i,T_1) \overset{a.s.}{=} \operatorname{Poisson}(p_i\Lambda_i)\,, \tag{4.32}$$

where

$$\mathcal{L}(\Lambda_i\,|\,T_1) \overset{a.s.}{=} \operatorname{Gamma}\Big(\frac{-l_i}{\ln(1-p_i)}T_1, 1-p_i\Big), \tag{4.33}$$

$$\mathcal{L}(T_{j-1}\,|\,T_j) \overset{a.s.}{=} \operatorname{Gamma}\Big(\frac{-T_j}{\ln(1-q_j)}, \frac{1-q_j}{q_j}\Big), \qquad j\in\{2,\dots,k\}, \tag{4.34}$$

$$\mathcal{L}(T_k) \overset{a.s.}{=} \operatorname{Gamma}\Big(\alpha, \frac{1-p}{p}\Big), \tag{4.35}$$

and $\Lambda_1,\dots,\Lambda_m$ *are conditionally independent given* T_1*.*

Then the random vectors M *and* N *have the same distribution.*

Proof. We apply Remark 2.1. We compute the characteristic function of the conditional distribution of N given T_1. Conditioning on $\Lambda_1,\dots,\Lambda_m,T_1$ and using the conditional independence of $N_1,\dots,N_m$ given $\Lambda_1,\dots,\Lambda_m,T_1$ and Equation (4.32) and the conditional independence of $\Lambda_1,\dots,\Lambda_m$ given T_1 for $s\in\mathbb{R}^m$ gives

$$\varphi_{N\,|\,T_1}(s) \overset{\text{a.s.}}{=} \mathbb{E}\Big[\prod_{i=1}^m \mathbb{E}\big[\mathrm{e}^{\mathrm{i}\,s_iN_i}\,|\,\Lambda_i,T_1\big]\,\Big|\,T_1\Big] \overset{\text{a.s.}}{=} \prod_{i=1}^m \mathbb{E}\big[\exp\!\big(-p_i\Lambda_i(1-\mathrm{e}^{\mathrm{i}\,s_i})\big)\,|\,T_1\big]\,.$$

Using the conditional distribution of $\Lambda_1,\dots,\Lambda_m$ given T_1 given by Equation (4.33) and rewriting the characteristic function as an exponential term yields

$$\begin{aligned}\varphi_{N\,|\,T_1}(s) &\overset{\text{a.s.}}{=} \prod_{i=1}^m\Big(\frac{1-p_i}{1-p_i+p_i(1-\mathrm{e}^{\mathrm{i}\,s_i})}\Big)^{\frac{-l_i}{\ln(1-p_i)}T_1}\\ &\overset{\text{a.s.}}{=} \exp\Big(-T_1\sum_{i=1}^m\frac{l_i}{\ln(1-p_i)}\ln\Big(\frac{1-p_i}{1-p_i+p_i(1-\mathrm{e}^{\mathrm{i}\,s_i})}\Big)\Big).\end{aligned}$$

Rewriting the logarithm and using $\sum_{i=1}^m l_i=1$ provides

$$\begin{aligned}\varphi_{N\,|\,T_1}(s) &\overset{\text{a.s.}}{=} \exp\Big(-T_1\sum_{i=1}^m\frac{l_i}{\ln(1-p_i)}\big(\ln(1-p_i)-\ln(1-p_i\,\mathrm{e}^{\mathrm{i}\,s_i})\big)\Big)\\ &\overset{\text{a.s.}}{=} \exp\Big(-T_1\Big(1-\sum_{i=1}^m\frac{\ln(1-p_i\,\mathrm{e}^{\mathrm{i}\,s_i})}{\ln(1-p_i)}l_i\Big)\Big).\end{aligned}$$

Note that by the theorem of total probability and the independence of $\{B_{i,j}\}_{j\in\mathbb{N}}$ and $\{K_{i,j}\}_{j\in\mathbb{N}}$ for $i\in\{1,\dots,m\}$ it follows by the distribution of $K_{i,1}$ and $B_{i,1}$ that for the characteristic function of the distribution of the random vector $BK_1=(B_{1,1}K_{1,1},\dots,B_{m,1}K_{m,1})$

with $s \in \mathbb{R}^m$

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(\sum_{i=1}^m B_{i,1}K_{i,1}\,\mathrm{i}\,s_i\Big)\Big] &= \sum_{l=1}^m \mathbb{E}\Big[\exp\Big(\sum_{i=1}^m B_{i,1}K_{i,1}\,\mathrm{i}\,s_i\Big)\,\Big|\,B_{l,1}=1\Big]\ \mathbb{P}[B_{l,1}=1]\\ &= \sum_{i=1}^m \frac{\ln(1-p_i\,\mathrm{e}^{\mathrm{i}\,s_i})}{\ln(1-p_i)} l_i\end{aligned}$$

holds. Hence

$$\varphi_{N\,|\,T_1}(s) \overset{\text{a.s.}}{=} \exp\big(-T_1(1-\varphi_{BK_1}(s))\big). \tag{4.36}$$

Let now $k=1$ and $M=(M_1,\dots,M_m)$ with $M_i=\sum_{j_1=1}^{M^1} B_{i,j_1}K_{i,j_1}$ for $i\in\{1,\dots,m\}$. Using Equation (4.35), an application of Lemma 4.23 with $\lambda=1$ immediately shows that M and N have the same distribution.

We now prove the corollary for $k\geq 2$. Let M and N be in the demanded form. Equation (4.36) still holds and we apply Lemma 4.25 $k-1$ times with $\lambda=1$ using Equation (4.34). Thus we obtain

$$\varphi_{N\,|\,T_k}(s) \overset{\text{a.s.}}{=} \exp(-T_k(1-\varphi_H(s))),$$

where $\{H_{j_1}\}_{j_1\in\mathbb{N}}=\{(H_{1,j_1},\dots,H_{m,j_1})\}_{j_1\in\mathbb{N}}$ is a collection of i.i.d. random vectors such that

$$H_{i,j_1}=\sum_{j_2=1}^{M_1^k}\cdots\sum_{j_k=1}^{M^2_{1,\dots,j_{k-1}}} B_{i,1,\dots,j_k}K_{i,1,\dots,j_k}, \qquad i\in\{1,\dots,m\}.$$

Then apply Lemma 4.23 with $\lambda=1$ using Equation (4.35) and setting $M_i=\sum_{j_1=1}^{M^1} H_{i,j_1}$ for $i\in\{1,\dots,m\}$, hence M and N have the same distribution. q.e.d.

Example 4.37. This corollary includes the result of Giese [23, Section 10.2] as a special case. Note that we use the the term mixture where Giese uses compound.

Again, Giese chooses the default cause intensities to be gamma-distributed. By mixing m independent default cause intensities $\Lambda_1,\dots,\Lambda_m$ with a common gamma-distributed random variable, he constructs another dependence structure. This background variable is denoted by T and is $\text{Gamma}\big(\frac{1}{\hat\sigma^2},\hat\sigma^2\big)$-distributed. According to his calibration condition [23, Equation (10.2)] we have $\hat\sigma^2\in(0,1)$. Hence Giese obtains the following mixture distribution

$$\mathcal{L}(\Lambda_i\,|\,T) \overset{\text{a.s.}}{=} \text{Gamma}(\alpha_i T,\beta_i)\,, \qquad i\in\{1,\dots,m\}.$$

Note that $\Lambda_1,\dots,\Lambda_m$ are independent given T. Using calibration condition [23, Equation (10.12)] we set $\beta_i\in(0,1)$ for $i\in\{1,\dots,m\}$. By [23, Equation (10.10)] with an adjustment to our parameterization of the gamma distribution we have $\alpha_i=\beta_i$ because in the Credit Risk$^+$ model the mean of the default cause intensity Λ_i is 1 for $i\in\{1,\dots,m\}$. In order to fit into our framework, we should consider claim numbers $N_1,\dots,N_m$ such that

$$\mathcal{L}(N_i\,|\,\Lambda_1,\dots,\Lambda_m) \overset{\text{a.s.}}{=} \mathcal{L}(N_i\,|\,\Lambda_i) \overset{\text{a.s.}}{=} \text{Poisson}((1-\beta_i)\Lambda_i)\,.$$

Since these default cause intensities and hence the claim numbers are no longer independent, we need to find an alternative representation with independent components but with the same distribution in order to apply Panjer's recursion.

This problem can be easily solved since it is a straightforward application of Corollary 4.30 in the case $k = 1$: We set

$$M_i = \sum_{j=1}^{M^1} B_{i,j} K_{i,j}, \qquad i \in \{1, \dots, m\},$$

where $M^1 \sim \text{NegBin}\big(\frac{1}{\hat{\sigma}^2}, \frac{1}{\hat{\sigma}^2+1}\big)$ and $\{K_{i,j}\}_{j\in\mathbb{N}}$ are m independent sequences of i.i.d. random variables with $K_{i,1} \sim \text{Log}(1-\beta_i)$ for $i \in \{1, \dots, m\}$, and $\{B_j\}_{j\in\mathbb{N}}$ is a sequence of i.i.d. random vectors such that $B_1 \sim \text{Multinomial}(1; l_1, \dots, l_m)$. The parameters $l_1, \dots, l_m$ should be determined by Equation (4.33) under the constraints $l_1 + \cdots + l_m = 1$ and $\mathbb{E}[\Lambda_i] = 1$. If a solution does not exist, the condition $\mathbb{E}[\Lambda_i] = 1$ should be released. Then the random vectors N and M have the same distribution. Thus the dependence structure between the marginal random variables (as projections) is preserved. The distribution of the sum of the random claim numbers is also the same since addition is a continuous map.

Thus we can apply Panjer's recursion in order to calculate the distribution of the random sum

$$S = \sum_{i=1}^{m} \sum_{h=1}^{N_i} X_{i,h} \stackrel{\mathrm{d}}{=} \sum_{j=1}^{M^1} \sum_{i=1}^{m} \sum_{h=1}^{B_{i,j}K_{i,j}} X_{i,h},$$

where $\{X_{i,h}\}_{h\in\mathbb{N}}$ are sequences of i.i.d. random variables independent of the random variable M and the sequence $\{K_{i,j}\}_{j\in\mathbb{N}}$ for each $i \in \{1, \dots, m\}$ which are in Panjer(a, b, k) classes. The calculation can be done by an iterated application of Panjer's recursion, cf. Algorithm 5.15.

Chapter 5

Generalizations of Recursions for Compound Distributions

In this chapter we present a generalization of Panjer's recursion. The claim sizes may depend on each other and may be multivariate. The distribution of the claim numbers may be linked by several distributions and are only in special cases in Panjer(a, b, k) classes. We present a generalization in the same manner of de Pril's recursion. We also give algorithms that show how Panjer's recursion is applicable in our model with the dependence structures given in Chapters 3 and 4. It might be suggested that the evaluation of the distribution of Equation (1.1) requires an n-fold convolution, but our algorithm, based on the ideas of [21, Section 5.5], circumvents these convolutions by an iterated application of Panjer's recursion and a convex combination. In the course of the algorithm we use the fact that a negative binomial distribution is a compound Poisson distribution. The convex combination uses the fact that the convolution of compound Poisson distributions provides another compound Poisson distribution. In the introduction of this part of the thesis we cite further works on generalizations of Panjer's recursion.

5.1 A Generalization of Panjer's Recursion and Algorithms

It is possible to relax the assumption of independent and identically distributed claim sizes. The claim sizes may be considered as an infinite exchangeable sequence of random variables. A basic approach to this idea can be found in [28], although the author does not describe the recursion formula in detail. We only demand that the claim sizes are i.i.d. given a certain σ-algebra $\mathcal{F}$. Thus we prove a generalization of Panjer's recursion, where the claim sizes are allowed to depend on each other by a dependence structure that is not further specified. In addition, we let the claim sizes be multivariate. The distribution of the claim numbers is linked by Equation (5.2). We slightly extend the notation of [61], cf. also [57]. For this, we use the following notation: if $j \in \mathbb{N}_0^d$, then we write $0 < j$ if $0 < j_r$ for each $r \in \{1, \ldots, d\}$, and we write $0 \le j$ if $0 \le j_r$ for each $r \in \{1, \ldots, d\}$.

Theorem 5.1. *Fix $l \in \mathbb{N}$. Let $\{X_h\}_{h\in\mathbb{N}}$ be $\mathbb{N}_0^d$-valued random vectors and let $\mathcal{F}$ be a σ-algebra such that $\{X_h\}_{h\in\mathbb{N}}$ given $\mathcal{F}$ is a sequence of i.i.d. random vectors. Let $\{q_n\}_{n\in\mathbb{N}_0}$ and $\{\tilde{q}_{i,n}\}_{n\in\mathbb{N}_0}$ denote the conditional probability distributions given $\mathcal{F}$ of $\mathbb{N}_0$-valued random variables N and $\tilde{N}_i$ for $i \in \{1, \ldots, l\}$, respectively, which are conditionally independent of $\{X_h\}_{h\in\mathbb{N}}$ given $\mathcal{F}$. Let $\{p_n\}_{n\in\mathbb{N}_0^d}$ and $\{\tilde{p}_{i,n}\}_{n\in\mathbb{N}_0^d}$ denote the conditional probability* *distributions given $\mathcal{F}$ of the random sums $S = X_1 + \cdots + X_N$ and $\tilde{S}_{(i)} = X_1 + \cdots + X_{\tilde{N}_i}$ for $i \in \{1, \dots, l\}$, respectively.*

(a) Assume that there exist $k \in \mathbb{N}_0$ and $\mathbb{R}$-valued $\mathcal{F}$-measurable functions $a_1, \dots, a_l, b_1, \dots, b_l$ such that

$$q_n \overset{a.s.}{=} \sum_{i=1}^{l} \Big(a_i + \frac{b_i}{n}\Big) \tilde{q}_{i,n-i} \qquad \textit{for all } n \in \mathbb{N} \textit{ with } n \geq k + l \tag{5.2}$$

and all the probabilities given $\mathcal{F}$ not used on the right side of Equation (5.2) are zero, i.e.

$$\tilde{q}_{i,0} \overset{a.s.}{=} \dots \overset{a.s.}{=} \tilde{q}_{i,k+l-i-1} \overset{a.s.}{=} 0 \qquad \textit{for all } i \in \{1, \dots, \min\{l, k+l-1\}\}. \tag{5.3}$$

Then

$$p_0 \overset{a.s.}{=} \mathbb{E}\big[\big(\mathbb{P}[X_1 = 0 \,|\, \mathcal{F}]\big)^N \,\big|\, \mathcal{F}\big]$$

with the convention $0^0 := 1$ and

$$p_n \overset{a.s.}{=} \sum_{m=1}^{k+l-1} \mathbb{P}[S_m = n \,|\, \mathcal{F}]\, q_m + \sum_{i=1}^{l} \sum_{\substack{0 \leq j \leq n \\ j \in \mathbb{N}_0^d}} \Big(a_i + \frac{b_i \langle c_n, j\rangle}{i \langle c_n, n\rangle}\Big) \mathbb{P}[S_i = j \,|\, \mathcal{F}]\, \tilde{p}_{i,n-j} \tag{5.4}$$

for all $n \in \mathbb{N}_0^d \setminus \{0\}$ and c_n an $\mathbb{R}^d$-valued $\mathcal{F}$-measurable function with $\langle c_n, n\rangle \neq 0$ a.s. and $S_i = X_1 + \cdots + X_i$.

(b) Assume that there exist $\mathcal{F}$-measurable functions $\nu_1, \dots, \nu_l \in [0, 1]$ a.s. with $\nu_1 + \cdots + \nu_l \leq 1$ a.s. such that $q_n \overset{a.s.}{=} \sum_{i=1}^{l} \nu_i \tilde{q}_{i,n}$ for all $n \in \mathbb{N}$. Then $p_n \overset{a.s.}{=} \sum_{i=1}^{l} \nu_i \tilde{p}_{i,n}$ for all $n \in \mathbb{N}_0^d \setminus \{0\}$.

Proof. The proof is an adaptation of the proofs of [21, Theorem 4.5] which contributes the claim numbers being linked by Equation (5.2) and a multivariate Panjer recursion given in [61, Theorem 1]. The idea of the dependence between the claim sizes has already been considered in [28].

In order to be precise we should also distinguish between a zero-random variable and a set where the random variable is zero. For reasons of readability this is omitted. Since $\{X_h\}_{h \in \mathbb{N}}$ and N are conditionally independent given $\mathcal{F}$ and since $\{X_h\}_{h \in \mathbb{N}}$ given $\mathcal{F}$ is a sequence of i.i.d. random vectors, we obtain

$$\begin{aligned}
\mathbb{P}[S = 0 \,|\, \mathcal{F}] &\overset{\text{a.s.}}{=} \mathbb{P}[N = 0 \,|\, \mathcal{F}] + \sum_{i=1}^{\infty} \mathbb{P}[X_1 = 0, \dots, X_i = 0 \,|\, \mathcal{F}]\; \mathbb{P}[N = i \,|\, \mathcal{F}] \\
&\overset{\text{a.s.}}{=} \mathbb{P}[N = 0 \,|\, \mathcal{F}] + \sum_{i=1}^{\infty} \mathbb{P}[X_1 = 0 \,|\, \mathcal{F}]^i\, \mathbb{P}[N = i \,|\, \mathcal{F}] \overset{\text{a.s.}}{=} \mathbb{E}\big[\big(\mathbb{P}[X_1 = 0 \,|\, \mathcal{F}]\big)^N \,\big|\, \mathcal{F}\big].
\end{aligned}$$

Let now $n \in \mathbb{N}_0^d \setminus \{0\}$. For fixed $i \in \{1, \dots, l\}$ and for every $m \in \mathbb{N}$ with $m \geq i$ let $S_m = X_1 + \cdots + X_m = S_{m-i} + S_{i,m}$ with $S_{i,m} = X_{m-i+1} + \cdots + X_m$. Since $\{X_h\}_{h \in \mathbb{N}}$ given $\mathcal{F}$ are i.i.d., we obtain

$$\begin{aligned}
1_{\{S_m = n\}} \langle c_n, n\rangle &\overset{\text{a.s.}}{=} \mathbb{E}\big[\langle c_n, S_m\rangle 1_{\{S_m = n\}} \,\big|\, \mathcal{F}\big] \overset{\text{a.s.}}{=} \sum_{h=1}^{m} \mathbb{E}\big[\langle c_n, X_h\rangle 1_{\{S_m = n\}} \,\big|\, \mathcal{F}\big] \\
&\overset{\text{a.s.}}{=} m\, \mathbb{E}\big[\langle c_n, X_1\rangle 1_{\{S_m = n\}} \,\big|\, \mathcal{F}\big] \overset{\text{a.s.}}{=} m\, \mathbb{E}\Big[\frac{\langle c_n, S_{i,m}\rangle}{i} 1_{\{S_m = n\}} \,\Big|\, \mathcal{F}\Big].
\end{aligned}$$

Hence after taking the conditional expected value given $\mathcal{F}$, we obtain

$$\begin{aligned}\mathbb{P}[S_m = n \,|\, \mathcal{F}] \Big(a_i + \frac{b_i}{m}\Big) &\overset{\text{a.s.}}{=} \mathbb{E}\Big[a_i + \frac{b_i\langle c_n, S_{i,m}\rangle}{i\langle c_n, n\rangle} 1_{\{S_m=n\}} \,\Big|\, \mathcal{F}\Big] \\ &\overset{\text{a.s.}}{=} \sum_{\substack{0\le j\le n \\ j\in\mathbb{N}_0^d}} \Big(a_i + \frac{b_i\langle c_n, j\rangle}{i\langle c_n, n\rangle}\Big)\, \mathbb{P}[S_{i,m} = j, S_m = n \,|\, \mathcal{F}]. \qquad (5.5)\end{aligned}$$

For every $m \ge i$ the sums S_{m-i} and $S_{i,m}$ are conditionally independent given $\mathcal{F}$ since $X_1, \dots, X_m$ are conditionally independent given $\mathcal{F}$. By an analogous argument $S_{i,m}$ and S_i have the same distribution given $\mathcal{F}$ because both $S_{i,m}$ and S_i have i summands, hence

$$\begin{aligned}\mathbb{P}[S_{i,m} = j, S_m = n \,|\, \mathcal{F}] &\overset{\text{a.s.}}{=} \mathbb{P}[S_{i,m} = j, S_{m-i} = n - j \,|\, \mathcal{F}] \\ &\overset{\text{a.s.}}{=} \mathbb{P}[S_i = j \,|\, \mathcal{F}]\; \mathbb{P}[S_{m-i} = n - j \,|\, \mathcal{F}]. \qquad (5.6)\end{aligned}$$

Now we are prepared for the actual proof. Applying the conditional monotone convergence theorem and using the conditional independence of S_m and N given $\mathcal{F}$ for every $m \in \mathbb{N}$ yields

$$\mathbb{P}[S = n \,|\, \mathcal{F}] \overset{\text{a.s.}}{=} \sum_{m\in\mathbb{N}} \mathbb{P}[S_m = n, N = m \,|\, \mathcal{F}] \overset{\text{a.s.}}{=} \sum_{m=1}^{k+l-1} \mathbb{P}[S_m = n \,|\, \mathcal{F}]\, q_m + A_n, \qquad (5.7)$$

where the abbreviation A_n can be rewritten using Equation (5.2)

$$A_n := \sum_{m=k+l}^{\infty} \mathbb{P}[S_m = n \,|\, \mathcal{F}]\, q_m \overset{\text{a.s.}}{=} \sum_{m=k+l}^{\infty} \sum_{i=1}^{l} \mathbb{P}[S_m = n \,|\, \mathcal{F}] \Big(a_i + \frac{b_i}{m}\Big) \tilde{q}_{i,m-i}.$$

Inserting Equation (5.6) into Equation (5.5) and then into the above equation, we get

$$\begin{aligned}A_n &\overset{\text{a.s.}}{=} \sum_{m=k+l}^{\infty} \sum_{i=1}^{l} \sum_{\substack{0\le j\le n \\ j\in\mathbb{N}_0^d}} \Big(a_i + \frac{b_i\langle c_n, j\rangle}{i\langle c_n, n\rangle}\Big) \mathbb{P}[S_i = j \,|\, \mathcal{F}]\; \mathbb{P}[S_{m-i} = n - j \,|\, \mathcal{F}]\, \tilde{q}_{i,m-i} \\ &\overset{\text{a.s.}}{=} \sum_{i=1}^{l} \sum_{\substack{0\le j\le n \\ j\in\mathbb{N}_0^d}} \Big(a_i + \frac{b_i\langle c_n, j\rangle}{i\langle c_n, n\rangle}\Big) \mathbb{P}[S_i = j \,|\, \mathcal{F}] \sum_{m=k+l}^{\infty} \mathbb{P}[S_{m-i} = n - j \,|\, \mathcal{F}]\, \tilde{q}_{i,m-i},\end{aligned}$$

where the interchange of the summations is admissible because the series in the second line converge for every $i \in \{1, \dots, l\}$ and $j \in \mathbb{N}_0^d$ with $0 \le j \le n$. This holds as follows: due to Equation (5.3) and an index shift and by the conditional independence of $\{X_h\}_{h\in\mathbb{N}}$ and N given $\mathcal{F}$

$$\begin{aligned}\sum_{m=k+l}^{\infty} \mathbb{P}[S_{m-i} = n - j \,|\, \mathcal{F}]\, \tilde{q}_{i,m-i} &\overset{\text{a.s.}}{=} \sum_{m=i}^{\infty} \mathbb{P}[S_{m-i} = n - j \,|\, \mathcal{F}]\, \tilde{q}_{i,m-i} \\ &\overset{\text{a.s.}}{=} \sum_{m=0}^{\infty} \mathbb{P}[S_m = n - j, \tilde{N}_i = m \,|\, \mathcal{F}] \overset{\text{a.s.}}{=} \mathbb{P}[\tilde{S}_{(i)} = n - j \,|\, \mathcal{F}] \overset{\text{a.s.}}{=} \tilde{p}_{i,n-j}.\end{aligned}$$

Inserting this into A_n and A_n into Equation (5.7) provides the first part of the claim.

The second part of the theorem can be proven as in the proof of [21, Theorem 4.5] with respect to $\mathcal{F}$. q.e.d.

Note that if $l = 1$, this theorem is the multivariate Panjer recursion given $\mathcal{F}$ with claim numbers in a Panjer(a, b, k) class. We observe dependence between the claim sizes in two types: The components of the random vectors $\{(X_{1,h}, \ldots, X_{d,h})\})_{h\in\mathbb{N}}$ may depend on each other, and the X_h may depend on each other for different $h \in \mathbb{N}$ since $\{X_h\}_{h\in\mathbb{N}}$ only needs to be conditionally independent given $\mathcal{F}$. If the σ-algebra $\mathcal{F}$ is finite, then the unconditional distribution of S can easily be computed numerically, cf. following remark.

Remark 5.8. The unconditional distribution $\mathbb{P}[S = n]$ can be obtained by using the tower property for expected values:

$$\mathbb{P}[S = n] \,=\, \mathbb{E}[\,\mathbb{P}[S = n\,|\,\mathcal{F}]\,]\,.$$

Hence we also need to insert Equation (5.4) into the above equation, which can be computed if $\langle c_n, n\rangle \neq 0$ a.s.

A possible application of the multivariate Panjer recursion could be a special case of the extended CreditRisk$^+$ model. If a default occurs, this default may lead to future payments that should be considered. Then the extended CreditRisk$^+$ model is a multiperiod model. Thus it makes sense to allocate the development of possible claims to a multivariate random variable X_1. This model is insofar appropriate for such requirements since future payments (may) depend on present or past payments, and the components of X_1 need not be independent. This framework is also useful in the collective risk model in the context of insurances. Health insurances provide a typical example of a first claim that causes a payment and, depending on the recovery process of the insured person, could lead to future payments. The claim of a pension also causes future payments of an initially unknown duration.

We now give algorithms for the different dependence scenarios we introduced and show how Panjer's recursion can be used for the evaluation of the distribution of S. They are based on the knowledge that a negative binomial distribution can be written as a compound Poisson distribution where the severity distribution is a logarithmic distribution, cf. also [21, Section 5.5].

We adapt the algorithm in [21, Section 5.5] for the evaluation of the random sum S in Equation (1.1). Panjer's recursion in Theorem 5.1 simplifies a lot since we consider $\{X_{i,h}\}_{h\in\mathbb{N}}$ to be i.i.d. and univariate, hence we do not consider a σ-algebra $\mathcal{F}$.

Algorithm 5.9. Consider the setting of Theorem 3.1. Assume that $R_1, \ldots, R_n$ are independent now. Then $Q_{j,0}, \ldots, Q_{j,n}$ are independent for each $j \in \mathcal{J}$. Assume further that according to the extended CreditRisk$^+$ model the random variables $R_1, \ldots, R_n$ have a distribution such that $R_l \sim \text{Gamma}(\alpha_l, \beta_l)$ with $\alpha_l, \beta_l > 0$ for $l \in \{1, \ldots, n\}$. We apply Theorem 3.1 and adopt the equivalent representation of the claim numbers into the random sum S in Equation (1.1):

$$S = \sum_{i=1}^{m}\sum_{h=1}^{N_i} X_{i,h} \overset{\mathrm{d}}{=} \sum_{i=1}^{m}\sum_{h=1}^{M_i} X_{i,h} = \sum_{j\in\mathcal{J}} 1_{\{J=j\}} \sum_{l=0}^{n}\sum_{h=1}^{Q_{j,l}}\sum_{i=1}^{m} B_{i,j,l,h} X_{i,l,h},$$

where $\{X_{i,h}\}_{h\in\mathbb{N}}$ and $\{X_{i,l,h}\}_{i\in\mathbb{N}}$ with $l \in \{0, \ldots, n\}$ and $i \in \{1, \ldots, m\}$ are independent sequences of i.i.d. $\mathbb{N}_0$-valued random variables with identical distributions. Hence, the distribution of S can be calculated iteratively: Let for $h \in \mathbb{N}$

$$W_{(j,l,h)} = \sum_{i=1}^{m} B_{i,j,l,h} X_{i,l,h}, \qquad j \in \mathcal{J}, \quad l \in \{0, \ldots, n\}.$$

Since $\{B_{i,j,l,h}X_{i,l,h}\}_{h\in\mathbb{N}}$ is i.i.d. and the distribution of $B_{j,l,1}$ is given in Equation (3.2), the distribution of $W_{(j,l,1)}$ can be calculated directly by the law of total probability for $p \in \mathbb{N}_0$ and $j \in \mathcal{J}$ and $l \in \{0, \dots, n\}$, thus

$$\mathbb{P}[W_{(j,l,1)} = p] = \sum_{i=1}^{m} \mathbb{P}[X_{i,l,1} = p]\, \mathbb{P}[B_{i,j,l,1} = 1] = \sum_{i=1}^{m} \mathbb{P}[X_{i,l,1} = p]\, p_{i,j,l} \tag{5.10}$$

under the natural assumption that $\{B_{i,j,l,h}\}_{h\in\mathbb{N}}$ and $\{X_{i,l,h}\}_{h\in\mathbb{N}}$ are independent. Let

$$q_{j,l} = \frac{\sum_{i=1}^{m} \lambda_{i,j} a_{i,l}^{j}}{\beta_l + \sum_{i=1}^{m} \lambda_{i,j} a_{i,l}^{j}}, \qquad j \in \mathcal{J}, \quad l \in \{1, \dots, n\}.$$

By Lemma 2.8 with $m = 1$ in that framework and $T \equiv 1$ for each $j \in \mathcal{J}$ and $l \in \{1, \dots, n\}$ the distribution of $Q_{j,l}$ is given by

$$\mathcal{L}(Q_{j,l}) = \mathrm{NegBin}(\alpha_l, q_{j,l})\,.$$

If $\sum_{i=1}^{m} \lambda_{i,j} a_{i,l}^{j} = 0$, then according to Equation (3.3) we have $\mathcal{L}(Q_{j,l}) = \delta_0$, hence for these j, l there is no random summation. Note that according to Remark 2.7 $Q_{j,l} \overset{\mathrm{d}}{=} \sum_{h=1}^{L_{j,l}} K_{(j,l,h)}$ for each $j \in \mathcal{J}$ and $l \in \{1, \dots, n\}$ with

$$Q_{j,l} \sim \mathrm{CPoi}\Big(\alpha_l \ln\Big(\frac{1}{1 - q_{j,l}}\Big), \mathrm{Log}(q_{j,l})\Big). \tag{5.11}$$

Define now according to [21, Remark 5.11] for $r \in \mathbb{N}$

$$S_{(j,l,r)} = \sum_{h=K_{(j,l,1)}+\dots+K_{(j,l,r-1)}+1}^{K_{(j,l,1)}+\dots+K_{(j,l,r)}} W_{(j,l,h)}, \qquad j \in \mathcal{J}, \quad l \in \{1, \dots, n\}. \tag{5.12}$$

According to [21, Remark 5.11] the sequence $\{S_{(j,l,r)}\}_{r\in\mathbb{N}}$ is i.i.d. and since $K_{(j,l,1)}$ is in a Panjer$(q_{j,l}, -q_{j,l}, 1)$ class, $\mathcal{L}\big(S_{(j,l,1)}\big)$ can be evaluated with a numerically stable Panjer recursion, cf. Theorem 5.1. Note that $q_{j,l} = 0$ is not possible in this case since by the above arguments for $\mathcal{L}(Q_{j,l})$ the distribution Poisson$(0) = \delta_0$ would suppress a random summation. For $l = 0$ and each $j \in \mathcal{J}$ we have $\mathcal{L}(Q_{j,0}) = \mathrm{Poisson}\big(\sum_{i=1}^{m} \lambda_{i,j} a_{i,0}^{j} R_0\big)$. Thus the total portfolio loss if $J = j$ can be written as

$$S_j = \sum_{h=1}^{Q_{j,0}} W_{(j,0,h)} + \sum_{l=1}^{n} \sum_{r=1}^{L_{j,l}} S_{(j,l,r)}, \qquad j \in \mathcal{J}.$$

Since $\{X_{i,0,h}\}_{h\in\mathbb{N}}, \dots, \{X_{i,n,h}\}_{h\in\mathbb{N}}$ are independent sequences, the corresponding probability-generating function provides with $G_{W_{(j,0,1)}}(z) = \mathbb{E}[z^{W_{(j,0,1)}}]$ and $G_{S_{(j,l,1)}}(z) = \mathbb{E}[z^{S_{(j,l,1)}}]$

$$\begin{aligned} G_{S_j}(z) &= \exp\Big(-\sum_{i=1}^{m} \lambda_{i,j} a_{i,0}^{j} R_0 (1 - G_{W_{(j,0,1)}}(z))\Big) \\ &\quad \times \prod_{l=1}^{n} \exp\Big(-\alpha_l \ln\Big(\frac{1}{1 - q_{j,l}}\Big)(1 - G_{S_{(j,l,1)}}(z))\Big) \\ &= \exp(-\mu_j(1 - G_j(z))) \end{aligned}$$

for $|z| \leq 1$ with

$$\mu_j = \sum_{i=1}^{m} \lambda_{i,j} a_{i,0}^j R_0 + \sum_{l=1}^{n} \alpha_l \ln\Big(\frac{1}{1-q_{j,l}}\Big),$$

and

$$G_j(z) = \frac{\sum_{i=1}^m \lambda_{i,j} a_{i,0}^j R_0}{\mu_j} G_{W_{(j,0,1)}}(z) + \sum_{l=1}^{n} \frac{\alpha_l \ln\left(\frac{1}{1-q_{j,l}}\right)}{\mu_j} G_{S_{(j,l,1)}}(z) \tag{5.13}$$

is a mixture distribution of $\mathcal{L}\big(W_{(j,0,1)}\big)$ and $\mathcal{L}\big(S_{(j,1,1)}\big), \dots, \mathcal{L}\big(S_{(j,n,1)}\big)$ for $j \in \mathcal{J}$. Hence $|\mathcal{J}|$ final numerically stable Panjer recursions for Poisson(μ_j) (cf. Theorem 5.1) yield the distributions of S_j for each $j \in \mathcal{J}$. To obtain the distribution of the random sum S in a straightforward manner, in a last step we condition on the possible values of J

$$\mathbb{P}[S = p] = \mathbb{P}\Big[\sum_{j\in\mathcal{J}} 1_{\{J=j\}} S_j = p\Big] = \sum_{j\in\mathcal{J}} \mathbb{P}[S_j = p]\ \mathbb{P}[J = j], \qquad p \in \mathbb{N}_0.$$

In a similar way, it is also possible to consider the random sums with respect to one default cause intensity $S_i = \sum_{h=1}^{N_i} X_{i,h}$ for $i \in \{1,\dots,m\}$ and compute the multivariate distribution of $S' = (S_1,\dots,S_m)$. For this purpose we apply the numerically stable multivariate Panjer recursion.

Remark 5.14. Algorithm 5.9 can be adapted to the multivariate case under the same assumptions on the risk factors $R_1,\dots,R_n$. By Theorem 3.1 the equivalent representation of the claim numbers in the random sum S_i for $i \in \{1,\dots,m\}$ is

$$S_i = \sum_{h=1}^{N_i} X_{i,h} \stackrel{\mathrm{d}}{=} \sum_{h=1}^{M_i} X_{i,h} = \sum_{j\in\mathcal{J}} 1_{\{J=j\}} \sum_{l=0}^{n} \sum_{h=1}^{Q_{j,l}} B_{i,j,l,h} X_{i,l,h},$$

where $\{X_{i,h}\}_{h\in\mathbb{N}}$ and $\{X_{i,l,h}\}_{h\in\mathbb{N}}$ with $i \in \{1,\dots,m\}$ and $l \in \{0,\dots,n\}$ are in l independent sequences of i.i.d. random variables with identical distributions. Thus the main difference to the univariate case is that we do not aggregate the claim sizes $\{B_{i,j,l,h}X_{i,l,h}\}_{h\in\mathbb{N}}$ for each $i \in \{1,\dots,m\}$, but consider it as a vector. We only need to replace in Equation (5.13) $W_{(j,0,1)}$ by $(BX)_{j,0,1}$ with $(BX)_{j,l,h} = (B_{1,j,l,h}X_{1,l,h},\dots,B_{m,j,l,h}X_{m,l,h})$ and $S_{(j,l,1)}$ by $T_{(j,l,1)}$ with $T_{(j,l,r)} = (T_{(1,j,l,r)},\dots,T_{(m,j,l,r)})$, where for $i \in \{1,\dots,m\}$

$$T_{(i,j,l,r)} = \sum_{h=K_{(j,l,1)}+\cdots+K_{(j,l,r-1)}+1}^{K_{(j,l,1)}+\cdots+K_{(j,l,r)}} B_{i,j,l,h} X_{i,l,h}, \qquad j \in \mathcal{J}, \quad l \in \{1,\dots,n\}.$$

Then we apply the multivariate Panjer recursion in Theorem 5.1 for c_n being the unit vector e_i for $i \in \{1,\dots,m\}$ twice as in Algorithm 5.9.

In a constellation such as in Lemma 4.1 the evaluation of the distribution of the random sum S is an adaptation of the algorithm in [21, Section 5.5] under certain assumptions and slighty different from the other dependence scenario.

Algorithm 5.15. Consider the setting of Lemma 4.1. According to the approach of Giese [23, Section 10.2] assume that the random variables $T_1,\dots,T_n$ have a distribution such that $T_l \sim \text{Gamma}(\sigma_l,\nu_l)$ with $\sigma_l,\nu_l > 0$ for $l \in \{1,\dots,n\}$. Assume further that $T_1,\dots,T_n$ are independent. We apply Lemma 4.1 and adopt the equivalent representation of the claim numbers into the random sum S in Equation (1.1):

$$S = \sum_{i=1}^{m}\sum_{h=1}^{N_i} X_{i,h} \overset{\mathrm{d}}{=} \sum_{i=1}^{m}\sum_{h=1}^{M_i} X_{i,h} = \sum_{j\in\mathcal{J}} 1_{\{J=j\}} \sum_{l=0}^{n}\sum_{h=1}^{P_{j,l}}\sum_{k=1}^{Y_{j,l,h}}\sum_{i=1}^{m} B_{i,j,l,h,k}X_{i,l,h,k},$$

where $\{X_{i,h}\}_{h\in\mathbb{N}}$ and $\{X_{i,l,h,k}\}_{h,k\in\mathbb{N}}$ are independent sequences of i.i.d. random variables with values in $\mathbb{N}_0$ and identical distribution for each $i \in \{1,\dots,m\}$ and $l \in \{0,\dots,n\}$. Hence the distribution of S can be calculated iteratively. In contrast to Algorithm 5.9 we need to insert some notation and steps. Let for $h,k \in \mathbb{N}$

$$Z_{(j,l,h,k)} = \sum_{i=1}^{m} B_{i,j,l,h,k}X_{i,l,h,k}, \qquad j\in\mathcal{J}, \quad l\in\{0,\dots,n\}.$$

Since $\{B_{i,j,l,h,k}X_{i,l,h,k}\}_{h,k\in\mathbb{N}}$ are i.i.d., the distribution of $Z_{(j,l,1,1)}$ can be calculated directly by the law of total probability for $p\in\mathbb{N}_0$ and $j\in\mathcal{J}$ and $l\in\{0,\dots,n\}$ as in Equation (5.10) under the natural assumption that $\{B_{i,j,l,h,k}\}_{h,k\in\mathbb{N}}$ and $\{X_{i,l,h,k}\}_{h,k\in\mathbb{N}}$ are independent for every $j\in\mathcal{J}$ and $l\in\{0,\dots,n\}$. Since $\{Y_{j,l,h}\}_{h\in\mathbb{N}}$ are i.i.d. and $Y_{j,l,1}$ has a logarithmic distribution given in Equation (4.6) and hence is in a Panjer$(q_{j,l}, -q_{j,l}, 1)$ class, the random sums with $h\in\mathbb{N}$

$$W_{(j,l,h)} = \sum_{k=1}^{Y_{j,l,h}} Z_{(j,l,h,k)}, \qquad j\in\mathcal{J}, \quad l\in\{0,\dots,n\}, \tag{5.16}$$

can be evaluated by Panjer's recursion for each $j\in\mathcal{J}$, $l\in\{0,\dots,n\}$, and $h=1$ (cf. Theorem 5.1). Let

$$s_{j,l} = \frac{-\alpha_l \ln(1-q_{j,l})}{\nu_l - \alpha_l\ln(1-q_{j,l})}, \qquad j\in\mathcal{J}, \quad l\in\{1,\dots,n\}.$$

Since T_l has a gamma distribution for $l\in\{1,\dots,n\}$, according to Lemma 2.8 with $m=1$ in that framework and $T\equiv 1$

$$\mathcal{L}(P_{j,l}) = \mathrm{NegBin}(\sigma_l, s_{j,l}).$$

If $q_{j,l}=0$ for some j,l, then there is no random summation for these indices. Note that according to Remark 2.7 $P_{j,l} \overset{\mathrm{d}}{=} \sum_{s=1}^{L_{j,l}} K_{(j,l,s)}$ with

$$P_{j,l} \sim \mathrm{CPoi}\Big(\sigma_l \ln\Big(\frac{1}{1-s_{j,l}}\Big), \mathrm{Log}(s_{j,l})\Big).$$

Insert Equation (5.16) into Equation (5.12) and let instead $K_{(j,l,1)} \sim \mathrm{Log}(s_{j,l})$. By the same reasoning as in Algorithm 5.9 and in [21, Remark 5.11] $\mathcal{L}\big(S_{(j,l,1)}\big)$ can be evaluated with a numerically stable Panjer recursion for every $j\in\mathcal{J}$ and $l\in\{1,\dots,n\}$, cf. Theorem 5.1. For $l=0$ and each $j\in\mathcal{J}$ we have $\mathcal{L}(P_{j,0}) = \mathrm{Poisson}\big(-\alpha_0\ln(1-q_{j,0})T_0\big)$. Then the total portfolio loss if $J=j$ can be written as

$$S_j = \sum_{h=1}^{P_{j,0}} W_{(j,0,h)} + \sum_{l=1}^{n}\sum_{r=1}^{L_{j,l}} S_{(j,l,r)}, \qquad j\in\mathcal{J}.$$

By the same argumentation as in Algorithm 5.9 the probability-generating function of the distribution of S_j can be written as

$$G_{S_j}(z) = \exp(-\mu_j(1 - G_j(z)))$$

for $|z| \leq 1$ with

$$\mu_j = -\alpha_0 \ln(1 - q_{j,0})T_0 + \sum_{l=1}^{n} \sigma_l \ln\Big(\frac{1}{1 - s_{j,l}}\Big),$$

and

$$G_j(z) = \frac{-\alpha_0 \ln(1 - q_{j,0})T_0}{\mu_j} G_{W_{(j,0,1)}}(z) + \sum_{l=1}^{n} \frac{\sigma_l \ln\big(\frac{1}{1-s_{j,l}}\big)}{\mu_j} G_{S_{(j,l,1)}}(z)$$

is a mixture distribution of $\mathcal{L}\big(W_{(j,0,1)}\big)$ and $\mathcal{L}\big(S_{(j,1,1)}\big), \dots, \mathcal{L}\big(S_{(j,n,1)}\big)$. The application of Panjer's recursion and the law of total probability works as in Algorithm 5.9.

Remark 5.17. The algorithms do not only work this way if $R_1, \dots, R_n$ and $T_1, \dots, T_n$, respectively, are gamma-distributed. They also work if $R_1, \dots, R_n$ or $T_1, \dots, T_n$ have a τ-tempered α-stable distribution. By an application of [21, Lemma 5.10] the severity distribution of the corresponding compound Poisson distribution of $\sum_{h=1}^{L_{j,l}} K_{j,l,h}$ is an extended negative binomial distribution. Thus apply [21, Algorithm 5.12]. A special case of this class of distributions is e.g. the inverse Gaussian distribution, which has also been described in [64, p. 91]. Gerhold, Schmock, and Warnung [21, Example 5.21] also reveal how to evaluate such a distribution.

Remark 5.18. If $T_1, \dots, T_n$ are not independent, it is possible to work with the framework of Algorithm 5.15 if the dependence structure of the $T_1, \dots, T_n$ satisfies the structure given in Assumption 2.18. Then an alternative representation of the random variables $P_{j,1}, \dots, P_{j,n}$ exists for $j \in \mathcal{J}$ given in Equation (4.4) by Theorem 3.1. Thus we can apply Algorithm 5.9.

Remark 5.19. Using the convex combination for the evaluation of the portfolio distribution provides a claim size with a logarithmic distribution that might require high computational effort in the evaluation of Panjer's recursion. This convex combination also requires a high computational effort in the evaluation of Panjer's recursion with the Poisson distribution. Yet, this approach is preferable since the evaluation of Panjer's recursion for a negative binomial distribution followed by several convolutions clearly demands more resources.

As we have seen, not only default cause intensities driving the claim numbers may depend on each other, but claim sizes might, too. We now consider a special such case. Theoretically, we could let every claim size be possibly dependent on every other claim size. But since we could have theoretically infinitely many claims, we restrict ourselves to dependence between claim sizes only with respect to the default cause intensities. This could also be generalized to risk groups, cf. [57].

Under the following assumption it is possible to evaluate the distribution of S in Equation (1.1) if the claim sizes are dependent.

Assumption 5.20. *Let $\mathcal{J} \neq \varnothing$ be an arbitrary finite set. Let $A_j = (a^j_{i,l}) \in \mathbb{N}^{m \times n}$ for $j \in \mathcal{J}$ and $i \in \{1, \dots, m\}$ and $l \in \{1, \dots, n\}$ and let J be a $\mathcal{J}$-valued random variable. Let further $A_J = \sum_{j \in \mathcal{J}} 1_{\{J=j\}} A_j$. Let $\Lambda_1, \dots, \Lambda_m$ be independent, infinitely divisible, and non-negative random variables and let $N_1, \dots, N_m$ be random variables conditionally independent given $\Lambda_1, \dots, \Lambda_m$ such that*

$$\mathcal{L}(N_i \,|\, \Lambda_1, \dots, \Lambda_m) \stackrel{a.s.}{=} \mathcal{L}(N_i \,|\, \Lambda_i) \stackrel{a.s.}{=} \text{Poisson}(\lambda_i \Lambda_i)\,, \qquad i \in \{1, \dots, m\}.$$

Define n independent sequences of i.i.d. discrete random variables $\{Z_{l,h}\}_{h\in\mathbb{N}}$ for $l \in \{1, \dots, n\}$ independent of J. Set

$$(Y_{1,h}, \dots, Y_{m,h})^\top = A_J(Z_{1,h}, \dots, Z_{n,h})^\top, \qquad h \in \mathbb{N},$$

i.e.,

$$Y_{i,h} = \sum_{j\in\mathcal{J}} 1_{\{J=j\}} \sum_{l=1}^{n} a_{i,l}^{j} Z_{l,h}$$

for $h \in \mathbb{N}$ and $i \in \{1, \dots, m\}$. Now let $(S_1, \dots, S_m) = (\sum_{h=1}^{N_1} Y_{1,h}, \dots, \sum_{h=1}^{N_m} Y_{m,h})$.

We assume that the claim sizes have a dependence structure given in Assumption 5.20. Then there is the following adaptation of the algorithm in [21, Section 5.5] in order to compute the distribution of $S = \sum_{i=1}^{m} S_i$:

Algorithm 5.21. In accordance with the extended CreditRisk$^+$ model assume that the default cause intensities have a distribution such that $\Lambda_i \sim \text{Gamma}(\alpha_i, \beta_i)$ with $\alpha_i, \beta_i > 0$ for $i \in \{1, \dots, m\}$. We insert the representation of $\{Y_{i,h}\}_{h\in\mathbb{N}}$ of Assumption 5.20 into the random sum S in Equation (1.1):

$$S = \sum_{i=1}^{m}\sum_{h=1}^{N_i} X_{i,h} \overset{\mathrm{d}}{=} \sum_{i=1}^{m}\sum_{h=1}^{N_i} Y_{i,h} = \sum_{j\in\mathcal{J}} 1_{\{J=j\}} \sum_{i=1}^{m}\sum_{l=1}^{n}\sum_{h=1}^{N_i} a_{i,l}^{j} Z_{l,h}.$$

Hence, the distribution of S can be calculated iteratively. Note that since $a_{i,j} \in \mathbb{N}$, the random variable $a_{i,j}Z_{j,1}$ still has a discrete distribution. Let

$$q_i = \frac{\lambda_i}{\beta_i + \lambda_i}, \qquad i \in \{1, \dots, m\}.$$

By Lemma 2.8 with $m = 1$ in that framework and $T \equiv 1$

$$N_i \sim \text{NegBin}(\alpha_i, q_i)\,, \qquad i \in \{1, \dots, m\}.$$

Recall that according to Remark 2.7 $N_i \overset{\mathrm{d}}{=} \sum_{s=1}^{L_i} K_{i,s}$ with

$$N_i \sim \text{CPoi}\big(\alpha_i \ln\big(1 + \lambda_i/\beta_i\big), \text{Log}(q_i)\big)\,, \qquad i \in \{1, \dots, m\}.$$

According to [21, Remark 5.11] let for $r \in \mathbb{N}$ and $j \in \mathcal{J}$

$$S_{(i,j,l,r)} = \sum_{s=K_{i,1}+\dots+K_{i,r-1}+1}^{K_{i,1}+\dots+K_{i,r}} a_{i,l}^{j} Z_{l,s}, \qquad l \in \{1, \dots, n\}, \quad i \in \{1, \dots, m\},$$

and evaluate the distribution of $S_{(i,j,l,1)}$ with a numerically stable Panjer recursion (cf. Theorem 5.1) for $\text{Log}(q_i)$, which is in a Panjer$(q_i, -q_i, 1)$ class. Then the total portfolio loss if $J = j$ can be written as

$$S_j = \sum_{i=1}^{m}\sum_{l=1}^{n}\sum_{r=1}^{L_i} S_{(i,j,l,r)}, \qquad j \in \mathcal{J},$$

and the corresponding probability-generating function is

$$G_{S_j}(z) = \prod_{i=1}^{m}\prod_{l=1}^{n} \exp\big(-\alpha_i \ln\big(1 + \lambda_i/\beta_i\big)(1 - G_{S_{(i,j,l,1)}}(z))\big) = \exp(-\mu(1 - G_j(z))),$$

for $|z| \leq 1$ with

$$\mu = n \sum_{i=1}^{m} \alpha_i \ln\bigl(1 + \lambda_i/\beta_i\bigr) \qquad \text{and} \qquad G_j(z) = \sum_{i=1}^{m} \sum_{l=1}^{n} \frac{\alpha_i \ln\bigl(1+\lambda_i/\beta_i\bigr)}{\mu} G_{S_{(i,j,l,1)}}(z)$$

is a mixture distribution of $\mathcal{L}\bigl(S_{(1,j,1,1)}\bigr), \ldots, \mathcal{L}\bigl(S_{(m,j,n,1)}\bigr)$. Hence $|\mathcal{J}|$ final numerically stable Panjer recursions for Poisson(μ) and the distribution belonging to G_j for $j \in \mathcal{J}$ yield the distributions of S_j. Finally, a conditioning argument provides the result.

Remark 5.22. It is also possible to evaluate the distribution of the random vector $(S_1, \ldots, S_m)$ in Assumption 5.20. This is described in [64, Chapters 20.2 and 20.6]

Thus we are able to observe a certain dependence structure between claim sizes, which may happen in a realistic scenario, and are still able to apply a variant of Panjer's recursion.

5.2 A Generalization of De Pril's Recursion

As for Panjer's recursion it is possible to derive a generalization of de Pril's recursion that evaluates higher moments of a random variable with compound distribution. This recursion was introduced in [13]. De Pril's recursion was also generalized in [21, Theorem 8.2 and Theorem 8.1]. We relax the assumption of i.i.d. claim sizes and replace it with the assumption that the claim sizes are i.i.d. given a σ-algebra $\mathcal{F}$. In addition, we consider multivariate claim sizes. We put the result into the following lemma:

Lemma 5.23. *Let the assumptions of Theorem 5.1 be satisfied.*

(a) Let c be an $\mathbb{R}^d$-valued $\mathcal{F}$-measurable function and $b \in \mathbb{N}$. Assume further that $\langle c, X_1\rangle^n$ and $\langle c, \tilde{S}_{(i)}\rangle^n$ are σ-integrable with respect to $\mathcal{F}$ for every $n \in \{1, \ldots, b\}$. Then for every $n \in \{1, \ldots, b\}$

$$\begin{aligned}\mathbb{E}[\langle c, S\rangle^n \,|\, \mathcal{F}] &\overset{a.s.}{=} \sum_{p=1}^{k+l-1} q_p \,\mathbb{E}[\langle c, S_p\rangle^n \,|\, \mathcal{F}] \\ &\quad + \sum_{i=1}^{l} \sum_{s=0}^{n} \binom{n}{s} \Bigl(a_i + \frac{b_i s}{in}\Bigr) \mathbb{E}\bigl[\langle c, \tilde{S}_{(i)}\rangle^{n-s} \,\big|\, \mathcal{F}\bigr] \,\mathbb{E}\bigl[\langle c, S_i\rangle^s \,\big|\, \mathcal{F}\bigr]\end{aligned}$$

holds, where $S_i = X_1 + \cdots + X_i$.

(b) Let the assumptions of Theorem 5.1 (b) be satisfied. Then

$$\mathbb{E}[\langle c, S\rangle^n \,|\, \mathcal{F}] \overset{a.s.}{=} \sum_{i=1}^{l} \nu_i \,\mathbb{E}\bigl[\langle c, \tilde{S}_i\rangle^n \,\big|\, \mathcal{F}\bigr]$$

holds for all $n \in \{1, \ldots, b\}$.

Proof. This proof is an adaptation of [21, Theorem 8.2] and is essentially the same; it is repeated only for convenience of the reader. By Jensen's inequality $\langle c, S_i\rangle^n$ is σ-integrable with respect to $\mathcal{F}$ for $j \in \mathbb{N}$:

$$\langle c, S_i\rangle^n = j^n \Bigl(\frac{\langle c, X_1\rangle + \cdots + \langle c, X_i\rangle}{j}\Bigr)^n \leq j^{n-1}(\langle c, X_1\rangle^n + \cdots + \langle c, X_i\rangle^n) \text{ a.s.}$$

The conditional expected value given $\mathcal{F}$ can be rewritten as a limit

$$\mathbb{E}[\langle c, S\rangle^n \,|\, \mathcal{F}] \overset{\text{a.s.}}{=} \lim_{M\to\infty} \sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}} \langle c, m\rangle^n p_m,$$

where p_m is the conditional probability mass function of S given $\mathcal{F}$ in Equation (5.4) and $M\to\infty$ means $M_r\to\infty$ for every $r\in\{1,\dots,d\}$. We apply Theorem 5.1 and note that Equation (5.4) is valid when multiplied by $\langle c, m\rangle$ and set $c_n = c$. Hence inserting Equation (5.4) yields

$$\begin{aligned}
\sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}} \langle c, m\rangle^n p_m &\overset{\text{a.s.}}{=} \sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}} \langle c, m\rangle^n \Bigg(\sum_{p=1}^{k+l-1} \mathbb{P}[S_p = m \,|\, \mathcal{F}]\, q_p \\
&\qquad + \sum_{i=1}^{l} \sum_{\substack{0\le j\le m\\ j\in\mathbb{N}_0^d}} \Big(a_i + \frac{b_i\langle c, j\rangle}{i\langle c, m\rangle}\Big) \mathbb{P}[S_i = j \,|\, \mathcal{F}]\, \tilde{p}_{i,m-j} \Bigg) \\
&\overset{\text{a.s.}}{=} \sum_{p=1}^{k+l-1} q_p \sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}} \langle c, m\rangle^n\, \mathbb{P}[S_p = m \,|\, \mathcal{F}] + \sum_{i=1}^{l} E_{i,M},
\end{aligned} \tag{5.24}$$

where with an interchange of the order of summation

$$\begin{aligned}
E_{i,M} &\overset{\text{a.s.}}{=} \sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}} \langle c, m\rangle^n \sum_{\substack{0\le j\le m\\ j\in\mathbb{N}_0^d}} \Big(a_i + \frac{b_i\langle c, j\rangle}{i\langle c, m\rangle}\Big) \mathbb{P}[S_i = j \,|\, \mathcal{F}]\, \tilde{p}_{i,m-j} \\
&\overset{\text{a.s.}}{=} \sum_{\substack{0\le j\le M\\ j\in\mathbb{N}_0^d}} \sum_{\substack{m_r=\max\{1,j_r\}\\ r\in\{1,\dots,d\}}}^{M_r} \Big(a_i\langle c, m\rangle^n + \frac{b_i\langle c, j\rangle}{i} \langle c, m\rangle^{n-1}\Big) \mathbb{P}[S_i = j \,|\, \mathcal{F}]\, \tilde{p}_{i,m-j}.
\end{aligned}$$

For $j_r = 0$ we may add the term for $m_r = 0$ which is zero. Shifting the summation index m_r down by j_r for each $r\in\{1,\dots,d\}$, we obtain

$$E_{i,M} \overset{\text{a.s.}}{=} \sum_{\substack{0\le j\le M\\ j\in\mathbb{N}_0^d}} \sum_{\substack{0\le m\le M-j\\ m\in\mathbb{N}_0^d}} \Big(a_i\langle c, m+j\rangle^n + \frac{b_i\langle c, j\rangle}{i} \langle c, m+j\rangle^{n-1}\Big) \mathbb{P}[S_i = j \,|\, \mathcal{F}]\, \tilde{p}_{i,m}.$$

Noting $\langle c, m+j\rangle = \langle c, m\rangle + \langle c, j\rangle$, applying the binomial formula and shifting the index up by 1, we have

$$\langle c, j\rangle\langle c, m+j\rangle^{n-1} = \sum_{s=0}^{n-1} \binom{n-1}{s} \langle c, m\rangle^{n-1-s} \langle c, j\rangle^{s+1} = \sum_{s=1}^{n} \frac{s}{n}\binom{n}{s} \langle c, m\rangle^{n-s}\langle c, j\rangle^{s}.$$

Adding the zero and using the binomial formula also for $\langle c, m+j\rangle^n$ and changing the order of summation and the order of the summands provides

$$E_{i,M} \overset{\text{a.s.}}{=} \sum_{s=0}^{n} \binom{n}{s} \Big(a_i + \frac{b_i s}{in}\Big) \sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}} \langle c, m\rangle^{n-s} \tilde{p}_{i,m} \sum_{\substack{0\le j\le M-m\\ j\in\mathbb{N}_0^d}} \langle c, j\rangle^s\, \mathbb{P}[S_i = j \,|\, \mathcal{F}].$$

Inserting this into Equation (5.24) yields

$$\sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}} \langle c,m\rangle^n p_m \overset{\text{a.s.}}{=} \sum_{p=1}^{k+l-1} q_p \sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}} \langle c,m\rangle^n\, \mathbb{P}[S_p=m\,|\,\mathcal{F}]$$
$$+\sum_{i=1}^{l}\sum_{s=0}^{n}\binom{n}{s}\Big(a_i+\frac{b_i s}{in}\Big)\sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}} \langle c,m\rangle^{n-s}\tilde{p}_{i,m} \sum_{\substack{0\le j\le M-m\\ j\in\mathbb{N}_0^d}} \langle c,j\rangle^s\, \mathbb{P}[S_i=j\,|\,\mathcal{F}]. \tag{5.25}$$

Letting $M\to\infty$ and noting that $\langle c,X_1\rangle^n$ and $\langle c,\tilde{S}_{(i)}\rangle^n$ are σ-integrable with respect to $\mathcal{F}$ for $n\in\{1,\dots,b\}$, we see that the expected values given $\mathcal{F}$ in Equations (5.25) and (5.24) are finite, hence the claim follows.

The second part of the proof can be proven as in [21, Theorem 8.2(b)] with respect to $\mathcal{F}$. q.e.d.

There is another generalization of de Pril's recursion, based on [21, Theorem 8.1].

Corollary 5.26. *Let c be an $\mathbb{R}^d$-valued $\mathcal{F}$-measurable function. Let $S=\sum_{h=1}^{N} X_h$, where $\{X_h\}_{h\in\mathbb{N}}$ is a sequence independent of the random variable N, consisting of i.i.d. d-dimensional random vectors that are conditionally independent given a σ-algebra $\mathcal{F}$. Assume that the distribution of N belongs to the Panjer(a,b,k) class. Define $S_k=X_1+\dots+X_k$ and note $\mathbb{E}\big[\langle c,S\rangle^0\,|\,\mathcal{F}\big]=1$ by convention.*

(a) If $a<1$ and $\langle c,X_1\rangle^s$ is σ-integrable with respect to $\mathcal{F}$ for $s\in\mathbb{N}$, then $\langle c,S_k\rangle^n$ is σ-integrable with respect to $\mathcal{F}$ and

$$\mathbb{E}[\langle c,S\rangle^n\,|\,\mathcal{F}] \overset{a.s.}{=} \frac{1}{1-a}\bigg(\mathbb{P}[N=k\,|\,\mathcal{F}]\,\mathbb{E}[\langle c,S_k\rangle^n\,|\,\mathcal{F}]$$
$$+\sum_{r=1}^{n}\binom{n}{r}\Big(a+\frac{br}{n}\Big)\mathbb{E}\big[\langle c,S\rangle^{n-r}\,|\,\mathcal{F}\big]\,\mathbb{E}[\langle c,X_1\rangle^r\,|\,\mathcal{F}]\bigg)$$

holds for every $n\in\{1,\dots,s\}$.

(b) If $a=1$ and $-b>2$ as well as if $\langle c,X_1\rangle^s$ is σ-integrable with respect to $\mathcal{F}$ for an $s\in\mathbb{N}\setminus\{1\}$ and $\mathbb{E}[\langle c,X_1\rangle\,|\,\mathcal{F}]>0$, then $\langle c,S_k\rangle^{n+1}$ is σ-integrable with respect to $\mathcal{F}$ and

$$\mathbb{E}[\langle c,S\rangle^n\,|\,\mathcal{F}] \overset{a.s.}{=} \frac{1}{-b-1-n}\,\mathbb{E}[\langle c,X_1\rangle\,|\,\mathcal{F}]\bigg(\mathbb{P}[N=k\,|\,\mathcal{F}]\,\mathbb{E}\big[\langle c,S_k\rangle^{n+1}\,|\,\mathcal{F}\big]$$
$$+\sum_{r=1}^{n}\binom{n}{r}\Big(\frac{n+1}{r+1}+b\Big)\mathbb{E}\big[\langle c,S\rangle^{n-r}\,|\,\mathcal{F}\big]\,\mathbb{E}\big[\langle c,X_1\rangle^{r+1}\,|\,\mathcal{F}\big]\bigg)$$

for every $n\in\{1,\dots,s-1\}$ with $n<-b-1$.

Proof. (a) The proof is an adaptation of the proof in [21, Theorem 8.1]. Since N belongs to the Panjer(a,b,k) class, the proof of Theorem 5.1 is applicable, and by exploiting

Equation (5.25) and $q_0 = \cdots = q_{k-1} = 0$ we obtain

$$\begin{aligned}\sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}} \langle c,m\rangle^n p_m \overset{\text{a.s.}}{=} q_k &\sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}} \langle c,m\rangle^n\ \mathbb{P}[S_k = m\,|\,\mathcal{F}]\\ &+\sum_{r=1}^{n}\binom{n}{r}\Big(a+\frac{br}{n}\Big)\sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}}\langle c,m\rangle^{n-r}p_m \sum_{\substack{0\le j\le M-m\\ j\in\mathbb{N}_0^d}}\langle c,j\rangle^r\ \mathbb{P}[X_1=j\,|\,\mathcal{F}]\\ &+\binom{n}{0}(a+0)\sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}}\langle c,m\rangle^n p_m\sum_{\substack{0\le j\le M-m\\ j\in\mathbb{N}_0^d}}\langle c,j\rangle^0\ \mathbb{P}[X_1=j\,|\,\mathcal{F}].\end{aligned}$$

Moving the term for $r = 0$ to the left side yields

$$\begin{aligned}\sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}}\big(1-\mathbb{P}[X_1\le M-m\,|\,\mathcal{F}]\,a\big)\langle c,m\rangle^n p_m \overset{\text{a.s.}}{=} q_k &\sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}} \langle c,m\rangle^n\ \mathbb{P}[S_k = m\,|\,\mathcal{F}]\\ &+\sum_{r=1}^{n}\binom{n}{r}\Big(a+\frac{br}{n}\Big)\sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}}\langle c,m\rangle^{n-r}p_m \sum_{\substack{0\le j\le M-m\\ j\in\mathbb{N}_0^d}}\langle c,j\rangle^r\ \mathbb{P}[X_1=j\,|\,\mathcal{F}].\end{aligned}$$

The rest of the proof of this part is exactly as in [21, Theorem 8.1(a)] except that we need to apply the conditional versions of Fatou's lemma and the dominated convergence theorem, respectively.

(b) In case $a = 1$ and $-b > 2$ we have

$$\begin{aligned}\sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}}\mathbb{P}[X_1> M-m\,|\,\mathcal{F}]\,\langle c,m\rangle^n p_m \overset{\text{a.s.}}{=} q_k &\sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}} \langle c,m\rangle^n\ \mathbb{P}[S_k = m\,|\,\mathcal{F}]\\ &+\sum_{r=1}^{n}\binom{n}{r}\Big(1+\frac{br}{n}\Big)\sum_{\substack{0\le m\le M\\ m\in\mathbb{N}_0^d}}\langle c,m\rangle^{n-r}p_m \sum_{\substack{0\le j\le M-m\\ j\in\mathbb{N}_0^d}}\langle c,j\rangle^r\,\mathbb{P}[X_1=j\,|\,\mathcal{F}].\end{aligned}$$

The rest of this part of the proof is also exactly as in [21, Theorem 8.1(b)], we need only apply the conditional versions of Fatou's lemma and the dominated convergence theorem, respectively.

q.e.d.

Chapter 6

Risk Contributions

Knowing the distribution of the portfolio loss S makes it possible to compute the corresponding law-invariant risk measure $\rho(S)$. For references on risk measures see [2] and [14]. The objective of risk contributions is to allocate the risk of the total portfolio to the risk of a subportfolio. Knowing the risk that comes from the total portfolio, it is also possible to diversify the risks into the subportfolios. An axiomatic approach to such a capital allocation can be found in [41]. We follow the approach of Kalkbrener [41] and for $i \in \{1, \ldots, m\}$ consider a subportfolio S_i of the portfolio S, where the risk capital distributed to this subportfolio only depends on S_i and S. We consider the capital allocation with respect to expected shortfall. For this purpose we give an introduction to conditional expected shortfall. This also allows us to consider risk contributions in a multivariate framework, which becomes interesting if e.g. a bank will have a recovery rate over a period of time after a credit default or if an insurance company observes a claim where the insured person's recovery takes time and several medical treatments. But first we consider a one-period model and use these results to derive a result for the multi-period model.

6.1 Introduction and Basic Definitions

For a better understanding of the risk measures used, we first need to introduce some notation. Because we also consider risk contributions in a multivariate framework, we introduce this notation more generally as needed. We will not need this notation in this generality in Section 6.2, but in Section 6.3. The following definitions of this section are taken from [34]. We quote some results here:

Definition 6.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra. Let X be an $\mathcal{F}$-measurable $\overline{\mathbb{R}}$-valued random variable and δ a $\mathcal{G}$-measurable $[0, 1]$-valued random variable. Then, define the lower δ-quantile $q_{\mathcal{G},\delta}(X)$ of X given $\mathcal{G}$ as the essential infimum of all $\mathcal{G}$-measurable random variables $Z\colon \Omega \to \overline{\mathbb{R}}$ satisfying $\mathbb{P}[X \le Z \,|\, \mathcal{G}] \ge \delta$ a.s.

As shown in [34], $q_{\mathcal{G},\delta}(X)$ is $\mathcal{G}$-measurable and satisfies $\mathbb{P}[X \le q_{\mathcal{G},\delta}(X)] \ge \delta$ a.s. We need to lay the notational groundwork for the risk measure expected shortfall:

Definition 6.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra. Let X be an $\mathcal{F}$-measurable $\overline{\mathbb{R}}$-valued random variable and δ be a $\mathcal{G}$-measurable $[0, 1]$-valued random variable. Then, define the adjusted indicator function $f_{\mathcal{G},\delta,X}\colon \Omega \to [0, 1]$ by

$$f_{\mathcal{G},\delta,X} := 1_{\{X > q_{\mathcal{G},\delta}(X)\}} + \beta_{\mathcal{G},\delta,X} 1_{\{X = q_{\mathcal{G},\delta}(X)\}},$$

where $\beta_{\mathcal{G},\delta,X}\colon \Omega \to [0,1]$ is a $\mathcal{G}$-measurable random variable satisfying

$$\beta_{\mathcal{G},\delta,X} \overset{\text{a.s.}}{=} \begin{cases} \frac{\mathbb{P}[X \le q_{\mathcal{G},\delta}(X)\,|\,\mathcal{G}]-\delta}{\mathbb{P}[X = q_{\mathcal{G},\delta}(X)\,|\,\mathcal{G}]} & \text{on the event } \{\mathbb{P}[X = q_{\mathcal{G},\delta}(X)\,|\,\mathcal{G}] > 0\}, \\ 0 & \text{otherwise.} \end{cases}$$

In the following definitions we will need the notion of the upper $\mathcal{G}$-measurable envelope $X^{\mathcal{G}}$ of X.

Definition 6.3. Let a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$ as well as an $\mathcal{F}$-measurable $\overline{\mathbb{R}}$-valued random variable X be given. Define $X^{\mathcal{G}}$ as the upper $\mathcal{G}$-measurable envelope of X, i.e., as the essential infimum of all $\mathcal{G}$-measurable random variables $Z\colon \Omega \to \overline{\mathbb{R}}$ satisfying $\mathbb{P}[X \le Z] = 1$.

Remark 6.4. Note that according to [29, Chapter I, § 4] we consider a general version of the conditional expectation, which is defined for random variables that are σ-integrable with respect to a σ-algebra.

Then conditional expected shortfall is defined as follows:

Definition 6.5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Let X be an $\mathcal{F}$-measurable $\overline{\mathbb{R}}$-valued random variable and δ be a $\mathcal{G}$-measurable $[0,1]$-valued random variable. Then the conditional expected shortfall of X at level δ given $\mathcal{G}$ is defined by

$$\mathrm{ES}_\delta[X\,|\,\mathcal{G}] = \begin{cases} X^{\mathcal{G}} & \text{on } \{\delta = 1\}, \\ \frac{1}{1-\delta}\,\mathbb{E}[f_{\mathcal{G},\delta,X} X\,|\,\mathcal{G}] & \text{on } \{0 < \delta < 1\}, \\ \operatorname{ess\,inf}_{\delta' \in (0,1)} \mathrm{ES}_{\delta'}[X\,|\,\mathcal{G}] & \text{on } \{\delta = 0\}, \end{cases}$$

where $f_{\mathcal{G},\delta,X}$ is given in Definition 6.2.

Let $\mathcal{L}_0(\mathbb{P})$ denote the vector space of all $\overline{\mathbb{R}}$-valued random variables $X\colon \Omega \to \overline{\mathbb{R}}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{L}^-_{\mathcal{G},1}(\mathbb{P})$ denote the cone of those $X \in \mathcal{L}_0(\mathbb{P})$ for which the negative part X^- is σ-integrable with respect to $\mathcal{G}$. Accordingly we have the following definition for the corresponding risk contributions:

Definition 6.6. For a portfolio loss $L \in \mathcal{L}_0(\mathbb{P})$ and a $\mathcal{G}$-measurable level δ with $\delta \in [0,1]$ consider a subportfolio loss $X \in \mathcal{L}_0(\mathbb{P})$ with $X 1_{\{L \ge q_{\mathcal{G},\delta}(L)\}} 1_{\{\delta > 0\}} \in \mathcal{L}^-_{\mathcal{G},1}(\mathbb{P})$. Then, the conditional expected shortfall contribution of the subportfolio loss X to L at level δ given $\mathcal{G}$ is defined by

$$\mathrm{ES}_\delta[X, L\,|\,\mathcal{G}] = \begin{cases} X^{\mathcal{G}} & \text{on } \{\delta = 1\}, \\ \frac{1}{1-\delta}\,\mathbb{E}[f_{\mathcal{G},\delta,L} X\,|\,\mathcal{G}] & \text{on } \{0 < \delta < 1\}, \\ \operatorname{ess\,inf}_{\delta' \in (0,1)} \mathrm{ES}_{\delta'}[X, L\,|\,\mathcal{G}] & \text{on } \{\delta = 0\}, \end{cases}$$

where $f_{\mathcal{G},\delta,L}$ is given in Definition 6.2.

In case $\mathcal{G}$ is a trivial σ-algebra, i.e., $\mathbb{P}[G] \in \{0,1\}$ for all $G \in \mathcal{G}$, the δ-quantile given $\mathcal{G}$ simplifies to the already known δ-quantile. Denote the lower δ-quantile for an $\overline{\mathbb{R}}$-valued random variable X and level $\delta \in [0,1]$ by

$$q_\delta(X) = \min\{x \in \overline{\mathbb{R}}\,|\,\mathbb{P}[X \le x] \ge \delta\}.$$

6.2 Expected Shortfall Contributions

We start our considerations with the unconditioned case, i.e., a trivial σ-algebra $\mathcal{G}$. We fix a level $\delta \in (0,1)$. For $\delta = 0$ and X^- integrable we have $\mathrm{ES}_\delta[X \,|\, \mathcal{G}] = \mathbb{E}[X]$. This reflects a model with only one time-period. In this thesis we consider the risk contributions of the subportfolios coming from a default cause intensity. That is, consider m claim numbers N_i with the corresponding default cause intensity Λ_i for $i \in \{1,\dots,m\}$. According to Equation (1.1) the portfolio loss is:

$$S = \sum_{i=1}^{m} S_i, \tag{6.7}$$

and we consider the random sums $S_i = \sum_{h=1}^{N_i} X_{i,h}$ for $i \in \{1,\dots,m\}$ as subportfolios, and therefore the following expected shortfall contributions

$$\mathrm{ES}_\delta[S_i, S] \;=\; \frac{\mathbb{E}\big[S_i 1_{\{S>q_\delta(S)\}}\big] + \beta_{\delta,S}\,\mathbb{E}\big[S_i 1_{\{S=q_\delta(S)\}}\big]}{1-\delta}, \tag{6.8}$$

where

$$\beta_{\delta,S} = \frac{\mathbb{P}[S \le q_\delta(S)] - \delta}{1-\delta}.$$

Remark 6.9. If the claim sizes $\{X_{i,h}\}_{h\in\mathbb{N}}$ are $\mathbb{N}_0$-valued for $i \in \{1,\dots,m\}$, this capital allocation can be evaluated using an extended Panjer recursion, cf. also [57]. The lower δ-quantile $q_\delta(S)$ can be computed by evaluating the single probabilities $\mathbb{P}[S = n]$, $n = 0,1,\dots$ and adding up until $\mathbb{P}[S \le n] \ge \delta$. The term $\beta_{\delta,S}$ may be evaluated in the same manner. Further note that

$$\mathbb{E}\big[S_i 1_{\{S>q_\delta(S)\}}\big] \;=\; \mathbb{E}[S_i] \;-\; \mathbb{E}\big[S_i 1_{\{S\le q_\delta(S)\}}\big]$$

and $\mathbb{E}[S_i 1_{\{S\le q_\delta(S)\}}] = \sum_{n=1}^{q_\delta(S)} \mathbb{E}[S_i 1_{\{S=n\}}]$ holds and that $\mathbb{E}[S_i]$ can be computed using Wald's identity. It remains to compute $\mathbb{E}[S_i 1_{\{S=n\}}]$ for $n \in \mathbb{N}_0$. Note that our assumptions imply that S has a distribution on $\mathbb{N}_0$, hence the δ-quantile of S is also a natural number.

The expected shortfall contribution of S_i on $\{S = n\}$, i.e., $\mathbb{E}[S_i 1_{\{S=n\}}]$, can be evaluated recursively. Since we considered different types of dependence structures between the claim numbers, the recursive structure differs for each dependence structure. We require the claim sizes to be discrete random variables, i.e., we allow them to take negative values. The evaluation of the term $\mathbb{E}\big[R_l 1_{\{S=p-\mu, J=j\}}\big]$ will be given in Section 6.4. The proofs are an adaptation of [57] and [65, Lemma 1], respectively.

Theorem 6.10. *Let Assumption 2.18 be satisfied. Let the random variables $N_1,\dots,$ N_m be conditionally independent given $J, R_1,\dots,R_n$ and satisfy Equation* (2.20)*. Let for $i \in \{1,\dots,m\}$ the independent sequences $\{X_{i,h}\}_{h\in\mathbb{N}}$ of i.i.d. discrete random variables be independent of all previously mentioned random variables and let the negative part $X_{i,1}^-$ be integrable. Define for $p \in \mathbb{R}$ and $i \in \{1,\dots,m\}$ the set*

$$I_{i,p} := \{\mu \in \mathbb{R}\setminus\{0\} \,|\, \mathbb{P}[X_{i,1} = \mu, S = p-\mu] > 0\}.$$

For every subportfolio S_i with $i \in \{1,\dots,m\}$,

$$\mathbb{E}\big[S_i 1_{\{S=p\}}\big] \;= \sum_{j\in\mathcal{J}} \lambda_{i,j} \sum_{l=0}^{n} a_{i,l}^{j} \sum_{\mu\in I_{i,p}} \mu\; \mathbb{P}[X_{i,l,1} = \mu]\; \mathbb{E}\big[R_l 1_{\{S=p-\mu, J=j\}}\big]\,, \qquad p \in \mathbb{R}.$$

Remark 6.11. If for each $i \in \{1, \ldots, m\}$ the random variables $\{X_{i,h}\}_{h\in\mathbb{N}}$ are $\mathbb{N}_0$-valued, then the expectation can be only non-zero for $p \in \mathbb{N}$ and it suffices to consider $I_{i,p} = \{1, \ldots, p\}$.

Proof of Theorem 6.10. By Theorem 3.1 the distribution of $(N_1, \ldots, N_m)$ is equal to the distribution of

$$M = \sum_{j\in\mathcal{J}} 1_{\{J=j\}} \sum_{l=0}^{n} \sum_{h=1}^{Q_{j,l}} B_{j,l,h},$$

where $(Q_{j,0}, \ldots, Q_{j,n})$ and $\{B_{j,l,h}\}_{h\in\mathbb{N}, l\in\{0,\ldots,n\}}$ are independent for each $j \in \mathcal{J}$ and given by Equations (3.2) and (3.3). Thus we consider the representation $S' = \sum_{i=1}^{m} S'_i$ with

$$S'_i = \sum_{j\in\mathcal{J}} 1_{\{J=j\}} \sum_{l=0}^{n} \sum_{h=1}^{Q_{j,l}} B_{i,j,l,h} X_{i,l,h},$$

where for each $i \in \{1, \ldots, m\}$ and $l \in \{0, \ldots, n\}$ the sequence $\{X_{i,l,h}\}_{h\in\mathbb{N}}$ is an independent copy of $\{X_{i,h}\}_{h\in\mathbb{N}}$. It is a natural assumption that $\{X_{i,l,h}\}_{h\in\mathbb{N}}$ is independent of $Q_{j,l}$, $\{B_{i,j,l,h}\}_{h\in\mathbb{N}}$, and J for $i \in \{1, \ldots, m\}$ and $l \in \{0, \ldots, n\}$. Because of this alternative representation $(S_1, \ldots, S_m) \stackrel{\mathrm{d}}{=} (S'_1, \ldots, S'_m)$ holds and thus also $(S_i, S) \stackrel{\mathrm{d}}{=} (S'_i, S')$. Using the definition of S'_i yields for each $i \in \{1, \ldots, m\}$

$$\mathbb{E}\big[S_i 1_{\{S=p\}}\big] = \mathbb{E}\big[S'_i 1_{\{S'=p\}}\big] = \sum_{j\in\mathcal{J}} \sum_{l=0}^{n} \mathbb{E}\Bigg[\sum_{h=1}^{Q_{j,l}} B_{i,j,l,h} X_{i,l,h} 1_{\{S'=p\}} 1_{\{J=j\}}\Bigg].$$

Then, using the law of total probability twice and that $\{B_{i,j,l,h} X_{i,l,h}\}_{h\in\mathbb{N}}$ are i.i.d. provides

$$\begin{aligned}
\mathbb{E}\big[S_i 1_{\{S=p\}}\big] &= \sum_{j\in\mathcal{J}} \sum_{l=0}^{n} \sum_{k=1}^{\infty} k \; \mathbb{E}\big[B_{i,j,l,k} X_{i,l,k} 1_{\{S'=p\}} 1_{\{J=j\}} 1_{\{Q_{j,l}=k\}}\big] \\
&= \sum_{j\in\mathcal{J}} \sum_{l=0}^{n} \sum_{k=1}^{\infty} k \sum_{\mu\in I_{i,p}} \mu \; \mathbb{P}[S' = p, J = j, Q_{j,l} = k, B_{i,j,l,k} X_{i,l,k} = \mu]. \qquad (6.12)
\end{aligned}$$

Define, for each $j \in \mathcal{J}$ and $l \in \{0, \ldots, n\}$, $\hat{S}_{j,l} = S' - \sum_{h=1}^{Q_{j,l}} \sum_{i=1}^{m} B_{i,j,l,h} X_{i,l,h}$ and

$$\hat{M}_{j,l}^{k} = \sum_{h=1}^{k-1} \sum_{i=1}^{m} B_{i,j,l,h} X_{i,l,h}, \qquad k \in \mathbb{N}.$$

Note that $\{B_{i,j,l,k} X_{i,l,k} = \mu\}$ with $\mu \in I_{i,p}$ implies $B_{i,j,l,k} = 1$ and $B_{i',j,l,k} = 0$ for $i' \neq i$. Thus we subtract and add, respectively, exactly those $B_{i,j,l,h} X_{i,l,h}$ we need. Hence

$$\begin{aligned}
&\{S' = p, J = j, Q_{j,l} = k, B_{i,j,l,k} X_{i,l,k} = \mu\} \\
&\qquad = \{\hat{S}_{j,l} + \hat{M}_{j,l}^{k} + B_{i,j,l,k} X_{i,l,k} = p, J = j, Q_{j,l} = k, B_{i,j,l,k} X_{i,l,k} = \mu\}.
\end{aligned}$$

Since $\{B_{i,j,l,k} X_{i,l,k} = \mu\}$ is independent of $\hat{S}_{j,l}$, $\hat{M}_{j,l}^{k}$, $Q_{j,l}$, and J because $\sum_{i=1}^{m} B_{i,j,l,h} X_{i,l,h}$ is added and subtracted, respectively, it follows that

$$\begin{aligned}
&\mathbb{P}[\hat{S}_{j,l} + \hat{M}_{j,l}^{k} + B_{i,j,l,k} X_{i,l,k} = p, J = j, Q_{j,l} = k, B_{i,j,l,k} X_{i,l,k} = \mu] \\
&\qquad = \mathbb{P}[\hat{S}_{j,l} + \hat{M}_{j,l}^{k} = p - \mu, J = j, Q_{j,l} = k] \; \mathbb{P}[B_{i,j,l,k} X_{i,l,k} = \mu]. \qquad (6.13)
\end{aligned}$$

By the conditional Poisson distribution of $Q_{j,l}$ given R_l for $j \in \mathcal{J}$ and $l \in \{0, \dots, n\}$ it follows that for $k \in \mathbb{N}$

$$\begin{aligned}\mathbb{P}[Q_{j,l} = k \,|\, R_l] &\overset{\text{a.s.}}{=} \frac{(\sum_{d=1}^m \lambda_{d,j} a_{d,l}^j R_l)^k}{k!} \exp\Big(-\sum_{d=1}^m \lambda_{d,j} a_{d,l}^j R_l\Big) \\ &\overset{\text{a.s.}}{=} \frac{\sum_{d=1}^m \lambda_{d,j} a_{d,l}^j R_l}{k} \, \mathbb{P}[Q_{j,l} = k-1 \,|\, R_l].\end{aligned}$$

Since $\{B_{i,j,l,h} X_{i,l,h}\}_{h\in\mathbb{N}}$ is a sequence of i.i.d. random variables, $\hat{M}_{j,l}^k$ are identically distributed for $k \in \mathbb{N}$. Further, due to the independence of $Q_{j,l}$ of $\hat{S}_{j,l}$ and $\hat{M}_{j,l}^k$

$$\begin{aligned}&\mathbb{P}[\hat{S}_{j,l} + \hat{M}_{j,l}^k = p - \mu, J = j, Q_{j,l} = k] \\ &\quad= \mathbb{E}\big[\,\mathbb{P}[\hat{S}_{j,l} + \hat{M}_{j,l}^k = p - \mu, J = j \,|\, R_1, \dots, R_n] \; \mathbb{P}[Q_{j,l} = k \,|\, R_l]\,\big] \\ &\quad= \frac{\sum_{d=1}^m \lambda_{d,j} a_{d,l}^j}{k} \; \mathbb{E}\big[R_l \, \mathbb{P}[\hat{S}_{j,l} + \hat{M}_{j,l}^k = p - \mu, J = j, Q_{j,l} = k-1 \,|\, R_l]\big] \qquad (6.14)\end{aligned}$$

holds. Note that

$$\begin{aligned}&\{\hat{S}_{j,l} + \hat{M}_{j,l}^k = p - \mu, J = j, Q_{j,l} = k-1, B_{i,j,l,k} X_{i,l,k} = \mu\} \\ &\qquad\qquad= \{S' = p - \mu, J = j, Q_{j,l} = k-1, B_{i,j,l,k} X_{i,l,k} = \mu\},\end{aligned}$$

and that for $h \in \mathbb{N}$ by Equation (3.2)

$$\mathbb{P}[B_{i,j,l,h} X_{i,l,h} = \mu] \;=\; \mathbb{P}[X_{i,l,h} = \mu] \, p_{i,j,l}. \qquad (6.15)$$

Hence we obtain because $p_{i,j,l} \sum_{d=1}^m \lambda_{d,j} a_{d,l}^j = \lambda_{i,j} a_{i,l}^j$ and $(M, R_1, \dots, R_n) \overset{\text{d}}{=} (N, R_1, \dots, R_n)$ and by insertion of Equations (6.14) and (6.15) into Equation (6.13) and Equation (6.13) into Equation (6.12)

$$\begin{aligned}\mathbb{E}\big[S_i 1_{\{S=p\}}\big] &= \sum_{j\in\mathcal{J}} \sum_{l=0}^n \sum_{k=1}^\infty \sum_{\mu\in I_{i,p}} \frac{\sum_{d=1}^m \lambda_{d,j} a_{d,l}^j}{k} p_{i,j,l} k \mu \; \mathbb{P}[X_{i,l,1} = \mu] \\ &\quad\times \mathbb{E}\big[R_l 1_{\{S'=p-\mu\}} 1_{\{J=j\}} 1_{\{Q_{j,l}=k-1\}}\big] \\ &= \sum_{j\in\mathcal{J}} \lambda_{i,j} \sum_{l=0}^n a_{i,l}^j \sum_{\mu\in I_{i,p}} \mu \; \mathbb{P}[X_{i,l,1} = \mu] \; \mathbb{E}\big[R_l 1_{\{S=p-\mu\}} 1_{\{J=j\}}\big]\,.\end{aligned}$$

q.e.d.

Now we consider a different construction of dependence between default cause intensities, namely the dependence by common continuous mixture distributions. The proof is very similar to that of Theorem 6.10, but it differs in some essential parts.

Lemma 6.16. *Let $T_1, \dots, T_n$ be strictly positive random variables and $\alpha_l, \beta_l > 0$, for $l \in \{1, \dots, n\}$ and $\lambda_{i,j} \geq 0$ for $i \in \{1, \dots, m\}$ and $j \in \mathcal{J}$ and let the Assumptions (a) and (b) of Lemma 4.1 be satisfied. Let for $i \in \{1, \dots, m\}$ the independent sequences $\{X_{i,h}\}_{h\in\mathbb{N}}$ of i.i.d. discrete random variables be independent of all previously mentioned random variables and let the negative part $X_{i,1}^-$ be integrable. Define for $p \in \mathbb{R}$ and $i \in \{1, \dots, m\}$ the set*

$$I_{i,p} := \{\mu \in \mathbb{R} \setminus \{0\} \,|\, \mathbb{P}[X_{i,1} = \mu, S = p - \mu] > 0\}.$$

Let $\{S_{i,j,l,h}\}_{h\in\mathbb{N}}$ *be independent sequences of i.i.d. random variables such that*

$$S_{i,j,l,h} = \sum_{k=1}^{Y_{j,l,h}} B_{i,j,l,h,k} X_{i,l,k}, \qquad h \in \mathbb{N},$$

for each $j \in \mathcal{J}$ *and* $l \in \{0, \dots, n\}$. *Then for every subportfolio* S_i *with* $i \in \{1, \dots, m\}$ *and dependence structure of the default cause intensities given in Assumption 2.18*

$$\begin{aligned}\mathbb{E}\big[S_i 1_{\{S=p\}}\big] &= \sum_{j\in\mathcal{J}} \sum_{l=1}^{n} (-\alpha_l) \ln(1-q_{j,l}) \sum_{\mu\in I_{i,p}} \mu \, \mathbb{P}[S_{i,j,l,1}=\mu] \, \mathbb{E}\big[T_l 1_{\{S=p-\mu, J=j\}}\big] \\ &\quad + \sum_{j\in\mathcal{J}} \sum_{\mu\in I_{i,p}} \sum_{d=1}^{m} \lambda_{d,j} a_{d,0}^{j} R_0 \mu \, \mathbb{P}[S_{i,j,0,1}=\mu] \, \mathbb{P}[S=p-\mu, J=j]\end{aligned}$$

holds for all $p \in \mathbb{R}$.

Proof. The proof goes along the lines of the proof of Theorem 6.10. By Lemma 4.1 the distribution of $(N_1, \dots, N_m)$ is equal to the distribution of

$$M = \sum_{j\in\mathcal{J}} 1_{\{J=j\}} \sum_{l=0}^{n} \sum_{h=1}^{P_{j,l}} \sum_{k=1}^{Y_{j,l,h}} B_{j,l,h,k},$$

where $(P_{j,0}, \dots, P_{j,n})$ and $\{(Y_{j,0,h}, \dots, Y_{j,n,h})\}_{h\in\mathbb{N}}$ and $\{(B_{j,0,h,k}, \dots, B_{j,n,h,k})\}_{h,k\in\mathbb{N}}$ are independent for each $j \in \mathcal{J}$ and given by Equations (4.4), (4.6) and (4.7). Thus we consider the representation $S' = \sum_{i=1}^{m} S'_i$ with

$$S'_i = \sum_{j\in\mathcal{J}} 1_{\{J=j\}} \sum_{l=0}^{n} \sum_{h=1}^{P_{j,l}} S_{i,j,l,h},$$

where for each $i \in \{1, \dots, m\}$ and $l \in \{0, \dots, n\}$ the random variables $\{X_{i,l,k}\}_{k\in\mathbb{N}}$ are independent copies of $\{X_{i,h}\}_{h\in\mathbb{N}}$. It is a natural assumption that $\{X_{i,l,k}\}_{k\in\mathbb{N}}$ is independent of $(P_{j,0}, \dots, P_{j,n})$ and $\{(Y_{j,0,h}, \dots, Y_{j,n,h})\}_{h\in\mathbb{N}}$ and $\{(B_{j,0,h,k}, \dots, B_{j,n,h,k})\}_{h,k\in\mathbb{N}}$ for each $i \in \{1, \dots, m\}$ and $l \in \{0, \dots, n\}$. Because of the equivalent representation $(S_1, \dots, S_m) \stackrel{\mathrm{d}}{=} (S'_1, \dots, S'_m)$ holds and thus also $(S_i, S) \stackrel{\mathrm{d}}{=} (S'_i, S')$. Using the definition of S'_i yields for each $i \in \{1, \dots, m\}$

$$\mathbb{E}\big[S_i 1_{\{S=p\}}\big] = \mathbb{E}\big[S'_i 1_{\{S'=p\}}\big] = \sum_{j\in\mathcal{J}} \sum_{l=0}^{n} \mathbb{E}\Bigg[\sum_{h=1}^{P_{j,l}} S_{i,j,l,h} 1_{\{S'=p\}} 1_{\{J=j\}}\Bigg].$$

Using the law of total probability twice and that $\{B_{i,j,l,h,k} X_{i,l,k}\}_{h,k\in\mathbb{N}}$ and $\{Y_{j,l,h}\}_{h\in\mathbb{N}}$ are i.i.d. provides

$$\begin{aligned}\mathbb{E}\big[S_i 1_{\{S=p\}}\big] &= \sum_{j\in\mathcal{J}} \sum_{l=0}^{n} \sum_{r=1}^{\infty} r \, \mathbb{E}\big[S_{i,j,l,r} 1_{\{S'=p\}} 1_{\{J=j\}} 1_{\{P_{j,l}=r\}}\big] \\ &= \sum_{j\in\mathcal{J}} \sum_{l=0}^{n} \sum_{r=1}^{\infty} r \sum_{\mu\in I_{i,p}} \mu \, \mathbb{P}[S'=p, J=j, P_{j,l}=r, S_{i,j,l,r}=\mu]. \qquad (6.17)\end{aligned}$$

Let, for each $j \in \mathcal{J}$ and $l \in \{0, \dots, n\}$, $\hat{S}_{j,l} = S' - \sum_{h=1}^{P_{j,l}} \sum_{i=1}^{m} S_{i,j,l,h}$ and

$$\hat{P}^r_{j,l} = \sum_{h=1}^{r-1} \sum_{i=1}^{m} S_{i,j,l,h}, \qquad r \in \mathbb{N}.$$

Note that $\{S_{i,j,l,r} = \mu\}$ with $\mu \in I_{i,p}$ implies $B_{i,j,l,r,k} = 1$ and consequently $B_{i',j,l,r,k} = 0$ for $i' \neq i$, thus we subtract and add, respectively, exactly those $S_{i,j,l,h}$ we therefore need, hence

$$\begin{aligned} &\{S' = p, J = j, P_{j,l} = r, S_{i,j,l,r} = \mu\} \\ &\qquad\qquad = \{\hat{S}_{j,l} + \hat{P}^r_{j,l} + S_{i,j,l,r} = p, J = j, P_{j,l} = r, S_{i,j,l,r} = \mu\}. \end{aligned}$$

Since $\{S_{i,j,l,r} = \mu\}$ is independent of $\hat{S}_{j,l}$, $\hat{P}^r_{j,l}$, $P_{j,l}$, and J because we subtract or add $S_{i,j,l,h}$,respectively, it follows for $j \in \mathcal{J}$ and $l \in \{0, \dots, n\}$ that

$$\begin{aligned} &\mathbb{P}[\hat{S}_{j,l} + \hat{P}^r_{j,l} + S_{i,j,l,r} = p, J = j, P_{j,l} = r, S_{i,j,l,r} = \mu] \\ &\qquad = \mathbb{P}[\hat{S}_{j,l} + \hat{P}^r_{j,l} = p - \mu, J = j, P_{j,l} = r]\ \mathbb{P}[S_{i,j,l,r} = \mu]. \end{aligned} \tag{6.18}$$

By the conditional Poisson distribution of $P_{j,l}$ given T_l for $j \in \mathcal{J}$ and $l \in \{1, \dots, n\}$ it follows that for $r \in \mathbb{N}$

$$\begin{aligned} \mathbb{P}[P_{j,l} = r \,|\, T_l] &\overset{\text{a.s.}}{=} \frac{(-\alpha_l \ln(1 - q_{j,l}) T_l)^r}{r!} \exp\bigl(-(-\alpha_l \ln(1 - q_{j,l}) T_l)\bigr) \\ &\overset{\text{a.s.}}{=} \frac{-\alpha_l \ln(1 - q_{j,l}) T_l}{r}\ \mathbb{P}[P_{j,l} = r - 1 \,|\, T_l], \end{aligned}$$

and for $l = 0$ and $r \in \mathbb{N}$ we have

$$\begin{aligned} \mathbb{P}[P_{j,0} = r] &= \frac{(\sum_{d=1}^{m} \lambda_{d,j} a^j_{d,0} R_0)^r}{r!} \exp\Bigl(-\sum_{d=1}^{m} \lambda_{d,j} a^j_{d,0} R_0\Bigr) \\ &= \frac{\sum_{d=1}^{m} \lambda_{d,j} a^j_{d,0} R_0}{r}\ \mathbb{P}[P_{j,0} = r - 1]. \end{aligned}$$

Since $\{B_{i,j,l,r,k} X_{i,l,k}\}_{k,r \in \mathbb{N}}$ and $\{Y_{j,l,r}\}_{r \in \mathbb{N}}$ are sequences of i.i.d. random variables, $\hat{P}^r_{j,l}$ are identically distributed for $r \in \mathbb{N}$. Further, due to the independence of $\hat{P}^r_{j,l}$ and $\hat{S}_{j,l}$ of $P_{j,l}$ for $l \in \{1, \dots, n\}$

$$\begin{aligned} &\mathbb{P}[\hat{S}_{j,l} + \hat{P}^r_{j,l} = p - \mu, J = j, P_{j,l} = r] \\ &\quad = \mathbb{E}\bigl[\mathbb{P}[\hat{S}_{j,l} + \hat{P}^r_{j,l} = p - \mu, J = j \,|\, T_1, \dots, T_n]\ \mathbb{P}[P_{j,l} = r \,|\, T_l]\bigr] \\ &\quad = \frac{-\alpha_l \ln(1 - q_{j,l})}{r}\ \mathbb{E}\bigl[T_l\ \mathbb{P}[\hat{S}_{j,l} + \hat{P}^r_{j,l} = p - \mu, J = j, P_{j,l} = r - 1 \,|\, T_l]\bigr], \end{aligned} \tag{6.19}$$

and for $l = 0$ because $P_{j,0}$ is independent of $\hat{S}_{j,0}$ and $P^r_{j,0}$

$$\begin{aligned} &\mathbb{P}[\hat{S}_{j,0} + \hat{P}^r_{j,0} = p - \mu, J = j, P_{j,0} = r] \\ &\quad = \mathbb{P}[\hat{S}_{j,0} + \hat{P}^r_{j,0} = p - \mu, J = j]\ \mathbb{P}[P_{j,0} = r] \\ &\quad = \frac{\sum_{d=1}^{m} \lambda_{d,j} a^j_{d,0} R_0}{r}\ \mathbb{P}[\hat{S}_{j,0} + \hat{P}^r_{j,0} = p - \mu, J = j, P_{j,0} = r - 1] \end{aligned} \tag{6.20}$$

holds. Note that for $j \in \mathcal{J}$ and $l \in \{0, \dots, n\}$

$$\begin{aligned}\{\hat{S}_{j,l} + \hat{P}^r_{j,l} = p - \mu, J = j, P_{j,l} = r - 1, S_{i,j,l,r} = \mu\}& \\ = \{S' = p - \mu, J = j, P_{j,l} = r - 1, S_{i,j,l,r} = \mu\}&.\end{aligned}$$

Hence we obtain because $(M, T_1, \dots, T_n) \overset{\mathrm{d}}{=} (N, T_1, \dots, T_n)$, and by insertion of Equations (6.19) and (6.20) into Equation (6.18) and Equation (6.18) into Equation (6.17)

$$\begin{aligned}\mathbb{E}\big[S_i 1_{\{S=p\}}\big] &= \sum_{j\in\mathcal{J}} \sum_{l=1}^{n} \sum_{r=1}^{\infty} \sum_{\mu\in I_{i,p}} r \frac{(-\alpha_l)\ln(1-q_{j,l})}{r} \mu \; \mathbb{P}[S_{i,j,l,r} = \mu] \\ &\quad \times \mathbb{E}\Big[T_l 1_{\{S'=p-\mu\}} 1_{\{J=j\}} 1_{\{P_{j,l}=r-1\}}\Big] \\ &\quad + \sum_{j\in\mathcal{J}} \sum_{r=1}^{\infty} \sum_{\mu\in I_{i,p}} r \frac{\sum_{d=1}^{m} \lambda_{d,j} a^j_{d,0} R_0}{r} \mu \; \mathbb{P}[S_{i,j,0,r} = \mu] \\ &\quad \times \mathbb{P}[S' = p - \mu, J = j, P_{j,0} = r - 1] \\ &= \sum_{j\in\mathcal{J}} \sum_{l=1}^{n} (-\alpha_l)\ln(1-q_{j,l}) \sum_{\mu\in I_{i,p}} \mu \; \mathbb{P}[S_{i,j,l,1} = \mu] \; \mathbb{E}\big[T_l 1_{\{S=p-\mu, J=j\}}\big] \\ &\quad + \sum_{j\in\mathcal{J}} \sum_{\mu\in I_{i,p}} \sum_{d=1}^{m} \lambda_{d,j} a^j_{d,0} R_0 \mu \; \mathbb{P}[S_{i,j,0,1} = \mu] \; \mathbb{P}[S = p - \mu, J = j].\end{aligned}$$

q.e.d.

6.3 Conditional Expected Shortfall Contributions in a Multiperiod Model

In this section we consider non-trivial σ-algebras $\mathcal{G}$ that also reflect a model with d periods and a filtration $\mathcal{G}_0, \dots, \mathcal{G}_{d-1}$ for $d \in \mathbb{N}$. Hence we consider conditional expected shortfall contributions. Then we consider multivariate claim sizes $\{(X_{1,i,h}, \dots, X_{d,i,h})\}_{h\in\mathbb{N}}$ for $i \in \{1, \dots, m\}$. Each component of such a claim size corresponds to a time period. This implies that we choose a filtration of σ-algebras $\mathcal{G}_r$ with $r \in \{0, \dots, d-1\}$. In time period r we condition on the claims of the $r-1$ preceding time periods. This means that at time period r the preceding claim sizes and numbers are known. Because we are still interested in the entire portfolio loss S, we aggregate over all time periods, i.e. we consider

$$S^\star = \sum_{i=1}^{m} \sum_{h=1}^{N_i} \sum_{r=1}^{d} X_{r,i,h}.$$

In the same manner we are interested in aggregated subportfolios. This requires some further notation and also a decomposition of the claim numbers according to the period in which they occur. We make this precise in the following lemma.

Lemma 6.21. *Let Assumption 2.18 be satisfied and let $N_1, \dots, N_m$ be random variables as in Equation (2.20) and let $\{(X_{1,i,h}, \dots, X_{d,i,h})\}_{h\in\mathbb{N}}$ be independent sequences independent of $(N_1, \dots, N_m, J, R_1, \dots, R_n)$, consisting of i.i.d. $\mathbb{R}^d$-valued random vectors for $i \in \{1, \dots, m\}$.*

Define for $i \in \{1, \ldots, m\}$ *and* $r \in \{1, \ldots, d\}$

$$N_{i,r} = \sum_{h=1}^{N_i} 1_{\{X_{1,i,h}=\cdots=X_{r-1,i,h}=0, X_{r,i,h}\neq 0\}}. \tag{6.22}$$

Then the following holds:

(a) The random variables $N_{1,1}, \ldots, N_{m,1}, \ldots, N_{d,1}, \ldots, N_{m,d}$ *are conditionally independent given* $J, R_1, \ldots, R_n$

(b) For $i \in \{1, \ldots, m\}$ *and* $r \in \{1, \ldots, d\}$ *the random variable* $N_{i,r}$ *is given by the conditional distribution*

$$\begin{aligned}\mathcal{L}(N_{i,r}\,|\,J, R_1, \ldots, R_n) &\overset{a.s.}{=} \mathcal{L}(N_{i,r}\,|\,J, \Lambda_i)\\ &\overset{a.s.}{=} \text{Poisson}(\lambda_{i,J}\Lambda_i\ \mathbb{P}[X_{1,i,1} = \cdots = X_{r-1,i,1} = 0, X_{r,i,1} \neq 0]\,).\end{aligned} \tag{6.23}$$

Furthermore define for $i \in \{1, \ldots, m\}$ *and* $r \in \{1, \ldots, d\}$

$$S_{i,r} = \sum_{h=1}^{N_{i,r}} Y_{i,r,h},$$

where $\{Y_{i,r,h}\}_{h\in\mathbb{N}}$ *is a sequence independent of* $(N_{i,1}, \ldots, N_{i,d})$*, consisting of i.i.d. d-dimensional random vectors with* $\mathcal{L}(Y_{i,r,1}) = \mathcal{L}(X_{i,1}\,|\,X_{1,i,1} = \cdots = X_{r-1,i,1} = 0, X_{r,i,1} \neq 0)$ *if* $\mathbb{P}[X_{1,i,1} = \cdots = X_{r-1,i,1} = 0, X_{r,i,1} \neq 0] > 0$*, otherwise take another arbitrary distribution. Then for* $i \in \{1, \ldots, m\}$

(c) the random variables $S_{i,1}, \ldots, S_{i,d}$ *are conditionally independent given* $J, R_1, \ldots, R_n$ *and*

(d) the random variables S_i *and* $S_{i,1} + \cdots + S_{i,d}$ *are equal in distribution.*

Proof. We apply Remark 2.1. We first compute the probability-generating function of the conditional distribution of $\bar{N} = (N_{1,1}, \ldots, N_{m,1}, \ldots, N_{1,d}, \ldots, N_{m,d})$ given $J, R_1, \ldots, R_n$ for $z \in [0,1]^{dm}$ with $z = (z_{1,1}, \ldots, z_{m,1}, \ldots, z_{1,d}, \ldots, z_{m,d})$. Using the conditional independence of $N_1, \ldots, N_m$ given $J, R_1, \ldots, R_n$ and that $\{X_{i,h}\}_{h\in\mathbb{N}}$ are independent of N_i for each $i \in \{1, \ldots, m\}$ and i.i.d., we observe

$$\begin{aligned}G_{\bar{N}\,|\,(J,R_1,\ldots,R_n)}(z) &\overset{\text{a.s.}}{=} \prod_{i=1}^{m} \mathbb{E}\Bigg[\prod_{r=1}^{d} z_{i,r}^{\sum_{h=1}^{N_i} 1_{\{X_{1,i,h}=\cdots=X_{r-1,i,h}=0,X_{r,i,h}\neq 0\}}}\,\Bigg|\,J, R_1, \ldots, R_n\Bigg]\\ &\overset{\text{a.s.}}{=} \prod_{i=1}^{m} \mathbb{E}\Bigg[\prod_{r=1}^{d}\Big(\mathbb{E}\Big[z_{i,r}^{1_{\{X_{1,i,1}=\cdots=X_{r-1,i,1}=0,X_{r,i,1}\neq 0\}}}\Big]\Big)^{N_i}\,\Bigg|\,J, R_1, \ldots, R_n\Bigg].\end{aligned}$$

Note that $1_{\{X_{1,i,1}\neq 0\}}, \ldots, 1_{\{X_{1,i,1}=\cdots=X_{d-1,i,1}=0,X_{d,i,1}\neq 0\}}$ are independent for each $i \in \{1, \ldots, m\}$ because the sets $\{X_{1,i,h} = \cdots = X_{r-1,i,h} = 0, X_{r,i,h} \neq 0\}$ are disjoint. By the conditional distribution of N_i given $J, R_1, \ldots, R_n$ and the distribution of the Bernoulli random variable $1_{\{X_{1,i,1}=\cdots=X_{r-1,i,1}=0,X_{r,i,1}\neq 0\}}$, we obtain

$$\begin{aligned}G_{\bar{N}\,|\,(J,R_1,\ldots,R_n)}(z) &\overset{\text{a.s.}}{=} \prod_{i=1}^{m}\prod_{r=1}^{d} \exp\big(-\lambda_{i,J}\Lambda_i\big(1 - \mathbb{E}\big[z_{i,r}^{1_{\{X_{1,i,1}=\cdots=X_{r-1,i,1}=0,X_{r,i,1}\neq 0\}}}\big]\big)\big)\\ &\overset{\text{a.s.}}{=} \prod_{i=1}^{m}\prod_{r=1}^{d} \exp(-\lambda_{i,J}\Lambda_i\ \mathbb{P}[X_{1,i,1} = \cdots = X_{r-1,i,1} = 0, X_{r,i,1} \neq 0](1 - z_{i,r})),\end{aligned}$$

which proves the claim for the conditional distribution of $N_{i,r}$ given $J, R_1, \dots, R_n$ and the conditional independence of $N_{1,1}, \dots, N_{m,1}, \dots, N_{1,d}, \dots, N_{m,d}$ given $J, R_1, \dots, R_n$.

Now we calculate the characteristic function of the conditional distribution of $S_{i,1} + \dots + S_{i,d}$ given $J, R_1, \dots, R_n$ with $Y_{i,r,h} = (Y_{1,i,r,h}, \dots, Y_{d,i,r,h})$ and $i \in \{1, \dots, m\}$ and $r \in \{1, \dots, d\}$ for $z \in \mathbb{R}^d$

$$\varphi_{S_{i,1}+\dots+S_{i,d}\,|\,(J,R_1,\dots,R_n)}(z) \overset{\text{a.s.}}{=} \mathbb{E}\Bigg[\prod_{s=1}^{d} \exp\Bigg(\sum_{r=1}^{d}\sum_{h=1}^{N_{i,r}} Y_{s,i,r,h}\,\mathrm{i}\,z_s\Bigg)\,\Bigg|\,J, R_1, \dots, R_n\Bigg].$$

Because the sets $\{X_{1,i,h} = \dots = X_{r-1,i,h} = 0, X_{r,i,h} \neq 0\}$ are disjoint for $r \in \{1, \dots, d\}$, the random variables $N_{i,1}, \dots, N_{i,d}$ are conditionally independent given $J, R_1, \dots, R_n$, hence we have

$$\varphi_{S_{i,1}+\dots+S_{i,d}\,|\,(J,R_1,\dots,R_n)}(z) \overset{\text{a.s.}}{=} \prod_{r=1}^{d} \mathbb{E}\Bigg[\exp\Bigg(\sum_{h=1}^{N_{i,r}}\sum_{s=1}^{d} Y_{s,i,r,h}\,\mathrm{i}\,z_s\Bigg)\,\Bigg|\,J, R_1, \dots, R_n\Bigg].$$

Using that the sequences $\{Y_{i,r,h}\}_{h\in\mathbb{N}}$ are i.i.d. and independent of $(N_{i,1}, \dots, N_{i,d})$ for $r \in \{1, \dots, d\}$ and applying the conditional distribution of $N_{i,r}$ given $J, R_1, \dots, R_n$ for $r \in \{1, \dots, d\}$ yields

$$\begin{aligned}
\varphi_{S_{i,1}+\dots+S_{i,d}\,|\,(J,R_1,\dots,R_n)}(z) &\overset{\text{a.s.}}{=} \prod_{r=1}^{d} \mathbb{E}\Bigg[\Bigg(\mathbb{E}\Bigg[\exp\Bigg(\sum_{s=1}^{d} Y_{s,i,r,1}\,\mathrm{i}\,z_s\Bigg)\Bigg]\Bigg)^{N_{i,r}}\,\Bigg|\,J, R_1, \dots, R_n\Bigg] \\
&\overset{\text{a.s.}}{=} \prod_{r=1}^{d} \exp\big(-\lambda_{i,J}\Lambda_i\,\mathbb{P}[X_{1,i,1} = \dots = X_{r-1,i,1} = 0, X_{r,i,1} \neq 0]\,(1 - \varphi_{Y_{i,r,1}}(z))\big), \quad (6.24)
\end{aligned}$$

where $\varphi_{Y_{i,r,1}}(z)$ denotes the characteristic function of the distribution of $Y_{i,r,1}$. Hence $S_{i,1}, \dots, S_{i,d}$ are conditionally independent given $J, R_1, \dots, R_n$. The characteristic function of the distribution of $Y_{i,r,1}$ can be evaluated as follows for $z \in \mathbb{R}^d$

$$\begin{aligned}
\varphi_{Y_{i,r,1}}(z) &= \mathbb{E}\Bigg[\prod_{s=1}^{d} \mathrm{e}^{\mathrm{i}\,z_s X_{s,i,1}}\,\Bigg|\,X_{1,i,1} = \dots = X_{r-1,i,1} = 0, X_{r,i,1} \neq 0\Bigg] \\
&= \frac{\mathbb{E}\big[\prod_{s=1}^{d} \mathrm{e}^{\mathrm{i}\,z_s X_{s,i,1}}\,1_{\{X_{1,i,1}=\dots=X_{r-1,i,1}=0, X_{r,i,1}\neq 0\}}\big]}{\mathbb{P}[X_{1,i,1} = \dots = X_{r-1,i,1} = 0, X_{r,i,1} \neq 0]}. \quad (6.25)
\end{aligned}$$

Inserting this term into Equation (6.24) and using $\sum_{r=1}^{d} \mathbb{P}[X_{1,i,1} = \dots = X_{r-1,i,1} = 0, X_{r,i,1} \neq 0] = 1$ provides

$$\begin{aligned}
&\varphi_{S_{i,1}+\dots+S_{i,d}\,|\,(J,R_1,\dots,R_n)}(z) \\
&\quad\overset{\text{a.s.}}{=} \exp\Bigg(-\lambda_{i,J}\Lambda_i\Bigg(1 - \sum_{r=1}^{d} \mathbb{E}\Bigg[\prod_{s=1}^{d} \mathrm{e}^{\mathrm{i}\,z_s X_{s,i,1}}\,1_{\{X_{1,i,1}=\dots=X_{r-1,i,1}=0, X_{r,i,1}\neq 0\}}\Bigg]\Bigg)\Bigg) \\
&\quad\overset{\text{a.s.}}{=} \exp\Bigg(-\lambda_{i,J}\Lambda_i\Bigg(1 - \mathbb{E}\Bigg[\prod_{s=1}^{d} \mathrm{e}^{\mathrm{i}\,z_s X_{s,i,1}}\Bigg]\Bigg)\Bigg).
\end{aligned}$$

Computing the characteristic function of the distribution of $S_{i,1} + \dots + S_{i,d}$ gives for $z \in \mathbb{R}^d$

$$\varphi_{S_{i,1}+\dots+S_{i,d}}(z) = \mathbb{E}\Bigg[\prod_{s=1}^{d} \mathrm{e}^{\mathrm{i}\,z_s \sum_{r=1}^{d}\sum_{h=1}^{N_{i,r}}\sum_{s=1}^{d} Y_{s,i,r,h}}\Bigg] = \mathbb{E}\big[\varphi_{S_{i,1}+\dots+S_{i,d}\,|\,(J,R_1,\dots,R_n)}(z)\big].$$

Noting Assumption 2.18 and $S_i = \sum_{h=1}^{N_i} X_{i,h}$, it is easy to see that $S_i \stackrel{\mathrm{d}}{=} S_{i,1} + \cdots + S_{i,d}$.
q.e.d.

This is a useful representation that we will employ for a recursion of the conditional expected shortfall. It is, however, a problem that for each $i \in \{1, \ldots, m\}$ the random variables $N_{i,1}, \ldots, N_{i,d}$ are not independent of each other but only conditionally given $J, R_1, \ldots, R_n$. The reason is that each of the $N_{i,r}$ with $r \in \{1, \ldots, d\}$ depends on the same Λ_i. Therefore it is necessary to replace the $N_{i,1}, \ldots, N_{i,d}$ with conditional distributions that use a different mixing distribution. This is done by a change of the density. We put it into the following corollary:

Corollary 6.26. *Let $k_1, \ldots, k_{r-1}, p \in \mathbb{N}$ for $r \in \{2, \ldots, d\}$. Fix $i \in \{1, \ldots, m\}$. Let the conditions of Lemma 6.21 be satisfied. Then for each $r \in \{2, \ldots, d\}$*

$$\begin{aligned}\mathbb{P}[N_{i,r} = p \,|\, N_{i,r-1} = k_{r-1}, \ldots, N_{i,1} = k_1]& \\ &= \frac{\mathbb{E}\big[\mathrm{e}^{-\lambda_{i,J}\Lambda_i p_{i,r}} \frac{(\lambda_{i,J}\Lambda_i p_{i,r})^p}{p!} \prod_{l=1}^{r-1} \mathrm{e}^{-\lambda_{i,J}\Lambda_i p_{i,l}} \frac{(\lambda_{i,J}\Lambda_i p_{i,l})^{k_l}}{k_l!}\big]}{\mathbb{E}\big[\prod_{l=1}^{r-1} \mathrm{e}^{-\lambda_{i,J}\Lambda_i p_{i,l}} \frac{(\lambda_{i,J}\Lambda_i p_{i,l})^{k_l}}{k_l!}\big]},\end{aligned}$$

where for each $l \in \{1, \ldots, r\}$

$$p_{i,l} = \mathbb{P}[X_{1,i,1} = \cdots = X_{l-1,i,1} = 0, X_{l,i,1} \neq 0].$$

Proof. By using the definition of the conditional probability we compute for each $r \in \{2, \ldots, d\}$

$$\mathbb{P}[N_{i,r} = p \,|\, N_{i,r-1} = k_{r-1}, \ldots, N_{i,1} = k_1] = \frac{\mathbb{P}[N_{i,r} = p, N_{i,r-1} = k_{r-1} \ldots, N_{i,1} = k_1]}{\mathbb{P}[N_{i,r-1} = k_{r-1} \ldots, N_{i,1} = k_1]}.$$

Conditioning on $J, R_1, \ldots, R_n$ and using the conditional independence of $N_{i,1}, \ldots, N_{i,r}$ given $J, R_1, \ldots, R_n$ yields

$$\begin{aligned}\mathbb{P}[N_{i,r} = p \,|\, N_{i,r-1} = k_{r-1}, \ldots, N_{i,1} = k_1]& \\ &= \frac{\mathbb{E}\big[\,\mathbb{P}[N_{i,r} = p \,|\, J, R_1, \ldots, R_n] \prod_{l=1}^{r-1} \mathbb{P}[N_{i,l} = k_l \,|\, J, R_1, \ldots, R_n]\,\big]}{\mathbb{E}\big[\prod_{l=1}^{r-1} \mathbb{P}[N_{i,l} = k_l \,|\, J, R_1, \ldots, R_n]\,\big]}.\end{aligned}$$

Using the conditional distributions of $N_{i,1}, \ldots, N_{i,r}$ given $J, R_1, \ldots, R_n$ given in Equation (6.23) provides

$$\begin{aligned}\mathbb{P}[N_{i,r} = p \,|\, N_{i,r-1} = k_{r-1}, \ldots, N_{i,1} = k_1]& \\ &= \frac{\mathbb{E}\big[\mathrm{e}^{-\lambda_{i,J}\Lambda_i p_{i,r}} \frac{(\lambda_{i,J}\Lambda_i p_{i,r})^p}{p!} \prod_{l=1}^{r-1} \mathrm{e}^{-\lambda_{i,J}\Lambda_i p_{i,l}} \frac{(\lambda_{i,J}\Lambda_i p_{i,l})^{k_l}}{k_l!}\big]}{\mathbb{E}\big[\prod_{l=1}^{r-1} \mathrm{e}^{-\lambda_{i,J}\Lambda_i p_{i,l}} \frac{(\lambda_{i,J}\Lambda_i p_{i,l})^{k_l}}{k_l!}\big]}.\end{aligned}$$

q.e.d.

The next corollary proves that these conditional distributions provide the same distribution for the random sum S_i for $i \in \{1, \ldots, m\}$ and can therefore be used for our model.

Corollary 6.27. *Let Assumption 2.18 and the assumptions of Lemma 6.21 be satisfied. For each $i \in \{1, \dots, m\}$ define $S_i = \sum_{h=1}^{N_i} X_{i,h}$. Further, for each $i \in \{1, \dots, m\}$ let $P_{i,1}$ be a random variable such that $\mathcal{L}(P_{i,1}) = \mathcal{L}(N_{i,1})$ and for $r \in \{2, \dots, d\}$ let $P_{i,r}$ be a random variable such that $\mathcal{L}(P_{i,r}) = \mathcal{L}(N_{i,r} \,|\, N_{i,r-1}, \dots, N_{i,1})$. Then let for each $r \in \{1, \dots, d\}$*

$$\bar{S}_{i,r} = \sum_{h=1}^{P_{i,r}} Y_{i,r,h}.$$

Then S_i and $\bar{S}_{i,1} + \dots + \bar{S}_{i,d}$ are equal in distribution for each $i \in \{1, \dots, m\}$.

Proof. Let $i \in \{1, \dots, m\}$. We apply Remark 2.1 and compute the characteristic function of $\bar{S}_{i,1} + \dots + \bar{S}_{i,d}$ for $z \in \mathbb{R}^d$ using that $\{Y_{i,r,h}\}_{h \in \mathbb{N}}$ are i.i.d. and independent of $P_{i,r}$ for each $r \in \{1, \dots, d\}$

$$\begin{aligned}
\varphi_{\bar{S}_{i,1}+\dots+\bar{S}_{i,d}}(z) &= \mathbb{E}\Bigg[\prod_{s=1}^{d} \exp\Bigg(\sum_{r=1}^{d} \sum_{h=1}^{P_{i,r}} Y_{s,i,r,h}\, \mathrm{i}\, z_s\Bigg)\Bigg] \\
&= \mathbb{E}\Bigg[\prod_{r=1}^{d} \Bigg(\mathbb{E}\Bigg[\exp\Bigg(\sum_{s=1}^{d} Y_{s,i,r,1}\, \mathrm{i}\, z_s\Bigg)\Bigg]\Bigg)^{P_{i,r}}\Bigg].
\end{aligned}$$

Applying for each $r \in \{1, \dots, d\}$ the distribution of $P_{i,r}$ and using the conditional independence of $N_{i,r}$ given $N_{i,1}, \dots, N_{i,r-1}$ for each $r \in \{1, \dots, d\}$ gives

$$\varphi_{\bar{S}_{i,1}+\dots+\bar{S}_{i,d}}(z) = \mathbb{E}\big[\big(\varphi_{Y_{i,1,1}}(z)\big)^{N_{i,1}}\big] \prod_{r=2}^{d} \mathbb{E}\big[\big(\varphi_{Y_{i,r,1}}(z)\big)^{N_{i,r}} \,\big|\, N_{i,r-1}, \dots, N_{i,1}\big],$$

where $\varphi_{Y_{i,r,1}}(z)$ is the characteristic function of the distribution of $Y_{i,r,1}$. An application of the law of total probability provides

$$\begin{aligned}
\varphi_{\bar{S}_{i,1}+\dots+\bar{S}_{i,d}}(z) = {} & \mathbb{E}\big[\big(\varphi_{Y_{i,1,1}}(z)\big)^{N_{i,1}}\big] \\
& \times \prod_{r=2}^{d} \sum_{k_{r-1} \in \mathbb{N}_0} \cdots \sum_{k_1 \in \mathbb{N}_0} \mathbb{E}\big[\big(\varphi_{Y_{i,r,1}}(z)\big)^{N_{i,r}} \,\big|\, N_{i,r-1} = k_{r-1}, \dots, N_{i,1} = k_1\big] \\
& \times \mathbb{P}[N_{i,r-1} = k_{r-1}, \dots, N_{i,1} = k_1].
\end{aligned}$$

Conditioning on $J, R_1, \dots, R_n$ yields

$$\begin{aligned}
\varphi_{\bar{S}_{i,1}+\dots+\bar{S}_{i,d}}(z) = {} & \mathbb{E}\big[\big(\varphi_{Y_{i,1,1}}(z)\big)^{N_{i,1}}\big] \\
& \times \prod_{r=2}^{d} \sum_{k_{r-1} \in \mathbb{N}_0} \cdots \sum_{k_1 \in \mathbb{N}_0} \mathbb{E}\big[\big(\varphi_{Y_{i,r,1}}(z)\big)^{N_{i,r}} \,\big|\, N_{i,r-1} = k_{r-1}, \dots, N_{i,1} = k_1\big] \\
& \times \mathbb{E}\big[\mathbb{P}[N_{i,r-1} = k_{r-1}, \dots, N_{i,1} = k_1 \,|\, J, R_1, \dots, R_n]\big].
\end{aligned}$$

An application of Corollary 6.26 provides

$$\begin{aligned}
\varphi_{\bar{S}_{i,1}+\cdots+\bar{S}_{i,d}}(z) &= \mathbb{E}\big[\big(\varphi_{Y_{i,1,1}}(z)\big)^{N_{i,1}}\big] \\
&\quad\times \prod_{r=2}^{d} \sum_{k_{r-1}\in\mathbb{N}_0} \cdots \sum_{k_1\in\mathbb{N}_0} \sum_{p\in\mathbb{N}_0} \big(\varphi_{Y_{i,r,1}}(z)\big)^p \, \mathbb{E}\Big[\mathrm{e}^{-\lambda_{i,J}\Lambda_i p_{i,r}} \frac{(\lambda_{i,J}\Lambda_i p_{i,r})^p}{p!} \\
&\quad\times \prod_{l=1}^{r-1} \mathrm{e}^{-\lambda_{i,J}\Lambda_i p_{i,l}} \frac{(\lambda_{i,J}\Lambda_i p_{i,l})^{k_l}}{k_l!}\Big] \\
&= \mathbb{E}\big[\big(\varphi_{Y_{i,1,1}}(z)\big)^{N_{i,1}}\big] \prod_{r=2}^{d} \sum_{p\in\mathbb{N}_0} \mathbb{E}\Big[\mathrm{e}^{-\lambda_{i,J}\Lambda_i p_{i,r}} \frac{(\lambda_{i,J}\Lambda_i p_{i,r}\varphi_{Y_{i,r,1}}(z))^p}{p!}\Big].
\end{aligned}$$

Since $\mathbb{E}\big[\exp(-\lambda_{i,J}\Lambda_i p_{i,r}(1-\varphi_{Y_{i,r,1}}(z)))\big] = \mathbb{E}[(\varphi_{Y_{i,r,1}}(z))^{N_{i,r}}]$ for $r \in \{1,\ldots,d\}$, this can be simplified as follows

$$\varphi_{\bar{S}_{i,1}+\cdots+\bar{S}_{i,d}}(z) = \mathbb{E}\big[\big(\varphi_{Y_{i,1,1}}(z)\big)^{N_{i,1}}\big] \prod_{r=2}^{d} \mathbb{E}\big[\big(\varphi_{Y_{i,r,1}}(z)\big)^{N_{i,r}}\big]. \tag{6.28}$$

The characteristic function of the distribution of $Y_{i,r,1}$ can be evaluated as in Equation (6.25). A comparison with Equation (6.24) shows that the expected value of Equation (6.24) yields the right-hand side of Equation (6.28), hence $\bar{S}_{i,1}+\cdots+\bar{S}_{i,d}$ and S_i are equal in distribution for each $i \in \{1,\ldots,m\}$. q.e.d.

This corollary shows that the random sums S_i can be decomposed into components corresponding to the individual periods such that

$$S_i \overset{\mathrm{d}}{=} \bar{S}_{i,1}+\cdots+\bar{S}_{i,d},$$

where S_i is a d-dimensional random vector for each $i \in \{1,\ldots,m\}$. Accordingly, the aggregated subportfolios are given by

$$\bar{S}_i^\star = \sum_{r=1}^{d} \sum_{h=1}^{P_{i,r}} \sum_{s=1}^{d} Y_{s,i,r,h},$$

and it is easily seen that $\bar{S}_{i,r} \overset{\mathrm{d}}{=} S_{i,r}$. Accordingly we define $\bar{S}^\star = \sum_{i=1}^m \bar{S}_i^\star$. The corresponding σ-algebras can be chosen such that

$$\mathcal{G}_r = \sigma(P_{i,1},\ldots,P_{i,r},\{Y_{i,1,h}\}_{h\in\mathbb{N}},\ldots,\{Y_{i,r,h}\}_{h\in\mathbb{N}}, i\in\{1,\ldots,m\}). \tag{6.29}$$

for $r \in \{1,\ldots,d-1\}$. For $r=0$ we set $\mathcal{G}_0 = \{\emptyset,\Omega\}$. We therefore consider exactly the claim numbers and claim sizes that have occurred in the preceding r time periods.

Using these preparations, we can compute the conditional expected shortfall contributions as follows. First, let $r=0$ and we have

$$\mathrm{ES}_\delta\big[\bar{S}_i^\star, \bar{S}^\star \,|\, \mathcal{G}_0\big] = \mathrm{ES}_\delta\big[\bar{S}_i^\star, \bar{S}^\star\big],$$

which is the standard expected shortfall risk contribution.

Let now $r \in \{1,\ldots,d-1\}$. Hence $\mathcal{G}_r$ is given by Equation (6.29) and non-empty. Let $\bar{S}_{i,r'} = (\bar{S}_{1,i,r'},\ldots,\bar{S}_{d,i,r'})$ such that $\bar{S}_{s,i,r'} = \sum_{h=1}^{P_{i,r'}} Y_{s,i,r',h}$ for $r', s \in \{1,\ldots,d\}$. By [34] we can use the linearity of the conditional expected shortfall, and because $\bar{S}_{s,i,1},\dots,\bar{S}_{s,i,r}$ are $\mathcal{G}_r$-measurable for $s\in\{1,\dots,d\}$, we obtain

$$\begin{aligned}\mathrm{ES}_\delta\big[\bar{S}_i^\star,\bar{S}^\star\,|\,\mathcal{G}_r\big] &\overset{\text{a.s.}}{=} \sum_{s=1}^d \mathrm{ES}_\delta\Big[\sum_{r'=1}^d \bar{S}_{s,i,r'},\bar{S}^\star\Big|\,\mathcal{G}_r\Big]\\ &\overset{\text{a.s.}}{=} \sum_{s=1}^d\big(\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r}+\mathrm{ES}_\delta\big[\bar{S}_{s,i,r+1}+\cdots+\bar{S}_{s,i,d},\bar{S}^\star\,|\,\mathcal{G}_r\big]\big).\end{aligned}$$

Applying Definition 6.6 yields

$$\begin{aligned}\mathrm{ES}_\delta\big[\bar{S}_i^\star,\bar{S}^\star\,|\,\mathcal{G}_r\big] \overset{\text{a.s.}}{=} \sum_{s=1}^d\Big(&\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r}\\ &+\frac{1}{1-\delta}\big(\mathbb{E}\big[1_{\{\bar{S}^\star>q_{\mathcal{G}_r,\delta}(\bar{S}^\star)\}}(\bar{S}_{s,i,r+1}+\cdots+\bar{S}_{s,i,d})\,\big|\,\mathcal{G}_r\big]\\ &+\beta_{\mathcal{G}_r,\delta,\bar{S}^\star}\,\mathbb{E}\big[1_{\{\bar{S}^\star=q_{\mathcal{G}_r,\delta}(\bar{S}^\star)\}}(\bar{S}_{s,i,r+1}+\cdots+\bar{S}_{s,i,d})\,\big|\,\mathcal{G}_r\big]\big)\Big)\end{aligned}$$

because $\beta_{\mathcal{G}_r,\delta,\bar{S}^\star}$ is $\mathcal{G}_r$-measurable. If we assume that the claim sizes $\{(X_{1,h},\dots,X_{d,h})\}_{h\in\mathbb{N}}$ are $\mathbb{N}_0$-valued, then $\beta_{\mathcal{G}_r,\delta,\bar{S}^\star}$ can be evaluated with an application of [34] and using an extended Panjer recursion, cf. Algorithm 5.9. Since $\bar{S}_{s,i,1},\dots,\bar{S}_{s,i,r}$ are $\mathcal{G}_r$-measurable for $s\in\{1,\dots,d\}$, we can follow the arguments of [34] and we obtain for the δ-quantile of $\bar{S}^\star$ given $\mathcal{G}_r$

$$q_{\mathcal{G}_r,\delta}(\bar{S}^\star)\overset{\text{a.s.}}{=}\sum_{s=1}^d(\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r})+q_\delta\Big(\bar{S}^\star-\sum_{s=1}^d(\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r})\Big). \tag{6.30}$$

Because of Lemma 6.21 and Corollary 6.27 we have $\bar{S}^\star\overset{\text{d}}{=}\sum_{i=1}^m\sum_{r=1}^d\sum_{h=1}^{N_{i,r}}\sum_{s=1}^d Y_{s,i,r,h}$. Since the claim sizes are i.i.d. and independent of each other, and if the claim sizes are $\mathbb{N}_0$-valued, then $q_\delta\big(\bar{S}^\star-\sum_{s=1}^d(\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r})\big)$ can be evaluated using an extended Panjer recursion (cf. Algorithm 5.9) if the single probabilities $\mathbb{P}[\bar{S}^\star-\sum_{s=1}^d(\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r})=n]$ for $n\in\mathbb{N}_0$, are evaluated and summed up until $\mathbb{P}[\bar{S}^\star-\sum_{s=1}^d(\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r})\le n]\ge\delta$.

Since $\bar{S}_{s,i,r+1}+\cdots+\bar{S}_{s,i,d}$ and $\bar{S}^\star-\sum_{s=1}^d(\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r})$ are independent of $\mathcal{G}_r$ and $\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r}$ are $\mathcal{G}_r$-measurable for $s\in\{1,\dots,d\}$ (note Equation (6.30)), we observe

$$\begin{aligned}\mathrm{ES}_\delta\big[\bar{S}_i^\star,\bar{S}^\star\,|\,\mathcal{G}_r\big] \overset{\text{a.s.}}{=} \sum_{s=1}^d\Big(&\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r}\\ &+\frac{1}{1-\delta}\big(\mathbb{E}\big[1_{\{\bar{S}^\star>q_{\mathcal{G}_r,\delta}(\bar{S}^\star)\}}(\bar{S}_{s,i,r+1}+\cdots+\bar{S}_{s,i,d})\big]\\ &+\beta_{\mathcal{G}_r,\delta,\bar{S}^\star}\,\mathbb{E}\big[1_{\{\bar{S}^\star=q_{\mathcal{G}_r,\delta}(\bar{S}^\star)\}}(\bar{S}_{s,i,r+1}+\cdots+\bar{S}_{s,i,d})\big]\big)\Big),\end{aligned}$$

where the application of Equation (6.30) has been omitted to facilitate reading.

As in the case with a trivial σ-algebra $\mathcal{G}_0$ we note that

$$\begin{aligned}\mathbb{E}\big[1_{\{\bar{S}^\star>q_{\mathcal{G}_r,\delta}(\bar{S}^\star)\}}(\bar{S}_{s,i,r+1}+\cdots+\bar{S}_{s,i,d})\big] &= \mathbb{E}\big[\bar{S}_{s,i,r+1}+\cdots+\bar{S}_{s,i,d}\big]\\ &-\mathbb{E}\big[1_{\{\bar{S}^\star-\sum_{s=1}^d(\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r})\le q_\delta(\bar{S}^\star-\sum_{s=1}^d(\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r}))\}}(\bar{S}_{s,i,r+1}+\cdots+\bar{S}_{s,i,d})\big].\end{aligned}$$

Further

$$\begin{aligned}
&\mathbb{E}\big[1_{\{\bar{S}^\star-\sum_{s=1}^d(\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r})\le q_\delta(\bar{S}^\star-\sum_{s=1}^d(\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r}))\}}(\bar{S}_{s,i,r+1}+\cdots+\bar{S}_{s,i,d})\big]\\
&\quad=\sum_{n=1}^{q_\delta(\bar{S}^\star-\sum_{s=1}^d(\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r}))}\mathbb{E}\big[(\bar{S}_{s,i,r+1}+\cdots+\bar{S}_{s,i,d})1_{\{\bar{S}^\star-\sum_{s=1}^d(\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r})=n\}}\big]
\end{aligned}$$

holds. Note that if $\bar{S}^\star-\sum_{s=1}^d(\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r}))$ is an $\mathbb{N}_0$-valued random variable, then $q_\delta(\bar{S}^\star-\sum_{s=1}^d(\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r})))$ is also $\mathbb{N}_0$-valued.

The last term can be evaluated recursively. We put it into the following corollary:

Corollary 6.31. *Let Assumption 2.18 be satisfied. Let the random variables $N_1,\dots,N_m$ be conditionally independent given $J,R_1,\dots,R_n$ and satisfy Equation* (2.20)*. Let for $i\in\{1,\dots,m\}$ the independent sequences $\{(X_{1,i,h},\dots,X_{d,i,h})\}_{h\in\mathbb{N}}$ of i.i.d. discrete random vectors be independent of all previously mentioned random variables and let the negative part $X_{i,1}^-$ be integrable. Let $\bar{S}_{s,i,r}=\sum_{h=1}^{P_{i,r}}Y_{s,i,r,h}$ for $s,r\in\{1,\dots,d\}$ and let the distributions of $P_{i,r}$ and $Y_{i,r,1}$ be given as in Corollary 6.27. Define for $p\in\mathbb{R}$, $s,r\in\{1,\dots,d\}$, $t\in\{r+1,\dots,d\}$ and $i\in\{1,\dots,m\}$ the set*

$$I_{i,r,s,t,p}:=\bigg\{\mu\in\mathbb{R}\setminus\{0\}\,\bigg|\,\mathbb{P}\bigg[Y_{s,i,t,1}=\mu,\bar{S}^\star-\sum_{s=1}^d(\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r})=p-\mu\bigg]>0\bigg\}.$$

Then for every $i\in\{1,\dots,m\}$ and $s,r\in\{1,\dots,d\}$

$$\begin{aligned}
&\sum_{t=r+1}^d\mathbb{E}\big[\bar{S}_{s,i,t}1_{\{\bar{S}^\star-\sum_{s=1}^d(\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r})=p\}}\big]\\
&\qquad=\sum_{t=r+1}^d\mathbb{P}[X_{1,i,1}=\cdots=X_{r-1,i,1}=0,X_{r,i,1}\neq0]\sum_{j\in\mathcal{J}}\lambda_{i,j}\sum_{l=0}^n a_{i,l}^j\\
&\qquad\quad\times\sum_{\mu\in I_{i,r,s,t,p}}\mu\,\mathbb{P}[Y_{s,i,t,1}=\mu]\,\mathbb{E}\big[R_l1_{\{\bar{S}^\star-\sum_{s=1}^d(\bar{S}_{s,i,1}+\cdots+\bar{S}_{s,i,r})=p-\mu,J=j\}}\big]
\end{aligned}$$

holds for all $p\in\mathbb{R}$.

Proof. By application of Lemma 6.21 and Corollary 6.27

$$(\bar{S}_{1,1},\dots,\bar{S}_{1,d},\dots,\bar{S}_{m,1},\dots,\bar{S}_{m,d})\stackrel{\mathrm{d}}{=}(S_{1,1},\dots,S_{1,d},\dots,S_{m,1},\dots,S_{m,d})$$

holds and thus also $(\bar{S}_{i,r},\bar{S}^\star)\stackrel{\mathrm{d}}{=}(S_{i,r},S^\star)$ for each $i\in\{1,\dots,m\}$ and $r\in\{1,\dots,d\}$. Note that by Theorem 3.1 $N_i\stackrel{\mathrm{d}}{=}M_i$ holds for each $i\in\{1,\dots,m\}$ with M_i given in Theorem 3.1. Let now for each $r\in\{1,\dots,d\}$

$$M_{i,r}=\sum_{h=1}^{M_i}1_{\{X_{1,i,h}=\cdots=X_{r-1,i,h}=0,X_{r,i,h}\neq0\}},$$

and obviously $N_{i,r}\stackrel{\mathrm{d}}{=}M_{i,r}$ holds, cf. also Equation (6.22). Then the equivalent representation of $M_{i,r}$ for $r\in\{1,\dots,d\}$ is given by

$$M_{i,r}=\sum_{j\in\mathcal{J}}1_{\{J=j\}}\sum_{l=0}^n\sum_{h=1}^{Q_{j,l,r}}B_{i,j,l,h},$$

where for each $j \in \mathcal{J}$, $l \in \{0, \dots, n\}$ and $r \in \{1, \dots, d\}$

$$\begin{aligned}\mathcal{L}(Q_{j,l,r} \,|\, J, R_0, \dots, R_n) &\overset{\text{a.s.}}{=} \mathcal{L}(Q_{j,l,r} \,|\, R_l) \\ &\overset{\text{a.s.}}{=} \text{Poisson}\bigg(\sum_{i=1}^{m} \lambda_{i,j} a_{i,l}^{j} R_l \; \mathbb{P}[X_{1,i,1} = \dots = X_{r-1,i,1} = 0, X_{r,i,1} \neq 0]\bigg),\end{aligned}$$

and the remaining random variables are given as in Theorem 3.1. It can be clearly seen that this claim holds after a comparison of the proofs of Lemma 6.21 and Theorem 3.1. Thus also $\bar{S}_{s,i,t} \overset{\text{d}}{=} \sum_{h=1}^{M_{i,t}} Y_{s,i,t,h}$ for $t \in \{1, \dots, d\}$ holds. Insertion yields

$$\begin{aligned}&\sum_{t=r}^{d} \mathbb{E}\big[\bar{S}_{s,i,t} 1_{\{\bar{S}^\star - \sum_{s=1}^{d}(\bar{S}_{s,i,1} + \dots + \bar{S}_{s,i,r}) = p\}}\big] \\ &\qquad = \sum_{t=r}^{d} \mathbb{E}\bigg[\sum_{j \in \mathcal{J}} 1_{\{J=j\}} \sum_{l=0}^{n} \sum_{h=1}^{Q_{j,l,t}} B_{i,j,l,h} Y_{s,i,t,h} 1_{\{\bar{S}^\star - \sum_{s=1}^{d}(\bar{S}_{s,i,1} + \dots + \bar{S}_{s,i,r}) = p\}}\bigg].\end{aligned}$$

The rest of the proof is analogous to that of Theorem 6.10. It is only important to note that due to $t \in \{r, \dots, d\}$ we have $Y_{s,i,t,h} \neq 0$, hence for $\mu \in I_{i,r,s,t,p}$ the set $\{B_{i,j,l,h} Y_{s,i,t,h} = \mu\}$ is well-defined. q.e.d.

6.4 Derivation of the Algorithm for Theorem 6.10

We derived a formula for $\mathbb{E}\big[S_i 1_{\{S=p\}}\big]$ but it still remains to compute $\mathbb{E}\big[R_l 1_{\{S'=p\}} 1_{\{J=j\}}\big]$ for $p \in \mathbb{N}_0$, $j \in \mathcal{J}$, and $l \in \{1, \dots, n\}$ and

$$S' = \sum_{j \in \mathcal{J}} 1_{\{J=j\}} \sum_{l=0}^{n} \sum_{h=1}^{Q_{j,l}} \sum_{i=1}^{m} B_{i,j,l,h} X_{i,l,h}$$

as can be derived from Theorem 3.1. If $R_1, \dots, R_n$ are gamma-distributed, then we proceed as in [57]. If $R_1, \dots, R_n$ have a τ-tempered α-stable distributions, then the approach is different. We put this into the following theorem:

Theorem 6.32. *Let $\nu \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Let $\alpha_l \in (0,1)$, $\sigma > 0$ and $\tau_l \geq 0$ for $l \in \{1, \dots, n\}$ and let $\gamma = (\gamma_1 \dots, \gamma_n) \in \{0,1\}^n$. Let the assumptions of Theorem 6.10 be satisfied and let $\{X_{l,h}\}_{h \in \mathbb{N}}$ be independent of $Q_{j,l}$ and $\{B_{j,l,h}\}_{h \in \mathbb{N}}$ for $j \in \mathcal{J}$ and $l \in \{0, \dots, n\}$. Define for every $j \in \mathcal{J}$ and $l \in \{1, \dots, n\}$*

$$\bar{\lambda}_{j,l} = \sum_{i=1}^{m} \lambda_{i,j} a_{i,l}^{j}.$$

Define the parameters

$$q_{j,l} = \frac{\bar{\lambda}_{j,l}}{\tau_l + \bar{\lambda}_{j,l}} \quad \textit{and} \quad \gamma_{\alpha_l,\sigma_l} = \frac{\sigma_l^{\alpha_l}}{\cos(\alpha_l \pi/2)}, \qquad j \in \mathcal{J}, \quad l \in \{0, \dots, n\}$$

and

$$\delta_1 = \sum_{l=1}^{n} \gamma_l (\alpha_l - 1) \ln(1 - q_{j,l}) \qquad \textit{and} \qquad \delta_2 = \sum_{l=1}^{n} \gamma_{\alpha_l,\sigma_l}((\bar{\lambda}_{j,l} + \tau_l)^{\alpha_l} - \tau_l^{\alpha_l}).$$

Let for $l \in \{1, \dots, n\}$ the random variable R_l have a distribution such that $R_l \sim F_{\alpha_l,\sigma_l,\tau_l,0}$. Let further $\{W_{j,l,h}\}_{h\in\mathbb{N}}$ be independent sequences of i.i.d. random variables such that

$$W_{j,l,h} = \sum_{i=1}^{m} B_{i,j,l,h} X_{i,l,h}, \qquad h \in \mathbb{N}, \quad j \in \mathcal{J}, \quad l \in \{0, \dots, n\}.$$

Let $\{\tilde{L}_{j,l,h}\}_{h\in\mathbb{N}}$ be independent sequences of i.i.d. random variables for $j \in \mathcal{J}$ and $l \in \{1, \dots, n\}$ such that

$$\tilde{L}_{j,l,1} \stackrel{d}{=} \sum_{r=1}^{L_{j,l}} W_{j,l,r}, \tag{6.33}$$

where $L_{j,l} \sim \mathrm{Log}(q_{j,l})$ is independent of $\{W_{j,l,h}\}_{h\in\mathbb{N}}$. Let the following random variables and sequences be independent of each other, respectively. Let $\{Q^1_{j,h}\}_{h\in\mathbb{N}}$ be a sequence of i.i.d. random variables such that

$$\mathcal{L}\big(Q^1_{j,1}\big) = \frac{1}{\delta_1} \sum_{l=1}^{n} \gamma_l(\alpha_l - 1)\ln(1 - q_{j,l})\,\mathcal{L}\big(\tilde{L}_{j,l,1}\big), \qquad j \in \mathcal{J}. \tag{6.34}$$

Let $\{N_{j,l,h}\}_{h\in\mathbb{N}}$ be independent sequences of i.i.d. random variables such that

$$N_{j,l,1} \sim \mathrm{ExtNegBin}(-\alpha_l, 1, q_{j,l})\,, \qquad j \in \mathcal{J}, \quad l \in \{1, \dots, n\}. \tag{6.35}$$

Define the sequence $\{Z_{j,l,h}\}_{h\in\mathbb{N}}$ of i.i.d. random variables such that $Z_{j,l,h} \stackrel{d}{=} \sum_{r=1}^{N_{j,l,h}} W_{j,l,r}$ for $h \in \mathbb{N}$. Let further $\{Q^2_{j,h}\}_{h\in\mathbb{N}}$ be a sequence of i.i.d. random variables such that

$$\mathcal{L}\big(Q^2_{j,1}\big) = \frac{1}{\delta_2} \sum_{l=1}^{n} \gamma_{\alpha_l,\sigma_l}((\bar{\lambda}_{j,l} + \tau_l)^{\alpha_l} - \tau_l^{\alpha_l})\,\mathcal{L}(Z_{j,l,1})\,, \qquad j \in \mathcal{J}. \tag{6.36}$$

Furthermore, define for $j \in \mathcal{J}$ the random variables

$$M'_j \sim \mathrm{Poisson}\big(\delta_1 + \delta_2 + \bar{\lambda}_{j,0} R_0\big)\,. \tag{6.37}$$

Now let $S'_j = \sum_{h=1}^{M'_j} X'_{j,h}$ for $j \in \mathcal{J}$ where $\{X'_{j,h}\}_{h\in\mathbb{N}}$ is a sequence independent of M'_j, consisting of i.i.d. random variables such that

$$\mathcal{L}\big(X'_{j,1}\big) = \frac{1}{\delta_1 + \delta_2 + \bar{\lambda}_{j,0} R_0} \big(\delta_1\,\mathcal{L}\big(Q^1_{j,1}\big) + \delta_2\,\mathcal{L}\big(Q^2_{j,1}\big) + \bar{\lambda}_{j,0} R_0\,\mathcal{L}(W_{j,0,1})\big). \tag{6.38}$$

Let $\gamma_l \in \{0, 1\}$ for $l \in \{1, \dots, n\}$. Then

$$\mathbb{E}\big[R_1^{\gamma_1} \dots R_n^{\gamma_n} 1_{\{S'=\nu\}} 1_{\{J=j\}}\big] = C_\gamma\ \mathbb{P}[S'_j = \nu]\ \mathbb{P}[J = j]$$

holds, where $C_\gamma = \prod_{l=1}^{n} \big(\alpha_l \gamma_{\alpha_l,\sigma_l} \tau_l^{\alpha_l - 1}\big)^{\gamma_l}$.

Proof. We proceed first as in [57]. Assume that $J, R_1, \dots, R_n$ are independent. Consider a weighted probability-generating function of the distribution of S' for $z \in [0, 1]$

$$\begin{aligned} G_{S',\gamma}(z) &= \sum_{p\in\mathbb{N}_0} \sum_{j\in\mathcal{J}} \mathbb{E}\big[R_1^{\gamma_1} \dots R_n^{\gamma_n} 1_{\{S'=p\}} 1_{\{J=j\}}\big]\, z^p \\ &= \mathbb{E}\big[R_1^{\gamma_1} \dots R_n^{\gamma_n} z^{S'}\big] = \mathbb{E}\big[\mathbb{E}\big[R_1^{\gamma_1} \dots R_n^{\gamma_n} z^{S'} \,\big|\, J\big]\big]. \end{aligned}$$

Condition on $J, R_1, \dots, R_n$, hence $S_l = \sum_{j\in\mathcal{J}} 1_{\{J=j\}} \sum_{h=1}^{Q_{j,l}} \sum_{i=1}^{m} B_{i,j,l,h} X_{i,l,h}$ are conditionally independent given $J, R_1, \dots, R_n$ for $l \in \{0, \dots, n\}$, thus

$$\mathbb{E}\big[R_1^{\gamma_1} \dots R_n^{\gamma_n} z^{S'} \,\big|\, J, R_1, \dots, R_n\big] \overset{\text{a.s.}}{=} R_1^{\gamma_1} \dots R_n^{\gamma_n} \prod_{l=0}^{n} \mathbb{E}\big[z^{S_l} \,\big|\, J, R_1, \dots, R_n\big]. \tag{6.39}$$

Let $G_{j,l}(z) = \mathbb{E}\big[z^{W_{j,l,1}}\big]$ be the probability-generating function of the distribution of $W_{j,l,1}$. Then the conditional distribution of $Q_{j,l}$ given $J, R_1, \dots, R_n$ yields for $l \in \{0, \dots, n\}$ (cf. Equation (3.3))

$$\mathbb{E}\big[z^{S_l} \,|\, J, R_1, \dots, R_n\big] \overset{\text{a.s.}}{=} \mathbb{E}\big[z^{S_l} \,|\, J, R_l\big] \overset{\text{a.s.}}{=} \exp\Big(-\sum_{i=1}^{m} \lambda_{i,J} a_{i,l}^{J} R_l (1 - G_{J,l}(z))\Big). \tag{6.40}$$

Inserting Equation (6.40) into Equation (6.39) yields

$$\begin{aligned}\mathbb{E}\big[R_1^{\gamma_1} \dots R_n^{\gamma_n} z^{S'} \big| J, R_1, \dots, R_n\big]& \\ \overset{\text{a.s.}}{=} \exp\big(-\bar{\lambda}_{J,0} R_0 (1 - G_{J,0}(z))\big) &\prod_{l=1}^{n} R_l^{\gamma_l} \exp\big(-\bar{\lambda}_{J,l} R_l (1 - G_{J,l}(z))\big).\end{aligned} \tag{6.41}$$

Take the conditional expectation given J of Equation (6.41) and use the independence of $J, R_1, \dots, R_n$

$$\mathbb{E}\big[R_1^{\gamma_1} \dots R_n^{\gamma_n} z^{S'} \,\big|\, J\big] \overset{\text{a.s.}}{=} \exp\big(-\bar{\lambda}_{J,0} R_0 (1 - G_{J,0}(z))\big) \prod_{l=1}^{n} \mathbb{E}\big[R_l^{\gamma_l} \exp\big(-\bar{\lambda}_{J,l} R_l (1 - G_{J,l}(z))\big) \,\big|\, J\big].$$

Let us introduce the notation of [21, Equation (5.32)]

$$I_{\alpha,\sigma}(-m, s) := (-1)^m \frac{d^m}{ds^m} \exp(-\gamma_{\alpha,\sigma} s^{\alpha}), \qquad m \in \mathbb{N}, \quad s > 0$$

with $\gamma_{\alpha,\sigma} = \frac{\sigma^{\alpha}}{\cos(\alpha\pi/2)}$ and $I_{\alpha,\sigma}(0, s) := \exp(-\gamma_{\alpha,\sigma} s^{\alpha})$ for $s \geq 0$. Then for $R_l \sim F_{\alpha_l,\sigma_l,\tau_l,0}$ with $l \in \{1, \dots, n\}$ according to [21, Corollary 5.16]

$$\mathbb{E}\big[R_l^{\gamma_l} \exp\big(-\bar{\lambda}_{J,l} R_l (1 - G_{J,l}(z))\big) \,|\, J\big] \overset{\text{a.s.}}{=} \frac{I_{\alpha_l,\sigma_l}(-\gamma_l, \bar{\lambda}_{J,l}(1 - G_{J,l}(z)) + \tau_l)}{I_{\alpha_l,\sigma_l}(0, \tau_l)}.$$

If $\gamma_l = 0$, then

$$\mathbb{E}\big[\exp\big(-\bar{\lambda}_{J,l} R_l (1 - G_{J,l}(z))\big) \,|\, J\big] \overset{\text{a.s.}}{=} \exp\big(-\gamma_{\alpha_l,\sigma_l}\big(\big(\bar{\lambda}_{J,l}(1 - G_{J,l}(z)) + \tau_l\big)^{\alpha_l} - \tau_l^{\alpha_l}\big)\big),$$

cf. also [21, Equation (5.25)]. Analogously, if $\gamma_l = 1$, then

$$\begin{aligned}&\mathbb{E}\big[R_l \exp\big(-\bar{\lambda}_{J,l} R_l (1 - G_{J,l}(z))\big) \,|\, J\big] \\ &\overset{\text{a.s.}}{=} \alpha_l \gamma_{\alpha_l,\sigma_l} \big(\bar{\lambda}_{J,l}(1 - G_{J,l}(z)) + \tau_l\big)^{\alpha_l - 1} \exp\big(-\gamma_{\alpha_l,\sigma_l}\big(\big(\bar{\lambda}_{J,l}(1 - G_{J,l}(z)) + \tau_l\big)^{\alpha_l} - \tau_l^{\alpha_l}\big)\big).\end{aligned}$$

Transferring this into a common exponential, this provides for the weighted probability-generating function for $z \in [0,1]$

$$\begin{aligned} G_{S',\gamma}(z) &= \sum_{j\in\mathcal{J}} \mathbb{E}\big[R_1^{\gamma_1},\dots,R_n^{\gamma_n} z^{S'} \,\big|\, J=j\big]\ \mathbb{P}[J=j] \\ &= \sum_{j\in\mathcal{J}} \exp\Big(-\bar\lambda_{j,0} R_0 (1-G_{j,0}(z)) \\ &\quad + \sum_{l=1}^{n} \gamma_l \ln\big(\alpha_l \gamma_{\alpha_l,\sigma_l} \big(\bar\lambda_{j,l}(1-G_{j,l}(z))+\tau_l\big)^{\alpha_l-1}\big) \\ &\quad - \gamma_{\alpha_l,\sigma_l}\big(\big(\bar\lambda_{j,l}(1-G_{j,l}(z))+\tau_l\big)^{\alpha_l} - \tau_l^{\alpha_l}\big)\Big)\ \mathbb{P}[J=j]. \end{aligned} \tag{6.42}$$

Let us treat the logarithmic term. We observe for each $j \in \mathcal{J}$ and $l \in \{1,\dots,n\}$

$$\begin{aligned} \ln(\bar\lambda_{j,l}(1-G_{j,l}(z))+\tau_l) &= \ln\Big(\tau_l\Big(\frac{1}{\tau_l}\bar\lambda_{j,l}\big(1-G_{j,l}(z)\big)+1\Big)\Big) \\ &= \ln(\tau_l) + \ln\Big(1-\bar\lambda_{j,l}\frac{G_{j,l}(z)-1}{\tau_l}\Big). \end{aligned} \tag{6.43}$$

Then

$$\tilde G_{j,l}(z) = \frac{\ln(1-q_{j,l}G_{j,l}(z))}{\ln(1-q_{j,l})}$$

is the probability-generating function of the distribution of the random sum $\tilde L_{j,l,1}$. Thus we obtain for every $j \in \mathcal{J}$ and $l \in \{1,\dots,n\}$

$$1-\bar\lambda_{j,l}\frac{G_{j,l}(z)-1}{\tau_l} = \frac{\tau_l+\bar\lambda_{j,l}}{\tau_l}\Big(1-\frac{\bar\lambda_{j,l}}{\tau_l+\bar\lambda_{j,l}}G_{j,l}(z)\Big) = \frac{1}{1-q_{j,l}}(1-q_{j,l}G_{j,l}(z)),$$

and hence

$$\ln\Big(1-\bar\lambda_{j,l}\frac{G_{j,l}(z)-1}{\tau_l}\Big) = (\tilde G_{j,l}(z)-1)\ln(1-q_{j,l}). \tag{6.44}$$

Then the following holds for the weighted probability-generating function by insertion of Equations (6.43) and (6.44) into Equation (6.42)

$$\begin{aligned} G_{S',\gamma}(z) &= \sum_{j\in\mathcal{J}} \exp\Big(-\bar\lambda_{j,0} R_0 (1-G_{j,0}(z)) \\ &\quad + \sum_{l=1}^{n} \gamma_l\big(\ln(\alpha_l\gamma_{\alpha_l,\sigma_l}) + (\alpha_l-1)\ln(\bar\lambda_{j,l}(1-G_{j,l}(z))+\tau_l)\big) \\ &\quad - \gamma_{\alpha_l,\sigma_l}((\bar\lambda_{j,l}(1-G_{j,l}(z))+\tau_l)^{\alpha_l} - \tau_l^{\alpha_l})\Big)\ \mathbb{P}[J=j] \\ &= C_\gamma \sum_{j\in\mathcal{J}} \exp\Big(-\bar\lambda_{j,0} R_0 (1-G_{j,0}(z)) \\ &\quad + \sum_{l=1}^{n} \gamma_l(\alpha_l-1)(\tilde G_{j,l}(z)-1)\ln(1-q_{j,l}) \\ &\quad - \gamma_{\alpha_l,\sigma_l}\big(\big(\bar\lambda_{j,l}(1-G_{j,l}(z))+\tau_l\big)^{\alpha_l} - \tau_l^{\alpha_l}\big)\Big)\ \mathbb{P}[J=j]. \end{aligned} \tag{6.45}$$

The exponential term in this sum is the probability-generating function $G_j(z)$ of the distribution of the sum of independent random sums

$$S'_j = \sum_{h=1}^{K_j} W_{j,0,h} + \sum_{l=1}^{n} \sum_{h=1}^{M_{j,l}} \tilde{L}_{j,l,h} + \sum_{l=1}^{n} \sum_{r=1}^{P_{j,l}} \sum_{h=1}^{N_{j,l,r}} W_{j,l,h}, \qquad j \in \mathcal{J}, \tag{6.46}$$

where $K_j \sim \text{Poisson}\big(\bar{\lambda}_{j,0} R_0\big)$ is independent of $\{W_{j,l,h}\}_{h\in\mathbb{N}}$. Furthermore,

$$M_{j,l} \sim \text{Poisson}(\gamma_l(\alpha_l - 1)\ln(1 - q_{j,l}))$$

are independent for $l \in \{1,\dots,n\}$ and $\{\tilde{L}_{j,l,h}\}_{h\in\mathbb{N}}$ are independent sequences independent of $M_{j,l}$, consisting of i.i.d. random variables such that $\tilde{L}_{j,l,1}$ has the distribution given by Equation (6.33) for each $l \in \{1,\dots,n\}$. The choice of the last random sum might not be so obvious. A comparison of the last term of Equation (6.45) with [21, Equation (5.25)] shows that this is a Poisson distribution mixed with a τ-tempered α-stable distribution. An application of [21, Lemma 5.10] provides our representation of the last summand in Equation (6.46), note our notation of the extended negative binomial distribution. Hence,

$$P_{j,l} \sim \text{Poisson}\big(\gamma_{\alpha_l,\sigma_l}((\bar{\lambda}_{j,l} + \tau_l)^{\alpha_l} - \tau_l^{\alpha_l})\big)$$

are independent for $l \in \{1,\dots,n\}$ and $\{N_{j,l,h}\}_{h\in\mathbb{N}}$ are independent sequences independent of $P_{j,l}$, consisting of i.i.d. random variables such that Equation (6.35) is satisfied and $\{W_{j,l,h}\}_{h\in\mathbb{N}}$ are independent sequences independent of $P_{j,l}$ and $\{N_{j,l,h}\}_{h\in\mathbb{N}}$, consisting of i.i.d. random variables such that $W_{j,l,1}$ has a distribution given by $G_{j,l}(z)$.

We observe here that we have twice a sum of random variables with compound Poisson distributions. According to [49, Proposition 3.3.4] we see that

$$S'_{j,1} = \sum_{l=1}^{n} \sum_{h=1}^{M_{j,l}} \tilde{L}_{j,l,h}, \qquad j \in \mathcal{J},$$

also has a compound Poisson distribution, i.e., $S'_{j,1} \overset{\text{d}}{=} \sum_{h=1}^{T_j} Q^1_{j,h}$ where we have $T_j \sim \text{Poisson}(\delta_1)$ and $\mathcal{L}(Q^1_{j,1})$ is given by Equation (6.34).

According to [49, Proposition 3.3.4] we also see that the other random sum

$$S'_{j,2} = \sum_{l=1}^{n} \sum_{r=1}^{P_{j,l}} \sum_{h=1}^{N_{j,l,r}} W_{j,l,h}, \qquad j \in \mathcal{J},$$

has a compound Poisson distribution, too, i.e., $S'_{j,2} \overset{\text{d}}{=} \sum_{h=1}^{U_j} Q^2_{j,h}$, with $U_j \sim \text{Poisson}(\delta_2)$ $\mathcal{L}(Q^2_{j,1})$ given by Equation (6.36).

Finally, since both $S'_{j,1}$ and $S'_{j,2}$ have a compound Poisson distribution, once more we apply [49, Proposition 3.3.4] and we observe for each $j \in \mathcal{J}$

$$S'_j = \sum_{h=1}^{K_j} W_{j,0,h} + S'_{j,1} + S'_{j,2} \overset{\text{d}}{=} \sum_{h=1}^{M'_j} X'_{j,h}$$

where the distribution of M'_j is given by Equation (6.37) and the distribution of $\{X_{j,h}\}_{h\in\mathbb{N}}$ by Equation (6.38).

Hence the weighted probability-generating function simplifies to

$$G_{S',\gamma}(z) = C_\gamma \sum_{j\in\mathcal{J}} G_j(z)\ \mathbb{P}[J=j],$$

and thus by conditioning on J we obtain

$$\begin{aligned}\mathbb{E}\big[R_1^{\gamma_1}\dots R_n^{\gamma_n} 1_{\{S'=\nu\}}\big] &= \sum_{j\in\mathcal{J}} \mathbb{E}\big[R_1^{\gamma_1}\dots R_n^{\gamma_n} 1_{\{S'=\nu\}} 1_{\{J=j\}}\big] \\ &= C_\gamma \sum_{j\in\mathcal{J}} \mathbb{P}[S'_j=\nu]\ \mathbb{P}[J=j], \qquad \nu\in\mathbb{N}_0.\end{aligned}$$

By comparison of the coefficients we have

$$\mathbb{E}\big[R_1^{\gamma_1}\dots R_n^{\gamma_n} 1_{\{S'=\nu\}} 1_{\{J=j\}}\big] = C_\gamma\ \mathbb{P}[S'_j=\nu]\ \mathbb{P}[J=j].$$

q.e.d.

Remark 6.47. Note that all random variables introduced in the theorem allow for an iterated application of Panjer's recursion. First, for each $j\in\mathcal{J}$ and $l\in\{0,\dots,n\}$ the distribution of $W_{j,l,1}$ can be determined by the law of total probability. The distribution of $\tilde{L}_{j,l,1}$ can be evaluated by an extended Panjer's recursion (cf. Theorem 5.1) for each $j\in\mathcal{J}$ and $l\in\{1,\dots,n\}$, hence the distribution of $Q^1_{j,1}$ is also known. Since the distribution of $Z_{j,l,1}$ can be computed with an extended Panjer recursion for each $j\in\mathcal{J}$ and $l\in\{1,\dots,n\}$, the distribution of $Q^2_{j,1}$ is also known. Because the distribution of $X'_{j,1}$ is a convex combination of distributions that can be evaluated with an extended Panjer recursion, the distribution of S'_j can be computed by a numerically stable Panjer recursion for each $j\in\mathcal{J}$.

Chapter 7

Numerical Illustration

In this chapter we give examples of the impacts of different dependence structures in our model on the distribution of the portfolio loss and on the corresponding risk contributions. More precisely, this means we consider the impacts of different correlations. We will see that given certain constraints originating from the extended CreditRisk$^+$ model, the resulting distributions differ.

7.1 Two Risk Factors

We give an example of the application of Algorithm 5.9 that implements the dependence structures given in Theorem 3.1. We consider three cases of dependence structures and we use the following assumptions and parameters: in each case we consider two independent and gamma-distributed risk factors, i.e., $R_i \sim \text{Gamma}(\alpha, \beta)$ for $i = 1, 2$ and $\alpha, \beta > 0$. This ensures that each risk factor has the same influence. A further constraint of our example is a condition of the extended CreditRisk$^+$ model, i.e., we assume for each default cause intensity $\mathbb{E}[\Lambda_1] = \mathbb{E}[\Lambda_2] = 1$. In order to have a closer comparison we further assume that the variances of the default cause intensities Λ_i are the same in each case of correlation. The distribution of the random sum S in each of these cases can be given explicitly. Consider the corresponding probability-generating functions. The three cases are the following:

(a) Independent default cause intensities

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Applying Theorem 3.1 and using $|\mathcal{J}| = 1$ and thus omitting the index j for $z \in [0, 1]$, we observe

$$\begin{aligned} G_S(z) &= \mathbb{E}\Big[z^{\sum_{i=1}^m \sum_{h=1}^{N_i} X_{i,h}}\Big] = \mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{2} \lambda_i \sum_{l=0}^{2} a_{i,l} R_l (1 - G_{X_{i,1}}(z))\Big)\Big] \\ &= \mathbb{E}\big[\exp\big(-\lambda_1 R_1 (1 - G_{X_{1,1}}(z)) - \lambda_2 R_2 (1 - G_{X_{2,1}}(z))\big)\big] \\ &= \Big(\frac{\beta_1}{\beta_1 + \lambda_1 (1 - G_{X_{1,1}}(z))}\Big)^{\alpha_1} \Big(\frac{\beta_2}{\beta_2 + \lambda_2 (1 - G_{X_{2,1}}(z))}\Big)^{\alpha_2}, \end{aligned}$$

which determines the convolution of two compound negative binomial distributions.

(b) Negatively correlated default cause intensities (cf. Example 3.14) with $a, b > 0$

$$A_1 = \begin{pmatrix} 0 & b & 0 \\ a & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \end{pmatrix}.$$

We choose two dependence scenarios, i.e. $\mathcal{J} = \{1, 2\}$, and we let J be a random variable such that $\mathbb{P}[J = 1] = \mathbb{P}[J = 2] = 1/2$. Applying Theorem 3.1 and using $\mathcal{J} = \{1, 2\}$ for $z \in [0, 1]$, we observe

$$\begin{aligned}
G_S(z) &= \mathbb{E}\Big[\exp\Big(-\sum_{j\in\mathcal{J}} 1_{\{J=j\}} \sum_{i=1}^{2} \lambda_{i,j} \sum_{l=0}^{2} a_{i,l}^{j} R_l (1 - G_{X_{i,1}}(z))\Big)\Big] \\
&= \frac{1}{2}\,\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{2} \lambda_{i,1} \sum_{l=0}^{2} a_{i,l}^{1} R_l (1 - G_{X_{i,1}}(z))\Big)\Big] \\
&\quad + \frac{1}{2}\,\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{2} \lambda_{i,2} \sum_{l=0}^{2} a_{i,l}^{2} R_l (1 - G_{X_{i,1}}(z))\Big)\Big] \\
&= \frac{1}{2}\,\mathbb{E}\big[\exp\big(-(\lambda_{2,1} a R_0 + \lambda_{1,1} b R_1)(1 - G_{X_{1,1}}(z))\big)\big] \\
&\quad + \frac{1}{2}\,\mathbb{E}\big[\exp\big(-(\lambda_{1,2} a R_0 + \lambda_{2,2} b R_2)(1 - G_{X_{2,1}}(z))\big)\big] \\
&= \frac{1}{2} \exp\big(-\lambda_{2,1} a R_0 (1 - G_{X_{1,1}}(z))\big) \Big(\frac{\beta_1}{\beta_1 + \lambda_{1,1} b (1 - G_{X_{1,1}}(z))}\Big)^{\alpha_1} \\
&\quad + \frac{1}{2} \exp\big(-\lambda_{1,2} a R_0 (1 - G_{X_{2,1}}(z))\big) \Big(\frac{\beta_2}{\beta_2 + \lambda_{2,2} b (1 - G_{X_{2,1}}(z))}\Big)^{\alpha_2},
\end{aligned}$$

which determines the mixture of two compound negative binomial distributions each of which is convoluted with a compound Poisson distribution. We give some explanations for the choice of these matrices in the remark below.

(c) Positively correlated default cause intensities

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Assume for simplicity $G_{X_{1,1}}(z) = G_{X_{2,1}}(z)$ for $z \in [0, 1]$. Applying Theorem 3.1 and using $|\mathcal{J}| = 1$ and also omitting the index j for $z \in [0, 1]$, we obtain

$$\begin{aligned}
G_S(z) &= \mathbb{E}\Big[z^{\sum_{i=1}^{m}\sum_{h=1}^{N_i} X_{i,h}}\Big] = \mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{2} \lambda_i \sum_{l=0}^{2} a_{i,l} R_l (1 - G_{X_{i,1}}(z))\Big)\Big] \\
&= \mathbb{E}\big[\exp\big(-\lambda_1 R_1 (1 - G_{X_{1,1}}(z)) - \lambda_2 R_1 (1 - G_{X_{2,1}}(z))\big)\big] \\
&= \Big(\frac{\beta_1}{\beta_1 + (\lambda_1 + \lambda_2)(1 - G_{X_{1,1}}(z))}\Big)^{\alpha_1},
\end{aligned}$$

which determines a compound negative binomial distribution.

Remark 7.1. The parameters a and b in Case (b) should be chosen such that $\mathbb{E}[\Lambda_i] = 1$ and the variance of Λ_i for $i = 1, 2$ in the case of negative correlation is the same as in the case of positive correlation and the independent case. We also assume that $R_0 = \mathbb{E}[R_1]$. In the independent case we obtain for the expected value $\mathbb{E}[\Lambda_1] = \mathbb{E}[R_1] = \frac{\alpha}{\beta}$ and for the variance $\mathrm{Var}(\Lambda_1) = \mathrm{Var}(R_1) = \frac{\alpha}{\beta^2}$. Hence we get the following constraint using the expected value and that R_1 and J are independent

$$\mathbb{E}[\Lambda_1] \;=\; \mathbb{E}\big[aR_0 1_{\{J=2\}} + bR_1 1_{\{J=1\}}\big] \;=\; \frac{a}{2}\,\mathbb{E}[R_0] + \frac{b}{2}\,\mathbb{E}[R_1] \;=\; \frac{\alpha}{\beta},$$

hence $a + b = 2$. Because $R_1 \overset{\mathrm{d}}{=} R_2$, this also holds for Λ_2. Concerning the variance, we observe the following by insertion of the definition of Λ_1

$$\begin{aligned}\mathrm{Var}(\Lambda_1) \;&=\; \mathbb{E}\big[(aR_0 1_{\{J=2\}} + bR_1 1_{\{J=1\}})^2\big] - \big(\mathbb{E}\big[aR_0 1_{\{J=2\}} + bR_1 1_{\{J=1\}}\big]\big)^2\\ &=\; \mathbb{E}\big[(aR_0)^2 1_{\{J=2\}} + (bR_1)^2 1_{\{J=1\}}\big] - \Big(\frac{a}{2}\frac{\alpha}{\beta} + \frac{b}{2}\frac{\alpha}{\beta}\Big)^2.\end{aligned}$$

According to [57] on properties of the gamma distribution we have $\mathbb{E}\big[R_1^2\big] = \frac{\alpha(\alpha+1)}{\beta^2}$, hence

$$\begin{aligned}\mathrm{Var}(\Lambda_1) \;&=\; \frac{a^2}{2}\frac{\alpha^2}{\beta^2} + \frac{b^2}{2}\frac{\alpha(\alpha+1)}{\beta^2} - \frac{\alpha^2}{\beta^2}\Big(\frac{a^2}{4} + \frac{ab}{2} + \frac{b^2}{4}\Big)\\ &=\; \frac{b^2}{2}\frac{\alpha}{\beta^2} + \frac{\alpha^2}{\beta^2}\Big(\frac{a-b}{2}\Big)^2 = \frac{\alpha}{\beta^2}\end{aligned}$$

by assumption. By algebraic transformations and using $a = 2 - b$ we have

$$1 = \frac{b^2}{2} + \alpha(1-b)^2 \qquad \Leftrightarrow \qquad b^2 - \frac{4\alpha}{1+2\alpha}b + \frac{2(\alpha-1)}{1+2\alpha} = 0.$$

Solving the quadratic equation yields

$$b_1 = \frac{2\alpha + \sqrt{2+2\alpha}}{1+2\alpha} \qquad \text{and} \qquad b_2 = \frac{2\alpha - \sqrt{2+2\alpha}}{1+2\alpha}$$

If $\alpha \in (0,1)$, then b_2 is negative, but $b_1 \in [0,2]$ for all $\alpha > 0$. Hence b_1 should be preferred. Thus for given α the matrices A_1 and A_2 are determined.

We specialize the remaining parameters as follows: we choose the risk factors such that $\mathbb{E}[R_1] = \mathbb{E}[R_2] = 1$, e.g., $\alpha = \beta = 2$. Thus we have

$$b = \frac{4+\sqrt{6}}{5} \qquad \text{and} \qquad a = \frac{6-\sqrt{6}}{5}.$$

As Poisson parameters we take $\lambda_{i,j} = 20$ with $i, j = 1, 2$. The parameter R_0 is chosen such that $R_0 = \mathbb{E}[R_1] = 1$. Finally, for the distribution of the claim sizes we choose $X_{i,1} \sim \mathrm{NegBin}(4, 0.4)$ for $i = 1, 2$. Note that this distribution is chosen arbitrarily and that it does not have to be in a $\mathrm{Panjer}(a, b, k)$ class since it is the claim size distribution. We compute the first 300 values of the respective probability mass functions.

For a better comparison we put the probability mass functions into one graph, see Figure 7.1. We see interesting differences between the probability mass functions. Taking the probability mass function with independent default cause intensities as an initial point, we observe that the probability mass function for negatively correlated default cause intensities is a bit more light-tailed with a taller maximum, whereas the probability mass function for positively correlated default cause intensities has a mass with a smaller maximum, and its mass is distributed more to the tails. We also observe that the probability mass function for

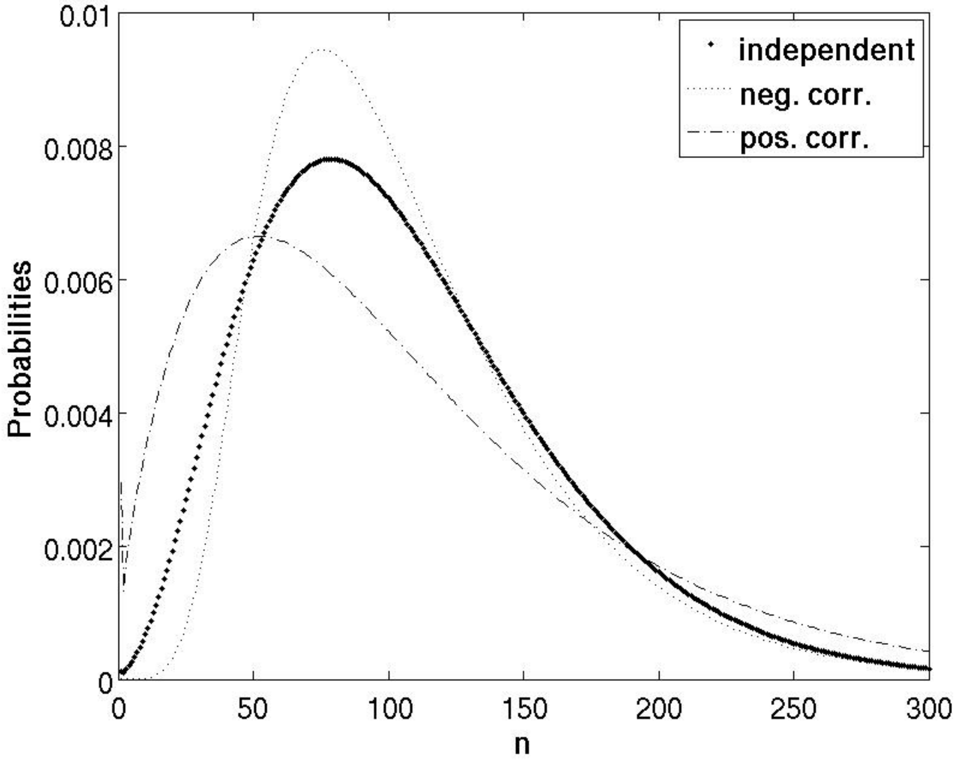

Figure 7.1: Probability mass functions of a portfolio loss for different dependence structures with two risk factors and claim size $X_{i,1} \sim \mathrm{NegBin}(4, 0.4)$.

the default cause intensities with positive correlation has a heavier tail than the other two probability mass functions.

We also compute risk contributions given in Theorem 6.10 and [57] for these parameters, and their behaviour matches the observations for the distributions with these dependence scenarios. A negative correlation between default cause intensities seems to produce a lower risk than that of independent default cause intensities, and a positive correlation between default cause intensities seems to produce a higher risk. This holds for all levels $\delta = 0.8$, 0.9, 0.95, 0.99 for which we computed expected shortfall risk contribution. The results are given in Table 7.1.

It is also possible to obtain results that show much more clearly the influence of different dependence scenarios. They also show that the types of the curves of the probability mass functions are independent of the claim sizes. This can be achieved by setting the claim sizes $X_{i,1} \equiv 1$ and keeping the other parameters. The result can be seen in Figure 7.2 with the first 150 values of the corresponding probability mass functions. The curves in this figure make clear that its behaviour is dominated by the correlation between the default cause intensities and less by the claim sizes. In order to be able to compare this behaviour more closely, we consider the variances. For this, we use Equation (3.12) in Remark 3.10. In case of independence we obtain

$$\mathrm{Var}(N_1) \;=\; \mathbb{E}[\lambda_{1,J}\Lambda_1] \;+\; \mathrm{Var}(\lambda_{1,J}\Lambda_1) \;= 20 + 400\,\mathrm{Var}(\Lambda_1) = 20 + 200 = 220 = \mathrm{Var}(N_2)$$

	Case (a)	Case (b)	Case (c)
level δ	S_1, S_2	S_1, S_2	S_1, S_2
0.8	97.5538	95.1297	114.8308
0.9	111.8582	110.5307	139.1222
0.95	125.7218	124.0371	160.8160
0.99	156.4339	158.4093	211.9374

Table 7.1: Risk contributions of S_1 and S_2 according to Equation (6.8) *for different levels δ and dependence structures with claim size $X_{i,1} \sim \mathrm{NegBin}(4, 0.4)$.*

by symmetry. Thus the variance is

$$\mathrm{Var}(N) \;=\; \mathrm{Var}(N_1) \;+\; \mathrm{Var}(N_2) \;= 220 + 220 = 440.$$

In case of negative correlation between the default cause intensities we obtain

$$\mathrm{Var}(N_1) \;=\; \mathbb{E}[\lambda_{1,J}\Lambda_1] \;+\; \mathrm{Var}(\lambda_{1,J}\Lambda_1) \;= 20 + 400\,\mathrm{Var}(\Lambda_1) = 220 = \mathrm{Var}(N_2)$$

by symmetry. By Equation (3.13) we have, using the definitions of Λ_1 and Λ_2 and the independence of J, R_1, R_2

$$\begin{aligned}
\mathrm{cov}(N_1, N_2) &= \mathrm{cov}(\lambda_{1,J}\Lambda_1, \lambda_{2,J}\Lambda_2) \\
&= 400\big(\mathbb{E}\big[abR_1 1_{\{J=1\}} + abR_2 1_{\{J=2\}}\big] \\
&\quad - \mathbb{E}\big[a1_{\{J=2\}} + bR_1 1_{\{J=1\}}\big]\;\mathbb{E}\big[a1_{\{J=1\}} + bR_2 1_{\{J=2\}}\big]\big) \\
&= 400\Big(\frac{ab}{2}\;\mathbb{E}[R_1] \;+\; \frac{ab}{2}\;\mathbb{E}[R_2] \;-\; \Big(\frac{a}{2} + \frac{b}{2}\;\mathbb{E}[R_1]\Big)\Big(\frac{a}{2} + \frac{b}{2}\;\mathbb{E}[R_2]\Big)\Big) \\
&= -400\Big(\frac{a-b}{2}\Big)^2.
\end{aligned}$$

Using the values for b and $a = 2 - b$ we obtain

$$\mathrm{cov}(\Lambda_1, \Lambda_2) = -(1-b)^2 = -\Big(\frac{5 - 4 - \sqrt{6}}{5}\Big)^2 = \frac{2\sqrt{6} - 7}{25}.$$

Thus the variance is

$$\mathrm{Var}(N) \;=\; \mathrm{Var}(N_1) \;+\; \mathrm{Var}(N_2) \;+ 2\,\mathrm{cov}(N_1, N_2) = 220 + 220 - 32(2\sqrt{6} - 7) \approx 372.7673.$$

In case of positive correlation between the default cause intensities we obtain

$$\mathrm{Var}(N_1) \;=\; \mathbb{E}[\lambda_{1,J}\Lambda_1] \;+\; \mathrm{Var}(\lambda_{1,J}\Lambda_1) \;= 20 + 400\,\mathrm{Var}(\Lambda_1) = 220 = \mathrm{Var}(N_2)$$

by symmetry. By Equation (3.13) we have

$$\mathrm{cov}(N_1, N_2) = \mathrm{cov}(\lambda_{1,J}\Lambda_1, \lambda_{2,J}\Lambda_2) = 400\,\mathrm{cov}(\Lambda_1, \Lambda_2) = 400\,\mathrm{Var}(R_1) = 200.$$

Thus the variance is

$$\mathrm{Var}(N) \;=\; \mathrm{Var}(N_1) \;+\; \mathrm{Var}(N_2) \;+ 2\,\mathrm{cov}(N_1, N_2) = 220 + 220 + 400 = 840.$$

Correspondingly, there are the following risk contributions in Table 7.2.

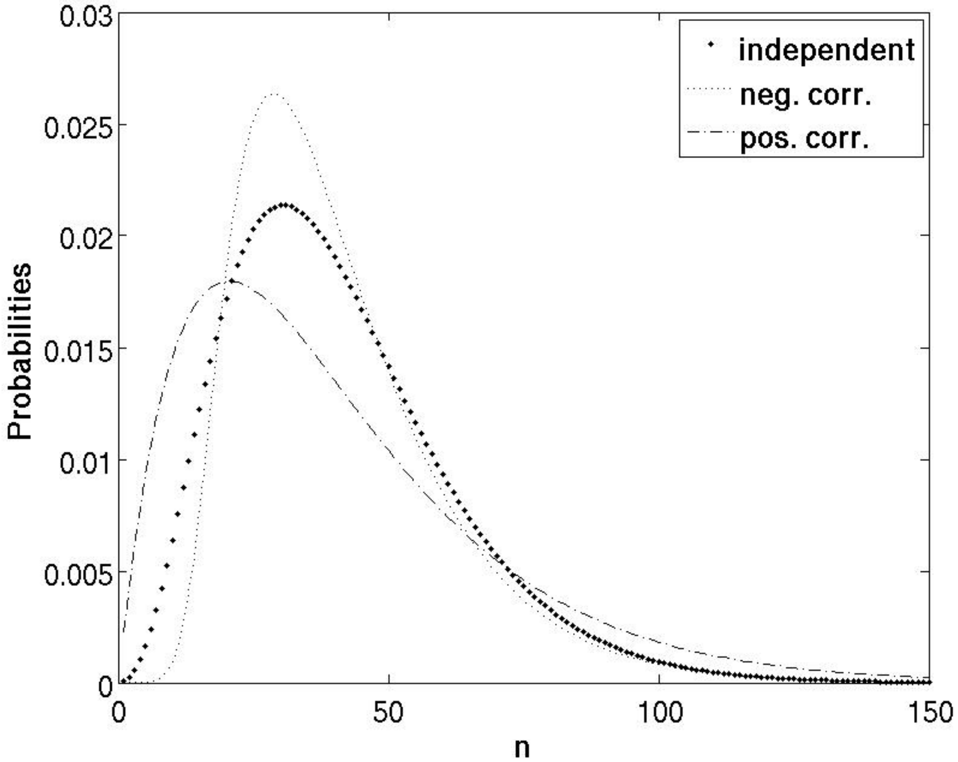

Figure 7.2: Probability mass functions of a portfolio loss for different dependence structures with two risk factors and claim size $X_{i,1} \equiv 1$.

7.2 Three Risk Factors

In this example we also apply Algorithm 5.9, but only consider two cases, namely a case with independent default cause intensities and a case with both positive and negative correlation between the default cause intensities. Again, we assume for each default cause $\mathbb{E}[\Lambda_i] = 1$ with $i \in \{1, \dots, 3\}$. We now want to study the impacts of positive and negative correlation between default cause intensities against each other in contrast to independent default cause intensities.

We let the risk factors be independent and gamma-distributed with $R_i \sim \text{Gamma}(\alpha_i, \beta_i)$ with $\alpha_i = 2$ and $\beta_i = 14$ for $i = 1, 2, 3$. The Poisson parameter is chosen such that $\lambda_{i,j} = 15$ for $i = 1, 2, 3$ and $j = 1, 2$. In the second case we let $\mathcal{J} = \{1, 2\}$ and consider a random variable J with two values such that $\mathbb{P}[J = 1] = \mathbb{P}[J = 2] = 1/2$ holds. If the default cause intensities are independent, we let

$$A = \begin{pmatrix} \frac{1}{40} & \frac{21}{4} & 0 & 0 \\ \frac{1}{40} & 0 & \frac{21}{4} & 0 \\ \frac{1}{40} & 0 & 0 & \frac{21}{4} \end{pmatrix}.$$

In the second case with both positively and negatively correlated default cause intensities

	Case (a)	Case (b)	Case (c)
level δ	S_1, S_2	S_1, S_2	S_1, S_2
0.8	35.4147	34.0681	42.8699
0.9	41.1260	39.4924	50.5834
0.95	46.8182	45.7197	59.6559
0.99	58.3337	56.0463	77.9308

Table 7.2: Risk contributions of S_1 and S_2 according to Equation (6.8) *for different levels δ and dependence structures with claim size $X_{i,1} \equiv 1$.*

	A	B_1 and B_2	
level δ	S_1, S_2, S_3	S_1 and S_2	S_3
0.8	61.0196	68.0356	74.6372
0.9	67.4310	78.0559	87.7557
0.95	74.6255	87.9749	100.7708
0.99	87.8464	110.7696	130.7648

Table 7.3: Risk contributions of S_1 to S_3 according to Equation (6.8) *for different levels δ and dependence structures.*

we let

$$B_1 = \begin{pmatrix} \frac{1}{40} & \frac{21}{8} & 0 & \frac{21}{8} \\ \frac{1}{40} & 0 & 0 & \frac{21}{4} \\ \frac{1}{40} & 0 & 0 & \frac{21}{4} \end{pmatrix}, \qquad B_2 = \begin{pmatrix} \frac{1}{40} & 0 & 0 & \frac{21}{4} \\ \frac{1}{40} & 0 & \frac{21}{8} & \frac{21}{8} \\ \frac{1}{40} & 0 & 0 & \frac{21}{4} \end{pmatrix}.$$

In both cases we let the parameter $R_0 = 10$. Finally, as distribution of the claim sizes we choose $X_{i,1} \sim \text{NegBin}(4, 0.4)$ for $i = 1, 2, 3$. We compute the first 300 values of the respective probability mass functions. We also combine both probability mass functions in one graph, see Figure 7.3. In this figure it is interesting to note that a positive correlation between default cause intensities seems to have a stronger impact on the distribution than a negative correlation if we take the case of independent default cause intensities as a reference point, see also e.g. Figure 7.2.

For these cases of dependence scenarios we also compute the corresponding risk contributions regarding the respective default cause intensities. The results are given with respect to different levels $\delta = 0.8$, 0.9, 0.95, 0.99 in Table 7.3.

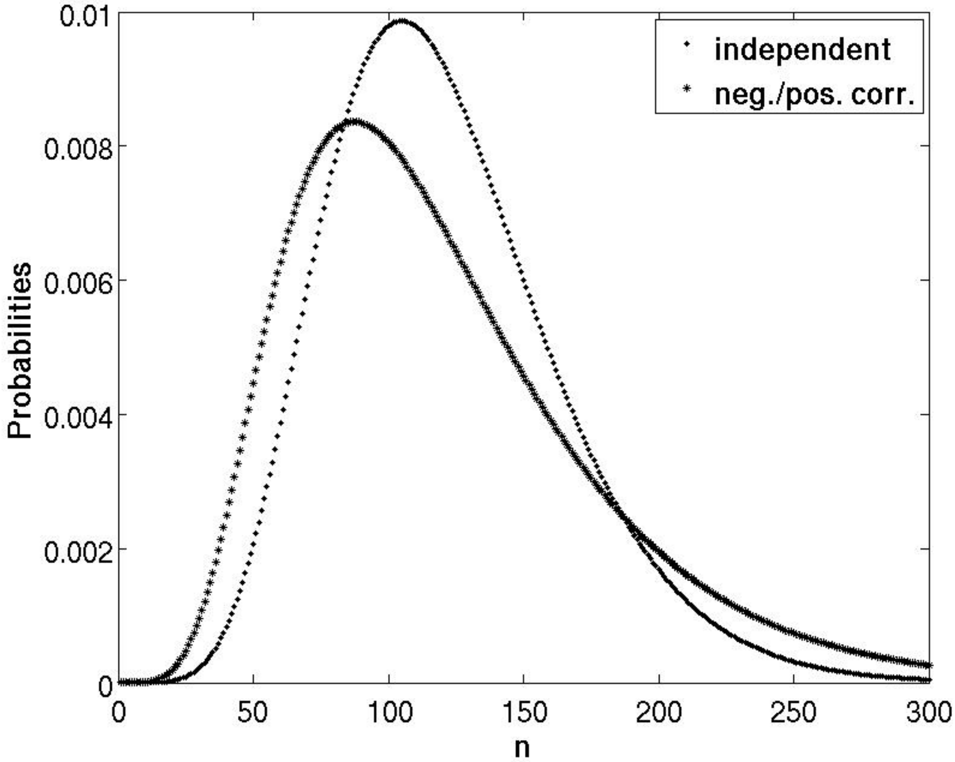

Figure 7.3: Probability mass functions of portfolio loss for different dependence structures with three risk factors and claim size $X_{i,1} \sim \mathrm{NegBin}(4, 0.4)$.

Part II

An Approximation via Panjer's Recursion for Poisson Mixture Models

Chapter 8

Introduction

It is quite common in the collective risk model to use Poisson mixture models for the distribution of N in a random sum

$$S = \sum_{n=1}^{N} X_n, \tag{8.1}$$

where $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of independently and identically distributed random variables and N is an $\mathbb{N}_0$-valued non-negative random variable independent of the sequence $\{X_n\}_{n\in\mathbb{N}}$. In actuarial science the distribution of N is called claim frequency or claim counting distribution, the distribution of X_1 is called severity distribution. In this part of the thesis we focus more on the distribution of N rather than on dependence structures as in the first part. The intensity of the Poisson random variable N is chosen to be stochastic, and this construction can represent effects such as heavy tails. Hence, for some non-negative random variable Λ, we consider

$$\mathcal{L}(N\,|\,\Lambda) \stackrel{\text{a.s.}}{=} \text{Poisson}(\Lambda)\,.$$

Not only in classical risk theory or in insurance models but also in risk management such as the extended CreditRisk$^+$ model, it is a major issue to evaluate the distribution of S. In [27, Chapter 2] the CreditRisk$^+$ model is described to have risk factors driving the risk of default with a gamma distribution. As the gamma distribution is a rather light-tailed distribution, it is interesting to investigate alternative distributions, possibly with heavier tails, that allow for the application of recursive methods like Panjer's recursion in order to evaluate the distribution of a random sum given in Equation (8.1). Since the distribution of a risk factor corresponds to the mixing distribution of a Poisson mixture distribution in our model, we also generalize CreditRisk$^+$, and this is a good reason to call such a risk factor default cause intensity. We should also mention that this approach can be applied to approximate value-at-risk, cf. e.g. Equation (10.1) and also Lemma 10.2. Though not the fastest method, recursions such as Panjer's offer the opportunity to compute distributions without a stochastic error as is the case in Monte Carlo methods or aliasing errors as in the fast Fourier transformation (possibly with exponential tilting). Panjer's recursion was already introduced in the introduction of the first part of this thesis. The recursion that characterizes a distribution in a Panjer(a, b, k) class is given by Equation (1.2) and the recursion used for the evaluation of a random sum of the type (8.1) is given by Equations (1.3) and (1.4). In that part the reader may also find further literature on Panjer's recursion.

We consider a more general class of distributions for the default cause intensity, namely the class of generalized gamma convolutions, which provide $\mathbb{R}_+$-valued random variables. It is the smallest class of distributions with support on $\mathbb{R}_+$ that contains all gamma distributions and is closed under convolution and weak limits. Examples for generalized gamma convolutions are the Pareto distribution, the beta distribution of the second kind, the (generalized) inverse Gaussian distribution, and the lognormal distribution. Generalized gamma convolutions are exhaustively treated in a monograph by Bondesson [8] as a continuation of Thorin's work, who studied whether the lognormal distribution is infinitely divisible or not and then introduced generalized gamma convolutions, cf. [67] and [66]. Steutel and van Harn [60] also contribute to this topic in their book on infinitely divisible probability distributions. James, Roynette, and Yor [36] treat generalized gamma convolutions as distributions of processes. Gerber [20] considers special generalized gamma convolutions, namely the generalized gamma distribution, as mixing distributions of Poisson mixture random variables and computes its distributions using Panjer's recursion. Contributions to the multivariate case are given by Barndorff–Nielsen, Maejima, and Sato [3]. A recent working paper by Pérez-Abreu and Stelzer [51] also establishes new results on the representation of multivariate generalized gamma convolutions.

The evaluation of the distribution of mixed compound Poisson distributions has already been treated in [74], but with other mixing distributions. Hesselager [32] also generalizes this recursion for some mixed compound Poisson distributions. The mixing distributions are also in the class of distributions we consider. Wang and Sobrero [72] extend the results of Hesselager to a more general class of distributions. Willmot [76] also provides the recursive evaluation of certain mixed Poisson probabilities that are apart from Panjer's recursion. The results we mention here all provide exact evaluations of the respective compound Poisson distributions. Unfortunately, it is no longer possible to calculate the distribution of S exactly with the general class of distributions we consider, but we give an upper bound for the error with respect to the total variation distance.

The remainder of this part of the thesis is organized as follows: In Chapter 9 we first introduce finite convolutions of gamma distributed random variables as mixing distributions of Poisson mixture distributions and then extend them to generalized gamma convolutions. Since this means that we work with infinite series, we cannot compute the distribution of Equation (8.1) exactly, but must approximate it. In this chapter we also present our approach for the approximation of this Poisson mixture distribution which is based on finite convolutions. It is a well-known result that a Poisson distribution mixed over a gamma distribution provides a negative binomial distribution, and hence Panjer's recursion can be applied. Further we exploit the convolution property of the Poisson distribution mixed over a finite gamma convolution. This will be crucial for our approximation.

Since we renounce an exact result using the approximation, Chapter 10 provides an upper bound of the error of the approximation. This error bound is given with respect to the total variation distance. Additionally, with regard to applications we also discuss two possibilities to approximate a generalized gamma convolution which lead to an algorithm to evaluate such a distribution. Here a decision must be taken whether to favor a more efficient approximation with respect to the number of steps that requires a possibly numerical integration or a less efficient integration with respect to the number of steps that circumvents an integration and thus might be more stable. This approximation together with an iterated application of Panjer's recursion is merged into an algorithm.

In Chapter 11 we present a generalization to the multivariate case, i.e., we also extend the CreditRisk$^+$ model. This means that we present a characterization of a multivariate generalized gamma convolution that we use for an alternative multivariate generalization of Bondesson's closure theorem. Using this result with an adaptation of notation, the results from the univariate case carry over to the multivariate case.

In Chapter 12 we present two examples that show the advantages of our approach in comparison to different versions of the fast Fourier transformation. It will become obvious that it is not clear a priori which of these versions of the fast Fourier transformation is preferable in contrast to our algorithm that provides stable results. Additionally, there are cases in which it is not possible to find an error estimate for fast Fourier transformations where our error bound applies.

Chapter 9

Gamma Convolutions

As already mentioned in the introduction, we consider a further class of distributions that allows us to apply Panjer's recursion for Poisson mixture models in the course of an approximation. We mix Poisson distributions with generalized gamma convolutions. As the name indicates, generalized gamma convolutions arise as the weak limit of sums of independent gamma distributed random variables (cf. e.g. [8, p. 29]). Therefore we first consider finite convolutions of gamma distributions that are mixed over Poisson distributions. Afterwards we introduce generalized gamma convolutions and present a few properties.

9.1 Finite Gamma Convolutions

We start our considerations with finite gamma convolutions. For the sake of comparability, we state the Laplace transform of a finite sum $Y = \sum_{j=0}^{n} Y_j$ of independent random variables Y_j with $Y_j \sim \mathrm{Gamma}(\alpha_j, \beta_j)$ for $j \in \{1, \ldots, n\}$ and $Y_0 \sim \delta_a$ for $a \geq 0$ which has the form

$$\mathbb{E}\big[\mathrm{e}^{-sY}\big] = \mathrm{e}^{-as} \prod_{j=1}^{n} \Big(\frac{\beta_j}{\beta_j + s}\Big)^{\alpha_j} = \exp\Big(-as + \sum_{j=1}^{n} \alpha_j \ln\Big(\frac{\beta_j}{\beta_j + s}\Big)\Big) \tag{9.1}$$

for $s > -\min\{\beta_1, \ldots, \beta_n\}$. Thus we will see in how far finite gamma convolutions are special cases of generalized gamma convolutions as its canonical representation will show. We prove a corollary giving a different representation of a Poisson mixture distribution where the mixing distribution is a finite gamma convolution. Note that we give it in the multivariate case.

Corollary 9.2. *Fix $k_i \in \mathbb{N}$ for $i \in \{1, \ldots, m\}$ and $m \in \mathbb{N}$. Let $\alpha_{i,j}, \beta_{i,j} > 0$ and $a_i \geq 0$ for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, k_i\}$. Let $\Lambda = (\Lambda_1, \ldots, \Lambda_m)$ be a random vector with*

$$\Lambda_i = a_i + Y_{i,1} + \cdots + Y_{i,k_i} \qquad \text{for } i \in \{1, \ldots, m\},$$

where $Y_{i,j} \sim \mathrm{Gamma}(\alpha_{i,j}, \beta_{i,j})$ are independent random variables for $j \in \{1, \ldots, k_i\}$ and $i \in \{1, \ldots, m\}$. Then consider the random vector $N = (N_1, \ldots, N_m)$ with conditionally independent components $N_1, \ldots, N_m$ given $\Lambda_1, \ldots, \Lambda_m$ such that for $i \in \{1, \ldots, m\}$

$$\mathcal{L}(N_i | \Lambda_1, \ldots, \Lambda_m) \overset{a.s.}{=} \mathcal{L}(N_i | \Lambda_i) \overset{a.s.}{=} \mathrm{Poisson}(\lambda_i \Lambda_i)\,, \tag{9.3}$$

where $\lambda_i \geq 0$ with $\lambda_1 + \cdots + \lambda_m > 0$. Further, let $\{B_{i,j,l}\}_{l \in \mathbb{N}}$ be $k_1 + \cdots + k_m$ independent sequences of i.i.d. random variables with $B_{i,j,1} \sim \mathrm{Ber}(p_{i,j})$ such that

$$p_{i,j} = \frac{\lambda_i / \beta_{i,j}}{\sum_{d=1}^{m} \lambda_d / \beta_{d,j}}$$

for $i \in \{1, \dots, m\}$ *and* $j \in \{1, \dots, k_i\}$*. Additionally, assume that* $R_{i,j} \sim \mathrm{NegBin}(\alpha_{i,j}, q_j)$ *such that*

$$q_j = \frac{\sum_{d=1}^m \lambda_d/\beta_{d,j}}{1 + \sum_{d=1}^m \lambda_d/\beta_{d,j}}$$

are independent of each other and of the sequences $\{B_{i,j,l}\}_{l\in\mathbb{N}}$ *for* $i \in \{1, \dots, m\}$ *and* $j \in \{1, \dots, k_i\}$*. Let* $P_i \sim \mathrm{Poisson}(\lambda_i a_i)$ *be independent of each other and of* $R_{i,j}$ *and* $\{B_{i,j,l}\}_{l\in\mathbb{N}}$ *for* $i \in \{1, \dots, m\}$ *and* $j \in \{1, \dots, k_i\}$*. Now let* $M = (M_1, \dots, M_m)$ *be a random vector such that*

$$M_i = P_i + \sum_{j=1}^{k_i} \sum_{l=1}^{R_{i,j}} B_{i,j,l} \qquad \text{for } i \in \{1, \dots, m\}.$$

Then N *and* M *have the same distribution.*

Proof. In order to prove this corollary, we apply Remark 2.1. We start with the calculation of the probability-generating function of the distribution of the random vector N: conditioning on $\Lambda_1, \dots, \Lambda_m$, using the conditional independence of $N_1, \dots, N_m$ given $\Lambda_1, \dots, \Lambda_m$ and Equation (9.3) yields for $0 \le z_i < 1 + \beta_{i,j}/\lambda_i$ for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, k_i\}$

$$G_N(z) = \mathbb{E}\biggl[\prod_{i=1}^m z_i^{N_i}\biggr] = \mathbb{E}\biggl[\prod_{i=1}^m \mathbb{E}\bigl[z_i^{N_i} \,\big|\, \Lambda_i\bigr]\biggr] = \mathbb{E}\biggl[\prod_{i=1}^m \exp(-\lambda_i \Lambda_i (1 - z_i))\biggr].$$

Using the definition of $\Lambda_1, \dots, \Lambda_m$ and the distribution and the independence of $Y_{i,j}$ for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, k_i\}$, we have by Equation (9.1)

$$\begin{aligned} G_N(z) &= \mathbb{E}\biggl[\prod_{i=1}^m \exp\biggl(-\lambda_i \biggl(a_i + \sum_{j=1}^{k_i} Y_{i,j}\biggr)(1 - z_i)\biggr)\biggr] \\ &= \prod_{i=1}^m \exp(-\lambda_i a_i (1 - z_i)) \prod_{j=1}^{k_i} \exp\biggl(\alpha_{i,j} \ln\Bigl(\frac{\beta_{i,j}}{\beta_{i,j} + \lambda_i(1 - z_i)}\Bigr)\biggr). \end{aligned}$$

Next, we consider the distribution of the random vector M and calculate its probability-generating function. Taking into account that all the components are independent of each other, we observe for $0 \le z_i < 1 + \beta_{i,j}/\lambda_i$ with $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, k_i\}$

$$G_M(z) = \prod_{i=1}^m \mathbb{E}\bigl[z_i^{M_i}\bigr] = \prod_{i=1}^m \mathbb{E}\Bigl[z_i^{P_i + \sum_{j=1}^{k_i} \sum_{l=1}^{R_{i,j}} B_{i,j,l}}\Bigr] = \prod_{i=1}^m \mathbb{E}\bigl[z_i^{P_i}\bigr] \prod_{j=1}^{k_i} \mathbb{E}\Bigl[z_i^{\sum_{l=1}^{R_{i,j}} B_{i,j,l}}\Bigr].$$

Using that $\{B_{i,j,l}\}_{l\in\mathbb{N}}$ are i.i.d. and are independent of $R_{i,j}$ and have a Bernoulli distribution and that P_i have a Poisson distribution, we observe

$$\begin{aligned} G_M(z) &= \prod_{i=1}^m \exp(-\lambda_i a_i (1 - z_i)) \prod_{j=1}^{k_i} \mathbb{E}\bigl[\bigl(\mathbb{E}\bigl[z_i^{B_{i,j,1}}\bigr]\bigr)^{R_{i,j}}\bigr] \\ &= \prod_{i=1}^m \exp(-\lambda_i a_i (1 - z_i)) \prod_{j=1}^{k_i} \mathbb{E}\bigl[\bigl(1 - p_{i,j} + z_i p_{i,j}\bigr)^{R_{i,j}}\bigr]. \end{aligned}$$

By the distribution of $R_{i,j}$ for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, k_i\}$ we see that

$$G_M(z) = \prod_{i=1}^m \exp(-\lambda_i a_i (1 - z_i)) \prod_{j=1}^{k_i} \biggl(\frac{1 - q_j}{1 - q_j(1 + p_{i,j}(z_i - 1))}\biggr)^{\alpha_{i,j}}.$$

A simplification yields

$$
\begin{aligned}
G_M(z) &= \prod_{i=1}^{m} \exp(-\lambda_i a_i(1-z_i)) \prod_{j=1}^{k_i} \Big(\frac{1}{1+\lambda_i/\beta_{i,j}(1-z_i)}\Big)^{\alpha_{i,j}} \\
&= \prod_{i=1}^{m} \prod_{j=1}^{k_i} \exp\Big(\alpha_{i,j} \ln\Big(\frac{\beta_{i,j}}{\beta_{i,j}+\lambda_i(1-z_i)}\Big)\Big).
\end{aligned}
$$

Hence the assertion follows. q.e.d.

Remark 9.4. Due to this representation result we may conclude that it is also possible to apply Panjer's recursion for a collective risk model with several business lines and a mixing random vector $\Lambda = (\Lambda_1 \dots, \Lambda_m)$ whose components have distributions of finite convolutions of gamma distributions. In this case the recursion is exact.

Remark 9.5. With this multivariate version it is possible to generalize the CreditRisk$^+$ model for several default cause intensities that influence the claim numbers on different levels.

Remark 9.6. In Corollary 9.2 we consider a very special case of a multivariate gamma distribution (and let us put $a_i = 0$ for $i \in \{1, \dots, m\}$). As the reader can verify in [44, Chapter 48.3], multivariate gamma distributions usually have correlated components. The approach of Prékopa and Szántai [44, Chapter 48.3.4] offers the possibility to construct independent components by choosing the identity matrix for the matrix A.

Remark 9.7. Because we consider a random vector $\Lambda = (\Lambda_1, \dots, \Lambda_m)$ this corollary differs from Remark 2.9 that uses a univariate random variable Λ as a mixing distribution which produces dependence between the components. That is also why we consider Bernoulli random variables instead of merging them into a multinomial distribution which would produce dependence.

9.2 Generalized Gamma Convolutions

Now we generalize finite convolutions of gamma distributions. Generalized gamma convolutions were first treated in [66, 67]. The construction is based on discrete measures $\{U_n\}_{n\in\mathbb{N}}$ that converge vaguely to the limit measure U and hence the corresponding distribution converges weakly to its limit distribution.

Definition 9.8. A locally finite non-negative measure on the half line $(0, \infty)$ equipped with its Borel σ-algebra satisfying

$$
\int_{(0,1]} |\ln t| \, U(dt) < \infty \qquad \text{and} \qquad \int_{(1,\infty)} \frac{1}{t} \, U(dt) < \infty \tag{9.9}
$$

is called a Thorin measure.

Furthermore, the conditions on U ensure that U is well-defined and that it is a Radon measure, as demanded in [36, Definition 1.0]. We state a definition of the canonical representation of generalized gamma convolutions as in [8, p. 29].

Definition 9.10. A generalized gamma convolution is a probability distribution F on $\mathbb{R}_+ = [0,\infty)$ with $a \geq 0$ and Thorin measure U such that its Laplace transform is of the form

$$L(s) = \int_0^\infty \mathrm{e}^{-sx}\ F(dx) = \exp\Big(-as - \int_{(0,\infty)} \ln\Big(\frac{t+s}{t}\Big)\, U(dt)\Big), \qquad s \geq 0. \tag{9.11}$$

The integral conditions on U ensure the existence of the integral in Equation (9.11): This holds owing to the estimates $\ln(\frac{t+s}{t}) \leq \ln(1+s) + |\ln t|$ for $t \in (0,1]$ and $\ln(\frac{t+s}{t}) \leq s/t$ for $t \in (1,\infty)$. The parameter a reflects the left-extremity of the distribution. Note that a finite gamma convolution such as in Equation (9.1) is also a generalized gamma convolution if we set in the representation of the generalized gamma convolution the parameter $a = 0$ and choose $U = \sum_{j=1}^n \alpha_j \delta_{\beta_j}$. This is crucial for the construction and approximation of generalized gamma convolutions as generalized gamma convolutions arise as the weak limit of a sequence of finite gamma convolutions.

The next corollary shows that each Thorin measure U, that satisfies the integral conditions and hence provides the well-definedness of the Laplace transform, determines a probability distribution. By the uniqueness theorem in [66] such a probability distribution is uniquely determined by the Thorin measure U and the parameter a.

Corollary 9.12. *For every $a \geq 0$ and Thorin measure U, the right-hand side of Equation* (9.11) *is the Laplace transform of a probability distribution.*

Proof. The proof of this claim is an application of Bernstein's theorem, cf. [56, Theorem 1.4]. Note that the Laplace transform of a generalized gamma convolution is given by Equation (9.11). Thus the complete monotonicity of L has to be shown. Consider the function $f(s) = -as - \int_{(0,\infty)} \ln\big(\frac{t+s}{t}\big) U(dt)$ for $s \geq 0$. The first derivative is

$$f'(s) = -a - \int_{(0,\infty)} \frac{t}{t+s}\frac{1}{t}\, U(dt) = -f_1(s), \qquad s > 0,$$

with

$$f_1(s) = a + \int_{(0,\infty)} \frac{1}{t+s}\, U(dt)$$

because integration and differentiation may be interchanged by [5, 16.2 Lemma]. We claim that for $j \in \mathbb{N}$ with $j \geq 2$

$$f^{(j)}(s) = (-1)^j f_j(s), \qquad s > 0,$$

with

$$f_j(s) = \int_{(0,\infty)} \frac{(j-1)!}{(t+s)^j}\, U(dt)$$

holds, as a short proof by induction shows. It is easy to see that $\{f_j(s)\}_{j\in\mathbb{N}}$ is non-negative for $s > 0$. In a next step we apply Faà di Bruno's formula to calculate the n-th derivative of $L(s) = \mathrm{e}^{f(s)}$ and obtain

$$L^{(n)}(s) = \mathrm{e}^{f(s)} \sum \frac{n!}{m_1!m_2!\dots m_n!} \prod_{j=1}^n \Big(\frac{f^{(j)}(s)}{j!}\Big)^{m_j}, \qquad s > 0, \tag{9.13}$$

where the sum runs over all n-tuples of non-negative integers $(m_1, \dots, m_n)$ such that

$$\sum_{j=1}^{n} j m_j = n. \tag{9.14}$$

Writing $(-1)^n = \prod_{j=1}^{n}(-1)^{jm_j}$ by Equation (9.14) and noting that $(-1)^{jm_j}(f^j(s))^{m_j} = (f_j(s))^{m_j}$ for all $j \in \{1, \dots, n\}$, Equation (9.13) proves that

$$\begin{aligned}(-1)^n L^{(n)}(s) &= (-1)^n \,\mathrm{e}^{f(s)} \sum \frac{n!}{m_1!m_2!\dots m_n!} \prod_{j=1}^{n} \Big(\frac{f^{(j)}(s)}{j!}\Big)^{m_j} \\ &= \mathrm{e}^{f(s)} \sum \frac{n!}{m_1!m_2!\dots m_n!} \prod_{j=1}^{n} \Big(\frac{f_j(s)}{j!}\Big)^{m_j} \geq 0\end{aligned}$$

for $s \geq 0$. Because this holds for every $n \in \mathbb{N}$, L is completely monotone, and since $L(0) = 1$, this determines uniquely a probability distribution. As these conclusions are independent of the choice of the Thorin measure U and the parameter a, every Laplace transform of the form (9.11) determines a probability distribution. q.e.d.

Since the multiplication of the Laplace transform by e^{-as} for $a > 0$ corresponds to the translation by a, we can also represent such a distribution as the convolution with a Dirac measure δ_a (cf. [67, Equation (2.5)]), which is the weak limit of the gamma distributions $\mathrm{Gamma}(an, n)$ for $n \to \infty$ because these have expectation a and variance $\frac{a}{n}$.

We state some clarifying notation for the Thorin measure U.

Notation 9.15. For the application of the theory of generalized gamma convolutions it is convenient to identify the Thorin measure U with the non-decreasing and right-continuous function U

$$U(t) = U((0, t]), \qquad t \in (0, \infty).$$

This is reasonable because the first condition of Equation (9.9) and the local finiteness of U guarantee $U((0, t]) < \infty$.

Remark 9.16. As [36, Remark 2.4.3)] also states, finding the Thorin measure of a generalized gamma convolution is tedious. Bondesson provides an inversion theorem [8, Theorem 3.1.4] and a theorem on the representation of the Thorin measure [8, Theorem 4.3.2]. Nevertheless, these result do not always provide explicit expressions, although [36, Theorem 2.3.3)] may lead to a tractable expression.

Let $\mathcal{T}$ be the class of distribution functions of generalized gamma convolutions. We assume that for each $F_n \in \mathcal{T}$ with $n \in \mathbb{N}$ the parameter a_n and the Thorin measure U_n characterize F_n. Our understanding of a reformulation in [8, p. 35] of [8, Theorem 3.1.5] is the following:

Theorem 9.17. *The sequence $\{F_n\}_{n\in\mathbb{N}} \in \mathcal{T}$ tends weakly to $F \in \mathcal{T}$ if and only if*

(a) the sequence $\{U_n\}_{n\in\mathbb{N}} \to U$ vaguely on $(0, \infty)$ as $n \to \infty$,

(b) $0 = \lim_{A\to\infty} \limsup_{n\to\infty} \big| a_n - a + \int_{(A,\infty)} t^{-1}\, U_n(dt) \big|$,

(c) $\lim_{\varepsilon\searrow 0} \limsup_{n\to\infty} \int_{(0,\varepsilon)} (\ln t^{-1})\, U_n(dt) = 0$.

According to [8, p. 35] Condition (c) ensures that F is non-defective.

Remark 9.18. The proof of this theorem seems to contain an expression with a false sign. In the notation of the proof it should be

$$\log \psi_n(-1) = -\int_{[0,\infty]} \nu_n(dt).$$

Yet the sequence $\{\nu_n\}_{n\in\mathbb{N}}$ is still a sequence of bounded measures on $[0,\infty]$.

Thus by applying Theorem 9.17 we can approximate generalized gamma convolutions by finite gamma convolutions. As mentioned above, generalized gamma convolutions provide a rich class of distributions. To make the reader more familiar with generalized gamma convolutions, we give some examples:

Example 9.19. Since a translation of a random variable by a can be approximated by a gamma distribution, a generalized gamma convolution may be characterized in most cases by a parameter $a = 0$. But there are distributions that require $a \neq 0$, as is also described in [60, p. 349]. Let $U_n(t) = n1_{[n,\infty)}(t)$ for $t \in (0,\infty)$ and $a_n = 0$ for $n \in \mathbb{N}$. By application of Theorem 9.17, a sequence exists $\{F_n\}_{n\in\mathbb{N}} \in \mathcal{T}$ that weakly tends to some $F \in \mathcal{T}$ because $U_n \to U = 0$ vaguely for $n \in \mathbb{N}$,

$$a = \lim_{A\to\infty}\lim_{n\to\infty}\left(a_n + \int_{(A,\infty)} t^{-1}\,U_n(dt)\right) = \lim_{A\to\infty}\lim_{n\to\infty}\left(0 + \frac{n}{n}\right) = 1,$$

and because $U_n \to 0$ as $n \to \infty$, integrating around 0 over it yields 0. Thus the limit distribution F is characterized by $a = 1$ and $U = 0$. Plugging this into the formula of the Laplace transform gives $L(s) = \mathrm{e}^{-s}$, which is the Laplace transform of a degenerate random variable.

Example 9.20 (Relationship to the τ-tempered α-stable distribution)**.** This example shows that our approach is a generalization of [21, Chapter 5.3]. For further explanations on stable distributions read Section 3.2.1. Recall that the Laplace transform of a random variable $Y \sim S_\alpha(\sigma,1,0)$ is given by

$$\mathbb{E}[\exp(-sY)] = \exp(-\gamma_{\alpha,\sigma}s^\alpha) \qquad \text{for } s \geq 0, \qquad \text{where } \gamma_{\alpha,\sigma} := \frac{\sigma^\alpha}{\cos(\alpha\pi/2)},$$

cf. [55, Proposition 1.2.12] (note $\alpha \in (0,1)$).

According to [60, Proposition VI.5.7] a stable distribution of an $\mathbb{R}_+$-valued random variable is a generalized gamma convolution for $\alpha \in (0,1)$ and $\sigma > 0$, where

$$u_{\alpha,\sigma}(x) = \gamma_{\alpha,\sigma}\alpha\frac{\sin(\alpha\pi)}{\pi}x^{-(1-\alpha)}, \qquad x > 0,$$

is a density of $U_{\alpha,\sigma}$. Steutel and van Harn [60, Proposition VI.5.26] note that if π is the Laplace transform of a generalized gamma convolution, then so is the function $s \mapsto \pi(a+s)/\pi(a)$ for every $a \geq 0$. A straightforward application shows that τ-tempered α-stable distributions are also generalized gamma convolutions:

$$\pi(\tau+s)/\pi(\tau) = \exp(-\gamma_{\alpha,\sigma}(s+\tau)^\alpha)/\exp(-\gamma_{\alpha,\sigma}\tau^\alpha) = \exp(-\gamma_{\alpha,\sigma}((s+\tau)^\alpha - \tau^\alpha)),$$

which is the Laplace transform of a τ-tempered α-stable distribution for $s \geq 0$, cf. [21, Equation (5.25)]. By knowledge of the density $u_{\alpha,\sigma}$ characterizing the stable distribution, we can also derive the density $u_{\alpha,\sigma,\tau}$ of the Thorin measure $U_{\alpha,\sigma,\tau}$ of the τ-tempered α-stable distribution since we know its Laplace transform

$$\begin{aligned}\exp(-\gamma_{\alpha,\sigma}((s+\tau)^\alpha-\tau^\alpha)) &= \exp\Big(-\int_{(0,\infty)}\Big(\ln\Big(\frac{t+s+\tau}{t}\Big)-\ln\Big(\frac{t+\tau}{t}\Big)\Big)\,u_{\alpha,\sigma}(t)dt\Big)\\ &= \exp\Big(-\int_{(0,\infty)}\ln\Big(\frac{t+s+\tau}{t+\tau}\Big)\,u_{\alpha,\sigma}(t)dt\Big).\end{aligned}$$

Using the substitution $y = t+\tau$ yields

$$\begin{aligned}\exp(-\gamma_{\alpha,\sigma}((s+\tau)^\alpha-\tau^\alpha)) &= \exp\Big(-\int_{(\tau,\infty)}\ln\Big(\frac{y+s}{y}\Big)\,u_{\alpha,\sigma}(y-\tau)dy\Big)\\ &= \exp\Big(-\int_{(0,\infty)}\ln\Big(\frac{y+s}{y}\Big)\,u_{\alpha,\sigma,\tau}(y)dy\Big),\end{aligned}$$

where

$$u_{\alpha,\sigma,\tau}(t) = \begin{cases} u_{\alpha,\sigma}(t-\tau) & \text{if } t>\tau,\\ 0 & \text{otherwise.}\end{cases}$$

Hence the Thorin measure $U_{\alpha,\sigma,\tau}$ is

$$U_{\alpha,\sigma,\tau}(x) = \begin{cases} \gamma_{\alpha,\sigma}\frac{\sin(\alpha\pi)}{\pi}(x-\tau)^\alpha & \text{if } x>\tau,\\ 0 & \text{otherwise.}\end{cases} \tag{9.21}$$

Thus we could also apply the approximation algorithm we present. But in this case it is possible to evaluate the distribution exactly.

If the mixing random variable has distribution $\Lambda \sim F_{\alpha,\sigma,\tau,0}$, then according to [21, Lemma 5.10] a random variable $N \sim \text{Poisson}(\lambda\Lambda)$ for $\lambda > 0$ can be represented as a random sum such that $N \stackrel{\text{d}}{=} \sum_{h=1}^{M} Y_h$ where $M \sim \text{Poisson}\big(\gamma_{\alpha,\sigma}\big((\lambda+\tau)^\alpha-\tau^\alpha\big)\big)$ and $\{Y_h\}_{h\in\mathbb{N}}$ is a sequence independent of M, consisting of i.i.d. random variables such that $Y_1 \sim \text{ExtNegBin}(-\alpha,1,p)$ with $p=\frac{\lambda}{\tau+\lambda}$. Since the extended negative binomial distribution $\text{ExtNegBin}(-\alpha,1,p)$ is in the $\text{Panjer}(1-p,(-\alpha-1)(1-p),1)$ class, such a random sum enables an iterated application of Panjer's recursion. Since the extended negative binomial distribution might be unstable for extreme parameters, [21, Algorithm 5.12], which is numerically stable, should be used instead.

Example 9.22. There are well-known distributions that fit into the framework of τ-tempered α-stable distributions. The Lévy distribution with $\alpha = 1/2$ and $\tau = 0$ (cf. [21, Example 5.19]) leads to the following Thorin measure

$$U_{1/2,\sigma,0}(x) = \frac{\sqrt{2\sigma x}}{\pi}, \qquad x>0.$$

A density of $F_{1/2,\sigma,0,0}$ is given by

$$f(x) = \Big(\frac{\sigma}{2\pi x^3}\Big)^{1/2}\exp\Big(-\frac{\sigma}{2x}\Big), \qquad x>0,$$

cf. [55, Equation (1.1.15)]. The inverse Gaussian distribution with mean $\mu > 0$ and shape-parameter $\tilde\sigma > 0$ yields with $\sigma = \mu^2/\tilde\sigma^2$ and $\alpha = 1/2$ and $\tau = 1/(2\tilde\sigma^2)$ (cf. [21, Example 5.21]) the Thorin measure

$$U_{1/2,\sigma,\tau}(x) = \begin{cases} \frac{\tilde\sigma}{\mu\pi}\big(2(x-1/(2\tilde\sigma^2)\big)^{1/2} & \text{if } x>\tau,\\ 0 & \text{otherwise,}\end{cases}$$

and $U_{1/2,\sigma,\tau}(0) = 0$.

Example 9.23 (The Pareto distribution)**.** We consider the following cumulative distribution function of a Pareto(g,h) distribution

$$P(x) = 1 - (1 + (x/g))^{-h}, \qquad x \geq 0, \quad g,h > 0.$$

According to [67] the Pareto(g,h) distribution is not only infinitely divisible, but also a generalized gamma convolution. The inversion theorem in [8, Theorem 3.1.4] states a connection between the moment generating function of the distribution of a random variable and the corresponding Thorin measure U. Hence, we obtain with [67, Equation (3.17)]

$$q(x) = \frac{\Gamma(h)}{\pi}\,\mathrm{e}^{gx}\left((gx)^{-h} + \sum_{j=1}^{\infty}\frac{(gx)^{j-h}}{(h-1)\cdots(h-j)}\right) - \cot(h\pi)$$

the Thorin measure

$$U(x) = \frac{1}{\pi}\,\mathrm{arccot}(q(x))\,,$$

where $x > 0$ and $h \in (0,\infty) \setminus \mathbb{N}$.

Example 9.24 (The beta distribution)**.** Let $\alpha, \beta > 0$. Let $X \sim \text{Gamma}(\alpha,1)$ and $Y \sim \text{Gamma}(\beta,1)$ be independent random variables. Then by [8, Theorem 5.1.1] the distribution of $Z = \frac{X}{X+Y}$ is also a generalized gamma convolution. It is the Beta(α,β) distribution and has a probability density function

$$f_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}, \qquad x \in (0,1),$$

cf. also [38, Equation (25.1)]. The derivation of the Thorin measure U is similar to the that of the beta distribution of the second kind in the next example and is therefore omitted. Note that we left out the values for the density for $x = 0$ and $x = 1$ since this would cause a gap in the definition if $\alpha = 1$ or $\beta = 1$, respectively.

Example 9.25 (The beta distribution of the second kind)**.** The beta distribution of the second kind with shape parameters $\alpha, \beta > 0$ is also a generalized gamma convolution. It arises as the ratio of two independent gamma distributed random variables. If $X \sim \text{Gamma}(\alpha,1)$ and $Y \sim \text{Gamma}(\beta,1)$ are independent, then according to [8, Theorem 5.1.1] the distribution of $Z = X/Y$ is a generalized gamma convolution and has a Beta$'(\alpha,\beta)$ distribution with density

$$f_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1+x)^{-(\alpha+\beta)}, \qquad x > 0.$$

Unfortunately, the Thorin measure U cannot be given in a very elegant form, but only with respect to the confluent hypergeometric distribution. The derivation of the Thorin measure U can be found in [24]. The Thorin measure U is given by

$$U(x) = \frac{1}{\pi}\int_0^x \mathrm{Im}\,\frac{U^{+\prime}(\alpha, 1-\beta, -z)}{U^{+}(\alpha, 1-\beta, -z)}dz$$

where

$$\mathrm{Im}\,\frac{U^{+\prime}(\alpha, 1-\beta, -z)}{U^{+}(\alpha, 1-\beta, -z)} = \pi\beta z^{\beta-1}\,\mathrm{e}^{-z}\,\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)\Gamma(\beta+1)}\,\frac{1}{|U^{+}(\alpha, 1-\beta, -z)|^2}$$

and for $a > 0$, $b \in \mathbb{R}$, and $\operatorname{Re} z \geq 0$

$$\Gamma(a)U(a,b,z) = \int_0^\infty \mathrm{e}^{-tz}\, t^{a-1}(1+t)^{b-a-1} dt.$$

According to [24] $U^+(a,b,-x)$ is the limit for $U(a,b,z)$ as z approaches $-x$ from the upper half plane for positive real x.

Example 9.26 (The lognormal distribution)**.** Thorin [66] shows that the lognormal distribution is a generalized gamma convolution, but there is no closed form for the Thorin measure U. Nevertheless, let us state his results. Let $N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y \mathrm{e}^{-t^2/2}\, dt$ denote the distribution function of the standard normal distribution function. Thorin considers the following representation of the distribution function of the lognormal distribution with $\alpha, \beta > 0$

$$\Lambda(x) = \Lambda(x;\alpha,\beta) = \begin{cases} N\left(\frac{\ln(\alpha x)}{\beta}\right) & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The moment generating function of Λ is defined as $\lambda(s) = \int_{(0,\infty)} \mathrm{e}^{sx}\, d\Lambda(x)$ for $\operatorname{Re} s \leq 0$, and its approximation as

$$\lambda_n(s) = \int_{(0,\infty)} \mathrm{e}^{sx}\, d\Lambda_n(x) = \int_{(0,\infty)} \frac{d\Lambda(x)}{(1-(sx/n))^n}.$$

Then by [66, Theorem 4.1] the approximating measure $\{U_n\}_{n\in\mathbb{N}}$, is given by

$$U_n(y) = \frac{1}{\pi} \arctan \frac{\operatorname{Im} \lambda_n^+(y)}{\operatorname{Re} \lambda_n^+(y)}, \qquad y > 0.$$

By help of [66, Theorem 5.2] we see that $\{U_n\}_{n\in\mathbb{N}}$ converges vaguely to the limit measure U such that

$$U(y) = \frac{1}{\pi} \arctan \frac{\operatorname{Im} \lambda^+(y)}{\operatorname{Re} \lambda^+(y)}, \qquad y > 0.$$

Remark 9.27. A similar Poisson mixture model leading to Panjer's recursion was already treated in [20], but with another, equivalent definition (cf. [36, Definition 1.0]) if we choose the right measure. According to [60, Example VI.12.8] a random variable with a generalized gamma distribution is defined as $X \stackrel{\mathrm{d}}{=} Y^{1/\alpha}$ with $\alpha \neq 0$ and $Y \sim \mathrm{Gamma}(r,1)$ for $r > 0$. If $|\alpha| \leq 1$, then it is known that the distribution of X also is a generalized gamma convolution. If $\alpha > 1$, then X is not infinitely divisible and hence its distribution not a generalized gamma convolution. It is not known what holds in the case $\alpha < -1$. Thus the model in [20] and this model have some common cases, and the model in [20] shows how to evaluate the respective distributions exactly.

As a generalized gamma convolution is the weak limit of convolutions of gamma distributions, it can be shown without much effort that for each generalized gamma convolution there exists a sequence that converges weakly to this distribution:

Corollary 9.28. *For every generalized gamma convolution F characterized by $a \geq 0$ and Thorin measure U there exists a weakly convergent sequence $\{F_n\}_{n\in\mathbb{N}}$ of finite gamma convolutions characterized by the parameter a and Thorin measure U_n with*

$$U_n = \sum_{j=1}^n \alpha_j^{(n)} \delta_{\beta_j^{(n)}}, \tag{9.29}$$

and $\{U_n\}_{n\in\mathbb{N}}$ converges vaguely to U as $n \to \infty$.

Remark 9.30. The representation (9.29) implies that the corresponding distribution satisfies $F_n = \delta_a \star \mathrm{Gamma}\big(\alpha_1^{(n)}, \beta_1^{(n)}\big) \star \cdots \star \mathrm{Gamma}\big(\alpha_n^{(n)}, \beta_n^{(n)}\big)$. Let $Y_j^{(n)} \sim \mathrm{Gamma}\big(\alpha_j^{(n)}, \beta_j^{(n)}\big)$ be independent for $j \in \{1, \dots, n\}$ and $n \in \mathbb{N}$, then

$$\Lambda_n = a + \sum_{j=1}^{n} Y_j^{(n)} \sim F_n. \tag{9.31}$$

Proof of Corollary 9.28. Since a generalized gamma convolution is characterized by a Thorin measure U, we construct a sequence of Thorin measures $\{U_n\}_{n\in\mathbb{N}}$ and show that this sequence converges vaguely to U. A Thorin measure U_n uniquely determines a distribution F_n. We apply Theorem 9.17.

Let $\mathcal{Z}_n = (\beta_k^{(n)})_{0\le k\le n}$ be a partition on a compact subset of $(0,\infty)$ where we have $0 < \beta_0^{(n)} < \beta_1^{(n)} < \cdots < \beta_n^{(n)} < \infty$ and $\beta_0^{(n)} \searrow 0$ and $\beta_n^{(n)} \to \infty$ as $n \to \infty$. The mesh of $\mathcal{Z}$ is defined as $\mathrm{mesh}(\mathcal{Z}_n) = \max_{j=1,\dots,n}(\beta_j^{(n)} - \beta_{j-1}^{(n)}) \to 0$ as $n \to \infty$. For each $j \in \{1, \dots, n\}$ let $\alpha_j^{(n)} = U(\beta_j^{(n)}) - U(\beta_{j-1}^{(n)})$. Consider now the Condition (a) of Theorem 9.17. Let $f\colon (0,\infty) \to \mathbb{R}$ be a continuous function with a compact support. By the definition of the Lebesgue-Stieltjes integral and because $\mathrm{supp}(f) \subset (\beta_0^{(n)}, \beta_n^{(n)}]$

$$\int_{(0,\infty)} f(t)\, U_n(dt) = \sum_{j=1}^{n} f(\beta_j^{(n)})(U(\beta_j^{(n)}) - U(\beta_{j-1}^{(n)})) \to \int_{(0,\infty)} f(t)\, U(dt)$$

converges vaguely as the mesh $\mu(\mathcal{Z})$ gets smaller and smaller because U is monotone (it is non-decreasing) for $n \to \infty$.

Consider now the Condition (b) of Theorem 9.17. For $j \in \{1 \dots, n\}$ we have

$$\frac{1}{\beta_j^{(n)}}\big(U(\beta_j^{(n)}) - U(\beta_{j-1}^{(n)})\big) = \frac{1}{\beta_j^{(n)}} \int_{(\beta_{j-1}^{(n)}, \beta_j^{(n)}]} U(dt).$$

Because $1/t$ is decreasing for $t > 0$, we estimate

$$\int_{(A,\infty)} \frac{1}{t}\, U_n(dt) = \sum_{\substack{j=1 \\ \beta_j^{(n)} > A}}^{n} \frac{1}{\beta_j^{(n)}} \alpha_j^{(n)} \le \int_{(A-\mathrm{mesh}(\mathcal{Z}_n),\infty)} \frac{1}{t}\, U(dt).$$

Because this integral tends to zero as $A \to \infty$, the left-extremity of the limit distribution is also a.

Finally, consider Condition (c). For $j \in \{1 \dots, n\}$ we have

$$\ln\!\Big(\frac{1}{\beta_j^{(n)}}\Big)\big(U(\beta_j^{(n)}) - U(\beta_{j-1}^{(n)})\big) = \ln\!\Big(\frac{1}{\beta_j^{(n)}}\Big) \int_{(\beta_{j-1}^{(n)}, \beta_j^{(n)}]} U(dt).$$

Because $\ln(1/t)$ is decreasing for $t > 0$, we estimate

$$\int_{(0,\varepsilon)} \ln\!\Big(\frac{1}{t}\Big)\, U_n(dt) = \sum_{\substack{j=1 \\ \beta_j^{(n)} < \varepsilon}}^{n} \ln\!\Big(\frac{1}{\beta_j^{(n)}}\Big) \alpha_j^{(n)} \le \int_{(0,\varepsilon-\mathrm{mesh}(\mathcal{Z}_n))} \ln\!\Big(\frac{1}{t}\Big)\, U(dt).$$

Because this integral tends to zero as $\varepsilon \to 0$, the assertion follows. q.e.d.

The next lemma describes the process of approximation of the Poisson mixture distribution mixed over a generalized gamma convolution.

Lemma 9.32. *Fix $\lambda \geq 0$. Let F be a generalized gamma convolution characterized by $a \geq 0$ and Thorin measure U with $\Lambda \sim F$ and let $\{\Lambda_n\}_{n\in\mathbb{N}}$ be a sequence of random variables given by Equation (9.31) that converges weakly to Λ as $n \to \infty$. Let $\{N_n\}_{n\in\mathbb{N}}$ be a sequence of random variables such that*

$$\mathcal{L}(N_n \,|\, \Lambda_n) \overset{a.s.}{=} \mathrm{Poisson}(\lambda\Lambda_n) \qquad \textit{for each } n \in \mathbb{N}. \tag{9.33}$$

Furthermore, for each $n \in \mathbb{N}$ let

$$R_j^{(n)} \sim \mathrm{NegBin}\left(\alpha_j^{(n)}, \frac{\lambda}{\beta_j^{(n)} + \lambda}\right), \qquad j \in \{1, \dots, n\}$$

and $P \sim \mathrm{Poisson}(\lambda a)$ be independent. Define a sequence of random variables $\{M_n\}_{n\in\mathbb{N}}$ by

$$M_n = P + \sum_{j=1}^{n} R_j^{(n)}, \tag{9.34}$$

Then $M_n \overset{d}{=} N_n$ for all $n \in \mathbb{N}$ and $\{N_n\}_{n\in\mathbb{N}}$ converges weakly to some random variable N such that

$$\mathcal{L}(N \,|\, \Lambda) \overset{a.s.}{=} \mathrm{Poisson}(\lambda\Lambda)\,. \tag{9.35}$$

Proof. The existence of the sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$ is proven in Corollary 9.28 and this corollary also proves the representation of the corresponding sequence of distributions $\{F_n\}_{n\in\mathbb{N}}$ that converges vaguely to the distribution F of Λ. The equality of the distributions of the random variables M_n and N_n for each $n \in \mathbb{N}$ is an immediate result of Corollary 9.2 for $m = 1$ because the random variables $\{B_{i,j,l}\}_{l\in\mathbb{N}}$ simplify to a δ_1 distribution.

Due to the weak convergence of the sequence of the distributions $\{F_n\}_{n\in\mathbb{N}}$ it follows that the sequence of the distributions of the random variables $\{N_n\}_{n\in\mathbb{N}}$ converges weakly, too, since

$$\begin{aligned} \mathbb{P}[N_n = k] \;&=\; \mathbb{E}[\,\mathbb{P}[N_n = k \,|\, \Lambda_n]\,] \;=\; \mathbb{E}\Big[\frac{\Lambda_n^k}{k!}\,\mathrm{e}^{-\Lambda_n}\Big] \\ &\to \mathbb{E}\Big[\frac{\Lambda^k}{k!}\,\mathrm{e}^{-\Lambda}\Big] \;=\; \mathbb{E}[\,\mathbb{P}[N = k \,|\, \Lambda]\,] \;=\; \mathbb{P}[N = k] \end{aligned}$$

as $n \to \infty$ because the function $f(x) = \frac{x^k}{k!}\,\mathrm{e}^{-x}$ is continuous and bounded for $x \in [0, \infty)$. Since $\mathcal{L}(N_n) \overset{w}{\to} \mathcal{L}(N)$ and $M_n \overset{d}{=} N_n$ for all $n \in \mathbb{N}$, we conclude that also $\mathcal{L}(M_n)$ converges weakly to $\mathcal{L}(N)$ as $n \to \infty$ due to the uniqueness of the limit. q.e.d.

Remark 9.36. It is interesting to note that a generalized analogue exists showing that a Poisson-gamma mixture distribution is a negative binomial distribution. Bondesson [8, p. 126/127, Chapter 8] defines the so-called generalized negative binomial convolution as the weak limit of a sequence of finite convolutions of negative binomial distributions. This class of distributions is the discrete analogue to the class of generalized gamma convolutions which has compact support on $\mathbb{R}_+$. There is a correspondence between generalized negative binomial convolutions and generalized gamma convolutions: A distribution is a generalized negative binomial convolution if and only if its probability-generating function $G(z)$ equals the moment generating function $M(z-1)$ of a generalized gamma convolution, which can be found in [8, p. 126/127, Chapter 8].

In this context, since a negative binomial distribution is also a compound Poisson distribution, it is interesting to note the following result in [60, Theorem III.3.3]: a distribution on $\mathbb{R}_+$ is infinitely divisible if and only if it is the weak limit of compound Poisson distributions, which matches the distribution of N.

Chapter 10

Error Bounds and an Approximation

By approximating the random variable N given in Equation (9.35) by a sequence $\{N_n\}_{n\in\mathbb{N}}$, the sequence of random sums $S_n = \sum_{h=1}^{N_n} X_h$ for $n \in \mathbb{N}$ approximates the random sum $S = \sum_{h=1}^{N} X_h$. We derive an upper bound of the distance between two Poisson distributions each of which is mixed over a generalized gamma convolution with respect to the total variation distance. A reference for the total variation distance can be found in [22]. This estimate can be applied to the error estimate between the sequence $\{S_n\}_{n\in\mathbb{N}}$ and the limit random variable S. Generalized gamma convolutions are self-decomposable (cf. [60, Proposition VI.5.5]) and hence infinitely divisible (cf. [60, Theorem V.2.5]). Thus, since we consider Poisson mixture distributions, we can find a representation as a compound Poisson distribution. This will be crucial for the proof of our error estimate of the approximation. Further we present how the approximation can be realized.

10.1 An Upper Estimate for the Total Variation Distance

In this section we establish an upper bound for the total variation distance between two Poisson mixture distributions each of which is mixed over a generalized gamma convolution. To this end, we use Lemma 9.32 that provides $M_n \overset{\mathrm{d}}{=} N_n$ for $n \in \mathbb{N}$. Note that Theorem 10.8 presents a result in the total variation distance. Since we only need weak convergence, the Prohorov metric is sufficient in this case. According to [35, Theorem 3.8] the Prohorov metric $\mathrm{d_P}$ metricizes the weak topology because $\mathbb{N}_0$ is a metric separable space and thus metricizes weak convergence of random variables. Because of the argumentation in [35, Equation (4.13)] the inequality

$$\mathrm{d_P} \le \mathrm{d_{TV}}$$

holds, hence the weak convergence follows by this estimate. We need this estimate because the total variation distance does not metricize the weak topology but has other convenient properties. These considerations have a direct application in risk management where the risk measure VaR is used. The value-at-risk at level $\delta \in (0,1)$ of a random loss S, which is the δ-quantile of S, is an approximation of the value-at-risk of S' in the sense that

$$\begin{aligned} \big|\, \mathbb{P}[S \le q_\delta(S)] \,-\, \mathbb{P}[S' \le q_\delta(S)] \,\big| &\le \sup_{m\in\mathbb{R}} \big|\, \mathbb{P}[S \le m] \,-\, \mathbb{P}[S' \le m] \,\big| \\ &= \mathrm{d_{KS}}(\mathcal{L}(S)\,,\mathcal{L}(S')) \le \mathrm{d_{TV}}\big(\mathcal{L}(S)\,,\mathcal{L}(S')\big)\,, \end{aligned} \tag{10.1}$$

where $\mathrm{d_{KS}}$ is the Kolmogorov–Smirnov distance (cf. [22]), and the claim sizes $\{X_h\}_{h\in\mathbb{N}}$ are also allowed to take real values. The last inequality holds by [22, Equations (5) and (6)].

We put this estimate more exactly into the following lemma:

Lemma 10.2. *Let X and Y be real-valued random variables and denote the Kolmogorov–Smirnov distance of their distribution by $d = \mathrm{d_{KS}}(\mathcal{L}(X)\,, \mathcal{L}(Y))$. Then the quantiles of X and Y satisfy*

(a) $q_\delta(X) \le q_{\delta+d}(Y)$ for every level $\delta \in (0, 1-d)$ and

(b) $q_{\delta-d}(X) \le q_\delta(Y)$ for every level $\delta \in (d, 1)$.

Proof. (a) For a level $\delta \in (0, 1-d)$ we estimate the quantile, and by adding and subtracting $\mathbb{P}[X \le q_{\delta+d}(Y)]$, we estimate

$$\begin{aligned}\delta + d &\le \mathbb{P}[Y \le q_{\delta+d}(Y)]\\ &\le \mathbb{P}[X \le q_{\delta+d}(Y)] + \big|\, \mathbb{P}[Y \le q_{\delta+d}(Y)] - \mathbb{P}[X \le q_{\delta+d}(Y)]\,\big|.\end{aligned}$$

Due to $\mathrm{d_{KS}}(\mathcal{L}(X)\,, \mathcal{L}(Y)) = \sup_{m\in\mathbb{R}}\big|\, \mathbb{P}[X \le m] - \mathbb{P}[Y \le m]\,\big|$, we estimate

$$\delta + d \le \mathbb{P}[X \le q_{\delta+d}(Y)] + d,$$

hence $\mathbb{P}[X \le q_{\delta+d}(Y)] \ge \delta$, therefore $q_\delta(X) \le q_{\delta+d}(Y)$.

(b) Note that the assumptions of the lemma are symmetric in X and Y. Applying (a) and interchanging X and Y and letting $\delta' := \delta - d$ yields $q_{\delta'}(Y) \le q_{\delta'+d}(X)$, which completes the proof.

q.e.d.

We continue with a representation result that we need for the comparison between our approximation and the limiting compound Poisson distribution:

Lemma 10.3. *Fix $n \in \mathbb{N}$. Let $\alpha_j, \beta_j > 0$ for $j \in \{1, \dots, n\}$ and let $\lambda \ge 0$. Let $Y_j \sim$ $\mathrm{Gamma}(\alpha_j, \beta_j)$ be independent random variables for $j \in \{1, \dots, n\}$. Let Λ be a non-negative random variable such that $\Lambda = a + Y_1 + \cdots + Y_n$ for $a \ge 0$. Let N be a random variable such that*

$$\mathcal{L}(N\,|\,\Lambda) \overset{a.s.}{=} \mathrm{Poisson}(\lambda\Lambda)\,.$$

Then

$$\mathcal{L}(N) = \mathrm{CPoi}(\mu, F)\,,$$

where

$$\mu = \lambda a + \sum_{j=1}^{n} \alpha_j \ln\Big(\frac{1}{1-p_j}\Big) \quad \text{and} \quad F = \frac{\lambda a}{\mu}\delta_1 + \frac{1}{\mu}\sum_{j=1}^{n} \alpha_j \ln\Big(\frac{1}{1-p_j}\Big) F_j, \tag{10.4}$$

where $F_j = \mathrm{Log}(p_j)$ with $p_j = \frac{\lambda}{\beta_j+\lambda}$ for $j \in \{1, \dots, n\}$.

Proof. Due to the convolution property of the Poisson distribution

$$\mathrm{Poisson}(\lambda\Lambda) = \mathrm{Poisson}(\lambda a) \star \mathrm{Poisson}(\lambda Y_1) \star \cdots \star \mathrm{Poisson}\big(\lambda Y_n\big)$$

holds, and for each $j \in \{1, \dots, n\}$ there exist some random variables L_j conditionally independent given $Y_1, \dots, Y_n$ such that $\mathcal{L}(L_j\,|\,Y_j) \overset{\text{a.s.}}{=} \mathrm{Poisson}(\lambda Y_j)$. Then $L_j \sim \mathrm{NegBin}(\alpha_j, p_j)$

holds for $j \in \{1, \dots, n\}$ because of Lemma 2.8 for $m = 1$ and $T \equiv 1$. By Remark 2.7 we may write

$$L_j \overset{\mathrm{d}}{=} \sum_{h=1}^{K_j} Z_{j,h}, \qquad j \in \{1, \dots, n\}, \tag{10.5}$$

where $K_j \sim \mathrm{Poisson}\big(\alpha_j \ln\big(\frac{1}{1-p_j}\big)\big)$ and $\{Z_{j,h}\}_{h\in\mathbb{N}}$ are n sequences independent of K_j, consisting of i.i.d. random variables such that $Z_{j,1} \sim \mathrm{Log}(p_j)$ for $j \in \{1, \dots, n\}$. Finally we apply [58, Theorem 1.1.15] and obtain the claim. q.e.d.

The following result gives us a powerful tool to find an upper estimate for differences between distributions of Poisson random sums with respect to the total variation distance to be found in [70, Corollary 3.2]:

Corollary 10.6. *Let $\alpha, \beta > 0$. Let $\{X_h\}_{h\in\mathbb{N}}$ and $\{Y_h\}_{h\in\mathbb{N}}$ be i.i.d. sequences of discrete random variables with distributions F and G, respectively. Let M and N be Poisson variables with means α and β, respectively, and M and N are independent of the sequences $\{X_h\}_{h\in\mathbb{N}}$ and $\{Y_h\}_{h\in\mathbb{N}}$, respectively. Let $S = \sum_{h=1}^{M} X_h$ and $T = \sum_{h=1}^{N} Y_h$. Then*

$$\mathrm{d_{TV}}(\mathcal{L}(S)\,, \mathcal{L}(T)) \le \min\{\big|\sqrt{\alpha} - \sqrt{\beta}\big|, \big|\alpha - \beta\big|\} + \min\{\alpha, \beta\}\, \mathrm{d_{TV}}(F, G)\,.$$

For the sake of simplicity we introduce the following notation:

Notation 10.7. Let $\lambda \ge 0$. Define

$$\kappa_\lambda(t) = \ln(t + \lambda) - \ln(t), \qquad t \in (0, \infty).$$

Now we are prepared for our estimate:

Theorem 10.8. *Let $\lambda \ge 0$. Let Λ denote a random variable distributed as a generalized gamma convolution characterized by the Thorin measure U and parameter $a \ge 0$ and let Ψ denote a random variable distributed as a generalized gamma convolution characterized by the Thorin measure V and parameter $b \ge 0$. Let*

$$\int_{(0,\infty)} \Big(\frac{\lambda}{t+\lambda}\Big)^n U(dt) \ge \int_{(0,\infty)} \Big(\frac{\lambda}{t+\lambda}\Big)^n V(dt) \qquad \textit{for all } n \in \mathbb{N} \tag{10.9}$$

be satisfied. Let the random variables N and M have Poisson mixture distributions such that

$$\mathcal{L}(N\,|\,\Lambda) \overset{a.s.}{=} \mathrm{Poisson}(\lambda\Lambda) \qquad \textit{and} \qquad \mathcal{L}(M\,|\,\Psi) \overset{a.s.}{=} \mathrm{Poisson}(\lambda\Psi)\,.$$

Then the total variation distance satisfies

$$\mathrm{d_{TV}}(\mathcal{L}(N)\,, \mathcal{L}(M)) \le \frac{3}{2}\big|\mu' - \nu'\big| + \frac{\mu - \nu}{2} + \frac{\lambda\,|a\nu' - b\mu'|}{2\nu'},$$

where

$$\mu = \int_{(0,\infty)} \kappa_\lambda(t)\, U(dt) \qquad \textit{and} \qquad \nu = \int_{(0,\infty)} \kappa_\lambda(t)\, V(dt),$$

as well as $\mu' = \mu + a\lambda$ and $\nu' = \nu + b\lambda$.

Remark 10.10. If $a = b = 0$ and $U(T) \ge V(T)$ for all $T > 0$, then

$$\mathrm{d_{TV}}(\mathcal{L}(N)\,, \mathcal{L}(M)) \le 2(\mu - \nu).$$

Remark 10.11. As we will see, a sufficient condition is

$$\tilde{U}(T) = \int_{(0,T]} \frac{\lambda}{t+\lambda} U(dt) \geq \int_{(0,T]} \frac{\lambda}{t+\lambda} V(dt) = \tilde{V}(T) \qquad \text{for all } T > 0. \tag{10.12}$$

Indeed, since $\tilde{U}(T)$ and $\tilde{V}(T)$ define measures,

$$\int_{(0,\infty)} \Big(\frac{\lambda}{t+\lambda}\Big)^{n-1} \tilde{U}(dt) = \int_{(0,\infty)} \Big(\frac{\lambda}{t+\lambda}\Big)^{n} U(dt)$$

and

$$\int_{(0,\infty)} \Big(\frac{\lambda}{t+\lambda}\Big)^{n-1} \tilde{V}(dt) = \int_{(0,\infty)} \Big(\frac{\lambda}{t+\lambda}\Big)^{n} V(dt)$$

hold for every $n \in \mathbb{N}$ because $T > 0$ is arbitrary. Due to $\tilde{U}(T) \geq \tilde{V}(T)$ for all $T > 0$ Equation (10.9) follows. $\tilde{U}(T) \geq \tilde{V}(T)$ for all $T > 0$ is also satisfied if $U(T) \geq V(T)$ for all $T > 0$.

Remark 10.13. The parameters μ and ν in Theorem 10.8 are well-defined, i.e., they are finite. This holds as follows; we have $\kappa_\lambda(t) = \ln(t+\lambda) - \ln t \leq \ln(1+\lambda) + |\ln t|$ for $t \in (0,1]$, and $\kappa_\lambda(t) \leq \lambda/t$ for $t \in (1,\infty)$. Hence the claim follows by Equation (9.9) because if the integral is finite for subsets, then it is also finite for the union of these subsets.

Remark 10.14. In the discretized case of the generalized gamma convolution, i.e. a finite gamma convolution, this convolution is characterized by a discrete measure $V = \sum_{i=1}^n \alpha_i \delta_{\beta_i}$ and hence

$$\int_{(0,\infty)} \kappa_\lambda(t)\, V(dt) = \sum_{i=1}^n \alpha_i \ln(1 + \lambda/\beta_i) = \nu$$

holds.

Proof of Theorem 10.8. Since a Poisson random variable mixed with an infinitely divisible non-negative random variable has a compound Poisson distribution (cf. [64, Corollary 4.1]), we assume $N \overset{\mathrm{d}}{=} \sum_{j=1}^K P_j$ and $M \overset{\mathrm{d}}{=} \sum_{j=1}^L Q_j$ with the following assumptions: Let $\{P_j\}_{j\in\mathbb{N}}$ be a sequence of i.i.d. random variables, and the sequence is independent of $K \sim \text{Poisson}(\mu')$. Further, let $\{Q_j\}_{j\in\mathbb{N}}$ be a sequence of i.i.d. random variables, and the sequence is independent of $L \sim \text{Poisson}(\nu')$. The first part of the upper bound is a direct application of Corollary 10.6 and provides

$$\mathrm{d_{TV}}(\mathcal{L}(M), \mathcal{L}(N)) \leq \min\{\big|\sqrt{\mu'} - \sqrt{\nu'}\big|, \big|\mu' - \nu'\big|\} + \min\{\mu', \nu'\}\, \mathrm{d_{TV}}(\mathcal{L}(P_1), \mathcal{L}(Q_1)). \tag{10.15}$$

It remains to compute μ', ν' and $\mathrm{d_{TV}}(\mathcal{L}(P_1), \mathcal{L}(Q_1))$. We first determine the distribution of the random variables P_1 and Q_1 with the help of the probability-generating function. We adopt the approach of [64, p. 90] that gives us the following formula for the Poisson parameter μ' and the probability-generating function $G_{P_1}(z)$ of the severity distribution P_1:

$$\mu' = -\ln L_{\lambda\Lambda}(1),$$
$$G_{P_1}(z) = \frac{\ln\Big(\frac{L_{\lambda\Lambda}(1)}{L_{\lambda\Lambda}(1-z)}\Big)}{\ln L_{\lambda\Lambda}(1)},$$

where L is the Laplace transform of a generalized gamma convolution as in Equation (9.11). Plugging this in yields for the Poisson parameter μ'

$$\begin{aligned}\mu' = -\ln L_\Lambda(\lambda) &= -\ln\bigg(\exp\Big(-a\lambda - \int_0^\infty \ln\Big(\frac{t+\lambda}{t}\Big)\, U(dt)\Big)\bigg) \\ &= a\lambda + \int_{(0,\infty)} \kappa_\lambda(t)\, U(dt) = \mu + a\lambda. \end{aligned} \tag{10.16}$$

We compute $\nu' = \nu + b\lambda$ in the same manner. In the next step we have

$$\begin{aligned}\ln\Big(\frac{L_{\lambda\Lambda}(1)}{L_{\lambda\Lambda}(1-z)}\Big) &= \ln\Big(\exp\Big(-a\lambda z - \int_{(0,\infty)} \ln\Big(\frac{t+\lambda}{t}\Big)\,U(dt) \\ &\quad + \int_{(0,\infty)} \ln\Big(\frac{t+\lambda(1-z)}{t}\Big)\,U(dt)\Big)\Big) \\ &= -a\lambda z - \int_{(0,\infty)} \ln\Big(\frac{t+\lambda}{t}\Big)\,U(dt) + \int_{(0,\infty)} \ln\Big(\frac{t+\lambda(1-z)}{t}\Big)\,U(dt),\end{aligned}$$

which implies

$$\begin{aligned}G_{P_1}(z) &= \frac{1}{-\mu'}\Big(-a\lambda z + \int_{(0,\infty)} \ln\Big(\frac{t+\lambda(1-z)}{t+\lambda}\Big)\,U(dt)\Big) \\ &= \frac{1}{\mu'}\Big(a\lambda z + \int_{(0,\infty)} \ln\Big(\frac{t+\lambda}{t+\lambda(1-z)}\Big)\,U(dt)\Big).\end{aligned}$$

Further, $\mathbb{P}[P_1 = n] = \frac{G_{P_1}^{(n)}(0)}{n!}$ for $n \in \mathbb{N}$ holds. The derivatives of $G_{P_1}(z)$ are

$$G_{P_1}^{(n)}(z) = \frac{1}{\mu'}\Big(a\lambda 1_{\{n=1\}} + \int_{(0,\infty)} \frac{\lambda^n (n-1)!}{(t+\lambda(1-z))^n}\,U(dt)\Big),$$

for $n \in \mathbb{N}$, as a short proof by induction shows because integration and differentiation may be interchanged according to [5, 16.2 Lemma]. Hence

$$\begin{aligned}\mathbb{P}[P_1 = n] &= \frac{1}{n!\mu'}\Big(a\lambda 1_{\{n=1\}} + \int_{(0,\infty)} \frac{\lambda^n (n-1)!}{(t+\lambda)^n}\,U(dt)\Big) \\ &= \frac{1}{n\mu'}\int_{(0,\infty)} \Big(\frac{\lambda}{t+\lambda}\Big)^n U(dt) + \frac{a\lambda}{n!\mu'} 1_{\{n=1\}}.\end{aligned} \tag{10.17}$$

The distribution of Q_1 can be calculated in the same way by substituting Λ by Ψ and plugging in V and b, and we obtain

$$\mathbb{P}[Q_1 = n] = \frac{1}{n\nu'}\int_{(0,\infty)} \Big(\frac{\lambda}{t+\lambda}\Big)^n V(dt) + \frac{b\lambda}{n!\nu'} 1_{\{n=1\}}, \qquad n \in \mathbb{N}.$$

By adding a “zero-term”, the difference between the probability mass functions of P_1 and Q_1 for $n \in \mathbb{N}$ evaluates to

$$\begin{aligned}\mathbb{P}[P_1 = n] - \mathbb{P}[Q_1 = n] &= \frac{\lambda(a\nu' - b\mu')}{n!\mu'\nu'} 1_{\{n=1\}} \\ &\quad + \frac{1}{n}\Big(\frac{1}{\mu'}\int_{(0,\infty)} \Big(\frac{\lambda}{t+\lambda}\Big)^n U(dt) - \frac{1}{\nu'}\int_{(0,\infty)} \Big(\frac{\lambda}{t+\lambda}\Big)^n V(dt)\Big) \\ &= \frac{\lambda(a\nu' - b\mu')}{n!\mu'\nu'} 1_{\{n=1\}} \\ &\quad + \frac{1}{n\mu'\nu'}\Big(\nu'\Big(\int_{(0,\infty)} \Big(\frac{\lambda}{t+\lambda}\Big)^n U(dt) - \int_{(0,\infty)} \Big(\frac{\lambda}{t+\lambda}\Big)^n V(dt)\Big) \\ &\quad + (\nu' - \mu')\int_{(0,\infty)} \Big(\frac{\lambda}{t+\lambda}\Big)^n V(dt)\Big).\end{aligned} \tag{10.18}$$

Due to the assumption in Equation (10.9) we calculate the absolute value for $n \in \mathbb{N}$

$$\begin{aligned}\big|\,\mathbb{P}[P_1 = n] \,-\, \mathbb{P}[Q_1 = n]\,\big| \le{}& \frac{\lambda\,|a\nu' - b\mu'|}{n!\mu'\nu'} 1_{\{n=1\}}\\ &+ \frac{1}{n\mu'\nu'}\Big(\nu'\Big(\int_{(0,\infty)}\Big(\frac{\lambda}{t+\lambda}\Big)^n U(dt) - \int_{(0,\infty)}\Big(\frac{\lambda}{t+\lambda}\Big)^n V(dt)\Big)\\ &+ |\mu' - \nu'|\int_{(0,\infty)}\Big(\frac{\lambda}{t+\lambda}\Big)^n V(dt)\Big). \qquad (10.19)\end{aligned}$$

By application of [5, 11.5 Corollary] summation and integration may be interchanged in the next estimate because $(\lambda/(\lambda+t))^n$ is non-negative. By using Equations (10.18) and (10.19) we obtain for the total variation distance

$$\begin{aligned}\mathrm{d_{TV}}(\mathcal{L}(P_1)\,,\mathcal{L}(Q_1)) &= \frac{1}{2}\sum_{n=1}^{\infty}\big|\,\mathbb{P}[P_1 = n] \,-\, \mathbb{P}[Q_1 = n]\,\big|\\ &\le \frac{1}{2}\Big(\frac{\lambda\,|a\nu' - b\mu'|}{\mu'\nu'}\\ &\quad + \frac{1}{\mu'}\Big(\int_{(0,\infty)}\sum_{n=1}^{\infty}\frac{1}{n}\Big(\frac{\lambda}{t+\lambda}\Big)^n U(dt) - \int_{(0,\infty)}\sum_{n=1}^{\infty}\frac{1}{n}\Big(\frac{\lambda}{t+\lambda}\Big)^n V(dt)\Big)\\ &\quad + \left|\frac{\mu' - \nu'}{\mu'\nu'}\right|\int_{(0,\infty)}\sum_{n=1}^{\infty}\frac{1}{n}\Big(\frac{\lambda}{t+\lambda}\Big)^n V(dt)\Big).\end{aligned}$$

The logarithm can be expanded as

$$\sum_{n=1}^{\infty}\frac{1}{n}\Big(\frac{\lambda}{t+\lambda}\Big)^n = -\ln\Big(1 - \frac{\lambda}{t+\lambda}\Big) = -\ln\Big(\frac{t}{t+\lambda}\Big), \qquad t > 0. \qquad (10.20)$$

Since $-\ln\big(\frac{t}{t+\lambda}\big) = \ln\big(\frac{t+\lambda}{t}\big)$, we obtain

$$\begin{aligned}\mathrm{d_{TV}}(\mathcal{L}(P_1)\,,\mathcal{L}(Q_1)) &\le \frac{1}{2}\Big(\frac{\lambda\,|a\nu' - b\mu'|}{\mu'\nu'}\\ &\quad + \frac{1}{\mu'}\Big(\int_{(0,\infty)}\ln\Big(\frac{t+\lambda}{t}\Big)\,U(dt) - \int_{(0,\infty)}\ln\Big(\frac{t+\lambda}{t}\Big)\,V(dt)\Big)\\ &\quad + \left|\frac{\mu' - \nu'}{\mu'\nu'}\right|\int_{(0,\infty)}\ln\Big(\frac{t+\lambda}{t}\Big)\,V(dt)\Big)\\ &= \frac{1}{2}\Big(\frac{\lambda\,|a\nu' - b\mu'|}{\mu'\nu'} + \frac{\mu - \nu}{\mu'} + \left|\frac{\mu' - \nu'}{\mu'\nu'}\right|\nu\Big)\\ &\le \frac{1}{2}\Big(\frac{\lambda\,|a\nu' - b\mu'|}{\mu'\nu'} + \frac{\mu - \nu}{\mu'} + \left|\frac{\mu' - \nu'}{\mu'}\right|\Big). \qquad (10.21)\end{aligned}$$

Since w.l.o.g. $\min\{\big|\sqrt{\mu'} - \sqrt{\nu'}\big|, \big|\mu' - \nu'\big|\} \le \big|\mu' - \nu'\big|$ and $\min\{\mu', \nu'\} \le \mu'$, we have by insertion into Equation (10.15)

$$\mathrm{d_{TV}}(\mathcal{L}(M)\,,\mathcal{L}(N)) \le \frac{3}{2}\big|\mu' - \nu'\big| + \frac{1}{2}(\mu - \nu) + \frac{1}{2}\frac{\lambda\big|a\nu' - b\mu'\big|}{\nu'},$$

which completes the proof. q.e.d.

It is interesting to note that the total variation distance between random sums with the same severity distribution only depends on the total variation distance between the claim number distribution:

Corollary 10.22. *Let the assumptions of Theorem 10.8 be satisfied. Let $\{X_h\}_{h\in\mathbb{N}}$ be a sequence independent of N and M, consisting of i.i.d. non-negative discrete random variables. Let*

$$S := \sum_{h=1}^{N} X_h \qquad \textit{and} \qquad T := \sum_{h=1}^{M} X_h .$$

Then the total variation distance is

$$\mathrm{d_{TV}}(\mathcal{L}(S)\,,\mathcal{L}(T)) \le \frac{3}{2}\left|\mu' - \nu'\right| + \frac{\mu-\nu}{2} + \frac{\lambda\,|a\nu' - b\mu'|}{2\nu'}$$

with the same notation as in Theorem 10.8.

Proof. We estimate the total variation distance between the random sums by applying the law of total probability and then interchanging the order of summation

$$\begin{aligned}
\mathrm{d_{TV}}(\mathcal{L}(S)\,,\mathcal{L}(T)) &= \frac{1}{2}\sum_{k=0}^{\infty}\left|\mathbb{P}\left[\sum_{h=1}^{M} X_h = k\right] - \mathbb{P}\left[\sum_{h=1}^{N} X_h = k\right]\right| \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\left|\sum_{m=0}^{\infty}\mathbb{P}\left[\sum_{h=1}^{m} X_h = k\right]\big(\mathbb{P}[M=m] - \mathbb{P}[N=m]\big)\right| \\
&\le \frac{1}{2}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\mathbb{P}\left[\sum_{h=1}^{m} X_h = k\right]\big|\,\mathbb{P}[M=m] - \mathbb{P}[N=m]\,\big| \\
&= \frac{1}{2}\sum_{m=0}^{\infty}\big|\,\mathbb{P}[M=m] - \mathbb{P}[N=m]\,\big| = \mathrm{d_{TV}}(\mathcal{L}(N)\,,\mathcal{L}(M))\,.
\end{aligned}$$

This estimate can be also found in [70, Lemma 3.1]. An application of Theorem 10.8 provides the claim. q.e.d.

Remark 10.23. Using the estimate in Theorem 10.8, it is possible to obtain an upper and lower bound of the limit distribution, i.e., an approximating distribution that underestimates the risk and an approximating distribution that overestimates it. The difference may be estimated by the triangle inequality applied to the total variation distance. This holds as follows: Let μ, ν, λ be probability measures on a countable state space Ω. Then

$$\begin{aligned}
\mathrm{d_{TV}}(\mu,\nu) &= \frac{1}{2}\sum_{x\in\Omega}|\mu(x) - \nu(x)| = \frac{1}{2}\sum_{x\in\Omega}|\mu(x) - \lambda(x) + \lambda(x) - \nu(x)| \\
&\le \frac{1}{2}\sum_{x\in\Omega}\big(|\mu(x) - \lambda(x)| + |\lambda(x) - \nu(x)|\big) = \mathrm{d_{TV}}(\mu,\lambda) + \mathrm{d_{TV}}(\lambda,\nu)\,.
\end{aligned}$$

Remark 10.24. The problem of evaluating a compound Poisson mixture distribution with a recursion was already treated in [64, Chapter 3.3]. The authors give a recursive algorithm for the evaluation of such a compound Poisson mixture distribution and assume only the mixing random variable to be positive while relaxing the assumption of infinite divisibility. It is an alternative to Panjer's recursion (which is used only partly). Yet, in the n-th step, the algorithm requires the calculation of the n-th derivative of the Laplace transform of the mixing distribution. As the reader may surmise, computing the n-th derivative of the Laplace transform of a generalized gamma convolution in Equation (9.11) is rather tedious work. It includes the application of the product rule and chain rule which can be done by Faà di Bruno's formula. For growing n this quickly becomes a cumbersome term. Our algorithm avoids the need to calculate derivatives as Panjer's recursion usually does.

The Laplace transform of the mixing distribution also contains the problem of computing an integral with respect to its Thorin measure. This integral might not be evaluable analytically but only numerically. Hence, there would be some inaccuracy to cope with. Unfortunately, the authors do not give an error estimate for this scenario.

10.2 Approximation of the Thorin Measure

In this section we consider approximations using the upper estimate given in Theorem 10.8. In order to further extend the examination of the error bounds for applications, we apply Theorem 10.8 to the Thorin measure V in Remark 10.14 which is a vague approximation of the Thorin measure U as in Theorem 9.17. In the following lemma we include in our consideration a case where U is not continuous but has several jumps. In this case the approximation V is exact at these points and hence the step sizes cancel out, but the approximation may have more steps than jumps. If U is continuous, we choose arbitrary points for the step sizes (and hence the respective jump sizes are zero). The following lemma provides an upper estimate for practical use in applications.

Lemma 10.25. *Let $\lambda > 0$, $n \in \mathbb{N}$ and U denote the Thorin measure of a generalized gamma convolution. Let $0 = \beta_0 < \beta_1 < \beta_2 < \cdots < \beta_n$ with jump sizes $\Delta U(\beta_1), \dots, \Delta U(\beta_n)$, where $\Delta U(t) := U(t) - U(t-)$. Let*

$$-\Delta U(\beta_i) < \alpha_i \le \frac{\beta_i + \lambda}{\lambda} \int_{(\beta_{i-1},\beta_i)} \frac{\lambda}{t+\lambda}\, U(dt)$$

for $i \in \{1, \dots, n\}$. Then for

$$V(t) := \sum_{i=1}^{n} \big(\alpha_i + \Delta U(\beta_i)\big) 1_{[\beta_i,\infty)}(t), \qquad t \in [0,\infty), \tag{10.26}$$

and

$$\nu := \int_{(0,\infty)} \kappa_\lambda(t)\, V(dt) \qquad \text{and} \qquad \mu := \int_{(0,\infty)} \kappa_\lambda(t)\, U(dt)$$

the following holds

$$0 \le \mu - \nu = \int_{(\beta_n,\infty)} \kappa_\lambda(t)\, U(dt) + \sum_{i=1}^{n} \Big(\int_{(\beta_{i-1},\beta_i)} \kappa_\lambda(t)\, U(dt) - \kappa_\lambda(\beta_i)\alpha_i \Big). \tag{10.27}$$

Proof. For all $t > 0$ the function $\kappa_\lambda(t) \ge 0$ is decreasing. Thus the integral is computed as follows since V does not contribute any mass between the points β_i for $i \in \{1, \ldots, n\}$:

$$\begin{aligned}
\mu - \nu &= \int_{(0,\infty)} \kappa_\lambda(t)\,(U - V)(dt) \\
&= \int_{(\beta_n,\infty)} \kappa_\lambda(t)\,U(dt) + \sum_{i=1}^{n} \int_{(\beta_{i-1},\beta_i)} \kappa_\lambda(t)\,U(dt) + \sum_{i=1}^{n} \kappa_\lambda(\beta_i)\big(\Delta U(\beta_i) - \Delta V(\beta_i)\big) \\
&= \int_{(\beta_n,\infty)} \kappa_\lambda(t)\,U(dt) + \sum_{i=1}^{n} \Big(\int_{(\beta_{i-1},\beta_i)} \kappa_\lambda(t)\,U(dt) - \kappa_\lambda(\beta_i)\alpha_i\Big) \qquad (10.28)
\end{aligned}$$

since $\Delta U(\beta_i) - \Delta V(\beta_i) = -\alpha_i$ for $i \in \{1, \ldots, n\}$. It is easy to see that if

$$\alpha_i \le \frac{1}{\kappa_\lambda(\beta_i)} \int_{(\beta_{i-1},\beta_i)} \kappa_\lambda(t)\,U(dt),$$

the summands are non-negative, and if $-\Delta U(\beta_i) < \alpha_i$, then V is a non-negative measure. This does not yet provide an upper bound for the approximation error. In order that α_i also satisfies $\mu - \nu \ge 0$ according to Theorem 10.8 consider the following: $\tilde{U}(T) \ge \tilde{V}(T)$ in Equation (10.12) implies for all β_k with $k \in \{1, \ldots, n\}$

$$\begin{aligned}
\int_{(0,\beta_k]} \frac{\lambda}{t+\lambda}\,(U - V)(dt) &= \sum_{i=1}^{k} \int_{(\beta_{i-1},\beta_i)} \frac{\lambda}{t+\lambda}\,U(dt) + \sum_{i=1}^{k} \frac{\lambda}{\beta_i+\lambda}\big(\Delta U(\beta_i) - \Delta V(\beta_i)\big) \\
&= \sum_{i=1}^{k} \Big(\int_{(\beta i-1,\beta_i)} \frac{\lambda}{t+\lambda}\,U(dt) - \frac{\lambda}{\beta_i+\lambda}\alpha_i\Big) \ge 0
\end{aligned}$$

if $0 \le \alpha_i \le \frac{\beta_i+\lambda}{\lambda} \int_{(\beta_{i-1},\beta_i)} \frac{\lambda}{t+\lambda}\,U(dt)$ for $i \in \{1, \ldots, n\}$.

Hence α_i with $i \in \{1, \ldots, n\}$ has to be smaller or equal to the minimum of the two upper bounds. We claim that

$$\frac{\beta_i+\lambda}{\lambda} \int_{(\beta_{i-1},\beta_i)} \frac{\lambda}{t+\lambda}\,U(dt) \le \frac{1}{\kappa_\lambda(\beta_i)} \int_{(\beta_{i-1},\beta_i)} \kappa_\lambda(t)\,U(dt).$$

Let $\int_{(\beta_{i-1},\beta_i)} \frac{\lambda}{t+\lambda}\,U(dt) = \frac{\lambda}{\beta_i+\lambda}\alpha_i$. Then, by Equation (10.20) and since $\frac{\lambda}{t+\lambda}$ is decreasing for $t \ge 0$,

$$\begin{aligned}
\int_{(\beta_{i-1},\beta_i)} \ln\Big(\frac{t+\lambda}{t}\Big)U(dt) &= \int_{(\beta_{i-1},\beta_i)} \sum_{n=1}^{\infty} \frac{1}{n}\Big(\frac{\lambda}{t+\lambda}\Big)^n U(dt) \\
&\ge \int_{(\beta_{i-1},\beta_i)} \sum_{n=1}^{\infty} \frac{1}{n}\Big(\frac{\lambda}{\beta_i+\lambda}\Big)^{n-1}\Big(\frac{\lambda}{t+\lambda}\Big)U(dt) \\
&= \sum_{n=1}^{\infty} \frac{1}{n}\Big(\frac{\lambda}{\beta_i+\lambda}\Big)^{n-1} \frac{\lambda}{\beta_i+\lambda}\alpha_i = \alpha_i \ln\Big(\frac{\beta_i+\lambda}{\beta_i}\Big).
\end{aligned}$$

Hence the assertion follows. q.e.d.

Remark 10.29. The proof also shows that an optimal choice of α_i is

$$\alpha_i = \frac{\beta_i+\lambda}{\lambda} \int_{(\beta_{i-1},\beta_i)} \Big(\frac{\lambda}{t+\lambda}\Big)U(dt), \qquad i \in \{1, \ldots, n\}. \qquad (10.30)$$

The previous lemma gives us only an upper bound for the total variation distance. The following corollary proves that the approximation in Remark 10.29 converges vaguely.

Corollary 10.31. *Let $\lambda > 0$ and $n(m) = n2^{m-1} - (2^{m-1} - 1)$ and U denote the Thorin measure of a generalized gamma convolution. Let $0 = \beta_0^{(1)} < \beta_1^{(1)} < \cdots < \beta_{n(1)}^{(1)} < \infty$ with jump sizes $\Delta U(\beta_1^{(1)}), \dots, \Delta U(\beta_{n(1)}^{(1)})$, where $\Delta U(t) := U(t) - U(t-)$. Choose $\{\alpha_i^{(m)}\}_{m\in\mathbb{N}}$ for $i \in \{1, \dots, n(m)\}$ as in Equation (10.30). For an appropriate choice of the step sizes $\{\beta_i^{(m)}\}_{m\in\mathbb{N}}$ for $i \in \{1, \dots, n(m)\}$ with $\beta_{n(m)}^{(m)} \to \infty$ and $\beta_1^{(m)} \searrow 0$ as $m \to \infty$ the sequence $\{V_m\}_{m\in\mathbb{N}}$ is determined and converges vaguely to U as $m \to \infty$. The difference in Equation (10.27) decays to zero.*

Proof. By the choice of

$$\alpha_i^{(m)} = \frac{\beta_i^{(m)} + \lambda}{\lambda} \int_{(\beta_{i-1}^{(m)}, \beta_i^{(m)})} \frac{\lambda}{t+\lambda} U(dt), \qquad i \in \{1, \dots, n(m)\}, \quad m \in \mathbb{N},$$

the approximating sequence $\{V_m\}_{m\in\mathbb{N}}$ becomes

$$V_m = \sum_{i=1}^{n(m)} \left(\frac{\beta_i^{(m)} + \lambda}{\lambda} \int_{(\beta_{i-1}^{(m)}, \beta_i^{(m)})} \frac{\lambda}{t+\lambda} U(dt) + \Delta U(\beta_i^{(m)}) \right) \delta_{\beta_i^{(m)}}. \tag{10.32}$$

To prove the vague convergence consider an arbitrary continuous function $f\colon (0,\infty) \to \mathbb{R}$ with compact support. Then we have

$$\begin{aligned} \int_{(0,\infty)} f(t)\, V_m(dt) &= \sum_{i=1}^{n(m)} f(\beta_i^{(m)}) \left(\frac{\beta_i^{(m)} + \lambda}{\lambda} \int_{(\beta_{i-1}^{(m)}, \beta_i^{(m)})} \frac{\lambda}{t+\lambda}\, U(dt) + \Delta U(\beta_i^{(m)}) \right) \\ &= \sum_{i=1}^{n(m)} f(\beta_i^{(m)}) \left(\frac{\beta_i^{(m)} + \lambda}{\lambda} \left(\int_{(0,\beta_i^{(m)})} \frac{\lambda}{t+\lambda}\, U(dt) - \int_{(0,\beta_{i-1}^{(m)}]} \frac{\lambda}{t+\lambda}\, U(dt) \right) + \Delta U(\beta_i^{(m)}) \right) \end{aligned}$$

where we note $\Delta U(0) = 0$. Let $\tilde{U}(T) = \int_{(0,T]} \frac{\lambda}{t+\lambda} U(dt)$ for $T > 0$. Noting that $\Delta U(\beta_i^{(m)})$ fills the gaps between the intervals $(\beta_{i-1}^{(m)}, \beta_i^{(m)})$ for $i \in \{1, \dots, n(m)\}$ and $m \in \mathbb{N}$, we observe

$$\begin{aligned} \int_{(0,\infty)} f(t)\, V_m(dt) &= \sum_{i=1}^{n(m)} f(\beta_i^{(m)}) \frac{\beta_i^{(m)} + \lambda}{\lambda} \left(\tilde{U}(\beta_i^{(m)}-) - \tilde{U}(\beta_{i-1}^{(m)}) + \frac{\lambda}{\beta_i^{(m)} + \lambda} \Delta U(\beta_i^{(m)}) \right) \\ &\to \int_{(0,\infty)} f(t) \frac{t+\lambda}{\lambda}\, \tilde{U}(dt) = \int_{(0,\infty)} f(t) U(dt) \end{aligned} \tag{10.33}$$

as $m \to \infty$ because of the definition of the Riemann-Stieltjes integral.

The integral in the upper estimate of Equation (10.27) can be arbitrarily small dependent on the choice of the parameters $\beta_1^{(1)}$ and $\beta_{n(1)}^{(1)}$. This holds as follows; we may assume that $\beta_1^{(1)}, \dots, \beta_{n(1)}^{(1)}$ satisfy $\beta_1^{(1)} < \beta_{n(1)}^{(1)} < \infty$ for the Thorin measure U such that the following holds for some arbitrary $\varepsilon > 0$

$$\int_{(0,\beta_1^{(1)})} \kappa_\lambda(t)\, U(dt) \le \varepsilon/3 \qquad \text{and} \qquad \int_{(\beta_{n(1)}^{(1)},\infty)} \kappa_\lambda(t)\, U(dt) \le \varepsilon/3, \tag{10.34}$$

respectively. This holds because of an analogous argumentation for both integrals. The well-defined state follows by Remark 10.13. Integrating at 0 and ∞ has measure zero and $\kappa_\lambda(t)$ is non-negative for $\lambda > 0$ and $t > 0$, thus the integrals are increasing.

The sum in Equation (10.27) can be transformed as follows with respect to Equation (10.30) using that $\lambda/(t+\lambda)$ decreases for $t > 0$

$$\begin{aligned}
&\sum_{i=2}^{n(m)} \left(\int_{(\beta_{i-1}^{(m)},\beta_i^{(m)})} \kappa_\lambda(t)\, U(dt) - \kappa_\lambda(\beta_i^{(m)})\alpha_i^{(m)} \right) \\
&\quad \le \sum_{i=2}^{n(m)} \left(\kappa_\lambda(\beta_{i-1}^{(m)})\big(U(\beta_i^{(m)}-) - U(\beta_{i-1}^{(m)})\big) - \kappa_\lambda(\beta_i^{(m)}) \frac{\beta_i^{(m)}+\lambda}{\lambda} \int_{(\beta_{i-1}^{(m)},\beta_i^{(m)})} \frac{\lambda}{t+\lambda}\, U(dt) \right) \\
&\quad \le \sum_{i=2}^{n(m)} \big(\kappa_\lambda(\beta_{i-1}^{(m)})\big(U(\beta_i^{(m)}-) - U(\beta_{i-1}^{(m)})\big) - \kappa_\lambda(\beta_i^{(m)})\big(U(\beta_i^{(m)}-) - U(\beta_{i-1}^{(m)})\big)\big).
\end{aligned}$$

Simplifying this term and extracting the upper estimate of $U(\beta_i^{(m)}-) - U(\beta_{i-1}^{(m)})$, that is $\max_{i=2\dots,n(m)}\big(U(\beta_i^{(m)}-) - U(\beta_{i-1}^{(m)})\big)$, yields

$$\begin{aligned}
&\sum_{i=2}^{n(m)} \left(\int_{(\beta_{i-1}^{(m)},\beta_i^{(m)})} \kappa_\lambda(t)\, U(dt) - \kappa_\lambda(\beta_i^{(m)})\alpha_i^{(m)} \right) \\
&\qquad\qquad \le \sum_{i=2}^{n(m)} \big(\kappa_\lambda(\beta_{i-1}^{(m)}) - \kappa_\lambda(\beta_i^{(m)})\big)\big(U(\beta_i^{(m)}-) - U(\beta_{i-1}^{(m)})\big) \\
&\qquad\qquad \le \max_{i=2\dots,n(m)} \big(U(\beta_i^{(m)}-) - U(\beta_{i-1}^{(m)})\big) \sum_{i=2}^{m(n)} \big(\kappa_\lambda(\beta_{i-1}^{(m)}) - \kappa_\lambda(\beta_i^{(m)})\big) \\
&\qquad\qquad = \max_{i=2\dots,n(m)} \big(U(\beta_i^{(m)}-) - U(\beta_{i-1}^{(m)})\big)\big(\kappa_\lambda(\beta_1^{(m)}) - \kappa_\lambda(\beta_{n(m)}^{(m)})\big).
\end{aligned}$$

Reducing $\max_{i=2\dots,n(m)}\big(U(\beta_i^{(m)}-) - U(\beta_{i-1}^{(m)})\big)$ by halves reduces the error coming from this sum by halves. Choosing $\beta_i^{(m+1)}$ for $i \in \{2,\dots,n(m+1)\}$ with $i \bmod 2 \equiv 0$ such that $\beta_i^{(m+1)}$ satisfies

$$U(\beta_i^{(m+1)}) = \frac{U(\beta_{(i+1)/2}^{(m)}-) + U(\beta_{(i-1)/2}^{(m)})}{2},$$

and keeping the other $\beta_i^{(m+1)}$ leads to a decay. Thus there exists an $N = N(\varepsilon)$ such that for $n(m) \ge N$

$$\sum_{i=2}^{n(m)} \left(\int_{(\beta_{i-1}^{(m)},\beta_i^{(m)})} \kappa_\lambda(t)\, U(dt) - \kappa_\lambda(\beta_i^{(m)})\alpha_i^{(m)} \right) - \kappa_\lambda(\beta_1^{(1)})\alpha_1^{(1)} \le \varepsilon/3,$$

which denotes the middle term in Equation (10.27) with respect to Equation (10.34). Because $\varepsilon > 0$ was chosen arbitrarily, the assertion follows. q.e.d.

A sequence of probability distributions defined by the sequence $\{V_m\}_{m\in\mathbb{N}}$ that converges vaguely converges weakly to a generalized gamma convolution as the next corollary proves.

Corollary 10.35. *Let a Thorin measure U be given. Let the sequence of Thorin measures $\{V_m\}_{m\in\mathbb{N}}$ be given as in Equation (10.32). Then the sequence of generalized gamma convolutions defined by $\{V_m\}_{m\in\mathbb{N}}$ converges weakly to the generalized gamma convolution defined by the Thorin measure U.*

Proof. For the proof we apply Theorem 9.17. Condition (a) has already been shown in Corollary 10.31. We proceed as in the proof of Corollary 9.28. For Condition (b) following Equation (10.33) we observe

$$\begin{aligned}\int_{(A,\infty)} \frac{1}{t}\, V_m(dt) &= \sum_{\substack{i=1\\ \beta_i^{(m)}>A}}^{n(m)} \frac{1}{\beta_i^{(m)}} \frac{\beta_i^{(m)}+\lambda}{\lambda}\big(\tilde{U}(\beta_i^{(m)}) - \tilde{U}(\beta_{i-1}^{(m)})\big)\\ &= \sum_{\substack{i=1\\ \beta_i^{(m)}>A}}^{n(m)} \frac{1}{\beta_i^{(m)}}\big(U(\beta_i^{(m)}) - U(\beta_{i-1}^{(m)})\big) = \sum_{\substack{i=1\\ \beta_i^{(m)}>A}}^{n(m)} \frac{1}{\beta_i^{(m)}} \int_{(\beta_{i-1}^{(m)},\beta_i^{(m)}]} U(dt).\end{aligned}$$

As in Corollary 9.28 this has a upper estimate that converges to zero as $A\to\infty$. An analogous argumentation holds for Condition (c) and $\int_{(0,\varepsilon)} \ln(t^{-1})\, V_m(dt)$. Hence the assertion follows. q.e.d.

The evaluation of an integral for each $\alpha_i^{(m)}$ for $i\in\{1,\dots,n(m)\}$ as in Corollary 10.31 can be circumvented. The upper bound can be rewritten:

Remark 10.36. Consider the special case when $\alpha_i = U(\beta_i-) - U(\beta_{i-1})$ denotes the U-measure of (β_{i-1},β_i) for $i\in\{1,\dots,n\}$. Then, since $\kappa_\lambda(t)$ is decreasing,

$$\begin{aligned}\alpha_i = U(\beta_i-) - U(\beta_{i-1}) &\le (U(\beta_i-) - U(\beta_{i-1}))\frac{\kappa_\lambda(\beta_{i-1})}{\kappa_\lambda(\beta_i)}\\ &\le \frac{1}{\kappa_\lambda(\beta_i)} \int_{(\beta_{i-1},\beta_i)} \kappa_\lambda(t)\, U(dt).\end{aligned}$$

Hence this estimate simplifies to

$$\begin{aligned}0 \le \mu - \nu \le &\int_{(0,\beta_1)} (\kappa_\lambda(t) - \kappa_\lambda(\beta_1))\, U(dt) + \int_{(\beta_n,\infty)} \kappa_\lambda(t)\, U(dt)\\ &+ \sum_{i=2}^{n} \alpha_i\big(\kappa_\lambda(\beta_{i-1}) - \kappa_\lambda(\beta_i)\big).\end{aligned}$$

Corollary 10.31 shows that the upper bound of the smallness difference between μ and ν can be chosen arbitrarily. However, this requires an evaluation of an integral, which could become very uncomfortable. The next corollary finds a remedy. It is an approximation that circumvents an approximation of the integral by approximating the measure U.

This corollary provides for each $n\in\mathbb{N}$ a decreasing estimate because μ and ν are well-defined (see Remark 10.13), respectively. It formalizes Remark 10.36.

Corollary 10.37. *Let $\lambda>0$ and $n(m) = n2^{m-1} - (2^{m-1}-1)$ and U denote the Thorin measure of a generalized gamma convolution. Let $0=\beta_0^{(1)} < \beta_1^{(1)} < \cdots < \beta_{n(1)}^{(1)} < \infty$ with jump sizes $\Delta U(\beta_1),\dots,\Delta U(\beta_n)$, where $\Delta U(t) := U(t) - U(t-)$. Let $\alpha_i^{(1)} = U(\beta_i^{(1)}-) - U(\beta_{i-1}^{(1)})$ for $i\in\{1,\dots,n(m)\}$. Let $\{V_m\}_{m\in\mathbb{N}}$ be chosen as in Equation* (10.26)*. Then for an appropriate choice of the sequence $\{\alpha_i^{(m)}\}_{m\in\mathbb{N}}$ for $i\in\{1,\dots,n(m)\}$ and of $\beta_1^{(1)}$ and $\beta_{n(1)}^{(1)}$ the difference in Equation* (10.27) *converges to zero and the sequence $\{V_m\}_{m\in\mathbb{N}}$ converges vaguely to U as $m\to\infty$ if $\beta_{n(m)}^{(m)}\to\infty$ and $\beta_1^{(m)} \searrow 0$ as $m\to\infty$.*

Proof. For the vague convergence consider a continuous function $f\colon (0,\infty) \to \mathbb{R}$ with compact support. Then we have using $V_m = \sum_{i=1}^{n(m)} (\alpha_i^{(m)} + \Delta U(\beta_i^{(m)}))\delta_{\beta_i^{(m)}}$ and $\alpha_i^{(m)} = U(\beta_i^{(m)}-) - U(\beta_{i-1}^{(m)})$

$$\begin{aligned}\int_{(0,\infty)} f(t)\,V(dt) &= \sum_{i=1}^{n(m)} f(\beta_i^{(m)})(U(\beta_i^{(m)}-) - U(\beta_{i-1}^{(m)}) + \Delta U(\beta_i^{(m)}))\\ &= \sum_{i=1}^{n(m)} f(\beta_i^{(m)})(U(\beta_i^{(m)}) - U(\beta_{i-1}^{(m)})) \to \int_{(0,\infty)} f(t)\,U(dt)\end{aligned}$$

as $m \to \infty$ because of the definition of the Riemann-Stieltjes integral.

The integral in the upper estimate of Equation (10.27) can be arbitrarily small dependent on the choice of the parameters $\beta_1^{(1)}$ and $\beta_{n(1)}^{(1)}$. This holds as follows: we may assume that $\beta_1^{(1)}, \dots, \beta_{n(1)}^{(1)}$ satisfy $\beta_1^{(1)} < \beta_{n(1)}^{(1)} < \infty$ for the Thorin measure U such that the following holds for some arbitrary $\varepsilon > 0$ shifting the first term of the sum in Equation (10.27) to the lower estimate

$$\int_{(0,\beta_1^{(1)})} (\kappa_\lambda(t) - \kappa_\lambda(\beta_1^{(1)}))\,U(dt) \le \varepsilon/3 \qquad \text{and} \qquad \int_{(\beta_{n(1)}^{(1)},\infty)} \kappa_\lambda(t)\,U(dt) \le \varepsilon/3, \quad (10.38)$$

respectively. This holds because of an analogous argumentation for both integrals. The well-defined state follows by Remark 10.13. Integrating at 0 and ∞, respectively, has the measure 0 and $\kappa_\lambda(t)$ is non-negative for $\lambda > 0$ and $t > 0$, thus the integral are increasing.

The sum in Equation (10.27) can be transformed as follows for $i \in \{2, \dots, n(m)\}$ and $m \in \mathbb{N}$ using $\alpha_i^{(m)} = U(\beta_i^{(m)}-) - U(\beta_{i-1}^{(m)})$

$$\begin{aligned}&\sum_{i=2}^{n(m)} \left(\int_{(\beta_{i-1}^{(m)},\beta_i^{(m)})} \kappa_\lambda(t)\,U(dt) - \kappa_\lambda(\beta_i^{(m)})\alpha_i^{(m)} \right)\\ &\qquad \le \sum_{i=2}^{n(m)} \left(\kappa_\lambda(\beta_{i-1}^{(m)})\big(U(\beta_i^{(m)}-) - U(\beta_{i-1}^{(m)})\big) - \kappa_\lambda(\beta_i^{(m)})\alpha_i^{(m)} \right)\\ &\qquad = \sum_{i=2}^{n(m)} \alpha_i^{(m)}\big(\kappa_\lambda(\beta_{i-1}^{(m)}) - \kappa_\lambda(\beta_i^{(m)})\big).\end{aligned}$$

Estimating with $\max_{i=2,\dots,n(m)} \alpha_i^{(m)}$ and factoring this out yields

$$\begin{aligned}&\sum_{i=2}^{n(m)} \left(\int_{(\beta_{i-1}^{(m)},\beta_i^{(m)})} \kappa_\lambda(t)\,U(dt) - \kappa_\lambda(\beta_i^{(m)})\alpha_i^{(m)} \right)\\ &\qquad \le \max_{i=2,\dots,n(m)} \alpha_i^{(m)} \sum_{i=2}^{n(m)} \big(\kappa_\lambda(\beta_{i-1}^{(m)}) - \kappa_\lambda(\beta_i^{(m)})\big)\\ &\qquad = \big(\kappa_\lambda(\beta_1^{(m)}) - \kappa_\lambda(\beta_{n(m)}^{(m)})\big) \max_{i=2,\dots,n(m)} \alpha_i^{(m)}.\end{aligned}$$

If we successively reduce the increments $\alpha_2^{(m)}, \dots, \alpha_{n(m)}^{(m)}$ of V_m by halves for $m \in \mathbb{N}$ (i.e., we also reduce the error resulting from this sum by halves), we have

$$\alpha_i^{(m)} = \frac{U(\beta_{n(1)}^{(1)}-) - U(\beta_1^{(1)}) - \sum_{i=1}^{n(1)} \Delta U(\beta_i^{(1)})}{2^{m-1}}, \qquad i \in \{2, \dots, n(m)\},$$

and hence there exists an $N = N(\varepsilon)$ such that for $n(m) \geq N$

$$\sum_{i=2}^{n(m)} \left(\int_{(\beta_{i-1}^{(m)}, \beta_i^{(m)})} \kappa_\lambda(t)\, U(dt) - \kappa_\lambda(\beta_i^{(m)}) \alpha_i^{(m)} \right) \leq \varepsilon/3.$$

Hence the assertion follows. q.e.d.

Remark 10.39. A sequence of Thorin measures $\{V_m\}_{m\in\mathbb{N}}$ given in Corollary 10.37 that converges vaguely to a Thorin measure U defines a sequence of generalized gamma convolutions that converges weakly to a generalized gamma convolution defined by the Thorin measure U, as an application of Corollary 9.28 shows.

Corollary 10.31 and Corollary 10.37 lead to the following conclusions: Given a Thorin measure U we determine a random variable Λ (cf. Definition 9.10). Because the sequence $\{V_m\}_{m\in\mathbb{N}}$ constructed in Corollaries 10.31 and 10.37, respectively, converges vaguely to U, it follows by Corollary 10.35 and Theorem 9.17, respectively, that the sequence $\{\Lambda_m\}_{m\in\mathbb{N}}$ of random variables converges weakly to Λ as $m \to \infty$.

To make the approximation more efficient, we state the following assumption in accordance with Corollary 10.31:

Assumption 10.40. *Apply the notation of Corollary 9.28 and let $n(m) = n2^{m-1} - (2^{m-1} - 1)$. To make the approximation of the Thorin measure U by $\{V_m\}_{m\in\mathbb{N}}$ more efficient, we assume the following:*

(a) If U has m jumps (with $m + 2 = n$), say at the points t_k for $k \in \{1, \dots, m\}$ and if $t_0 = 0$, then we set

$$\beta_1^{(1)} = c, \quad \beta_{k+1}^{(1)} = t_k, \quad \beta_n^{(1)} = C,$$

such that c and C satisfy Equation (10.34) *and then compute $\alpha_i^{(1)}$ for $i \in \{1, \dots, n\}$ according to Equation* (10.30)*. If U does not have jumps, then we suggest taking the following values*

$$\beta_1^{(1)} = c \quad \text{and} \quad \beta_2^{(1)} = C,$$

and then calculate the corresponding $\alpha_i^{(1)}$ for $i = 1, 2$.

(b) For $m \in \mathbb{N}$ we assume for $i \in \{1, \dots, n(m+1)\}$

$$\beta_i^{(m+1)} = \begin{cases} \beta_{\frac{i+1}{2}}^{(m)} & \text{if } i \bmod 2 \equiv 1 \text{ and } i < n(m+1) \\ U^{-1}\Big(\frac{1}{2}\Big(U\big(\beta_{\frac{i}{2}}^{(m)}\big) + U\big(\beta_{\frac{i+2}{2}}^{(m)}\big)\Big)\Big) & \text{if } i \bmod 2 \equiv 0 \text{ and } i < n(m+1) \\ \beta_{n(m)}^{(m)} & \text{if } i = n(m+1), \end{cases} \tag{10.41}$$

and that $\alpha_i^{(m+1)}$ satisfy Equation (10.30)*.*

We now derive an algorithm that shows how to approximate recursively the random sum $S = \sum_{h=1}^{N} X_h$, where $\mathcal{L}(N\,|\,\Lambda) \overset{\text{a.s.}}{=} \text{Poisson}(\lambda\Lambda)$ and Λ is a generalized gamma convolution with parameter a and Thorin measure U, and $\{X_h\}_{h\in\mathbb{N}}$ is a sequence independent of (N, Λ), consisting of i.i.d. random variables, which are either non-negative and discrete or discrete approximations of continuous non-negative random variables:

Algorithm 10.42. Given a generalized gamma convolution F with $\Lambda \sim F$ characterized by $a \geq 0$ and Thorin measure U, Corollary 10.31 describes how to construct a sequence $\{V_m\}_{m \in \mathbb{N}}$ that converges vaguely to U. For each $m \in \mathbb{N}$ the measure V_m defines a generalized gamma convolution F_n and by Corollary 10.35 the corresponding distribution converges weakly, by Theorem 10.8 and Lemma 10.25 the quality of convergence of the sequence of random variables $\{N_m\}_{m \in \mathbb{N}}$ follows. Corollary 10.31 also describes the difference between the random variables N and N_m. An application of Lemma 9.32 provides a alternative representation that also converges weakly. Equation (10.27) gives an upper bound of the difference between μ and ν with respect to a series which also converges to 0. We proceed as follows:

- Choose an error bound ε in the total variation distance. To obtain a reasonable result pay attention to $\varepsilon < \int_0^\infty \kappa_\lambda(t)\, U(dt)$. Determine by Assumption 10.40 (a) the first parameters $\alpha_i^{(1)}, \beta_i^{(1)}$ for $i \in \{1, \dots, n(1)\}$.

- If Equation (10.27) is satisfied, we are done. Otherwise, determine $\alpha_i^{(m)}$ and $\beta_i^{(m)}$ for $m \in \mathbb{N}$ and $i \in \{1, \dots, n(m)\}$ by Assumption 10.40 (b) and check Equation (10.27) iteratively until the desired exactness is reached.

- By having found appropriate parameters $\alpha_i^{(m)}$ and $\beta_i^{(m)}$ for $i \in \{1, \dots, n(m)\}$ with $n(m) = n2^{m-1} - (2^{m-1} - 1)$, the respective distribution of the random variable $M_{n(m)}$ in Equation (9.34) is determined, which is a convolution of $n(m)$ negative binomial distributions. Then we can compute an appropriate approximation of the distribution of S by $S_{n(m)}$ which is

$$S_{n(m)} \overset{\mathrm{d}}{=} \sum_{h=1}^{M_{n(m)}} X_h = \sum_{h=1}^{P} X_h + \sum_{i=1}^{n(m)} \sum_{h=1}^{R_i^{(n(m))}} X_{i,h},$$

  where $\{X_{i,h}\}_{h \in \mathbb{N}}$ are $n(m)$ independent sequences independent of the random variables $R_i^{(n(m))}$ for $i \in \{1, \dots, n(m)\}$, consisting of i.i.d. random variables and $X_{i,1}$ has the same distribution as X_1 for $i \in \{1, \dots, n(m)\}$. Additionally $P \sim \text{Poisson}(\lambda a)$ holds.

- Let

$$q_i^{(m)} = \frac{\lambda}{\beta_i^{(m)} + \lambda} \qquad \text{for } i \in \{1, \dots, n(m)\}.$$

  By Remark 2.7 we have $R_i^{(n(m))} \overset{\mathrm{d}}{=} \sum_{h=1}^{L_i^{(n(m))}} Y_{i,h}^{(n(m))}$ for $i \in \{1, \dots, n(m)\}$ such that

$$R_i^{(n(m))} \sim \mathrm{CPoi}\bigg(\alpha_i^{(m)} \ln\bigg(\frac{1}{1 - q_i^{(m)}}\bigg), \mathrm{Log}\big(q_i^{(m)}\big)\bigg).$$

  We proceed as in [21, Section 5.5]. For the reader's convenience we repeat the single steps. Now let

$$S_{(i,h)}^{(n(m))} = \sum_{k=Y_{(1,h)}^{(n(m))} + \dots + Y_{(i-1,h)}^{(n(m))} + 1}^{Y_{(1,h)}^{(n(m))} + \dots + Y_{(i,h)}^{(n(m))}} X_{i,k}$$

  for $i \in \{1, \dots, n(m)\}$ and $h \in \mathbb{N}$. By [21, Remark 5.11]

$$S_{(i,1)}^{(n(m))} + \dots + S_{(i,L_i^{(n(m))})}^{(n(m))} \overset{\mathrm{d}}{=} X_{i,1} + \dots + X_{i,R_i^{(n(m))}}$$

holds. The sequence $\{S^{(n(m))}_{(i,h)}\}_{h\in\mathbb{N}}$ is i.i.d. and the distribution of $S^{(n(m))}_{(i,1)}$ can be calculated by a numerically stable Panjer recursion since $Y^{(n(m))}_{(i,1)}$ is in the Panjer$(q_i, -q_i, 1)$ class.

- Further the random sum $S_{n(m)}$ can be represented as

$$S_{n(m)} = \sum_{i=1}^{P} X_i + \sum_{i=1}^{n(m)} \sum_{h=1}^{L_i^{(n(m))}} S^{(n(m))}_{(i,h)},$$

  and the probability-generating function of its distribution is given by

$$\begin{aligned} G_{S_{n(m)}}(z) &= \exp\bigl(\lambda a(G_{X_1}(z) - 1)\bigr) \prod_{i=1}^{n(m)} \exp\biggl(\alpha_i^{(m)} \ln\Bigl(\frac{1}{1-q_i^{(m)}}\Bigr)\bigl(G_{S^{(m)}_{(i,1)}}(z) - 1\bigr)\biggr) \\ &= \exp\bigl((\lambda a + \nu)(G(z) - 1)\bigr), \end{aligned}$$

  where $|z| \le 1$ and $\nu = \sum_{i=1}^{n(m)} \alpha_i^{(m)} \ln\Bigl(\frac{1}{1-q_i^{(m)}}\Bigr)$ and

$$G(z) = \frac{\lambda a}{\lambda a + \nu} G_{X_1}(z) + \sum_{i=1}^{n(m)} \frac{\alpha_i^{(m)} \ln\Bigl(\frac{1}{1-q_i^{(m)}}\Bigr)}{\lambda a + \nu} G_{S^{(n(m))}_{(i,1)}}(z). \tag{10.43}$$

- Hence calculating the distribution of $S^{(n(m))}_{(1,1)}, \dots, S^{(n(m))}_{(n(m),1)}$ and of X_1 provides the coefficients of the respective probability-generating functions and the coefficients of $G(z)$, which are the probabilities of the severity distribution of a compound Poisson distribution with counting distribution Poisson$(\lambda a + \nu)$. Thus we only conduct $n(m)+1$ Panjer recursions and replace the $n(m)$ convolutions by a convex combination and another numerically stable Panjer recursion.

Remark 10.44. This algorithm is very efficient with respect to the speed of approximation, but might require a huge memory capacity. Remembering old values of the $\beta_i^{(m)}$ has the advantage that in each step we only need to compute for half of the $\beta_i^{(m)}$ the value in Equation (10.41). If the Thorin measure U is strictly monotone and continuous, the inverse U^{-1} can be evaluated directly, otherwise U^{-1} should be understood as the generalized left-inverse. If the inverse is known, the computation is very fast, otherwise the numerical approximation of the inverse could become very time-consuming. This approximation can be circumvented, if in each step $\frac{1}{2}\bigl(\beta_i^{(m)} + \beta_{i+1}^{(m)}\bigr)$ is taken, respectively, and then the $\alpha_i^{(m)}$ are computed. This also converges vaguely, but the efficiency of approximation is abandoned.

Chapter 11

An Extension of the CreditRisk$^+$ Model to Multivariate Generalized Gamma Convolutions

We now provide a generalization of the CreditRisk$^+$ model with several risk factors. Consider the random sum

$$S = \sum_{i=1}^{m} \sum_{h=1}^{N_i} X_{i,h}, \tag{11.1}$$

where the loss sizes $\{X_{i,h}\}_{h\in\mathbb{N}}$ for $i \in \{1, \ldots, m\}$ are m independent sequences of i.i.d. random variables and the loss numbers $(N_1, \ldots, N_m)$ are random variables, each with a Poisson mixture distribution independent of all the losses $\{X_{i,h}\}_{h\in\mathbb{N}}$. We assume that the mixing distributions are generalized gamma convolutions. Hence we model a common mixing random vector with a multivariate distribution. It is therefore necessary to consider a further characterization of generalized gamma convolutions in one dimension because this is crucial for the construction of multivariate generalized gamma convolutions.

11.1 Introduction to Wiener–Gamma integrals

In Chapter 9.2 we introduced generalized gamma convolutions as the limit of sums of independent gamma-distributed random variables. Now we consider once again generalized gamma convolutions and introduce the notion of Wiener–Gamma integrals. This connection is presented in [36], and we state here some results that are important for us. For further reading on gamma processes cf. [68]. Recall that a gamma process $(\gamma_t)_{t\geq 0}$ follows a Gamma$(t, 1)$ distribution for $t > 0$, cf. [36, Equation (4)]

$$\mathbb{P}[\gamma_t \in da] = \frac{\mathrm{e}^{-a}}{\Gamma(t)} a^{t-1} da, \qquad a \geq 0.$$

Thus we use Wiener–Gamma integrals as in [36, Equation (5)]

$$\Lambda^h := \int_{(0,\infty)} h(s)\, d\gamma_s,$$

where $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel measurable function such that

$$\int_{(0,\infty)} \ln(1+h(u))\,du < \infty.$$

This condition ensures that the Laplace transform of Λ^h is well-defined, cf. [68, Equation (7)]. Such a Wiener–Gamma integral allows for another approach to generalized gamma convolutions because they coincide as can be found in [36, Proposition 1.1].

Proposition 11.2. *The class of positive generalized gamma convolution random variables coincide with the class of Wiener–Gamma integrals. More precisely:*

(a) If $\Lambda^h = \int_{(0,\infty)} h(s)\,d\gamma_s$, then

$$\mathbb{E}\big[\mathrm{e}^{-t\Lambda^h}\big] = \exp\Big(-\int_{(0,\infty)} \ln\Big(1+\frac{t}{x}\Big)\,U_h(dx)\Big), \qquad t \geq 0$$

where U_h denotes the image of Lebesgue's measure on $\mathbb{R}_+$ under the application $s \mapsto \frac{1}{h(s)}$. In other terms:

$$\int_{(0,\infty)} \mathrm{e}^{-\frac{x}{h(s)}}\,ds = \int_{(0,\infty)} e^{-xz}\,U_h(dz), \qquad x > 0.$$

We note that h may vanish on some measurable set.

(b) Let Λ denote a generalized gamma convolution with Thorin measure U. Let $F_U(x) := \int_{(0,x]} U(dy)$ for $x \geq 0$ and denote by F_U^{-1} its right-continuous inverse, in the sense of the composition of functions. Then

$$\Lambda \overset{d}{=} \Lambda^h, \qquad \text{with} \quad h(s) = \frac{1}{F_U^{-1}(s)}.$$

These results can be generalized to higher dimensions. Let $S_{\|\cdot\|}$ be the unit sphere on a finite dimensional Euclidean space B with respect to the norm $\|\cdot\|$ and equipped with its Borel σ-algebra. Let K be a cone. Write $S_{K,\|\cdot\|} = S_{\|\cdot\|} \cap K$. Examples for K are $K = \mathbb{R}_+$ or $\mathbb{R}_+^m$. As in [51, Section 4.1] we let the dual cone K' of K be defined as $K' = \{y \in B \,|\, \langle y, s\rangle \geq 0 \text{ for every } s \in K\}$. Let us first introduce the definition of a multivariate gamma distribution, cf. [51, Definition 4.1, Proposition 3.3]:

Definition 11.3. Let μ be a probability distribution on K. Let α be a finite measure on the unit sphere $S_{K,\|\cdot\|}$. Let $\beta\colon S_{K,\|\cdot\|} \to (0,\infty)$ be a Borel-measurable function satisfying the condition

$$\int_{S_{K,\|\cdot\|}} \ln\Big(1+\frac{1}{\beta(v)}\Big)\,\alpha(dv) < \infty. \tag{11.4}$$

Then μ is called an m-dimensional gamma distribution with characteristic quantities α and β, abbreviated by $\Gamma_K(\alpha,\beta)$ if the Laplace transform of μ satisfies

$$L(s) = \exp\Big(\int_{S_{K,\|\cdot\|}} \int_{\mathbb{R}_+} \big(\mathrm{e}^{-r\langle v,s\rangle} - 1\big)\frac{\mathrm{e}^{-\beta(v)r}}{r}\,dr\,\alpha(dv)\Big) \tag{11.5}$$

for all $s \in K'$.

Note that by [51, Proposition 3.3] Equation (11.4) ensures that Equation (11.5) is well-defined. Now we present some results of [51, Section 4.3] concerning Itô–Wiener–Gamma integrals. Let $\gamma = (\gamma_t)_{t\geq 0}$ be a K-valued gamma process such that $\mu_{\alpha,\beta} = \Gamma_K(\alpha, \beta)$ is the distribution of γ_1. Let $N_\gamma(ds, dx)$ be the random measure on $\mathbb{R}_+ \times K$ associated to the K-valued jumps of γ and $\eta_{\mu_{\alpha,\beta}}$ which is the Lévy measure of γ_1 given by

$$\eta_{\mu_{\alpha,\beta}}(E) = \int_{S_{\mathbb{R}^m_+,\|\cdot\|}} \int_{(0,\infty)} 1_E(rv) \frac{\mathrm{e}^{-\beta(v)r}}{r}\, dr\, \alpha(dv), \qquad E \in \mathcal{B}(K).$$

Further (cf. [51, Section 4.3]), let $h\colon \mathbb{R}_+ \times S_{K,\|\cdot\|} \to \mathbb{R}_+$ be a measurable function such that

$$\int_{S_{K,\|\cdot\|}} \int_{(0,\infty)} \ln\Big(1 + \frac{h(w,v)}{\beta(v)}\Big)\, dw\, \alpha(dv) < \infty,$$

in which case we say that h belongs to $L(\Gamma_K(\alpha,\beta))$. According to [51, Section 4.3] it can be proven that the following Itô–Wiener–Gamma integral type is well defined

$$\Lambda^h = a + \int_{(0,\infty)} \int_K h\Big(s, \frac{x}{\|x\|}\Big) x\, N(ds, dx) \tag{11.6}$$

in the framework of integration with respect to infinitely divisible independently scattered random measures (cf. [52, Lemma 2.3]) with $a \in K$. By [51, Proposition 4.2] the integral in Equation (11.6) is well-defined if and only if h belongs to $L(\Gamma_K(\alpha,\beta))$. This is equivalent to the following two conditions

$$\int_{S_{K,\|\cdot\|}} \int_{(0,1/2)} |\ln(t)|\; G_v(dt)\, \alpha(dv) < \infty$$

and

$$\int_{S_{K,\|\cdot\|}} \int_{(1/2,\infty)} \frac{1}{t}\, G_v(dt)\, \alpha(dv) < \infty,$$

where $G_v(dt)$ is the measure on $\mathbb{R}_+$ which is the image of the Lebesgue measure on $\mathbb{R}_+$ under the change of variable $s \mapsto \beta(v)/h(s,v)$. It can be easily seen that there is a direct analogy to the existence of the Thorin measure U in Equation (9.9). Finally, the following proposition is important for us, cf. [51, Proposition 4.3]:

Proposition 11.7. *Let $h \in L(\Gamma_K(\alpha,\beta))$. Then the distribution of the K-valued random variable Λ^h is infinitely divisible and has Laplace transform*

$$\begin{aligned} L_{\Lambda^h}(s) &= \exp\Big(-\langle s, a\rangle - \int_{S_{K,\|\cdot\|}} \int_{(0,\infty)} \int_{(0,\infty)} \big(1 - \mathrm{e}^{-rh(z,v)\langle s,v\rangle}\big) \frac{\mathrm{e}^{-\beta(v)r}}{r}\, dz\, dr\, \alpha(dv)\Big) \\ &= \exp\Big(-\langle s, a\rangle - \int_{S_{K,\|\cdot\|}} \int_{(0,\infty)} \ln\Big(1 + \frac{\langle s, v\rangle}{t}\Big)\, G_v(dt)\, \alpha(dv)\Big), \qquad s \in K', \end{aligned} \tag{11.8}$$

where $a \in K$ and G_v is a Thorin measure on $\mathbb{R}_+$ for α-a.e. v which is the image of the Lebesgue measure on $\mathbb{R}_+$ under the change of variable $w \mapsto \beta(v)/h(w,v)$ (cf. Definition 9.10). Moreover, the Lévy measure of Λ^h is

$$\eta(E) = \int_{S_{K,\|\cdot\|}} \int_{(0,\infty)} 1_E(rv) \frac{k_v(r)}{r}\, dr\, \alpha(dv), \qquad E \in \mathcal{B}(K),$$

where

$$k_v(r) = \int_{(0,\infty)} \mathrm{e}^{-rt}\, G_v(dt).$$

11.2 Multivariate Generalized Gamma Convolutions

In this section we introduce a version of multivariate gamma distributions and – the main topic of this section – multivariate generalized gamma convolutions. For further reading on multivariate infinitely divisible distributions cf. [3], for multivariate generalized gamma convolutions cf. [51]. We will use multivariate generalized gamma convolutions as mixing distributions for the Poisson mixture distribution of the claim numbers later in this chapter. As in the case of multivariate gamma distributions a unique extension of generalized gamma convolutions to higher dimensions does not exist, hence we consider two approaches.

A definition of the multivariate normal distribution motivates the following definition of multivariate generalized gamma convolutions: a random vector Y is said to have a multivariate normal distribution in a Hilbert space H if for every $h \in H$ the scalar product $\langle h, Y\rangle$ is normally distributed (cf. [11, p. 53]). According to [8, Chapter 3.6, p. 46], we may extend Definition 9.10 to an m-dimensional generalized gamma convolution as follows:

Definition 11.9. The distribution of a random vector $(\Lambda_1, \dots, \Lambda_m)$ with non-negative components is said to be an m-dimensional generalized gamma convolution (*in the weak sense*) if the distribution of $c_1\Lambda_1 + \dots + c_m\Lambda_m$ is a generalized gamma convolution whenever $c_1, \dots, c_m \geq 0$.

Using this definition, it is possible to model default cause intensities that satisfy the requirements of Equation (11.1), as the next example demonstrates.

Example 11.10. If $\Lambda_1, \dots, \Lambda_m$ are independent random variables each of which is distributed as a generalized gamma convolution, then $c_0 + c_1\Lambda_1 + \dots + c_m\Lambda_m$ whenever $c_0, \dots, c_m \geq 0$ is a random variable with a generalized gamma convolution because generalized gamma convolutions are closed under convolution and scaling.

This is a very intuitive definition for an easy understanding. However, for an analytical tractability there is a more useful characterization of a multivariate generalized gamma convolution.

Let K now be a proper cone of $\mathbb{R}^m$. As [51, Section 4.1] states, the cone K is called proper if $x = 0$ whenever x and $-x$ are in K. In accordance to our model assume that $K = \mathbb{R}_+^m = [0, \infty)^m$. With regard to Section 11.1 we are prepared for the actual characterization. Multivariate generalized gamma convolutions are given by [51, Definition 4.4]:

Definition 11.11. The class of K-valued generalized gamma convolutions is the collection of all infinitely divisible distributions on K with Lévy measure η_μ having a polar decomposition

$$\eta_\mu(E) = \int_{S_{K,\|\cdot\|}} \int_{(0,\infty)} 1_E(rv) \frac{k_v(r)}{r} \, dr \, \alpha(dv), \qquad E \in \mathcal{B}(K),$$

where $k_v(r)$ is a measurable function in v and completely monotone in r for α-a.e. v.

As in the univariate case, Proposition 11.7 states that the class of distributions of Itô–Wiener–Gamma integrals Λ^h coincides with the class of K-valued generalized gamma convolutions. Note that in most cases we will use the characterization given in Equation (11.8).

Remark 11.12. It is not necessary to specify the norm we use. By [51, Proposition 3.4] we need only change the parameterization if we choose another norm.

These two definitions lead to a similar construction and interpretation of a multivariate generalized gamma convolution. Since there are many approaches of finding a multivariate gamma distribution, cf. [44, Chapter 48], a unique construction of a multivariate generalized gamma convolution does not exist either. Cheriyan and Ramabhadran's extension (cf. e.g. [44, Chapter 48.3.1]) seems to offer a good starting point: Consider n independent gamma-distributed random variables such that $Y_j \sim \text{Gamma}(\alpha_j, \beta_j)$ with $\alpha_j, \beta_j > 0$ for $j \in \{1, \ldots, n\}$. Define $Z = (Z_1, \ldots, Z_m)^\top$ where $Z_i = \sum_{j=1}^n c_{i,j} Y_j$ and $c_{i,j} \in \{0, 1\}$ for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. For simplicity we assume that for each $j \in \{1, \ldots, n\}$ the random variable Y_j should appear in at least one Z_i for $i \in \{1, \ldots, m\}$. Then the Laplace transform of the distribution of Z is computed as follows

$$L(s) = \mathbb{E}\big[\mathrm{e}^{-\langle s,Z\rangle}\big] = \mathbb{E}\Big[\mathrm{e}^{-\sum_{i=1}^m s_i Z_i}\Big] = \mathbb{E}\Big[\mathrm{e}^{-\sum_{i=1}^m s_i \sum_{j=1}^n c_{i,j} Y_j}\Big], \qquad s \in \mathbb{R}_+^m.$$

Using the independence of $Y_1, \ldots, Y_n$ and their distribution

$$L(s) = \prod_{j=1}^n \mathbb{E}\Big[\mathrm{e}^{-\sum_{i=1}^m s_i c_{i,j} Y_j}\Big] = \prod_{j=1}^n \Big(\frac{\beta_j}{\beta_j + \sum_{i=1}^m s_i c_{i,j}}\Big)^{\alpha_j}, \qquad s \in \mathbb{R}_+^m.$$

Bringing this into an integral form yields

$$\begin{aligned} L(s) &= \exp\Big(\sum_{j=1}^n \alpha_j \ln\Big(\frac{\beta_j}{\beta_j + \sum_{i=1}^m s_i c_{i,j}}\Big)\Big) \\ &= \exp\Big(-\int_{S_{\mathbb{R}_+^m, \|\cdot\|}} \int_{(0,\infty)} \ln\Big(\frac{t + \langle s, v\rangle}{t}\Big)\, G_v(dt)\, \alpha(dv)\Big), \end{aligned} \tag{11.13}$$

where $\alpha(\{c_j\}) = \alpha_j$ and $\beta(c_j) = \beta_j$ and $h(w, v) = 1_{\{w=1\}}(w, v)$ with $c_j = (c_{1,j}, \ldots, c_{m,j})$, cf. Equation (11.8). If we take the supremum norm, then every c_j for $j \in \{1, \ldots, n\}$ is on the unit sphere and that is why the null-vector should be excluded. Note that according to [8, Theorem 3.3.2] this is also the Laplace transform of a generalized gamma convolution for $s_i = s$, which is in accordance with Definition 11.9 of a multivariate generalized gamma convolution in the sense of Bondesson.

Using this definition it is possible to generalize Bondesson's closure theorem [8, Theorem 3.1.5] to higher dimensions. This result has already been proven in [3, Theorem F, p. 27]. We give an alternative proof. Here and in the following 1 in the inner product $\langle 1, v\rangle$ denotes the vector $1 = (1, \ldots, 1)$. Along the lines of his argumentation, for reasons of convenience, we define a signed measure $\nu_v(dt)$ on $[0, \infty]$ by

$$\nu_v(\{0\}) = 0 \quad \text{and} \quad \nu_v(\{\infty\}) = \langle b_a, v\rangle\langle 1, v\rangle, \qquad v \in S_{\mathbb{R}_+^m, \|\cdot\|}, \quad a \in \mathbb{R}_+^m, \tag{11.14}$$

and $b_a \in \mathbb{R}^m$ depends on a which will be made clear in the proof of the lemma. According to Bondesson's choice, we let

$$\nu_v(dt) = \ln(1 + t^{-1}\langle 1, v\rangle)\, G_v(dt), \qquad 0 < t < \infty. \tag{11.15}$$

Because this signed measure can be decomposed into the difference of two finite non-negative measures, the convergence results of non-negative measures carry over. Further we let

$$g(s, v, t) = \frac{1}{\ln(1 + t^{-1}\langle 1, v\rangle)} \ln\Big(\frac{t + \langle s, v\rangle}{t}\Big), \qquad s \in \mathbb{R}_+^m, \quad v \in S_{\mathbb{R}_+^m, \|\cdot\|}, \quad t > 0.$$

Then the Laplace transform of a multivariate generalized gamma convolution is

$$L(s) = \exp\Big(-\langle s, a\rangle - \int_{S_{\mathbb{R}^m_+,\|\cdot\|}} \int_{(0,\infty)} g(s,v,t)\,\nu_v(dt)\,\alpha(dv)\Big).$$

Then our generalization of Bondesson's closure theorem is the following:

Lemma 11.16. *Let $\{F_n\}_{n\in\mathbb{N}}$ be an m-dimensional sequence of multivariate generalized gamma convolutions, defined by the sequences of measures $\{(\nu_n)_v\}_{n\in\mathbb{N}}$ and $\{\alpha_n\}_{n\in\mathbb{N}}$. If $F_n \to F$ weakly as $n\to\infty$, then F is a multivariate generalized gamma convolution and the corresponding measure ν_v is a vague limit on $[0,\infty]$ and α is the corresponding vague limit on $S_{\mathbb{R}^m_+,\|\cdot\|}$. Conversely, if $(\nu_n)_v \to \nu_v$ vaguely as $n\to\infty$ on $[0,\infty]$ and $\alpha_n\to\alpha$ vaguely as $n\to\infty$ on $S_{\mathbb{R}^m_+,\|\cdot\|}$ and $\nu_v(\{0\})=0$, then $F_n\to F$ weakly as $n\to\infty$, where F is a multivariate generalized gamma convolution which corresponds to ν_v and α.*

Proof. We follow Bondesson's proof of his closure theorem, cf. [8, p. 34/35]. Let $F_n\to F$ weakly. Denote by L_n the Laplace transform of F_n. Then $\ln(L_n(s))\to\ln(L(s))$ as $n\to\infty$ for $s\in\mathbb{R}^m_+$. Further

$$\begin{aligned}\ln(L_n(1)) &= -\int_{S_{\mathbb{R}^m_+,\|\cdot\|}} \int_{[0,\infty]} \frac{\ln\big(\frac{t+\langle 1,v\rangle}{t}\big)}{\ln(1+t^{-1}\langle 1,v\rangle)} \ln(1+t^{-1}\langle 1,v\rangle)\,(G_n)_v(dt)\,\alpha_n(dv)\\ &= -\int_{S_{\mathbb{R}^m_+,\|\cdot\|}} \int_{[0,\infty]} (\nu_n)_v(dt)\,\alpha_n(dv)\end{aligned}$$

holds. Since $L_n(1)$ is defined for $n\in\mathbb{N}$ and the integrals are bounded (see also the definition of h), it follows that this is bounded. Hence $\{(\nu_n)_v\}_{n\in\mathbb{N}}$ is a sequence of bounded measures on $[0,\infty]$ and $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence of bounded measures on $S_{\mathbb{R}^m_+,\|\cdot\|}$. By Helly's selection theorem (cf. [42, Theorem 5.19]) there exists a subsequence $\{L_{n_k}(s)\}_{k\in\mathbb{N}}$ for $s\in\mathbb{R}^m_+$ that converges pointwise in all points of continuity. According to the continuous mapping theorem (cf. [43, Theorem 13.25]) the weak convergence is preserved if the measure of all points of discontinuity is zero, which is satisfied here. Note that $\{L_{n_k}(s)\}_{k\in\mathbb{N}}$ uniquely determines a subsequence $\{F_{n_k}\}_{k\in\mathbb{N}}$ of probability distributions. Further we should note that $\mathbb{R}_+$ is a polish space, hence by [43, Theorem 13.34] the weak convergence of $\{F_n\}_{n\in\mathbb{N}}$ implies that $\{F_n\}_{n\in\mathbb{N}}$ (and $\{F_{n_k}\}_{k\in\mathbb{N}}$) is also tight. By [12, Theorem 22.22] it follows that the limit determines a probability distribution.

The convergence of the subsequence $\{L_{n_k}(s)\}_{k\in\mathbb{N}}$ for $s\in\mathbb{R}^m_+$ induces convergent subsequences $\{(\nu_{n_k})_v\}_{k\in\mathbb{N}}$ with limit ν_v and $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ with limit α. Consider now the vague convergence. $t\mapsto g(s,v,t)$ is continuous on $[0,\infty]$ for $v\in S_{\mathbb{R}^m_+,\|\cdot\|}$ and $s\in\mathbb{R}^m_+$, where

$$\lim_{t\to 0} g(s,v,t) = \lim_{t\to 0}\frac{\ln(1+\langle s,v\rangle/t)}{\ln(1+\langle 1,v\rangle/t)} = 1$$

and

$$\begin{aligned}\lim_{t\to\infty} g(s,v,t) &= \lim_{t\to\infty}\Big(\frac{1}{t+\langle s,v\rangle}-\frac{1}{t}\Big)\Big/\Big(\frac{1}{t+\langle 1,v\rangle}-\frac{1}{t}\Big)\\ &= \lim_{t\to\infty}\frac{\langle s,v\rangle t(t+\langle 1,v\rangle)}{t(t+\langle s,v\rangle)\langle 1,v\rangle} = \frac{\langle s,v\rangle}{\langle 1,v\rangle}.\end{aligned}$$

Furthermore, consider $\int_{[0,\infty]} g(s,v,t)\,\nu_v(dt)$. For $s \in \mathbb{R}_+^m$ the map

$$v \mapsto \int_{[0,\infty]} g(s,v,t)\,\nu_v(dt)$$

is defined on the compact set $S_{\mathbb{R}_+^m,\|\cdot\|}$. By Lusin's theorem (cf. e.g. [5, 26.7 Theorem]) for every $\varepsilon > 0$ there exists a compact set $K \subset S_{\mathbb{R}_+^m,\|\cdot\|}$ such that $\alpha(S_{\mathbb{R}_+^m,\|\cdot\|} \setminus K) < \varepsilon$ and the restriction $v \mapsto \int_{[0,\infty]} g(s,v,t)\,\nu_v(dt)\big|_K$ is continuous. Because the unit sphere is compact, this is especially satisfied for $K = S_{\mathbb{R}_+^m,\|\cdot\|}$ and thus this map is continuous. Therefore, according to [79, p. 4] the image of the map $v \mapsto \int_{[0,\infty]} g(s,v,t)\,\nu_v(dt)$ is compact. Altogether we have that $\{(\nu_{n_k})_v\}_{k\in\mathbb{N}}$ is a vaguely convergent subsequence and $t \mapsto g(s,v,t)$ is continuous on a compact support and that $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ is a vaguely convergent subsequence and $v \mapsto \int_{[0,\infty]} g(s,v,t)\,\nu_v(dt)$ is continuous on a compact support. Hence, for every $s \in \mathbb{R}_+^m$ we have

$$\begin{aligned}
\ln(L_{n_k}(s)) &= -\int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \int_{[0,\infty]} g(s,v,t)\,(\nu_{n_k})_v(dt)\,\alpha_{n_k}(dv) \\
&\to -\int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \int_{[0,\infty]} g(s,v,t)\,\nu_v(dt)\,\alpha(dv) \\
&= -\int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \nu_v(\{0\})\,\alpha(dv) - \int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \nu_v(\{\infty\})\frac{\langle s,v\rangle}{\langle 1,v\rangle}\,\alpha(dv) \\
&\quad - \int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \int_{(0,\infty)} g(s,v,t)\,\nu_v(dt)\,\alpha(dv)
\end{aligned}$$

as $k \to \infty$. The integral over $\nu_v(\{\infty\})\langle s,v\rangle/\langle 1,v\rangle$ can be written as follows noting $\nu_v(\{\infty\}) = \langle b_a,v\rangle\langle 1,v\rangle$

$$\int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \langle b_a,v\rangle\langle s,v\rangle\,\alpha(dv) = \sum_{i=1}^m s_i \int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \langle b_a,v\rangle v_i\,\alpha(dv),$$

where we now assume that for $i \in \{1,\dots,m\}$

$$a_i = \int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \langle b_a,v\rangle v_i\,\alpha(dv)$$

holds. Hence the vector $b_a = (b_1,\dots,b_m)$ should be chosen such that we obtain a. Note that α is a product measure and hence Fubini's theorem can be applied. For each $i \in \{1,\dots,m\}$ we can write

$$a_i = \sum_{j=1}^m b_j \int_{\substack{v_m\in[0,1]\\ \|v\|=1}} \cdots \int_{\substack{v_1\in[0,1]\\ \|v\|=1}} v_j v_i\,\alpha_1(dv_1)\dots\alpha_m(dv_m).$$

Because the integrals do not depend on $b_1,\dots,b_m$, this is a linear equation system. We prove that for given $a \in \mathbb{R}_+^m$ we can construct solutions that satisfy the given linear equation system. As the unit vectors $e_1,\dots,e_m$ are a basis of $\mathbb{R}_+^m$, it suffices that we consider solutions of this linear equation system where $a = e_i$. Having such a solution, we get any arbitrary a by linear combinations. The reader can confirm that finding a solution for a unit vector e_i

can be done by Gaussian elimination. If the matrix that is determined by the integrals is not invertible, then we have (up to several) levels of free choice. Thus for given b_a we obtain

$$\int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \langle b_a, v\rangle\langle s, v\rangle\, \alpha(dv) = \langle s, a\rangle.$$

Since $\nu_v(\{0\}) = 0$, F is a multivariate generalized gamma convolution. As can be easily shown, the vague limits ν_v and α are uniquely determined, and by Billingsley [6, Theorem 2.6] $\{(\nu_v)_n\}_{n\in\mathbb{N}}$ and $\{\alpha_n\}_{n\in\mathbb{N}}$ are vaguely convergent.

Conversely, the claim follows by an application of [5, 30.8 Theorem] and a translation of the portmanteau theorem [6, Theorem 2.1], cf. [6, p. 26]. q.e.d.

It is possible to generalize Definition 11.9 up to infinite dimensions that also justifies this definition in finite dimensions:

Remark 11.17. According to [17, Chapter 8, (2.7) Theorem] weak convergence in $\mathbb{R}^\infty$ is equivalent to convergence of finite dimensional distributions. Note therefore that $\mathbb{R}^\infty$ is also separable and complete (cf. the proof of [17, Chapter 8, (2.7) Theorem] and [17, Chapter 8, (2.6) Lemma]), and hence by [6, Theorem 1.3] each probability measure on $\mathbb{R}^\infty$ is tight. Hence Definition 11.9 can be extended to infinite dimensions. We can then use Corollary 11.16 to construct infinite-dimensional generalized gamma convolutions.

Using Lemma 11.16 it is possible to give an existence result of a sequence of distributions that converges weakly to a given multivariate generalized gamma convolution. For reasons of convenience we use the supremum norm.

Corollary 11.18. *Let F be an m-dimensional generalized gamma convolution with $\Lambda \sim F$ defined by $a \in \mathbb{R}_+^m$ and a Thorin measure G_v on $\mathbb{R}_+$, which is the image of the Lebesgue measure on $\mathbb{R}_+$ under the change of variable $t \mapsto \beta(v)/h(t, v)$ for $v \in S_{\mathbb{R}_+^m,\|\cdot\|}$ and $t \in \mathbb{R}_+$, and a finite measure α on $S_{\mathbb{R}_+^m,\|\cdot\|}$. Then there exists a weakly convergent sequence $\{F_n\}_{n\in\mathbb{N}}$ of m-dimensional generalized gamma convolutions with $\Lambda_n \sim F_n$ for $n \in \mathbb{N}$ characterized by $\{(G_n)_v\}_{n\in\mathbb{N}}$ and $\{\alpha_n\}_{n\in\mathbb{N}}$ where*

$$\alpha_n = \sum_{j=1}^{n} (\alpha(\{v_j^{(n)}\}) - \alpha(\{v_{j-1}^{(n)}\}))\delta_{v_j^{(n)}}$$

for some isotonic sequence $v_n^{(n)} = (v_{1,n}^{(n)}, \dots, v_{m,n}^{(n)}) \in S_{\mathbb{R}_+^m,\|\cdot\|}$ and

$$(G_n)_v = \sum_{j=1}^{n} (G_v(b_j^{(n)}) - G_v(b_{j-1}^{(n)}))\delta_{b_i^{(n)}}$$

for a partition $\mathcal{Z}_n = (b_j^{(n)})_{0\le j\le n}$ on a compact subset of $(0, \infty)$ for all $n \in \mathbb{N}$. Let $\{(\nu_n)_v\}_{n\in\mathbb{N}}$ be defined according to Equations (11.14) and (11.15). Let b_a be given. Then these sequences converge vaguely to ν_v and α as $n \to \infty$, respectively. Then $\Lambda_n = (\Lambda_{1,n}, \dots, \Lambda_{m,n})$ with $\Lambda_{i,n} = a_i + \sum_{j=1}^{n} v_{i,j}^{(n)} Y_j$ for $i \in \{1, \dots, m\}$, and $Y_j \sim \mathrm{Gamma}(\alpha_j, \beta_j)$ for $j \in \{1, \dots, n\}$ and $n \in \mathbb{N}$ are independent.

Proof. We proceed as in Corollary 9.28. Let $\mathcal{Z}_n = (b_j^{(n)})_{0\le j\le n}$ be a partition where $0 < b_0^{(n)} < b_1^{(n)} < \cdots < b_n^{(n)} < \infty$ and $b_0^{(n)} \searrow 0$ and $b_n^{(n)} \to \infty$ as $n \to \infty$. The mesh of $\mathcal{Z}$ is defined as $\mathrm{mesh}(\mathcal{Z}_n) = \max_{j=1,\dots,n}(b_j^{(n)} - b_{j-1}^{(n)}) \to 0$ as $n \to \infty$. Let $f\colon (0, \infty) \to \mathbb{R}$ be a continuous function with a compact support. By the definition of the Lebesgue-Stieltjes integral and because $\operatorname{supp}(f) \subset (b_0^{(n)}, b_n^{(n)}]$ for $v \in S_{\mathbb{R}_+^m,\|\cdot\|}$

$$\int_{[0,\infty]} f(t)\,(\nu_n)_v(dt) = \sum_{j=1}^{n} f(b_j^{(n)}) \ln(1 + (b_j^{(n)})^{-1}\langle 1, v\rangle)(G_v(b_j^{(n)}) - G_v(b_{j-1}^{(n)}))$$
$$\to \int_{[0,\infty]} f(t)\,\nu_v(dt)$$

converges vaguely for $n \to \infty$ as the mesh $\mu(\mathcal{Z})$ gets smaller and smaller because G_v is monotone (it is non-decreasing). Let f be a continuous function on $S_{\mathbb{R}_+^m,\|\cdot\|}$ with compact support. Then by the definition of the Lebesgue-Stieltjes integral

$$\int_{S_{\mathbb{R}_+^m,\|\cdot\|}} f(v)\,\alpha_n(dv) = \sum_{j=1}^{n} f(v_j^{(n)})(\alpha(v_j^{(n)}) - \alpha(v_{j-1}^{(n)})) \to \int_{S_{\mathbb{R}_+^m,\|\cdot\|}} f(v)\,\alpha(dv)$$

converges vaguely as $n \to \infty$. Thus by an application of Lemma 11.16 the constructed sequence $\{F_n\}_{n\in\mathbb{N}}$ converges weakly to F. q.e.d.

11.3 Applications to the Extended CreditRisk$^+$ Model

In this section we use the results of the preceding section in order to extend the CreditRisk$^+$ model for several business lines to a broader class of distributions – the multivariate generalized gamma convolutions. As already mentioned above, the idea goes back to Cheriyan and Ramabhadran's extension (cf. e.g. [44, Chapter 48.3.1])

Applying this corollary it is possible to find an alternative representation for the approximation.

Corollary 11.19. *Let $\lambda = (\lambda_1 \ldots, \lambda_m)$ with $\lambda_i \geq 0$ for $i \in \{1, \ldots, m\}$. Define a sequence $\{(\Lambda_{1,n}, \ldots, \Lambda_{m,n})\}_{n\in\mathbb{N}}$ of random vectors given by independent random variables Y_j for $j \in \{1, \ldots, n\}$ and $n \in \mathbb{N}$ as in Corollary 11.18, and this sequence converges weakly to a random vector Λ. Let $\{(N_{1,n}, \ldots, N_{m,n})\}_{n\in\mathbb{N}}$ be a sequence of random vectors. Let $N_{1,n}, \ldots, N_{m,n}$ be conditionally independent given $\Lambda_{1,n}, \ldots, \Lambda_{m,n}$ for each $n \in \mathbb{N}$ such that*

$$\mathcal{L}(N_{i,n}\,|\,\Lambda_{1,n}, \ldots, \Lambda_{m,n}) \stackrel{a.s.}{=} \mathcal{L}(N_{i,n}\,|\,\Lambda_{i,n}) \stackrel{a.s.}{=} \operatorname{Poisson}(\lambda_i \Lambda_{i,n})\,, \qquad i \in \{1, \ldots, m\}.$$

On the other hand, consider n independent sequences of i.i.d. random vectors $\{B_{j,h}^{(n)}\}_{h\in\mathbb{N}}$, such that

$$B_{j,1}^{(n)} \sim \operatorname{Multinomial}\big(1; p_{1,j}^{(n)}, \ldots, p_{m,j}^{(n)}\big),$$

where $p_{i,j}^{(n)} \in [0,1]$ with $\sum_{i=1}^m p_{i,j}^{(n)} = 1$ satisfies the condition $p_{i,j}^{(n)} \sum_{d=1}^m \lambda_d v_{d,j}^{(n)} = \lambda_i v_{i,j}^{(n)}$ for each $j \in \{1, \ldots, n\}$ and $n \in \mathbb{N}$ and $i \in \{1, \ldots, m\}$. Let $\{Q_j^{(n)}\}_{j\in\{1,\ldots,n\}}$ be independent random variables for each $n \in \mathbb{N}$ which satisfy

$$\mathcal{L}\big(Q_j^{(n)} \mid Y_1, \ldots, Y_n\big) \stackrel{a.s.}{=} \mathcal{L}\big(Q_j^{(n)} \mid Y_j\big) \stackrel{a.s.}{=} \operatorname{Poisson}\bigg(\sum_{i=1}^m \lambda_i v_{i,j}^{(n)} Y_j\bigg).$$

Assume further for each $n \in \mathbb{N}$ that $(Q_1^{(n)}, \ldots, Q_n^{(n)})$ and the sequences $\{B_{j,h}^{(n)}\}_{h\in\mathbb{N}}$ are independent for each $j \in \{1, \ldots, n\}$. Let the random vector P with independent components $P_i \sim \operatorname{Poisson}(\lambda_i a_i)$ for $i \in \{1, \ldots, m\}$ be independent of $(Q_1^{(n)}, \ldots, Q_n^{(n)})$ and the sequences

$\{B_{j,h}^{(n)}\}_{h\in\mathbb{N}}$ *for each* $j \in \{1,\dots,n\}$ *and* $n \in \mathbb{N}$. *Finally, let* $\{M_n\}_{n\in\mathbb{N}}$ *be a sequence of* $\mathbb{N}_0^m$*-valued random vectors such that*

$$M_n = P + \sum_{j=1}^{n} \sum_{h=1}^{Q_j^{(n)}} B_{j,h}^{(n)}, \qquad n \in \mathbb{N}.$$

Then M_n *and* $(N_{1,n},\dots,N_{m,n})$ *have the same distribution for each* $n \in \mathbb{N}$ *and* $\{N_n\}_{n\in\mathbb{N}}$ *converge weakly to some random vector such that*

$$\mathcal{L}(N\,|\,\Lambda) \stackrel{a.s.}{=} \mathrm{Poisson}(\lambda\Lambda)\,.$$

Proof. The equality in distribution follows by an application of Theorem 3.1 and the convolution property of the Poisson distribution. The weak convergence of the sequence $\{(\Lambda_{1,n},\dots,\Lambda_{m,n})\}_{n\in\mathbb{N}}$ follows by Corollary 11.18. The weak convergence of the sequence $\{N_n\}_{n\in\mathbb{N}}$ follows as in the proof of Corollary 9.32. q.e.d.

We now derive an upper bound with respect to the total variation distance between two Poisson mixture distributions with m-dimensional generalized gamma convolutions as mixing distributions. The proof goes along the lines of Theorem 10.8.

Lemma 11.20. *Let* $m \in \mathbb{N}$ *and* $\lambda = (\lambda_1,\dots,\lambda_m)$ *with* $\lambda_i \geq 0$ *for* $i \in \{1,\dots,m\}$ *and* λ *is not the null-vector. Let* $g \in L(\Gamma_{\mathbb{R}_+^m}(\alpha,\beta))$ *and* $h \in L(\Gamma_{\mathbb{R}_+^m}(\gamma,\delta))$. *Let* $(\Lambda_1,\dots,\Lambda_m)$ *denote a random vector distributed as an* m*-dimensional multivariate generalized gamma convolution characterized by the Thorin measure* G *and the function* g *and parameter* a *with non-negative components, and let* $(\Psi_1,\dots,\Psi_m)$ *denote a random vector distributed as an* m*-dimensional multivariate generalized gamma convolution characterized by the Thorin measure* H *and the function* h *and parameter* b *with non-negative components. Let*

$$\int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \int_{(0,\infty)} \Bigl(\frac{\langle\lambda,v\rangle}{t+\langle\lambda,v\rangle}\Bigr)^n G_v(dt)\,\alpha(dv) \geq \int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \int_{(0,\infty)} \Bigl(\frac{\langle\lambda,v\rangle}{t+\langle\lambda,v\rangle}\Bigr)^n H_v(dt)\,\gamma(dv) \tag{11.21}$$

for all $n \in \mathbb{N}$ *be satisfied. Further let* $N = (N_1,\dots,N_m)$ *and* $M = (M_1,\dots,M_m)$ *be random vectors with conditionally independent components given* $\Lambda_1,\dots,\Lambda_m$ *and* $\Psi_1,\dots,\Psi_m$, *respectively, such that*

$$\mathcal{L}(N_i\,|\,\Lambda_1,\dots,\Lambda_m) \stackrel{a.s.}{=} \mathcal{L}(N_i\,|\,\Lambda_i) \stackrel{a.s.}{=} \mathrm{Poisson}(\lambda_i\Lambda_i)$$

and

$$\mathcal{L}(M_i\,|\,\Psi_1,\dots,\Psi_m) \stackrel{a.s.}{=} \mathcal{L}(M_i\,|\,\Psi_i) \stackrel{a.s.}{=} \mathrm{Poisson}(\lambda_i\Psi_i)$$

for $i \in \{1,\dots,m\}$. *Then the total variation distance satisfies*

$$\mathrm{d_{TV}}(\mathcal{L}(N)\,,\mathcal{L}(M)) \leq \frac{3}{2}\bigl|\mu'-\nu'\bigr| + \frac{1}{2}\biggl(\mu-\nu+\frac{\langle\lambda,|a\nu'-b\mu'|\rangle}{\nu'}\biggr),$$

where

$$\mu = \int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \int_{(0,\infty)} \kappa_{\langle\lambda,v\rangle}(t)\,G_v(dt)\,\alpha(dv)$$

and

$$\nu = \int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \int_{(0,\infty)} \kappa_{\langle\lambda,v\rangle}(t)\,H_v(dt)\,\gamma(dv),$$

and $\mu' = \mu + \langle\lambda,a\rangle$ *and* $\nu' = \nu + \langle\lambda,b\rangle$, *and* $\kappa_{\langle\lambda,v\rangle}$ *is defined as in Notation 10.7.*

Proof. As already mentioned before, this proof is similar to the proof of Theorem 10.8. A multivariate generalized gamma convolution is infinitely divisible, cf. [51, Proposition 4.3]. A Poisson distribution mixed over an infinitely divisible distribution is also infinitely divisible, as can be found in [64, Theorem 4.1(v)], and can be easily generalized to the multivariate case. Denote by $\mathbb{N}^m$ the set of m-dimensional natural numbers where each component is positive. Since a Poisson distribution mixed with an infinitely divisible non-negative random vector is a compound Poisson distribution with severity distribution in $\mathbb{N}^m$ (cf. [62, Theorem 1]), we assume $N \stackrel{\mathrm{d}}{=} \sum_{h=1}^{K} X_h$ and $M \stackrel{\mathrm{d}}{=} \sum_{h=1}^{L} Y_h$ with the following assumptions: Let $\{X_h\}_{h\in\mathbb{N}}$ be m-dimensional sequences of i.i.d. $\mathbb{N}^m$-valued random vectors independent of $K \sim \mathrm{Poisson}(\mu')$. Let further $\{Y_h\}_{h\in\mathbb{N}}$ be m-dimensional sequences of i.i.d. $\mathbb{N}^m$-valued random vectors independent of $L \sim \mathrm{Poisson}(\nu')$.

The results in [70], especially [70, Theorem 3.1], can be generalized without much effort using the results in [73] insofar as we may consider multivariate severity distributions. The first part of the upper bound is therefore a direct application of a generalization of Corollary 10.6 and provides

$$\mathrm{d_{TV}}(\mathcal{L}(M)\,,\mathcal{L}(N)) \le \min\{\big|\sqrt{\mu'}-\sqrt{\nu'}\big|, \big|\mu'-\nu'\big|\} + \min\{\mu',\nu'\}\,\mathrm{d_{TV}}(\mathcal{L}(X_1)\,,\mathcal{L}(Y_1))\,. \tag{11.22}$$

It remains to compute μ', ν' and $\mathrm{d_{TV}}(\mathcal{L}(X_1)\,,\mathcal{L}(Y_1))$. The Poisson parameters and the probability-generating functions of the respective distributions can be derived along the lines of the approach on [64, p. 90]. As we consider the multivariate case, we give the derivation here: For completeness let us first state the probability-generating function of the distribution of N for $z \in [0,1]^m$

$$\begin{aligned} G_N(z) &= \mathbb{E}\Big[\prod_{i=1}^{m} \mathbb{E}\big[z_i^{N_i}\,\big|\,\Lambda_i\big]\Big] = \mathbb{E}\Big[\prod_{i=1}^{m} \exp\big(-\lambda_i\Lambda_i(1-z_i)\big)\Big] \\ &= \exp\Big(-\langle\lambda(1-z),a\rangle - \int_{S_{\mathbb{R}_+^m,\|\cdot\|}}\int_{(0,\infty)} \ln\Big(\frac{t+\langle\lambda(1-z),v\rangle}{t}\Big)\,G_v(dt)\,\alpha(dv)\Big). \end{aligned}$$

The probability-generating function of the distribution of M is similarly given. The Poisson parameter of the compound Poisson distribution of N is given as follows:

$$\mathbb{P}[N_1=0,\dots,N_m=0] = \mathbb{P}[K=0] = \mathrm{e}^{-\mu'}\,\frac{{\mu'}^0}{0!} = \mathrm{e}^{-\mu'},$$

hence

$$\mu' = -\ln G_N(0) = \langle\lambda,a\rangle + \int_{S_{\mathbb{R}_+^m,\|\cdot\|}}\int_{(0,\infty)} \ln\Big(\frac{t+\langle\lambda,v\rangle}{t}\Big)\,G_v(dt)\,\alpha(dv).$$

Accordingly, we obtain ν'. Noting the Poisson distribution of the random variable K, the probability-generating function of the distribution of N is given as follows for $z\in[0,1]^m$

$$G_N(z) = G_K(G_{X_1}(z)) = \exp(-\mu'(1-G_{X_1}(z))),$$

hence we have, using $G_N(z) = L_\Lambda(\lambda(1-z))$,

$$G_{X_1}(z) = \frac{\ln G_N(z)}{\mu'} + 1 = \frac{\ln\Big(\frac{L_\Lambda(\lambda)}{L_\Lambda(\lambda(1-z))}\Big)}{\ln L_\Lambda(\lambda)}.$$

Thus we have for $z \in [0,1]^m$

$$\ln\Big(\frac{L_\Lambda(\lambda)}{L_\Lambda(\lambda(1-z))}\Big) = \ln\Big(\exp\Big(-\langle \lambda z, a\rangle - \int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \int_{(0,\infty)} \ln\Big(\frac{t+\langle\lambda,v\rangle}{t}\Big)\, G_v(dt)\,\alpha(dv) + \int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \int_{(0,\infty)} \ln\Big(\frac{t+\langle\lambda(1-z),v\rangle}{t}\Big)\, G_v(dt)\,\alpha(dv)\Big)\Big).$$

The probability-generating function of the distribution of X_1 is therefore given by

$$\begin{aligned} G_{X_1}(z) &= \frac{1}{-\mu'}\Big(-\langle \lambda z, a\rangle + \int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \int_{(0,\infty)} \ln\Big(\frac{t+\langle\lambda(1-z),v\rangle}{t+\langle\lambda,v\rangle}\Big)\, G_v(dt)\,\alpha(dv)\Big) \\ &= \frac{1}{\mu'}\Big(\langle \lambda z, a\rangle + \int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \int_{(0,\infty)} \ln\Big(\frac{t+\langle\lambda,v\rangle}{t+\langle\lambda(1-z),v\rangle}\Big)\, G_v(dt)\,\alpha(dv)\Big). \end{aligned}$$

As [80, 1.1.2 Corollary] states, the probability mass function of $X_1 = (X_{1,1},\dots,X_{m,1})$ is given by

$$\mathbb{P}[X_{1,1}=n_1,\dots,X_{m,1}=n_m] = \frac{1}{n_1!\dots n_m!}\frac{\partial^{n_1+\dots+n_m} G_{X_1}(0)}{\partial z_1^{n_1}\dots\partial z_m^{n_m}}.$$

Letting $\{n=1\} = \{n_1=1,\dots,n_m=1\}$, the partial derivatives are

$$\frac{\partial^{n_1+\dots+n_m} G_{X_1}(z)}{\partial z_1^{n_1}\dots\partial z_m^{n_m}} = \frac{1}{\mu'}\Big(\langle\lambda,a\rangle 1_{\{n=1\}} + \int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \int_{(0,\infty)} \prod_{i=1}^m \frac{\langle\lambda,v\rangle^{n_i}(n_i-1)!}{(t+\langle\lambda(1-z),v\rangle)^{n_i}}\, G_v(dt)\,\alpha(dv)\Big), \tag{11.23}$$

as can be proven without much effort by induction as integration and differentiation can be exchanged, cf. e.g. [5, 16.2 Lemma]. Thus we obtain with $\{X_1 = n\} = \{X_{1,1} = n_1,\dots,X_{m,1}=n_m\}$

$$\mathbb{P}[X_1=n] = \frac{1}{\mu'}\Big(\frac{\langle\lambda,a\rangle 1_{\{n=1\}}}{n_1!\dots n_m!} + \int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \int_{(0,\infty)} \prod_{i=1}^m \frac{1}{n_i}\Big(\frac{\langle\lambda,v\rangle}{t+\langle\lambda,v\rangle}\Big)^{n_i} G_v(dt)\,\alpha(dv)\Big).$$

The respective formula holds for $\mathbb{P}[Y_1=n]$, i.e.,

$$\mathbb{P}[Y_1=n] = \frac{1}{\nu'}\Big(\frac{\langle\lambda,b\rangle 1_{\{n=1\}}}{n_1!\dots n_m!} + \int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \int_{(0,\infty)} \prod_{i=1}^m \frac{1}{n_i}\Big(\frac{\langle\lambda,v\rangle}{t+\langle\lambda,v\rangle}\Big)^{n_i} H_v(dt)\,\gamma(dv)\Big).$$

We can proceed as in Equations (10.18) and (10.19) by exploiting Equation (11.21) and obtain

$$\begin{aligned} \big|\,\mathbb{P}[X_1=n] - \mathbb{P}[Y_1=n]\,\big| \le{}& \frac{\langle\lambda,|a\nu'-b\mu'|\rangle}{\mu'\nu' n_1!\dots n_m!}1_{\{n=1\}} \\ &+ \frac{1}{\mu'\nu'}\Big(\nu'\Big(\int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \int_{(0,\infty)} \prod_{i=1}^m \frac{1}{n_i}\Big(\frac{\langle\lambda,v\rangle}{t+\langle\lambda,v\rangle}\Big)^{n_i} G_v(dt)\,\alpha(dv) \\ &- \int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \int_{(0,\infty)} \prod_{i=1}^m \frac{1}{n_i}\Big(\frac{\langle\lambda,v\rangle}{t+\langle\lambda,v\rangle}\Big)^{n_i} H_v(dt)\,\gamma(dv)\Big) \\ &+ |\mu'-\nu'| \int_{S_{\mathbb{R}_+^m,\|\cdot\|}} \int_{(0,\infty)} \prod_{i=1}^m \frac{1}{n_i}\Big(\frac{\langle\lambda,v\rangle}{t+\langle\lambda,v\rangle}\Big)^{n_i} H_v(dt)\,\gamma(dv)\Big) \end{aligned} \tag{11.24}$$

By application of [5, 11.5 Corollary] summation and integration may be interchanged in the next estimate because $(\langle\lambda,v\rangle/(\langle\lambda,v\rangle+t))^{n_i}$ is non-negative for $i\in\{1,\dots,m\}$. By using Equation (11.24), for the total variation distance we obtain

$$\begin{aligned}
\mathrm{d_{TV}}(\mathcal{L}(X_1)\,,\mathcal{L}(Y_1)) &= \frac{1}{2}\sum_{\substack{n\in\mathbb{N}^m\\ n=(n_1,\dots,n_m)}}\big|\,\mathbb{P}[X_1=n]\;-\;\mathbb{P}[Y_1=n]\,\big|\\
&\le \frac{1}{2}\Bigg(\frac{\langle\lambda,|a\nu'-b\mu'|\rangle}{\mu'\nu'}\\
&\quad+\frac{1}{\mu'}\Bigg(\int_{S_{\mathbb{R}^m_+,\|\cdot\|}}\int_{(0,\infty)}\sum_{\substack{n\in\mathbb{N}^m\\ n=(n_1,\dots,n_m)}}\prod_{i=1}^m\frac{1}{n_i}\Big(\frac{\langle\lambda,v\rangle}{t+\langle\lambda,v\rangle}\Big)^{n_i}G_v(dt)\,\alpha(dv)\\
&\quad-\int_{S_{\mathbb{R}^m_+,\|\cdot\|}}\int_{(0,\infty)}\sum_{\substack{n\in\mathbb{N}^m\\ n=(n_1,\dots,n_m)}}\prod_{i=1}^m\frac{1}{n_i}\Big(\frac{\langle\lambda,v\rangle}{t+\langle\lambda,v\rangle}\Big)^{n_i}H_v(dt)\,\gamma(dv)\Bigg)\\
&\quad+\left|\frac{\mu'-\nu'}{\mu'\nu'}\right|\int_{S_{\mathbb{R}^m_+,\|\cdot\|}}\int_{(0,\infty)}\sum_{\substack{n\in\mathbb{N}^m\\ n=(n_1,\dots,n_m)}}\prod_{i=1}^m\frac{1}{n_i}\Big(\frac{\langle\lambda,v\rangle}{t+\langle\lambda,v\rangle}\Big)^{n_i}H_v(dt)\,\gamma(dv)\Bigg).
\end{aligned}$$

A comparison with the derivative in the integrals in Equation (11.23) shows that

$$\frac{\partial^{n_1+\dots+n_m}}{\partial z_1^{n_1}\dots\partial z_m^{n_m}}\ln\Big(\frac{t+\langle\lambda(1-z),v\rangle}{t+\langle\lambda,v\rangle}\Big)=\prod_{i=1}^m\frac{\langle\lambda,v\rangle^{n_i}(n_i-1)!}{(t+\langle\lambda(1-z),v\rangle)^{n_i}},$$

hence we see that this series is the multivariate Taylor expansion at 0 applied to 1, and thus

$$T(1,0)=\sum_{\substack{n\in\mathbb{N}^m\\ n=(n_1,\dots,n_m)}}\prod_{i=1}^m\frac{1}{n_i!}\Big((1-0)\frac{\langle\lambda,v\rangle}{t+\langle\lambda(1-1),v\rangle}\Big)^{n_i}(n_i-1)!=\ln\Big(\frac{t+\langle\lambda,v\rangle}{t}\Big).$$

Hence we obtain

$$\begin{aligned}
\mathrm{d_{TV}}(\mathcal{L}(X_1)\,,\mathcal{L}(Y_1)) &\le \frac{1}{2}\Bigg(\frac{\langle\lambda,|a\nu'-b\mu'|\rangle}{\mu'\nu'}\\
&\quad+\frac{1}{\mu'}\Bigg(\int_{S_{\mathbb{R}^m_+,\|\cdot\|}}\int_{(0,\infty)}\ln\Big(\frac{t+\langle\lambda,v\rangle}{t}\Big)G_v(dt)\,\alpha(dv)\\
&\quad-\int_{S_{\mathbb{R}^m_+,\|\cdot\|}}\int_{(0,\infty)}\ln\Big(\frac{t+\langle\lambda,v\rangle}{t}\Big)H_v(dt)\,\gamma(dv)\Bigg)\\
&\quad+\left|\frac{\mu'-\nu'}{\mu'\nu'}\right|\int_{S_{\mathbb{R}^m_+,\|\cdot\|}}\int_{(0,\infty)}\ln\Big(\frac{t+\langle\lambda,v\rangle}{t}\Big)H_v(dt)\,\gamma(dv)\Bigg)\\
&\le \frac{1}{2}\Bigg(\frac{\langle\lambda,|a\nu'-b\mu'|\rangle}{\mu'\nu'}+\frac{\mu-\nu}{\mu'}+\left|\frac{\mu'-\nu'}{\mu'}\right|\Bigg).
\end{aligned}$$

Since w.l.o.g. $\min\{\big|\sqrt{\mu'}-\sqrt{\nu'}\big|,\big|\mu'-\nu'\big|\}\le\big|\mu'-\nu'\big|$ and $\min\{\mu',\nu'\}\le\mu'$, we have by insertion into Equation (11.22)

$$\mathrm{d_{TV}}(\mathcal{L}(M)\,,\mathcal{L}(N))\le\frac{3}{2}\big|\mu'-\nu'\big|+\frac{1}{2}(\mu-\nu)+\frac{1}{2}\frac{\langle\lambda,|a\nu'-b\mu'|\rangle}{\mu'\nu'},$$

which completes the proof. q.e.d.

We thereby obtain the following algorithm for the multivariate case:

Algorithm 11.25. Due to Corollary 11.18 an approximation of a multivariate generalized gamma convolution is possible and Lemma 11.20 gives an error bound with respect to the total variation distance which also provides an estimate for value-at-risk, cf. Equation (10.1). The approximation of the Thorin measure G_v can be done as in Algorithm 10.42. α can be approximated as in Corollary 11.18. Hence the approximation of a random sum $S = \sum_{i=1}^{m} \sum_{h=1}^{N_i} X_{i,h}$ by some random sum $S_{n(p)} = \sum_{i=1}^{m} \sum_{h=1}^{N_{i,n(p_i)}} X_{i,h}$ should be clear, where for each component $i \in \{1, \dots, m\}$ there are p_i steps of approximation necessary and $n(p) = \sum_{i=1}^{m} n(p_i)$. Let for simplicity $a_i = 0$ for $i \in \{1, \dots, m\}$. The random variables $N_{i,n(p_i)}$ have a Poisson distribution mixed over finite gamma convolutions for each $i \in \{1, \dots, m\}$. By Corollary 11.19 we obtain an alternative representation of $S_{n(p)}$, i.e.

$$S_{n(p)} = \sum_{i=1}^{m} \sum_{h=1}^{N_{i,n(p_i)}} X_{i,h} \overset{\mathrm{d}}{=} \sum_{i=1}^{m} \sum_{j=1}^{n(p)} \sum_{h=1}^{Q_j^{(n(p))}} B_{i,j,h}^{(n(p))} X_{i,j,h},$$

where $\{X_{i,h}\}_{h \in \mathbb{N}}$ and $\{X_{i,j,h}\}_{j,h \in \mathbb{N}}$ are sequences with identical distributions for each $i \in \{1, \dots, m\}$. Then the distribution of $S_{n(p)}$ can be evaluated as in Algorithm 5.9; the distribution of $\sum_{i=1}^{m} B_{i,j,1}^{(n(p))} X_{i,j,1}$ may be determined by the law of total probability. Then we proceed along the lines of Algorithm 10.42. Let

$$q_j^{n(p)} = \frac{\sum_{d=1}^{m} \lambda_d v_{d,j}^{n(p)}}{\beta_j^{n(p)} + \sum_{d=1}^{m} \lambda_d v_{d,j}^{n(p)}} \qquad \text{for } j \in \{1, \dots, n(p)\}.$$

Since $Q_j^{(n(p))} \sim \mathrm{NegBin}\big(\alpha_j^{n(p)}, q_j^{n(p)}\big)$, by Remark 2.7 this distribution is equivalent to a compound Poisson distribution as follows

$$Q_j^{(n(p))} \overset{\mathrm{d}}{=} \sum_{h=1}^{P_j^{(n(p))}} L_{j,h}^{(n(p))},$$

where

$$P_j^{(n(p))} \sim \mathrm{Poisson}\left(\alpha_j^{n(p)} \ln\left(\frac{1}{1 - q_j^{n(p)}}\right)\right)$$

and $\{L_{j,h}^{(n(p))}\}_{h \in \mathbb{N}}$ is a sequence independent of $P_j^{(n(p))}$ for each $j \in \{1, \dots, n(p)\}$, consisting of i.i.d. random variables such that $L_{j,1}^{(n(p))} \sim \mathrm{Log}\big(q_j^{n(p)}\big)$.

Thus

$$S_{n(p)} \overset{\mathrm{d}}{=} \sum_{j=1}^{n(p)} \sum_{h=1}^{P_j^{(n(p))}} S_{j,h}$$

holds, where $S_{j,h} = \sum_{k=1}^{L_{j,h}^{(n(p))}} \sum_{i=1}^{m} X_{i,j,k} B_{i,j,k}^{n(p)}$ for $h \in \mathbb{N}$. The distribution of $S_{j,1}$ may be evaluated by a numerically stable Panjer recursion. Thus $S_{n(p)}$ is the sum of independent compound Poisson sums. This can be rewritten as follows for $j \in \{1, \dots, n(p)\}$

$$T_j = \sum_{h=1}^{P_j^{(n(p))}} S_{j,h}$$

has a compound Poisson distribution and

$$S_{n(p)} = \sum_{j=1}^{n(p)} T_j$$

has a compound Poisson distribution with $S_{n(p)} \overset{\mathrm{d}}{=} \sum_{j=1}^{R} K_j$ where

$$R \sim \text{Poisson}(\gamma) \qquad \text{and} \qquad \mathcal{L}(K_1) \overset{\text{a.s.}}{=} \frac{1}{\gamma} \sum_{j=1}^{n(p)} \alpha_j^{n(p)} \ln\left(\frac{1}{1 - q_j^{n(p)}} \right) \mathcal{L}(S_{j,1})$$

with

$$\gamma = \sum_{j=1}^{n(p)} \alpha_j^{n(p)} \ln\left(\frac{1}{1 - q_j^{n(p)}} \right).$$

Hence the distribution of $S_{n(p)}$ can be evaluated by a numerically stable Panjer recursion.

Remark 11.26. Using this algorithm, it is also possible to compute an approximation of conditional expected shortfall. Using the alternative representation of an appropriate approximation, without any further adaptations we can apply the results presented in Sections 6.2 and 6.3.

Chapter 12

Numerical Examples of the Limitations of the Fourier Transform

For illustration purposes we also give some examples that show the usefulness of our new algorithm. As already mentioned, in our model we consider a compound Poisson mixture distribution with the following constellation:

$$S = \sum_{h=1}^{N} X_h,$$

where $\{X_h\}_{h\in\mathbb{N}}$ is a sequence independent of N, consisting of i.i.d. random variables. The random variable N has a Poisson mixture distribution such that

$$\mathcal{L}(N\,|\,\Lambda) \overset{\text{a.s.}}{=} \text{Poisson}(\lambda\Lambda)\,,$$

where the distribution of Λ is a generalized gamma convolution with Thorin measure U and parameter a. After choosing an error bound $\varepsilon > 0$ in Corollary 10.31, we can use this measure U to approximate it with a discretization V_m for $m \in \mathbb{N}$ as described in Algorithm 10.42 and evaluate the approximating distribution. We compare this approach to the fast Fourier transform (FFT). For the reader's convenience we repeat the algorithm here as it is described in [19]: let $p_n = \mathbb{P}[S = n]$ and $f_n = \mathbb{P}[X_1 = n]$ for $n \in \mathbb{N}_0$ and p_n^M the approximation of p_n by FFT. First, note that the characteristic function of the distribution of S can be written with respect to the characteristic function of the claim size distribution

$$\varphi_S(t) = \mathbb{E}\big[\mathrm{e}^{\mathrm{i}\,tS}\big] = G_N(\varphi_{X_1}(t)), \qquad t \in \mathbb{R},$$

where G_N denotes the probability-generating function of the distribution of N. Then choose a truncation point $M \in \mathbb{N}$. The discrete Fourier transform $\hat{f} = (\hat{f}_0, \dots, \hat{f}_{M-1})$ is defined by

$$\hat{f}_j = \sum_{k=0}^{M-1} f_k\, \mathrm{e}^{\mathrm{i}\,2\pi jk/M}, \qquad j \in \{0, \dots, M-1\}.$$

Then compute the inverse discrete Fourier transform of $\hat{p}^M = G_N(\hat{f})_{j\in\{0,\dots,M-1\}}$ by

$$p_j^M = \frac{1}{M} \sum_{k=0}^{M-1} \hat{p}_k^M\, \mathrm{e}^{-\,\mathrm{i}\,2\pi jk/M}, \qquad j \in \{0, \dots, M-1\}.$$

The computational effort required by FFT is far lower than for our algorithm, but the quality of the result might be critical due to aliasing errors, which seem to be tremendous in some cases. Especially heavy-tailed distributions require a truncation point M that is much higher than the numbers of the first n atoms. A substantial improvement is FFT with a change of measure, the exponential tilting, cf. [26] and [19]. Therefore, for some $\theta > 0$ consider the following tilting operator

$$E_\theta f = (\mathrm{e}^{-\theta j} f_j)_{j\in\{0,\ldots,M-1\}}.$$

Then FFT with exponential tilting requires the tilting of the sequence $f \mapsto E_\theta f$; compute first the discrete Fourier transform of $E_\theta f$ and then the inverse discrete Fourier transform of $G_N(\hat{E_\theta f})$. Finally untilt by applying $E_{-\theta}$.

However, in both cases the probability-generating function of the distribution of N has to be evaluated. This means

$$G_N(z) = \mathbb{E}\big[z^N\big] = \mathbb{E}\big[\mathrm{e}^{-\lambda\Lambda(1-z)}\big] = L_\Lambda(\lambda(1-z)), \qquad z \in [0,1],$$

which is the Laplace transform of a generalized gamma convolution, cf. Equation (9.11). Hence in each step an integral has to be evaluated.

Grübel and Hermesmeier [26, Equation (2.12)] give an a posteriori error estimate for the fast Fourier transform with truncation point M: if the second moment of the distribution of the claim number N is finite, then

$$\sum_{n=0}^{M-1} \big|p_n^M - p_n\big| = \sum_{n=M}^{\infty} p_n - \mathbb{E}[N] \sum_{n=M}^{\infty} f_n + O\bigg(\Big(\sum_{n=M}^{\infty} f_n\Big)^2\bigg) \tag{12.1}$$

holds.

As [19] points out, FFT is usually faster than Panjer's recursion (for $n \geq 256$). As is commonly known (cf. [19]), FFT usually needs just $O(n \log n)$ operations while Panjer's recursion requires $O(n^2)$ operations. Since we have to do this $m+1$ times (depending on the accuracy) in order to obtain the probability-generating function of the linear combination of the actual probability-generating function, cf. Equation (10.43), we end with $O((m+1)n^2)$ operations. The estimation of an appropriate V_m takes $O(m^2)$ operations. Hence, the complexity is $O(n^3)$.

In what follows, we consider two cases that provide interesting results.

12.1 A Comparison with the Exact Result

We consider the τ-tempered α-stable distribution as in Example 9.20. The exact evaluation of such a compound Poisson mixture distribution can be found in [21, Algorithm 5.12]. We compare this result to results obtained by fast Fourier transform, fast Fourier transform with exponential tilting, and our algorithm. Exemplary computations show that both FFT and FFT with exponential tilting can produce significant errors whereas the error that arises from our algorithm can be controlled in any case.

12.1.1 Fast Fourier Transform

The class of τ-tempered α-stable distributions covers a rich class of distribution families. One well-known distribution class is the Lévy distribution where $\alpha = 1/2$ and $\tau = 0$. For

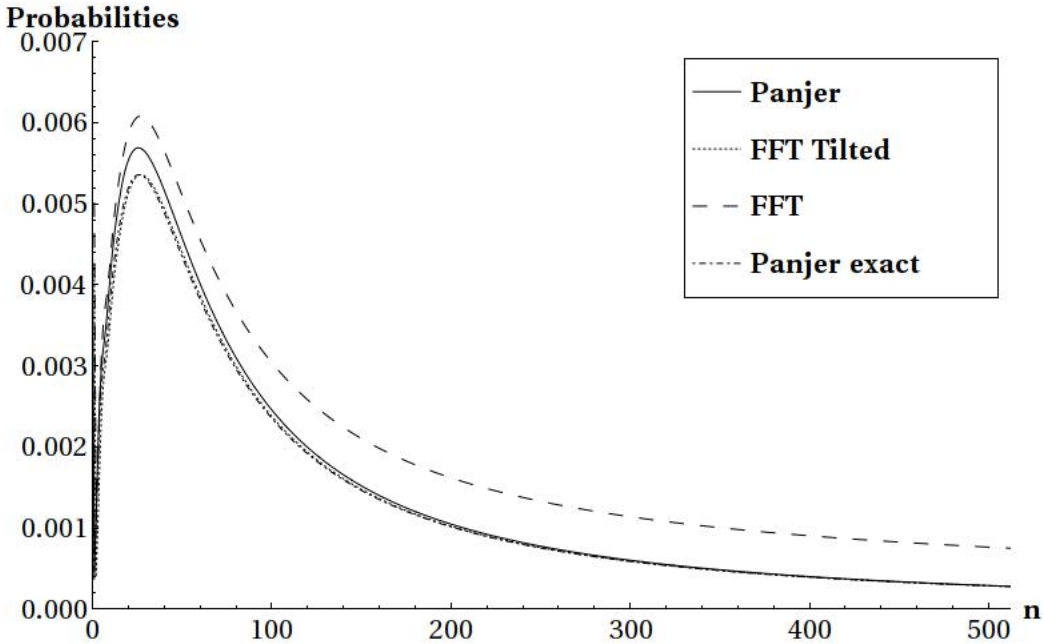

Figure 12.1: Approximations of Poisson($5\Lambda_{0,0}$) $\vee$ Poisson(5) *with upper error bound* $\varepsilon = 0.1$ *for Panjer approximation in Algorithm 10.42. FFT Tilted and Panjer exact almost coincide, cf. Table 12.1.*

Difference exact value and approximating Panjer recursion	0.0320578
Difference exact value and FFT	0.29829869350
Difference exact value and FFT with exponential tilting	0.00000000369

Table 12.1: Absolute differences between the probabilities w.r.t. the Lévy distribution.

further information on the Lévy distribution cf. Example 9.22. This distribution is especially interesting because a random variable $X \sim S_{1/2}(\sigma, 1, 0)$ does not have an expected value (it is not finite) and the second moments are infinite (cf. [55, Property 1.2.16]), too. Hence the error estimate (12.1) for FFT is not applicable. Yet, our algorithm still provides an error estimate. For an example of this distribution family we choose $\sigma = 3$, $M = 512$, $\lambda = 5$ and $X_1 \sim$ Poisson(5). The upper bound of the error is chosen as $\varepsilon = 0.1$ and the number of steps of the approximation results in $n = 8193$. The tilting parameter is set to $\theta = 20/M$ as [19] suggests. The result for $j \in \{0, \ldots, 511\}$ is shown in Figure 12.1. Obviously, FFT does not perform very well, in contrast to FFT with exponential tilting and our algorithm. The absolute differences for the computed values are depicted in Table 12.1. Although the distribution of X_1 is very light-tailed, FFT does not perform well. This is in contrast to [19] that says that the wrap-around error might be an issue for considerable tail mass. If the distribution of X_1 is changed to have heavier tails, then the quality of the performance of FFT is even worse. The reason for this misbehaviour might be the fact that the Lévy distribution does not have a mean, i.e., the tail mass might be considerable.

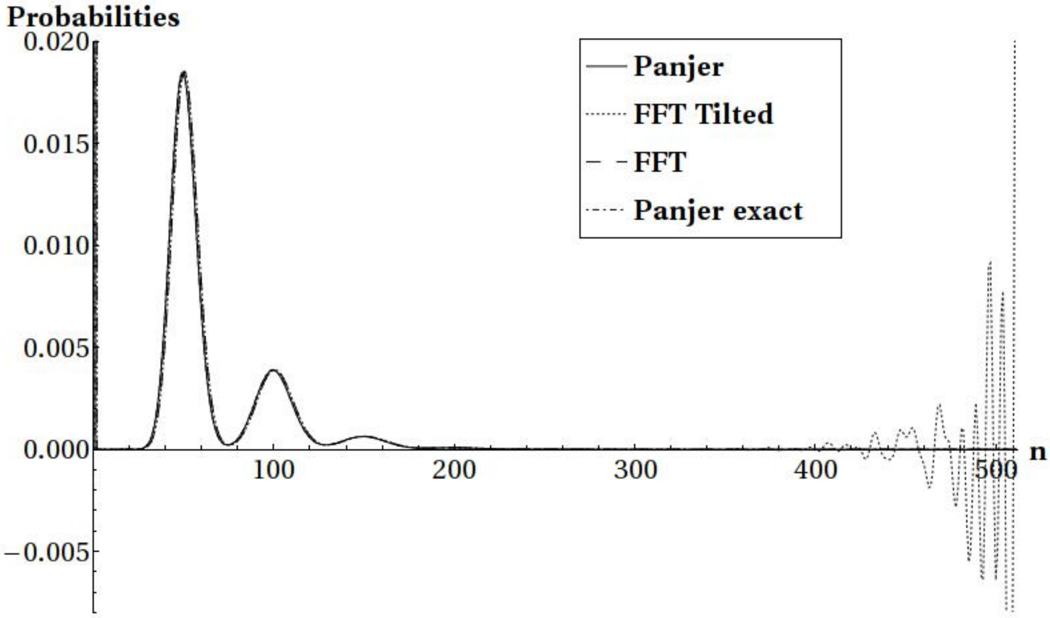

Figure 12.2: Approximations of Poisson($5\Lambda_{\tau,0}$) $\vee$ Poisson(50) *with error bound* $\varepsilon = 0.01$ *for Panjer approximation in Algorithm 10.42. Panjer, Panjer exact, and FFT almost coincide, cf. Table 12.3.*

12.1.2 Fast Fourier Transform with Exponential Tilting

As has already been pointed out in [19], heavy-tailed severity distributions can become a considerable issue for the fast Fourier transform as Equation (12.1) shows. Grübel and Hermesmeier [26] suggest applying the fast Fourier transform with exponential tilting in this case. Experiments have shown however that even then there are cases in which the fast Fourier transform with exponential tilting becomes quite unstable. The parameters are chosen as follows. The τ-tempered α-stable distribution $\Lambda_{\tau,0} \sim F_{\alpha,\sigma,\tau,0}$ is determined by the parameters $\alpha = 0.8$, $\sigma = 1.2$, and $\tau = 10\,000\,000$. The Poisson parameter of the random variable N is $\lambda = 5$, and for the severity distribution we choose $X_1 \sim$ Poisson(50). The truncation point is $M = 512$ and the tilting parameter $\theta = 20/M$. The upper bound of the error is $\varepsilon = 0.01$ and this requires an approximation of the measure in Equation (9.21) with $n = 2$ steps.

The approximation then reads

$$U_2 = 6729.66\delta_{1.00945*10^7} + 1.08712 \cdot 10^{17}\delta_{9.22336*10^{17}}$$

Since this is a very light-tailed distribution (more than 55% of the total mass of the distribution are on the event of no default), Figure 12.2 depicts only the atoms $j \in \{1, \ldots, 511\}$. We have $\mathbb{E}[N] \approx 29.8121$ which supports the correctness of the result because the mass should then be concentrated around $j = 30$.

The respective probabilities of no default are shown in Table 12.2. The absolute differences between the three approximations and the exact result are given in Table 12.3. It is striking that the fast Fourier transform with exponential tilting becomes quite unstable for large j. Due to the negative values this is even not a probability mass function. Tests,

Exact value	0.550878
Approximation by Panjer's Recursion	0.552853
FFT	0.550878
FFT with exponential tilting	0.550878

Table 12.2: Probabilities of the event of no default of an extreme distribution of $\Lambda_{\tau,0}$ with $\tau = 10\,000\,000$.

Difference exact value and approximating Panjer recursion	0.00395013
Difference exact value and FFT	0.00000000906361
Difference exact value and FFT with exponential tilting	0.404005

Table 12.3: Absolute differences between the probabilities w.r.t. an extreme distribution of $\Lambda_{\tau,0}$ with $\tau = 10\,000\,000$.

where the random variable $\Lambda_{\tau,0}$ is replaced by its expected value, suggest that this instability originates from the stochastic parameter of the Poisson distribution.

The fast Fourier transform with exponential tilting seems to be subject to some numerical instability. It was quite easy to find other examples, where the result was not a probability mass function either.

Finding the source of instability is rather tedious work. The probability-generating function of the distribution of N can be evaluated analytically, which excludes errors from numerical integration. An analysis of the discrete Fourier transform, where $f_k = \mathbb{P}[X_1 = k]$ for $k \in \mathbb{N}_0$, reveals the following for each $j \in \{0, \dots, M-1\}$

$$\hat{f}_j = \sum_{k=0}^{M-1} f_k \, \mathrm{e}^{\mathrm{i}\, 2\pi jk/M}$$

and

$$\begin{aligned} \hat{f}_{M-j} &= \sum_{k=0}^{M-1} f_k \, \mathrm{e}^{\mathrm{i}\, 2\pi (M-j)k/M} = \sum_{k=0}^{M-1} f_k \, \mathrm{e}^{\mathrm{i}\, 2\pi k} \, \mathrm{e}^{-\mathrm{i}\, 2\pi jk/M} \\ &= \sum_{k=0}^{M-1} f_k \, \mathrm{e}^{-\mathrm{i}\, 2\pi jk/M} . \end{aligned}$$

Thus we conclude for $j \in \{1, \dots, M-1\}$

$$\operatorname{Re}(\hat{f}_j) = \operatorname{Re}(\hat{f}_{M-j}) \quad \text{and} \quad \operatorname{Im}(\hat{f}_j) = -\operatorname{Im}(\hat{f}_{M-j}). \tag{12.2}$$

The application of the probability-generating function of the distribution of N preserves these relations and for $j \in \{1, \dots, M-1\}$ we obtain

$$\operatorname{Re}(G_N(\hat{f}_j)) = \operatorname{Re}(G_N(\hat{f}_{M-j})) \quad \text{and} \quad \operatorname{Im}(G_N(\hat{f}_j)) = -\operatorname{Im}(G_N(\hat{f}_{M-j})). \tag{12.3}$$

The analysis of a single probability that is obtained by the inverse Fourier transform shows that the summands of it satisfy for $j \in \{1, \dots, M-1\}$ and $k \in \{0, \dots, M-1\}$

$$\begin{aligned}\operatorname{Re}(G_N(\hat{f}_k)\,\mathrm{e}^{-\mathrm{i}\,2\pi jk/M}) &= \operatorname{Re}(G_N(\hat{f}_{M-k})\,\mathrm{e}^{-\mathrm{i}\,2\pi jk/M}) \quad \text{and}\\ \operatorname{Im}(G_N(\hat{f}_k)\,\mathrm{e}^{-\mathrm{i}\,2\pi jk/M}) &= -\operatorname{Im}(G_N(\hat{f}_{M-k})\,\mathrm{e}^{-\mathrm{i}\,2\pi jk/M}).\end{aligned} \tag{12.4}$$

Conducting the fast Fourier transform with exponential tilting and analyzing the single steps for these relations shows that Equation (12.2) still holds. An analysis of Equation (12.3) reveals that the problem should be found in the probability-generating function of the Poisson mixture distribution. The probability-generating function is

$$\begin{aligned}G_N(z) = \mathbb{E}\big[z^N\big] &= \mathbb{E}\big[\,\mathbb{E}\big[z^N\,|\,\Lambda\big]\,\big] = \mathbb{E}\big[\mathrm{e}^{-\lambda\Lambda(1-z)}\big]\\ &= \exp(-\gamma_{\alpha,\sigma}((\lambda(1-z)+\tau)^\alpha - \tau^\alpha)).\end{aligned}$$

For very large τ this difference becomes unstable, as the difference operation is generally an unstable operation in numerical mathematics. There are several approaches to avoid the computation of such a difference. A comparison with the exact result shows that using numerical integration and setting the working precision up to 30 digits provides much better results. A more elegant solution is the use of the binomial series

$$\begin{aligned}(\lambda+\tau)^\alpha - \tau^\alpha &= \tau^\alpha\Big(\Big(\frac{\lambda+\tau}{\tau}\Big)^\alpha - 1\Big) = \tau^\alpha\Big(\Big(1+\frac{\lambda}{\tau}\Big)^\alpha - 1\Big)\\ &= \tau^\alpha \sum_{k=1}^{\infty} \binom{\alpha}{k}\Big(\frac{\lambda}{\tau}\Big)^k.\end{aligned}$$

If τ is large enough, then this series converges quite quickly. Equation (12.3) still holds, but a comparison of these two approaches of computing the probability-generating function illustrates that the respective equations differ by an order of up to 10^{-4}. Accordingly, the respective Equations (12.4) no longer agree either. This is an error that might also happen with fast Fourier transform, but using fast Fourier transform with exponential tilting requires multiplying by $\mathrm{e}^{\theta j}$ for $j \in \{0, \dots, M-1\}$. In the worst case we multiply by $\mathrm{e}^{20} \approx 4.85 \cdot 10^8$. This could be the reason for instability. The improvement of the calculations due to the newly set working precision can be found in Figure 12.3. The difference between the exact values and the value obtained by FFT with exponential tilting then adds up to only $3.40782 \cdot 10^{-8}$. However, it is important to note that this approach does not work in case $\tau = 0$. This holds especially for the Lévy distribution. But this should not be a problem since we do not have this sort of instability in this case.

12.2 A Comparison without Exact Result

As another meaningful example we take the Pareto distribution as a mixing distribution. The Pareto distribution could be an interesting alternative as a default cause intensity to the gamma distribution, which is quite often used as a default cause intensity. The reason why it might be a natural extension is that it is an exponential transformation of a gamma distribution, cf. [49, p. 98]. Therefore, for a portfolio with a large number of components, the sum of the single default probabilities might became very large. This might make it necessary to switch from a rather light-tailed distribution as the gamma distribution to a more heavy-tailed distribution. For further information on the Pareto distribution cf.

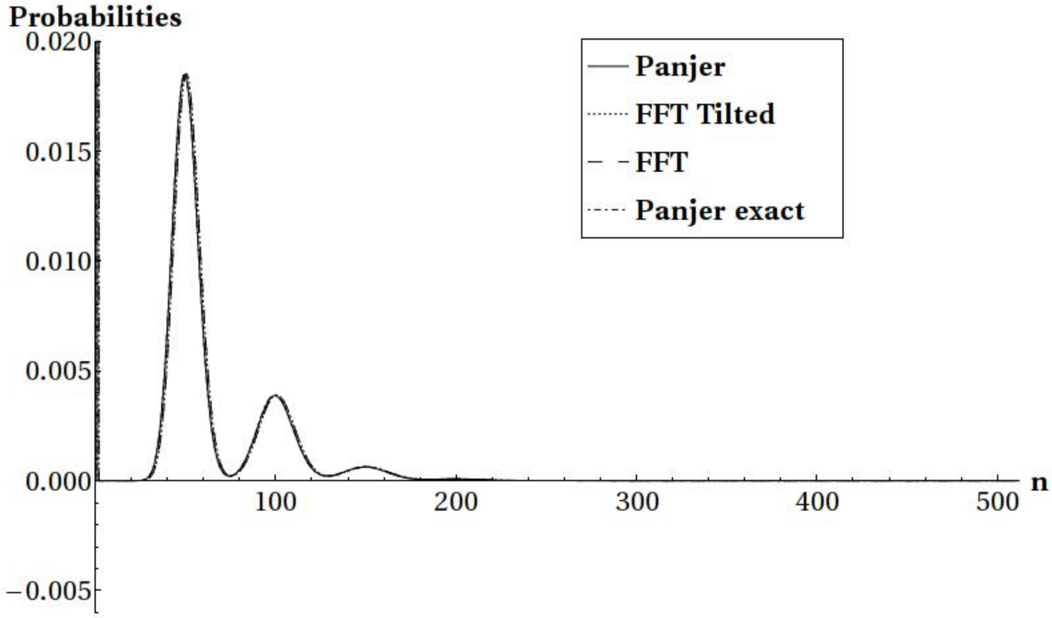

Figure 12.3: Approximations of Poisson$(5\Lambda_{\tau,0}) \vee$ Poisson(50) *with error bound* $\varepsilon = 0.01$ *for Panjer approximation in Algorithm 10.42 and higher precision. Now all algorithms coincide.*

Example 9.23. The expected value of Λ does not exist for $h \in (0,1)$ and the variance does not exist for $h \in (0,2)$ (cf. [37, p. 577]). Hence the error estimate in Equation (12.1) does not apply in this case.

We give two examples where either FFT or FFT with exponential tilting do not seem to work properly, but our algorithm does. Although there is always one FFT that works, determining which one works is tedious. Our algorithm is independent of that.

12.2.1 Fast Fourier Transform

Let the sequence of i.i.d. random variables $\{X_h\}_{h\in\mathbb{N}}$ have a distribution such that $X_1 \sim$ Poisson(25). This is a rather light-tailed distribution. Let $\lambda = 20$ and $\Lambda \sim$ Pareto$(0.6, 1.4)$. We choose the accuracy of the approximation to be $\varepsilon = 0.01$. The approximation of U by $\{V_n\}_{n\in\mathbb{N}}$ results in $n = 513$ steps. Since U admits a density (cf. [67, Equation (3.19)]), we do not need to deal with points of discontinuity. We compute the probabilities of the first $M = 512$ atoms and choose the tilting parameter $\theta = 20/M$ as [19] suggests. The results for $j \in \{1,\dots,511\}$ are depicted in Figure 12.4, the values of the probability of no default are shown in Table 12.4. The differences between the approximation via Panjer's recursion and the fast Fourier transforms can be seen in Table 12.5. As the difference between the approximation via Panjer's recursion and FFT is much bigger than the chosen error bound of $\varepsilon = 0.01$, we should suppose that FFT provides a very critical result and needs a truncation point that is much higher than the number of computed probabilities.

This is a rather surprising result because the fast Fourier transform should work well with light tailed severities as can be concluded from Equation (12.1). Both FFT and FFT with exponential tilting show slightly complex valued numbers.

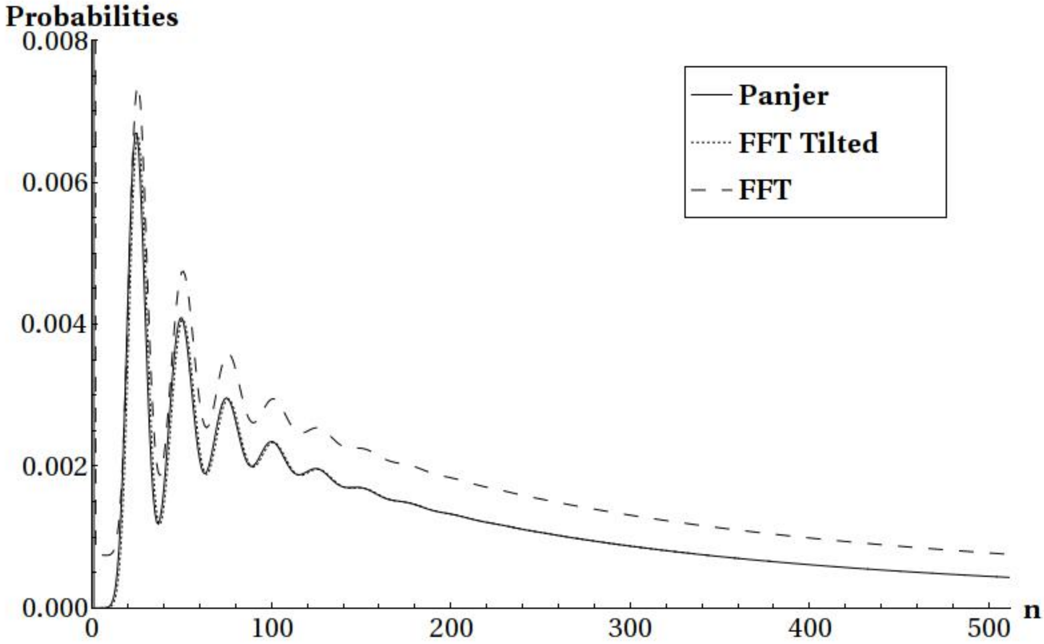

Figure 12.4: Approximations of Poisson$(20\Lambda) \vee$ Poisson(25) *where* $\Lambda \sim$ Pareto$(0.6, 1.4)$ *with error bound* $\varepsilon = 0.01$ *for Panjer approximation in Algorithm 10.42. Panjer and FFT Tilted almost coincide, cf. Table 12.5.*

Approximation by Panjer's Recursion	0.0991706
FFT	0.0989961
FFT with exponential tilting	0.0982446

Table 12.4: Probabilities of the event of no default w.r.t. the Pareto$(0.6, 1.4)$ *distribution.*

12.2.2 Fast Fourier Transform with Exponential Tilting

Let now the sequence of i.i.d. random variables $\{X_h\}_{h\in\mathbb{N}}$ have a distribution such that $X_1 \sim$ Poisson(30). This is also a rather light-tailed distribution. Let $\lambda = 20$ and $\Lambda \sim$ Pareto$(0.5, 2.5)$. We again choose the accuracy of the approximation of the Thorin measure U to be $\varepsilon = 0.01$. The approximation of U by V_n results in $n = 257$ steps and the approximating points are chosen according to Equation (10.41). We have $\beta_1 = 0.120384$ and $\beta_{257} = 24.5486$, so the approximation takes place on a small interval only. The approximating sequence can be seen in Figure 12.5 together with the sequence of the difference $U - V_n$.

Again, we approximate the distribution by our algorithm using Panjer's recursion, FFT and FFT with exponential tilting. The results can be found in Figure 12.6 for $j \in \{1, \dots, 511\}$. The values for $j = 0$ are presented in Table 12.6. It is quite clear that the graph of the probability mass function of the distribution computed by a fast Fourier transform with exponential tilting does not depict a probability mass function. The cumulated absolute differences between the results are given in Table 12.7. This example demonstrates that the FFT with exponential tilting is very sensitive. Analyzing the internal values of the FFT with

Difference Panjer's Recursion and FFT	0.247445
Difference Panjer's Recursion and FFT with exponential tilting	0.003546

Table 12.5: Absolute differences between the probabilities w.r.t. the Pareto$(0.6, 1.4)$ *distribution.*

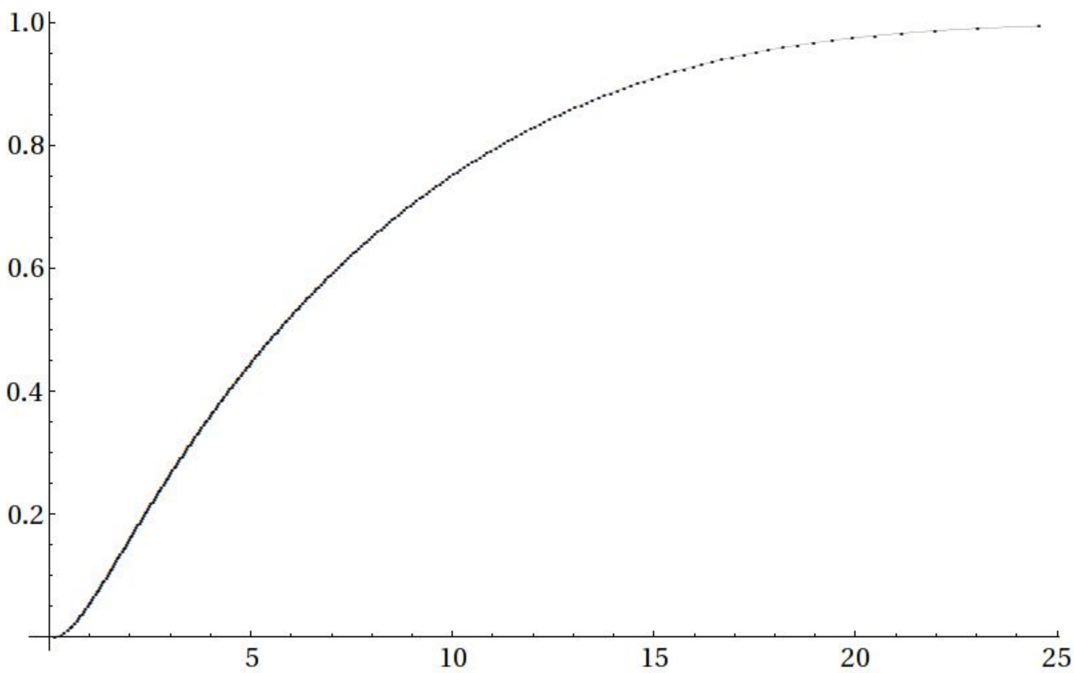

Figure 12.5: Approximations of the Thorin measure U of $\Lambda \sim$ Pareto$(0.5, 2.5)$.

exponential tilting shows that the evaluation of the transformed probability mass function of the claim sizes applied to the probability-generating function of the distribution of N is critical. The computations shown in Figure 12.6 have been done with a default working precision, which seems inadequate. The default working precision is machine precision that yields 16 significant digits (cf. [16, p. 276]). Setting the working precision to 17 significant digits yields a significant improvement, which can be seen in Figure 12.7. The cumulated absolute differences can be found in Table 12.8.

These examples have shown several advantages of our new algorithm; we are not dependent on the existence of moments and therefore have an error estimate of our approximation in each case. Moreover, our algorithm is numerically much more stable that fast Fourier transforms. We also only need to evaluate integrals once. This is especially useful if one changes the distribution of the claim sizes because in this case, the FFT has to evaluate the integrals once again. We also saw that FFT with exponential tilting is very sensitive in terms of this integration.

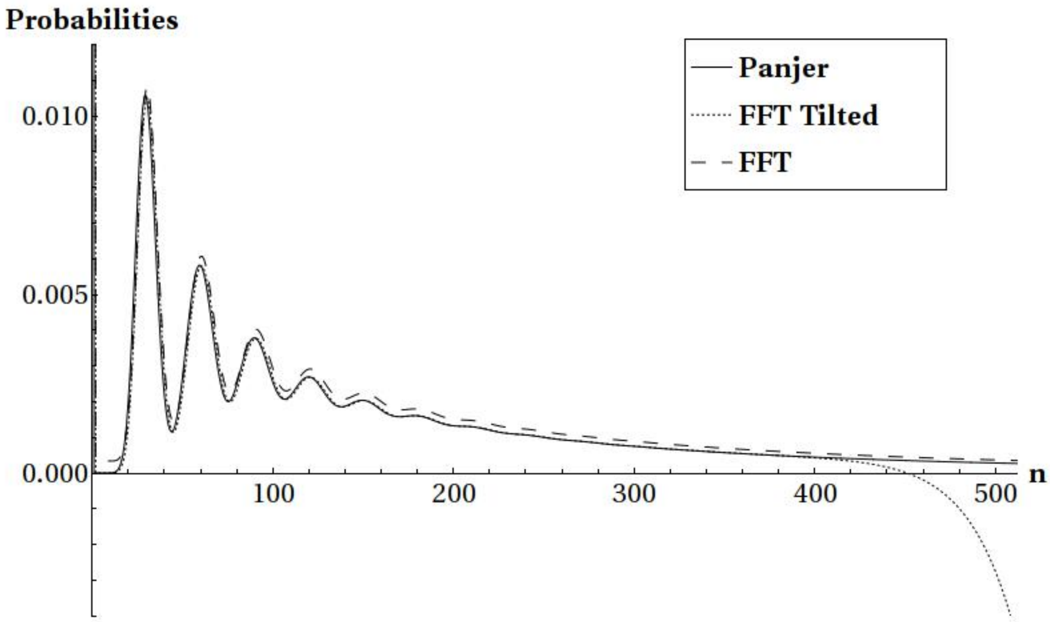

Figure 12.6: Approximations of Poisson(20Λ) $\vee$ Poisson(30) *where* $\Lambda \sim$ Pareto(0.5, 2.5) *with error bound* $\varepsilon = 0.01$ *for Panjer approximation in Algorithm 10.42. For small* n *Panjer and FFT Tilted are quite close, for larger* n *Panjer and FFT are closer. Altogether, both FFT and FFT Tilted are critical approximations for this distribution, cf. Table 12.7.*

Approximation by Panjer's Recursion	0.190276
FFT	0.188799
FFT with exponential tilting	0.188452

Table 12.6: Probabilities of the event of no default w.r.t. the Pareto(0.5, 2.5) *distribution.*

Difference Panjer's Recursion and FFT	0.0871913
Difference Panjer's Recursion and FFT with exponential tilting	0.127476

Table 12.7: Absolute differences between the probabilities w.r.t. the Pareto(0.5, 2.5) *distribution.*

Difference Panjer's Recursion and FFT	0.0871913
Difference Panjer's Recursion and FFT with exponential tilting	0.0045705

Table 12.8: Absolute differences between the probabilities w.r.t. the Pareto(0.5, 2.5) *distribution with higher working precision.*

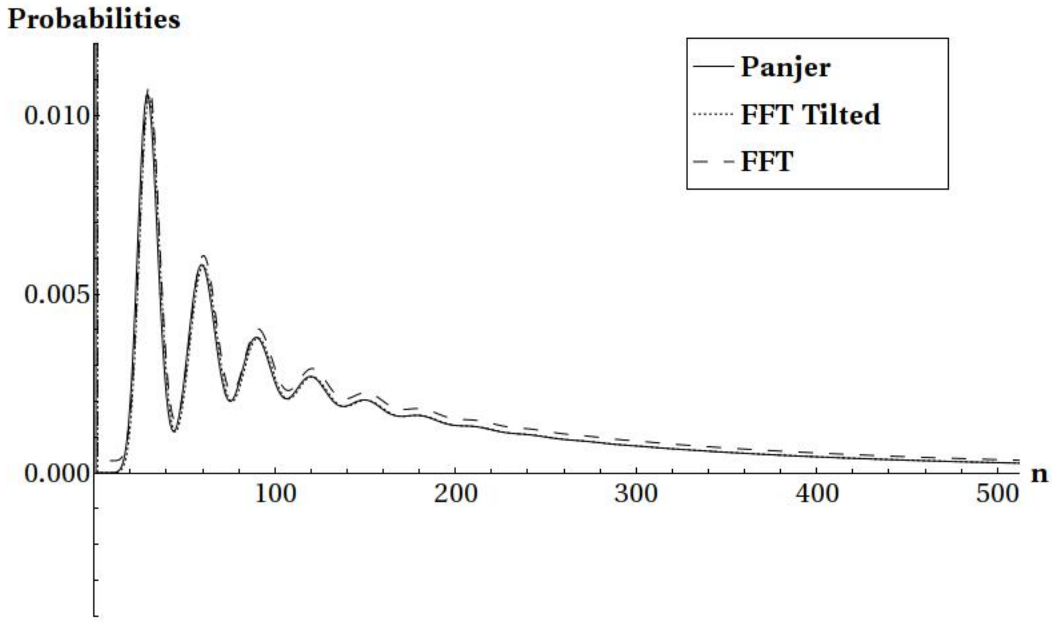

Figure 12.7: Approximations of $\mathrm{Poisson}(20\Lambda) \vee \mathrm{Poisson}(30)$ *where* $\Lambda \sim \mathrm{Pareto}(0.5, 2.5)$ *with error bound* $\varepsilon = 0.01$ *for Panjer approximation in Algorithm 10.42 and higher working precision. Now Panjer and FFT Tilted almost coincide, cf. Table 12.8.*

Glossary

$\mathrm{Ber}(p)$	Bernoulli distribution with $0 \le p \le 1$
$\mathrm{Beta}(\alpha, \beta)$	beta distribution with $\alpha, \beta > 0$
$\mathrm{Beta}'(\alpha, \beta)$	beta distribution of the second kind with $\alpha, \beta > 0$
$\mathrm{CLog}(q, F)$	compound logarithmic distribution with parameter $q \in (0,1)$ and distribution F, cf. also Definition 2.2
$\mathrm{CNegBin}(\alpha, p, F)$	compound negative binomial distribution with parameters $\alpha \ge 0$ and $p \in [0,1)$ and distribution F, cf. also Definition 2.2
$\mathrm{CPoi}(\mu, F)$	compound Poisson distribution with Poisson parameter $\mu \ge 0$ and distribution F, cf. also Definition 2.2
$\mathrm{d_{TV}}(F, G)$	total variation distance between the distributions F and G
$\mathrm{ExtNegBin}(\alpha, k, p)$	extended negative binomial distribution, cf. also Definition 4.14
$\mathrm{Gamma}(\alpha, \beta)$	gamma distribution with parameters $\alpha, \beta > 0$, a probability density is $f_{\alpha,\beta}(x) = \frac{\alpha^\beta}{\Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-\beta x}$ for $x > 0$
$L_X(t)$	Laplace transform of the distribution of a random variable X
$\mathcal{L}(X)$	law of a random variable X
$\mathrm{Log}(q)$	logarithmic distribution with $q \in (0,1)$, and its probability mass function is given by $f(k) = \frac{-1}{\ln(1-p)} \frac{p^k}{k}$ for $k \in \mathbb{N}$
$M_X(t)$	moment generating function of the distribution of a random variable X
$\mathrm{MultHyperGeom}(n, k_1, \dots, k_m)$	multivariate hypergeometric distribution with parameters $n \in \mathbb{N}$ and $k_1, \dots, k_m \in \mathbb{N}_0$, cf. also [39, Equation (39.1)]
$\mathrm{MultLog}(p_1, \dots, p_m)$	multivariate logarithmic distribution, cf. also Definition 2.13
$\mathrm{Multinomial}(n; p_1, \dots, p_m)$	multinomial distribution with $n \in \mathbb{N}$ and $0 \le p_1, \dots, p_m \le 1$
$\mathbb{N}_0$	natural numbers with zero, i.e., $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$

$\mathbb{N}$	natural numbers, i.e., $\mathbb{N} = \{1, 2, 3, \dots\}$
$\mathrm{NegBin}(\alpha, p)$	negative binomial distribution with parameters $\alpha > 0$ and $0 \le p < 1$, cf. also Definition 2.6
$\mathrm{NegMult}(\alpha, p_1, \dots, p_m)$	negative multinomial distribution with parameters $\alpha > 0$ and $0 \le p_1, \dots, p_m < 1$, cf. also Definition 2.6
$\mathrm{Pareto}(g, h)$	Pareto distribution with $g, h > 0$ with cumulative distribution function $F(x) = 1 - (1 + (x/g))^{-h}$ for $x \ge 0$
$G_X(z)$	probability-generating function of the distribution of a random variable X
$\varphi_X(t)$	characteristic function of the distribution of a random variable X
$\mathrm{Poisson}(\lambda)$	Poisson distribution with mean $\lambda \ge 0$
$\bar{\mathbb{R}}$	real numbers with infinite values, i.e., $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$
$\mathbb{R}_+$	non-negative real numbers, i.e., $\mathbb{R}_+ = [0, \infty)$
$S_\alpha(\sigma, \beta, \mu)$	stable distribution with $\alpha \in (0, 2]$, $\sigma > 0$, $\beta \in [-1, 1]$ and $\mu \in \mathbb{R}$, cf. also Section 3.2.1
$F_{\alpha,\sigma,\tau,m}$	generalized stable distribution with $\alpha \in (0, 1)$, $\sigma > 0$, $\tau \ge 0$, and $m \in \mathbb{N}_0$, cf. also Section 3.2.1; $m = 0$ implies a τ-tempered α-stable distribution
$U(dt)$	Thorin measure U, cf. also Definition 9.10

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in preparation C. Rudolph and U. Schmock, *An Approximation via Panjer's Recursion for Poisson Mixture Models*