The Cori-Vauquelin-Schaeffer Bijection and its Applications in the Theory of Scaling Limits of Random Labelled Quadrangulations and Trees

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Introduction

The aim of the first part of this work is to establish a relationship between quadrangulations and certain classes of trees in order to gain deeper insights into the nature of the very same structures when chosen uniformly and at random at a large scale. The relationship in question is the so-called Cori-Vauquelin-Schaeffer Bijection (CVS bijection) which lays the foundation for most of the other results. In the second part of this thesis we study two probabilistic objects that can in a way be viewed as continuous analogues of discrete trees and quadrangulations. These objects are known as the Brownian Continuum Random Tree and the Brownian Map.

In the first chapter we define and discuss briefly the fundamental, discrete structures that are used throughout the entire work. We introduce the notion of graphtheoretical maps and the important subclass of quadrangulations. Afterwards, we look at two special classes of trees - well-labelled and embedded trees - and their associated contour process which will turn out to be crucial later on.

Chapter 2 is dedicated to the construction of the CVS bijection between well-labelled trees and quadrangulations as well as between embedded trees and pointed quadrangulations. By looking at its workings in more detail, we will see that the CVS bijection not only establishes a one-to-one relation between the objects in question, but also preserves certain metric characteristics. This will allow us to transfer properties of trees to quadrangulations and vice versa.

Chapter 3 gives an elementary introduction into the theory of stochastic processes and convergence in distribution in order to be able to work with random walks and their continuous analogue in form of the Brownian motion. Brownian motion - being a special real-valued process - plays an important role in Donsker's theorem which postulates the existence of a limit of a sequence of certain random walks. This statement lays the foundation for the main theorem of this chapter which allows - roughly speaking - to describe a sequence of contour processes (as discussed in Chapter 2) in terms of a stochastic processes that is based on the Brownian motion.

The following chapter (Chapter 4) is concerned with certain metric spaces that try to reflect properties of discrete trees in a "continuous" way. These so-called real trees lead us to the study of the Brownian Continuum Random Tree - a probabilistic object that was first introduced by Aldous [2, 3, 4] around 1991 and which appears as limit of a sequence of properly rescaled, discrete random trees. The theorems concerned with this result will allow us to obtain scaling limit results for embedded trees and pointed quadrangulations in the last two chapters of this thesis.

Chapter 5 is entirely dedicated to the discussion of metric properties of large, random pointed quadrangulations. Our investigation makes us come across a rather unintuitive object called the Brownian snake that acts like a path-valued stochastic process and that plays an important role in the limiting description of the radius and profile of a sequence of uniformly distributed pointed quadrangulations. We will also touch briefly the topic of Integrated Super-Brownian Excursion which can be linked with the label distribution of a sequence of uniformly distributed embedded trees.

The final chapter (Chapter 6) raises the question whether there exists an object, similar in nature to the Brownian Continuum Random Tree, which appears as the limit of a sequence of uniformly chosen pointed quadrangulations. This limit is identified as the Brownian Map. At the end, we establish an equivalent description of this structure with the help of the Brownian Continuum Random Tree. This also closes the circle in terms of connecting the apparently different worlds of "continuous" trees and maps/quadrangulations as previously initiated by the CVS bijection for discrete trees and quadrangulations.

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Finally, my wholehearted gratitude for my mother's unselfish guidance, caringness and protection during my entire life cannot be measured in numbers or repaid by earthly deeds. Thank you very much.

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1 Preliminaries

In this first chapter we start by collecting the building blocks on which the rest of this thesis is based upon. This includes elementary concepts and notions from graph theory like planar maps, quadrangulations and certain classes of trees. These structures and their mutual interrelation will attract our attention during this entire work.

After having familiarized ourselves with these preliminaries we will introduce contour processes and associate them with trees.

1.1 Graphs, Maps and Quadrangulations

We start by recalling basic facts of graph theory.

Definition 1. An **undirected graph** is a triple (V, E, ι) where V is the set of vertices, E the set of edges and the mapping

$$\iota\colon \begin{cases} E & \to \{\{u, v\} : u, v \in V\} \\ e & \mapsto \{u, v\} \end{cases}$$

is the incidence relation between edges and unordered pairs of vertices.

The set of vertices V of a graph $G = (V, E, \iota)$ can also be written as $V(G)$ and the set of edges as $E(G)$ to avoid ambiguity. Likewise the incidence relation can be denoted with ι_G to emphasize the dependency on the graph G.

Definition 2. Let $G = (V, E, \iota)$ be an undirected graph. We say

- an edge e is a **multi-edge**, if there exists an edge $e' \neq e$ with $\iota(e) = \iota(e')$
- an edge e is **incident** to vertices u and v, if $\iota(e) = \{u, v\}$
- vertices u and v are **adjacent**, if $u \neq v$ and there exists and edge e with $\iota(e) = \{u, v\}$
- an edge e is a **loop**, if $|\iota(e)| = 1$.

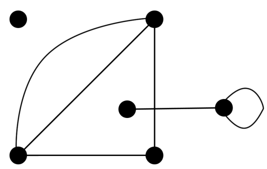


Figure 1.1: Undirected graph with a multi-edge and a loop

Definition 3. Let G be an undirected graph and $v \in V(G)$. Denote with $l_G(v)$ the number of loops incident to v and with $e_G(v)$ the number of edges (that are no loops) incident to v. The **degree** of v is then defined as

$$\deg_G(v) := 2 \cdot l_G(v) + e_G(v).$$

Lemma 1. *For every undirected graph $G = (V, E, \iota)$ the following formula holds*

$$\sum_{v \in V(G)} deg_G(v) = 2 \cdot |E(G)|.$$

Proof. Each edge contributes exactly 1 to the degree of the vertices it is incident to. If the edge is a loop then it is incident to only one vertex but the argument still remains the same. □

Definition 4. A **directed graph** G is a triple $G = (V, E, \iota)$ where again $V = V(G)$ is the set of vertices, $E = E(G)$ the set of edges and the mapping $\iota = \iota_G$ defined as

$$\iota\colon \begin{cases} E & \to V \times V \\ e & \mapsto (u, v) \end{cases}$$

is the incidence relation between edges and ordered pairs of vertices. For $e \in E$ and $\iota(e) = (u, v)$ we call u the **tail** and v the **head** of the edge e. In this context we will also use the notation $e = (u, v)$ to indicate the relation between a directed edge, its tail u and head v. Furthermore if $e = (u, v)$ we set $e^- := u$ and $e^+ := v$. Also, the notation of Definition 2 can be adopted for directed graphs almost without changes.

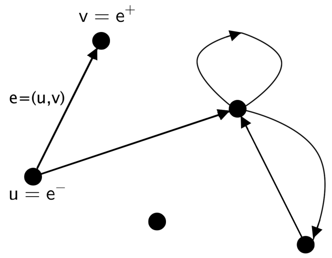


Figure 1.2: Naming conventions for a directed graph with a multi-edge and a loop

Definition 5. Let $G = (V, E, \iota)$ be a directed graph. For every $v \in V(G)$ set

$$\deg_G^+(v) := |\{e \in E(G) | \exists w : \iota(e) = (v, w)\}|,$$

$$\deg_G^-(v) := |\{e \in E(G) | \exists w : \iota(e) = (w, v)\}|,$$

$$\deg_G(v) := \deg_G^+ + \deg_G^-.$$

$\deg_G^+$ is the **outer degree** of G, $\deg_G^-$ is the **inner degree** of G and $\deg_G$ is the **degree** of G.

As with undirected graphs there is a simple relation between the number of edges and the outer/inner degree.

Lemma 2. *For every directed graph $G = (V, E, \iota)$ the following formula holds*

$$|E(G)| = \sum_{v \in V(G)} deg_G^+(v) = \sum_{v \in V(G)} deg_G^-(v).$$

Proof. Every edge contributes exactly the value 1 to the terms on the right side. □

Definition 6. Let $G = (V, E, \iota)$ be a graph and $v_0, ..., v_n \in V(G),\ e_1, ..., e_n \in E(G)$. If G is undirected we say that a sequence

$$p := (v_0, e_{1,} v_1, e_2, v_2, e_3, v_3, ..., e_n, v_n)$$

is a **path** from v_0 to v_n of length n if all vertices (except possibly the first and last) are distinct and

$$\iota(e_i) = \{v_{i-1}, v_i\} \tag{1.1.1}$$

for all $i \in \{1, ..., n\}$. Likewise if G is directed then p is a **directed path** if all vertices (except possibly the first and last) are distinct and

$$\iota(e_i) = (v_{i-1}, v_i)$$

for every $i \in \{1, ..., n\}$. If G is undirected and $v_0 = v_n$, we call p a **cycle**. Likewise if G is directed and $v_0 = v_n$, we call p a **directed cycle**. Two vertices $u, v \in V(G)$ are **connected** if there exists a (directed) path in G starting with u and ending at v. If every pair of vertices is connected, then we say that G is connected.

It is often useful to regard graphs as topological spaces, in which vertices are points and edges are disjoint paths between their endpoints, rather than combinatorial objects. However, to avoid excessive formality we will not distinguish between a graph and the corresponding topological graph.

We now recall the famous Jordan curve theorem.

Definition 7. A **Jordan curve** is the image of a continuous map ϕ from $[0,1]$ into the plane such that $\phi(0) = \phi(1)$ and the restriction of ϕ to $[0,1)$ is injective. A **Jordan arc** is the image of an injective continuous map of a closed interval into the plane.

Theorem 1 (Jordan's curve theorem)**.** *Let C be a Jordan curve in the plane. Then its complement consists of exactly two connected components. One of these components is bounded (the interior) and the other unbounded (the exterior). The curve C is the boundary of each component.*

Definition 8. An **embedding** of a graph G is an injective function ϕ from G into the $2-$sphere $\mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. More explicitly, an embedding maps the vertices of G to distinct points on $\mathbb{S}^2$ and edges of G to simple paths (between the image of their head and tail) on $\mathbb{S}^2$. If G is connected then ϕ is continuous.

Definition 9. A **map** is an embedding of a connected graph on $\mathbb{S}^2$. A **planar map** is a map such that no two edges cross each other. A **rooted map** is a map with a distinguished oriented edge e_*, which is called the **root edge**. The tail of e_* is called the **root vertex**. The class of all **rooted planar maps** is denoted by $\mathcal{M}$ and the subclass $\mathcal{M}_n$ contains all rooted planar maps with $n \geq 1$ edges.

The faces of a planar map m are the connected components of the complement of the union of the edges of m. This description can be made more accurately with the help of Jordans curve theorem:
Suppose again m is a planar map and consider its image $\pi(m)$ under the stereographic projection $\pi : \mathbb{S}^2 \backslash \{(0,0,1)\} \to \mathbb{R}^2$, where $\pi(x, y, z) := (\frac{x}{1-z}, \frac{y}{1-z})$. Notice that we can always rotate $\mathbb{S}^2$ appropriately so that the embedding m avoids the point $(0,0,1)$. Choosing a cycle C in $\pi(m)$, the union of the edges (as Jordan arcs) of C form a closed Jordan curve. Denote with P the points of the plane without the edges of $\pi(m)$. Jordan's curve theorem now allows the definition of an equivalence relation on P :

Definition 10. For points $u, v \in P$ define a relation $\sim_P$ on P by $u \sim_P v$ if, and only if, there exists a Jordan arc from u to v in P. Clearly, this is an equivalence relation. The image of the equivalence classes of $\sim_P$ under π^{-1} are called the **faces** of m, denoted by $F(m)$. For $f \in F(m)$ the number of edge-sides bordering f is called the **degree** of f and denoted with $\deg_m(f)$. It can happen that both sides of an edge are incident to the same face in which case the edge is counted twice in the degree of the face.

Notice that we use the same function name for the degree of vertices, edges and faces in order to simplify the notation.

Lemma 3. *For every planar map $m \in \mathcal{M}$ the following formula holds*

$$\sum_{f \in F(m)} deg_m(f) = 2 \cdot |E(m)|.$$

Proof. Each edge contributes the value 1 to the degree of exactly two different faces. $\square$

One of the most fundamental results concerning planar maps is Euler's Polyhedron formula. A proof can be found in almost any book about graph theory, for instance in **[33, Theorem 6.1]**.

Theorem 2 (Euler's Polyhedron formula)**.** *Let $m \in \mathcal{M}$ be a planar map. Then*

$$|V(m)| + |F(m)| - |E(m)| = 2.$$

In this thesis we will always work with maps seen up to "deformation", or more precisely:

Definition 11. If m and m' are maps, we say that they are **isomorphic** if there exists an orientation-preserving homeomorphism h on $\mathbb{S}^2$ such that $V(m') = h(V(m))$ and $E(m') = h(E(m))$. The mapping h is called a **(map-)isomorphism**.

Note that a map isomorphism not only preserves the incidence relation between vertices and edges but also the incidence relation between edges and faces.

Definition 12. Rooted maps m and m' with root edges e_* and e'_* are **isomorphic** if there exists a map isomorphism h (as in Definition 11) which additionally satisfies $h(e_*) = e'_*$.

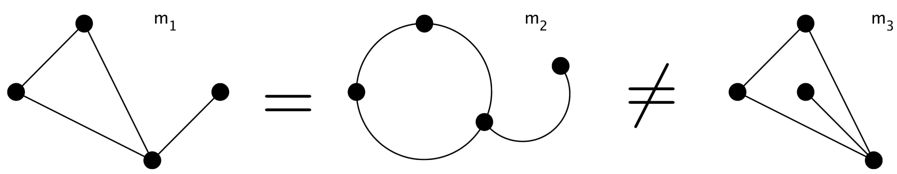


Figure 1.3: Isometric planar maps m_1 and m_2 which are both not isometric to m_3.

From now on we will always identify isomorphic maps. We also want to mention that the identification of maps up to isomorphisms allows the enumeration of these combinatorical structures. Therefore, we actually view the elements of $\mathcal{M} = \bigcup_{n \geq 1} \mathcal{M}_n$ as isometry classes of rooted planar maps.

Definition 13. Let m be a planar map. We denote with $d_m(u, v)$ the number of edges of the shortest path connecting u and v for all $u, v \in V(m)$ and call this the (natural) **distance** on m.

Note that $d_m : V(m) \times V(m) \to \mathbb{N}$ is a metric because m is connected.

Definition 14. Let $m \in \mathcal{M}$. Then the **maximal graph distance** from a vertex $v \in V(m)$ is defined as

$$R_m(v) := \max_{w \in V(m)} d_m(v, w).$$

The maximal graph distance from a vertex $v \in V(m)$ can also be interpreted as the radius of m centered at v. We will study this important quantity in Chapter 5 when we discuss metric properties of large random trees and quadrangulations.

Definition 15. Let $\mathcal{Q}$ be the class of all **rooted quadrangulations,** i.e. rooted planar maps, where each face has degree four. The subclass $\mathcal{Q}_n$ contains all rooted quadrangulations with n faces. A **rooted and pointed quadrangulations** is a rooted quadrangulation where a vertex v_* is additionally marked. We call v_* the **pointed vertex** of the quadrangulation and denote by $\mathcal{Q}^\bullet$ the class of all rooted and pointed quadrangulations. Again the subclass $\mathcal{Q}_n^\bullet$ contains the rooted and pointed quadrangulations with n faces.

We will sometimes omit the prefixes "rooted" and "rooted and pointed" if it is clear from the context which kind of quadrangulation is considered or if it is of no particular relevance at the point of study.

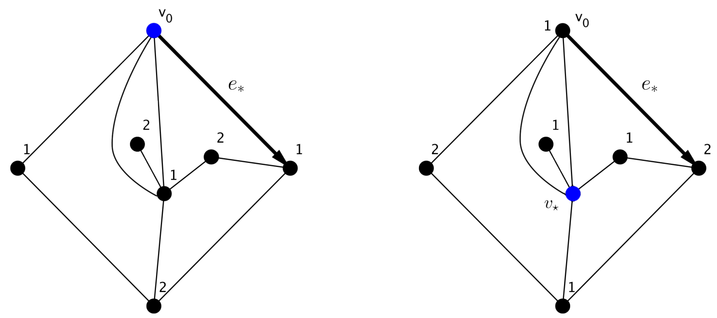


Figure 1.4: Rooted and rooted and pointed quadrangulation

Lemma 4. *Let $m \in \mathcal{M}_n$. Then $m \in \mathcal{Q}_n$ if and only if $|E(m)| = 2n$ and $|V(m)| = n+2$.*

Proof. Since every face of $m \in \mathcal{Q}_n$ is bounded by exactly four edges and every edge seperates exactly two faces, it follows with Theorem 2 that m has $2n$ edges and $n+2$ vertices. Controversely if we have a planar map m with $2n$ edges and $n+2$ vertices then again m must have n faces due to Theorem 2. $\square$

Lemma 5. *Let $q \in \mathcal{Q}$, v_0 its root vertex and let u_1, u_2 and w_1, w_2 be the opposite vertices of an arbitrary face of q. Then we have $d_q(u_1, v_0) = d_q(u_2, v_0)$ or $d_q(w_1, v_0) = d_q(w_2, v_0)$.*

Proof. This is clear since the labels of adjacent vertices can only differ by one. □

Definition 16. With the notation of Lemma 5 we call f **simple** if only one equality is satisfied and **confluent** otherwise.

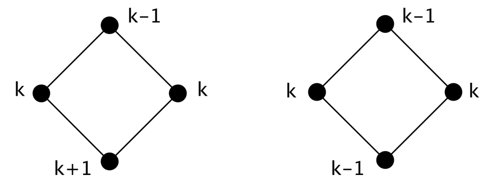


Figure 1.5: A simple and a confluent face

Definition 17. Let $q \in \mathcal{Q}$ and $q' \in \mathcal{Q}^\bullet$. Suppose v_0 is the root vertex of q and v_* the pointed vertex of q'. The **profile** of q is the sequence $(H_k^q)_{k\geq 1}$, where H_k^q denotes the number of vertices with distance k to v_0. Likewise, the **profile** of q' is the sequence $(H_k^{q'})_{k\geq 1}$, where $H_k^{q'}$ denotes the number of vertices with distance k to v_* in q'.

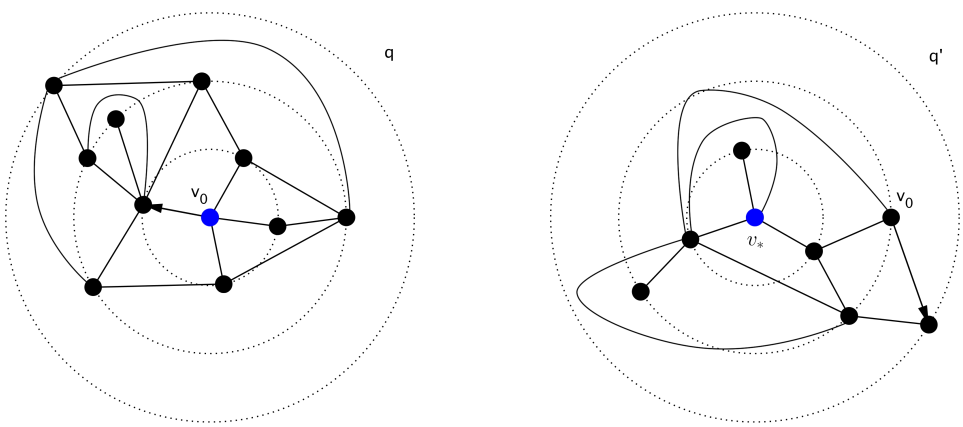


Figure 1.6: A rooted quadrangulation q with profile $(H_k^q)_{k\geq 1} = (4, 5, 1, 0, 0, ...)$ and a rooted and pointed quadrangulation q' with profile $(H_k^{q'})_{k\geq 1} = (3, 3, 1, 0, 0, ...)$

Notice that the profile $(H_k^q)_{k\geq 1}$ measures the "volumes of the spheres" induced by the natural distance on q, see Figure 1.6. It can therefore be seen as a measure on $\mathbb{N}$ with total volume

$$\sum_{k\geq 1} H_k^q = n + 2 \tag{1.1.2}$$

due to Lemma 4.

1.2 Plane, Well-labelled and Embedded Trees

In this section we introduce two classes of labelled trees, namely well-labelled and embedded trees where the choice of the labels are based on a simple rule. Both of these classes of trees are build upon so called plane trees. Well-labelled and embedded trees are of particular interest to us because of their close relation to quadrangulations.

Definition 18. A **tree** is a connected, undirected graph without cycles. A **forest** is a disjoint union of trees. The vertices of a tree, also called **nodes**, with degree 1 are called **leaves** and the remaining ones **internal vertices** or **internal nodes**. A **rooted tree** is a tree where a particular node is explicitly marked. This node is then called the **root vertex** or **root node**.

Definition 19. In a rooted tree, the **level** of a vertex v is the length of the unique path from the root vertex to v. If v immediately precedes a vertex w on the path from the root to w, then v is the **parent** (or **ancestor)** of w and w the **child** (or **descendent)** of v.

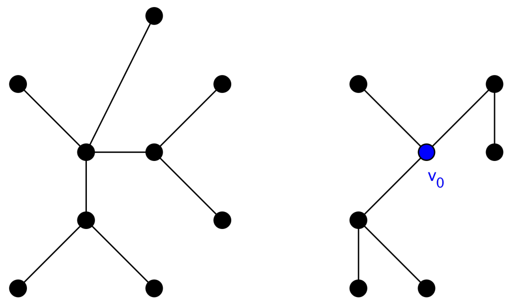


Figure 1.7: A tree and a rooted tree with root vertex v_0

Notice that a rooted tree τ with root vertex v_0 induces a (natural) partial order $\preccurlyeq$ on its vertex set $V(\tau)$ with least element v_0. This partial order is defined for every $v, w \in V(\tau)$ by

$$v \preccurlyeq w \quad \text{if the unique path from } v_0 \text{ to } w \text{ contains } v. \tag{1.2.1}$$

Definition 20. A **plane tree** is a planar map whose underlying graph structure is a rooted tree. We denote the class of plane trees by $\mathcal{P}$ and the subclass of plane trees with n edges by $\mathcal{P}_n$.

Proposition 1. *A plane tree has exactly one face.*

Proof. The absence of cycles in a plane tree τ is due to Jordan's curve theorem equivalent to the fact that τ has only one face. □

Plane trees can also be described recursively by interpreting them as combinatorial structures: a plane tree is a root vertex attached to a sequence of plane trees. This can be written formally as

$$\mathcal{P}_n = \{\bullet\} \times SEQ(\mathcal{P}_n) \tag{1.2.2}$$

where $\bullet$ represents the root node and the sequence construction $SEQ(\mathcal{C})$ of a class $\mathcal{C}$ is defined as

$$SEQ(\mathcal{C}) := \{\epsilon\} + \mathcal{C} + (\mathcal{C} \times \mathcal{C}) + (\mathcal{C} \times \mathcal{C} \times \mathcal{C}) + \cdots$$

with ϵ being a neutral structure of size 0. For further studies on the topic of symbolic enumeration methods, we refer the reader to [31, I.1.].

Lemma 6. *The number of plane trees with n edges is given by the n-th Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. Suppose p_n is the number of plane trees with n vertices and denote with $P(z) := \sum_{n\geq 0} p_n z^n$ its generating function. Equation (1.2.2) implies

$$P(z) = \frac{z}{1 - P(z)}$$

which is of the form

$$z = \frac{P}{\Phi(P)}$$

with $\Phi(P) = \frac{1}{1-P}$. This allows us to apply Lagrange's inversion formula (see [31, Theorem A.2]) to compute the n-th coefficient of $P(z)$

$$p_n = [z^n]P(z) = \frac{1}{n}[\omega^{n-1}]\Phi(\omega)^n = \frac{1}{n}[\omega^{n-1}](1-\omega)^{-n}. \tag{1.2.3}$$

Using the generalized binomial theorem

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \qquad \forall \alpha \in \mathbb{C}$$

we get

$$(1-\omega)^{-n} = \sum_{k=0}^{\infty}\binom{-n}{k}(-1)^k\omega^k.$$

This immediately gives

$$[\omega^{n-1}](1-\omega)^{-n} = \binom{-n}{n-1}(-1)^{n-1} =$$

$$= \frac{\overbrace{(-n)(-n-1)(-n-2)\cdots(-n-(n-1)+1)}^{n-1 \text{ factors}}}{(n-1)!}(-1)^{n-1} =$$

$$= \frac{(-1)^{n-1}n(n+1)(n+2)\cdots(2n-2)}{(n-1)!}(-1)^{n-1} = \binom{2n-2}{n-1}. \tag{1.2.4}$$

Putting (1.2.3) and (1.2.4) together we get the number of plane trees with n vertices

$$p_n = \frac{1}{n}\binom{2n-2}{n-2} = C_{n-1}.$$

The final result follows from the observation that a plane tree with $n+1$ vertices has n edges. □

Next we discuss well-labelled and embedded trees.

Definition 21. A **well-labelled** tree is a plane tree, where

1. all vertices have positive integer labels,
2. the labels of two adjacent vertices differ at most by one and
3. the label of the root vertex is one.

We denote the class of well-labelled trees by $\mathcal{W}$ and the subclass of well-labelled trees with n edges by $\mathcal{W}_n$.

Definition 21 suggests that we can associate for all $\tau \in \mathcal{W}$ a **label function** $\ell : V(\tau) \to \mathbb{N}_{>0}$ such that

1. $\ell(v_0) = 1$ where v_0 is the root vertex of τ and
2. $|\ell(u) - \ell(v)| \leq 1$ for all adjacent vertices $u, v \in V(\tau)$.

Definition 22. An **embedded tree** with increments 0 and ± 1 is a plane tree, where

1. all its vertices have integer labels,
2. the labels of adjacent vertices differ by 0 or by ± 1,
3. the label of the root vertex is zero.

We denote the class of embedded trees with increments 0 and ± 1 by $\mathcal{E}$ and the subclass of embedded trees with increments 0 and ± 1 and with n edges by $\mathcal{E}_n$. Also, when talking about embedded trees, we always mean embedded trees with increments 0 and ± 1 because these are the only ones which are of interest to us.

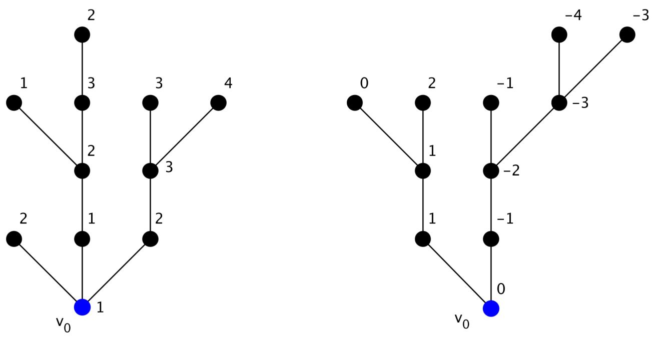


Figure 1.8: Well-labelled and embedded trees with root vertex v_0

Just as with well-labelled trees, we can associate a **label function** $\ell : V(\tau) \to \mathbb{Z}$ that satisfies

1. $\ell(v_0) = 0$ where v_0 is the root vertex of τ and
2. $|\ell(u) - \ell(v)| \leq 1$ for all adjacent vertices $u, v \in V(\tau)$.

Lemma 7. *The number of embedded trees with n edges is given by*

$$|\mathcal{E}_n| = \frac{3^n}{n+1}\binom{2n}{n}.$$

Proof. Suppose $\tau_n \in \mathcal{E}_n$ and choose an arbitrary edge $e \in V(\tau_n)$. The labels of the head and tail of e are either equal or differ atmost by one according to the definition of an embedded tree. Therefore the number of labellings of a plane tree that yield an embedded tree is 3^n. Thus Lemma 6 gives us

$$|\mathcal{E}_n| = 3^n \cdot C(n) = \frac{3^n}{n+1}\binom{2n}{n}.$$

□

Observe that we cannot use the same argument to count the possible labellings that turn a plane tree into a well-labelled tree because of the positivity condition of well-labelled trees. We will address this problem in a different way in Section 2.4.

Definition 23. Let τ be a labelled tree and define a mapping $\mathcal{N}$ for the class of all labelled trees by translating all labels in τ such that the minimum label is equal to zero and the remaining labels are shifted accordingly. We call $\mathcal{N}(\tau)$ the **normalized** tree.

Definition 24. Let $\tau_w \in \mathcal{W}$ and $\tau_e \in \mathcal{E}$. The **label distribution** of τ_w is the sequence $(\lambda_k^{\tau_w})_{k\geq 1}$, where λ_k is the number of nodes in τ_w with label k. Likewise, the **label distribution** of τ_e is the sequence $(\Lambda_k^{\tau_e})_{k\in\mathbb{Z}}$, where Λ_k is the number of nodes in τ_e with label k.

1.3 The Contour Process of a Plane Tree

Among the many existing encodings for trees one that will be of special interest to us is the so called contour exploration or contour process.

Definition 25. Let $\tau_n \in \mathcal{P}_n$ with root vertex v_0. Denote further with $(e_0, e_1, ..., e_{2n-1})$ the sequence of oriented edges bounding the only face of τ_n, starting with the edge incident to v_0. We call this sequence the **contour exploration** or **contour process** of τ_n.

For a plane tree $\tau_n \in \mathcal{P}_n$ and its contour exploration $(e_0, e_1, ..., e_{2n-1})$ we denote by $u_i := e_i^-$ the i-th visited vertex in the contour exploration and set

$$D_{\tau_n}(i) := d_{\tau_n}(u_i, v_0)$$

for all $i \in \{0, ..., 2n-1\}$ where $d_{\tau_n}(u_i, v_0)$ is the natural distance on τ_n when viewed as planar map (see Definition 13). We also set $e_{2n} := e_0$ and $u_{2n} := u_0$. The function D_{τ_n} can be extended by linear interpolation

$$C_{\tau_n}(s) := (1 - \{s\})D_{\tau_n}(\lfloor s \rfloor) + \{s\}D_{\tau_n}(\lfloor s \rfloor + 1)$$

for all $0 \leq s \leq 2n$ where $\{s\} := s - \lfloor s \rfloor$ is the fractional part of s. C_{τ_n} is a non-negative path of length $2n$ starting and ending at 0 with slope ± 1 between integer values. We call C_{τ_n} the **contour function** of τ_n and denote the set of all contour functions of length $2n$ with $\mathcal{C}_n$. An example of a plane tree together with its contour function is depicted in Figure 1.9.

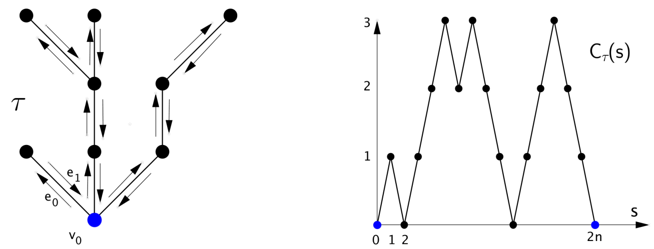


Figure 1.9: A plane tree with its contour process and its contour function

It can be shown that the number of contour functions matches the Catalan numbers C_n (see [31, Chapter I, Example I.16]) which is also the number of plane trees with n edges according to Lemma 6. Therefore the equality of cardinalities $|\mathcal{C}_n| = C_n = |\mathcal{P}_n|$ guarantees the existence of a bijective mapping between contour functions of length $2n$ and plane trees with n edges. More explicitly there holds:

Lemma 8. *The mapping*

$$f: \begin{cases} \mathcal{P}_n & \to \mathcal{C}_n \\ \tau_n & \mapsto C_{\tau_n} \end{cases}$$

is a bijection.

We will deepen our study of the contour process in Chapter 3, especially in relation to random walks.

Definition 26. Let $\tau_n \in \mathcal{P}_n$ with contour exploration $E_{\tau_n} := (e_0, e_1, ..., e_{2n-1})$. A **corner** is a sector between two consecutive edges of E_{τ_n} around a vertex. If τ_n is labelled then the label of a corner c, denoted by $\ell(c)$, is defined as the label of the corresponding vertex. Also, we use the notation c^- for the vertex associated with the corner c. If we speak of the corner of a directed edge e we mean the corner associated with the vertex e^-, the edge e and the edge right before e in the contour exploration.

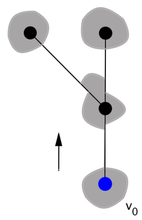


Figure 1.10: Corners of a plane tree

Remark 1. A vertex of degree k defines k corners and the number of corners of a plane tree with n edges is $2n$ because every edge induces two corners.

Definition 27. Let $\tau_n \in \mathcal{W}_n \cup \mathcal{E}_n$ with contour exploration $(e_0, e_1, ..., e_{2n-1})$. Define the **successor** s of a corner $i \in \{0, ..., 2n-1\}$ by

$$s(i) := \inf\{j > i : \ell(j) = \ell(i) - 1\}.$$

We also use the notation $s(e_i) := e_{s(i)}$.

The successor of a corner will play a crucial role in the next chapter when we discuss the CVS bijection.

2 CVS Bijection

In this chapter we will explore the Cori-Vauquelin-Schaeffer bijection (CVS bijection in short) between certain classes of quadrangulations and trees. This correspondence serves in the forthcoming sections as an essential tool to understand limit properties, namely global properties of distances, of quadrangulations (and therefore also of their corresponding tree structures). We restrict our study to the special case of quadrangulations, but similar bijections exist for p-angulations (see [6]).

At first, we discuss how a rooted quadrangulation can be encoded as a well-labelled tree in such a way that certain metric properties are preserved. This result was shown by Schaeffer in his PhD Thesis [34] in the year 1998. This process can then be modified to obtain a mapping that transforms a rooted and pointed quadrangulation into an embedded tree. Afterwards, we will discuss the corresponding inverse procedures yielding alltogether the following results:

Theorem 3 (Schaeffer)**.** *There exists a bijection between rooted quadrangulations with n faces and well-labelled trees with n edges such that the profile of a rooted quadrangulation is mapped onto the label distribution of the corresponding well-labelled tree.*

Theorem 4. *There exists a bijection between rooted and pointed quadrangulations with n faces and two copies of the set of embedded trees with n edges such that the profile of a rooted and pointed quadrangulation is mapped onto the label distribution of the corresponding normalized embedded tree. (By "two copies" we mean the set $\{1,-1\} \times \mathcal{E}_n$.)*

The idea to introduce a pointing in a quadrangulation and thus lift the positivity condition of well-labelled trees was first established by Chassaing and Schaeffer in [10].

2.1 Encoding of Quadrangulations as Trees

Our analyis follows along the lines of [10, Chapters 3.2-3.3] and [14, Chapter 5.1]. Suppose $q \in \mathcal{Q}_n$ and denote with $e_* = (v_0, v_1)$ its root edge. Start by labelling the vertices of q by the distance from the root vertex v_0. We construct a mapping Φ from $\mathcal{Q}_n$ into $\mathcal{W}_n$ by performing the following steps:

(T_1) Obtain a new map q' by dividing all confluent faces of q into two triangular faces by an edge joining the two vertices with maximal label.

(T_2) Define a subset of edges of q' in the following way (see Figure 2.1):

   a) In each confluent face of q, the edge that was added to form q' is selected.

b) For each simple face f of q: Let v be the vertex with maximal label in f. Choose the edge $\{v, w\}$ in f such that f is on the left side of the directed edge (v, w).

The edges selected in a) and b) are the edges of $\Phi(q)$.

(T_3) Choose the head v_1 as the root vertex of $\Phi(q)$ and discard v_0 as well as the previous root edge e_* of q.

This completes the construction of $\Phi(q)$.

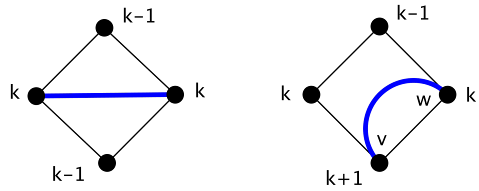


Figure 2.1: Local encoding rules (T_1) and (T_2)

Proposition 2. *Φ sends a rooted quadrangulation with n faces on a well-labelled tree with n edges.*

Proof. We show that every vertex $v \in V(q)\backslash\{v_0\}$ is also a vertex of $V(\Phi(q))$ hence $V(q)\backslash\{v_0\} \subseteq V(\Phi(q))$. Since $v \neq v_0$ there exists a vertex $w \in V(q)$ adjacent to v. Without loss of generality we can assume $d(v) - d(w) = 1$, where $d(v) := d_q(v, v_0)$. The edge (v, w) can be incident to

1. at least a confluent face or
2. at least a simple face (in which v has maximal label) or
3. two simple faces in which v has intermediate label.

All possible configurations can be seen (up two equivalent cases) in Figures 2.2-2.4.

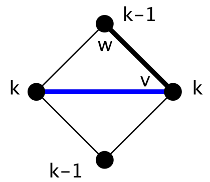


Figure 2.2: Configuration for case 1

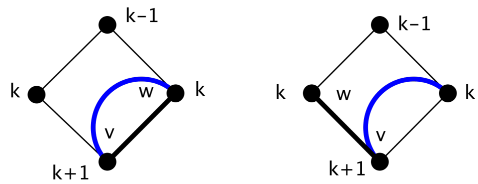


Figure 2.3: Possible configurations for case 2

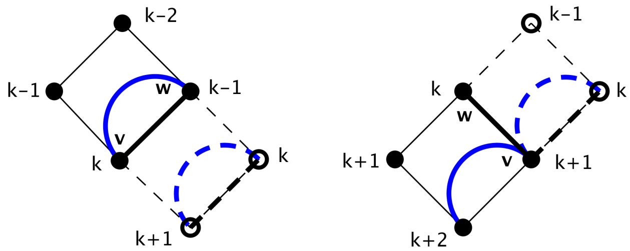


Figure 2.4: Possible configurations for case 3

The selection rules guarantee that v is incident to the newly selected edge (blue), i.e. $v \in V(\Phi(q))$. Therefore $V(q)\backslash\{v_0\} \subseteq V(\Phi(q))$.

Next we observe

- $|V(\Phi(q))| = n + 1$ because $v_0 \notin V(\Phi(q))$ and Lemma 4
- $|E(\Phi(q))| = n$ because for all $f \in F(q)$ there is exactly one edge selected.

Because q and q' are planar it follows that each connected component of $\Phi(q)$ is planar. Therefore $\Phi(q)$ is a forest of trees.
Suppose now that there exists a cycle C in $\Phi(q)$. According to the labelling rules all the labels of C are equal or there is a path $(k+1, k, k+1)$ as depicted in Figure 2.5. The selection rules guarantee the existence of vertices v and w inside and outside of C with $d(v) = d(w) = \min\{d(u) : u \in V(C)\} - 1$. The shortest path from either v or w to the root vertex v_0 has to intersect the cycle because of Theorem 1. This leads to a contradiction to the definition of the labelling because the labels of the shortest path from either v or w to v_0 have to decrease on every vertex. Therefore $\Phi(q)$ is a single tree and by construction also well-labelled. $\square$

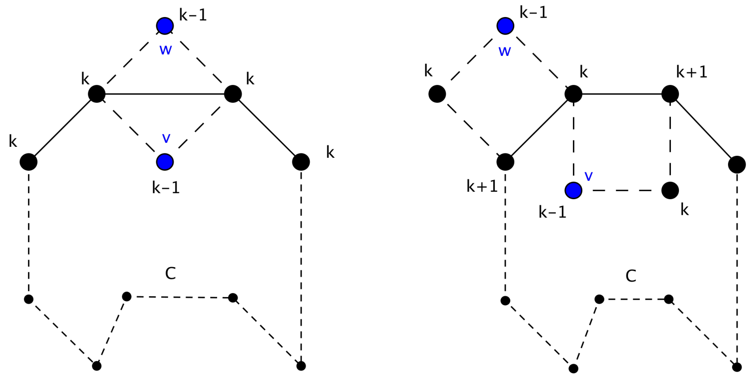


Figure 2.5: Impossibility of cycles

An example to illustrate Theorem 2 can be seen in Figures 2.6-2.7.

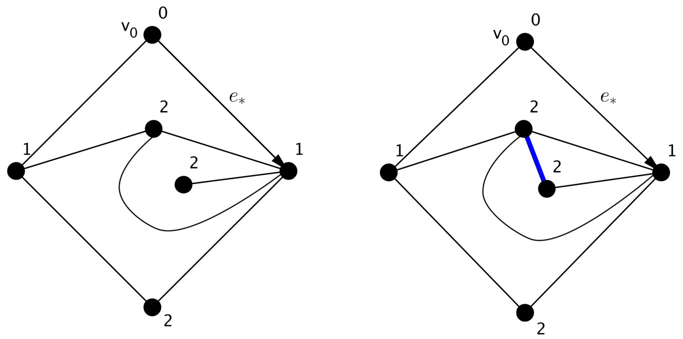


Figure 2.6: A quadrangulation and the new map after (T_1)

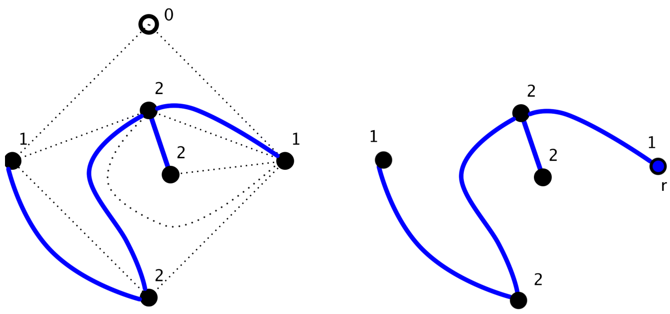


Figure 2.7: (T_2) and the final tree $\Phi(q)$ with root vertex r after (T_3)

The algorithm used to construct Φ can be slightly extended to obtain a mapping from $\mathcal{Q}_n^\bullet$ onto $\mathcal{E}_n$ together with an additional parameter $\epsilon \in \{1, -1\}$:

Suppose $q \in \mathcal{Q}_n^\bullet$ with root edge $e_* = (v_0, v_1)$ and perform the following steps:

(T_1') Apply steps $(T_1) - (T_3)$ as in the construction of Φ. This yields a labelled plane tree τ with root vertex v_1.

(T_2') Shift all labels of τ such that the new root vertex v_1 gets label 0. This gives us an embedded tree τ'.

(T_3') To ensure that the root edge of q can be recovered in the inverse procedure, we need to encode the direction of e :

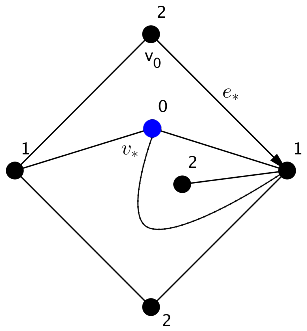
a) If $e = (s(e_0)^-, e_0^-)$ in τ' (where s is the successor function), then set $\epsilon := 1$.

b) If $e = (e_0^-, s(e_0)^-)$ in τ', then set $\epsilon := -1$.

This procedure gives us the desired mapping

$$\tilde{\Phi} : \mathcal{Q}_n^\bullet \to \{1, -1\} \times \mathcal{E}_n.$$

An example is depicted in Figures 2.8-2.9.



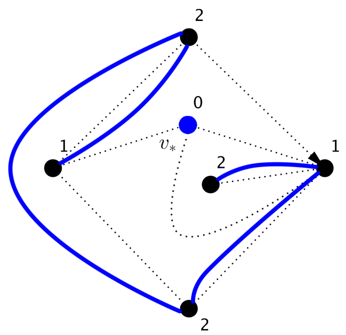


Figure 2.8: A pointed, rooted quadrangulation and the labelled plane tree after (T_1')

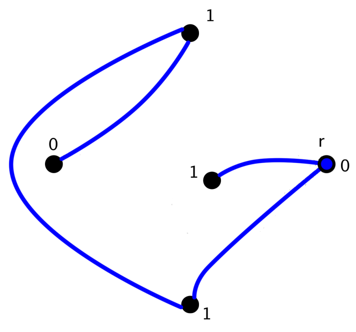


Figure 2.9: The final tree $\tilde{\Phi}(\tau)$ after (T_2') and (T_3') with root vertex r and $\epsilon = 1$

2.2 Encoding of Trees as Quadrangulations

We now discuss the construction of a mapping $\Psi : \mathcal{W}_n \to \mathcal{Q}_n$ that will be the inverse of the previously studied mapping Φ. Afterwards we modify this construction to obtain a mapping $\tilde{\Psi} : \mathcal{E}_n \times \{1, -1\} \to \mathcal{Q}_n^\bullet$ which will be the inverse of $\tilde{\Phi}$. This section is motivated by [10, Chapter 3.4.] and [29, Chapter 2.3].

Suppose $\tau \in \mathcal{W}_n$ with root vertex v_0 and label function ℓ. Furthermore let $(e_0, ..., e_{2n-1})$ be the contour exploration of τ. Also, we sometimes regard ℓ as a function $\ell : E(\tau) \to \mathbb{Z}$ and define $\ell(e) := \ell(e^-)$ so that $\ell(e)$ is the label of the tail of the edge e.

Define the image $\Psi(\tau)$ in four steps:

(Q_1) Place a vertex v_* with label 0 in the only face of τ (see Remark 1).

(Q_2) Add edges between the corners e_i and $s(e_i)$ for all $i \in \{0, ..., 2n-1\}$, where s is the successor function.

(Q_3) Delete all edges of the original tree τ.

(Q_4) Take the edge arriving from v_* at the corner before the root vertex v_0 of τ as the new root edge e_* so that $e_* = (v_*, v_0)$.

This completes the construction of $\Psi(q)$.

An example of this encoding process can be seen in Figures 2.10-2.11.

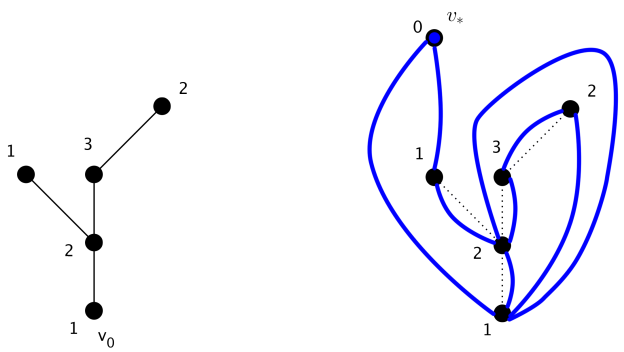


Figure 2.10: A well-labelled tree and the resulting planar map after (Q_1) and (Q_2)

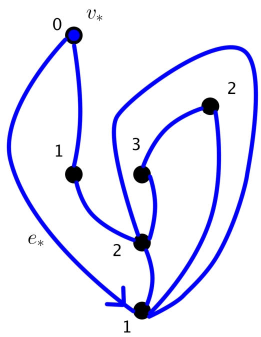


Figure 2.11: $\Psi(\tau)$ with root edge e_* after (Q_3) and (Q_4)

It is at first not obvious if (Q_2) can be applied in a planarity-preserving way. We close this gap now before continuing to examine the structure of $\Psi(\tau)$.

Lemma 9. *The edges added in step (Q_2) can be drawn in such a way that the newly created graph is a planar map.*

Proof. We show that having pairwise distinct corners $c_1, ..., c_4$ (that are ordered according to the contour exploration) together with the planarity-violating conditions $c_3 = s(c_1)$ and $c_4 = s(c_2)$ cannot occur (see Figure 2.12). If this were the case then we would have $\ell(c_3) < \ell(c_1)$ and $\ell(c_4) < \ell(c_2)$. Furthermore $s(c_1) = c_3$ implies $\ell(c_2) \geq \ell(c_1)$. Likewise $s(c_2) = c_4$ implies $\ell(c_3) \geq \ell(c_2)$. Putting this together we have $\ell(c_2) \leq \ell(c_3) < \ell(c_1) \leq \ell(c_2)$ which is impossible. $\square$

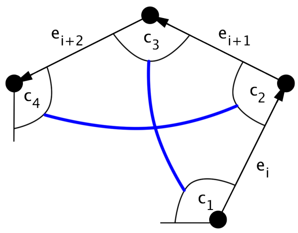


Figure 2.12: Impossible edge-configuration

Lemma 10. *For every $\tau_n \in \mathcal{W}_n$ the image $\Psi(\tau_n)$ is a rooted quadrangulation with n faces.*

Proof. At first we observe that $\Psi(\tau_n)$ is connected because for every corner c there exists a (finite) path $(c, s(c), s(s(c), ..., c_0)$ to the root corner c_0. Next we want to deduce that every face of $\Psi(\tau_n)$ is quadrangular. Actually we show that every face of $\Psi(\tau_n)$ is either simple or confluent as described by the encoding rules in Section 2.1. Since all faces of $\Psi(\tau_n)$ are by construction build "around" τ_n, we consider an arbitrary edge of τ_n which corresponds to two directed edges e and $\bar{e}$ in the contour exploration of τ_n. There are three possible cases (see Figure 2.13):

1. $\ell(e^+) = \ell(e^-) - 1$: The encoding rules imply that $s(e)$ is incident to e^+ and thus we have an edge from the corner of e^- to the corner of e^+. Next we look at the corner c' of the head of the directed edge $\bar{e}$. We have $\ell(c') = \ell(e^-) = \ell(\bar{e}) + 1$. $s(c')$ is the first corner coming after c' with label $\ell(s(c')) = \ell(e)$. $s(s(c'))$ is the first corner coming after c' with $\ell(s(s(c'))) = \ell(e) - 2$. This implies $s(\bar{e}) = s(s(c'))$ which leads to the form of a simple face.

2. $\ell(e^+) = \ell(e^-) + 1$: This is equivalent to $\ell(e^-) = \ell(e^+) - 1$ and therefore we can apply case 1 by interchanging the roles of e and $\bar{e}$.

3. $\ell(e^+) = \ell(e^-)$: Let c' be the corner of the head of e and c'' be the corner of the head of $\bar{e}$. The identity $\ell(e) = \ell(c') = \ell(\bar{e}) = \ell(c'')$ implies $s(e) = s(c')$ and $s(\bar{e}) = s(c'')$. Therefore we have a face of the confluent type with the diagonal edge $\{e, \bar{e}\}$.

This shows that every edge of $\Psi(\tau_n)$ is in fact quadrangular. Also $\Psi(\tau_n)$ has $2n$ edges (one for each corner of τ_n) and $n + 2$ vertices, so it must have n faces due to Euler's formula. Finally, $\Psi(\tau_n)$ is a rooted quadrangulation because of step (Q_4). $\square$

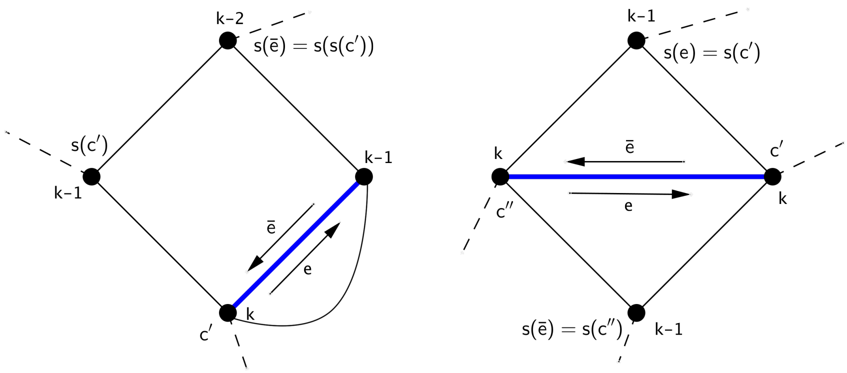


Figure 2.13: Cases 1 and 3 occuring in the proof of Lemma 10

Since the algorithms used to construct Φ and Ψ are inverse to one another, we have the following result:

Corollary 1 (CVS bijection for well-labelled trees)**.** *The mapping Ψ is a bijection between $\mathcal{W}_n$ and $\mathcal{Q}_n$. Its inverse is given by Φ.*

For a detailled argumentation why $\Psi^{-1} = \Phi$ holds, we refer the reader to the (rather technical) proof of Proposition 2 in [10].

As already mentioned, the above procedure can be modified to obtain a mapping $\tilde{\Psi}$ from $\{1, -1\} \times \mathcal{E}_n$ onto $\mathcal{Q}_n^\bullet$: Suppose $\tau \in \mathcal{E}_n$ and $\epsilon \in \{1, -1\}$. Since we are now dealing with labels which are not restricted to positive values, we have to modify (Q_1) in the following way:

(Q_1') Place a vertex v_* in the only face of τ (see Proposition 1) and set

$$\ell(v_*) := \min\{\ell(u) : u \in V(\tau)\} - 1.$$

(Q_2') Apply (Q_2), but with the additional setting of $s(i) := \infty$ if $\ell(e_i) = \ell(v_*)$.

(Q_3') Apply (Q_3).

(Q_4') If $\epsilon = 1$, we let the root edge e_* be oriented from the corner $s(e_0)$ to e_0 so that $e_* = (s(e_0)^-, e_0^-)$. Otherwise we set $e_* := (e_0^-, s(e_0)^-)$.

An example can be seen in Figures 2.14-2.15.

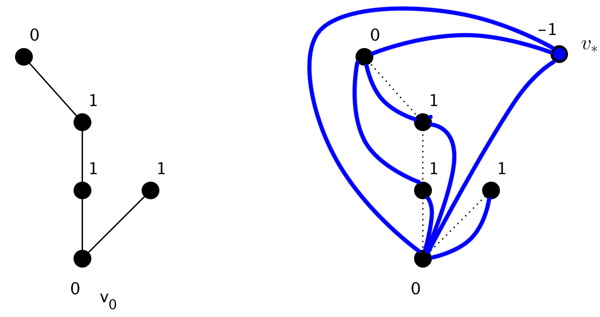


Figure 2.14: An embedded tree with $\epsilon = 1$ and the planar map after (Q_1') and (Q_2')

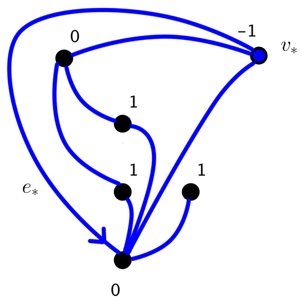


Figure 2.15: $\tilde{\Psi}(\tau)$ with root edge e_* after (Q_3') and (Q_4')

Notice that (Q_1') is a generalization of (Q_1) in the sense that if $\tau \in \mathcal{W}_n$ with root vertex w_*, there always holds $\min\{\ell(u) : u \in V(\tau)\} = 1$ and therefore $l(w_*) = 0$ as it is the case in (Q_1). (Q_1') is necessary because we are now dealing with labels that are not restricted to positive values anymore.
Thus we have defined a mapping

$$\tilde{\Psi} : \{1, -1\} \times \mathcal{E}_n \to \mathcal{Q}_n^\bullet.$$

In analogy to Corollary 1 we obtain the following result:

Corollary 2 (CVS bijection for embedded trees). *The mapping $\tilde{\Psi}$ is a bijection between $\{1, -1\} \times \mathcal{E}_n$ and $\mathcal{Q}_n^\bullet$. Its inverse is given by $\tilde{\Phi}$.*

A final word on the notation: consider an embedded tree $\tau \in \mathcal{E}_n$ and $\epsilon \in \{1, -1\}$. Let $q \in \mathcal{Q}_n^\bullet$ be the associated pointed quadrangulation constructed via the CVS bijection. Then we will often inaccurately write $\tilde{\Psi}(\tau) = q$ instead of $\tilde{\Psi}(\epsilon, \tau) = q$ if we are not interested in the direction of the root edges of τ and q. The same abbreviation will be used for the inverse mapping $\tilde{\Phi}$, i.e. we may write $\tilde{\Phi}(q) = \tau$ instead of $\tilde{\Phi}(q) = (\epsilon, \tau)$.

A short summary of the relations between vertices, edges, faces and corners established by the CVS bijection is listed in the table below:

Tree		Quadrangulation
vertex with label k	$\longleftrightarrow$	vertex at distance k to the root/pointed vertex
edge	$\longleftrightarrow$	face
corners c and $s(c)$ with labels i and j	$\longleftrightarrow$	edge with labels i and j

2.3 Properties of the CVS Bijection

In the previous chapter we have discussed the CVS bijection between rooted quadrangulations and well-labelled trees as well as a slightly modified version for rooted and pointed quadrangulations and embedded trees. The outstanding property of the CVS bijection is that distances from the root/pointed vertex in the quadrangulation are given by the labels on the associated tree. We are going to discuss this property for embedded trees only, but a similar result holds for well-labelled trees.

Lemma 11. *Let $\tau \in \mathcal{E}_n$ with associated label function ℓ and let $q \in \mathcal{Q}_n^\bullet$ be the corresponding quadrangulation according to the CVS bijection, i.e. $q = \tilde{\Psi}(\tau)$. Then we have*

$$d_q(u, v) \geq |\ell(u) - \ell(v)|$$

for all $u, v \in V(\tau) \cup \{v_*\} = V(q)$, where v_* is the pointed vertex in q.

Proof. We choose a path $(w_i)_{(i=1,\dots,d)}$ from u to v such that $d = d(u,v)$. Note that $w_1 = u$ and $w_d = v$. Since $|\ell(c) - \ell(s(c))| = 1$ for every corner c on the path we have

$$d_q(u,v) = d = \sum_{i=2}^{d} |\ell(w_i) - \ell(w_{i-1})| \geq |\sum_{i=2}^{d} \ell(w_i) - \ell(w_{i-1})| \geq$$

$$\geq \ell(w_d) - \ell(w_1)| = |\ell(u) - \ell(v)|.$$

□

This lower bound of the distance function gives rise to the following result (see [29, Proposition 2.3.7]):

Proposition 3. *Suppose $\tau \in \mathcal{E}_n$ together with its associated label function ℓ. Let $q \in \mathcal{Q}_n^\bullet$ be the corresponding quadrangulation due to the CVS bijection, i.e. $q = \tilde{\Psi}(\tau)$. Then we have*

$$d_q(v, v_*) = \ell(v) - min\ell + 1$$

for all $v \in V(q)\backslash\{v_*\}$, where v_* is the pointed vertex in q and $min\ \ell := min\{\ell(v) : v \in V(\tau)\}$.

Proof. Choose $v \in V(\tau) = V(q)\backslash\{v_*\}$ and let c_v be a corner in τ that is incident to v. The path

$$(c, s(c), s^2(c), ..., c_0)$$

has length $\ell(c) - \ell(c_0) = \ell(v) - \ell(v_*)$ by construction. Since $d_q(v, v_*)$ is the path with minimal distance we get

$$d_q(v, v_*) \leq \ell(v) - \ell(v_*) = \ell(v) - \min\ell + 1.$$

Controversely, choose a path $(v_i)_{(i=1,\dots,d)}$ from v to v_* with $d = d_q(v, v_*)$ in q. Then Lemma 11 shows that

$$d_q(v, v_*) \geq |\ell(v) - \ell(v_*)| = \ell(v) - \min\ell + 1.$$

□

Unfortunately there is no similar expression for the distance between arbitrary vertices v and w, but Proposition 3 will turn out to be of special importance when we discuss metric properties of random quadrangulations in Chapter 5. A consequence of Proposition 3 is, that the profile $(H_k^q)_{k\geq 1}$ of a rooted and pointed quadrangulation q can be interpreted in terms of the label function ℓ of the associated embedded tree, i.e.

$$H_k^q = |\{v \in V(q) : d_q(v, v_*) = k\}| = |\{v \in V(\tau) : \ell(v) - \min\ell + 1 = k\}| \qquad (2.3.1)$$

for every $k \geq 1$. We dive further into the structural properties of the profile in Chapter 5 when we discuss limiting properties of large quadrangulations.

Corollary 3. *The CVS bijection $\tilde{\Psi}$ works in such a way that the number of vertices in the quadrangulation with distance k from the pointed vertex is equal to the number of vertices of label k in the normalized embedded tree.*

Proof. This is just and alternative description of Proposition 3 and equation (2.3.1). □

Corollary 3 says that the profile of a rooted and pointed quadrangulation is mapped onto the label distribution of a normalized embedded tree by the CVS bijection. Notice that a similar result holds for Φ :

Proposition 4. *Suppose $\tau \in \mathcal{W}_n$ together with its associated label function ℓ. Let $q \in \mathcal{Q}_n$ be the corresponding quadrangulation due to the CVS bijection, i.e. $q = \Psi(\tau)$. Then we have*

$$d_q(v, v_0) = \ell(v)$$

for all $v \in V(q)\backslash\{v_0\}$, where v_0 is the root vertex in q.

This again entails the following property:

Corollary 4. *The CVS bijection Ψ works in such a way that the number of vertices in the quadrangulation with distance k from the root vertex is equal to the number of vertices of label k in the associated well-labelled tree.*

We already mentioned that it is not possible to give an immediate expression of the distance function between arbitrary vertices on a pointed quadrangulation in terms of the label function of the associated embedded tree. Still it is possible to obtain two useful bounds that come as close as possible.

Proposition 5. *Suppose $\tau \in \mathcal{E}_n$ with label function ℓ and associated pointed quadrangulation q via the CVS bijection, i.e. $\tilde{\Psi}(\tau) = q$. Let $u, v \in V(q)\backslash\{v_*\}$ where v_* is the pointed vertex in q. Remembering the notation in Definition 26, we let c_u and c_v be two corners of τ such that $u = c_u^-$ and $v = c_v^-$.*

1. *Then there holds*

$$d_q(u, v) \leq \ell(u) + \ell(v) - 2 \min_{c \in [c_u, c_v]} \ell(c) + 2,$$

where $[c_u, c_v]$ denotes the set of all corners encountered when starting from c_u, following the contour exploration and stopping at c_v.

2. *Then there holds*

$$d_q(u, v) \geq \ell(u) + \ell(v) - 2 \min_{w \in [[u,v]]} \ell(w),$$

where $[[u, v]]$ is the set of all vertices lying on the unique path from u to v in τ.

Proof. To follow the proof more easily, we advice the reader to keep an eye on Figures 2.16 and 2.17. We start with the first point: let $m := \min_{c\in[c_u,c_v]}\{\ell(c)\}$ and denote with c' the corner in $[c_u, c_v]$ that assumes the minimal label, i.e. $\ell(c') = m$. Because the successor function is monotonically decreasing with step size -1, we see that $c' = s^{\ell(c_u)-m}$. Using the same argument we get $s(c') = s^{\ell(c_v)-m+1}$. These equalities give us paths p_1 and p_2 with

$$p_1 = \{c_u \to s(c_u) \to \cdots \to s^{\ell(c_u)-m} = c' \to s^{\ell(c_u)-m+1} = s(c')\}$$

from c_u to the successor of c' of length $\ell(c_u) - m + 1$ and

$$p_2 = \{c_v \to s(c_v) \to \cdots \to s^{\ell(c_v)-m+1} = s(c')\}$$

from c_v to the successor of c' of length $\ell(c_v) - m + 1$ in q. Concatenating p_1 and p_2 at the common corner $s(c')$ yields a path p_3 of length

$$\ell(c_u) + \ell(c_v) - 2m + 2 = \ell(u) + \ell(v) - 2m + 2.$$

The first claim of the proposition now holds because $d_q(u, v)$ is the length of a minimal path between u and v in q and therefore less than or equal to the length of p_3.

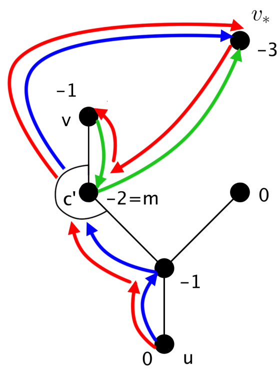


Figure 2.16: An examplified embedded tree as in the first point of the proof of Proposition 5 with the paths p_1(blue) and p_2(green) constructed according to the CVS bijection together with the properly concatenated path p_3(red).

We now turn to the second point. Let $w \in [[u, v]]$ be such that $\ell(w) = \min\{\ell(w') : w' \in [[u, v]]\}$. If $w = u$ or $w = v$ then the statement follows trivially from Lemma 11. Otherwise, we can write τ as the union $\tau = \tau_1 \cup \tau_2$ with $\tau_1 \cap \tau_2 = \{w\}$ and such that $u \in V(\tau_1)$, $u \notin V(\tau_2)$ and $v \in V(\tau_2)$, $v \notin V(\tau_1)$. Consider a minima path γ from u to v in q. There are now two possibilities:
Case 1: v_* belongs to γ : then we have $d_q(u, v) = d_q(u, v_*) + d_q(v_*, v)$ and the result follows from Lemma 11.
Case 2: v_* does not belong to γ : from the choice of the subtrees τ_1 and τ_2 of τ, we can find corners c_1 in τ_1 and c_2 in τ_2 such that c_1^- and c_2^- are consecutive vertices on γ and

the corners c_1 and c_2 are connected by an edge of q. Therefore we have $c_2 = s(c_1)$ or $c_1 = s(c_2)$. We only consider the first case (the other one is treated in a similar way). The successor function property together with the fact that the contour exploration of τ must visit w between any visit of u and v guarantees $\ell(w) \geq \ell(c_2)$. Then, again due to Lemma 11, we have

$$\begin{aligned} d_q(u,v) &= d_q(u,c_2^-) + d_q(c_2^-,v) \\ &\geq |\ell(u) - \ell(c_2^-)| + |\ell(v) - \ell(c_2^-)| \\ &\geq |\ell(u) - \ell(c_2^-) + \ell(v) - \ell(c_2^-)| \\ &= \ell(u) + \ell(v) - 2\ell(c_2^-) \\ &\geq \ell(u) + \ell(v) - 2\ell(w). \end{aligned}$$

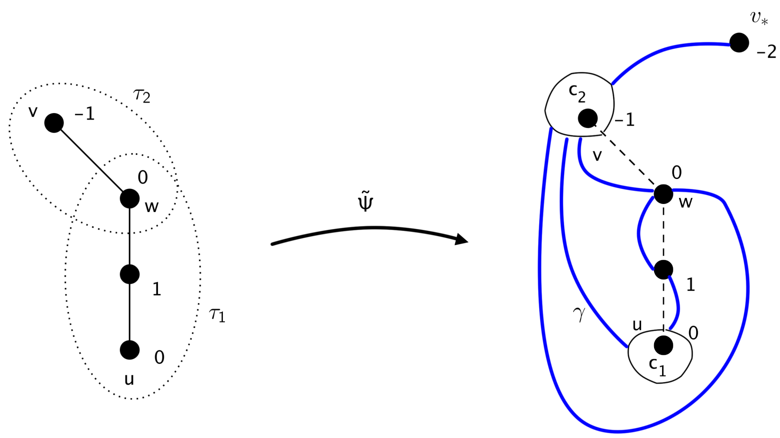


Figure 2.17: Illustration of the proof of the second point of Proposition 5.

□

We want to end this section with an interesting geometric construction involving three vertices as described in [8].

Definition 28. Let $q \in \mathcal{Q}_n$ and consider three distinct vertices $v_1, v_2, v_3 \in V(q)$. A **separating loop** is a loop L on q that passes through v_3 and **separates** v_1 from v_2, in the sense that any path from v_1 to v_2 necessarily intersects L (see Figure 2.18). A **minimal separating loop** is a separating loop of minimal length.

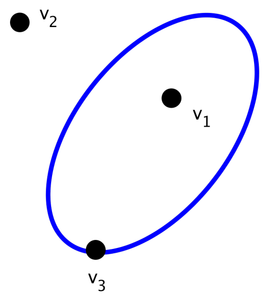


Figure 2.18: A schematic picture of a separating loop.

Lemma 12. *Let q be an unlabelled quadrangulation with n faces. Consider three distinct vertices $v_1, v_2, v_3 \in V(q)$ and denote with l_{123} the length of the minimal separating loop that contains v_3 and that separates v_1 and v_2. In this setting, we transform q into a labelled, rooted quadrangulation in the following way: let $v_0 := v_3$ be the root vertex of q and start labelling q in terms of the natural distance from v_0. This yields a rooted quadrangulation (with arbitrary choice of the root edge) that we also denote with q. Suppose $\tau \in \mathcal{W}_n$ is the well-labelled tree with label function ℓ associated with q via the CVS bijection, i.e. $\tau = \Phi(q)$. Then we have*

$$l_{123} = 2 \min_{w \in [[u,v]]} \ell(w),$$

where $[[u, v]]$ is the set of all vertices lying on the unique path from u to v in τ.

Proof. Let us first show that $l_{123} \geq 2m$ with $m := \min_{w \in [[u,v]]} \ell(w)$: consider an arbitrary loop L separating v_1 and v_2. L must intersect the unique path from u to v in τ at some vertex z. By decomposing L into a path L_1 from z to v_3 and a path L_2 from v_3 back to z, we see that the lenghts of both L_1 and L_2 must be larger or equal to the value of $\ell(z)$ because $\ell(z)$ is the minimal distance from z to v_3 in q. This implies

$$l_{123} = |L_1| + |L_2| = 2\ell(z) \geq 2m,$$

where $|L_i|$ denotes the number of vertices in L_i for $i \in \{1, 2\}$. Conversely, we can find a separating loop L (see Figure 2.19) by considering a vertex z with minimal label $\ell(z) = m$ on the unique path P between u and v (in τ) by picking two corners of z on opposite sites of P and considering the successor chain of these two corners. Both of these chains have length m and end at v_3 because the successor function is monotonically decreasing. Because l_{123} is the length of a minimal separating loop, we obtain $l_{123} \leq 2m$ and hence $l_{123} = 2m$.

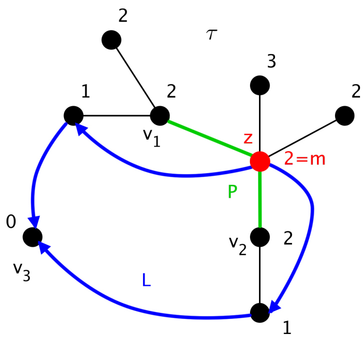


Figure 2.19: Construction of a minimal separating loop L in the proof of Lemma 12.

$\square$

2.4 Enumeration of Rooted Quadrangulations

The first breakthough in enumerating (planar) maps was due to Tutte [36] in 1963. His method used a recursive description of maps with the help of generating functions. This resulted in the so-called "quadratic-method". With this approach he obtained for example the formula

$$\frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n} \tag{2.4.1}$$

for the number of maps with n edges. The downside of Tutte's method is that it does not reveal insights into the structure of the respective objects. On the other hand, a bijective approach gives a deeper understanding of the combinatorial properties of the objects in question.

In this short section we use the CVS bijection to justify that formula (2.4.1) holds for the number of rooted quadrangulations with n faces (which is also the number of well-labelled trees with n edges) as well.

Finally, we want to mention that the enumerative theory of maps has not only many connections to different branches of mathematics but to theoretical physics as well. For example, maps have been used in the study of quantum gravity as discrete models for random geometries (see [22]).

Corollary 5. *For every $n \in \mathbb{N}$ there holds*

$$|\mathcal{Q}_n| = |\mathcal{W}_n| = \frac{2 \cdot 3^n}{n+2} \cdot C_n,$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

Proof. First we notice that $|\mathcal{Q}_n^\bullet| = (n+2)\cdot|\mathcal{Q}_n|$ since every non-pointed quadrangulation has $n+2$ vertices and each one of them induces a distinct element of $\mathcal{Q}_n^\bullet$. Then we have $|\{1,-1\}\times\mathcal{E}_n| = |Q_n^\bullet|$ due to the CVS bijection. This give us

$$|\{1,-1\}\times\mathcal{E}_n| = (n+2)\cdot|Q_n| \tag{2.4.2}$$

With Remark 7 we get that $|\{1,-1\}\times\mathcal{E}_n|$ is also equal to $2\cdot 3^n|\mathcal{P}_n|$. Since the number of plane trees is given by the Catalan numbers according to Lemma 6 we obtain

$$|\{1,-1\}\times\mathcal{E}_n| = 2\cdot 3^n\cdot C_n \tag{2.4.3}$$

Putting (2.4.2) and (2.4.3) together we conclude

$$|\mathcal{Q}_n| = \frac{2\cdot 3^n}{n+2}\cdot C_n.$$

□

Corollary 5 together with Lemma 7 entails the following relation between the number of well-labelled and embedded trees with n edges:

Corollary 6. *For every $n\in\mathbb{N}$ there holds*

$$|\mathcal{W}_n| = \frac{2}{n+2}\cdot|\mathcal{E}_n|.$$

Of course, these enumerative results lead directly to corresponding probabilistic descriptions:

Corollary 7. *Let $q_n\in\mathcal{Q}_n$ and $w_n\in\mathcal{W}_n$ be chosen uniformly at random. Let further $q_n^\bullet\in\mathcal{Q}_n^\bullet$ and $(\epsilon_n,\tau_n)\in\{1,-1\}\times\mathcal{E}_n$ be chosen uniformly at random such that ϵ_n is independent of τ_n. Then q_n has the same distribution as $\Psi(w_n)$ and $q_n^\bullet$ has the same distribution as $\tilde{\Psi}(\epsilon_n,\tau_n)$.*

Proof. Choose a fixed quadrangulation $q\in\mathcal{Q}_n$. The fact that w_n is uniformly distributed in $\mathcal{W}_n$ is transfered by the CVS bijection and therefore the quadrangulation $\Psi(w_n)$ can be seen as uniformly distributed in $\mathcal{Q}_n$. This gives us

$$\mathbb{P}\{q_n = q\} = \frac{1}{|\mathcal{Q}_n|} = \frac{1}{|\Psi(\mathcal{W}_n)|} = \mathbb{P}\{\Psi(w_n) = q\}.$$

Likewise, choose a fixed, pointed quadrangulation $q^\bullet\in\mathcal{Q}_n^\bullet$. Again, $\tilde{\Psi}(\tau_n)$ is uniformly distributed in $\mathcal{Q}_n^\bullet$ and we obtain

$$\mathbb{P}\{q_n^\bullet = q^\bullet\} = \frac{1}{|\mathcal{Q}_n^\bullet|} = \frac{1}{|\tilde{\Psi}(\{1,-1\}\times\mathcal{E}_n)|} = \mathbb{P}\{\tilde{\Psi}(\epsilon_n,\tau_n) = q^\bullet\}.$$

□

3 Convergence of the Contour Process

In Section 1.3 we gave a brief introduction to the one-to-one relationship between a plane tree and its associated contour process. The goal of this chapter is to use this knowledge in order to show that certain metric properties of large random plane trees can be described in terms of the so-called Brownian excursion. Inevitably, we are led into the study of random walks and Brownian motion. This alone is a considerably large field of exploration and so we restrict our attention to what is relevant for our purpose. Brownian motion will be of special interest because it turns out to be a limiting object in the context of (properly rescaled) random walks. This is the statement of Donsker's theorem in Section 3.3.

3.1 Convergence in Distribution and Stochastic Processes

The major tools for studying properties of large random trees are to be found in the theory of stochastic processes and in particular convergence in distribution. Both are central concepts of probability theory which we cover now briefly. A more elaborate discussion about these topics can be found in almost any book about stochastic processes, for example [32].

Definition 29. Suppose (E, d) is a metric space endowed with the Borel $\sigma-$field $\mathcal{B}$ constructed from the topology that is induced by the metric d. Let X_n and X be E-valued random variables. Then we say that X_n **converges in distribution** to X, if, for every bounded continuous function $g : E \to \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}[g(X_n)] = \mathbb{E}[g(X)].$$

We write $X_n \xrightarrow{d} X$ for convergence in distribution.

The following theorem justifies the naming "convergence in distribution".

Theorem 5. *Suppose (E, d) is a metric space endowed with the Borel $\sigma-$field constructed from the topology that is induced by the metric d. Let X_n and X be E-valued random variables with distribution functions $F_n(x) := \mathbb{P}\{X_n \le x\}$ and $F(x) := \mathbb{P}\{X \le x\}$ for $x \in E$. Then the following assertions are equivalent:*

1. $X_n \xrightarrow{d} X$
2. $\lim_{n\to\infty} F_n(x) = F(x)$ *for all $x \in E$ such that F is continuous in x.*
3. $\limsup_{n\to\infty} \mathbb{P}\{X_n \in K\} \le \mathbb{P}\{X \in K\}$ *for all closed sets $K \subseteq E$.*

Sometimes it is useful to relate convergence in distribution with pointwise convergent random variables. More precisely, the following is true duo to Skorokhod, see [5, Theorem 6.7]:

Theorem 6 (Skorokhod's representation theorem)**.** *Suppose (E, d) is a metric space. Let X_n and X be E-valued random variables such that $X_n \overset{d}{\to} X$. Then there exist random variables Y_n and Y on a common probability space $(\Omega, \mathcal{B}, \mathbb{P})$ such that*

1. *X_n has the same probability distribution as Y_n and likewise X has the same probability distribution as Y,*
2. *$\mathbb{P}\{Y_n \to Y\} = 1$, i.e. $Y_n(\omega) \to Y(w)$ almost surely for every $\omega \in \Omega$ as $n \to \infty$.*

We recall the basic definition of a stochastic process:

Definition 30. Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space, $(Z, \mathcal{Z})$ an arbitrary space with σ-field $\mathcal{Z}$ and I an indexset. Then a **stochastic process** $\{X_i : i \in I\}$ is a family of random variables defined on $(\Omega, \mathcal{B}, \mathbb{P})$ such that

$$X_i\colon \begin{cases} \Omega & \to Z \\ \omega & \mapsto X_i(\omega) \end{cases} \tag{3.1.1}$$

is $\mathcal{B}$-$\mathcal{Z}$ measurable for all $i \in I$.

If not otherwise mentioned, we will only be concerned with real-valued stochastic processes, i.e. processes with $(Z, \mathcal{Z}) = (\mathbb{R}, \sigma(\mathbb{R}))$ where $\sigma(\mathbb{R})$ is the Borel σ-field on $\mathbb{R}$ induced by the Euclidean topology.

Definition 31. Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space and $(Z, \mathcal{Z})$ an arbitrary space with σ-field $\mathcal{Z}$ and I an indexset as in Definition 30. Let $X := \{X_i : \Omega \to Z : i \in I\}$ be a stochastic process and set

$$(\Delta_X(\omega))(i) := X_i(\omega)$$

for every $\omega \in \Omega$ and $i \in I$. The **law** of the stochastic process X is the pushforward measure $\mathcal{L}_X := \mathbb{P} \circ \Delta_X^{-1}$ defined on the space of all functions from I into Z.

Another very important concept in probability theory are Markov processes. They are special stochastic processes whose long term behaviour is completely determined by the transition probabilities between the various states of the process and not on where the process started or what happened "in the past". We formalize this heuristic description now:

Definition 32. Let $(\Omega, \mathcal{B})$ be a measurable space. A **filtration** $\mathcal{F} := \{F_t : t \geq 0\}$ is a sequence of $\sigma-$fields with $F_t \subseteq \mathcal{B}$ satisfying

$$t_1 \leq t_2 \Rightarrow F_{t_1} \subseteq F_{t_2}.$$

Notice that a stochastic process $\{X_i : i \geq 0\}$ naturally induces a filtration by setting $F_t := \sigma(\{X_i^{-1}(B) : i \leq t, B \in \mathcal{B}\})$ for all $t \geq 0$.

Definition 33. Let $\{X_t : t \geq 0\}$ be a stochastic process on the probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with values in $(Z, \mathcal{Z})$. We call $\{X_t : t \geq 0\}$ a **Markov process** with respect to the filtration $\mathcal{F}$, if for all $0 \leq s < t$ and $A \in \mathcal{Z}$

$$\mathbb{P}\{X_t \in A | F_s\} = \mathbb{P}\{X_t \in A | X_s\}$$

almost surely.

A special class of stochastic processes are the so-called Gaussian processes.

Definition 34. A real-valued stochastic process $\{X_t : t \geq 0\}$ is a **Gaussian process** if for any choice of distinct values $t_1, ..., t_k \geq 0$, the random vector $X := (X_{t_1}, ..., X_{t_k})$ has a multivariate normal distribution with mean vector $\mu = \mathbb{E}[X]$ and covariance function $\Sigma = \mathrm{Cov}(X, X)$.

A Gaussian process is completely determined by its mean

$$\mu(t) := \mathbb{E}[X_t]$$

and covariance function

$$\Sigma(s, t) := \mathrm{Cov}(X_s, X_t)$$

respectively. In fact the following result holds, which is a consequence of Kolmogorov's extension theorem:

Proposition 6. *Let μ be a real-valued function on $\mathbb{R}_+$ and Σ a non-negative definite function on $\mathbb{R}_+ \times \mathbb{R}_+$. Then there exists a Gaussian process with mean function μ and covariance function Σ.*

3.2 Random Walks

We shall now be concerned with sums of discrete random variables and their properties. These objects are called random walks and are a well-known topic in probability theory. Random walks and especially their limiting behaviour is important in our context because there is a close relationship to the contour process of large, random plane trees. In this section we set up the necessary notations and derive some basic propositions on which we can build up.

Definition 35. Let $\{X_k : k \geq 1\}$ be discrete, independent and identically distributed (in short i.i.d.) random variables with values in $\mathbb{Z}^d$. A stochastic process $\{S_n : n \geq 0\}$ is called a **random walk** on $\mathbb{Z}^d$, if $S_0 = 0$ and

$$S_n = X_1 + X_2 + \cdots + X_n$$

for all $n \in \mathbb{N}$. In this case we say that the random walk is supported by the random variables $\{X_k : k \geq 1\}$.

Random walks can for example be interpreted as the movement of a particle from position S_{n-1} to S_n during the time interval $[n-1, n]$. These movements are independent and identically distributed for all $n \in N$ because $S_n - S_{n-1} = X_n$.

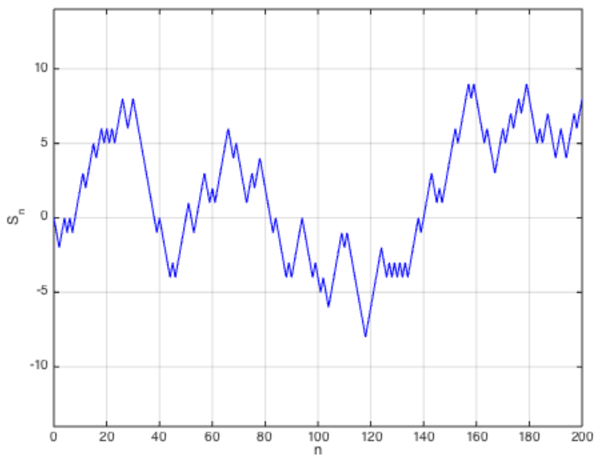


Figure 3.1: One-dimensional random walk

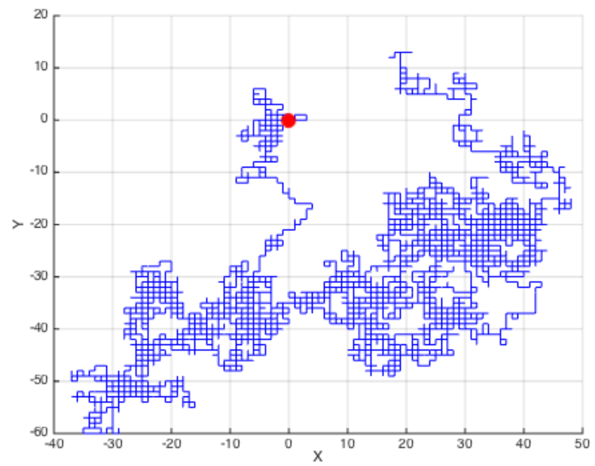


Figure 3.2: Two-dimensional random walk

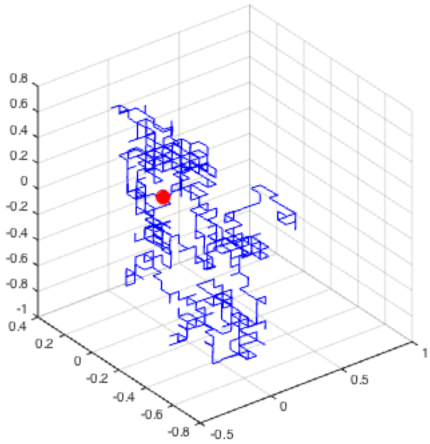


Figure 3.3: Three-dimensional random walk

Note that in this work we are only interested in the lattice $\mathbb{Z}$, i.e. we always have $d = 1$ in Definition 35.

Definition 36. A random walk $\{S_n : n \geq 0\}$ supported by i.i.d. random variables $\{X_k : k \geq 1\}$ is called **simple** if

1. $X_k \in \{1, -1\}$
2. $\mathbb{P}\{X_k = 1\} = p$ and $\mathbb{P}\{X_k = -1\} = 1 - p$ where $0 < p < 1$

for all $k \geq 1$.
If $p = \frac{1}{2}$ the simple random walk is called **symmetric**, otherwise **asymmetric** (see Figure 3.4).

We state two simple properties:

Lemma 13. *Let $\{S_n : n \geq 0\}$ be a simple random walk supported by $\{X_k : k \geq 1\}$. Then for all $n \in \mathbb{N}$*

1. $\mathbb{E}[S_n] = n(2p - 1)$
2. $\mathbb{V}[S_n] = 4np(1 - p)$

Proof. The first point follows easily from

$$\mathbb{E}[X_k] = 1 \cdot p + (-1)(1 - p) = 2p - 1 \tag{3.2.1}$$

together with the linearity of the expectation and the fact that the random variables X_k are i.i.d. for all $k \geq 1$:

$$\mathbb{E}[S_n] = \mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = n(2p-1).$$

The second point is a direct consequence of Steiner's translation theorem and (3.2.1):

$$\mathbb{V}[S_n] = \mathbb{V}[X_1] + \cdots \mathbb{V}[X_n] = n\mathbb{V}[X_1] =$$

$$= n(\mathbb{E}[X_1^2] - \mathbb{E}[X_1]^2) = n(1-(2p-1)^2) = 4np(1-p)$$

□

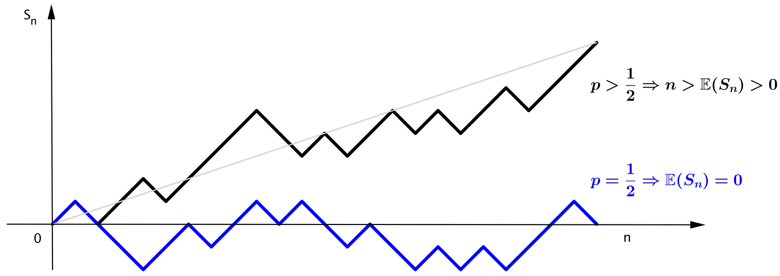


Figure 3.4: Symmetric (blue) and asymmetric (black) simple random walks.

Lemma 14. *Let $\{S_n : n \geq 0\}$ be a simple random walk. Then for every $k \in \mathbb{N}$ the probability density function is given by*

$$\mathbb{P}\{S_{2n} = 2k\} = \binom{2n}{n+k} p^{n+k}(1-p)^{n-k}.$$

Proof. $S_{2n} = X_1 + \cdots + X_{2n}$ equals $2k$ if and only if there are exactly $n+k$ indices $i_1, ..., i_{n+k} \subseteq \{1, ..., 2n\}$ such that $X_{i_l} = 1$ for all $l = 1, ..., n+k$. The probability density function for $S_{2n} = 2k$ is now given by the binomial distribution

$$\mathbb{P}\{S_{2n} = 2k\} = \binom{2n}{n+k} p^{n+k}(1-p)^{2n-(n+k)} = \binom{2n}{n+k} p^{n+k}(1-p)^{n-k}.$$

□

If we allow the supporting random variables $\{X_k : k \geq 1\}$ of a random walk $\{S_n : n \geq 0\}$ to take values in $\mathbb{R}$, then the central limit theorem states that the distribution of

$$\frac{S_n}{\sqrt{n}} = \frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

approaches that of a normal distribution with mean zero and variance σ^2 as n tends towards ∞. More precisely, for $-\infty < r < s < \infty$,

$$\lim_{n\to\infty} \mathbb{P}\left\{ r \leq \frac{S_n}{\sqrt{n}} \leq s \right\} = \frac{1}{\sqrt{2\pi}\sigma} \int_r^s e^{-\frac{y^2}{2\sigma^2}} dy. \tag{3.2.2}$$

Forgetting for a moment that we chose r and s independent of n, one might be drawn to use equation (3.2.2) to approximate the probability density function of $\{S_n : n \geq 0\}$ for large n :

$$\mathbb{P}\{S_n = k\} = \mathbb{P}\left\{ \frac{k}{\sqrt{n}} \leq \frac{S_n}{\sqrt{n}} < \frac{k+1}{\sqrt{n}} \right\} \approx \frac{1}{\sqrt{2\pi}\sigma} \int_{\frac{k}{\sqrt{n}}}^{\frac{k+1}{\sqrt{n}}} e^{-\frac{y^2}{2\sigma^2}} dy. \tag{3.2.3}$$

Nonetheless, this should be understood as a heuristic argument only because the terms $\frac{k}{\sqrt{n}}$ and $\frac{k+1}{\sqrt{n}}$ depend on n. In the next section we shall see that the central limit theorem can be extended to certain random functions based on random walks. That is the statement of Donsker's theorem which will serve as a major tool to establish a limit law for the contour process of a random plane tree.

The next lemma relates symmetric, simple random walks to contour functions of random plane trees. This should be understood as a motivation for more elaborate connections between these two objects.

Suppose $\tau_n \in \mathcal{P}_n$ is a random plane tree with n edges and $C_{\tau_n} \in \mathcal{C}_n$ the contour function of length $2n$ asssociated with the contour process of τ_n as defined in Section 1.3. C_{τ_n} is a uniformly distributed random variable in $\mathcal{C}_n$ because the mapping that associates τ_n with its contour function is a bijection according to Lemma 8.
Suppose further $\{S_n : n \geq 0\}$ is a symmetric, simple random walk. By linear interpolation we can extend $\{S_n : n \geq 0\}$ to a random continuous function

$$S\colon \begin{cases} \mathbb{R}_{\geq 0} & \to \mathbb{R} \\ t & \mapsto S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor)(S_{\lfloor t \rfloor + 1} - S_{\lfloor t \rfloor}) \end{cases} \tag{3.2.4}$$

where $\lfloor t \rfloor := \max\{k \in \mathbb{Z} : k \leq t\}$. The restriction $S_{|A_n}$ with $A_n := \{t \in \mathbb{R}_{\geq 0} : t \leq 2n \wedge S(t) \geq 0 \wedge S(2n) = 0\}$ is then a uniformly distributed contour function of length $2n$ and hence also an element of $\mathcal{C}_n$. Lemma 15 summarizes this observation.

Lemma 15. *Let S be the linear interpolation of a symmetric, simple random walk as in (3.2). Then the contour function C_{τ_n} associated with a random plane tree τ_n with n edges has the same distribution as $S_{|A_n}$.*

3.3 Brownian Motion and Donsker's Theorem

We would now like to discuss an extension of the central limit theorem to random functions

$$\tilde{S}_n(t) := \frac{S(nt)}{\sqrt{n}} \tag{3.3.1}$$

for all $t \in [0,1]$, where S is the continuous, linear interpolation of a symmetric, simple random walk $\{S_n : n \geq 0\}$ as defined in (3.2). Donsker's theorem (Theorem 7) shows that the macroscopic picture of such random functions can be described by the so-called (standard) Brownian motion. In reverse this means that (standard) Brownian motion can be obtained by taking the limit (in terms of convergence in distribution) of certain random walks.

An example of a Brownian motion is the process of chaotic displacements of particles suspended in a liquid or gas. These chaotic displacements are the result of collisions with the molecules of the medium. One possible mathematical model of such a motion is the so-called Wiener process.

Definition 37. A real-valued stochastic process $\{B(t) : t \geq 0\}$ is called a **Brownian motion** (or **Wiener process**) with start in $x \in \mathbb{R}$ if the following holds:

1. $\mathbb{P}[B(0) = x] = 1$,
2. the process has independent increments, i.e. for all times $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$ the increments $B(t_n) - B(t_{n-1}),\ B(t_{n-1}) - B(t_{n-2}),\ \ldots,\ B(t_2) - B(t_1)$ are independent random variables,
3. for all $t \geq 0$ and $h > 0$, the increments $B(t+h) - B(t)$ are normally distributed with expectation zero and variance h,
4. the function $t \mapsto B(t)$ is almost surely continuous.

We say that $\{B(t) : t \geq 0\}$ is a **standard Brownian motion** if it starts with $x = 0$.

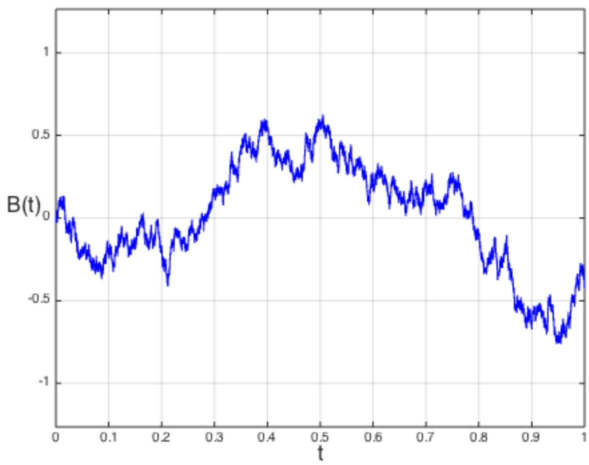


Figure 3.5: Brownian motion in dimension one

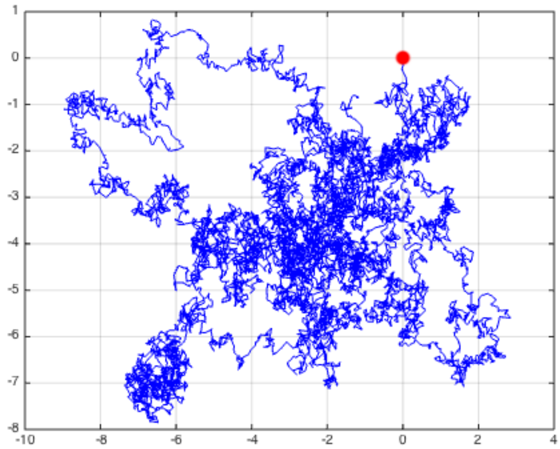


Figure 3.6: Brownian motion in dimension two

Brownian motion can be studied in arbitrary dimensions, but we are here only interested in dimension one (which we also assume from now onwards). Note that we do not know yet if such a process actually exists. Luckily, it does and one can find a possible construction in [25, Chapter 3.2].

A very simple, but import property of Brownian motion is its scaling invariance, which can be derived immediately from Definition 37.

Lemma 16 (Scaling invariance)**.** *Suppose* $\{B(t) : t \geq 0\}$ *is a standard Brownian motion and let* $a > 0$*. Then the process* $\{X(t) : t \geq 0\}$ *defined by* $X(t) := \frac{1}{a}B(a^2 t)$ *is also a standard Brownian motion.*

A natural question to ask is, how much time a Brownian motion spends in $A \subseteq \mathbb{R}$. In order to answer this, we introduce the occupation measure μ_t of a standard Brownian motion $\{B(s) : s \geq 0\}$ by

$$\mu_t(A) := \int_0^t \mathbb{1}_A(B(s))\, ds \tag{3.3.2}$$

for $t \geq 0$ and arbitrary Borel sets $A \subseteq \mathbb{R}$. It can be shown that the time a Brownian motion spends in a certain Borel subset of $\mathbb{R}$ is comparable to its Lebesgue measure. More precisely, almost surely μ_t is absolutely continuous with respect to the Lebesgue measure (see [30, Theorem 3.25]).

The key to the proof of Donsker's theorem lies in the fact that a symmetric, simple random walk $\{S_n : n \geq 0\}$ can be "embedded" into a Brownian motion $\{B(t) : t \geq 0\}$ in the sense that there exists a sequence $\{T_n : n \geq 0\}$ such that $S_n = B(T_n)$. This technique is called the Skorokhod embedding (see [30, Chapter 5.3]). A consequence is Lemma 17. For a proof we refer again to [30, Lemma 5.24].

Lemma 17. *Let $\{S_n : n \geq 0\}$ be a symmetric, simple random walk. Suppose further $\{B(t) : t \geq 0\}$ is a Brownian motion. Then, for any normalized, random variable X, i.e. $\mathbb{E}[X] = 0$ and $\mathbb{V}[X] = 1$, there exists a sequence $0 = T_0 \leq T_1 \leq T_2 \leq ...$ such that*

1. *$\{B(T_n) : n \geq 0\}$ has the distribution of $\{S_n : n \geq 0\}$ with increments given by the law of X,*
2. *the sequence $\{\tilde{S}_n : n \geq 0\}$ constructed from this random walk as in (3.3.1) satisfies*

$$\lim_{n\to\infty} \mathbb{P}\left\{\left|\left|\frac{B(nt)}{\sqrt{n}} - \tilde{S}_n(t)\right|\right|_\infty > \epsilon\right\} = 0,$$

where the supremum norm $||.||_\infty$ is taken over all $t \in [0, 1]$.

Theorem 7 (Donsker's theorem)**.** *Let $\{S_n : n \geq 0\}$ be a symmetric, simple random walk supported by i.i.d. random variables $\{X_k : k \geq 1\}$ and suppose that they are normalized, i.e. $\mathbb{E}[X_k] = 0$ and $\mathbb{V}[X_k] = 1$ for all $k \geq 1$. Then on the space of continuous functions on $[0, 1]$ together with the metric induced by the supremum norm $||.||_\infty$, the sequence $\{\tilde{S}_n : n \geq 1\}$, as defined in (3.3.1), converges in distribution to a standard Brownian motion $B := \{B(t) : t \in [0, 1]\}$, i.e.*

$$\tilde{S}_n \xrightarrow{d} B. \tag{3.3.3}$$

Proof. Suppose $\{B(t) : t \in [0, 1]\}$ is a standard Brownian motion. Lemma 16 with $a := \sqrt{n}$ shows that the sequence $\{\tilde{B}_n(t) : t \in [0, 1]\}$ defined by $\tilde{B}_n(t) := \frac{B(nt)}{\sqrt{n}}$ is also a standard Brownian motion. Denote with $\mathcal{C}[0, 1]$ the set of real-valued, continuous functions on $[0, 1]$ and suppose $K \subseteq \mathcal{C}[0, 1]$ is closed. Define for $\epsilon > 0$

$$K_\epsilon := \{f \in \mathcal{C}[0, 1] : \exists g \in K : ||f - g||_\infty \leq \epsilon\}.$$

Then

$$\mathbb{P}\{\tilde{S}_n \in K\} \leq \mathbb{P}\{\tilde{B}_n \in K_\epsilon\} + \mathbb{P}\{||\tilde{S}_n - \tilde{B}_n||_\infty > \epsilon\}.$$

Again the scaling invariance, Lemma 16, entails $\mathbb{P}\{\tilde{B}_n \in K_\epsilon\} = \mathbb{P}\{B \in K_\epsilon\}$ and therefore

$$\mathbb{P}\{\tilde{S}_n \in K\} \leq \mathbb{P}\{B \in K_\epsilon\} + \mathbb{P}\{||\tilde{S}_n - \tilde{B}_n||_\infty > \epsilon\}. \tag{3.3.4}$$

According to the second point in Lemma 17, $\lim_{n\to\infty} \mathbb{P}\{||\tilde{S}_n - \tilde{B}_n||_\infty > \epsilon\}$ equals 0 and because K is closed we have

$$\lim_{\epsilon\to 0} \mathbb{P}\{B \in K_\epsilon\} = \mathbb{P}\left\{B \in \bigcap_{\epsilon>0} K_\epsilon\right\} = \mathbb{P}\{B \in K\}. \tag{3.3.5}$$

Putting (3.3.4) and (3.3.5) together, we get

$$\limsup_{n\to\infty} \mathbb{P}\{\tilde{S}_n \in K\} \leq \mathbb{P}\{B \in K\}.$$

Theorem 5 then provides $\tilde{S}_n \xrightarrow{d} B$. $\square$

Note that the assumption of normalized supporting random variables $\{X_k : k \geq 1\}$ in Theorem 7 is no loss of generality for X_k with finite variance, since we can always consider the normalization

$$\frac{X_k - \mathbb{E}[X_k]}{\sqrt{\mathbb{V}[X_k]}}.$$

Donsker's theorem is also often called Donsker's invariance principle because the limit (3.3.3) is independend of the exact distribution of the supporting random variables $\{X_k : k \geq 1\}$.

3.4 Brownian Excursion and Random Plane Trees

So far we have seen that the continuous functions $\tilde{S}_n$, originating from a properly rescaled random walk, converge in distribution to a standard Brownian motion. Motivated by Lemma 15 one might expect a similar behaviour for the contour function C_{τ_n} of a plane tree τ_n. What needs to be considered, is, that C_{τ_n} assumes only non-negative values in contrast to $\tilde{S}_n$. In dealing with this issue, we first introduce Brownian excursion. Our discussion in this section follows [29, Chapter 3.1].

Brownian excursion is a process which can be described in several ways. One possibility is the so-called Vervaat transform [37]. Another one is to define it as a certain Markov process (see for example [14, p.111-112]). Our approach is to model Brownian excursion as a rescaled, standard Brownian motion with compact support conditioned to remain non-negative.

Suppose $\{B_t : t \geq 0\}$ is a standard Brownian motion and set

$$d := \inf\{t \geq 1 : B_t = 0\},$$

$$g := \sup\{t \leq 1 : B_t = 0\}$$

Because $\mathbb{P}\{B_0 = 0\} = 1$ and $\mathbb{P}\{B_1 \neq 0\} = 1$ almost surely we get (together with the continuity of $t \mapsto B_t$ almost surely) $g < 1 < d$ almost surely. This allows the following Definition:

Definition 38. The stochastic process $\{B_t : t \in [g, d]\}$ is called the **Brownian excursion**. The **normalized Brownian excursion** is the stochastic process $\mathbb{e} := \{\mathbb{e}_t : t \in [0, 1]\}$ with

$$\mathbb{e}_t := \frac{|B_{g+t(d-g)}|}{\sqrt{d-g}}.$$

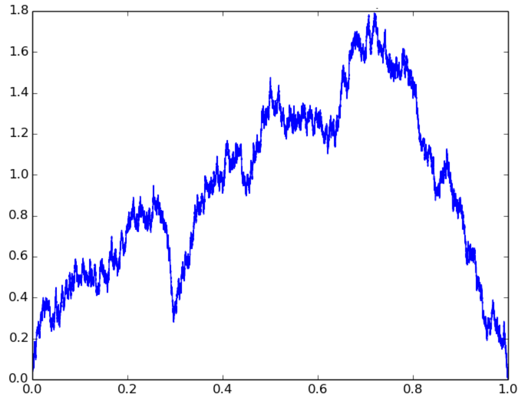


Figure 3.7: Brownian excursion

Suppose now C_{τ_n} is the contour function associated with the contour process of a plane tree τ_n with n edges. Similar to (3.3.1) we set

$$\tilde{C}_{\tau_n}(t) := \frac{C_{\tau_n}(2nt)}{\sqrt{2n}} \tag{3.4.1}$$

for $t \in [0,1]$ and call this the **normalized contour function** of τ_n. Then, as a consequence of Donsker's theorem and motivated by Lemma 15, the following holds:

Theorem 8. *Let $(\tau_n)_{n\geq 0}$ be a sequence of random plane trees with $\tau_n \in \mathcal{P}_n$ and associated contour functions $C_{\tau_n} \in \mathcal{C}_n$. Then on the space of continuous, non-negative real-valued functions on $[0,1]$ together with the metric induced by the supremum norm $||.||_\infty$, the sequence $\{\tilde{C}_{\tau_n} : n \geq 1\}$ of normalized contour functions, as defined in (3.4.1), converges in distribution to a normalized Brownian excursion $\mathbb{e} = \{\mathbb{e}_t : t \in [0,1]\}$, i.e.*

$$\tilde{C}_{\tau_n} \xrightarrow{d} \mathbb{e}.$$

For a proof we refer to Le Gall and Miermont [21, Theorem 2.10]. Theorem 8 is the key to unveil geometric properties of large random trees as we shall see now.

Corollary 8. *Let $(\tau_n)_{n\geq 0}$ be a sequence of random plane trees, i.e. $\tau_n \in \mathcal{P}_n$ with root vertex $v_0^n \in V(\tau_n)$. Then the following convergences in distribution hold, where $\mathbb{e} = \{\mathbb{e}_t : t \in [0,1]\}$ is the normalized Brownian excursion:*

1. *The maximal graph distance from the root vertex v_0^n as declared in Definition 14 satisfies*

$$\frac{R_{\tau_n}(v_0^n)}{\sqrt{2n}} \xrightarrow{d} \sup_{t\in[0,1]} \mathbb{e}_t.$$

2. For a uniformly chosen vertex $u_n \in V(\tau_n)$ there holds

$$\frac{d_{\tau_n}(u_n, v_0^n)}{\sqrt{2n}} \xrightarrow{d} \mathbb{e}_U,$$

where U is a uniform random variable in $[0,1]$ independent of $\mathbb{e}$.

Proof. The first point follows immediately from Theorem 8 by observing that the maximal graph distance from the root vertex v_0^n of τ_n is nothing other than

$$R_{\tau_n}(v_0^n) = \sup_{t\in[0,2n]} \tilde{C}_{\tau_n}(t),$$

where $\tilde{C}_{\tau_n}$ is the normalized contour function associated with τ_n. In order to prove the second point we would also like to interpret $d_{\tau_n}(u_n, v_0^n)$ in terms of $\tilde{C}_{\tau_n}$ so that we are again in the position to apply Theorem 8.
First we notice that it is sufficient to prove the convergence when u_n is replaced by a uniformly chosen vertex of τ_n that is distinct from the root vertex v_0^n.
Suppose $t \in [0, 2n)$ and let $(w_i^n)_{i=0,\dots,n}$ be the vertex sequence associated with the contour exploration of τ_n. We define a function $\langle . \rangle$ by setting

$$\langle t\rangle := \begin{cases} \lceil t\rceil, & \text{if } C_{\tau_n} \text{ has positive slope immediately after } t,\\ \lfloor t\rfloor, & \text{otherwise,}\end{cases}$$

where $\lfloor t\rfloor := \max\{k \in \mathbb{Z} : k \le t\}$ and $\lceil t\rceil := \min\{k \in \mathbb{Z} : k \ge t\}$. Then, if $u_n \in V(\tau_n)\backslash\{v_0^n\}$, we have $w^n_{\langle t\rangle} = u_n$ if and only if t is a time when the contour process around τ_n explores one of the two directed edges immediately preceding or succeeding u_n (see Figure 3.8).

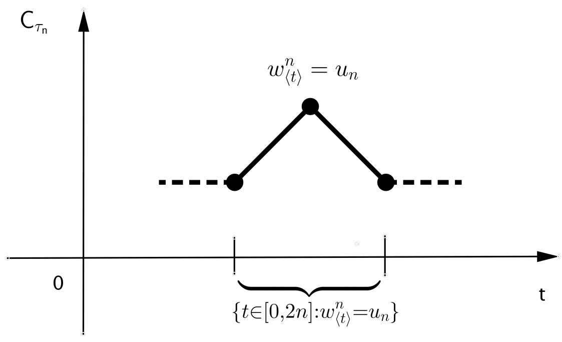


Figure 3.8: Situation for $w^n_{\langle t\rangle} = u_n$

Therefore the Lebesgue measure of $\{t \in [0, 2n) : w^n_{\langle t\rangle} = u_n\}$ equals 2. It follows that if U is a uniformly chosen random variable in $[0,1]$ and independent of τ_n, then $w^n_{\langle 2nU\rangle}$ is uniform in $V(\tau_n)\backslash\{v_0^n\}$. Hence, it suffices to prove the desired result with $w^n_{\langle 2nU\rangle}$ instead of u_n. Thus we have

$$\frac{d_{\tau_n}(w^n_{\langle 2nU\rangle}, v_0^n)}{\sqrt{2n}} = \frac{C_{\tau_n}(\langle 2nU\rangle)}{\sqrt{2n}} = \frac{C_{\tau_n}(2n\frac{\langle 2nU\rangle}{2n})}{\sqrt{2n}} = \tilde{C}_{\tau_n}\left(\frac{\langle 2nU\rangle}{2n}\right).$$

The requirements of Theorem 8 are satisfied because $U \in [0,1] \Leftrightarrow 2nU \in [0,2n] \Leftrightarrow \frac{\langle 2nU\rangle}{2n} \in [0,1]$ and therefore

$$\frac{d_{\tau_n}(w^n_{\langle 2nU\rangle}, v_0^n)}{\sqrt{2n}} = \tilde{C}_{\tau_n}\left(\frac{\langle 2nU\rangle}{2n}\right) \xrightarrow{d} \mathbb{e}_U.$$

□

It can be shown that the distribution of $\mathbb{e}_U$ in Theorem 8 obeys the following law

$$\mathbb{P}\{\mathbb{e}_U > x\} = e^{\frac{-x^2}{2}},$$

for $x \geq 0$. This is the so-called Rayleigh distribution. Due to Chung [12] one finds the distribution of $\sup_{t\in[0,1]} \mathbb{e}_t > x$ to be

$$\mathbb{P}\left\{\sup_{t\in[0,1]} \mathbb{e}_t > x\right\} = 2\sum_{j=1}^{\infty}(4j^2x^2-1)e^{-2j^2x^2} = \frac{\sqrt{2}\pi^{\frac{5}{2}}}{x^3}\sum_{j=1}^{\infty} j^2 e^{-\frac{\pi^2 j^2}{2x^2}}, \tag{3.4.2}$$

for $x > 0$. (3.4.2) is called Theta distribution.

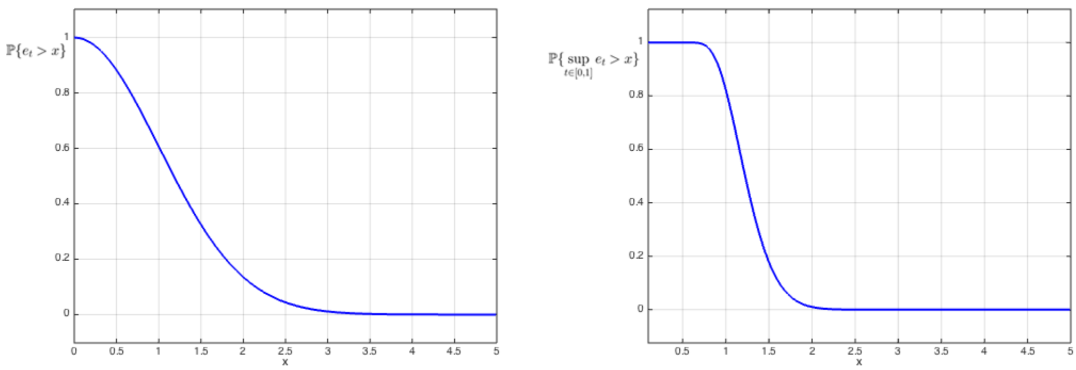


Figure 3.9: Rayleigh (left) and approximated Theta (right) distributions

4 The Brownian Continuum Random Tree

In the previous chapter we have studied metric properties of large random trees with the help ouf their corresponding contour processes. Such investigations have a long history and yielded indeed very interesting results. A cornerstone was laid by Aldous [2, 3, 4] around 1991. He didn't just investigate asymptotic properties of random trees, but was also interested in a possible limiting object of a sequence of random trees itself. This "limiting tree" is the so-called Brownian continuum random tree (CRT) which falls into a class of trees that are commonly referred to as real-trees. The aim of this chapter is to construct the CRT and to discuss its relationship with large random plane trees.

4.1 Excursion Functions

Before we discuss the Brownian CRT in depth, we want to ground our survey in the knowledge that we have already established. This also motivates the construction of the CRT. First observe that the relationship between a plane tree and its contour process is of great importance. Roughly speaking we have seen so far, that (after normalization) the asymptotic behaviour of a sequence of contour functions can be interpreted in terms of the Brownian excursion. In taking this leap, we somehow left the discrete world of the contour process and stepped into the realm of an object whose laws are governed by probability theory. Nevertheless, one might suggest that Brownian excursion can be used to describe a certain "limiting-tree" in the same way as the contour process describes a finite plane tree. Actually, this object turns out to be the Brownian CRT.

We start by giving an equivalent description of the natural distance on a plane tree and follow the development in [29, Chapter 3.2]:

Lemma 18. *Let $\tau_n \in \mathcal{P}_n$ with its associated contour exploration $(e_0^n, ..., e_{2n-1}^n)$ and contour function $C_{\tau_n} \in \mathcal{C}_n$. Define the function $\check{C}_{\tau_n}$ by setting*

$$\check{C}_{\tau_n}(i,j) := \inf\{C_{\tau_n}(k) : k \in \mathbb{N},\ i \le k \le j\} \tag{4.1.1}$$

for $i, j \in \{0, ..., 2n-1\}$ such that $i \le j$. Then the natural distance d_{τ_n} on τ_n can be written as

$$d_{\tau_n}(v_i^n, v_j^n) = C_{\tau_n}(i) + C_{\tau_n}(j) - 2\check{C}_{\tau_n}(i,j) \tag{4.1.2}$$

where $v_r^n := (e_r^n)^-$ as in the discussion after Definition 25.

Proof. Denote with $v_i^n \wedge v_j^n$ the highest common ancestor in τ_n, i.e. $v_i^n \wedge v_j^n$ is the vertex on the highest level in τ_n on the unique path between v_i^n and v_j^n that is smaller than (or equal to) the levels of v_i^n and v_j^n (see Figure 4.1). This means $v_i^n \wedge v_j^n = v_k^n$ if and only if $C_{\tau_n}(k) = \check{C}_{\tau_n}(i,j)$. Therefore $v_i^n \wedge v_j^n$ is on level $\check{C}_{\tau_n}(i,j)$. Also, the unique path from v_i^n to v_j^n, which goes from v_i^n down to $v_i^n \wedge v_j^n$ and then from $v_i^n \wedge v_j^n$ up to v_j^n, has length $d_{\tau_n}(v_i^n, v_j^n) = C_{\tau_n}(i) + C_{\tau_n}(j) - 2\check{C}_{\tau_n}(i,j)$. Notice that $v_i^n \wedge v_j^n = v_i^n$ if and only if $v_i^n \preccurlyeq v_j^n$, where $\preccurlyeq$ is the partial order on τ_n as defined in (1.2.1). $\square$

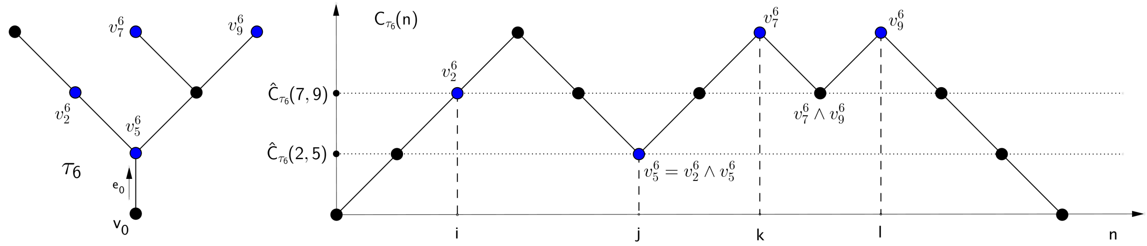


Figure 4.1: Example for $\hat{C}_{\tau_n}$

Observe that (4.1.2) does not depend on the choice of the indices i and j, meaning that if we choose i' and j' such that $v_i^n = v_{i'}^n$ and $v_j^n = v_{j'}^n$, we get the same result because the contour function depends only on the level of the vertices and not on their order in the contour exploration.

Lemma 18 suggests how a function might look like if it is to act as a contour function of a tree-like object. We extend this idea to describe "continuous" trees.

Definition 39. Let $g : [0,1] \to \mathbb{R}_+$ be a non-negative, continuous function with $g(0) = g(1) = 0$. We call g an **excursion function**. For $s, t \in [0,1]$, let

$$\check{g}(s,t) := \inf\{g(u) : \min\{s,t\} \le u \le \max\{s,t\}\}$$

and set

$$d_g(s,t) := g(s) + g(t) - 2\check{g}(s,t). \tag{4.1.3}$$

Definition 40. Let (X, d) be an arbitrary space with $d : X \times X \to \mathbb{R}_+$ satisfying $d(x,x) = 0$ and $d(x,y) = d(y,x)$ for every $x, y \in X$. We say that (X, d) satisfies the **four point condition** if for every $s, t, u, v \in X$,

$$d(s,t) + d(u,v) \le \max\{d(s,u) + d(t,v), d(s,v) + d(t,u)\}.$$

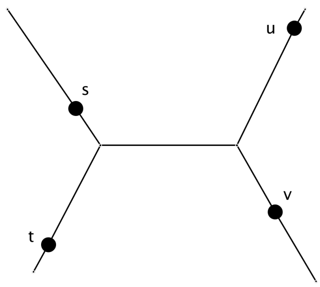


Figure 4.2: The four point condition holds on a tree (with leaves s, t, u and v): $d(s,t) + d(u,v) \leq d(s,u) + d(t,v) = d(s,v) + d(t,u)$

We observe the following property of the four point condition:

Lemma 19. *The four point condition implies the triangular inequality.*

Proof. Let (X, d) be as in Definition 40 and choose $s, t, u \in X$ arbitrarily. By setting $v := u$ we get

$$
\begin{aligned}
d(s,t) &= d(s,t) + d(u,v) \\
&\leq \max\{d(s,u) + d(t,u), d(s,u) + d(t,u)\} \\
&= d(s,u) + d(t,u) = d(s,u) + d(u,t)
\end{aligned}
$$

which is the triangular inequality. $\square$

The previous lemma states that the four point condition is a generalization of the triangular inequality. It is also closely related to real trees as we shall see soon.

Lemma 20. *Let g be an excursion function and d_g as in (4.1.3). Then for every $s, t, u, v \in [0,1]$, the four point condition for d_g is equivalent to*

$$\breve{g}(s,u) + \breve{g}(u,v) \geq \min\{\breve{g}(s,u) + \breve{g}(t,v), \breve{g}(s,v) + \breve{g}(t,u)\}.$$

Proof. This follows by applying simple transformations:

$$
\begin{aligned}
d_g(s,t) + d_g(u,v) &\leq \max\{d_g(s,u) + d_g(t,v), d_g(s,v) + d_g(t,u)\} \Leftrightarrow \\
g(s) + g(t) - 2\breve{g}(s,t) + g(u) + g(v) - 2\breve{g}(u,v) &\leq \max\left\{ \begin{array}{c} g(s)+g(u)-2\breve{g}(s,u)+g(t)+g(v)-2\breve{g}(t,v), \\ g(s)+g(v)-2\breve{g}(s,v)+g(t)+g(u)-2\breve{g}(t,u) \end{array} \right\} \Leftrightarrow \\
-\breve{g}(s,t) - \breve{g}(u,v) &\leq \max\{-\breve{g}(s,u) - \breve{g}(t,v), -\breve{g}(s,v) - \breve{g}(t,u)\} \Leftrightarrow \\
-(\breve{g}(s,t) + \breve{g}(u,v)) &\leq -\max\{\breve{g}(s,u) + \breve{g}(t,v), \breve{g}(s,v) + \breve{g}(t,u)\} \Leftrightarrow \\
\breve{g}(s,t) + \breve{g}(u,v) &\geq \min\{\breve{g}(s,u) + \breve{g}(t,v), \breve{g}(s,v) + \breve{g}(t,u)\}.
\end{aligned}
$$

$\square$

Lemma 21. *The function d_g satisfies the four point condition for every excursion function g.*

Proof. Accorindg to Lemma 20 it is sufficient to verify

$$\breve{g}(s,t)+\breve{g}(u,v) \geq \min\{\breve{g}(s,u)+\breve{g}(t,v), \breve{g}(s,v)+\breve{g}(t,u)\}$$

for every $s,t,u,v \in [0,1]$. This is not very difficult to prove, but a little tedious. We differentiate the following cases:

1. $s \leq t \leq u \leq v$: We have $\breve{g}(s,u) \leq \breve{g}(s,t)$ and $\breve{g}(t,v) \leq \breve{g}(u,v)$. This implies $\breve{g}(s,u)+\breve{g}(t,v) \leq \breve{g}(s,t)+\breve{g}(u,v)$ and hence $\min\{\breve{g}(s,u)+\breve{g}(t,v), \breve{g}(s,v)+\breve{g}(t,u)\} \leq \breve{g}(s,t)+\breve{g}(u,v)$.

2. $s \leq u \leq t \leq v$: We further distinguish:

   a) $\breve{g}(s,v) = \breve{g}(u,t)$: Then this is also equal to $\breve{g}(s,t) = \breve{g}(u,v)$ and the result follows.

   b) $\breve{g}(s,v) = \breve{g}(s,u)$ and $\breve{g}(u,t) \leq \breve{g}(t,v)$: Then $\breve{g}(s,t) = \breve{g}(s,v)$ and $\breve{g}(u,v) = \breve{g}(u,t)$ and again the result follows.

   c) $\breve{g}(s,v) = \breve{g}(s,u)$ and $\breve{g}(t,v) \leq \breve{g}(u,t)$: Then $\breve{g}(s,t) = \breve{g}(s,u)$ and $\breve{g}(u,v) = \breve{g}(t,v)$ from which the result follows. The remaining cases in Case 2 are symmetric to those already discussed.

3. $s \leq u \leq v \leq t$:

   a) $\breve{g}(s,t) = \breve{g}(u,v)$: Then these quantities are equal to $\breve{g}(s,u) = \breve{g}(t,v)$ from which the claim follows.

   b) $\breve{g}(s,t) = \breve{g}(s,u) \leq \breve{g}(u,v) \leq \breve{g}(v,t)$: Then $\breve{g}(s,t) = \breve{g}(s,v)$ and $\breve{g}(u,v) = \breve{g}(u,t)$.

   c) $\breve{g}(s,t) = \breve{g}(s,u) \leq \breve{g}(v,t) \leq \breve{g}(u,v)$: From this the result follows directly.

All remaining case are symmetric to the ones discussed above.

□

Proposition 7. *For every excursion function g the space $([0,1], d_g)$ is a pseudometric space.*

Proof. The symmetry of d_g and $d_g(s,s) = 0$ for every $s \in [0,1]$ are obvious. The triangular inequality is a direct consequence of Lemma 21 together with Lemma 19. □

4.2 Real Trees

Aldous approach in finding a limit for certain sequences of finite trees was to identify those trees as continuous functions or compact subsets of the Banach space of absolutely summable sequences $\ell^1 = \{(x_i)_{i\in\mathbb{N}} : \sum_{n\in\mathbb{N}} |x_i| < \infty\}$. Without going into more detail, this view reminds one of the use of coordinates instead of working with more abstract objects. Another possibility is to interpret trees as topological spaces or, more precisely as metric spaces. This might seem unfamiliar at first, so let us motivate this view: Suppose τ is an arbitrary tree embedded into the plane such that $\tau \subseteq \mathbb{R}^2$ and all edges are straight lines with length one. Let us also assume that the edges only intersect at their incident vertices. In this way we can provide a natural metric on τ by defining the distance between two vertices as the length of the unique path through the tree that connects them. The uniqueness of this path is guaranteed by the tree-structure. This leads to the definition of so-called real trees. An elobarete exploration of the theory of real trees can be found in [15, Chapter 3.3].

Definition 41. A metric space $(\mathcal{T}, d)$ is a **real tree** if the following properties hold for every $x, y \in \mathcal{T}$,

(R_1) There is a unique isometric map $\phi_{x,y} : [0, d(x,y)] \to \mathcal{T}$ such that $\phi_{x,y}(0) = x$ and $\phi_{x,y}(d(x,y)) = y$.

(R_2) If $\tilde{\phi} : [0,1] \to \mathcal{T}$ is a continuous injective map with $\tilde{\phi}(0) = x$ and $\tilde{\phi}(1) = y$, then

$$\tilde{\phi}([0,1]) = \phi_{x,y}([0, d(x,y)]).$$

Notice that Definition 41 actually reassembles the properties of a tree (in the graph theoretical sense) in a continuous way: (R_1) guarantees the uniqueness of a "path" between x and y in $\mathcal{T}$ and (R_2) insists that $\mathcal{T}$ is free of "loops". Since we are also interested in rooted trees, we extend this definition in the following way:

Definition 42. A **rooted real tree** $(\mathcal{T}, d, \rho)$ is a real tree with a distinguished point $\rho \in \mathcal{T}$ that we call **root** in analogy to the root vertex of plane trees.

Quite an interesting description of a real tree can be given in terms of the four point condition. For a proof, see for instance [13, 11].

Proposition 8. *Let (X, d) be a connected metric space. Then (X, d) is a real tree if and only if it satisfies the four point condition.*

Suppose now that g is an excursion function as in Definition 39. According to Lemma 21 the space $([0,1], d_g)$ is a pseudometric space. We define a binary relation $\approx_g$ on $[0,1]$ in the following way: For $s, t \in [0,1]$ we let $s \approx_g t$ if and only if $d_g(s,t) = 0$, i.e. $\approx_g = \{(s,t) \in [0,1]^2 : d_g(s,t) = 0\}$. Obviously, $\approx_g$ is an equivalence relation. The idea is that every $s \in [0,1]$ can be interpreted as a vertex of a tree where we identify s and t if and only if

$$g(s) = g(t) = \check{g}(s,t).$$

More precisely, let $\mathcal{T}_g := [0,1]/\approx_g$ be the quotient set and denote with

$$\pi_g : [0,1] \to \mathcal{T}_g \tag{4.2.1}$$

the canonical projection. We can equip $\mathcal{T}_g$ with a distance function by setting

$$d_{\mathcal{T}_g}(\pi_g(x), \pi_g(y)) := d_g(x, y). \tag{4.2.2}$$

The function $d_{\mathcal{T}_g}$ is well-defined: Suppose $\pi_g(x) = \pi_g(x')$ and $\pi_g(y) = \pi_g(y')$. We have to show that $d_{\mathcal{T}_g}(\pi_g(x), \pi_g(y)) = d_{\mathcal{T}_g}(\pi_g(x'), \pi_g(y'))$. The triangular inequality implies

$$d_g(x, y) \leq d_g(x, x') + d_g(x', y) = d_g(x', y) \leq d_g(x', y') + d_g(y', y) = d_g(x', y')$$

and

$$d_g(x', y') \leq d_g(x', x) + d_g(x, y') = d_g(x, y') \leq d_g(x, y) + d_g(y, y') = d_g(x, y).$$

Putting these together we get $d_g(x, y) \leq d_g(x', y') \leq d_g(x, y)$ and therefore

$$d_{\mathcal{T}_g}(\pi_g(x), \pi_g(y)) = d_g(x, y) = d_g(x', y') = d_{\mathcal{T}_g}(\pi_g(x'), \pi_g(y')).$$

Lemma 22. *For every excursion function g the space $(\mathcal{T}_g, d_{\mathcal{T}_g})$ is a metric space.*

Proof. It is clear that $d_{\mathcal{T}_g}$ is a pseudometric because all properties of d_g are inherited. What is left open is to show that $d_{\mathcal{T}_g}(\pi_g(x), \pi_g(y)) = 0$ implies $\pi_g(x) = \pi_g(y)$. This follows from the definition of $\approx_g$ because $d_{\mathcal{T}_g}(\pi_g(x), \pi_g(y)) = 0$ is equal to $d_g(x, y) = 0$ which entails $x \approx_g y$ and hence $\pi_g(x) = \pi_g(y)$. □

We can also equip $(\mathcal{T}_g, d_{\mathcal{T}_g})$ with a partial order $\preccurlyeq$ by setting $s \preccurlyeq t$ for $s, t \in \mathcal{T}_g$ if and only if

$$g(s) = \check{g}(s, t)$$

Proposition 9. *For every excursion function g the space $(\mathcal{T}_g, d_{\mathcal{T}_g})$ is a compact real tree.*

Proof. First observe that the canonical projection π_g between the metric spaces $([0, 1], d_g)$ and $(\mathcal{T}_g, d_{\mathcal{T}_g})$ is obviously continuous because $d_{\mathcal{T}_g}(\pi_g(x), \pi_g(y)) = d_g(x, y)$. Therefore, $(\mathcal{T}_g, d_{\mathcal{T}_g})$ is compact and connected as a continuous image of the compact and connected space $([0, 1], d_g)$. The result follows now from Lemma 21 and Proposition 8. □

Proposition 9 tells us that we can construct a compact real tree with the help of an excursion function. We say that the real tree $\mathcal{T}_g$ is coded by the excursion function g. Notice also that we can interpret $(\mathcal{T}_g, d_{\mathcal{T}_g})$ as rooted real tree by identifiying the equivalence class $\rho_{\mathcal{T}_{\mathbb{e}}} := \pi_g(0) = \pi_g(1)$ as the root.

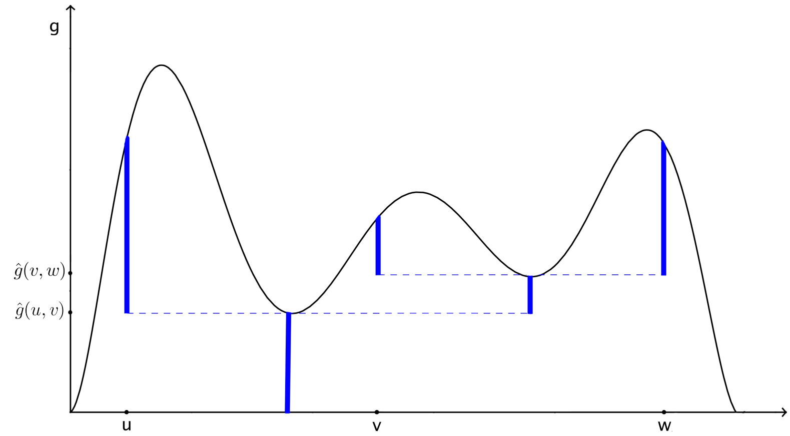


Figure 4.3: Coding of a subtree (blue) of a real tree by its excursion function. This image can be seen as a continuous analog of Figure 4.1.

To simplify the notation we abbreviate the distance function $d_{\mathcal{T}_g}$ of a real tree $(\mathcal{T}_g, d_{\mathcal{T}_g})$ with d_g from now on. This is justified by (4.2.2). It should be clear from the context if d_g is a distance on $[0,1]$ or on the real tree $\mathcal{T}_g$. We also set $\rho_g := \rho_{\mathcal{T}_g}$ for the root.

In analogy to the graph theoretical notions of "degree" and "leaf", we can define the following:

Definition 43. Let $(\mathcal{T}, d)$ be a real tree. For every $x \in \mathcal{T}$, the **degree** $\deg(x)$ of x is the number of connected components of $\mathcal{T}\backslash\{x\}$. The set $\mathrm{Lf}(\mathcal{T})$ is the set of all $x \in \mathcal{T}$ such that $\mathcal{T}\backslash\{x\}$ is connected, i.e. $\deg(x) = 1$. A point $x \in \mathrm{Lf}(\mathcal{T})$ is called a **leaf** of $\mathcal{T}$. The complements of leaves, denoted by $\mathrm{Sk}(\mathcal{T})$, is the **skeleton** of $\mathcal{T}$.

Observe also that every real tree $\mathcal{T}_g = [0,1]/\approx_g$ induces a natural probability measure on $\mathcal{T}_g$ by setting

$$\lambda_g := \lambda \circ \pi_g^{-1} \tag{4.2.3}$$

where λ denotes the Lebesgues-measure on $[0,1]$ and π_g is again the canonical projection from $[0,1]$ to $\mathcal{T}_g$. This measure allows us to convert random points in $[0,1]$ to random variables in $\mathcal{T}_g$, i.e. if $t \in [0,1]$ is uniformly chosen at random, then we can view $\pi_g(t)$ as uniformly, λ_g-distributed random variable in $\mathcal{T}_g$.

In the world of real trees there is also a natural analogue corresponding to the fact, that for any two distinct vertices in a discrete tree, there exists a unique path between them.

Definition 44. Let $(\mathcal{T}, d)$ be a real tree. Any isometric embedding $\phi_{x,y} : [0, d(x,y)] \to \mathcal{T}$ with $\phi_{x,y}(0) = x$ and $\phi_{x,y}(d(x,y)) = y$ is called a **geodesic path** and its image $\phi_{x,y}([0, d(x,y)])$ a **geodesic segment** between x and y in $(\mathcal{T}, d)$.

Notice that the existence of geodesic paths follows immediately from the definition of a real tree (in the same way as the existence of a unique path between to distinct vertices in a discrete tree follows from its graph theoretical definition). Furthermore, a geodesic path between x and y is uniquely determined by x and y. The following lemma gives an explicit description of the geodesic paths from/to the root:

Lemma 23. *Let $(\mathcal{T}_g, d_{\mathcal{T}_g}, \rho_g)$ be a rooted real tree and set for every $t \in [0,1]$ with $0 \leq r \leq g(t)$,*

$$\gamma_t^+(r) := \inf\{s \geq t : g(s) < g(t) - r\},$$
$$\gamma_t^-(r) := \sup\{s \leq t : g(s) < g(t) - r\}$$

where $\inf \emptyset := 1$ and $\sup \emptyset := 0$. Denote this common value with $\Gamma_t(r) := \pi_g(\gamma_t^-(r)) = \pi_g(\gamma_t^+(r))$ for every $r \in [0, g(t)]$. Then Γ_t is the (unique) geodesic path from $\pi_g(t)$ to the root ρ_g.

Proof. At first we observe $d_g(\gamma_t^-(r), \gamma_t^+(r)) = 0$ for every r. Therefore, $d_{\mathcal{T}_g}(\pi_g(\gamma_t^-(r)), \pi_g(\gamma_t^+(r))) = d_g(\gamma_t^-(r), \gamma_t^+(r)) = 0$ and hence $\pi_g(\gamma_t^-(r)) = \pi_g(\gamma_t^+(r))$. We denote this common value with $\Gamma_t(r)$ as in the claim of the lemma. Γ_t is a function from $[0, d_{\mathcal{T}_g}(\pi_g(t), \pi_g(0))] = [0, g(t)]$ into $\mathcal{T}_g$, since $d_{\mathcal{T}_g}(\pi_g(t), \pi_g(0)) = d_g(t, 0) = g(t)$. It satisfies $\Gamma_t(0) = \pi_g(\gamma_t^+(0)) = \pi_g(t)$ and

$$\begin{aligned}
\Gamma_t(d_{\mathcal{T}_g}(\pi_g(t), \pi_g(0))) &= \Gamma_t(d_g(t, 0)) \\
&= \Gamma_t(g(t) + g(0) - 2\check{g}(t, 0)) \\
&= \Gamma_t(g(t) + g(0) - 2g(0)) \\
&= \Gamma_t(g(t)) \\
&= \pi_g(\gamma_t^-(g(t))) \\
&= \pi_g(0),
\end{aligned}$$

as required by the property (R_1) (see Definition 41). What remains open is to check the isometry condition of Γ_t:

$$\begin{aligned}
d_{\mathcal{T}_g}(\Gamma_t(r), \Gamma_t(r')) &= d_{\mathcal{T}_g}(\pi_t(\gamma_t^+(r)), \pi_t(\gamma_t^+(r'))) \\
&= d_g(\pi_t(\gamma_t^+(r)), \pi_t(\gamma_t^+(r'))) \\
&= g(\gamma_t^+(r)) + g(\gamma_t^+(r')) - 2\check{g}(\gamma_t^+(r), \gamma_t^+(r')) \\
&= g(t) - r + g(t) + r' - 2(g(t) - r') \\
&= r' - r,
\end{aligned}$$

for every $0 \leq r \leq r' \leq g(t)$. This completes the proof. $\square$

Finally, we use the theory about real trees to define the Brownian CRT:

Definition 45. The Brownian CRT is the compact rooted real tree $(\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}}, \rho_{\mathbb{e}})$ where $\mathbb{e}$ is the normalized Brownian excursion as in Definition 38.

Notice that the real tree $(\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}}, \rho_{\mathbb{e}})$ is well-defined because the normalized Brownian excursion is a random continuous function satisfiying all the properties of an excursion function.

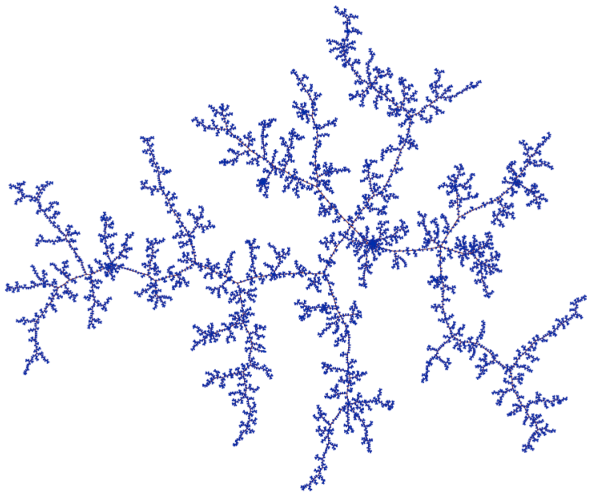


Figure 4.4: Visualization of a large random tree, which is an approximation of the Brownian CRT (by Igor Kortchemski).

4.3 The Gromov-Hausdorff Topology

In the previous section we have constructed the Brownian CRT $\mathcal{T}_{\mathbb{e}}$ with the help of the normalized Brownian excursion $\mathbb{e}$. We already know that the normalized Brownian excursion $\mathbb{e}$ is a stochastic process on the space of continuous real-valued functions on $[0,1]$ endowed with the Borel $\sigma-$field obtained by the Euclidean topology on $[0,1]$. In order to view the compact real tree $(\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}})$ itself as a random variable, we have to provide a meaningful $\sigma-$field on the set of all compact real trees. The aim of this section is to construct such a $\sigma-$field with the help of the so-called Gromov-Hausdorff topology. By following [15, Chapter 4.2-4.3], we take a step back and look at the problem from a more general viewpoint at first.

Definition 46. Let (X, d) be a metric space and $A, B \subseteq X$ non-empty subsets. The **Hausdorff distance** is defined by

$$d_H(A,B) := \max\left\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\right\}.$$

We give an equivalent description of the Hausdorff distance that might be more intuitiv: Denote with $B_r(x)$ the open ball of radius $r > 0$ centered at x. Then the Hausdorff distance is given by

$$d_H(A,B) = \inf\{r > 0 : A \subseteq U_r(B) \wedge B \subseteq U_r(A)\},$$

where $U_r(S) := \bigcup_{x \in S} B_r(x)$ for $S \subseteq X$. Heuristically speaking, the Hausdorff distances measures how far two subsets of a metric space are from each other (see Figure 4.5). On the space of non-empty subsets of a metric space, d_H is a pseudometric. If we condition the non-empty subsets to be closed, then d_H is a metric. We will now use the Hausdorff distance to introduce a distance between compact metric spaces.

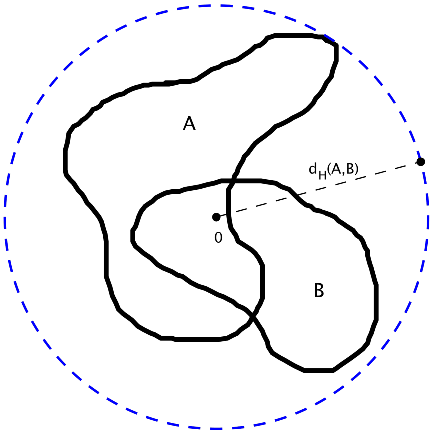


Figure 4.5: The Hausdorff distance between the sets A and B

Definition 47. Let (X, d_X) and (Y, d_Y) be compact metric spaces. The **Gromov-Hausdorff distance** $d_{GH}(X,Y) = d_{GH}((X,d_X),(Y,d_Y))$ between these spaces is defined by

$$d_{GH}(X,Y) := \inf d_H(\phi_X(X), \phi_Y(Y)), \tag{4.3.1}$$

where the infimum is taken over all metric spaces (Z, d_Z) and all isometric embeddings $\phi_X : X \to Z$ and $\phi_Y : Y \to Z$ from X and Y into Z (see Figure 4.6).

Remembering property (R1) of a real tree, we should mention that the mappings ϕ_X and ϕ_Y are actually sets of mappings $\phi_X = \{\phi^X_{a,b} : [0, d_X(a,b)] \to X : a, b \in X\}$ and likewise $\phi_Y = \{\phi^Y_{a,b} : [0, d_Y(a,b)] \to Y : a, b \in Y\}$. For the upcoming discussion we will stick with the slightly incorrect but simpler notation as in (4.3.1) and ask the reader to keep this in mind.

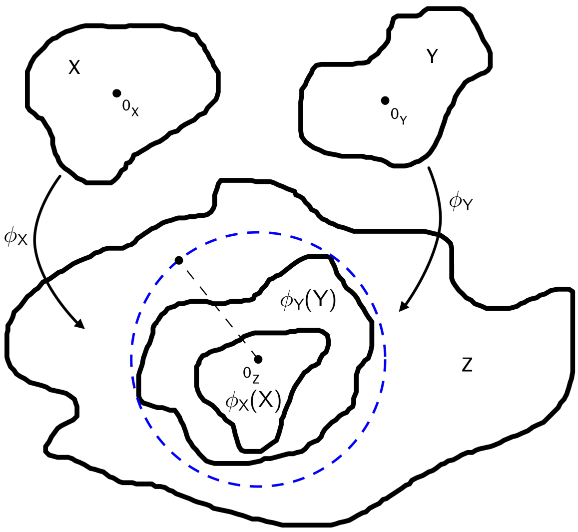


Figure 4.6: One step in the evaluation of the Gromov-Hausdorff distance between the compact metric spaces (X, d_X) and (Y, d_Y) by computing the Hausdorff metric between the isometric copies $\phi_X(X)$ and $\phi_Y(Y)$ in (Z, d_Z).

The Definition of the Gromov-Hausdorff distance makes it necessary to consider all possible embedding spaces (Z, d_Z). This is quite unhandy. In fact it is enough to consider the disjoint union $X\dot{\cup}Y$ of X and Y and all possible metrices $d_{X\dot{\cup}Y}$ on $X\dot{\cup}Y$ whose restrictions to X and Y coincide with d_X and d_Y. In that way, d_{GH} is equal to

$$d_{GH}(X, Y) = \inf\{r > 0 : d_H(X, Y) < r\}$$

where $d_H(X, Y)$ is evaluated in the space $(X\dot{\cup}Y, d_{X\dot{\cup}Y})$ under the conditions mentioned above. Although this is an improvement, it still leaves the question of how to find an optimal metric on $X\dot{\cup}Y$ unanswered.

Definition 48. Let X and Y be two sets. A **correspondence** between X and Y is a subset $\mathcal{R} \subseteq X \times Y$ such that for every $x \in X$ there exists at least one $y \in Y$ for which $(x, y) \in \mathcal{R}$ and similarly for every $y \in Y$ there exists an $x \in X$ for which $(x, y) \in \mathcal{R}$. We let $\mathrm{Cor}(X, Y)$ be the set of all correspondences between X and Y.

Definition 48 is equivalent to saying that the restrictions of the two canonical projections $X \times Y \to X$ and $X \times Y \to Y$ to $\mathcal{R}$ are surjective. Correspondences will give us a more natural description of the Gromov-Hausdorff distance.

Definition 49. Let $\mathcal{R}$ be a correspondence between metric spaces (X, d_X) and (Y, d_Y). The **distortion** $\mathrm{dis}(\mathcal{R}) = \mathrm{dis}(\mathcal{R}, d_X, d_Y)$ of $\mathcal{R}$ is defined as

$$\mathrm{dis}(\mathcal{R}) := \sup\{|d_X(x, y) - d_Y(x', y')| : (x, x'), (y, y') \in \mathcal{R}\}.$$

Theorem 9. *For any two metric spaces* (X, d_X) *and* (Y, d_Y),

$$d_{GH}(X,Y) = \frac{1}{2}\inf\{dis(\mathcal{R}) : \mathcal{R} \in Cor(X,Y)\}.$$

For a proof we refer to [15, Theorem 4.11].

Definition 50. Let $(\mathbf{T}, d_{GH})$ be the space of isometry classes of compact real trees equipped with the Gromov-Hausdorff distance.

We collect some important properties of $(\mathbf{T}, d_{GH})$. For a proof of these results we refer to [15, Theorem 4.14] and [15, Theorem 4.23].

Theorem 10. $(\mathbf{T}, d_{GH})$ *is a metric space.*

Theorem 11. $(\mathbf{T}, d_{GH})$ *is complete and separable.*

Since we are mainly interested in rooted trees, we now extend the definition of aWe now define the rooted Gromov-Hausdorff distance for $(\mathbf{T}, d_{GH})$ in a similar way to Definition 47.

Definition 51. Let (X, d_X, ρ_X) and (Y, d_Y, ρ_Y) be two rooted real trees. The **rooted Gromov-Hausdorff** distance $d_{GH}^r(X,Y) = d_{GH}^r((X, d_X, \rho_X), (Y, d_Y, \rho_Y))$ between these two spaces is defined by

$$d_{GH}^r(X,Y) := \inf \max\{d_H(\phi_X(X), \phi_Y(Y)), d_Z(\phi_X(\rho_X), \phi_Y(\rho_Y))\}, \tag{4.3.2}$$

where the infimum is taken over all metric spaces (Z, d_Z) and all isometric embeddings $\phi_X : X \to Z$ and $\phi_Y : Y \to Z$ from X and Y (as unrooted trees) into Z.

In analogy to Theorem 9 we can describe the rooted Gromov-Hausdorff distance in terms of the distortion

$$d_{GH}^r(X,Y) = \frac{1}{2}\inf\{\mathrm{dis}(\mathcal{R}) : \mathcal{R} \in \mathrm{Cor}(X,Y), (\rho_X, \rho_Y) \in \mathcal{R}\}. \tag{4.3.3}$$

Definition 52. Let $(\mathbf{T}^r, d_{GH}^r)$ be the metric space of isometry classes of compact rooted real trees equipped with the rooted Gromov-Hausdorff metric.

Notice that we have defined d_{GH}^r for pairs of rooted real trees in Definition 51 and not for pairs of isometry classes of compact rooted real trees. Nevertheless the right side of (4.3.2) does not depend on the element of the isometry class of (X, d_X, ρ_X) and (Y, d_Y, ρ_Y) respectively and is therefore well-defined.

For a proof of the next result we refer again to [15, Theorem 4.31].

Theorem 12. $(\mathbf{T}^r, d_{GH}^r)$ *is complete and separable.*

We want to mention at this point that (4.3.3) allows us to define a Gromov-Hausdorff distance for arbitrary triples (X, d_X, ρ_X) and (Y, d_Y, ρ_Y) where (X, d_X) and (Y, d_Y) are metric spaces and ρ_X and ρ_Y are elements of X and Y respectively. This is possible because (4.3.3) does not depend on the mappings ϕ_X and ϕ_Y that are required in order to turn a metric space into a real tree (property (R_1)). In its most general setting, we can even look at triplets (X, d_X, ρ_X), where $\rho_X = (\rho_X^1, \ldots, \rho_X^k) \in X^k$. This gives rise to the following definition that we need in a later chapter:

Definition 53. A $k-$**pointed metric space** is a triple (X, d_X, ρ_X) where (X, d_X) is a metric space and $\rho_X = (\rho_X^1, \ldots, \rho_X^k) \in X^k$. We denote with $(\mathbf{M}_k, d_{GH}^k)$ the space of isometry classes of $k-$pointed compact metric spaces (preserving the distinguished points) equipped with the $k-$**pointed Gromov-Hausdorff distance** d_{GH}^k defined by

$$d_{GH}^k(X, Y) := \frac{1}{2} \inf\{\mathrm{dis}(\mathcal{R}) : \mathcal{R} \in \mathrm{Cor}(X, Y) \wedge (\rho_X^i, \rho_Y^i) \in \mathcal{R} \, \forall i = 1, \ldots, k\},$$

where (Y, d_Y, ρ_Y) is another $k-$pointed metric space. In the case of $k = 1$ we simply call the triple (X, d_X, ρ_X) a **pointed metric space** and d_{GH}^1 the **pointed Gromov-Hausdorff distance**.

It can be shown similarily to Theorem 11 that $(\mathbf{M}_k, d_{GH}^k)$ is complete and seperable.

The next lemma relates real trees and their associated excursion functions, see [20, Lemma 2.3].

Lemma 24. *Let g_1 and g_2 be two excursion functions from $\mathbb{R}_+ \to \mathbb{R}_+$ with compact support and $g_1(0) = g_2(0)$. Then*

$$d_{GH}^r(\mathcal{T}_{g_1}, \mathcal{T}_{g_2}) \leq 2||g_1 - g_2||_\infty,$$

where $\mathcal{T}_{g_i}$ are the rooted real trees associated with g_i for $i = 1, 2$ and $|| \cdot ||_\infty$ denotes the supremum norm.

Proof. We construct a correspondence $\mathcal{R}_{g_1}^{g_2}$ between $\mathcal{T}_{g_1}$ and $\mathcal{T}_{g_2}$ by setting

$$\mathcal{R} := \{(x_1, x_2) : x_1 = \pi_{g_1}(t) \text{ and } x_2 = \pi_{g_2}(t) \text{ for some } t \geq 0\}.$$

Clearly, $\mathcal{R}$ is a correspondence by the surjectivity of π_{g_1} and π_{g_2}. Suppose now $(\sigma_1, \sigma_2), (\eta_1, \eta_2) \in \mathcal{R}$ with $s, t \geq 0$ such that $\pi_{g_1}(s) = \sigma_1, \pi_{g_2}(s) = \sigma_2$ and $\pi_{g_1}(t) = \eta_1, \pi_{g_2}(t) = \eta_2$. Recalling $d_g(x, y) = g(x) + g(y) - 2\check{g}(x, y)$, we compute

$$\begin{aligned} |d_{g_1}(\sigma_1, \eta_1) - d_{g_2}(\sigma_2, \eta_2)| &= |g_1(\sigma_1) - g_2(\sigma_2) + g_1(\eta_1) - g_2(\eta_2) - \\ &\quad 2(\check{g}_1(\sigma_1, \eta_1) - \check{g}_2(\sigma_2, \eta_2))|. \end{aligned} \tag{4.3.4}$$

The functions $\check{g}_i(\sigma_i, \eta_i) = \inf\{g_i(\zeta) : \min(\sigma_i, \eta_i) \leq \zeta \leq \max(\sigma_i, \eta_i)\}$ are bounded by

$$\check{g}_i(\sigma_i, \eta_i) \leq g_i(\sigma_i) \text{ and } \check{g}_i(\sigma_i, \eta_i) \leq g_i(\eta_i)$$

for $i = 1, 2$. Thus we can estimate (4.3.4) with the triangular inequality

$$\begin{aligned} |d_{g_1}(\sigma_1, \eta_1) - d_{g_2}(\sigma_2, \eta_2)| &\leq ||g_1 - g_2||_\infty + ||g_1 - g_2||_\infty + 2||g_1 - g_2||_\infty \\ &= 4||g_1 - g_2||_\infty. \end{aligned}$$

Taking the supremum over all $(\sigma_1, \sigma_2), (\eta_1, \eta_2) \in \mathcal{R}$ yields $\mathrm{dis}(\mathcal{R}) \leq 4||g_1 - g_2||_\infty$. Because $(\rho_{g_1}, \rho_{g_2}) = (\pi_{g_1}(0), \pi_{g_2}(0)) \in \mathcal{R}$, we can use equation (4.3.3) to conclude

$$\begin{aligned} d_{GH}^r(\mathcal{T}_{g_1}, \mathcal{T}_{g_2}) &= \frac{1}{2} \inf\{\mathrm{dis}(\mathcal{R}) : \mathcal{R} \in \mathrm{Cor}(\mathcal{T}_{g_1}, \mathcal{T}_{g_2}), (\rho_{g_1}, \rho_{g_2}) \in \mathcal{R}\} \\ &\leq \frac{1}{2} \mathrm{dis}(\mathcal{R}_{g_1}^{g_2}) \\ &\leq 2||g_1 - g_2||_\infty. \end{aligned}$$

$\square$

In short, Lemma 24 tells us that two (rooted) real trees are close if their excursion functions are close with respect to the corresponding metrics. Observe that Lemma 24 is equally true if we interpret $\mathcal{T}_{g_1}$ and $\mathcal{T}_{g_2}$ as unrooted real trees. Yet another formulation of Lemma 24 is to say that the mapping

$$\Gamma : g \mapsto (\mathcal{T}_g, d_g, \rho_g) \tag{4.3.5}$$

from the space of all excursion functions to $(\mathbf{T}^r, d_{GH}^r)$ is (Lipschitz-) continuous. This allows us to identify the Brownian CRT $\mathcal{T}_{\mathbb{e}}$ as a random variable, being the image under Γ of the normalized Brownian excursion $\mathbb{e}$. More precisely, $(\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}}, \rho_{\mathbb{e}})$ is a random variable taking values in the space $\mathbf{T}^r$ of isometry classes of compact rooted real trees endowed with the Borel $\sigma-$field associated with the rooted Gromov-Hausdorff topology, i.e. the topology induced by the rooted Gromov-Hausdorff distance. This allows us to introduce probabilty theory into the world of real trees.

4.4 Convergence of Plane Trees as Metric Spaces

Equipped with the knowledge about real trees discussed in the previous section, we can now state the following result that allows us to view the CRT as a continuous limit of properly rescaled, discrete random trees (see [29, Theorem 3.3.4]):

Theorem 13. *Let $(\tau_n)_{n\geq 1}$ be a sequence of random plane trees where τ_n is uniformly distributed over $\mathcal{P}_n$ with associated root vertex $v_0^n \in V(\tau_n)$ and natural distance d_{τ_n}. Then the following convergence in distribution holds in the space $(\mathbf{T}^r, d_{GH}^r)$:*

$$\left(V(\tau_n), \frac{d_{\tau_n}}{\sqrt{2n}}, v_0^n\right) \xrightarrow{d} (\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}}, \rho_{\mathbb{e}}). \tag{4.4.1}$$

Proof. We first observe that $\left(V(\tau_n), \frac{d_{\tau_n}}{\sqrt{2n}}, v_0^n\right)$ can not only be viewed as a random variable in the space of all compact metric spaces, but also as a compact rooted real tree. Likewise we can interpret $(\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}}, \rho_{\mathbb{e}})$ as a random variable in the space $\mathbf{T}^r$ duo to Lemma 24 together with equation (4.3.5) and the succeding remarks. Therefore the limit in (4.4.1) makes sense. Next, let $(e_0^n, ..., e_{2n-1}^n)$ be the contour exploration associated with τ_n. We define the mapping

$$h\colon \begin{cases} \{0, 1, ..., 2n\} \times [0, 1] & \to V(\tau_n) \times \mathcal{T}_{\mathbf{e}} \\ (i, t) & \mapsto (u_i^n, \pi_{\mathbf{e}}(t)) \end{cases} \tag{4.4.2}$$

with $u_i^n := (e_i^n)^-$ (see Definition 25) and $\pi_{\mathbb{e}}$ as in (4.2.1). We build a sequence of correspondences $(\mathcal{R}_n)_{n\in\mathbb{N}}$ between $V(\tau_n)$ and $\mathcal{T}_{\mathbb{e}}$ by setting

$$\mathcal{R}_n := h(\{(i, t) : i \in \{0, 1, ..., 2n\} \text{ and } t \in [0, 1] \text{ such that } i = \lfloor 2nt \rfloor\}).$$

The surjectivity of h guarantees that $\mathcal{R}_n$ is indeed a correspondence for every $n \in \mathbb{N}$. We evaluate the distortion of $\mathcal{R}_n$:

$$
\begin{aligned}
\mathrm{dis}(\mathcal{R}_n) &= \sup\left\{\left|\frac{1}{\sqrt{2n}}d_{\tau_n}(x,x')-d_{\mathbb{e}}(y,y')\right| : (x,y),(x',y')\in\mathcal{R}_n\right\}\\
&= \sup\left\{\left|\frac{1}{\sqrt{2n}}d_{\tau_n}(x,x')-d_{\mathbb{e}}(y,y')\right| : \exists i_x, i_{x'}, t_y, t_{y'} \text{ with}\right.\\
&\qquad\qquad (x,y)=h(i_x,t_y) \text{ and } (x',y')=h(i_{x'},t_{y'}) \text{ such that}\\
&\qquad\qquad \left. i_x=\lfloor 2nt_y\rfloor \text{ and } i_{x'}=\lfloor 2nt_{y'}\rfloor\right\}\\
&= \sup\left\{\left|\frac{1}{\sqrt{2n}}d_{\tau_n}(x,x')-d_{\mathbb{e}}(y,y')\right| : \exists i_x, i_{x'}, t_y, t_{y'} \text{ with}\right.\\
&\qquad\qquad (x,y)=(u^n_{i_x},\pi_{\mathbb{e}}(t_y)) \text{ and}\\
&\qquad\qquad (x',y')=(u^n_{i_{x'}},\pi_{\mathbb{e}}(t_{y'})) \text{ such that}\\
&\qquad\qquad \left. i_x=\lfloor 2nt_y\rfloor \text{ and } i_{x'}=\lfloor 2nt_{y'}\rfloor\right\}\\
&= \sup\left\{\left|\frac{1}{\sqrt{2n}}d_{\tau_n}(x,x')-d_{\mathbb{e}}(y,y')\right| : \exists t_y, t_{y'} \text{ with}\right.\\
&\qquad\qquad (x,y)=(u^n_{\lfloor 2nt_y\rfloor},\pi_{\mathbb{e}}(t_y)) \text{ and}\\
&\qquad\qquad \left.(x',y')=(u^n_{\lfloor 2nt_{y'}\rfloor},\pi_{\mathbb{e}}(t_{y'}))\right\}\\
&= \sup\left\{\left|\frac{1}{\sqrt{2n}}d_{\tau_n}(u^n_{\lfloor 2nr\rfloor},u^n_{\lfloor 2ns\rfloor})-d_{\mathbb{e}}(\pi_{\mathbb{e}}(r),\pi_{\mathbb{e}}(s))\right| : r,s\in[0,1]\right\}\\
&= \sup\left\{\left|\frac{1}{\sqrt{2n}}\Big(C_{\tau_n}(\lfloor 2nr\rfloor)+C_{\tau_n}(\lfloor 2ns\rfloor)-2\check{C}_{\tau_n}(\lfloor 2nr\rfloor,\lfloor 2ns\rfloor)\Big)\right.\right.\\
&\qquad\qquad \left.\left.-d_{\mathbb{e}}(\pi_{\mathbb{e}}(r),\pi_{\mathbb{e}}(s))\right| : r,s\in[0,1]\right\}. \qquad (4.4.3)
\end{aligned}
$$

Notice that the previous equality is due to (4.1.2). Remembering that $d_{\mathbb{e}}(\pi_{\mathbb{e}}(r),\pi_{\mathbb{e}}(s))$ on $\mathcal{T}_{\mathbb{e}}$ is equal to $d_{\mathbb{e}}(r,s)$ on $[0,1]$ and using the normalized contour function $\tilde{C}_{\tau_n}$ to express C_{τ_n}, we see that (4.4.3) is equal to

$$
\begin{aligned}
& \sup\left\{\left|\tilde{C}_{\tau_n}(\frac{\lfloor 2nr\rfloor}{2n})+\tilde{C}_{\tau_n}(\frac{\lfloor 2ns\rfloor}{2n})-2\check{\tilde{C}}_{\tau_n}(\frac{\lfloor 2nr\rfloor}{2n},\frac{\lfloor 2ns\rfloor}{2n}))-d_{\mathbb{e}}(r,s)\right| : r,s\in[0,1]\right\}\\
= & \sup\left\{\left|d_{\tilde{C}_{\tau_n}}(\frac{\lfloor 2nr\rfloor}{2n},\frac{\lfloor 2ns\rfloor}{2n})-d_{\mathbb{e}}(r,s)\right| : r,s\in[0,1]\right\}.
\end{aligned}
$$

By using Skorokhod's representation theorem (Theorem 6) we can assume that we are working on a probablity space in which the convergence of Theorem 8 holds almost surely rather than in distribution. This gives us $\mathrm{dis}(\mathcal{R}_n)\to 0$ as $n\to\infty$. The fact that $h(0,0)=(u^n_0,\rho_{\mathbb{e}})\in\mathcal{R}_n$ together with identity (4.3.3) concludes the proof. □

Actually, the Brownian CRT is not only the scaling limit of random plane trees but also of many other different classes of combinatorial trees. For instance, if $(\tau_n)_{n\geq 0}$ is a sequence of uniformly distributed rooted Cayley trees[1] with n vertices, then one obtains

$$\left(V(\tau_n), \frac{d_{\tau_n}}{\sqrt{4n}}\right) \xrightarrow{d} (\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}})$$

in the space $(\mathbf{T}, d_{GH})$. Likewise, a sequence of uniformly distributed (rescaled) binary trees with $2n$ edges converges in distribution to the CRT. By way of this, one can view the CRT as a kind of universal limiting object in the setting of combinatorial trees.

Next we notice that the choice of the root $\rho_{\mathbb{e}}$ of the CRT is of no significance when viewed from a probabilistic standpoint:

Proposition 10. *For every $t \in [0,1]$, the $\mathbf{T}^r$−valued random variables $(\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}}, \rho_{\mathbb{e}})$ and $(\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}}, \rho_{\mathbb{e}}(t))$ have the same distribution.*

Proof. Consider a plane tree $\tau_n \in \mathcal{P}_n$ with contour exploration $(e_0, ..., e_{2n-1})$ and $u_i^n := e_i^-$. By definition the root vertex of τ_n is u_0^n. For every $k \in \{0, 2n-1\}$ we can re-root τ_n and obtain the same tree with new root vertex u_k^n. Obviously, this procedure is a bijective mapping on $\mathcal{P}_n$. Therefore, $(V(\tau_n), d_{\tau_n}, u_0^n)$ has the same distribution as $(V(\tau_n), d_{\tau_n}, u_k^n)$ for every $k \in \{0, 2n-1\}$. The same argument as in Theorem 13 entails

$$\left(V(\tau_n), \frac{d_{\tau_n}}{\sqrt{2n}}, u^n_{\lfloor 2nt \rfloor}\right) \xrightarrow{d} (\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}}, \rho_{\mathbb{e}}(t))$$

in the space $(\mathbf{T}^r, d^r_{GH})$. □

By randomizing t in Proposition 10, we obtain the following result:

Corollary 9 (Re-rooting Invariance)**.** *Let U be a $\lambda_{\mathbb{e}}$−distributed random variable (see (4.2.3)) in the Brownian CRT. Then the two $\mathbf{T}^r$−valued random variables $(\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}}, \rho_{\mathbb{e}})$ and $(\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}}, U)$ have the same distribution.*

Finally, we want to mention that the Brownian CRT has an interesting property concerning its leaves (see Definition 43):

Proposition 11. *For every $t \in [0,1]$, the point $\pi_{\mathbb{e}}(t)$ is almost surely a leaf. In particular, the set of leaves of $\mathcal{T}_{\mathbb{e}}$ is uncountable and $\lambda_{\mathbb{e}}(Lf(\mathcal{T}_{\mathbb{e}})) = 1$. Moreover, for every $x \in Sk(\mathcal{T}_{\mathbb{e}})$ there holds $deg(x) \in \{2,3\}$ almost surely.*

4.5 The Hausdorff Dimension of the Brownian CRT

In this short section we want to introduce another terminology, the so-called Hausdorff dimension as well as its relation to the Brownian CRT. The Hausdorff dimension is a number that is based on the topological structure of the underlying space. It enjoys special attention in the field of self-similar or "fractal" sets. We start with a definition:

[1] A Cayley tree is a tree in which every non-leaf node has a fixed number of edges attached to it.

Definition 54. Let (X, ρ) be a metric space, $A \subseteq X$, $d \in [0, \infty]$ and $\delta \in (0, \infty)$. Set

$$H_\delta^d(A) := \inf\{\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^d : \forall (U_j)_{j=1}^{\infty} : U_j \subseteq X \wedge A \subseteq \bigcup_{j=1}^{\infty} U_i \wedge \operatorname{diam}(U_i) < \delta\}$$

where $\operatorname{diam}(U) := \sup\{\rho(x, y) : x, y \in U\}$ for every $U \subseteq X, U \neq \emptyset$ and $\operatorname{diam}(\emptyset) := 0$. Then the $d-$**dimensional Hausdorff measure** $H^d(A)$ of A is defined by

$$H^d(A) := \sup_{\delta > 0} H_\delta^d(A) = \lim_{\delta \to 0} H_\delta^d(A).$$

If X is a subset of $\mathbb{R}^d$ with $d \in \mathbb{N}^+$, then the Hausdorff measure H^d is a finite, translation-invariant Borel-measure. A standard theorem of measure theory implies that there exists a constant $c_d > 0$ such that $H^d = c_d \lambda_d$ where λ_d is the $d-$dimensional Lebesgue measure. More precisely, this constant can be expressed in terms of the $d-$dimensional ball $B_{1/2}^d(0) \subseteq \mathbb{R}^d$ with radius $1/2$, i.e. $c_d = \lambda_d(B_{1/2}^d(0))^{-1}$. The $d-$dimensional Hausdorff measure can therefore be seen as the scaled $d-$dimensional Lebesgue measure. The case $d = 0$ also yields a well-known object: If A is a finite subset of X, i.e. $A = \{a_1, \ldots, a_n\}$ for some $n \in \mathbb{N}$ and $a_i \in X$, we can set $U_i := a_i$ for $i = 1, \ldots, n$ and $U_i := \infty$ for $i > n$. This gives for every $\delta > 0$,

$$\begin{aligned} H_\delta^0(A) &= \inf\{\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^0 : \forall (U_j)_{j=1}^{\infty} : U_j \subseteq X \wedge A \subseteq \bigcup_{j=1}^{\infty} U_j \wedge \operatorname{diam}(U_j) < \delta\} \\ &\leq \sum_{i=1}^{n} \operatorname{diam}(\{a_i\})^0 + \sum_{i=n+1}^{\infty} 0 = n. \end{aligned}$$

Taking the limit $\delta \to 0$ yields $H^0(A) = \lim_{\delta \to 0} H_\delta^0(A) \leq n$. On the other hand, let $\delta > 0$ be such, that $\min_{i \neq j} \rho(a_i, a_j) = 2\delta$ and choose an arbitrary covering $(U_i)_{i=1}^n$ of A with $\operatorname{diam}(U_i) < \delta$ for $i = 1, \ldots, n$. Then it follows

$$\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^0 \geq \sum_{i=1}^{\infty} \operatorname{diam}(U_i \cap A)^0 = \sum_{i=1}^{n} \operatorname{diam}(\{a_i\})^0 = n.$$

By taking the infimum over all possible coverings $(U_i)_{i=1}^n$ of A with $\operatorname{diam}(U_i) < \delta$ we obtain $H^0(A) = \lim_{\delta \to 0} H_\delta^0(A) \geq n$ and therefore $H^0(A) = n$. If A is not finite, then $H^0(A)$ is infinity. In summary, $H^0(A)$ is the counting measure if $d = 0$.

The next result is a direct consequence of the definition of the Hausdorff measure.

Lemma 25. *Let (X, ρ) be a metric space, $A \subseteq X$ and $(U_i)_{i \geq 1}$ a $\delta-$covering of A, i.e. diam$(U_i) < \delta$ for every $i \geq 1$. Then for every $t > s$, we have*

$$H_\delta^t(A) \leq \delta^{t-s} H_\delta^s(A).$$

Proof. The assumption $\mathrm{diam}(U_i) < \delta$ implies $\mathrm{diam}(U_i)^{t-s} < \delta^{t-s}$ and therefore

$$\mathrm{diam}(U_i)^t = \mathrm{diam}(U_i)^{t-s}\mathrm{diam}(U_i)^s < \delta^{t-s}\mathrm{diam}(U_i)^s.$$

Summing over all U_i yields $\sum_{i\geq 1} \mathrm{diam}(U_i)^t \leq \delta^{t-s} \sum_{i\geq 1} \mathrm{diam}(U_i)^s$. The claim follows after taking the infimum over all coverings. □

A direct consequence of the previous result is:

Proposition 12. *Let (X, ρ) be a metric space, $A \subseteq X$ and $0 \leq s < t < \infty$. Then there holds*

1. $H^s(A) < \infty \Rightarrow H^t(A) = 0$.
2. $H^t(A) > 0 \Rightarrow H^s(A) = \infty$.

Proof. Under the assumption $H^s(A) < \infty$ we obtain

$$H^t(A) = \lim_{\delta\to 0} H^t_\delta(A) \leq \lim_{\delta\to 0} \delta^{t-s} H^s_\delta(A) = H^s(A) \cdot \lim_{\delta\to 0} \delta^{t-s} = 0$$

by using Lemma 25. The second point follows quiet similar: again due to Lemma 25 we get $\frac{1}{\delta^{t-s}} H^t_\delta(A) \leq H^s_\delta(A)$ for every $\delta > 0$. Taking the limit $\delta \to 0$ yields $H^s(A) = \infty$. □

The previous Proposition guarantees the existence of a unique value where the Hausdorff measure "jumps" from ∞ to 0, see Figure 4.7:

Corollary 10. *Let (X, ρ) be a metric space. For every $A \subseteq X$ there exists a unique value $d \in \mathbb{R}^+_0 \cup \{\infty\}$ such that $H^t(A) = 0$ if $t > d$ and $H^s(A) = \infty$ if $s < d$.*

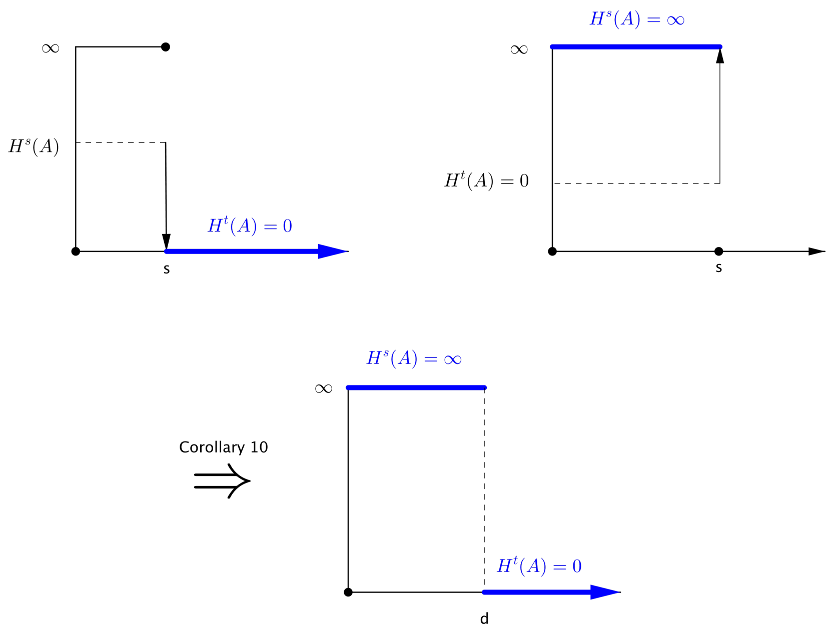


Figure 4.7: The two sketches on top outline the statement of Proposition 12 and the one at the bottom shows the consequence according to Corollary 10.

This leads to the following definition:

Definition 55. Let (X, ρ) be a metric space and $A \subseteq X$. Then the **Hausdorff dimension** $\dim_H(A)$ of A is the unique value d from Corollary 10 satisfying

$$\dim_H(A) := d = \inf\{t \geq 0 : H^t(A) = 0\} = \sup\{s \geq 0 : H^s(A) = \infty\}.$$

The Hausdorff dimension is often used to analyze certain "fractal" sets. The reader might get a feeling concerning its meaning by looking at Figure 4.8.

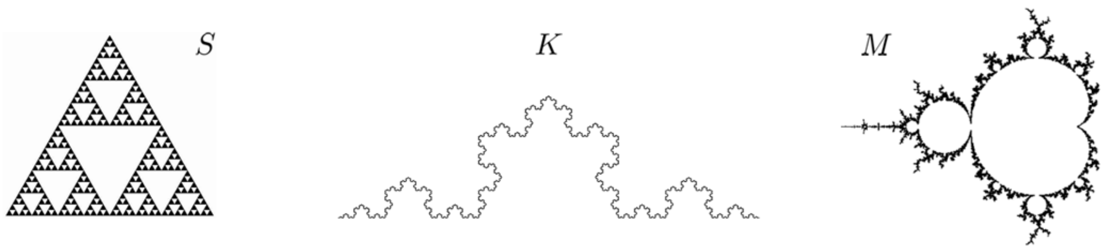


Figure 4.8: The Sierpinski triangle S, the Koch curve K and the Mandelbrot set M have Hausdorff dimensions $\dim_H(S) = 1.5849$, $\dim_H(K) = 1.2619$ and $\dim_H(M) = 2$.

In view of our discussion, the following holds:

Theorem 14. *Almost surely, the Hausdorff dimension* $\dim_H(\mathcal{T}_{\mathbb{e}})$ *of the Brownian CRT* $(\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}})$ *equals* 2.

A proof can be found in [27, Proposition 3.4].

5 Scaling Limit Results

In the previous section we showed that one can derive the convergence of a sequence of random plane trees towards the Brownian Continuum Random Tree with the help of the contour process. We would like to extend this result to embedded trees. The first problem one encounters, is that embedded trees are not uniquely determined by their contour process alone because the contour process contains no information about the labels. We address this issue by introducing the contour pair of an embedded tree. This leads us to the study of a random object called the Brownian snake that plays an important role in describing the limit behaviour of a sequence of random embedded trees in terms of their contour pairs.

Our further discussion will then focus on metric properties, especially the radius, of large random quadrangulations. In order to state these results, we combine the CVS bijection (Chapter 2) with the limit results obtained for random contour pairs. This will also lead to the appearance of a certain random measure, called the Integrated Super-Brownian Excursion, that we discuss briefly.

We follow mainly the works of [10, Chapter 6],[14, Chapter 5.2.2] and [29, Chapter 3.4].

5.1 Contour Pairs of Embedded Trees

The contour pair is a natural way to encode an embedded tree or, more generally speaking, any labelled tree. But we are only interested in embedded trees. We start with the following definition:

Definition 56. Suppose $\tau_n \in \mathcal{E}_n$ with associated label function ℓ_{τ_n} and let

$$L_{\tau_n}(i) := \ell_{\tau_n}(u_i^n)$$

be the label of the $i-$th visited vertex in the contour exploration (see Definition 25). By linear interpolation we can extend L_{τ_n} to a continuous function on $[0, 2n]$ that we also denote with L_{τ_n}. We call L_{τ_n} as function from $[0, 2n] \to \mathbb{R}$ the **label contour function** of τ_n. We denote with $\mathcal{L}_n$ the set of all label contour functions operating on $\mathcal{E}_n$. The **contour pair** of τ_n is then defined as the pair (C_{τ_n}, L_{τ_n}) where $C_{\tau_n} \in \mathcal{C}_n$ is the associated contour function and $L_{\tau_n} \in \mathcal{L}_n$ the associated label contour function.

In this way we can extend the statement of Lemma 8 to embedded trees (see Figure 5.1):

Proposition 13. *The contour pair construction is a bijection between the sets $\mathcal{E}_n$ and $\mathcal{C}_n \times \mathcal{L}_n$.*

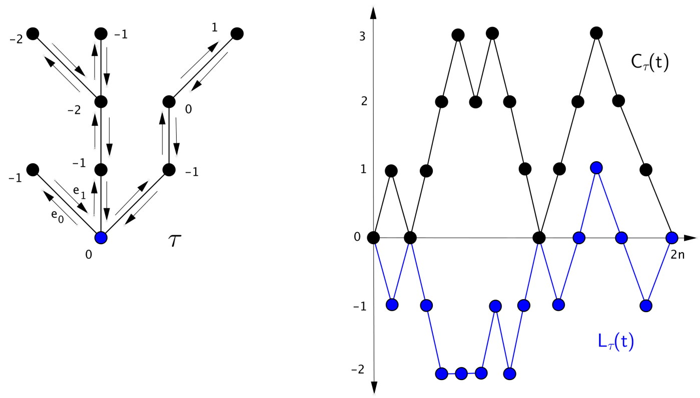


Figure 5.1: An embedded tree τ with its uniquely determined contour pair (C_τ, L_τ).

We define similar to (3.4.1) a scaled version of the label contour function by setting

$$\tilde{L}_{\tau_n}(t) := \left(\frac{9}{8n}\right)^{1/4} L_{\tau_n}(2nt) \tag{5.1.1}$$

for $t \in [0,1]$, $\tau_n \in \mathcal{E}_n$ and $L_{\tau_n} \in \mathcal{L}_n$. We call $\tilde{L}_{\tau_n}$ the **normalized label contour function** of τ_n.

In Theorem 8 we have already seen that a sequence of uniformly at random chosen normalized contour functions $(\tilde{C}_{\tau_n})_{n\geq 1}$ converges in distribution to the normalized Brownian excursion $\mathbb{e}$. We would like to extend this result to the **normalized contour pair** $(\tilde{C}_{\tau_n}, \tilde{L}_{\tau_n})$ that is itself a random variable taking values in $\mathcal{C}([0,1],\mathbb{R})^2$.[1]

At this point we want to dig a little deeper and study the label function ℓ_{τ_n} from a different angle. Consider an embedded tree $\tau_n \in \mathcal{E}_n$ with contour exploration $(e_0^n, e_1^n, ..., e_{2n-1}^n)$ and $u_i^n = (e_i^n)^-$. Suppose v is a uniformly chosen vertex in $V(\tau_n)\backslash\{u_0^n\}$ and define the random variable

$$Y_v := \ell_{\tau_n}(v) - \ell_{\tau_n}(v^-),$$

where v^- denotes the ancestor of v in the contour exploration of τ_n. Obviously, the random variables $\{Y_v : v \in V(\tau_n)\backslash\{u_0^n\}\}$ are uniform in $\{-1,0,1\}$ and independent. If we denote by $u \prec v$ the fact that u is an ancestor of v in the contour exploration of τ_n, then we obtain

[1] $\mathcal{C}([0,1],\mathbb{R})$ is the space of all continuous $\mathbb{R}-$valued functions on $[0,1]$.

$$\ell_{\tau_n}(v) = \sum_{u \prec v, u \neq u_0^n} Y_u.$$

Therefore, ℓ_{τ_n} can be seen as a random walk supported by the random variables $\{Y_v : v \in V(\tau_n)\backslash\{u_0^n\}\}$. This might suggest according to the results about random walks that a possible limit of a sequence of randomly chosen, normalized label contour functions $(\tilde{L}_{\tau_n})_{n\geq 1}$, which are build on top of the label functions $(\ell_{\tau_n})_{n\geq 1}$, is (again) strongly connected to the Brownian motion.

A final word on the choice of the scaling factor $(\frac{9}{8n})^{1/4}$ in (5.1.1) that is, of course, not chosen arbitrarily. We have just seen, that the label function ℓ_{τ_n} can be written as a sum of independend, random variables $Y := \{Y_v : v \in V(\tau_n)\backslash\{u_0^n\}\}$. Every $Y_v \in Y$ satisfies $\mathbb{E}[Y_v] = 0$ and

$$\mathbb{V}[Y_v] = \mathbb{E}[Y_v^2] - \mathbb{E}[Y_v]^2 = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}.$$

Also, the asymptotic length of a branch of τ_n going from u_0^n to $u_{\lfloor 2nt \rfloor}^n$ equals $\sqrt{2n\mathbb{e}_t}$ according to Theorem 8. This information is reflected in the scaling factor $(\frac{9}{8n})^{1/4}$ which can be seen by rewritting it in the form

$$\left(\frac{9}{8n}\right)^{1/4} = \left(\frac{9}{4 \cdot 2n}\right)^{1/4} = \frac{1}{(\frac{4}{9})^{1/4}(2n)^{1/4}} = \frac{1}{(\frac{2}{3})^{1/2}(2n)^{1/4}} = \frac{1}{\sqrt{(\frac{2}{3})}\sqrt{\sqrt{2n}}}.$$

By the central limit theorem, one might then expect the convergence of the normalized label contour functions

$$\tilde{L}_{\tau_n}\left(\frac{\lfloor 2nt \rfloor}{2n}\right) = \left(\frac{9}{8n}\right)^{1/4} L_{\tau_n}(\lfloor 2nt \rfloor) = \left(\frac{9}{8n}\right)^{1/4} \ell_{\tau_n}(u_{\lfloor 2nt \rfloor}^n) = \frac{\ell_{\tau_n}(u_{\lfloor 2nt \rfloor}^n)}{\sqrt{(\frac{2}{3})}\sqrt{\sqrt{2n}}}$$

to a centered Gaussian random variable. Again, this motivates a possible convergence of the normalized contour pair $(\tilde{C}_{\tau_n}, \tilde{L}_{\tau_n})$ which we aim to find in the next section.

5.2 Brownian Snake

During the previous two chapters, Brownian motion and Brownian excursion were crucial in understanding the convergence of the contour process as well as the Brownian CRT. In order to study the limiting behaviour of the normalized contour pair $(\tilde{C}_{\tau_n}, \tilde{L}_{\tau_n})$, we will once more introduce an object that is build upon Brownian motion and Brownian excursion. We start with a formal definition:

Definition 57. Let $\mathbb{e}$ be the normalized Brownian excursion. The **Brownian snake** is the path-valued stochastic process

$$W := \{W_s : s \in [0,1]\}$$

where $W_s = \{W_s(t) : t \in [0, \mathbbm{e}_s]\}$ obeys the following properties:

1. for all $s \in [0,1]$, $t \mapsto W_s(t)$ is a standard Brownian motion defined for $t \in [0, \mathbbm{e}_s]$.
2. $\{W_s : s \in [0,1]\}$ is a continuous Markov process satisfying for $s_1 < s_2$ the following properties: Let $m := \check{\mathbbm{e}}_{s_1,s_2}$ (see Definition 39), then
   a) $\{W_{s_1}(t) : t \in [0,m]\} = \{W_{s_2}(t) : t \in [0,m]\}$ and
   b) $\{W_{s_2}(m+t) : t \in [0, \mathbbm{e}_{s_2} - m]\}$ is a standard Brownian motion starting from $W_{s_2}(m)$ and independent of W_{s_1}.

This looks a bit cumbersome at first, but we advice the reader to take a look at Figure 5.2 which breaths some life into this technical definition. Loosely speaking, for every $s \in [0,1]$, W_s is a spatial path with random lifetime $\mathbbm{e}_s$ evolving according to the standard Brownian motion. We can therefore view the Brownian snake as a “branching” Brownian motion. Alternatively it can be seen as an embedding of the CRT or, equally true, as a process on the CRT. In the present work we will not be concerned with the full Brownian snake, but only with its “head”.

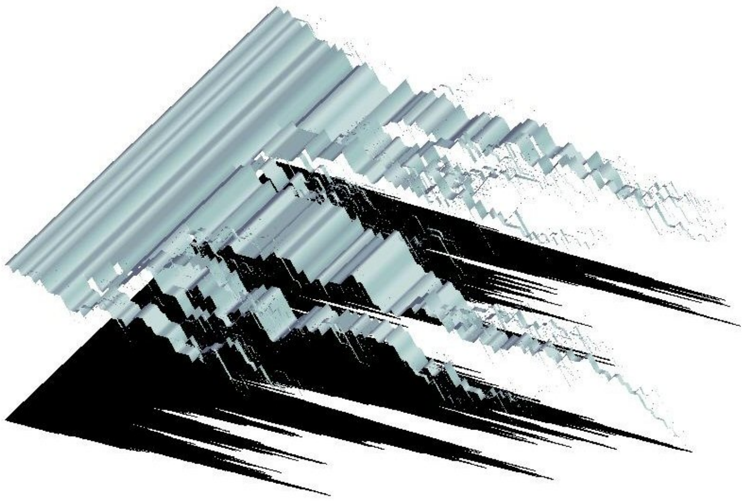


Figure 5.2: A visualization of the Brownian snake by Jérémie Bettinelli. The shadowy part represents the normalized Brownian excursion. For every $s \in [0,1]$, we observe the path W_s by cutting the surface at the right depth and by looking at the edge of the cut piece.

Definition 58. Let $W := \{W_s : s \in [0,1]\}$ be the Brownian snake. Define for $s \in [0,1]$ and $\hat{W}_s := W_s(\mathbbm{e}_s))$ the contour pair $X_s := \big(\mathbbm{e}_s, \hat{W}_s\big)$. Then the stochastic process

$$X := \{X_s : s \in [0,1]\} \tag{5.2.1}$$

is called the **head of the Brownian snake**. The normalized Brownian excursion $\mathbb{e}$ is in this context also called the life-time process of the head of the Brownian snake.

The full Brownian snake can be reconstructed from its head in the same way as an embedded tree can be reconstructed from its contour pair, i.e. $W_s(t) = \hat{W}_{\sup\{s' \le s : \mathbb{e}_{s'} = t\}}$. We refer to the works of Le Gall ([16] for example) or Marckert and Mokkadem [26] for further information on the Brownian snake.

Another way of introducing the head of the Brownian snake without considering the full snake is in terms of a certain Gaussian process. Although this is not as explicit as Le Gall's approach (Defintion 57), it fits nicely into our discussion of excursion functions and real trees. Consider an excursion function $g : [0,1] \to \mathbb{R}_+$ as studied in Section 4.1 with its associated real tree $(\mathcal{T}_g, d_g)$. We define a Gaussian process (see Definition 34) $Z_g := \{Z_g(t) : t \in [0,1]\}$ with mean function $\mu(t) = \mathbb{E}[Z_g(t)] = 0$ and covariance function

$$\Sigma(s,t) = \mathrm{Cov}(Z_g(s), Z_g(t)) = \check{g}(s,t) \tag{5.2.2}$$

for every $s, t \in [0,1]$. This implies $\mathbb{E}[Z_g(s) \cdot Z_g(t)] = \check{g}(s,t)$ because

$$\mathrm{Cov}(Z_g(s), Z_g(t)) = \mathbb{E}[Z_g(s) \cdot Z_g(t)] - \underbrace{\mathbb{E}[Z_g(s)]}_{=0} \cdot \underbrace{\mathbb{E}[Z_g(t)]}_{=0} = \check{g}(s,t).$$

Hence

$$\begin{aligned}
\mathbb{V}[(Z_g(s) - Z_g(t)] &= \mathbb{E}[(Z_g(s) - Z_g(t))^2] \\
&= \mathbb{E}[Z_g(s)^2] - 2\mathbb{E}[Z_g(s) \cdot Z_g(t)] + \mathbb{E}[Z_g(t)^2] \\
&= g(s) - 2\check{g}(s,t) + g(t) \\
&= d_g(s,t).
\end{aligned} \tag{5.2.3}$$

Notice that the existence of the Gaussian process Z_g is not trivial, but guaranteed by Proposition 6. To subsequently motivate the definition of Z_g, one should view each "time" $t \in [0,1]$ as a vertex of the real tree $\mathcal{T}_g$ and $Z_g(t)$ as the position of this vertex. Furthermore, we would like to recall the function $\check{C}_{\tau_n}$ from equation (4.1.1) associated with a plane tree $\tau_n \in \mathcal{P}_n$. We have seen that $\check{C}_{\tau_n}(i,j)$ can be interpreted as the value of the contour function C_{τ_n} of the highest common ancestor of the vertices v_i^n and v_j^n according to the contour exploration of τ_n. Also, we identified for every excursion function g a function $\check{g}$ whose definition and interpretation was motivated by $\check{C}_{\tau_n}$. In fact this construction was a generalization of a contour function for "continuous" trees. Therefore, the formula (5.2) for the covariance of $Z_g(s)$ and $Z_g(t)$ can be viewed as the value of the "contour function" g of the highest common ancestor of the "vertices" s and t in the real tree $\mathcal{T}_g$.

Turning back to equation (5.2.3), we see that $d_g(s,t) = 0$ implies

$$\mathbb{P}\{Z_g(s) - Z_g(t) = 0\} = 1.$$

Hence the following result holds:

Lemma 26. *For every $s, t \in [0,1]$, $d_g(s,t) = 0$ implies $Z_g(s) = Z_g(t)$ alsmost surely.*

Remembering the definition of the real tree $\mathcal{T}_g = [0,1]/\approx_g$ with $\approx_g = \{(s,t) \in [0,1]^2 : d_g(s,t) = 0\}$, Lemma 26 now allows us to interpret Z_g as a stochastic process on $\mathcal{T}_g$ or, more precisely, as Brownian motion indexed by the real tree $\mathcal{T}_g$. It therefore makes sense to identify the process Z_g in terms of

$$\{Z_g(t) : t \in [0,1]\} \mathrel{\widehat{=}} \{Z_g(x) : x \in \mathcal{T}_g\}$$

as a Gaussian process on $\mathcal{T}_g$ starting at the root ρ_g. This view allows the alternative form of (5.2.2) as

$$\mathrm{Cov}(Z_g(x), Z_g(y)) = d_g(\rho_g, x \wedge y)$$

for every $x, y \in \mathcal{T}_g$, where $x \wedge y$ denotes the highest common ancestor of x and y in $\mathcal{T}_g$. In the particular case where the excursion function g is piecewise monotone, $\mathcal{T}_g$ can be considered as a discrete tree and Z_g can be constructed by running independent, standard Brownian motions along the edges of $\mathcal{T}_g$.

Finally, we specialize g to the normalized Brownian excursion $\mathbb{e}$ which yields the Gaussian process $Z_{\mathbb{e}}$. Notice that $Z_{\mathbb{e}} = \{Z_{\mathbb{e}}(t) : t \in [0,1]\}$ obviously does not directly coincide with our former definition of the head of the Brownian snake $X = \{(\mathbb{e}_s, \hat{W}_s) : s \in [0,1]\}$, but rather with $\{\hat{W}_s : s \in [0,1]\}$. That is the reason why $Z_{\mathbb{e}}$ is also called the **terminal point process** of the head of the Brownian snake.

We are now in the position to state the main result of this section that is due to Chassaing and Schaeffer [9]:

Theorem 15. *For every $n \geq 1$, let τ_n be uniformly distributed over $\mathcal{E}_n$ with associated label function ℓ_{τ_n}. Let $(\tilde{C}_{\tau_n}, \tilde{L}_{\tau_n})$ be the normalized contour pair associated with τ_n. Then the following convergence in distribution holds in the space $\mathcal{C}([0,1], \mathbb{R})^2$:*

$$(\tilde{C}_{\tau_n}, \tilde{L}_{\tau_n}) \xrightarrow{d} (\mathbb{e}, Z_{\mathbb{e}}),$$

where $\mathbb{e}$ is the normalized Brownian excursion and $Z_{\mathbb{e}}$ is the terminal point process of the head of the Brownian snake.

The previous theorem is an extension of Theorem 8 which made a convergence statement about plane trees only. We will make use of Theorem 15 extensively in the next section.

5.3 Radius and Profile

In this section we want to combine the results established so far in order to obtain metric limiting properties of random quadrangulations. More precisely, we want to combine the CVS-Bijection together with the convergence result of the contour pair as established in the previous section to gain an understanding of how the radius of large, random quadrangulations behaves. Our discussion will also make us come across a random measure called the Integrated Super-Brownian Excursion which is closely related to the radius of a quadrangulation.

The first property of the next theorem is due to Chassaing and Schaeffer [9] and the second is due to Le Gall [17].

Theorem 16. *For $n \geq 1$, let q_n be uniformly distributed over $\mathcal{Q}_n^\bullet$ with pointed vertex v_n^*. Let further $Z_{\mathbb{e}}$ be the terminal point process of the head of the Brownian snake. Then*

1. *we have*
$$\left(\frac{9}{8n}\right)^{1/4} R_{q_n}(v_n^*) \xrightarrow{d} \sup Z_{\mathbb{e}} - \inf Z_{\mathbb{e}}.$$

2. *if u_n is another vertex chosen uniformly in $V(q_n)$ and independent of v_n^*, then*
$$\left(\frac{9}{8n}\right)^{1/4} d_{q_n}(u_n, v_n^*) \xrightarrow{d} \sup Z_{\mathbb{e}}.$$

Proof. Let τ_n be uniformly chosen at random in $\mathcal{E}_n$ with label function ℓ_{τ_n} and associated normalized label contour function $\tilde{L}_{\tau_n}$. Let further $\epsilon_n \in \{1, -1\}$ also be uniformly chosen and independent of τ_n. Then by Corollary 7 we can assume that $q_n = \tilde{\Psi}(\epsilon, \tau_n)$, where $\tilde{\Psi}$ is the CVS bijection (Corollary 2) between $\{1, -1\} \times \mathcal{E}_n$ and $\mathcal{Q}_n^\bullet$. The interpretation of the natural distance d_{q_n} on q_n with the help of the label function ℓ_{τ_n} of the associated embedded tree τ_n (Proposition 3) by way of the CVS bijection gives us

$$\begin{aligned}
\left(\frac{9}{8n}\right)^{1/4} R_{q_n}(v_n^*) &= \left(\frac{9}{8n}\right)^{1/4} \max_{w_n \in V(q_n)} d_{q_n}(w_n, v_n^*) \\
&= \left(\frac{9}{8n}\right)^{1/4} \max_{w_n \in V(q_n)} (\ell_{\tau_n}(w_n) - \min(\ell_{\tau_n}) + 1) \\
&= \left(\frac{9}{8n}\right)^{1/4} (\max \ell_{\tau_n} - \min \ell_{\tau_n} + 1) \\
&= \left(\frac{9}{8n}\right)^{1/4} (\max L_{\tau_n} - \min L_{\tau_n} + 1) \\
&= \sup \tilde{L}_{\tau_n} - \inf \tilde{L}_{\tau_n} + \left(\frac{9}{8n}\right)^{1/4}.
\end{aligned}$$

Property 1 is then obtained by Theorem 15.

As for Property 2, instead of choosing u_n uniformly in $V(q_n)$, we choose it without loss of generality uniformly in $V(\tau_n)\backslash\{v_0^n\} = V(q_n)\backslash\{\{v_n^*\}\cup\{v_0^n\}\}$, where v_0^n is the root vertex of τ_n. We now follow the proof of Corollary 8: if U is a uniformly chosen random variable in $[0,1]$ and independent of τ_n, then $w^n_{\langle 2nU\rangle}$ is uniform in $V(\tau_n)\backslash\{v_0^n\}$, where $(w_i^n)_{i=0,\ldots,n}$ is the sequence of vertices associated with the contour process of τ_n and v_0^n is the root vertex in τ_n. Hence it is sufficient to prove the result with $w^n_{\langle 2nU\rangle}$ instead of u_n. As in the proof of Corollary 8, the requirements of Theorem 8 are satisfied and we obtain

$$\begin{aligned}\left(\frac{9}{8n}\right)^{1/4} d_{q_n}(w^n_{\langle 2nU\rangle}, v_n^*) &= \left(\frac{9}{8n}\right)^{1/4} (\ell_{\tau_n}(w^n_{\langle 2nU\rangle}) - \min \ell_{\tau_n} + 1) \\ &= \left(\frac{9}{8n}\right)^{1/4} (L_{\tau_n}(\langle 2nU\rangle) - \min L_{\tau_n} + 1) \\ &= \tilde{L}_{\tau_n}(\langle 2nU\rangle) - \inf \tilde{L}_{\tau_n} + \left(\frac{9}{8n}\right)^{1/4},\end{aligned}$$

which converges in distribution to $Z_{\mathbb{e}}(U) - \inf Z_{\mathbb{e}}$ due to Theorem 15. This completes the proof of the second point of the theorem, because $Z_{\mathbb{e}}(U) - \inf Z_{\mathbb{e}}$ has the same distribution as $\sup Z_{\mathbb{e}}$ which is a consequence of the re-rooting invariance (Corollary 9). $\square$

The previous theorem allows us to state an interesting result concerning the limiting behaviour of the (scaled) profiles of a sequence of random quadrangulations. Suppose $q_n \in \mathcal{Q}_n^\bullet$ with pointed vertex v_n^* and associated profile $(H_k^{q_n})_{k\in\mathbb{Z}}$. For $k \in \mathbb{N}$, we interpret the profile as a function

$$I_{q_n} : k \mapsto H_k^{q_n},$$

i.e.

$$I_{q_n}(k) = |\{u \in V(q_n) : d_{q_n}(u, v_n^*) = k\}|.$$

Then upon scaling, I_{q_n} converges to a certain random measure. This result is (again) due to Chassaing and Schaeffer [9]:

Theorem 17. *For $n \geq 1$, let q_n be uniformly distributed over $\mathcal{Q}_n^\bullet$ with pointed vertex $v_n^* \in V(q_n)$. Then the following convergence in distribution holds for the weak topology on the space of probability measures on $\mathbb{R}_+$:*

$$\frac{I_{q_n}((\frac{9}{8n})^{1/4}(.))}{n+2} \xrightarrow{d} \mathcal{I},$$

where $\mathcal{I}$ is the occupation measure of $Z_{\mathbb{e}}$ above its infimum, defined as follows: for every non-negative, measurable $g : \mathbb{R}_+ \to \mathbb{R}$,

$$\int g\, d\mathcal{I} := \int_0^1 g(Z_{\mathbb{e}}(s) - \inf Z_{\mathbb{e}})\, ds. \tag{5.3.1}$$

Proof. Let $g : \mathbb{R}_+ \to \mathbb{R}$ be a bounded continuous function. We evaluate the integral of g with respect to the measure $\mu_{q_n} := \frac{I_{q_n}((\frac{9}{8n})^{1/4}(.))}{n+2}$:

$$\begin{aligned}
\int g\, d\mu_{q_n} &= \frac{1}{n+2}\sum_{k\in\mathbb{N}} I_{q_n}(k) g((\frac{9}{8n})^{1/4}k) \\
&= \frac{1}{n+2}\sum_{k\in\mathbb{N}} |\{w \in V(q_n) : d_{q_n}(w, v_n^*) = k\}| \cdot g((\frac{9}{8n})^{1/4}k) \\
&= \frac{1}{n+2}\sum_{k\in\mathbb{N}} \sum_{w\in V(q_n), d_{q_n}(w,v_n^*)=k} 1 \cdot g((\frac{9}{8n})^{1/4}k) \\
&= \frac{1}{n+2}\sum_{w\in V(q_n)} g((\frac{9}{8n})^{1/4} d_{q_n}(w, v_n^*)) \\
&= \mathbb{E}_{u_n}[g((\frac{9}{8n})^{1/4} d_{q_n}(u_n, v_n^*))] \\
&\xrightarrow{d} \mathbb{E}_U[g(Z_{\mathbb{e}}(U) - \inf Z_{\mathbb{e}})] \\
&= \int_0^1 g(Z_{\mathbb{e}}(s) - \inf Z_{\mathbb{e}})\, ds \\
&= \int g\, d\mathcal{I},
\end{aligned}$$

where $\mathbb{E}_{u_n}$ and $\mathbb{E}_U$ means that we take the expectation only with respect to u_n and U in the corresponding expressions (u_n and U as in the proof of Theorem 16, Point 2). The convergence in distribution is then (again) duo to Point 2 of the proof of Theorem 16. □

At this point we want to mention another interesting random measure that is similar in structure to $\mathcal{I}$ and closely related to embedded trees:

Definition 59. Let $Z_{\mathbb{e}}$ be the terminal point process of the head of the Brownian snake. Then the (one-dimensional) **Integrated Super-Brownian Excursion (ISE)** is the random measure μ_{ISE} defined by

$$\mu_{\mathrm{ISE}}(A) := \int_0^1 \mathbb{1}_A(Z_{\mathbb{e}}(s))\, ds$$

for every Borel set $A \subseteq \mathbb{R}$ and supported by a random interval $0 \in [L, R] \subseteq \mathbb{R}$.

The ISE was introduced by Aldous [1] as a limit of certain branching processes. We do not want to go into further detail about the meaning and applications of the ISE because that is a complex topic on its own. We just want to hint at some of its properties: First, its structure is similar to the occupation measure μ_t of the Brownian motion as defined in (3.3.2). By way of this, it can be seen as the occupation measure of the terminal point process of the head of the Brownian snake. Furthermore, integration with respect to μ_{ISE} satisfies

$$\int g\,d\mu_{\mathrm{ISE}} = \int_0^1 g(Z_{\mathbb{e}}(s))\,ds \tag{5.3.2}$$

for any measurable function g. Notice the similarity of (5.3.2) and (5.3.1). Actually, they are shifted versions of one another, i.e. $\mathcal{I}$ is the image of μ_{ISE} under the shift $x \mapsto x - \inf Z_{\mathbb{e}}$. The support of $\mathcal{I}$ is therefore the shifted support $[0, R-L]$ of μ_{ISE}.

Also, as the support $[L, R]$ of the ISE satisfies $L = \inf Z_{\mathbb{e}}$ and $R = \sup Z_{\mathbb{e}}$, the limiting behaviour of the radius of a sequence of random quadrangulations as obtained in Theorem 16 can be seen as the width of the support of μ_{ISE} , i.e.:

Corollary 11. *With the notation of Theorem 16, the sequence of random variables $(n^{-1/4}R_{q_n}(v_n^*) : n \geq 1)$ converges in distribution (up to rescaling) to the width of the support $[L, R]$ of μ_{ISE},*

$$\frac{R_{q_n}(v_n^*)}{n^{1/4}} \xrightarrow{d} \left(\frac{8}{9}\right)^{1/4} (R-L).$$

The last result about the ISE that we want to mention and that is interesting in terms of our study is, that the ISE appears as the limiting measure of a sequence of probability measures defined on top of the label distributions of randomly choosen embedded trees (see [1]):

Theorem 18. *For $n \geq 1$, let τ_n be uniformly distributed over $\mathcal{E}_n$ with associated label function ℓ_{τ_n} and label distribution $(\Lambda_k^{\tau_n})_{k\in\mathbb{Z}}$ as in Definition 24. Consider the random probability measure on $\mathbb{R}$ defined by*

$$\mu_n := \frac{1}{n+1} \sum_{v\in V(\tau_n)} \Lambda^{\tau_n}_{\ell_{\tau_n}(v)} \delta_{c\cdot\ell_{\tau_n}(v)\cdot n^{-1/4}}, \tag{5.3.3}$$

where δ_x denotes the Dirac measure at x and the constant c equals $\sqrt{3}$. Then the following convergence in distribution holds in the space of probability measures on $\mathbb{R}$:

$$\mu_n \xrightarrow{d} \mu_{ISE}.$$

In fact, Theorem 18 not only holds for embedded trees, but also for the classes of all labelled plane trees with increments ± 1 and binary trees as well. For labelled plane trees with increments ± 1 the constant c in (5.3.3) equals $\sqrt{2}$ and for binary trees it equals 1.

We end our little side trip into the Integrated Super-Brownian Excursion with two references that deal with the subject in more depth, namely [35] and [16].

6 The Brownian Map

In the previous chapter we were able to derive some interesting conclusions about random, pointed quadrangulations and distances between their vertices by combining the CVS bijection with scaling limit results about random embedded trees. We have also seen that a sequence of randomly chosen and properly rescaled embedded trees converges in distribution towards a "limiting tree" that we identified as the Brownian CRT. It is now tempting in view of the structural similarity between embedded trees and pointed quadrangulations established by the CVS bijection that there also exists a "limiting map" of a sequence of randomly chosen quadrangulations - again, up to rescaling. We outline the necessary steps in order to deal with this problem:

At first, we extend the distance function d_{q_n} of a pointed quadrangulation $q_n \in \mathcal{Q}_n^\bullet$ to a continuous function D_{q_n} on the interval $[0, 2n]$. Furthermore, we can introduce another distance function, named by $D^\circ_{\tau_n}$, with the help of the label contour function L_{τ_n} associated with the embedded tree $\tau_n := \tilde{\Phi}(q_n)$ by way of the CVS bijection. By relating a properly rescaled version of D_{q_n} with a rescaled version of $D^\circ_{\tau_n}$ we are led to the observation that any sequence of such rescaled functions D_{q_n} converges along a suitable subsequence in distribution towards a random, continous real-valued function on $[0, 1]$ (see Theorem 19 and the limit in (6.1.5)). This limiting function, denoted by D, allows us to define in analogy to the construction of the Brownian CRT, the quotient space

$$\mathcal{S} := [0,1]/\approx_D,$$

where $\approx_D$ is defined similar to the equivalence relation $\approx_{\mathbb{e}}$. Esssentially, this space equipped with the quotient distance $D_{\mathcal{S}}$ induced by D, turns out to be the limit in distribution of a sequence of uniformly distributed pointed quadrangulations q_n, i.e. the convergence

$$\left(V(q_n), \left(\frac{9}{8n}\right)^{1/4} d_{q_n}\right) \xrightarrow{d} (\mathcal{S}, D_{\mathcal{S}})$$

for the Gromov-Hausdorff topology on the set of compact metric spaces holds even without having to take an appropriate subsequence. We call $(\mathcal{S}, D_{\mathcal{S}})$ the Brownian Map because it is an analogon of the Brownian CRT $(\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}})$ in the world of pointed quadrangulations. In fact, there is a close relation between $\mathcal{S}$ and $\mathcal{T}_{\mathbb{e}}$ that we outline at the end of this chapter. Our forthcoming in-depth study follows the work of Le Gall [18].

6.1 Convergence of Quadrangulations as Metric Spaces

Consider for the ongoing discussion a pointed quadrangulation $q_n \in \mathcal{Q}_n^\bullet$ with distance function d_{q_n} and pointed vertex v_*. Let $\tau_n \in \mathcal{E}_n$ be the image of q_n under the CVS bijection, i.e. $\tilde{\Phi}(q_n) = \tau_n$ and denote with ℓ_{τ_n} its associated label function. Let further $(e_0^n, ..., e_{2n-1}^n)$ be the contour exploration of τ_n and set $u_i^n := (e_i^n)^-$ as always. We define a pseudo-metric on $\{0, ..., 2n\}$[1] that we denote with D_{q_n} by

$$D_{q_n}(i,j) := d_{q_n}(u_i^n, u_j^n).$$

The proofs of the metric properties obtained in the previous chapter were all rooted in the fact that the distance d_{q_n} was expressible in terms of the contour pair of the (normalized) contour pair (C_{τ_n}, L_{τ_n}). Unfortunately, this does not hold for the pseudo-metric D_{q_n}. The only distances that we are able to handle are the distances to the pointed vertex v_* according to Proposition 3:

$$\begin{aligned} D_{q_n}(i,0) &= d_{q_n}(u_i^n, u_0^n) \\ &= d_{q_n}(u_i^n, v_*) \\ &= \ell_{\tau_n}(u_i^n) - \min \ell_{\tau_n} + 1 \\ &= L_{\tau_n}(i) - \min L_{\tau_n} + 1, \end{aligned} \tag{6.1.1}$$

for every $i \in \{0, ..., 2n\}$. To work around this problem, we introduce another distance function on $\{0, ..., 2n\}$ with the help of the label contour function L_{τ_n} of the associated embedded tree τ_n by setting

$$D_{\tau_n}^\circ(i,j) := L_{\tau_n}(i) + L_{\tau_n}(j) - 2\max\left\{ \min_{i \le k \le j} L_{\tau_n}(k), \min_{j \le k \le i} L_{\tau_n}(k) \right\} + 2,$$

where the ranges of the minima are to be understood as “cyclic”, i.e. if $j < i$, the condition $i \le k \le j$ means that $k \in \{i, i+1, ..., 2n\} \cup \{0, 1, ..., j\}$ and similarly for the condition $j \le k \le i$ if $i < j$.

Lemma 27. *For every $i, j \in \{0, ..., 2n\}$ there holds*

$$D_{q_n}(i,j) \le D_{\tau_n}^\circ(i,j).$$

Proof. We exclude the trivial case $i = j$ and assume without loss of generality $i < j$. Then with $u := u_i^n$ and $v := u_j^n$, we get according to (and with the notation of) Proposition 5, point 1.:

[1] Remember that we always set $e_{2n}^n := e_0^n$ and $u_{2n}^n := u_0^n$.

$$\begin{aligned}
D_{q_n}(i,j) &= d_{q_n}(u,v) \\
&= \ell_{\tau_n}(u) + \ell_{\tau_n}(v) - 2 \min_{c \in [c_u, c_v]} \ell_{\tau_n}(c) + 2 \\
&= L_{\tau_n}(i) + L_{\tau_n}(j) - 2 \min_{i \le k \le j} L_{\tau_n}(k) + 2 \\
&\le L_{\tau_n}(i) + L_{\tau_n}(j) - 2 \max \Big\{ \min_{i \le k \le j} L_{\tau_n}(k), \min_{j \le k \le i} L_{\tau_n}(k) \Big\} + 2 \\
&= D^{\circ}_{\tau_n}(i,j).
\end{aligned}$$

□

We now extend the distance function D_{q_n} to non-integer values by linear interpolation. If $s, t \in [0, 2n]$, we set

$$\begin{aligned}
D_{q_n}(s,t) &:= (\lceil s \rceil - s)(\lceil t \rceil - t) D_{q_n}(\lfloor s \rfloor, \lfloor t \rfloor) \\
&+ (\lceil s \rceil - s)(t - \lfloor t \rfloor) D_{q_n}(\lfloor s \rfloor, \lceil t \rceil) \\
&+ (s - \lfloor s \rfloor)(\lceil t \rceil - t) D_{q_n}(\lceil s \rceil, \lfloor t \rfloor) \\
&+ (s - \lfloor s \rfloor)(t - \lfloor t \rfloor) D_{q_n}(\lceil s \rceil, \lceil t \rceil),
\end{aligned}$$

where $\lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \le x\}$ and $\lceil x \rceil := \lfloor x \rfloor + 1$. In the same way we extend $D^{\circ}_{\tau_n}$ to a function on $[0, 2n]^2$.
Thus, D_{q_n} is continuous on $[0, 2n]^2$ and satisfies the triangular inequality (which is not the case for $D^{\circ}_{\tau_n}$). Also, the bound $D_{q_n} \le D^{\circ}_{\tau_n}$ still holds. We define in analogy to (5.1.1) a rescaled version of these functions by letting for $s, t \in [0, 1]$,

$$\tilde{D}_{q_n}(s,t) := \left(\frac{9}{8n} \right)^{1/4} D_{q_n}(2ns, 2nt)$$

and

$$\tilde{D}^{\circ}_{\tau_n}(s,t) := \left(\frac{9}{8n} \right)^{1/4} D^{\circ}_{\tau_n}(2ns, 2nt). \tag{6.1.2}$$

Corollary 12. *For every $n \ge 1$, let $\tilde{D}^{\circ}_{\tau_n}$ be constructed from a uniformly at random chosen pointed quadrangulation $q_n \in \mathcal{Q}^{\bullet}_n$ as in (6.1.2) and denote with τ_n the associated embedded tree by way of the CVS bijection. Then the following convergence in distribution holds for the uniform topology on $\mathcal{C}([0,1]^2, \mathbb{R})$:*

$$\tilde{D}^{\circ}_{\tau_n} \xrightarrow{d} D^{\circ},$$

with

$$D^{\circ}(s,t) = Z_{\mathbb{e}}(s) + Z_{\mathbb{e}}(t) - 2 \max \Big\{ \min_{s \le \zeta \le t} Z_{\mathbb{e}}(\zeta), \min_{t \le \zeta \le s} Z_{\mathbb{e}}(\zeta) \Big\}$$

and where $Z_{\mathbb{e}}$ is the terminal point process of the head of the Brownian snake.

Proof. Let $s, t \in [0,1]$. The result follows from Theorem 15 by inserting into the definition of $\tilde{D}^\circ_{\tau_n}$:

$$\begin{aligned}
\tilde{D}^\circ_{\tau_n}(s,t) &= \left(\frac{9}{8n}\right)^{1/4} D^\circ_{\tau_n}(2ns, 2nt) \\
&= \left(\frac{9}{8n}\right)^{1/4} \left(L_{\tau_n}(2ns) + L_{\tau_n}(2nt) - 2\max\left\{\min_{2ns\le\xi\le 2nt} L_{\tau_n}(\xi), \min_{2nt\le\xi\le 2ns} L_{\tau_n}(\xi)\right\} + 2\right) \\
&= \tilde{L}_{\tau_n}(s) + \tilde{L}_{\tau_n}(t) - 2\max\left\{\min_{s\le\frac{\xi}{2n}\le t} \tilde{L}_{\tau_n}\left(\frac{\xi}{2n}\right), \min_{t\le\frac{\xi}{2n}\le s} \tilde{L}_{\tau_n}\left(\frac{\xi}{2n}\right)\right\} + \left(\frac{18}{n}\right)^{1/4} \\
&\xrightarrow{d} Z_{\mathbb{e}}(s) + Z_{\mathbb{e}}(t) - 2\max\left\{\min_{s\le\zeta\le t} Z_{\mathbb{e}}(\zeta), \min_{t\le\zeta\le s} Z_{\mathbb{e}}(\zeta)\right\} \\
&= D^\circ(s,t),
\end{aligned}$$

where the minima are again to be understood as "cyclic" in the sense that if $t < s$ the condition $s \le \zeta \le t$ means that $\zeta \in [s,1] \cup [0,t]$ (and likewise for $\frac{\xi}{2n}$). $\square$

In order to state the main result of this section, we need some new terminology:

Definition 60. Let (X, d) be a metric space and denote with $\mathcal{P}(X)$ the set of all Borel[2] probability measures on X. A probability measure $P \in \mathcal{P}(X)$ is said to be **tight** if for every $\epsilon > 0$ there exists a compact subset $K \subseteq X$ such that $P(K) \ge 1 - \epsilon$. A family of probability measures $\mathcal{M} \subseteq \mathcal{P}(X)$ is **tight** if for every $\epsilon > 0$ there exists a compact set $K \subseteq X$ such that $\inf_{P\in\mathcal{M}} \ge 1 - \epsilon$.

Theorem 19. *For every $n \ge 1$, let q_n be uniformly distributed over $\mathcal{Q}^\bullet_n$. Denote with $\mathcal{L}$ the family of laws of the processes $\{\tilde{D}_{q_n}(s,t) : s,t \in [0,1]\}$ as n varies. Then $\mathcal{L}$ is relatively compact for the weak topology on the space $\mathcal{P}(\mathcal{C}([0,1]^2, \mathbb{R}))$.*

Proof. Let $s, t, s', t' \in [0,1]$ and set $\tau_n := \tilde{\Phi}(q_n)$. Then the triangular inequality for $\tilde{D}_{q_n}$ together with $\tilde{D}_{q_n} \le \tilde{D}^\circ_{\tau_n}$ yields

$$\begin{aligned}
|\tilde{D}_{q_n}(s,t) - \tilde{D}_{q_n}(s',t')| &= |\tilde{D}_{q_n}(s,t) - \tilde{D}_{q_n}(t,s') + \tilde{D}_{q_n}(t,s') - \tilde{D}_{q_n}(s',t')| \\
&\le |\tilde{D}_{q_n}(s,t) - \tilde{D}_{q_n}(t,s')| + |\tilde{D}_{q_n}(t,s') - \tilde{D}_{q_n}(s',t')| \\
&\le |\tilde{D}_{q_n}(s,s')| + |\tilde{D}_{q_n}(t,t')| \\
&= \tilde{D}_{q_n}(s,s') + \tilde{D}_{q_n}(t,t') \\
&\le \tilde{D}^\circ_{\tau_n}(s,s') + \tilde{D}^\circ_{\tau_n}(t,t').
\end{aligned}$$

For a fixed $\delta > 0$, this leads to the important estimate

$$\sup_{\substack{|s-s'|\le\delta \\ |t-t'|\le\delta}} \{\tilde{D}_{q_n}(s,t) - \tilde{D}_{q_n}(s',t')\} \le 2 \sup_{|r-r'|\le\delta} \tilde{D}^\circ_{\tau_n}(r,r'). \tag{6.1.3}$$

[2] A Borel measure is a measure defined on all Borel sets of the underlying topological space.

Define for $\epsilon > 0$

$$K_{\delta,\epsilon} := \{f \in \mathcal{C}([0,1]^2, \mathbb{R}) : \sup_{|r-r'|\leq\delta} f(r,r') \geq \epsilon\}.$$

Then the convergence in Corollary 12 together with the fact that the sets $K_{\delta,\epsilon}$ are closed entails due to Theorem 5, point 3., that

$$\limsup_{n\to\infty} \mathbb{P}\{\tilde{D}^\circ_{\tau_n} \in K_{\delta,\epsilon}\} \leq \mathbb{P}\{D^\circ \in K_{\delta,\epsilon}\}.$$

The right side approaches 0 as $\delta \to 0$ by the continuity of D° and because of $D^\circ(r,r) = 0$. Now, taking $\eta > 0$ and letting $\epsilon = \epsilon_k = 2^{-k}$, we can choose $\delta = \delta_{k,\eta}$ such that

$$\sup_{n\geq 1} \mathbb{P}\{\tilde{D}^\circ_{\tau_n} \in K_{\delta_{k,\eta},2^{-k}}\} \leq \frac{\eta}{2^k}$$

for every $k \geq 1$. Taking the limit $k \to 1$ entails

$$\mathbb{P}\left\{\bigcap_{k\geq 1}\left\{\tilde{D}^\circ_{\tau_n} \in K_{\delta_{k,\eta},2^{-k}}\right\}\right\} = \mathbb{P}\left\{\bigcap_{k\geq 1}\left\{\tilde{D}^\circ_{\tau_n} : \sup_{|r-r'|\leq\delta_{k,\eta}} \tilde{D}^\circ_{\tau_n}(r,r') \geq 2^{-k}\right\}\right\} \leq \frac{\eta}{2} < \eta \tag{6.1.4}$$

and hence

$$\mathbb{P}\left\{\bigcap_{k\geq 1}\left\{\tilde{D}^\circ_{\tau_n} : \sup_{|r-r'|\leq\delta_{k,\eta}} \tilde{D}^\circ_{\tau_n}(r,r') \leq 2^{-k}\right\}\right\} \geq 1-\eta$$

for every $n \geq 1$. Let K denote the set of all functions $f \in \mathcal{C}([0,1]^2, \mathbb{R})$ such that $f(0,0) = 0$ and for every $k \geq 1$,

$$\sup_{\substack{|s-s'|\leq\delta_{k,\eta} \\ |t-t'|\leq\delta_{k,\eta}}} |f(s,t) - f(s',t')| \leq 2^{-k}.$$

The equations (6.1.3) and (6.1.4) imply $\tilde{D}_{q_n} \in K$ with probability at least $1-\eta$. Furthermore, K satisfies the conditions of the Arzelà-Ascoli Theorem (Theorem 24) and is therefore compact. This means that $\mathcal{L}$ is tight in the space of probability measures on $\mathcal{C}([0,1]^2, \mathbb{R})$. According to Prohorov's Theorem (Theorem 25) this is equivalent to the fact that $\mathcal{L}$ is relatively compact. □

The relative compactness statement in the previous theorem allows us to obtain for every sequence $(\tilde{D}_{q_n})_{n\geq 1}$ a convergent subsequence $(\tilde{D}_{q_{n(k)}})_{k\geq 1}$ such that the limit in distribution

$$\tilde{D}_{q_{n(k)}} \xrightarrow{d} D \tag{6.1.5}$$

exists for the uniform topology on $\mathcal{C}([0,1]^2, \mathbb{R})$. As a consequence of this, we find the following corollary to be true:

Corollary 13. *For every $n \geq 1$, let q_n be uniformly distributed over $\mathcal{Q}_n^\bullet$ with its associated embedded tree $\tau_n \in \mathcal{E}_n$ by way of the CVS bijection. Then the closure of the family of laws of the triplets $\big(\tilde{C}_{\tau_n}, \tilde{L}_{\tau_n}, \tilde{D}_{q_n}\big)_{n\geq 1}$ is compact in the space $\mathcal{P}(\mathcal{C}([0,1],\mathbb{R})^2 \times \mathcal{C}([0,1]^2,\mathbb{R}))$.*

This implies in particular that for every sequence of triplets $\big(\tilde{C}_{\tau_n}, \tilde{L}_{\tau_n}, \tilde{D}_{q_n}\big)$ there exists a subsequence $(n(k))_{k\geq 1}$ of $n \geq 1$ such that

$$(\tilde{C}_{\tau_{n(k)}}, \tilde{L}_{\tau_{n(k)}}, \tilde{D}_{q_{n(k)}}) \xrightarrow{d} (\mathbb{e}, Z_{\mathbb{e}}, D) \tag{6.1.6}$$

where $\mathbb{e} = \{\mathbb{e}_t : t \in [0,1]\}$ is the normalized Brownian Excursion, $Z_{\mathbb{e}} = \{Z_{\mathbb{e}}(t) : t \in [0,1]\}$ is the terminal point process of the head of the Brownian snake and D is the limiting function in (6.1.5). For the remaining part of this section, we fix one such subsequence $(n(k))_{k\geq 1}$ as in (6.1.6).

The following proposition states some important properties of the random function D that we will rely on in the next chapter.

Proposition 14. *For every $s, t \in [0,1]$, the following properties hold almost surely:*

1. $([0,1], D)$ is a metric space.
2. $s \approx_{\mathbb{e}} t$ implies $D(s,t) = 0$, where $\approx_{\mathbb{e}}$ is the equivalence relation on $[0,1]$ that was used to construct the Brownian CRT $\mathcal{T}_{\mathbb{e}} = [0,1]/\approx_{\mathbb{e}}$.
3. $D(s,t) \leq D^\circ(s,t)$.

Proof. The first result summarizes our previous discussion. Let us look at the second point. With the notation of Corollary 13, we assume using Skorokhod's representation theorem, that we are working on a probability space where $(\mathbb{e}, Z_{\mathbb{e}}, D)$ is the limit of $(\tilde{C}_{\tau_{n(k)}}, \tilde{L}_{\tau_{n(k)}}, \tilde{D}_{q_{n(k)}})$ in the almost sure sense. Suppose $s \approx_{\mathbb{e}} t$ with $s < t$ and $\{u_i^n : i \in \{0,...,2n\}\} = V(\tau_n)$, where $u_i^n := (e_i^n)^-$ with $(e_0^n, ..., e_{2n-1}^n)$ is the contour exploration of τ_n. By definition, $s \approx_{\mathbb{e}} t$ means $d_{\mathbb{e}}(s,t) = 0$. Because of the almost sure convergence of $\tilde{C}_{\tau_{n(k)}}$ towards $\mathbb{e}$, we can find integers $0 \leq s_{n(k)} < t_{n(k)} \leq 2n$ such that

$$\frac{s_{n(k)}}{2n(k)} \to s, \ \frac{t_{n(k)}}{2n(k)} \to t \quad \text{as} \quad k \to \infty$$

and $u^n_{s_{n(k)}} = u^n_{s_{n(k)}}$. This implies

$$\begin{aligned}
D(s,t) &= \lim_{k\to\infty} \tilde{D}_{q_{n(k)}}(s,t) \\
&= \lim_{k\to\infty} \Big(\frac{9}{8n(k)}\Big)^{1/4} D_{q_{n(k)}}(2n(k)s, 2n(k)t) \\
&= \lim_{k\to\infty} \Big(\frac{9}{8n(k)}\Big)^{1/4} D_{q_{n(k)}}\Big(2n(k)\frac{s_{n(k)}}{2n(k)}, 2n(k)\frac{t_{n(k)}}{2n(k)}\Big) \\
&= \lim_{k\to\infty} \Big(\frac{9}{8n(k)}\Big)^{1/4} D_{q_{n(k)}}(s_{n(k)}, t_{n(k)}) \\
&= \lim_{k\to\infty} \Big(\frac{9}{8n(k)}\Big)^{1/4} d_{q_{n(k)}}(u^n_{s_{n(k)}}, u^n_{t_{n(k)}}) \\
&= 0.
\end{aligned}$$

The case where $s = t$ is trivial. The third point of the proposition is similarly obtained by using $\tilde{D}_{q_n} \leq \tilde{D}^\circ_{\mathcal{T}_n}$ and the convergence of $\tilde{D}^\circ_{\mathcal{T}_n}(s,t)$ to $D^\circ(s,t)$. □

We can now approach the main result of this section. For that, we define a relation $\approx_D$ on $[0,1]$ that is similar in nature to the one used in the construction process of the Brownian CRT: let $s \approx_D t$ if and only if $D(s,t) = 0$. Proposition 14, property 1., guarantees that $\approx_D$ is an equivalence relation. Let further

$$\mathcal{S} := [0,1]/\approx_D \tag{6.1.7}$$

be the associated quotient space and denote with $\pi_{\mathcal{S}}$ the canonical projection from $[0,1]$ into S. We can equip S with the quotient distance

$$D_{\mathcal{S}}(\pi_{\mathcal{S}}(s), \pi_{\mathcal{S}}(t)) := D(s,t).$$

The triangular inequality for D assures that $D_{\mathcal{S}}$ is well-defined. Furthermore, $(\mathcal{S}, D_{\mathcal{S}})$ is a metric space because all properties of D are inherited. Recalling the definition of a pointed metric space in (53), we can now state the following result:

Theorem 20. *For $n \geq 1$, let q_n be uniformly distributed over $\mathcal{Q}^\bullet_n$ with associated distance function d_{q_n} and pointed vertex v^*_n and denote as always with $Z_{\mathbb{e}}$ the terminal point process of the head of the Brownian snake. Suppose further $s_* \in [0,1]$ such that $Z_{\mathbb{e}}(s_*) = \inf Z_{\mathbb{e}}$ and set $x_* := \pi_{\mathcal{S}}(s_*)$ Then the pointed metric space $(\mathcal{S}, D_{\mathcal{S}}, x_*)$ is the limit in distribution of the pointed metric spaces $(V(q_n), (9/8n)^{1/4} d_{q_n}, v^*_n)$ for the pointed Gromov-Hausdorff topology on $\mathbf{M}_1$ along the subsequence $(n(k))_{k\geq 1}$, i.e.*

$$\Big(V(q_{n(k)}), \Big(\frac{9}{8n(k)}\Big)^{1/4} d_{q_{n(k)}}, v^*_{n(k)}\Big) \xrightarrow{d} (\mathcal{S}, D_{\mathcal{S}}, x_*).$$

Moreover, there holds almost surely for every $x \in \mathcal{S}$ and $s \in [0,1]$ such that $\pi_{\mathcal{S}}(s) = x$,

$$D_{\mathcal{S}}(x, x_*) = D(s, s_*) = Z_{\mathbb{e}}(s) - Z_{\mathbb{e}}(s_*).$$

Proof. To simplify the reading this proof, we use with a slight abuse of notation the abbreviation $k := n(k)$. Let us first assume by Skorokhod's representation theorem (Theorem 6) that the convergence in (6.1.6) holds in the almost sure sense, i.e.

$$(\tilde{C}_{\tau_k}, \tilde{L}_{\tau_k}, \tilde{D}_{q_k}) \longrightarrow (\mathbb{e}, Z_{\mathbb{e}}, D) \tag{6.1.8}$$

as $k \to \infty$ with probability one, where $\tau_k = \tilde{\Phi}(q_k)$. For every k, let $i_*^k \in \{0, ..., 2n\}$ such that $L_{\tau_k}(i_*^k) = \min L_{\tau_k}$. The procedure applied within the CVS bijection guarantees $d_{q_k}(v_k^*, u_{i_*^k}^k) = 1$ which implies

$$|d_{q_k}(v_k^*, v) - d_{q_k}(u_{i_*^k}^k, v)| \leq 1 \tag{6.1.9}$$

for every $v \in V(q_k)$. We construct a relation $\mathcal{R}_k$ between $V(q_k)$ and $\mathcal{S}$ by setting

$$\mathcal{R}_k := \{(u_{\lfloor 2ks \rfloor}^k, \pi_{\mathcal{S}}(s)) : s \in [0,1]\} \cup \{(v_k^*, x_*)\}.$$

Because the canonical projection $\pi_{\mathcal{S}}$ is surjective, $\mathcal{R}_k$ is also a correspondence. We compute its distortion with respect to the metrics on $V(q_k)$ and $\mathcal{S}$:

$$\mathrm{dis}(\mathcal{R}_k) \quad = \quad \sup\left\{\left|\left(\frac{9}{8k}\right)^{1/4} d_{q_k}(x, x') - D_{\mathcal{S}}(y, y')\right| : (x,y), (x', y') \in \mathcal{R}_k\right\} \tag{6.1.10}$$

According to the definition of $\mathcal{R}_k$, we have to distinguish two cases for the values of the pairs (x, y) and (x', y') in (6.1.10). First, we estimate the distortion of $\mathcal{R}_k$ for $(x, y) = (v_k^*, x_*)$ and $(x', y') \in \{(u_{\lfloor 2ks \rfloor}^k, \pi_{\mathcal{S}}(s)) : s \in [0, 1]\}$ by using (6.1.9):

$$\begin{aligned} & \sup\left\{\left|\left(\frac{9}{8k}\right)^{1/4} d_{q_k}(v_k^*, u_{\lfloor 2ks \rfloor}^k) - D_{\mathcal{S}}(x_*, \pi_{\mathcal{S}}(s))\right| : s \in [0,1]\right\} \\ \leq & \sup\left\{\left|\left(\frac{9}{8k}\right)^{1/4} (1 + d_{q_k}(u_{i_*^k}^k, u_{\lfloor 2ks \rfloor}^k)) - D_{\mathcal{S}}(x_*, \pi_{\mathcal{S}}(s))\right| : s \in [0,1]\right\} \\ = & \left(\frac{9}{8k}\right)^{1/4} + \sup\left\{\left|\tilde{D}_{q_k}\left(\frac{i_*^k}{2k}, \frac{\lfloor 2ks \rfloor}{2k}\right) - D(s_*, s)\right| : s \in [0,1]\right\} \end{aligned}$$

The last expression converges to zero due to (6.1.8) and the fact that $\frac{i_*^k}{2k}$ converges to s_*. Similarly, for $(x, y), (x', y') \in \{(u_{\lfloor 2ks \rfloor}^k, \pi_{\mathcal{S}}(s)) : s \in [0, 1]\}$ we can bound the distortion of $\mathcal{R}_k$ from above by

$$\begin{aligned} & \sup\left\{\left|\left(\frac{9}{8k}\right)^{1/4} d_{q_k}(u_{\lfloor 2ks \rfloor}^k, u_{\lfloor 2kt \rfloor}^k) - D_{\mathcal{S}}(\pi_{\mathcal{S}}(s), \pi_{\mathcal{S}}(st))\right| : s, t \in [0,1]\right\} \\ = & \sup\left\{\left|\tilde{D}_{q_k}(\frac{\lfloor 2ks \rfloor}{2k}, \frac{\lfloor 2kt \rfloor}{2k}) - D(s,t)\right| : s, t \in [0,1]\right\} \end{aligned}$$

which also tends to zero. Therefore, $\mathrm{dis}(\mathcal{R}_k) \to 0$ as $k \to \infty$ almost surely. Since $d_{GH}^{\bullet}(X, Y) = \frac{1}{2} \inf\{\mathrm{dis}(\mathcal{R}) : \mathcal{R} \in \mathrm{Cor}(X, Y), (\rho_X, \rho_Y) \in \mathcal{R}\}$, we can conclude that the pointed metric spaces $(V(q_k), (9/8k)^{1/4}d_{q_k}, v_k^*)$ also converge almost surely to $(\mathcal{S}, D_{\mathcal{S}}, x_*)$ along the subsequence $k = n(k)$ in the pointed Gromov-Hausdorff topology. In order to show the second statement of the theorem, we use once more the uniform convergence in (6.1.8) together with (6.1.1) which yields for $x \in \mathcal{S}$ and $s \in [0,1]$ with $\pi_{\mathcal{S}}(s) = x$,

$$\begin{aligned}
D_{\mathcal{S}}(x_*, x) &= D(s_*, s) \\
&= \lim_{k\to\infty} \tilde{D}_{q_k}\left(\frac{i_*^k}{2k}, \frac{\lfloor 2ks\rfloor}{2k}\right) \\
&= \lim_{k\to\infty} \left(\frac{9}{8k}\right)^{1/4} D_{q_k}\left(2k\frac{i_*^k}{2k}, 2k\frac{\lfloor 2ks\rfloor}{2k}\right) \\
&= \lim_{k\to\infty} \left(\frac{9}{8k}\right)^{1/4} d_{q_k}(u^k_{i_*^k}, u^k_{\lfloor 2ks\rfloor}) \\
&= \lim_{k\to\infty} \left(\frac{9}{8k}\right)^{1/4} d_{q_k}(u^k_{\lfloor 2ks\rfloor}, u^k_{i_*^k}) \\
&= \lim_{k\to\infty} \left(\frac{9}{8k}\right)^{1/4} (L_{\tau_k}(\lfloor 2ks\rfloor) - \min L_{\tau_k} + 1) \\
&= \lim_{k\to\infty} \left(\tilde{L}_{\tau_k}\left(\frac{\lfloor 2ks\rfloor}{2k}\right) - \min \tilde{L}_{\tau_k} + \left(\frac{9}{8k}\right)^{1/4}\right) \\
&= Z_{\mathbb{e}}(s) - \inf Z_{\mathbb{e}} \\
&= Z_{\mathbb{e}}(s) - Z_{\mathbb{e}}(s_*).
\end{aligned}$$

$\square$

In Theorem 13 we have seen, that a sequence of random plane trees converges in distribution towards a random limiting tree that we identified as the Brownian Continuum Random Tree $(\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}}, \rho_{\mathbb{e}})$. The convergence result in the last theorem can be seen as an analogue for pointed quadrangulations. Indeed, the proofs of both statements are quite similar in nature. One might therefore be tempted to call the pointed metric space $(\mathcal{S}, D_{\mathcal{S}}, x_*)$ the "Brownian Continuum Random Map/Quadrangulation". However, we have to keep in mind that the whole construction depends on a fixed subsequence $(n(k))_{k\geq 1}$ of $n \geq 1$. This fact clouds the beauty of Theorem 20 in the sense, that we are not (yet) able to obtain the limit independently of the chosen subsequence $(n(k))_{k\geq 1}$. We address this obstacle in the next section.

A last point that might also be worth mentioning is, that we do not have an explicit description of the metric $D_{\mathcal{S}}$ of the limiting pointed metric space $(\mathcal{S}, D_{\mathcal{S}}, x_*)$ contrary to the metric $d_{\mathbb{e}}$ of the Brownian CRT which can be computed directly with the help of the normalized Brownian Excursion $\mathbb{e}$ (see (4.1.3)).

6.2 Uniqueness and Properties of the Brownian Map

We now approach the problem mentioned at the end of the last section. Let us recall that the inequality $D \leq D^\circ$ (see Proposition 14, point 3.) implies for every $s, t \in [0,1]$,

$$D_{\mathcal{S}}(\pi_{\mathcal{S}}(s), \pi_{\mathcal{S}}(t)) = D(s,t) \leq D^{\circ}(s,t).$$

Consider an integer $k \geq 1$ and $s_i, t_i \in [0,1]$ for every $i \in \{1, ..., k\}$ such that $s_1 = s, t_k = t$ and $t_i \approx_{\mathbb{e}} s_{i+1}$ for every $i \in \{1, ..., k-1\}$. Then by the triangular inequality and again by Proposition 14, we obtain

$$\begin{aligned}
D_{\mathcal{S}}(\pi_{\mathcal{S}}(s), \pi_{\mathcal{S}}(t)) &= D(s,t) \\
&\leq D(s_1,t_1) + D(t_1,s_2) + D(s_2,t_2) + D(t_2,s_3) + \cdots + D(s_k,t_k) \\
&= \sum_{i=1}^{k} D(s_i,t_i) + D(t_1,s_2) + D(t_2,s_3) + \cdots + D(t_{k-1},s_k) \\
&= \sum_{i=1}^{k} D(s_i,t_i) \leq \sum_{i=1}^{k} D^{\circ}(s_i,t_i).
\end{aligned}$$

Notice that for $k = 1$ we obtain the original bound $D_{\mathcal{S}}(\pi_{\mathcal{S}}(s), \pi_{\mathcal{S}}(t)) \leq D^{\circ}(s,t)$. Letting, for $s, t \in [0,1]$

$$D^*(s,t) := \inf\Big\{ \sum_{i=1}^{k} D^{\circ}(s_i,t_i) : k \geq 1, s = s_1, t = t_k, t_i \approx_{\mathbb{e}} s_{i+1}, 1 \leq i \leq k-1 \Big\},$$

we see that

$$D_{\mathcal{S}}(\pi_{\mathcal{S}}(s), \pi_{\mathcal{S}}(t)) \leq D^*(s,t).$$

Also, most of the metric properties of D° are transferred to D^* except that $D^*(s,t) = 0$ does not necessarily imply $s = t$. Therefore D^* is a pseudo-metric on $[0,1]$. D^* can be seen as resulting from "gluing" the space $[0,1]$ along the equivalence relation $\approx_{\mathbb{e}}$. In analogy to the definition of the quotient space $\mathcal{S}$ in (6.1.7), we can set

$$\mathcal{S}^* := [0,1]/ \approx_{D^*}$$

where the equivalence relation $\approx_{D^*}$ is naturally defined by $s \approx_{D^*} t$ if and only if $D^*(s,t) = 0$. We can endow $\mathcal{S}^*$ with the quotient distance by setting

$$D_{\mathcal{S}^*}(\pi_{\mathcal{S}^*}(s), \pi_{\mathcal{S}^*}(t)) := D^*(s,t),$$

where $\pi_{\mathcal{S}^*}$ denotes the canonical projection from $[0,1]$ into $\mathcal{S}^*$. Marckert and Mokkadem conjectured that $(\mathcal{S}^*, D_{\mathcal{S}^*})$ is the unique scaling limit of the metric spaces $(V(q_n), (9/8n)^{1/4} d_{q_n})$ as in Theorem 20. This is indeed the case:

Theorem 21. *Almost surely, it holds that the functions $D_{\mathcal{S}}$ and $D_{\mathcal{S}^*}$ are equal. Consequently, the convergence*

$$\Big(V(q_n), \Big(\frac{9}{8n}\Big)^{1/4} d_{q_n}\Big) \xrightarrow{d} (\mathcal{S}, D_{\mathcal{S}}) = (\mathcal{S}^*, D_{\mathcal{S}^*})$$

for the Gromov-Hausdorff topology on the set of compact metric spaces holds without having to take an appropriate subsequence.

A proof of this result can be found in [28, Theorem 1]. Theorem 21 allows us to call the space $(\mathcal{S}, D_{\mathcal{S}}) = (\mathcal{S}^*, D_{\mathcal{S}^*})$ *the* **Brownian Map**.

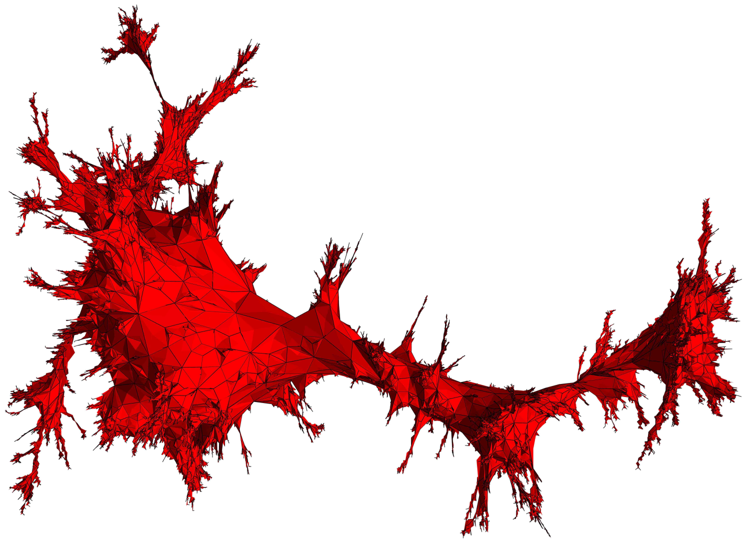


Figure 6.1: Visualization of a large random quadrangulation, which is an approximation of the Brownian Map (by Jérémie Bettinelli).

We now want to briefly discuss some important properties of the Brownian Map. At first, there is a re-rooting property quiet similar to the one obtained for the Brownian CRT (Corollary 9).

Proposition 15. *Let $k \geq 1$ and $\{U_i \in [0,1] : 1 \leq i \leq k+1\}$ independent uniform random variables. Then $(\mathcal{S}, D_{\mathcal{S}}, x_*, \pi_{\mathcal{S}}(U_1), \ldots, \pi_{\mathcal{S}}(U_k))$ and $(\mathcal{S}, D_{\mathcal{S}}, \pi_{\mathcal{S}}(U_1), \ldots, \pi_{\mathcal{S}}(U_{k+1}))$ have the same distribution, seen as random variables in the set $\mathbf{M}_{k+1}$ of $k+1-$pointed metric spaces.*

The method of proof for this result is the same as for Theorem 20 but extended to $k+1-$pointed metric spaces.

Our previous study of the Brownian CRT has shown, that its Hausdorff dimension equals 2, almost surely. A similar result can be obtained for the Brownian Map:

Theorem 22. *Almost surely, the Hausdorff dimension $\dim_H(\mathcal{S})$ of the Brownian Map $(\mathcal{S}, D_{\mathcal{S}})$ equals 4.*

For a proof of this result we refer to the discussion in [29, Theorem 4.4.3].

Finally, we want to give an alternative description of the Brownian Map by establishing a relation with the Brownian CRT. For this, recall that originally we introduced the Brownian Map as a quotient space $\mathcal{S} = [0,1]/\approx_D$ where the equivalence relation $\approx_D$ was defined by $s \approx_D t$ if and only if $D_{\mathcal{S}}(\pi_{\mathcal{S}}(s), \pi_{\mathcal{S}}(t)) = D(s,t) = 0$. Notice, that this construction was based on a suitable subsequence. We also saw in Theorem 21 that $(S, D_{\mathcal{S}})$ coincides (along any suitable subsequence) with the metric space $(\mathcal{S}^*, D_{\mathcal{S}^*})$ which was defined by $\mathcal{S}^* = [0,1]/\approx_{D^*}$ and which does not depend on a fixed subsequence. The equivalence relation used to construct $\mathcal{S}^*$ satisfies $s \approx_{D^*} t$ if and only if $D_{\mathcal{S}^*}(\pi_{\mathcal{S}^*}(s), \pi_{\mathcal{S}^*}(t)) = 0$. The third equivalence relation that was important in our study was $\approx_{\mathbb{e}}$, which appeared in the definition of the Brownian CRT $\mathcal{T}_{\mathbb{e}} = [0,1]/\approx_{\mathbb{e}}$. Proposition 14, point 2., showed, that for every $s, t \in [0,1]$, $D(s,t)$ depends only on $\pi_{\mathbb{e}}(s)$ and $\pi_{\mathbb{e}}(t)$ and not on a any special representatives. For $a, b \in \mathcal{T}_{\mathbb{e}}$ and any $s, t \in [0,1]$ satisfying $a = \pi_{\mathbb{e}}(s)$ and $b = \pi_{\mathbb{e}}(t)$, we can therefore define

$$D_{\mathbb{e}}(a,b) := D(s,t),$$

which yields a pseudo-distance on $\mathcal{T}_{\mathbb{e}}$. We can use it to define (yet another) equivalence relation $\approx_{D_{\mathbb{e}}}$ on $\mathcal{T}_{\mathbb{e}}$ by setting $a \approx_{D_{\mathbb{e}}} b$ if and only if $D_{\mathbb{e}}(a,b) = 0$. Let

$$\mathcal{S}_{\mathbb{e}} := \mathcal{T}_{\mathbb{e}}/\approx_{D_{\mathbb{e}}}$$

and denote with $\pi_{\mathcal{S}_{\mathbb{e}}}$ the canonical projection from $\mathcal{T}_{\mathbb{e}}$ into $\mathcal{S}_{\mathbb{e}}$. Let further

$$D_{\mathcal{S}_{\mathbb{e}}}(\pi_{\mathcal{S}_{\mathbb{e}}}(a), \pi_{\mathcal{S}_{\mathbb{e}}}(b)) := D_{\mathbb{e}}(a,b)$$

be the quotient distance on $\mathcal{S}_{\mathbb{e}}$ for every $a, b \in \mathcal{T}_{\mathbb{e}}$. Equipped with these new notations, the following holds:

Proposition 16. *The Brownian Map $(\mathcal{S}, D_{\mathcal{S}})$ is isometric to $(\mathcal{S}_{\mathbb{e}}, D_{\mathcal{S}_{\mathbb{e}}})$.*

Proof. We define a mapping ϕ from $\mathcal{S} = [0,1]/\approx_D$ into $\mathcal{S}_{\mathbb{e}} = \mathcal{T}_{\mathbb{e}}/\approx_{D_{\mathbb{e}}}$ by

$$\phi(a) := \pi_{\mathcal{S}_{\mathbb{e}}} \circ \pi_{\mathbb{e}}(s)$$

for every $a \in \mathcal{S}$ with $a = \pi_{\mathcal{S}}(s)$ for an appropriate $s \in [0,1]$. Then, for $a, b \in \mathcal{S}$ with $a = \pi_{\mathcal{S}}(s)$ and $b = \pi_{\mathcal{S}}(t)$, the calculation

$$\begin{aligned} D_{\mathcal{S}_{\mathbb{e}}}(\phi(a), \phi(b)) &= D_{\mathcal{S}_{\mathbb{e}}}(\pi_{\mathcal{S}_{\mathbb{e}}} \circ \pi_{\mathbb{e}}(s), \pi_{\mathcal{S}_{\mathbb{e}}} \circ \pi_{\mathbb{e}}(t)) \\ &= D_{\mathbb{e}}(\pi_{\mathbb{e}}(s), \pi_{\mathbb{e}}(t)) \\ &= D(s,t) \\ &= D_{\mathcal{S}}(\pi_{\mathcal{S}}(s), \pi_{\mathcal{S}}(t)) \\ &= D_{\mathcal{S}}(a,b) \end{aligned}$$

shows, that ϕ is an isometry. □

The description of the Brownian Map in terms of a quotient of the Brownian CRT allows us to further explore the equivalence relation $\approx_{D_{\mathbb{e}}}$ and hence the metric $D_{\mathcal{S}}$ of which we know very little so far. Theorem 20 showed, that it is possible to handle distances from the pointed vertex $x_* \in \mathcal{S}$ with the help of the terminal point process of the head of the

Brownian snake, i.e. for $x \in \mathcal{S}$ and $x = \pi_{\mathcal{S}}(s)$, we have (with the notation of Theorem 20)

$$D_{\mathcal{S}}(x, x_*) = D(s, s_*) = Z_{\mathbb{e}}(s) - Z_{\mathbb{e}}(s_*).$$

Suppose $a, b \in \mathrm{Lf}(\mathcal{T}_{\mathbb{e}})$ and let $s, t \in [0,1)$ such that $\pi_{\mathbb{e}}(s) = a$ and $\pi_{\mathbb{e}}(t) = b$. Assume that $s < t$ and set

$$\begin{aligned} [a,b]_{\mathbb{e}} &:= \pi_{\mathbb{e}}([s,t]) \\ [b,a]_{\mathbb{e}} &:= \pi_{\mathbb{e}}([t,1] \cup [0,s]). \end{aligned}$$

It can be shown that $[a,b] \cap [b,a]$ is the geodesic segment between a and b in $\mathcal{T}_{\mathbb{e}}$ consisting of the union of the geodesic segments between a and the root $\rho_{\mathbb{e}} \in \mathcal{T}_{\mathbb{e}}$ as well as $\rho_{\mathbb{e}}$ and b, i.e. $[a,b] \cap [b,a] = \Gamma_s \cup \Gamma_t$ with the notation of Lemma 23.

Theorem 23. *Almost surely, for every distinct $a, b \in \mathcal{T}_{\mathbb{e}}$,*

$$a \approx_{D_{\mathbb{e}}} b \;\Leftrightarrow\; \begin{cases} a, b \in Lf(\mathcal{T}_{\mathbb{e}}) \text{ and} \\ Z_{\mathbb{e}}(a) = Z_{\mathbb{e}}(b) = \max\{\min_{c \in [a,b]_{\mathbb{e}}} Z_{\mathbb{e}}(c), \min_{c \in [b,a]_{\mathbb{e}}} Z_{\mathbb{e}}(c)\} \end{cases} \tag{6.2.1}$$

The proof of this theorem is quiet arduous and can be found in [29, Theorem 5.1.1].

Appendix

In this short appendix we collect two well-known theorems that are used occasionally and which do not fit in any other chapter.

Theorem 24 (Arzelà-Ascoli)**.** *Let $(X, \mathcal{T})$ be a compact topological space and let M be a closed subset of $\mathcal{C}(X, \mathbb{R})$ such that*

1. *For every $x \in X$, $\sup_{f \in M} |f(x)|$ is bounded*
2. *For every $x \in X$ and $\epsilon > 0$ there exists a neighbourhood U around x such that $|f(y) - f(x)| < \epsilon$ for every $y \in U, f \in M$.*

Then M is compact if and only if 1. and 2. hold.

Theorem 25 (Prohorov's Theorem)**.** *Let (X, d) be a complete and separable metric space, and let $\mathcal{M} \subseteq \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the set of Borel probability measures on X. Then the following statements are equivalent:*

1. *$\mathcal{M}$ is tight.*
2. *For every $\epsilon > 0$, there exists a compact subset $K \subseteq X$ such that*

$$\inf_{P \in \mathcal{M}} P(K^\epsilon) \geq 1 - \epsilon,$$

   where $K^\epsilon := \{x \in X : \inf_{y \in K} d(x, y) < \epsilon\}$.

3. *$\mathcal{M}$ is relatively compact.*

Conclusion and Outlook

In this thesis, we studied an interesting mapping between certain classes of discrete trees (well-labellend and embedded trees) and (pointed) quadrangulations known as the CVS bijection (Corollary 1 and Corollary 2). This correspondence turned out to be special in the sense that it preserves certain metric properties between the investigated objects which we used to obtain limiting expressions for the radius and profile of large random pointed quadrangulations (Theorem 16 and Theorem 18). The CVS bijection also helped us in the exploration of the limiting behaviour of large random quadrangulations. Our discussion of these scaling limit results culminated in the discovery of two random "continuous" structures which are called the Brownian Continuum Random Tree (CRT) and the Brownian Map. We demonstrated that these two objects - being the limits in distribution of a sequence of uniformly distributed embeeded trees and pointed quadrangulations respectively (Theorem 13 and Theorem 21) - are closely linked (Proposition 16). This observation can be seen as a "continuous" analogue to the fact that discrete trees (well-labellend and embedded trees) and (pointed) quadrangulations are connected by way of the CVS bijection.

We also want to use the opportunity to take a quick look beyond the scope of this thesis. The first point worth mentioning is, that the main result of the first part of our work (the CVS bijection in its versions for well-labelled and embedded trees) can be generalized to multi-pointed quadrangulations. This was done by Bouttier, Di Francesco and Guitter [24] and can be used to obtain further properties of the Brownian Map as for example the joint law of distances between three randomly chosen vertices in the Brownian Map, see [7]. Apart from that, there exist several other one-to-one correspondences between similar classes of trees and maps. One of them is a recent bijection due to Ambjørn and Budd [23] which offers astounding applications in the field of two-dimensional quantum gravity.

Another point which is of relevance to us is concerned with Theorem 20 and Theorem 21 respectively: we have seen that a sequence of uniformly distributed pointed quadrangulations converges in distribution towards the Brownian Map $(\mathcal{S}, D_{\mathcal{S}})$. Le Gall [18, Theorem 3.4] showed that it is possible to extend this result to $2p$-angulations (planar maps where every face has $2p$ adjacent edges and again with the convention that if an edge lies entirely inside a face it is counted twice):

Theorem. *Almost surely there holds for every integer $p \geq 2$,*

$$\Big(V(m_n), \Big(\frac{9}{4p(p-1)n}\Big)^{1/4} d_{m_n}\Big) \xrightarrow{d} (\mathcal{S}, D_{\mathcal{S}})$$

for the Gromov-Hausdorff topology on the set of compact metric spaces, where for every

$n \geq 1$, m_n is a $2p-$angulation with n faces and d_{m_n} denotes its associated natural distance.

Finally, we want to mention that the Brownian Map is almost surely homeomorphic to the $2-$sphere $\mathbb{S}^2$ due to Le Gall and Paulin [19]. This topological equivalence of the Brownian Map to $\mathbb{S}^2$ allows one to view it as a "random surface" which again is of particular interest in theoretical physics. The proof of this so-called Sphericity Theorem uses the description of the Brownian Map as quotient of the Brownian CRT as stated in Proposition 16.

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