DIPLOMARBEIT

On Optimal Reservoir Usage for Hydro Power Plants

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Das Thema dieser Arbeit wurde von mir bisher weder im In- noch Ausland einer Beurteilerin/einem Beurteiler zur Begutachtung in irgendeiner Form als Prüfungsarbeit vorgelegt. Diese Arbeit stimmt mit der von den Begutachterinnen/Begutachtern beurteilten Arbeit überein.

Contents

1. Introduction

The goal of this thesis is the optimal control of reservoir usage for hydro power plants. We are especially interested in pumped hydroelectricity storage facilities, because the additional capability of pumping water into the reservoir for storage offers more possibilities for optimisation.

The problem revolves around the question whether to release water to generate electricity, do nothing, or, if available, to pump water into the reservoir. The control we are looking for will depend on the actual electricity price and the water level of the reservoir. Therefore, we need to model the electricity spot price dynamics. There are a lot of possible ways to do so due to the complexity of the electricity market, see for instance Carmona and Coulon [3]. Electricity is non-storable and has transportation constraints, leading to high dependency, among others, on local demand, supply and weather conditions at every precise moment (see Lucia and Schwartz [6, p. 2]).

We will use the approach of Benth, Kallsen and Meyer-Brandis [1, pp. 2–3] and use a mean-reverting model including seasonality and spikes. The dynamics will be modelled as a sum of non-Gaussian Ornstein–Uhlenbeck processes giving the normal variation and spike behaviour of the prices. Although we will not use jump processes and instead use Brownian motions like Lucia and Schwartz [6, p. 11].

The thesis is organised as follows:

In Chapter 2 we describe the problem and examine the different types of hydro power plants and take a look at the electricity market. Of special interest are pumped hydroelectric storage plants and the electricity spot price.

In Chapter 3 we give a mathematical formulation of the problem. At first we define our control process, which represents the water usage of a hydro power plant, followed by a model for the water level of the reservoir and the electricity spot price. Then we finally present the optimization problem, or more precisely the stochastic control problem. We also define a performance criterion to measure different controls of reservoir usage. This criterion leads to the value function by taking the supremum over all controls. To simplify the stochastic control problem, we proceed with a time-invariant case.

In the second half of the chapter we present the dynamic programming principle. Together with the condition that the value function is smooth enough, this principle will be used to derive the HJB equation. Under the same smoothness condition we also prove that the solution of the HJB equation is the value function and that the optimal control exists. In the last section of the chapter we give conditions, depending on the water level and electricity spot price, under which the optimal control assumes one of four values.

In Chapter 4 we use a numeric method to solve the HJB equation, and therefore approximate the value function and the optimal control. We do this for parameters which model a pumped hydroelectric storage plant currently under construction, Obervermuntwerk II, owned by Vorarlberger Illwerke. We present an algorithm written in Matlab, based on the finite difference method. At the end of the chapter we illustrate an approximation of the value function and the optimal control. We conclude the chapter by proving that the algorithm is a convergent approximation to the HJB equation.

The whole algorithm is presented in the appendix.

2. Statement of the Problem

The Problem consists of the optimisation of a hydro power plant. Optimisation in the sense of generating the highest profit by finding the optimal control for reservoir usage.

The power plant can generate electricity by releasing water through a turbine and a generator, or, if available, use electricity to pump water into the reservoir for storage. There are constrictions in the form of the maximum amount of water release- and pump-able at one time and the capacity of the reservoir.

To achieve the highest profit, one needs to know when to hold back water reserves and when to generate or use electricity by releasing or pumping water, and in which quantities. This depends on the actual price of said energy. Due to the fact that electricity prices are stochastic, we are confronted with a stochastic control problem.

Before presenting the mathematical formulation, which will happen in Chapter 3, we take a closer look at different types of hydro power plants and the electricity market, particularly at pumped hydro storage facilities and the spot market.

2.1. Hydro Power Plants

Hydro power plants can be categorised under different aspects. In the light of our problem it is best to do so under a technical factor, like hydraulic engineering:

1. *Run-of-the-river Hydroelectric Power Plants.* They have a very small reservoir capacity, or non at all. Therefore, electricity is generated only by natural stream flow and any oversupply is wasted.

2. *Hydroelectric Storage Plants.* Most hydro facilities are of this kind. They make use of a reservoir, most commonly provided by a dam, to store water and generate electricity when demanded.

3. *Pumped Hydroelectric Storage (PHS) Plants.* PHS plants make use of two reservoirs with different elevation to either supply electricity during high peak demands or using electricity during low peak demand by moving water between them.

(see Giesecke and Mosonyi [4, p. 83])

There are more types of hydroelectricity generating plants, like tidal power, glacier power or wave power plants which we will not examine closer. However, we will provide a more accurate description of PHS Plants in the following section, because those are the most complex to optimise among the three listed.

It is noteworthy that some hydro power plants can be legally obligated to keep the water level of their (upper) reservoir within given limits for reasons like their impact on the environment (see Giesecke and Mosonyi [4, pp. 44,46]). For example if the reservoir is a lake and its wildlife must be protected.

Usually, there is an extensive examination of the flow patterns of rivers feeding hydro power plants. Typically, datasets span at least ten, but most of the time more than twenty-five years, giving us a good picture on how to model natural stream flow (see Giesecke and Mosonyi [4, p. 45]).

2.1.1. Pumped Hydroelectricity Storage (PHS) Plants

A PHS plant is normally equipped with turbines, generators and pumps, which are connected to an upper and a lower reservoir, see Figure 2.1. During off-peak hours, like at night or weekends, relatively cheap energy is used to operate the pumps to move water from the lower reservoir to the upper reservoir in order to store energy. During periods of high demand, water is released back into the lower reservoir through a turbine to generate power at higher price.
(see Yang [13, pp. 3–4])

There are two main types of PHS facilities:

1. *Pure or off-stream PHS*, which rely entirely on water that was previously pumped into the upper reservoir.

2. *Combined, hybrid, or pump-back PHS*, which make use of both pumped water and natural stream flow water.

(see Yang [13, p. 4])

An important fact to mention is that energy is lost while pumping water. The *round-trip efficiency*, meaning electricity generated through releasing water divided by the electricity used to pump said water, of facilities with older designs can be lower than 60%, while a state-of-the-art PHS system may achieve over 80% efficiency (see Yang [13, p. 5]).

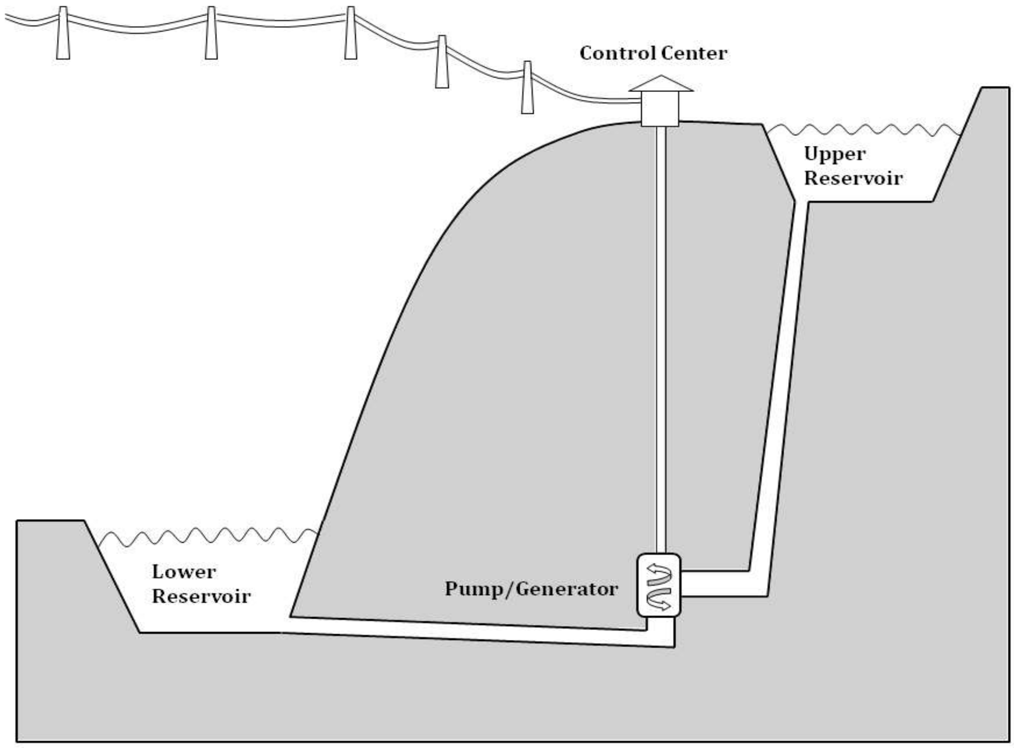

Figure 2.1.: PHS Diagram (see Yang [13, p. 4]).

PHS plants have good controllability due to short start-up times (see Giesecke and Mosonyi [4, p. 99]).

2.2. Electricity Marked

Electricity has a limited transportability and limited storability. It may be considered a flow commodity. The non-storability of electricity leads users to perceive electricity delivered at different times and dates as a distinct commodity.
(see Lucia and Schwartz [6, p. 2])

Therefore, prices depend strongly on the energy demand, influenced by factors like business activity, temporal weather conditions and the like, in every precise moment. Additionally, the transportation constraints for electricity, caused by limited transmission capacity and transportation losses, make prices of energy local. They dependent on characteristics like the local generation plants, local climate and weather conditions, and market rules together with their derived uses of electricity.
(see Lucia and Schwartz [6, p. 2])

In regard to our problem, we are especially interested in the spot market because the operation of hydro power stations depend heavily on the price of electricity at the precise moment it is generated.

Electricity spot prices have several key characteristics that can be observed more or less distinctly in all electricity markets. Let us recall the three most important stylized facts:

Usually, electricity prices have very sharp spikes. This is caused by inelastic demand coupled with an exponentially increasing curve of marginal costs. A sudden change of demand or supply, for example caused by temporal weather conditions or technical problems at a local power plant, causes strong jumps of electricity prices. In light of these sudden changes, another important observation is that afterwards electricity prices tend to revert back to a normal level rather swiftly, making mean reverting processes suitable to model spot prices. Finally, spot prices exhibit seasonal behaviour, namely over one day, one week and over the year.
(see Benth, Kallsen and Meyer-Brandis [1, p. 2])

Often only one or two components are used to model seasonal behaviour. For example, Lucia and Schwartz [6, p. 7] use the arithmetic average of all hourly electricity prices for a given day as the system or spot price, which immediately ignores the intra-day price patterns. Our model will include all three components of seasonal behaviour. Although, we will study a simplified case from Section 3.4 onwards that does not include seasonality.

3. Mathematical Formulation and the HJB Equation

In this chapter we will provide a mathematical formulation on optimal reservoir usage for a hydro power plant, assumed to be a PHS facility. It is structured as followed: In Section 3.1 we provide a control function representing the water used by the power plant. Followed by a model for the water level of the upper reservoir. In Section 3.2 we model the spot price of electricity.

In Section 3.3 we finally present the whole problem, formulated as a stochastic control problem. In Section 3.4 we simplify the problem by considering a time-invariant state process. This simplified case will be the subject-matter for the rest of the thesis.

In Section 3.5 we introduce the dynamic programming principle, which will be used to derive the HJB equation in the following Section 3.6. In the last section of this chapter we show that the optimal control function assumes only four values.

The relevant data for the controller are the spot price, denoted by $S(t)$, and the water level, denoted by $H^u(t)$. The joined process $X^u(t) := (H^u(t), S(t))$ is the state system and will be introduced in Section 3.3.

3.1. The Control Process and the Water Level

The process $u(t) = u(t, w) : [s, \infty) \times \Omega \to [-\alpha_1, \alpha_2]$, with $s \geq 0$, $\alpha_1 \geq 0$ and $\alpha_2 > 0$, is our *control process*, assumed to be càdlàg and adapted (see Øksendal and Sulem [8, p. 69]).

It represents the volume flow rate between the two reservoirs. Negative values of u mean that water is pumped from the lower to the upper reservoir, while positive values of u represent the water flow from the upper to the lower reservoir. If it were a hydro storage plant instead of an PHS facility, we would restrict u to be non negative. In the case of a run-of-the river hydro plant u would be simply equal to the natural stream flow, which would not offer us any means of optimisation beside not generating electricity while the spot price is negative.

The Process $H^u(t) : [s, \infty) \to [0, \bar{H}]$, $\bar{H} \in \mathbb{R}_+$, represents the *water level* at time t and measures the by the hydro power plant usable amount of water in the upper reservoir. $\bar{H}$ is the *maximal amount of usable water.* It may be lower than the natural capacity of the reservoir, because there can be a legal fluctuation of the water level. An example for a legal fluctuation is given in section 2.1.

The Process is assumed to have the following form:

$$H^u(t) = x_1 + \int_s^t z(p)\ dp - \int_s^t u(p)\ dp - A(t),$$

where x_1 gives the starting water level, $z : \mathbb{R}_+ \to \mathbb{R}$ models the influx of water and A is the overflow channel, see below. Furthermore, we allow for a variable start point s due to the fact that $X^u(t)$ is time-variant.

The variable $x_1 = H^u(s) \in [0, \bar{H}]$ gives the water level at start time s. The function $z(t)$ models the influx of water, precisely the *volume flow rate of the natural stream* at time t. For simplicity, we assume $z(t)$ is a deterministic non negative function. Considering the extensive stream flow pattern examination mentioned in Section 2.1, this is a plausible assumption.

The integral over the volume flow rate $u(t)$ gives the amount of water that was moved from or to the upper reservoir until time t. Finally, $A(t)$ models the *overflow channel*:

$$A(t) = \int_s^t (z(p) - u(p))\mathbb{1}_{\{H^u(p)=\bar{H}\ \wedge\ z(p)>u(p)\}}\ dp.$$

$A(t)$ is a non decreasing function that only increases while $H^u(t)$ is at the maximal water level $\bar{H}$, and the volume flow rate $u(t)$ of water used is less than the volume flow rate $z(t)$ of water provided by natural stream flow.

Therefore, $A(t)$ guarantees that $H^u(t)$ fulfils the upper bounding condition given by the maximum water level $H^u(t)$ of the reservoir. To assure that $H^u(t)$ is non negative, we need the following *condition for our control function* $u(t)$:

$$u(t) \leq z(t) \quad \text{on } \{t \geq s : H^u(t) = 0\}. \tag{3.1}$$

Altogether, $H^u(t)$ can be simplified as follows:

$$
\begin{aligned}
H^u(t) &= x_1 + \int_s^t z(p)\ dp - \int_s^t u(p)\ dp - \int_s^t (z(p) - u(p)) \mathbb{1}_{\left\{H^u(p)=\bar{H}\ \wedge\ z(p)>u(p)\right\}}\ dp \\
&= x_1 + \int_s^t (z(p) - u(p)) \mathbb{1}_{\left\{H^u(p)<\bar{H}\ \vee\ z(p)\leq u(p)\right\}}\ dp.
\end{aligned}
$$

3.2. Electricity Spot Price

We denote the *spot price* of electricity at time t by $S(t)$ and model S by

$$
dS(t) = d\Lambda(t) + dY(t)
$$

with the starting point $x_2 := \Lambda(s) + Y(s) \in \mathbb{R}$, $s \geq 0$. $\Lambda(t)$ is the deterministic, periodic function modelling the *seasonal behaviour* and is given by

$$
\Lambda(t) = \mu + \sum_{j=1}^{3} c_j \sin(b_j(t - d_j))
$$

for constants μ, b, c and d, similar as in Benth, Benth and Koekebakker [2, p. 85]. The time-inhomogeneity of Λ transfers of course to time-inhomogeneity in the stochastic control problem which we will set up in the following section. Although, in Section 3.4 we will assume $X^u(t)$ to be time-invariant in order to simplify the presentation of this discussion.

The diffusion stochastic process $Y(t)$ is described by the following dynamics:

$$
Y(t) = \sum_{i=1}^{n} Y_i(t),
$$

where

$$
dY_i(t) = -\lambda_i Y_i(t)\ dt + \sigma_i\ dW_i(t)
$$

with given starting points $Y_i(s) = y_i \in \mathbb{R}$, $i = 1, \ldots, n$ and $n \in \mathbb{N}$. The λ_i and σ_i are positive constants. The processes $W_i(t)$ are independent standard Brownian motions. Therefore, $Y_i(t)$ follows a *stationary mean-reverting process*, or *Ornstein–Uhlenbeck process*, with a zero *long-run mean*, a *speed of mean-reversion* λ_i and a *degree of volatility* σ_i.

This resembles closely the model of Benth, Kallsen and Meyer-Brandis [1, p. 3], but we use standard Brownian motions instead of jump processes.

An example with three Ornstein–Uhlenbeck processes given by Benth, Kallsen and Meyer-Brandis [1, p. 5] describes what roles different Ornstein–Uhlenbeck processes can play in a model of the spot price. It reads as follows: "The first Ornstein–Uhlenbeck process is for normal variations, with a slow mean reversion. The second process is for more rare variations, with a stronger mean reversion due to occurrence of market-news altering the supply or demand, and therefore non-seasonal. The third process is for seasonal spikes and has strong mean reversion."

3.3. The Stochastic Control Problem

In this section we provide the state system $X^u(t)$, a performance criterion $V^u(s,x)$ and the corresponding value function $V(s,x)$, leading to the stochastic control problem.

The process $X^u(t) = (H^u(t), S(t))$ describing our system consists of the water level $H^u(t)$ and the spot price $S(t)$ given as before:

$$H^u(t) = x_1 + \int_s^t (z(p) - u(p))\mathbb{1}_{\{H^u(p)<\bar{H} \ \vee \ z(p)\leq u(p)\}} \ dp,$$

$$\Rightarrow dH^u(t) = (z(t) - u(t))\mathbb{1}_{\{H^u(t)<\bar{H} \ \vee \ z(t)\leq u(t)\}} \ dt.$$

$$dS(t) = d\Lambda(t) + dY(t).$$

For simplicity, lets assume our model for the spot price uses only one Ornstein–Uhlenbeck process, giving it the form:

$$dY(t) = -\lambda Y(t) \ dt + \sigma \ dW(t),$$

and because $Y(t) = S(t) - \Lambda(t)$:

$$\Rightarrow dS(t) = \lambda\Big(\frac{1}{\lambda}\frac{d}{dt}\Lambda(t) + \Lambda(t) - S(t)\Big) \ dt + \sigma \ dW(t)$$

.

Altogether, our *state system* has the form $X^u(t) = (H^u(t), S(t))$ and therefore:

$$dX^u(t) = \begin{pmatrix} (z(t) - u(t))\mathbb{1}_{\{H^u(t)<\bar{H} \ \vee \ z(t)\leq u(t)\}} \\ \lambda(\frac{1}{\lambda}\frac{d}{dt}\Lambda(t) + \Lambda(t) - S(t)) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW(t) \qquad (3.2)$$

with a given starting point $X^u(s) = x = (x_1, x_2) \in [0, \bar{H}] \times \mathbb{R}$ and $t \geq s$.

The *interest rate* $r > 0$ is assumed to be constant. Now we will present a *performance criterion* $V^u(s, x)$, which allows us to measure the performance of the hydro power plant:

$$V^u(s, x) := \mathbb{E}\Big[\int_s^\infty e^{-rt} \varphi(u(t)) S(t) \; dt \;\Big|\; X^u(s) = x \Big] \tag{3.3}$$

for $s \geq 0$, $x \in [0, \bar{H}] \times \mathbb{R}$ and with φ modelling the performance and efficiency of the turbines, generators and pumps, see below. This criterion measures the performance starting at time s with the starting point x, meaning the water level is at x_1 and the spot price at x_2 at time s. We simply integrate over the product of water moved between the upper and lower reservoir, and the actual spot price.

We choose not to use a finite time horizon, because otherwise the optimal control would be to release as much water as possible to generate electricity as we get close to the end point, with a positive electricity price being the only condition.

The function

$$\varphi(u) = \big(\zeta \; \mathbb{1}_{\{u \geq 0\}} + \xi \; \mathbb{1}_{\{u < 0\}}\big) \, u \tag{3.4}$$

with $\zeta > 0$ and $\xi > 0$, represents the *performance and efficiency of the turbines, generators and pumps*. The amount of electricity generated with water volume flow rate u is given by ζu, and the amount of electricity needed to pump with water volume flow rate u for storage is given by ξu. The variable ζ is lower than ξ, because of the round-trip efficiency mentioned in Section 2.1.1.

The last information we need is that we say that the control process u is admissible and write $u \in \mathcal{A}$ if (3.2) has a unique strong solution in $[0, \bar{H}] \times \mathbb{R}$, u satisfies the condition (3.1) and the performance criterion (3.3) is well defined (see Øksendal and Sulem [8, p. 69]).

Now, the *stochastic control problem* is to find the *value function* $V(s, x)$ and an *optimal control* $u^* \in \mathcal{A}$, such that

$$V(s, x) := \sup_{u \in \mathcal{A}} V^u(s, x) = V^{u^*}(s, x).$$

(see Øksendal and Sulem [8, p. 70])

3.4. The Time-Invariant Stochastic Control Problem

To simplify the presentation of this discussion, we consider from now on a time-invariant state system and therefore value functions not depending on the time in order to skip the partial derivative with respect to the time. This case will be the subject-matter for the rest of the thesis. The *time-invariant state system* $X^u(t) = (H^u(t), S(t))$ has the form:

$$dX^u(t) = \begin{pmatrix} (\beta - u(t))\mathbb{1}_{\{H^u(t)<\bar{H} \ \vee \ \beta \leq u(t)\}} \\ \lambda(\mu - S(t)) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW(t) \tag{3.5}$$

with $X^u(0) = x \in [0, \bar{H}] \times \mathbb{R}$, $u \in \mathcal{A}$ and $t \geq 0$. The natural stream flow rate $z(t)$ is now modelled to be constant β and the seasonal behaviour $\Lambda(t)$ is modelled to be constant μ.

The *time-invariant performance criterion* $V^u(x)$ has the following form:

$$V^u(x) := V^u(0, x) = \mathbb{E}\Big[\int_0^\infty e^{-rt}\varphi(u(t))S(t)\ dt \ \Big|\ X^u(0) = x\Big]. \tag{3.6}$$

The *time-invariant stochastic control problem* is to find the *value function* $V(x)$ and an *optimal control* $u^* \in \mathcal{A}$ for our time-invariant state system (3.5) and time-invariant performance criterion (3.6), such that

$$V(x) := \sup_{u\in\mathcal{A}} V^u(x) = V^{u^*}(x). \tag{3.7}$$

3.5. The Dynamic Programming Principle

We consider the time-invariant stochastic control problem, see (3.7). In order to solve the problem, we will formulate the HJB equation in the next section. To reach that equation, we will first present the dynamic programming principle.

Proposition 3.5.1 (Dynamic Programming Principle)**.** *We have*

$$V(x) = \sup_{u\in\mathcal{A}} \mathbb{E}\Big[\int_0^\epsilon e^{-rt}\varphi(u(t))S(t)\ dt + e^{-r\epsilon}V(X^u(\epsilon))\Big]$$

for any $x \in [0, \bar{H}] \times \mathbb{R}$ *and* $\epsilon > 0$.
(see Schmidli [9, p. 31])

Proof. Let $\kappa > 0$. For any $x \in [0, \bar{H}] \times \mathbb{R}$ choose a control $\tilde{v}_x$ such that

$$V(x) < V^{\tilde{v}_x}(x) + \kappa.$$

This can be done in a measurable way, see the *Measurable Selection Theorem* [12]. Take an arbitrary control v and define

$$v_\epsilon(t) := \begin{cases} v(t) & \text{if } t \in [0, \epsilon], \\ \tilde{v}_{X^v(\epsilon)}(t-\epsilon) & \text{if } t > \epsilon. \end{cases}$$

Then, conditioning on $\mathcal{F}_\epsilon$ and using the Markov property, gives us the following:

$$\begin{aligned}
V(x) &\geq V^{v_\epsilon}(x) \\
&= \mathbb{E}\Big[\int_0^\epsilon e^{-rt}\varphi(v(t))S(t)\ dt + \int_\epsilon^\infty e^{-rt}\varphi(\tilde{v}_{X^v(\epsilon)}(t-\epsilon))S(t)\ dt\Big] \\
&= \mathbb{E}\Big[\int_0^\epsilon e^{-rt}\varphi(v(t))S(t)\ dt + e^{-r\epsilon}\mathbb{E}\Big[\int_\epsilon^\infty e^{-r(t-\epsilon)}\varphi(\tilde{v}_{X^v(\epsilon)}(t-\epsilon))S(t)\ dt\ \Big|\ \mathcal{F}_\epsilon\Big]\Big] \\
&= \mathbb{E}\Big[\int_0^\epsilon e^{-rt}\varphi(v(t))S(t)\ dt + \\
&\quad + e^{-r\epsilon}\mathbb{E}\Big[\int_\epsilon^\infty e^{-r(t-\epsilon)}\varphi(\tilde{v}_{X^v(\epsilon)}(t-\epsilon))S(t)\ dt\ \Big|\ X^{v_\epsilon}(\epsilon) = X^{v_\epsilon}(\epsilon)\Big]\Big] \\
&= \mathbb{E}\Big[\int_0^\epsilon e^{-rt}\varphi(v(t))S(t)\ dt + \\
&\quad + e^{-r\epsilon}\mathbb{E}\Big[\int_0^\infty e^{-rt}\varphi(\tilde{v}_{X^v(\epsilon)}(t))S(t)\ dt\ \Big|\ X^{v_\epsilon}(0) = x\Big]\ \Big|_{x=X^{v_\epsilon}(\epsilon)}\Big] \\
&= \mathbb{E}\Big[\int_0^\epsilon e^{-rt}\varphi(v(t))S(t)\ dt + e^{-r\epsilon}V^{\tilde{v}_{X^v(\epsilon)}}(X^{v_\epsilon}(\epsilon))\Big] \\
&> \mathbb{E}\Big[\int_0^\epsilon e^{-rt}\varphi(v(t))S(t)\ dt + e^{-r\epsilon}V(X^v(\epsilon))\Big] - \kappa.
\end{aligned}$$

Because the right-hand side does not directly depend on $\tilde{v}$, the (weak) inequality must also hold fur $\kappa = 0$.

$$V(x) \geq \sup_{v \in \mathcal{A}} \mathbb{E}\Big[\int_0^\epsilon e^{-rt}\varphi(v(t))S(t)\ dt + e^{-r\epsilon}V(X^v(\epsilon))\Big].$$

Starting with an arbitrary control yields

$$
\begin{aligned}
V^u(x) &= \mathbb{E}\Big[\int_0^\epsilon e^{-rt}\varphi(u(t))S(t)\ dt + e^{-r\epsilon}V^u(X^u(\epsilon))\Big] \\
&\le \mathbb{E}\Big[\int_0^\epsilon e^{-rt}\varphi(u(t))S(t)\ dt + e^{-r\epsilon}V(X^u(\epsilon))\Big] \\
&\le \sup_{\bar{u}\in\mathcal{A}} \mathbb{E}\Big[\int_0^\epsilon e^{-rt}\varphi(\bar{u}(t))S(t)\ dt + e^{-r\epsilon}V(X^{\bar{u}}(\epsilon))\Big].
\end{aligned}
$$

Taking the supremum over u yields

$$
V(x) \le \sup_{u\in\mathcal{A}} \mathbb{E}\Big[\int_0^\epsilon e^{-rt}\varphi(u(t))S(t)\ dt + e^{-r\epsilon}V(X^u(\epsilon))\Big].
$$

Both inequalities together give us the dynamic programming principle, also called *Bellman's "Principle of Optimality"*.
(see Schmidli [9, pp. 30–31]) □

This principle says that if the behaviour from zero to ϵ is optimal, and it is also after ϵ optimal, then it is overall optimal.

3.6. The Hamilton–Jacobi–Bellman (HJB) Equation

We consider the time-invariant stochastic control problem, see (3.7). We derive the HJB equation from the dynamic programming principle using Itô's formula and show that the solution of the HJB equation is the value function corresponding to the problem. Furthermore, we will give proof that the optimal control exists.

From now on we will use the notation $V_{x_1}(x) = \frac{\partial}{\partial x_1}V(x)$, $V_{x_2}(x) = \frac{\partial}{\partial x_2}V(x)$ and $V_{x_2x_2}(x) = \frac{\partial^2}{\partial x_2 x_2}V(x)$. We also use the same notation for a generic function $F : [0,\bar{H}] \times \mathbb{R} \to \mathbb{R}$ if it is smooth enough.

The following lemma shows us that a given function f attains a maximum in u, and that at the maximum, as u depends on x, f is as a function of x measurable. This function will later turn out to be part of the HJB equation. The second Lemma shows that the partial differential $V_{x_1}(x)$ is non-negative.

Lemma 3.6.1. *Assume that F is in $C^{1,2}([0,\bar{H}] \times \mathbb{R}, \mathbb{R})$ and $F_{x_1} \ge 0$. Then there is a measurable function $u : [0,\bar{H}] \times \mathbb{R} \to [-\alpha_1, \alpha_2]$ with $u \le \beta$ for $x_1 = 0$ such that $f : [-\alpha_1, \alpha_2] \to \mathbb{R}$,*

$$f(u) := \varphi(u)x_2 - rF(x) + (\beta - u)F_{x_1}(x)\mathbb{1}_{\{x_1<\bar{H} \ \vee \ \beta \le u\}} + \lambda(\mu - x_2)F_{x_2}(x) + \frac{1}{2}\sigma^2 F_{x_2x_2}(x),$$

attains a maximum in

$$u(x) = \begin{cases} -\alpha_1 & \text{for } x \in [0,\bar{H}] \times (-\infty, 0] \\ & \text{ or } x \in \{(y_1,y_2) \in [0,\bar{H}) \times (0,\infty) : \xi y_2 < F_{x_1}(y_1,y_2)\}, \\ 0 & \text{for } x \in \{(y_1,y_2) \in [0,\bar{H}) \times (0,\infty) : \xi y_2 \ge F_{x_1}(y_1,y_2) \wedge \zeta y_2 \le F_{x_1}(y_1,y_2)\}, \\ \alpha_2 & \text{for } x \in \{(y_1,y_2) \in (0,\bar{H}] \times (0,\infty) : \zeta y_2 > F_{x_1}(y_1,y_2)\}, \\ \beta & \text{otherwise.} \end{cases}$$

Proof. The terms of interest are those which depend on u:

$$\varphi(u)x_2 + (\beta - u)F_{x_1}(x_1,x_2)\mathbb{1}_{\{x_1<\bar{H} \ \vee \ \beta\le u\}} \quad \longrightarrow \max_{u\in[-\alpha_1,\alpha_2]}. \tag{3.8}$$

Recall that $\varphi(u)$ is equal to $\left(\zeta\ \mathbb{1}_{\{u>0\}} + \xi\ \mathbb{1}_{\{u<0\}}\right) u$, see (3.3).

Case 1: Let us start with $x_1 \in [0,\bar{H}]$ and $x_2 \in (-\infty, 0]$. It follows easily that we maximise (3.8) with $u = -\alpha_1$.

- $x \in [0,\bar{H}] \times (-\infty, 0], \qquad \Rightarrow \ u = -\alpha_1.$

For $x_1 = \bar{H}$ and $x_2 = 0$ (3.8) is maximised for $u \in [-\alpha_1, \beta]$. For convenience we have chosen $u = -\alpha_1$.

Case 2: Now we set $x_1 \in (0,\bar{H})$ and $x_2 \in (0,\infty)$. We can simplify (3.8) as follows:

$$u\left(\left(\zeta\ \mathbb{1}_{\{u>0\}} + \xi\ \mathbb{1}_{\{u<0\}}\right) x_2 - F_{x_1}(x_1,x_2)\right) \quad \longrightarrow \max_{u\in[-\alpha_1,\alpha_2]}. \tag{3.9}$$

In order to maximise (3.9), we can choose one of three values. If $\xi x_2 < F_{x_1}(x_1,x_2)$, then $u = -\alpha_1$. For $\zeta x_2 > F_{x_1}(x_1,x_2)$, $u = \alpha_2$. Otherwise, $u = 0$. For convenience we also choose $u = 0$ if $\xi x_2 = F_{x_1}(x_1,x_2)$ or $\zeta x_2 = F_{x_1}(x_1,x_2)$.

Altogether, this results in the following values for u:

- $x \in \{(y_1,y_2) \in (0,\bar{H}) \times (0,\infty) : \xi y_2 < F_{x_1}(y_1,y_2)\}, \qquad \Rightarrow \ u = -\alpha_1.$

- $x \in \{(y_1, y_2) \in (0, \bar{H}) \times (0, \infty) : \xi y_2 \geq F_{x_1}(y_1, y_2) \wedge \zeta y_2 \leq F_{x_1}(y_1, y_2)\}, \quad \Rightarrow \; u = 0.$

- $x \in \{(y_1, y_2) \in (0, \bar{H}) \times (0, \infty) : \zeta y_2 > F_{x_1}(y_1, y_2)\}, \quad \Rightarrow \; u = \alpha_2.$

Case 3: We set $x_1 = 0$ and $x_2 \in (0, \infty)$. The only difference to the previous case is the condition $u \leq \beta$:

- $x \in \{(y_1, y_2) \in [0, 0] \times (0, \infty) : \xi y_2 < F_{x_1}(y_1, y_2)\}, \quad \Rightarrow \; u = -\alpha_1.$

- $x \in \{(y_1, y_2) \in [0, 0] \times (0, \infty) : \xi y_2 \geq F_{x_1}(y_1, y_2) \wedge \zeta y_2 \leq F_{x_1}(y_1, y_2)\}, \quad \Rightarrow \; u = 0.$

- $x \in \{(y_1, y_2) \in [0, 0] \times (0, \infty) : \zeta y_2 > F_{x_1}(y_1, y_2)\}, \quad \Rightarrow \; u = \beta.$

Case 4: We set $x_1 = \bar{H}$ and $x_2 \in (0, \infty)$. This changes (3.8) to:

$$\varphi(u)x_2 + (\beta - u)F_{x_1}(x_1, x_2)\mathbb{1}_{\{\beta \leq u\}} \quad \longrightarrow \max_{u \in [-\alpha_1, \alpha_2]}.$$

For $u \leq \beta$ this becomes $\max_{u \in [-\alpha_1, \beta]}\{\varphi(u)x_2\}$. Because x_2 is positive, u maximises this expression by being equal to β. That means that u is at least β, leaving us with

$$u(\zeta x_2 - F_{x_1}(x_1, x_2)) \quad \longrightarrow \max_{u \in [\beta, \alpha_2]},$$

which leads to the following values for u:

- $x \in \{(y_1, y_2) \in [\bar{H}, \bar{H}] \times (0, \infty) : \zeta y_2 \leq F_{x_1}(y_1, y_2)\}, \quad \Rightarrow \; u = \beta.$

- $x \in \{(y_1, y_2) \in [\bar{H}, \bar{H}] \times (0, \infty) : \zeta y_2 > F_{x_1}(y_1, y_2)\}, \quad \Rightarrow \; u = \alpha_2.$

For convenience we have chosen $u = \beta$ for $\zeta y_2 = F_{x_1}(y_1, y_2)$. □

Lemma 3.6.2. *Assume $V \in C^{1,0}([0, \bar{H}] \times \mathbb{R}, \mathbb{R})$. Then $V_{x_1} \geq 0$.*

Proof. We compare the value function at different starting water levels $x_1, \tilde{x}_1$ and identical starting electricity price x_2. Since we may use the same strategy at higher water levels, we get $V^u(x_1, x_2) \leq V^u(\tilde{x}_1, x_2)$ for $x_1 \leq \tilde{x}_1$ and thus

$$V(x_1,x_2) = \sup_{u\in\mathcal{A}_{x_1}} V^u(x_1,x_2) \le \sup_{u\in\mathcal{A}_{x_1}} V^u(\tilde{x}_1,x_2) \le \sup_{u\in\mathcal{A}_{\tilde{x}_1}} V^u(\tilde{x}_1,x_2) = V(\tilde{x}_1,x_2).$$

□

Now we derive the HJB equation. Recall

$$dX^u(t) = \begin{pmatrix} (\beta - u(t))\mathbb{1}_{\{H^u(t)<\bar{H} \ \vee \ \beta \le u(t)\}} \\ \lambda(\mu - S(t)) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW(t)$$

with start point $X^u(0) = x \in [0,\bar{H}] \times \mathbb{R}$, $u \in \mathcal{A}$ and $t \ge t$, see (3.5).

Proposition 3.6.3 (HJB equation)**.** *Let $V \in C^{1,2}([0,\bar{H}) \times \mathbb{R}, \mathbb{R})$. Then V satisfies the HJB equation:*

$$0 = \sup_{u\in[-\alpha_1,\alpha_2]} \Big\{ \varphi(u)x_2 - rV(x) + (\beta - u)V_{x_1}(x)\mathbb{1}_{\{x_1<\bar{H} \ \vee \ \beta\le u\}} + \lambda(\mu - x_2)V_{x_2}(x) + \frac{1}{2}\sigma^2 V_{x_2x_2}(x) \Big\}$$

for any $x \in [0,\bar{H}] \times \mathbb{R}$.

Proof. Let $u \in \mathcal{A}$, $\epsilon > 0$ and $x \in [0,\bar{H}] \times \mathbb{R}$. We take $e^{-r\epsilon}V(X^u(\epsilon))$ and apply *Itô's formula*:

$$\begin{aligned} e^{-r\epsilon}V(X^u(\epsilon)) &= e^0 V(X^u(0)) + \int_0^\epsilon (-r)e^{-rt}V(X^u(t))\,dt + \int_0^\epsilon e^{-rt}V_{x_1}(X^u(t))\,dX_1^u(t) + \\ &\quad + \int_0^\epsilon e^{-rt}V_{x_2}(X^u(t))\,dX_2^u(t) + \frac{1}{2}\int_0^\epsilon e^{-rt}V_{x_2x_2}(X^u(t))\,d\langle X_2^u, X_2^u\rangle_t \\ &= V(x) + \int_0^\epsilon e^{-rt}\Big[-rV(X^u(t)) + (\beta - u(t))V_{x_1}(X^u(t))\mathbb{1}_{\{H^u(t)<\bar{H} \ \vee \ \beta\le u(t)\}} + \\ &\quad + \lambda(\mu - S(t))V_{x_2}(X^u(t)) + \frac{1}{2}\sigma^2 V_{x_2x_2}(X^u(t))\Big]\,dt + \\ &\quad + \int_0^\epsilon e^{-rt}\sigma V_{x_2}(X^u(t))\,dW(t). \end{aligned}$$

Replacing $e^{-r\epsilon}V(X^u(\epsilon))$ in the dynamic programming principle (cf. Proposition 3.5.1) with the above yields

$$
\begin{aligned}
\cancel{V(x)} = \sup_{u \in \mathcal{A}} \mathbb{E}\Big[\int_0^{\epsilon} & e^{-rt} \varphi(u(t)) S(t) \; dt + \cancel{V(x)} + \int_0^{\epsilon} e^{-rt} \Big[- rV(X^u(t)) + \\
& + (\beta - u(t)) V_{x_1}(X^u(t)) \mathbb{1}_{\{H^u(t) < \bar{H} \;\vee\; \beta \leq u(t)\}} + \lambda(\mu - S(t)) V_{x_2}(X^u(t)) + \\
& + \frac{1}{2} \sigma^2 V_{x_2 x_2}(X^u(t)) \Big] \; dt \Big],
\end{aligned}
$$

with $\mathbb{E}\big[\int_0^{\epsilon} e^{-rt} \sigma V_{x_2}(X^u(t)) \; dW(t)\big]$ disappearing because the integral with the Brownian motion as integrator is equal to zero under the expectation.

We know that V_{x_1} is non negative and $u \leq \beta$ for $x_1 = 0$, see Lemma 3.6.2 and condition (3.1). This allows us to choose an u such as in Lemma 3.6.1 and set $u^*(t) := u(X^u(t))$. This leads to

$$
\begin{aligned}
0 = \mathbb{E}\Big[\int_0^{\epsilon} & e^{-rt} \Big[\varphi(u^*(t)) S(t) - rV(X^{u^*}(t)) + \\
& + (\beta - u^*(t)) V_{x_1}(X^{u^*}(t)) \mathbb{1}_{\{H^{u^*}(t) < \bar{H} \;\vee\; \beta \leq u^*(t)\}} + \lambda(\mu - S(t)) V_{x_2}(X^{u^*}(t)) + \\
& + \frac{1}{2} \sigma^2 V_{x_2 x_2}(X^{u^*}(t)) \Big] \; dt \Big],
\end{aligned}
$$

because the maximum is assumed.

With Fubini's theorem we interchange the expectation and the integral. In the next step we want to divide by ϵ and send ϵ to zero, and therefore take a closer look at the jump process u^* and the indicator function at time zero:

$$
\varphi(u^*(0)) S(0) + (\beta - u^*(0)) V_{x_1}(X^{u^*}(0)) \mathbb{1}_{\{H^{u^*}(0) < \bar{H} \;\vee\; \beta \leq u^*(0)\}}. \tag{3.10}
$$

The process $u^*(0)$ jumps when $\varphi(u^*(0))S(0)$ is equal to $u^*(0)V_{x_1}(X^{u^*}(0))$, or may jump from $-\alpha_1$ to β when $(S(0), H^{u*}(0))$ is equal to $(0, \bar{H})$. In the first case the jumps of u^* have no influence at the time they occur, because every term with u^* is negated. In the second case the jump of u^* also has no influence, because (3.10) is equal to zero for both $-\alpha_1$ and β.

Now we examine the indicator function, because its value does not only depend on u^*. There is only one case where the indicator function may change values at time zero without u^* jumping. It happens when $H^{u^*}(0) = \bar{H}$ and $H^{u^*}(\delta) < \bar{H}$ for any $\delta > 0$. It follows that $\beta < u^*(0)$ must hold, because more water needs to be released than comes in at time zero. This yields

$$
\lim_{\delta \to 0} \mathbb{1}_{\{H^{u^*}(\delta) < \bar{H} \;\vee\; \beta \leq u^*(\delta)\}} = 1 = \mathbb{1}_{\{H^{u^*}(0) < \bar{H} \;\vee\; \beta \leq u^*(0)\}}.
$$

Finally, we can divide by ϵ and send ϵ to zero. This results in

$$\begin{aligned}0 &= \mathbb{E}\Big[\varphi(u^*(0))S(0) - rV(X^{u^*}(0)) + (\beta - u^*(0))V_{x_1}(X^{u^*}(0))\mathbb{1}_{\{H^{u^*}(0)<\bar{H} \ \vee \ \beta\leq u^*(0)\}} + \\ &\quad + \lambda(\mu - S(0))V_{x_2}(X^{u^*}(0)) + \frac{1}{2}\sigma^2 V_{x_2x_2}(X^{u^*}(0))\Big] \\ &= \varphi(u^*(0))x_2 - rV(x) + (\beta - u^*(0))V_{x_1}(x)\mathbb{1}_{\{x_1<\bar{H} \ \vee \ \beta\leq u^*(0)\}} + \lambda(\mu - x_2)V_{x_2}(x) + \\ &\quad + \frac{1}{2}\sigma^2 V_{x_2x_2}(x).\end{aligned}$$

Choosing the supremum over u instead of u^* again leads to the HJB equation:

$$\begin{aligned}0 &= \sup_{u\in\mathcal{A}}\Big\{\varphi(u(0))x_2 - rV(x) + (\beta - u(0))V_{x_1}(x)\mathbb{1}_{\{x_1<\bar{H} \ \vee \ \beta\leq u(0)\}} + \lambda(\mu - x_2)V_{x_2}(x) + \\ &\quad + \frac{1}{2}\sigma^2 V_{x_2x_2}(x)\Big\} \\ &= \sup_{\tilde{u}\in[-\alpha_1,\alpha_2]}\Big\{\varphi(\tilde{u})x_2 - rV(x) + (\beta - \tilde{u})V_{x_1}(x)\mathbb{1}_{\{x_1<\bar{H} \ \vee \ \beta\leq\tilde{u}\}} + \lambda(\mu - x_2)V_{x_2}(x) + \frac{1}{2}\sigma^2 V_{x_2x_2}(x)\Big\}\end{aligned}$$

for any $x \in [0, \bar{H}] \times \mathbb{R}$. □

We showed that if the value function is smooth enough, it satisfies the HJB equation. In general, it does not follow that a solution of the HJB equation is the value function. The subsequent Verification Theorem shows, however, that in our case the solution to the HJB equation is indeed the value function, and also proves that the optimal control exists.

Theorem 3.6.4 (Verification Theorem)**.** *Let $F \in C^{1,2}([0, \bar{H}] \times \mathbb{R}, \mathbb{R})$ and $F_{x_1} \geq 0$. If F satisfies the HJB equation (cf. Proposition 3.6.3), then F is the value function V. In this case $u^*(t) := u(X^u(t))$, for a function u such as in Lemma 3.6.1, is an optimal control.*

Proof. Let $u \in \mathcal{A}$ and $x \in [0, \bar{H}] \times \mathbb{R}$. We take $e^{-r\epsilon}F(X^u(\epsilon))$ and apply Itô's formula:

$$
\begin{aligned}
e^{-r\epsilon}F(X^u(\epsilon)) &= e^0 F(X^u(0)) + \int_0^\epsilon (-r)e^{-rt}F(X^u(t))\ dt + \int_0^\epsilon e^{-rt}F_{x_1}(X^u(t))\ dX_1^u(t) + \\
&\quad + \int_0^\epsilon e^{-rt}F_{x_2}(X^u(t))\ dX_2^u(t) + \frac{1}{2}\int_0^\epsilon e^{-rt}F_{x_2x_2}(X^u(t))\ d\langle X_2^u, X_2^u\rangle_t \\
&= F(x) + \int_0^\epsilon e^{-rt}\Big[-rF(X^u(t)) + (\beta - u(t))F_{x_1}(X^u(t))\mathbb{1}_{\{H^u(t)<\bar{H}\ \vee\ \beta\leq u(t)\}} + \\
&\quad + \lambda(\mu - S(t))F_{x_2}(X^u(t)) + \frac{1}{2}\sigma^2 F_{x_2x_2}(X^u(t)) + \varphi(u(t))S(t)\Big]\ dt + \\
&\quad + \int_0^\epsilon e^{-rt}\sigma F_{x_2}(X^u(t))\ dW(t) - \int_0^\epsilon e^{-rt}\varphi(u(t))S(t)\ dt \\
&\leq F(x) + \int_0^\epsilon e^{-rt}\sigma F_{x_2}(X^u(t))\ dW(t) - \int_0^\epsilon e^{-rt}\varphi(u(t))S(t)\ dt.
\end{aligned}
$$

In the second step we expanded the term with $\int_0^\epsilon e^{-rt}\varphi(u(t))S(t)\ dt$. The inequality in the last step follows because F satisfies the HJB equation. Applying the expectation yields

$$
\mathbb{E}\big[e^{-r\epsilon}F(X^u(\epsilon))\big] \leq F(x) - \mathbb{E}\Big[\int_0^\epsilon e^{-rt}\varphi(u(t))S(t)\ dt\Big],
$$

with $\mathbb{E}\big[\int_0^\epsilon e^{-rt}\sigma F_{x_2}(X^u(t))\ dW(t)\big]$ disappearing because the integral with the Brownian motion as integrator is equal to zero under the expectation.

Sending ϵ to infinity and bringing the expectation from the right side of the inequality to the left results in

$$
\mathbb{E}\Big[\int_0^\infty e^{-rt}\varphi(u(t))S(t)\ dt\Big] \leq F(x).
$$

The expectation is the time-invariant performance criterion $V^u(x)$, see (3.6). This leads to the inequality

$$
V^u(x) \leq F(x),
$$

which holds for all $u \in \mathcal{A}$. Therefore, it follows that

$$
V(x) \leq F(x).
$$

The inequality came into the calculation because F satisfies the HJB equation and we use a random control $u \in \mathcal{A}$. Now we choose u such as in Lemma 3.6.1 and set $u^*(t) := u(X^u(t))$. Then

$$V(x) \le F(x) = V^{u^*}(x) \le V(x).$$

Now we prove that for u such as in Lemma 3.6.1, $u^*(t) := u(X^u(t))$ is an optimal control. We take $e^{-r\epsilon}V(X^{u^*}(\epsilon))$ and apply Itô‘s formula:

$$\begin{aligned}
e^{-r\epsilon}V(X^{u^*}(\epsilon)) = V(x) + \int_0^\epsilon e^{-rt}\Big[&- rV(X^{u^*}(t)) + \\
&+ (\beta - u^*(t))V_{x_1}(X^{u^*}(t))\mathbb{1}_{\{H^{u^*}(t)<\bar{H} \ \vee \ \beta \le u^*(t)\}} + \\
&+ \lambda(\mu - S(t))V_{x_2}(X^{u^*}(t)) + \frac{1}{2}\sigma^2 V_{x_2x_2}(X^{u^*}(t)) + \varphi(u^*(t))S(t)\Big]\, dt + \\
&+ \int_0^\epsilon e^{-rt}\sigma V_{x_2}(X^{u^*}(t))\, dW(t) - \int_0^\epsilon e^{-rt}\varphi(u^*(t))S(t)\, dt \\
= V(x) + \int_0^\epsilon & e^{-rt}\sigma V_{x_2}(X^{u^*}(t))\, dW(t) - \int_0^\epsilon e^{-rt}\varphi(u^*(t))S(t)\, dt.
\end{aligned}$$

Sending ϵ to infinity and applying the expectation yields

$$V(x) = \mathbb{E}\Big[\int_0^\infty e^{-rt}\varphi(u^*(t))S(t)\, dt\Big],$$

which proves our statement, see (3.7). □

3.7. The Optimal Control

We consider the time-invariant stochastic control problem, see (3.7), and will show under which conditions the optimal control for reservoir usage of a hydro power plant attains one of the four values $\{-\alpha_1, 0, \beta, \alpha_2\}$.

Corollary 3.7.1. *Assume $V \in C^{1,2}([0,\bar{H}] \times \mathbb{R}, \mathbb{R})$. There is a function $u : [0,\bar{H}] \times \mathbb{R} \to \{-\alpha_1, 0, \beta, \alpha_2\}$ such that*

$$
u(x) = \begin{cases} -\alpha_1 & \text{for } x \in [0,\bar{H}] \times (-\infty, 0] \\ & \text{ or } x \in \{(y_1,y_2) \in [0,\bar{H}) \times (0,\infty) : \xi y_2 < V_{x_1}(y_1,y_2)\}, \\ 0 & \text{for } x \in \{(y_1,y_2) \in [0,\bar{H}) \times (0,\infty) : \xi y_2 \geq V_{x_1}(y_1,y_2) \wedge \zeta y_2 \leq V_{x_1}(y_1,y_2)\}, \\ \alpha_2 & \text{for } x \in \{(y_1,y_2) \in (0,\bar{H}] \times (0,\infty) : \zeta y_2 > V_{x_1}(y_1,y_2)\}, \\ \beta & \text{otherwise,} \end{cases}
$$

and $u^*(t) := u(X^u(t))$ *is an optimal control.*

Proof. We know that V_{x_1} is non negative and $u \leq \beta$ for $x_1 = 0$, see Lemma 3.6.2 and condition (3.1). Thus the values u attains follow immediately from Lemma 3.6.1.

That $u^*(t) := u(X^u(t))$ is an optimal control is guaranteed by Theorem 3.6.4. □

Altogether, we found an optimal control attaining four values, which depends on the water level and the spot price. Pump water ($u = -\alpha_1$), do nothing ($u = 0$), release water equal to the natural stream flow ($u = \beta$), or release water at full capacity ($u = \alpha_2$). Such a control is illustrated in Figure 3.1. Also, see Figure 4.1 from numerical analysis.

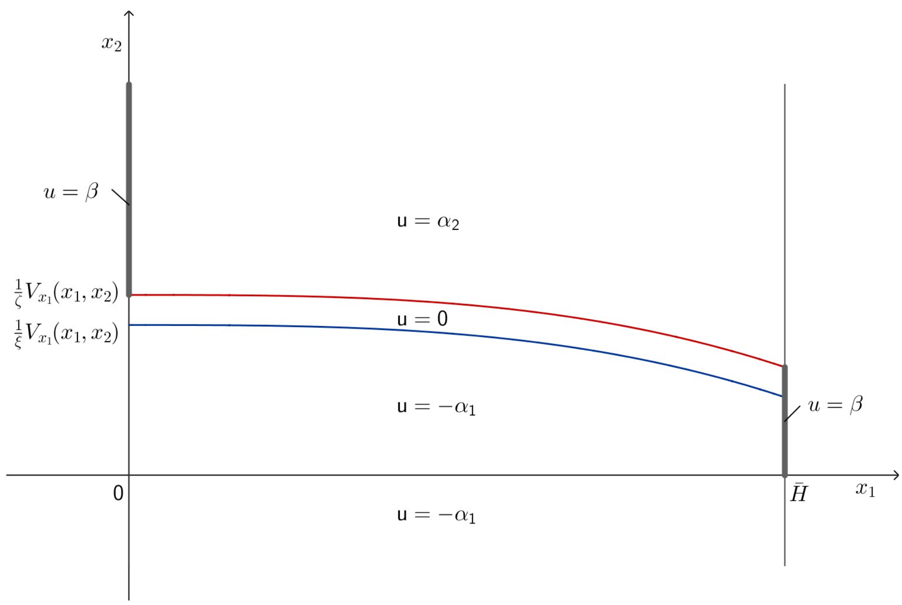

Figure 3.1.: The four values the optimal control attains for reservoir usage of a hydro power plant.

The form of the curves $x_2 = \frac{1}{\zeta} V_{x_1}(x_1, x_2)$ and $x_2 = \frac{1}{\xi} V_{x_1}(x_1, x_2)$ in Figure 3.1 are guesses, since they depend on the partial derivative of the value function. However, assuming the value function V is concave in x_1 for a fixed positive x_2, together with it being non-decreasing in x_1, we can say that the slope V_{x_1} is steep for small x_1 and levels out for bigger x_1. This leads us to assume that the curves $x_2 = \frac{1}{\zeta} V_{x_1}(x_1, x_2)$ and $x_2 = \frac{1}{\xi} V_{x_1}(x_1, x_2)$ have large values for small x_1 and lower values for bigger x_1.

This coincides with our expectation that with a low water level in the reservoir, we accept higher electricity prices in order to raise the level of the reservoir through pumping water than for water levels closer to the maximum storage capacity.

4. Numerics

In this chapter we numerically solve the time-invariant stochastic control problem, see (3.7). We start with taking a closer look at our parameters. In order to use realistic parameters, we will orient us on a real pumped hydroelectric storage plant, Obervermuntwerk II, which is currently under construction by Vorarlberger Illwerke and will begin operation in 2018.

In the second section we show how we obtain our initial values. In the third section we explain how the algorithm operates. It will be written using Matlab and the whole code can be found in the appendix. In the fourth section we present our solution, the approximation of the value function and the optimal control. In the last section we look at the convergence of our algorithm.

4.1. The Parameters

We need the following parameters. The maximal usable amount of water in the upper reservoir $\bar{H}$ and the constant natural stream flow rate β. The maximal flow rate from the lower to the upper reservoir α_1, the maximal flow rate from the upper to the lower reservoir α_2, the performance and efficiency of the generators and pumps ξ and the performance and efficiency of the generators and turbines ζ. We also need the long-run mean μ, the speed of mean-reversion λ, the degree of volatility σ and the yearly discount rate r.

As our time unit we use $t = 1$ day, because the time invariance does not allow us to model the difference between night and day and therefore we use the average daily spot price.

Obervermuntwerk II will use the Silvretta-Reservoir as an upper reservoir, which has a capacity of $38\,600\,000\,m^3$ (see Vorarlberger Illwerke AG [11]). In order to find the value for a constant natural stream flow rate, we assume that the available amount of water in an average year is twice the capacity of the reservoir. This leads to an average natural stream flow rate of 2.448 cubic meters per second (m^3/s). In Section 4.4 we will also look at the impact a higher stream flow rate has. The reservoir is also used by the hydro power plant Obervermuntwerk I, and therefore we will choose to halve both values:

- $\bar{H} = 193$ measured in $100\,000\,m^3$.

- $\beta = 1.224 * 0.036 * 24 = 1.057536$ measured in $100\,000\, m^3/d$ (per day).

The pumps use 360 MW (megawatt) with a maximal water flow rate of $135\, m^3/s$ and the turbines produce 360 MW with a maximal water flow rate of $150\, m^3/s$ (see Vorarlberger Illwerke AG [10]):

- $\alpha_1 = 135 * 0.036 * 24 = 116.64$ measured in $100\,000\, m^3/d$.

- $\alpha_2 = 150 * 0.036 * 24 = 129.6$ measured in $100\,000\, m^3/d$.

It follows that the round-trip efficiency is 90%, and that it takes approximately 1.5 days to empty the reservoir and 1.6 days to fill it.

We choose the long-run mean $\mu = 40$ EUR/MWh (megawatt hour), the speed of mean-reversion $\lambda = 20$ per year and the degree of volatility $\sigma = 10$ per year based on estimates. We set the yearly discount rate r at 3%. In order to keep $\mu = 40$ we translate EUR/MWh to EUR/3600MJ (megajoul) and MW to 3600MJ/day, which leads to ξ and ζ being multiplied with 24:

- $\mu = 40$ measured in EUR/3600MJ.

- $\lambda = \frac{20}{365}$.

- $\sigma = \frac{10}{\sqrt{365}}$.

- $r_1 = (1.03)^{\frac{1}{365}} - 1$. Discrete discount rate.

- $r_2 = \frac{\log(1.03)}{365}$. Continuous discount rate.

- $\xi = \frac{360*24}{\alpha_1}$.

- $\zeta = \frac{360*24}{\alpha_2}$.

4.2. Initial Values

Before we present our algorithm, we show how to acquire initial values for a full reservoir and any spot price. To be precise, we approximate the value function $V(x_1, x_2)$ for $x_1 = \bar{H}$ and $x_2 \in \mathbb{R}$. The code is written in Matlab and can be found in the appendix.

Step 1: We consider a deterministic setting, i.e., where the degree of volatility σ of the spot price $S(t)$ is equal to zero.

The solution of our Ornstein–Uhlenbeck process defined by

$$dS(t) = \lambda\left(\mu - S(t)\right)\ dt + \sigma\ dW(t),$$

is given by

$$S(t) = S(0)e^{-\lambda t} + \mu\big(1 - e^{-\lambda t}\big) + \int_0^t \sigma e^{\lambda(s-t)}\ dW(s).$$

(see Lucia and Schwartz [6, p. 13])

In our deterministic setting the integral term disappears because $\sigma = 0$, therefore

$$S(t) = S(0)e^{-\lambda t} + \mu\big(1 - e^{-\lambda t}\big), \tag{4.1}$$

Step 2: We compute the time-invariant performance criterion for the control α_2 while water is in the reservoir, and β when it is empty. This leads to a function V_{α_2}, defined for $[0, \bar{H}] \times \mathbb{R}_+$, which is equal to the deterministic value function for a spot price above or equal to a curve g.

First we compute an auxiliary function

$$F(s, z) = \mathbb{E}\Big[\int_0^z e^{-rt} S(t)\ dt\Big],$$

with $s := S(0)$.

The solution (4.1) of $S(t)$ leads to

$$\begin{aligned} F(s, z) &= \int_0^z s e^{-(r+\lambda)t} + \mu\big(e^{-rt} - e^{-(r+\lambda)t}\big)\ dt \\ &= \frac{1}{r+\lambda}\Big(s\big(1 - e^{-(r+\lambda)z}\big) + \frac{\mu\lambda}{r}\big(1 - e^{-rz}\big) + \mu\big(e^{-(r+\lambda)z} - e^{-rz}\big)\Big). \end{aligned} \tag{4.2}$$

The time until the reservoir is empty for a water level x_1 is given by

$$\eta(x_1) = \frac{x_1}{\alpha_2 - \beta}, \tag{4.3}$$

and the expected stock price at time η follows from (4.1):

$$S(\eta(x_1)) = \mu + (s - \mu)e^{-\lambda\eta(x_1)}.$$

Recalling the time-invariant performance criterion (3.6) together with the control α_2 until η and β afterwards, leads to

$$V_{\alpha_2}(x_1, s) = \zeta\alpha_2 F(s, \eta(x_1)) + e^{-r\eta(x_1)}\zeta\beta F(S(\eta(x_1)), \infty) \tag{4.4}$$

with $(x_1, s) \in [0, \bar{H}] \times \mathbb{R}_+$.

Step 3: We maximise u in the HJB equation for V_{α_2}. The area where $u \neq \alpha_2$ is below or equal to a curve g, and the area where $u = \alpha_2$ is above g. Together with Corollary 3.7.1 it follows that

$$\zeta\ g(y_1) = \frac{\partial}{\partial x_1} V_{\alpha_2}(y_1, g(y_1)) \tag{4.5}$$

for all $y_1 \in [0, \bar{H}]$. This equation allows us to derive g.

We start by inserting (4.2) into (4.4), leading to

$$\begin{aligned} V_{\alpha_2}(x_1, s) = \frac{\zeta}{r+\lambda}\Big(& s\big(\alpha_2 + (\beta - \alpha_2)e^{-(r+\lambda)\eta(x_1)}\big) + \frac{\alpha_2\mu\lambda}{r} + \\ & + \mu(\beta - \alpha_2)\Big(\frac{\lambda}{r}e^{-r\eta(x_1)} + e^{-r\eta(x_1)} - e^{-(r+\lambda)\eta(x_1)}\Big)\Big). \end{aligned}$$

By recalling (4.3) we easily derive the derivative

$$\begin{aligned} \frac{\partial}{\partial x_1} V_{\alpha_2}(x_1, s) = \frac{\zeta}{r+\lambda}\big(& s(r+\lambda)e^{-(r+\lambda)\eta(x_1)} + \\ & + \mu\big(\lambda e^{-r\eta(x_1)} + re^{-r\eta(x_1)} - (r+\lambda)e^{-(r+\lambda)\eta(x_1)}\big)\big). \end{aligned} \tag{4.6}$$

Inserting (4.6) into (4.5) leads to the following explicit expression for g, because (4.6) is an affine transformation in the second variable:

$$g(x_1) = \frac{\mu}{1 - e^{-(r+\lambda)\frac{x_1}{\alpha_2 - \beta}}} \left(e^{-r\frac{y_1}{\alpha_2 - \beta}} - e^{-(r+\lambda)\frac{x_1}{\alpha_2 - \beta}} \right)$$

with $x_1 \in [0, \bar{H}]$.

Step 4: We define the time it takes for a non-negative starting spot price below g to reach g and call it γ. Until time γ the optimal control in the deterministic case for the maximum water level is β.

The time $\gamma(x_1, s)$ is obtained from (4.1), because $S(\gamma(x_1, s)) = g(x_1)$:

$$g(x_1) = se^{-\lambda\gamma(x_1,s)} + \mu\big(1 - e^{-\lambda\gamma(x_1,s)}\big),$$
$$\Rightarrow \gamma(x_1, s) = \log\Big(\frac{\mu - s}{\mu - g(x_1)}\Big)\frac{1}{\lambda}$$

with $(x_1, s) \in [0, \bar{H}] \times \mathbb{R}_+$.

Step 5: We compute the time-invariant performance criterion for $\bar{H}$ and the control β until time γ and add the with time γ discounted function $V_{\alpha_2}(\bar{H}, g(\bar{H}))$. This leads to a function $V_\beta(s)$, defined for $[0, g(\bar{H})]$, which is equal to the deterministic value function on $\bar{H} \times [0, g(\bar{H})]$:

$$V_\beta(s) = \zeta\beta F(s, \gamma(\bar{H}, s)) + e^{-r\gamma(\bar{H},s)} V_{\alpha_2}(\bar{H}, g(\bar{H}))$$

with $s \in [0, g(\bar{H})]$.

Step 6: Similar to Step 4 we define the time it takes for a starting spot price below zero to reach zero and call it ν. Until time ν the optimal control is $-\alpha_1$.

The time ν is again obtained from (4.1):

$$\nu(s) = \log\Big(1 - \frac{s}{\mu}\Big)\frac{1}{\lambda}.$$

Step 7: We compute the time-invariant performance criterion for $\bar{H}$ and the control $-\alpha_1$ until time ν and add the with time ν discounted function $V_\beta(\bar{H}, g(\bar{H}))$. This leads to a function $V_{\alpha_1}(s)$, defined for $(-\infty, 0)$, which is equal to the deterministic value function on $\bar{H} \times (-\infty, 0)$:

$$V_{\alpha_1}(s) = -\xi\alpha_1 F(s, \nu(s)) + e^{-r\nu(s)} V_\beta(0)$$

with $s \in (-\infty, 0)$.

Step 8: Now we compute the value function in the stochastic setting, i.e. $\sigma > 0$. We obtain the deterministic value function V_{det} from Steps 1-5. Then we convolute V_{det} with the normal distribution. This leads to the value function $V(x_1, x_2)$ defined for $x_1 = \bar{H}$ and $x_2 \in \mathbb{R}$:

$$V(\bar{H}, x_2) = \int_{-\infty}^{\infty} V_{det}(\bar{H}, x_2 - y) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}\, dy \approx \sum_{j=-8000}^{8000} V_{det}(\bar{H}, x_2 - \frac{j}{800}) \frac{\frac{1}{800}}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\frac{j}{800})^2}{2\sigma^2}}.$$

We approximate the convolution with a sum from -10 to 10 and step size $\frac{1}{800}$. We start at -10 and stop at 10 because the exponential function becomes very small.

4.3. Algorithm

In order to solve the time-invariant stochastic control problem, we need to solve the HJB equation (cf. Proposition 3.6.3):

$$0 = \sup_{u \in [-\alpha_1, \alpha_2]} \Big\{ \varphi(u) x_2 - rV(x) + (\beta - u) V_{x_1}(x) \mathbb{1}_{\{x_1 < \bar{H} \,\vee\, \beta \le u\}} + \lambda(\mu - x_2) V_{x_2}(x) + \frac{1}{2}\sigma^2 V_{x_2 x_2}(x) \Big\}$$

for any $x \in [0, \bar{H}] \times \mathbb{R}$.

Our algorithm is based on the finite difference method for non-linear PDE, which proceeds by replacing the derivatives in a differential equation with finite difference approximations based only on values of the function itself at discrete points. This results in a large but finite algebraic system of equations to be solved instead of the differential equation. The simplest finite difference approximation is the difference quotient.
(see LeVeque [5, p. 3])

The code is written in Matlab and can be found in the appendix.

Step 1: We start by discretizing a rectangular sub-area. This results in a grid for the value function $V(x)$ and the control u where each grid point correspondents with a water level x_1 and a spot price x_2. The algorithm consists of two nested for-loops. The outer loop is for the water level and moves from the maximal to the minimal level, and the inner loop is for the spot price and moves from a chosen minimal price of €-20 to a maximal price of €80 .

Step 2: The algorithm operates by approximating V with values of V from a higher water level by using Taylor's Theorem about the point $(x_1 + \epsilon_1, x_2)$:

$$V(x_1, x_2) = V(x_1 + \epsilon_1, x_2) - \epsilon_1 V_{x_1}(x_1 + \epsilon_1, x_2) + o_1(\epsilon_1) \tag{4.7}$$

with $\epsilon_1 > 0$ as the *step size for the water level.*

The "little-oh" notation means that the *error of the approximation*, also called *truncation error*, decays to zero at least as fast as ϵ_1 as $\epsilon_1 \to 0$ (see LeVeque [5, pp. 5,247]).

We know $V(x_1 + \epsilon_1, x_2)$ either from our initial values or from a previous iteration, while $V_{x_1}(x_1 + \epsilon_1, x_2)$ will be approximated in the following steps.

Step 3: The partial differential $V_{x_1}(x_1 + \epsilon_1, x_2)$ is obtained through the HJB equation:

$$\begin{aligned} V_{x_1}(x_1 + \epsilon_1, x_2) = \frac{1}{\beta - u(x_1 + \epsilon_1, x_2)} \Big(& - \varphi(u(x_1 + \epsilon_1, x_2))x_2 + rV(x_1 + \epsilon_1, x_2) - \\ & - \lambda(\mu - x_2)V_{x_2}(x_1 + \epsilon_1, x_2) - \frac{1}{2}\sigma^2 V_{x_2x_2}(x_1 + \epsilon_1, x_2)\Big) \end{aligned} \tag{4.8}$$

for $u \neq \beta$.

The optimal control is only equal to β for $x_1 \in \{0, \bar{H}\}$. Our algorithm does not need $u(0, \cdot)$ for water level zero, because we use $u(0 + \epsilon_1, \cdot)$ to approximate $V(0, \cdot)$. At maximal water level $\bar{H}$ the optimal control assumes the values $\{-\alpha_1, \beta, \alpha_2\}$, see Corollary 3.7.1. We replace β with zero, causing a small error. Fortunately, we will see that the optimal control assumes β only for a few spot prices.

Now we look at the case where the control is $-\alpha_1$ and the water level is maximal, which causes the indicator function in the HJB equation to be equal to zero. First, we approximate V_{x_1} as if the indicator is equal to one, in essence assuming that the maximal capacity of the upper reservoir is a little bigger than $\bar{H}$. Then we approximate V_{x_1} with zero and run the algorithm with both approximations. This results in no significant difference, meaning both approximations are usable.

The values unknown are $V_{x_2x_2}(x_1 + \epsilon_1, x_2)$, $V_{x_2}(x_1 + \epsilon_1, x_2)$ and $u(x_1 + \epsilon_1, x_2)$, and will be approximated in the following steps.

Step 4: The partial differentials $V_{x_2}(x_1+\epsilon_1, x_2)$ and $V_{x_2x_2}(x_1+\epsilon_1, x_2)$ are approximated with the centred difference quotients of the first and second order:

$$V_{x_2}(x_1+\epsilon_1, x_2) = \frac{V(x_1+\epsilon_1, x_2+\epsilon_2) - V(x_1+\epsilon_1, x_2-\epsilon_2)}{2\epsilon_2} + O_2(\epsilon_2^2), \tag{4.9}$$

and

$$V_{x_2x_2}(x_1+\epsilon_1, x_2) = \frac{V(x_1+\epsilon_1, x_2+\epsilon_2) - 2V(x_1+\epsilon_1, x_2) + V(x_1+\epsilon_1, x_2-\epsilon_2)}{\epsilon_2^2} + O_2(\epsilon_2^2) \tag{4.10}$$

with $\epsilon_2 > 0$ as the *step size for the spot prize.*

The "big-oh" notation means that the truncation error decays to zero at least as fast as ϵ_2^2 as $\epsilon_2^2 \to 0$. This is easily deduced by expanding the function value of $V(x)$ in a Taylor series as done by LeVeque [5, pp. 5,7,247].

For the chosen minimal and maximal spot price we cannot use the centred difference quotients, and instead use the one sided difference quotients. Both for the first and the second order approximations the truncation error decays to zero as fast as ϵ_2 as $\epsilon_2 \to 0$, which is again deduced by expanding the function value of $V(x)$ in a Taylor series.

Step 5: The non-linearity of the HJB equation leads to three choices for $u(x_1+\epsilon_1, x_2)$. Our algorithm computes V_{x_1} for control $-\alpha_1$ while the spot price is negative, and then for all three controls $-\alpha_1$, β and α_2 when the spot price changes to positive values. It chooses the control that maximises the HJB equation. While in the inner loop, meaning a fixed water level and a monotone rising spot price, we also stop considering the controls we already changed away from. We do this because the optimal control cannot change from on control to another and back again for a fixed water level and a monotone rising spot price, see Lemma 3.6.2 and Corollary 3.7.1. Lemma 3.6.2 also tells us that V_{x_1} must be non-negative.

4.4. Solution

The algorithm plots two graphics. Figure 4.1 shows the optimal control depending on the water level and spot price. For the minimal water level zero and a spot price of 40 EUR/MWh and above, the optimal control is to release water equal to the natural stream flow β. For a rising water level the optimal control changes to α_2, while the required minimal value of the spot price drops. At the maximum water level $\bar{H}$ the value is 2.49 EUR/MWh, which cannot be clearly seen in Figure 4.1 because the slope is too steep.

The area where the optimal control is waiting corresponds to the round-trip efficiency. It expanses for a lower and shrinks for a higher efficiency.

For water level zero and a spot price below 35.43 EUR/MWh the optimal control is to pump water. This value also drops for a rising water level. For water level $\bar{H}$ the value is 1.69 EUR/MWh.

For water level $\bar{H}$ and a spot price between 2.49 and 1.69 EUR/MWh the optimal control is β.

Raising the natural stream flow would lead to lower spot prices required both for changing from releasing water to waiting, and from waiting to pumping water. We get closer to a model of a Run-of-the-river Hydroelectric Power Plant the higher we assume the natural stream flow to be.

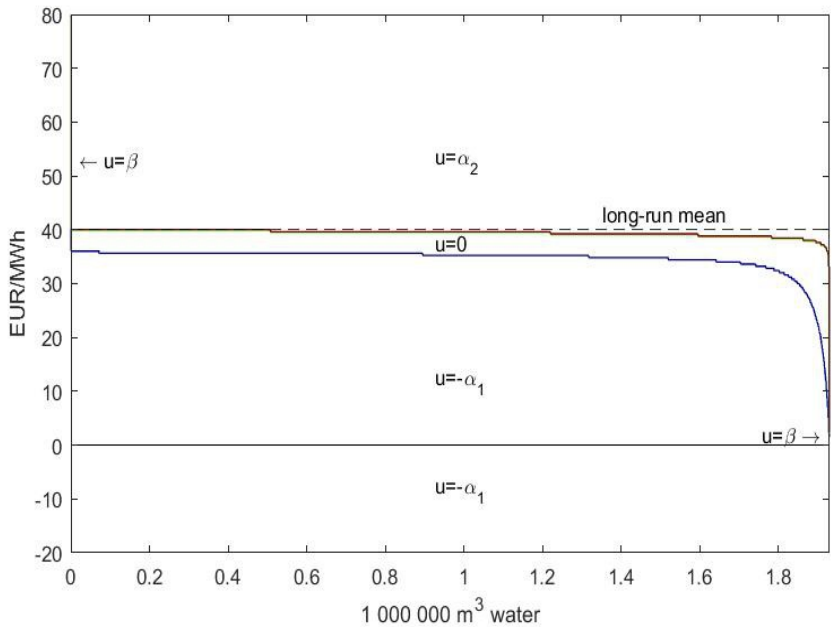

Figure 4.1.: The approximation of the optimal control for the parameters from Section 4.1.

Figure 4.2 shows the value function, and Figure 4.3 shows it rotated 90 degrees. In the following we discuss the behaviour of the value function for rising spot prices and fixed water levels:

- For spot prices above 40 EUR/MWh the value function increases. The slope is steeper for higher water levels, because we can benefit longer from the high spot prices. At minimal water level we still have a rising slope, since we can release the water from the natural stream flow.

- For spot prices between 40 and 0 EUR/MWh the value function changes from decreasing at minimal water level, to increasing at maximal water level. For low water levels we choose to pump water, therefore we are saving money when the spot price decreases. At the maximal water level we mainly release water, meaning we profit from rising prices. The area where the optimal control is waiting the value function hardly changes.

- For negative spot prices the value function decreases, because we receive money for filling our reservoir. The value function hardly changes even for different water levels, since it only takes approximately 1.6 days to fill the reservoir.

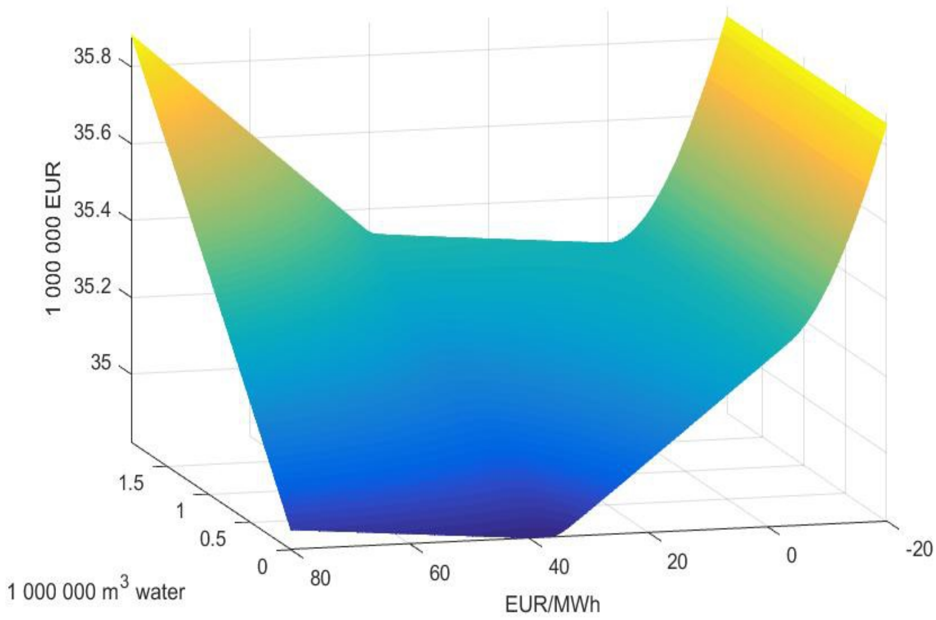

Figure 4.2.: The approximation of the value function for the parameters from Section 4.1.

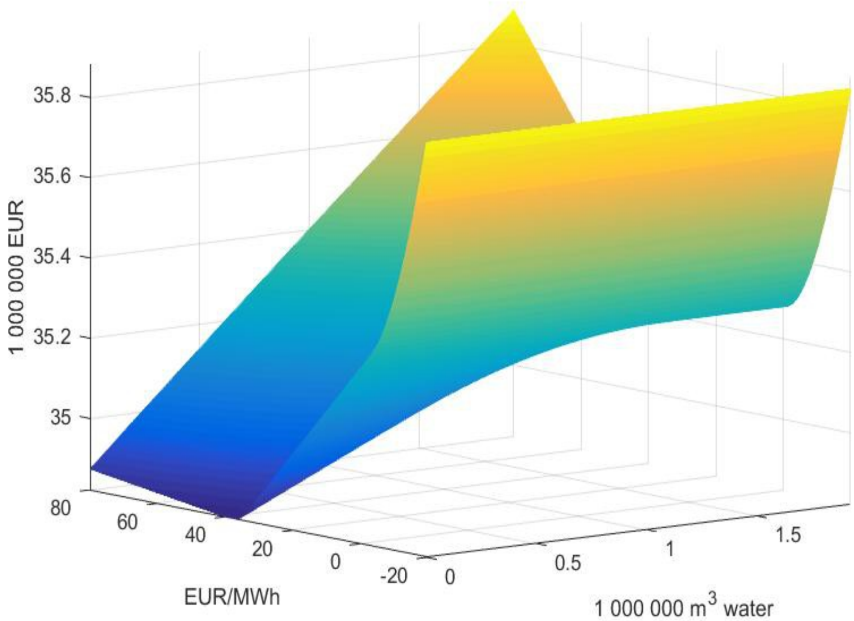

Figure 4.3.: The approximation of the value function. Figure 4.2 rotated 90 degrees.

4.5. Convergence

In order to prove convergence we wish to use the framework provided by Morton and Mayers [7, pp. 151–161]. They give a definition for consistency and stability, and prove that consistency and stability of a scheme is sufficient for convergence in the Lax Equivalence Theorem.

From now on we treat the water level x_1 as the *time variable* and the spot price x_2 as the *space variable*. Thus our problem, solving a non-linear PDE, falls under the general form of problems considered by Morton and Mayers [7, pp. 151–152], only with initial values at time $\bar{H}$ and the time variable decreasing.

We denote $x_i \in G$, $i \in \{1, 2\}$, if x_i corresponds with a grid line from our discretization done in Step 1 of our algorithm, see Section 4.3. The step size for the time variable ϵ_1 and the step size for the space variable ϵ_2 together give the *grid size*. We denote the solution of the HJB equation, the value function, as always with V and the *approximation of the solution* obtained by the algorithm described in Section 4.3 with $\hat{V}$.

In the framework of Morton and Mayers [7, p. 7] our problem is considered a *parabolic equation in one space variable*. Although their definition does not fit our HJB equation exactly, we only use this characterization of the problem to indicate why we choose one of two norms introduced in this framework over the other. Both norms are used to define stability and convergence, but Morton and Mayers [7, pp. 152–154,61] call the first norm appropriate for parabolic problems. That norm is the *maximum norm* given by

$$\|V(x_1, \cdot)\|_\infty := \max\{|V(x_1, x_2)|,\ x_2 \in G\}.$$

for $x_1 \in G$. This norm gives the maximal value of the at the grid points of the grid line x_1 evaluated function V.

To prove convergence, we first show that our algorithm is consistent and stable.

Our algorithm is *consistent* if the truncation error given by the approximation converges to zero for the step size converging to zero (see Morton and Mayers [7, p. 157]).

In essence, *stability* means that for small changes in the initial values there are only small changes in the solution of our algorithm. We denote a *different approximated solution* to the HJB equation obtained from different initial values with $\hat{W}$, and thus for stability have to prove that

$$\|\hat{V}(x_1, \cdot) - \hat{W}(x_1, \cdot)\|_\infty \leq K\ \|\hat{V}(\bar{H}, \cdot) - \hat{W}(\bar{H}, \cdot)\|_\infty$$

for a constant K and all $x_1 \in G$.
(see Morton and Mayers [7, p. 158])

Lemma 4.5.1. *The algorithm described in Section 4.3 is consistent and stable for a fixed step size ϵ_2.*

Proof. In our case the condition for consistence holds, see (4.7). Indeed consistency will usually hold for any finite difference scheme.
(see Morton and Mayers [7, p. 160])

Stability, on the other hand, is not so easily obtained.

First we consider the case where the control $\hat{u} \in \{-\alpha_1, 0, \alpha\}$ is the same for $\hat{V}$ and $\hat{W}$ at the grid point $(x_1 + \epsilon_1, x_2)$. We denote $\hat{u} := \hat{u}(x_1 + \epsilon_1, x_2)$ and

$$\delta_j := \hat{V}(x_1 + \epsilon_1, x_1 + j\epsilon_2) - \hat{W}(x_1 + \epsilon_1, x_1 + j\epsilon_2)$$

for $j \in \{-1, 0, 1\}$.

Let $x_1, x_2 \in G$. Using (4.7), (4.8), (4.9) and (4.10) allows us to compute

$$\begin{aligned}\hat{V}(x_1,x_2) - \hat{W}(x_1,x_2) &= \delta_0 - \epsilon_1 \frac{1}{\beta - \hat{u}}\Big(r\delta_0 - \lambda(\mu - x_2)\frac{\delta_1 - \delta_{-1}}{2\epsilon_2} - \frac{1}{2}\sigma^2 \frac{\delta_1 - 2\delta_0 + \delta_{-1}}{\epsilon^2}\Big) \\ &= \Big(1 - \frac{\epsilon_1 r}{\beta - \hat{u}} - \frac{\epsilon_1 \sigma^2}{(\beta - \hat{u})\epsilon_2^2}\Big)\delta_0 + \frac{\epsilon_1}{\beta - \hat{u}}\Big(\frac{\lambda(\mu - x_2)}{2\epsilon_2} + \frac{\sigma^2}{2\epsilon_2^2}\Big)\delta_1 + \\ &\quad + \frac{\epsilon_1}{\beta - \hat{u}}\Big(-\frac{\lambda(\mu - x_2)}{2\epsilon_2} + \frac{\sigma^2}{2\epsilon_2^2}\Big)\delta_{-1}.\end{aligned}$$

The coefficients in front of δ_1 and δ_{-1} can have different signs depending on ϵ_2 and x_2, but one term will always be eliminated. Now we estimate the absolute value

$$\begin{aligned}|\hat{V}(x_1,x_2) - \hat{W}(x_1,x_2)| &\leq \Big(1 - \frac{\epsilon_1 r}{\beta - \hat{u}} - \frac{\epsilon_1 \sigma^2}{(\beta - \hat{u})\epsilon_2^2}\Big)|\delta_0| + \frac{\epsilon_1}{\beta - \hat{u}}\Big(\frac{\lambda(\mu - x_2)}{2\epsilon_2} + \frac{\sigma^2}{2\epsilon_2^2}\Big)|\delta_1| + \\ &\quad + \frac{\epsilon_1}{\beta - \hat{u}}\Big(-\frac{\lambda(\mu - x_2)}{2\epsilon_2} + \frac{\sigma^2}{2\epsilon_2^2}\Big)|\delta_{-1}| \\ &\leq \Big(1 + \frac{\epsilon_1 r}{|\beta - \hat{u}|} + \frac{\epsilon_1 \sigma^2}{|\beta - \hat{u}|\epsilon_2^2} + \frac{\epsilon_1}{|\beta - \hat{u}|}c(x_2,\epsilon_2)\Big)\delta\end{aligned}$$

with $\delta := \max\{|\delta_1|, |\delta_0|, |\delta_{-1}|\}$ and $c(x_2,\epsilon_2) := \max\{\frac{\lambda|\mu - x_2|}{\epsilon_2}, \frac{\sigma^2}{\epsilon_2^2}\}$. The number of steps with length ϵ_1 needed to reach x_1 from $\bar{H}$ is given by $\frac{\bar{H} - x_1}{\epsilon_1}$. This leads to

$$\begin{aligned}&|\hat{V}(x_1,x_2) - \hat{W}(x_1,x_2)| \leq \\ &\leq \Big(1 + \frac{\epsilon_1}{|\beta - \hat{u}|}(r + \frac{\sigma^2}{\epsilon_2^2} + c(x_2,\epsilon_2))\Big)^{\frac{\bar{H} - x_1}{\epsilon_1}} |\hat{V}(\bar{H},x_2) - \hat{W}(\bar{H},x_2)|. \qquad (4.11)\end{aligned}$$

The inequality (4.11) holds for any $x_1, x_2 \in G$ and therefore

$$\|\hat{V}(x_1,\cdot) - \hat{W}(x_1,\cdot)\|_\infty \leq e^{\frac{\bar{H}}{|\beta - \hat{u}|}(r + \frac{\sigma^2}{\epsilon_2^2} + c(\epsilon_2))} \|\hat{V}(\bar{H},\cdot) - \hat{W}(\bar{H},\cdot)\|_\infty$$

as $\epsilon_1 \to 0$ and with $c(\epsilon_2) := \max\{\frac{\lambda 60}{\epsilon_2}, \frac{\sigma^2}{\epsilon_2^2}\}$, because we set $x_2 \in [-20, 80]$ in Step 1 of Section 4.3. The exponential function is our constant K.

Now we look at the case where $\hat{u}$ is not equal for $\hat{V}$ and $\hat{W}$. From Corollary 3.7.1 and Step 3 in Section 4.3 we know that when the optimal control changes from one control to another at point (y_1, y_2), then $V_{x_1}(y_1, y_2)$ has the same value for both controls. The optimal control also changes at most two times for either a fixed x_1 and any x_2 or vice versa. This leads us to assume that the non-linearity given by $\hat{u}$ has no influence on stability.

Thus our algorithm is consistent and stable for a fixed value for ϵ_2. □

Now we prove the *convergence* of our algorithm.

Theorem 4.5.2. *The algorithm described in Section 4.3 is a convergent approximation to the HJB equation (cf. Proposition 3.6.3) in the sense that*

$$\lim_{\epsilon_2 \to 0} \lim_{\epsilon_1 \to 0} \|V(x_1, x_2) - \hat{V}(x_1, x_2)\|_\infty = 0.$$

Proof. The by Morton and Mayers [7, p. 159]) introduced *Lax Equivalence Theorem* states that consistency and stability is sufficient for convergence. The theorem is formulated for linear problems, but as stated in the proof of Lemma 4.5.1 we consider the non-linearity to be no issue in our case. Thus it follows from Lemma 4.5.1 that our algorithm is a convergent approximation to a *partial integro-differential equation* for $\epsilon_1 \to 0$. The name "integro" is due to the central difference quotients, which can be written as integrals with the Dirac measure as integrator.

Now we consider the partial integro-differential equation and note that it converges to the HJB equation for $\epsilon_2 \to 0$, because the centred difference quotients when taken to the limit as the step size ϵ_2 approaches zero are equal to the derivative. Thus we have shown that our algorithm is a convergent approximation to the HJB equation, although not for step sizes $\epsilon_1, \epsilon_2 \to 0$, but for $\epsilon_2 \to 0$ after $\epsilon_1 \to 0$. □

For our error analysis we have to consider both limits. Let us assume we want the error of our algorithm approximating the HJB equation to be equal to or smaller than $z > 0$. Then we first have to look at the error given by the approximation of the HJB equation with a partial integro-differential equation. We can choose ϵ_2 so that the truncation error of (4.9) and (4.10) are for example less than or equal to $\frac{z}{2}$. Then we have to choose an ϵ_1 so that the error given by our algorithm approximating the partial integro-differential equation is less than or equal to $\frac{z}{2}$. This error can be obtained through the stability constant K and the truncation error of (4.7).

Together the error will then be less than or equal to z.

A. Matlab Code

V_alpha2.m:

```
function [ y ] = V_alpha2( barH,s,r,alpha2,beta,zeta,lambda,mu )
% Computes V(barH,s) in the deterministic case if s is large enough.

% Computes expectation of \int_0^z S_t e^{-rt} dt where s = S_0:
F= @(s,z) (s*(1-exp(-(r+lambda)*z))+mu*lambda/r*(1-exp(-r*z))
+mu*(exp(-(lambda+r)*z)-exp(-r*z))) / (r+lambda);

eta = barH/(alpha2-beta); % Time until reservoir is empty.
S_eta = mu+(s-mu)*exp(-lambda*eta); % Expected stock price at empty reservoir.
y=zeta*alpha2*F(s,eta)+exp(-r*eta)*zeta*beta*F(S_eta,Inf);
end
```

V_det.m:

```
function [ y ] = V_det( barH,s,r,alpha1,alpha2,beta,zeta,lambda,mu,xi )
% Computes V(barH,s) in the deterministic case.

% Computes expectation of \int_0^z S_t e^{-rt} dt where s = S_0:
F = @(s,z) (s*(1-exp(-(r+lambda)*z))+mu*lambda/r*(1-exp(-r*z))+
mu*(exp(-(lambda+r)*z)-exp(-r*z))) / (r+lambda);
% The optimal control is equal to alpha_2 above g and at full reservoir:
g_barH = mu* (exp(-r*barH/(alpha2-beta))-exp(-(r+lambda)*barH/(alpha2-beta)))
/ (1-exp(-(r+lambda)*barH/(alpha2-beta)));

% Time until spot price >= g_barH, with current spot price equal to zero:
gamma = log( (mu)/(mu-g_barH) )/lambda;
% V(barH,0):
V_beta_0 = zeta*beta*F(0,gamma)
+exp(-r*gamma)*V_alpha2( barH,g_barH,r,alpha2,beta,zeta,lambda,mu );

y = s;
for j=1:length(s)
    if s(j) >= g_barH
        y(j) = V_alpha2( barH,s(j),r,alpha2,beta,zeta,lambda,mu );
    elseif s(j) < 0
```

```
        % Time until spot price >= 0, with current spot price equal to s(j):
        nu = log(1-s(j)/mu)/lambda;
        y(j)=-alpha1*xi*F(s(j),nu) + exp(-r*nu)*V_beta_0; %V_alpha1
    else
        % Time until spot price >= g_barH, with current spot price equal to s(j):
        gamma = log( (mu-s(j))/(mu-g_barH) )/lambda;
        y(j)= zeta*beta*F(s(j),gamma)
        +exp(-r*gamma)*V_alpha2( barH,g_barH,r,alpha2,beta,zeta,lambda,mu ); % V_beta
    end
end
end
```

conv.m:

```
function [ y ] = conv( z,sigma,V_det_hash,s_hash_1 )
% Adds the stochasticity with sigma^2 = 100/365.

M=-10:1/800:10;
st=M(2)-M(1); % Step size.

% Convolution between V_det and a normal distribution:
y= sum( V_det_hash(1+fix((z-M-s_hash_1)*800)).*exp(-M.^2/sigma^2/2) )
*st/sqrt(2*pi*sigma^2);
end
```

ApproxVu.m:

```
% Approximation for the value function and the optimal control.
% HJB equation:
% 0=sup_{u in [-alpha_1, alpha_2]}{phi(u)*x_2-r*V(x)+(beta-u)*V_{x_1}(x)*
% *indicator_{x_1<barH v beta <= u}+lambda*(a-x_2)*V_{x_2}(x)+
% +sigma^2/2*V_{x_2x_2}(x)}
% N data points for the water level with step length sN, M data points for the spot price
% with step length sM.
% V_x_2 and V_x_2x_2 are approximated with the difference quotient and V_x_1 with the
% HJB equation.
% V is approximated by the Taylor series.

N=1250; % Number of data points for the water level.
M=250; % Number of data points for the spot price.
barH=193; % Maximal water level of the upper reservoir in 100 000 m^3.
mu=40; % The long-run mean of the spot price in EUR/MWh (megawatt hour)=
% EUR/3600MJ (megajoule).
barS=80; % Maximal starting spot price in EUR/MWh=EUR/3600MJ.
ubarS=-20; % Minimal starting spot price in EUR/MWh=EUR/3600MJ.
```

```
pumps=360*24; % Performance and efficiency of the pumps (*24: MW (megawatt) ->
% 3600MJ/day).
turbines=360*24; % Performance and efficiency of the turbines (*24: MW (megawatt) ->
3600MJ/day).
lambda=20/365; % Speed of mean-reversion (/365: t=1 year -> t=1 day).
sigma=10/sqrt(365); % The degree of volatility (/sqrt(365): t=1 year -> t=1 day).
alpha1=135*0.036*24; % Maximal flow rate from the lower to the upper reservoir (*24*0.36:
m^3/s -> 100 000 m^3/day).
alpha2=150*0.036*24; % Maximal flow rate from the upper to the lower reservoir (*24*0.36:
m^3/s -> 100 000 m^3/day).
beta=1.224*0.036*24; % Natural stream flow rate (*24*0.36: m^3/s -> 100 000 m^3/day).
r=0.03; % Yearly discount rate.
r1=(1+r)^(1/365)-1 ; % Discrete discount rate (t=1 year -> t=1 day).
r2=log(1+r)/365; % Continuous discount rate (t=1 year -> t=1 day).

V=zeros(M,N); % Allocation for the value matrix.
u=ones(M,N)*-alpha1; % Allocation for the optimal control.

xi=pumps/alpha1;
zeta=turbines/alpha2;
if xi<zeta
    disp('xi must not be smaller than zeta')
    return
else
    round_trip_efficiency = zeta/xi
end

x1=barH*(0:N-1)/(N-1); % Vector with N data points for the water level, from min to max.
x2=ubarS*(1-(0:M-1)/(M-1))+barS*(0:M-1)/(M-1); % Vector with M data points for the spot
price, from min to max.
sN=(x1(N)-x1(1))/(N-1); % Step size for the data points of the water level.
sM=(x2(M)-x2(1))/(M-1); % Step size for the data points of the spot prices.
neg=find(x2>0,1); % Point where the spot price changes from positive to negative.
mean=find(x2>mu,1); % Point where the spot price is approximately mu.
u(M,:)=alpha2; % We know that for the maximal spot price the control is alpha2.

% Computing values which will be accessed multiple times in order to speed up the algorithm:
s_hash=-200:1/800:200;
V_det_hash=V_det( barH,s_hash,r2,alpha1,alpha2,beta,zeta,lambda,mu,xi );
for j=1:length(x2)
    V(j,N)=conv(x2(j),sigma,V_det_hash,s_hash(1));
end

for h=N-1:-1:1 % Water level.
    U=[-alpha1,0,alpha2]; % Candidates for the optimal control.
```

```
    Phi=[xi,0,zeta];
    j=1;
    V_x2=(V(2,h+1)-V(1,h+1))/sM;
    V_x2x2=(V(3,h+1)-2*V(2,h+1)+V(1,h+1))/sM^2;
    V_x1=(alpha1*xi*x2(1)+r1*V(1,h+1)-lambda*(mu-x2(1))*V_x2-sigma^2/2*V_x2x2)
    / (beta+alpha1);
    V_x1=max([0,V_x1]);
    V(1,h)=V(1,h+1)-sN*V_x1;
    for s=2:neg-1 % Negative spot price.
        V_x2=(V(s+1,h+1)-V(s-1,h+1))/(2*sM);
        V_x2x2=(V(s+1,h+1)-2*V(s,h+1)+V(s-1,h+1))/sM^2;
        V_x1=(alpha1*xi*x2(s)+r1*V(s,h+1)-lambda*(mu-x2(s))*V_x2-sigma^2/2*V_x2x2)
        / (beta+alpha1);
        V_x1=max([0,V_x1]);
        V(s,h)=V(s,h+1)-sN*V_x1;
    end
    for s=neg:M-1 % Positive spot price.
        V_x2=(V(s+1,h+1)-V(s-1,h+1))/(2*sM);
        V_x2x2=(V(s+1,h+1)-2*V(s,h+1)+V(s-1,h+1))/sM^2;
        while j<=3
            V_x1_U=(-U.*Phi*x2(s)+r1*V(s,h+1)-lambda*(mu-x2(s))*V_x2-sigma^2/2
            *V_x2x2) ./ (beta-U);
            if j==3
                u(s,h+1)=alpha2;
                V_x1=V_x1_U(end);
                break
            end
            [~,I]=max(U.*Phi*x2(s)+(beta-U)*V_x1_U(1)); % Argmax.
            if I==1
                u(s,h+1)=U(1);
                V_x1=V_x1_U(1);
                break
            end
            U(1)=[]; % Excluding the control that does not maximise the HJB anymore.
            Phi(1)=[];% Excluding the control that does not maximise the HJB anymore.
            j=j+1;
        end
        V_x1=max([0,V_x1]);
        V(s,h)=V(s,h+1)-sN*V_x1;
    end
    V_x2=(V(M,h+1)-V(M-1,h+1))/sM;
    V_x2x2=(V(M,h+1)-2*V(M-1,h+1)+V(M-2,h+1))/sM^2;
    V_x1=(-alpha2*zeta*x2(M)+r1*V(M,h+1)-lambda*(mu-x2(M))*V_x2-sigma^2/2
    *V_x2x2) / (beta-alpha2);
    V_x1=max([0,V_x1]);
    V(M,h)=V(M,h+1)-sN*V_x1;
end
```

```
% The graphics for u and V:
x=(0:barH/(N-1):barH)/100;
y=(ubarS:(barS-ubarS)/(M-1):barS);
[X, Y] = meshgrid(x,y);

figure(1);
contour(X,Y,u,3,'LineWidth',0.8)
colormap(jet)
xlabel('1 000 000 m^3 water')
ylabel('EUR/MWh')
text(0.7,0.63,'long-run mean','Units','normalized')
text(0.48,0.73,'u=\alpha_2','Units','normalized')
text(0.01,0.73,'\leftarrow u=\beta','Units','normalized')
text(0.48,0.576,'u=0','Units','normalized')
text(0.48,0.32,'u=-\alpha_1','Units','normalized')
text(0.91,0.22,'u=\beta \rightarrow','Units','normalized')
text(0.48,0.12,'u=-\alpha_1','Units','normalized')

hold on
plot(get(gca,'xlim'), [40 40],'–k','LineWidth',0.6); % A line for the long-run mean.
plot(get(gca,'xlim'), [0 0],'k','LineWidth',0.6); % A line for zero.
hold off

figure(2);
mesh(X,Y,V/1000000)
axis tight
xlabel('1 000 000 m^3 water')
ylabel('EUR/MWh')
zlabel('1 000 000 EUR')
```

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