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Binary Operations and Floating Bodies in Spherical Convexity

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Univ.Prof. Dr. Franz Schuster

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Mathematik und Geoinformation
der Technischen Universität Wien

durch
Dipl.-Ing. Florian Besau
Matrikelnummer: e0625493
Fickeysstrasse 7/2/98, 1110 Wien

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Abstract

This thesis contains contributions to the theory of spherical convex bodies, i.e., closed convex sets on the unit n-sphere.

On one hand projection covariant binary operations on the set of spherical convex bodies are investigated. In Euclidean convexity Minkowski addition, a projection covariant binary operation, together with the volume gives rise to the Brunn–Minkowski theory. This theory lies at the very core of classical Euclidean convexity and provides a unifying framework for various extremal and uniqueness problems for convex bodies in $\mathbb{R}^n$. However, in spherical convexity there is no known natural analogue to Minkowski addition. Together with Franz Schuster, all projection covariant binary operations on the set of spherical convex bodies contained in a fixed open hemisphere are characterized and it is shown, that the convex hull is essentially the only non-trivial projection covariant binary operations between pairs of convex bodies contained in open hemispheres.

On the other hand a spherical analogue of the Euclidean convex floating body is introduced and investigated. Together with Elisabeth Werner, a new notion of spherical convex floating bodies is defined and the volume difference of a spherical convex body and its floating body is investigated. Remarkably, this volume difference gives rise to a new spherical area measure, the floating area. This floating area can be seen as a spherical analogue of the classical affine surface area from affine differential geometry. We start an investigation of the properties of this new spherical quantity.

Kurzfassung

Diese Abeit enthält Beiträge zur Theorie sphärisch konvexer Körper, das sind abgeschlossene konvexe Teilmengen der n-dimensionalen Sphäre.

Einerseits werden projektionskovariante binäre Operationen auf der Menge der sphärisch konvexen Körper untersucht. In der Euklidischen Konvexität führt die Kombination von Minkowski Addition – eine projektionskovariante binäre Operation – und Volumen zur Brunn–Minkowski Theorie. Diese zentrale Theorie liefert einen vereinheitlichenden Rahmen zur Lösung verschiedenster Extremal und Eindeutigkeits Probleme im $\mathbb{R}^n$. In der sphärischen Konvexgeometrie ist dagegen kein natürliches Analogon zur Minkowski Addition bekannt. Zusammen mit Franz Schuster, werden alle projektionskovarianten binären Operationen auf der Menge der sphärisch konvexen Körper die in einer fixen offenen Halbsphäre liegen charakterisiert und es wird gezeigen, dass die konvexe Hülle im Wesentlichen die einzige nicht-triviale projektionskovariante binäre Operationen ist, auf paaren von sphärisch konvexen Körpern die in offenen Halbsphäre enthalten sind.

Andererseits wird in dieser Arbeit ein sphärisches Analogon zum konvexen Schwimmkörper aus der Euklidischen Konvexität eingeführt und untersucht. Zusammen mit Elisabeth Werner, wird der neue Begriff eines sphärisch konvexen Schwimmkörpers eingeführt und die Volumendifferenz eines sphärisch konvexen Körpers und seines Schwimmkörpers untersucht. Bemerkenswerterweise führt diese Volumendifferenz zu einem gänzlich neuen sphärischen Oberflächenmaß, der Schwimmoberfläche, welches als ein Analogon zur klassischen Affinoberfläche der affinen Differentialgeometrie gesehen werden kann. Wir beginnen eine Untersuchung der Eigenschaften dieser neuen sphärischen Größe.

Contents

CHAPTER 1

Introduction

In Euclidean convex geometry the main object of interest are compact convex subsets, i.e. convex bodies, of a Euclidean vector space. Convex bodies have been studied since antiquity, for instance the Platonic solids, and most likely long before then. Convexity has ever since been a prosperous branch of mathematics and is still thriving and growing. Nowadays, we see convexity connect to many fields in mathematics producing numerous exciting and non-trivial results.

It is well known (see e.g. [**70**]) that the notion of convex bodies extends well to spaces of constant sectional curvature (space forms). This includes of course the n-sphere and hyperbolic n-space. However, the development of the theory of spherical or hyperbolic convex bodies has lagged behind the theory of Euclidean convex bodies.

In recent years the research on non-Euclidean convex geometry has picked up momentum [**4**, **8**, **21**, **24**, **29**, **30**, **64**, **66**, **76**, **89**]. In particular the integral geometry of spherical convex bodies has witnessed tremendous progress [**1**–**3**, **9**, **31**, **41**, **74**, **83**].

In this thesis we contribute to spherical convexity in two different ways. First, we consider binary operations between spherical convex bodies that are projection covariant. This is motivated of course by the classical and important notion of Minkowski addition in Euclidean convexity. The combination of Minkowski addition and volume gives rise to the classical Brunn–Minkowski theory, which is a fundamental part of classical convex geometry and convex analysis. It provides a framework that unifies various problems on convex bodies, see e.g. [**28**, **34**, **75**].

Recently, a new investigation into the fundamental characteristics of known binary operations between sets in Euclidean convexity was started by Gardner, Hug and Weil [**25**]. They were able to characterize Minkowski addition as the only projection covariant binary operation between convex bodies that also satisfies the identity property. In fact, they achieved even more by eventually characterizing all projection covariant operations between origin-symmetric convex bodies and establishing a relation to Orlicz additions. Orlicz additions are an important generalization of Minkowski addition which give rise to the Orlicz-Brunn-Minkowski theory [**26**].

Motivated by these developments and the fact that there is a natural notion of projection on the n-sphere, in a joint work with Franz E. Schuster [**10**], we consider binary operations between spherical convex bodies that are projection covariant. We prove that the convex hull is essentially the only non-trivial projection covariant operation on pairs of convex bodies that are contained in open hemispheres. Furthermore, if one restricts the domain to a fixed open hemisphere, then we show a one-to-one correspondence between binary operations that are projection covariant in the spherical and the Euclidean setting. Thus, we find a multitude of projection covariant operations between convex bodies contained in a fixed open hemisphere, which also prove to be continuous.

The other topic investigated in this thesis are spherical convex floating bodies. The notion of floating bodies goes back to paper of Dupin from 1822 [**19**], but the principles trace back to antiquity: Consider a solid body in a fluid. The body will float on the surface of the fluid when the weight of the fluid that it displaced is equal to the weight of the solid. This is generally known as principle of flotation and was already studied by Archimedes in his series of books "On floating bodies". Now if the solid is light enough and we roll it around in the fluid then there is a core part which will always stay above surface. If we picture the solid as a convex body, then this core part will be the intersection of all half-spaces that cut off a set of constant volume – the convex floating body.

Although the notion of floating body is very classical it seems all the more surprising to see it appear as seminal new and fundamental tool in affine convex geometry. Dupin's floating body already appeared in 1923 when Blaschke introduced affine surface area in (equi-)affine differential geometry [**12**], a now classical and important notion [**5**, **6**, **39**, **40**, **84**, **85**]. Extending affine surface area to general convex bodies in all dimensions proved to be much more difficult compared to other notions from differential geometry, like surface area measures and curvature measures. However, successively such extensions were established [**43**, **52**]. One of the first extensions to all convex bodies in all dimensions was achieved in 1990 by Schütt and Werner [**81**] who introduced the convex floating body and used it to extend affine surface area.

Affine surface area and its generalizations (see e.g. [**37**, **48**, **54**, **65**]) have been characterized in the setting of valuations [**35**, **49**, **50**] and applications are manifold, see e.g. the best and random approximation of convex bodies by convex polytopes [**13**, **14**, **32**–**34**, **45**, **71**, **78**, **80**, **82**], concentration of measure [**23**, **58**], and information theory [**7**, **17**, **59**, **60**, **62**, **67**, **87**, **88**]. Furthermore, the fundamental affine isoperimetric inequality [**12**, **68**, **73**] is related to many other inequalities [**51**, **56**] and implies the Blaschke–Santaló inequality [**28**]. It proved to be the key ingredient in the solution of numerous problems, see e.g. [**15**, **27**, **36**, **48**, **55**, **77**, **84**].

In the second part of this thesis, in a joint work with Elisabeth M. Werner [**11**], we introduce a new notion which seems natural in spherical convexity, the spherical convex floating body. We are able to relate this floating body to a curvature integral, which again seems natural, but has not yet appeared in the literature. We call it floating area and it bears striking similarities with affine surface area from Euclidean convexity. For example, the floating area not only arises from the spherical convex floating body in a similar way that the affine surface area arises from the convex floating body, but also the properties of both notions are similar. For instance, the floating area vanishes on spherical polytopes, is upper semicontinuous and a valuation.

This thesis is structured as follows: In the second and third chapter we recall basic and classical results from Euclidean and spherical convexity respectively. In the fourth chapter we state and prove our results on spherical binary operations that are projection covariant and in the final chapter we introduce and investigate the spherical convex floating body and the floating area.

CHAPTER 2

Background Material from Euclidean Convex Geometry

In this chapter we collect basic material about convex bodies in $\mathbb{R}^n$, $n \geq 2$. As a general reference for the facts in this chapter we recommend [**75**].

A *convex body* is a non-empty compact convex subset of $\mathbb{R}^n$ and the set of convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}(\mathbb{R}^n)$. Let $\mathcal{K}_e(\mathbb{R}^n)$ be the set of origin symmetric convex bodies and let $\mathcal{K}_{\mathrm{o}}(\mathbb{R}^n)$ denote the set of convex bodies containing the origin (not necessarily in the interior). We denote by $\mathcal{K}_0(\mathbb{R}^n)$ the set of convex bodies with non-empty interior.

The scalar product in $\mathbb{R}^n$ is denote by $\cdot$ and $\|.\|$ denotes the Euclidean norm. A convex body $K \in \mathcal{K}(\mathbb{R}^n)$ is uniquely determined by its *support function* defined by

$$h_K(x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n.$$

Support functions are 1-*homogeneous*, that is, $h_K(\lambda x) = \lambda h_K(x)$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$, and are therefore often regarded as functions on $\mathbb{S}^{n-1}$. They are also *subadditive*, that is, $h_K(x + y) \leq h_K(x) + h_K(y)$ for all $x, y \in \mathbb{R}^n$. Conversely, every 1-homogeneous and subadditive function on $\mathbb{R}^n$ is the support function of a convex body [**75**, Theorem 1.7.1]. Clearly, $K \in \mathcal{K}_e(\mathbb{R}^n)$ if and only if h_K is even.

The *Minkowski sum* of subsets X and Y of $\mathbb{R}^n$ is defined by

$$X + Y = \{x + y : x \in X, y \in Y\}.$$

If $K, L \in \mathcal{K}(\mathbb{R}^n)$, then $K + L$ can be equivalently defined (see [**75**, Theorem 1.7.5]) as the convex body such that

$$h_{K+L} = h_K + h_L.$$

The *Hausdorff distance* $\delta(X, Y)$ between compact subsets X and Y of $\mathbb{R}^n$ is defined by

$$\delta(X, Y) = \min\{\lambda \geq 0 : X \subseteq Y + \lambda\, B^n(0,1) \text{ and } Y \subseteq X + \lambda\, B^n(0,1)^n\},$$

where $B^n(x, r)$ denotes a closed ball of radius r and center $x \in \mathbb{R}^n$.

If $K, L \in \mathcal{K}(\mathbb{R}^n)$, then $\delta(K, L)$ can be alternatively defined by

$$\delta(K, L) = \|h_K - h_L\|_\infty, \tag{2.1}$$

where $\|\cdot\|_\infty$ denotes the L_∞ norm on $\mathbb{S}^{n-1}$, see [**75**, Lemma 1.8.13].

2.1. *M*-Addition

In this section we recall the definition of the L_p Minkowski addition and, more generally, the M-addition of convex bodies as well as the characterizing properties established in [**25**].

The standard orthonormal basis for $\mathbb{R}^n$ will be $\{e_1, \ldots, e_n\}$. Otherwise, we usually denote the coordinates of $x \in \mathbb{R}^n$ by $x_1, \ldots, x_n$. We call a subset of $\mathbb{R}^n$ 1-*unconditional* if it is symmetric with respect to each coordinate hyperplane.

For $p \in [1, \infty]$, the L_p *Minkowski sum* of convex bodies $K, L \in \mathcal{K}_o(\mathbb{R}^n)$ was first defined by Firey [**22**] by

$$h^p_{K+_pL} = h^p_K + h^p_L,$$

for $p < \infty$, and by

$$h_{K+_\infty L} = \max\{h_K, h_L\}.$$

Note that $K +_\infty L$ is just the usual convex hull in $\mathbb{R}^n$ of K and L.

Lutwak [**53, 54**] showed that the L_p Minkowski addition leads to a very powerful extension of the classical Brunn–Minkowski theory. Since the 1990's this L_p Brunn–Minkowski theory has provided new tools for unsolved problems and established new connections between convex geometry and other fields (see, e.g., [**16, 47, 56, 57, 59, 72, 84, 86–88**] and the references therein). An extension of the L_p Minkowski addition to arbitrary sets in $\mathbb{R}^n$ was given only recently in [**61**].

An even more general way of combining two subsets of $\mathbb{R}^n$ is the still more recent *M-addition*: If M is an arbitrary subset of $\mathbb{R}^2$, then the *M-sum* of $X, Y \subseteq \mathbb{R}^n$ is defined by

$$X \oplus_M Y = \bigcup_{(a,b)\in M} a\,X + b\,Y = \{ax + by : (a,b) \in M, x \in X, y \in Y\}. \tag{2.2}$$

Protasov [**69**] first introduced M-addition for centrally symmetric convex bodies and a 1-unconditional convex body M in $\mathbb{R}^2$. He also proved that $\oplus_M : \mathcal{K}_e(\mathbb{R}^n) \times \mathcal{K}_e(\mathbb{R}^n) \to \mathcal{K}_e(\mathbb{R}^n)$ for such M.

Gardner, Hug and Weil [**25**] rediscovered M-addition in the more general form (2.2) in their investigation of projection covariant binary operations between convex bodies in $\mathbb{R}^n$. Among several results on this seminal operation, they proved the following:

Theorem 2.1 ([**25**]). *Let $M \subseteq \mathbb{R}^2$. Then $\oplus_M : \mathcal{K}(\mathbb{R}^n) \times \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$ if and only if $M \in \mathcal{K}(\mathbb{R}^2)$ and M is contained in one of the 4 quadrants of $\mathbb{R}^2$. In this case, let $\varepsilon_i = \pm 1$, $i = 1, 2$, denote the sign of the ith coordinate of a point in the interior of this quadrant and let*

$$M^+ = \{(\varepsilon_1 a, \varepsilon_2 b) : (a,b) \in M\}$$

be the reflection of M contained in $[0,\infty)^2$. If $K, L \in \mathcal{K}(\mathbb{R}^n)$, then

$$h_{K\oplus_M L}(x) = h_{M^+}(h_{\varepsilon_1 K}(x), h_{\varepsilon_2 L}(x)), \qquad x \in \mathbb{R}^n. \tag{2.3}$$

Example 2.2. For some $1 \le p \le \infty$, let

$$M = \{(a,b) \in [0,1]^2 : a^{p'} + b^{p'} \le 1\},$$

where $1/p + 1/p' = 1$. Then $\oplus_M = +_p$ is L_p Minkowski addition on $\mathcal{K}_o(\mathbb{R}^n)$.

The following basic properties of M-addition are of particular interest for us. They are immediate consequences of either definition (2.2) or (2.3).

Proposition 2.3. *Suppose that* $M \in \mathcal{K}(\mathbb{R}^2)$ *is contained in* $[0,\infty)^2$. *Then* $\oplus_M : \mathcal{K}(\mathbb{R}^n) \times \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$ *has the following properties:*

- Continuity
  $K_i \to K$, $L_i \to L$ *implies* $K_i \oplus_M L_i \to K \oplus_M L$ *as* $i \to \infty$ *in the Hausdorff metric;*
- $\mathrm{GL}(n)$ covariance
  $(AK) \oplus_M (AL) = A(K \oplus_M L)$ *for all* $A \in \mathrm{GL}(n)$*;*
- Projection covariance
  $(K|V) \oplus_M (L|V) = (K \oplus_M L)|V$ *for every linear subspace* V *of* $\mathbb{R}^n$*.*

It is easy to show that continuity and $\mathrm{GL}(n)$ covariance imply projection covariance. That the converse statement is also true, follows from a deep result of Gardner, Hug, and Weil which states the following:

Theorem 2.4 ([**25**])**.** *An operation* $*\colon \mathcal{K}(\mathbb{R}^n)\times\mathcal{K}(\mathbb{R}^n)\to\mathcal{K}(\mathbb{R}^n)$ *is projection covariant if and only if there exists a nonempty closed convex set* $\overline{M}$ *in* $\mathbb{R}^4$ *such that, for all* K, $L \in \mathcal{K}(\mathbb{R}^n)$,

$$h_{K*L}(x) = h_{\overline{M}}(h_{-K}(x), h_K(x), h_{-L}(x), h_L(x)), \qquad x \in \mathbb{R}^n. \tag{2.4}$$

Consequently, every such operation is continuous and $\mathrm{GL}(n)$ *covariant.*

Note that it is an open problem whether the binary operation on $\mathcal{K}(\mathbb{R}^n)$ defined by (2.4) is M-addition for some (convex) subset M of $\mathbb{R}^2$. However, Gardner, Hug, and Weil [**25**] proved that an operation between *o-symmetric* convex bodies is projection covariant if and only if it is M-addition for some 1-unconditional convex body in $\mathbb{R}^2$.

2.2. The Convex Floating Body and Affine Surface Area

In this section we recall well known results about the convex floating body and its connection to affine surface area.

Definition (Convex Floating Body [**81**])**.** Let $K \in \mathcal{K}_0(\mathbb{R}^n)$. For $\delta > 0$ the *convex floating body* $K_{[\delta]}$ is defined as the intersection of all closed halfspaces $\mathbb{H}^-$ such that the hyperplanes cut off a set of volume less or equal δ, that is,

$$K_{[\delta]} = \bigcap \Big\{ \mathbb{H}^- : \mathrm{vol}_n(K \cap \mathbb{H}^+) \le \delta \Big\}.$$

Basic properties of the convex floating body are collected in the following

Proposition 2.5 ([**81**])**.** *Let $K \in \mathcal{K}_0(\mathbb{R}^n)$ and $\delta > 0$ such that $K_{[\delta]}$ exists.*

(i) Through every $x \in \mathrm{bd}\, K_{[\delta]}$ there exists at least one hyperplane $\mathbb{H}$ that cuts off of K a set of volume δ.

(ii) A hyperplane $\mathbb{H}$ that cuts off of K a set of volume δ touches $K_{[\delta]}$ in exactly one point, the barycenter of $K \cap \mathbb{H}$.

(iii) $K_{[\delta]}$ is strictly convex.

(iv) Let $\delta_0 = \max\{\delta : \mathrm{vol}_n(K_{[\delta]}) > 0\}$. Then $K_{[\delta_0]}$ is only one point and for all $0 < \delta < \delta_0$, $K_{[\delta]}$ exists and has non-empty interior.

(v) For a linear transformation $A \in \mathrm{GL}(\mathbb{R}^n)$ and a vector $y \in \mathbb{R}^n$ we have $(AK + y)_{[\delta]} = AK_{[|\det A|\,\delta]} + y$.

We can parametrize the halfspaces in the definition of the convex floating body over $\mathbb{S}^{n-1}$ in the following way:

Corollary 2.6. *If $K \in \mathcal{K}_0(\mathbb{R}^n)$, then for any $v \in \mathbb{S}^{n-1}$ there exists $s(v,\delta) \in \mathbb{R}$ such that*

$$\delta = \mathrm{vol}_n\Big(K \cap \mathbb{H}^+_{v,s(v,\delta)}\Big),$$

where $\mathbb{H}^+_{v,s(v,\delta)} = \{x \in \mathbb{R}^n : x \cdot v \ge s(v,\delta)\}$. Moreover, we have

$$K_{[\delta]} = \bigcap_{v \in \mathbb{S}^{n-1}} \mathbb{H}^-_{v,s(v,\delta)}. \tag{2.5}$$

Proof. This follows from Proposition 2.5 (i). □

The *generalized Gauss–Kronecker curvature* of a convex body $K \in \mathcal{K}(\mathbb{R}^n)$ at a boundary point x is denoted by $H^{\mathbb{R}^n}_{n-1}(K,x)$ and exists for $\mathcal{H}^{n-1}$-almost all boundary points, see e.g. [**37**, Lemma 2.3].

Definition (Affine Surface Area)**.** Let $K \in \mathcal{K}_0(\mathbb{R}^n)$. Then the *affine surface area* $\operatorname{as}(K)$ of K is defined by

$$\operatorname{as}(K) = \int_{\operatorname{bd} K} H_{n-1}^{\mathbb{R}^n}(K,x)^{\frac{1}{n+1}}\, dx. \tag{2.6}$$

Affine surface area $\operatorname{as}(K)$ is finite for all convex bodies K. This can be seen in the following way: For a convex body $K \in \mathcal{K}_0(\mathbb{R}^n)$ and a boundary point $x \in \operatorname{bd} K$ we denote by $r_K(x)$ the maximal radius of a Euclidean ball that is contained in K and touches the boundary of K in x, in other words,

$$r_K(x) = \sup\{r \geq 0 : \exists y \in K \text{ such that } B^n(y,r) \subseteq K \text{ and } x \in \operatorname{bd} B^n(y,r)\},$$

where $B^n(y,r)$ denotes the closed Euclidean ball of radius r and center y.

By Blaschke's Rolling Theorem (see e.g. [**75**, Corollary 3.2.13]) we know that $r_K > 0$ for $\mathcal{H}^{n-1}$-almost all boundary points. In fact Schütt and Werner proved in [**81**] the following:

Theorem 2.7 ([**81**])**.** *Let $K \in \mathcal{K}_0(\mathbb{R}^n)$. Then for all $\alpha \in [0,1)$*

$$\int_{\operatorname{bd} K} r_K(x)^{-\alpha}\, dx < \infty. \tag{2.7}$$

Since the Gauss–Kronecker curvature at a boundary point x of K is less or equal to the curvature of any ball contained in K that touches the boundary in x we have $H_{n-1}^{\mathbb{R}^n}(K,x) \leq r_K(x)^{-(n-1)}$ for $\mathcal{H}^{n-1}$-almost all $x \in \operatorname{bd} K$. We conclude

$$\operatorname{as}(K) \leq \int_{\operatorname{bd} K} r_K(x)^{-\frac{n-1}{n+1}}\, dx < \infty.$$

The limit of the volume difference of a convex body and its floating body converges to the affine surface area of the body in the following way:

Theorem 2.8 ([**81**])**.** *Let $K \in \mathcal{K}_0(\mathbb{R}^n)$. Then*

$$\operatorname{as}(K) = \frac{1}{c_n} \lim_{\delta \to 0^+} \frac{\operatorname{vol}_n(K) - \operatorname{vol}_n(K_{[\delta]})}{\delta^{\frac{2}{n+1}}}, \tag{2.8}$$

where $c_n = \frac{1}{2}\left(\frac{n+1}{\kappa_{n-1}}\right)^{\frac{2}{n+1}}$.

By (2.8) and the covariance of the floating body under affine transformations that preserve volume we conclude that for all $A \in \mathrm{SL}(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$,

$$\mathrm{as}(AK + y) = \mathrm{as}(K).$$

We see that, as the name suggests, affine surface area is invariant under volume preserving affine transformations.

The proof of Theorem 2.8 is built on the following results. We choose to include them here since we will need them to prove Theorem 5.1. Note that the convex hull of two points $x, y \in \mathbb{R}^n$ is denoted by $\mathrm{conv}(x, y)$.

Theorem 2.9 ([**81**]). *Let $K \in \mathcal{K}_0(\mathbb{R}^n)$ and $0 \in \mathrm{int}\, K$. For $x \in \mathrm{bd}\, K$ and $\delta > 0$, let $\{x_\delta\} = \mathrm{bd}\, K_{[\delta]} \cap \mathrm{conv}(0, x)$. Then there exists $C > 0$ and $\delta_0 > 0$ such that for all $\delta < \delta_0$ we have*

$$\frac{\|x - x_\delta\|}{\delta^{\frac{2}{n+1}}} \leq C r_K(x)^{-\frac{n-1}{n+1}}$$

for $\mathcal{H}^{n-1}$-almost all $x \in \mathrm{bd}\, K$.

Theorem 2.10 ([**81**]). *Let $K \in \mathcal{K}_0(\mathbb{R}^n)$ with $0 \in \mathrm{int}\, K$. For $\delta > 0$ small enough, we have for $\mathcal{H}^{n-1}$-almost all $x \in \mathrm{bd}\, K$,*

$$\lim_{\delta \to 0^+} \frac{1}{n} \frac{x \cdot N_x^K}{\delta^{\frac{2}{n+1}}} \left(1 - \left(\frac{\|x_\delta\|}{\|x\|} \right)^n \right) = c_n H_{n-1}^{\mathbb{R}^n}(K, x)^{\frac{1}{n+1}}, \tag{2.9}$$

where $\{x_\delta\} = \mathrm{bd}\, K_{[\delta]} \cap \mathrm{conv}(0, x)$ and N_x^K denotes the outer unit normal vector at $x \in \mathrm{bd}\, K$.

Note that for the left hand side in (2.9) we have

$$\lim_{\delta \to 0^+} \frac{1}{n} \frac{x \cdot N_x^K}{\delta^{\frac{2}{n+1}}} \left(1 - \left(\frac{\|x_\delta\|}{\|x\|} \right)^n \right) = \lim_{\delta \to 0^+} \left(\frac{x}{\|x\|} \cdot N_x^K \right) \frac{\|x - x_\delta\|}{\delta^{\frac{2}{n+1}}}. \tag{2.10}$$

CHAPTER 3

Basic Facts from Spherical Convex Geometry

We denote the n-dimensional Euclidean unit sphere by $\mathbb{S}^n$, $n \geq 2$. The natural *spherical distance* between points u and v von $\mathbb{S}^n$ is given by $d(u,v) = \arccos(u \cdot v)$, where $\cdot$ denotes the Euclidean scalar product.

The *Hausdorff distance* between closed sets $A, B \subseteq \mathbb{S}^n$ is given by

$$\delta_s(A,B) = \min\{0 \leq \lambda \leq \pi : A \subseteq B_\lambda \text{ and } B \subseteq A_\lambda\},$$

where A_λ denotes the set of all points with distance at most λ from A. A closed *spherical cap* is denoted by $C_u(\lambda) = \{u\}_\lambda$ for $u \in \mathbb{S}^n$.

The *interior* of $A \subseteq \mathbb{S}^n$ is denoted by $\operatorname{int} A$, the *closure* is $\operatorname{cls} A$ and the *boundary* is $\operatorname{bd} A$. The *radial extension* $\operatorname{rad} A$ is given by

$$\operatorname{rad} A = \{\lambda x : \lambda \geq 0, x \in A\} \subseteq \mathbb{R}^{n+1}.$$

A set $A \subseteq \mathbb{S}^n$ is called *(spherical) convex* if $\operatorname{rad} A$ is convex. We say $K \subseteq \mathbb{S}^n$ is a *convex body* if K is closed and convex and non-empty. Let $\mathcal{K}(\mathbb{S}^n)$ denote the space of convex bodies in $\mathbb{S}^n$ with the Hausdorff distance. If a convex body does not contain a pair of antipodal points, we call it *proper*. The subspace of proper convex bodies is denoted by $\mathcal{K}^p(\mathbb{S}^n)$. Furthermore, the subset of bodies with non-empty interior is denoted by $\mathcal{K}_0(\mathbb{S}^n)$ (resp. for proper convex bodies by $\mathcal{K}_0^p(\mathbb{S}^n)$).

A *k-sphere* S, $0 \leq k \leq n$, is the intersection of a $(k+1)$-dimensional linear subspace L of $\mathbb{R}^{n+1}$ with $\mathbb{S}^n$. For $u \in \mathbb{S}^n$ we denote by $\mathbb{S}_u$ the hypersphere that is given by $\mathbb{S}_u = \{v \in \mathbb{S}^n : u \cdot v = 0\}$. The *open hemisphere* with center in u is denoted by $\mathbb{S}_u^+$. We set $\mathbb{S}_u^- = \mathbb{S}_{-u}^+$. The closure of $\mathbb{S}_u^+$ is a closed hemisphere denoted by $\overline{\mathbb{S}}_u^+$. Obviously k-spheres and closed hemispheres are examples for non-proper convex bodies.

For the following alternative definitions of proper convex bodies in $\mathbb{S}^n$, we refer to [**18**].

Proposition 3.1. *The following statements about a closed set $K \subseteq \mathbb{S}^n$ are equivalent:*

(a) The set K is a proper convex body.

(b) The set K is an intersection of open hemispheres.

(c) There are no antipodal points in K and for every two points $u, v \in K$, the minimal geodesic connecting u and v is contained in K.

We remark, that a set $K \subseteq \mathbb{S}^n$ is a convex body if and only if K is the intersection of *closed* hemispheres.

The *convex hull* $\operatorname{conv} A$ of $A \subseteq \mathbb{S}^n$ is the intersection of all convex sets in $\mathbb{S}^n$ that contain A. The convex hull of two sets A, B is denoted by $\operatorname{conv}(A, B) = \operatorname{conv}(A \cup B)$. Also the convex hull of two points $u, v \in \mathbb{S}^n$ is denoted by $\operatorname{conv}(u, v) = \operatorname{conv}(\{u, v\})$.

Example 3.2. Let $u \in \mathbb{S}^n$ and consider a spherical cap $K = C_u(\frac{\pi}{4})$. For $v \in \mathbb{S}_u$ we set $z = \frac{u+v}{\sqrt{2}} \in \operatorname{bd} K$ and put $L = \{-z\}$. Then

$$\operatorname{conv}(K, L) = \mathbb{S}_v^+ \cup \{\pm z\}.$$

Thus, the convex hull of two convex bodies is in general not closed and therefore no convex body. However, if K and L are convex bodies such that there is an open hemisphere $\mathbb{S}_u^+ \supseteq K \cup L$, then $\operatorname{conv}(K, L) \in \mathcal{K}(\mathbb{S}^n)$.

For fixed $u \in \mathbb{S}^n$ we denote by $\mathcal{K}_u^p(\mathbb{S}^n)$ the subspace of (proper) convex bodies that are contained in the open hemisphere centered at u. Then

$$\mathcal{K}^p(\mathbb{S}^n) = \bigcup_{u \in \mathbb{S}^n} \mathcal{K}_u^p(\mathbb{S}^n).$$

Note again that, for $K, L \in \mathcal{K}_u^p(\mathbb{S}^n)$, we have $\operatorname{conv}(K, L) \in \mathcal{K}_u^p(\mathbb{S}^n)$.

The k-dimensional *Hausdorff measure* is denoted by $\mathcal{H}^k$ and vol_n denotes the usual volume measure on $\mathbb{S}^n$. Of course, the n-dimensional Hausdorff measure restricted to $\mathbb{S}^n$ coincides with vol_n. The volume of the unit ball $B^n(0,1)$ is denoted by $\kappa_n = \mathrm{vol}_n(B^n(0,1))$ and the perimeter of $\operatorname{bd} B^n(0,1) = \mathbb{S}^{n-1}$ is given by $\omega_{n-1} = \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = n\kappa_n$.

The *polar body* K° of a convex body $K \in \mathcal{K}(\mathbb{S}^n)$ is defined by

$$K^\circ = \{v \in \mathbb{S}^n : v \cdot w \leq 0 \text{ for all } w \in K\}$$

and is again a convex body. The following lemma collects well-known facts about the polar body:

Lemma 3.3. *Let $K \in \mathcal{K}(\mathbb{S}^n)$. Then*

(i) $(K^\circ)^\circ = K$.

(ii) We have

$$K^\circ = \bigcap_{w \in K} \overline{\mathbb{S}}_w^- = \left\{v \in \mathbb{S}^n : K \subseteq \overline{\mathbb{S}}_v^-\right\}.$$

In particular,

$$\operatorname{int} K^\circ = \bigcap_{w \in K} \mathbb{S}_w^- = \left\{v \in \mathbb{S}^n : K \subseteq \mathbb{S}_v^-\right\} = \mathbb{S}^n \backslash K_{\frac{\pi}{2}},$$

where $K_{\frac{\pi}{2}}$ is the set of all points of distance at most $\frac{\pi}{2}$ to K.

(iii) $K \in \mathcal{K}_0^p(\mathbb{S}^n)$ if and only if $K^\circ \in \mathcal{K}_0^p(\mathbb{S}^n)$.

(iv) $(K \cap L)^\circ = \operatorname{cls}\operatorname{conv}(K^\circ, L^\circ)$.

(v) If $K \in \mathcal{K}_0^p(\mathbb{S}^n)$, then $\operatorname{int} K \cap \operatorname{int}(-K^\circ) \neq \emptyset$. In particular, for $K \in \mathcal{K}_0^p(\mathbb{S}^n)$ there exists $u \in \operatorname{int} K$ such that $K \subseteq \mathbb{S}_u^+$.

Let S be a k-sphere for some $k \in \{0, \ldots, n\}$ and let $K \in \mathcal{K}(\mathbb{S}^n)$. Then the *spherical projection* $K|S$ is defined by

$$K|S = \operatorname{conv}(K, S^\circ) \cap S = (\operatorname{rad}(K)|V) \cap \mathbb{S}^n,$$

where $S = V \cap \mathbb{S}^n$ and S° is the $(n-k-1)$-sphere orthogonal to S, that is, $S^\circ = V^\perp \cap \mathbb{S}^n$. In general the spherical projection of a convex body may not be closed (see Example 3.2). However, if $K \in \mathcal{K}^p_u(\mathbb{S}^n)$, then $K|S \in \mathcal{K}^p_u(\mathbb{S}^n)$ for any k-sphere S such that $u \in S$.

The spherical projection of a point w to a hypersphere $\mathbb{S}_v$, for some $v \neq \pm w$, is given by

$$w|\mathbb{S}_v = \frac{w - \cos(d(v,w))v}{\sin(d(v,w))}. \tag{3.1}$$

We have $w = \cos(d(v,w))v + \sin(d(v,w))(w|\mathbb{S}_v)$.

Example 3.4 (k-Lunes)**.** There are essentially three different types of convex bodies: proper convex bodies, k-spheres and *k-lunes*. A k-lune is the convex hull of a k-sphere S and proper convex body $K \subseteq S^\circ$. A closed hemisphere $\overline{\mathbb{S}}^+_u$ is an $(n-1)$-lune since $\overline{\mathbb{S}}^+_u = \operatorname{conv}(\mathbb{S}_u, \{u\})$.

The space $(\mathcal{K}(\mathbb{S}^n), \delta_s)$ is compact (see e.g. [**31**]) and the closure of $\mathcal{K}^p(\mathbb{S}^n)$ with respect to δ_s is given by $\mathcal{K}^p(\mathbb{S}^n)$ and all k-lunes.

3.1. Spherical Support Functions and the Gnomonic Projection

We introduce spherical support functions of proper convex sets contained in a fixed hemisphere (cf. [**44**] for a related construction). For non-antipodal $u, v \in \mathbb{S}^n$, we write $\mathbb{S}^1_{u,v}$ for the unique great circle containing u and v.

Definition (Spherical Support Function)**.** For $u \in \mathbb{S}^n$ and a proper convex body $K \in \mathcal{K}^p_u(\mathbb{S}^n)$, the *spherical support function* $h_u(K,\cdot)\colon \mathbb{S}_u \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ of K is defined by

$$h_u(K,v) = \max\{\operatorname{sgn}(v \cdot w)\, d(u, w|\mathbb{S}^1_{u,v}) : w \in K\}.$$

Recall that the (Euclidean) support function of a convex body L in $\mathbb{R}^n$ encodes the signed distances of the supporting planes to L from the origin. In other words, we have for every $v \in \mathbb{S}^{n-1}$,

$$L|\operatorname{span}\{v\} = \{tv : t \in [-h_K(-v), h_K(v)]\}.$$

The intuitive meaning of the spherical support function of a proper convex body $K \in \mathcal{K}^p_u(\mathbb{S}^n)$ is similar. It yields the oriented angle between u and the supporting hyperspheres to K. More precisely, we have for every $v \in \mathbb{S}_u$,

$$K|\mathbb{S}^1_{u,v} = \{\cos(\alpha)u + \sin(\alpha)v : \alpha \in [-h_u(K,-v), h_u(K,v)]\}.$$

In particular, for $K, L \in \mathcal{K}^p_u(\mathbb{S}^n)$, $K \subseteq L$ if and only if $h_u(K,\cdot) \le h_u(L,\cdot)$.

Let $u \in \mathbb{S}^n$. For $v \in \mathbb{S}_u$ and $\delta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we set $z = \cos(\delta)v - \sin(\delta)u$ and

$$\overline{\mathbb{S}}^+_{u,v,\delta} = \overline{\mathbb{S}}^+_z = \{w \in \mathbb{S}^n : v \cdot w \ge \tan(\delta)(u \cdot w)\}. \tag{3.2}$$

Let $K \in \mathcal{K}(\mathbb{S}^n)$ such that $K \subseteq \mathbb{S}^+_u$. For $v \in \mathbb{S}_u$, the supporting hyperspheres of K are parametrized by $\mathbb{S}_{u,v,h_u(K,v)}$. In particular, we have that

$$z = \cos(h_u(K,v))v - \sin(h_u(K,v))u \in \operatorname{bd} K^\circ$$

and thus z is an outer unit normal vector for any point $w \in \operatorname{bd} K \cap \mathbb{S}_{u,v,h_u(K,v)}$. We note that $v = z|\mathbb{S}_u$ and

$$-u \cdot z = \sin(h_u(K, z|\mathbb{S}_u)). \tag{3.3}$$

We conclude that $K = \bigcap_{v\in\mathbb{S}_u} \overline{\mathbb{S}}^-_{u,v,h_u(K,v)} = \bigcap_{z\in\operatorname{bd} K^\circ} \overline{\mathbb{S}}^-_z$.

In the following we denote by $\mathbb{R}^n_u$ (instead of $u^\perp$) the hyperplane in $\mathbb{R}^{n+1}$ orthogonal to $u \in \mathbb{S}^n$.

Definition (Gnomonic Projection)**.** For $u \in \mathbb{S}^n$, the *gnomonic projection* $g_u \colon \mathbb{S}^+_u \to \mathbb{R}^n_u$ is defined by

$$g_u(v) = \frac{v}{u \cdot v} - u.$$

In the literature, the gnomonic projection is often considered as a map to the tangent plane at u. However, for our purposes it is more convenient if the range of g_u contains the origin.

In the following lemma we collect a number of well-known properties of the gnomonic projection which are immediate consequences of its definition.

Lemma 3.5. *For $u \in \mathbb{S}^n$, the following statements hold:*

(a) The gnomonic projection $g_u \colon \mathbb{S}^+_u \to \mathbb{R}^n_u$ is a bijection with inverse

$$g_u^{-1}(x) = \frac{x+u}{\|x+u\|}, \qquad x \in \mathbb{R}^n_u.$$

(b) If $S \subseteq \mathbb{S}^n$ is a k-sphere, $0 \le k \le n-1$, such that $S \cap \mathbb{S}^+_u$ is non-empty, then $g_u(S \cap \mathbb{S}^+_u)$ is a k-dimensional affine subspace of $\mathbb{R}^n_u$. Conversely, g_u^{-1} maps k-dimensional affine subspaces of $\mathbb{R}^n_u$ to k-spheres in $\mathbb{S}^+_u$.

(c) The gnomonic projection maps $\mathcal{K}^p_u(\mathbb{S}^n)$ bijectively to $\mathcal{K}(\mathbb{R}^n_u)$.

If $K \in \mathcal{K}^p_u(\mathbb{S}^n)$, then, by Lemma 3.5 (c), the set $g_u(K)$ is a convex body in $\mathcal{K}(\mathbb{R}^n_u)$. The next lemma relates the (Euclidean) support function of $g_u(K)$ with the spherical support function of K.

Lemma 3.6. *For $u \in \mathbb{S}^n$ and every $K \in \mathcal{K}^p_u(\mathbb{S}^n)$, we have*

$$h_{g_u(K)} = \tan h_u(K, \cdot). \tag{3.4}$$

In particular, K is uniquely determined by $h_u(K, \cdot)$.

Proof. For $v \in \mathbb{S}_u$ and $w \in \mathbb{S}^+_u$, an elementary calculation shows that

$$\frac{v \cdot w}{u \cdot w} = \tan\Big(\operatorname{sgn}(v \cdot w)\, d(u, w|\mathbb{S}^1_{u,v})\Big).$$

Therefore, the definition of g_u and the monotonicity of the tangent yield

$$h_{g_u(K)}(v) = \max_{x \in g_u(K)} \{v \cdot x\} = \max_{w \in K} \left\{\frac{v \cdot w}{u \cdot w}\right\} = \tan h_u(K, v).$$

□

By Lemma 3.6, a function $h\colon \mathbb{S}_u \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is the spherical support function of a convex body $K \in \mathcal{K}^p_u(\mathbb{S}^n)$ if and only if the 1-homogeneous extension of $\tan h$ to $\mathbb{R}^n_u$ is the support function of a convex body in $\mathbb{R}^n_u$.

For $K \in \mathcal{K}^p_0(\mathbb{S}^n)$ and $u \in \mathbb{S}^n$ such that $u \in \operatorname{int} K$, the *radial function* $\rho_u(K, \cdot)\colon \mathbb{S}_u \to [0, \pi]$ is defined by

$$\rho_u(K, v) = \max\{\alpha \in [0, \pi] : \cos(\alpha)u + \sin(\alpha)v \in K\}. \tag{3.5}$$

If $K \subseteq \mathbb{S}^+_u$, then the Euclidean radial function $\rho_{g_u(K)}\colon \mathbb{R}^n_u \to \mathbb{R}$ is given by

$$\tan(\rho(K, v)) = \rho_{g_u(K)}(v), \quad v \in \mathbb{S}_u.$$

The Euclidean polar body $g_u(K)^* = \{y \in \mathbb{R}^n_u : x \cdot y \le 1 \text{ for all } x \in g_u(K)\}$ is related to K° by $g_u(K)^* = g_{-u}(K^\circ)$. Moreover,

$$\tan(h_u(K, v)) = h_{g_u(K)}(v) = \frac{1}{\rho_{g_u(K)^*}(v)} = \cot(\rho_{-u}(K^\circ, v)),$$

or, equivalently,

$$h_u(K, v) + \rho_{-u}(K^\circ, v) = \frac{\pi}{2}, \quad v \in \mathbb{S}_u. \tag{3.6}$$

Using spherical support functions, we define a metric γ_u on $\mathcal{K}_u^p(\mathbb{S}^n)$ by

$$\gamma_u(K,L) = \max_{v \in \mathbb{S}_u} |h_u(K,v) - h_u(L,v)|.$$

Since for $K \in \mathcal{K}_u^p(\mathbb{S}^n)$ and $\varepsilon > 0$, the set K_ε of all points with distance at most ε from K is *not* necessarily convex, it is not difficult to see that the restriction of δ_s to $\mathcal{K}_u^p(\mathbb{S}^n)$ does *not* coincide with γ_u (in contrast to the Euclidean setting). However, our next result shows that γ_u and δ_s induce the same topology on $\mathcal{K}_u^p(\mathbb{S}^n)$. Since we could not find a reference for this basic result, we include a proof for the readers convenience.

Proposition 3.7. *For $u \in \mathbb{S}^n$, the metrics γ_u and δ_s induce the same topology on $\mathcal{K}_u^p(\mathbb{S}^n)$.*

Proof. Let $K \in \mathcal{K}_u^p(\mathbb{S}^n)$ and $\varepsilon > 0$ sufficiently small. We denote by $B_{\gamma_u}(K,\varepsilon)$ the metric ball with respect to γ_u of radius ε and center K and $B_{\delta_s}(K,\varepsilon)$ is defined similarly. We first show that there exists $r(K,\varepsilon) > 0$ such that

$$B_{\delta_s}(K, r(K,\varepsilon)) \subseteq B_{\gamma_u}(K,\varepsilon). \tag{3.7}$$

To this end, let $w \in \mathbb{S}_u^+$. Since $C_w(\varepsilon)$ is a spherical cap of radius ε, it is not difficult to show that

$$\max_{v \in \mathbb{S}_u}(h_u(C_w(\varepsilon),v) - h_u(\{w\},v)) = \arcsin\left(\frac{\sin\varepsilon}{u \cdot w}\right),$$

where this maximum is attained for $v \in \mathbb{S}_u \cap \mathbb{S}_w$. Therefore, if we define

$$c(w,\varepsilon) = \arcsin(u \cdot w \, \sin\varepsilon),$$

then

$$h_u(C_w(c(w,\varepsilon)),v) \leq h_u(\{w\},v) + \varepsilon \tag{3.8}$$

for all $v \in \mathbb{S}_u$. We now define

$$r(K,\varepsilon) = \min_{w \in K} c(w,\varepsilon/2).$$

Note that, by the compactness of K, we have $r(K,\varepsilon) > 0$. Since (3.8) holds for all $w \in \mathbb{S}_u^+$, we obtain

$$\begin{aligned}\max_{w \in K} h_u(C_w(r(K,\varepsilon)),v) &\leq \max_{w \in K} h_u(C_w(c(w,\varepsilon/2)),v) \\ &\leq \max_{w \in K} h_u(\{w\},v) + \frac{\varepsilon}{2} = h_u(K,v) + \frac{\varepsilon}{2}\end{aligned}$$

for all $v \in \mathbb{S}_u$. Using

$$\max_{w\in K} h_u(C_w(r(K,\varepsilon)),v) = h_u(\mathrm{conv}(K_{r(K,\varepsilon)}),v), \qquad v \in \mathbb{S}_u,$$

we conclude that

$$h_u(\mathrm{conv}(K_{r(K,\varepsilon)}),v) \le h_u(K,v) + \frac{\varepsilon}{2} \tag{3.9}$$

for all $v \in \mathbb{S}_u$. Moreover,

$$r(\{w\}_{r(\{w\},\varepsilon)}, 2\varepsilon) = \min\{c(w',\varepsilon) : w' \in C_w(c(w,\varepsilon/2))\} \ge c(w,\varepsilon/2). \tag{3.10}$$

This follows from an elementary calculation and the fact, that $c(w',\varepsilon)$ attains its minimum in $C_w(c(w,\varepsilon/2))$ when $d(w',u) = d(w,u) + c(w,\varepsilon/2)$.

Now, let $L \in \mathcal{K}_u^p(\mathbb{S}^n)$ such that $\delta_s(K,L) \le r(K,\varepsilon)$. Then, from

$$L \subseteq K_{r(K,\varepsilon)} \subseteq \mathrm{conv}(K_{r(K,\varepsilon)}) \tag{3.11}$$

and (3.9), we obtain on the one hand

$$h_u(L,v) \le h_u(\mathrm{conv}(K_{r(K,\varepsilon)}),v) \le h_u(K,v) + \frac{\varepsilon}{2} \le h_u(K,v) + \varepsilon$$

for all $v \in \mathbb{S}_u$. On the other hand, from (3.11) and (3.10) we deduce that

$$\begin{aligned} r(L,2\varepsilon) &\ge \min_{w\in K_{r(K,\varepsilon)}} c(w,\varepsilon) \ge \min_{w\in K} r(C_w(r(\{w\},\varepsilon)),2\varepsilon) \\ &\ge \min_{w\in K} c(w,\varepsilon/2) = r(K,\varepsilon) \end{aligned}$$

and, thus, $K \subseteq L_{r(K,\varepsilon)} \subseteq L_{r(L,2\varepsilon)} \subseteq \mathrm{conv}(L_{r(L,2\varepsilon)})$. Consequently, by (3.9),

$$h_u(K,v) \le h_u(\mathrm{conv}(L_{r(L,2\varepsilon)}),v) \le h_u(L,v) + \varepsilon$$

for all $v \in \mathbb{S}_u$, which concludes the proof of (3.7).

It remains to show that there also exists $\overline{r}(K,\varepsilon) > 0$ such that

$$B_{\gamma_u}(K,\overline{r}(K,\varepsilon)) \subseteq B_{\delta_s}(K,\varepsilon). \tag{3.12}$$

To this end, let again $w \in \mathbb{S}_u^+$. By our definition of spherical support functions, we have for sufficiently small $\lambda > 0$,

$$\min_{v\in\mathbb{S}_u} (h_u(C_w(\lambda),v) - h_u(\{w\},v)) = \lambda,$$

where this minimum is attained for $v \in \mathbb{S}_u \cap \mathbb{S}^1_{u,w}$. Consequently, we obtain

$$h_u(\{w\}, v) + \lambda \leq h_u(C_w(\lambda), v)$$

for all $v \in \mathbb{S}_u$. Since this holds for all $w \in \mathbb{S}^+_u$, we conclude that

$$h_u(K, v) + \lambda = \max_{w \in K} h_u(\{w\}, v) + \lambda \leq \max_{w \in K} h_u(C_w(\lambda), v) = h_u(\operatorname{conv}(K_\lambda), v)$$

for all $v \in \mathbb{S}_u$. Therefore, if $L \in \mathcal{K}^p_u(\mathbb{S}^n)$ such that $\gamma_u(K, L) \leq \lambda$, then

$$L \subseteq \operatorname{conv}(K_\lambda) \qquad \text{and} \qquad K \subseteq \operatorname{conv}(L_\lambda). \tag{3.13}$$

We want to choose $\lambda = \overline{r}(K, \varepsilon)$ in such a way that

$$\operatorname{conv}(K_{\overline{r}(K,\varepsilon)}) \subseteq K_\varepsilon \qquad \text{and} \qquad \operatorname{conv}(L_{\overline{r}(K,\varepsilon)}) \subseteq L_\varepsilon. \tag{3.14}$$

In order to compute $\overline{r}(K, \varepsilon)$ let $v, w \in \mathbb{S}^+_u$ and denote by $J^w_v \in \mathcal{K}^p_u(\mathbb{S}^n)$ the spherical segment connecting v and w. An elementary calculation shows that

$$\operatorname{conv}((J^w_v)_{\overline{c}(J^w_v,\varepsilon)}) \subseteq (J^w_v)_\varepsilon, \tag{3.15}$$

where

$$\overline{c}(J^w_v, \varepsilon) = \arcsin\left(\sin\varepsilon \cos\left(\frac{d(v, w)}{2}\right)\right).$$

We define

$$\overline{r}(K, \varepsilon) = \min_{v,w \in K} \overline{c}(J^w_v, \varepsilon/2).$$

Then, by (3.15),

$$\operatorname{conv}(K_{\overline{r}(K,\varepsilon)}) = \bigcup_{v,w \in K} \operatorname{conv}((J^w_v)_{\overline{r}(K,\varepsilon)}) \subseteq \bigcup_{v,w \in K} \operatorname{conv}((J^w_v)_{\overline{c}(J^w_v,\varepsilon/2)}) \tag{3.16}$$

$$\subseteq \bigcup_{v,w \in K} (J^w_v)_{\varepsilon/2} = K_{\varepsilon/2} \subseteq K_\varepsilon. \tag{3.17}$$

This proves the first inclusion of (3.14). To see the second inclusion, note that

$$\overline{r}((J^w_v)_{\varepsilon/2}, 2\varepsilon) = \min_{v',w' \in (J^w_v)_{\varepsilon/2}} \overline{c}(J^{w'}_{v'}, \varepsilon) \geq \overline{c}(J^w_v, \varepsilon/2).$$

which follows from an elementary calculation and the fact, that $\overline{c}(J_{v'}^{w'}, \varepsilon)$ attains its minimum in $(J_v^w)_{\varepsilon/2}$ when $d(v', w') = d(v, w) + \varepsilon$. Thus, for $L \in \mathcal{K}_u^p(\mathbb{S}^n)$ such that $\gamma_u(K, L) \leq \overline{r}(K, \varepsilon)$, it follows from (3.13), (3.16), (3.17) that $L \subseteq \operatorname{conv}(K_{\overline{r}(K,\varepsilon)}) \subseteq K_{\varepsilon/2}$ and we conclude

$$\begin{aligned}\overline{r}(L, 2\varepsilon) &\geq \min_{v,w\in K_{\varepsilon/2}} \overline{c}(J_v^w, \varepsilon) = \min_{v,w\in K} \overline{r}((J_v^w)_{\varepsilon/2}, 2\varepsilon)\\ &\geq \min_{v,w\in K} \overline{c}(J_v^w, \varepsilon/2) = \overline{r}(K, \varepsilon).\end{aligned}$$

Hence, using again (3.16) and (3.17), where K is replaced by L,

$$\operatorname{conv}(L_{\overline{r}(K,\varepsilon)}) \subseteq \operatorname{conv}(L_{\overline{r}(L,2\varepsilon)}) \subseteq L_\varepsilon.$$

This proves the second inclusion of (3.14) and, thus, (3.12). □

Note that if $K, L \in \mathcal{K}_u^p(\mathbb{S}^n)$, then, by (2.1) and Lemma 3.6,

$$\delta(g_u(K), g_u(L)) = \max_{v\in\mathbb{S}_u} |\tan h_u(K, v) - \tan h_u(L, v)|.$$

Thus, from Proposition 3.7 and the continuity of the tangent we obtain the following.

Corollary 3.8. *The gnomonic projection is a homeomorphism between* $(\mathcal{K}_u^p(\mathbb{S}^n), \delta_s)$ *and* $(\mathcal{K}(\mathbb{R}_u^n), \delta)$.

3.2. Boundary Structure of Spherical Convex Bodies

In this section we develop the technical framework to transform integrals on $\mathbb{S}^n$ and the boundary of spherical convex bodies via the gnomonic projection or the Gauss map. We will consider subsets of the sphere as subsets in $\mathbb{R}^{n+1}$ and use the area formula on rectifiable sets, where we explicitly calculate the (approximate) tangential Jacobian. For a reference on the area formula and tangential Jacobian we refer to F. Maggi [**63**] or S. G. Krantz and H. R. Parks [**42**], which will provide sufficient background for the tools we use (for a more extensive exposition see, e.g., H. Federer [**20**]).

We begin with an outline of what follows: Using the area formula (see Theorem 11.6 in [**63**]) we show for the diffeomorphism $g_u\colon \mathbb{S}_u^+ \to \mathbb{R}_u^n$, that for measurable $\omega \subseteq \mathbb{S}_u^+$ and measurable $f\colon \mathbb{R}_u^n \to \mathbb{R}$ we have

$$\int_\omega f \circ g_u \, d\mathcal{H}_{\mathbb{S}^n}^{n-1} = \int_{g_u(\omega)} f \, J^{\mathbb{S}_u^+}(g_u) \, d\mathcal{H}_{\mathbb{R}_u^n}^{n-1}.$$

Here $J^{\mathbb{S}_u^+}(g_u)$ denotes the *tangential Jacobian* of g_u on $\mathbb{S}_u^+$ which is defined by

$$J^{\mathbb{S}_u^+}(g_u)(v) = \sqrt{\det(d(g_u)_v^* d(g_u)_v)},$$

where $d(g_u)_v$ denotes the *tangential derivative* (differential) in v and $d(g_u)_v^*$ denotes the adjoint. In the following proposition we will explicitly calculate this expression.

Proposition 3.9. *Let $u \in \mathbb{S}^n$. Then g_u is a diffeomorphism between $\mathbb{S}_u^+$ and $\mathbb{R}_u^n$. For $v \in \mathbb{S}_u^+$ we identify the tangent space $T_v\mathbb{S}_u^+$ with $\mathbb{R}_v^n$ and $T_{g_u(v)}\mathbb{R}_u^n$ with $\mathbb{R}_u^n$. Then the differential $d(g_u)_v \colon T_v\mathbb{S}_u^+ \to T_{g_u(v)}\mathbb{R}_u^n$ is given by*

$$d(g_u)_v(X_v) = \frac{1}{u \cdot v} X_v - \frac{u \cdot X_v}{(u \cdot v)^2} v, \tag{3.18}$$

for all $X_v \in \mathbb{R}_v^n$. The tangential Jacobian of g_u is given by

$$J^{\mathbb{S}_u^+}(g_u)(v) = \sqrt{\det((d(g_u)_v)^* d(g_u)_v)} = \frac{1}{(u \cdot v)^{n+1}} \tag{3.19}$$

and the tangential Jacobian of the inverse g_u^{-1} in $x \in \mathbb{R}_u^n$ is

$$J^{\mathbb{R}_u^n}(g_u^{-1})(x) = \frac{1}{J^{\mathbb{S}_u^+}(g_u)(g_u^{-1}(x))} = \frac{1}{(1 + \|x\|^2)^{\frac{n+1}{2}}}. \tag{3.20}$$

Proof. An elementary calculation leads to formula (3.18). Also the inverse can be explicitly calculated. Thus g_u is a diffeomorphism. In order to prove (3.19), we first note that $d(g_u)_v$ can be expressed as a matrix on $\mathbb{R}^{n+1}$ of rank n by

$$d(g_u)_v = \frac{1}{u \cdot v}\left(\mathrm{Id}_{n+1} - \frac{1}{u \cdot v} v \otimes u\right),$$

where $v \otimes u$ denotes the matrix determined by $(v \otimes u)X_v = (u \cdot X_v)v$. Thus,

$$(d(g_u)_v)^* d(g_u)_v = \frac{1}{(u \cdot v)^2}\left(\mathrm{Id}_{\mathbb{R}_v^n} + \frac{u - (u \cdot v)v}{u \cdot v} \otimes \frac{u - (u \cdot v)v}{u \cdot v}\right).$$

Using the well known formula $\det(\mathrm{Id} + z \otimes z) = 1 + \|z\|^2$, we conclude

$$J^{\mathbb{S}_u^+}(g_u)(v) = \frac{1}{(u \cdot v)^n}\sqrt{1 + \frac{\|u - (u \cdot v)v\|^2}{(u \cdot v)^2}} = \frac{1}{(u \cdot v)^{n+1}}.$$

□

The analytic properties of the boundary of a spherical convex body K are similar to the properties of the boundary of a Euclidean convex body. This is obvious when considering the gnomonic projection around a boundary point $w \in \operatorname{bd} K$, that is, for an arbitrary but fixed $\varepsilon \in (0, \frac{\pi}{2})$ we consider $L = g_w(K \cap C_w(\varepsilon))$. Then L is a Euclidean convex body with $g_w(w) = 0$ and $\operatorname{bd} L = \operatorname{bd} g_w(K \cap C_w(\varepsilon))$. Since g_w is a diffeomorphism on $C_w(\varepsilon)$ we conclude that all regularity results of $\operatorname{bd} L$ in 0 from Euclidean convex geometry hold for $w \in \operatorname{bd} K$.

Proposition 3.10. *Let $K \in \mathcal{K}(\mathbb{S}^n)$.*

(i) The boundary of K is an $\mathcal{H}^{n-1}$-rectifiable set.

(ii) If K has non-empty interior, then for $\mathcal{H}^{n-1}$-almost all boundary points w there exists a unique outer unit normal $N_w^K \in \mathbb{S}_w$, where we identify the tangent space $T_w\mathbb{S}^n$ with $\mathbb{R}_w^n$.

Note that the gnomonic projection is not conformal and thus $d(g_u)_w(N_w^K)$ is in general not the outer normal vector $N_{g_u(w)}^{g_u(K)}$ of $\operatorname{bd} g_u(K)$. The relation between the outer normal vector of $\operatorname{bd} K$ and $\operatorname{bd} g_u(K)$ is given by

$$N_{g_u(w)}^{g_u(K)} = N_w^K|\mathbb{S}_u. \tag{3.21}$$

Now let $u \in \mathbb{S}^n$ and $K \in \mathcal{K}_0^p(\mathbb{S}^n)$ such that $K \subseteq \mathbb{S}_u^+$. Then there is $\beta \in (0, \frac{\pi}{2})$ such that $K \subseteq \operatorname{int} C_u(\beta)$. Since g_u is differentiable and therefore Lipschitz on $C_u(\beta)$ and because $\operatorname{bd} K$ is $\mathcal{H}^{n-1}$-rectifiable, we conclude that the approximate tangential derivative $d^{\operatorname{bd} K} g_u$ exists $\mathcal{H}^{n-1}$-almost everywhere. Furthermore, since g_u maps tangential hypersphere to w to the affine hyperplane tangent to $g_u(w)$, we have $d(g_u)_w(T_w \operatorname{bd} K) = T_{g_u(w)} \operatorname{bd} g_u(K)$ and conclude

$$d^{\operatorname{bd} K}(g_u)_w(X_w) = d(g_u)_w(X_w),$$

for all $X_w \in T_w \operatorname{bd} K$. We can write $d^{\operatorname{bd} K}(g_u)_w$ as a matrix on $\mathbb{R}^{n+1}$ of rank $n-1$ in the form

$$d^{\operatorname{bd} K}(g_u)_w = \frac{1}{u \cdot w}\left(\mathrm{Id}_{n+1} - \frac{1}{u \cdot w} w \otimes u\right) - d(g_u)_w(N_w^K) \otimes N_w^K.$$

Thus,

$$(d^{\operatorname{bd} K}(g_u)_w)^* d^{\operatorname{bd} K}(g_u)_w = \frac{1}{(u \cdot w)^2}(\mathrm{Id}_{T_w \operatorname{bd} K} + \tilde{z} \otimes \tilde{z}),$$

where $\tilde{z} = \frac{u-(u\cdot w)w-(u\cdot N_w^K)N_w^K}{u\cdot w}$. Hence, the tangential Jacobian

$$J^{\mathrm{bd}\,K}(g_u)(w) = \sqrt{\det((d^{\mathrm{bd}\,K}(g_u)_w)^* d^{\mathrm{bd}\,K}(g_u)_w)}$$

is given by

$$J^{\mathrm{bd}\,K}(g_u)(w) = \frac{\sqrt{1-(u\cdot N_w^K)^2}}{(u\cdot w)^n} = \frac{\cos(h_u(K, N_w^K|\mathbb{S}_u))}{\cos(d(u,w))^n}, \tag{3.22}$$

for $\mathcal{H}^{n-1}$-almost all $w \in \mathrm{bd}\,K$, where the last equation used (3.3) for $z = N_w^K|\mathbb{S}_u$. For the inverse g_u^{-1} we obtain

$$J^{\mathrm{bd}\,g_u(K)}(g_u^{-1})(x) = \frac{1}{J^{\mathrm{bd}\,K}(g_u)(g_u^{-1}(x))} = \frac{\sqrt{1+\left(x\cdot N_x^{g_u(K)}\right)^2}}{(1+\|x\|^2)^{\frac{n}{2}}}, \tag{3.23}$$

for $\mathcal{H}^{n-1}$-almost all $x \in \mathrm{bd}\,g_u(K)$.

The *radial map* $R_K\colon \mathrm{bd}\,K \to \mathbb{S}_u$ is defined by $R_K(w) = w|\mathbb{S}_u$. Since the Euclidean radial map $R_L\colon \mathrm{bd}\,L \to \mathbb{S}^{n-1}$ is defined by $R_L(x) = \frac{x}{\|x\|}$. The tangential Jacobian of R_L is given by $J^{\mathrm{bd}\,L}(R_L)(x) = \frac{x\cdot N_x^L}{\|x\|^n} = \frac{h_L(N_x^L)}{\|x\|^n}$ for $\mathcal{H}^{n-1}$-almost all $x \in \mathrm{bd}\,L$.

The radial maps R_K and $R_{g_u(K)}$ are related by $R_{g_u(K)}(g_u(w)) = R_K(w)$. We conclude for the approximate tangential Jacobian of R_K,

$$\begin{aligned} J^{\mathrm{bd}\,K}(R_K)(w) &= J^{\mathrm{bd}\,g_u(K)}(R_{g_u(K)})(g_u(w))J^{\mathrm{bd}\,K}(g_u)(w) \\ &= \frac{\sin(h_u(K, N_w^K|\mathbb{S}_u))}{\sin(d(u,w))^n} = \frac{-u\cdot N_w^K}{\sin(d(u,w))^n}, \end{aligned} \tag{3.24}$$

where we used the fact that $h_{g_u(K)}\left(N_{g_u(w)}^{g_u(K)}\right) = \tan(h_u(K, N_w^K|\mathbb{S}_u))$ (see (3.4), (3.3) and (3.21)).

A boundary point $w \in \mathrm{bd}\,K$ is called *regular* if it has a unique outer unit normal N_w^K. The set of regular boundary points of K is denoted by $\mathrm{reg}\,K$. As in the Euclidean setting, the Gauss map $N^K\colon \mathrm{reg}\,K \to \mathbb{S}^n$ is in general not Lipschitz, see e.g. [**37**]. However, as was pointed out in [**37**], if, for $\alpha > 0$, one restricts N^K to $(\mathrm{bd}\,K)_\alpha$, defined by

$$(\mathrm{bd}\,K)_\alpha = \{w \in \mathrm{bd}\,K : \exists v \in \mathbb{S}^n \text{ such that } w \in C_v(\alpha) \subseteq K\},$$

then $N^K|_{(\mathrm{bd}\,K)_\alpha}$ is Lipschitz.

Furthermore, for

$$(\operatorname{bd} K)_+ := \bigcup_{n \in \mathbb{N}} (\operatorname{bd} K)_{\frac{1}{n}},$$

we have $\mathcal{H}^{n-1}(\operatorname{bd} K \backslash (\operatorname{bd} K)_+) = 0$, see [**37**, Lemma 2.2]. It follows that for $\mathcal{H}^{n-1}$-almost all boundary points the approximate Jacobian of N^K exists and is therefore given by $J^{\operatorname{bd} K}(N^K)(w) = H_{n-1}^{\mathbb{S}^n}(K,w)$, where $H_{n-1}^{\mathbb{S}^n}(K,w)$ denotes the Gauss–Kronecker curvature. Again this can be seen easily when considering the gnomonic projection around a boundary point $w \in \operatorname{bd} K$, since

$$\begin{aligned} J^{\operatorname{bd} K}(N^K)(w) &= J^{\operatorname{bd} g_w(K \cap C_w(\varepsilon))}(N^{g_w(K \cap C_w(\varepsilon))})(0) J^{\operatorname{bd} K}(g_w)(w) \\ &= H_{n-1}^{\mathbb{R}^n_w}(g_w(K \cap C_w(\varepsilon)), 0) \end{aligned}$$

and the fact that $H_{n-1}^{\mathbb{R}^n_w}(g_w(K \cap C_w(\varepsilon)), 0) = H_{n-1}^{\mathbb{S}^n}(K,w)$. The latter is obvious since $d(g_w)_w = \mathrm{Id}_{\mathbb{R}^n_w}$.

In particular, for a proper convex body with non-empty interior there exists $u \in \operatorname{int} K$ such that $K \subseteq \mathbb{S}^+_u$ (see Lemma 3.3(v)) and we may express $H_{n-1}^{\mathbb{S}^n}(K,\cdot)$ by $H_{n-1}^{\mathbb{R}^n_u}(g_u(K),\cdot)$.

Lemma 3.11. *Let $K \in \mathcal{K}_0^p(\mathbb{S}^n)$ and $u \in \mathbb{S}^n$ such that $K \subseteq \mathbb{S}^+_u$. Then*

$$H_{n-1}^{\mathbb{S}^n}(K,w) = H_{n-1}^{\mathbb{R}^n_u}(g_u(K), g_u(w)) \left(\frac{\cos(h_u(K, N_w^K | \mathbb{S}_u))}{\cos(d(u,w))} \right)^{n+1} \tag{3.25}$$

for $\mathcal{H}^{n-1}$-almost all $w \in \operatorname{bd} K$.

Proof. Since g_u is a geodesic diffeomorphism we have $H_{n-1}^{\mathbb{S}^n}(K,w) = 0$ if and only if $H_{n-1}^{\mathbb{R}^n_u}(g_u(K), g_u(w)) = 0$. Thus, in this case, (3.25) holds. So assume $H_{n-1}^{\mathbb{S}^n}(K,w) > 0$. By (3.21), we have $N_w^K | \mathbb{S}_u = N_{g_u(w)}^{g_u(K)}$, which implies that

$$R_{K^\circ} \circ N_w^K = N^{g_u(K)} \circ g_u(w),$$

where $N^{g_u(K)} \colon \operatorname{bd} g_u(K) \to \mathbb{S}_u$ is the Euclidean Gauss map of $g_u(K)$. For the outer unit normal $N^{g_u(K)}$ of a Euclidean convex body L and a regular boundary point x of L, we have $J^{\operatorname{bd} L}(N^L)(x) = H_{n-1}^{\mathbb{R}^n_u}(L,x)$ (see [**37**, Lemma 2.3]). Thus, we conclude

$$J^{\mathbb{S}_u}(R_{K^\circ})(N_w^K) H_{n-1}^{\mathbb{S}^n}(K,w) = J^{\operatorname{bd} g_u(K)}(N^{g_u(K)})(g_u(w)) J^{\operatorname{bd} K}(g_u)(w),$$

and, by (3.24) and (3.22),

$$\frac{\sin(h_{-u}(K^\circ, N^{K^\circ}_{N^K_w}|\mathbb{S}_u))}{\sin(d(-u, N^K_w))^n} H^{\mathbb{S}^n}_{n-1}(K,w) = H^{\mathbb{R}^n_u}_{n-1}(g_u(K), g_u(w)) \frac{\cos(h_u(K, N^K_w|\mathbb{S}_u))}{\cos(d(u,w))^n}.$$

Since $H^{\mathbb{S}^n}_{n-1}(K,w) > 0$, we have that N^K_w is the unique outer unit normal to w and this also implies that w is the unique outer unit normal to $N^K_w \in K^\circ$, thus, $(N^{K^\circ} \circ N^K)(w) = w$. By the duality formula (3.6), we obtain (3.25). We note that $\rho_{-u}(K^\circ, N^K_w|\mathbb{S}_u) = d(-u, N^K_w)$. □

Remark 3.12. This theorem relates to [**38**, Theorem 2.2] in the following way: If $w \in \operatorname{bd} K$ such that $H^{\mathbb{S}^n}_{n-1}(K,w) > 0$, then

$$\begin{aligned} 1 &= H^{\mathbb{S}^n}_{n-1}(K,w) H^{\mathbb{S}^n}_{n-1}(K^\circ, N^K_w) \\ &= \left(\frac{\tan(d(u,w))}{\tan(h_u(K, N^K_w|\mathbb{S}_u))}\right)^{n+1} H^{\mathbb{R}^n_u}_{n-1}(g_u(K), g_u(w))\, H^{\mathbb{R}^n_u}_{n-1}(g_{-u}(K^\circ), g_{-u}(N^K_w)) \end{aligned}$$

thus,

$$\begin{aligned} H_{n-1}(g_u(K), g_u(w)) &= \left(\frac{\tan(d(w,u))}{\tan(h_u(K, N^K_w|\mathbb{S}_u))}\right)^{n+1} \frac{1}{H_{n-1}(g_u(K)^*, g_{-u}(N^K_w))} \\ &= \left(\frac{g_u(w)}{\|g_u(w)\|} \cdot N^{g_u(K)}_{g_u(w)}\right)^{n+1} D_{n-1} h_{g_u(K)^*}\left(\frac{g_u(w)}{\|g_u(w)\|}\right), \end{aligned}$$

where for the second equality we used [**37**, Lemma 2.5].

The following lemma can be considered as a spherical version of the splitting of Lebesgue integrals by orthogonal subspaces.

Lemma 3.13 ([**76**])**.** *Let S be a k-sphere, $0 \le k \le n-1$, and let $f\colon \mathbb{S}^n \to \mathbb{R}$ be a non-negative measurable function. Then*

$$\int_{\mathbb{S}^n} f(w)\, dw = \int_S \int_{\operatorname{conv}(S^\circ, v)} \sin(d(S^\circ, u))^k f(u)\, du\, dv.$$

For a hypersphere $\mathbb{S}_u$, we derive

$$\int_{\mathbb{S}^n} f(w)\, dw = \int_{\mathbb{S}_u} \int_0^\pi \sin(t)^{n-1} f(\cos(t)u + \sin(t)v)\, dt\, dv.$$

Furthermore, for $K \in \mathcal{K}_u^p(\mathbb{S}^n)$ and $v \in \mathbb{S}_u$ we have

$$\mathrm{vol}_n(K) = \int\limits_{-h_u(K,-v)}^{h_u(K,v)} \int\limits_{K \cap \mathbb{S}_{u,v,t}} \frac{\cos(d(u,w))}{\cos(t)}\, dw\, dt.$$

This follows by (3.19), (3.20) and (3.22), from

$$\begin{aligned}
\mathrm{vol}_n(K) &= \int\limits_{g_u(K)} J^{\mathbb{R}^n_u}(g_u^{-1})(x)\, dx \\
&= \int\limits_{-\tan(h_u(K,-v))}^{\tan(h_u(K,v))} \int\limits_{K \cap \mathbb{S}_{z(s)}} \frac{J^{\mathbb{S}_{z(s)}}(g_u)(w)}{J^{\mathbb{S}_u^+}(g_u)(w)}\, dw\, ds \\
&= \int\limits_{-h_u(K,-v)}^{h_u(K,v)} \int\limits_{K \cap \mathbb{S}_{u,v,t}} \frac{\cos(d(w,u))^{n+1}\cos(t)}{\cos(t)^2\cos(d(w,u))^n}\, dw\, dt,
\end{aligned}$$

where we put $z(s) = \cos(\arctan(s))v - \sin(\arctan(s))u$.

For a Euclidean convex body $L \in \mathcal{K}_0(\mathbb{R}^n)$ with $0 \in \mathrm{int}\, L$ we can express the volume of L by integrating the cone volume measure over the boundary of L, i.e.,

$$\mathrm{vol}_n(L) = \frac{1}{n} \int\limits_{\mathrm{bd}\, L} x \cdot N_x^L\, dx = \int\limits_{\mathrm{bd}\, L} \frac{x \cdot N_x^L}{\|x\|^n} \int\limits_0^{\|x\|} r^{n-1}\, dr\, dx.$$

The following proposition is a spherical version of this for $K \in \mathcal{K}_0^p(\mathbb{S}^n)$, where we fix a reference point $u \in \mathrm{int}\, K$.

Proposition 3.14. *Let $K \in \mathcal{K}_0^p(\mathbb{S}^n)$ and $u \in \mathrm{int}\, K$ such that $K \subseteq \mathbb{S}_u^+$. Then*

$$\mathrm{vol}_n(K) = \int\limits_{\mathrm{bd}\, K} \frac{-u \cdot N_w^K}{\sin(d(u,w))^n} \int\limits_0^{d(u,w)} \sin(t)^{n-1}\, dt\, dw.$$

In particular, for $K, L \in \mathcal{K}_0^p(\mathbb{S}^n)$ such that $u \in \mathrm{int}\, L$ and $L \subseteq K \subseteq \mathbb{S}_u^+$,

$$\mathrm{vol}_n(K) - \mathrm{vol}_n(L) = \int\limits_{\mathrm{bd}\, K} \frac{-u \cdot N_w^K}{\sin(d(u,w))^n} \int\limits_{d(u,w_L)}^{d(u,w)} \sin(t)^{n-1}\, dt\, dw,$$

where we set $\{w_L\} = \mathrm{bd}\, L \cap \mathrm{conv}(u,w)$ for $w \in \mathrm{bd}\, K$.

Proof. Using Lemma 3.13 and the fact that for the hypersphere $\mathbb{S}_u$ we have $\mathbb{S}_u^\circ = \{-u, u\}$, we obtain

$$\mathrm{vol}_n(K) = \int\limits_{\mathbb{S}_u} \int\limits_{\mathrm{conv}(\mathbb{S}_u^\circ, v)} \sin(d(z, \mathbb{S}_u^\circ))^{n-1} I_K(z)\, d\mathcal{H}^1(z)\, d\mathcal{H}^{n-1}(v). \qquad (*)$$

Now for any $z \in \mathrm{conv}(\mathbb{S}_u^\circ, v)$ we can write $z = \cos(t)u + \sin(t)v$ where t is determined by $t = d(\mathbb{S}_u^\circ, z) = d(u, z)$. From definition (3.5) of $\rho_u(K, v)$, we conclude

$$(*) = \int\limits_{\mathbb{S}_u} \int\limits_0^{\rho_u(K,v)} \sin(t)^{n-1}\, dt\, dv = \int\limits_{\mathrm{bd}\, K} J^{\mathrm{bd}\, K}(R_K)(w) \int\limits_0^{\rho_u(K, R_K(w))} \sin(t)^{n-1}\, dt\, dw,$$

where we used the area formula for the spherical radial map $R_K \colon \mathrm{bd}\, K \to \mathbb{S}_u$ defined by $R_K(w) = w|\mathbb{S}_u$. By (3.24) and since, for any $w \in \mathrm{bd}\, K$, we have $\rho_u(K, R_K(w)) = d(u, w)$, we are done.

The second statement of the proposition follows easily. $\square$

CHAPTER 4

Binary Operations in Spherical Convexity

Abstract. Characterizations of binary operations between convex bodies on the Euclidean unit sphere are established. The main result shows that the convex hull is essentially the only non-trivial projection covariant operation between pairs of convex bodies contained in open hemispheres. Moreover, it is proved that any continuous and projection covariant binary operation between all proper spherical convex bodies must be trivial. The results in this chapter are published in a joint work with Franz E. Schuster in [**10**].

For fixed $u \in \mathbb{S}^n$ we call a binary operation $*\colon \mathcal{K}^p(\mathbb{S}^n) \times \mathcal{K}^p(\mathbb{S}^n) \to \mathcal{K}^p(\mathbb{S}^n)$ *u-projection covariant* if for all k-spheres S, $0 \leq k \leq n-1$, with $u \in S$ and for all K, $L \in \mathcal{K}^p_u(\mathbb{S}^n)$, we have

$$(K|S) * (L|S) = (K * L)|S.$$

We call $*$ *projection covariant* if $*$ is u-projection covariant for all $u \in \mathbb{S}^n$.

The main objective of this chapter is to characterize projection covariant operations between spherical convex bodies. Our first result shows that such operations between *all* proper convex bodies in $\mathbb{S}^n$ are of a very simple form.

Theorem 4.1. *An operation* $*\colon \mathcal{K}^p(\mathbb{S}^n)\times\mathcal{K}^p(\mathbb{S}^n)\to\mathcal{K}^p(\mathbb{S}^n)$ *between proper convex bodies is projection covariant and continuous with respect to the Hausdorff metric if and only if either* $K*L=K$, *or* $K*L=-K$, *or* $K*L=L$, *or* $K*L=-L$ *for all* $K,L\in\mathcal{K}^p(\mathbb{S}^n)$.

We call the binary operations from Theorem 4.1 *trivial*. The following example shows, that the continuity assumption in Theorem 4.1 cannot be omitted.

Example 4.2. Consider the set $\mathcal{C}\subseteq\mathcal{K}^p(\mathbb{S}^n)\times\mathcal{K}^p(\mathbb{S}^n)$ of all pairs (K,L) such that both K and L are contained in some open hemisphere, that is,

$$\mathcal{C}=\bigcup_{u\in\mathbb{S}^n}(\mathcal{K}^p_u(\mathbb{S}^n)\times\mathcal{K}^p_u(\mathbb{S}^n)).$$

Define an operation $*\colon \mathcal{K}^p(\mathbb{S}^n)\times\mathcal{K}^p(\mathbb{S}^n)\to\mathcal{K}^p(\mathbb{S}^n)$ by

$$K*L=\begin{cases}K & \text{if } (K,L)\in\mathcal{C},\\ L & \text{if } (K,L)\notin\mathcal{C}.\end{cases}$$

Clearly, $*$ is not continuous but by our definition it is projection covariant.

The proof of Theorem 4.1 relies on ideas of Gardner, Hug, and Weil [**25**]. The critical tool to transfer their techniques to the sphere is the gnomonic projection (see Section 3.1) which establishes the following correspondence between projection covariant operations on $\mathcal{K}(\mathbb{R}^n)$, the space of compact, convex sets in $\mathbb{R}^n$, and u-projection covariant operations on $\mathcal{K}^p_u(\mathbb{S}^n)$:

Theorem 4.3. *For every fixed* $u\in\mathbb{S}^n$, *there is a one-to-one correspondence between* u*-projection covariant operations* $*\colon \mathcal{K}^p_u(\mathbb{S}^n)\times\mathcal{K}^p_u(\mathbb{S}^n)\to\mathcal{K}^p_u(\mathbb{S}^n)$ *and projection covariant operations* $\overline{*}\colon \mathcal{K}(\mathbb{R}^n)\times\mathcal{K}(\mathbb{R}^n)\to\mathcal{K}(\mathbb{R}^n)$. *Moreover, every such* u*-projection covariant operation* $*$ *is continuous in the Hausdorff metric.*

Note that by Theorem 4.3 every projection covariant operation $*$ on $\mathcal{C}$ is also automatically continuous.

Finally, as our main result, we prove that the only *non-trivial* projection covariant operation on the set $\mathcal{C}$ is essentially the spherical convex hull.

Theorem 4.4. *An operation* $*\colon \mathcal{C}\to\mathcal{K}^p(\mathbb{S}^n)$ *is non-trivial and projection covariant if and only if either* $K*L=\operatorname{conv}(K\cup L)$ *or* $K*L=-\operatorname{conv}(K\cup L)$ *for all* $(K,L)\in\mathcal{C}$.

We prove Theorem 4.3 in the next section. Sections 4.2 and 4.3 contain the proofs of Theorems 4.1 and 4.4. Motivated by investigations of Gardner, Hug, and Weil [**25**] in $\mathbb{R}^n$, we discuss *section* covariant operations between spherical *star sets* in the concluding section of this chapter.

4.1. Proof of Theorem 4.3

Using Proposition 3.8 and other basic properties of the gnomonic projection, we can prove the following refinement of Theorem 4.3.

Theorem 4.5. *For every fixed $u \in \mathbb{S}^n$, the gnomonic projection g_u induces a one-to-one correspondence between operations $*\colon \mathcal{K}^p_u(\mathbb{S}^n) \times \mathcal{K}^p_u(\mathbb{S}^n) \to \mathcal{K}^p_u(\mathbb{S}^n)$ which are u-projection covariant and operations $\overline{*}\colon \mathcal{K}(\mathbb{R}^n_u) \times \mathcal{K}(\mathbb{R}^n_u) \to \mathcal{K}(\mathbb{R}^n_u)$ which are projection covariant. Moreover, every such u-projection covariant operation $*$ is continuous.*

Proof. First assume that $*$ is u-projection covariant and define an operation $\overline{*}\colon \mathcal{K}(\mathbb{R}^n_u) \times \mathcal{K}(\mathbb{R}^n_u) \to \mathcal{K}(\mathbb{R}^n_u)$ by

$$K \,\overline{*}\, L = g_u(g_u^{-1}(K) * g_u^{-1}(L))$$

for $K, L \in \mathcal{K}(\mathbb{R}^n_u)$. Since for every k-sphere S containing u, there exists a linear subspace V in $\mathbb{R}^n_u$ such that

$$g_u^{-1}(K|V) = g_u^{-1}(K)|S$$

for all $K \in \mathcal{K}(\mathbb{R}^n_u)$, we obtain

$$\begin{aligned}(K|V) \,\overline{*}\, (L|V) &= g_u(g_u^{-1}(K|V) * g_u^{-1}(L|V)) = g_u((g_u^{-1}(K)|S) * (g_u^{-1}(L)|S)) \\ &= g_u((g_u^{-1}(K) * g_u^{-1}(L))|S) = g_u(g_u^{-1}(K) * g_u^{-1}(L))|V \\ &= (K \,\overline{*}\, L)|V.\end{aligned}$$

for all $K, L \in \mathcal{K}(\mathbb{R}^n_u)$. Thus, $\overline{*}$ is projection covariant.

Now, let $\overline{*}\colon \mathcal{K}(\mathbb{R}^n_u) \times \mathcal{K}(\mathbb{R}^n_u) \to \mathcal{K}(\mathbb{R}^n_u)$ be projection covariant and define $*\colon \mathcal{K}^p_u(\mathbb{S}^n) \times \mathcal{K}^p_u(\mathbb{S}^n) \to \mathcal{K}^p_u(\mathbb{S}^n)$ by

$$K * L = g_u^{-1}(g_u(K) \,\overline{*}\, g_u(L))$$

for $K, L \in \mathcal{K}^p_u(\mathbb{S}^n)$. Using a similar argument as before, it is easy to show that $*$ is u-projection covariant.

The continuity of an operation $*\colon \mathcal{K}^p_u(\mathbb{S}^n) \times \mathcal{K}^p_u(\mathbb{S}^n) \to \mathcal{K}^p_u(\mathbb{S}^n)$ which is u-projection covariant is now a direct consequence of Theorem 2.4 and Proposition 3.8. □

Recall that the set $\mathcal{C} \subseteq \mathcal{K}^p(\mathbb{S}^n) \times \mathcal{K}^p(\mathbb{S}^n)$ is defined by

$$\mathcal{C} = \bigcup_{u \in \mathbb{S}^n} (\mathcal{K}^p_u(\mathbb{S}^n) \times \mathcal{K}^p_u(\mathbb{S}^n)).$$

By Theorem 4.5, the restriction of an operation $*\colon \mathcal{C} \to \mathcal{K}^p(\mathbb{S}^n)$ which is projection covariant to convex bodies contained in a fixed open hemisphere is continuous. Therefore, we obtain:

Corollary 4.6. *Every projection covariant operation $*\colon \mathcal{C} \to \mathcal{K}^p(\mathbb{S}^n)$ is continuous.*

4.2. Auxiliary results

We continue in this section with our preparations for the proofs of Theorems 4.1 and 4.4. We prove three auxiliary results which will be used at different stages in Section 4.3. We begin by establishing first constraints on projection covariant operations $*$ on $\mathcal{C}$.

Lemma 4.7. *If $*\colon \mathcal{C} \to \mathcal{K}^p(\mathbb{S}^n)$ is projection covariant, then either*

$$K * L \subseteq \operatorname{conv}(K \cup L) \tag{4.1}$$

for all $(K, L) \in \mathcal{C}$ or

$$K * L \subseteq -\operatorname{conv}(K \cup L) \tag{4.2}$$

for all $(K, L) \in \mathcal{C}$.

Proof. For $u \in \mathbb{S}^n$, the 0-sphere $\mathbb{S}^\circ_u$ is just $\{-u, u\}$. By the projection covariance of $*$, we have

$$(\{u\} * \{u\})|\mathbb{S}^\circ_u = (\{u\}|\mathbb{S}^\circ_u) * (\{u\}|\mathbb{S}^\circ_u) = \{u\} * \{u\}.$$

Thus, $\{u\} * \{u\} \subseteq \{-u, u\}$. However, since $\{u\} * \{u\} \in \mathcal{K}^p(\mathbb{S}^n)$, we must have either $\{u\} * \{u\} = \{u\}$ or $\{u\} * \{u\} = \{-u\}$. Let

$$P = \{u \in \mathbb{S}^n : \{u\} * \{u\} = \{u\}\} \quad \text{and} \quad N = \{u \in \mathbb{S}^n : \{u\} * \{u\} = \{-u\}\}.$$

Clearly, $P \cap N = \emptyset$ and $P \cup N = \mathbb{S}^n$.

Since, by Corollary 4.6, $*$ is continuous, we obtain for every sequence $u_i \in P$ with limit $u \in \mathbb{S}^n$,

$$\{u\} * \{u\} = \{\lim u_i\} * \{\lim u_i\} = \lim(\{u_i\} * \{u_i\}) = \lim\{u_i\} = \{u\}.$$

Thus, $u \in P$ which shows that P is closed. In the same way, we see that N is closed. Consequently, we have either $P = \mathbb{S}^n$ or $N = \mathbb{S}^n$.

First assume that $P = \mathbb{S}^n$ and let $(K, L) \in \mathcal{C}$. Then there exists $u \in \mathbb{S}^n$ such that $K, L \subseteq \mathbb{S}_u^+$ or, equivalently, $\operatorname{conv}(K \cup L) \subseteq \mathbb{S}_u^+$. By the projection covariance of $*$, we have

$$(K * L)|\mathbb{S}_u^\circ = (K|\mathbb{S}_u^\circ) * (L|\mathbb{S}_u^\circ) = \{u\} * \{u\} = \{u\}.$$

Thus, $K * L \subseteq \mathbb{S}_u^+$ and we conclude that

$$K * L \subseteq \bigcap \{\mathbb{S}_u^+ : u \in \mathbb{S}^n \text{ such that } \operatorname{conv}(K \cup L) \subseteq \mathbb{S}_u^+\} = \operatorname{conv}(K \cup L)$$

for all $(K, L) \in \mathcal{C}$.

Conversely, if $N = \mathbb{S}^n$, then we obtain $(K * L)|\mathbb{S}_u^\circ = \{-u\}$ and, therefore, $K * L \subseteq \mathbb{S}_u^-$ whenever $\operatorname{conv}(K \cup L) \subseteq \mathbb{S}_u^+$. This yields

$$K * L \subseteq \bigcap \{\mathbb{S}_u^- : u \in \mathbb{S}^n \text{ such that } \operatorname{conv}(K \cup L) \subseteq \mathbb{S}_u^+\} = -\operatorname{conv}(K \cup L)$$

for all $(K, L) \in \mathcal{C}$. $\square$

Our next lemma concerns spherical support functions of a spherical segment contained in an open hemisphere.

Lemma 4.8. *For $u \in \mathbb{S}^n$, $v \in \mathbb{S}_u^+$, $w \in \mathbb{S}_u \cap \mathbb{S}_v$, and $-\frac{\pi}{2} < \alpha \leq \beta < \frac{\pi}{2}$ let*

$$I_u^w(\alpha, \beta) = \{\cos(\lambda)u + \sin(\lambda)w : \lambda \in [\alpha, \beta]\}.$$

Then,

$$\tan h_v(I_u^w(\alpha, \beta), w) = \frac{\tan \beta}{u \cdot v} \quad \text{and} \quad \tan h_v(I_u^w(\alpha, \beta), -w) = -\frac{\tan \alpha}{u \cdot v}.$$

Proof. First note that by our definition of the spherical support function

$$h_u(I_u^w(\alpha, \beta), w) = \beta \quad \text{and} \quad h_u(I_u^w(\alpha, \beta), -w) = -\alpha.$$

Let

$$A = g_v(I_u^w(\alpha,\alpha)) = \frac{\cos(\alpha)u + \sin(\alpha)w}{(u\cdot v)\cos\alpha} - v,$$
$$B = g_v(I_u^w(\beta,\beta)) = \frac{\cos(\beta)u + \sin(\beta)w}{(u\cdot v)\cos\beta} - v.$$

By Lemma 3.5 (b), $g_v(I_u^w(\alpha,\beta))$ is the line segment in $\mathbb{R}_v^n$ in direction w with endpoints A and B. Thus, by Lemma 3.6 and the definition of (Euclidean) support functions, we obtain

$$\tan h_v(I_u^w(\alpha,\beta),w) = h(g_v(I_u^w(\alpha,\beta)),w) = w\cdot B = \frac{\tan\beta}{u\cdot v},$$
$$\tan h_v(I_u^w(\alpha,\beta),-w) = h(g_v(I_u^w(\alpha,\beta)),-w) = -w\cdot A = -\frac{\tan\alpha}{u\cdot v}.$$

□

In view of Lemma 4.7, Theorem 4.5, and Theorem 2.4, the following result will be useful in the proof of Theorem 4.4.

Lemma 4.9. *Let $M \subseteq \mathbb{R}^4$ be closed and convex. If for all $a,b,c,d \in \mathbb{R}$ such that $-a \le b$ and $-c \le d$,*

$$h_M(a,b,c,d) \le \max\{b,d\}, \tag{4.3}$$

then

$$M \subseteq \{(\lambda_2, \lambda_1+\lambda_2, \lambda_3, 1-\lambda_1+\lambda_3) \in \mathbb{R}^4 : \lambda_1 \in [0,1], \lambda_2 \le 0, \lambda_3 \le 0\}.$$

Proof. For $z = (-1,1,-1,1)$, we obtain from (4.3) that

$$h(M,z) \le 1 \qquad \text{and} \qquad h(M,-z) \le -1.$$

Since $-h(M,-z) \le h(M,z)$, we conclude that $-h(M,-z) = h(M,z) = 1$ or, equivalently,

$$M \subseteq \{x \in \mathbb{R}^4 : -x_1 + x_2 - x_3 + x_4 = 1\}. \tag{4.4}$$

By (4.3), we also have $h_M(1,0,0,0) \le 0$ and $h_M(0,0,1,0) \le 0$. Thus,

$$M \subseteq \{x \in \mathbb{R}^4 : x_1 \le 0, x_3 \le 0\}. \tag{4.5}$$

Finally, we deduce from (4.3) that

$$h_M(-1,1,0,0) \leq 1 \qquad \text{and} \qquad h_M(1,-1,0,0) \leq 0,$$

as well as

$$h_M(0,0,-1,1) \leq 1 \qquad \text{and} \qquad h_M(0,0,1,-1) \leq 0.$$

Consequently,

$$M \subseteq \{x \in \mathbb{R}^4 : 0 \leq x_2 - x_1 \leq 1 \text{ and } 0 \leq x_4 - x_3 \leq 1\}. \tag{4.6}$$

Combining (4.4), (4.5), and (4.6), completes the proof. □

The importance for us of the set

$$E := \{(\lambda_2, \lambda_1 + \lambda_2, \lambda_3, 1 - \lambda_1 + \lambda_3) \in \mathbb{R}^4 : \lambda_1 \in [0,1], \lambda_2 \leq 0, \lambda_3 \leq 0\}$$

follows from

$$h_E(h_{-K}(x), h_K(x), h_{-L}(x), h_L(x)) = h_{\mathrm{conv}(K \cup L)}(x).$$

4.3. Proofs of Theorem 4.1 and 4.4

After these preparations, we are now in a position to first proof Theorem 4.4 and then complete the proof of Theorem 4.1. In order to enhance the readability of several formulas below, we write $\tan(x_1, \ldots, x_k)$ for the vector $(\tan x_1, \ldots, \tan x_k)$ and $\arctan(x_1, \ldots, x_k)$ is defined similarly.

Theorem 4.10. *An operation $*\colon \mathcal{C} \to \mathcal{K}^p(\mathbb{S}^n)$ is projection covariant if and only if it is either $K * L = \mathrm{conv}(K \cup L)$ or $K * L = -\mathrm{conv}(K \cup L)$ for all $(K, L) \in \mathcal{C}$ or it is trivial, that is, $K * L = K$, or $K * L = -K$, or $K * L = L$, or $K * L = -L$ for all $(K, L) \in \mathcal{C}$.*

Proof. By Lemma 4.7, we may assume that

$$K * L \subseteq \mathrm{conv}(K \cup L) \tag{4.7}$$

holds for all $(K, L) \in \mathcal{C}$ (otherwise, replace $*$ by $*^-\colon \mathcal{C} \to \mathcal{K}^p(\mathbb{S}^n)$ defined by $K *^- L = -(K * L)$). In particular, for every $u \in \mathbb{S}^n$, the range of the restriction of $*$ to $\mathcal{K}^p_u(\mathbb{S}^n) \times \mathcal{K}^p_u(\mathbb{S}^n)$ lies in $\mathcal{K}^p_u(\mathbb{S}^n)$.

In the proof of Theorem 4.5 we have seen that, for every $u \in \mathbb{S}^n$, there exists a (unique) projection covariant operation $\overline{*}_u \colon \mathcal{K}(\mathbb{R}^n_u) \times \mathcal{K}(\mathbb{R}^n_u) \to \mathcal{K}(\mathbb{R}^n_u)$ such that

$$\overline{K} \;\overline{*}_u\, \overline{L} = g_u(g_u^{-1}(\overline{K}) * g_u^{-1}(\overline{L}))$$

for all $\overline{K}$, $\overline{L} \in \mathcal{K}(\mathbb{R}^n_u)$. Thus, by Theorem 2.4, there exists a nonempty closed convex set $M_u \subseteq \mathbb{R}^4$ such that

$$h_{\overline{K}\;\overline{*}_u \overline{L}}(v) = h_{M_u}(h_{\overline{K}}(-v), h_{\overline{K}}(v), h_{\overline{L}}(-v), h_{\overline{L}}(v))$$

for all $v \in \mathbb{S}_u$. Therefore, Lemma 3.6 yields

$$\begin{aligned} \tan h_u(K * L, v) &= h_{g_u(K*L)}(v) = h_{g_u(K)\,\overline{*}_u\, g_u(L)}(v) && (4.8)\\ &= h_{M_u}(h_{g_u(K)}(-v), h_{g_u(K)}(v), h_{g_u(L)}(-v), h_{g_u(L)}(v))\\ &= h_{M_u}(\tan(h_u(K,-v), h_u(K,v), h_u(L,-v), h_u(L,v))) && (4.9) \end{aligned}$$

for all $K, L \in \mathcal{K}^p_u(\mathbb{S}^n)$. Thus, since $-h_{\overline{K}}(-v) \leq h_{\overline{K}}(v)$ for every $\overline{K} \in \mathcal{K}(\mathbb{R}^n_u)$ and every $v \in \mathbb{S}_u$, the restriction of $*$ to $\mathcal{K}^p_u(\mathbb{S}^n) \times \mathcal{K}^p_u(\mathbb{S}^n)$ is completely determined by the values $h_{M_u}(a, b, c, d)$, where $-a \leq b$ and $-c \leq d$. Next, we want to show that for such $a, b, c, d \in \mathbb{R}$,

$$h_{M_u}(a, b, c, d) = h_{M_v}(a, b, c, d) \qquad (4.10)$$

whenever $v \in \mathbb{S}^+_u$. To this end, let $-\frac{\pi}{2} < \alpha \leq \beta < \frac{\pi}{2}$ and $-\frac{\pi}{2} < \varphi \leq \psi < \frac{\pi}{2}$. For every $u \in \mathbb{S}^n$ and $w \in \mathbb{S}_u$, the u-projection covariance of $*$ implies that there exist σ, τ such that $-\frac{\pi}{2} < \sigma \leq \tau < \frac{\pi}{2}$ and

$$I^w_u(\alpha, \beta) * I^w_u(\varphi, \psi) = I^w_u(\sigma, \tau), \qquad (4.11)$$

where we have used the notation from Lemma 4.8 for spherical segments I^w_u. Since, for $-\frac{\pi}{2} < \xi \leq \zeta < \frac{\pi}{2}$, we have $h_u(I^w_u(\xi, \zeta), -w) = -\xi$ and $h_u(I^w_u(\xi, \zeta), w) = \zeta$, we obtain on the one hand from (4.11), (4.8), and (4.9),

$$\begin{aligned} \tan \tau &= \tan h_u(I^w_u(\sigma, \tau), w) = \tan h_u(I^w_u(\alpha, \beta) * I^w_u(\varphi, \psi), w)\\ &= h_{M_u}(\tan(-\alpha, \beta, -\varphi, \psi)). \end{aligned}$$

For $v \in \mathbb{S}^+_u$ and $w \in \mathbb{S}_u \cap \mathbb{S}_v$, we obtain from Lemma 4.8 and again (4.11), (4.8), and (4.9),

$$\begin{aligned} \tan \tau &= (u \cdot v) \tan h_v(I^w_u(\sigma, \tau), w) = (u \cdot v) \tan h_v(I^w_u(\alpha, \beta) * I^w_u(\varphi, \psi), w)\\ &= (u \cdot v) h_{M_v}\left(\frac{\tan(-\alpha, \beta, -\varphi, \psi)}{u \cdot v}\right) = h_{M_v}(\tan(-\alpha, \beta, -\varphi, \psi)) \end{aligned}$$

which proves (4.10). Since $u \in \mathbb{S}^n$, $v \in \mathbb{S}_u^+$, and $\alpha, \beta, \varphi, \psi$ were arbitrary, we conclude from (4.8), (4.9), and (4.10) that there exists a nonempty closed convex set $M \subseteq \mathbb{R}^4$, *independent* of $u \in \mathbb{S}^n$, such that

$$\tan h_u(K * L, v) = h_M(\tan(h_u(K, -v), h_u(K, v), h_u(L, -v), h_u(L, v))) \tag{4.12}$$

for all $K, L \in \mathcal{K}_u^p(\mathbb{S}^n)$ and $v \in \mathbb{S}_u$.

To complete the proof, we have to show that for $-a \leq b$ and $-c \leq d$, the support function h_M satisfies one of the following three conditions:

(i) $h_M(a, b, c, d) = b$, that is, $K * L = K$ for $(K, L) \in \mathcal{C}$;

(ii) $h_M(a, b, c, d) = d$, that is, $K * L = L$ for $(K, L) \in \mathcal{C}$;

(iii) $h_M(a, b, c, d) = \max\{b, d\}$, that is, $K * L = \operatorname{conv}(K \cup L)$ for $(K, L) \in \mathcal{C}$.

From (4.7) and (4.12), we deduce that

$$h_M(a, b, c, d) \leq \max\{b, d\} \tag{4.13}$$

whenever $-a \leq b$ and $-c \leq d$. Moreover, since

$$-h_u(K * L, -v) \leq h_u(K * L, v)$$

for all $K, L \in \mathcal{K}_u^p(\mathbb{S}^n)$ and $v \in \mathbb{S}_u$, we deduce from (4.12) that

$$-h_M(b, a, d, c) \leq h_M(a, b, c, d). \tag{4.14}$$

Next, we want to show that for all $-\frac{\pi}{2} < \alpha \leq \beta < \frac{\pi}{2}$, $-\frac{\pi}{2} < \varphi \leq \psi < \frac{\pi}{2}$, and $-\frac{\pi}{2} + \max\{\beta, \psi\} < \eta < \frac{\pi}{2} + \min\{\alpha, \varphi\}$, we have

$$\arctan h_M(\tan \Lambda) = \arctan h_M(\tan(\Lambda + \Theta)) + \eta, \tag{4.15}$$

where $\Lambda = (-\alpha, \beta, -\varphi, \psi)$ and $\Theta = (\eta, -\eta, \eta, -\eta)$. In order to prove (4.15), let $u \in \mathbb{S}^n$, $v \in \mathbb{S}_u$ and define

$$u' = \cos(\eta)u - \sin(\eta)v \qquad \text{and} \qquad v' = \cos(\eta)v + \sin(\eta)u.$$

Note that u' and v' are rotations of u and v in the plane $\operatorname{span}\{u, v\}$ by an angle $-\eta$. Therefore, for every $\lambda \in [0, 2\pi)$,

$$\cos(\lambda)u' + \sin(\lambda)v' = \cos(\lambda - \eta)u + \sin(\lambda - \eta)v.$$

Hence,

$$I_{u'}^{v'}(\alpha,\beta) = I_u^v(\alpha-\eta,\beta-\eta) \subseteq \mathbb{S}_u^+. \tag{4.16}$$

Now, let

$$\begin{aligned}\sigma &= -h_u(I_u^v(\alpha-\eta,\beta-\eta) * I_u^v(\varphi-\eta,\psi-\eta), -v),\\ \tau &= h_u(I_u^v(\alpha-\eta,\beta-\eta) * I_u^v(\varphi-\eta,\psi-\eta), v),\end{aligned}$$

and

$$\begin{aligned}\sigma' &= -h_{u'}(I_{u'}^{v'}(\alpha,\beta) * I_{u'}^{v'}(\varphi,\psi), -v'),\\ \tau' &= h_{u'}(I_{u'}^{v'}(\alpha,\beta) * I_{u'}^{v'}(\varphi,\psi), v').\end{aligned}$$

By the u-projection covariance and the u'-projection covariance of $*$ and (4.16), we obtain

$$\begin{aligned}I_{u'}^{v'}(\sigma',\tau') &= I_{u'}^{v'}(\alpha,\beta) * I_{u'}^{v'}(\varphi,\psi) = I_u^v(\alpha-\eta,\beta-\eta) * I_u^v(\varphi-\eta,\psi-\eta)\\ &= I_u^v(\sigma,\tau) = I_{u'}^{v'}(\sigma+\eta,\tau+\eta).\end{aligned}$$

Thus, $\tau' = \tau+\eta$. Using (4.12) and the definitions of τ and τ', we obtain (4.15).

From applications of (4.15) with $\Lambda = \pm(-\alpha,\alpha,\alpha,-\alpha)$ and $\eta = \pm\alpha$, where $\alpha \in [0,\frac{\pi}{4})$, we obtain

$$\arctan(h_M(-1,1,1,-1)\tan\alpha) = \arctan(h_M(0,0,1,-1)\tan(2\alpha)) + \alpha, \tag{4.17}$$
$$\arctan(h_M(-1,1,1,-1)\tan\alpha) = \arctan(h_M(-1,1,0,0)\tan(2\alpha)) - \alpha, \tag{4.18}$$

and

$$\arctan(h_M(1,-1,-1,1)\tan\alpha) = \arctan(h_M(0,0,-1,1)\tan(2\alpha)) - \alpha, \tag{4.19}$$
$$\arctan(h_M(1,-1,-1,1)\tan\alpha) = \arctan(h_M(1,-1,0,0)\tan(2\alpha)) + \alpha. \tag{4.20}$$

On the one hand, using (4.17) and (4.18), it is not difficult to show that either

$$h_M(-1,1,1,-1) = 1, \quad h_M(0,0,1,-1) = 0, \quad h_M(-1,1,0,0) = 1, \tag{4.21}$$

or

$$h_M(-1,1,1,-1) = -1, \quad h_M(0,0,1,-1) = -1, \quad h_M(-1,1,0,0) = 0. \tag{4.22}$$

On the other hand, by (4.19) and (4.20), we have either

$$h_M(1,-1,-1,1) = 1, \quad h_M(0,0,-1,1) = 1, \quad h_M(1,-1,0,0) = 0, \quad (4.23)$$

or

$$h_M(1,-1,-1,1) = -1, \quad h_M(0,0,-1,1) = 0, \quad h_M(1,-1,0,0) = -1. \quad (4.24)$$

Note that, since $-h_M(1,-1,-1,1) \leq h_M(-1,1,1,-1)$, (4.22) and (4.24) cannot both be satisfied. Also recall that by Lemma 4.9, we have

$$M \subseteq E = \{(\lambda_2, \lambda_1 + \lambda_2, \lambda_3, 1 - \lambda_1 + \lambda_3) : \lambda_1 \in [0,1], \lambda_2, \lambda_3 \leq 0\}.$$

and let

$$\begin{aligned} E_0 &= \{(\lambda_2, \lambda_2, \lambda_3, 1 + \lambda_3) : \lambda_2 \leq 0, \lambda_3 \leq 0\}, \\ E_1 &= \{(\lambda_2, 1 + \lambda_2, \lambda_3, \lambda_3) : \lambda_2 \leq 0, \lambda_3 \leq 0\}. \end{aligned}$$

If (4.21) holds, then $h_M(-1,1,0,0) = 1$ and, since $M \subseteq E$, we have

$$1 = \max\{\lambda_1 \in [0,1] : (\lambda_2, \lambda_1 + \lambda_2, \lambda_3, 1 - \lambda_1 + \lambda_3) \in M\}.$$

Thus, there are λ_2, $\lambda_3 \leq 0$, such that $(\lambda_2, 1+\lambda_2, \lambda_3, \lambda_3) \in M$ or, equivalently, $M \cap E_1$ is nonempty. Similarly, it follows from (4.23) that $M \cap E_0$ is nonempty.

If (4.22) holds, we have $h_M(-1,1,0,0) = 0$ and we deduce that

$$0 = \max\{\lambda_1 \in [0,1] : (\lambda_2, \lambda_1 + \lambda_2, \lambda_3, 1 - \lambda_1 + \lambda_3) \in M\}$$

which yields $M \subseteq E_0$. Analogously, (4.24) implies $M \subseteq E_1$.

Next, an application of (4.15) with $\Lambda = (0, \alpha, 0, \alpha)$ and $\eta = \alpha$, where again $\alpha \in [0, \frac{\pi}{4})$, yields

$$\arctan(h_M(0,1,0,1) \tan\alpha) = \arctan(h_M(1,0,1,0) \tan\alpha) + \alpha$$

Clearly, this is possible if and only if either

$$h_M(0,1,0,1) = 1, \quad h_M(1,0,1,0) = 0, \quad (4.25)$$

or

$$h_M(0,1,0,1) = 0, \quad h_M(1,0,1,0) = -1. \quad (4.26)$$

However, (4.26) contradicts (4.14) and is therefore not possible.

From (4.25) and the fact that $M \subseteq E$, we infer

$$0 = \max\{\lambda_2 + \lambda_3 : \lambda_2, \lambda_3 \leq 0 \text{ and } (\lambda_2, \lambda_1 + \lambda_2, \lambda_3, 1 - \lambda_1 + \lambda_3) \in M\}$$

which implies

$$M \cap \{(0, \lambda_1, 0, 1 - \lambda_1) : \lambda_1 \in [0, 1]\} \neq \emptyset. \tag{4.27}$$

For the final part of the proof, we distinguish three cases:

(i) (4.21) and (4.24) hold, in particular, $M \subseteq E_1$;

(ii) (4.22) and (4.23) hold, in particular, $M \subseteq E_0$;

(iii) (4.21) and (4.23) hold.

In case (i), $M \subseteq E_1$ and (4.27) imply that $e_2 \in M$. Using (4.13), we conclude that $h_M(a, b, c, d) = b$, that is, $K * L = K$ for $(K, L) \in \mathcal{C}$.

Similarly, in case (ii), $M \subseteq E_0$ and (4.27) imply that $e_4 \in M$. Using again (4.13), we obtain $h_M(a, b, c, d) = d$, that is, $K * L = L$ for $(K, L) \in \mathcal{C}$.

It remains to show that in case (iii), we have e_2, $e_4 \in M$ which implies that $h_M(a, b, c, d) = \max\{b, d\}$ or $K * L = \operatorname{conv}(K \cup L)$ for $(K, L) \in \mathcal{C}$ by (4.13). To this end, we apply again (4.15) with $\Lambda = (0, \alpha, 0, 0)$ and $\eta = \alpha$, where $\alpha \in [0, \frac{\pi}{4})$ to obtain

$$\arctan(h_M(0, 1, 0, 0) \tan \alpha) = \arctan(h_M(1, 0, 1, -1) \tan \alpha) + \alpha$$

This is possible if and only if either

$$h_M(0, 1, 0, 0) = 1, \quad h_M(1, 0, 1, -1) = 0, \tag{4.28}$$

or

$$h_M(0, 1, 0, 0) = 0, \quad h_M(1, 0, 1, -1) = -1. \tag{4.29}$$

Assume that (4.29) holds. Then, by (4.13), (4.14), and the subadditivity of h_M, we obtain

$$-1 \leq h_M(1, 1, 1, -1) \leq h_M(1, 0, 1, -1) + h_M(0, 1, 0, 0) = -1 \tag{4.30}$$

Hence, $h_M(1, 1, 1, -1) = -1$.

Now, consider the convex bodies $\overline{K} = [-e_2, e_2]$ and $\overline{L} = \{e_1\}$ in $\mathbb{R}^n$. Then, $h_{\overline{K}}(x) = |e_2 \cdot x|$ and $h_{\overline{L}}(x) = e_1 \cdot x$ for $x \in \mathbb{R}^n$, and we obtain from (4.21) and $h_M(1,1,1,-1) = -1$,

$$\begin{aligned} h_M(h_{\overline{K}}(e_1), h_{\overline{K}}(-e_1), h_{\overline{L}}(e_1), h_{\overline{L}}(-e_1)) &= h_M(0,0,1,-1) = 0, \\ h_M(h_{\overline{K}}(e_1+e_2), h_{\overline{K}}(-e_1-e_2), h_{\overline{L}}(e_1+e_2), h_{\overline{L}}(-e_1-e_2)) &= h_M(1,1,1,-1) = -1, \\ h_M(h_{\overline{K}}(e_1-e_2), h_{\overline{K}}(e_2-e_1), h_{\overline{L}}(e_1-e_2), h_{\overline{L}}(e_2-e_1)) &= h_M(1,1,1,-1) = -1. \end{aligned}$$

Since $h_M(h_{-\overline{K}}, h_{\overline{K}}, h_{-\overline{L}}, h_{\overline{L}})$ defines a support function of a convex body $\overline{Z}$ in $\mathbb{R}^n$, we infer

$$0 = h_{\overline{Z}}(-2e_1) \geq h_{\overline{Z}}(-e_1 - e_2) + h_{\overline{Z}}(-e_1 + e_2) = -2$$

which contradicts the subadditivity of $h_{\overline{Z}}$. Thus, (4.29) cannot hold.

Another application of (4.15) with $\Lambda = (0, \alpha, \alpha, 0)$ and $\eta = -\alpha$, where $\alpha \in [0, \frac{\pi}{4})$, yields

$$\arctan(h_M(0,1,1,0)\tan\alpha) = \arctan(h_M(\tan(-\alpha, 2\alpha, 0, \alpha))) - \alpha.$$

Consequently,

$$h_M\left(-\frac{\tan\alpha}{\tan(2\alpha)}, 1, 0, \frac{\tan\alpha}{\tan(2\alpha)}\right) = \frac{\tan(\arctan(h_M(0,1,1,0)\tan\alpha) + \alpha)}{\tan(2\alpha)}.$$

By letting $\alpha \to \frac{\pi}{4}$ and using (4.28), we deduce that $h_M(0,1,1,0) = 1$. Since $M \subseteq E$, this yields

$$1 = \max\{\lambda_1 + \lambda_2 + \lambda_3 : (\lambda_2, \lambda_1 + \lambda_2, \lambda_3, (1 - \lambda_1) + \lambda_3) \in M\}$$

which, in turn, implies that $e_2 \in M$.

The proof that $e_4 \in M$ is now very similar. We first use (4.15) with $\Lambda = (0,0,0,\alpha)$ and $\eta = \alpha$ to deduce that

$$h_M(0,0,0,1) = 1, \quad h_M(1,-1,1,0) = 0.$$

Using this and another application of (4.15) with $\Lambda = (\alpha, 0, 0, \alpha)$ and $\eta = -\alpha$, finally leads to $h_M(1,0,0,1) = 1$. From this and $M \subseteq E$, follows $e_4 \in M$ which completes the proof. □

Using Theorem 4.10, we can now also complete the proof of Theorem 4.1:

Theorem 4.11. *An operation* $*\colon \mathcal{K}^p(\mathbb{S}^n)\times\mathcal{K}^p(\mathbb{S}^n)\to\mathcal{K}^p(\mathbb{S}^n)$ *is projection covariant and continuous if and only if either* $K*L=K$*, or* $K*L=-K$*, or* $K*L=L$*, or* $K*L=-L$ *for all* $K,L\in\mathcal{K}^p(\mathbb{S}^n)$*.*

Proof. By Theorem 4.10, it is sufficient to prove that the convex hull does not admit a continuous extension from $\mathcal{C}$ to a map from $\mathcal{K}^p(\mathbb{S}^n)\times\mathcal{K}^p(\mathbb{S}^n)$ to $\mathcal{K}^p(\mathbb{S}^n)$. In order to show this, let $u\in\mathbb{S}^n$, $v\in\mathbb{S}_u$, and consider the spherical segments $K=I_u^v(-\frac{\pi}{2},0)$ and $L_\varepsilon=I_u^v(0,\frac{\pi}{2}-\varepsilon)$, where $\varepsilon>0$. Then $(K,L_\varepsilon)\in\mathcal{C}$ converges in the Hausdorff metric to $(K,L_0)\in\mathcal{K}^p(\mathbb{S}^n)\times\mathcal{K}^p(\mathbb{S}^n)$ as $\varepsilon\to 0^+$. However,

$$\lim_{\varepsilon\to 0^+}\operatorname{conv}(K\cup L_\varepsilon)=\lim_{\varepsilon\to 0^+} I_u^v\Big(-\frac{\pi}{2},\frac{\pi}{2}-\varepsilon\Big)=I_u^v\Big(-\frac{\pi}{2},\frac{\pi}{2}\Big)\notin\mathcal{K}^p(\mathbb{S}^n).$$ □

We remark, that the convex hull on $\mathcal{K}^p(\mathbb{S}^n)\times\mathcal{K}^p(\mathbb{S}^n)$ in general does not map to $\mathcal{K}(\mathbb{S}^n)$, see Example 3.2. Furthermore, it is not difficult to see that the closure of the convex hull $\operatorname{cls}\operatorname{conv}\colon \mathcal{K}^p(\mathbb{S}^n)\times\mathcal{K}^p(\mathbb{S}^n)\to\mathcal{K}(\mathbb{S}^n)$ is *not* continuous.

There are still open questions when considering the closed convex hull $\overline{\operatorname{conv}}=\operatorname{cls}\operatorname{conv}$ as a binary operation on $\mathcal{K}(\mathbb{S}^n)$. This operation is closed projection covariant, that is, for any k-sphere S, we have

$$\operatorname{cls}(\overline{\operatorname{conv}}(K,L)|S)=\overline{\operatorname{conv}}(\operatorname{cls}(K|S),\operatorname{cls}(L|S))$$

for all $K,L\in\mathcal{K}(\mathbb{S}^n)$, where one admits the empty set as a convex body (this is necessary since the projection of a spherical convex body may be empty in general). Of course, on $\mathcal{C}$ the closed convex hull is the same as the ordinary convex hull. Characterizing all closed projection covariant binary operations on $\mathcal{K}(\mathbb{S}^n)$ is still an open problem.

4.4. Section covariant operations

In this final section of the chapter, first we briefly recall a characterization of rotation and section covariant operations between Euclidean star sets established in [**25**]. Then, we discuss basic properties of spherical star sets in order to eventually prove a corresponding result to Theorem 4.3 for rotation and section covariant operations between them.

A subset L of $\mathbb{R}^n$ is called *star-shaped* with respect to o if every line through the origin intersects L in a (possibly degenerate) closed line segment. A *star set* in $\mathbb{R}^n$ is a compact set that is star-shaped with respect to o. The radial function $\rho_L \colon \mathbb{R}^n \backslash \{o\} \to [0, \infty)$ of a star set L is defined by

$$\rho_L(x) = \max\{\lambda \geq 0 : \lambda x \in L\}, \qquad x \in \mathbb{R}^n \backslash \{o\}.$$

Radial functions are -1-homogeneous, that is, $\rho_L(\lambda x) = \lambda^{-1} \rho_L(x)$ for all $x \in \mathbb{R}^n \backslash \{o\}$ and $\lambda > 0$, and are therefore often regarded as functions on $\mathbb{S}^{n-1}$. If ρ_L is positive and continuous, we call L a *star body*. If $K \in \mathcal{K}(\mathbb{R}^n)$ contains the origin in its interior, then K is a star body and we have

$$\rho_{K^*} = \frac{1}{h_K} \qquad \text{and} \qquad h_{K^*} = \frac{1}{\rho_K}, \tag{4.31}$$

where K^* denotes the *polar body* of K defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}.$$

The *radial distance* $\widetilde{\delta}(K, L)$ between two star sets K and L in $\mathbb{R}^n$ is defined by

$$\widetilde{\delta}(K, L) = \|\rho_K - \rho_L\|_\infty. \tag{4.32}$$

We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of all star sets in $\mathbb{R}^n$ endowed with the radial distance. The *radial sum* $K \mathbin{\widetilde{+}} L$ of K, $L \in \mathcal{S}(\mathbb{R}^n)$ is defined as the star set such that

$$\rho_{K \widetilde{+} L} = \rho_K + \rho_L.$$

More generally, for any $p > 0$, the L_p *radial sum* $K \mathbin{\widetilde{+}_p} L$ of K, $L \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\rho^p_{K \widetilde{+}_p L} = \rho^p_K + \rho^p_L.$$

Lutwak [**54**] showed that in the same way as the L_p Minkowski addition leads to the L_p Brunn–Minkowski theory, L_p radial addition leads to a dual L_p Brunn–Minkowski theory (see also [**28**] and the references therein).

While L_p radial addition is *not* projection covariant, the L_p radial sum of star sets is *section covariant*, that is,

$$(K \cap V) \mathbin{\widetilde{+}_p} (L \cap V) = (K \mathbin{\widetilde{+}_p} L) \cap V$$

for every linear subspace V of $\mathbb{R}^n$. It is also $\mathrm{GL}(n)$ covariant and therefore, in particular, covariant with respect to rotations.

A complete classification of all binary operations between star sets in $\mathbb{R}^n$ that are rotation and section covariant was established by Gardner, Hug, and Weil and can be stated as follows:

Theorem 4.12 ([**25**])**.** *An operation* $*\colon \mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *is rotation and section covariant if and only if there exists a function* $f\colon [0,\infty)^4 \to \mathbb{R}$ *such that, for all* K, $L \in \mathcal{S}(\mathbb{R}^n)$,

$$\rho_{K*L}(v) = f(\rho_{-K}(v), \rho_K(v), \rho_{-L}(v), \rho_L(v)), \qquad v \in \mathbb{S}^{n-1}.$$

We turn now to star sets in $\mathbb{S}^n$. We call a subset L of $\mathbb{S}^n$ a *(*spherical) star set with respect to $u \in L$ if $L \cap \mathbb{S}^1_{u,v}$ is a (possibly degenerate) closed spherical segment for all $v \in \mathbb{S}_u$. We denote by $\mathcal{S}_u(\mathbb{S}^n)$ the class of all spherical star sets with respect to u and we write $\mathcal{S}^p_u(\mathbb{S}^n)$ for the subclass of *proper star sets* with respect to u, that is, star sets with respect to u contained in $\mathbb{S}^+_u$.

Note that, by the definition of the spherical radial function $\rho_u(L,\cdot)$, see (3.5), for every $v \in \mathbb{S}_u$, we have

$$\cos(\rho_u(L,v))u + \sin(\rho_u(L,v))v \in \partial L.$$

The counterparts to Lemma 3.5 (c) and Lemma 3.6 in the setting of spherical star sets are the contents of our next lemma.

Lemma 4.13. *For* $u \in \mathbb{S}^n$*, the following statements hold:*

(a) The gnomonic projection maps $\mathcal{S}^p_u(\mathbb{S}^n)$ *bijectively to* $\mathcal{S}(\mathbb{R}^n_u)$*.*

(b) For every $L \in \mathcal{S}^p_u(\mathbb{S}^n)$*, we have*

$$\rho_{g_u(L)}(v) = \tan \rho_u(L,v), \qquad v \in \mathbb{S}_u.$$

Proof. Statement (a) is an immediate consequence of Lemma 3.5 (a) and (b). From Lemma 3.5 (a) and the definitions of radial and spherical radial functions, we obtain

$$\begin{aligned}
\rho_{g_u(L)}(v) &= \max\{\lambda \ge 0 : \lambda v \in g_u(L)\} \\
&= \max\left\{\lambda \ge 0 : \frac{u+\lambda v}{\|u+\lambda v\|} = \frac{1}{\sqrt{1+\lambda^2}}\, u + \frac{\lambda}{\sqrt{1+\lambda^2}}\, v \in L\right\} \\
&= \tan \max\{\alpha \in [0,\tfrac{\pi}{2}) : \cos(\alpha)u + \sin(\alpha)v \in L\} \\
&= \tan \rho_u(L,v)
\end{aligned}$$

which proves (b). □

A function $\rho\colon \mathbb{S}_u \to [0, \frac{\pi}{2})$ is the spherical radial function of a star set $L \in \mathcal{S}_u^p(\mathbb{S}^n)$ if and only if the -1-homogeneous extension of $\tan \rho$ to $\mathbb{R}_u^n$ is the radial function of a star set in $\mathbb{R}_u^n$, see Lemma 4.13 (b).

We call a proper star set $L \in \mathcal{S}_u^p(\mathbb{S}^n)$ a *(spherical) star body* with respect to $u \in \mathbb{S}^n$ if $\rho_u(L, \cdot)$ is positive and continuous. Clearly, every proper convex body $K \in \mathcal{K}_u^p(\mathbb{S}^n)$ containing u in its interior is a star body with respect to u. In order to establish a counterpart to (4.31), we recall that, for $K \in \mathcal{K}^p(\mathbb{S}^n)$ with non-empty interior, the *(spherical) polar body* $K^\circ \in \mathcal{K}^p(\mathbb{S}^n)$ is defined by

$$K^\circ = \{v \in \mathbb{S}^n : v \cdot w \leq 0 \text{ for all } w \in K\}.$$

Note that if $K \in \mathcal{K}_u^p(\mathbb{S}^n)$ contains u in its interior, then $K^\circ \in \mathcal{K}_{-u}^p(\mathbb{S}^n)$ contains $-u$ in its interior.

Proposition 4.14. *If $u \in \mathbb{S}^n$ and $K \in \mathcal{K}_u^p(\mathbb{S}^n)$ contains u in its interior, then*

$$g_u(K)^* = g_{-u}(K^\circ) \tag{4.33}$$

and

$$h_u(K, \cdot) + \rho_{-u}(K^\circ, \cdot) = \frac{\pi}{2}. \tag{4.34}$$

Proof. By the definitions of the Euclidean and spherical polar bodies and the gnomonic projection, we have

$$\begin{aligned} g_u(K)^* &= \{x \in \mathbb{R}_u^n : x \cdot y \leq 1 \text{ for all } y \in g_u(K)\} \\ &= \{x \in \mathbb{R}_u^n : x \cdot g_u(w) \leq 1 \text{ for all } w \in K\} \\ &= \{x \in \mathbb{R}_u^n : w \cdot x \leq w \cdot u \text{ for all } w \in K\} \\ &= \left\{x \in \mathbb{R}_u^n : w \cdot \frac{x-u}{\|x-u\|} \leq 0 \text{ for all } w \in K\right\} \\ &= g_{-u}(\{v \in \mathbb{S}^n : v \cdot w \leq 0 \text{ for all } w \in K\}) = g_{-u}(K^\circ). \end{aligned}$$

which proves (4.33). Lemma 3.6, (4.31), (4.33), and Lemma 4.13 (b), now yield

$$\tan h_u(K, \cdot) = h_{g_u(K)} = \frac{1}{\rho_{g_u(K)^*}} = \frac{1}{\rho_{g_{-u}(K^\circ)}} = \frac{1}{\tan \rho_{-u}(K^\circ, \cdot)}$$

which is equivalent to (4.34). □

Using spherical radial functions, we define a metric $\widetilde{\gamma}_u$ on $\mathcal{S}^p_u(\mathbb{S}^n)$ by

$$\widetilde{\gamma}_u(K,L) = \sup_{v\in\mathbb{S}_u} |\rho_u(K,v) - \rho_u(L,v)|.$$

Note that if $K,L \in \mathcal{S}^p_u(\mathbb{S}^n)$, then by (4.32) and Lemma 4.13 (b),

$$\widetilde{\delta}(g_u(K), g_u(L)) = \sup_{v\in\mathbb{S}_u} |\tan\rho_u(K,v) - \tan\rho_u(L,v)|.$$

Thus, from the continuity of the tangent we obtain the following.

Theorem 4.15. *The gnomonic projection is a homeomorphism between* $(\mathcal{S}^p_u(\mathbb{S}^n), \widetilde{\gamma}_u)$ *and* $(\mathcal{S}(\mathbb{R}^n_u), \widetilde{\delta})$.

For fixed $u \in \mathbb{S}^n$ we call a binary operation $*\colon \mathcal{S}^p_u(\mathbb{S}^n) \times \mathcal{S}^p_u(\mathbb{S}^n) \to \mathcal{S}^p_u(\mathbb{S}^n)$ *u-section covariant* if for all k-spheres S, $1 \le k \le n-1$, with $u \in S$ and for all $K,L \in \mathcal{S}^p_u(\mathbb{S}^n)$, we have

$$(K\cap S) * (L \cap S) = (K * L) \cap S.$$

The operation $*$ is called *u-rotation covariant* if $(\vartheta K) * (\vartheta L) = \vartheta(K * L)$ for all $\vartheta \in \mathrm{SO}(n+1)$ which fix u. Our next result is a version of Theorem 4.3 (or Theorem 4.5, respectively) in the setting of star sets.

Theorem 4.16. *For* $u \in \mathbb{S}^n$*, the gnomonic projection* g_u *induces a one-to-one correspondence between operations* $*\colon \mathcal{S}^p_u(\mathbb{S}^n) \times \mathcal{S}^p_u(\mathbb{S}^n) \to \mathcal{S}^p_u(\mathbb{S}^n)$ *which are u-rotation and u-section covariant and rotation and section covariant operations* $\overline{*}\colon \mathcal{S}(\mathbb{R}^n_u) \times \mathcal{S}(\mathbb{R}^n_u) \to \mathcal{S}(\mathbb{R}^n_u)$. *Moreover, any such operation* $*$ *is continuous if and only if* $\overline{*}$ *is continuous.*

Proof. First assume that $*$ is u-rotation and u-section covariant and define an operation $\overline{*}\colon \mathcal{S}(\mathbb{R}^n_u) \times \mathcal{S}(\mathbb{R}^n_u) \to \mathcal{S}(\mathbb{R}^n_u)$ by

$$K \,\overline{*}\, L = g_u(g_u^{-1}(K) * g_u^{-1}(L))$$

for $K,L \in \mathcal{S}(\mathbb{R}^n_u)$. As in the proof of Theorem 4.5, it follows that $\overline{*}$ is section covariant. The rotation covariance of $\overline{*}$ is a consequence of the u-rotation covariance of $*$ and the fact that $\vartheta g_u(L) = g_u(\vartheta L)$ for all $L \in \mathcal{S}^p_u(\mathbb{S}^n)$ and $\vartheta \in \mathrm{SO}(n+1)$ which fix u.

Conversely, if $\overline{*}\colon \mathcal{S}(\mathbb{R}^n_u) \times \mathcal{S}(\mathbb{R}^n_u) \to \mathcal{S}(\mathbb{R}^n_u)$ is rotation and section covariant, then define $*\colon \mathcal{S}^p_u(\mathbb{S}^n) \times \mathcal{S}^p_u(\mathbb{S}^n) \to \mathcal{S}^p_u(\mathbb{S}^n)$ by $K * L = g_u^{-1}(g_u(K) \,\overline{*}\, g_u(L))$ for $K,L \in \mathcal{S}^p_u(\mathbb{S}^n)$.

As before, it is easy to show that $*$ is u-rotation and u-section covariant and, by Theorem 4.15, the operation $*$ is continuous if and only if $\overline{*}$ is continuous. □

We conclude with a corollary to Theorem 4.12 of Gardner, Hug, and Weil and Theorem 4.16.

Corollary 4.17. *For fixed $u \in \mathbb{S}^n$, an operation $* \colon \mathcal{S}^p_u(\mathbb{S}^n) \times \mathcal{S}^p_u(\mathbb{S}^n) \to \mathcal{S}^p_u(\mathbb{S}^n)$ is u-rotation and u-section covariant if and only if there exists a function $f \colon [0, \frac{\pi}{2})^4 \to [0, \frac{\pi}{2})$ such that, for all $K, L \in \mathcal{S}^p_u(\mathbb{S}^n)$ and $v \in \mathbb{S}_u$,*

$$\rho_u(K * L, v) = f(\rho_u(K, -v), \rho_u(K, v), \rho_u(L, -v), \rho_u(L, v)).$$

CHAPTER 5

Floating Bodies in Spherical Convexity

Abstract. For a convex body on the Euclidean unit sphere the spherical convex floating body is introduced. The asymptotic behavior of the volume difference of a spherical convex body and its spherical floating body is investigated. This gives rise to a new spherical area measure, the floating area. Remarkably, this floating area turns out to be a spherical analogue to the classical affine surface area from affine differential geometry. Several properties of the floating area are established. The results in this chapter are published in a joint work with Elisabeth M. Werner in [**11**].

In Euclidean convex geometry the convex floating body is defined by the intersection of all halfspaces such that the hyperplanes cut off a set of constant volume. This motivates the following

Definition (Spherical Convex Floating Body)**.** Let $K \in \mathcal{K}_0(\mathbb{S}^n)$, $K \neq \mathbb{S}^n$. For $\delta > 0$ small enough we define the *(spherical) convex floating body* $K_{[\delta]}$ as the intersection of all closed hemispheres $\overline{\mathbb{S}}^-$ such that the hyperspheres cut off a set of volume less or equal δ, that is,

$$K_{[\delta]} = \bigcap \Big\{ \overline{\mathbb{S}}^- : \mathrm{vol}_n\Big(K \cap \overline{\mathbb{S}}^+\Big) \leq \delta \Big\}, \tag{5.1}$$

where $\overline{\mathbb{S}}^+$ is the complementary closed hemisphere to $\overline{\mathbb{S}}^-$, that is, $\overline{\mathbb{S}}^+ = -\overline{\mathbb{S}}^-$.

In our main theorem of this chapter we consider the volume difference of a spherical convex body K and its floating body. We show that the limit, as δ goes to zero, converges to the total curvature over the boundary $\operatorname{bd} K$ of K when integrating the spherical Gauss–Kronecker curvature $H_{n-1}^{\mathbb{S}^n}(K,\cdot)$ raised to the power $\frac{1}{n+1}$ (see Section 3.2 for details on the spherical Gauss–Kronecker curvature).

Theorem 5.1. *If $K \in \mathcal{K}_0^p(\mathbb{S}^n)$ is a proper convex body with non-empty interior, then*

$$\lim_{\delta\to 0^+} \frac{\operatorname{vol}_n(K) - \operatorname{vol}_n(K_{[\delta]})}{\delta^{\frac{2}{n+1}}} = c_n \int_{\operatorname{bd} K} H_{n-1}^{\mathbb{S}^n}(K,w)^{\frac{1}{n+1}}\, d\mathcal{H}^{n-1}(w), \tag{5.2}$$

where $c_n = \frac{1}{2}\left(\frac{n+1}{\kappa_{n-1}}\right)^{\frac{2}{n+1}}$ and κ_{n-1} is the volume of the $(n-1)$-dimensional Euclidean unit ball.

We will prove Theorem 5.1 in Section 5.2.

The curvature integral appearing on the right-hand side of (5.2) shares striking similarities with the affine surface area from affine differential geometry, see (2.6). Since, to our knowledge, there is no property similar to the affine invariance of Euclidean affine surface area in the spherical setting, we are reluctant to call this new spherical area measure a spherical affine surface area.

Definition (Floating Area)**.** For a convex body $K \in \mathcal{K}_0(\mathbb{S}^n)$ with non-empty interior the *floating area* $\Omega(K)$ is defined by

$$\Omega(K) = \int_{\operatorname{bd} K} H_{n-1}^{\mathbb{S}^n}(K,\cdot)^{\frac{1}{n+1}}\, d\mathcal{H}^{n-1}$$

if K is proper and by $\Omega(K) = 0$ otherwise.

In Section 5.3 we investigate properties of the floating area.

5.1. The Spherical Convex Floating Body

In this section we collect results about the spherical convex floating body which we will need in Section 5.2 to prove Theorem 5.1.

The definition of $K_{[\delta]}$, see (5.1), immediately yields $K_{[\delta]}$ to be convex, since it is an intersection of closed hemispheres. Furthermore, as we will show, it exists if δ is small enough. First, we show that for a proper convex body the intersection can be parametrized with respect to a fixed hypersphere.

Lemma 5.2. *Let $K \in \mathcal{K}_0^p(\mathbb{S}^n)$, $\delta \in (0, \mathrm{vol}_n(K))$ and $u \in \operatorname{int} K$ such that $K \subseteq \mathbb{S}_u^+$. For $v \in \mathbb{S}_u$ there exists a unique $s(v,\delta) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ determined by*

$$\mathrm{vol}_n\Big(K \cap \overline{\mathbb{S}}^+_{u,v,s(v,\delta)}\Big) = \delta,$$

where $\overline{\mathbb{S}}^+_{u,v,s(v,\delta)} = \{w \in \mathbb{S}^n : v \cdot w \geq \tan(s(v,\delta))(u \cdot w)\}$. Moreover,

$$K_{[\delta]} = \bigcap_{v \in \mathbb{S}_u} \overline{\mathbb{S}}^-_{u,v,s(v,\delta)} \tag{5.3}$$

and $s(v,\delta)$ is continuous in both arguments and strictly decreasing in δ.

Proof. We consider the function f defined by $f(v,s) = \mathrm{vol}_n(K \cap \overline{\mathbb{S}}^+_{u,v,s})$. Then f is continuous in both arguments and strictly decreasing in s. By (3.3), we have, for $z = \cos(h_u(K,v))v - \sin(h_u(K,v))u$, that $\mathbb{S}_z$ is a supporting hypersphere to K. Therefore $f(v, h_u(K,v)) = 0$. Similarly, we have that $f(v, -h_u(K,-v)) = \mathrm{vol}_n(K)$. We therefore conclude that there exists a unique $s(v,\delta) \in (-h_u(K,-v), h_u(K,v)) \subseteq (-\frac{\pi}{2}, \frac{\pi}{2})$ such that $f(v, s(v,\delta)) = \delta$. Thus $s(v,\delta)$ is continuous in both arguments and strictly decreasing in δ.

To prove (5.3) we only have to show that

$$K_{[\delta]} \supseteq \bigcap_{v \in \mathbb{S}_u} \overline{\mathbb{S}}^-_{u,v,s(v,\delta)}. \tag{5.4}$$

Let $z \in \mathbb{S}^n$ such that $\mathrm{vol}_n(K \cap \overline{\mathbb{S}}^+_z) \leq \delta$. For $z \neq \pm u$, we set $v = (z|\mathbb{S}_u)$. Then there is a unique $s' \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that $z = \cos(s')v - \sin(s')u$. We conclude

$$f(s') = \mathrm{vol}_n\Big(K \cap \overline{\mathbb{S}}^+_{u,v,s'}\Big) = \mathrm{vol}_n\Big(K \cap \overline{\mathbb{S}}^+_z\Big) \leq \delta = f(s(v,\delta)).$$

Thus, $s' \geq s(v,\delta)$, and therefore,

$$\begin{aligned}\overline{\mathbb{S}}^+_u \cap \overline{\mathbb{S}}^-_z &= \overline{\mathbb{S}}^+_u \cap \overline{\mathbb{S}}^-_{u,v,s'} = \{w \in \mathbb{S}^n : u \cdot w \geq 0 \text{ and } v \cdot w \leq \tan(s')(u \cdot w)\} \\ &\supseteq \overline{\mathbb{S}}^+_u \cap \overline{\mathbb{S}}^-_{u,v,s(v,\delta)}.\end{aligned}$$

Since $\overline{\mathbb{S}}_u^+ \supseteq K \supseteq K_{[\delta]}$, we obtain

$$\begin{aligned} K_{[\delta]} &= \overline{\mathbb{S}}_u^+ \cap K_{[\delta]} = \bigcap \Big\{ \overline{\mathbb{S}}_u^+ \cap \overline{\mathbb{S}}_z^- : \operatorname{vol}_n(K \cap \overline{\mathbb{S}}_z^+) \le \delta \Big\} \\ &\supseteq \bigcap \Big\{ \overline{\mathbb{S}}_u^+ \cap \overline{\mathbb{S}}_{u,v,s(v,\delta)}^- : v \in \mathbb{S}_u \Big\} = \overline{\mathbb{S}}_u^+ \cap \left(\bigcap_{v \in \mathbb{S}_u} \overline{\mathbb{S}}_{u,v,s(v,\delta)}^- \right). \end{aligned}$$

Since $\overline{\mathbb{S}}_{u,v,s(v,\delta)}^- \cap \overline{\mathbb{S}}_{u,-v,s(-v,\delta)}^- \subseteq \overline{\mathbb{S}}_u^+$, we conclude (5.4). $\square$

The following theorem relates the Euclidean floating body of the gnomonic projection of a proper spherical convex body to the spherical convex floating body.

Theorem 5.3. *Let $K \in \mathcal{K}_0^p(\mathbb{S}^n)$ such that $C_u(\alpha) \subseteq \operatorname{int} K$ and $K \subseteq C_u(\beta)$ for some $u \in \mathbb{S}^n$ and $\alpha, \beta \in [0, \frac{\pi}{2})$. Then, for $\delta > 0$ small enough, we have*

$$g_u(K)_{\left[\frac{\delta}{\cos(\beta)^{n+1}}\right]} \subseteq g_u\Big(K_{[\delta]}\Big) \subseteq g_u(K)_{\left[\frac{\delta}{\cos(\alpha)^{n+1}}\right]}.$$

In particular, this shows that $K_{[\delta]}$ exists if δ is small enough.

Proof. Set $L = g_u(K)$. By (5.3), we have $K_{[\delta]} = \bigcap_{v \in \mathbb{S}_u} \overline{\mathbb{S}}_{u,v,s(v,\delta)}^-$, where $s(v,\delta)$ is determined by $\delta = \operatorname{vol}_n\Big(K \cap \overline{\mathbb{S}}_{u,v,s(v,\delta)}^+\Big)$. The gnomonic projection g_u maps $\mathbb{S}_{u,v,s(v,\delta)}$ to the hyperplane

$$\mathbb{H}_{v,\tan(s(v,\delta))} = \{x \in \mathbb{R}_u^n : x \cdot v = \tan(s(v,\delta))\}.$$

For δ small enough, we have, for $v \in \mathbb{S}_u$, that

$$\emptyset = g_u\Big(\Big(K \cap \mathbb{S}_{u,v,s(v,\delta)}\Big) \cap C_u(\alpha)\Big) = \Big(L \cap \mathbb{H}_{v,\tan(s(v,\delta))}^+\Big) \cap B(0, \tan(\alpha)).$$

By (3.20), we conclude

$$\begin{aligned} \delta = \operatorname{vol}_n\Big(K \cap \overline{\mathbb{S}}_{u,v,s(v,\delta)}^+\Big) &= \int\limits_{L \cap \mathbb{H}_{v,\tan(s(v,\delta))}^+} \frac{dx}{(1 + \|x\|^2)^{\frac{n+1}{2}}} \\ &\le \cos(\alpha)^{n+1} \operatorname{vol}_n\Big(L \cap \mathbb{H}_{v,\tan(s(v,\delta))}^+\Big). \end{aligned}$$

Now let $\widetilde{s}(v,\delta)$ such that $\delta = \cos(\alpha)^{n+1} \operatorname{vol}_n\Big(L \cap \mathbb{H}_{v,\widetilde{s}(v,\delta)}^+\Big)$. Then obviously $\widetilde{s}(v,\delta) \ge \tan(s(v,\delta))$ and $\mathbb{H}_{v,\tan(s(v,\delta))}^- \subseteq \mathbb{H}_{v,\widetilde{s}(v,\delta)}^-$. By (2.5), we deduce

$$g_u\Big(K_{[\delta]}\Big) = \bigcap_{v \in \mathbb{S}_u} \mathbb{H}_{v,\tan(s(v,\delta))}^- \subseteq \bigcap_{v \in \mathbb{S}_u} \mathbb{H}_{v,\widetilde{s}(v,\delta)}^- = L_{\left[\frac{\delta}{\cos(\alpha)^{n+1}}\right]}.$$

For the converse, since $g_u(K) \subseteq g_u(C_u(\beta)) = B(0, \tan(\beta))$, we first note that

$$\begin{aligned}\delta = \operatorname{vol}_n\Big(K \cap \overline{\mathbb{S}}^+_{u,v,s(v,\delta)}\Big) &= \int\limits_{L \cap \mathbb{H}^+_{v,\tan(s(v,\delta))}} \frac{dx}{(1+\|x\|^2)^{\frac{n+1}{2}}} \\ &\geq \cos(\beta)^{n+1} \operatorname{vol}_n\Big(L \cap \mathbb{H}^+_{\tan(s(v,\delta))}\Big).\end{aligned}$$

Now let $\widetilde{S}(v,\delta)$ such that $\delta = \cos(\beta)^{n+1} \operatorname{vol}_n\Big(L \cap \mathbb{H}^+_{v,\widetilde{S}(v,\delta)}\Big)$. Then we have $\widetilde{S}(v,\delta) \leq \tan(s(v,\delta))$ and therefore $\mathbb{H}^-_{v,\tan(s(v,\delta))} \supseteq \mathbb{H}^-_{v,\widetilde{S}(v,\delta)}$. We conclude

$$g_u\Big(K_{[\delta]}\Big) = \bigcap_{v \in \mathbb{S}_u} \mathbb{H}^-_{v,\tan(s(v,\delta))} \supseteq \bigcap_{v \in \mathbb{S}_u} \mathbb{H}^-_{v,\widetilde{S}(v,\delta)} = L_{\left[\frac{\delta}{\cos(\beta)^{n+1}}\right]}.$$

□

In the following three lemmas we establish properties of the spherical convex floating body as δ goes to 0. First we show that the boundary points of the floating body converge to boundary points of the convex body.

Lemma 5.4. *If $K \in \mathcal{K}_0(\mathbb{S}^n)$ and $\delta_1 \leq \delta_2$, then $K_{[\delta_1]} \supseteq K_{[\delta_2]}$. In particular, we have* $\operatorname{int} K = \bigcup_{\delta>0} K_{[\delta]}$.

Furthermore, let $K \in \mathcal{K}_0^p(\mathbb{S}^n)$ and $u \in \operatorname{int} K$ such that $K \subseteq \mathbb{S}^+_u$. For $w \in \operatorname{bd} K$, put $\{w_\delta\} = \operatorname{bd} K_{[\delta]} \cap \operatorname{conv}(u,w)$. Then $\lim_{\delta \to 0} w_\delta = w$.

Proof. The monotonicity of the floating body is obvious from its definition.

First we prove $\operatorname{int} K = \bigcup_{\delta>0} K_{[\delta]}$. Let $\delta > 0$ small enough such that $K_{[\delta]}$ is non-empty. If $w \in K_{[\delta]}$, then every hypersphere through w cuts off a set of K of volume greater or equal to δ. Thus $K_{[\delta]} \subseteq \operatorname{int} K$ and we conclude $\bigcup_{\delta>0} K_{[\delta]} \subseteq \operatorname{int} K$.
In order to prove the converse, we assume that there is $w \in \operatorname{int} K$ such that $w \notin \bigcap_{\delta>0} K_{[\delta]}$. For every $v \in \mathbb{S}_w$ we have

$$\operatorname{vol}_n\Big(K \cap \overline{\mathbb{S}}^+_v\Big) > 0. \tag{5.5}$$

Since $w \notin \bigcap_{\delta>0} K_{[\delta]}$ we conclude that for every $\delta > 0$ there exists $v(\delta) \in \mathbb{S}_w$ such that $\operatorname{vol}_n\Big(K \cap \overline{\mathbb{S}}^+_{v(\delta)}\Big) < \delta$. By compactness of $\mathbb{S}_w$ and continuity there exists $v_0 \in \mathbb{S}_w$ such that $\operatorname{vol}_n\Big(K \cap \overline{\mathbb{S}}^+_{v_0}\Big) = 0$. This is a contradiction to (5.5).

Finally, let $K \in \mathcal{K}_0^p(\mathbb{S}^n)$ and $u \in \operatorname{int} K$ such that $K \subseteq \mathbb{S}_u^+$. We have

$$\bigcup_{\delta>0} \operatorname{conv}(u, w_\delta) = \bigcup_{\delta>0} K_{[\delta]} \cap \operatorname{conv}(u, w) = \operatorname{conv}(u, w) \backslash \{w\}.$$

We conclude $\lim_{\delta\to 0} d(w_\delta, w) = 0$. □

In the next lemma we show that the outer normals of the spherical convex floating body converge to the outer normals of the convex body.

Lemma 5.5. *Let $K \in \mathcal{K}_0^p(\mathbb{S}^n)$, $u \in \operatorname{int} K$ and $w \in \operatorname{bd} K$ be a regular boundary point. For $\delta > 0$ such that $K_{[\delta]} \neq \emptyset$, we set $\{w_\delta\} = \operatorname{bd} K \cap \operatorname{conv}(u, w)$. Then*

$$\lim_{\delta\to 0^+} N_{w_\delta}^{K_{[\delta]}} = N_w^K, \tag{5.6}$$

where $N_{w_\delta}^{K_{[\delta]}}$ is an outer unit normal to $K_{[\delta]}$ in w_δ such that

$$\delta = \operatorname{vol}_n\left(K \cap \overline{\mathbb{S}}^+_{N_{w_\delta}^{K_{[\delta]}}}\right).$$

In particular, for all $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that

$$N_w^K \cdot N_{w_\delta}^{K_{[\delta]}} \geq 1 - \varepsilon, \tag{5.7}$$

for all $\delta \leq \delta(\varepsilon)$ and, if $K \subseteq \operatorname{int} \mathbb{S}_u^+$, then for all $\delta \leq \delta(\varepsilon)$

$$(N_w^K | \mathbb{S}_u) \cdot (N_{w_\delta}^{K_{[\delta]}} | \mathbb{S}_u) \geq 1 - \varepsilon. \tag{5.8}$$

Proof. Suppose (5.6) is not true. Then, by compactness, there exists a subsequence $(\delta_i)_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} \delta_i = 0$, $\lim_{i\to\infty} N_{w_{\delta_i}}^{K_{[\delta_i]}} = v_0$ and $v_0 \neq N_w^K$. By the choice of $N_{w_{\delta_i}}^{K_{[\delta_i]}}$, we have $\operatorname{vol}_n\left(K \cap \overline{\mathbb{S}}^+_{N_{w_{\delta_i}}^{K_{[\delta_i]}}}\right) = \delta_i$. We conclude that $\operatorname{vol}_n(K \cap \overline{\mathbb{S}}^+_{v_0}) = 0$. By Lemma 5.4, we have $\lim_{i\to\infty} w_{\delta_i} = w$, thus v_0 is a normal to $\operatorname{bd} K$ in w. This contradicts the assumption that w is a regular boundary point and therefore has a unique outer unit normal $N_w^K \neq v_0$.

The other statements, (5.7) and (5.8), follow directly from (5.6). □

Let $w \in \operatorname{bd} K$ be a boundary point such that $H_{n-1}^{\mathbb{S}^n}(K,w) > 0$. Then $K \cap \overline{\mathbb{S}}^+_{w,N_w^K,-\Delta}$ is decreasing in Δ and for $\Delta = 0$ equals just $\{w\}$. Thus for Δ small enough, $K \cap \overline{\mathbb{S}}^+_{w,N_w^K,-\Delta}$ is contained in some arbitrarily small cap around w. By continuity this will still be true if we allow directions $v \in \mathbb{S}_w$ close to N_w^K. Let w_δ be a boundary point of the floating body $K_{[\delta]}$ that converges to a boundary point $w \in \operatorname{bd} K$. Then, by Lemma 5.5, the normals to w_δ converge to N_w^K. Thus, if we consider directions $v \in \mathbb{S}_w$ close to N_w^K, then, for δ small enough, w_δ will already be determined by these directions. Hence, if we replace K by $K' = K \cap C_w(\varepsilon)$ for arbitrarily small ε, then for δ small enough, we will have $w_\delta^{K'} = w_\delta^K$. We will prove this rigorously in the following lemma.

Lemma 5.6. *Let $K \in \mathcal{K}_0^p(\mathbb{S}^n)$ and $w \in \operatorname{bd} K$ such that $H_{n-1}^{\mathbb{S}^n}(K,w) > 0$. For $\varepsilon > 0$ set $K' = K \cap C_w(\varepsilon)$.*

(i) There exists Δ_ε such that for all $\Delta < \Delta_\varepsilon$ we have

$$K' \cap \overline{\mathbb{S}}^+_{w,N_w^K,-\Delta} = K \cap \overline{\mathbb{S}}^+_{w,N_w^K,-\Delta}.$$

(ii) There exists ξ_ε and η_ε such that, for all $v \in \mathbb{S}_w$ with $d(v, N_w^K) < \xi_\varepsilon$ and $\Delta < \eta_\varepsilon$, we have

$$K' \cap \overline{\mathbb{S}}^+_{w,v,-\Delta} = K \cap \overline{\mathbb{S}}^+_{w,v,-\Delta}.$$

(iii) Let $u \in \operatorname{int} K'$. There exists δ_ε such that, for all $\delta < \delta_\varepsilon$, we have

$$K'_{[\delta]} \cap \operatorname{conv}(u,w) = K_{[\delta]} \cap \operatorname{conv}(u,w).$$

In particular, we have $w_\delta^{K'} = w_\delta^K$ for all $\delta < \delta_\varepsilon$.

Proof. (i) Assume this does not hold. Then there exists an $\varepsilon > 0$ such that for all $\Delta > 0$, we have

$$\begin{aligned}\emptyset &\neq K \cap \overline{\mathbb{S}}^+_{w,N_w^K,-\Delta} \backslash \left(K' \cap \overline{\mathbb{S}}^+_{w,N_w^K,-\Delta}\right)\\ &= (K \backslash C_w(\varepsilon)) \cap \overline{\mathbb{S}}^+_{w,N_w^K,-\Delta} \subseteq (K \backslash \operatorname{int} C_w(\varepsilon)) \cap \overline{\mathbb{S}}^+_{w,N_w^K,-\Delta}.\end{aligned}$$

For $\Delta_1 \leq \Delta_2$, we have $(K\backslash \operatorname{int} C_w(\varepsilon)) \cap \overline{\mathbb{S}}^+_{w,N_w^K,-\Delta_1} \subseteq (K\backslash \operatorname{int} C_w(\varepsilon)) \cap \overline{\mathbb{S}}^+_{w,N_w^K,-\Delta_2}$, and, by compactness, we conclude that $\emptyset \neq (K\backslash \operatorname{int} C_w(\varepsilon)) \cap \overline{\mathbb{S}}^+_{N_w^K}$. Since $\mathbb{S}_{N_w^K}$ is a supporting hyperplane at w, this implies that there exists $v \in K \cap \mathbb{S}_{N_w^K}$ such that $d(v,w) \geq \varepsilon$. Since K is convex, the whole segment $\operatorname{conv}(v,w)$ is contained in K. Considering $L = g_w(K \cap C_w(\varepsilon))$, we see that the boundary of L contains the segment $g_w(C_w(\varepsilon) \cap \mathrm{conv}(w, v))$, and we conclude that $H^{\mathbb{R}^n_w}_{n-1}(L, 0) = 0$. This implies, by Lemma 3.11, that $H^{\mathbb{S}^n}_{n-1}(K, w) = 0$ which is a contradiction.

(ii) We may assume $\varepsilon < \frac{\pi}{3}$. By (i), there exists $\Delta_{\varepsilon/2}$ such that $\Delta_{\varepsilon/2} < \frac{\varepsilon}{2}$ and

$$(K \backslash C_w(\varepsilon/2)) \cap \overline{\mathbb{S}}^+_{w,N^K_w,-\Delta_{\varepsilon/2}} = \emptyset. \tag{5.9}$$

We set $\eta_\varepsilon = \frac{\Delta_{\varepsilon/2}}{2} < \frac{\varepsilon}{4}$ and

$$\sin(\xi_\varepsilon) = \frac{\tan(\Delta_{\varepsilon/2}) - \tan(\eta_\varepsilon)}{\sqrt{\tan(\varepsilon)^2 - \tan(\eta_\varepsilon)^2}} > 0.$$

We show that, for all $v \in \mathbb{S}_w$ such that $d(v, N^K_w) < \xi_\varepsilon$, we have

$$(K \backslash C_w(\varepsilon)) \cap \overline{\mathbb{S}}^+_{w,v,-\eta_\varepsilon} = \emptyset. \tag{5.10}$$

This implies $K' \cap \overline{\mathbb{S}}^+_{u,v,-\Delta} = K \cap \overline{\mathbb{S}}^+_{u,v,-\Delta}$ for all $\Delta \leq \eta_\varepsilon$.

Assume (5.10) is not true. Then there exists $z \in (K \backslash C_w(\varepsilon)) \cap \overline{\mathbb{S}}^+_{w,v,-\eta_\varepsilon}$. Since K is convex, the whole segment $\mathrm{conv}(z, w)$ is contained in K. Since $d(z, w) > \varepsilon$, there exists $\{z'\} = \mathrm{bd}\, C_w(\varepsilon) \cap \mathrm{conv}(z, w)$. We will show that $z' \in \overline{\mathbb{S}}^+_{w,N^K_w,-\Delta_{\varepsilon/2}}$. This will be a contradiction to (5.9), since by construction $z' \in K \backslash C_w(\varepsilon/2)$. We have to show that

$$\frac{-z' \cdot N^K_w}{z' \cdot w} \leq \tan(\Delta_{\varepsilon/2}). \tag{5.11}$$

Since $z' \in \mathrm{bd}\, C_w(\varepsilon)$, we have $z' = \cos(\varepsilon)w + \sin(\varepsilon)(z'|\mathbb{S}_w)$ and conclude

$$\frac{-z' \cdot N^K_w}{z' \cdot w} = \tan(\varepsilon) \sin\Big(d(z'|\mathbb{S}_w, N^K_w) - \frac{\pi}{2}\Big),$$

and, since $z' \in \overline{\mathbb{S}}^+_{w,v,-\eta_\varepsilon}$, we obtain

$$\tan(\eta_\varepsilon) \geq \frac{-z' \cdot v}{z' \cdot w} = \tan(\varepsilon) \sin\Big(d(z'|\mathbb{S}_w, v) - \frac{\pi}{2}\Big).$$

Thus $d(z'|\mathbb{S}_w, v) \leq \frac{\pi}{2} + \arcsin\Big(\frac{\tan(\eta_\varepsilon)}{\tan(\varepsilon)}\Big)$ and the triangle inequality yields

$$d(z'|\mathbb{S}_w, N^K_w) \leq d(z'|\mathbb{S}_w, v) + d(v, N^K_w) \leq \frac{\pi}{2} + \arcsin\left(\frac{\tan(\eta_\varepsilon)}{\tan(\varepsilon)}\right) + \xi_\varepsilon.$$

Finally, we obtain (5.11) from

$$\begin{aligned}\frac{-z'\cdot N_w^K}{z'\cdot w} &\leq \tan(\varepsilon)\sin\left(\arcsin\left(\frac{\tan(\eta_\varepsilon)}{\tan(\varepsilon)}\right)+\xi_\varepsilon\right)\\ &= \sin(\xi_\varepsilon)\sqrt{\tan(\varepsilon)^2-\tan(\eta_\varepsilon)^2}+\cos(\xi_\varepsilon)\tan(\eta_\varepsilon)\\ &\leq \sin(\xi_\varepsilon)\sqrt{\tan(\varepsilon)^2-\tan(\eta_\varepsilon)^2}+\tan(\eta_\varepsilon)\leq \tan(\Delta_{\varepsilon/2}).\end{aligned}$$

(iii) Since $K'\subseteq C_w(\varepsilon)$, by (5.3), we can write $K'_{[\delta]}=\bigcap_{v\in\mathbb{S}_w}\overline{\mathbb{S}}^-_{w,v,s^{K'}(v,\delta)}$. Here $s^{K'}(v,\delta)$ is uniquely determined by $\delta=\mathrm{vol}_n\Big(K'\cap\overline{\mathbb{S}}^+_{w,v,s^{K'}(v,\delta)}\Big)$ and is continuous in both arguments. By (ii), there exists ξ_ε and η_ε such that, for all $v\in\mathbb{S}_w$ with $d(v,N_w^K)<\xi_\varepsilon$, we have

$$\delta^{K'}(v,-\Delta)=\mathrm{vol}_n(K'\cap\overline{\mathbb{S}}^+_{w,v,-\Delta})=\mathrm{vol}_n(K\cap\overline{\mathbb{S}}^+_{w,v,-\Delta})=\delta^K(v,-\Delta),$$

for all $\Delta<\eta_\varepsilon$. Hence $s^K(v,\delta)$ exists for $v\in\mathbb{S}_w$ such that $d(v,N_w^K)<\xi_\varepsilon$ and δ small. Thus, there exist δ_1 such that for all $\delta<\delta_1$ and $v\in\mathbb{S}_w$ such that $d(v,N_w^{K'})=d(v,N_w^K)<\xi_\varepsilon$, we have $s^{K'}(v,\delta)=s^K(v,\delta)$.

Claim: *There exists δ_2 such that, for all $\delta<\delta_2$, we have*

$$K_{[\delta]}\cap\mathrm{conv}(u,w)=\bigcap\{\overline{\mathbb{S}}^-_{w,v,s^K(v,\delta)}:v\in\mathbb{S}_w,\ d(v,N_w^K)\leq\xi_\varepsilon\}\cap\mathrm{conv}(u,w).$$

Assume that this is not true. Then, for all $\delta>0$, we have

$$\begin{aligned}\mathrm{conv}(u,w_\delta^K)&=K_{[\delta]}\cap\mathrm{conv}(u,w)\\ &\subsetneq\bigcap\{\overline{\mathbb{S}}^-_{w,v,s^K(v,\delta)}:d(v,N_w^K)\leq\xi_\varepsilon\}\cap\mathrm{conv}(u,w).\end{aligned}$$

Thus, for any normal z to $K_{[\delta]}$ in w_δ^K, we have $d(z|\mathbb{S}_w,N_w^K)\geq\xi_\varepsilon>0$ for all $\delta>0$. This is a contradiction to Lemma 5.5.

With a similar argument for K' we also find a δ_3 such that, for all $\delta<\delta_3$,

$$K'_{[\delta]}\cap\mathrm{conv}(u,w)=\bigcap\{\overline{\mathbb{S}}^-_{w,v,s^{K'}(v,\delta)}:v\in\mathbb{S}_w,\ d(v,N_w^{K'})\leq\xi_\varepsilon\}\cap\mathrm{conv}(u,w).$$

Setting $\delta_\varepsilon=\min\{\delta_1,\delta_2,\delta_3\}$, we conclude, for all $\delta<\delta_\varepsilon$,

$$\begin{aligned}K'_{[\delta]}\cap\mathrm{conv}(u,w)&=\bigcap\Big\{\overline{\mathbb{S}}^-_{w,v,-s^{K'}(v,\delta)}:d(v,N_w^{K'})<\xi_1\Big\}\cap\mathrm{conv}(u,w)\\ &=\bigcap\Big\{\overline{\mathbb{S}}^-_{w,v,-s^K(v,\delta)}:d(v,N_w^K)<\xi_1\Big\}\cap\mathrm{conv}(u,w)\\ &=K_{[\delta]}\cap\mathrm{conv}(u,w).\end{aligned}$$

The second statement of (iii) is obvious since $K_{[\delta]}\cap\mathrm{conv}(u,w)=\mathrm{conv}(u,w_\delta^K)$ and $K'_{[\delta]}\cap\mathrm{conv}(u,w)=\mathrm{conv}(u,w_\delta^{K'})$. □

5.2. Proof of Theorem 5.1

We are now ready to proof Theorem 5.1. By Lemma 3.3, there exists $u \in \operatorname{int} K$ such that $K \subseteq \mathbb{S}^+_u$. Using Proposition 3.14, we may write the left hand side of (5.2) as

$$\frac{\operatorname{vol}_n(K) - \operatorname{vol}_n(K_{[\delta]})}{\delta^{\frac{2}{n+1}}} = \int\limits_{\operatorname{bd} K} \underbrace{\delta^{-\frac{2}{n+1}} \frac{-u \cdot N^K_w}{\sin(d(w,u))^n} \int\limits_{d(w_\delta,u)}^{d(w,u)} \sin(t)^{n-1}\, dt}_{f(w,\delta)}\, dw. \tag{5.12}$$

The proof of Theorem 5.1 will now be completed in two steps. We will first show that the integrand $f(w,\delta)$ in the integral on the right hand side of (5.12) is bounded from above uniformly in δ almost everywhere by an integrable function.

Lemma 5.7. *Let $K \in \mathcal{K}^p_0(\mathbb{S}^n)$ and $u \in \operatorname{int} K$ such that $K \subseteq \mathbb{S}^+_u$. There exists $\delta_0 > 0$ and an integrable function $g\colon \operatorname{bd} K \to \mathbb{R}$ such that, for all $\delta < \delta_0$,*

$$\delta^{-\frac{2}{n+1}} \frac{-u \cdot N^K_w}{\sin(d(w,u))^n} \int\limits_{d(w_\delta,u)}^{d(w,u)} \sin(t)^{n-1}\, dt \le g(w) \tag{5.13}$$

for $\mathcal{H}^{n-1}$-almost every $w \in \operatorname{bd} K$, where $\{w_\delta\} = \operatorname{bd} K_{[\delta]} \cap \operatorname{conv}(u,w)$.

Proof. Since $u \in \operatorname{int} K$ and $K \subseteq \mathbb{S}^+_u$, there is $0 < \alpha < \beta < \frac{\pi}{2}$ such that $C_u(\alpha) \subseteq \operatorname{int} K$ and $K \subseteq C_u(\beta)$. Therefore, $\sin(d(u,w)) \ge \sin(\alpha)$, $-u \cdot N^K_w \le 1$ and we conclude

$$\delta^{-\frac{2}{n+1}} \frac{-u \cdot N^K_w}{\sin(d(w,u))^n} \int\limits_{d(w_\delta,u)}^{d(w,u)} \sin(t)^{n-1}\, dt \le \frac{1}{\sin(\alpha)} \frac{d(w,w_\delta)}{\delta^{\frac{2}{n+1}}}.$$

We will show that there exists $C > 0$ and δ_0 such that, for all $\delta < \delta_0$,

$$\frac{1}{\sin(\alpha)} \frac{d(w,w_\delta)}{\delta^{\frac{2}{n+1}}} \le C r_{g_u(K)}(g_u(w))^{-\frac{n-1}{n+1}} \tag{5.14}$$

for $\mathcal{H}^{n-1}$-almost all $w \in \operatorname{bd} K$. Then the right hand side of (5.14) is integrable: By (3.23), the fact that $1 + (x \cdot N^{g_u(K)}_x)^2 \le 1 + \|x\|^2$ and

$1 + \|x\|^2 \geq 1$, we have

$$\int\limits_{\operatorname{bd} K} r_{g_u(K)}(g_u(w))^{-\frac{n-1}{n+1}}\, dw = \int\limits_{\operatorname{bd} g_u(K)} r_{g_u(K)}(x)^{-\frac{n-1}{n+1}} \frac{\sqrt{1+(x\cdot N_x^{g_u(K)})^2}}{(1+\|x\|^2)^{\frac{n}{2}}}\, dx$$
$$\leq \int\limits_{\operatorname{bd} g_u(K)} r_{g_u(K)}(x)^{-\frac{n-1}{n+1}}\, dx,$$

which is finite by Theorem 2.7.

In order to prove (5.14), we set $L = g_u(K)$, $x = g_u(w)$. By (3.4), we have $\tan(d(u,w)) = \|g_u(w)\|$ and $\tan(d(u,w_\delta)) = \|g_u(w_\delta))\|$. We derive

$$\frac{d(w,w_\delta)}{\delta^{\frac{2}{n+1}}} = \frac{\arctan(\|x\|) - \arctan(\|g_u(w_\delta)\|)}{\delta^{\frac{2}{n+1}}} \leq \frac{\|x - g_u(w_\delta)\|}{\delta^{\frac{2}{n+1}}}.$$

By Theorem 5.3 and as $g_u(w_\delta)$ is on the line $g_u(\operatorname{conv}(u,w)) = \operatorname{conv}(0,x)$, we have $g_u(K_{[\delta]}) \subseteq L_{[\tilde\delta]}$, where $\tilde\delta = \frac{\delta}{\cos(\alpha)^{n+1}}$. Setting $\{x_{\tilde\delta}\} = \operatorname{bd} L_{[\tilde\delta]} \cap \operatorname{conv}(0,x)$, we conclude that $\|g_u(w_\delta)\| \geq \|x_{\tilde\delta}\|$. Therefore,

$$\frac{1}{\sin(\alpha)} \frac{d(w,w_\delta)}{\delta^{\frac{2}{n+1}}} \leq \frac{1}{\cos(\alpha)^2 \sin(\alpha)} \frac{\|x - x_{\tilde\delta}\|}{\tilde\delta^{\frac{2}{n+1}}}.$$

By Theorem 2.9, there exists $\tilde\delta_0$ and $\tilde C > 0$ such that, for all $\tilde\delta < \tilde\delta_0$,

$$\frac{\|x - x_{\tilde\delta}\|}{\tilde\delta^{\frac{2}{n+1}}} \leq \tilde C r_L(x)^{-\frac{n-1}{n+1}},$$

for $\mathcal{H}^{n-1}$-almost all $x \in \operatorname{bd} L$. Thus, for all $\delta < \tilde\delta_0 \cos(\alpha)^{n+1}$, we have

$$\frac{1}{\sin(\alpha)} \frac{d(w,w_\delta)}{\delta^{\frac{2}{n+1}}} \leq \frac{\tilde C}{\cos(\alpha)^2 \sin(\alpha)} r_{g_u(K)}(g_u(w))^{-\frac{n-1}{n+1}}$$

for $\mathcal{H}^{n-1}$-almost all $w \in \operatorname{bd} K$. □

Using this lemma and the dominated convergence theorem, we can express the limit as δ tends to zero of the right hand side of (5.12) by the integral over the point-wise limit of the integrand.

Theorem 5.8. *Let $K \in \mathcal{K}_0^p(\mathbb{S}^n)$ and $u \in \operatorname{int} K$ such that $K \subseteq \mathbb{S}_u^+$. Then, for $\mathcal{H}^{n-1}$-almost all $w \in \operatorname{bd} K$, we have*

$$\lim_{\delta \to 0^+} \delta^{-\frac{2}{n+1}} \frac{-u \cdot N_w^K}{\sin(d(u,w))^n} \int\limits_{d(u,w_\delta)}^{d(u,w)} \sin(t)^{n-1}\, dt = c_n H_{n-1}^{\mathbb{S}^n}(K,w)^{\frac{1}{n+1}}, \qquad (5.15)$$

where $\{w_\delta\} = \operatorname{bd} K_{[\delta]} \cap \operatorname{conv}(u,w)$ and $c_n = \frac{1}{2}\left(\frac{n+1}{\kappa_{n-1}}\right)^{\frac{2}{n+1}}$.

Proof. For $t \in [d(u, w_\delta), d(u, w)]$, we have

$$\frac{\sin(d(u, w_\delta))}{\sin(d(u, w))} \leq \frac{\sin(t)}{\sin(d(u, w))} \leq 1.$$

Furthermore, $\lim_{\delta \to 0} d(u, w_\delta) = d(u, w)$ and $d(u, w) - d(u, w_\delta) = d(w, w_\delta)$. We therefore conclude

$$\lim_{\delta \to 0^+} \frac{-u \cdot N_w^K}{\sin(d(u, w))^n} \frac{1}{\delta^{\frac{2}{n+1}}} \int\limits_{d(u,w_\delta)}^{d(u,w)} \sin(t)^{n-1} = \lim_{\delta \to 0^+} \frac{-u \cdot N_w^K}{\sin(d(u, w))} \frac{d(w, w_\delta)}{\delta^{\frac{2}{n+1}}}. \tag{5.16}$$

We first show that for a regular boundary point w of K with positive curvature, the right hand side of (5.16) only depends on the local structure of $\operatorname{bd} K$ at w.

Claim: *Let $\varepsilon \in \left(0, \frac{\pi}{2}\right)$ and $w \in \operatorname{bd} K$ such that $H_{n-1}^{\mathbb{S}^n}(K, w) > 0$. If we set $K' = K \cap C_w(\varepsilon)$ and let $u' \in \operatorname{int} K' \cap \operatorname{conv}(u, w)$ be such that $K' \subseteq \mathbb{S}^+_{u'}$, then, for δ small enough, we have*

$$\frac{-u \cdot N_w^K}{\sin(d(u, w))} \frac{d(w, w_\delta^K)}{\delta^{\frac{2}{n+1}}} = \frac{-u' \cdot N_w^K}{\sin(d(u', w))} \frac{d(w, w_\delta^{K'})}{\delta^{\frac{2}{n+1}}}. \tag{5.17}$$

By (3.1), we have

$$u'|\mathbb{S}_w = \frac{u' - \cos(d(w, u'))w}{\sin(d(w, u'))}.$$

Thus, we can write $u' = \cos(d(w, u'))w + \sin(d(w, u'))(u'|\mathbb{S}_w)$ and, similarly, $u = \cos(d(w, u))w + \sin(d(w, u))(u|\mathbb{S}_w)$. Hence, since $\mathbb{S}_w^\circ = \{\pm w\}$, we have $u'|\mathbb{S}_w = \operatorname{conv}(u', \{\pm w\}) \cap \mathbb{S}_w = u|\mathbb{S}_w$. Therefore,

$$\frac{u \cdot N_w^K}{\sin(d(u, w))} = (u|\mathbb{S}_w) \cdot N_w^K = \frac{u' \cdot N_w^K}{\sin(d(u', w))}.$$

Using Lemma 5.6 (iii) we conclude (5.17).

Now we can prove (5.15) for regular boundary points with positive curvature.

Claim: *Let $w \in \operatorname{bd} K$ such that $H_{n-1}^{\mathbb{S}^n}(K, w) > 0$. Then*

$$\lim_{\delta \to 0^+} \frac{-u \cdot N_w^K}{\sin(d(u, w))} \frac{d(w, w_\delta)}{\delta^{\frac{2}{n+1}}} = c_n H_{n-1}^{\mathbb{S}^n}(K, w)^{\frac{1}{n+1}}.$$

By the previous claim, we may assume that $K \subseteq C_w(\varepsilon)$ for arbitrarily small $\varepsilon > 0$ and $u \in \operatorname{int} K$ such that $K \subseteq \mathbb{S}^+_u$. Set $L = g_w(K)$, and $\xi = -\frac{g_w(u)}{\|g_w(u)\|}$. Since $\lim_{\delta\to 0^+} g_w(w_\delta) = 0$, we obtain

$$\begin{aligned}\lim_{\delta\to 0^+} \frac{-u\cdot N^K_w}{\sin(d(u,w))}\frac{d(w,w_\delta)}{\delta^{\frac{2}{n+1}}} &= \lim_{\delta\to 0^+}\left(\xi\cdot N^K_w\right)\frac{\arctan(\|g_w(w_\delta)\|)}{\delta^{\frac{2}{n+1}}}\\ &= \lim_{\delta\to 0^+}\left(\xi\cdot N^K_w\right)\frac{\|g_w(w_\delta)\|}{\delta^{\frac{2}{n+1}}}.\end{aligned}$$

Using $\alpha = 0$, $\beta = \varepsilon$ and $u = w$ in Theorem 5.3, we conclude

$$\left\| x_{\frac{\delta}{\cos(\varepsilon)^{n+1}}}\right\| \geq \|g_w(w_\delta)\| \geq \|x_\delta\|$$

for δ small (note that the origin $0 = g_w(w)$ is a boundary point of L). Since $N^K_w = N^L_0$, Theorem 2.10 implies

$$\lim_{\delta\to 0^+}\left(\xi\cdot N^K_w\right)\frac{\|x_\delta\|}{\delta^{\frac{2}{n+1}}} = c_n H^{\mathbb{R}^n_w}_{n-1}(L,0)^{\frac{1}{n+1}}$$

and, by Lemma 3.11, we have $H^{\mathbb{R}^n_w}_{n-1}(L,0) = H^{\mathbb{S}^n}_{n-1}(K,w)$. We conclude

$$\frac{c_n}{\cos(\varepsilon)^2}H^{\mathbb{S}^n}_{n-1}(K,w)^{\frac{1}{n+1}} \geq \lim_{\delta\to 0^+}\frac{-u\cdot N^K_w}{\sin(d(u,w))}\frac{d(w,w_\delta)}{\delta^{\frac{2}{n+1}}} \geq c_n H^{\mathbb{S}^n}_{n-1}(K,w)^{\frac{1}{n+1}}.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small the claim follows.

To finish the proof we only need to consider regular boundary points with vanishing curvature.

Claim: *Let $K \subseteq C_u(\beta)$ for some $\beta \in \left(0, \frac{\pi}{2}\right)$ and $u \in \operatorname{int} K$. Then, for $w \in \operatorname{bd} K$ such that $H^{\mathbb{S}^n}_{n-1}(K,w) = 0$,*

$$\lim_{\delta\to 0^+}\frac{-u\cdot N^K_w}{\sin(d(u,w))}\frac{d(w,w_\delta)}{\delta^{\frac{2}{n+1}}} = 0.$$

We consider $L = g_u(K)$ and $x = g_u(w)$. Then

$$\begin{aligned}\lim_{\delta\to 0^+}\frac{-u\cdot N^K_w}{\sin(d(u,w))}\frac{d(w,w_\delta)}{\delta^{\frac{2}{n+1}}} &= \lim_{\delta\to 0^+}\left(\frac{x}{\|x\|}\cdot N^L_x\right)\frac{\arctan(\|x\|)-\arctan(\|g_u(w_\delta)\|)}{\delta^{\frac{2}{n+1}}}\\ &\leq \lim_{\delta\to 0^+}\left(\frac{x}{\|x\|}\cdot N^L_x\right)\frac{\|x-g_u(w_\delta)\|}{\delta^{\frac{2}{n+1}}}.\end{aligned}$$

By Theorem 5.3, we deduce

$$\|x - g_u(w_\delta)\| \leq \left\| x - x_{\frac{\delta}{\cos(\beta)^{n+1}}} \right\|.$$

As before, with Theorem 2.10 and Lemma 3.11, we conclude

$$0 \leq \lim_{\delta \to 0^+} \frac{-u \cdot N_w^K}{\sin(d(u,w))} \frac{d(w, w_\delta)}{\delta^{\frac{2}{n+1}}} \leq \lim_{\delta \to 0^+} \left(\frac{x}{\|x\|} \cdot N_x^L \right) \frac{\left\| x - x_{\frac{\delta}{\cos(\beta)^{n+1}}} \right\|}{\delta^{\frac{2}{n+1}}} = 0.$$

□

This concludes the proof of Theorem 5.1.

5.3. The Floating Area

In this final section we will investigate some of the properties of the floating area. First we note that we may localize the floating area to a measure on the Borel σ-algebra $\mathcal{B}(\mathbb{S}^n)$ on $\mathbb{S}^n$ in the following way.

Definition (The Floating Measure)**.** For $K \in \mathcal{K}(\mathbb{S}^n)$ and $\omega \in \mathcal{B}(\mathbb{S}^n)$ we define the *floating measure* $\Omega(K, \omega)$ by

$$\Omega(K, \omega) = \int\limits_{\omega \cap \operatorname{bd} K} H_{n-1}^{\mathbb{S}^n}(K, w)^{\frac{1}{n+1}} \, dw.$$

The *floating area* $\Omega(K)$ of K is given by $\Omega(K) = \Omega(K, \mathbb{S}^n)$.

This notion is well defined since, by Theorem 5.1, the floating measure exists for all proper spherical convex bodies and is finite. For non-proper convex bodies the floating measure is identically zero. This is shown next.

Theorem 5.9. *The floating measure of a spherical polytope or a non-proper spherical convex body C is trivial, that is, $\Omega(C, \cdot) \equiv 0$.*

Proof. This is obvious since in both cases the Gauss–Kronecker curvature of C is zero for $\mathcal{H}^{n-1}$-almost all boundary points. Note that for $K = \mathbb{S}^n$ the floating measure is also trivial since $\operatorname{bd} \mathbb{S}^n = \emptyset$. □

As is the case for the affine surface area, the floating measure is a valuation.

Theorem 5.10. *Let $K, L \in \mathcal{K}(\mathbb{S}^n)$ such that $K \cup L \in \mathcal{K}(\mathbb{S}^n)$. Then, for all $\omega \in \mathcal{B}(\mathbb{S}^n)$, we have*

$$\Omega(K,\omega) + \Omega(L,\omega) = \Omega(K \cup L,\omega) + \Omega(K \cap L,\omega).$$

Proof. This can be proven similar to the Euclidean case by C. Schütt [**79**]. We will give a short outline of the argument. We decompose

$$\begin{aligned}
\operatorname{bd}(K\cup L) &= (\operatorname{bd}K \cap \operatorname{bd}L) \cup (\operatorname{bd}K \cap L^c) \cup (K^c \cap \operatorname{bd}L),\\
\operatorname{bd}(K\cap L) &= (\operatorname{bd}K \cap \operatorname{bd}L) \cup (\operatorname{bd}K \cap \operatorname{int}L) \cup (\operatorname{int}L \cap \operatorname{bd}L),\\
\operatorname{bd}K &= (\operatorname{bd}K \cap \operatorname{bd}L) \cup (\operatorname{bd}K \cap \operatorname{int}L) \cup (\operatorname{bd}K \cap L^c),\\
\operatorname{bd}L &= (\operatorname{bd}K \cap \operatorname{bd}L) \cup (\operatorname{int}K \cap \operatorname{bd}L) \cup (K^c \cap \operatorname{bd}L),
\end{aligned}$$

where $K^c = \mathbb{S}^n \backslash K$ and $L^c = \mathbb{S}^n \backslash L$. Thus the integrals cancel for all sets but $\operatorname{bd}K \cap \operatorname{bd}L$. So we are done, once we show that

$$\begin{aligned}
&\int\limits_{\operatorname{bd}K\cap\operatorname{bd}L} H_{n-1}^{\mathbb{S}^n}(K\cup L, w)^{\frac{1}{n+1}}\,dw + \int\limits_{\operatorname{bd}K\cap\operatorname{bd}L} H_{n-1}^{\mathbb{S}^n}(K\cap L, w)^{\frac{1}{n+1}}\,dw\\
&= \int\limits_{\operatorname{bd}K\cap\operatorname{bd}L} H_{n-1}^{\mathbb{S}^n}(K, w)^{\frac{1}{n+1}}\,dw + \int\limits_{\operatorname{bd}K\cap\operatorname{bd}L} H_{n-1}^{\mathbb{S}^n}(L, w)^{\frac{1}{n+1}}\,dw.
\end{aligned}$$

This follows from the fact that for $\mathcal{H}^{n-1}$-almost all $w \in \operatorname{bd}K \cap \operatorname{bd}L$ we have

$$\begin{aligned}
H_{n-1}^{\mathbb{S}^n}(K\cup L, w) &= \min\{H_{n-1}^{\mathbb{S}^n}(K, w), H_{n-1}^{\mathbb{S}^n}(L, w)\},\\
H_{n-1}^{\mathbb{S}^n}(K\cap L, w) &= \max\{H_{n-1}^{\mathbb{S}^n}(K, w), H_{n-1}^{\mathbb{S}^n}(L, w)\}.
\end{aligned}$$

This local result follows from the Euclidean case by applying the gnomonic projection in w. □

Next, we will show that the floating area is upper semicontinuous with respect to the spherical Hausdorff metric δ_s. It is easy to see, that the floating area Ω cannot be continuous. We may consider a sequence of spherical polytopes $(P_j)_{j\in\mathbb{N}}$ that converges to a spherical cap $C_u(\frac{\pi}{4})$. Then $\Omega(C_u(\frac{\pi}{4})) = \omega_{n-1}2^{-\frac{n-1}{2}} > 0$, but for all $j \in \mathbb{N}$ we have $\Omega(P_j, \mathbb{S}^n) = 0$.

Theorem 5.11. *The floating area is upper semicontinuous. Thus, for any sequence $(K_j)_{j\in\mathbb{N}}$ of convex bodies converging to a convex body K in the spherical Hausdorff distance, we have $\limsup_{j\to\infty} \Omega(K_j, \mathbb{S}^n) \le \Omega(K, \mathbb{S}^n)$.*

Proof. The proof of this theorem is along the lines of the proof of the upper semicontinuity of curvature integrals in the Euclidean setting by M. Ludwig [**46**]. Again, we will only give an outline. We denote the *m-th support measure* of a spherical convex body K by $\Theta_m(K,\cdot)$. It is a uniquely determined finite Borel measure on $\mathbb{S}^n \times \mathbb{S}^n$ (see e.g. [**76**] for precise definitions). Furthermore it is weakly continuous in the first argument, that is, $K_j \to K$ in the Hausdorff metric implies $\Theta_m(K_j,\cdot) \overset{w}{\to} \Theta_m(K,\cdot)$.

The *m-th curvature measure* $C_m(K,\cdot)$ of $K \in \mathcal{K}(\mathbb{S}^n)$ is defined, for $\omega \in \mathcal{B}(\mathbb{S}^n)$, by $C_m(K,\omega) = \Theta_m(K, \omega \times \mathbb{S}^n)$. $C_m(K,\omega)$ is concentrated on $\operatorname{bd} K$. The Hausdorff measure restricted to $\operatorname{bd} K$ is denoted by $\mathcal{H}^{n-1}_{\operatorname{bd} K}$ and is a Radon measure. Thus we may split $C_m(K,\cdot) = C^a_m(K,\cdot) + C^s_m(K,\cdot)$ such that $C^a_m(K,\cdot)$ is absolutely continuous with respect to $\mathcal{H}^{n-1}_{\operatorname{bd} K}$ and $C^s_m(K,\cdot)$ is the singular part. Moreover, for the absolutely continuous part we have

$$C^a_m(K,\omega) = \int\limits_{\omega \cap \operatorname{bd} K} H_{n-1-m}(K,w)\, dw,$$

for all $\omega \in \mathcal{B}(\mathbb{S}^n)$, and the singular part is concentrated on a null set, that is, there exists $\omega_0 \in \mathcal{B}(\mathbb{S}^n)$ such that $C^s_m(K, \omega \backslash \omega_0) = 0$ for all $\omega \in \mathcal{B}(\mathbb{S}^n)$.

Since Θ_m is weakly continuous in the first argument, so is C_m and we conclude for $m = 0$,

$$\limsup_{j\to\infty} \int\limits_{\omega \cap \operatorname{bd} K_j} H^{\mathbb{S}^n}_{n-1}(K_j,w)\, dw \le \limsup_{j\to\infty} C_0(K_j,\omega) \le C_0(K,\omega)$$

for all $\omega \in \mathcal{B}(\mathbb{S}^n)$. Using the arguments from [**46**, Section 4] and adapting the terminology, upper semicontinuity of the floating area follows. □

Using the Gauss map $N^K_{\cdot}$, we find an equivalent expression for the floating area of K as a curvature integral over the boundary of the polar of K.

Theorem 5.12. *Let $K \in \mathcal{K}(\mathbb{S}^n)$. Then*

$$\Omega(K) = \int\limits_{\operatorname{bd} K^\circ} H^{\mathbb{S}^n}_{n-1}(K^\circ, w)^{\frac{n}{n+1}}\, dw.$$

Proof. The proof is similar to the proof of Theorem 2.8 in [**37**] by D. Hug. Even more can be said: Let $\omega \in \mathcal{B}(\mathbb{S}^n)$ and denote by $\sigma(K,\omega)$ the set of $v \in \mathbb{S}^n$ such that v is an outer unit normal to some boundary point $w \in \operatorname{bd} K \cap \omega$. Clearly, $\sigma(K,\omega) \subseteq \operatorname{bd} K^\circ$. It follows that

$$\Omega(K,\omega) = \int\limits_{\sigma(K,\omega)} H_{n-1}^{\mathbb{S}^n}(K^\circ, w)^{\frac{n}{n+1}} \, dw.$$

Since $\sigma(K,\mathbb{S}^n) = \operatorname{bd} K^\circ$, this implies the statement. □

5.3.1. Isoperimetric Inequality

By Theorem 5.11, the floating area is upper semicontinuous on $\mathcal{K}(\mathbb{S}^n)$. Since $\mathcal{K}(\mathbb{S}^n)$ with the Hausdorff metric is compact (see, e.g., [**31**]), we may immediately conclude the existence of maximizers $C \in \mathcal{K}(\mathbb{S}^n)$ such that

$$\sup\{\Omega(K) : K \in \mathcal{K}(\mathbb{S}^n) \text{ such that } \operatorname{vol}_n(K) = c\} \leq \Omega(C)$$

for a fixed $c \in [0, \frac{\omega_n}{2}]$ and $\operatorname{vol}_n(C) = c$.

We believe that the only maximizers of the floating area are geodesic balls:

Conjecture 5.13. Let $K \in \mathcal{K}(\mathbb{S}^n)$. Then

$$\Omega(K) \leq \Omega(C^K), \tag{5.18}$$

where C^K is a spherical cap such that $\operatorname{vol}_n(C^K) = \operatorname{vol}_n(K)$. Equality holds if and only if K is spherical cap.

This conjecture is still open, but we are able to prove the following inequality.

Theorem 5.14. *Let $K \in \mathcal{K}(\mathbb{S}^n)$. Then*

$$\Omega(K) \leq P(K^\circ)^{\frac{1}{n+1}} P(K)^{\frac{n}{n+1}}, \tag{5.19}$$

where $P(K)$ denotes the perimeter of K. Equality holds if and only if K is a spherical cap.

Proof. Using Hölder's inequality, we obtain

$$\Omega(K) = \int\limits_{\operatorname{bd} K} H_{n-1}^{\mathbb{S}^n}(K,w)^{\frac{1}{n+1}} \, dw \leq \left(\int\limits_{\operatorname{bd} K} H_{n-1}^{\mathbb{S}^n}(K,w) \, dw \right)^{\frac{1}{n+1}} \left(\int\limits_{\operatorname{bd} K} dw \right)^{\frac{n}{n+1}}.$$

Since $N^K(\operatorname{bd} K) \subseteq \operatorname{bd} K^\circ$ and $J^{\operatorname{bd} K}(N^K)(w) = H_{n-1}^{\mathbb{S}^n}(K,w)$ for $\mathcal{H}^{n-1}$-almost all $w \in \operatorname{bd} K$, this implies (5.19).

That equality holds precisely for spherical caps follows from the fact, that equality holds in Hölder's inequality if and only if $H_{n-1}^{\mathbb{S}^n}(K,.)$ is constant. □

Another inequality for the floating area can be derived using the gnomonic projection and the affine isoperimetric inequality.

Theorem 5.15. *Let $K \in \mathcal{K}_0^p(\mathbb{S}^n)$. Then, for $u \in \operatorname{int} K$ and $0 < \alpha \leq \beta < \frac{\pi}{2}$ such that $C_u(\alpha) \subseteq K \subseteq C_u(\beta)$, we have*

$$\frac{\Omega(K)}{\omega_{n-1}} \leq \left(\frac{\cos(\alpha)^2 \tan(\beta)^n}{\tan(\alpha)^{n-1}} \frac{P(K)}{\omega_{n-1}} \right)^{\frac{n-1}{n+1}}. \tag{5.20}$$

Equality holds for spherical caps for which $\alpha = \beta$.

Proof. Using the gnomonic projection in u, (3.23) and (3.25), we obtain

$$\begin{aligned}\Omega(K) &= \int_{\operatorname{bd} g_u(K)} H_{n-1}^{\mathbb{S}^n}(K, g_u^{-1}(x))^{\frac{1}{n+1}} J^{\operatorname{bd} g_u(K)}(g_u^{-1})(x)\, dx \\ &= \int_{\operatorname{bd} g_u(K)} H_{n-1}^{\mathbb{R}^n_u}(g_u(K), x)^{\frac{1}{n+1}} \frac{1}{(1+\|x\|^2)^{\frac{n-1}{2}}}\, dx.\end{aligned}$$

Since $\tan(\alpha) \leq \|x\| \leq \tan(\beta)$ for all $x \in \operatorname{bd} g_u(K)$, we conclude

$$\cos(\beta)^{n-1} \operatorname{as}(g_u(K)) \leq \Omega(K) \leq \cos(\alpha)^{n-1} \operatorname{as}(g_u(K)).$$

Using the classical *affine isoperimetric inequality* for $L \in \mathcal{K}_0(\mathbb{R}^n)$,

$$\frac{\operatorname{as}(L)}{\omega_{n-1}} \leq \left(\frac{n \operatorname{vol}_n(L)}{\omega_{n-1}} \right)^{\frac{n-1}{n+1}},$$

gives

$$\frac{\Omega(K)}{\omega_{n-1}} \leq \left(\cos(\alpha)^{n+1} \frac{n \operatorname{vol}_n(g_u(K))}{\omega_{n-1}} \right)^{\frac{n-1}{n+1}}.$$

For the volume of the gnomonic projection we derive the inequality

$$\begin{aligned}\operatorname{vol}_n(g_u(K)) &\overset{(3.19)}{=} \int\limits_K \frac{dv}{\cos(d(v,u))^{n+1}} = \int\limits_{\operatorname{bd} K} \frac{-u \cdot N_w^K}{\sin(d(u,w))^n} \int\limits_0^{d(u,w)} \frac{\tan(t)^{n-1}}{\cos(t)^2}\, dt \\ &\leq \int\limits_{\operatorname{bd} K} \frac{1}{\sin(d(u,w))^{n-1}} \frac{\tan(d(u,w))^n}{n}\, dw \leq \frac{P(K)}{n} \frac{\tan(\beta)^n}{\sin(\alpha)^{n-1}},\end{aligned}$$

where we used the fact that $-u \cdot N_w^K = \sin(d(u, \mathbb{S}_{N_w^K})) \leq \sin(d(u,w))$. This concludes the prove of (5.20). It is easy to check that equality holds for spherical caps of radius $\alpha = \beta$. □

Inequality (5.20) is weaker than our conjectured inequality (5.18): For any $K \in \mathcal{K}_0^p(\mathbb{S}^n)$ and $u \in \operatorname{int} K$ such that $C_u(\alpha) \subseteq K \subseteq C_u(\beta)$ we have

$$\mathrm{vol}_n(C_u(\alpha)) \leq \mathrm{vol}_n(K) \leq \mathrm{vol}_n(C_u(\beta)).$$

Thus, for the spherical cap $C^K = C_u(\alpha_K)$ such that $\mathrm{vol}_n(C^K) = \mathrm{vol}_n(K)$, we have $\alpha \leq \alpha_K \leq \beta$. We therefore conclude for the right hand side of inequality (5.20) that

$$\left(\cos(\alpha_K)\sin(\alpha_K)\frac{P(K)}{\omega_{n-1}}\right)^{\frac{n-1}{n+1}} \leq \left(\frac{\cos(\alpha)^2\tan(\beta)^n}{\tan(\alpha)^{n-1}}\frac{P(K)}{\omega_{n-1}}\right)^{\frac{n-1}{n+1}}. \tag{5.21}$$

Our conjectured inequality (5.18) would imply

$$\Omega(K) \leq \Omega(C^K) = \cos(\alpha_K)^{\frac{n-1}{n+1}}\sin(\alpha_K)^{n\frac{n-1}{n+1}}\omega_{n-1},$$

which in turn would imply (5.20) by (5.21) and the isoperimetric inequality

$$P(K) \geq P(C^K) = \sin(\alpha_K)^{n-1}\omega_{n-1}.$$

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Index

Curriculum Vitae

FLORIAN BESAU
`florian.besau@tuwien.ac.at`
Vienna University of Technology
Wiedner Hauptstr. 8–10
1040 Vienna, Austria
`+43 1 58801 104762`

Personal information

Date of birth: April 8, 1986
Place of birth: Vienna, Austria
Nationality: Austria

Education

November 2012 – January 2015
Doctoral Programme in Technical Mathematics
Dotoral thesis: *Binary operations and floating bodies in spherical convexity.*
Vienna University of Technology
Advisor: Univ.Prof. Dr. Franz Schuster

March 2010 – October 2012
Master Programme in Mathematics in Computer Science
Master thesis: *Optimal transport on the n-sphere.*
Vienna University of Technology
Advisor: Univ.Prof. Dr. Franz Schuster
Co-Advisor: Dr. Gabriel Maresch

October 2006 – February 2010
Bachelor Programme in Mathematics in Computer Science
Vienna University of Technology

Professional experience

November 2012 – Dezember 2014
Research Assistant
Institute for Discrete Mathematics and Geometry
Vienna University of Technology
ERC Grant Project: *Isoperimetric Inequalities and Integral Geometry*

Publications

F. Besau and F. E. Schuster, *Binary operations in spherical convex geometry*, ArXiv e-print (July 2014), available at `1407.1153`

F. Besau and E. M. Werner, *The spherical convex floating body*, ArXiv e-print (November 2014), available at `1411.7664`