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DISSERTATION

Chern–Simons Holography

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Chern–Simons Holography

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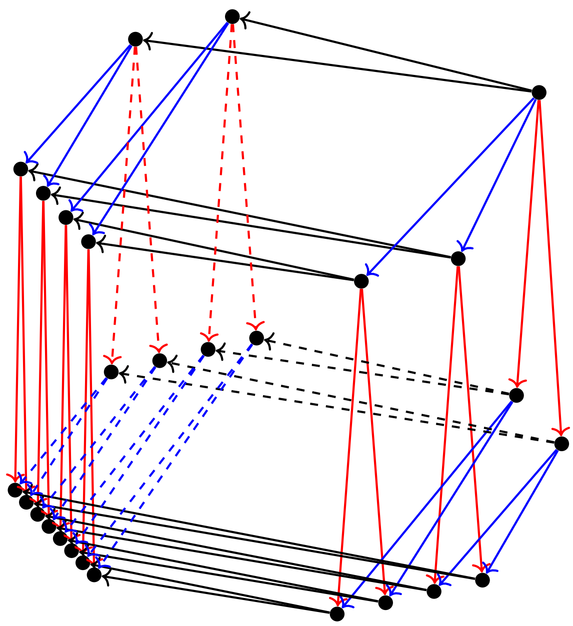

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Abstract

The holographic principle originates from the observation that black hole entropy is proportional to the horizon area and not, as expected from a quantum field theory perspective, to the volume. This principle has found a concrete realization in the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence. It is interesting to ponder whether the key insights about holography so far are specific to AdS/CFT or if they are general lessons for quantum gravity and (non)relativistic field theories.

Relativistic and nonrelativistic geometries play a fundamental role in advances of holography beyond AdS spacetimes, e.g., for strongly coupled systems in condensed matter physics. Holography for higher spin theories is comparably well understood and they are therefore good candidates to gain further insights. In three spacetime dimensions they are distinguished by technical simplicity, the possibility to write the theory in Chern–Simons form and the option to consistently truncate the infinite higher spin fields to any integer spin greater than two.

Here we will show progress that has been made to construct relativistic and nonrelativistic theories in spin-three gravity. These theories describe a coupled spin two and three field and are based on Chern–Simons theories with kinematical gauge algebras of which the Poincaré, Galilei and Carroll algebra are prominent examples. To have a spin-three theory where all fields are dynamical it is sometimes necessary, as will be shown, to extend the gauge algebras accordingly.

We will also discuss concepts which are useful in these constructions. Guidance is provided by combining Lie algebra contractions and, a procedure that will be reviewed extensively, double extensions.

Acknowledgments

It is unfortunate that I will not be able to acknowledge every person that has shaped and contributed to this work or the enjoyable time I had in preparing it. However, the following people deserve special thanks for their direct or indirect role in the development of this text.

Professional Community

I am grateful to my supervisors Daniel Grumiller and Mirah Gary for their continuous support and encouragement. Their guidance from my master's thesis to the completion of this PhD thesis was exceptional and the patience in answering my never-ending questions admirable. It was always a pleasure to come to "work". This is very much due to the enjoyable and friendly atmosphere that was created in our work group. Daniel was available for discussions and advice practically every time and as spontaneously as sometimes necessary. His enthusiasm, creativity, and commitment for physics and science in general is something I hope to be able to emulate myself.

Without the "office crew" consisting of Maria Irakleidou, Jakob Salzer, and Friedrich Schöller the last three years would not have been as gratifying as they were. The thorough discussions about physics and beyond shaped my understanding of nearly any aspect of life.

I am pleased to thank Jan Rosseel for fruitful collaborations and for introducing me to double extensions which play a fundamental role in this work. Also his valuable advise concerning social aspects of theoretical physics is very much appreciated.

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Contents

Notes to the Reader

In der Kürze liegt die Würze.
Brevity is the soul of wit.

This PhD thesis is based, some parts verbatim, on my master's thesis and the following publications:

[1] H. Afshar, A. Bagchi, S. Detournay, D. Grumiller, S. Prohazka, and M. Riegler, "Holographic Chern-Simons Theories," *Lect. Notes Phys.* **892** (2015) 311–329, arXiv:1404.1919 [hep-th].

[2] M. Gary, D. Grumiller, S. Prohazka, and S.-J. Rey, "Lifshitz Holography with Isotropic Scale Invariance," *JHEP* **1408** (2014) 001, arXiv:1406.1468 [hep-th].

[3] V. Breunhölder, M. Gary, D. Grumiller, and S. Prohazka, "Null warped AdS in higher spin gravity," *JHEP* **12** (2015) 021, arXiv:1509.08487 [hep-th].

[4] D. Grumiller, A. Perez, S. Prohazka, D. Tempo, and R. Troncoso, "Higher Spin Black Holes with Soft Hair," *JHEP* **10** (2016) 119, arXiv:1607.05360 [hep-th].

[5] E. Bergshoeff, D. Grumiller, S. Prohazka, and J. Rosseel, "Three-dimensional Spin-3 Theories Based on General Kinematical Algebras," *JHEP* **01** (2017) 114, arXiv:1612.02277 [hep-th].

[6] S. Prohazka, J. Salzer, and F. Schöller, "Linking Past and Future Null Infinity in Three Dimensions," *Phys. Rev.* **D95** no. 8, (2017) 086011, arXiv:1701.06573 [hep-th].

[7] M. Ammon, D. Grumiller, S. Prohazka, M. Riegler, and R. Wutte, "Higher-Spin Flat Space Cosmologies with Soft Hair," *JHEP* **05** (2017) 031, arXiv:1703.02594 [hep-th].

I do not intend to reproduce all the details of the aforementioned publications. It was the goal of the publications to provide sufficient information and repeating everything without further insights felt to be an uninteresting and especially useless endeavor. So, I will only discuss the parts of these publications that seemed necessary for my explanations or for other reasons.

My actual objective is to highlight and explain the underlying concepts used in these publications. At various stages I will review and combine information that would usually not find its way into a publication. The motivation is to get a deeper understanding and a bird eye's view upon them, in such a way that the results become somehow obvious once the fundamental theory is understood.

This led to some results that to my best knowledge are absent in the literature, e.g., I am not aware of a place where contractions of double extensions are studied.

I try to explain the abstract concepts and ideas first, often with a preference for simple examples over lengthy explanations. Furthermore, I did try to set my work into context with the literature that I think might be useful for future investigations.

At first sight and in seeming opposition to the proverb stated in the beginning of this note I will be redundant at various places and more explicit than necessary. The reason for this is that often it was useful for me. For instance having an abstract equation written explicitly in a basis, although an easy exercise to an expert, takes time and leaves place for errors. Of course this work is not exempt from errors, wrong and missing citations or other misconceptions and I am therefore grateful for any e-mail to prohazka@hep.itp.tuwien.ac.at that points them out. I furthermore hope that these sometimes explicit calculations and collections of formulas (mostly in the appendices) are useful to others.

Chapter 1

Introduction

Symmetries have always been a successful guiding principle in physics. Already Galilei realized that the everyday physical laws are invariant under transformations like rotations, time and space translations but also more general ones like the so called inertial transformations. Under the assumptions [8] that space is rotation invariant and boots form a noncompact subgroup (and another natural technical assumption) and sufficient knowledge of Lie algebras Galilei would have seen that his physical worldview might be an approximation of a more fundamental one. He would have arrived at the Poincaré algebra of which the Galilei algebra emerges as a contraction. See Figure 1.1, where at each corner sits a so called "kinematical algebra" which we will further discuss in Chapter 9.

The difference between the laws of physics how Galilei would have seen them and the relativistic ones can be made obvious by introducing the speed of light. We know nowadays that the speed of light is a finite constant approximately given by $c = 3 \cdot 10^8\,\mathrm{m/s}$. For nonrelativistic theories there is no reason for the speed of light to be finite and they are often described as approximate theories in the $c \to \infty$ limit of more fundamental relativistic ones (see Figure 1.1).

Nonrelativistic Theories

For many everyday phenomena the finiteness of the speed of light is of no relevant consequence and can be safely ignored. Interesting examples with technological interest are strongly coupled condensed matter systems (for a review see [9]) or the fractional quantum Hall effect [10–13]. In both cases nonrelativistic geometries play an influential role.

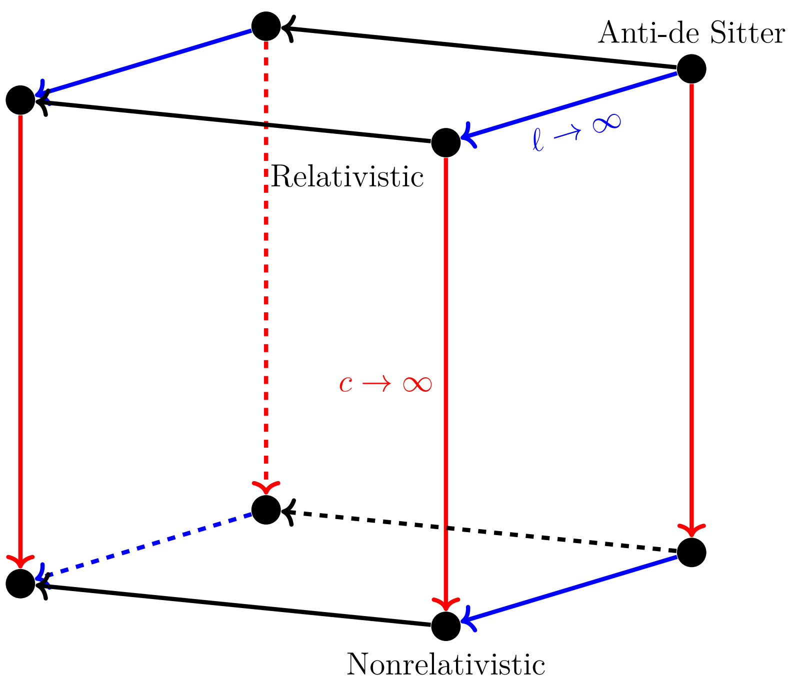

Figure 1.1: This figure shows that the relativistic symmetries can be understood as a contraction from the Anti-de Sitter symmetries where the universe is negatively curved with radius ℓ. From relativistic systems we can send the speed of light to infinity to arrive at the nonrelativistic ones.

AdS/CFT

Interestingly, new ways to analyze strongly coupled systems have been found [14–17] and are best understood on another place of the cube, to be specific, at the Anti-de Sitter corner (see Figure 1.1). Here one needs to introduce an additional constant which equals the curvature of the universe. These new techniques are due to the holographic principle [18, 19] which states that a quantum gravitational theory admits a dual description in terms of a non-gravitational quantum field theory in lower spacetime dimension. It is considered a key element of any approach to quantum gravity.

This principle found its realization in the famous AdS/CFT (Anti-de Sitter/Conformal field theory) correspondence [14–17]. But neither AdS spacetimes nor CFTs are strictly necessary for the holographic principle to be true. This begs the question if the tools used in AdS/CFT can lead to insights at other corners of the cube. For that it is useful to start with theories where the duality has been tested in detail and is comparably well understood. For that higher spin gravity seems like a good candidate which has passed various nontrivial checks in different dimensions (for reviews see [20, 21]).

Higher Spin Theories

A very interesting class of theories where holography is realizable is higher spin theory. Most of the work in this thesis is focused on $2 + 1$ dimensions where the theory admits a Chern–Simons formulation [22]. Much of the simplicity comes then from the fact that there might be a two-dimensional conformal field theory on the boundary. Due to the large amount of symmetries in two-dimensions these conformal field theories provide a high degree of analytic control and are therefore distinguished theories for the exploration of conceptual questions that seem far out of reach in higher dimensions. The higher spin bulk theory can be understood as a generalization of pure (2+1)-dimensional Einstein gravity in the Chern–Simons formulation [23,24] accompanied by bosonic higher spin fields, or as a simplified version of the Fradkin–Vasiliev theory [25]. These theories provide new insights with respect to possible dualities [26–29], higher spin generalizations of black holes [30], singularity resolution thereof [31], thermodynamics [32–34], entanglement entropy [35,36], holography [37] and string theory [38,39]. Therefore, this seems like an interesting starting point to look for generalizations.

This work centers around which of the above mentioned features are specific to AdS and which can be generalized. The discussion will be focused towards spacetimes that have the possibility to describe boundary theories with applications in, e.g., condensed matter physics [40, 41].

Two such spacetimes (Lifshitz and Null-warped) were realized explicitly in higher spin gravity, and consistent boundary conditions and the asymptotic symmetry algebra were provided [2, 3]. This showed that it is possible to realize spacetimes beyond AdS in higher spin gravity.

What was missing so far was a systematic procedure to go from higher spin Anti-de Sitter to (non)relativistic higher spin theories. Concepts that will provide this transition will be investigated in this thesis (see also [5]). It can be seen on the cover of this thesis that symmetry was again a useful guide in deriving these (non)relativistic higher spin geometries. Since nonrelativistic geometries play a central role in non-AdS holography [42–45] the hope is that their higher spin geometry generalization lead to an equal important generalization.

Outline

The Chapters 1 to 4 can be seen as introduction to the main part given by Chapters 7 to 10 after which conclusion, outlook and appendices follow. The introductory chapters are without reference to a specific gauge algebra and therefore of general interest. Furthermore, various statements generalize to any gauge theory that is based on a Lie algebra valued one-form. In the main part we will focus on specific examples of higher spin theories and follow closely the publications [1–7]. The appendices can in principle be omitted, but they fix the notation (see also the Index at the end) and provide useful additional information.

Chapter 2 The theory that this work is centered around, the Chern–Simons theory, is introduced. It is usually based on a gauge algebra with a symmetric invariant nondegenerate bilinear form (invariant metric) and each of these requirements is examined for its importance.

Chapter 3 Due to a structure theorem it is known how Lie algebras that posses such an invariant metric are constructed and it is therefore of interest to review the ingredients. Besides the direct sum of one-dimensional and simple Lie algebras, double extensions are introduced. This is beneficial for later considerations of kinematical algebras, since they are based on these concepts.

Chapter 4 For the study of approximate physical theories contractions are a useful tool since one is automatically guided by considerations of the original theory. Lie algebra contractions of different generality are discussed. Contractions are used later in Chapter 9 and 10 for the classification of (spin-3) kinematical algebras.

Chapter 5 A contracted Lie algebra that is useful for gauge theories should be accompanied by an (also contracted) invariant metric. For self-dual algebras a special invariant metric preserving contraction is defined.

Chapter 6 The global charges of Chern–Simons theories with boundary provide information concerning possible boundary theories and are therefore reviewed.

Chapter 7 After a short review of higher spin theories the standard $\mathcal{W}_3$ boundary conditions are introduced as $\hat{\mathfrak{u}}(1)$ composite objects [4, 7].

Chapter 8 Consistent boundary conditions for Lifshitz [2] and null-warped [3] spin-3 gravity and difficulties concerning their interpretation are reviewed.

Chapter 9 Kinematical algebras are analyzed and boundary conditions for Carroll gravity [5] are proposed.

Chapter 10 Using contractions spin-3 kinematical algebras are classified [5]. For spin-3 Carroll gravity the invariant metric preserving contractions of Chapter 5 show their usefulness whereas the considerations of Chapter 3 concerning double extensions provide spin-3 Galilei gravity with an invariant metric.

Chapter 11 Conclusions and a discussion of interesting open problems and possible future projects are provided.

Appendix A A summary of the conventions is provided in this appendix.

Appendix B A brief review of Lie algebra concepts that are used in the main part of this thesis is given, partially to fix the notation.

Appendix C Some explicit calculations for symmetry discussions for CS actions are provided.

Appendix D A useful and extensive overview of the various Lie algebras and their invariant metrics that underlie spin-2 and spin-3 gravity is given.

Chapter 2

Chern–Simons Theory

We start by introducing the theory that forms the foundation of this work, the Chern–Simons theory. It is based on a Lie algebra with an invariant metric. The importance of the properties of this symmetric nondegenerate invariant bilinear form will be examined.

2.1 Chern–Simons Action

The Lagrange density of the three-dimensional Chern–Simons (CS) theory [46] (see also [47,48] and [49]) is given by

$$\mathrm{CS}[A] = \langle dA \wedge A + \frac{2}{3} A \wedge A \wedge A \rangle \tag{2.1}$$

$$\equiv \langle dA \wedge A + \frac{1}{3} [A, A] \wedge A \rangle \tag{2.2}$$

$$= \langle \mathtt{T}_a \mathtt{T}_b \rangle (dA^a \wedge A^b + \frac{1}{3} f_{cd}{}^a A^c \wedge A^d \wedge A^b) \tag{2.3}$$

with some connection A. We also write $A = A_\mu\, dx^\mu = A^a \mathtt{T}_a = A^a{}_\mu\, \mathtt{T}_a \otimes dx^\mu \in \mathfrak{g} \otimes TM_3^*$, which shows that A is a Lie algebra valued one-form[1]. We define the commutator between Lie algebra valued one-forms by $[A, B] \equiv A^a \wedge B^b\, [\mathtt{T}_a, \mathtt{T}_b]$ where $[\mathtt{T}_a, \mathtt{T}_b] = f_{ab}{}^c \mathtt{T}_c$. The symmetric nondegenerate invariant bilinear

[1] By writing the Lagrangian density in this form we implicitly assume that the G bundle is trivial. The connection can otherwise not be regarded as a Lie algebra valued one-form although a suitable generalized definition exists (see, e.g., [50]). For connected, simply connected Lie groups on a three manifold the G bundle is necessarily trivial. So specifying the Lie group and not just the Lie algebra differentiates between the Lie groups whose Lie algebra is $\mathfrak{g}$ and provides additional information. We will ignore this subtleties in the following and restrict mainly to discussions of the Lie algebra. For more information see [50].

form, also called invariant metric, is denoted by $\langle \mathtt{T}_a \mathtt{T}_b \rangle$ (see Definition 2.1 in Section 2.2). Often this is written as $\mathrm{tr}(\mathtt{T}_a \mathtt{T}_b)$, but as will become clear this form can be defined without any reference to a matrix representation and a trace thereof. Therefore, this notation is reserved for places where the matrices are actually defined. Using the Lagrangian density the action is given by

$$I_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int_{M_3} \mathrm{CS}[A] \tag{2.4}$$

where M_3 denotes an oriented three-dimensional manifold.

As we have just defined the Chern–Simons (bulk) theory this leaves still some freedom:

1. The three-dimensional manifold is the spacetime and it is mostly assumed that we can decompose it as $M_3 = \mathbb{R} \times \Sigma$. The time part $\mathbb{R}$ might get identified periodically when black holes are discussed in an Euclidean setup. The space part Σ is for holographic purposes assumed to have an (asymptotic) boundary, see Figure 2.1.

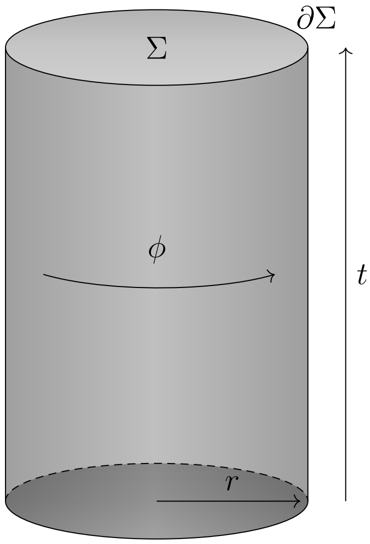

Figure 2.1: The three-dimensional manifold M_3.

2. Our goal is to describe three-dimensional gravitational theories using the Chern–Simons description [23,24]. The Lie algebra $\mathfrak{g}$ specifies then which one. The Chern–Simons theory based on $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ for example leads to three-dimensional gravity with negative cosmological constant, whereas $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{isl}(2,\mathbb{R})$ corresponds to positive and

vanishing cosmological constant, respectively. Lie algebras that have these Lie algebras as a subalgebra can be understood as a generalization of Einstein gravity, e.g., $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})$ is a generalization including an additional spin 3 field.

3. Additionally to the Lie algebra one needs to specify the invariant metric. Once a Lie algebra is chosen it might happen that the invariant metric has some freedom, outside of the overall scaling, that one needs to specify. Another possibility is that the Lie algebra might not posses an invariant metric. So, there is actually a tight connection between the Lie algebra and its possible invariant metric. To specify one without the other makes little sense.

   The importance of the various conditions of the invariant metric for a well defined Chern–Simons theory will be discussed in the next section. The kind of Lie algebras that have an invariant metric are reviewed in Section 3.

4. One point that might not seem obvious from the definition of the action is the importance of boundary conditions. Without specifying these the action is not well defined and from a holographic point of few the boundary conditions determine the possible boundary theories. They will be discussed in Section 6.

One point that differentiates Chern–Simons theory from other theories like electrodynamics and general relativity is that it is independent of any spacetime metric. It is thus a topological quantum field theory of Schwarz type, for a review see [51].

Another property of CS theories (in three dimensions) is that it has no local degrees of freedom in the bulk or in other words, there are no “Chern–Simons waves” propagating inside the spacetime. This fits nicely with the fact that pure three-dimensional gravity, for any value of the cosmological constant, also has no gravitational waves [52].

Variation and Equations of Motion

To get the equations of motion we vary the CS Lagrangian density

$$\begin{aligned} \delta\mathrm{CS}[A] &= \langle d\,\delta A \wedge A + (dA + 2A\wedge A)\wedge \delta A\rangle && (2.5)\\ &= \langle 2F\,\delta A\rangle - d\langle A\wedge \delta A\rangle\,. && (2.6) \end{aligned}$$

Here we have defined the Lie algebra valued two-form $F = dA + A\wedge A \equiv dA + \frac{1}{2}[A, A]$, which is the curvature of the connection. Given suitable boundary conditions, meaning that the boundary term in (2.6) vanishes when integrated, leads to the equations of motion that the curvature is flat $F = 0$, or more explicitly,

$$F^a_{\mu\nu} = \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + f^a{}_{bc} A^b{}_\mu A^c{}_\nu = 0\,. \tag{2.7}$$

Solutions to the equations of motion can locally be written as $A = g^{-1}\,dg$ for a group element g, see Appendix C.1.

2.2 Invariant Metric

We will now define what an invariant metric is. Afterwards will be examined why and to which extend each of its properties are really necessary for a well defined CS theory. This is of special importance since each part of the definition of the invariant metric is an additional restriction on the possible Lie algebras. For example, any Lie algebra would be possible for a well defined CS theory if we left out the condition of non-degeneracy. So it is of interest if one could relax some conditions and still get a well defined theory.

Definition 2.1. An **invariant metric** is a bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g}\times\mathfrak{g}\to K$ on a Lie algebra $\mathfrak{g}$ with field K which has the following three properties:

1. Symmetry
$$\langle X, Y\rangle = \langle Y, X\rangle \quad \text{for all} \quad X, Y \in \mathfrak{g}\,. \tag{2.8}$$

2. Non-degeneracy
$$\text{If} \quad \langle X, Y\rangle = 0 \quad \text{for all} \quad Y\in\mathfrak{g} \quad \text{then} \quad X = 0\,. \tag{2.9}$$

3. Invariance
$$\langle [Z, X], Y\rangle + \langle X, [Z, Y]\rangle = 0 \quad \text{for all} \quad X, Y, Z \in \mathfrak{g}\,. \tag{2.10}$$

A **symmetric self-dual Lie algebra**[2] is a Lie algebra possessing an invariant metric.

When there is no risk of confusion the comma between the two arguments of the bilinear form will be omitted. Given two symmetric self-dual algebras with their invariant metrics $(\mathfrak{g}_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathfrak{g}_2, \langle\cdot,\cdot\rangle_2)$ we can obtain a new symmetric self-dual algebra by using a direct sum of Lie algebras and the

[2] For details concerning the nomenclature see Remark 2.2. in [53].

orthogonal direct product metric $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \langle\cdot,\cdot\rangle_1 \dot{+} \langle\cdot,\cdot\rangle_2)$. A Lie algebra which can be written as such a direct sum is **decomposable**, if not it is **indecomposable**. Examples for indecomposable symmetric self-dual Lie algebras are simple and one-dimensional Lie algebras, whereas semisimple ones are decomposable. That there are symmetric self-dual Lie algebras beyond these examples will be shown in Section 3.

Using the basis $\mathtt{T}_a$ for the Lie algebra $[\mathtt{T}_a, \mathtt{T}_b] = f_{ab}{}^c \mathtt{T}_c$ and $\mathtt{T}_{ab} \equiv \langle \mathtt{T}_a, \mathtt{T}_b \rangle$ the conditions on the invariant metric in components are given by

$$\mathtt{T}_{ab} = \mathtt{T}_{ba} \qquad \text{(Symmetry)}, \tag{2.11}$$

$$\det(\mathtt{T}_{ab}) \neq 0 \qquad \text{(Non-degeneracy)}, \tag{2.12}$$

$$f_{ab}{}^d \mathtt{T}_{dc} + f_{ac}{}^d \mathtt{T}_{db} = 0 \qquad \text{(Invariance)}. \tag{2.13}$$

Not every Lie algebra admits an invariant metric, e.g., the three-dimensional Galilei algebra or the two-dimensional algebra $[\mathtt{T}_1, \mathtt{T}_2] = \mathtt{T}_1$, $[\mathtt{T}_1, \mathtt{T}_1] = [\mathtt{T}_2, \mathtt{T}_2] = 0$ do not. We will now analyze why symmetry, nondegeneracy and invariance are important properties for CS theories.

Symmetry

We start with the CS Lagrangian and ignore the condition that the invariant metric should be symmetric. One is then always able to decompose the bilinear form into symmetric and antisymmetric parts

$$\langle \mathtt{T}_a \mathtt{T}_b \rangle = \frac{1}{2}(\langle \mathtt{T}_a \mathtt{T}_b \rangle + \langle \mathtt{T}_b \mathtt{T}_a \rangle) + \frac{1}{2}(\langle \mathtt{T}_a \mathtt{T}_b \rangle - \langle \mathtt{T}_b \mathtt{T}_a \rangle) \tag{2.14}$$

$$= \langle \mathtt{T}_a \mathtt{T}_b \rangle_{\mathrm{S}} + \langle \mathtt{T}_a \mathtt{T}_b \rangle_{\mathrm{AS}}\,. \tag{2.15}$$

If we apply this to the CS Lagrangian the symmetric part reduces to the well known CS Lagrangian (2.1), the antisymmetric part reduces to a total derivative

$$\langle dA \wedge A + \frac{1}{3}[A, A] \wedge A \rangle_{\mathrm{AS}} = \frac{1}{2}\, d\langle A \wedge A \rangle_{\mathrm{AS}}\,. \tag{2.16}$$

The first term of the left hand side of (2.16) leads to the total derivative and the second one vanishes using the antisymmetry and the invariance of the bilinear form.

So, in principle, one could relax the symmetry condition, but one would merely change the theory by a total derivative[3] or equivalently, the equations of motion would stay unaltered.

[3] This might have implications for boundary theories.

Non-degeneracy

If we ignore the condition of non-degeneracy in the definition of the invariant metric then there exists a vector subspace $V \subset \mathfrak{g}$ of the Lie algebra to which the whole Lie algebra is orthogonal, i.e., $\langle V, \mathfrak{g}\rangle = 0$. An immediate consequence is that the fields that are part of V have no kinetic term $\langle A \wedge dA\rangle$ and are therefore not dynamical.

So non-degeneracy is necessary if we want a theory where all fields have a kinetic term.

Invariance

We will illustrate the importance of the invariance of the metric for non-abelian gauge theories by applying (part of) a gauge transformation $g = e^Z$ to $\langle X, Y\rangle$,

$$\langle g^{-1}Xg, g^{-1}Yg\rangle = \langle X, Y\rangle - \langle [Z, X], Y\rangle - \langle X, [Z, Y]\rangle + \mathcal{O}(Z^2)\,. \tag{2.17}$$

The invariance of the metric $\langle [Z, X], Y\rangle + \langle X, [Z, Y]\rangle = 0$ (equation (2.10)) is sufficiency that these kind of “gauge transformations” vanish. Not having this invariance property might lead to additional constraints for the possible Lie algebras.

If one inserts for X and Y the curvature and the Hodge dual curvature of a connection this calculation basically shows also the importance of the invariance of the metric for the gauge invariance of the Yang–Mills action. Invariance is also important for similar calculations concerning the CS theory in Section 6, as well as for other gauge theories like, e.g., the Wess–Zumino–Witten (WZW) model.

Summary

For a well defined Chern–Simons theory where all fields have a kinetic term it seems reasonable to search for Lie algebras with invariant metric, i.e., for symmetric self-dual Lie algebras. In the next chapter we will discuss what kind of Lie algebras posses such an invariant metric.

Chapter 3

Symmetric Self-dual Lie Algebras

Lie algebras that posses an invariant metric play a fundamental role for gauge theories in physics, e.g., as possible gauge algebras for Yang–Mills, CS and WZW theories.

We will now examine what kind of Lie algebras admit such a metric. The discussions in this chapter are general and independent of any specific gauge theory. Appendix B contains further details and definitions.

3.1 Reductive Lie Algebras and the Killing Form

Given a Lie algebra $\mathfrak{g}$ over a field real or complex field K one can always construct the **Killing form** $\kappa : \mathfrak{g} \times \mathfrak{g} \to K$ by defining

$$\kappa(X, Y) \equiv \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \quad \text{or using a basis} \quad \kappa(\mathtt{T}_a, \mathtt{T}_b) = f_{ac}{}^{d} f_{bd}{}^{c} \,. \quad (3.1)$$

The definition of the adjoint action (see appendix B.1) and the invariance of the trace under cyclic permutations shows that the Killing form is a symmetric invariant bilinear form on the Lie algebra. However, as stated by Cartan's criterion, in general the Killing form might be degenerate.

Theorem 3.1 (Cartan's criterion)**.** *A Lie algebra is semisimple if and only if its Killing form is non-degenerate.*

So it follows that only for the semisimple Lie algebras the Killing form automatically provides us with an invariant metric. For simple Lie algebras this invariant metric is even unique up to overall normalization.

Example 3.2 ($\mathfrak{sl}(2,\mathbb{R})$)**.** An example for a simple Lie algebra is $\mathfrak{sl}(2,\mathbb{R})$ given by the commutation relations

$$[\mathtt{L}_a, \mathtt{L}_b] = (a-b)\mathtt{L}_{a+b} \tag{3.2}$$

where $a = -1, 0, +1$. As just discussed, since the Lie algebra is simple we can construct the Killing form that then automatically provides us with an invariant metric. An explicit calculation gives the Killing form (for further details see Section D.1)

$$\kappa(\mathtt{L}_a\,\mathtt{L}_b) = \left(\begin{array}{c|ccc} & \mathtt{L}_{-1} & \mathtt{L}_0 & \mathtt{L}_{+1} \\ \hline \mathtt{L}_{-1} & 0 & 0 & -4 \\ \mathtt{L}_0 & 0 & 2 & 0 \\ \mathtt{L}_{+1} & -4 & 0 & 0 \end{array}\right) \tag{3.3}$$

which indeed fulfills all requirements of a well defined invariant metric. So does any invariant metric proportional to it.

For a semisimple Lie algebra one could now add a second $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra as a direct sum. So additionally to (3.2) we have now

$$[\widetilde{\mathtt{L}}_a, \widetilde{\mathtt{L}}_b] = (a-b)\widetilde{\mathtt{L}}_{a+b} \qquad\qquad [\widetilde{\mathtt{L}}_a, \mathtt{L}_b] = 0 \tag{3.4}$$

for which we get the Killing form $\kappa(\widetilde{\mathtt{L}}_a\,\widetilde{\mathtt{L}}_b) = \kappa(\mathtt{L}_a\,\mathtt{L}_b)$ and $\kappa(\mathtt{L}_a\,\widetilde{\mathtt{L}}_b) = 0$. For the direct sum we have two parameters in the invariant metric, one for each factor, that we can freely choose.

A generalization of semisimple Lie algebras is given by **reductive** Lie algebras which are direct sums of simple and abelian Lie algebras. Since the commutator of an abelian Lie algebra vanishes, so does their Killing form. Nevertheless is it possible to construct an invariant metric for reductive Lie algebras.

Example 3.3 ($\mathfrak{u}(1)$)**.** The abelian Lie algebra $\mathfrak{u}(1)$, which is the unique one-dimensional algebra, is given by the commutation relation $[\mathtt{T},\mathtt{T}] = 0$. Even though the Killing form is $\kappa(\mathtt{T},\mathtt{T}) = 0$ we can define an invariant metric by $\langle \mathtt{T},\mathtt{T}\rangle = \mu$, where μ is a nonzero real constant.

A reductive Lie algebra would then be for example the direct sum $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$, with the same invariant metrics as on their factors and $\langle \mathtt{L}_a, \mathtt{T}\rangle = 0$.

One might ask if one could take direct sums of Lie algebras and find an invariant metric that makes it indecomposable. This would mean for Example (3.2) that $\kappa(\mathtt{L}_a\,\widetilde{\mathtt{L}}_b) \neq 0$ or for Example 3.3 that $\langle \mathtt{L}_a, \mathtt{T}\rangle \neq 0$.

That this is not possible for the direct sum with a simple Lie algebra can be easily shown. Suppose we have a symmetric self-dual Lie algebra $\mathfrak{s} \oplus \mathfrak{g}$ which is a direct sum of a simple one $\mathfrak{s}$ with another arbitrary Lie algebra $\mathfrak{g}$. Then the invariant metric is orthogonal since $\langle \mathfrak{s}, \mathfrak{g} \rangle = \langle [\mathfrak{s}, \mathfrak{s}], \mathfrak{g} \rangle = \langle \mathfrak{s}, [\mathfrak{g}, \mathfrak{s}] \rangle = 0$ [53]. We have used that simple Lie algebras are perfect ($[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$), that the metric is invariant and afterwards that the Lie algebras are a direct sum. For abelian Lie algebras one can always find an isomorphism that also diagonalizes the invariant metric and therefore makes it decomposable. So, reductive Lie algebras are also always decomposable.

Many important gauge theories are based on reductive gauge algebras, e.g., electrodynamics with $\mathfrak{u}(1)$ and the Standard Model of particle physics with $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$.

3.2 Double Extensions

An interesting question is of course if there are Lie algebras possessing an invariant metric besides the reductive ones. We answer this in the affirmative, via the construction of double extensions, and discuss the construction of any such symmetric self-dual Lie algebra in the next section. This section is based on the work of Medina and Revoy [54], but we will follow closely [53]. For the notation see Appendix B or the Index (the symbol $\dot{+}$ means direct sum as vector space).

Definition 3.4 (Double extension [54])**.** Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ be a Lie algebra with an invariant metric on which a Lie algebra $\mathfrak{h}$ acts on via antisymmetric derivations, i.e.,

$$h \cdot [x, y]_{\mathfrak{g}} = [h \cdot x, y]_{\mathfrak{g}} + [x, h \cdot y]_{\mathfrak{g}} \quad \text{and} \quad \langle h \cdot x, y \rangle_{\mathfrak{g}} + \langle x, h \cdot y \rangle_{\mathfrak{g}} = 0\,. \tag{3.5}$$

Then we can define on the vector space $\mathfrak{g} \dot{+} \mathfrak{h} \dot{+} \mathfrak{h}^*$ the Lie algebra $\mathfrak{d}$, called the **double extension** of $\mathfrak{g}$ by $\mathfrak{h}$, by

$$\begin{aligned}&[(x, h, \alpha), (x', h', \alpha')] = \\ &\qquad ([x, x']_{\mathfrak{g}} + h \cdot x' - h' \cdot x, [h, h']_{\mathfrak{h}}, \beta(x, x') + \mathrm{ad}^*_h \cdot \alpha' - \mathrm{ad}^*_{h'} \cdot \alpha)\end{aligned} \tag{3.6}$$

where $x, x' \in \mathfrak{g}$, $h, h' \in \mathfrak{h}$, $\alpha, \alpha' \in \mathfrak{h}^*$. The skew-symmetric bilinear form $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}^*$ fulfills

$$\langle h \cdot x, y \rangle_{\mathfrak{g}} = \langle h, \beta(x, y) \rangle\,. \tag{3.7}$$

Double extensions are symmetric self-dual Lie algebras since they always carry an invariant metric defined by

$$\langle (x,h,\alpha),(x',h',\alpha')\rangle = \langle x,x'\rangle_{\mathfrak{g}} + \langle h,h'\rangle_{\mathfrak{h}} + \alpha(h') + \alpha'(h) \tag{3.8}$$

where $\langle\cdot,\cdot\rangle_{\mathfrak{h}}$ is a (possibly degenerate) invariant symmetric bilinear form on $\mathfrak{h}$.

The stars following $\mathfrak{h}^*$ and ad^* denote dual space and coadjoint representation, respectively. We will denote double extensions by $D(\mathfrak{g},\mathfrak{h})$ or the mnemonic $(\mathfrak{g}\oplus_c \mathfrak{h}^*) \ni \mathfrak{h}$. This also explains the name double extension since $\mathfrak{g}$ is central extended by $\mathfrak{h}^*$ which then split extends $\mathfrak{h}$.

Any nontrivial double extension, meaning that $\mathfrak{h}$ is nontrivial, is nonsemisimple. This is due to the abelian ideal $[\alpha,\alpha']=0$. If $\mathfrak{h}$ is also nonabelian we have a new class of symmetric self-dual Lie algebras.

Before we discuss further details of double extensions is it convenient to write it in a basis. For $\mathfrak{g}$ we fix the basis $\{\mathtt{G}_i\}$ in which the invariant metric of $\mathfrak{g}$ is given by $\Omega^{\mathfrak{g}}_{ij}$ and the commutation relations by $[\mathtt{G}_i,\mathtt{G}_j]_{\mathfrak{g}} = f_{ij}{}^k\mathtt{G}_k$. For the Lie algebra $\mathfrak{h}$ the basis $\{\mathtt{H}_\alpha\}$ has the Lie bracket $[\mathtt{H}_\alpha,\mathtt{H}_\beta]_{\mathfrak{h}} = f_{\alpha\beta}{}^\gamma\mathtt{H}_\gamma$ which acts via antisymmetric derivations

$$\mathtt{H}_\alpha\cdot\mathtt{G}_i = f_{\alpha i}{}^j\mathtt{G}_j \qquad (f_{\alpha i}{}^j = -f_{i\alpha}{}^j)\,. \tag{3.9}$$

That it is a derivation can be read of from $\mathtt{H}_\alpha\cdot[\mathtt{G}_i,\mathtt{G}_j] = [\mathtt{H}_\alpha\cdot\mathtt{G}_i,\mathtt{G}_j]+[\mathtt{G}_i,\mathtt{H}_\alpha\cdot\mathtt{G}_j]$, and is equivalent to

$$f_{\alpha k}{}^l f_{ij}{}^k = f_{\alpha i}{}^k f_{kj}{}^l + f_{ik}{}^l f_{\alpha j}{}^k \tag{3.10}$$

whereas the antisymmetry condition $\langle[\mathtt{H}_\alpha,\mathtt{G}_i],\mathtt{G}_j\rangle_{\mathfrak{g}} + \langle\mathtt{G}_i,[\mathtt{H}_\alpha,\mathtt{G}_j]\rangle_{\mathfrak{g}} = 0$ leads to

$$f_{\alpha i}{}^k\Omega^{\mathfrak{g}}_{kj} + f_{\alpha j}{}^k\Omega^{\mathfrak{g}}_{ki} = 0\,. \tag{3.11}$$

Its canonical dual basis is given by $\{\mathtt{H}^\alpha\}$.

Then the Lie algebra $\mathfrak{d} = D(\mathfrak{g},\mathfrak{h})$ defined on the vector space $\mathfrak{g}\dot{+}\mathfrak{h}\dot{+}\mathfrak{h}^*$ by

$$[\mathtt{G}_i,\mathtt{G}_j] = f_{ij}{}^k\mathtt{G}_k + f_{\alpha i}{}^k\Omega^{\mathfrak{g}}_{kj}\mathtt{H}^\alpha \tag{3.12}$$
$$[\mathtt{H}_\alpha,\mathtt{G}_i] = f_{\alpha i}{}^j\mathtt{G}_j \tag{3.13}$$
$$[\mathtt{H}_\alpha,\mathtt{H}_\beta] = f_{\alpha\beta}{}^\gamma\mathtt{H}_\gamma \tag{3.14}$$
$$[\mathtt{H}_\alpha,\mathtt{H}^\beta] = -f_{\alpha\gamma}{}^\beta\mathtt{H}^\gamma \tag{3.15}$$
$$[\mathtt{H}^\alpha,\mathtt{G}_j] = 0 \tag{3.16}$$
$$[\mathtt{H}^\alpha,\mathtt{H}^\beta] = 0 \tag{3.17}$$

is a double extension of $\mathfrak{g}$ by $\mathfrak{h}$. It has the invariant metric

$$\Omega^{\mathfrak{d}}_{ab} = \begin{array}{c c} & \begin{array}{ccc} \mathtt{G}_j & \mathtt{H}_\beta & \mathtt{H}^\beta \end{array} \\ \begin{array}{c} \mathtt{G}_i \\ \mathtt{H}_\alpha \\ \mathtt{H}^\alpha \end{array} & \begin{pmatrix} \Omega^{\mathfrak{g}}_{ij} & 0 & 0 \\ 0 & h_{\alpha\beta} & \delta_\alpha{}^\beta \\ 0 & \delta^\alpha{}_\beta & 0 \end{pmatrix} \end{array} \tag{3.18}$$

where $h_{\alpha\beta}$ is some arbitrary (possibly degenerate) invariant symmetric bilinear form on $\mathfrak{h}$.

Even though the notation $D(\mathfrak{g}, \mathfrak{h})$ might suggest otherwise further information is necessary to fully define the double extension. Since that can be clearly illustrated via the explicit Lie bracket realizations (3.12) to (3.17), the necessary expression is provided in the parentheses:

1. The Lie algebra $\mathfrak{g}$ ($f_{ij}{}^k$).
2. An invariant metric on $\mathfrak{g}$ ($\Omega^{\mathfrak{g}}_{ij}$).
3. The Lie algebra $\mathfrak{h}$ ($f_{\alpha\beta}{}^\gamma$).
4. The action (antisymmetric derivation) of $\mathfrak{h}$ on $\mathfrak{g}$ ($f_{\alpha i}{}^j$).

As can be seen the remaining structure is mandated by the given one.

To fix the invariant metric of the double extension one has to additionally provide the invariant symmetric bilinear form on $\mathfrak{h}$ ($h_{\alpha\beta}$). This part of the bilinear form can be freely chosen without disturbing the properties of the full invariant metric.

It might be illuminating to check that the double extended Lie algebra is indeed well defined. Antisymmetry for the right hand side of the $[\mathtt{G}_i, \mathtt{G}_j]$ commutator, see equation (3.12), follows from the definition of $\mathfrak{g}$ and the antisymmetry condition (3.11). Otherwise antisymmetry follows from the definition of the derivation and of the Lie algebra $\mathfrak{h}$.

To verify that the Jacobi identity is satisfied we will use that $f_{ijk} \equiv f_{ij}{}^l \Omega^{\mathfrak{g}}_{lk}$ and $f_{ij\alpha} \equiv f_{\alpha i}{}^k \Omega^{\mathfrak{g}}_{kj}$ are totally antisymmetric in ijk and $ij\alpha$, respectively. Then

$$\underset{ijk}{\circlearrowleft} [[\mathtt{G}_i, \mathtt{G}_j], \mathtt{G}_k] = \underset{ijk}{\circlearrowleft} f_{ij}{}^l f_{\alpha l}{}^m \Omega^{\mathfrak{g}}_{mk} \mathtt{H}^\alpha \tag{3.19}$$

$$= (f_{ljk} f_{\alpha i}{}^l + f_{ilk} f_{\alpha j}{}^l + f_{ki}{}^l f_{lj\alpha} + f_{jk}{}^l f_{li\alpha}) \mathtt{H}^\alpha \tag{3.20}$$

$$= 0 \tag{3.21}$$

where we used in the first line that $\mathfrak{g}$ itself satisfies Jacobi's identity, which leaves us with the remaining terms. In the second line the sum is expanded and the derivation condition (3.10) is used on the first term. The total antisymmetry of f_{ijk} and $f_{ij\alpha}$ can then be used to show that the first (second) and last (third) term cancel. The other identities terms can be verified in a similar manner.

3.3 Indecomposable Symmetric Self-dual Lie Algebras

In the last section we have seen that double extensions provide an additional way to construct Lie algebras with invariant metrics. The proof that all symmetric self-dual Lie algebras can be obtained by direct sums and double extensions of simple and one-dimensional Lie algebras is due to the structure theorem of Medina and Revoy [54]. Here, we add two additional refinements (3a and 3c) which, to my best knowledge, were first presented in [53].

Theorem 3.5. *Every indecomposable Lie algebra which permits an invariant metric, i.e., every indecomposable symmetric self-dual Lie algebra is either:*

1. *A simple Lie algebra.*
2. *A one-dimensional Lie algebra.*
3. *A double extended Lie algebra $D(\mathfrak{g}, \mathfrak{h})$ where:*
   a) *$\mathfrak{g}$ has no factor $\mathfrak{p}$ for which $H^1(\mathfrak{p}, \mathbb{R}) = H^2(\mathfrak{p}, \mathbb{R}) = 0$.*
   b) *$\mathfrak{h}$ is either simple or one-dimensional.*
   c) *$\mathfrak{h}$ acts on $\mathfrak{g}$ via outer derivations.*

Since every decomposable Lie algebra can be obtained from the indecomposable ones this theorem describes how all of them can be generated.

In Theorem 3.5 we have presented further restrictions on double extensions that are necessary to make them indecomposable. They are not sufficient as will be shown in Example 3.8. Before, the restrictions on indecomposable double extensions of Theorem 3.5 will be further discussed.

The condition 3a is necessary since otherwise the factor $\mathfrak{p}$ would also factor out of the double extension, i.e., $D(\mathfrak{p} \oplus \mathfrak{g}, \mathfrak{h}) = \mathfrak{p} \oplus D(\mathfrak{g}, \mathfrak{h})$ which makes it decomposable. This is basically due to the restriction 3c, since for such a factor $\mathfrak{p}$ all derivations are inner and these also factor out of double extensions [53]. Lie algebras $\mathfrak{p}$ with $H^1(\mathfrak{p}, \mathbb{R}) = H^2(\mathfrak{p}, \mathbb{R}) = 0$ are sometimes called pluperfect [53]. Partly because $H^1(\mathfrak{p}, \mathbb{R}) = 0$ is equivalent to the condition that $\mathfrak{p}$ is perfect ($[\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}$). The second condition $H^2(\mathfrak{p}, \mathbb{R}) = 0$

is equivalent to the condition that $\mathfrak{p}$ does not admit any nontrivial one-dimensional central extensions, see Appendix B.6. Semisimple Lie algebras are pluperfect and are therefore not allowed as factors if the resulting double extension should be indecomposable [53]. This restricts the class of Lie algebras that one could double extend to the abelian and the ones that have already been double extended.

It should be emphasized that there is no restriction concerning decomposability on $\mathfrak{g}$. Example 3.11 shows the double extension of a degenerate symmetric self-dual Lie algebra to an indecomposable one.

One special case are double extensions of trivial Lie algebras. For these the resulting symmetric self-dual Lie algebra is a semidirect sum since $D(0,\mathfrak{h}) = (0 \oplus_c \mathfrak{h}^*) \ni \mathfrak{h} = \mathfrak{h}^* \ni \mathfrak{h}$.

Example 3.6 (Poincaré)**.** The three-dimensional Poincaré algebra is a double extension of a trivial Lie algebra by $\mathfrak{so}(2,1)$, i.e., $D(0,\mathfrak{so}(2,1))$. This can be seen explicitly from the commutation relations

$$[\mathtt{J}_A, \mathtt{J}_B] = \epsilon_{AB}{}^C \mathtt{J}_C \qquad \left[\mathtt{J}_A, \mathtt{P}^B\right] = -\epsilon_{AC}{}^B \mathtt{P}^C \qquad \left[\mathtt{P}^A, \mathtt{P}^B\right] = 0 \tag{3.22}$$

and the invariant metric

$$\langle \mathtt{J}_A, \mathtt{J}_B \rangle = \eta_{AB} \qquad \langle \mathtt{J}_A, \mathtt{P}^B \rangle = \delta_A{}^B \qquad \langle \mathtt{P}^A, \mathtt{P}^B \rangle = 0\,. \tag{3.23}$$

3.4 Low-Dimensional Symmetric Self-dual Lie Algebras

To get more familiar with the above mentioned constructions the lowest dimensional symmetric self-dual Lie algebras will be discussed. Since the dimension 1 was already discussed in Example 3.3 we proceed with dimension 2.

Dimension 2

Example 3.7 ($\mathfrak{u}(1) \oplus \mathfrak{u}(1)$)**.** For the direct sum of two $\mathfrak{u}(1)$ algebras one has the commutation relations $[\mathtt{G}_i, \mathtt{G}_j] = 0$ for $i, j = 1, 2$ and the most general invariant metric is given by

$$\Omega = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \quad \text{for} \quad ab - c^2 \neq 0\,. \tag{3.24}$$

Since it is a symmetric matrix we can always find an invertible matrix that diagonalizes it. This in turn shows that there exists an isomorphism that makes it decomposable. Depending on the three parameters a, b and c the metric might be positive definite.

Similar reasoning generalizes to higher-dimensional abelian Lie algebras, which always admit an invariant metric and are decomposable.

Example 3.8 ($D(0, \mathfrak{u}(1))$)**.** We will now double extend a trivial $\mathfrak{g}$ by an abelian algebra $\mathfrak{u}(1)$. This example fulfills all the necessary requirements of Theorem 3.5 for a double extension to be indecomposable. In the end it will fail to be so. Of course, in the case at hand the conditions 3a and 3c are rather trivial.

Since $\mathfrak{u}(1)$ is abelian it follows that the double extension is also abelian, $[\mathtt{H}, \mathtt{H}] = [\mathtt{H}, \mathtt{H}^*] = [\mathtt{H}^*, \mathtt{H}^*] = 0$. This is the same Lie algebra as discussed in Example 3.7. As discussed, double extensions admit an invariant metric given by $\langle \mathtt{H}, \mathtt{H}^* \rangle = c$. In this example we could also add $\langle \mathtt{H}, \mathtt{H} \rangle = a$ and, what is more uncommon for double extensions, the $\langle \mathtt{H}^*, \mathtt{H}^* \rangle = b$ term. We will ignore these two terms subsequently. Using the isomorphism $\mathtt{H}^{\pm} = 1/2(\mathtt{H} \pm \mathtt{H}^*)$ we can show that the double extension is decomposable. The commutation relations remain that of an abelian algebra and the invariant metric is given by $\langle \mathtt{H}^{\pm}, \mathtt{H}^{\pm} \rangle = \pm 1$ and $\langle \mathtt{H}^+, \mathtt{H}^- \rangle = 0$. So the double extension is decomposable $D(0, \mathfrak{u}(1)) \simeq (\mathfrak{u}(1) \oplus \mathfrak{u}(1), \langle -, - \rangle_+ \dotplus \langle -, - \rangle_-)$.

As for generic double extension, with possibly $a \neq 0$ but $b = 0$, we see that the invariant metric is non positive definite.

Even though two Lie algebras might be isomorphic this might not be true when in addition their invariant metrics as additional structure are taken into consideration. An example for this phenomena would be the just mentioned abelian Lie algebras, where once we take the invariant metric (3.24) with $c = 0$ and once with $a = b = 0$. Even though the Lie algebras are isomorphic the metric is positive definite and indefinite, respectively.

Example 3.9 (Nonabelian)**.** In two dimensions up to isomorphism there is exactly one nonabelian Lie algebra. The nonzero commutator is $[\mathtt{G}_1, \mathtt{G}_2] = c_1 \mathtt{G}_1 + c_2 \mathtt{G}_2$ with the restriction that not both c_1 and c_2 are allowed to vanish. The most general invariant metric is then proportional to

$$\Omega = \begin{pmatrix} c_2^2 & -c_1 c_2 \\ -c_1 c_2 & c_1^2 \end{pmatrix} \tag{3.25}$$

which is always degenerate.

That it is the only nonabelian Lie algebra of dimension two can be shown with the isomorphism $\mathtt{G}_i = T_i{}^j \widetilde{\mathtt{G}}_j$ (see B.1), which leads with

$$\widetilde{c}_2 T_1{}^1 + \widetilde{c}_1 T_1{}^2 = -c_2 \tag{3.26}$$

$$\widetilde{c}_2 T_2{}^1 + \widetilde{c}_1 T_2{}^2 = c_1 \tag{3.27}$$

to $[\widetilde{\mathtt{G}}_1, \widetilde{\mathtt{G}}_2] = \widetilde{c}_1 \widetilde{\mathtt{G}}_1 + \widetilde{c}_2 \widetilde{\mathtt{G}}_2$. Except for $\widetilde{c}_1 = \widetilde{c}_2 = 0$ we can always find a invertible ${T_i}^j$ that fulfills (3.26) and (3.27).

Dimension 3

The smallest simple Lie algebras have dimension three. They have an invariant metric that is proportional to the Killing form and we therefore do not need to discuss it further. Furthermore, we have the three-dimensional abelian Lie algebra which was discussed in Example 3.7.

Example 3.10 ($D(\mathfrak{u}(1), \mathfrak{u}(1))$**.** This is the only possible three-dimensional double extension, but it leads to an abelian Lie algebra as we will show.

We start with $\mathfrak{g} = \mathfrak{u}(1)$ since it has an invariant metric, as is required for double extensions, see Example 3.3. It is not perfect and therefore $H^1(\mathfrak{u}(1), \mathbb{R}) \neq 0$, which is another requirement for a possible double extension. However there do not exist antisymmetric outer derivations since the most general derivation $\mathtt{D} \cdot \mathtt{G} = c\,\mathtt{G}$ (which is outer for $c \neq 0$),using the antisymmetry condition, leads to $\langle \mathtt{D} \cdot \mathtt{G}, \mathtt{G} \rangle + \langle \mathtt{G}, \mathtt{D} \cdot \mathtt{G} \rangle = 2c \langle \mathtt{G}, \mathtt{G} \rangle = 2ca$, which is nonzero for outer derivations and thus not antisymmetric.

Dimension 4

In exist no four-dimensional simple Lie algebras. Again there is the abelian algebra given by the direct sum of four $\mathfrak{u}(1)$ algebras. Additionally there are reductive Lie algebras given by the direct sum of the three-dimensional simple Lie algebras with $\mathfrak{u}(1)$.

In four dimensions exists the first indecomposable double extension. The Lie algebra $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$ is actually the only option for which an indecomposable double extension is possible.

Example 3.11 ($D(\mathfrak{u}(1) \oplus \mathfrak{u}(1), \mathfrak{u}(1))$**.** We start with the decomposable direct sum $\mathfrak{g} = \mathfrak{u}(1) \oplus \mathfrak{u}(1)$ explicitly given by the commutators $[\mathtt{G}_i, \mathtt{G}_j] = 0$ and the invariant metric $\langle \mathtt{G}_i, \mathtt{G}_j \rangle = \delta_{ij}$[1]. All conditions for a possibly indecomposable double extension are fulfilled since there exists an outer antisymmetric derivations given by $[\mathtt{H}, \mathtt{G}_i] = {\epsilon_i}^j \mathtt{G}_j$, where ${\epsilon_1}^2 = -{\epsilon_2}^1 = 1$. The double extension is then given by

$$[\mathtt{G}_i, \mathtt{G}_j] = {\epsilon_i}^k \delta_{kj} \mathtt{H}^* \tag{3.28}$$

$$[\mathtt{H}, \mathtt{G}_i] = {\epsilon_i}^j \mathtt{G}_j \tag{3.29}$$

$$[\mathtt{H}, \mathtt{H}^*] = 0 \tag{3.30}$$

[1] We restrict the invariant metric to specific values.

with the invariant metric

$$\Omega^{\mathfrak{d}}_{ab} = \begin{array}{c c} & \begin{array}{ccc} \mathtt{G}_j & \mathtt{H} & \mathtt{H}^* \end{array} \\ \begin{array}{c} \mathtt{G}_i \\ \mathtt{H} \\ \mathtt{H}^* \end{array} & \begin{pmatrix} \delta_{ij} & 0 & 0 \\ 0 & h & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array} . \tag{3.31}$$

This algebra has been used by Nappi and Witten to construct a nonsemisimple WZW model [55].

Summary

All symmetric self-dual Lie algebras are given by application of direct sums and/or double extensions to simple and/or $\mathfrak{u}(1)$ Lie algebras. The indecomposable ones are of the type described in Theorem 3.5.

Chapter 4

Contractions of Lie Algebras

Contractions go back to the works of Segal [56] and Inönü and Wigner [57]. While also mathematically interesting, in physics their importance comes from the fact that they are related to approximations. The probably most famous example is the contraction from the Poincaré group to the Galilei group [57], i.e., going from relativistic to nonrelativistic physics.

We will discuss contractions on the level of Lie algebras and in relation to invariant metrics, but often useful insights for other interesting structures like the Lie group and representations follow.

We start by introducing contractions, generalized and simple Inönü–Wigner contractions and briefly discuss their relations. Afterwards the effect of contraction on invariant metrics will be investigated.

We will follow partially [58, 59][1] where further details can be found.

4.1 Contractions

We will start with the most general Lie algebra contraction definition. For that we start with a Lie algebra $\mathfrak{g}$ with an underlying vector space V over $\mathbb{R}$.

Definition 4.1 (Contraction)**.** Let $T(\epsilon)$, with $0 < \epsilon \leq 1$, be a family of continuous non-singular linear maps on V. Then the Lie algebras

$$\mathfrak{g}_{T(\epsilon)} = (V, [\cdot,\cdot]_{T(\epsilon)}) \quad \text{for} \quad \epsilon > 0, \tag{4.1}$$

where

$$[x, y]_{T(\epsilon)} = T^{-1}(\epsilon)[T(\epsilon)x, T(\epsilon)y] \quad \text{with} \quad x, y \in V \tag{4.2}$$

[1]Lemma 2.2. in [59] is not correct and therefore the proof of Theorem 3.1. does not work [58, 60].

are isomorphic to $\mathfrak{g} = (V, [\cdot, \cdot])$. If the limit

$$[x, y]_T \equiv \lim_{\epsilon \to 0} [x, y]_{T(\epsilon)} \tag{4.3}$$

exists for all $x, y \in V$, then $[\cdot, \cdot]_T$ is a Lie product and the Lie algebra $\mathfrak{g}_T = (V, [\cdot, \cdot]_T)$ is called the **contraction** of $\mathfrak{g}$ by $T(\epsilon)$, in short,

$$\mathfrak{g} \xrightarrow{T(\epsilon)} \mathfrak{g}_T \,. \tag{4.4}$$

When a basis is fixed $T(\epsilon)$ is a matrix and we can define the limit on the structure constants by

$$(f_T)_{ab}{}^c \equiv \lim_{\epsilon \to 0} T(\epsilon)_a{}^d T(\epsilon)_b{}^e T^{-1}(\epsilon)_f{}^c f_{de}{}^f \,. \tag{4.5}$$

When the specific contraction is clear we will sometimes leave out the $T(\epsilon)$ or just write an ϵ.

Two contractions always exist:

1. $\mathfrak{g}_T \simeq \mathfrak{g}$: Contractions where the contracted Lie algebra is isomorphic to the original one are called **improper**. Such a contraction can be defined using just an identity matrix for $T(\epsilon)$.

2. Abelian $\mathfrak{g}_T$: This **trivial** contraction also always exists. One just has to set $T(\epsilon) = \mathrm{diag}(\epsilon, \ldots, \epsilon)$, which leads to

$$[x, y]_T = \lim_{\epsilon \to 0} \epsilon^{-1} [\epsilon x, \epsilon y] = \lim_{\epsilon \to 0} \epsilon [x, y] = 0 \,. \tag{4.6}$$

This definition of a contraction is more general than the one for Inönü–Wigner contractions [57] (IW-contractions) as well as for Saletan contractions [61], because we do not restrict to the existence of the limit

$$T(0) \equiv \lim_{\epsilon \to 0} T(\epsilon) \,. \tag{4.7}$$

In all cases the dimension of the Lie algebra stays unaltered.

One justification for this generalization is that this restriction is not necessary for the existence of the contracted Lie algebra. But it might be useful to have an equivalent contraction where $T(0)$ is well defined. Because there might arise situations where one wants to interpret the quantities $T(0)x$ and not only the contracted Lie algebra bracket. Therefore, it might be preferential if all components of $T(0)$ are finite, or are composed in such a way that the structure of interest is finite. One such a situation where this is the case will be seen in Section 5. Here by **equivalent** we mean contractions that lead from two isomorphic Lie algebras $\mathfrak{g} \simeq \mathfrak{g}'$ again to two isomorphic Lie algebras $\mathfrak{g}_T \simeq \mathfrak{g}'_S$, i.e.,

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "T(\epsilon)"] \arrow[d, leftrightarrow, "\simeq"'] & \mathfrak{g}_T \arrow[d, leftrightarrow, "\simeq"] \\ \mathfrak{g}' \arrow[r, "S(\epsilon)"] & \mathfrak{g}'_S \end{tikzcd} .$$

4.2 Generalized IW-contractions

A subclass of contractions are the generalized Inönü–Wigner contractions. Due to the diagonal form they are very useful for explicit calculations and they go back to work of Doebner and Melsheimer [62].

Definition 4.2. A contraction $\mathfrak{g} \xrightarrow{T(\epsilon)} \mathfrak{g}_T$ is called a **generalized Inönü–Wigner contraction** (gIW-contraction) if the matrix $T(\epsilon)$ has the form

$$T(\epsilon)_a{}^b = \delta_a{}^b \epsilon^{n_b} \quad \text{where} \quad n_b \in \mathbb{R};\ \epsilon > 0;\ a, b = 1, 2, \ldots, \dim(\mathfrak{g}) \tag{4.8}$$

for some basis $\mathtt{G}_1, \ldots, \mathtt{G}_N$.

Another way to write gIW-contractions is $T(\epsilon) = \mathrm{diag}(\epsilon^{n_1}, \ldots, \epsilon^{n_{\dim \mathfrak{g}}})$. There are no sums over the exponents n_a, which can be restricted to integer values without loss of generality . This includes negative exponents which, as already discussed, render the $\epsilon \to 0$ limit of $T(\epsilon)$ (see (4.8)) non-existent. Furthermore, the matrices $T(\epsilon)$ for gIW-contractions are not necessarily linear in ϵ, which differentiates them from IW-contractions [57] and Saletan contractions [61].

For a generic gIW-contraction our definition leads to

$$[\mathtt{G}_a, \mathtt{G}_b]_{T(\epsilon)} = \epsilon^{n_a + n_b - n_c} f_{ab}{}^c \mathtt{G}_c\,. \tag{4.9}$$

It is a contraction, i.e., well defined in the $\epsilon \to 0$ limit, if and only if $n_a + n_b - n_c \geq 0$ for nonzero $f_{ab}{}^c$. For such a well defined contraction we then get

$$(f_T)_{ab}{}^c = \begin{cases} f_{ab}{}^c & \text{if } n_a + n_b = n_c \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

4.3 Simple IW-contractions

A special class of gIW-contractions are the ones originally defined by Inönü and Wigner [57].

Definition 4.3. A **(simple) Inönü–Wigner contraction** ((s)IW-contraction) is a generalized Inönü–Wigner contraction where all n_b in

$$T(\epsilon)_a{}^b = \delta_a{}^b \epsilon^{n_b} \quad (\epsilon > 0) \tag{4.11}$$

are either 0 or 1.

An immediate consequence of this definition is that $T(0)$ always exists for sIW-contractions. This is of course always true when all $n_a \geq 0$.

The condition for the existence of sIW-contractions can be translated into conditions for Lie subalgebras. Suppose we start with a Lie algebra $\mathfrak{g}$ that is a (non-intersecting) vector space direct sum $\mathfrak{g} = \mathfrak{h} \dot{+} \mathfrak{i}$. We then set for $\mathfrak{h}$ all $n_a = 0$ and for $\mathfrak{i}$ all $n_a = 1$. We see, using (4.9), that the commutator of two elements of $\mathfrak{h}$ is not allowed to close into $\mathfrak{i}$ for the contraction $(\mathfrak{h} \dot{+} \mathfrak{i}) \xrightarrow{T(\epsilon)} (\mathfrak{h} \dot{+} \mathfrak{i})_T$ to be well defined because

$$[\mathfrak{h}, \mathfrak{h}]_{T(\epsilon)} = \mathfrak{h} \dot{+} \epsilon^{-1}\mathfrak{i} \tag{4.12}$$

is not well defined in the $\epsilon \to 0$ limit. So a Lie algebra can be contracted with a sIW-contraction with respect to a Lie subalgebra (e.g., $\mathfrak{h}$ above) and only with respect to a Lie subalgebra [57]. This subalgebra specifies the contracted Lie algebra (e.g., $(\mathfrak{h} \dot{+} \mathfrak{i})_T$) uniquely up to isomorphism [61]. This property makes sIW-contractions, although less general than arbitrary contractions and gIW-contractions, conceptually much easier and gives an easy criterion for when the contraction exists.

So explicitly the whole contraction $(\mathfrak{h} \dot{+} \mathfrak{i}) \xrightarrow{T(\epsilon)} (\mathfrak{h} \dot{+} \mathfrak{i})_T$ with respect to the subalgebra $\mathfrak{h}$ is given by

$$\begin{aligned}
[\mathfrak{h}, \mathfrak{h}]_{T(\epsilon)} &= \mathfrak{h} & & & [\mathfrak{h}, \mathfrak{h}]_T &= \mathfrak{h} & (4.13)\\
[\mathfrak{h}, \mathfrak{i}]_{T(\epsilon)} &= \epsilon\mathfrak{h} \dot{+} \mathfrak{i} & &\xrightarrow{T(\epsilon)} & [\mathfrak{h}, \mathfrak{i}]_T &= \mathfrak{i} & (4.14)\\
[\mathfrak{i}, \mathfrak{i}]_{T(\epsilon)} &= \epsilon\mathfrak{h} \dot{+} \epsilon^2\mathfrak{i} & & & [\mathfrak{i}, \mathfrak{i}]_T &= 0\,. & (4.15)
\end{aligned}$$

The ideal $\mathfrak{i}$ of $(\mathfrak{h} \dot{+} \mathfrak{i})_T$ is abelian and therefore any proper sIW-contraction leads to a nonsemisimple Lie algebra. The subalgebra $\mathfrak{h}$ stays unaltered under the contraction and is isomorphic to the quotient algebra $(\mathfrak{h} \dot{+} \mathfrak{i})_T/\mathfrak{i}$.

It should be noted, that the sIW-contractions do not exhaust all possible contractions. A Saletan contraction where no equivalent sIW-contraction exists was already constructed by Saletan in [61]. Even for the wider class of Saletan contractions, which are still linear in ϵ and include the sIW-contractions, no contraction from $\mathfrak{so}(3)$ to the Heisenberg algebra exists. On the other hand a gIW-contraction from $\mathfrak{so}(3)$ to the Heisenberg algebra exists, but there are other contractions where no equivalent diagonal gIW-contraction is possible. This phenomena starts with four-dimensional Lie algebras [58]. One might hope that every gIW-contraction is decomposable in sIW-contractions. In full generality this is not the case (for more details see, e.g., [58]).

Leaving this very general considerations aside a lot of physically interesting contractions, see e.g., [8, 57] are given by sIW and gIW-contractions and we will in the following restrict to these cases.

One remark should be added concerning this general discussion, especially because it will become important later. Here we have discussed Lie algebras merely on the level of an abstract mathematical structure without any reference to physics. When a physical Lie algebra is discussed specific Lie algebra generators have an interpretation, e.g., as generator of time translations. So just because two Lie algebras are isomorphic does not mean they are physically the same. Exchanging the interpretation of a rotation and a time translation leads to the same Lie algebras, but obviously our physical interpretation would change drastically. Suddenly elements that commuted with time translations do not anymore and are therefore not conserved. One such example are the Poincaré and para-Poincaré algebras that will be discussed later.

4.4 Contractions and Central Extensions

There is an interesting interplay between contractions and central extensions. The Lie algebra $\mathfrak{a}$ will be abelian and more details concerning Lie algebra cohomology are given in Appendix B. One consequence is the following diagram

$$\begin{tikzcd}
\mathfrak{g} \oplus \mathfrak{a} \arrow[r, "\text{Contraction}"] \arrow[d, "\text{Change of coboundary}"'] & \mathfrak{g}_T \oplus \mathfrak{a} \arrow[d, phantom, "\not\parallel"] \\
\mathfrak{g} \oplus \mathfrak{a} \arrow[r, "\text{Contraction}"'] & \mathfrak{g}_T \oplus_c \mathfrak{a}
\end{tikzcd}$$

which means that trivial central extension ($\mathfrak{g} \oplus \mathfrak{a}$) might lead after contraction to nontrivial ones ($\mathfrak{g} \oplus_c \mathfrak{a}$) [61, 63]. The reason for that being that the coboundary that makes a central extension trivial might be gone after the contraction. One famous example for this effect is that one can centrally extend the Poincaré algebra and can contract it to a nontrivial central extended Galilei algebra, the Bargmann algebra. This central term is of importance since it is the mass of the system.

Example 4.4. We start by trivially centrally extending the two-dimensional Lie algebra with the nonzero commutator $[\mathtt{X}, \mathtt{Y}] = \mathtt{X}$. We now change by a coboundary, which means that the central extension $\mathtt{Z}$ is still trivial, to get

$$[\mathtt{X}, \mathtt{Y}] = \mathtt{X} + \mathtt{Z}\,. \tag{4.16}$$

This can be implemented by shifting $\mathtt{X}$ by $\mathtt{Z}$. The contraction with $n_{\mathtt{X}} = 0$ and $n_{\mathtt{Y}} = n_{\mathtt{Z}} = 1$, which leads to

$$[\mathtt{X}, \mathtt{Y}]_\epsilon = \epsilon\mathtt{X} + \mathtt{Z} \tag{4.17}$$

shows now that that the central extension is not trivial anymore in the $\epsilon \to 0$ limit . The reason for this effect is that the necessary coboundary is gone or in other words that the shift by $\mathtt{X}$ is not possible anymore.

These considerations also have an influence on invariant metrics since nontrivial central extensions can then render a degenerate invariant metric nondegenerate. Further examples where this is of interest are given by the three-dimensional Galilei algebra, which can be centrally extended to the Extended Bargmann algebra. Similar to the above considerations contractions from a trivially extended (Anti)-de Sitter algebra or Poincaré algebra are possible, see Section 9.2.

Chapter 5

Contractions and Invariant Metrics

We now want to set Lie algebra contractions in relation to existence of invariant metrics. Instead of trying to give a complete discussion we will focus on examples that are of relevance for our later considerations.

5.1 Contraction of Invariant Metric

We start with a Lie algebra $\mathfrak{g}$ and a contraction $T(\epsilon)$. Given this contraction of the Lie algebra we can induce one on the invariant metric by

$$\langle x, y\rangle_{T(\epsilon)} = \langle T(\epsilon)x, T(\epsilon)y\rangle \, . \tag{5.1}$$

We see that a divergent $T(0)$ might lead to a divergent contracted invariant metric. Of course one could always ignore that an invariant metric exists for the original Lie algebra, contract it and look afterwards for invariant metrics. While this is certainly an option it might, e.g., for a theory given by an action, be beneficial to also have a contraction on the level of the invariant metric. For CS theories the contraction on the level of the invariant metric basically corresponds to the limit on the level of the action. Therefore, it might lead to additional insights and input for the contracted theory.

The considerations of Sections 4.2 and 4.3 can be adapted in a straightforward manner for invariant metrics and will therefore not be explicitly carried out.

5.2 Contraction to Inhomogeneous Lie Algebras

Here we will show how the invariant metric of the inhomogeneous Lie algebras can be derived via contractions from a direct sum of two simple Lie algebras $\mathfrak{g} \oplus \tilde{\mathfrak{g}}$. This is of special importance since this is how the Poincaré algebra and its higher spin generalizations in three spacetime dimensions are contracted. The Lie algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are isomorphic and since they are simple automatically admit an invariant metric. A basis for the first and second summand is given by $\mathtt{G}_a$ and $\tilde{\mathtt{G}}_a$, respectively. The commutation relations are of the form

$$[\mathtt{G}_a, \mathtt{G}_b] = f_{ab}{}^c \mathtt{G}_c \qquad [\mathtt{G}_a, \tilde{\mathtt{G}}_b] = 0 \qquad [\tilde{\mathtt{G}}_a, \tilde{\mathtt{G}}_b] = f_{ab}{}^c \tilde{\mathtt{G}}_c \tag{5.2}$$

with the most general invariant metric

$$\langle \mathtt{G}_a \mathtt{G}_b \rangle = \mu\, \Omega_{ab} \qquad \langle \mathtt{G}_a \tilde{\mathtt{G}}_b \rangle = 0 \qquad \langle \tilde{\mathtt{G}}_a \tilde{\mathtt{G}}_b \rangle = \tilde{\mu}\, \Omega_{ab} \tag{5.3}$$

where $\mu\tilde{\mu} \neq 0$. Defining

$$\mathtt{G}^{\pm}_a = \mathtt{G}_a \pm \tilde{\mathtt{G}}_a \tag{5.4}$$

leads to

$$[\mathtt{G}^+_a, \mathtt{G}^+_b] = f_{ab}{}^c \mathtt{G}^+_c \qquad [\mathtt{G}^+_a, \mathtt{G}^-_b] = f_{ab}{}^c \mathtt{G}^-_c \qquad [\mathtt{G}^-_a, \mathtt{G}^-_b] = f_{ab}{}^c \mathtt{G}^+_c \tag{5.5}$$

with the invariant metric

$$\langle \mathtt{G}^+_a \mathtt{G}^+_b \rangle = \mu^+\, \Omega_{ab} \qquad \langle \mathtt{G}^+_a \tilde{\mathtt{G}}^-_b \rangle = \mu^-\, \Omega_{ab} \qquad \langle \tilde{\mathtt{G}}^-_a \tilde{\mathtt{G}}^-_b \rangle = \mu^+\, \Omega_{ab}\,. \tag{5.6}$$

where $\mu^{\pm} = \mu \pm \tilde{\mu}$. The generators $\mathtt{G}^+_a$ span a Lie subalgebra with respect to which we now make a sIW-contraction. This leads to the Lie algebra $\mathfrak{g}_\epsilon$

$$[\mathtt{G}^+_a, \mathtt{G}^+_b]_\epsilon = f_{ab}{}^c \mathtt{G}^+_c \qquad [\mathtt{G}^+_a, \mathtt{G}^-_b]_\epsilon = f_{ab}{}^c \mathtt{G}^-_c \qquad [\mathtt{G}^-_a, \mathtt{G}^-_b]_\epsilon = \epsilon^2 f_{ab}{}^c \mathtt{G}^+_c \tag{5.7}$$

and the, for $\epsilon \to 0$ degenerate, bilinear form

$$\langle \mathtt{G}^+_a \mathtt{G}^+_b \rangle_\epsilon = \mu^+\, \Omega_{ab} \qquad \langle \mathtt{G}^+_a \mathtt{G}^-_b \rangle_\epsilon = \epsilon\mu^-\, \Omega_{ab} \qquad \langle \mathtt{G}^-_a \mathtt{G}^-_b \rangle_\epsilon = \epsilon^2 \mu^+\, \Omega_{ab}\,. \tag{5.8}$$

This degeneracy is to be expected since we have basically contracted the Killing form which for nonsemisimple Lie algebra should be degenerate. We know on the other hand that this can not be the most general invariant metric since the contracted algebra is a trivial double extension $D(0, \mathfrak{g}^+_0) = \mathfrak{g}^-_0 \ni \mathfrak{g}^+_0$ and therefore symmetric self-dual. We know from our earlier considerations that we can add $\langle \mathtt{G}_a^+ \mathtt{G}_b^- \rangle_0 = \mu^- \, \Omega_{ab}$ to make it nondegenerate. One might ask, if it is possible to also get this term using our current discussion. It works if one recognizes that one could rescale $\mu^- \mapsto \epsilon^{-1}\mu^-$ to cancel the ϵ term in (5.8) leading to the final contracted Lie algebra with an invariant metric

$$[\mathtt{G}_a^+, \mathtt{G}_b^+]_0 = f_{ab}{}^c \mathtt{G}_c^+ \qquad [\mathtt{G}_a^+, \mathtt{G}_b^-]_0 = f_{ab}{}^c \mathtt{G}_c^- \qquad [\mathtt{G}_a^-, \mathtt{G}_b^-]_0 = 0 \tag{5.9}$$

$$\langle \mathtt{G}_a^+ \mathtt{G}_b^+ \rangle_0 = \mu^+ \, \Omega_{ab} \qquad \langle \mathtt{G}_a^+ \mathtt{G}_b^- \rangle_0 = \mu^- \, \Omega_{ab} \qquad \langle \mathtt{G}_a^- \mathtt{G}_b^- \rangle_0 = 0 \, . \tag{5.10}$$

5.3 Invariant Metric Preserving Contraction

There exists a special class of contractions, which we will call invariant metric preserving, that lead from a double extended Lie algebra to another one. Therefore, it leaves the properties of the invariant metric untouched. This is done in a fashion that is naturally adapted to double extensions. and it is not just of theoretical importance. As we will see in Example 5.3, these contractions explain why the contraction of Poincaré to Carroll (higher spin) algebras in $2+1$ dimensions leaves the degeneracy of the invariant metric untouched. This gives another explanation for the algebras discussed in [5, 64]. To the best of my knowledge this special kind of contraction has not yet been discussed in the literature.

One starts with a double extended Lie algebra $D(\mathfrak{g}, \mathfrak{h} \dotplus \widetilde{\mathfrak{h}})$, where $\mathfrak{h}$ should be a Lie subalgebra. This allows us to perform a sIW-contraction on $\mathfrak{h} \dotplus \widetilde{\mathfrak{h}}$ with respect to the subalgebra $\mathfrak{h}$, since this is a subalgebra of the whole double extension. Now, this would not leave the invariant metric invariant since the important part for nondegeneracy $\langle \widetilde{\mathfrak{h}}, \widetilde{\mathfrak{h}}^* \rangle_{T(\epsilon)} = \epsilon \langle \widetilde{\mathfrak{h}}, \widetilde{\mathfrak{h}}^* \rangle$, would degenerate. But, this already hints towards the solution that we have to do the "dual", i.e., the inverse transformation on the dual space $\widetilde{\mathfrak{h}}^*$. Given the knowledge of double extensions this seems a very natural thing to do. We will now write this contraction explicitly in a basis.

Using the contraction $T(\epsilon)\widetilde{\mathfrak{h}} = \epsilon \widetilde{\mathfrak{h}}$ and $T(\epsilon)\widetilde{\mathfrak{h}}^* = \epsilon^{-1}\widetilde{\mathfrak{h}}^*$ where the remaining parts stay unaltered we write it, in hopefully obvious notation (we omit the subscript $T(\epsilon)$ for the Lie brackets)

$$
\begin{aligned}
[\mathtt{G}_i, \mathtt{G}_j] &= f_{ij}{}^{k}\mathtt{G}_k + f_{\alpha i}{}^{k}\Omega^{\mathfrak{g}}_{kj}\mathtt{H}^{\alpha} + \epsilon f_{\tilde{\alpha} i}{}^{k}\Omega^{\mathfrak{g}}_{kj}\widetilde{\mathtt{H}}^{\tilde{\alpha}} && (5.11)\\
[\mathtt{H}_\alpha, \mathtt{G}_i] &= f_{\alpha i}{}^{j}\mathtt{G}_j && (5.12)\\
[\widetilde{\mathtt{H}}_{\tilde{\alpha}}, \mathtt{G}_i] &= \epsilon f_{\tilde{\alpha} i}{}^{j}\mathtt{G}_j && (5.13)\\
[\mathtt{H}_\alpha, \mathtt{H}_\beta] &= f_{\alpha\beta}{}^{\gamma}\mathtt{H}_\gamma + \cancel{\epsilon^{-1} f_{\alpha\beta}{}^{\tilde{\gamma}}\widetilde{\mathtt{H}}_{\tilde{\gamma}}} && (5.14)\\
[\mathtt{H}_\alpha, \widetilde{\mathtt{H}}_{\tilde{\beta}}] &= \epsilon f_{\alpha\tilde{\beta}}{}^{\gamma}\mathtt{H}_\gamma + f_{\alpha\tilde{\beta}}{}^{\tilde{\gamma}}\widetilde{\mathtt{H}}_{\tilde{\gamma}} && (5.15)\\
[\widetilde{\mathtt{H}}_{\tilde{\alpha}}, \widetilde{\mathtt{H}}_{\tilde{\beta}}] &= \epsilon^2 f_{\tilde{\alpha}\tilde{\beta}}{}^{\gamma}\mathtt{H}_\gamma + \epsilon f_{\tilde{\alpha}\tilde{\beta}}{}^{\tilde{\gamma}}\widetilde{\mathtt{H}}_{\tilde{\gamma}} && (5.16)\\
[\mathtt{H}_\alpha, \mathtt{H}^\beta] &= -f_{\alpha\gamma}{}^{\beta}\mathtt{H}^\gamma - \epsilon f_{\alpha\tilde{\gamma}}{}^{\beta}\widetilde{\mathtt{H}}^{\tilde{\gamma}} && (5.17)\\
[\mathtt{H}_\alpha, \widetilde{\mathtt{H}}^{\tilde{\beta}}] &= -\cancel{\epsilon^{-1} f_{\alpha\gamma}{}^{\tilde{\beta}}\mathtt{H}^\gamma} - f_{\alpha\tilde{\gamma}}{}^{\tilde{\beta}}\widetilde{\mathtt{H}}^{\tilde{\gamma}} && (5.18)\\
[\widetilde{\mathtt{H}}_{\tilde{\alpha}}, \mathtt{H}^\beta] &= -\epsilon f_{\tilde{\alpha}\gamma}{}^{\beta}\mathtt{H}^\gamma - \epsilon^2 f_{\tilde{\alpha}\tilde{\gamma}}{}^{\beta}\widetilde{\mathtt{H}}^{\tilde{\gamma}} && (5.19)\\
[\widetilde{\mathtt{H}}_{\tilde{\alpha}}, \widetilde{\mathtt{H}}^{\tilde{\beta}}] &= -f_{\tilde{\alpha}\gamma}{}^{\tilde{\beta}}\mathtt{H}^\gamma - \epsilon f_{\tilde{\alpha}\tilde{\gamma}}{}^{\tilde{\beta}}\widetilde{\mathtt{H}}^{\tilde{\gamma}}\,. && (5.20)
\end{aligned}
$$

The two crossed terms indicate elements that would render the contraction not well defined. But the first, in (5.14), is no obstruction, because we require $\mathfrak{h}$ to be a subalgebra. This is just the usual condition for sIW-contractions. It is nice that this property automatically also renders the second crossed term nonexistent and therefore the whole contraction is well defined.

The corresponding invariant metric is given by

$$
\Omega^{\mathfrak{d}}_{ab} = \begin{array}{c} \\ \mathtt{G}_i \\ \mathtt{H}_\alpha \\ \widetilde{\mathtt{H}}_{\tilde{\alpha}} \\ \mathtt{H}^\alpha \\ \widetilde{\mathtt{H}}^{\tilde{\alpha}} \end{array}
\begin{array}{c}
\begin{array}{ccccc} \mathtt{G}_j & \mathtt{H}_\beta & \widetilde{\mathtt{H}}_{\tilde{\beta}} & \mathtt{H}^\beta & \widetilde{\mathtt{H}}^{\tilde{\beta}} \end{array}\\
\begin{pmatrix}
\Omega^{\mathfrak{g}}_{ij} & 0 & 0 & 0 & 0\\
0 & h_{\alpha\beta} & \epsilon h_{\alpha\tilde{\beta}} & \delta_\alpha{}^\beta & 0\\
0 & \epsilon h_{\tilde{\alpha}\beta} & \epsilon^2 h_{\tilde{\alpha}\tilde{\beta}} & 0 & \delta_{\tilde{\alpha}}{}^{\tilde{\beta}}\\
0 & \delta^\alpha{}_\beta & 0 & 0 & 0\\
0 & 0 & \delta^{\tilde{\alpha}}{}_{\tilde{\beta}} & 0 & 0
\end{pmatrix}
\end{array}
\qquad (5.21)
$$

and one can see that this special kind of contraction leaves it nondegenerate.

Given that after the $\epsilon \to 0$ limit we have again a symmetric self-dual Lie algebra one might ask what kind of double extension this contraction leads. It is of the form $D(\mathfrak{g} \oplus D(0, \widetilde{\mathfrak{h}}), \underset{\sim}{\mathfrak{h}})$. Notice that according to Theorem 3.5 the decomposability of $\mathfrak{g} \oplus D(0, \mathfrak{h})$ is no problem for the indecomposability of the new double extension. One nice feature of this contraction is that, like for sIW-contractions, just the specification of a subalgebra gives a very easy criterion for a well defined contraction. So we have proven the following theorem.

Theorem 5.1 (Invariant metric preserving contraction)**.** *Let the double extended Lie algebra $D(\mathfrak{g}, \mathfrak{h} \dot{+} \widetilde{\mathfrak{h}})$ have a Lie subalgebra $\mathfrak{h}$. Then a contraction of the form $T(\epsilon)\widetilde{\mathfrak{h}} = \epsilon\widetilde{\mathfrak{h}}$ and $T(\epsilon)\widetilde{\mathfrak{h}}^* = \epsilon^{-1}\widetilde{\mathfrak{h}}^*$ with the remaining elements unaltered, see (5.11) to (5.20), is a contraction that leads to a double extension $D(\mathfrak{g} \oplus D(0, \widetilde{\mathfrak{h}}), \mathfrak{h})$ explicitly given by*

$$[\mathtt{G}_i, \mathtt{G}_j] = f_{ij}{}^k \mathtt{G}_k + f_{\alpha i}{}^k \Omega^{\mathfrak{g}}_{kj} \mathtt{H}^\alpha \tag{5.22}$$
$$[\widetilde{\mathtt{H}}_{\widetilde{\alpha}}, \mathtt{G}_i] = 0 \tag{5.23}$$
$$[\widetilde{\mathtt{H}}_{\widetilde{\alpha}}, \widetilde{\mathtt{H}}_{\widetilde{\beta}}] = 0 \tag{5.24}$$
$$[\widetilde{\mathtt{H}}_{\widetilde{\alpha}}, \widetilde{\mathtt{H}}^{\widetilde{\beta}}] = -f_{\widetilde{\alpha}\gamma}{}^{\widetilde{\beta}} \mathtt{H}^\gamma \tag{5.25}$$
$$[\mathtt{H}_\alpha, \mathtt{G}_i] = f_{\alpha i}{}^j \mathtt{G}_j \tag{5.26}$$
$$[\mathtt{H}_\alpha, \widetilde{\mathtt{H}}_{\widetilde{\beta}}] = f_{\alpha\widetilde{\beta}}{}^{\widetilde{\gamma}} \widetilde{\mathtt{H}}_{\widetilde{\gamma}} \tag{5.27}$$
$$[\mathtt{H}_\alpha, \widetilde{\mathtt{H}}^{\widetilde{\beta}}] = -f_{\alpha\widetilde{\gamma}}{}^{\widetilde{\beta}} \widetilde{\mathtt{H}}^{\widetilde{\gamma}} \tag{5.28}$$
$$[\mathtt{H}_\alpha, \mathtt{H}_\beta] = f_{\alpha\beta}{}^\gamma \mathtt{H}_\gamma \tag{5.29}$$
$$[\mathtt{H}_\alpha, \mathtt{H}^\beta] = -f_{\alpha\gamma}{}^\beta \mathtt{H}^\gamma \tag{5.30}$$
$$\tag{5.31}$$

with the invariant metric

$$\Omega_{ab} = \begin{array}{cc} & \begin{array}{ccccc} \mathtt{G}_j & \mathtt{H}_\beta & \widetilde{\mathtt{H}}_{\widetilde{\beta}} & \mathtt{H}^\beta & \widetilde{\mathtt{H}}^{\widetilde{\beta}} \end{array} \\ \begin{array}{c} \mathtt{G}_i \\ \mathtt{H}_\alpha \\ \widetilde{\mathtt{H}}_{\widetilde{\alpha}} \\ \mathtt{H}^\alpha \\ \widetilde{\mathtt{H}}^{\widetilde{\alpha}} \end{array} & \begin{pmatrix} \Omega^{\mathfrak{g}}_{ij} & 0 & 0 & 0 & 0 \\ 0 & h_{\alpha\beta} & 0 & \delta_\alpha{}^\beta & 0 \\ 0 & 0 & 0 & 0 & \delta_{\widetilde{\alpha}}{}^{\widetilde{\beta}} \\ 0 & \delta^\alpha{}_\beta & 0 & 0 & 0 \\ 0 & 0 & \delta^{\widetilde{\alpha}}{}_{\widetilde{\beta}} & 0 & 0 \end{pmatrix} \end{array}. \tag{5.32}$$

We will call this type of contractions **invariant metric preserving**.

Ignoring the double extension structure and since we rescale $\widetilde{\mathfrak{h}}^*$ by inverse powers, this contraction is interpreted as a gIW-contraction. Taking the full structure into account one could see this contraction as a sIW-contraction and its dual.

One observation will be useful, when we want to explain why the sIW-contractions from the Poincaré to Carroll algebras lead to such a contraction.

Corollary 5.2. *For trivial double extensions, i.e., $D(0, \mathfrak{h} \dot{+} \widetilde{\mathfrak{h}})$ the contraction described in Theorem 5.3 equals to a sIW-contraction with respect to the subalgebra $\mathfrak{h} \dot{+} \widetilde{\mathfrak{h}}^*$*

This explains that even though sIW-contractions were done in [5] the invariant metric stayed nondegenerate.

Example 5.3 (Poincaré to Carroll)**.** The Poincaré algebra in 2+1 dimension is a trivial double extension $D(0, \mathfrak{h})$ where $\mathfrak{h} = \{\mathtt{J}, \mathtt{G}_a\}$ and $\mathfrak{h}^* = \{\mathtt{H}, \mathtt{P}_a\}$, see Table 5.1. There exists a sIW-contraction, with respect to the subalgebra $\{\mathtt{J}, \mathtt{P}_a\}$ to the Carroll algebra [8]. Similar to considerations of Section 5.2 we could have found the invariant metric of the Carroll algebra. But in this case it is equivalent to an invariant metric preserving contraction, with the notation of before $\mathtt{H}_\alpha = \mathtt{J}$, $\widetilde{\mathtt{H}}_{\widetilde{\alpha}} = \mathtt{G}_a$ and $\mathtt{H}^\alpha = \mathtt{H}$, $\widetilde{\mathtt{H}}^{\widetilde{\alpha}} = \mathtt{P}_a$.

This discussion generalizes to the contractions of the higher spin versions of Poincaré and Carroll.

	$\mathfrak{poi}$	$\mathfrak{car}$
$[\mathtt{J}, \mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J}, \mathtt{P}_a]$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a, \mathtt{G}_b]$	$-\epsilon_{ab}\mathtt{J}$	0
$[\mathtt{G}_a, \mathtt{H}]$	$-\epsilon_{am}\mathtt{P}_m$	0
$[\mathtt{G}_a, \mathtt{P}_b]$	$-\epsilon_{ab}\mathtt{H}$	$-\epsilon_{ab}\mathtt{H}$
$\langle \mathtt{H}, \mathtt{J} \rangle$	$-\mu^-$	$-\mu^-$
$\langle \mathtt{P}_a, \mathtt{G}_b \rangle$	$\mu^-\delta_{ab}$	$\mu^-\delta_{ab}$
$\langle \mathtt{J}, \mathtt{J} \rangle$	$-\mu^+$	$-\mu^+$
$\langle \mathtt{G}_a, \mathtt{G}_b \rangle$	$\mu^+\delta_{ab}$	0

Table 5.1: Poincaré and Carroll algebra and their invariant metrics.

Chapter 6

Charges and Boundary Conditions

According to the AdS/CFT dictionary asymptotic symmetries of the bulk theory correspond to global symmetries of the boundary theory. So to get information concerning possible boundary theories the asymptotic symmetry algebra is a very useful tool. To construct it one first needs to define differentiable gauge transformations. From there global charges can be defined, which one then quotients by the true (proper) gauge transformations.

Although they are not of direct importance to the considerations of these sections some possibly useful and explicit calculations in relation to symmetries of CS theories are summarized in Appendix C.

6.1 Global Charges

To construct global charges for CS theories we follow the approach pioneered by Regge and Teitelboim [65] (see also [66]) and first applied to CS theories by Bañados [67]. I will follow Section 3 of [27] (which is based on [67–69]), [34] and [70] where more information can be found.

We start by $2+1$ decomposing[1] the CS action (for the notation see

[1] We ignore terms at $t = \pm\infty$.

Appendix A.3)

$$
\begin{aligned}
I_{\mathrm{CS}}[A] &= \frac{k}{4\pi}\int_{M_3}\langle A\wedge dA + \frac{2}{3}A\wedge A\wedge A\rangle & (6.1)\\
&= \frac{k}{4\pi}\int_{\mathbb{R}\times\Sigma}\langle d_N\widetilde{A}\widetilde{A} + 2A_N\widetilde{F} + \widetilde{d}(A_N\widetilde{A})\rangle & (6.2)\\
&= \frac{k}{4\pi}\int_{\mathbb{R}\times\Sigma}\langle \dot{A}_iA_j + A_tF_{ij}\rangle\, dt\, dx^i\, dx^j + \frac{k}{4\pi}\int_{\mathbb{R}\times\partial\Sigma}\mathrm{tr}(A_t\widetilde{A}_i)\, dt\, dx^i\,. & (6.3)
\end{aligned}
$$

This action principle is that of a constrained system in Hamiltonian form, i.e., it has the form $\int(\dot{q}p - u\gamma(q,p))\,dt$. The $\dim(\mathfrak{g})$ Lagrange multipliers A_0 enforce the first-class constraints and the (bulk) Hamiltonian consists only of these. There are $2\cdot\dim(\mathfrak{g})$ canonical/dynamical fields A_i and via the standard formula (e.g., [71]), and since there are no second-class constraints, we get

$$
\begin{aligned}
2\cdot\begin{pmatrix}\text{Number of physical}\\ \text{degrees of freedom}\end{pmatrix} &= \begin{pmatrix}\text{Canonical}\\ \text{variables}\end{pmatrix} - 2\cdot\begin{pmatrix}\text{First-class}\\ \text{constraints}\end{pmatrix} & (6.4)\\
&= 2\dim(\mathfrak{g}) - 2\dim(\mathfrak{g}) & (6.5)\\
&= 0\,. & (6.6)
\end{aligned}
$$

So there are no (local) degrees of freedom (in the bulk).

The equal-time Poisson bracket for two differentiable functionals $M[A_i]$ and $N[A_i]$ is defined by

$$\{M,N\} = \frac{2\pi}{k}\int_\Sigma dx^i\wedge dx^j\left\langle\frac{\delta M}{\delta A_i(x)}\frac{\delta N}{\delta A_j(x)}\right\rangle. \tag{6.7}$$

Using the Poisson bracket first-class constraints generate gauge transformations (if they are differentiable) by defining the gauge generator

$$\bar{G}[\lambda] = \frac{k}{2\pi}\int_\Sigma\langle\lambda\widetilde{F}\rangle\,. \tag{6.8}$$

The variation of this gauge transformation shows that gauge generator is not differentiable as can be seen from the nonvanishing boundary term in

$$\delta\bar{G}[\lambda] = \frac{k}{2\pi}\int_\Sigma\langle\delta\lambda\,\widetilde{F} - \widetilde{\delta}_\lambda\widetilde{A}\wedge\delta\widetilde{A}\rangle + \frac{k}{2\pi}\int_{\partial\Sigma}\langle\lambda\delta\widetilde{A}\rangle \tag{6.9}$$

where

$$\widetilde{\delta}_\lambda\bullet \equiv \widetilde{d} + [\bullet,\lambda]\,. \tag{6.10}$$

Only differentiable gauge transformations are allowed to enter the Poisson bracket so one needs to add the boundary term $\delta Q[\lambda]$. Assuming that λ is independent of dynamical fields[2] $\delta Q[\lambda]$ can be integrated in field space and leads to

$$G[\lambda] = \bar{G}[\lambda] + Q[\lambda] \tag{6.11}$$

$$= \frac{k}{2\pi}\int_\Sigma \langle \lambda \widetilde{F}\rangle - \frac{k}{2\pi}\int_{\partial\Sigma}\langle \lambda \widetilde{A}\rangle \tag{6.12}$$

$$= \frac{k}{4\pi}\int_\Sigma dx^i \wedge dx^j \langle \lambda F_{ij}\rangle - \frac{k}{2\pi}\int_{\partial\Sigma} dx^i \langle \lambda A_i\rangle\,. \tag{6.13}$$

We can now plug this differentiable gauge generator into the Poisson algebra

$$\{G[\lambda], G[\sigma]\} = G[[\lambda,\sigma]] + \frac{k}{2\pi}\int_{\partial\sigma} dx^i \langle \lambda\, \partial_i \sigma\rangle\,. \tag{6.14}$$

One has to differentiate between two categories of differentiable gauge transformations [65, 66]:

- **Proper** or **true** gauge transformations are defined by $G[\lambda] = 0$ on the constrained surface $\tilde{F} = 0$. This implies that generically $Q[\lambda] = 0$ since this term does not automatically vanish on-shell. On the other hand $\bar{G}[\lambda] = 0$ vanishes automatically since this is the part that consists of the constraints. These are the true gauge symmetries of the system in the sense that they are a redundancy of the description. Or said in a more drastic fashion, proper gauge transformations do physically nothing. They form an ideal subalgebra of the differentiable gauge transformations.

- **Improper** gauge transformations are nonzero on the constraint surface and therefore $G[\lambda] = Q[\lambda] \neq 0$. These are no true gauge transformations and they lead to the global symmetries of the theory. They change the physical state of the system and are the origin of the boundary degrees of freedom.

When the constraints are solved and the gauge is fixed, the $Q[\lambda]$ give the global charges of the theory, which in turn generate the asymptotic symmetry algebra (when the quotient by the proper gauge symmetries is taken)[3]. The global symmetries are then generated by

$$\delta_\lambda M = \{Q(\lambda), M\} \tag{6.15}$$

[2] Here might appear a problem with integrability if this is not the case.
[3] This is possible since the proper gauge symmetries are an ideal.

and on the reduced phase space lead to

$$\{Q[\lambda], Q[\sigma]\} = Q[[\lambda, \sigma]] + \frac{k}{2\pi} \int_{\partial\sigma} dx^i \langle \lambda \, \partial_i \sigma \rangle \, . \tag{6.16}$$

6.2 Boundary Conditions

Once an action principle is fixed the procedure to establish boundary conditions "is one of trial and error" [34]. This means no bullet proof recipe is known, but one minimum requirement is that the extremized action gives the desired equations of motion up to surface terms at (spatial) infinity

$$\delta I_{CS}[A] = \frac{k}{2\pi} \int_{M_3} \langle F \wedge \delta A \rangle - \frac{k}{4\pi} \int_{\partial M_3} \langle A \wedge \delta A \rangle \tag{6.17}$$

$$\begin{aligned} &= \frac{k}{2\pi} \int_{\mathbb{R}\times\Sigma} dt \wedge \langle (\partial_t \widetilde{A} - \widetilde{\delta}_{A_t} \widetilde{A}) \delta \widetilde{A} + \widetilde{F} \delta A_t \rangle \\ &\quad + \frac{k}{4\pi} \int_{\mathbb{R}\times\partial\Sigma} dt \wedge \langle \widetilde{A} \delta A_t - A_t \delta \widetilde{A} \rangle \, . \end{aligned} \tag{6.18}$$

Using the boundary conditions the final action should be differentiable, i.e., extremized without additional boundary terms. Furthermore, the boundary conditions should allow for all solutions of interest.

For CS theories and ignoring any specific physical requirements one might have, there exist always (up to topological obstructions) boundary conditions that are related to the WZW model [49, 72].

Chapter 7

AdS Higher Spin Gravity

We will first review higher spin theories[1], with emphasis towards $(2+1)$-dimensional spacetimes. A nice and more complete review can be found in [27]. Afterwards we discuss, following closely [4], the $\mathfrak{u}(1)$ higher spin boundary conditions.

7.1 Higher Spin Theories

The equations for non interacting massless particles of integer spin in $(3+1)$ dimensions on a flat background were found by Frondsdal [74][2]. For $s = 0, 1, 2$ they reduce to the well known Klein–Gordon equation, Maxwell equation and to linearized general relativity. It is comparably easy to write down these free higher spin fields. But coupling these for $s > 2$ to gravity leads to various no-go theorems (for a review see [75]). Fradkin and Vasiliev [25] showed that consistent higher spin gauge theories involving gravity need to be defined on a curved background. They were first formulated by Vasiliev [76] (and are reviewed in [77–79]). These theories involve an infinite tower of massless fields and can be constructed on (A)dS spaces.

One interesting aspect of higher spin gauge fields is that they might be connected to string theory in the tensionless limit in which the massive excitations of string theory become massless. It is conjectured that string theory is a broken phase of a higher spin gauge theory. For more details see [80] and references therein.

Furthermore the holographic principle finds a realization in the form of the proposal made by Klebanov and Polyakov [81] and Sezgin and Sundell [82, 83]. They conjectured that there exists a duality in the large N limit of

[1] We partially follow [1, 73].

[2] We will restrict our explanations to integer spin for the sake of simplicity.

the critical 3-dimensional $O(N)$ model and the minimal bosonic higher spin theory in AdS_4. This holographic proposal got supported by calculations of Giombi and Yin [84] and is reviewed in [20].

In 2+1 spacetime dimensions the situation changes significantly. Massless gauge fields with "spin"[3] $s > 1$ posses no local degrees of freedom anymore. This makes theories in $2+1$ dimensions interesting in various aspects. While there is still enough structure to be nontrivial the technical difficulties that arise in the higher-dimensional cases are often circumvented.

This is already the case in the famous result by Brown and Henneaux [85] which can be seen as a precursor of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. They showed that three-dimensional Einstein–Hilbert gravity with a negative cosmological constant and Brown–Henneaux boundary conditions leads to asymptotic symmetries given by the infinite-dimensional conformal algebra in two dimensions. These are two copies of the Virasoro algebra (see Section D.5) with a nonvanishing central charge. Equivalent results were derived in the CS formulation, based on $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ [67].

The central charge appears again in the analysis of another unexpected result, the Bañados–Teitelboim–Zanelli (BTZ) black hole [86, 87]. Even though there are no local degrees of freedom in three-dimensional gravity, for the case of negative cosmological constant these black holes exist. Using the central charge it was shown that it is possible to calculate the asymptotic density of states and the entropy [88]. So a microscopic interpretation for the states of the black hole is possible and the holographic principle is realized.

To add interacting fields with spin $s > 2$, in contrast to the higher-dimensional case in $2+1$ dimension, no infinite number of higher spin fields are needed (at least in the classical theory) [89]. The Brown–Henneaux analysis has been generalized to higher spin fields [26–29]. In the case of the coupling of a spin-3 field to gravity the asymptotic symmetries are given by $\mathcal{W}_3 \oplus \mathcal{W}_3$ algebras [26, 27]. For a review of $\mathcal{W}$ algebras see [90] and for the explicit commutation relations see Section D.5. Fields of spin $s = 3, 4, \ldots, N$ coupled to gravity are given by a Chern-Simons theory with gauge algebra $\mathfrak{sl}(n,\mathbb{R}) \oplus \mathfrak{sl}(n,\mathbb{R})$ (see Appendix D for the commutation relations) and have in the case of an AdS_3 background the asymptotic symmetries $\mathcal{W}_N \oplus \mathcal{W}_N$ [28, 29]. Using the infinite-dimensional higher spin algebras $\mathfrak{hs}[\lambda] \oplus \mathfrak{hs}[\lambda]$ as gauge algebra we get gravity coupled to spin fields $s = 3, 4, \ldots, \infty$ and again for AdS_3 asymptotic symmetries $\mathcal{W}_\infty[\lambda] \oplus \mathcal{W}_\infty[\lambda]$ [28]. The $\mathfrak{hs}[\lambda]$ algebra can be truncated to $\mathfrak{sl}(N,\mathbb{R})$ for integer N, see Appendix D.4.

[3]"Spin" in $D = 3$ refers to the transformation properties of the field under Lorentz transformations.

Another aspect that is advantageous in $2+1$ dimensions is that the dual to AdS_3 is given by CFT_2 and extensions thereof. Two-dimensional conformal field theories are well understood and offer a high degree of analytic control. It was proposed by Gaberdiel and Gopakumar [37] that the $\mathfrak{hs}[\lambda]$ theory on AdS_3 is dual to the large-N limit of $\mathcal{W}_N$ minimal models on the CFT side. As a hint for the validity of this proposal can be seen that this limit on the CFT side leads, like in the bulk theory, also to a $\mathcal{W}_\infty$ algebra. The duality is reviewed in [21] and new developments can be found in [38, 39].

The BTZ black hole can also be generalized to higher spin black holes [30]. Since higher spin gauge theories have an extended gauge symmetry with respect to general relativity new questions concerning gauge invariant characterization and black hole thermodynamics arise (for a review of the proposed answers see [91, 92]).

Before background and boundary conditions beyond AdS_3 will be discussed it is useful to review the standard spin-3 ones. There exist excellent resources where they are derived from first principles [27, 34, 93] and therefore we will choose a different route. We will construct them following [4] where they are composed out of $\mathfrak{u}(1)$ boundary conditions [94].

7.2 $\mathcal{W}_3$ via $\hat{\mathfrak{u}}(1)$ Boundary Conditions

Higher spin gravity in $2+1$ dimensions can be generically described in terms of the difference of two Chern–Simons actions for independent gauge fields $A^\pm$ that take values in $\mathfrak{sl}(N,\mathbb{R})$, so that the action reads

$$I = I_{\mathrm{CS}}\Big[A^+\Big] - I_{\mathrm{CS}}\Big[A^-\Big]\,, \tag{7.1}$$

with

$$I_{\mathrm{CS}}[A] = \frac{k_N}{4\pi}\int_{M_3} \mathrm{tr}\Big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\Big)\,, \tag{7.2}$$

where $\mathrm{tr}(\cdots)$ stands for the trace in the fundamental representation of $\mathfrak{sl}(N,\mathbb{R})$ (see Appendix D.3). The level in (7.2) relates to the Newton constant and the AdS radius according to $k_N = \frac{k}{2\epsilon_N} = \frac{\ell}{8G\epsilon_N}$, whose normalization is determined by $\epsilon_N = \frac{N(N^2-1)}{12}$.

The gauge fields are related to a suitable generalization of the zuvielbein and the spin connection, defined through

$$A^\pm = \omega \pm \frac{e}{\ell} \tag{7.3}$$

and hence, the spacetime metric and the higher spin fields can be reconstructed from

$$g_{\mu\nu} = \frac{1}{\epsilon_N}\mathrm{tr}(e_\mu e_\nu) \qquad\qquad \Phi_{\mu_1\ldots\mu_s} = \frac{1}{\epsilon_N^{(s)}}\mathrm{tr}(e_{(\mu_1}\ldots e_{\mu_s)})\,. \tag{7.4}$$

Asymptotic Structure

The asymptotic structure of AdS gravity coupled to higher spin fields in three-dimensional spacetimes was investigated in [26, 27], where it was shown that the asymptotic symmetries are spanned by two chiral copies of $\mathcal{W}$ algebras (see also [28,29]). In order to accommodate the different higher spin black hole solutions in [30,31], and [34,95], the asymptotic behavior has to be extended so as to incorporate chemical potentials associated to the global charges. The one in [30,96] successfully accommodates the black hole solution with higher spin fields of [30], while the set of boundary conditions in [34,95] do for the higher spin black holes described therein. It is worth pointing out that the asymptotic symmetries of both sets are different.

Here we construct an inequivalent set of boundary conditions, which reduces to the one recently introduced in [94] when the higher spin fields are switched off. The asymptotic behavior of the $\mathfrak{sl}(3,\mathbb{R})$ gauge fields is proposed to be given by

$$A^\pm = b_\pm^{-1}\Big(d + a^\pm\Big)b_\pm \tag{7.5}$$

so that the dependence on the radial coordinate is completely contained in the group elements

$$b_\pm = \exp\bigg(\pm\frac{1}{\ell\zeta^\pm}\mathtt{L}_1\bigg)\cdot\exp\bigg(\pm\frac{\rho}{2}\mathtt{L}_{-1}\bigg)\,. \tag{7.6}$$

The auxiliary connection reads

$$a^\pm = \Big(\pm\mathcal{J}^\pm\, d\varphi + \zeta^\pm\, dt\Big)\mathtt{L}_0 + \Big(\pm\mathcal{J}^\pm_{(3)}\, d\varphi + \zeta^\pm_{(3)}\, dt\Big)\mathtt{W}_0 \tag{7.7}$$

where $\mathtt{L}_i, \mathtt{W}_n$, with $i = -1, 0, 1$, and $n = -2, -1, 0, 1, 2$, span the $sl(3,\mathbb{R})$ algebra (see Appendix D.2). Following [95], it can be seen that $\mathcal{J}^\pm$ and $\mathcal{J}^\pm_{(3)}$ stand for arbitrary functions of (advanced) time and the angular coordinate that correspond to the dynamical fields, while $\zeta^\pm$ and $\zeta^\pm_{(3)}$ describe their associated Lagrange multipliers that can be assumed to be fixed at the boundary without variation ($\delta\zeta^\pm = \delta\zeta^\pm_{(3)} = 0$). We shall refer to $\zeta^\pm$, $\zeta^\pm_{(3)}$ as chemical potentials.

The field equations, implying the local flatness of the gauge fields, then reduce to

$$\dot{\mathcal{J}}^{\pm} = \pm\zeta' \qquad \dot{\mathcal{J}}^{\pm}_{(3)} = \pm\zeta'_{(3)} \;, \tag{7.8}$$

where dot and prime denote derivatives with respect to t and φ, respectively.

Asymptotic Symmetries and Canonical Generators

In the canonical approach [65], the variation of the conserved charges

$$Q[\epsilon^{+}, \epsilon^{-}] = \mathcal{Q}^{+}[\epsilon^{+}] - \mathcal{Q}^{-}[\epsilon^{-}] \tag{7.9}$$

associated to gauge symmetries spanned by $\epsilon^{\pm} = \epsilon^{\pm}_{i}\mathtt{L}_i + \epsilon^{\pm}_{(3)n}\mathtt{W}_n$, that maintain the asymptotic form of the gauge fields, is determined by

$$\delta\mathcal{Q}^{\pm}\left[\epsilon^{\pm}\right] = \mp\frac{k}{4\pi}\int d\varphi\left(\eta^{\pm}\delta\mathcal{J}^{\pm} + \frac{4}{3}\eta^{\pm}_{(3)}\delta\mathcal{J}^{\pm}_{(3)}\right) \;, \tag{7.10}$$

with $\eta^{\pm} = \epsilon^{\pm}_{0}$, and $\eta^{\pm}_{(3)} = \epsilon^{\pm}_{(3)0}$. According to (7.7), the asymptotic symmetries fulfill $\delta_{\epsilon^{\pm}} a^{\pm} = d\epsilon^{\pm} + [a^{\pm}, \epsilon^{\pm}] = \mathcal{O}(\delta a^{\pm})$, provided that the transformation law of the dynamical fields reads

$$\delta\mathcal{J}^{\pm} = \pm\eta^{\pm\prime} \qquad \delta\mathcal{J}^{\pm}_{(3)} = \pm\eta^{\pm\prime}_{(3)} \tag{7.11}$$

and the parameters are time-independent ($\dot{\eta}^{\pm} = \dot{\eta}^{\pm}_{(3)} = 0$). One has to take the quotient over the remaining components of $\epsilon^{\pm}$, since they just span trivial gauge transformations that neither appear in the variation of the global charges nor in the transformation law of the dynamical fields.

The surface integrals that correspond to the conserved charges associated with the asymptotic symmetries then readily integrate as

$$\mathcal{Q}^{\pm}\left[\eta^{\pm}, \eta^{\pm}_{(3)}\right] = \mp\frac{k}{4\pi}\int d\varphi\left(\eta^{\pm}(\varphi)\mathcal{J}^{\pm}(\varphi) + \frac{4}{3}\eta^{\pm}_{(3)}(\varphi)\mathcal{J}^{\pm}_{(3)}(\varphi)\right) \;, \tag{7.12}$$

which are manifestly independent of the radial coordinate ρ. Consequently, the boundary could be located at any fixed value $\rho = \rho_0$. Hereafter, we assume that $\rho_0 \to \infty$, since this choice has the clear advantage of making our analysis to cover the entire spacetime in bulk.

The algebra of the global charges can then be obtained directly from the computation of their Poisson brackets; or as a shortcut, by virtue of $\delta_Y Q[X] = \{Q[X], Q[Y]\}$, from the variation of the dynamical fields in (7.11). Expanding in Fourier modes

$$\mathcal{J}^{\pm}(\varphi) = \frac{2}{k}\sum_{n=-\infty}^{\infty} J^{\pm}_{n} e^{\pm in\varphi} \qquad \mathcal{J}^{\pm}_{(3)}(\varphi) = \frac{3}{2k}\sum_{n=-\infty}^{\infty} J^{(3)\pm}_{n} e^{\pm in\varphi} \tag{7.13}$$

leads to the asymptotic symmetry algebra which is described by a set of $\hat{\mathfrak{u}}(1)$ currents whose nonvanishing brackets are given by

$$i\Big\{J_n^{\pm},J_m^{\pm}\Big\}=\frac{1}{2}kn\delta_{m+n,0}\qquad\qquad i\Big\{J_n^{(3)\pm},J_m^{(3)\pm}\Big\}=\frac{2}{3}kn\delta_{m+n,0}\;,\tag{7.14}$$

with levels $\frac{1}{2}k$, and $\frac{2}{3}k$, respectively.

(Higher Spin) Soft Hair

Following the spin-2 construction [94], we consider now all vacuum descendants $|\psi(q)\rangle$ labeled by a set q of non-negative integers $N^{\pm}$, $N^{\pm}_{(3)}$, $n_i^{\pm}$, $n_i^{(3)\,\pm}$, $m_i^{\pm}$ and $m_i^{(3)\,\pm}$

$$\Big|\psi(q)\rangle=N(q)\prod_{i=1}^{N^{\pm}}\Big(J^{\pm}_{-n_i^{\pm}}\Big)^{m_i^{\pm}}\prod_{i=1}^{N^{\pm}_{(3)}}\Big(J^{(3)\,\pm}_{-n_i^{(3)\,\pm}}\Big)^{m_i^{(3)\,\pm}}\Big|0\rangle\,.\tag{7.15}$$

Here $N(q)$ is some normalization constant such that $\langle\psi(q)|\psi(q)\rangle=1$ and the vacuum state[4] is defined through highest weight conditions, $J_n^{\pm}|0\rangle=J_n^{(3)\,\pm}|0\rangle=0$ for non-negative n.

We want to check now if all vacuum descendants $|\psi(q)\rangle$ have the same energy as the vacuum and are thus soft hair (our discussion easily generalizes from soft hair descendants of the vacuum to soft hair descendants of any higher spin black hole state). To this end we consider the surface integral associated with the generator in time, given by

$$H:=Q(\partial_t)=\frac{k}{4\pi}\int d\varphi\Big(\zeta^+\mathcal{J}^++\zeta^-\mathcal{J}^-+\frac{4}{3}\zeta^+_{(3)}\mathcal{J}^+_{(3)}+\frac{4}{3}\zeta^-_{(3)}\mathcal{J}^-_{(3)}\Big)\,.\tag{7.16}$$

For constant chemical potentials $\zeta^{\pm}$, $\zeta^{\pm}_{(3)}$ the field equations (7.8) imply that the dynamical fields become time-independent, and the total Hamiltonian reduces to

$$H=\zeta^+J_0^++\zeta^-J_0^-+\zeta^+_{(3)}J_0^{(3)+}+\zeta^-_{(3)}J_0^{(3)-}\;,\tag{7.17}$$

which clearly commutes with the whole set of asymptotic symmetry generators spanned by $J_n^{\pm}$ and $J_m^{(3)\pm}$. One then concludes that for an arbitrary fixed value of the total energy, configurations endowed with different sets of nonvanishing $\hat{\mathfrak{u}}(1)$ charges turn out to be inequivalent, because they can not be related to each other through a pure gauge transformation. Since excitations (7.15) associated with the generators $J_n^{\pm}$, $J_m^{(3)\pm}$ preserve the total energy and cannot be gauged away, they are (higher spin) soft hair in the sense of Hawking, Perry and Strominger [97].

[4]The vacuum state considered here resembles Poincaré-AdS rather than global AdS. The state corresponding to global AdS is gapped by an imaginary amount of the zero mode charges from the vacuum state considered here.

Highest Weight Gauge and the Emergence of Composite $\mathcal{W}_3$ Symmetries

Quite remarkably, it can be seen that spin-2 and spin-3 charges naturally emerge as composite currents constructed out from the $\hat{\mathfrak{u}}(1)$ ones. Actually, the full set of generators of the $\mathcal{W}_3$ algebra arises from suitable composite operators of the $\hat{\mathfrak{u}}(1)$ charges through a twisted Sugawara construction. Here we show this explicitly through the comparison of the new set of boundary conditions proposed in the previous section with the ones that accommodate the higher spin black holes in [34, 95], whose asymptotic symmetries are described by two copies of the $\mathcal{W}_3$ algebra. In order to carry out this task it is necessary to express both sets in terms of the same variables. The asymptotic behavior described by (7.5) and (7.7) is formulated so that the auxiliary connections $a^{\pm}$ are written in the diagonal gauge, while the set in [34,95] was formulated in the so-called highest weight gauge. Consequently, what we look for can be unveiled once the gauge fields in (7.5) and (7.7) are expressed in terms of the variables that are naturally adapted to the gauge fields $\hat{A}^{\pm}$ in the highest weight gauge.

For a generic choice of Lagrange multipliers, which are still unspecified, the asymptotic form of the gauge fields in the highest weight gauge reads [34, 95]

$$\hat{A}^{\pm} = \hat{b}_{\pm}^{-1}(d + \hat{a}^{\pm})\hat{b}_{\pm} \ , \tag{7.18}$$

where the radial dependence can be captured by the choice $\hat{b}_{\pm} = e^{\pm\rho \mathtt{L}_0}$, and

$$\hat{a}_{\varphi}^{\pm} = \mathtt{L}_{\pm 1} - \frac{2\pi}{k}\mathcal{L}_{\pm}\mathtt{L}_{\mp 1} - \frac{\pi}{2k}\mathcal{W}_{\pm}\mathtt{W}_{\mp 2} \qquad\qquad \hat{a}_t^{\pm} = \Lambda^{\pm}[\mu_{\pm}, \nu_{\pm}] \ , \tag{7.19}$$

with

$$\begin{aligned}\Lambda^{\pm} = \pm\Big[&\mu_{\pm}\mathtt{L}_{\pm 1} + \nu_{\pm}\mathtt{W}_{\pm 2} \mp \mu'_{\pm}\mathtt{L}_0 \mp \nu'_{\pm}\mathtt{W}_{\pm 1} + \tfrac{1}{2}\Big(\mu''_{\pm} - \tfrac{4\pi}{k}\mu_{\pm}\mathcal{L}_{\pm} + \tfrac{8\pi}{k}\mathcal{W}_{\pm}\nu_{\pm}\Big)\mathtt{L}_{\mp 1} \\ &- \Big(\tfrac{\pi}{2k}\mathcal{W}_{\pm}\mu_{\pm} + \tfrac{7\pi}{6k}\mathcal{L}'_{\pm}\nu'_{\pm} + \tfrac{\pi}{3k}\nu_{\pm}\mathcal{L}''_{\pm} + \tfrac{4\pi}{3\kappa}\mathcal{L}_{\pm}\nu''_{\pm} - \tfrac{4\pi^2}{k^2}\mathcal{L}_{\pm}^2\nu_{\pm} - \tfrac{1}{24}\nu''''_{\pm}\Big)\mathtt{W}_{\mp 2} \\ &+ \tfrac{1}{2}\Big(\nu''_{\pm} - \tfrac{8\pi}{k}\mathcal{L}_{\pm}\nu_{\pm}\Big)\mathtt{W}_0 \mp \tfrac{1}{6}\Big(\nu'''_{\pm} - \tfrac{8\pi}{k}\nu_{\pm}\mathcal{L}'_{\pm} - \tfrac{20\pi}{k}\mathcal{L}_{\pm}\nu'_{\pm}\Big)\mathtt{W}_{\mp 1}\Big] \ ,\end{aligned} \tag{7.20}$$

where $\mathcal{L}_{\pm}$, $\mathcal{W}_{\pm}$ and $\mu_{\pm}$, $\nu_{\pm}$ stand for arbitrary functions of t, φ.

One then needs to find suitable permissible gauge transformations spanned by group elements $g_{\pm}$, for which $\hat{a}^{\pm} = g_{\pm}^{-1}(d + a^{\pm})g_{\pm}$. These group elements indeed exist and are given, as well as necessary consistency conditions, explicitly in [4]. The gauge fields $a^{\pm}$ and $\hat{a}^{\pm}$ are then mapped to each other provided

$$\mathcal{L}_{\pm} = \pm\frac{k}{4\pi}\Big(\frac{1}{2}\big(\mathcal{J}^{\pm}\big)^{2} + \frac{2}{3}\big(\mathcal{J}^{\pm}_{(3)}\big)^{2} + \mathcal{J}^{\pm\prime}\Big) \tag{7.21}$$

$$\mathcal{W}_{\pm} = \mp\frac{k}{6\pi}\Big(-\frac{8}{9}\big(\mathcal{J}^{\pm}_{(3)}\big)^{3} + 2\big(\mathcal{J}^{\pm}\big)^{2}\mathcal{J}^{\pm}_{(3)} + \mathcal{J}^{\pm}_{(3)}\mathcal{J}^{\pm\prime} + 3\mathcal{J}^{\pm}\mathcal{J}^{\pm\prime}_{(3)} + \mathcal{J}^{\pm\prime\prime}_{(3)}\Big) \tag{7.22}$$

from which one recognizes the Miura transformation between the variables, see e.g. [90].

Note that the functions $\mathcal{L}_{\pm}$, $\mathcal{W}_{\pm}$, that are naturally defined in the highest weight gauge, depend on the global charges $\mathcal{J}^{\pm}$, $\mathcal{J}^{\pm}_{(3)}$ as in eqs. (7.21), (7.22). In sum, our proposal for boundary conditions once expressed in the highest weight gauge, is such that the Lagrange multipliers $\mu_{\pm}$ and $\nu_{\pm}$ depend on the dynamical variables.

Indeed, for a generic choice of Lagrange multipliers in the highest weight gauge, the field equations read [34]

$$\dot{\mathcal{L}}_{\pm} = \pm2\mathcal{L}_{\pm}\mu'_{\pm} \pm \mu_{\pm}\mathcal{L}'_{\pm} \mp \frac{k}{4\pi}\mu'''_{\pm} \mp 2\nu_{\pm}\mathcal{W}'_{\pm} \mp 3\mathcal{W}_{\pm}\nu'_{\pm} \tag{7.23}$$

$$\begin{aligned}\dot{\mathcal{W}}_{\pm} = &\pm3\mathcal{W}_{\pm}\mu'_{\pm} \pm \mu_{\pm}\mathcal{W}'_{\pm} \pm \frac{2}{3}\nu_{\pm}\Big(\mathcal{L}'''_{\pm} - \frac{16\pi}{k}\mathcal{L}^{2\prime}_{\pm}\Big) \pm 3\Big(\mathcal{L}''_{\pm} - \frac{64\pi}{9k}\mathcal{L}^{2}_{\pm}\Big)\nu'_{\pm} \\ &\pm 5\nu''_{\pm}\mathcal{L}'_{\pm} \pm \frac{10}{3}\mathcal{L}_{\pm}\nu'''_{\pm} \mp \frac{k}{12\pi}\nu^{(5)}_{\pm}\ ,\end{aligned} \tag{7.24}$$

which by virtue of the definition of our boundary conditionsreduce to the remarkably simple ones, given by $\dot{\mathcal{J}}^{\pm} = \pm\zeta'$, $\dot{\mathcal{J}}^{\pm}_{(3)} = \pm\zeta'_{(3)}$, which were directly obtained in the diagonal gauge (see eq. (7.8)).

It is also worth highlighting that eqs. (7.21), (7.22) can be regarded as the higher spin gravity version of the twisted Sugawara construction. In fact, as show in [4] the currents $\mathcal{L}_{\pm}$, $\mathcal{W}_{\pm}$ fulfill the $\mathcal{W}_3$ algebra.

$$\delta\mathcal{L}_{\pm} = \pm2\mathcal{L}_{\pm}\varepsilon'_{\pm} \pm \varepsilon_{\pm}\mathcal{L}'_{\pm} \mp \frac{k}{4\pi}\varepsilon'''_{\pm} \mp 2\chi_{\pm}\mathcal{W}'_{\pm} \mp 3\mathcal{W}_{\pm}\chi'_{\pm} \tag{7.25}$$

$$\begin{aligned}\delta\mathcal{W}_{\pm} = &\pm3\mathcal{W}_{\pm}\varepsilon'_{\pm} \pm \varepsilon_{\pm}\mathcal{W}'_{\pm} \pm \frac{2}{3}\chi_{\pm}\Big(\mathcal{L}'''_{\pm} - \frac{16\pi}{k}\mathcal{L}^{2\prime}_{\pm}\Big) \pm 3\Big(\mathcal{L}''_{\pm} - \frac{64\pi}{9k}\mathcal{L}^{2}_{\pm}\Big)\chi'_{\pm} \\ &\pm 5\chi''_{\pm}\mathcal{L}'_{\pm} \pm \frac{10}{3}\mathcal{L}_{\pm}\chi'''_{\pm} \mp \frac{k}{12\pi}\chi^{(5)}_{\pm}\ .\end{aligned} \tag{7.26}$$

It is then apparent that $\mathcal{L}_{\pm}$ and $\mathcal{W}_{\pm}$ turn out to be composite anomalous spin-2 and spin-3 currents, respectively. In other words, the asymptotic $\mathcal{W}_3$ algebra obtained in [34, 95] for a different set of boundary conditions, being defined through requiring the Lagrange multipliers in the highest weight gauge to be fixed without variation ($\delta\mu_{\pm} = \delta\nu_{\pm} = 0$), is recovered as a composite one that emerges from the $\hat{\mathfrak{u}}(1)$ currents.

Despite of the fact that the spin-2 and spin-3 currents $\mathcal{L}_\pm$, $\mathcal{W}_\pm$ fulfill the $\mathcal{W}_3$ algebra, their associated global charges generate the $\hat{\mathfrak{u}}(1)$ current algebras discussed in section (7.2). This is so because, by virtue of the consistency conditions and (7.21), (7.22) the variation of the global charges readily reduces to

$$\delta \mathcal{Q}^\pm = \mp \int d\varphi (\varepsilon_\pm \delta \mathcal{L}_\pm - \chi_\pm \delta \mathcal{W}_\pm) = \mp \frac{k}{4\pi} \int d\varphi \Big(\eta^\pm \delta \mathcal{J}^\pm + \frac{4}{3} \eta^\pm_{(3)} \delta \mathcal{J}^\pm_{(3)} \Big) \,, \tag{7.27}$$

so that they satisfy the current algebras in (7.14). Indeed, this result just reflects the fact that the gauge transformation that maps our asymptotic conditions in the highest weight and diagonal gauges is a permissible one in the sense of [34]. Therefore, the global charges associated with our asymptotic conditions, although written in the highest weight gauge manifestly do not fulfill the $\mathcal{W}_3$ algebra. This is because the Lagrange multipliers $\mu_\pm$, $\nu_\pm$, are not chosen to be fixed at infinity without variation as in [34, 95], but instead, here they explicitly depend on the global charges. What is actually kept fixed at the boundary without variation is the set of Lagrange multipliers that is naturally defined in the diagonal gauge ($\delta\zeta^\pm = \delta\zeta^\pm_{(3)} = 0$).

Higher Spin Black Holes with Soft Hair

As shown the simpler subset of our boundary conditions, obtained by choosing the Lagrange multipliers $\zeta^\pm$, $\zeta^\pm_{(3)}$ to be constants, possesses the noticeable property of making the global charges $J^\pm_n$, $J^{(3)\pm}_m$ to behave as (higher spin) soft hair. An additional remarkable feature that also occurs in this case is the fact that regularity of the whole spectrum of Euclidean solutions that fulfill our boundary conditions holds everywhere, regardless the value of the global charges.

An interesting effect occurs for the branch of higher spin black holes that is continuously connected to the BTZ black hole [86, 87], corresponding to $m = 0$, $n = 1$. Indeed, for this branch the entropy is found to depend just on the zero modes of the electric-like $\hat{\mathfrak{u}}(1)$ charges of the purely gravitational sector, i.e.,

$$S = 2\pi \Big(J^+_0 + J^-_0 \Big) \,. \tag{7.28}$$

Nonetheless, the information about the presence of the higher spin fields is subtle hidden within the purely gravitational global charges, as can be seen from the map between the $\hat{\mathfrak{u}}(1)$ and $\mathcal{W}_3$ currents. In fact, for the spherically symmetric higher spin black hole, by virtue of (7.21), (7.22), the relationship between the zero modes of the purely gravitational $\hat{\mathfrak{u}}(1)$ charges and the zero modes of the $\mathcal{W}_3$ ones reads

$$J_0^\pm = \sqrt{2\pi k \mathcal{L}_\pm} \cos\left[\frac{1}{3} \arcsin\left(\frac{3}{8}\sqrt{\frac{3k}{2\pi\mathcal{L}_\pm^3}}\mathcal{W}_\pm\right)\right] . \tag{7.29}$$

Therefore, replacing (7.29) into (7.28) one recovers the following expression for the higher spin black hole entropy in terms of the spin-2 and spin-3 charges, which reads

$$\begin{aligned} S = 2\pi\sqrt{2\pi k}\Bigg(&\sqrt{\mathcal{L}_+} \cos\left[\frac{1}{3} \arcsin\left(\frac{3}{8}\sqrt{\frac{3k}{2\pi\mathcal{L}_+^3}}\mathcal{W}_+\right)\right] \\ &+\sqrt{\mathcal{L}_-} \cos\left[\frac{1}{3} \arcsin\left(\frac{3}{8}\sqrt{\frac{3k}{2\pi\mathcal{L}_-^3}}\mathcal{W}_-\right)\right]\Bigg) , \end{aligned} \tag{7.30}$$

in full agreement with the result obtained in [34].

This analysis was generalized to arbitrary spin [4] as well as to the case of flat space [98] and flat space higher spin [7]. For more on “Black Hole Horizon Fluff” see [99].

Chapter 8

Non-AdS Higher Spin Gravity

In Chapter 7 we have discussed higher spin theories based on $\mathfrak{sl}(3,\mathbb{R})\oplus\mathfrak{sl}(3,\mathbb{R})$ algebras and (higher spin generalized) AdS spacetimes. We want to stick to the same underlying Lie algebra (this will be changed in the following chapters), but we want to generalize to backgrounds and boundary conditions beyond AdS.

In many applications it is necessary to generalize holography to spacetimes more general than asymptotic AdS, for a review see e.g., [9]. Examples for which the spacetime can be constructed in higher spin theories are [100]:

Null-warped AdS spacetimes which arise in proposed holographic duals of nonrelativistic CFTs describing cold atoms [40, 41].

Schrödinger spacetimes, which generalize null warped AdS by introducing an arbitrary scaling exponent [101].

Lifshitz spacetimes, which arise in gravity duals of Lifshitz-like fixed points [102] and also have a scaling exponent parametrizing spacetime anisotropy.

A variational principle for 3-dimensional higher spin gravity that accommodates spacetimes like asymptotically $\mathrm{AdS}_2\times\mathbb{R}$, $\mathbb{H}_2\times\mathbb{R}$, Schrödinger, Lifshitz or warped AdS spacetimes was proposed and the connections that generate this backgrounds presented [100]. For the case of $\mathbb{H}_2\times\mathbb{R}$ realized in $\mathfrak{sl}(3,\mathbb{R})$ HS gravity in the non-principal embedding the asymptotic symmetry algebra turned out to be the direct sum of the $\mathcal{W}_3^{(2)}\oplus\hat{\mathfrak{u}}(1)$ [103]. We now want to investigate following [2] the case of Lifshitz higher spin theories. Since the situation for null-warped AdS [3] follows similar considerations we will only provide a short overview.

8.1 Lifshitz Higher Spin

A variety of condensed matter systems exhibits anisotropic scaling near a renormalization group fixed point. Classical Lifshitz fixed points, in which the system scales anisotropically in different spatial directions, are extensively explored. Quantum Lifshitz fixed points, in which time and space scale anisotropically, with relative scaling ratio z, are particularly common in strongly correlated systems [104–114]. Many-body field theories describing such anisotropic fixed points were proposed to be holographically dual to gravity in the background of Lifshitz geometries, where time and space scale asymptotically with the same ratio z [102].

Lifshitz Spacetime in Three Dimensions

The $(2+1)$-dimensional Lifshitz spacetime [102] is described by the line element

$$ds^2_{\mathrm{Lif}_z} = \ell^2\Big(- r^{2z}\, dt^2 + \frac{dr^2}{r^2} + r^2\, dx^2\Big)\,. \tag{8.1}$$

The Lifshitz spacetime (8.1) is invariant under the anisotropic scaling ($z \in \mathbb{R}$):

$$t \to \lambda^z t \qquad x \to \lambda x \qquad r \to \lambda^{-1} r\,. \tag{8.2}$$

For $z = 1$, the scaling is isotropic and the spacetime (8.1) reduces to Poincaré patch AdS_3.

It is often useful to consider a change of coordinates to the radial variable $\rho = \ln r$. The spacetime (8.1) now becomes

$$ds^2_{\mathrm{Lif}_z} = \ell^2\Big(-e^{2z\rho}\, dt^2 + d\rho^2 + e^{2\rho}\, dx^2\Big)\,. \tag{8.3}$$

The asymptotic region is approached for $\rho \to \infty$.

The Lifshitz spacetime (8.3) possesses spacetime isometries. These Lifshitz isometries are generated by the Killing vector fields

$$\xi_{\mathbb{H}} = \partial_t \qquad \xi_{\mathbb{P}} = \partial_x \qquad \xi_{\mathbb{D}} = -zt\,\partial_t + \partial_\rho - x\,\partial_x \tag{8.4}$$

whose isometry algebra is the Lifshitz algebra $\mathfrak{lif}(z,\mathbb{R})$

$$[\xi_{\mathbb{H}},\, \xi_{\mathbb{P}}] = 0 \qquad [\xi_{\mathbb{D}},\, \xi_{\mathbb{H}}] = z\,\xi_{\mathbb{H}} \qquad [\xi_{\mathbb{D}},\, \xi_{\mathbb{P}}] = \xi_{\mathbb{P}} \tag{8.5}$$

The Killing vector $\xi_{\mathbb{H}}$ ($\xi_{\mathbb{P}}$) [$\xi_{\mathbb{D}}$] generates time translations (spatial translations) [anisotropic dilatations]. The Lifshitz spacetime with $z = 1$ corresponds to the Poincaré patch of the isotropic AdS_3 spacetime. With enhanced $(1+1)$-dimensional Lorentz (boost) invariance, the isometry algebra gets enlarged to $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ associated with two copies of chiral and anti-chiral excitations. Conversely, the Lifshitz algebra $\mathfrak{lif}(1,\mathbb{R})$ is a subalgebra of the $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ isometry algebra of the AdS$_3$ spacetime.

Since the Lifshitz spacetime does not fulfill the vacuum Einstein equations, matter contributions are necessary. Known realizations so far involve, e.g., p-form gauge fields [102]. For example, AdS Einstein gravity coupled to two 1-form abelian gauge fields $F_2 = dA_1,\ G_2 = dC_1$,

$$I = \frac{1}{16\pi G_3}\int d^3x\sqrt{-g}\Big[R(g) + \frac{2}{\ell^2} + \frac{1}{4}||F_2||^2 + \frac{1}{4\alpha}||G_2||^2 + \frac{1}{2}*(A_1\wedge G_2)\Big]\,, \tag{8.6}$$

admits the Lifshitz spacetime as a classical solution, where the scaling ratio z is determined by

$$z = \alpha \pm \sqrt{\alpha^2-1} \qquad (\alpha \geq 1)\,. \tag{8.7}$$

Some other constructions require either a massive gauge field [115], a massive graviton [116, 117] or Hořava–Lifshitz gravity [118].

Here, we take a different route and realize the Lifshitz spacetime by coupling AdS$_3$ Einstein gravity to a spin-3 field with full higher-spin gauge symmetry. In the next section, we construct an explicit example of $(2+1)$-dimensional $z=2$ Lifshitz spacetime (8.3) with non-trivial spin-3 background field. We shall then carefully examine boundary conditions for the gravitational and spin-3 excitations over this Lifshitz spacetime.

Lifshitz Boundary Conditions

In order to find the Lifshitz spacetime, we decompose again as in (7.5) but with the group element $b_\pm = e^{\pm\rho \mathtt{L}_0}$.

To fix a variational principle, we take $\delta A_t^+ = 0 = \delta A_t^-$ at asymptotic infinity $\rho\to\infty$, where this time we denote our boundary coordinates by t and x. With the boundary term

$$\frac{k}{4\pi}\int_{\mathbb{R}\times\partial\Sigma}\mathrm{tr}\Big(A_t A_x - A_t^- A_x^-\Big)\,dt\,dx \tag{8.8}$$

added to the bulk action (7.2), such a variational principle is well-posed [100]. We take as a background that leads to the Lifshitz spacetime the connections

$$\hat{a}^+ = \tfrac{4}{9}\mathtt{W}_{+2}\,dt + \mathtt{L}_{+1}\,dx \tag{8.9a}$$

$$\hat{a}^- = \mathtt{W}_{-2}\,dt + \mathtt{L}_{-1}\,dx\,. \tag{8.9b}$$

The specific numerical coefficients are chosen to cancel factors arising from traces.

Using the standard definition of the metric in terms of the zuvielbein (7.4) leads to the geometry

$$ds^2_{\mathrm{Lif}_2} = \ell^2\Big(-e^{4\rho}\,dt^2 + d\rho^2 + e^{2\rho}\,dx^2\Big)\,. \tag{8.10}$$

We thus obtain as a classical configuration the $(2+1)$-dimensional Lifshitz spacetime (8.3) with $z = 2$. The classical solution also involves the totally symmetric spin-3 gauge field. For our configuration, we find that the Lifshitz spacetime is supported by a nontrivial spin-3 background gauge field

$$\phi_{\mu\nu\lambda}\;dx^\mu\,dx^\nu\,dx^\lambda = -\frac{5\ell^3}{4}\,e^{4\rho}\;dt\,(dx)^2\,. \tag{8.11}$$

From now on we set $\ell = 1$ to reduce clutter. The spin-3 gauge field is invariant under the transformations generated by the Killing vector fields (8.4). We conclude that the classical configuration (8.10), (8.11) respects the Lifshitz algebra $\mathfrak{lif}(2,\mathbb{R})$. The above construction of the Lifshitz spacetime is quite elementary and simple.

Let us next examine the algebra of the symmetry currents for the Lifshitz system we have constructed. To this end, we first need to impose boundary conditions consistent with the background Lifshitz spacetime geometry. Note that we take the ansatz used in [73, 103], which differs from the asymptotic behavior $A - \hat{A} = \mathcal{O}(1)$ used in [27, 119], where $\hat{A}$ was a fixed background connection. The fluctuations, which are already on-shell, turn out to take the following form

$$\begin{aligned} a^+ = &\Big(\tfrac{8\pi}{9k}t\mathcal{W}(x)\mathtt{L}_0 - \tfrac{\pi}{2k}\mathcal{L}(x)\mathtt{L}_{-1}\Big)\;dx \\ &+\Big(-\tfrac{32\pi}{81k}t^2\mathcal{W}(x)\mathtt{W}_{+2} + \tfrac{8\pi}{9k}t\mathcal{L}(x)\mathtt{W}_{+1} + \tfrac{2\pi}{9k}\mathcal{W}(x)\mathtt{W}_{-2}\Big)\;dx \end{aligned} \tag{8.12}$$

$$\begin{aligned} a^- = &\Big(-\tfrac{2\pi}{k}t\overline{\mathcal{W}}(x)\mathtt{L}_0 - \tfrac{\pi}{2k}\overline{\mathcal{L}}(x)\mathtt{L}_{+1}\Big)\;dx \\ &+\Big(-\tfrac{2\pi}{k}t^2\overline{\mathcal{W}}(x)\mathtt{W}_{-2} - \tfrac{2\pi}{k}t\overline{\mathcal{L}}(x)\mathtt{W}_{-1} + \tfrac{2\pi}{9k}\overline{\mathcal{W}}(x)\mathtt{W}_{+2}\Big)\;dx \end{aligned} \tag{8.13}$$

The set of all boundary functions $\mathcal{L}$, $\overline{\mathcal{L}}$, $\mathcal{W}$ and $\overline{\mathcal{W}}$ specify the set of all admissible fluctuations about the Lifshitz background.

A interesting and possibly disturbing feature of these boundary conditions is the polynomial time dependence. In general, time-dependent boundary conditions lead to non-conservation of canonical charges. However, due to the specific form of the boundary conditions, and as can be seen explicitly in [2] all t-dependence is canceled in the boundary charge density and hence the canonical charges are conserved.

Below, we address some immediate consequences of the above boundary conditions, which all point to the fact that consistency of the boundary conditions is a highly non-trivial result.

Using (7.4), we also extract fluctuations of spin-2 and spin-3 fields. Up to the sub-leading terms, fluctuations of the spin-2 field take the form (for notational simplification, we suppress the x-dependence of all component functions hereafter)

$$g_{tt} = \boldsymbol{-e^{4\rho}} \tag{8.14a}$$
$$g_{t\rho} = 0 \tag{8.14b}$$
$$g_{tx} = t^2 e^{4\rho}\Big(\pi\overline{\mathcal{W}} + \tfrac{4\pi}{9}\mathcal{W}\Big) + \tfrac{\pi}{4}\overline{\mathcal{W}} + \tfrac{\pi}{9}\mathcal{W} \tag{8.14c}$$
$$g_{\rho\rho} = \mathbf{1} \tag{8.14d}$$
$$g_{\rho x} = t\left(\tfrac{\pi}{2}\overline{\mathcal{W}} + \tfrac{2\pi}{9}\mathcal{W}\right) \tag{8.14e}$$
$$\begin{aligned} g_{xx} = &\boldsymbol{e^{2\rho}} - t^4 e^{4\rho}\tfrac{16\pi^2}{81}\overline{\mathcal{W}}\mathcal{W} - t^2 e^{2\rho}\tfrac{\pi^2}{9}\overline{\mathcal{L}}\mathcal{L} \\ &+ \tfrac{\pi}{6}\overline{\mathcal{L}} + \tfrac{\pi}{6}\mathcal{L} + t^2\tfrac{8\pi^2}{81}\overline{\mathcal{W}}\mathcal{W} + \tfrac{\pi^2}{36}e^{-2\rho}\overline{\mathcal{L}}\mathcal{L} - \tfrac{\pi^2}{81}e^{-4\rho}\overline{\mathcal{W}}\mathcal{W}\,, \end{aligned} \tag{8.14f}$$

while fluctuations of the spin-3 field take the form

$$\begin{aligned} \phi_{txx} = &\boldsymbol{-\tfrac{5}{12}e^{4\rho}} + t^2 e^{4\rho}\left(\tfrac{\pi^2}{3k^2}\mathcal{L}^2 - \tfrac{3\pi^2}{4k^2}\overline{\mathcal{L}}^2\right) \\ &+ e^{2\rho}\left(\tfrac{\pi}{3k}\mathcal{L} - \tfrac{3\pi}{4k}\overline{\mathcal{L}}\right) + \tfrac{\pi^2}{12k^2}\mathcal{L}^2 - \tfrac{3\pi^2}{16k^2}\overline{\mathcal{L}}^2 \end{aligned} \tag{8.15a}$$
$$\phi_{\rho xx} = te^{2\rho}\left(\tfrac{2\pi}{3k}\mathcal{L} - \tfrac{3\pi}{2k}\overline{\mathcal{L}}\right) + t\left(\tfrac{\pi^2}{3k^2}\mathcal{L}^2 - \tfrac{3\pi^2}{4k^2}\overline{\mathcal{L}}^2\right)$$
$$\begin{aligned} \phi_{xxx} = &t^4 e^{4\rho}\left(\tfrac{2\pi^3}{k^3}\overline{\mathcal{L}}^2\mathcal{W} - \tfrac{2\pi^3}{k^3}\mathcal{L}^2\overline{\mathcal{W}}\right) + t^2 e^{4\rho}\left(\tfrac{9\pi}{2k}\overline{\mathcal{W}} - \tfrac{8\pi}{9k}\mathcal{W}\right) \\ &+ t^2 e^{2\rho}\left(\tfrac{2\pi^2}{k^2}\mathcal{L}\overline{\mathcal{W}} - \tfrac{2\pi^2}{k^2}\overline{\mathcal{L}}\mathcal{W}\right) + t^2\left(\tfrac{\pi^3}{k^3}\mathcal{L}^2\overline{\mathcal{W}} - \tfrac{\pi^3}{k^3}\overline{\mathcal{L}}^2\mathcal{W}\right) - \tfrac{\pi}{2k}\overline{\mathcal{W}} + \tfrac{\pi}{2k}\mathcal{W} \\ &+ e^{-2\rho}\left(\tfrac{\pi^2}{2k^2}\overline{\mathcal{L}}\mathcal{W} - \tfrac{\pi^2}{2k^2}\mathcal{L}\overline{\mathcal{W}}\right) + e^{-4\rho}\left(\tfrac{\pi^3}{8k^3}\overline{\mathcal{L}}^2\mathcal{W} - \tfrac{\pi^3}{8k^3}\mathcal{L}^2\overline{\mathcal{W}}\right) \end{aligned} \tag{8.15b}$$
$$\phi_{\mu\nu\lambda} = 0 \quad \text{otherwise}\,. \tag{8.15c}$$

The boldfaced terms denote background geometry, while the remaining terms correspond to state-dependent contributions to the spin-2 and spin-3 fields.

It is also interesting to observe that, although the background geometry is Lifshitz, the boundary conditions also admit spin-2 field configurations that have asymptotically stronger divergent contributions in ρ than the background geometry. For example, it is possible to have configurations whose g_{tt} and g_{xx} have the same asymptotic growth, $\sim e^{4\rho}$. Nevertheless, as we are going to show below, all the configurations allowed by our boundary conditions correspond to finite energy excitations, in the sense that all the canonical charges associated with these configurations are finite (as well as integrable and conserved). It should be stressed that this feature crucially relies on higher-spin gauge symmetry that acts nontrivially on the spin-2 metric field: the would-be infinite energy density in Einstein-gravity for configurations of $\sim e^{4\rho}$ asymptotic growth is canceled off by the spin-3 gauge transformations in higher-spin gravity.

The computation of the asymptotic symmetries and the canonical charges is quite lengthy and cumbersome. Therefore we refer to [2] for the details and jump directly to the answers. The canonical charges are conserved and well defined and the asymptotic symmetry algebra is given by two commuting $\mathcal{W}_3$ algebras (see Appendix D.5), i.e., $\mathcal{W}_3 \oplus \mathcal{W}_3$ with the same central charge as Brown–Henneaux [85]. This are not the symmetries one might expect of a nonrelativistic Lifshitz system and one might therefore ask what to the aforementioned Lifshitz symmetries happened. But, as remarked in [2] and by virtue of the relation between gauge symmetries and diffeomorphisms, $\epsilon = \xi^\mu A_\mu$, $\bar{\epsilon} = \xi^\mu A^-_\mu$ [24] (see Section C.5) one can see that the Lifshitz symmetries (8.5) get enhanced. With the identification $\mathcal{W}_{-2} \leftrightarrow \mathbb{H}$, $\mathcal{L}_{-1} \leftrightarrow \mathbb{P}$, $\mathcal{L}_0 \leftrightarrow \mathbb{D}$ and the use of (D.57), it becomes obvious that we have the isometry subalgebra $\mathfrak{lif}(2, \mathbb{R})$ as a subalgebra of $\mathcal{W}_3$.

Another work focusing on aspects of Lifshitz black holes [119] also found boundary conditions that lead to a $\mathcal{W}_3$ algebra, as pointed out in [92]. In fact, their field configurations turn out to be a special case of a general class of solutions of spin-3 gravity in the presence of chemical potentials [95, 120].

Built upon their work and ours, we put forth the conjecture that for generic higher-spin Lifshitz holography the asymptotic symmetry algebra gets ubiquitously enhanced to a class of $\mathcal{W}$-algebras.

It was pointed out in [121], that when considering gravitational theories in the first order formalism it can sometimes happen that the spin connection is not uniquely determined by the zuvielbein. In such cases the second order formulation is difficult to interpret as a gravitational theory in the traditional sense. While this is not an obstruction to studying such theories, it can make the interpretation more difficult and our Lifshitz theory is plagued by this issues. Further remarks concerning the degeneracy of the nonrelativistic solutions can be found in [121].

8.2 Null-warped Higher Spin

In [3] three-dimensional spin-3 gravity was equipped with a set of boundary conditions called "asymptotically null warped AdS". Null warped AdS is a special case of a large class of geometries studied by a number of researchers mainly in the context of topologically massive gravity [47, 48, 122], see e.g. [123–129]. The asymptotic symmetry algebra for the higher spin generalization found in [3] was found to be a chiral copy of the $\mathcal{W}_3^{(2)}$ Polyakov–Bershadsky algebra reminiscent of the situation in topologically massive gravity with strict null warped AdS boundary conditions (see [129]). Again, the "usual" null warped isometry algebra get enhanced to a much bigger one.

Furthermore, was it shown that the invertibility issues [121] are not a problem for the null warped AdS case. Given the asymptotic symmetries it seemed natural to check if our boundary conditions can be mapped to asymptotically AdS boundary conditions that also lead to a $\mathcal{W}_3^{(2)}$ algebra [34] which was indeed the case. We refer to [3] for further details concerning the introduction of chemical potentials, the derivation of entropy, free energy, and the holographic response functions.

Summary

As seen in this chapter, it is nontrivial to get boundary conditions where the asymptotic symmetry algebra does not get enhanced to one that could be considered as relativistic. It can be observed that in both cases the resulting asymptotic symmetry algebra is related to the gauge algebra. The $\mathcal{W}_3$ algebras as well as the $\mathcal{W}_3^{(2)}$ algebra arise naturally in connection to the two inequivalent embedding of $\mathfrak{sl}(2,\mathbb{R})$ into $\mathfrak{sl}(3,\mathbb{R})$ and their highest weight boundary conditions (for details concerning this differentiation see, e.g., [34]).

This considerations already hint towards a way to make other symmetries than (A)dS manifest. A change of gauge algebra seems like a reasonable starting point to get asymptotic symmetry algebras of different kinematics and will be discussed in the next chapters.

Chapter 9

Kinematical Spin-2 Theories

Due to the principle of relativity, the notion of kinematical or spacetime symmetry algebras, which contain all symmetries that relate different inertial frames, is a crucial ingredient in the construction of physical theories. Bacry and Lévy-Leblond have classified all possibilities for kinematical algebras [8], consisting of spacetime translations, spatial rotations and boosts, under some reasonable assumptions. Apart from the relativistic Poincaré and (A)dS algebras, this classification also contains the Galilei and Carroll algebras (and generalizations thereof that include a cosmological constant), that appear as kinematical algebras in the nonrelativistic ($c \to \infty$) and ultra-relativistic ($c \to 0$) limit. Even though fundamental theories are relativistic, the Galilei and Carroll algebras continue to play an important role in current explorations of string theory, holography and also phenomenology.

For instance, nonrelativistic symmetries underlie Newton–Cartan geometry, a differential geometric framework for nonrelativistic spacetimes that has found recent applications in holography [40–44, 102, 130–132], Hořava–Lifshitz gravity [45, 133, 134] and in the construction of effective field theories for strongly interacting condensed matter systems [12, 13, 135–140].

On the other hand, ultra-relativistic Carroll symmetries have recently been studied in relation to their connection [141] with the Bondi–Metzner–Sachs (BMS) algebra of asymptotic symmetries of flat spacetime [142, 143]. As such, Carroll symmetries play a role in attempts to construct holographic dualities in asymptotically flat spacetimes [144–152], as symmetries of the S-matrix in gravitational scattering [153] and in the recent notion of soft hair on black hole horizons [97, 154].

The kinematical algebras that have been classified by Bacry and Lévy-Leblond pertain to theories that contain bosonic fields with spins up to 2. One can also consider theories in which massless higher spin fields are coupled to gravity [155]. These so-called higher spin gauge theories have been formulated in (A)dS spacetimes (see [77–79] for reviews) and have featured prominently in the AdS/CFT literature, as a class of theories for which holographic dualities can be constructed rigorously [20, 21, 37, 84, 156–160], essentially because they are a weak-weak type of duality, i.e., CFTs with unbroken higher spin currents are free [161]. They typically contain an infinite number of higher spin fields. As a consequence, their spacetime symmetries are extended to infinite-dimensional algebras that include higher spin generalizations of spacetime translations, spatial rotations and boosts. Higher spin gauge theories have thus far mostly been considered in relativistic (A)dS spacetimes, with relativistic CFT duals[1].

Since both higher spin gauge theories as well as non- and ultra-relativistic spacetime symmetries have played an important role in recent developments in holography, it is natural to ask whether one can combine the two. In order to answer this question, one needs to know which non- and ultra-relativistic kinematical algebras can appear as symmetries of higher spin theories. This will first be discussed without the additional higher spin symmetries and in Chapter 10 including them.

Chapter 9 and Chapter 10 are based on [5]. There is a slight change of terminology, which hopefully does not lead to confusion. In order to be consistent with the introductory material presented in the beginning the term “contraction procedure” is substituted by just “contraction” or special cases thereof.

9.1 Kinematical Algebras

Before discussing spin-3, it is convenient to start with giving a short review of the spin-2 case [8]. Since both the spin-2 and spin-3 cases make use of the sIW-contractions thoroughly reviewed in Section 4.3 we just fix the notation that will be used throughout the next sections.

Starting from a Lie algebra $\mathfrak{g}$, one can choose a subalgebra $\mathfrak{h}$ and consider the decomposition $\mathfrak{g} = \mathfrak{h} \dot{+} \mathfrak{i}$. As already discussed $\mathfrak{h}$ will be the subalgebra with respect to which we will sIW-contract the original Lie algebra leading to

$$[\,\mathfrak{h}\,,\,\mathfrak{h}\,] \subset \mathfrak{h} \qquad\qquad [\,\mathfrak{h}\,,\,\mathfrak{i}\,] \subset \mathfrak{i} \qquad\qquad [\,\mathfrak{i}\,,\,\mathfrak{i}\,] = 0\,. \tag{9.1}$$

Remember that a nontrivial sIW-contraction is uniquely specified by a suitable choice of the subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

[1]See however [2, 3, 100, 103, 119, 121, 162] for attempts to consider higher spin theories in non-AdS backgrounds with nonrelativistic CFT duals.

Not all possible subalgebras, however, lead to interesting contractions that can, e.g., be interpreted as kinematical algebras. For spin-2, the question which contractions of the isometry algebras of AdS or dS lead to kinematical algebras, has been addressed by Bacry and Lévy–Leblond [8]. In particular, they have shown that there are only four different sIW-contractions of the AdS or dS algebras that lead to kinematical algebras. These have been called “space-time”, “speed-space”, “speed-time” and “general” in [8]. Effectively, the first three of these contractions can be described by either taking a limit of the (A)dS radius ℓ or the speed of light c. Specifically, the space-time contraction corresponds to $\ell \to \infty$, the speed-time contraction corresponds to $c \to 0$ and the speed-space contraction corresponds to $c \to \infty$. However, in this work we suppress factors of ℓ and c. The general contraction can also be obtained as consecutive sIW-contractions of the other three and therefore does not provide us with a new algebra. Moreover, it has been shown that there are in total 8 possible kinematical algebras[2] that can be obtained by combining different sIW-contractions of the AdS or dS isometry algebras. We have summarized the four sIW-contractions in the following Table 9.1, by indicating the subalgebra $\mathfrak{h}$ with respect to which the contraction is taken, as well as the generators that form the abelian ideal $\mathfrak{i}$.

Contraction	$\mathfrak{h}$	$\mathfrak{i}$
Space-time	$\{\mathtt{J}, \mathtt{G}_a\}$	$\{\mathtt{H}, \mathtt{P}_a\}$
Speed-space	$\{\mathtt{J}, \mathtt{H}\}$	$\{\mathtt{G}_a, \mathtt{P}_a\}$
Speed-time	$\{\mathtt{J}, \mathtt{P}_a\}$	$\{\mathtt{G}_a, \mathtt{H}\}$
General	$\{\mathtt{J}\}$	$\{\mathtt{H}, \mathtt{P}_a, \mathtt{G}_a\}$

Table 9.1: The four different IW contractions classified in [8].

The names of the eight kinematical algebras of [8], along with the symbols we will use to denote them, are given in Table 9.2. The sIW-contractions and the resulting Lie algebras that we have discussed so far can be conveniently summarized as a cube, see Figure 9.1 and all the commutation relations of the resulting Lie algebras, together with the most general invariant metric of (A)dS, are collected in Appendix D.6.

[2]The possible kinematical algebras considered in [8] are all possible spacetime symmetry algebras that obey the assumptions that space is isotropic and therefore their generators have the correct ($\mathtt{H}$ is a scalar, $\mathtt{P}, \mathtt{J}, \mathtt{G}$ are vectors) transformation behavior under rotations. Furthermore, parity and time-reversal are automorphisms and boosts are non-compact.

Name	Symbol
(Anti)-de Sitter	$(\mathfrak{A})\mathfrak{d}\mathfrak{S}$
Poincaré	$\mathfrak{poi}$
Para-Poincaré	$\mathfrak{ppoi}$
Newton–Hooke	$\mathfrak{nh}$
Galilei	$\mathfrak{gal}$
Para-Galilei	$\mathfrak{pgal}$
Carroll	$\mathfrak{car}$
Static	$\mathfrak{st}$

Table 9.2: Names of the kinematical algebras and the symbols that denote them.

For discussions concerning the invariant metric we have copied some of them in Table 9.1. The $(\mathfrak{A})\mathfrak{d}\mathfrak{S}_{(\mp)}$ Lie algebras are real (semi)simple Lie algebras are have therefore an invariant metric proportional to the Killing form. The contraction to the $\mathfrak{poi}$ algebra is of the form discussed in Section 4.2 and leads therefore also to a Lie algebra with invariant metric. Another kinematical algebra that is automatically equipped with an invariant metric is given by the $\mathfrak{car}$ algebra. This is due to the invariant metric preserving contraction of $\mathfrak{poi}$ to $\mathfrak{car}$ (see Section 5.3) and was shown explicitly in Example 5.3. So, the (A)dS, Poincaré and Carroll algebra permit an invariant metric, but the Newton–Hooke and Galilei algebra do not.

	$(\mathfrak{A})\mathfrak{d}\mathfrak{S}_{(\mp)}$	$\mathfrak{poi}$	$\mathfrak{nh}$	$\mathfrak{gal}$	$\mathfrak{ebarg}$
$[\mathtt{J}\,,\,\mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J}\,,\,\mathtt{P}_a]$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a\,,\,\mathtt{G}_b]$	$-\epsilon_{ab}\mathtt{J}$	$-\epsilon_{ab}\mathtt{J}$	0	0	$\epsilon_{ab}\mathtt{H}^*$
$[\mathtt{G}_a\,,\,\mathtt{H}]$	$-\epsilon_{am}\mathtt{P}_m$	$-\epsilon_{am}\mathtt{P}_m$	$-\epsilon_{am}\mathtt{P}_m$	$-\epsilon_{am}\mathtt{P}_m$	$-\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a\,,\,\mathtt{P}_b]$	$-\epsilon_{ab}\mathtt{H}$	$-\epsilon_{ab}\mathtt{H}$	0	0	$\epsilon_{ab}\mathtt{J}^*$
$[\mathtt{H}\,,\,\mathtt{P}_a]$	$\pm\epsilon_{am}\mathtt{G}_m$	0	$\pm\epsilon_{am}\mathtt{G}_m$	0	0
$[\mathtt{P}_a\,,\,\mathtt{P}_b]$	$\mp\epsilon_{ab}\mathtt{J}$	0	0	0	0

Table 9.3: The commutation relations of the (Anti)-de Sitter, Poincaré, Newton–Hooke, Galilei and Extended Bargmann algebras.

We will now analyze what needs to be done to get an extension of the Galilei algebra that is symmetric self-dual (see Table 9.1). Here knowledge

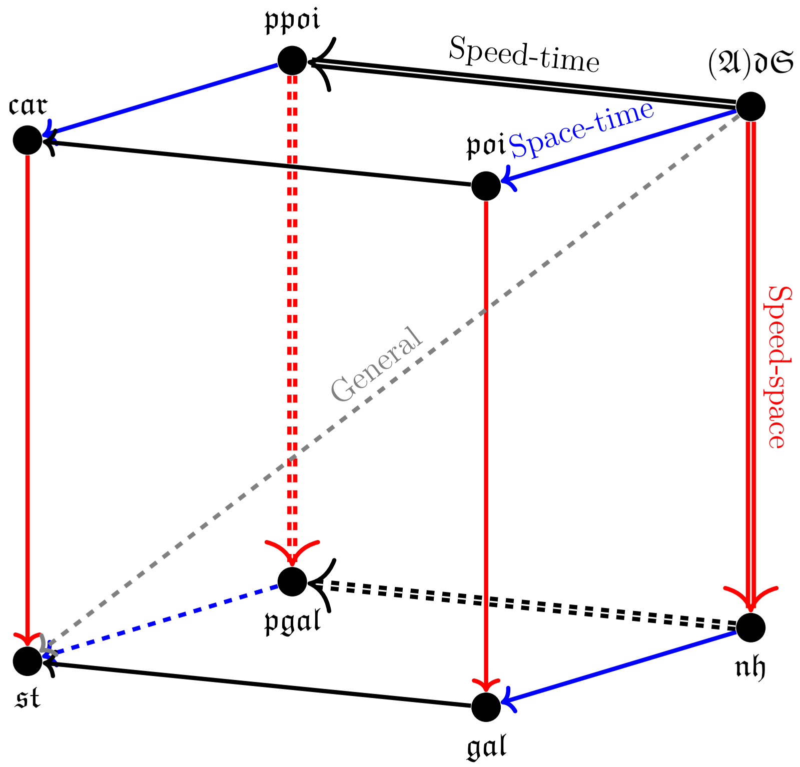

Figure 9.1: This cube summarizes the contractions starting from $(\mathfrak{A})\mathfrak{dS}$. The lines represent contractions and the dots represent the resulting contracted Lie algebra. We consider contractions starting from AdS and dS simultaneously. Each dot can therefore represent one Lie algebra, if the contractions from AdS and dS lead to the same algebra, or two Lie algebras, if the contractions from AdS and dS lead to two different results. We have indicated this in the cube by using single lines, for contraction that lead to the same contraction, and double lines otherwise. Dashed lines have no specific meaning except that they should convey the feeling of a three-dimensional cube.

about double extensions is useful. Restricting to $\mathtt{P}_a$ and $\mathtt{G}_b$ and recognizing that $\langle \mathtt{P}_a \, , \mathtt{G}_b \rangle = \delta_{ab}$ is an invariant metric on this restricted Lie algebra leads to the insight that we can double extend $\mathfrak{g} = \mathfrak{u}(1)^4 = \{\mathtt{P}_a, \mathtt{G}_b\}$ by $\mathtt{H}$ and $\mathtt{J}$ which leads to two nontrivial central extensions $\mathtt{H}^*$ and $\mathtt{J}^*$, respectively. This algebra will be called Extended Bargmann algebra or $\mathfrak{ebarg}$. "Bargmann algebra" because the importance of the central extension $\mathtt{J}^*$, which is possible in any spacetime dimension and is interpreted as mass, has been emphasized by Bargmann [163]. "Extended Bargmann algebra" because of the second central extension, which is not possible for higher dimensions (for a discussion and references concerning possible interpretations see, e.g., the introduction of [164]). In three spacetime dimension there actually is a third nontrivial central extension possible. Since it is not necessary to get an invariant metric and does not correspond to a central extension of the group [165] we will ignore it in the following.

The projective unitary irreducible representations of this extended Galilei group were analyzed in [166]. The invariant metric that the Extended Bargmann algebra possesses was used in [164] to define "Galilean quantum gravity" using a CS formulation. Furthermore, the coadjoint orbits of the group were discussed. In [45] is was shown that this theory is related to projectable Hořava–Lifshitz gravity with a local $\mathfrak{u}(1)$ gauge symmetry and without a cosmological constant. There also exists an extension to Extended Bargmann supergravity [167]. We will now study if we can arrive at the Extended Bargmann algebra using contractions.

9.2 Extended Kinematical Algebras

We have already discussed in Section 4.4 that trivial central extensions can lead to nontrivial ones upon contraction. Since we want to start our investigations from $(\mathfrak{A})\mathfrak{d}\mathfrak{S}$ algebras which are (semi)simple our only option is to centrally extend trivially. With hindsight we shift $\mathtt{J} \to \mathtt{J} - \mathtt{H}^*$ and $\mathtt{H} \to \mathtt{H} - \mathtt{J}^*$ where the starred generators denote the trivial central extensions. The shift applied to the commutation relations and to the invariant metric, also normalized with hindsight, can be seen in Table 9.4.

The contraction that leads from (A)dS to the Poincaré algebra is given by a sIW-contraction with respect to the subalgebra spanned by $\{\mathtt{J}, \mathtt{G}_a, \mathtt{H}^*\}$. Or with the notation of (4.9) where we just denote the subscript of a of each n_a (e.g., $\mathtt{H} = 1$ means that $n_{\mathtt{H}} = 1$): $\mathtt{J} = \mathtt{G}_a = \mathtt{H}^* = 0$, $\mathtt{H} = \mathtt{P}_a = \mathtt{J}^* = 1$ and $\mu^- = -1$. The $\mathfrak{poi}$ algebra does still not allow for nontrivial central extension.

Now the interesting contractions are the ones from the centrally extended relativistic algebras (A)dS and Poincaré to the extended nonrelativistic Newton–Hooke and Galilei algebra. They indeed lead to nontrivial central extended ones which posses an invariant metric. The gIW-contraction is in both cases given by: $\mathtt{J} = \mathtt{H} = 0$, $\mathtt{G}_a = \mathtt{P}_a = 1$, $\mathtt{J}^* = \mathtt{H}^* = 2$ and $\mu^- = -2$.

For completeness we also provide the gIW-contraction $\mathfrak{nh} \oplus_c \mathfrak{u}(1)^2 \to \mathfrak{ebarg}$: $\mathtt{J} = \mathtt{J}^* = 0$, $\mathtt{P}_a = -\mathtt{G}_a = 1$, $\mathtt{H} = -\mathtt{H}^* = 2$ and $\mu^- = 0$.

	$(\mathfrak{A})\mathfrak{dS}_{(\mp)}$ $\oplus\mathfrak{u}(1)^2$	$\mathfrak{poi}$ $\oplus\mathfrak{u}(1)^2$	$\mathfrak{nh}$ $\oplus_c\mathfrak{u}(1)^2$	$\mathfrak{ebarg}$
$[\,\mathtt{J}\,,\,\mathtt{G}_a\,]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\,\mathtt{J}\,,\,\mathtt{P}_a\,]$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$
$[\,\mathtt{G}_a\,,\,\mathtt{G}_b\,]$	$-\epsilon_{ab}(\mathtt{J}-\mathtt{H}^*)$	$-\epsilon_{ab}(\mathtt{J}-\mathtt{H}^*)$	$\epsilon_{ab}\mathtt{H}^*$	$\epsilon_{ab}\mathtt{H}^*$
$[\,\mathtt{G}_a\,,\,\mathtt{H}\,]$	$-\epsilon_{am}\mathtt{P}_m$	$-\epsilon_{am}\mathtt{P}_m$	$-\epsilon_{am}\mathtt{P}_m$	$-\epsilon_{am}\mathtt{P}_m$
$[\,\mathtt{G}_a\,,\,\mathtt{P}_b\,]$	$-\epsilon_{ab}(\mathtt{H}-\mathtt{J}^*)$	$-\epsilon_{ab}(\mathtt{H}-\mathtt{J}^*)$	$\epsilon_{ab}\mathtt{J}^*$	$\epsilon_{ab}\mathtt{J}^*$
$[\,\mathtt{H}\,,\,\mathtt{P}_a\,]$	$\pm\epsilon_{am}\mathtt{G}_m$	0	$\pm\epsilon_{am}\mathtt{G}_m$	0
$[\,\mathtt{P}_a\,,\,\mathtt{P}_b\,]$	$\mp\epsilon_{ab}(\mathtt{J}-\mathtt{H}^*)$	0	$\pm\epsilon_{ab}\mathtt{H}^*$	0
$\langle\mathtt{P}_a\,,\mathtt{G}_b\rangle$	$\mu^-\delta_{ab}$	$\mu^-\delta_{ab}$	$\mu^-\delta_{ab}$	$\mu^-\delta_{ab}$
$\langle\mathtt{J}^*\,,\mathtt{H}^*\rangle$	μ^-	μ^-	0	0
$\langle\mathtt{J}\,,\mathtt{J}^*\rangle$	μ^-	μ^-	μ^-	μ^-
$\langle\mathtt{H}\,,\mathtt{H}^*\rangle$	μ^-	μ^-	μ^-	μ^-

Table 9.4: The central extended Lie algebras of (A)dS, Poincaré, Newton–Hooke and Galilei and their invariant metrics. The central extension of $(\mathfrak{A})\mathfrak{dS}$ and $\mathfrak{poi}$ are trivial. For $\mathfrak{nh}$ and $\mathfrak{ebarg}$ they are nontrival and necessary to permit an invariant metric. Nondegeneracy of the invariant metric demands that $\mu^- \neq 0$.

9.3 Carroll Gravity

In this section we address whether there are interesting infinite extensions of the algebras discussed above, in the same way that the global conformal algebra in two dimensions gets extended to the Virasoro algebra by imposing Brown–Henneaux boundary conditions [85]. We will focus here on a specific simple example. In fact, as a first step we consider spin-2 Carroll gravity, defined by a Chern–Simons gauge theory with the connection

$$A = \tau\,\mathtt{H} + e^a\,\mathtt{P}_a + \omega\,\mathtt{J} + B^a\,\mathtt{G}_a \tag{9.2}$$

takes values in the spin-2 Carroll algebra ($a = 1, 2$), whose non-vanishing commutation relations read

$$[\mathtt{J},\,\mathtt{P}_a] = \epsilon_{ab}\,\mathtt{P}_b\,, \tag{9.3a}$$

$$[\mathtt{J},\,\mathtt{G}_a] = \epsilon_{ab}\,\mathtt{G}_b\,, \tag{9.3b}$$

$$[\mathtt{P}_a,\,\mathtt{G}_b] = -\epsilon_{ab}\,\mathtt{H}\,, \tag{9.3c}$$

where we use the convention $\epsilon_{12} = +1$ for the antisymmetric ϵ-symbol. The invariant metric has the non-vanishing entries

$$\langle \mathtt{H}, \mathtt{J} \rangle = -1 \qquad \langle \mathtt{P}_a, \mathtt{G}_b \rangle = \delta_{ab} \,. \tag{9.4}$$

Our main goal is not just to find some infinite extension of the algebra (9.3) (this always exists at least in the form of the loop algebra of the underlying gauge algebra, see e.g. [72]; for AdS_3 gravity such boundary conditions were investigated recently in [168]), but rather to find an extension that has a "nice" geometric interpretation along the lines of the Brown–Henneaux boundary conditions. This means that we want to achieve a suitable Drinfeld–Sokolov type of reduction where not all algebraic components of the connection are allowed to fluctuate. The words "nice" and "suitable" here mean that, in particular, we want that the appropriate Carroll background geometry as part of our spectrum of physical states is allowed by our boundary conditions, and that all additional states are fluctuations around this background. First, we recall some basic aspects of Carroll geometry.

The Carroll-zweibein for the flat background geometry in some Fefferman–Graham like coordinates should take the form

$$e^1_\varphi = \rho \qquad e^2_\rho = 1 \qquad e^1_\rho = e^2_\varphi = 0 \tag{9.5}$$

so that the corresponding two-dimensional line-element reads

$$ds^2_{(2)} = e^a e^b \delta_{ab} = \rho^2 \, d\varphi^2 + d\rho^2 \,. \tag{9.6}$$

We shall refer to ρ as radial coordinate and to φ as angular coordinate, assuming $\varphi \sim \varphi + 2\pi$. Moreover, on the background the time-component should be fixed as

$$\tau = dt \,. \tag{9.7}$$

Below we shall allow subleading (in ρ) fluctuations in the two-dimensional line-element (9.6) and leading fluctuations in the time-component (9.7).

We proceed now by stating the result for the boundary conditions that define our example of Carroll gravity and discuss afterwards the rationale behind our choices as well as the consistency of the boundary conditions by proving the finiteness, integrability, non-triviality and conservation of the canonical boundary charges. We follow the general recipe reviewed e.g. in [103, 169]. First, we bring the connection (9.2) into a convenient gauge [67]

$$A = b^{-1}(\rho) \Big(d + a(t, \varphi) \Big) \, b(\rho) \tag{9.8}$$

where the group element

$$b(\rho) = e^{\rho \mathtt{P}_2} \tag{9.9}$$

is fixed as part of the specification of our boundary conditions, $\delta b = 0$. The boundary connection a does not depend on the radial coordinate ρ and is given by

$$a_\varphi = -\mathtt{J} + h(t,\, \varphi)\, \mathtt{H} + p_a(t,\, \varphi)\, \mathtt{P}_a + g_a(t,\, \varphi)\, \mathtt{G}_a\,, \tag{9.10a}$$

$$a_t = \mu(t,\, \varphi)\, \mathtt{H}\,, \tag{9.10b}$$

where μ is arbitrary but fixed, $\delta\mu = 0$, while all other functions are arbitrary and can vary. This means that the allowed variations of the boundary connection are given by

$$\delta a = \delta a_\varphi \; d\varphi = \Big(\delta h\, \mathtt{H} + \delta p_a\, \mathtt{P}_a + \delta g_a\, \mathtt{G}_a\Big)\; d\varphi\,. \tag{9.11}$$

The full connection in terms of the boundary connection is then given by

$$A = a + \mathtt{P}_2\, d\rho + \rho\, [a,\, \mathtt{P}_2] \tag{9.12}$$

and acquires its non-trivial radial dependence through the last term,

$$\rho\, [a,\, \mathtt{P}_2] = \rho\, (\mathtt{P}_1 - g_1(t,\, \varphi)\, \mathtt{H})\; d\varphi. \tag{9.13}$$

Only the φ-component of the connection is then allowed to vary.

$$\delta A = \delta a + \rho\, [\delta a,\, \mathtt{P}_2] = \Big(\delta h\, \mathtt{H} + \delta p_a\, \mathtt{P}_a + \delta g_a\, \mathtt{G}_a - \rho\, \delta g_1\, \mathtt{H}\Big)\; d\varphi \tag{9.14}$$

The above boundary conditions lead to Carroll-geometries of the form

$$ds^2_{(2)} = \Big[\Big(\rho + p_1(t,\, \varphi)\Big)^2 + p_2(t,\, \varphi)^2\Big]\; d\varphi^2 + 2p_2(t,\, \varphi)\; d\varphi\, d\rho + d\rho^2 \tag{9.15}$$

and

$$\tau = \mu(t,\, \varphi)\; dt + \Big(h(t,\, \varphi) - \rho\, g_1(t,\, \varphi)\Big)\; d\varphi\,. \tag{9.16}$$

Thus, we see that to leading order in ρ the background line-element (9.6) is recovered from (9.15), plus subleading (state-dependent) fluctuations captured by the functions $p_a(t,\, \varphi)$. As we shall see in the next paragraph the functions p_a and g_a are t-independent on-shell. In the metric-formulation our boundary conditions can be phrased as

$$ds^2_{(2)} = \Big(\rho^2 + \mathcal{O}(\rho)\Big)\; d\varphi^2 + \mathcal{O}(1)\; d\rho\, d\varphi + d\rho^2 \tag{9.17}$$

and

$$\tau = \mu(t,\, \varphi)\; dt + \mathcal{O}(\rho)\; d\varphi\,. \tag{9.18}$$

Note that while the asymptotic form of the two-dimensional line-element (9.17) may have been guessed easily, the specific form of the time-component (9.18) is much harder to guess, particularly the existence of a shift-component proportional to $d\varphi$ that grows linearly in ρ. Fortunately, the Chern–Simons formulation together with the gauge choice (9.8) minimizes the amount of guesswork needed to come up with meaningful boundary conditions.

We consider now the impact of the equations of motion on the free functions in the boundary connection (9.10). Gauge-flatness $F = 0$ implies

$$\partial_t a_\varphi - \partial_\varphi a_t + [a_t,\, a_\varphi] = \partial_t a_\varphi - \partial_\varphi a_t = 0\,. \tag{9.19}$$

As a consequence, we get the on-shell conditions (which also could be called "holographic Ward identities")

$$\partial_t p_a = \partial_t g_a = 0 \qquad \partial_t h = \partial_\varphi \mu\,. \tag{9.20}$$

Thus, most of the functions in the boundary connection (9.10) are time-independent, with the possible exception of h and μ.

The boundary-condition preserving transformations, $\delta_{\hat\lambda} A = d\hat\lambda + [A,\, \hat\lambda] = \mathcal{O}(\delta A)$, generated by $\hat\lambda = b^{-1}\lambda b$ have to obey the relations

$$\delta_\lambda a_t = \partial_t \lambda + [a_t,\, \lambda] = \partial_t \lambda = 0\,, \tag{9.21a}$$
$$\delta_\lambda a_\varphi = \partial_\varphi \lambda + [a_\varphi,\, \lambda] = \mathcal{O}(\delta a_\varphi)\,, \tag{9.21b}$$

where $\mathcal{O}(\delta a_\varphi)$ denotes all the allowed variations displayed in (9.11). It is useful to decompose λ with respect to the algebra (9.2).

$$\lambda = \lambda^{\mathtt{H}}\,\mathtt{H} + \lambda^{\mathtt{P}_a}\,\mathtt{P}_a + \lambda^{\mathtt{J}}\,\mathtt{J} + \lambda^{\mathtt{G}_a}\,\mathtt{G}_a\,. \tag{9.22}$$

The first line in (9.21) establishes the time-independence of λ, while the second line yields the consistency condition

$$\partial_\varphi \lambda^{\mathtt{J}} = 0 \tag{9.23}$$

as well as the transformations rules

$$\delta_\lambda h = \partial_\varphi \lambda^{\mathtt{H}} - \Big(p_1 \lambda^{\mathtt{G}_2} - p_2 \lambda^{\mathtt{G}_1} + g_1 \lambda^{\mathtt{P}_2} - g_2 \lambda^{\mathtt{P}_1}\Big)\,, \tag{9.24a}$$
$$\delta_\lambda p_a = \partial_\varphi \lambda^{\mathtt{P}_a} - \epsilon_{ab}\Big(\lambda^{\mathtt{P}_b} - p_b \lambda^{\mathtt{J}}\Big)\,, \tag{9.24b}$$
$$\delta_\lambda g_a = \partial_\varphi \lambda^{\mathtt{G}_a} - \epsilon_{ab}\Big(\lambda^{\mathtt{G}_b} - g_b \lambda^{\mathtt{J}}\Big)\,. \tag{9.24c}$$

Applying the Regge–Teitelboim approach [65] to Chern–Simons theories yields the following background-independent result for the variation of the canonical boundary charges

$$\delta Q[\lambda] = \frac{k}{2\pi} \oint \langle \hat\lambda\, \delta A \rangle = \frac{k}{2\pi} \oint \langle \lambda\, \delta a_\varphi \rangle\; d\varphi \tag{9.25}$$

which in our case expands to

$$\delta Q[\lambda] = \frac{k}{2\pi} \oint \Big(-\lambda^{\mathtt{J}} \delta h + \lambda^{\mathtt{P}_a} \delta g_a + \lambda^{\mathtt{G}_a} \delta p_a \Big)\, d\varphi \,. \tag{9.26}$$

The canonical boundary charges are manifestly finite since the ρ-dependence drops out in (9.25); they are also integrable in field-space since our λ is state-independent.

$$Q[\lambda] = \frac{k}{2\pi} \oint \Big(-\lambda^{\mathtt{J}} h + \lambda^{\mathtt{P}_a} g_a + \lambda^{\mathtt{G}_a} p_a \Big)\, d\varphi \,. \tag{9.27}$$

The result (9.27) clearly is non-trivial in general. To conclude the proof that we have meaningful boundary conditions we finally check conservation in time, using the on-shell relations (9.20) as well as the time-independence of λ, see (9.21a):

$$\partial_t Q[\lambda]\Big|_{\text{EOM}} = -\frac{k}{2\pi} \oint \lambda^{\mathtt{J}} \partial_t h \; d\varphi = -\frac{k}{2\pi} \oint \lambda^{\mathtt{J}} \partial_\varphi \mu \; d\varphi = \frac{k}{2\pi} \oint \mu \partial_\varphi \lambda^{\mathtt{J}} \; d\varphi \,. \tag{9.28}$$

By virtue of (9.23) we see that the last integrand vanishes and thus we have established charge conservation on-shell:

$$\partial_t Q[\lambda]\Big|_{\text{EOM}} = 0 \,. \tag{9.29}$$

Since our canonical boundary charges (9.27) are finite, integrable in field space, non-trivial and conserved in time the boundary conditions (9.8)-(9.14) are consistent and lead to a non-trivial theory. For later purposes, it is useful to note that due to the constancy of $\lambda^{\mathtt{J}}$ only the zero mode charge associated with the function h can be non-trivial. This means that we can gauge-fix our connection using proper gauge transformations such that $h = \text{const}$.

We now introduce Fourier modes in order to be able to present the asymptotic symmetry algebra in a convenient form.[3]

$$\mathtt{P}^a_n := \frac{1}{2\pi} \oint d\varphi \, e^{in\varphi} g_a(t,\, \varphi)\Big|_{\text{EOM}} \,, \tag{9.30a}$$

$$\mathtt{G}^a_n := \frac{1}{2\pi} \oint d\varphi \, e^{in\varphi} p_a(t,\, \varphi)\Big|_{\text{EOM}} \,, \tag{9.30b}$$

$$\mathtt{J} := -\frac{1}{2\pi} \oint d\varphi \, h(t,\, \varphi)\Big|_{\text{EOM}} \,. \tag{9.30c}$$

[3]There is no meaning to the index positions in this section. The only reason why we write $\mathtt{P}^a_n$ and $\mathtt{G}^a_n$ instead of corresponding quantities with lower indices is that our current convention is easier to read.

A few explanations are in order. Due to our off-diagonal bilinear form (9.4) we associate the n^{th} Fourier mode of the functions g_a (p_a) with the generator $\mathtt{P}_n^a$ ($\mathtt{G}_n^a$). For the same reason we associate $\mathtt{J}$ with minus the zero mode of h. Finally, the subscript EOM means that all integrals are evaluated on-shell, in which case all t-dependence drops out (and in the last integral also all φ-dependence).

We make a similar Fourier decomposition of the gauge parameters λ^i, where i refers to the generators $\mathtt{P}_a$, $\mathtt{G}_a$ and $\mathtt{J}$; the parameter $\lambda^{\mathtt{H}}$ is not needed since it does not appear in the canonical boundary charges (9.27), so all gauge transformations associated with it are proper ones and can be used to make h constant.

$$\lambda_n^{\mathtt{P}_a} := \frac{1}{2\pi} \oint d\varphi\, e^{in\varphi} \lambda^{\mathtt{P}_a}(\varphi)\,, \tag{9.31a}$$

$$\lambda_n^{\mathtt{G}_a} := \frac{1}{2\pi} \oint d\varphi\, e^{in\varphi} \lambda^{\mathtt{G}_a}(\varphi)\,. \tag{9.31b}$$

Note that we have used (9.21) to eliminate all time-dependence and that $\lambda^{\mathtt{J}}$ is a constant according to (9.23) thus requiring no Fourier decomposition.

The variations (9.24) of the state-dependent functions then establish corresponding variations in terms of the Fourier components (9.30), (9.31).

$$\delta \mathtt{P}_n^a = -in\lambda_n^{\mathtt{G}_a} - \epsilon_{ab}\lambda_n^{\mathtt{G}_b} + \epsilon_{ab}\lambda^{\mathtt{J}}\mathtt{P}_n^b\,, \tag{9.32a}$$

$$\delta \mathtt{G}_n^a = -in\lambda_n^{\mathtt{P}_a} - \epsilon_{ab}\lambda_n^{\mathtt{P}_b} + \epsilon_{ab}\lambda^{\mathtt{J}}\mathtt{G}_n^b\,, \tag{9.32b}$$

$$\delta \mathtt{J} = \sum_{n\in\mathbb{Z}} \epsilon_{ab}\Big(\mathtt{G}_n^a\lambda_{-n}^{\mathtt{G}_b} + \mathtt{P}_n^a\lambda_{-n}^{\mathtt{P}_b}\Big)\,. \tag{9.32c}$$

From the variations (9.32) we can read off the asymptotic symmetry algebra, using the fact that the canonical generators generate gauge transformations via the Dirac bracket $\delta_{\lambda_1}Q[\lambda_2] = \{Q[\lambda_1],\, Q[\lambda_2]\}$.

Converting Dirac brackets into commutators then establishes the asymptotic symmetry algebra as the commutator algebra of the infinite set of generators $\mathtt{P}_n^a$, $\mathtt{G}_n^a$ and $\mathtt{J}$. The central element of this algebra will be associated with (minus) $\mathtt{H}$, concurrent with the notation of (9.3). Evaluating (9.32) yields[4]

$$[\mathtt{J},\, \mathtt{P}_n^a] = \epsilon_{ab}\,\mathtt{P}_n^b\,, \tag{9.33a}$$

$$[\mathtt{J},\, \mathtt{G}_n^a] = \epsilon_{ab}\,\mathtt{G}_n^b\,, \tag{9.33b}$$

$$[\mathtt{P}_n^a,\, \mathtt{G}_m^b] = -\Big(\epsilon_{ab} + in\delta_{ab}\Big)\,\mathtt{H}\,\delta_{n+m,\,0}\,, \tag{9.33c}$$

[4]Note that our definitions of Fourier-components (9.30), (9.31) require that we associate the negative Fourier components of the λ with the positive Fourier components of the generators so that, for instance, $[\mathtt{P}_n^b,\, \mathtt{J}] = \delta_{\lambda_{-n}^{\mathtt{P}_b}}\, \mathtt{J}$.

where all commutators not displayed vanish. We have thus succeeded in providing an infinite lift of the Carroll algebra (9.3), which is contained as a subalgebra of our asymptotic symmetry algebra (9.33) by restricting to the zero-mode generators $\mathtt{P}_a = \mathtt{P}_0^a$, $\mathtt{G}_a = \mathtt{G}_0^a$ in addition to $\mathtt{J}$ and $\mathtt{H}$. As a simple consistency check one may verify that the Jacobi identities indeed hold. The only non-trivial one to be checked is the identity $[[\mathtt{J}, \mathtt{P}_n^a], \mathtt{G}_m^b] + \text{cycl.} = 0$.

We conclude this section with a couple of remarks. The boundary conditions (9.8)-(9.10) by no means are unique and can be either generalized or specialized to looser or stricter ones, respectively. Another set of well defined boundary condition has been proposed in [64]. In particular, we have switched off nearly all chemical potentials in our specification of the time-component of the connection (9.10b), and it could be of interest to allow arbitrary chemical potentials. Apart from this issue there is only one substantial generalization of our boundary conditions, namely to allow for a state-dependent function in front of the generator $\mathtt{J}$ in the angular component of the connection (9.10a). As mentioned in the opening paragraph of this section, in that case the expected asymptotic symmetry algebra is the loop algebra of the Carroll algebra (9.3). In principle, it is possible to make our boundary conditions stricter, but that would potentially eliminate interesting physical states like some of the Carroll geometries (9.15), (9.16). Thus, while our choice (9.8)-(9.10) is not unique it provides an interesting set of boundary conditions for spin-2 Carroll gravity. Using the same techniques it should be straightforward to extend the discussion of this section to higher spin Carroll gravity and related theories discussed in this thesis.

Chapter 10

Kinematical Spin-3 Theories

The reason for restricting ourselves to three spacetime dimensions stems from the fact that, as far as higher spin gauge theory is concerned, this case is a lot simpler than its higher-dimensional counterpart. For instance, in three dimensions it is possible to consider higher spin gauge theory in flat spacetimes [170–174], unlike the situation in higher dimensions where a non-zero cosmological constant is required[1]. Moreover, and as already discussed, in three dimensions higher spin gauge theories with only a finite number of higher spin fields can be constructed [89]. In the relativistic case, such theories assume the form of Chern–Simons theories, for a gauge group that is a suitable finite-dimensional extension of the three-dimensional (A)dS and Poincaré algebras. For theories with integer spins ranging from 2 to N in AdS spacetime, the gauge algebra is given by $\mathfrak{sl}(N,\mathbb{R}) \oplus \mathfrak{sl}(N,\mathbb{R})$. Here, we will restrict ourselves for simplicity to "spin-3 theory" for which $N = 3$, although our analysis can straightforwardly be generalized to arbitrary N.

We will thus extend the discussion of kinematical algebras of [8] and review in Section 9 to theories in three spacetime dimensions that include a spin-3 field coupled to gravity. In particular, we will start from the observation made in [8] that all kinematical algebras can be obtained by taking sequential Inönü-Wigner (IW) contractions of the (A)dS algebras. We will then classify all possible sIW contractions[2] of the kinematical algebra of spin-3 theory in (A)dS$_3$, as well as all possible kinematical algebras that can be obtained by sequential contractions. Some of the kinematical algebras that are obtained in this way can be interpreted as spin-3 extensions of the Galilei and Carroll algebras. We will show that one can construct

[1] See however [175–177] for recent progress concerning higher spin theories in four-dimensional flat space.

[2] It should be emphasized that this does not classify the Lie algebras that result from the contraction.

Chern–Simons theories for (suitable extensions of) these algebras. These can then be interpreted as non- and ultra-relativistic three-dimensional spin-3 theories. We will in particular argue that these theories can be viewed as higher spin generalizations of Extended Bargmann gravity [45, 164, 167, 178] and Carroll gravity [179], two examples of non- and ultra-relativistic gravity theories that have been considered in the literature recently.

The kinematical algebras of spin-3 theories that we obtain are finite-dimensional. Relativistic three-dimensional kinematical algebras have infinite-dimensional extensions that are obtained as asymptotic symmetry algebras upon imposing suitable boundary conditions on metric and higher spin fields, such as the Virasoro algebra (for the AdS algebra) [85], the BMS algebra (for the Poincaré algebra) [180, 181] or $\mathcal{W}$-algebras (for their higher spin generalizations) [26, 27]. One such example for the Carroll algebra was discussed in Section 9.3. It is interesting to ask whether the found non- and ultra-relativistic algebras also have infinite-dimensional extensions that correspond to asymptotic symmetry algebras of their corresponding higher spin gravity theories.

This chapter is based on Section 2 and 3 of [5]. We will first, in section 10.1, classify all sIW contractions of the kinematical algebra of spin-3 theory in (A)dS$_3$. We then classify all kinematical algebras that can be obtained by combining these various contractions. In section 10.2, we restrict ourselves to the algebras that can be interpreted as non- and ultra-relativistic ones, for zero cosmological constant. We argue that in the ultra-relativistic cases, a Chern–Simons theory can be constructed in a straightforward manner. This is due to the considerations of Section 5.3. This is not true for the nonrelativistic cases. However, we demonstrate that the nonrelativistic kinematical algebras can be suitably extended in such a way that a Chern–Simons action can be written down. Here the knowledge of double extension will be useful. We then show via a linearized analysis that the non- and ultra-relativistic spin-3 Chern–Simons theories thus obtained can be viewed as spin-3 generalizations of Extended Bargmann gravity and Carroll gravity, respectively.

10.1 Kinematical Spin-3 Algebras

In this section, we will be concerned with three-dimensional kinematical spin-3 algebras, i.e., generalized spacetime symmetry algebras of theories of interacting, massless spin-2 and spin-3 fields. In particular, following Bacry and Lévy–Leblond [8] we will classify all such algebras that can be obtained by combining different sIW-contractions from the algebras that underlie spin-3 gravity in AdS_3 and dS_3. After recalling the latter, we will present all possible ways of contracting them, such that non-trivial kinematical spin-3 algebras are obtained, via a classification theorem. Combining different of these sIW-contractions leads to various kinematical spin-3 algebras, some of which will be discussed in the next section as a starting point for considering Carroll and Galilei spin-3 gravity Chern–Simons theories.

AdS_3 and dS_3 Spin-3 Algebras

Remember that Spin-3 gravity in (A)dS_3 [26, 27] can be written as a Chern–Simons theory for the Lie algebra $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})$ for AdS_3 or $\mathfrak{sl}(3,\mathbb{C})$ (viewed as a real Lie algebra) for dS_3. In the following we will often denote the higher spin algebra $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})$, realizing Spin-3 gravity in AdS_3, by $\mathfrak{hs}_3\mathfrak{AdS}$. Similarly, we indicate the higher spin algebra $\mathfrak{sl}(3,\mathbb{C})$, realizing Spin-3 gravity in dS_3, by $\mathfrak{hs}_3\mathfrak{dS}$. In both cases, the algebra consists of the generators of Lorentz transformations $\hat{\mathtt{J}}_A$ and translations $\hat{\mathtt{P}}_A$ along with "spin-3 rotations" $\hat{\mathtt{J}}_{AB}$ and "spin-3 translations" $\hat{\mathtt{P}}_{AB}$, that are traceless-symmetric in the (AB) indices $(A = 0, 1, 2)$ [3]:

$$\hat{\mathtt{J}}_{AB} = \hat{\mathtt{J}}_{BA} \qquad\qquad \eta^{AB}\hat{\mathtt{J}}_{AB} = 0 \tag{10.1}$$

$$\hat{\mathtt{P}}_{AB} = \hat{\mathtt{P}}_{BA} \qquad\qquad \eta^{AB}\hat{\mathtt{P}}_{AB} = 0\,. \tag{10.2}$$

Here, η^{AB} is the three-dimensional Minkowski metric. We will often refer to $\{\hat{\mathtt{J}}_A, \hat{\mathtt{P}}_A\}$ as the "spin-2 generators" or the "spin-2 part" and similarly to $\{\hat{\mathtt{J}}_{AB}, \hat{\mathtt{P}}_{AB}\}$ as the "spin-3 generators" or "spin-3 part". Their commutation relations are given by [26, 27]

$$\begin{aligned}
&\left[\,\hat{\mathtt{J}}_A\,,\,\hat{\mathtt{J}}_B\,\right] = \epsilon_{ABC}\,\hat{\mathtt{J}}^C\,, && \left[\,\hat{\mathtt{J}}_A\,,\,\hat{\mathtt{P}}_B\,\right] = \epsilon_{ABC}\,\hat{\mathtt{P}}^C\,,\\
&\left[\,\hat{\mathtt{P}}_A\,,\,\hat{\mathtt{P}}_B\,\right] = \pm\epsilon_{ABC}\,\hat{\mathtt{J}}^C\,, && \\
&\left[\,\hat{\mathtt{J}}_A\,,\,\hat{\mathtt{J}}_{BC}\,\right] = \epsilon^M{}_{A(B}\,\hat{\mathtt{J}}_{C)M}\,, && \left[\,\hat{\mathtt{P}}_A\,,\,\hat{\mathtt{P}}_{BC}\,\right] = \pm\,\epsilon^M{}_{A(B}\,\hat{\mathtt{J}}_{C)M}\,,\\
&\left[\,\hat{\mathtt{J}}_A\,,\,\hat{\mathtt{P}}_{BC}\,\right] = \epsilon^M{}_{A(B}\,\hat{\mathtt{P}}_{C)M}\,, && \left[\,\hat{\mathtt{P}}_A\,,\,\hat{\mathtt{J}}_{BC}\,\right] = \epsilon^M{}_{A(B}\,\hat{\mathtt{P}}_{C)M}\,,\\
&\left[\,\hat{\mathtt{J}}_{AB}\,,\,\hat{\mathtt{J}}_{CD}\,\right] = -\eta_{(A(C}\epsilon_{D)B)M}\,\hat{\mathtt{J}}^M\,, && \left[\,\hat{\mathtt{J}}_{AB}\,,\,\hat{\mathtt{P}}_{CD}\,\right] = -\eta_{(A(C}\epsilon_{D)B)M}\,\hat{\mathtt{P}}^M\,,\\
&\left[\,\hat{\mathtt{P}}_{AB}\,,\,\hat{\mathtt{P}}_{CD}\,\right] = \mp\eta_{(A(C}\epsilon_{D)B)M}\,\hat{\mathtt{J}}^M\,, &&
\end{aligned} \tag{10.3}$$

where the upper sign refers to $\mathfrak{hs}_3\mathfrak{AdS}$ and the lower sign to $\mathfrak{hs}_3\mathfrak{dS}$. Note that the first two lines constitute the isometry algebra of (A)dS_3, i.e.,

[3] We refer to Appendix A for index and other conventions used in this and upcoming sections.

$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ for AdS$_3$ and $\mathfrak{sl}(2,\mathbb{C})$, viewed as a real Lie algebra, for dS$_3$.

The above mentioned algebra is (semi)simple and therefore has an invariant metric. The most general one is given in Section D.7 but we will restrict here to

$$\langle \hat{\mathtt{P}}_A \,, \hat{\mathtt{J}}_B \rangle = \eta_{AB} \qquad \langle \hat{\mathtt{P}}_{AB} \,, \hat{\mathtt{J}}_{CD} \rangle = \eta_{A(C}\eta_{D)B} - \frac{2}{3}\eta_{AB}\eta_{CD} \,. \tag{10.4}$$

Note that this represents an invariant metric for both $\mathfrak{hs}_3\mathfrak{AdS}$ and $\mathfrak{hs}_3\mathfrak{dS}$. The existence of this metric allows one to construct Chern–Simons actions for the algebras $\mathfrak{hs}_3\mathfrak{AdS}$ and $\mathfrak{hs}_3\mathfrak{dS}$, that correspond to the actions for spin-3 gravity in (A)dS$_3$ [26, 27].

In the following, it will prove convenient to introduce a time-space splitting of the indices $A = \{0, a; a = 1, 2\}$. We will thereby use the following notation:

$$\mathtt{J} = \hat{\mathtt{J}}_0 \qquad \mathtt{G}_a = \hat{\mathtt{J}}_a \qquad \mathtt{H} = \hat{\mathtt{P}}_0 \qquad \mathtt{P}_a = \hat{\mathtt{P}}_a \tag{10.5}$$

$$\mathtt{J}_a = \hat{\mathtt{J}}_{0a} \qquad \mathtt{G}_{ab} = \hat{\mathtt{J}}_{ab} \qquad \mathtt{H}_a = \hat{\mathtt{P}}_{0a} \qquad \mathtt{P}_{ab} = \hat{\mathtt{P}}_{ab} \,. \tag{10.6}$$

Note that we have left out the generators $\hat{\mathtt{P}}_{00}$ and $\hat{\mathtt{G}}_{00}$ here. These generators are not independent, due to the tracelessness constraint (10.1) and in the following we will eliminate them in favor of $\mathtt{P}_{ab}$ and $\mathtt{G}_{ab}$. After these substitutions, the commutation relations of $\mathfrak{hs}_3(\mathfrak{A})\mathfrak{dS}$ in this new basis are given in the first column of Table D.3.

All Kinematical Spin-3 Algebras by Contracting $\mathfrak{hs}_3(\mathfrak{A})\mathfrak{dS}$

We now consider the spin-3 case where, following the spin-2 case (see Section 9.1), we will obtain a classification of all possible contractions[4] of $\mathfrak{hs}_3\mathfrak{AdS}$

[4] Here, we will classify different contractions, in the sense defined above as different choices of subalgebra $\mathfrak{h}$, i.e., we restrict to sIW-contractions. This does not mean that all these contractions lead to non-isomorphic Lie algebras. Indeed, in the analysis of [8], e.g., one can see that the space-time and speed-time contractions applied to the AdS$_3$ isometry algebra lead to two Lie algebras that are both isomorphic to the Poincaré algebra. We should however mention that these algebras are isomorphic in the mathematical sense; physically they can be regarded as non-equivalent as the isomorphism that relates them corresponds to an interchange of boost and translation generators. Note also that the different contractions that are classified here are not necessarily independent. As an example, one can check that the general sIW-contraction of Table 9.1 can be obtained by sequential space-time, speed-space and speed-time sIW-contractions in an arbitrary order.

and $\mathfrak{hs}_3\mathfrak{dS}$ by listing all their possible subalgebras. We start from $\mathfrak{hs}_3(\mathfrak{A})\mathfrak{dS}$ since these are semisimple algebras and can therefore not be viewed as a result of a sIW-contraction (since proper sIW-contractions always lead to algebras with an abelian ideal that are thus not semisimple). Now, in order to obtain contractions that can be identified as interesting kinematical spin-3 algebras, we will impose two restrictions:

- When restricted to the spin-2 part of the algebra, the sIW-contraction should correspond to those considered in Table 9.1. This ensures that the spin-2 parts of the algebras obtained by various combinations of these contractions correspond to the kinematical algebras of [8].

- Furthermore, we will also demand that in the resulting Lie algebra not all commutators of the spin-3 part are vanishing. This requirement is motivated by the fact that we are interested in using these contractions to describe fully interacting theories of massless spin-2 and spin-3 fields. Indeed, as we will show later on, for some of the algebras obtained here, one can construct a Chern–Simons action for spin-2 and spin-3 fields. Only when the commutators of the spin-3 part are not all vanishing, do the spin-3 fields contribute to the equations of motion of the spin-2 fields.

All ways of sIW-contracting $\mathfrak{hs}_3\mathfrak{A}\mathfrak{dS}$ and $\mathfrak{hs}_3\mathfrak{dS}$ that obey these two restrictions can then be summarized by the following theorem:

Theorem 10.1. *All possible sIW-contractions, that reduce to those considered in Table 9.1 when restricted to the spin-2 part and that are nonabelian on the subspace spanned by the spin-3 generators $\{\mathtt{J}_a, \mathtt{H}_a, \mathtt{G}_{ab}, \mathtt{P}_{ab}\}$, are given by 10 "democratic" contractions that are specified in Table 10.1 and 7 "traceless" contractions, given in Table 10.2. As in Table 9.1, we have specified these contractions by indicating the subalgebra $\mathfrak{h}$ with respect to which $\mathfrak{hs}_3(\mathfrak{A})\mathfrak{dS}$ is contracted, as well as by giving the resulting abelian ideal $\mathfrak{i}$.*

Contraction	#	$\mathfrak{h}$	$\mathfrak{i}$
Space-time	1	$\{\mathtt{J}, \mathtt{G}_a, \mathtt{J}_a, \mathtt{G}_{ab}\}$	$\{\mathtt{H}, \mathtt{P}_a, \mathtt{H}_a, \mathtt{P}_{ab}\}$
	2	$\{\mathtt{J}, \mathtt{G}_a, \mathtt{H}_a, \mathtt{P}_{ab}\}$	$\{\mathtt{H}, \mathtt{P}_a, \mathtt{J}_a, \mathtt{G}_{ab}\}$
Speed-space	3	$\{\mathtt{J}, \mathtt{H}, \mathtt{J}_a, \mathtt{H}_a\}$	$\{\mathtt{G}_a, \mathtt{P}_a, \mathtt{G}_{ab}, \mathtt{P}_{ab}\}$
	4	$\{\mathtt{J}, \mathtt{H}, \mathtt{G}_{ab}, \mathtt{P}_{ab}\}$	$\{\mathtt{G}_a, \mathtt{P}_a, \mathtt{J}_a, \mathtt{H}_a\}$
Speed-time	5	$\{\mathtt{J}, \mathtt{P}_a, \mathtt{J}_a, \mathtt{P}_{ab}\}$	$\{\mathtt{G}_a, \mathtt{H}, \mathtt{H}_a, \mathtt{G}_{ab}\}$
	6	$\{\mathtt{J}, \mathtt{P}_a, \mathtt{H}_a, \mathtt{G}_{ab}\}$	$\{\mathtt{G}_a, \mathtt{H}, \mathtt{J}_a, \mathtt{P}_{ab}\}$
General	7	$\{\mathtt{J}, \mathtt{J}_a\}$	$\{\mathtt{H}, \mathtt{P}_a, \mathtt{G}_a, \mathtt{H}_a, \mathtt{G}_{ab}, \mathtt{P}_{ab}\}$
	8	$\{\mathtt{J}, \mathtt{G}_{ab}\}$	$\{\mathtt{H}, \mathtt{P}_a, \mathtt{G}_a, \mathtt{J}_a, \mathtt{H}_a, \mathtt{P}_{ab}\}$
	9	$\{\mathtt{J}, \mathtt{H}_a\}$	$\{\mathtt{H}, \mathtt{P}_a, \mathtt{G}_a, \mathtt{J}_a, \mathtt{G}_{ab}, \mathtt{P}_{ab}\}$
	10	$\{\mathtt{J}, \mathtt{P}_{ab}\}$	$\{\mathtt{H}, \mathtt{P}_a, \mathtt{G}_a, \mathtt{J}_a, \mathtt{H}_a, \mathtt{G}_{ab}\}$

Table 10.1: All democratic sIW-contractions.

Contr.	#	$\mathfrak{h}$	$\mathfrak{i}$
Speed -space	$4a$	$\{\mathtt{J}, \mathtt{H}, \mathtt{G}_{ab}, \mathtt{P}_{12}, \mathtt{P}_{22} - \mathtt{P}_{11}\}$	$\{\mathtt{P}_{11} + \mathtt{P}_{22}\}$
	$4b$	$\{\mathtt{J}, \mathtt{H}, \mathtt{G}_{12}, \mathtt{G}_{22} - \mathtt{G}_{11}, \mathtt{P}_{ab}\}$	$\{\mathtt{G}_{11} + \mathtt{G}_{22}\}$
	$4c$	$\{\mathtt{J}, \mathtt{H}, \mathtt{G}_{12}, \mathtt{G}_{22} - \mathtt{G}_{11}, \mathtt{P}_{12}, \mathtt{P}_{22} - \mathtt{P}_{11}\}$	$\{\mathtt{G}_{11} + \mathtt{G}_{22}, \mathtt{P}_{11} + \mathtt{P}_{22}\}$
General	$8a$	$\{\mathtt{J}, \mathtt{G}_{12}, \mathtt{G}_{22} - \mathtt{G}_{11}\}$	$\{\mathtt{G}_{11} + \mathtt{G}_{22}, \mathtt{P}_{ab}\}$
	$10a$	$\{\mathtt{J}, \mathtt{P}_{12}, \mathtt{P}_{22} - \mathtt{P}_{11}\}$	$\{\mathtt{G}_{ab}, \mathtt{P}_{11} + \mathtt{P}_{22}\}$
	$8b$	$\{\mathtt{J}, \mathtt{G}_{12}, \mathtt{G}_{22} - \mathtt{G}_{11}, \mathtt{P}_{11} + \mathtt{P}_{22}\}$	$\{\mathtt{G}_{11} + \mathtt{G}_{22}, \mathtt{P}_{12}, \mathtt{P}_{22} - \mathtt{P}_{11}\}$
	$10b$	$\{\mathtt{J}, \mathtt{P}_{12}, \mathtt{P}_{22} - \mathtt{P}_{11}, \mathtt{G}_{11} + \mathtt{G}_{22}\}$	$\{\mathtt{G}_{12}, \mathtt{G}_{22} - \mathtt{G}_{11}, \mathtt{P}_{11} + \mathtt{P}_{22}\}$

Table 10.2: All traceless sIW-contractions, where we have to add in the $\mathfrak{i}$ column for the speed-space sIW-contractions $\{\mathtt{G}_a, \mathtt{P}_a, \mathtt{J}_a, \mathtt{H}_a\}$ and for the general sIW-contractions $\{\mathtt{H}, \mathtt{P}_a, \mathtt{G}_a, \mathtt{J}_a, \mathtt{H}_a\}$.

The complete proof of this theorem is given in the Appendix of [5]. For now, let us suffice by saying that the proof starts by noting that each of the subalgebras $\mathfrak{h}$ in Table 9.1 needs to be supplemented with spin-3 generators, in order to have a contraction with a nonabelian spin-3 part. The proof then proceeds by enumerating, for each of the contractions of Table 9.1, all possibilities in which spin-3 generators can be added to $\mathfrak{h}$ such that one still obtains a subalgebra, that leads to a contraction with a nonabelian spin-3 part. We refer to appendix D.7 for the explicit Lie algebras of the contracted Lie algebra given in Table 10.1.

Finally, let us comment on the terminology "democratic" and "traceless". This terminology stems from the fact that the three independent generators contained in $\mathtt{P}_{ab}$ ($\mathtt{G}_{ab}$) form a real, reducible representation of $\mathtt{J}$, that can be split into a tracefree symmetric part consisting of the generators $\{\mathtt{P}_{12}, \mathtt{P}_{22} - \mathtt{P}_{11}\}$ ($\{\mathtt{G}_{12}, \mathtt{G}_{22} - \mathtt{G}_{11}\}$) and a trace part $\mathtt{P}_{11} + \mathtt{P}_{11}$ ($\mathtt{G}_{11} + \mathtt{G}_{22}$). The democratic contractions are such that the subalgebra $\mathfrak{h}$ contains both tracefree symmetric and trace components of $\mathtt{P}_{ab}$ ($\mathtt{G}_{ab}$), if present. In some cases, it is not necessary to include the trace component in $\mathfrak{h}$ in order to obtain a valid subalgebra. This is the case for the democratic contractions , numbered 4, 8 and 10 in Table 10.1. Moving the trace component from $\mathfrak{h}$ to $\mathfrak{i}$ leads to the traceless cases 4a, 4b, 4c, 8a and 10a in Table 10.2. In the last two remaining cases both the tracefree symmetric part of $\mathtt{G}_{ab}$ ($\mathtt{P}_{ab}$) and the trace part of $\mathtt{P}_{ab}$ ($\mathtt{G}_{ab}$) belong to the subalgebra $\mathfrak{h}$. Doing this leads to the traceless cases 8b and 10b.

The democratic contractions can again be summarized as a cube, see Figure 10.1.

10.2 Carroll, Galilei and Extended Bargmann Theories

In the previous section, we have classified all possible (sIW-)contractions of the spin-3 AdS$_3$ and dS$_3$ algebras. Combining some of these contractions can lead to algebras whose spin-2 part corresponds to the Carroll or Galilei algebra. Here, we will study these cases in more detail. In particular, we will be concerned with constructing Chern–Simons theories for these spin-3 algebras, or suitable extensions thereof. This extends [179] where the case of spin-2 Carroll and spin-2 Galilei gravity is discussed.

In order to construct Chern–Simons actions for Carroll and Galilei spin-3 algebras, one therefore needs to know whether these algebras can be equipped with an invariant metric. We have already seen in Section 9 that this is not even for the spin-2 algebras always possible. In this respect, it is useful to remember that it is not always true that the contraction of a Lie algebra equipped with an invariant metric, also admits one. A counter-example was provided by the three-dimensional spin-2 Galilei algebra which arises as sIW contractions of the Poincaré algebra, that in three dimensions has an invariant metric. Naively, one can thus not construct a Chern–Simons action for the Galilei algebra how ever as shown, there exists an extension of the Galilei algebra, the so-called Extended Bargmann algebra, that can be equipped with an invariant metric and for which a Chern–Simons action

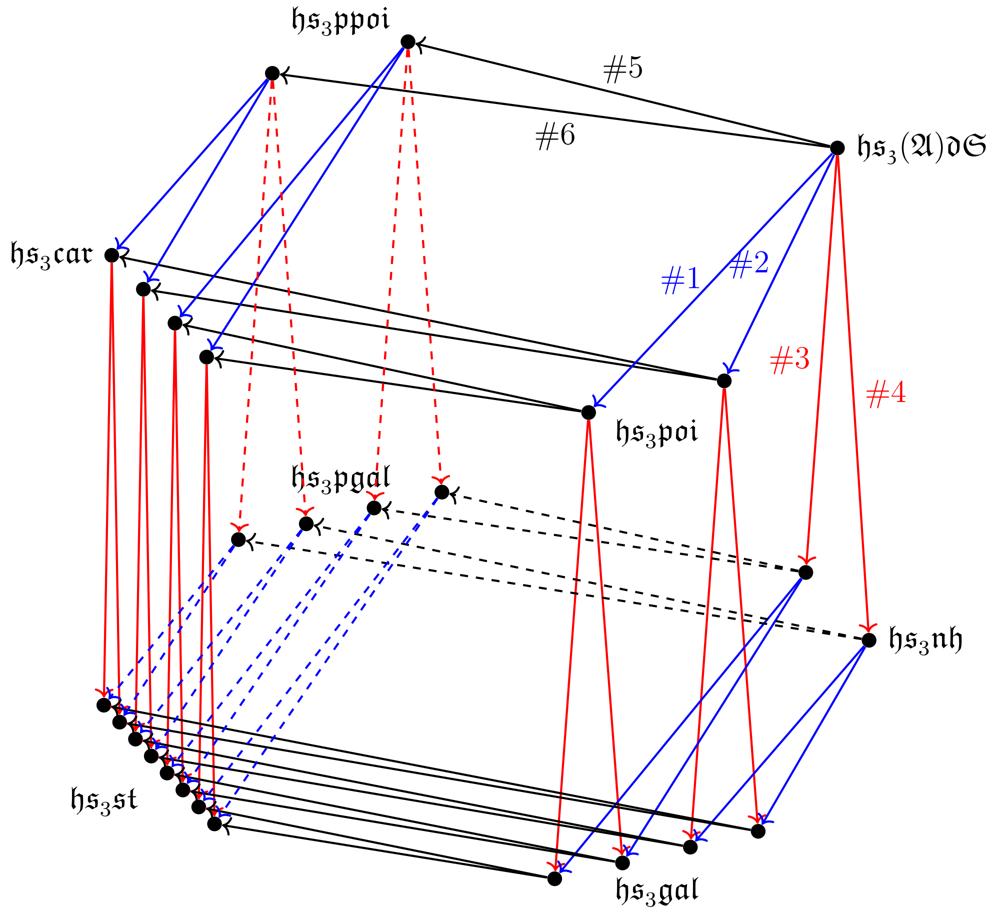

Figure 10.1: This figure summarizes the sequential democratic contractions of Table 10.1. There are 2 space-time (blue; #1,#2), 2 speed-space (red; #3,#4) and 2 speed-time (black; #5,#6) contractions and combining them leads to the full cube. The commutators of the algebras corresponding to the dots are given in Table D.3-D.13. In comparison to Figure 9.1, we have for clarity omitted the double lines and the diagonal lines that indicate the direct sIW-contractions to the static algebras.

can be constructed.

In this section, we will show that similar results hold in the spin-3 case. In particular, we will see that the spin-3 versions of the Carroll algebra admit an invariant metric and that a Chern–Simons action can be straightforwardly constructed. The spin-3 versions of the Galilei algebra, like their spin-2 versions, do not have an invariant metric. However, using double extensions we can extended them to Lie algebras with an invariant metric. In contrast to the spin-2 case this double extension is not just given by nontrivial central extensions. We will explicitly construct these "spin-3 Extended Bargmann" algebras and their associated Chern–Simons actions. In this way, we will obtain spin-3 versions of Carroll gravity [149, 179] and Extended Bargmann gravity [45, 164, 167, 178].

We will first treat the case of spin-3 Carroll gravity, while the spin-3 Extended Bargmann gravity case will be discussed afterwards. In both cases, we will also study the equations of motion, at the linearized level. This will allow us to interpret the Chern-Simons actions for these theories as suitable spin-3 generalizations of the actions of Carroll and Extended Bargmann gravity, in a first order formulation. In particular, this linearized analysis will show that some of the gauge fields appearing in these actions can be interpreted as generalized vielbeine, while others can be viewed as generalized spin connections. The latter in particular appear only algebraically in the equations of motion and are therefore dependent fields that can be expressed in terms of other fields. We will give these expressions. In some cases, we will see that not all spin connection components become dependent. We will argue that the remaining independent spin connection components can be viewed as Lagrange multipliers that implement certain constraints on the geometry. For simplicity, we will restrict ourselves to Carroll and Galilei spin-3 gravity theories. The analysis provided here can be straightforwardly extended to include a cosmological constant.

Spin-3 Carroll Gravity

There are four distinct ways of contracting $\mathfrak{hs}_3(\mathfrak{A})\mathfrak{dS}$, such that a spin-3 algebra whose spin-2 part coincides with the Carroll algebra is obtained. These four ways correspond to combining the contractions 1 and 5, 1 and 6, 2 and 5 or 2 and 6 of Table 10.1, respectively. We will denote the resulting algebras as $\mathfrak{hs}_3\mathfrak{car1}$, $\mathfrak{hs}_3\mathfrak{car2}$, $\mathfrak{hs}_3\mathfrak{car3}$ and $\mathfrak{hs}_3\mathfrak{car4}$. Their structure constants are summarized in Table D.6. Note that $\mathfrak{hs}_3\mathfrak{car3}$ and $\mathfrak{hs}_3\mathfrak{car4}$ each come in two versions, since we apply the sIW-contractions to AdS and dS simultaneously. These versions differ in the signs of some of their structure constants, as can be seen from Table D.6. The existence of these different versions when applying the contractions 2 and 5 (or 2 and 6) stems from the fact that the combination of these contractions leads to different algebras, depending on whether one starts from $\mathfrak{hs}_3\mathfrak{AdS}$ or from $\mathfrak{hs}_3\mathfrak{dS}$. By contrast, applying contraction 1 and 5 (or 1 and 6) on $\mathfrak{hs}_3\mathfrak{AdS}$ and $\mathfrak{hs}_3\mathfrak{dS}$ leads to the same result, namely $\mathfrak{hs}_3\mathfrak{car1}$ (or $\mathfrak{hs}_3\mathfrak{car2}$).

All these spin-3 algebras have an invariant metric. This can either be seen using the just mentioned contractions or using the invariant metric preserving contractions discussed in Section 5.3. The invariant metric preserving contractions are specified by just a subalgebra $\mathfrak{h}$ of the original algebra and for these cases are given by:

- $\mathfrak{hs}_3\mathfrak{poi1} \to \mathfrak{hs}_3\mathfrak{car1}$: $\mathfrak{h} = \{\mathtt{J}, \mathtt{J}_a\}$,
- $\mathfrak{hs}_3\mathfrak{poi1} \to \mathfrak{hs}_3\mathfrak{car2}$: $\mathfrak{h} = \{\mathtt{J}, \mathtt{G}_{ab}\}$,
- $\mathfrak{hs}_3\mathfrak{poi2} \to \mathfrak{hs}_3\mathfrak{car3}$: $\mathfrak{h} = \{\mathtt{J}, \mathtt{P}_{ab}\}$,
- $\mathfrak{hs}_3\mathfrak{poi2} \to \mathfrak{hs}_3\mathfrak{car4}$: $\mathfrak{h} = \{\mathtt{J}, \mathtt{H}_a\}$.

By examining the structure constants of Table D.6 and D.7, one can see that $\mathfrak{hs}_3\mathfrak{car1}$ ($\mathfrak{hs}_3\mathfrak{car2}$) and $\mathfrak{hs}_3\mathfrak{car3}$ ($\mathfrak{hs}_3\mathfrak{car4}$) are related via the following interchange of generators

$$\mathtt{H}^a \leftrightarrow \mathtt{J}^a \qquad\qquad \mathtt{P}^{ab} \leftrightarrow \mathtt{G}^{ab} \tag{10.7}$$

plus potentially some sign changes in structure constants, as mentioned in the previous paragraph. The structure of the Chern–Simons theories will therefore be very similar for $\mathfrak{hs}_3\mathfrak{car1}$ ($\mathfrak{hs}_3\mathfrak{car2}$) and $\mathfrak{hs}_3\mathfrak{car3}$ ($\mathfrak{hs}_3\mathfrak{car4}$). In the following, we will restrict to the case of $\mathfrak{hs}_3\mathfrak{car1}$. The CS theory based on $\mathfrak{hs}_3\mathfrak{car2}$ is explicitly treated in [5].

Chern–Simons Theory for $\mathfrak{hs}_3\mathfrak{car1}$

The commutation relations of $\mathfrak{hs}_3\mathfrak{car1}$ are summarized in the first column of Table D.6. This algebra admits the following invariant metric

$$\langle \mathtt{H}\,, \mathtt{J}\rangle = -1 \qquad\qquad \langle \mathtt{P}_a \mathtt{G}_b\rangle = \delta_{ab} \tag{10.8}$$

$$\langle \mathtt{H}_a\,, \mathtt{J}_b\rangle = -\delta_{ab} \qquad\qquad \langle \mathtt{P}_{ab}\,, \mathtt{G}_{cd}\rangle = \delta_{a(c}\delta_{d)b} - \frac{2}{3}\delta_{ab}\delta_{cd}\,. \tag{10.9}$$

Using the commutation relations of $\mathfrak{hs}_3\mathfrak{car1}$ and the invariant metric (10.8), the Chern–Simons action ((2.4)) and its equations of motion can be explicitly written down. Here, we will be interested in studying the action and equations of motion, linearized around a flat background solution[5] given by

$$\bar{A}_\mu = \delta^0_\mu\,\mathtt{H} + \delta^a_\mu\,\mathtt{P}_a\,. \tag{10.10}$$

[5]For fields in this flat background solution, the curved μ index becomes equivalent to a flat one. In the following, we will therefore denote the time-like and spatial values of the μ index by 0 and a. The a index can moreover be freely raised and lowered using a Kronecker delta. We will often raise or lower spatial a indices on field components (even if it leads to equations with non-matching index positions on the left- and right-hand-sides), to make more clear which field components are being meant.

We will therefore assume that the gauge field is given by this background solution $\bar{A}_\mu$, plus fluctuations around this background

$$
\begin{aligned}
A_\mu = \left(\delta^0_\mu + \tau_\mu\right)\mathtt{H} + \left(\delta^a_\mu + e_\mu{}^a\right)\mathtt{P}_a + \omega_\mu\,\mathtt{J} + B_\mu{}^a\,\mathtt{G}_a \\
+ \tau_\mu{}^a\,\mathtt{H}_a + e_\mu{}^{ab}\,\mathtt{P}_{ab} + \omega_\mu{}^a\,\mathtt{J}_a + B_\mu{}^{ab}\,\mathtt{G}_{ab}\,.
\end{aligned} \tag{10.11}
$$

Here, τ_μ can be interpreted as a linearized time-like vielbein, $e_\mu{}^a$ as a linearized spatial vielbein, while ω_μ and $B_\mu{}^a$ can be viewed as linearized spin connections for spatial rotations and boosts respectively. Similarly, $\tau_\mu{}^a$, $e_\mu{}^{ab}$, $\omega_\mu{}^a$ and $B_\mu{}^{ab}$ can be interpreted as spin-3 versions of these linearized vielbeine and spin connections.

Using the expansion (10.11) in the Chern–Simons action and keeping only the terms quadratic in the fluctuations, one finds the following linearized action:

$$
\begin{aligned}
S_{\mathfrak{hs}_3\mathfrak{carl}} = \int d^3x\,\epsilon^{\mu\nu\rho}\Big(& -2\tau_\mu\partial_\nu\omega_\rho + 2e_\mu{}^a\partial_\nu B_\rho{}^a - 2\tau_\mu{}^a\partial_\nu\omega_\rho{}^a + 4e_\mu{}^{ab}\partial_\nu B_\rho{}^{ab} \\
& - \frac{4}{3}e_\mu{}^{aa}\partial_\nu B_\rho{}^{bb} - \delta^0_\mu\omega_\nu{}^a\omega_\rho{}^b\epsilon_{ab} - 2\delta^a_\mu\omega_\nu B_\rho{}^b\epsilon_{ab} - 4\delta^a_\mu\omega_\nu{}^c B_\rho{}^{cb}\epsilon_{ab}\Big)\,.
\end{aligned} \tag{10.12}
$$

The linearized equations of motion corresponding to this action are given by

$$
\begin{aligned}
0 &= R_{\mu\nu}(\mathtt{H}) \equiv \partial_\mu\tau_\nu - \partial_\nu\tau_\mu - \delta^a_\mu B_\nu{}^b\epsilon_{ab} + \delta^a_\nu B_\mu{}^b\epsilon_{ab} \\
0 &= R_{\mu\nu}(\mathtt{P}^a) \equiv \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \epsilon^{ab}\delta^b_\mu\omega_\nu - \epsilon^{ab}\delta^b_\nu\omega_\mu \\
0 &= R_{\mu\nu}(\mathtt{J}) \equiv \partial_\mu\omega_\nu - \partial_\nu\omega_\mu \\
0 &= R_{\mu\nu}(\mathtt{G}^a) \equiv \partial_\mu B_\nu{}^a - \partial_\nu B_\mu{}^a \\
0 &= R_{\mu\nu}(\mathtt{H}^a) \equiv \partial_{[\mu}\tau_{\nu]}{}^a - \epsilon^{ab}\delta^0_\mu\omega_\nu{}^b + \epsilon^{ab}\delta^0_\nu\omega_\mu{}^b - 2\delta^b_\mu B_\nu{}^{ac}\epsilon_{bc} + 2\delta^b_\nu B_\mu{}^{ac}\epsilon_{bc} \\
0 &= R_{\mu\nu}(\mathtt{P}^{ab}) \equiv \partial_{[\mu}e_{\nu]}{}^{ab} + \frac{1}{2}\delta^c_{[\mu}\omega_{\nu]}{}^{(a}\epsilon^{b)c} - \delta^c_\mu\omega_\nu{}^d\epsilon_{cd}\delta^{ab} + \delta^c_\nu\omega_\mu{}^d\epsilon_{cd}\delta^{ab} \\
0 &= R_{\mu\nu}(\mathtt{J}^a) \equiv \partial_\mu\omega_\nu{}^a - \partial_\nu\omega_\mu{}^a \\
0 &= R_{\mu\nu}(\mathtt{G}^{ab}) \equiv \partial_\mu B_\nu{}^{ab} - \partial_\nu B_\mu{}^{ab}
\end{aligned} \tag{10.13}
$$

The equations

$$
R_{\mu\nu}(\mathtt{H}) = 0 \quad R_{\mu\nu}(\mathtt{P}^a) = 0 \quad R_{\mu\nu}(\mathtt{H}^a) = 0 \quad R_{\mu\nu}(\mathtt{P}^{ab}) = 0 \tag{10.14}
$$

contain the spin connections ω_μ, $B_\mu{}^a$, $\omega_\mu{}^a$ and $B_\mu{}^{ab}$ only in an algebraic way. These equations can thus be solved to yield expressions for some of the spin connection components in terms of the vielbeine and their derivatives.

Let us first see how this works for the spin-2 spin connections ω_μ and $B_\mu{}^a$. The equation $R_{0a}(\mathtt{H}) = 0$ can be straightforwardly solved for $B_0{}^a$:

$$B_0{}^a = \epsilon^{ab}(\partial_0\tau_b - \partial_b\tau_0)\,. \tag{10.15}$$

Similarly, the equation $R_{ab}(\mathtt{H}) = 0$ (or equivalently $\epsilon^{ab}R_{ab}(\mathtt{H}) = 0$) can be solved for $B_c{}^c$ (the spatial trace of $B_\mu{}^a$):

$$B_c{}^c = \frac{1}{2}\epsilon^{ab}(\partial_a\tau_b - \partial_b\tau_a)\,. \tag{10.16}$$

From $R_{ab}(\mathtt{P}^c) = 0$ (or equivalently $\epsilon^{ab}R_{ab}(\mathtt{P}^c) = 0$) one finds the spatial part of ω_μ:

$$\omega_a = \frac{1}{2}\epsilon^{bc}(\partial_b e_{ca} - \partial_c e_{ba})\,. \tag{10.17}$$

Finally, let us consider the equation $R_{0a}(\mathtt{P}_b) = 0$. The anti-symmetric part of this equation $\epsilon^{ab}R_{0a}(\mathtt{P}_b) = 0$ can be solved for the time-like part of ω_μ:

$$\omega_0 = \frac{1}{2}\epsilon^{ab}(\partial_a e_{0b} - \partial_0 e_{ab})\,. \tag{10.18}$$

The symmetric part $R_{0(a}(\mathtt{P}_{b)}) = 0$ does not contain any spin connection and can be viewed as a constraint on the geometry

$$\partial_0 e_{(ab)} - \partial_{(a} e_{|0|b)} = 0\,. \tag{10.19}$$

In summary, we find that $R_{\mu\nu}(\mathtt{H}) = 0$ and $R_{\mu\nu}(\mathtt{P}^a) = 0$ lead to the constraint (10.19) as well as the following solutions for ω_μ and $B_\mu{}^a$

$$\begin{aligned}\omega_\mu &= \frac{1}{2}\delta^0_\mu\,\epsilon^{ab}(\partial_a e_{0b} - \partial_0 e_{ab}) + \frac{1}{2}\delta^a_\mu\,\epsilon^{bc}(\partial_b e_{ca} - \partial_c e_{ba})\,,\\ B_\mu{}^a &= \delta^0_\mu\,\epsilon^{ab}(\partial_0\tau_b - \partial_b\tau_0) + \frac{1}{4}\delta^a_\mu\,\epsilon^{bc}(\partial_b\tau_c - \partial_c\tau_b) + \delta^b_\mu\,\tilde{B}_b{}^a\,,\end{aligned} \tag{10.20}$$

where $\tilde{B}_b{}^a$ is an undetermined traceless tensor. The boost connection $B_\mu{}^a$ is thus not fully determined in terms of τ_μ and $e_\mu{}^a$.

A similar reasoning allows one to solve for certain components of the spin-3 connections $\omega_\mu{}^a$ and $B_\mu{}^{ab}$. In particular, the equation $R_{ab}(\mathtt{H}^c) = 0$ can be solved for $B_d{}^{da}$, a spatial trace of $B_\mu{}^{ab}$:

$$B_d{}^{da} = \frac{1}{4}\epsilon^{bc}(\partial_b\tau_c{}^a - \partial_c\tau_b{}^a)\,. \tag{10.21}$$

The equation $R_{ab}(\mathtt{P}^{cd}) = 0$ can be solved for the symmetric, spatial part of $\omega_\mu{}^a$:

$$\omega^{(ab)} = \epsilon^{cd}\Big(\partial_c e_d{}^{ab} - \partial_d e_c{}^{ab}\Big) - \frac{1}{3}\delta^{ab}\epsilon^{cd}(\partial_c e_d{}^{ee} - \partial_d e_c{}^{ee})\,. \tag{10.22}$$

The anti-symmetric, spatial part of $\omega_\mu{}^a$ can be found from $R_{0a}(\mathtt{H}^a) = 0$:

$$\epsilon^{ab}\omega_{ab} = \partial_0\tau_a{}^a - \partial_a\tau_0{}^a \,. \tag{10.23}$$

From the other equations contained in $R_{0b}(\mathtt{H}^a) = 0$ one then finds

$$B_0{}^{ab} = \frac{1}{4}\epsilon^{(a|c|}\Big(\partial_0\tau_c{}^{b)} - \partial_c\tau_0{}^{b)}\Big) + \frac{1}{4}\epsilon^{cd}\partial_{[c}e_{d]}{}^{ab} - \frac{1}{6}\delta^{ab}\epsilon^{cd}(\partial_c e_d{}^{ee} - \partial_d e_c{}^{ee}) \,. \tag{10.24}$$

The equation $R_{0a}(\mathtt{P}_{bc}) = 0$ can be divided into a part that is fully symmetric in the indices a, b, c and a part that is of mixed symmetry:

$$R_{0a}(\mathtt{P}_{bc}) = 0 \qquad \Leftrightarrow \qquad R^{\mathrm{S}}_{0a}(\mathtt{P}_{bc}) = 0 \quad \text{and} \quad R^{\mathrm{MS}}_{0a}(\mathtt{P}_{bc}) = 0 \,, \tag{10.25}$$

where

$$\begin{aligned} R^{\mathrm{S}}_{0a}(\mathtt{P}_{bc}) &= \frac{1}{3}(R_{0a}(\mathtt{P}_{bc}) + R_{0c}(\mathtt{P}_{ab}) + R_{0b}(\mathtt{P}_{ca})) \,, \\ R^{\mathrm{MS}}_{0a}(\mathtt{P}_{bc}) &= \frac{1}{3}(2R_{0a}(\mathtt{P}_{bc}) - R_{0c}(\mathtt{P}_{ab}) - R_{0b}(\mathtt{P}_{ca})) \,. \end{aligned} \tag{10.26}$$

The equation $R^{\mathrm{MS}}_{0a}(\mathtt{P}_{bc}) = 0$ can be solved for $\omega_0{}^a$, by noting that

$$R^{\mathrm{MS}}_{0a}(\mathtt{P}_{bc}) = 0 \qquad \Leftrightarrow \qquad \epsilon^{ab}R_{0a}(\mathtt{P}_{bc}) = 0 \,. \tag{10.27}$$

The solution one finds is given by

$$\omega_0{}^a = \frac{2}{5}\epsilon^{bc}(\partial_b e_0{}^{ca} - \partial_0 e_b{}^{ca}) \,. \tag{10.28}$$

The fully symmetric part $R^{\mathrm{S}}_{0a}(\mathtt{P}_{bc}) = 0$ can not be used to solve for other spin connection components. Rather, it should be viewed as a constraint on the geometry:

$$\begin{aligned} &\partial_0 e_b{}^{ac} - \partial_b e_0{}^{ac} + \partial_0 e_a{}^{bc} - \partial_a e_0{}^{bc} + \partial_0 e_c{}^{ab} - \partial_c e_0{}^{ab} \\ &\quad + \frac{2}{5}\delta_{ac}\Big(\partial_b e_0{}^{dd} - \partial_d e_0{}^{bd} + \partial_0 e_d{}^{bd} - \partial_0 e_b{}^{dd}\Big) \\ &\quad + \frac{2}{5}\delta_{bc}\Big(\partial_a e_0{}^{dd} - \partial_d e_0{}^{ad} + \partial_0 e_d{}^{ad} - \partial_0 e_a{}^{dd}\Big) \\ &\quad + \frac{2}{5}\delta_{ab}\Big(\partial_c e_0{}^{dd} - \partial_d e_0{}^{cd} + \partial_0 e_d{}^{cd} - \partial_0 e_c{}^{dd}\Big) = 0 \,. \end{aligned} \tag{10.29}$$

This constraint can be slightly simplified. By contracting it with δ^{bc}, one finds that

$$\partial_a e_0{}^{bb} - \partial_0 e_a{}^{bb} = 6\Big(\partial_b e_0{}^{ab} - \partial_0 e_b{}^{ab}\Big) \,. \tag{10.30}$$

Using this, one finds that (10.29) simplifies to

$$\partial_0 e_b{}^{ac} - \partial_b e_0{}^{ac} + \partial_0 e_a{}^{bc} - \partial_a e_0{}^{bc} + \partial_0 e_c{}^{ab} - \partial_c e_0{}^{ab} + \frac{1}{3}\delta_{bc}\Big(\partial_a e_0{}^{dd} - \partial_0 e_a{}^{dd}\Big)$$
$$+ \frac{1}{3}\delta_{ac}\Big(\partial_b e_0{}^{dd} - \partial_0 e_b{}^{dd}\Big) + \frac{1}{3}\delta_{ab}\Big(\partial_c e_0{}^{dd} - \partial_0 e_c{}^{dd}\Big) = 0\,. \tag{10.31}$$

One thus finds for the spin-3 sector, that the equations $R_{\mu\nu}(\mathtt{H}^a) = 0$ and $R_{\mu\nu}(\mathtt{P}^{ab}) = 0$ lead to the constraint (10.31) and the following solutions for $\omega_\mu{}^a$ and $B_\mu{}^{ab}$:

$$\begin{aligned}
\omega_\mu{}^a &= \frac{2}{5}\delta^0_\mu \epsilon^{bc}(\partial_b e_0{}^{ca} - \partial_0 e_b{}^{ca}) + \frac{1}{2}\delta^b_\mu\Big(\epsilon^{cd}\Big(\partial_c e_d{}^{ba} - \partial_d e_c{}^{ba}\Big) \\
&\qquad - \frac{1}{3}\delta^a_b \epsilon^{cd}(\partial_c e_d{}^{ee} - \partial_d e_c{}^{ee}) + \epsilon_{ba}(\partial_0 \tau_c{}^c - \partial_c \tau_0{}^c)\Big)\,, \\
B_\mu{}^{ab} &= \frac{1}{4}\delta^0_\mu\Big(\epsilon^{(a|c|}\Big(\partial_0 \tau_c{}^{b)} - \partial_c \tau_0{}^{b)}\Big) + \epsilon^{cd}\partial_{[c} e_{d]}{}^{ab} - \frac{2}{3}\delta^{ab}\epsilon^{cd}(\partial_c e_d{}^{ee} - \partial_d e_c{}^{ee})\Big) \\
&\qquad + \frac{1}{12}\delta^{(a}_\mu \epsilon^{|de|}\Big(\partial_d \tau_e{}^{b)} - \partial_e \tau_d{}^{b)}\Big) + \delta^c_\mu \tilde{B}_c{}^{ab}\,,
\end{aligned} \tag{10.32}$$

where $\tilde{B}_c{}^{ab}$ is an arbitrary tensor obeying $\tilde{B}_b{}^{ba} = 0$. As for the spin-2 sector, one thus finds that the spin-3 boost connection $B_\mu{}^{ab}$ can not be fully determined in terms of $\tau_\mu{}^a$ and $e_\mu{}^{ab}$.

It is interesting to see what role the undetermined components $\tilde{B}_b{}^a$ and $\tilde{B}_c{}^{ab}$ play. In particular, one can check how these components appear in the Lagrangian and what their equations of motion are. Upon partial integration in the action (10.12), one finds that the terms in the Lagrangian involving $B_\mu{}^a$ can be written as

$$\epsilon^{\mu\nu\rho} R_{\mu\nu}(\mathtt{P}_a) B_\rho{}^a\,. \tag{10.33}$$

The traceless spatial components $\tilde{B}_b{}^a$ of $B_\rho{}^a$ thus couple to

$$\epsilon^{cb} R_{0c}(\mathtt{P}_a) - \frac{1}{2}\delta^b_a \epsilon^{cd} R_{0c}(\mathtt{P}_d)\,. \tag{10.34}$$

This can however be rewritten as

$$-\frac{1}{2}\epsilon^{cb} R_{0(a}(\mathtt{P}_{b)})\,. \tag{10.35}$$

One thus sees that $\tilde{B}_b{}^a$ acts as a Lagrange multiplier for $R_{0(a}(\mathtt{P}_{b)}) = 0$, which led to the constraint (10.19). Similarly, one can check that $\tilde{B}_c{}^{ab}$ plays the role of Lagrange multiplier for the constraint (10.31).

Spin-3 Galilei and Extended Bargmann Gravity

In the previous section, we have studied Carroll spin-3 algebras, whose spin-2 part corresponds to the Carroll algebra. Using the contractions of Table 10.1, one can also obtain nonrelativistic spin-3 algebras, that contain the Galilei algebra. As in the Carroll case, there are four distinct ways of doing this, namely by successively applying the contractions 1 and 3, 1 and 4, 2 and 3 or 2 and 4 of Table 10.1. We have called the resulting algebras $\mathfrak{hs}_3\mathfrak{gal}1$, $\mathfrak{hs}_3\mathfrak{gal}2$, $\mathfrak{hs}_3\mathfrak{gal}3$ and $\mathfrak{hs}_3\mathfrak{gal}4$ respectively and summarized their commutation relations in Table D.8 and D.9. As in the Carroll case, $\mathfrak{hs}_3\mathfrak{gal}3$ and $\mathfrak{hs}_3\mathfrak{gal}4$ each come in two different versions, depending on whether one applies the combination of contraction on $\mathfrak{hs}_3\mathfrak{AdS}$ or $\mathfrak{hs}_3\mathfrak{dS}$. They are again structurally similar to $\mathfrak{hs}_3\mathfrak{gal}1$ and $\mathfrak{hs}_3\mathfrak{gal}2$. We will therefore restrict our discussion here to these two cases.

In contrast to the spin-3 Carroll algebras, whose invariant metrics arose from applying the relevant contraction on (10.4), a similar reasoning for the spin-3 Galilei algebras leads to degenerate bilinear forms. One can in fact show by direct computation that they can not be equipped with a nondegenerate symmetric invariant bilinear form. This is even true when one allows nontrivial central extensions. One algebra admits no nontrivial central extensions (the second cohomology group is trivial), whereas the other does admit three nontrivial extensions of which no combination of them can be used to define an invariant metric. In this sense the spin-3 version differs from the spin-2 one, see Section 9. It could be interesting to investigate these algebras, given explicitly in Table D.8, and their degenerate bilinear forms. For the spin-2 case, this has been done in [179]. Due to the degeneracy of the bilinear form, some of the fields appear without kinetic term in the action (see the discussion in Section 2.2 concerning non-degeneracy) and are therefore not dynamical. In the spin-2 case, one can nevertheless interpret these non-dynamical fields as Lagrange multipliers for geometrical constraints, similarly to what happens in the Carroll cases of the previous section. Although it would be interesting to see whether similar results hold for the higher spin case, we will not do this here and instead we will look at Chern–Simons theories where each field has a kinetic term. These can not be based on the spin-3 Galilei algebras, but interestingly, double extensions help to find Lie algebras that admit an invariant metric, i.e. a nondegenerate invariant symmetric bilinear form. Remarkably, in this way one ends up with a spin-3 version of the Extended Bargmann algebra, it the sense that the spin-2 subalgebra is the $\mathfrak{ebarg}$ discussed in Section 10.

Double extensions applied to the ordinary Galilei algebra in three dimensions and yields the so-called Extended Bargmann algebra [45, 164, 167, 178], that extends the Galilei algebra with two central extensions. Applying the theorem to $\mathfrak{hs}_3\mathfrak{gal1}$ and $\mathfrak{hs}_3\mathfrak{gal2}$ yields two spin-3 algebras, that we will denote, in hindsight, by $\mathfrak{hs}_3\mathfrak{ebarg1}$ and $\mathfrak{hs}_3\mathfrak{ebarg2}$ (since they have an Extended Bargmann spin-2 subalgebra).

The algebra $\mathfrak{hs}_3\mathfrak{ebarg1}$ can be obtained by looking for double extension for $\mathfrak{hs}_3\mathfrak{gal1}$. Indeed, with the choices $\mathfrak{g} = \{\mathtt{P}_a, \mathtt{G}_a, \mathtt{P}_{ab}, \mathtt{G}_{ab}\}$, $\mathfrak{h} = \{\mathtt{H}, \mathtt{J}, \mathtt{H}_a, \mathtt{J}_a\}$ and

$$\langle \mathtt{P}_a \,, \mathtt{G}_b \rangle_{\mathfrak{g}} = \delta_{ab} \,, \qquad \langle \mathtt{P}_{ab} \,, \mathtt{G}_{cd} \rangle_{\mathfrak{g}} = \delta_{a(c}\delta_{d)b} - \frac{2}{3}\delta_{ab}\delta_{cd} \,, \tag{10.36}$$

the assumptions of a double extension theorem are fulfilled and the algebra $\mathfrak{hs}_3\mathfrak{ebarg1}$ can be constructed. Denoting the generators of $\mathfrak{h}^*$ by $\{\mathtt{H}^*, \mathtt{J}^*, \mathtt{H}^*_a, \mathtt{J}^*_a\}$, the commutation relations of $\mathfrak{hs}_3\mathfrak{ebarg1}$ are given in Table 10.3. The invariant metric of $\mathfrak{hs}_3\mathfrak{ebarg1}$ is explicitly given by

$$\begin{aligned} \langle \mathtt{P}_a \,, \mathtt{G}_b \rangle &= \delta_{ab} \,, & \langle \mathtt{P}_{ab} \,, \mathtt{G}_{cd} \rangle &= \delta_{a(c}\delta_{d)b} - \frac{2}{3}\delta_{ab}\delta_{cd} \,, \\ \langle \mathtt{H} \,, \mathtt{H}^* \rangle &= 1 \,, & \langle \mathtt{J} \,, \mathtt{J}^* \rangle &= 1 \,, \\ \langle \mathtt{H}_a \,, \mathtt{H}^*_b \rangle &= \delta_{ab} \,, & \langle \mathtt{J}_a \,, \mathtt{J}^*_b \rangle &= \delta_{ab} \,. \end{aligned} \tag{10.37}$$

Similarly, starting from $\mathfrak{hs}_3\mathfrak{gal2}$ and double extending $\mathfrak{g} = \{\mathtt{P}_a, \mathtt{G}_a, \mathtt{H}_a, \mathtt{J}_a\}$ by $\mathfrak{h} = \{\mathtt{H}, \mathtt{J}, \mathtt{P}_{ab}, \mathtt{G}_{ab}\}$ and

$$\langle \mathtt{P}_a \,, \mathtt{G}_b \rangle_{\mathfrak{g}} = \delta_{ab} \,, \qquad \langle \mathtt{H}_a \,, \mathtt{J}_b \rangle_{\mathfrak{g}} = -\delta_{ab} \,, \tag{10.38}$$

the algebra $\mathfrak{hs}_3\mathfrak{ebarg2}$ can be constructed. Denoting the generators of $\mathfrak{h}^*$ by $\{\mathtt{H}^*, \mathtt{J}^*, \mathtt{P}^*_{ab}, \mathtt{G}^*_{ab}\}$, its commutation relations are given in Table 10.3.

This algebra admits the following invariant metric

$$\begin{aligned} \langle \mathtt{P}_a, \mathtt{G}_b \rangle &= \delta_{ab} \,, & \langle \mathtt{H}_a, \mathtt{J}_b \rangle &= -\delta_{ab} \,, \\ \langle \mathtt{H}, \mathtt{H}^* \rangle &= 1 \,, & \langle \mathtt{J}, \mathtt{J}^* \rangle &= 1 \,, \\ \langle \mathtt{P}_{ab}, \mathtt{P}^*_{cd} \rangle &= \delta_{a(c}\delta_{d)b} \,, & \langle \mathtt{G}_{ab}, \mathtt{G}^*_{cd} \rangle &= \delta_{a(c}\delta_{d)b} \,. \end{aligned} \tag{10.39}$$

Note that for both $\mathfrak{hs}_3\mathfrak{ebarg1}$ and $\mathfrak{hs}_3\mathfrak{ebarg2}$ the generators $\{\mathtt{H}, \mathtt{J}, \mathtt{P}_a, \mathtt{G}_a, \mathtt{H}^*, \mathtt{J}^*\}$ form a subalgebra that coincides with the Extended Bargmann algebra. The Chern–Simons theories based on these algebras can therefore be viewed as spin-3 extensions of Extended Bargmann gravity, studied in [45,164,167,178]. In [5] these spin-3 Extended Bargmann gravity theories were studied in detail at the linearized level.

	$\mathfrak{hs}_3\mathfrak{ebarg1}$		$\mathfrak{hs}_3\mathfrak{ebarg2}$
$[\mathtt{J}, \mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$[\mathtt{J}, \mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J}, \mathtt{P}_a]$	$\epsilon_{am}\mathtt{P}_m$	$[\mathtt{J}, \mathtt{P}_a]$	$\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a, \mathtt{H}]$	$-\epsilon_{am}\mathtt{P}_m$	$[\mathtt{G}_a, \mathtt{H}]$	$-\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a, \mathtt{G}_b]$	$\epsilon_{ab}\mathtt{H}^*$	$[\mathtt{G}_a, \mathtt{G}_b]$	$\epsilon_{ab}\mathtt{H}^*$
$[\mathtt{P}_a, \mathtt{G}_b]$	$\epsilon_{ab}\mathtt{J}^*$	$[\mathtt{G}_a, \mathtt{P}_b]$	$\epsilon_{ab}\mathtt{J}^*$
$[\mathtt{J}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{J}_m$	$[\mathtt{J}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{J}_m$
$[\mathtt{J}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$	$[\mathtt{J}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{J}, \mathtt{H}_a]$	$\epsilon_{am}\mathtt{H}_m$	$[\mathtt{J}, \mathtt{H}_a]$	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{J}, \mathtt{P}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$[\mathtt{J}, \mathtt{P}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{G}_a, \mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$	$[\mathtt{G}_a, \mathtt{G}_{bc}]$	$-\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{G}_a, \mathtt{H}_b]$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$	$[\mathtt{G}_a, \mathtt{P}_{bc}]$	$-\epsilon_{a(b}\mathtt{H}_{c)}$
$[\mathtt{H}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{H}_m$	$[\mathtt{H}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{H}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$[\mathtt{H}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{P}_a, \mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$	$[\mathtt{P}_a, \mathtt{G}_{bc}]$	$-\epsilon_{a(b}\mathtt{H}_{c)}$
$[\mathtt{J}_a, \mathtt{J}_b]$	$\epsilon_{ab}\mathtt{J}$	$[\mathtt{J}_a, \mathtt{G}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\mathtt{J}_a, \mathtt{G}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	$[\mathtt{J}_a, \mathtt{P}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$
$[\mathtt{J}_a, \mathtt{H}_b]$	$\epsilon_{ab}\mathtt{H}$	$[\mathtt{G}_{ab}, \mathtt{G}_{cd}]$	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$
$[\mathtt{J}_a, \mathtt{P}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$	$[\mathtt{G}_{ab}, \mathtt{H}_c]$	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}, \mathtt{H}_c]$	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$	$[\mathtt{G}_{ab}, \mathtt{P}_{cd}]$	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$
$[\mathtt{G}_{ab}, \mathtt{G}_{cd}]$	$\epsilon_{(a(c}\delta_{d)b)}\mathtt{H}^*$	$[\mathtt{G}_a, \mathtt{J}_b]$	$-\epsilon_{am}\mathtt{P}^*_{mb}$
$[\mathtt{P}_{ab}, \mathtt{G}_{cd}]$	$\epsilon_{(a(c}\delta_{d)b)}\mathtt{J}^*$	$[\mathtt{G}_a, \mathtt{H}_b]$	$-\epsilon_{am}\mathtt{G}^*_{mb}$
$[\mathtt{P}_a, \mathtt{G}_{bc}]$	$\epsilon_{a(b}\mathtt{J}^*_{c)}$	$[\mathtt{P}_a, \mathtt{J}_b]$	$-\epsilon_{am}\mathtt{G}^*_{mb}$
$[\mathtt{G}_a, \mathtt{G}_{bc}]$	$\epsilon_{a(b}\mathtt{H}^*_{c)}$	$[\mathtt{J}_a, \mathtt{J}_b]$	$-\epsilon_{ab}\mathtt{H}^*$
$[\mathtt{G}_a, \mathtt{P}_{bc}]$	$\epsilon_{a(b}\mathtt{J}^*_{c)}$	$[\mathtt{J}_a, \mathtt{H}_b]$	$-\epsilon_{ab}\mathtt{J}^*$
$[\mathtt{J}, \mathtt{H}^*_a]$	$\epsilon_{am}\mathtt{H}^*_m$	$[\mathtt{J}, \mathtt{P}^*_{ab}]$	$-\epsilon_{m(a}\mathtt{P}^*_{b)m}$
$[\mathtt{J}, \mathtt{J}^*_a]$	$\epsilon_{am}\mathtt{J}^*_m$	$[\mathtt{J}, \mathtt{G}^*_{ab}]$	$-\epsilon_{m(a}\mathtt{G}^*_{b)m}$
$[\mathtt{H}, \mathtt{H}^*_a]$	$\epsilon_{am}\mathtt{J}^*_m$	$[\mathtt{H}, \mathtt{P}^*_{ab}]$	$-\epsilon_{m(a}\mathtt{G}^*_{b)m}$
$[\mathtt{J}_a, \mathtt{J}^*]$	$-\epsilon_{am}\mathtt{J}^*_m$	$[\mathtt{G}_{ab}, \mathtt{J}^*]$	$-\epsilon_{m(a}\mathtt{G}^*_{b)m}$
$[\mathtt{J}_a, \mathtt{H}^*]$	$-\epsilon_{am}\mathtt{H}^*_m$	$[\mathtt{G}_{ab}, \mathtt{H}^*]$	$-\epsilon_{m(a}\mathtt{P}^*_{b)m}$
$[\mathtt{J}_a, \mathtt{J}^*_b]$	$\epsilon_{ab}\mathtt{J}^*$	$[\mathtt{G}_{ab}, \mathtt{G}^*_{cd}]$	$\epsilon_{(a(c}\delta_{d)b)}\mathtt{J}^*$
$[\mathtt{J}_a, \mathtt{H}^*_b]$	$\epsilon_{ab}\mathtt{H}^*$	$[\mathtt{G}_{ab}, \mathtt{P}^*_{cd}]$	$\epsilon_{(a(c}\delta_{d)b)}\mathtt{H}^*$
$[\mathtt{H}_a, \mathtt{H}^*]$	$-\epsilon_{am}\mathtt{J}^*_m$	$[\mathtt{P}_{ab}, \mathtt{H}^*]$	$-\epsilon_{m(a}\mathtt{G}^*_{b)m}$
$[\mathtt{H}_a, \mathtt{H}^*_b]$	$\epsilon_{ab}\mathtt{J}^*$	$[\mathtt{P}_{ab}, \mathtt{P}^*_{cd}]$	$\epsilon_{(a(c}\delta_{d)b)}\mathtt{J}^*$

Table 10.3: Nonzero commutators of $\mathfrak{hs}_3\mathfrak{ebarg1}$ and $\mathfrak{hs}_3\mathfrak{ebarg2}$. This algebras admit an invariant metric, given by equation (10.37) and (10.39), respectively.

Chapter 11

Conclusions

We will summarize the accomplished results and highlight areas that permit further investigations.

Algebraic Tools for CS Theories

In Chapter 2 we established that the natural set-up for CS theories is based on gauge algebras admitting an invariant metric. Besides the well known direct sum of abelian and simple Lie algebras which lead to reductive Lie algebras another construction needs to be added. With the addition of double extensions, see Definition 3.4, one fully exhausts the possible symmetric self-dual Lie algebras. This is due to the remarkable Theorem 3.5 of Medina and Revoy which states how every such indecomposable Lie algebra has to look like.

With this knowledge we reviewed Lie algebra contractions whose physical interpretation is that of an approximation. Therefore not only a lot can be learned from the original algebra, but they can also be used to classify possible physical systems in various limits.

The combination of invariant metrics with contractions, see Chapter 5, paired with the knowledge of double extensions is the ideal set-up for investigations of approximate CS theories. A new type of invariant metric preserving contraction, see Theorem 5.1, tailor made for double extensions, explains why (higher spin) Carroll algebras in $2+1$ dimensions stay equipped with an invariant metric in the limit from Poincaré.

The generalization to Lie superalgebras seems like a fruitful endeavor. Double extensions generalize [53] but the analog of the Medina and Revoy theorem is unproven.

Even for the Lie algebras a more systematic study of contractions of the various types of symmetric self-dual Lie algebras seems of interest. Especially since the importance of invariant metrics are not restricted to CS gauge theories.

From an algebraic point of view it might be of interest which (simple) Lie algebra contract to Lie algebras that are double extensions. This could for example explain from which (simple) Lie algebra one could arrive at the spin-3 Extended Bargmann algebras. Notice that this is different to the spin-2 case since we needed more than just central extensions. Another point for why this might be of importance is that the "inverse" of a contraction might lead, analog to the deformation from Galilei to Poincaré algebras, to more fundamental theories.

Boundary Conditions

In Chapter 6 the concepts of global charge and boundary conditions was reviewed, and afterwards applied to AdS (Chapter 7), in the form of the $\mathfrak{u}(1)$ boundary conditions, as well as to Lifshitz and null warped AdS (Chapter 8).

It might be interesting to re-investigate possibilities to consistently break the boundary conditions or the $\mathcal{W}$ algebras. Maybe contractions are useful for this task. On the more speculative side one might try to restrict boundary conditions mode wise.

In Section 10.2 consistent boundary conditions for Carroll Gravity were found but the generalization to the spin-3 versions has not been done. Actually, even for the extended kinematical spin-2 algebras consistent boundary conditions have not been established. See also the algebras proposed in [45]. Extended Newton–Hooke might provide an interesting intermediate step since it might have more "box-like" properties due to non-vanishing cosmological constant. This means it might be closer to AdS than, e.g., Poincaré, and for AdS/CFT generalizations better suited.

Kinematical Chern–Simons Theories

In Chapter 10 we have extended the work of Bacry and Lévy-Leblond [8] by classifying all possible kinematical algebras of three-dimensional theories of a spin-3 field coupled to gravity, that can be obtained via (sequential) simple Inönü-Wigner contractions of the algebras of spin-3 gravity in (A)dS. This classification can be found in Section 10.1 and the resulting possible kinematical algebras, along with their origin via contraction, are summarized in Figure 10.1. We have summarized the commutation relations of the algebras in Tables D.3-D.15. The algebras of Tables D.6 and D.8 are suitable generalizations of the Carroll and Galilei algebras, that correspond to the ultra-relativistic and nonrelativistic limits of the Poincaré algebra. We have argued that one can easily construct a Chern-Simons action for the spin-3 Carroll algebras (here invariant metric preserving contractions were useful), that leads to a spin-3 generalization of Carroll gravity. We have moreover shown that Chern-Simons actions can be written down for suitable extensions of the spin-3 Galilei algebras, that lead to spin-3 generalizations of Extended Bargmann gravity.

The constructed kinematical algebras are finite-dimensional. We have shown in Section 9.3 that the three-dimensional Carroll algebra admits an infinite-dimensional extension, that is the asymptotic symmetry algebra of Carroll gravity with suitable boundary conditions. This can be taken as a hint that similar results hold for the higher spin non- and ultra-relativistic algebras as well as for the spin-2 algebras whose infinite-dimensional extensions have not been addressed in the literature yet.

There are several questions that are worthwhile for future study. The non- and ultra-relativistic spin-3 gravity theories constructed here, are given in the Chern-Simons (i.e. first order 'zuvielbein') formulation. It is interesting to see whether a metric-like [182] formulation can be constructed and whether the linearized field equations can be rewritten as Fronsdal-like equations. The results for the linearized spin connections given in Section 10.2 and more exhaustively in [5] should be useful in this regard.

We have restricted our investigations to spin-3 theories. This analysis can be extended to theories with fields up to spin N, by considering sIW-contractions of $\mathfrak{sl}(N,\mathbb{R}) \oplus \mathfrak{sl}(N,\mathbb{R})$ or $\mathfrak{sl}(N,\mathbb{C})$ [29]. One can then study the non- and ultra-relativistic gravity theories that arise in this way and in particular investigate the types of boundary conditions that lead to interesting asymptotic symmetry algebras. It would be particularly interesting to see whether it is possible to construct non- and ultra-relativistic versions of non-linear $\mathcal{W}$-algebras.

Another research direction concerns the inclusion of fermionic fields with spins higher than or equal to 3/2. This will require a classification of contractions of Lie superalgebras and can lead to higher spin generalizations of three-dimensional Extended Bargmann supergravity [167].

Some of the results presented in this thesis are also useful for studies of Hořava–Lifshitz gravity, that has been proposed as a new framework for Lifshitz holography [45, 118, 134, 183–187]. Extended Bargmann gravity has been argued to correspond to a special case of Hořava–Lifshitz gravity [45]. In this paper, we have constructed spin-3 generalizations of Extended Bargmann gravity. It is conceivable that these can be interpreted as suitable spin-3 generalizations of Hořava–Lifshitz gravity. It would be interesting to check whether this is indeed the case and whether the construction presented here can be generalized to yield spin-3 generalizations of generic Hořava–Lifshitz gravity theories.

Finally, higher spin theory has recently been argued to describe some of the excitations in fractional quantum Hall liquids [188]. Newton–Cartan geometry and gravity, that are based on extensions of the Galilei algebra, have been very useful in constructing effective actions that can capture transport properties in studies of the fractional quantum Hall effect. It would be interesting to investigate whether the nonrelativistic higher spin gravity theories that can be constructed using the results of this paper, can play a similar role.

Appendix A

Conventions

A.1 Symmetrization and Indices

We adopt the convention that the symmetrization of a pair of indices a, b are denoted with parentheses (ab), while anti-symmetrization is denoted with square brackets $[ab]$. Symmetrization and anti-symmetrization is performed without normalization factor, i.e.,

$$T_{(ab)} = T_{ab} + T_{ba} \qquad T_{[ab]} = T_{ab} - T_{ba} \,. \tag{A.1}$$

Nested (anti-)symmetrizations are understood to be taken from the outermost ones to the innermost ones, e.g.

$$T_{(a(bc)d)} = T_{a(bc)d} + T_{d(bc)a} = T_{abcd} + T_{acbd} + T_{dbca} + T_{dcba} \,. \tag{A.2}$$

Vertical bars denote that the (anti-)symmetrization does not affect the enclosed indices, e.g.,

$$T_{[a|bc|d]} = T_{abcd} - T_{dbca} \,. \tag{A.3}$$

With our conventions this means that $T_{(a|(bc)|d)} = T_{(a(bc)d)}$.

Upper case Latin indices denote spacetime indices, while lower case ones denote spatial indices:

$$A, B, C, M, \ldots = 0, 1, 2 \,, \qquad a, b, c, m, \ldots = 1, 2 \,. \tag{A.4}$$

We take the following conventions for the metric

$$\eta_{AB} = \mathrm{diag}(-, +, +) \qquad \eta_{ab} = \delta_{ab} = \mathrm{diag}(+, +) \,. \tag{A.5}$$

For the Levi-Civita symbol, we adopt the following convention:

$$\epsilon_{012} = \epsilon_{12} = 1 \,, \qquad \epsilon_{0ab} = \epsilon_{ab} \,, \qquad \epsilon^{ab} = \epsilon_{ab} \,. \tag{A.6}$$

Any convention concerning Lie algebras and vector spaces is given in Appendix B. Definitions of various symbols can also be found using the Index at the end of the document.

A.2 Differential Forms

These useful identities for the a-form α and the b-form β with the normalization

$$\alpha = \frac{1}{a!}\,\alpha_{\mu_1\cdots\mu_a}\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_a} \tag{A.7}$$
$$dx^\mu\wedge dx^\nu = dx^\mu\otimes dx^\nu - dx^\nu\otimes dx^\mu \tag{A.8}$$

are taken from [189], [190] and [191]. We denote the exterior product by $\alpha\wedge\beta$, the Lie derivative by $\mathscr{L}_X\alpha$ and the contraction of the vector field X with α by $i_X\alpha$.

$$\alpha\wedge\beta = (-1)^{ab}\beta\wedge\alpha \tag{A.9}$$

$$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^a\alpha\wedge d\beta \tag{A.10}$$
$$d^2 = 0 \tag{A.11}$$
$$d\,\mathscr{L}_X = \mathscr{L}_X\,d \tag{A.12}$$

$$i_X(\alpha\wedge\beta) = i_X\alpha\wedge\beta + (-1)^a\alpha\wedge i_X\beta \tag{A.13}$$
$$i_X^2 = 0 \tag{A.14}$$
$$i_X\,\mathscr{L}_X = \mathscr{L}_X\,i_X \tag{A.15}$$

$$\mathscr{L}_X(\alpha\wedge\beta) = \mathscr{L}_X\alpha\wedge\beta + \alpha\wedge\mathscr{L}_X\beta \tag{A.16}$$
$$\mathscr{L}_X = d\circ i_X + i_X\circ d \tag{A.17}$$
$$[\mathscr{L}_X, i_Y]\alpha = i_{[X,Y]}\alpha \tag{A.18}$$
$$[\mathscr{L}_X, \mathscr{L}_Y]\alpha = \mathscr{L}_{[X,Y]}\alpha \tag{A.19}$$

A.3 $2+1$ Decomposition

$$A = A_N + \widetilde{A} = A_t\,dt + A_i\,dx^i \tag{A.20}$$
$$d = d_N + \widetilde{d} \tag{A.21}$$
$$\widetilde{F} = \widetilde{d}\widetilde{A} + \widetilde{A}\wedge\widetilde{A} = \frac{1}{2}F_{ij}\,dx^i\wedge dx^j \tag{A.22}$$

$$A_N^2 = \widetilde{A}^3 = 0 \tag{A.23}$$

$$d(A_N + \widetilde{A}) = d_N\widetilde{A} + \widetilde{d}A_N + \widetilde{d}\widetilde{A} \tag{A.24}$$

$$\widetilde{d}(A_N \wedge \widetilde{A}) = \widetilde{d}A_N \wedge \widetilde{A} - A_N \wedge \widetilde{d}\widetilde{A} \tag{A.25}$$

Appendix B

Lie Algebras

This appendix provides further introductory material for Lie algebras and fixes the notation that is used in the main sections. Since it is standard material this section is neither complete nor are all necessary details provided. The following references where used and provide further information concerning Lie algebra concepts [192–194], cohomology [195], abelian [195] and nonabelian extensions [196].

Any Lie algebra, if not mentioned otherwise is assumed to be real and finite-dimensional. Furthermore, if Lie algebra brackets or invariant metric components are not explicitly mentioned they are vanishing.

B.1 Basic Concepts of Lie Algebras

Definition B.1. A real or complex **Lie algebra** is a real or complex vector space with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ with the following properties:

1. $[\cdot,\cdot]$ is bilinear.

2. $[\cdot,\cdot]$ is skew-symmetric: $[X,Y] = -[Y,X]$ for all $X, Y \in \mathfrak{g}$.

3. The Jacobi identity holds

$$\underset{XYZ}{\circlearrowleft} [X,[Y,Z]] \equiv [X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0 \qquad \text{(B.1)}$$

for all $X, Y, Z \in \mathfrak{g}$.

If we choose a basis $\mathtt{T}_a \in \mathfrak{g}$, where $a = 1, \ldots, \dim \mathfrak{g}$, and use bilinearity the Lie algebra can be written as

$$[\mathtt{T}_a, \mathtt{T}_b] = f_{ab}{}^c \mathtt{T}_c \qquad \text{(B.2)}$$

where ${f_{ab}}^c$ are the structure constants of the Lie algebra $\mathfrak{g}$. They fully specify a Lie algebra. Skew-symmetry and the Jacobi identity yield

$$ {f_{ab}}^c = -{f_{ba}}^c \tag{B.3} $$

$$ \underset{abc}{\circlearrowleft} {f_{ab}}^d {f_{cd}}^e = 0\,. \tag{B.4} $$

A **homomorphism** is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ with

$$ \phi([X,Y]_{\mathfrak{g}}) = [\phi(X), \phi(Y)]_{\mathfrak{h}} \quad \text{for all} \quad X, Y \in \mathfrak{g}\,. \tag{B.5} $$

An **isomorphism** between the two Lie algebras is an injective and surjective homomorphism. For explicit calculations we fix for the Lie algebra $\mathfrak{g}$ the basis and the structure constants by $[\mathtt{G}_a, \mathtt{G}_b] = {g_{ab}}^c \mathtt{G}_c$ and for $\mathfrak{h}$ by $[\mathtt{H}_i, \mathtt{H}_j] = {h_{ij}}^k \mathtt{H}_k$. The linear map $\phi(\mathtt{G}_a) = {T_a}^i \mathtt{H}_i$ is then a homomorphism if

$$ {g_{ab}}^c {T_c}^k = {T_a}^i {T_b}^j {h_{ij}}^k\,. \tag{B.6} $$

For an isomorphism invertibility leads to ${(T^{-1})_i}^a {T_a}^j = \delta_i^j$ and therefore

$$ {h_{ij}}^k = {(T^{-1})_i}^a {(T^{-1})_j}^b {g_{ab}}^c {T_c}^k\,. \tag{B.7} $$

When $\mathfrak{a}$ and $\mathfrak{b}$ are subsets of $\mathfrak{g}$, we write

$$ [\mathfrak{a}, \mathfrak{b}] \equiv \mathrm{span}\{[X,Y] \,|\, X \in \mathfrak{a}, Y \in \mathfrak{b}\}\,. \tag{B.8} $$

Given a Lie algebra $\mathfrak{g}$ a subspace $\mathfrak{h}$ is a **subalgebra**, if $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, and an **ideal** if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$. If the commutator of all elements of the Lie algebra vanishes, $[\mathfrak{g}, \mathfrak{g}] = 0$, then it is called **abelian**. The maximal ideal $\mathfrak{z}$ for which $[\mathfrak{g}, \mathfrak{z}] = 0$ is called the **center** of the Lie algebra.

Given an ideal $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ the **quotient algebra** $\mathfrak{g}/\mathfrak{h}$ is the vector space quotient $\mathfrak{g}/\mathfrak{h}$ with the definition

$$ [X + \mathfrak{h}, Y + \mathfrak{h}] = [X, Y] + \mathfrak{h} \quad \text{for all} \quad X, Y \in \mathfrak{g}\,. \tag{B.9} $$

A Lie algebra $\mathfrak{g}$ is a **direct sum** of Lie algebras, denoted by $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, if it is a direct sum of vector spaces, denoted by $\mathfrak{g} = \mathfrak{g}_1 \dotplus \mathfrak{g}_2$ and fulfills

$$ [\mathfrak{g}_i, \mathfrak{g}_i] \subset \mathfrak{g}_i \quad \text{and} \quad [\mathfrak{g}_i, \mathfrak{g}_j] = 0 \quad \text{for} \quad i, j = 1, 2\,. \tag{B.10} $$

Semidirect sums are denoted by $\mathfrak{i} \ni \mathfrak{g}$ where $\mathfrak{i}$ is an ideal and $\mathfrak{g}$ is a subalgebra, see Appendix B.4.

A Lie algebra is **semisimple** if it has no non-zero commutative ideals and **simple** if it has dimension bigger than one and no ideals other than $\{0\}$ and the Lie algebra itself. Semisimple Lie algebras are direct sums of simple ones. They are **perfect**, which means that they obey $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Not all perfect Lie algebras are semisimple, e.g., there exist semidirect sums that are perfect.

The **adjoint representation** is given by $\mathrm{ad} : X \in \mathfrak{g} \mapsto \mathrm{ad}_X \in \mathrm{End}(\mathfrak{g})$[1]

[1] $\mathrm{End}(V)$ denote the endomorphisms of V.

where $\mathrm{ad}_X Y \equiv [X, Y]$ or in a basis $(\mathrm{ad}_{\mathtt{T}_a})^c{}_b = f_{ab}{}^c$. On the vector space dual $\mathfrak{g}^*$ the **coadjoint representation** is defined by

$$\langle \mathrm{ad}^*_X \alpha, Y\rangle \equiv -\langle \alpha, \mathrm{ad}_X Y\rangle \tag{B.11}$$

where $\alpha \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$ and $\langle \alpha, X\rangle$ is the value of the linear functional α evaluated on the vector X. The representation can be written in a basis as $(\mathrm{ad}^*_{\mathtt{T}_a})^c{}_b = -f_{ab}{}^c$.

Any linear mapping $D : \mathfrak{g} \to \mathfrak{g}$ for which

$$D[X, Y] = [DX, Y] + [X, DY] \tag{B.12}$$

is a **derivation** and is an element of the space of derivations $\mathrm{der}(\mathfrak{g})$. An example of a derivation is ad_X. An **inner derivation** can be written in this form, i.e., $D = \mathrm{ad}_X$ for some $X \in \mathfrak{g}$. Derivations for which this is not possible are called **outer derivations**.

B.2 Sequences

A **sequence** consists of objects O_n and homomorphisms f_n between them

$$\cdots \to O_n \xrightarrow{f_n} O_{n+1} \xrightarrow{f_{n+1}} O_{n+2} \to \cdots\,. \tag{B.13}$$

The sequence is **exact** if the image of each homomorphism is equal to the kernel of the next, i.e.,

$$\mathrm{im}(f_n) = \ker(f_{n+1}) \text{ for all } n\,. \tag{B.14}$$

A **short** exact sequence is an exact sequence with

$$0 \to A \to B \to C \to 0\,. \tag{B.15}$$

B.3 Lie Algebra Cohomology

Suppose we have a Lie algebra $\mathfrak{g}$ and a vector space V which is an $\alpha_{\mathfrak{g}}$-module[2]. An n-dimensional V-**cochain** ω_n for the Lie algebra $\mathfrak{g}$ is a skew-symmetric n-linear mapping

$$\omega_n : \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{n} \to V\,. \tag{B.16}$$

[2] This means $\mathfrak{g} \times V \to V : (X, v) \mapsto \alpha_X v$ which satisfies $\alpha_X(v_1 + v_2) = \alpha_X v_1 + \alpha_X v_2$; $\alpha_{X_1+X_2} v = \alpha_{X_1} v + \alpha_{X_2} v$ and $\alpha_{[X_1,X_2]} v = [\alpha_{X_1}, \alpha_{X_2}] v$.

The (abelian) group of all n-cochains will be denoted $C^n(\mathfrak{g}, V)$.

The **coboundary operator** $\delta_n : C^n(\mathfrak{g}, V) \to C^{n+1}(\mathfrak{g}, V)$ is defined by its action on the cochains by

$$\begin{aligned}(\delta\omega_n)(X_1, \dots, X_{n+1}) \equiv \sum_{i=1}^{n+1} (-)^{i+1} \alpha_{X_i}(\omega(X_1, \dots, \hat{X}_i, \dots X_{n+1})) \\ + \sum_{\substack{j,k=1 \\ j<k}}^{n+1} (-)^{j+k} \omega([X_j, X_k], X_1, \dots, \hat{X}_j, \dots, \hat{X}_k, \dots, X_{n+1})\end{aligned} \tag{B.17}$$

where the hat above the Lie algebra elements means that this element should be omitted. The coboundary operator has the property that $\delta^2 = 0$. This can be checked explicitly for the first few cases

$$(\delta\omega_0)(X) = \alpha_X \omega_0 \tag{B.18}$$

$$(\delta\omega_1)(X_1, X_2) = \alpha_{X_1}\omega_1(X_2) - \alpha_{X_2}\omega_1(X_1) - \omega_1([X_1, X_2]) \tag{B.19}$$

$$\begin{aligned}(\delta\omega_2)(X_1, X_2, X_3) &= \alpha_{X_1}\omega_2(X_2, X_3) + \alpha_{X_3}\omega_2(X_1, X_2) + \alpha_{X_2}\omega_2(X_3, X_1) \\ &\quad - \omega_2([X_1, X_2], X_3) - \omega_2([X_3, X_1], X_2) - \omega_2([X_2, X_3], X_1) \\ &= \underset{X_1X_2X_3}{\circlearrowleft} \left(\alpha_{X_1}\omega_2(X_2, X_3) - \omega_2([X_1, X_2], X_3)\right).\end{aligned} \tag{B.20}$$

Using the coboundary operator one can define the following sequence

$$0 \xrightarrow{\delta_{-1}} C^0(\mathfrak{g}, V) \xrightarrow{\delta_0} C^1(\mathfrak{g}, V) \xrightarrow{\delta_1} \cdots \tag{B.21}$$

and furthermore the quotient group $H^n_\alpha(\mathfrak{g}, V)$, called the n**-th cohomology group**, by

$$H^n_\alpha(\mathfrak{g}, V) \equiv \frac{\ker \delta_n}{\operatorname{im} \delta_{n-1}} = \frac{\{n - \text{cocycles}\}}{\{n - \text{coboundarys}\}}\,. \tag{B.22}$$

The cohomology group “measures” the amount at which the sequence fails to be exact. When α is trivial we will sometimes omit it and write $H^n \equiv H^n_0$.

B.4 A Sketch of Lie Algebra Extensions

We now use sequences between Lie algebras to define Lie algebra extensions.

Definition B.2. The Lie algebra $\mathfrak{e}$ is a Lie algebra **extension** of $\mathfrak{g}$ by $\mathfrak{h}$ if

$$0 \to \mathfrak{h} \xrightarrow{i} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \to 0 \tag{B.23}$$

is a short exact sequence. Two extensions are equivalent if there exists a Lie algebra isomorphism ϕ and the following diagram commutes

$$\begin{tikzcd} & & \mathfrak{e} \arrow[dd, "\phi"] \arrow[dr] & & \\ 0 \arrow[r] & \mathfrak{h} \arrow[ur] \arrow[dr] & & \mathfrak{g} \arrow[r] & 0 \, . \\ & & \mathfrak{e}' \arrow[ur] & & \end{tikzcd}$$

Since the homomorphism i has a trivial kernel it is an injective map. Furthermore, is the image isomorphic to the original algebra, so $\mathrm{im}(i) \simeq \mathfrak{h}$. When the context is clear and since they are isomorphic we will often use $\mathfrak{h}$ instead of $\mathrm{im}(i)$. Another consequence of the definition is that π is surjective and therefore (the image of) $\mathfrak{h}$ is an ideal in $\mathfrak{e}$. On the other hand there might not exist a subalgebra of $\mathfrak{e}$ that is isomorphic to $\mathfrak{g}$. But there exists a quotient that leads to $\mathfrak{e}/\mathfrak{h} \simeq \mathfrak{g}$. The linear mapping $\tau : \mathfrak{g} \to \mathfrak{e}$ with $\pi \circ \tau = \mathrm{id}_{\mathfrak{g}}$ induces the mappings

$$\alpha : \mathfrak{g} \to \mathrm{der}(\mathfrak{h}) \, , \qquad \alpha_X(H) = [\tau(X), H] \tag{B.24}$$

$$\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h} \, , \qquad \omega(X,Y) = [\tau(X), \tau(Y)] - \tau([X,Y]) \tag{B.25}$$

where ω is skew-symmetric and which satisfy

$$[\alpha_X, \alpha_Y] - \alpha_{[X,Y]} = \mathrm{ad}_{\omega(X,Y)} \tag{B.26}$$

$$\underset{XYZ}{\circlearrowleft} \left(\alpha_X \omega(Y,Z) - \omega([X,Y],Z)\right) = 0 \, . \tag{B.27}$$

They describe the Lie algebra structure on $\mathfrak{e} = \mathfrak{h} \dotplus \tau(\mathfrak{g})$ as

$$\begin{aligned} [H_1 + \tau(X_1), H_2 + \tau(X_2)] = [H_1, H_2] &+ \alpha_{X_1} H_2 - \alpha_{X_2} H_1 \\ &+ \tau([X_1, X_2]) + \omega(X_1, X_2) \, . \end{aligned} \tag{B.28}$$

On the other hand, we can start with two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and maps $\alpha : \mathfrak{g} \to \mathrm{der}(\mathfrak{h})$ and skew-symmetric $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$ fulfilling (B.26) and (B.27). Then on the vector space $\mathfrak{e} = \mathfrak{h} \dotplus \mathfrak{g}$ a Lie algebra structure is given by

$$\begin{aligned} [H_1 + X_1, H_2 + X_2]_{\mathfrak{e}} = [H_1, H_2]_{\mathfrak{h}} &+ \alpha_{X_1} H_2 - \alpha_{X_2} H_1 \\ &+ [X_1, X_2]_{\mathfrak{g}} + \omega(X_1, X_2) \, . \end{aligned} \tag{B.29}$$

So a general Lie algebra extension schematically has the form

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g} \dotplus \mathfrak{h} \qquad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \qquad [\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h} \, . \tag{B.30}$$

An extension is **trivial** if $\mathfrak{e} \simeq \mathfrak{h} \oplus \mathfrak{g}$, which means that it is just the direct sum discussed in Section B.1,

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g} \qquad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \qquad [\mathfrak{g}, \mathfrak{h}] = 0\,. \tag{B.31}$$

Equivalently, this means that $\alpha = \omega = 0$. A **split** extension is a Lie algebra extension with a homomorphism $\tau : \mathfrak{g} \to \mathfrak{e}$ and $\pi \circ \tau = \mathrm{id}_{\mathfrak{g}}$. Since τ is a homomorphism it follows from (B.25) that $\omega = 0$, so this extension can be written as

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g} \qquad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \qquad [\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}\,. \tag{B.32}$$

Since it is a semidirect sum it will be denoted by $\mathfrak{e} \simeq \mathfrak{h} \ni \mathfrak{g}$. The following theorem characterizes the extensions of simple or one-dimensional Lie algebras.

Theorem B.3. *If $\mathfrak{g}$ is simple or one-dimensional, every Lie algebra extension*

$$0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0 \tag{B.33}$$

splits [53, Prop. A.1].

A **central** extension is a Lie algebra extension where $\mathfrak{h}$ is in the center of $\mathfrak{e}$. It follows that $\mathfrak{h}$ is abelian and that $\alpha = 0$ (see equation (B.24)). It can be written as

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g} \dot{+} \mathfrak{h} \qquad [\mathfrak{h}, \mathfrak{h}] = 0 \qquad [\mathfrak{g}, \mathfrak{h}] = 0 \tag{B.34}$$

and we will denote it by $\mathfrak{g} \oplus_c \mathfrak{h}$. By definition a split central extension is a trivial extension. Therefore, as a consequence of Theorem B.3, we have the well known result (part of Whitehead's lemma) that a simple Lie algebra has no nontrivial central extension.

B.5 Abelian Lie Algebra Extension

For a Lie algebra extension by an abelian Lie algebra $\mathfrak{a}$, i.e., for the short exact sequence

$$0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \to 0 \tag{B.35}$$

we can make contact with Lie algebra cohomology discussed in Appendix B.3. Because for abelian extensions $\mathfrak{a}$ is an $\alpha_{\mathfrak{g}}$-module and therefore

$$[\alpha_X, \alpha_Y]H = \alpha_{[X,Y]}H\,. \tag{B.36}$$

The coboundary operator acting on ω vanishes ($\delta\omega = 0$) and therefore ω is a 2-cocylce. Inequivalent extensions differ by 2-coboundaries and we obtain the following theorem.

Theorem B.4. *For a given α, the classes of equivalent extensions $\mathfrak{e}$ of $\mathfrak{g}$ by the abelian algebra $\mathfrak{a}$ are in one-to-one correspondence with the elements of the second cohomology group $H^2_\alpha(\mathfrak{g},\mathfrak{a})$.*

B.6 Central Extensions

A special class of Lie algebra extensions are the central extensions. For central extensions the Lie algebra structure simplifies, and can be written as

$$[H_1 + X_1, H_2 + X_2]_{\mathfrak{e}} = [X_1, X_2]_{\mathfrak{g}} + \omega(X_1, X_2)\,. \tag{B.37}$$

Choosing the basis $\mathtt{T}_a$ for $\mathfrak{g}$ and the basis $\mathtt{Z}_\alpha$ for $\mathfrak{a}$ we can write the commutation relations in form

$$[\mathtt{T}_a, \mathtt{T}_b] = f_{ab}{}^c \mathtt{T}_c + \omega_{ab}{}^\alpha \mathtt{Z}_\alpha \qquad [\mathtt{T}_a, \mathtt{Z}_\alpha] = [\mathtt{Z}_\alpha, \mathtt{Z}_\beta] = 0 \tag{B.38}$$

The inequivalent central extensions are given by the second cohomology group $H^2_0(\mathfrak{g},\mathfrak{a})$. Therefore, ω is a 2-cocylce which means that it is antisymmetric

$$\omega(X, Y) = -\omega(Y, X) \qquad \omega_{ab}{}^\alpha = -\omega_{ba}{}^\alpha \tag{B.39}$$

and that $\delta\omega = 0$, which leads to

$$\underset{XYZ}{\circlearrowleft} \omega([X, Y], Z) = 0 \qquad \underset{abc}{\circlearrowleft} f_{ab}{}^d \omega_{dc}{}^\alpha = 0\,. \tag{B.40}$$

The last condition also ensures that the Jacobi identities of the whole Lie algebra are satisfied.

Central extensions are seen as equivalent if they differ by a 2-coboundary which is given by

$$\delta\eta(X, Y) = -\eta([X, Y]) \qquad \delta\eta_{ab}{}^\alpha = -f_{ab}{}^c \eta_c{}^\alpha\,. \tag{B.41}$$

So for a nontrivial central extension necessarily the cocycle should not be given by a cobounday, i.e., $\omega \neq \delta\eta$.

Example: Canonical Commutation Relations

We start with an abelian algebra $\mathfrak{g}_d$ with the basis

$$\mathtt{T}_a = (q_1, \ldots, q_d, p_1, \ldots, p_d). \tag{B.42}$$

So we have the commutation relations

$$[q_i, p_j] = [q_i, q_j] = [p_i, p_j] = 0 \tag{B.43}$$

or equivalently ${f_{ab}}^c = 0$. This means that every skew-symmetric ω leads to a 2-cocylce, see equation (B.40). Since all the 2-coboundarys are trivial, see equation (B.41), they are all inequivalent.

For the case of $d = 1$, spanned by q and p, the cohomology group is one-dimensional, $\dim H_0^2(\mathfrak{g}_1, \mathbb{R}) = 1$. And the commutation relations of the nontrivial central extension are the canonical commutation relations

$$[q, p] = \omega Z \qquad\qquad [q, q] = [p, p] = 0 \tag{B.44}$$

where $\omega \neq 0$. For arbitrary dimension d every skew symmetric ${\omega_{ab}}^\alpha$ is possible. So $\dim H_0^2(\mathfrak{g}_d, \mathbb{R}) = d(2d - 1)$.

Appendix C

Useful Formulas

C.1 Details: Solutions of $F = 0$

To show that $F = dA + A \wedge A = 0$ is solved by $A = g^{-1}\,dg$ one uses $d(g^{-1}g) = dg^{-1}g + g^{-1}\,dg = d(1) = 0$ to derive

$$dg^{-1} = -g^{-1}\,dgg^{-1}\,. \tag{C.1}$$

Then we just insert it in the third line of

$$\begin{aligned} dA &= d(g^{-1}\,dg) && \text{(C.2)}\\ &= dg^{-1} \wedge dg && \text{(C.3)}\\ &= -g^{-1}\,dg \wedge g^{-1}\,dg && \text{(C.4)}\\ &= -A \wedge A && \text{(C.5)} \end{aligned}$$

from which the flatness condition on the connection can be read of.

C.2 Finite Gauge Transformation

For finite gauge transformations

$$A \to g^{-1}Ag + g^{-1}\,dg \tag{C.6}$$

the action transforms as

$$\mathrm{CS}[A] \to \mathrm{CS}[A] - \frac{1}{3}\langle (g^{-1}\,dg)^3 \rangle - d\langle A \wedge g\,dg^{-1} \rangle\,. \tag{C.7}$$

This can be seen using the following useful formulas. As already seen in Section 2.1 we define

$$\langle A \wedge B \wedge C \wedge \rangle \equiv \frac{1}{2}\langle [A,B] \wedge C \rangle \tag{C.8}$$

Since (C.8) is symmetric under any permutation of A, B and C we will, if convenient, omit the wedge product. Using

$$d(gg^{-1}) = dgg^{-1} + g\,dg^{-1} = 0 \tag{C.9}$$
$$\alpha = dgg^{-1} = -g\,dg^{-1} \tag{C.10}$$
$$dg = \alpha g \quad dg^{-1} = -g^{-1}\alpha \tag{C.11}$$

and

$$A \to g^{-1}(A+d)g = g^{-1}(A+\alpha)g \tag{C.12}$$
$$dA \to g^{-1}(dA - \alpha^2 - \alpha A - A\alpha)g \tag{C.13}$$
$$\langle dA \wedge A\rangle \to \langle dAA + dA\alpha - \alpha^3 - 3A\alpha^2 - 2A^2\alpha\rangle \tag{C.14}$$
$$\langle A^3\rangle \to \langle A^3 + \alpha^3 + 3A\alpha^2 + 3A^2\alpha\rangle \tag{C.15}$$

leads to (C.7).

Equivalently, one can express the CS Lagrangian in terms of its curvature

$$\mathrm{CS}[A] = \langle F \wedge A - \frac{1}{6}[A,A]\wedge A\rangle \tag{C.16}$$

where one can use that curvature transforms as

$$F \to g^{-1}Fg\,. \tag{C.17}$$

C.3 Infinitesimal Gauge Transformations

The infinitesimal gauge (like) transformation

$$\delta_\lambda A = D\lambda \equiv d\lambda + [A,\lambda] \tag{C.18}$$

is an infinitesimal divergence symmetry of I_{CS}

$$\delta_\lambda \mathrm{CS}[A] = d\langle \lambda \wedge dA\rangle\,. \tag{C.19}$$

The explicitly calculations is given by

$$\begin{aligned}
\delta_\lambda CS[A] &= \langle dD\lambda \wedge A + dA\wedge D\lambda + 2A\wedge A\wedge D\lambda\rangle && \text{(C.20)}\\
&= \langle (([dA,\lambda] - [A,d\lambda])\wedge A + dA\wedge d\lambda + dA\wedge[A,\lambda] \\
&\quad + [A,A]\wedge d\lambda + [A,A]\wedge[A,\lambda]\rangle && \text{(C.21)}\\
&= \langle - dA\wedge[A,\lambda] - [A,A]\wedge d\lambda + dA\wedge d\lambda + dA\wedge[A,\lambda] \\
&\quad + [A,A]\wedge d\lambda - \lambda\wedge[A,[A,A]]\rangle && \text{(C.22)}\\
&= d\langle \lambda\wedge dA\rangle && \text{(C.23)}
\end{aligned}$$

where $[A,[A,A]] = 0$ using the Jacobi identity.

C.4 Infinitesimal Diffeomorphisms

That the CS action is invariant under diffeomorphisms is evident from the fact that it is a (covariant) differential form. Infinitesimal diffeomorphisms are given by the Lie derivative

$$\delta_\xi A = \mathscr{L}_\xi A = i_\xi(dA) + d(i_\xi A) \tag{C.24}$$

and lead to an infinitesimal divergence symmetry

$$\delta_\xi \mathrm{CS}[A] = \mathscr{L}_\xi \mathrm{CS}[A] = d(i_\xi \mathrm{CS}[A])\,. \tag{C.25}$$

C.5 Diffeomorphisms as Gauge Transformations

On-shell an infinitesimal diffeomorphism generated by ξ can be written as a gauge transformations defined by [24]

$$\delta_\lambda A = D\lambda \equiv d\lambda + [A, \lambda] \tag{C.26}$$

since with the gauge parameter given by $\lambda = i_\xi A = \xi^\mu A_\mu$ we get

$$\begin{aligned}
\delta_\xi A &= d\, i_\xi A + i_\xi\, dA && \text{(C.27)}\\
&= d\, i_\xi A - i_\xi(A \wedge A) + i_\xi F && \text{(C.28)}\\
&= d\, i_\xi A + [A, i_\xi A] + i_\xi F && \text{(C.29)}\\
&\overset{\text{o.s.}}{=} D(i_\xi A)\,. && \text{(C.30)}
\end{aligned}$$

Or said differently, gauge transformations with the gauge parameter $\lambda = i_\xi A$ should be regarded as diffeomorphisms.

Appendix D

Explicit Lie Algebra Relations

D.1 $\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{so}(2,1)$

The simple real Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is given by the commutation relations

$$[\mathtt{L}_a, \mathtt{L}_b] = (a-b)\mathtt{L}_{a+b} \tag{D.1}$$

where $a, b = -1, 0, +1$. A defining representation are tracefree 2×2 matrices

$$\mathtt{L}_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \mathtt{L}_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathtt{L}_{+1} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \tag{D.2}$$

for which the trace defines an invariant metric

$$\operatorname{tr}(\mathtt{L}_a\,\mathtt{L}_b) = \left(\begin{array}{c|ccc} & \mathtt{L}_{-1} & \mathtt{L}_0 & \mathtt{L}_{+1} \\ \hline \mathtt{L}_{-1} & 0 & 0 & -1 \\ \mathtt{L}_0 & 0 & \frac{1}{2} & 0 \\ \mathtt{L}_{+1} & -1 & 0 & 0 \end{array}\right). \tag{D.3}$$

This metric is, like every invariant metric of a simple Lie algebra, proportional to the Killing form, $\kappa_{ab} = 4\operatorname{tr}(\mathtt{L}_a\,\mathtt{L}_b)$. This can be verified by using the adjoint representation

$$\mathrm{ad}_{\mathtt{L}_{-1}} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathrm{ad}_{\mathtt{L}_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \mathrm{ad}_{\mathtt{L}_{+1}} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{D.4}$$

and the definition of the Killing form $\kappa_{ab} = \operatorname{tr}(\mathrm{ad}_{\mathtt{L}_a}\,\mathrm{ad}_{\mathtt{L}_b})$. Since the Killing form is nondegenerate this algebra is simple.

The Lie algebra $\mathfrak{so}(2,1)$ is defined by 3×3 matrices M, which have to satisfy $M = -\eta \cdot M^T \cdot \eta$ where the superscript T denotes transpose and

$\eta = \mathrm{diag}(-1,1,1)$. We use $\eta = \mathrm{diag}(-1,1,1)$ and $\epsilon_{012} = 1$ and simplify the above expressions to get

$$[J_A, J_B] = \epsilon_{ABC}\eta^{CD}J_D = \epsilon_{AB}{}^{C}J_C \tag{D.5}$$

$$\mathtt{J}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \mathtt{J}_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \mathtt{J}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{D.6}$$

Again the Lie algebra permits an invariant metric given by

$$\langle \mathtt{J}_A, \mathtt{J}_B \rangle = \eta_{AB} \tag{D.7}$$

which is related to the trace and the Killing form by

$$\langle \mathtt{J}_A, \mathtt{J}_B \rangle = 2\,\mathrm{tr}(\mathtt{J}_A, \mathtt{J}_B) = 2\kappa_{AB}\,. \tag{D.8}$$

The isomorphism between $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(2,1)$ is given by

$$\mathtt{J}_0 = -\frac{1}{2}(\mathtt{L}_{+1} + \mathtt{L}_{-1}) \qquad \mathtt{J}_1 = -\frac{1}{2}(\mathtt{L}_{+1} - \mathtt{L}_{-1}) \qquad \mathtt{J}_2 = -\mathtt{L}_0\,. \tag{D.9}$$

We should note that the invariant metric given using the isomorphism and the invariant metric (D.3) are related by $\langle \mathtt{J}_a, \mathtt{J}_b \rangle = 2\langle \mathtt{J}_a, \mathtt{J}_b \rangle_{\mathrm{iso}}$.

D.2 $\mathfrak{sl}(3,\mathbb{R})$

$$[\mathtt{L}_i, \mathtt{L}_j] = (i-j)\mathtt{L}_{i+j} \tag{D.10}$$

$$[\mathtt{L}_i, \mathtt{W}_m] = (2i-m)\mathtt{W}_{i+m} \tag{D.11}$$

$$[\mathtt{W}_m, \mathtt{W}_n] = -\frac{\sigma}{3}(m-n)\Big(2m^2 + 2n^2 - mn - 8\Big)\mathtt{L}_{m+n}. \tag{D.12}$$

with $i, j = -1, 0, 1$ and $m, n = -2, -1, 0, 1, 2$. With our conventions The constant σ is restricted to be positive for $\mathfrak{sl}(3,\mathbb{R})$ while negative σ would lead to $\mathfrak{su}(1,2)$. With our conventions $\sigma = 1$ for [4,5,7,34,93], and $\sigma_{\mathrm{here}} = -\sigma_{\mathrm{there}}$ for [27].

A matrix representation for is given by

$$\mathtt{L}_{-1} = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \quad \mathtt{L}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \mathtt{L}_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \end{pmatrix} \tag{D.13}$$

and

$$\mathtt{W}_{-2} = \sqrt{4\sigma}\begin{pmatrix}0&0&2\\0&0&0\\0&0&0\end{pmatrix} \qquad \mathtt{W}_{-1} = \frac{\sqrt{4\sigma}}{\sqrt{2}}\begin{pmatrix}0&1&0\\0&0&-1\\0&0&0\end{pmatrix} \quad \mathtt{W}_{0} = \frac{\sqrt{4\sigma}}{3}\begin{pmatrix}1&0&0\\0&-2&0\\0&0&1\end{pmatrix}$$

$$\mathtt{W}_{1} = \frac{\sqrt{4\sigma}}{\sqrt{2}}\begin{pmatrix}0&0&0\\-1&0&0\\0&1&0\end{pmatrix} \qquad \mathtt{W}_{2} = \sqrt{4\sigma}\begin{pmatrix}0&0&0\\0&0&0\\2&0&0\end{pmatrix}\,. \tag{D.14}$$

$$\langle \mathtt{T}_a \mathtt{T}_b \rangle = \left(\begin{array}{c|ccc|ccccc}
 & \mathtt{L}_{-1} & \mathtt{L}_{0} & \mathtt{L}_{1} & \mathtt{W}_{-2} & \mathtt{W}_{-1} & \mathtt{W}_{0} & \mathtt{W}_{1} & \mathtt{W}_{2} \\ \hline
\mathtt{L}_{-1} & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
\mathtt{L}_{0} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathtt{L}_{1} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
\mathtt{W}_{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4\sigma \\
\mathtt{W}_{-1} & 0 & 0 & 0 & 0 & 0 & 0 & -\sigma & 0 \\
\mathtt{W}_{0} & 0 & 0 & 0 & 0 & 0 & \frac{2}{3}\sigma & 0 & 0 \\
\mathtt{W}_{1} & 0 & 0 & 0 & 0 & -\sigma & 0 & 0 & 0 \\
\mathtt{W}_{2} & 0 & 0 & 0 & 4\sigma & 0 & 0 & 0 & 0
\end{array}\right) \tag{D.15}$$

The invariant metric is proportional to the trace and the Killing form in the following form $\langle \mathtt{T}_a \mathtt{T}_b \rangle = \frac{1}{4}\operatorname{tr}(\mathtt{T}_a \mathtt{T}_b) = \frac{1}{24}\kappa_{ab}$.

There is another useful form to write $\mathfrak{sl}(3,\mathbb{R})$ which makes its interpretation as spin-2 and spin-3 fields more obvious [27]. One introduces symmetric and traceless generators $\mathtt{J}_{AB}$, i.e.,

$$\mathtt{J}_{AB} = \mathtt{J}_{BA}\,, \qquad\qquad \eta^{AB}\mathtt{J}_{AB} = 0 \tag{D.16}$$

and defines the Lie algebra

$$[\mathtt{J}_A, \mathtt{J}_B] = \epsilon_{AB}{}^{C}\,\mathtt{J}_c \tag{D.17}$$

$$[\mathtt{J}_A, \mathtt{J}_{BC}] = \epsilon^{M}{}_{A(B}\,\mathtt{J}_{C)M} \tag{D.18}$$

$$[\mathtt{J}_{AB}, \mathtt{J}_{CD}] = -\sigma\eta_{(A(C}\epsilon_{D)B)M}\,\mathtt{J}^{M}\,, \tag{D.19}$$

It permits the invariant metric

$$\langle \mathtt{J}_A, \mathtt{J}_B \rangle = \eta_{AB} \tag{D.20}$$

$$\langle \mathtt{J}_A, \mathtt{J}_{BC} \rangle = 0 \tag{D.21}$$

$$\langle \mathtt{J}_{AB}, \mathtt{J}_{CD} \rangle = \sigma\left(\eta_{A(C}\eta_{D)B} - \tfrac{2}{3}\eta_{AB}\eta_{CD}\right). \tag{D.22}$$

The isomorphism to the basis given by (D.10) to (D.12) is given by (D.9) combined with

$$\mathtt{J}_{00} = \frac{1}{4}(\mathtt{W}_2 + \mathtt{W}_{-2} + 2\,\mathtt{W}_0)\,, \qquad \mathtt{J}_{01} = \frac{1}{4}(\mathtt{W}_2 - \mathtt{W}_{-2}) \tag{D.23}$$

$$\mathtt{J}_{11} = \frac{1}{4}(\mathtt{W}_2 + \mathtt{W}_{-2} - 2\,\mathtt{W}_0)\,, \qquad \mathtt{J}_{02} = \frac{1}{2}(\mathtt{W}_1 + \mathtt{W}_{-1}) \tag{D.24}$$

$$\mathtt{J}_{22} = \mathtt{W}_0\,, \qquad \mathtt{J}_{12} = \frac{1}{2}(\mathtt{W}_1 - \mathtt{W}_{-1})\,. \tag{D.25}$$

This transformation shows explicitly that $\mathtt{W}_m$ automatically satisfies the traceless condition

$$-\mathtt{J}_{00} + \mathtt{J}_{11} + \mathtt{J}_{22} = 0\,. \tag{D.26}$$

The invariant metric given by (D.20) to (D.22) is rescaled by two with respect to the invariant metric given by the isomorphism and using (D.15), e.g., $\langle \mathtt{J}_{AB}, \mathtt{J}_{CD}\rangle = 2\langle \mathtt{J}_{AB}, \mathtt{J}_{CD}\rangle_{\text{iso}}$.

D.3 Principal $\mathfrak{sl}(N,\mathbb{R})$

The conventions are the ones used in [197] with the difference that a conventional positive constant σ is introduced.

The $\mathfrak{sl}(N,\mathbb{R})$ with a principally embedded $\mathfrak{sl}(2,\mathbb{R})$ have generators of spin $s = 2, 3, \ldots, N$. The generators $\{\mathtt{L}_0, \mathtt{L}_{\pm 1}\}$ label the $sl(2,\mathbb{R})$ subalgebra, while the higher spin generators are denoted by $\mathtt{W}^{(s)}_m$ for $m = -(s-1), \ldots, 0, \ldots, s-1$. The algebra in this representation is

$$[\mathtt{L}_i, \mathtt{L}_j] = (i-j)\mathtt{L}_{i+j}\,, \tag{D.27}$$

$$[\mathtt{L}_i, \mathtt{W}^{(s)}_m] = (i(s-1)-m)\mathtt{W}^{(s)}_{i+m}\,. \tag{D.28}$$

and additional commutators for $[\mathtt{W}^{(s)}_m, \mathtt{W}^{(t)}_n]$. We take the N-dimensional generators of the principally embedded $sl(2,\mathbb{R})$, denoted as $\mathtt{L}_i$ to be

$$(\mathtt{L}_1)_{jk} = -\sqrt{j(N-j)}\,\delta_{j+1,k} \tag{D.29}$$

$$(\mathtt{L}_{-1})_{jk} = \sqrt{k(N-k)}\,\delta_{j,k+1} \tag{D.30}$$

$$(\mathtt{L}_0)_{jk} = \frac{1}{2}(N+1-2j)\,\delta_{j,k}\,, \tag{D.31}$$

or explicitly

$$\mathtt{L}_1 = -\begin{pmatrix} 0 & & \cdots & & & & 0 \\ \sqrt{N-1} & 0 & & & & & \\ 0 & \sqrt{2(N-2)} & 0 & & & & \\ & & \ddots & \ddots & & & \vdots \\ \vdots & & & \sqrt{k(N-k)} & 0 & & \\ & & & & \ddots & \ddots & \\ 0 & & \cdots & & & \sqrt{N-1} & 0 \end{pmatrix}, \tag{D.32}$$

$$\mathtt{L}_{-1} = \begin{pmatrix} 0 & \sqrt{N-1} & & \cdots & & & 0 \\ & 0 & \sqrt{2(N-2)} & & & & \\ & & \ddots & \ddots & & & \\ \vdots & & & 0 & \sqrt{k(N-k)} & & \vdots \\ & & & & \ddots & \ddots & \\ & & & & & 0 & \sqrt{N-1} \\ 0 & & & \cdots & & & 0 \end{pmatrix}, \tag{D.33}$$

$$\mathtt{L}_0 = \frac{1}{2}\begin{pmatrix} (N-1) & 0 & & \cdots & & 0 \\ 0 & (N-3) & & & & \\ & & \ddots & & & \\ \vdots & & & (N+1-2k) & & \vdots \\ & & & & \ddots & \\ & & & & -(N-3) & 0 \\ 0 & & & \cdots & 0 & -(N-1) \end{pmatrix}. \tag{D.34}$$

The normalization from this choice of generators is

$$\mathrm{tr}(\mathtt{L}_0\mathtt{L}_0) = \frac{1}{12}N(N^2-1)\,. \tag{D.35}$$

The representation for the higher spin generators follows from

$$\mathtt{L}^{(s)}_m = (\sqrt{4\sigma})^{1-\delta_{s,2}}(-1)^{s+m-1}\frac{(s+m-1)!}{(2s-2)!}\underbrace{[\mathtt{L}_{-1},[\mathtt{L}_{-1},\ldots[\mathtt{L}_{-1}}_{s-m-1 \text{ terms}},(\mathtt{L}_1)^{s-1}]\ldots]]\,. \tag{D.36}$$

$$= (\sqrt{4\sigma})^{1-\delta_{s,2}}(-1)^{s+m-1}\frac{(s+m-1)!}{(2s-2)!}(\mathrm{ad}_{\mathtt{L}_{-1}})^{s-m-1}(\mathtt{L}_1)^{s-1}\,. \tag{D.37}$$

where the $(\sqrt{4\sigma})^{1-\delta_{s,2}}$ term is added such that the definitions are still true for $s = 2$. The matrices obey the hermiticity property

$$\begin{aligned} \mathtt{L}_i^\dagger &= (-1)^i \mathtt{L}_{-i}, & \text{(D.38)} \\ (\mathtt{L}_m^{(s)})^\dagger &= (-1)^m \mathtt{L}_{-m}^{(s)}. & \text{(D.39)} \end{aligned}$$

The trace of the matrix representation given above is given by[1]

$$\operatorname{tr}(\mathtt{L}_m^{(s)} \mathtt{L}_n^{(t)}) = (4\sigma)^{1-\delta_{s,2}} t_m^{(s)} \delta^{s,t} \delta_{m,-n}\,, \tag{D.40}$$

with

$$t_m^{(s)} = (-1)^m \frac{(s-1)!^2 (s+m-1)!(s-m-1)!}{(2s-1)!(2s-2)!} N \prod_{i=1}^{s-1} (N^2 - i^2)\,. \tag{D.41}$$

The relationship between the Killing form κ and the invariant metric given by the trace in the fundamental $n \times n$ matrix representation for $\mathfrak{sl}(N, \mathbb{R})$ is

$$\kappa(x, y) = 2N \operatorname{tr}(xy)\,. \tag{D.42}$$

A normalization where the $\mathfrak{sl}(2, \mathbb{R})$ sector is in agreement with (D.3) is given by

$$\langle \mathtt{L}_m^{(s)} \mathtt{L}_n^{(t)} \rangle = \frac{24}{N(N^2-1)} \operatorname{tr}(\mathtt{L}_m^{(s)} \mathtt{L}_n^{(t)}) \tag{D.43}$$

D.4 $\mathfrak{hs}[\lambda]$

We define here the infinite-dimensional Lie algebra $\mathfrak{hs}[\lambda]$. The finite-dimensional algebra $\mathfrak{sl}(N, \mathbb{R})$ is then given by a Lie algebra quotient thereof. We will provide an invariant metric for both algebras as well as the commutators for spins $s \leq 4$ of $\mathfrak{hs}[\lambda]$.

The generators of $\mathfrak{hs}[\lambda]$ are given by

$$\mathtt{L}_n^{(s)}, \quad s \geq 2, \quad |n| < s\,. \tag{D.44}$$

With the notation used in the previous sections $\mathtt{L}_n^{(2)} = \mathtt{L}_n$ and $\mathtt{L}_n^{(3)} = \mathtt{W}_n$. Using the contraction described in the preceding subsection we can use the commutation relations of $\mathfrak{hs}[\lambda]$ [198–202][2]

$$[\mathtt{L}_n^{(s)}, \mathtt{L}_m^{(t)}] = \sum_{\substack{u=2 \\ \text{even}}}^{s+t-1} g_u^{st}(n, m; \lambda)\, \mathtt{L}_{n+m}^{(s+t-u)} \tag{D.45}$$

---

[1] It is called “Killing Cartan form” in [197], but this is not the Killing form as defined here.

[2] The commutation relations were explicitly given in [201]. Our structure constants are divided by four with respect to the ones given in [28], but we otherwise closely follow [28] (see also [26, 29, 203]).

where

$$g_u^{st}(n,m;\lambda) = \frac{q^{u-2}}{2(u-1)!}\phi_u^{st}(\lambda)N_u^{st}(n,m) \tag{D.46a}$$

$$N_u^{st}(n,m) = \sum_{k=0}^{u-1}(-1)^k\binom{u-1}{k}[s-1+n]_{u-1-k}[s-1-n]_k$$

$$\times [t-1+m]_k[t-1-m]_{u-1-k} \tag{D.46b}$$

$$\phi_u^{st}(\lambda) = {}_4F_3\left[\begin{array}{c} \frac{1}{2}+\lambda\ ,\ \frac{1}{2}-\lambda\ ,\ \frac{2-u}{2}\ ,\ \frac{1-u}{2} \\ \frac{3}{2}-s\ ,\ \frac{3}{2}-t\ ,\ \frac{1}{2}+s+t-u \end{array}\middle|\, 1\right]. \tag{D.46c}$$

The number q is a normalization factor that can be set to any fixed value (for more details see Appendix A in [28]). The falling factorial or Pochhammer symbol is given by

$$[a]_n = a(a-1)(a-2)\cdots(a-n+1) = \frac{a!}{(a-n)!} = \frac{\Gamma(a+1)}{\Gamma(a+1-n)} \tag{D.47}$$

the rising factorial or Pochhammer symbol is given by

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{(a+n-1)!}{(a-1)!} = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{D.48}$$

with $(a)_0 = [a]_0 = 1$. The generalized hypergeometric function ${}_mF_n(z)$ is defined by

$${}_mF_n\left[\begin{array}{c} a_1,\ldots,a_m \\ b_1,\ldots,b_n \end{array}\middle|\, z\right] = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\ldots(a_m)_k}{(b_1)_k(b_2)_k\ldots(b_n)_k}\frac{z^k}{k!}\,. \tag{D.49}$$

The infinite-dimensional Lie algebra $\mathfrak{hs}[\lambda]$ possesses an invariant metric given by

$$\langle \mathrm{L}_n^{(s)}\mathrm{L}_m^{(t)}\rangle \equiv \frac{3}{4q(\lambda^2-1)}g_{s+t-1}^{st}(n,m,\lambda) \tag{D.50a}$$

$$= N_s\frac{(-1)^{s-n-1}}{4(2s-2)!}\Gamma(s+n)\Gamma(s-n)\delta^{st}\delta_{n,-m}$$

with

$$N_s \equiv \frac{3\cdot 4^{s-3}\sqrt{\pi}q^{2s-4}\Gamma(s)}{(\lambda^2-1)\Gamma(s+\frac{1}{2})}(1-\lambda)_{s-1}(1+\lambda)_{s-1}\,. \tag{D.51}$$

The overall constant has been chosen so that

$$\langle \mathrm{L}_0^{(2)}\mathrm{L}_0^{(2)}\rangle = \frac{1}{2} \tag{D.52}$$

which ensures that the $\mathfrak{sl}(2,\mathbb{R})$ sector agrees with (D.3).

From $\mathfrak{hs}[\lambda]$ to $\mathfrak{sl}(N,\mathbb{R})$

Using $\mathfrak{hs}[\lambda]$ one can define $\mathfrak{sl}(N,\mathbb{R})$ as a Lie algebra quotient. This is only possible for $\lambda = N$ since this leads to an ideal χ_N [198, 204, 205] spanned by $\mathtt{L}^{(s)}_n$ with $s > N$. Using this ideal we can then define the finite-dimensional algebra $\mathfrak{sl}(N,\mathbb{R})$ by the quotient

$$\mathfrak{sl}(N,\mathbb{R}) = \mathfrak{hs}[N]/\chi_N\,. \tag{D.53}$$

The invariant metric, equation (D.50) with $\lambda = N$, stays an invariant metric for $\mathfrak{sl}(N,\mathbb{R})$. It is zero for higher spins. In the next section this can be seen explicitly.

Commutators of $\mathfrak{hs}[\lambda]$ for $s \leq 4$

We list here the commutators for $s \leq 4$ of $\mathfrak{hs}[\lambda]$ (with $q = 1/4$)[3]

$$[\mathtt{L}^{(2)}_n, \mathtt{L}^{(2)}_m] = (n-m)\mathtt{L}^{(2)}_{n+m} \tag{D.54a}$$

$$[\mathtt{L}^{(2)}_n, \mathtt{L}^{(3)}_m] = (2n-m)\mathtt{L}^{(3)}_{n+m} \tag{D.54b}$$

$$\begin{aligned}[\mathtt{L}^{(3)}_n, \mathtt{L}^{(3)}_m] = &-\frac{1}{60}(\lambda^2-4)(n-m)(2n^2-nm+2m^2-8)\mathtt{L}^{(2)}_{n+m}\\ &+2(n-m)\mathtt{L}^{(4)}_{n+m}\end{aligned} \tag{D.54c}$$

$$[\mathtt{L}^{(2)}_n, \mathtt{L}^{(4)}_m] = (3n-m)\mathtt{L}^{(4)}_{n+m} \tag{D.54d}$$

$$\begin{aligned}[\mathtt{L}^{(3)}_n, \mathtt{L}^{(4)}_m] = &-\frac{1}{70}(\lambda^2-9)(5n^3-5n^2m-17n+3nm^2+9m-m^3)\mathtt{L}^{(3)}_{n+m}\\ &+(3n-2m)\mathtt{L}^{(5)}_{n+m}\end{aligned} \tag{D.54e}$$

$$\begin{aligned}[\mathtt{L}^{(4)}_n, \mathtt{L}^{(4)}_m] = &(\lambda^2-4)(\lambda^2-9)(n-m)f(n,m)\mathtt{L}^{(2)}_{n+m}\\ &-\frac{1}{30}(\lambda^2-19)(n-m)(n^2-nm+m^2-7)\mathtt{L}^{(4)}_{n+m}\\ &+3(n-m)\mathtt{L}^{(6)}_{n+m}\end{aligned} \tag{D.54f}$$

with

$$f(n,m) = \tfrac{1}{8400}\Big[3n^4+3m^4-2nm(n-m)^2-39(n^2+m^2)+20nm+108\Big]. \tag{D.55}$$

[3] A `Mathematica` workbook that reproduces the commutation relations and might be useful for further checks is uploaded with [7] as an ancillary file on the `arxiv` server.

The invariant metric for $s \leq 4$ is given by the anti-diagonal matrices

$$\langle \mathtt{L}^{(2)}_n\, \mathtt{L}^{(2)}_m \rangle = \mathrm{adiag}(-1, \tfrac{1}{2}, -1) \tag{D.56a}$$

$$\langle \mathtt{L}^{(3)}_n\, \mathtt{L}^{(3)}_m \rangle = \frac{1}{20}(\lambda^2 - 4) \cdot \mathrm{adiag}(4, -1, \tfrac{2}{3}, -1, 4) \tag{D.56b}$$

$$\langle \mathtt{L}^{(4)}_n\, \mathtt{L}^{(4)}_m \rangle = \frac{1}{140}(\lambda^2 - 4)(\lambda^2 - 9) \cdot \mathrm{adiag}(-6, 1, \tfrac{2}{5}, \tfrac{3}{10}, \tfrac{2}{5}, 1, -6)\,. \tag{D.56c}$$

So the quotient agrees with $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(3,\mathbb{R})$ with $\sigma = 1/4$.

D.5 Virasoro and $\mathcal{W}_3$ Algebra

The $\mathcal{W}_3$ algebra at finite central charge, first introduced in [206] and reviewed in [90], is explicitly given by

$$[\mathcal{L}_n,\, \mathcal{L}_m] = (n-m)\,\mathcal{L}_{n+m} + \frac{c}{12}\,(n^3 - n)\,\delta_{n+m,\,0} \tag{D.57a}$$

$$[\mathcal{L}_n,\, \mathcal{W}_m] = (2n-m)\,\mathcal{W}_{n+m} \tag{D.57b}$$

$$\begin{aligned}[\mathcal{W}_n,\, \mathcal{W}_m] &= (n-m)(2n^2 + 2m^2 - nm - 8)\,\mathcal{L}_{n+m} \\ &\quad + \frac{c}{12}\,(n^2-4)(n^3-n)\,\delta_{n+m,\,0} + \frac{96}{c + \frac{22}{5}}\,(n-m)\,\Lambda_{n+m}\end{aligned} \tag{D.57c}$$

where

$$\Lambda_n = \sum_{p\in\mathbb{Z}} :(\mathcal{L}_{n-p}\mathcal{L}_p): - \frac{3}{10}(n+3)(n+2)\mathcal{L}_n\ . \tag{D.58}$$

The generators split into the Virasoro generators $\mathcal{L}_n$ and of spin-3 generators $\mathcal{W}_n$ both with integer n.

D.6 Kinematical Spin-2 Algebras

	$(\mathfrak{A})\mathfrak{dS}_{(\mp)}$	$\mathfrak{poi}$	$\mathfrak{nh}$	$\mathfrak{ppoi}$
$[\mathtt{J}\,,\,\mathtt{J}]$	0	0	0	0
$[\mathtt{J}\,,\,\mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J}\,,\,\mathtt{H}]$	0	0	0	0
$[\mathtt{J}\,,\,\mathtt{P}_a]$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a\,,\,\mathtt{G}_b]$	$-\epsilon_{ab}\mathtt{J}$	$-\epsilon_{ab}\mathtt{J}$	0	0
$[\mathtt{G}_a\,,\,\mathtt{H}]$	$-\epsilon_{am}\mathtt{P}_m$	$-\epsilon_{am}\mathtt{P}_m$	$-\epsilon_{am}\mathtt{P}_m$	0
$[\mathtt{G}_a\,,\,\mathtt{P}_b]$	$-\epsilon_{ab}\mathtt{H}$	$-\epsilon_{ab}\mathtt{H}$	0	$-\epsilon_{ab}\mathtt{H}$
$[\mathtt{H}\,,\,\mathtt{P}_a]$	$\pm\epsilon_{am}\mathtt{G}_m$	0	$\pm\epsilon_{am}\mathtt{G}_m$	$\pm\epsilon_{am}\mathtt{G}_m$
$[\mathtt{P}_a\,,\,\mathtt{P}_b]$	$\mp\epsilon_{ab}\mathtt{J}$	0	0	$\mp\epsilon_{ab}\mathtt{J}$

Table D.1: (Anti-)de Sitter, Poincaré, Newton–Hooke and para-Poincaré algebras. The upper sign is for AdS (and contractions thereof) and the lower sign for dS (and contractions thereof).

	$\mathfrak{car}$	$\mathfrak{gal}$	$\mathfrak{pgal}$	$\mathfrak{st}$
$[\mathtt{J}\,,\,\mathtt{J}]$	0	0	0	0
$[\mathtt{J}\,,\,\mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J}\,,\,\mathtt{H}]$	0	0	0	0
$[\mathtt{J}\,,\,\mathtt{P}_a]$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a\,,\,\mathtt{G}_b]$	0	0	0	0
$[\mathtt{G}_a\,,\,\mathtt{H}]$	0	$-\epsilon_{am}\mathtt{P}_m$	0	0
$[\mathtt{G}_a\,,\,\mathtt{P}_b]$	$-\epsilon_{ab}\mathtt{H}$	0	0	0
$[\mathtt{H}\,,\,\mathtt{P}_a]$	0	0	$\pm\epsilon_{am}\mathtt{G}_m$	0
$[\mathtt{P}_a\,,\,\mathtt{P}_b]$	0	0	0	0

Table D.2: Carroll, Galilei, para-Galilei and static algebra. The upper sign is for AdS (and contractions thereof) and the lower sign for dS (and contractions thereof).

The most general invariant metric for the $(\mathfrak{A})\mathfrak{dS}_{(\mp)}$ algebra is given by

$$\langle \mathtt{H}\,,\mathtt{J}\rangle = -\mu^- \qquad \langle \mathtt{P}_a\,,\mathtt{G}_b\rangle = \mu^-\delta_{ab} \tag{D.59}$$

$$\langle \mathtt{J}\,,\mathtt{J}\rangle = -\mu^+ \qquad \langle \mathtt{G}_a\,,\mathtt{G}_b\rangle = \mu^+\delta_{ab} \tag{D.60}$$

$$\langle \mathtt{H}\,,\mathtt{H}\rangle = \mp\mu^+ \qquad \langle \mathtt{P}_a\,,\mathtt{P}_b\rangle = \pm\mu^+\delta_{ab}\,. \tag{D.61}$$

The two real constants need to satisfy $\mu^+ \neq \pm\mu^-$ for the metric to be nondegenerate, see Section 5.2.

D.7 Democratic Spin-3 Algebras

This appendix contains tables with all the commutation relations of the spin-3 algebras that can be obtained via sequential application of the "democratic" sIW-contractions. We start each table with the spin-2 commutation relations, then proceed with the mixed spin commutation relations and conclude with the spin-3 commutation relations. The table caption contains information about what type of higher spin version we are dealing with (e.g. higher spin version of Poincaré, Galilei or Carroll). Under the heading 'Contraction #', we have indicated one possibility of obtaining the corresponding algebra as a sequential application of IW contraction procedures. The numbers in this heading refer to the contraction procedures of Table 10.1.

Contr. #	$\mathfrak{hs}_3(\mathfrak{A})\mathfrak{dS}_{(\mp)}$	$\mathfrak{hs}_3\mathfrak{poi1}$ 1	$\mathfrak{hs}_3\mathfrak{poi2}$ 2
$[\mathtt{J}, \mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J}, \mathtt{H}]$	0	0	0
$[\mathtt{J}, \mathtt{P}_a]$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a, \mathtt{G}_b]$	$-\epsilon_{ab}\mathtt{J}$	$-\epsilon_{ab}\mathtt{J}$	$-\epsilon_{ab}\mathtt{J}$
$[\mathtt{G}_a, \mathtt{H}]$	$-\epsilon_{am}\mathtt{P}_m$	$-\epsilon_{am}\mathtt{P}_m$	$-\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a, \mathtt{P}_b]$	$-\epsilon_{ab}\mathtt{H}$	$-\epsilon_{ab}\mathtt{H}$	$-\epsilon_{ab}\mathtt{H}$
$[\mathtt{H}, \mathtt{P}_a]$	$\pm\epsilon_{am}\mathtt{G}_m$	0	0
$[\mathtt{P}_a, \mathtt{P}_b]$	$\mp\epsilon_{ab}\mathtt{J}$	0	0
$[\mathtt{J}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{J}_m$	$\epsilon_{am}\mathtt{J}_m$	$\epsilon_{am}\mathtt{J}_m$
$[\mathtt{J}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{J}, \mathtt{H}_a]$	$\epsilon_{am}\mathtt{H}_m$	$\epsilon_{am}\mathtt{H}_m$	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{J}, \mathtt{P}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{G}_a, \mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$	$-(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$	$-(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$
$[\mathtt{G}_a, \mathtt{G}_{bc}]$	$-\epsilon_{a(b}\mathtt{J}_{c)}$	$-\epsilon_{a(b}\mathtt{J}_{c)}$	$-\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{G}_a, \mathtt{H}_b]$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$
$[\mathtt{G}_a, \mathtt{P}_{bc}]$	$-\epsilon_{a(b}\mathtt{H}_{c)}$	$-\epsilon_{a(b}\mathtt{H}_{c)}$	$-\epsilon_{a(b}\mathtt{H}_{c)}$
$[\mathtt{H}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{H}_m$	$\epsilon_{am}\mathtt{H}_m$	0
$[\mathtt{H}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	0
$[\mathtt{H}, \mathtt{H}_a]$	$\pm\epsilon_{am}\mathtt{J}_m$	0	$\pm\epsilon_{am}\mathtt{J}_m$
$[\mathtt{H}, \mathtt{P}_{ab}]$	$\mp\epsilon_{m(a}\mathtt{G}_{b)m}$	0	$\mp\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{P}_a, \mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$	0
$[\mathtt{P}_a, \mathtt{G}_{bc}]$	$-\epsilon_{a(b}\mathtt{H}_{c)}$	$-\epsilon_{a(b}\mathtt{H}_{c)}$	0
$[\mathtt{P}_a, \mathtt{H}_b]$	$\mp(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$	0	$\mp(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$
$[\mathtt{P}_a, \mathtt{P}_{bc}]$	$\mp\epsilon_{a(b}\mathtt{J}_{c)}$	0	$\mp\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{J}_a, \mathtt{J}_b]$	$\epsilon_{ab}\mathtt{J}$	$\epsilon_{ab}\mathtt{J}$	0
$[\mathtt{J}_a, \mathtt{G}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	0
$[\mathtt{J}_a, \mathtt{H}_b]$	$\epsilon_{ab}\mathtt{H}$	$\epsilon_{ab}\mathtt{H}$	$\epsilon_{ab}\mathtt{H}$
$[\mathtt{J}_a, \mathtt{P}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}, \mathtt{G}_{cd}]$	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$	0
$[\mathtt{G}_{ab}, \mathtt{H}_c]$	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}, \mathtt{P}_{cd}]$	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$
$[\mathtt{H}_a, \mathtt{H}_b]$	$\pm\epsilon_{ab}\mathtt{J}$	0	$\pm\epsilon_{ab}\mathtt{J}$
$[\mathtt{H}_a, \mathtt{P}_{bc}]$	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	0	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\mathtt{P}_{ab}, \mathtt{P}_{cd}]$	$\pm\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$	0	$\pm\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$

Table D.3: Higher spin versions of the (A)dS and Poincaré algebra. The upper sign is for AdS (and contractions thereof) and the lower sign for dS (and contractions thereof).

	$\mathfrak{hs}_3\mathfrak{nh}1$	$\mathfrak{hs}_3\mathfrak{nh}2$
Contraction #	3	4
$[\mathtt{J}, \mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J}, \mathtt{H}]$	0	0
$[\mathtt{J}, \mathtt{P}_a]$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a, \mathtt{G}_b]$	0	0
$[\mathtt{G}_a, \mathtt{H}]$	$-\epsilon_{am}\mathtt{P}_m$	$-\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a, \mathtt{P}_b]$	0	0
$[\mathtt{H}, \mathtt{P}_a]$	$\pm\epsilon_{am}\mathtt{G}_m$	$\pm\epsilon_{am}\mathtt{G}_m$
$[\mathtt{P}_a, \mathtt{P}_b]$	0	0
$[\mathtt{J}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{J}_m$	$\epsilon_{am}\mathtt{J}_m$
$[\mathtt{J}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{J}, \mathtt{H}_a]$	$\epsilon_{am}\mathtt{H}_m$	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{J}, \mathtt{P}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{G}_a, \mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$	0
$[\mathtt{G}_a, \mathtt{G}_{bc}]$	0	$-\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{G}_a, \mathtt{H}_b]$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$	0
$[\mathtt{G}_a, \mathtt{P}_{bc}]$	0	$-\epsilon_{a(b}\mathtt{H}_{c)}$
$[\mathtt{H}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{H}_m$	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{H}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{H}, \mathtt{H}_a]$	$\pm\epsilon_{am}\mathtt{J}_m$	$\pm\epsilon_{am}\mathtt{J}_m$
$[\mathtt{H}, \mathtt{P}_{ab}]$	$\mp\epsilon_{m(a}\mathtt{G}_{b)m}$	$\mp\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{P}_a, \mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$	0
$[\mathtt{P}_a, \mathtt{G}_{bc}]$	0	$-\epsilon_{a(b}\mathtt{H}_{c)}$
$[\mathtt{P}_a, \mathtt{H}_b]$	$\mp(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$	0
$[\mathtt{P}_a, \mathtt{P}_{bc}]$	0	$\mp\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{J}_a, \mathtt{J}_b]$	$\epsilon_{ab}\mathtt{J}$	0
$[\mathtt{J}_a, \mathtt{G}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\mathtt{J}_a, \mathtt{H}_b]$	$\epsilon_{ab}\mathtt{H}$	0
$[\mathtt{J}_a, \mathtt{P}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}, \mathtt{G}_{cd}]$	0	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$
$[\mathtt{G}_{ab}, \mathtt{H}_c]$	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}, \mathtt{P}_{cd}]$	0	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$
$[\mathtt{H}_a, \mathtt{H}_b]$	$\pm\epsilon_{ab}\mathtt{J}$	0
$[\mathtt{H}_a, \mathtt{P}_{bc}]$	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\mathtt{P}_{ab}, \mathtt{P}_{cd}]$	0	$\pm\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$

Table D.4: Higher spin versions of the Newton–Hooke algebra. The upper sign is for contractions of AdS and the lower sign for contractions of dS.

	$\mathfrak{hs}_3\mathfrak{ppoi1}$	$\mathfrak{hs}_3\mathfrak{ppoi2}$
Contraction #	5	6
$[\,\mathtt{J}\,,\,\mathtt{G}_a\,]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\,\mathtt{J}\,,\,\mathtt{H}\,]$	0	0
$[\,\mathtt{J}\,,\,\mathtt{P}_a\,]$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$
$[\,\mathtt{G}_a\,,\,\mathtt{G}_b\,]$	0	0
$[\,\mathtt{G}_a\,,\,\mathtt{H}\,]$	0	0
$[\,\mathtt{G}_a\,,\,\mathtt{P}_b\,]$	$-\epsilon_{ab}\mathtt{H}$	$-\epsilon_{ab}\mathtt{H}$
$[\,\mathtt{H}\,,\,\mathtt{P}_a\,]$	$\pm\epsilon_{am}\mathtt{G}_m$	$\pm\epsilon_{am}\mathtt{G}_m$
$[\,\mathtt{P}_a\,,\,\mathtt{P}_b\,]$	$\mp\epsilon_{ab}\mathtt{J}$	$\mp\epsilon_{ab}\mathtt{J}$
$[\,\mathtt{J}\,,\,\mathtt{J}_a\,]$	$\epsilon_{am}\mathtt{J}_m$	$\epsilon_{am}\mathtt{J}_m$
$[\,\mathtt{J}\,,\,\mathtt{G}_{ab}\,]$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\,\mathtt{J}\,,\,\mathtt{H}_a\,]$	$\epsilon_{am}\mathtt{H}_m$	$\epsilon_{am}\mathtt{H}_m$
$[\,\mathtt{J}\,,\,\mathtt{P}_{ab}\,]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\,\mathtt{G}_a\,,\,\mathtt{J}_b\,]$	$-(\epsilon_{am}\mathtt{G}_{bm}+\epsilon_{ab}\mathtt{G}_{mm})$	0
$[\,\mathtt{G}_a\,,\,\mathtt{G}_{bc}\,]$	0	$-\epsilon_{a(b}\mathtt{J}_{c)}$
$[\,\mathtt{G}_a\,,\,\mathtt{H}_b\,]$	0	$-(\epsilon_{am}\mathtt{P}_{bm}+\epsilon_{ab}\mathtt{P}_{mm})$
$[\,\mathtt{G}_a\,,\,\mathtt{P}_{bc}\,]$	$-\epsilon_{a(b}\mathtt{H}_{c)}$	0
$[\,\mathtt{H}\,,\,\mathtt{J}_a\,]$	$\epsilon_{am}\mathtt{H}_m$	0
$[\,\mathtt{H}\,,\,\mathtt{G}_{ab}\,]$	0	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\,\mathtt{H}\,,\,\mathtt{H}_a\,]$	0	$\pm\epsilon_{am}\mathtt{J}_m$
$[\,\mathtt{H}\,,\,\mathtt{P}_{ab}\,]$	$\mp\epsilon_{m(a}\mathtt{G}_{b)m}$	0
$[\,\mathtt{P}_a\,,\,\mathtt{J}_b\,]$	$-(\epsilon_{am}\mathtt{P}_{bm}+\epsilon_{ab}\mathtt{P}_{mm})$	$-(\epsilon_{am}\mathtt{P}_{bm}+\epsilon_{ab}\mathtt{P}_{mm})$
$[\,\mathtt{P}_a\,,\,\mathtt{G}_{bc}\,]$	$-\epsilon_{a(b}\mathtt{H}_{c)}$	$-\epsilon_{a(b}\mathtt{H}_{c)}$
$[\,\mathtt{P}_a\,,\,\mathtt{H}_b\,]$	$\mp(\epsilon_{am}\mathtt{G}_{bm}+\epsilon_{ab}\mathtt{G}_{mm})$	$\mp(\epsilon_{am}\mathtt{G}_{bm}+\epsilon_{ab}\mathtt{G}_{mm})$
$[\,\mathtt{P}_a\,,\,\mathtt{P}_{bc}\,]$	$\mp\epsilon_{a(b}\mathtt{J}_{c)}$	$\mp\epsilon_{a(b}\mathtt{J}_{c)}$
$[\,\mathtt{J}_a\,,\,\mathtt{J}_b\,]$	$\epsilon_{ab}\mathtt{J}$	0
$[\,\mathtt{J}_a\,,\,\mathtt{G}_{bc}\,]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\,\mathtt{J}_a\,,\,\mathtt{H}_b\,]$	$\epsilon_{ab}\mathtt{H}$	$\epsilon_{ab}\mathtt{H}$
$[\,\mathtt{J}_a\,,\,\mathtt{P}_{bc}\,]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$	0
$[\,\mathtt{G}_{ab}\,,\,\mathtt{G}_{cd}\,]$	0	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$
$[\,\mathtt{G}_{ab}\,,\,\mathtt{H}_c\,]$	0	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$
$[\,\mathtt{G}_{ab}\,,\,\mathtt{P}_{cd}\,]$	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$
$[\,\mathtt{H}_a\,,\,\mathtt{H}_b\,]$	0	$\pm\epsilon_{ab}\mathtt{J}$
$[\,\mathtt{H}_a\,,\,\mathtt{P}_{bc}\,]$	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\,\mathtt{P}_{ab}\,,\,\mathtt{P}_{cd}\,]$	$\pm\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$	0

Table D.5: Higher spin versions of para-Poincaré algebra. The upper sign is for contractions of AdS and the lower sign for contractions of dS.

	$\mathfrak{hs}_3\mathfrak{car1}$	$\mathfrak{hs}_3\mathfrak{car2}$
Contraction #	$1,5$	$1,6$
$[\mathtt{J}\,,\,\mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J}\,,\,\mathtt{H}]$	0	0
$[\mathtt{J}\,,\,\mathtt{P}_a]$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a\,,\,\mathtt{G}_b]$	0	0
$[\mathtt{G}_a\,,\,\mathtt{H}]$	0	0
$[\mathtt{G}_a\,,\,\mathtt{P}_b]$	$-\epsilon_{ab}\mathtt{H}$	$-\epsilon_{ab}\mathtt{H}$
$[\mathtt{H}\,,\,\mathtt{P}_a]$	0	0
$[\mathtt{P}_a\,,\,\mathtt{P}_b]$	0	0
$[\mathtt{J}\,,\,\mathtt{J}_a]$	$\epsilon_{am}\mathtt{J}_m$	$\epsilon_{am}\mathtt{J}_m$
$[\mathtt{J}\,,\,\mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{J}\,,\,\mathtt{H}_a]$	$\epsilon_{am}\mathtt{H}_m$	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{J}\,,\,\mathtt{P}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{G}_a\,,\,\mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{G}_{bm}+\epsilon_{ab}\mathtt{G}_{mm})$	0
$[\mathtt{G}_a\,,\,\mathtt{G}_{bc}]$	0	$-\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{G}_a\,,\,\mathtt{H}_b]$	0	$-(\epsilon_{am}\mathtt{P}_{bm}+\epsilon_{ab}\mathtt{P}_{mm})$
$[\mathtt{G}_a\,,\,\mathtt{P}_{bc}]$	$-\epsilon_{a(b}\mathtt{H}_{c)}$	0
$[\mathtt{H}\,,\,\mathtt{J}_a]$	$\epsilon_{am}\mathtt{H}_m$	0
$[\mathtt{H}\,,\,\mathtt{G}_{ab}]$	0	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{H}\,,\,\mathtt{H}_a]$	0	0
$[\mathtt{H}\,,\,\mathtt{P}_{ab}]$	0	0
$[\mathtt{P}_a\,,\,\mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{P}_{bm}+\epsilon_{ab}\mathtt{P}_{mm})$	$-(\epsilon_{am}\mathtt{P}_{bm}+\epsilon_{ab}\mathtt{P}_{mm})$
$[\mathtt{P}_a\,,\,\mathtt{G}_{bc}]$	$-\epsilon_{a(b}\mathtt{H}_{c)}$	$-\epsilon_{a(b}\mathtt{H}_{c)}$
$[\mathtt{P}_a\,,\,\mathtt{H}_b]$	0	0
$[\mathtt{P}_a\,,\,\mathtt{P}_{bc}]$	0	0
$[\mathtt{J}_a\,,\,\mathtt{J}_b]$	$\epsilon_{ab}\mathtt{J}$	0
$[\mathtt{J}_a\,,\,\mathtt{G}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\mathtt{J}_a\,,\,\mathtt{H}_b]$	$\epsilon_{ab}\mathtt{H}$	$\epsilon_{ab}\mathtt{H}$
$[\mathtt{J}_a\,,\,\mathtt{P}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$	0
$[\mathtt{G}_{ab}\,,\,\mathtt{G}_{cd}]$	0	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$
$[\mathtt{G}_{ab}\,,\,\mathtt{H}_c]$	0	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}\,,\,\mathtt{P}_{cd}]$	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$
$[\mathtt{H}_a\,,\,\mathtt{H}_b]$	0	0
$[\mathtt{H}_a\,,\,\mathtt{P}_{bc}]$	0	0
$[\mathtt{P}_{ab}\,,\,\mathtt{P}_{cd}]$	0	0

Table D.6: Higher spin versions of the Carroll algebra.

Contraction #	$\mathfrak{hs}_3\mathfrak{car}3$ 5, 2	$\mathfrak{hs}_3\mathfrak{car}4$ 6, 2
$[\mathtt{J}, \mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J}, \mathtt{H}]$	0	0
$[\mathtt{J}, \mathtt{P}_a]$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a, \mathtt{G}_b]$	0	0
$[\mathtt{G}_a, \mathtt{H}]$	0	0
$[\mathtt{G}_a, \mathtt{P}_b]$	$-\epsilon_{ab}\mathtt{H}$	$-\epsilon_{ab}\mathtt{H}$
$[\mathtt{H}, \mathtt{P}_a]$	0	0
$[\mathtt{P}_a, \mathtt{P}_b]$	0	0
$[\mathtt{J}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{J}_m$	$\epsilon_{am}\mathtt{J}_m$
$[\mathtt{J}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{J}, \mathtt{H}_a]$	$\epsilon_{am}\mathtt{H}_m$	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{J}, \mathtt{P}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{G}_a, \mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$	0
$[\mathtt{G}_a, \mathtt{G}_{bc}]$	0	$-\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{G}_a, \mathtt{H}_b]$	0	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$
$[\mathtt{G}_a, \mathtt{P}_{bc}]$	$-\epsilon_{a(b}\mathtt{H}_{c)}$	0
$[\mathtt{H}, \mathtt{J}_a]$	0	0
$[\mathtt{H}, \mathtt{G}_{ab}]$	0	0
$[\mathtt{H}, \mathtt{H}_a]$	0	$\pm\epsilon_{am}\mathtt{J}_m$
$[\mathtt{H}, \mathtt{P}_{ab}]$	$\mp\epsilon_{m(a}\mathtt{G}_{b)m}$	0
$[\mathtt{P}_a, \mathtt{J}_b]$	0	0
$[\mathtt{P}_a, \mathtt{G}_{bc}]$	0	0
$[\mathtt{P}_a, \mathtt{H}_b]$	$\mp(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$	$\mp(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$
$[\mathtt{P}_a, \mathtt{P}_{bc}]$	$\mp\epsilon_{a(b}\mathtt{J}_{c)}$	$\mp\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{J}_a, \mathtt{J}_b]$	0	0
$[\mathtt{J}_a, \mathtt{G}_{bc}]$	0	0
$[\mathtt{J}_a, \mathtt{H}_b]$	$\epsilon_{ab}\mathtt{H}$	$\epsilon_{ab}\mathtt{H}$
$[\mathtt{J}_a, \mathtt{P}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$	0
$[\mathtt{G}_{ab}, \mathtt{G}_{cd}]$	0	0
$[\mathtt{G}_{ab}, \mathtt{H}_c]$	0	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}, \mathtt{P}_{cd}]$	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$
$[\mathtt{H}_a, \mathtt{H}_b]$	0	$\pm\epsilon_{ab}\mathtt{J}$
$[\mathtt{H}_a, \mathtt{P}_{bc}]$	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\mathtt{P}_{ab}, \mathtt{P}_{cd}]$	$\pm\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$	0

Table D.7: Higher spin versions of the Carroll algebra. The upper sign is for contractions of AdS and the lower sign for contractions of dS.

Contraction #	$\mathfrak{hs}_3\mathfrak{gal1}$ 1, 3	$\mathfrak{hs}_3\mathfrak{gal2}$ 1, 4
$[\mathtt{J}, \mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J}, \mathtt{H}]$	0	0
$[\mathtt{J}, \mathtt{P}_a]$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a, \mathtt{G}_b]$	0	0
$[\mathtt{G}_a, \mathtt{H}]$	$-\epsilon_{am}\mathtt{P}_m$	$-\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a, \mathtt{P}_b]$	0	0
$[\mathtt{H}, \mathtt{P}_a]$	0	0
$[\mathtt{P}_a, \mathtt{P}_b]$	0	0
$[\mathtt{J}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{J}_m$	$\epsilon_{am}\mathtt{J}_m$
$[\mathtt{J}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{J}, \mathtt{H}_a]$	$\epsilon_{am}\mathtt{H}_m$	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{J}, \mathtt{P}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{G}_a, \mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$	0
$[\mathtt{G}_a, \mathtt{G}_{bc}]$	0	$-\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{G}_a, \mathtt{H}_b]$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$	0
$[\mathtt{G}_a, \mathtt{P}_{bc}]$	0	$-\epsilon_{a(b}\mathtt{H}_{c)}$
$[\mathtt{H}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{H}_m$	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{H}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{H}, \mathtt{H}_a]$	0	0
$[\mathtt{H}, \mathtt{P}_{ab}]$	0	0
$[\mathtt{P}_a, \mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$	0
$[\mathtt{P}_a, \mathtt{G}_{bc}]$	0	$-\epsilon_{a(b}\mathtt{H}_{c)}$
$[\mathtt{P}_a, \mathtt{H}_b]$	0	0
$[\mathtt{P}_a, \mathtt{P}_{bc}]$	0	0
$[\mathtt{J}_a, \mathtt{J}_b]$	$\epsilon_{ab}\mathtt{J}$	0
$[\mathtt{J}_a, \mathtt{G}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\mathtt{J}_a, \mathtt{H}_b]$	$\epsilon_{ab}\mathtt{H}$	0
$[\mathtt{J}_a, \mathtt{P}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}, \mathtt{G}_{cd}]$	0	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$
$[\mathtt{G}_{ab}, \mathtt{H}_c]$	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}, \mathtt{P}_{cd}]$	0	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$
$[\mathtt{H}_a, \mathtt{H}_b]$	0	0
$[\mathtt{H}_a, \mathtt{P}_{bc}]$	0	0
$[\mathtt{P}_{ab}, \mathtt{P}_{cd}]$	0	0

Table D.8: Higher spin versions of the Galilei algebra.

Contraction #	$\mathfrak{hs}_3\mathfrak{gal}3$ 3, 2	$\mathfrak{hs}_3\mathfrak{gal}4$ 4, 2
$[\mathtt{J}, \mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J}, \mathtt{H}]$	0	0
$[\mathtt{J}, \mathtt{P}_a]$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a, \mathtt{G}_b]$	0	0
$[\mathtt{G}_a, \mathtt{H}]$	$-\epsilon_{am}\mathtt{P}_m$	$-\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a, \mathtt{P}_b]$	0	0
$[\mathtt{H}, \mathtt{P}_a]$	0	0
$[\mathtt{P}_a, \mathtt{P}_b]$	0	0
$[\mathtt{J}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{J}_m$	$\epsilon_{am}\mathtt{J}_m$
$[\mathtt{J}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{J}, \mathtt{H}_a]$	$\epsilon_{am}\mathtt{H}_m$	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{J}, \mathtt{P}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{G}_a, \mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$	0
$[\mathtt{G}_a, \mathtt{G}_{bc}]$	0	$-\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{G}_a, \mathtt{H}_b]$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$	0
$[\mathtt{G}_a, \mathtt{P}_{bc}]$	0	$-\epsilon_{a(b}\mathtt{H}_{c)}$
$[\mathtt{H}, \mathtt{J}_a]$	0	0
$[\mathtt{H}, \mathtt{G}_{ab}]$	0	0
$[\mathtt{H}, \mathtt{H}_a]$	$\pm\epsilon_{am}\mathtt{J}_m$	$\pm\epsilon_{am}\mathtt{J}_m$
$[\mathtt{H}, \mathtt{P}_{ab}]$	$\mp\epsilon_{m(a}\mathtt{G}_{b)m}$	$\mp\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{P}_a, \mathtt{J}_b]$	0	0
$[\mathtt{P}_a, \mathtt{G}_{bc}]$	0	0
$[\mathtt{P}_a, \mathtt{H}_b]$	$\mp(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$	0
$[\mathtt{P}_a, \mathtt{P}_{bc}]$	0	$\mp\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{J}_a, \mathtt{J}_b]$	0	0
$[\mathtt{J}_a, \mathtt{G}_{bc}]$	0	0
$[\mathtt{J}_a, \mathtt{H}_b]$	$\epsilon_{ab}\mathtt{H}$	0
$[\mathtt{J}_a, \mathtt{P}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}, \mathtt{G}_{cd}]$	0	0
$[\mathtt{G}_{ab}, \mathtt{H}_c]$	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}, \mathtt{P}_{cd}]$	0	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$
$[\mathtt{H}_a, \mathtt{H}_b]$	$\pm\epsilon_{ab}\mathtt{J}$	0
$[\mathtt{H}_a, \mathtt{P}_{bc}]$	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\mathtt{P}_{ab}, \mathtt{P}_{cd}]$	0	$\pm\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$

Table D.9: Higher spin versions of the Galilei algebra. The upper sign is for contractions of AdS and the lower sign for contractions of dS.

Contraction #	$\mathfrak{hs}_3\mathfrak{pgal1}$ 3, 5	$\mathfrak{hs}_3\mathfrak{pgal2}$ 3, 6
$[\mathtt{J}, \mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J}, \mathtt{H}]$	0	0
$[\mathtt{J}, \mathtt{P}_a]$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a, \mathtt{G}_b]$	0	0
$[\mathtt{G}_a, \mathtt{H}]$	0	0
$[\mathtt{G}_a, \mathtt{P}_b]$	0	0
$[\mathtt{H}, \mathtt{P}_a]$	$\pm\epsilon_{am}\mathtt{G}_m$	$\pm\epsilon_{am}\mathtt{G}_m$
$[\mathtt{P}_a, \mathtt{P}_b]$	0	0
$[\mathtt{J}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{J}_m$	$\epsilon_{am}\mathtt{J}_m$
$[\mathtt{J}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{J}, \mathtt{H}_a]$	$\epsilon_{am}\mathtt{H}_m$	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{J}, \mathtt{P}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{G}_a, \mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$	0
$[\mathtt{G}_a, \mathtt{G}_{bc}]$	0	0
$[\mathtt{G}_a, \mathtt{H}_b]$	0	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$
$[\mathtt{G}_a, \mathtt{P}_{bc}]$	0	0
$[\mathtt{H}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{H}_m$	0
$[\mathtt{H}, \mathtt{G}_{ab}]$	0	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{H}, \mathtt{H}_a]$	0	$\pm\epsilon_{am}\mathtt{J}_m$
$[\mathtt{H}, \mathtt{P}_{ab}]$	$\mp\epsilon_{m(a}\mathtt{G}_{b)m}$	0
$[\mathtt{P}_a, \mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$
$[\mathtt{P}_a, \mathtt{G}_{bc}]$	0	0
$[\mathtt{P}_a, \mathtt{H}_b]$	$\mp(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$	$\mp(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$
$[\mathtt{P}_a, \mathtt{P}_{bc}]$	0	0
$[\mathtt{J}_a, \mathtt{J}_b]$	$\epsilon_{ab}\mathtt{J}$	0
$[\mathtt{J}_a, \mathtt{G}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\mathtt{J}_a, \mathtt{H}_b]$	$\epsilon_{ab}\mathtt{H}$	$\epsilon_{ab}\mathtt{H}$
$[\mathtt{J}_a, \mathtt{P}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$	0
$[\mathtt{G}_{ab}, \mathtt{G}_{cd}]$	0	0
$[\mathtt{G}_{ab}, \mathtt{H}_c]$	0	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}, \mathtt{P}_{cd}]$	0	0
$[\mathtt{H}_a, \mathtt{H}_b]$	0	$\pm\epsilon_{ab}\mathtt{J}$
$[\mathtt{H}_a, \mathtt{P}_{bc}]$	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\mathtt{P}_{ab}, \mathtt{P}_{cd}]$	0	0

Table D.10: Higher spin versions of the para-Galilei algebra. The upper sign is for contractions of AdS and the lower sign for contractions of dS.

Contraction #	$\mathfrak{hs}_3\mathfrak{pgal}3$ 4, 5	$\mathfrak{hs}_3\mathfrak{pgal}4$ 4, 6
$[\mathtt{J}, \mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J}, \mathtt{H}]$	0	0
$[\mathtt{J}, \mathtt{P}_a]$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a, \mathtt{G}_b]$	0	0
$[\mathtt{G}_a, \mathtt{H}]$	0	0
$[\mathtt{G}_a, \mathtt{P}_b]$	0	0
$[\mathtt{H}, \mathtt{P}_a]$	$\pm\epsilon_{am}\mathtt{G}_m$	$\pm\epsilon_{am}\mathtt{G}_m$
$[\mathtt{P}_a, \mathtt{P}_b]$	0	0
$[\mathtt{J}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{J}_m$	$\epsilon_{am}\mathtt{J}_m$
$[\mathtt{J}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{J}, \mathtt{H}_a]$	$\epsilon_{am}\mathtt{H}_m$	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{J}, \mathtt{P}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{G}_a, \mathtt{J}_b]$	0	0
$[\mathtt{G}_a, \mathtt{G}_{bc}]$	0	$-\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{G}_a, \mathtt{H}_b]$	0	0
$[\mathtt{G}_a, \mathtt{P}_{bc}]$	$-\epsilon_{a(b}\mathtt{H}_{c)}$	0
$[\mathtt{H}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{H}_m$	0
$[\mathtt{H}, \mathtt{G}_{ab}]$	0	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{H}, \mathtt{H}_a]$	0	$\pm\epsilon_{am}\mathtt{J}_m$
$[\mathtt{H}, \mathtt{P}_{ab}]$	$\mp\epsilon_{m(a}\mathtt{G}_{b)m}$	0
$[\mathtt{P}_a, \mathtt{J}_b]$	0	0
$[\mathtt{P}_a, \mathtt{G}_{bc}]$	$-\epsilon_{a(b}\mathtt{H}_{c)}$	$-\epsilon_{a(b}\mathtt{H}_{c)}$
$[\mathtt{P}_a, \mathtt{H}_b]$	0	0
$[\mathtt{P}_a, \mathtt{P}_{bc}]$	$\mp\epsilon_{a(b}\mathtt{J}_{c)}$	$\mp\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{J}_a, \mathtt{J}_b]$	0	0
$[\mathtt{J}_a, \mathtt{G}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\mathtt{J}_a, \mathtt{H}_b]$	0	0
$[\mathtt{J}_a, \mathtt{P}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$	0
$[\mathtt{G}_{ab}, \mathtt{G}_{cd}]$	0	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$
$[\mathtt{G}_{ab}, \mathtt{H}_c]$	0	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}, \mathtt{P}_{cd}]$	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$
$[\mathtt{H}_a, \mathtt{H}_b]$	0	0
$[\mathtt{H}_a, \mathtt{P}_{bc}]$	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\mathtt{P}_{ab}, \mathtt{P}_{cd}]$	$\pm\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$	0

Table D.11: Higher spin versions of the para-Galilei algebra. The upper sign is for contractions of AdS and the lower sign for contractions of dS.

Contraction #	$\mathfrak{hs}_3\mathfrak{st}1$ $1,3,5=7$	$\mathfrak{hs}_3\mathfrak{st}2$ $1,4,6=8$
$[\mathtt{J}, \mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J}, \mathtt{H}]$	0	0
$[\mathtt{J}, \mathtt{P}_a]$	0	0
$[\mathtt{G}_a, \mathtt{G}_b]$	0	0
$[\mathtt{G}_a, \mathtt{H}]$	0	0
$[\mathtt{G}_a, \mathtt{P}_b]$	0	0
$[\mathtt{H}, \mathtt{P}_a]$	0	0
$[\mathtt{P}_a, \mathtt{P}_b]$	0	0
$[\mathtt{J}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{J}_m$	$\epsilon_{am}\mathtt{J}_m$
$[\mathtt{J}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{J}, \mathtt{H}_a]$	$\epsilon_{am}\mathtt{H}_m$	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{J}, \mathtt{P}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{G}_a, \mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$	0
$[\mathtt{G}_a, \mathtt{G}_{bc}]$	0	$-\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{G}_a, \mathtt{H}_b]$	0	0
$[\mathtt{G}_a, \mathtt{P}_{bc}]$	0	0
$[\mathtt{H}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{H}_m$	0
$[\mathtt{H}, \mathtt{G}_{ab}]$	0	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{H}, \mathtt{H}_a]$	0	0
$[\mathtt{H}, \mathtt{P}_{ab}]$	0	0
$[\mathtt{P}_a, \mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$	0
$[\mathtt{P}_a, \mathtt{G}_{bc}]$	0	$-\epsilon_{a(b}\mathtt{H}_{c)}$
$[\mathtt{P}_a, \mathtt{H}_b]$	0	0
$[\mathtt{P}_a, \mathtt{P}_{bc}]$	0	0
$[\mathtt{J}_a, \mathtt{J}_b]$	$\epsilon_{ab}\mathtt{J}$	0
$[\mathtt{J}_a, \mathtt{G}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\mathtt{J}_a, \mathtt{H}_b]$	$\epsilon_{ab}\mathtt{H}$	0
$[\mathtt{J}_a, \mathtt{P}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$	0
$[\mathtt{G}_{ab}, \mathtt{G}_{cd}]$	0	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$
$[\mathtt{G}_{ab}, \mathtt{H}_c]$	0	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}, \mathtt{P}_{cd}]$	0	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$
$[\mathtt{H}_a, \mathtt{H}_b]$	0	0
$[\mathtt{H}_a, \mathtt{P}_{bc}]$	0	0
$[\mathtt{P}_{ab}, \mathtt{P}_{cd}]$	0	0

Table D.12: Higher spin versions of the static algebra. The upper sign is for contractions of AdS and the lower sign for contractions of dS.

Contraction #	$\mathfrak{hs}_3\mathfrak{st}3$ $2,3,6=9$	$\mathfrak{hs}_3\mathfrak{st}4$ $2,4,5=10$
$[\mathtt{J}\,,\,\mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J}\,,\,\mathtt{H}]$	0	0
$[\mathtt{J}\,,\,\mathtt{P}_a]$	0	0
$[\mathtt{G}_a\,,\,\mathtt{G}_b]$	0	0
$[\mathtt{G}_a\,,\,\mathtt{H}]$	0	0
$[\mathtt{G}_a\,,\,\mathtt{P}_b]$	0	0
$[\mathtt{H}\,,\,\mathtt{P}_a]$	0	0
$[\mathtt{P}_a\,,\,\mathtt{P}_b]$	0	0
$[\mathtt{J}\,,\,\mathtt{J}_a]$	$\epsilon_{am}\mathtt{J}_m$	$\epsilon_{am}\mathtt{J}_m$
$[\mathtt{J}\,,\,\mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{J}\,,\,\mathtt{H}_a]$	$\epsilon_{am}\mathtt{H}_m$	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{J}\,,\,\mathtt{P}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{G}_a\,,\,\mathtt{J}_b]$	0	0
$[\mathtt{G}_a\,,\,\mathtt{G}_{bc}]$	0	0
$[\mathtt{G}_a\,,\,\mathtt{H}_b]$	$-(\epsilon_{am}\mathtt{P}_{bm}+\epsilon_{ab}\mathtt{P}_{mm})$	0
$[\mathtt{G}_a\,,\,\mathtt{P}_{bc}]$	0	$-\epsilon_{a(b}\mathtt{H}_{c)}$
$[\mathtt{H}\,,\,\mathtt{J}_a]$	0	0
$[\mathtt{H}\,,\,\mathtt{G}_{ab}]$	0	0
$[\mathtt{H}\,,\,\mathtt{H}_a]$	$\pm\epsilon_{am}\mathtt{J}_m$	0
$[\mathtt{H}\,,\,\mathtt{P}_{ab}]$	0	$\mp\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{P}_a\,,\,\mathtt{J}_b]$	0	0
$[\mathtt{P}_a\,,\,\mathtt{G}_{bc}]$	0	0
$[\mathtt{P}_a\,,\,\mathtt{H}_b]$	$\mp(\epsilon_{am}\mathtt{G}_{bm}+\epsilon_{ab}\mathtt{G}_{mm})$	0
$[\mathtt{P}_a\,,\,\mathtt{P}_{bc}]$	0	$\mp\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{J}_a\,,\,\mathtt{J}_b]$	0	0
$[\mathtt{J}_a\,,\,\mathtt{G}_{bc}]$	0	0
$[\mathtt{J}_a\,,\,\mathtt{H}_b]$	$\epsilon_{ab}\mathtt{H}$	0
$[\mathtt{J}_a\,,\,\mathtt{P}_{bc}]$	0	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}\,,\,\mathtt{G}_{cd}]$	0	0
$[\mathtt{G}_{ab}\,,\,\mathtt{H}_c]$	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$	0
$[\mathtt{G}_{ab}\,,\,\mathtt{P}_{cd}]$	0	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$
$[\mathtt{H}_a\,,\,\mathtt{H}_b]$	$\pm\epsilon_{ab}\mathtt{J}$	0
$[\mathtt{H}_a\,,\,\mathtt{P}_{bc}]$	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\mathtt{P}_{ab}\,,\,\mathtt{P}_{cd}]$	0	$\pm\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$

Table D.13: Higher spin versions of the static algebra. The upper sign is for contractions of AdS and the lower sign for contractions of dS.

Contraction #	$\mathfrak{hs}_3\mathfrak{st}5$ 3, 1, 6	$\mathfrak{hs}_3\mathfrak{st}6$ 1, 4, 5
$[\mathtt{J}, \mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J}, \mathtt{H}]$	0	0
$[\mathtt{J}, \mathtt{P}_a]$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a, \mathtt{G}_b]$	0	0
$[\mathtt{G}_a, \mathtt{H}]$	0	0
$[\mathtt{G}_a, \mathtt{P}_b]$	0	0
$[\mathtt{H}, \mathtt{P}_a]$	0	0
$[\mathtt{P}_a, \mathtt{P}_b]$	0	0
$[\mathtt{J}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{J}_m$	$\epsilon_{am}\mathtt{J}_m$
$[\mathtt{J}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{J}, \mathtt{H}_a]$	$\epsilon_{am}\mathtt{H}_m$	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{J}, \mathtt{P}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{G}_a, \mathtt{J}_b]$	0	0
$[\mathtt{G}_a, \mathtt{G}_{bc}]$	0	0
$[\mathtt{G}_a, \mathtt{H}_b]$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$	0
$[\mathtt{G}_a, \mathtt{P}_{bc}]$	0	$-\epsilon_{a(b}\mathtt{H}_{c)}$
$[\mathtt{H}, \mathtt{J}_a]$	0	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{H}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	0
$[\mathtt{H}, \mathtt{H}_a]$	0	0
$[\mathtt{H}, \mathtt{P}_{ab}]$	0	0
$[\mathtt{P}_a, \mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$	0
$[\mathtt{P}_a, \mathtt{G}_{bc}]$	0	$-\epsilon_{a(b}\mathtt{H}_{c)}$
$[\mathtt{P}_a, \mathtt{H}_b]$	0	0
$[\mathtt{P}_a, \mathtt{P}_{bc}]$	0	0
$[\mathtt{J}_a, \mathtt{J}_b]$	0	0
$[\mathtt{J}_a, \mathtt{G}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\mathtt{J}_a, \mathtt{H}_b]$	$\epsilon_{ab}\mathtt{H}$	0
$[\mathtt{J}_a, \mathtt{P}_{bc}]$	0	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}, \mathtt{G}_{cd}]$	0	0
$[\mathtt{G}_{ab}, \mathtt{H}_c]$	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$	0
$[\mathtt{G}_{ab}, \mathtt{P}_{cd}]$	0	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$
$[\mathtt{H}_a, \mathtt{H}_b]$	0	0
$[\mathtt{H}_a, \mathtt{P}_{bc}]$	0	0
$[\mathtt{P}_{ab}, \mathtt{P}_{cd}]$	0	0

Table D.14: Higher spin versions of the static algebra which can not be directly contracted.

Contraction #	$\mathfrak{hs}_3\mathfrak{st}7$ 3, 2, 5	$\mathfrak{hs}_3\mathfrak{st}8$ 4, 2, 6
$[\mathtt{J}, \mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J}, \mathtt{H}]$	0	0
$[\mathtt{J}, \mathtt{P}_a]$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a, \mathtt{G}_b]$	0	0
$[\mathtt{G}_a, \mathtt{H}]$	0	0
$[\mathtt{G}_a, \mathtt{P}_b]$	0	0
$[\mathtt{H}, \mathtt{P}_a]$	0	0
$[\mathtt{P}_a, \mathtt{P}_b]$	0	0
$[\mathtt{J}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{J}_m$	$\epsilon_{am}\mathtt{J}_m$
$[\mathtt{J}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{J}, \mathtt{H}_a]$	$\epsilon_{am}\mathtt{H}_m$	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{J}, \mathtt{P}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{G}_a, \mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$	0
$[\mathtt{G}_a, \mathtt{G}_{bc}]$	0	$-\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{G}_a, \mathtt{H}_b]$	0	0
$[\mathtt{G}_a, \mathtt{P}_{bc}]$	0	0
$[\mathtt{H}, \mathtt{J}_a]$	0	0
$[\mathtt{H}, \mathtt{G}_{ab}]$	0	0
$[\mathtt{H}, \mathtt{H}_a]$	0	$\pm\epsilon_{am}\mathtt{J}_m$
$[\mathtt{H}, \mathtt{P}_{ab}]$	$\mp\epsilon_{m(a}\mathtt{G}_{b)m}$	0
$[\mathtt{P}_a, \mathtt{J}_b]$	0	0
$[\mathtt{P}_a, \mathtt{G}_{bc}]$	0	0
$[\mathtt{P}_a, \mathtt{H}_b]$	$\mp(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$	0
$[\mathtt{P}_a, \mathtt{P}_{bc}]$	0	$\mp\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{J}_a, \mathtt{J}_b]$	0	0
$[\mathtt{J}_a, \mathtt{G}_{bc}]$	0	0
$[\mathtt{J}_a, \mathtt{H}_b]$	$\epsilon_{ab}\mathtt{H}$	0
$[\mathtt{J}_a, \mathtt{P}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$	0
$[\mathtt{G}_{ab}, \mathtt{G}_{cd}]$	0	0
$[\mathtt{G}_{ab}, \mathtt{H}_c]$	0	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}, \mathtt{P}_{cd}]$	0	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$
$[\mathtt{H}_a, \mathtt{H}_b]$	0	0
$[\mathtt{H}_a, \mathtt{P}_{bc}]$	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\mathtt{P}_{ab}, \mathtt{P}_{cd}]$	0	0

Table D.15: Higher spin versions of the static algebra which can not be directly contracted. The upper sign is for contractions of AdS and the lower sign for contractions of dS.

Invariant Metric of $\mathfrak{hs}_3(\mathfrak{A})\mathfrak{dS}$

The most general invariant metric for both $\mathfrak{hs}_3\mathfrak{AdS}$ and $\mathfrak{hs}_3\mathfrak{dS}$, as well as their subalgebras $\mathfrak{AdS}$ and $\mathfrak{dS}$ in the notation given in (10.3) is

$$\langle \hat{\mathtt{P}}_A\,,\hat{\mathtt{J}}_B\rangle = \mu^-\eta_{AB} \qquad \langle \hat{\mathtt{P}}_{AB}\,,\hat{\mathtt{J}}_{CD}\rangle = \mu^-(\eta_{A(C}\eta_{D)B} - \tfrac{2}{3}\eta_{AB}\eta_{CD}) \tag{D.62}$$

and additionally

$$\langle \hat{\mathtt{J}}_A\,,\hat{\mathtt{J}}_B\rangle = \mu^+\eta_{AB} \qquad \langle \hat{\mathtt{J}}_{AB}\,,\hat{\mathtt{J}}_{CD}\rangle = \mu^+(\eta_{A(C}\eta_{D)B} - \tfrac{2}{3}\eta_{AB}\eta_{CD}) \tag{D.63}$$

$$\langle \hat{\mathtt{P}}_A\,,\hat{\mathtt{P}}_B\rangle = \pm\mu^+\eta_{AB} \quad \langle \hat{\mathtt{P}}_{AB}\,,\hat{\mathtt{P}}_{CD}\rangle = \pm\mu^+(\eta_{A(C}\eta_{D)B} - \tfrac{2}{3}\eta_{AB}\eta_{CD})\,. \tag{D.64}$$

where the upper sign is for the $\mathfrak{AdS}$ case. While for $\mathfrak{AdS}$ non-degeneracy requires $\mu^+ \neq \pm\mu^-$ the $\mathfrak{dS}$ case requires only that not both $\mu^\pm$ vanish,. The remaining products like, e.g., $\langle \hat{\mathtt{P}}_A\,,\hat{\mathtt{J}}_{BC}\rangle$ are vanishing.

Using the decomposition (10.5) leads to

$$\langle \mathtt{H}\,,\mathtt{J}\rangle = -\mu^- \qquad \langle \mathtt{H}_a\,,\mathtt{J}_b\rangle = -\mu^-\delta_{ab} \tag{D.65}$$

$$\langle \mathtt{P}_a\,,\mathtt{G}_b\rangle = \mu^-\delta_{ab} \qquad \langle \mathtt{P}_{ab}\,,\mathtt{G}_{cd}\rangle = \mu^-(\delta_{a(c}\delta_{d)b} - \tfrac{2}{3}\delta_{ab}\delta_{cd}) \tag{D.66}$$

$$\langle \mathtt{J}\,,\mathtt{J}\rangle = -\mu^+ \qquad \langle \mathtt{J}_a\,,\mathtt{J}_b\rangle = -\mu^+\delta_{ab} \tag{D.67}$$

$$\langle \mathtt{G}_a\,,\mathtt{G}_b\rangle = \mu^+\delta_{ab} \qquad \langle \mathtt{G}_{ab}\,,\mathtt{G}_{cd}\rangle = \mu^+(\delta_{a(c}\delta_{d)b} - \tfrac{2}{3}\delta_{ab}\delta_{cd}) \tag{D.68}$$

$$\langle \mathtt{H}\,,\mathtt{H}\rangle = \mp\mu^+ \qquad \langle \mathtt{H}_a\,,\mathtt{H}_b\rangle = \mp\mu^+\delta_{ab} \tag{D.69}$$

$$\langle \mathtt{P}_a\,,\mathtt{P}_b\rangle = \pm\mu^+\delta_{ab} \qquad \langle \mathtt{P}_{ab}\,,\mathtt{P}_{cd}\rangle = \pm\mu^+(\delta_{a(c}\delta_{d)b} - \tfrac{2}{3}\delta_{ab}\delta_{cd}) \tag{D.70}$$

Again, only nonzero elements are displayed.

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