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Refined Doob Inequalities for σ-Integrable Submartingales and Intertemporal Risk Constraints

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Kurzfassung

Ziel dieser Diplomarbeit ist es, die Martingaltheorie für σ-endliche Maßräume und σ-integrierbare Funktionen zu erweitern und bekannte Doob'sche Ungleichungen zu verallgemeinern und zu verbessern. Zu Beginn wird eine schwächere Form der Integrierbarkeit, die σ-Integrierbarkeit, vorgestellt, um die Existenz eines bedingten Erwartungswerts einer Funktion gegeben einem σ-endlichen Maß sowie dessen Eigenschaften zu beweisen. Infolgedessen wird der Begriff der σ-integrierbaren (Sub-/Super-)Martingale, welcher dieser Arbeit ihren Titel gibt, eingeführt.

Das Herzstück der Arbeit behandelt Erweiterungen und Verbesserungen der Doob'schen Maximalungleichungen sowie der Doob'schen L^p-Ungleichungen für σ-integrierbare Submartingale auf σ-endlichen Maßräumen. Die zugehörigen Beweise werden dabei mithilfe rein deterministischer Ungleichungen geführt. Es wird weiters versucht, sich von der Notwendigkeit der Adaptiertheit sowie jener eines Start- und Endpunktes der betrachteten Periode zu befreien. Anschließend wird diskutiert, unter welchen Gegebenheiten die unterschiedlichen Ungleichungen zu Gleichheiten werden können. Beispiele dienen hierbei der weiteren Veranschaulichung.

Das letzte Kapitel gibt Aufschluss darüber, wie die verbesserten und erweiterten Doob'schen Ungleichungen PraktikerInnen bei ihrer Arbeit in der Finanz- und Versicherungsmathematik unterstützen können. Beispielsweise ergeben sich durch die Ungleichungen minimale obere Schranken für den Erwartungswert des essenziellen Supremums des diskontierten Preisprozesses, welcher bekannterweise unter einem risikoneutralen Maß zum Martingal wird. Andererseits können die verbesserten Doob'schen Ungleichungen eingesetzt werden, um den Verlust einer Versicherungspolizze abzuschätzen. Die Besonderheit in beiden Fällen ist, dass das maximale Risiko zu jedem Zeitpunkt innerhalb einer beobachteten Periode abgeschätzt werden kann. Somit bieten die Erkenntnissse dieser Arbeit Möglichkeiten Risiken intertemporär zu kontrollieren.

Abstract

The main goal of this thesis is to expand the theory of martingales to σ-finite measure spaces and σ-integrable functions. First, we introduce a weakened form of integrability, the σ-integrability, in order to show the existence of conditional expectations of functions w.r.t. σ-finite measures and properties thereof. Furthermore, we introduce the eponymous term of this thesis, σ-integrable (sub-/super-)martingales.

The core of this thesis consists of various generalisations and improvements of Doob's maximum and L^p-inequalities for σ-integrable submartingales on σ-finite measure spaces. For the proofs we rely on purely deterministic inequalities. Furthermore, we free ourselves from the need for adaptedness and the need for a period's starting and endpoint. Last but not least, we discuss under what circumstances our improved inequalities hold with equality and give examples thereof.

The final chapter gives an outlook on how our improved versions of Doob's L^p-inequalities can help practitioners in the fields of financial and actuarial mathematics. For example, the findings of this thesis enable practitioners to determine upper bounds for the expectation of the essential supremum of the discounted price process (which is a martingale given a risk neutral measure). On the other hand, the findings of this thesis provide upper bounds for the expected essential supremum of the loss random variable. This enables practitioners to make informed statements concerning the expected loss of a insurance contract. In particular, in both cases practitioners can estimate the maximal risk at any time within a certain period, which may assist them in intertemporal risk control.

Acknowledgements

When I started studying Financial and Actuarial Mathematics at Vienna University of Technology I had no concrete idea of what I awaited me. I had already been studying Translation Studies for a year but wanted to broaden my horizon further by picking up another degree. I had always been quite fond of mathematics, so I decided to enroll for Financial and Actuarial Mathematics without knowing what a fascinating and complex world lay ahead. I soon realised that the mathematics taught at school had little to do with what we were to learn in the upcoming years. Sometimes I was close to quitting due to the stress two simultaneous degrees inflicted on me. However, my passion for mathematics, my love for great challenges, my reluctance to give up and the incredible group of friends I was fortunate enough to make kept me going until the end.

So here I am, six years later with many fond memories of my fantastic time at Vienna University of Technology. The time of me being involved in the theoretical world of financial and actuarial mathematics is coming to an end. Therefore, I decided to express my passion for mathematics (maybe) for the last time by expanding the theory of martingales to σ-integrable functions and σ-finite measure spaces. I hope to have made a small but significant contribution to the world of mathematics.

There are various people without whom all of this would not have been possible and I would like to take a moment to thank them properly. First of all, I thank my supervisor, Univ.-Prof. Dipl.-Math. Dr. rer. nat. habil. Uwe Schmock. Thank you very much for your guidance and for taking so much time out of your busy schedule to support me however you could!

Furthermore, I thank all my university colleagues from both Vienna University and Vienna University of Technology. You turned my time as a university student into a great experience and always supported and helped me, even when I was close to losing my head over dealing with two degrees simultaneously. I am incredibly grateful to have you in my life and I hope our bond will last us for the rest of our lives!

Finally, I am incredibly grateful and hugely indebted to my loving family! I thank my parents for supporting me emotionally and financially, which made it possible for me to turn my passions into a career. Thank you for always encouraging me and teaching me that I can achieve anything I dream of, if I work hard and diligently! I also owe thanks to my aunt, Dipl. Ing. Margarete Popp, who introduced me to financial and actuarial mathematics and who has always been a role model for me in many ways. The greatest thanks, however, goes to my mother who has always encouraged and supported me and has always devoted herself to caring for my sister and me, which has made us into the independent and intelligent young women we are today.

Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Diplomarbeit selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

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Chapter 1

Introduction

The theory of martingales was first introduced by Paul Lévy[1] in 1925. Around 30 years after its initial introduction, Joseph L. Doob[2] greatly contributed to its expansion to the theory of stochastic processes. Today martingales play a vital role in financial and actuarial mathematics and cannot be overlooked in education either. However, the theory of martingales is often treated solely on probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$, where *martingales* are commonly defined as integrable processes $M := (M_t)_{t\in T}$ that are adapted to a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in T}$, i.e. M_t is $\mathcal{F}_t$-measurable for all t in $T \subset \overline{\mathbb{R}}$, with values in $\mathbb{K}^d$ such that

$$M_s \overset{\text{a.s.}}{=} \mathbb{E}[M_t|\mathcal{F}_s] \tag{1.1}$$

for all $s \leq t$ in T. M is called a *submartingale*, if $\mathbb{K}^d = \mathbb{R}$ and

$$M_s \overset{\text{a.s.}}{\leq} \mathbb{E}[M_t|\mathcal{F}_s] \tag{1.2}$$

for all $s \leq t$ in T and *supermartingale*, if (1.2) is reversed[3]. The goal of this thesis is to expand the theory of martingales by adapting it to σ-finite measure spaces. René Schilling laid out the groundwork thereof in 2005 but focused on discrete time, whereas this thesis considers continuous time.

Chapter 2 first introduces the definition of σ-integrable functions. We use this generalisation of integrability to expand the theory of the conditional expectation to σ-finite measure spaces. We also show that many well-known properties of the conditional expectation w.r.t. probability measures still hold when considering σ-finite measures. Lastly, we define (sub-/super-)martingales on σ-finite measure spaces via the generalised version of

[1]* 15 September 1886 in Paris, † 15 December 1971 in Paris
[2]* 27 February 1910 in Cincinnati, Ohio, † 7 June 2004 in Urbana, Illinois
[3]See [13, Definition 4.1 and Definition 4.49]

the conditional expectation and introduce the eponymous term for this thesis, *σ-integrable (sub-/super-)martingales*, before proving that finite optional stopping also holds for our definition of submartingales.

Chapter 3 proves the central theorem of this thesis: Doob's classical L^p-inequality for submartingales for $p > 1$ on σ-finite measure spaces (see Theorem 3.2). An important tool for the proof are Doob's maximum inequalities, which also hold true on our generalised setting, as we show in Theorem 3.1. Moreover, Chapter 4 proves that Doob's maximum inequalities can be generalised and improved even further by relying on quite simple deterministic inequalities.

The core of this thesis is Chapter 5, which proves various improvements to Doob's classical L^p-inequality for σ-integrable submartingales on σ-finite measure spaces. Like in the previous chapter the proofs rely on rather basic deterministic inequalities. Throughout the chapter we try to free ourselves from the need of a starting and endpoint of a period within our time span T and prove sharper versions of Doob's inequalities for $p > 1$ (see Theorem 5.2), $p = 1$ (see Theorem 5.8) and $p \in (0, 1)$ (see Theorem 5.14) on our generalised setting. Last but not least, we will discuss under what circumstances our improved inequalities hold with equality and give examples thereof.

The final chapter gives an outlook on how our improved versions of Doob's L^p-inequalities can help practitioners in the fields of financial and actuarial mathematics. For example, our findings provide upper bounds for the expectation of the essential supremum of the discounted price process, which is a martingale given a risk neutral measure. Furthermore, the newly developed inequalities in this thesis can be used to find upper bounds for the expectation of the essential supremum of the loss random variable. Hence, they may help practitioners to make informed statements concerning the expected loss of a insurance contract. Finally, we summarise our findings and the novelties of our work to conclude this thesis.

Chapter 2

Martingales in σ-Finite Measure Spaces

2.1 The Conditional Expectation in σ-Finite Measure Spaces

This chapter introduces a generalised definition of the conditional expectation on σ-finite measure spaces based on the concept of σ-integrable functions. We will prove that the fundamental properties for the conditional expectation for probability spaces and random variables can be adapted to this setting as well. This newly developed theory will be essential in order to prove Doob's L^p-inequality for submartingales.

For the entirety of this thesis (unless stated otherwise) let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. In general, within this thesis we will mainly look at measurable functions that take values in $\mathbb{K}^d$, $d \in \mathbb{N}$. Therefore – unless indicated otherwise – let $f \in L^0(\Omega, \mathcal{F}, \mu; \mathbb{K}^d)$ take values in either $\mathbb{R}^d$ or $\mathbb{C}^d$. For $d > 1$ apply the newly developed theory to each $\mathbb{K}$-valued function f_i, $i = 1, \dots, d$ such that $f = (f_1, \dots, f_d)$.

For comparability reasons we wish to introduce a similar notation to the expected value of random variables as the integral w.r.t. the σ-finite measure μ. First, let f take values in $\mathbb{R}$ and define $\int f\, d\mu = \int f^+\, d\mu - \int f^-\, d\mu$ for $f \in L^0(\Omega, \mathcal{F}, \mu; \mathbb{R})$ with $f = f^+ - f^-$ such that $\min\{\int f^+\, d\mu, \int f^-\, d\mu\} < \infty$.

Notation: $\mathbb{E}_\mu[f] := \int_\Omega f\, d\mu$

If f takes values in $\mathbb{C}$, we will use the same notation and apply the definition above to the real and imaginary part of f. In the case of $\mathbb{K}^d$ consider the notation applied componentwise.

Later on we shall use a similar notation when it comes to conditional expectations. We will start off with introducing the definition of σ-integrable functions, which is a key aspect for this thesis.

2.1.1 σ-Integrable Functions

The following definition and the newly developed theory thereafter are inspired by the work of [7], who introduced a generalised definition for the conditional expectation w.r.t. probability measures and random variables. Functions that satisfy the properties below will play a key role in proving that the theory of the conditional expectation can be expanded to functions on σ-finite measure spaces.

Definition 2.1. Let $f \in L^0(\Omega, \mathcal{F}, \mu; \mathbb{K}^d)$ and $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Then f is called *σ-integrable* w.r.t. $\mathcal{G}$, if there exists a sequence $(\Omega_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ with $\Omega_n \nearrow \Omega$ as $n \to \infty$ such that

$$f\mathbb{1}_{\Omega_n} \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{K}^d), \quad n \in \mathbb{N}.$$

Before we go any further we would like to give some examples for σ-integrable functions. For this purpose we will specifically look at functions that are σ-integrable, but not integrable, because it follows immediately that integrable functions are also σ-integrable. Furthermore, we would like to give a quick overview of some properties of σ-integrable functions, which will help us later on.

Example 2.2. Consider the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ together with the finite measure $\mu(\{n\}) := n^{-2}$. Then the identity function $f : n \mapsto n$ for $n \in \mathbb{N}$ is not integrable, because $\int f\, d\mu = \sum_{n=1}^{\infty} \frac{1}{n^2} n = \infty$. However, f is σ-integrable w.r.t. $\mathcal{P}(\mathbb{N})$, because we have with $\Omega_n := \{1, \ldots, n\}$, $n \in \mathbb{N}$, a sequence as required in the definition above: simply note that $\Omega_n \nearrow \mathbb{N}$ as $n \to \infty$, $\Omega_n \in \mathcal{P}(\mathbb{N})$ and $\int f\mathbb{1}_{\Omega_n}\, d\mu = \sum_{i=1}^{n} \frac{1}{i^2} i < \infty$ for all $n \in \mathbb{N}$.

Example 2.3. Consider the measure space $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \lambda)$, where $\mathcal{B}_\mathbb{R}$ refers to the σ-algebra of one-dimensional Borel sets on the topological space $\mathbb{R}$ and λ to the Lebesgue–Borel measure. The function $f : \mathbb{R} \to \mathbb{R}$, where

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{x} & \text{otherwise,} \end{cases}$$

is not λ-integrable on $\mathbb{R}$, because the integral $\int_\mathbb{R} |f|\, d\lambda$ is infinite for two reasons: the singularity at 0 and the slow decay of $|f(x)|$ as $x \to \pm\infty$. However, f is σ-integrable w.r.t. $\mathcal{B}_\mathbb{R}$, since we have with $\Omega_n := (-n, -\frac{1}{n}] \cup$

$\{0\} \cup \left[\frac{1}{n}, n\right)$, for $n \in \mathbb{N}$, a sequence as required in Definition 2.1: simply note that $|f|$ is an even[1] function and that $\Omega_n \nearrow \mathbb{R}$ as $n \to \infty$, $\Omega_n \in \mathcal{B}_{\mathbb{R}}$ and

$$\int_{\mathbb{R}} |f| \mathbb{1}_{\Omega_n} \, d\lambda = \int_{\{0\}} |f| \, d\lambda + 2 \int_{[\frac{1}{n}, n)} |f| \, d\lambda = 2 \int_{\frac{1}{n}}^{n} \frac{1}{x} \, dx = 4 \ln(n) < \infty$$

for all $n \in \mathbb{N}$.

Lemma 2.4. *Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$ and let all functions below take values in $\mathbb{K}^d$ unless stated otherwise.*

(1) *Let $a \in \mathbb{K}$ and f, g be σ-integrable functions w.r.t. $\mathcal{G}$. Then $af + g$ is again a $\mathbb{K}^d$-valued σ-integrable function w.r.t. $\mathcal{G}$.*

(2) *If $\mu|_{\mathcal{G}}$ is σ-finite and f $\mathcal{G}$-measurable, then f is σ-integrable w.r.t. $\mathcal{G}$.*

(3) *Let f be σ-integrable w.r.t. $\mathcal{G}$. If g is a $\mathcal{G}$-measurable function, then gf is σ-integrable w.r.t. $\mathcal{G}$.*

(4) *f is σ-integrable w.r.t. $\mathcal{G}$ if, and only if, $|f|$ is σ-integrable w.r.t. $\mathcal{G}$.*

(5) *Let f be an $\mathcal{F}$-measurable function. If $|f| \le g$ μ-a.e., where g is an $\mathbb{R}$-valued σ-integrable function w.r.t. $\mathcal{G}$, then f is also σ-integrable w.r.t. $\mathcal{G}$. Thus, if $\mu|_{\mathcal{G}}$ is σ-finite and f is bounded, we can conclude that f is also σ-integrable w.r.t. $\mathcal{G}$.*

(6) *If f, g are $\mathbb{R}$-valued σ-integrable functions w.r.t. $\mathcal{G}$, then so are $f \vee g$ and $f \wedge g$.*

(7) *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of σ-integrable functions w.r.t. $\mathcal{G}$ such that $f_n \ge g$ μ-a.e. for all $n \in \mathbb{N}$ and a σ-integrable function g w.r.t. $\mathcal{G}$. If $\mu|_{\mathcal{G}}$ is σ-finite, the infimum $\inf_{n\in\mathbb{N}} f_n$ is also σ-integrable w.r.t. $\mathcal{G}$.*

Proof. (1) Since f and g are σ-integrable w.r.t. $\mathcal{G}$, there exist sequences $(\Phi_n)_{n\in\mathbb{N}}, (\Psi_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ with $\Phi_n, \Psi_n \nearrow \Omega$ as $n \to \infty$ such that $f\mathbb{1}_{\Phi_n}$ and $g\mathbb{1}_{\Psi_n}$ are μ-integrable for all $n \in \mathbb{N}$. Define $\Omega_n = \Phi_n \cap \Psi_n$. Then $(\Omega_n)_{n\in\mathbb{N}}$ is again a sequence in $\mathcal{G}$ with $\Omega_n \nearrow \Omega$ as $n \to \infty$. Furthermore, $(af + g)\mathbb{1}_{\Omega_n}$ is μ-integrable and therefore a σ-integrable function w.r.t. $\mathcal{G}$.

(2) Since $\mu|_{\mathcal{G}}$ is σ-finite, there exists a sequence $(\Omega_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ with $\mu(\Omega_n) < \infty$ for all $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$. By setting $\overline{\Omega}_n := \Omega_n \cap \{|f| \le n\}$ for $n \in \mathbb{N}$ we have found a sequence with $\overline{\Omega}_n \in \mathcal{G}$ for all $n \in \mathbb{N}$ (due to the $\mathcal{G}$-measurability of f) and $\overline{\Omega}_n \nearrow \Omega$ as $n \to \infty$. Since

$$\mathbb{E}_\mu[|f\mathbb{1}_{\overline{\Omega}_n}|] \le n\mu(\Omega_n) < \infty,$$

[1] i.e. $g := |f|$ has the property that $g(x) = g(-x)$

it follows that $f\mathbb{1}_{\overline{\Omega}_n} \in L^1(\Omega, \mathcal{F}, \mu)$ for all $n \in \mathbb{N}$. Therefore, f is σ-integrable w.r.t. $\mathcal{G}$.

(3) Let $(\Omega_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ be the corresponding sequence such that f is σ-integrable w.r.t. $\mathcal{G}$ and define $\overline{\Omega}_n := \Omega_n \cap \{|g| \leq n\}$ for $n \in \mathbb{N}$. Then it follows that $\overline{\Omega}_n \in \mathcal{G}$ for all $n \in \mathbb{N}$ (due to the $\mathcal{G}$-measurability of g), $\overline{\Omega}_n \nearrow \Omega$ as $n \to \infty$ and $gf\mathbb{1}_{\overline{\Omega}_n} \in L^1(\Omega, \mathcal{F}, \mu)$ for all $n \in \mathbb{N}$, because

$$\mathbb{E}_\mu[|gf|\mathbb{1}_{\overline{\Omega}_n}] \leq n\mathbb{E}_\mu[|f|\mathbb{1}_{\Omega_n}] < \infty.$$

Thus, gf is σ-integrable w.r.t. $\mathcal{G}$.

(4) Let Ω_n be the sequence as required in Definition 2.1 such that f is σ-integrable w.r.t. $\mathcal{G}$. Then $f\mathbb{1}_{\Omega_n} \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{K}^d)$ if, and only if, $|f\mathbb{1}_{\Omega_n}| \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{K}^d)$ by Lemma A.13. Therefore, $|f|$ is also σ-integrable w.r.t. $\mathcal{G}$.

(5) Let Ω_n be the sequence as required in Definition 2.1 such that g is σ-integrable w.r.t. $\mathcal{G}$. Then the claim follows immediately, because $\mathbb{E}_\mu[|f|\mathbb{1}_{\Omega_n}] \leq \mathbb{E}_\mu[|g|\mathbb{1}_{\Omega_n}] < \infty$. For $|f| \leq c \in \mathbb{R}$ the σ-integrability of f w.r.t. $\mathcal{G}$ follows immediately by (2).

(6) Note that $f \vee g$ and $f \wedge g$ are again measurable functions by Theorem A.5. Since $|f \vee g| \leq |f| + |g|$ and $|f \wedge g| \leq |f| + |g|$, both functions are σ-integrable w.r.t. $\mathcal{G}$ by (1), (4) and (5).

(7) Theorem A.5 shows that $\inf_{n\in\mathbb{N}} f_n$ is again a measurable function. Thus, the claim follows by (2) and (5), because $g \leq \inf_{n\in\mathbb{N}} f_n \leq f_n$ for all $n \in \mathbb{N}$. □

2.1.2 Generalisation of the Conditional Expectation

Using σ-integrability, which can be viewed as a weaker form of integrability, we wish to introduce a generalised version of the conditional expectation. For this purpose we will need the following lemma for multiple claims and proofs in the course of this thesis.

Lemma 2.5. *Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$ and $f, g \in L^0(\Omega, \mathcal{G}, \mu; \mathbb{R})$.*

(1) *If there exists a sequence $(\Omega_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ with $\Omega_n \nearrow \Omega$ as $n \to \infty$ such that $f\mathbb{1}_{\Omega_n}, g\mathbb{1}_{\Omega_n} \in L^1(\Omega, \mathcal{G}, \mu)$ and*

$$\mathbb{E}_\mu[(f\mathbb{1}_{\Omega_n})\mathbb{1}_G] \leq \mathbb{E}_\mu[(g\mathbb{1}_{\Omega_n})\mathbb{1}_G], \quad G \in \mathcal{G},\ n \in \mathbb{N} \tag{2.1}$$

then $f \leq g$ μ-a.e.

(2) *If f and g take values in $\mathbb{K}^d$ and (2.1) holds with equality, then $f = g$ μ-a.e.*

Proof. (1) Consider $A_n := \{f - g > \frac{1}{n}\}$ for $n \in \mathbb{N}$. Then $A_n \in \mathcal{G}$ for all $n \in \mathbb{N}$ due to the $\mathcal{G}$-measurability of f and g. Hence,

$$0 \geq \mathbb{E}_\mu[(g\mathbb{1}_{\Omega_n})\mathbb{1}_{A_n}] \geq \mathbb{E}_\mu\left[\frac{1}{n}\mathbb{1}_{\Omega_n}\mathbb{1}_{A_n}\right] = \frac{1}{n}\mu(\Omega_n \cap A_n),$$

which implies $\mu(\Omega_n \cap A_n) = 0$ for all $n \in \mathbb{N}$. Note that $\{f - g > 0\} = \bigcup_{n\in\mathbb{N}}(A_n \cap \Omega_n)$. Hence,

$$\mu(\{f - g > 0\}) = \mu\Big(\bigcup_{n\in\mathbb{N}}(A_n \cap \Omega_n)\Big) \leq \sum_{n\in\mathbb{N}} \mu(A_n \cap \Omega_n) = 0$$

by the σ-subadditivity of measures. Thus, $f \leq g$ μ-a.e.

(2) We can now derive the claim for $\mathbb{R}$-valued functions directly from what we have just shown, by considering $\mathbb{E}_\mu[(f\mathbb{1}_{\Omega_n})\mathbb{1}_G] \leq \mathbb{E}_\mu[(g\mathbb{1}_{\Omega_n})\mathbb{1}_G]$ and $\mathbb{E}_\mu[(f\mathbb{1}_{\Omega_n})\mathbb{1}_G] \geq \mathbb{E}_\mu[(g\mathbb{1}_{\Omega_n})\mathbb{1}_G]$ for all $G \in \mathcal{G}$ and all $n \in \mathbb{N}$. If f and g take values in $\mathbb{C}$, simply apply the claim separately to the real and imaginary part of the functions. The claim for the general case of $\mathbb{K}^d$-valued functions now follows directly by considering the components separately. This concludes the proof. □

With Definition 2.1 and Definition 2.6 below we can now introduce a generalised version of the conditional expectation w.r.t. a sub-σ-algebra of $\mathcal{F}$. This concept is integral to the general topic of this thesis and is based on the works of [7, § 1.4, p. 10–13] who introduced a similar generalisation of the conditional expectation regarding probability spaces and random variables.

Definition 2.6. Let f be σ-integrable w.r.t. a sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$, where $\mu|_\mathcal{G}$ is σ-finite. We define the delta-ring of all sets in $\mathcal{G}$ such that f is μ-integrable by[2]

$$\mathcal{R}_{f,\mathcal{G}} = \{G \in \mathcal{G} : \mathbb{E}_\mu[|f|\mathbb{1}_G] < \infty\}. \tag{2.2}$$

Theorem 2.7. Existence of conditional expectation
Let f be σ-integrable w.r.t. a sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$, where $\mu|_\mathcal{G}$ is σ-finite. Then there exists a μ-a.e. uniquely determined $g \in L^0(\Omega, \mathcal{G}, \mu; \mathbb{K}^d)$ such that $g\mathbb{1}_G \in L^1(\Omega, \mathcal{G}, \mu; \mathbb{K}^d)$ and

$$\mathbb{E}_\mu[f\mathbb{1}_G] = \mathbb{E}_\mu[g\mathbb{1}_G], \quad G \in \mathcal{R}_{f,\mathcal{G}}. \tag{2.3}$$

We call $\mathbb{E}_\mu[f|\mathcal{G}] := g$ the conditional expectation of f w.r.t. $\mathcal{G}$.

[2] Note that $\mathcal{R}_{f,\mathcal{G}}$ is a δ-*ring* for all $\mathcal{G} \subset \mathcal{F}$. See Definition A.6 for more information.

Remark 2.8. Let f be σ-integrable w.r.t. the sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$ and define $\mathcal{R}_{f,\mathcal{G}}$ like in the definition above. Then $\mathcal{R}_{f,\mathcal{G}} \subset \mathcal{R}_{\mathbb{E}_\mu[f|\mathcal{G}],\mathcal{G}}$ for the conditional expectation of f w.r.t. $\mathcal{G}$.

Indeed: This follows immediately from the definition of the conditional expectation in Theorem 2.7. It is important to note, however, that $\mathcal{R}_{f,\mathcal{G}}$ can be a strict subset of $\mathcal{R}_{\mathbb{E}_\mu[f|\mathcal{G}],\mathcal{G}}$, as the following example shows.

Example 2.9. Consider the setting of Example 2.3 and define $\mathcal{B}_\mathbb{R}^{\text{sym}} = \{B \in \mathcal{B}_\mathbb{R} : B = \{-x : x \in B\}\}$. Since f is an odd function[3], $\mathbb{E}_\mu[f \mathbb{1}_B] = 0$ for all $B \in \mathcal{B}_\mathbb{R}^{\text{sym}}$. This, however, is also true for the function $h \equiv 0$. By the a.e.-uniqueness of the conditional expectation and (2.3) this implies $\mathbb{E}_\mu[f|\mathcal{B}_\mathbb{R}^{\text{sym}}] = 0$ λ-a.e. and therefore, $\mathcal{R}_{\mathbb{E}_\mu[f|\mathcal{B}_\mathbb{R}^{\text{sym}}],\mathcal{B}_\mathbb{R}^{\text{sym}}} = \mathcal{B}_\mathbb{R}^{\text{sym}}$. However, $B_k := (-k, k) \in \mathcal{B}_\mathbb{R}^{\text{sym}}$ is not an element of $\mathcal{R}_{f,\mathcal{B}_\mathbb{R}^{\text{sym}}}$ for all $k \in \mathbb{N}$, since $\int |f| \mathbb{1}_{B_k}\, d\lambda = \infty$.

Proof. Theorem 2.7
First, let f be an $\mathbb{R}_+$-valued σ-integrable function w.r.t. $\mathcal{G}$ and $(\Omega_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ the corresponding sequence with $\mu(\Omega_n) < \infty$ for all $n \in \mathbb{N}$ and $\Omega_n \nearrow \Omega$ as $n \to \infty$ such that $\int_{\Omega_n} |f|\, d\mu < \infty$ for all $n \in \mathbb{N}$.

First, define

$$\nu(G) = \int_G f\, d\mu, \quad G \in \mathcal{G}.$$

Since $f \geq 0$ and by the monotone convergence theorem in Theorem A.18 it follows that ν is a σ-finite measure on $\mathcal{G}$ (and even finite on $\mathcal{R}_{f,\mathcal{G}}$). Furthermore, $\nu \ll \mu$ on $\mathcal{G}$ due to the definition above. According to the Radon–Nikodým theorem in Theorem A.14 there exists a μ-a.e. uniquely determined $\mathcal{G}$-measurable function $g : \Omega \to \mathbb{R}_+$ such that

$$\nu(G) = \int_G g\, d\mu, \quad G \in \mathcal{G}. \tag{2.4}$$

Now, let $G \in \mathcal{R}_{f,\mathcal{G}}$. It follows that $g\mathbb{1}_G \in L^1(\Omega, \mathcal{G}, \mu)$ since

$$\int_G g\, d\mu = \int_G f\, d\mu < \infty,$$

and thus, for $\mathbb{E}_\mu[f|\mathcal{G}] := g$ we have (2.3) for this case.

To treat the $\mathbb{R}$-valued case, consider $f = f^+ - f^-$. Since f is σ-integrable, then so are its negative and positive part, as Lemma 2.4(4) shows. We have just proved, that for f^+ and f^- Theorem 2.7 gives us the existence of two μ-a.e. uniquely determined $\mathcal{G}$-measurable functions g_+ and g_- with $g_\pm : \Omega \to \mathbb{R}_+$ such that $g_\pm \mathbb{1}_G \in L^1(\Omega, \mathcal{G}, \mu)$ and $\mathbb{E}_\mu[f^\pm \mathbb{1}_G] = \mathbb{E}_\mu[g_\pm \mathbb{1}_G]$ for all

[3] i.e. $-f(x) = f(-x)$.

$G \in \mathcal{R}_{f,\mathcal{G}}$. Due to the additivity of the integral in Theorem A.12(2) it follows that for $g := g_+ - g_-$ we have $g\mathbb{1}_G \in L^1(\Omega, \mathcal{G}, \mu)$ and $\mathbb{E}_\mu[f\mathbb{1}_G] = \mathbb{E}_\mu[g\mathbb{1}_G]$ for all $G \in \mathcal{R}_{f,\mathcal{G}}$.

Finally, it remains to show that g is a.e. unique. In order to do so, let h have the same properties as g. Then $\mathbb{E}_\mu[h\mathbb{1}_G] = \mathbb{E}_\mu[g\mathbb{1}_G]$ for all $G \in \mathcal{R}_{f,\mathcal{G}}$ by (2.3). This implies $g = h$ μ-a.e. by Lemma 2.5(2).

If f takes values in $\mathbb{C}$, the claim still holds, because we can apply what we have just proved to both the real and the imaginary part of f. The $\mathbb{K}^d$-valued case follows by considering the components separately. □

Remark 2.10. The assumption in Theorem 2.7 that $\mu|_\mathcal{G}$ is σ-finite is essential, because otherwise we could not rely on the Radon–Nikodým theorem in (2.4). For example, consider the measure space $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \lambda)$, where λ denotes the Borel–Lebesgue measure and $\mathcal{G} := \{\emptyset, \mathbb{R}\}$, which is a sub-$\sigma$-algebra of $\mathcal{B}_\mathbb{R}$. Then for $f \equiv c \in \mathbb{R}$ the integral $\int_\mathbb{R} f\, d\lambda$ is infinite and there does not exist a uniquely determined function $g : \mathbb{R} \to \mathbb{R}_+$ such that $\int_\mathbb{R} f\, d\lambda = \int_\mathbb{R} g\, d\lambda$.

There is, in fact, another way to prove Theorem 2.7 without the assumption that $\mu|_\mathcal{G}$ is σ-finite, which relies on the Bayes' formula (see Theorem A.31). However, we decided to base our theory on the proof above with all the necessary assumptions as it is more straightforward and similar to the proof of the existence of the conditional expectation for random variables and probability spaces. If interested in the alternative approach to proving (2.3), please refer to page 92.

Remark 2.11. *Generalisation of the conditional expectation.* Let f be an $\mathbb{R}$-valued $\mathcal{F}$-measurable function with decomposition $f = f^+ - f^-$, where $f^\pm := \max\{0, \pm f\}$, and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra such that $\mu|_\mathcal{G}$ is σ-finite.

(1) *For functions with σ-integrable negative part*: Assume that f^- is σ-integrable w.r.t. $\mathcal{G}$, but f^+ might not be. Define $f_n = \min\{n, f\}$ and note that f_n is bounded from above and therefore σ-integrable w.r.t. $\mathcal{G}$ for all $n \in \mathbb{N}$ (see Lemma 2.4(5)). Then, by Theorem 2.7, the conditional expectation $\mathbb{E}_\mu[f_n|\mathcal{G}]$ exists. As we will show in the next theorem, it follows that $\mathbb{E}_\mu[f_n|\mathcal{G}] \leq \mathbb{E}_\mu[f_{n+1}|\mathcal{G}]$ μ-a.e. for all $n \in \mathbb{N}$ due to the monotonicity of the conditional expectation (see Theorem 2.12(3) and its proof). Hence, we may conclude that

$$\mathbb{E}_\mu[f|\mathcal{G}] := \lim_{n\to\infty} \mathbb{E}_\mu[f_n|\mathcal{G}]$$

is μ-a.e. pointwise well-defined with values in $\mathbb{R} \cup \{+\infty\}$.

(2) *For functions with σ-integrable positive part*: Assume that f^+ is σ-integrable w.r.t. $\mathcal{G}$, but f^- might not be. Define $f_n = \max\{-n, f\}$

and note that f_n is bounded from below and therefore σ-integrable w.r.t. $\mathcal{G}$ for all $n \in \mathbb{N}$ (see Lemma 2.4(5)). Then, by Theorem 2.7, the conditional expectation $\mathbb{E}_\mu[f_n|\mathcal{G}]$ exists. As we will show in the next theorem, it follows that $\mathbb{E}_\mu[f_n|\mathcal{G}] \geq \mathbb{E}_\mu[f_{n+1}|\mathcal{G}]$ μ-a.e. for all $n \in \mathbb{N}$ due to the monotonicity of the conditional expectation (see Theorem 2.12(3) and its proof). Hence, we may conclude that

$$\mathbb{E}_\mu[f|\mathcal{G}] := \lim_{n\to\infty} \mathbb{E}_\mu[f_n|\mathcal{G}]$$

is μ-a.e. pointwise well-defined with values in $\mathbb{R} \cup \{-\infty\}$.

We will use this generalisation of the conditional expectation w.r.t. a sub-σ-algebra of $\mathcal{F}$ in situations where we cannot ensure the σ-integrability of the function. Examples of its use can be found in Theorem 2.12(5) and (8).

The main goal of Chapter 4 and Chapter 5 is to prove generalised and improved versions of Doob's maximum and L^p-inequalities for our newly developed understanding of the conditional expectation and σ-integrable submartingales, which we will introduce shortly. In order to do so, we need some of the fundamental properties listed in the theorem below.

Theorem 2.12. LIST OF PROPERTIES
Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$ such that $\mu|_\mathcal{G}$ is σ-finite. Unless stated otherwise, let f be a $\mathbb{K}^d$-valued σ-integrable function w.r.t. $\mathcal{G}$. Then the following properties hold true:

(1) *If $f \in L^0(\Omega, \mathcal{G}, \mu; \mathbb{K}^d)$, then $\mathbb{E}_\mu[f|\mathcal{G}] = f$ μ-a.e.*

(2) Law of total expectation: *If f is integrable, then $\mathbb{E}_\mu[\mathbb{E}_\mu[f|\mathcal{G}]] = \mathbb{E}_\mu[f]$.*

(3) Monotonicity: *Let $\mathbb{K}^d = \mathbb{R}$ and g be another $\mathbb{R}$-valued σ-integrable function w.r.t. $\mathcal{G}$. If $f \leq g$ μ-a.e., then $\mathbb{E}_\mu[f|\mathcal{G}] \leq \mathbb{E}_\mu[g|\mathcal{G}]$ μ-a.e.*

(4) Linearity: *Let $a \in \mathbb{K}$ and g be another $\mathbb{K}^d$-valued σ-integrable function w.r.t. $\mathcal{G}$. Then $af + g$ is again a $\mathbb{K}^d$-valued σ-integrable function w.r.t. $\mathcal{G}$ and*

$$\mathbb{E}_\mu[af + g|\mathcal{G}] = a\,\mathbb{E}_\mu[f|\mathcal{G}] + \mathbb{E}_\mu[g|\mathcal{G}] \quad \mu\text{-a.e.}$$

(5) Conditional monotone convergence theorem: *Let $\mathbb{K}^d = \mathbb{R}$ and $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence of $\mathcal{F}$-measurable functions with values in $\mathbb{R}$, which are not necessarily σ-integrable w.r.t. $\mathcal{G}$, such that $f_n \geq g$ μ-a.e. for all $n \in \mathbb{N}$ and an $\mathbb{R}$-valued σ-integrable function g w.r.t. $\mathcal{G}$. If*

$f_n \nearrow f$ μ-a.e. as $n \to \infty$ for a $\mathcal{F}$-measurable function f with values in $\mathbb{R}$, which is also not necessarily σ-integrable w.r.t. $\mathcal{G}$, then[4]

$$\lim_{n\to\infty} \mathbb{E}_\mu[f_n|\mathcal{G}] = \mathbb{E}_\mu[f|\mathcal{G}] \quad \mu\text{-a.e.} \tag{2.5}$$

(6) Take out what is known:

(a) If $g \in L^0(\Omega, \mathcal{G}, \mu; \mathbb{K}^{n\times d})$, then fg is σ-integrable w.r.t. $\mathcal{G}$ with values in $\mathbb{K}^n$ and

$$\mathbb{E}_\mu[gf|\mathcal{G}] = g\,\mathbb{E}_\mu[f|\mathcal{G}] \quad \mu\text{-a.e.}$$

(b) If f is a positive $\mathcal{F}$-measurable function, but not necessarily σ-integrable w.r.t. $\mathcal{G}$, that takes values in $\mathbb{R}$ and g is a positive $\mathcal{G}$-measurable function with values in $\mathbb{R}$, then

$$\mathbb{E}_\mu[gf|\mathcal{G}] = g\,\mathbb{E}_\mu[f|\mathcal{G}] \quad \mu\text{-a.e.}$$

(7) Tower property: *If $\mathcal{H} \subset \mathcal{G}$ is a further sub-σ-algebra of $\mathcal{F}$ and f is σ-integrable w.r.t. $\mathcal{H}$ (which implies the σ-integrability w.r.t. $\mathcal{G}$), then $\mathbb{E}_\mu[f|\mathcal{G}]$ is σ-integrable w.r.t. $\mathcal{H}$ and*

$$\mathbb{E}_\mu[\,\mathbb{E}_\mu[f|\mathcal{G}]\,|\mathcal{H}] = \mathbb{E}_\mu[f|\mathcal{H}] \quad \mu\text{-a.e.}$$

(8) Conditional version of Fatou's lemma: *Let $\mathbb{K}^d = \mathbb{R}$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{F}$-measurable functions with values in $\mathbb{R}$, which are not necessarily σ-integrable w.r.t. $\mathcal{G}$, such that $f_n \geq g$ μ-a.e. for all $n \in \mathbb{N}$ and an $\mathbb{R}$-valued σ-integrable function g w.r.t. $\mathcal{G}$. Then*[5]

$$\mathbb{E}_\mu\left[\liminf_{n\to\infty} f_n \middle| \mathcal{G}\right] \leq \liminf_{n\to\infty} \mathbb{E}_\mu[f_n|\mathcal{G}] \quad \mu\text{-a.e.}$$

If $f_n \leq g$ μ-a.e. for all $n \in \mathbb{N}$, then[6]

$$\mathbb{E}_\mu\left[\limsup_{n\to\infty} f_n \middle| \mathcal{G}\right] \geq \limsup_{n\to\infty} \mathbb{E}_\mu[f_n|\mathcal{G}] \quad \mu\text{-a.e.}$$

[4] Please refer to Remark 2.11(1) for the generalised definition of the conditional expectation of $\mathcal{F}$-measurable functions, which are not necessarily σ-integrable w.r.t. a sub-σ-algebra of $\mathcal{F}$.

[5] Note that $\tilde{f} := \liminf_{n\to\infty} f_n$ might not be σ-integrable w.r.t. $\mathcal{G}$. In that case define $\mathbb{E}_\mu[\tilde{f}|\mathcal{G}]$ according to Remark 2.11(1).

[6] The function $\hat{f} := \limsup_{n\to\infty} f_n$ might not be σ-integrable w.r.t. $\mathcal{G}$. In that case define $\mathbb{E}_\mu[\hat{f}|\mathcal{G}]$ according to Remark 2.11(2).

(9) Conditional version of Jensen's inequality: *Let $C \subset \mathbb{R}$ be an interval or $C \subset \mathbb{R}^d$ an open, convex set and assume that f takes values in C. If $\varphi : C \to \mathbb{R}$ is a convex function such that $\varphi(f)$ is σ-integrable w.r.t. $\mathcal{G}$, then*

$$\varphi(\mathbb{E}_\mu[f|\mathcal{G}]) \leq \mathbb{E}_\mu[\varphi(f)|\mathcal{G}] \quad \mu\text{-a.e.} \tag{2.6}$$

If φ is concave, the inequality is reversed.

(10) Conditional dominated convergence theorem: *Again, let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{F}$-measurable functions with values in $\mathbb{K}^d$ such that $|f_n| \leq g$ μ-a.e. for all $n \in \mathbb{N}$ and some $\mathbb{R}$-valued σ-integrable function g w.r.t. $\mathcal{G}$. If $f_n \to f$ μ-a.e. as $n \to \infty$ for an $\mathcal{F}$-measurable[7] function f with values in $\mathbb{K}^d$, then*[8]

$$\lim_{n\to\infty} \mathbb{E}_\mu[f_n|\mathcal{G}] = \mathbb{E}_\mu[f|\mathcal{G}] \quad \mu\text{-a.e.}$$

(11) Conditional version of Hölder's inequality: *Let f and g be $\mathcal{F}$-measurable functions with values in $\mathbb{K}^d$, which are not necessarily σ-integrable w.r.t. $\mathcal{G}$. Furthermore, let $p, q \in (1, \infty)$ with $1/p + 1/q = 1$. Then*[9]

$$\mathbb{E}_\mu[\,|\langle f, g\rangle|\,|\mathcal{G}] \leq (\mathbb{E}_\mu[|f|^p|\mathcal{G}])^{1/p}(\mathbb{E}_\mu[|g|^q|\mathcal{G}])^{1/q} \quad \mu\text{-a.e.}, \tag{2.7}$$

where $\langle\cdot,\cdot\rangle : \mathbb{K}^d \times \mathbb{K}^d \to \mathbb{R}$ denotes the inner product defined on $\mathbb{R}^d$ or $\mathbb{C}^d$. In (2.7) 0 times ∞ as well as ∞ times 0 means 0 and ∞ times $a > 0$ gives ∞.

Proof. Similarly to Definition 2.6 define $\mathcal{R}_{f,g,\mathcal{G}} = \{G \cap \bar{G} : G \in \mathcal{R}_{f,\mathcal{G}}, \bar{G} \in \mathcal{R}_{g,\mathcal{G}}\}$, where g is another σ-integrable function w.r.t. $\mathcal{G}$.

(1) This follows directly from Lemma 2.5(2) and the definition of the conditional expectation in Theorem 2.7 by observing that $\mathcal{G}$-measurable functions are also σ-integrable w.r.t $\mathcal{G}$ (see Lemma 2.4(2)).

(2) For integrable f we have $\Omega \in \mathcal{R}_{f,\mathcal{G}} = \mathcal{G}$ and thus, the claim follows directly from the definition of the conditional expectation in Theorem 2.7.

(3) It follows for $f \leq g$ μ-a.e. that

$$\mathbb{E}_\mu[\mathbb{E}_\mu[f|\mathcal{G}]\mathbb{1}_G] = \mathbb{E}_\mu[f\mathbb{1}_G] \leq \mathbb{E}_\mu[g\mathbb{1}_G] = \mathbb{E}_\mu[\mathbb{E}_\mu[g|\mathcal{G}]\mathbb{1}_G]$$

[7] The $\mathcal{F}$-measurability of f is a necessary assumption, because the μ-a.e.-limit of $\mathcal{F}$-measurable functions does not necessarily have to be $\mathcal{F}$-measurable itself.

[8] Please refer to Remark 2.11 for the generalised definition of the conditional expectation of $\mathcal{F}$-measurable functions, which are not necessarily σ-integrable w.r.t. a sub-σ-algebra of $\mathcal{F}$.

[9] Please refer to Remark 2.11 for the generalised definition of the conditional expectation of $\mathcal{F}$-measurable functions, which are not necessarily σ-integrable w.r.t. a sub-σ-algebra of $\mathcal{F}$.

for all $G \in \mathcal{R}_{f,g,\mathcal{G}}$. Lemma 2.5(2) then implies the claim.

(4) We will prove the claim for $\mathbb{K}^d = \mathbb{R}$, then it also holds for $\mathbb{K}^d = \mathbb{C}$ by the same arguments applied to the real and imaginary part as well as the general case $\mathbb{K}^d$ by applying the arguments componentwise. The fact that $af + g$ is a σ-integrable function w.r.t. $\mathcal{G}$ has already been shown in Lemma 2.4(1). Therefore, the claim holds true by Lemma 2.5(2) and the linearity of the integral in Theorem A.12, since

$$\begin{aligned}\mathbb{E}_\mu[\mathbb{E}_\mu[(af+g)|\mathcal{G}]\mathbb{1}_G] &\overset{2.7}{=} \mathbb{E}_\mu[(af+g)\mathbb{1}_G] \\ &= a\,\mathbb{E}_\mu[f\mathbb{1}_G] + \mathbb{E}_\mu[g\mathbb{1}_G] \\ &\overset{2.7}{=} a\,\mathbb{E}_\mu[\mathbb{E}_\mu[f|\mathcal{G}]\mathbb{1}_G] + \mathbb{E}_\mu[\mathbb{E}_\mu[g|\mathcal{G}]\mathbb{1}_G]\end{aligned}$$

for all $G \in \{G \cap \bar{G} \cap \tilde{G} : G \in \mathcal{R}_{f,\mathcal{G}}, \bar{G} \in \mathcal{R}_{g,\mathcal{G}}, \tilde{G} \in \mathcal{R}_{af+g,\mathcal{G}}\}$.

(5)[10] *Step 1*: Let us first assume that f and f_n are σ-integrable w.r.t. $\mathcal{G}$ for all $n \in \mathbb{N}$. Note that[11] $\mathcal{R}_{g,\mathcal{G}} \cap \mathcal{R}_{f,\mathcal{G}} \subset \mathcal{R}_{f_n,\mathcal{G}}$ for all $n \in \mathbb{N}$. This is true, since for $R \in \mathcal{R}_{g,\mathcal{G}} \cap \mathcal{R}_{f,\mathcal{G}}$ we have $\mathbb{E}_\mu[|g|\mathbb{1}_R] \leq \mathbb{E}_\mu[|f_n|\mathbb{1}_R] \leq \mathbb{E}_\mu[|f|\mathbb{1}_R] < \infty$, which implies $R \in \mathcal{R}_{f_n,\mathcal{G}}$. We will need this further on in order to apply Theorem 2.7 to f_n, $n \in \mathbb{N}$. Now, define $h_n = \mathbb{E}_\mu[f_n|\mathcal{G}]$ for all $n \in \mathbb{N}$ and note that $\mathcal{R}_{g,\mathcal{G}} \cap \mathcal{R}_{f,\mathcal{G}} \subset \mathcal{R}_{h_n,\mathcal{G}}$ for all $n \in \mathbb{N}$ by Remark 2.8, since $h_n \geq g$ μ-a.e. due to the monotonicity of the conditional expectation in (3). Furthermore, $(h_n)_{n\in\mathbb{N}}$ is a μ-a.e. increasing sequence of functions. For $h := \limsup_{n\to\infty} h_n$ it follows by Theorem A.5 that h is $\mathcal{G}$-measurable. Since $h_n \nearrow h$ μ-a.e. as $n \to \infty$ we may now use the dominated convergence theorem as stated in Theorem A.17 and deduce that

$$\int_G \mathbb{E}_\mu[f|\mathcal{G}]\,d\mu \overset{2.7}{=} \int_G f\,d\mu \overset{\text{A.17}}{=} \lim_{n\to\infty}\int_G f_n\,d\mu \overset{2.7}{=} \lim_{n\to\infty}\int_G h_n\,d\mu \overset{\text{A.17}}{=} \int_G h\,d\mu$$

for all $G \in \mathcal{R}_{g,\mathcal{G}} \cap \mathcal{R}_{f,\mathcal{G}} \subset \mathcal{R}_{h,\mathcal{G}}$. Thus, the claim follows by Lemma 2.5(2).

Step 2: In case f and f_n are only $\mathcal{F}$-measurable but not σ-integrable w.r.t. $\mathcal{G}$, define $f_n^{(m)} = f_n \wedge m$ for $m \in \mathbb{N}$. Then $f_n^{(m)} \geq g \wedge 0$ and $f_n^{(m)}$ is σ-integrable w.r.t. $\mathcal{G}$ for all $n \in \mathbb{N}$ and the same goes for $f \wedge m$ since both are bounded from above (see Remark 2.11(1)). Also, $\mathbb{E}_\mu[f_n^{(m)}|\mathcal{G}] \leq \mathbb{E}_\mu[f_n^{(m+1)}|\mathcal{G}]$ μ-a.e. for all $m, n \in \mathbb{N}$ by (3). Note that $f_n^{(m)} \nearrow (f \wedge m)$ for $n \to \infty$ and $(f \wedge m) \nearrow f$ for $m \to \infty$. Thus,

$$\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}_\mu[f_n^{(m)}|\mathcal{G}] = \lim_{m\to\infty}\mathbb{E}_\mu[f \wedge m|\mathcal{G}] \overset{2.11}{=} \mathbb{E}_\mu[f|\mathcal{G}] \quad \mu\text{-a.e.},$$

[10] This proof was inspired by [14, Section 9.8, Property (e), p. 89].

[11] Note that $\mathcal{R}_{f,\mathcal{G}} \subset \mathcal{R}_{f_n,\mathcal{G}}$ for all $n \in \mathbb{N}$ on its own does not hold true. Consider the measure space $(\mathbb{R}, \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel-σ-algebra on $\mathbb{R}$ and λ the Lebesgue–Borel measure on $\mathbb{R}$. Define a sequence of functions by $f_n(x) := -\frac{1}{n}|x|$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then f_n converges to the constant function $f \equiv 0$, for which $\mathcal{R}_{f,\mathcal{B}} = \mathcal{B}$. However, since f_n is not integrable over open intervals, $\mathcal{R}_{f,\mathcal{B}} \not\subset \mathcal{R}_{f_n,\mathcal{B}}$ for $n \in \mathbb{N}$.

where we used the conditional monotone convergence theorem proved for *Step 1* for the first equality and the generalised definition of the conditional expectation in Remark 2.11(1) for the last one. Again, by applying the conditional monotone convergence theorem proved for *Step 1* it follows that

$$\lim_{m\to\infty} \mathbb{E}_\mu[f_n^{(m)}|\mathcal{G}] = \mathbb{E}_\mu[f_n|\mathcal{G}] \quad \mu\text{-a.e.}, n \in \mathbb{N}.$$

Now simply apply Theorem A.33, which gives

$$\lim_{n\to\infty} \mathbb{E}_\mu[f_n|\mathcal{G}] = \mathbb{E}_\mu[f|\mathcal{G}] \quad \mu\text{-a.e.}$$

(6a)The σ-integrability of gf w.r.t $\mathcal{G}$ follows immediately from Lemma 2.4(3). Therefore, it remains to show that for all $G \in \mathcal{R}_{f,fg,\mathcal{G}}$ we have

$$\mathbb{E}_\mu[gf\mathbb{1}_G] = \mathbb{E}_\mu[\mathbb{E}_\mu[gf|\mathcal{G}]\mathbb{1}_G] = \mathbb{E}_\mu[g\,\mathbb{E}_\mu[f|\mathcal{G}]\mathbb{1}_G]. \tag{2.8}$$

In order to do so we will treat different cases for g and assume that g takes values in $\mathbb{R}$. For the complex-valued case we simply need to apply the steps below to the real- and the imaginary part of g. It remains to apply this to each component in the case that g takes values in $\mathbb{K}^{n\times d}$ and we are done.

Step 1: $g := \mathbb{1}_H$, $H \in \mathcal{R}_{f,\mathcal{G}}$. It follows immediately that (2.8) holds true because

$$\int_G \mathbb{E}_\mu[gf|\mathcal{G}]\,d\mu \overset{2.7}{=} \int_G gf\,d\mu \overset{2.7}{=} \int_{G\cap H} \mathbb{E}_\mu[f|\mathcal{G}]\,d\mu = \int_G g\,\mathbb{E}_\mu[f|\mathcal{G}]\,d\mu$$

for all $G \in \mathcal{R}_{f,\mathcal{G}}$ due to the definition of the conditional expectation. Thus, by Lemma 2.5(2) the claim follows for this case.

Step 2: $g := \sum_{k=1}^n \gamma_k \mathbb{1}_{G_k}$, for $\gamma_1, \dots, \gamma_n \in \mathbb{R}_+$, $G_1, \dots, G_n \in \mathcal{G}$ and $n \in \mathbb{N}$. Then (2.8) follows immediately from *Step 1* and due to the linearity of the integral as stated in Theorem A.12(1) and (2).

Step 3: $0 \le g \in L^0_+(\Omega, \mathcal{G}, \mu; \mathbb{R})$. By Lemma A.4 there exists a sequence of monotonously increasing non-negative simple functions (see Definition A.3) $(g_n)_{n\in\mathbb{N}}$ such that $g = \lim_{n\to\infty} g_n$. Thus, we may apply the conditional monotone convergence theorem proven in *Step 2*, which yields (2.8) for this case.

Step 4: $g \in L^0(\Omega, \mathcal{G}, \mu; \mathbb{R})$. For this case consider $g = g^+ - g^-$. Then (2.8) follows by *Step 3* and the linearity of the integral.

(6b) In case f is only $\mathcal{F}$-measurable but not σ-integrable w.r.t. $\mathcal{G}$, define $f_n = f \wedge n$ for $n \in \mathbb{N}$ and the conditional expectation of f w.r.t. $\mathcal{G}$ according to Remark 2.11. Then by (6a) we can conclude that $\mathbb{E}_\mu[gf_n|\mathcal{G}] = g\,\mathbb{E}_\mu[f_n|\mathcal{G}]$ μ-a.e. Since $gf_n \ge 0$ and $gf_n \nearrow gf$ for $n \to \infty$, the conditional monotone convergence theorem in (5) yields

$$g\,\mathbb{E}_\mu[f|\mathcal{G}] = g \lim_{n\to\infty} \mathbb{E}_\mu[f_n|\mathcal{G}] = \lim_{n\to\infty} \mathbb{E}_\mu[gf_n|\mathcal{G}] = \mathbb{E}_\mu[gf|\mathcal{G}] \quad \mu\text{-a.e.}$$

(7)[12] Since f is σ-integrable w.r.t. $\mathcal{H} \subset \mathcal{G}$, there exists a sequence $(\Omega_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ according to Definition 2.1. Due to (6) we then have that $\mathbb{E}_\mu[f\mathbb{1}_{\Omega_n}|\mathcal{G}] = \mathbb{1}_{\Omega_n}\mathbb{E}_\mu[f|\mathcal{G}]$ μ-a.e. for all $n \in \mathbb{N}$, which implies that $\mathbb{E}_\mu[f|\mathcal{G}]$ is also σ-integrable w.r.t. $\mathcal{H}$.

Furthermore, it follows from the definition of the conditional expectation in Theorem 2.7 that

$$\mathbb{E}_\mu[f\mathbb{1}_H] = \mathbb{E}_\mu[\mathbb{E}_\mu[f|\mathcal{H}]\mathbb{1}_H] \tag{2.9}$$

for all $H \in \mathcal{R}_{f,\mathcal{H}}$. Now note that $\mathcal{R}_{f,\mathcal{H}} \subset \mathcal{R}_{f,\mathcal{G}}$. Using (2.9) for $\mathbb{E}_\mu[f|\mathcal{G}]$ instead of f it follows that

$$\mathbb{E}_\mu[f\mathbb{1}_H] = \mathbb{E}_\mu[\mathbb{E}_\mu[f|\mathcal{G}]\mathbb{1}_H] = \mathbb{E}_\mu[\mathbb{E}_\mu[\,\mathbb{E}_\mu[f|\mathcal{G}]\,|\mathcal{H}]\mathbb{1}_H]$$

for all $H \in \mathcal{R}_{f,\mathbb{E}_\mu[f|\mathcal{G}],\mathcal{H}}$. Lemma 2.5(2) then implies the claim.

(8)[13] *Step 1*: Let us first assume that f_n is σ-integrable w.r.t. $\mathcal{G}$ for all $n \in \mathbb{N}$ and define $h_n = \inf_{k\geq n} f_k$, $n \in \mathbb{N}$, which is again σ-integrable w.r.t. $\mathcal{G}$ by Lemma 2.4(7). Then $h_n \geq g$ μ-a.e. for all $n \in \mathbb{N}$ and $h_n \leq f_k$, which by (3) implies $\mathbb{E}_\mu[h_n|\mathcal{G}] \leq \mathbb{E}_\mu[f_k|\mathcal{G}]$ μ-a.e. for $k \geq n$. Hence,

$$\mathbb{E}_\mu[h_n|\mathcal{G}] \leq \inf_{k\geq n} \mathbb{E}_\mu[f_k|\mathcal{G}] \quad \mu\text{-a.e.}, n \in \mathbb{N}. \tag{2.10}$$

Furthermore, we have

$$\lim_{n\to\infty} h_n = \lim_{n\to\infty} \inf_{k\geq n} f_k = \liminf_{n\to\infty} f_n.$$

Thus, it follows by the conditional monotone convergence theorem in (5) that

$$\mathbb{E}_\mu\Big[\liminf_{n\to\infty} f_n \Big| \mathcal{G}\Big] = \mathbb{E}_\mu\Big[\lim_{n\to\infty} h_n \Big| \mathcal{G}\Big] \overset{(5)}{=} \lim_{n\to\infty} \mathbb{E}_\mu[h_n|\mathcal{G}] \overset{(2.10)}{\leq} \liminf_{n\to\infty} \mathbb{E}_\mu[f_n|\mathcal{G}],$$

which holds true μ-a.e.

For the second part of Fatou's lemma consider $h_n := -f_n$, $n \in \mathbb{N}$. We may now apply the first part of the lemma, which gives us

$$\mathbb{E}_\mu\Big[\liminf_{n\to\infty} h_n \Big| \mathcal{G}\Big] \leq \liminf_{n\to\infty} \mathbb{E}_\mu[h_n|\mathcal{G}] \quad \mu\text{-a.e.},$$

which is

$$\mathbb{E}_\mu\Big[\liminf_{n\to\infty}(-f_n) \Big| \mathcal{G}\Big] \leq \liminf_{n\to\infty}(-\mathbb{E}_\mu[f_n|\mathcal{G}]) \quad \mu\text{-a.e.}$$

by the definition of h_n. We can rewrite this equation as

$$\mathbb{E}_\mu\Big[\limsup_{n\to\infty} f_n \Big| \mathcal{G}\Big] \geq \limsup_{n\to\infty} \mathbb{E}_\mu[f_n|\mathcal{G}] \quad \mu\text{-a.e.}$$

[12] The idea for this proof was inspired by [7, Theorem 1.22, p. 12–13].

[13] This proof was inspired by [14, Section 5.4, p. 52–53] and [9, Satz 14.10, p. 230–231].

due to the relation between the limit superior and limit inferior and are done with this proof.

Step 2: In case f_n is only $\mathcal{F}$-measurable but not σ-integrable w.r.t. $\mathcal{G}$ for all $n \in \mathbb{N}$, define $f_n^{(m)} = f_n \wedge m$ for $m \in \mathbb{N}$. Then $f_n^{(m)} \geq g \wedge 0$, $f_n^{(m)}$ is σ-integrable w.r.t. $\mathcal{G}$ for all $n \in \mathbb{N}$ (see Remark 2.11(1)) and $\mathbb{E}_\mu[f_n^{(m)}|\mathcal{G}] \leq \mathbb{E}_\mu[f_n^{(m+1)}|\mathcal{G}]$ μ-a.e. for all $m, n \in \mathbb{N}$ by (3). Now apply the conditional version of Fatou's lemma proved in *Step 1*, which gives us

$$\mathbb{E}_\mu\left[\liminf_{n\to\infty} f_n^{(m)}\middle|\mathcal{G}\right] \leq \liminf_{n\to\infty} \mathbb{E}_\mu[f_n^{(m)}|\mathcal{G}] \quad \mu\text{-a.e.} \tag{2.11}$$

By definition it follows that $f_n^{(m)} \nearrow f_n$ for $m \to \infty$. Furthermore, note that then $\liminf_{n\to\infty} f_n^{(m)} = (\liminf_{n\to\infty} f_n) \wedge m \nearrow \liminf_{n\to\infty} f_n$ for $m \to \infty$ and that $\mathbb{E}_\mu[f_n^{(m)}|\mathcal{G}] \leq \mathbb{E}_\mu[f_n|\mathcal{G}]$ μ-a.e. by the monotonicity in (3). Now we can apply the conditional monotone convergence theorem in (5) and arrive at

$$\begin{aligned}\mathbb{E}_\mu\left[\liminf_{n\to\infty} f_n\middle|\mathcal{G}\right] &\overset{(5)}{=} \lim_{m\to\infty} \mathbb{E}_\mu\left[\liminf_{n\to\infty} f_n^{(m)}\middle|\mathcal{G}\right] \overset{(2.11)}{\leq} \lim_{m\to\infty} \liminf_{n\to\infty} \mathbb{E}_\mu[f_n^{(m)}|\mathcal{G}] \\ &\leq \liminf_{n\to\infty} \mathbb{E}_\mu[f_n|\mathcal{G}] \quad \mu\text{-a.e.}\end{aligned}$$

For the second part of Fatou's lemma define $f_n^{(m)} = f_n \vee -m$ for $m \in \mathbb{N}$. Then $f_n^{(m)} \leq g \vee 0$, $f_n^{(m)}$ is σ-integrable w.r.t. $\mathcal{G}$ for all $n \in \mathbb{N}$ (see Remark 2.11(2)) and $\mathbb{E}_\mu[f_n^{(m)}|\mathcal{G}] \geq \mathbb{E}_\mu[f_n^{(m+1)}|\mathcal{G}]$ μ-a.e. for all $m, n \in \mathbb{N}$ by (3). Now apply the second part of the conditional version of Fatou's lemma proved in *Step 1* to $-f_n^{(m)}$, $n \in \mathbb{N}$. By definition it follows that $-f_n^{(m)} \nearrow -f_n$ for $m \to \infty$. Furthermore, note that then $\liminf_{n\to\infty}(-f_n^{(m)}) = (\liminf_{n\to\infty}(-f_n)) \wedge m \nearrow \liminf_{n\to\infty}(-f_n)$ for $m \to \infty$ and that $-\mathbb{E}_\mu[f_n^{(m)}|\mathcal{G}] \leq -\mathbb{E}_\mu[f_n|\mathcal{G}]$ μ-a.e. by the monotonicity in (3). Now we can apply the conditional monotone convergence theorem in (5) and Fatou's lemma as shown in *Step 1*. This yields

$$\begin{aligned}\mathbb{E}_\mu\left[\liminf_{n\to\infty}(-f_n)\middle|\mathcal{G}\right] &\overset{(5)}{=} \lim_{m\to\infty} \mathbb{E}_\mu\left[\liminf_{n\to\infty}(-f_n^{(m)})\middle|\mathcal{G}\right] \\ &\leq \lim_{m\to\infty} \liminf_{n\to\infty} \mathbb{E}_\mu[(-f_n^{(m)})|\mathcal{G}] \\ &\leq \liminf_{n\to\infty}(-\mathbb{E}_\mu[f_n|\mathcal{G}]) \quad \mu\text{-a.e.}\end{aligned}$$

Finally, we can rewrite this inequality using the relation between the limit superior and limit inferior and arrive at

$$\mathbb{E}_\mu\left[\limsup_{n\to\infty} f_n\middle|\mathcal{G}\right] \geq \limsup_{n\to\infty} \mathbb{E}_\mu[f_n|\mathcal{G}] \quad \mu\text{-a.e.}$$

(9) *Step 1*: First, let $C = [a, b]$ for $a, b \in \mathbb{R}$ and assume φ is continuous on C. Define the function $l_{\alpha,\beta}(z) := \alpha z + \beta$ for $\alpha, \beta \in \mathbb{R}$ and $z \in C$. Lemma A.32 tells us that $\varphi(x) = \sup\{l_{\alpha,\beta}(x) : \alpha, \beta \in \mathbb{R} \text{ satisfy } l_{\alpha,\beta} \leq \varphi\}$, $x \in C$. Thus, for $\alpha, \beta \in \mathbb{R}$ such that $l_{\alpha,\beta} \leq \varphi$ it follows that

$$l_{\alpha,\beta}(\mathbb{E}_\mu[f|\mathcal{G}]) = \alpha\,\mathbb{E}_\mu[f|\mathcal{G}] + \beta \overset{(4)}{=} \mathbb{E}_\mu[l_{\alpha,\beta}(f)|\mathcal{G}] \overset{(3)}{\leq} \mathbb{E}_\mu[\varphi(f)|\mathcal{G}] \quad \mu\text{-a.e.}$$

Hence,

$$\varphi(\mathbb{E}_\mu[f|\mathcal{G}]) \leq \mathbb{E}_\mu[\varphi(f)|\mathcal{G}] \quad \mu\text{-a.e.}$$

The reverse inequality for concave functions follows immediately from what we have just shown by simply considering $-\varphi$.

Now, assume that φ no longer needs to be continuous on C. W.l.o.g. assume that φ has a discontinuity at the outer left point $a \in C$. Consider a sequence of continuous, convex functions $\varphi_n : C \to \mathbb{R}$ defined by

$$\varphi_n(x) := \max\{\varphi(x), \varphi(a) - n(x - a)\}, \quad x \in C, n \in \mathbb{N}.$$

For this definition it follows that φ_n is continuous, convex and σ-integrable w.r.t. $\mathcal{G}$ for all $n \in \mathbb{N}$ and therefore, we can apply the conditional version of Jensen's inequality to every φ_n as we have just shown. Thus, $\varphi_n(\mathbb{E}_\mu[f|\mathcal{G}]) \leq \mathbb{E}_\mu[\varphi_n(f)|\mathcal{G}]$ for all $n \in \mathbb{N}$. Since $\varphi_n \searrow \varphi$ pointwise on C as $n \to \infty$, we can conclude that

$$\varphi(\mathbb{E}_\mu[f|\mathcal{G}]) = \lim_{n\to\infty} \varphi_n(\mathbb{E}_\mu[f|\mathcal{G}]) \leq \lim_{n\to\infty} \mathbb{E}_\mu[\varphi_n(f)|\mathcal{G}] \overset{(10)}{=} \mathbb{E}_\mu[\varphi(f)|\mathcal{G}] \quad \mu\text{-a.e.}$$

by using the conditional dominated convergence theorem in (10).

Step 2: The claim for convex open sets $C \subset \mathbb{R}^d$ follows via the definition of the Fenchel conjugate (see Definition A.37) and the Fenchel–Moreau theorem as stated in Theorem A.40. The Fenchel–Moreau theorem shows that we can express the function f via its biconjugate, which in turn is the supremum of affine and continuous functions. Therefore, we can reduce this case to what we have showed in *Step 1*.

(10)[14] Note that since $|f_n| \leq g$ μ-a.e., it follows that also the pointwise limit $|f| \leq g$ μ-a.e. and thus, f and f_n are σ-integrable w.r.t. $\mathcal{G}$. Now observe that $|f_n - f| \leq 2g$ μ-a.e., therefore, we can apply the second part of the conditional version of Fatou's lemma in (8), which gives us

$$\limsup_{n\to\infty} \mathbb{E}_\mu[|f_n - f|\,|\mathcal{G}] \leq \mathbb{E}_\mu\Big[\limsup_{n\to\infty} |f_n - f| \Big| \mathcal{G}\Big] = 0 \quad \mu\text{-a.e.}$$

[14]This proof was inspired by [14, p. 54–55, Section 5.9] and [9, p. 231, Satz 14.11].

Furthermore, by the linearity in (4) and the conditional version of Jensen's inequality in (9) applied to the function $x \mapsto |x|$ we have

$$|\mathbb{E}_\mu[f_n|\mathcal{G}] - \mathbb{E}_\mu[f|\mathcal{G}]| = |\mathbb{E}_\mu[f_n - f|\mathcal{G}]| \leq \mathbb{E}_\mu[|f_n - f|\,|\mathcal{G}] \quad \mu\text{-a.e.}$$

Therefore, the claim follows.

(11)[15] *Step 1*: Assume that $\langle f, g\rangle$, $|f|^p$ and $|g|^q$ are σ-integrable w.r.t. $\mathcal{G}$. Define $F = (\mathbb{E}_\mu[|f|^p\,|\,\mathcal{G}])^{1/p}$ and $G = (\mathbb{E}_\mu[|g|^q\,|\,\mathcal{G}])^{1/q}$ and note that $F, G \in L^0(\Omega, \mathcal{G}, \mu)$ according to the definition of the conditional expectation in Theorem 2.7. It follows that

$$\begin{aligned}\mathbb{E}_\mu\big[\big(|f|^p \mathbb{1}_{\{F=0\}}\big)\mathbb{1}_R\big] &= \mathbb{E}_\mu\big[\mathbb{E}_\mu\big[|f|^p \mathbb{1}_{\{F=0\}}\big|\mathcal{G}\big]\,\mathbb{1}_R\big] \\ &\overset{(6)}{=} \mathbb{E}_\mu\big[\mathbb{1}_{\{F=0\}}\underbrace{\mathbb{E}_\mu[|f|^p|\mathcal{G}]}_{=F^p\ \mu\text{-a.e.}}\mathbb{1}_R\big] = 0\end{aligned}$$

for all $R \in \mathcal{R}_{|f|^p,\mathcal{G}}$. Thus, $|f| = 0$ μ-a.e. on $\{F = 0\} \cap \Omega_n$ for all $n \in \mathbb{N}$ and a sequence $(\Omega_n)_{n\in\mathbb{N}} \in \mathcal{R}_{|f|^p,\mathcal{G}}$ such that $\Omega_n \nearrow \Omega$ as $n \to \infty$. However, since $\{F = 0\} = \bigcup_{n\in\mathbb{N}}(\{F = 0\} \cap \Omega_n)$, we can conclude that $|f| = 0$ μ-a.e. on $\{F = 0\}$. It can be shown in a similar manner that $|g| = 0$ μ-a.e. on $\{G = 0\}$. Since $|\langle f, g\rangle| \leq |f|\,|g|$ by the Cauchy-Schwarz inequality in Theorem A.34, this implies $\mathbb{E}_\mu[\,|\langle f, g\rangle|\,|\mathcal{G}] = 0$ μ-a.e. on $\{F = 0\} \cup \{G = 0\}$ and hence, the Hölder inequality holds.

On $\{F = \infty, G > 0\} \cup \{F > 0, G = \infty\}$ the right-hand side of (2.7) equals ∞ and the Hölder inequality is vacuously true.

Finally, let us consider $H := \{0 < F < \infty, 0 < G < \infty\}$. It remains to show

$$\mathbb{E}_\mu\left[\frac{\mathbb{E}_\mu[\,|\langle f, g\rangle|\,|\mathcal{G}]}{FG}\mathbb{1}_R\right] \leq \mathbb{E}_\mu[\mathbb{1}_R] \tag{2.12}$$

for all $R \in \{R \cap \bar{R} \cap \tilde{R} : R \in \mathcal{R}_{\frac{\langle f,g\rangle}{FG},\mathcal{G}}, \bar{R} \in \mathcal{R}_{\frac{|f|^p}{F^p},\mathcal{G}}, \tilde{R} \in \mathcal{R}_{\frac{|g|^q}{G^q},\mathcal{G}}\}$ with $R \subset H$. This is, of course, equivalent to showing (2.7) on H. By remembering the $\mathcal{G}$-measurability of F, G and R (2.12) follows quite quickly, because

$$\mathbb{E}_\mu\left[\frac{\mathbb{E}_\mu[\,|\langle f, g\rangle|\,|\mathcal{G}]}{FG}\mathbb{1}_R\right] \overset{(6)}{=} \mathbb{E}_\mu\left[\mathbb{E}_\mu\left[\frac{|\langle f, g\rangle|}{FG}\middle|\mathcal{G}\right]\mathbb{1}_R\right] \overset{2.7}{=} \mathbb{E}_\mu\left[\frac{|\langle f, g\rangle|}{FG}\mathbb{1}_R\right]$$

By the Cauchy-Schwarz inequality in Theorem A.34 and the Hölder inequality in its original form (see Theorem A.15) we arrive at

$$\mathbb{E}_\mu\left[\frac{|\langle f, g\rangle|}{FG}\mathbb{1}_R\right] \leq \mathbb{E}_\mu\left[\frac{|f|}{F}\mathbb{1}_R \cdot \frac{|g|}{G}\mathbb{1}_R\right] \leq \left(\mathbb{E}_\mu\left[\frac{|f|^p}{F^p}\mathbb{1}_R\right]\right)^{1/p}\left(\mathbb{E}_\mu\left[\frac{|g|^q}{G^q}\mathbb{1}_R\right]\right)^{1/q} \tag{2.13}$$

[15]This proof was inspired by the proof for the conditional Hölder inequality for probability spaces and random variables in `https://en.wikipedia.org/wiki/H%C3%B6lder%27s_inequality#Conditional_H%C3%B6lder_inequality`, Stand: 18.10.2019.

Focusing on the first term on the right-hand side of (2.13) we can again use the definition of the conditional expectation to prove that

$$\left(\mathbb{E}_\mu\left[\frac{|f|^p}{F^p}\mathbb{1}_R\right]\right)^{1/p} \overset{2.7}{=} \left(\mathbb{E}_\mu\left[\mathbb{E}_\mu\left[\frac{|f|^p}{F^p}\middle|\mathcal{G}\right]\mathbb{1}_R\right]\right)^{1/p} \overset{(6)}{=} \left(\mathbb{E}_\mu\left[\underbrace{\frac{\mathbb{E}_\mu[|f|^p\,|\,\mathcal{G}]}{F^p}}_{=1\,\mu\text{-a.e. on }R}\mathbb{1}_R\right]\right)^{1/p}$$

We can make the same observation for the second term on the right-hand side of (2.13), which finally gives us

$$\mathbb{E}_\mu\left[\frac{\mathbb{E}_\mu[|\langle f,g\rangle|\,|\,\mathcal{G}]}{FG}\mathbb{1}_R\right] \leq (\mathbb{E}_\mu[\mathbb{1}_R])^{1/p}(\mathbb{E}_\mu[\mathbb{1}_R])^{1/q} = \mathbb{E}_\mu[\mathbb{1}_R].$$

Therefore,

$$\frac{\mathbb{E}_\mu[\,|\langle f,g\rangle|\,|\mathcal{G}]}{FG} \leq 1 \quad \mu\text{-a-e. on } H,$$

by Lemma 2.5(1), which implies $\mathbb{E}_\mu[\,|\langle f,g\rangle|\,|\mathcal{G}] \leq FG$ μ-a-e. on H.

Step 2: For the more general setting of $\mathcal{F}$-measurable functions f and g define $f_n = f\mathbb{1}_{\{|f|\leq n\}}$ and $g_n = g\mathbb{1}_{\{|g|\leq n\}}$ for $n \in \mathbb{N}$ and note that they are both σ-integrable w.r.t. $\mathcal{G}$ (see Lemma 2.4(5)). Then $f_n \to f$ and $g_n \to g$ as $n \to \infty$. By applying the first part of the proof to f_n and g_n for all $n \in \mathbb{N}$ we arrive at

$$\mathbb{E}_\mu[\,|\langle f_n,g_n\rangle|\,|\mathcal{G}] \leq (\mathbb{E}_\mu[|f_n|^p|\mathcal{G}])^{1/p}(\mathbb{E}_\mu[|g_n|^q|\mathcal{G}])^{1/q} \quad \mu\text{-a.e.}$$

for all $n \in \mathbb{N}$. Since $x \mapsto |x|$, $x \mapsto x^p$ for $p > 1$ and the inner product are continuous functions, it follows that $|f_n|^p \nearrow |f|^p$, $|g_n|^q \nearrow |g|^q$ and $|\langle f_n, g_n\rangle| \nearrow |\langle f,g\rangle|$ as $n \to \infty$. By applying the conditional monotone convergence theorem twice we can conclude that

$$\begin{aligned}\mathbb{E}_\mu[\,|\langle f,g\rangle|\,|\mathcal{G}] &\overset{(5)}{=} \lim_{n\to\infty} \mathbb{E}_\mu[\,|\langle f_n,g_n\rangle|\,|\mathcal{G}] \\ &\leq \lim_{n\to\infty} (\mathbb{E}_\mu[|f_n|^p|\mathcal{G}])^{1/p}(\mathbb{E}_\mu[|g_n|^q|\mathcal{G}])^{1/q} \\ &\overset{(5)}{=} (\mathbb{E}_\mu[|f|^p|\mathcal{G}])^{1/p}(\mathbb{E}_\mu[|g|^q|\mathcal{G}])^{1/q} \quad \mu\text{-a.e.}\end{aligned}$$

This concludes the proof. □

This chapter showed that the definition of the conditional expectation can be generalised to hold on a σ-finite measure space. There are two ways in which this can be proved. Furthermore, the usual properties that hold for the conditional expectation (see Theorem A.28) remain true on σ-finite measure space and thus, in a more general setting than when restricted to random variables and probability measures. Our first steps in this chapter play a vital role in generalising the martingale theory as can be seen in the following chapters.

2.2 σ-Integrable Martingales

This chapter aims to introduce a generalised definition of martingales for σ-finite measures. Throughout this chapter, let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and $T \subset \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ an arbitrary index set[16], unless stated otherwise. Furthermore, define $t^* = \sup T$ and $\overline{T} = T \cup \{t^*\}$.

From now on, $\mathbb{F} := (\mathcal{F}_t)_{t\in T}$ shall refer to a filtration of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $s, t \in T$ with $s \leq t$ and let $(\Omega, \mathcal{F}, \mathbb{F}, \mu)$ be a σ-finite filtered measure space, meaning μ is σ-finite on[17] $\mathcal{F}_t$ for all $t \in T$.

Theorem 2.7 and the generalised definition of the conditional expectation will now help us to define martingales using the conditional expectation – like it is usually the case when considering random variables and probability measures.

Definition 2.13. Martingales as conditional expectations
Let $(\Omega, \mathcal{F}, \mathbb{F}, \mu)$ be a σ-finite filtered measure space and let $(f_t)_{t\in T}$ be a sequence of $\mathbb{K}^d$-valued integrable functions such that f_t is $\mathcal{F}_t$-measurable for all $t \in T$ and[18] $\mu|_{\mathcal{F}_s}$ is σ-finite for all $s \leq t$ in T. We then call $(f_t)_{t\in T}$ *martingale* (w.r.t. the filtration $\mathbb{F}$) if for all $s \leq t$ in T

$$\mathbb{E}_\mu[f_t|\mathcal{F}_s] = f_s \quad \mu\text{-a.e.} \tag{2.14}$$

Similarly, if $\mathbb{K}^d = \mathbb{R}$, $(f_t)_{t\in T}$ is called *submartingale* (w.r.t. the filtration $\mathbb{F}$) if for all $s \leq t$ in T

$$\mathbb{E}_\mu[f_t|\mathcal{F}_s] \geq f_s \quad \mu\text{-a.e.}, \tag{2.15}$$

and respectively *supermartingale* (w.r.t. the filtration $\mathbb{F}$)

$$\mathbb{E}_\mu[f_t|\mathcal{F}_s] \leq f_s \quad \mu\text{-a.e.} \tag{2.16}$$

In [12, Definition 17.1] we can find a similar definition for a discrete setting and $\mathbb{K}^d = \mathbb{R}$. We have adapted Schilling's definition to using an arbitrary index set $T \subset \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ to make an important observation.

Remark 2.14. Definition 2.13 with $\mathbb{K}^d = \mathbb{R}$ is equivalent to the following.

Let $(f_t)_{t\in T}$ be a sequence of μ-integrable functions such that f_t is $\mathcal{F}_t$-measurable for all $t \in T$. Then $(f_t)_{t\in T}$ is called *martingale* (w.r.t. the filtration $\mathbb{F}$), if for all $s \leq t$ in T

$$\int_F f_t \, d\mu = \int_F f_s \, d\mu, \quad F \in \mathcal{F}_s. \tag{2.17}$$

[16] e.g. for T think of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}_+$ and $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$ (we want for T to be a totally ordered set so that we my add an upper bound if necessary)

[17] If there is a smallest element in T, it suffices to ask μ to be σ-finite on the σ-algebra connected to said smallest element.

[18] If there is a smallest element in T, it suffices to assume that μ is finite on the sub-σ-algebra w.r.t. the smallest element of T.

We call $(f_t)_{t\in T}$ *submartingale* (w.r.t. $(\mathcal{F}_t)_{t\in T}$), if for all $s \le t$ in T

$$\int_F f_t \, d\mu \ge \int_F f_s \, d\mu, \quad F \in \mathcal{F}_s, \tag{2.18}$$

or *supermartingale* (w.r.t. $(\mathcal{F}_t)_{t\in T}$), if for all $s \le t$ in T

$$\int_F f_t \, d\mu \le \int_F f_s \, d\mu, \quad F \in \mathcal{F}_s. \tag{2.19}$$

Indeed: For all $F \in \mathcal{R}_{f_t,\mathcal{F}_s} = \mathcal{F}_s$ (see Definition 2.7) and $s \le t \in T$ we have

$$\int_F f_t \, d\mu = \mathbb{E}_\mu[f_t \mathbb{1}_F] \overset{2.7}{=} \mathbb{E}_\mu[\mathbb{E}_\mu[f_t|\mathcal{F}_s]\mathbb{1}_F] \overset{(2.14)}{=} \mathbb{E}_\mu[f_s \mathbb{1}_F] = \int_F f_s \, d\mu.$$

The defining properties of sub- and supermartingales in (2.18) and (2.19) follow in the same manner.

The following definition is a generalisation of Definition 2.13, because it remains meaningful even without the need for adaptedness and integrability. Furthermore, this definition is eponymous for this thesis.

Definition 2.15. σ-INTEGRABLE (SUB-/SUPER-)MARTINGALES
Let $(\Omega, \mathcal{F}, \mathbb{F}, \mu)$ be a σ-finite filtered measure space and let $f := (f_t)_{t\in T}$ be a sequence of $\mathbb{F} := (\mathcal{F}_t)_{t\in T}$-adapted functions with values in $\mathbb{K}^d$ such that f_t is σ-integrable w.r.t. $\mathcal{F}_s$ and[19] $\mu|_{\mathcal{F}_s}$ is σ-finite for all $s \le t$ in T. We call f a *σ-integrable martingale* (w.r.t. the filtration $\mathbb{F}$) if for all $s \le t$ in T

$$\mathbb{E}_\mu[f_t - f_s|\mathcal{F}_s] = 0 \quad \mu\text{-a.e.} \tag{2.20}$$

Now let $\mathbb{K}^d = \mathbb{R}$. Then we call f a *σ-integrable submartingale* (w.r.t. the filtration $\mathbb{F}$) if for all $s \le t$ in T

$$\mathbb{E}_\mu[f_t - f_s|\mathcal{F}_s] \ge 0 \quad \mu\text{-a.e.} \tag{2.21}$$

or a *σ-integrable supermartingale* (w.r.t. the filtration $\mathbb{F}$)

$$\mathbb{E}_\mu[f_t - f_s|\mathcal{F}_s] \le 0 \quad \mu\text{-a.e.} \tag{2.22}$$

The conditions (2.20), (2.21) and (2.22) remain meaningful even without the assumption that f is $\mathbb{F}$-adapted.

[19] If there is a smallest element in T, it suffices to assume that μ is finite on the sub-σ-algebra w.r.t. the smallest element of T.

We will continue to use the definition of σ-integrable (sub-/super-) martingales for the remainder of the thesis and highlight passages which hold specifically for (sub-/super-) martingales according to Definition 2.13. Based on Definition 2.15 we can rewrite some well-known elementary relations between martingales, sub- and supermartingales. For this purpose, let $(\Omega, \mathcal{F}, \mathbb{F}, \mu)$ be a σ-finite filtered measure space.

Remark 2.16. *Elemental properties and relations between σ-integrable sub-, super- and martingales.*

(1) $(f_t)_{t\in T}$ is a σ-integrable martingale if, and only if, each of its components is both a sub- and a supermartingale.

(2) $(f_t)_{t\in T}$ is a σ-integrable supermartingale if, and only if, $(-f_t)_{t\in T}$ is a σ-integrable submartingale.

(3) Consider the map $\kappa : T \to \mathbb{R} : t \mapsto \mathbb{E}_\mu[f_t]$. Then κ is

   (a) a constant function, if $f := (f_t)_{t\in T}$ is a martingale according to Definition 2.13;

   (b) an increasing function, if f is a submartingale according to Definition 2.13;

   (c) a decreasing function, if f is a supermartingale according to Definition 2.13.

   Indeed: The first claim follows immediately by taking expectations w.r.t. μ in (2.14) and by Theorem 2.12(2), where we use the integrability of martingales. Proving the claim for submartingales follows in the same manner. The last claim then follows by (2) applied to submartingales according to Definition 2.13.

(4) Let $T = \mathbb{N}$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of σ-integrable and real-valued functions such that f_n is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$. Define $g_n = \max\{f_1, \dots, f_n\}$ and $h_n = \min\{f_1, \dots, f_n\}$. Then $(g_n)_{n\in\mathbb{N}}$ is a σ-integrable submartingale and $(h_n)_{n\in\mathbb{N}}$ a σ-integrable supermartingale w.r.t. $\mathbb{F}$.

   Indeed: The σ-integrability of g_n and h_n for all $n \in \mathbb{N}$ follows by Lemma 2.4(6). Furthermore, for all $m \leq n$

$$\mathbb{E}_\mu[g_n|\mathcal{F}_m] \overset{2.12(3)}{\geq} \mathbb{E}_\mu[\max\{f_1, \dots, f_m\}|\mathcal{F}_m] \overset{2.12(1)}{=} g_m \quad \mu\text{-a.e.},$$

   and

$$\mathbb{E}_\mu[h_n|\mathcal{F}_m] \overset{2.12(3)}{\leq} \mathbb{E}_\mu[\min\{f_1, \dots, f_m\}|\mathcal{F}_m] \overset{2.12(1)}{=} h_m \quad \mu\text{-a.e.},$$

because of the monotonicity of the conditional expectation and the measurability assumption for f_m with $m \in \mathbb{N}$.

(5) Let $(f_t)_{t\in T}$ be a σ-integrable martingale with values in an interval $I \subset \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ a convex function such that $g_t := \varphi(f_t)$ is $\mathcal{F}_t$-measurable for all $t \in T$ and σ-integrable w.r.t. $\mathcal{F}_s$ for all $s \leq t$ in T. Then $(g_t)_{t\in T}$ is a σ-integrable submartingale.

Indeed: Because of Jensen's inequality in the conditional form and the σ-integrable martingale property it follows that

$$\mathbb{E}_\mu[g_t|\mathcal{F}_s] = \mathbb{E}_\mu[\varphi(f_t)|\mathcal{F}_s] \overset{2.12(9)}{\geq} \varphi(\mathbb{E}_\mu[f_t|\mathcal{F}_s]) \overset{(2.20)}{=} \varphi(f_s) = g_s \quad \mu\text{-a.e.}$$

for all $s \leq t \in T$.

(6) Let $(f_t)_{t\in T}$ be a σ-integrable submartingale with values in an interval $I \subset \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ a convex and increasing function such that $g_t := \varphi(f_t)$ is $\mathcal{F}_t$-measurable for all $t \in T$ and σ-integrable w.r.t. $\mathcal{F}_s$ for all $s \leq t$ in T. Then $(g_t)_{t\in T}$ is also a σ-integrable submartingale.

Indeed: Again, using the conditional version of Jensen's inequality it follows that

$$\mathbb{E}_\mu[g_t|\mathcal{F}_s] = \mathbb{E}_\mu[\varphi(f_t)|\mathcal{F}_s] \overset{2.12(9)}{\geq} \varphi(\underbrace{\mathbb{E}_\mu[f_t|\mathcal{F}_s]}_{\geq f_s}) \geq \varphi(f_s) = g_s \quad \mu\text{-a.e.}$$

for all $s \leq t \in T$, where the last inequality holds due to the submartingale property in (2.21) and the fact that φ is an increasing function.

When working with martingale theory, one cannot overlook their connection to stopping times. Thus, we would like to extend some well-known lemmata and theorems to our setting. In particular, we would like to prove that finite optional stopping can be generalised to our definition of σ-integrable (sub-) martingales. This will be crucial for some of the proofs in Chapter 3. Please, refer to Definition A.20 as well as Theorem A.22 in the Appendix if a revision of the definition of stopping times and some of their important properties is needed.

Lemma 2.17. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mu)$ be a σ-finite filtered measure space and f a σ-integrable function w.r.t. $\mathcal{F}_t$ for all $t \in T$ with values in $\mathbb{R}$. Furthermore, let τ a $\overline{T}$-valued stopping time w.r.t. $\mathbb{F}$ such that the image $\tau(\Omega)$ is countable. Then*

$$\mathbb{E}_\mu[f|\mathcal{F}_\tau] = \sum_{t\in\tau(\Omega)} \mathbb{E}_\mu[f|\mathcal{F}_t]\, \mathbb{1}_{\{\tau=t\}} \quad \mu\text{-a.e.}$$

Proof. Since $(f \sum_{n\in\mathbb{N}} \mathbb{1}_{\{\tau=t_n\}}) \le |f|$ we can apply the conditional dominated convergence theorem and conclude that

$$\mathbb{E}_\mu[f|\mathcal{F}_\tau] = \mathbb{E}_\mu\left[f \sum_{n\in\mathbb{N}} \mathbb{1}_{\{\tau=t_n\}} \middle| \mathcal{F}_\tau\right] \overset{2.12(10)}{=} \sum_{n\in\mathbb{N}} \mathbb{E}_\mu[f\mathbb{1}_{\{\tau=t_n\}}|\mathcal{F}_\tau] \quad \mu\text{-a.e.} \tag{2.23}$$

We will now look at each of the terms of the sum separately and prove that

$$\mathbb{E}_\mu[f\mathbb{1}_{\{\tau=t_n\}}|\mathcal{F}_\tau] = \mathbb{1}_{\{\tau=t_n\}}\mathbb{E}_\mu[f|\mathcal{F}_{t_n}] \quad \mu\text{-a.e.}$$

for all $n \in \mathbb{N}$.

For this purpose let $F \in \mathcal{R}_{f,\mathcal{F}_{t_n}}$ for $n \in \mathbb{N}$ (see Definition 2.6). At this point, let it be noted that the constant time and the pointwise minimum of stopping times are both stopping times by A.22(1) and (2), which implies that $\{\tau = t_n\} \cap F \in \mathcal{F}_{\tau\wedge t_n} \subset \mathcal{F}_\tau$ by A.22(4) and (8) for all $F \in \mathcal{F}_{t_n}$. Hence, by using that $\{\tau = t_n\} \in \mathcal{F}_\tau$ it follows that

$$\mathbb{E}_\mu[\mathbb{E}_\mu[f\mathbb{1}_{\{\tau=t_n\}}|\mathcal{F}_\tau]\mathbb{1}_F] \overset{2.12(6)}{=} \mathbb{E}_\mu[\mathbb{E}_\mu[f|\mathcal{F}_\tau]\mathbb{1}_{\{\tau=t_n\}\cap F}].$$

By the definition of the conditional expectation (simply observe that $\{\tau = t_n\} \cap F \in \mathcal{R}_{f,\mathcal{F}_\tau}$) we now have

$$\mathbb{E}_\mu[\mathbb{E}_\mu[f|\mathcal{F}_\tau]\mathbb{1}_{\{\tau=t_n\}\cap F}] = \mathbb{E}_\mu[f\mathbb{1}_{\{\tau=t_n\}\cap F}]$$

We can now use the same tricks again, since we have chosen $F \in \mathcal{R}_{f,\mathcal{F}_{t_n}}$ and deduce (from the fact that F is also an element of $\mathcal{R}_{f\mathbb{1}_{\{\tau=t_n\}},\mathcal{F}_{t_n}}$) that

$$\mathbb{E}_\mu[f\mathbb{1}_{\{\tau=t_n\}\cap F}] = \mathbb{E}_\mu[\mathbb{E}_\mu[f\mathbb{1}_{\{\tau=t_n\}}|\mathcal{F}_{t_n}]\mathbb{1}_F] \overset{2.12(6)}{=} \mathbb{E}_\mu[\mathbb{1}_{\{\tau=t_n\}}\mathbb{E}_\mu[f|\mathcal{F}_{t_n}]\mathbb{1}_F],$$

where we used that $\{\tau = t_n\} \in \mathcal{F}_{t_n}$ for the final equality. By plugging what we have just proven into (2.23) we arrive at

$$\mathbb{E}_\mu[f|\mathcal{F}_\tau] = \sum_{n\in\mathbb{N}} \mathbb{E}_\mu[f|\mathcal{F}_{t_n}]\mathbb{1}_{\{\tau=t_n\}} \quad \mu\text{-a.e.},$$

which is what we wanted to show. □

The proofs of the following two lemmata are inspired by [13, Lemma 3.44(a) and Lemma 3.51(a)], who proves the statements for probability spaces and stochastic processes.

Lemma 2.18. *Let $(S, \mathcal{S})$ be a measurable space and $f := (f_t)_{t\in T}$ a sequence of functions such that $f_t : \Omega \to S$ for all $t \in T$ and let $\tau : \Omega \to T$ be a stopping time w.r.t. $\mathbb{F}$. Define $f_\tau : \Omega \to S$ by $f_\tau(\omega) = f_{\tau(\omega)}(\omega)$ for every $\omega \in \Omega$. Then f_τ is $\mathcal{F}_\tau$-measurable, if $\tau(\Omega) \subset T$ is countable and if f_t is $\mathcal{F}_t$-measurable for all $t \in T$ (in this case f is called* adapted *to $\mathbb{F}$).*

Proof. The sets $\{\tau = t\}$ build a partition of Ω for $t \in T$, thus

$$f_\tau^{-1}(A) = \bigcup_{t\in\tau(\Omega)} f_\tau^{-1}(A) \cap \{\tau = t\} = \bigcup_{t\in\tau(\Omega)} f_t^{-1}(A) \cap \{\tau = t\}$$

for all $A \in \mathcal{S}$. For every $t \in \tau(\Omega)$ we have $f_t^{-1}(A) \cap \{\tau = t\} \in \mathcal{F}_{\tau\wedge t} \subset \mathcal{F}_\tau$ for all $t \in \tau(\Omega)$ by Theorem A.22(8) and (4). This implies that $f_\tau^{-1}(A) \in \mathcal{F}_\tau$, since $\tau(\Omega)$ is countable and $\mathcal{F}_\tau$ a filtration, as Remark A.21 shows. □

Lemma 2.19. *Let (S, ρ) be a metric space and $f := (f_t)_{t\in T}$ a sequence of functions such that $f_t : \Omega \to S$ is $\mathcal{F}_t$-measurable for all $t \in T$. Furthermore, let $\sigma : \Omega \to \overline{T}$ be a stopping time w.r.t. $\mathbb{F}$. For $A \in \mathcal{B}_S$ define the* first hitting time of A after σ *(also called* first entrance time*) by*

$$\tau = \inf\{t \in T : \sigma \leq t, f_t \in A\}, \tag{2.24}$$

where we define $\inf \emptyset = t^*$. If T is countable and every non-empty subset, which is bounded from below, contains its infimum (think of T finite, $T \subset \mathbb{Z}$ or $T = \{k - \frac{1}{n} : k \in \mathbb{Z}, n \in \mathbb{N}\}$), then τ is a stopping time w.r.t. $\mathbb{F}$.

Proof. The conditions ensure that τ takes values in $\overline{T}$. For $t \in T$ the set $\{u \in T : t < u, \sigma \leq u, f_u \in A\}$ contains its infimum by assumption, which cannot be t. Hence,

$$\{\tau \leq t\} = \bigcup_{\substack{s\in T\\ s\leq t}} \underbrace{\{\sigma \leq s, f_s \in A\}}_{\in \mathcal{F}_s \subset \mathcal{F}_t} \in \mathcal{F}_t,$$

where we used that σ is a stopping time and that T is countable. □

Theorem 2.20. Finite optional stopping for submartingales
Let $f := (f_t)_{t\in T}$ be a σ-integrable submartingale and σ, τ stopping times w.r.t. $\mathbb{F}$ such that τ and $\sigma \wedge \tau$ attain only finitely many values in T. Then f_τ and $f_{\sigma\wedge\tau}$ are σ-integrable, $f_{\sigma\wedge\tau}$ is measurable w.r.t. $\mathcal{F}_{\sigma\wedge\tau}$ and $\mathcal{F}_\sigma$ and

$$f_{\sigma\wedge\tau} \leq \mathbb{E}_\mu[f_\tau|\mathcal{F}_\sigma] \quad \mu\text{-a.e.} \tag{2.25}$$

Remark 2.21. Consider the setting of Theorem 2.20, then (2.25) is equivalent to

$$\mathbb{E}_\mu[f_\tau - f_{\sigma\wedge\tau}|\mathcal{F}_\sigma] \geq 0 \quad \mu\text{-a.e.} \tag{2.26}$$

This inequality is, in fact, meaningful without the $\mathcal{F}_\sigma$-measurability of $f_{\sigma\wedge\tau}$.

Proof. Theorem 2.20
In the spirit of Definition 2.15 we will prove the more general version of finite stopping for σ-integrable submartingales in (2.26). For this purpose let f fulfill the submartingale property as defined in (2.21), which means that f_t is σ-integrable w.r.t. $\mathcal{F}_s$ for all $s \leq t$ in T and $\mathbb{E}_\mu[f_t - f_s|\mathcal{F}_s] \geq 0$ μ-a.e. for all $s \leq t$ in T.

Due to the assumptions we may assume that τ and $\sigma \wedge \tau$ attain values in a finite set $S := \{t_0, \dots, t_n\}$ with $n \in \mathbb{N}$, where $t_0 < t_1 < \ldots < t_n$ w.l.o.g. The σ-integrability of f_τ w.r.t. $\mathcal{F}_{t_0}$ follows immediately from Lemma 2.4(1), (4), (5) and (6), since $|f_\tau| \leq |f_{t_0}| + \cdots + |f_{t_n}|$. The σ-integrability of $f_{\sigma\wedge\tau}$ w.r.t. $\mathcal{F}_{t_0}$ follows in the same manner. Note that f_τ and $f_{\sigma\wedge\tau}$ are then σ-integrable w.r.t. $\mathcal{F}_{t_i}$ for all $i \in \{0, \dots, n\}$, since $\mathbb{F}$ is a filtration. Let us make a redefinition of the supremum of T, which we will only use in this proof for ease of notation: $t_{n+1} := t^* = \sup T$. Furthermore, define $\bar{S} = S \cup \{t_{n+1}\}$.

Step 1: Treating only τ. We want to show, that for all $i \in \{0, \dots, n+1\}$ we have

$$\mathbb{E}_\mu[f_\tau - f_{t_i\wedge\tau}|\mathcal{F}_{t_i}] \geq 0 \quad \mu\text{-a.e.} \tag{2.27}$$

by using backward induction.

First, let $i \in \{n, n+1\}$, then $\tau = t_i \wedge \tau \leq t_i$, which implies that f_τ is $\mathcal{F}_{t_i}$-measurable by Theorem A.22(4). Thus, $\mathbb{E}_\mu[f_\tau - f_{t_i\wedge\tau}|\mathcal{F}_{t_i}] \geq 0$ μ-a.e., which proves (2.27) for this case.

Now suppose (2.27) holds for $i+1$ with $i \in \{0, \dots, n-1\}$ and let us prove it for i. By the tower property in Theorem 2.12(7) it follows that

$$\mathbb{E}_\mu[f_\tau - f_{t_{i+1}\wedge\tau}|\mathcal{F}_{t_i}] \geq 0 \quad \mu\text{-a.e.} \tag{2.28}$$

Furthermore, by adding and subtracting $f_{t_{i+1}\wedge\tau}$ within the conditional expectation in (2.28) we arrive at

$$\mathbb{E}_\mu[f_\tau - \underbrace{f_{t_i\wedge\tau} + f_{t_{i+1}\wedge\tau}}_{=(f_{t_{i+1}} - f_{t_i})\mathbb{1}_{\{\tau > t_i\}}} - f_{t_{i+1}\wedge\tau}|\mathcal{F}_{t_i}] \geq 0 \quad \mu\text{-a.e.}$$

This is, of course, equivalent to

$$\mathbb{E}_\mu[f_\tau - f_{t_{i+1}\wedge\tau}|\mathcal{F}_{t_i}] \geq \mathbb{E}_\mu[(f_{t_{i+1}} - f_{t_i})\mathbb{1}_{\{\tau > t_i\}}|\mathcal{F}_{t_i}] \quad \mu\text{-a.e.}$$

Since $\{\tau > t_i\} \in \mathcal{F}_{t_i}$ by Definition A.20 we can use (2.21) and conclude

$$\mathbb{E}_\mu[(f_{t_{i+1}} - f_{t_i})\mathbb{1}_{\{\tau > t_i\}}|\mathcal{F}_{t_i}] \overset{2.12(6)}{=} \mathbb{1}_{\{\tau > t_i\}}\mathbb{E}_\mu[(f_{t_{i+1}} - f_{t_i})|\mathcal{F}_{t_i}] \geq 0 \quad \mu\text{-a.e.}$$

Hence, we are done with this part.

Step 2: Treating discretised σ. Define

$$\tilde{\sigma} = \min\{t_i : i \in \{0, \dots, n+1\}, \sigma \leq t_i\}$$

to discretise σ. Then $\tilde{\sigma}$ is a $\overline{T}$-valued stopping time w.r.t. $\mathbb{F}$, since the same goes for σ and (see also Lemma 2.19)

$$\{\tilde{\sigma} \leq t\} = \bigcup_{\substack{s \in \bar{S} \\ s \leq t}} \underbrace{\{\sigma = s\}}_{\in \mathcal{F}_s \subset \mathcal{F}_t} \in \mathcal{F}_t.$$

It follows that

$$\mathbb{E}_\mu[f_\tau - f_{\tilde{\sigma}\wedge\tau} | \mathcal{F}_{\tilde{\sigma}}] \overset{2.17}{=} \sum_{i=0}^{n+1} \mathbb{E}_\mu[f_\tau - f_{\tilde{\sigma}\wedge\tau} | \mathcal{F}_{t_i}] \mathbb{1}_{\{\tilde{\sigma}=t_i\}} \quad \mu\text{-a.e.}$$

Since $f_{\tilde{\sigma}\wedge\tau}\mathbb{1}_{\{\tilde{\sigma}=t_i\}} = f_{t_i\wedge\tau}\mathbb{1}_{\{\tilde{\sigma}=t_i\}}$, we can deduce that

$$\begin{aligned}\mathbb{E}_\mu[f_\tau - f_{\tilde{\sigma}\wedge\tau} | \mathcal{F}_{t_i}] \mathbb{1}_{\{\tilde{\sigma}=t_i\}} &= \mathbb{E}_\mu[(f_\tau - f_{\tilde{\sigma}\wedge\tau})\mathbb{1}_{\{\tilde{\sigma}=t_i\}} | \mathcal{F}_{t_i}] \\ &= \underbrace{\mathbb{E}_\mu[f_\tau - f_{t_i\wedge\tau} | \mathcal{F}_{t_i}]}_{\geq 0\, \mu\text{-a.e. by (2.27)}} \mathbb{1}_{\{\tilde{\sigma}=t_i\}} \geq 0 \quad \mu\text{-a.e.}\end{aligned}$$

by using that $\{\tilde{\sigma} = t_i\} \in \mathcal{F}_{t_i}$ and Theorem 2.12(6) for each term in the sum. Thus, *Step 1* now gives the result

$$\mathbb{E}_\mu[f_\tau - f_{\tilde{\sigma}\wedge\tau} | \mathcal{F}_{\tilde{\sigma}}] \geq 0 \quad \mu\text{-a.e.}$$

Step 3: Removing the discretisation of σ. It follows from the definition of $\tilde{\sigma}$, that $\sigma \leq \tilde{\sigma}$ and $\sigma \wedge \tau = \tilde{\sigma} \wedge \tau$. This implies, $f_{\sigma\wedge\tau} = f_{\tilde{\sigma}\wedge\tau}$ and $\mathcal{F}_\sigma \subset \mathcal{F}_{\tilde{\sigma}}$ by Theorem A.22(4). Finally, by *Step 2* and the tower property as well as the monotonicity of the conditional expectation we can conclude that

$$\mathbb{E}_\mu[f_\tau - f_{\sigma\wedge\tau} | \mathcal{F}_\sigma] = \mathbb{E}_\mu[\underbrace{\mathbb{E}_\mu[f_\tau - f_{\sigma\wedge\tau} | \mathcal{F}_{\tilde{\sigma}}]}_{\geq 0\, \mu\text{-a.e.}} | \mathcal{F}_\sigma] \geq 0 \quad \mu\text{-a.e.}$$

This concludes the proof. □

Remark 2.22. Inequality (2.25) is reversed for supermartingales, because of Remark 2.16(2). Similarly, due to Remark 2.16(1), we have an equality in (2.25) for martingales.

In this chapter we were able to introduce a generalised definition for martingales using σ-integrable functions and σ-finite measure spaces that is similar to the more commonly known definition for probability measures and random variables. [12] provided the groundwork for this newly developed part of martingale theory, however, as he focused on a discrete setting, it was our goal to extend his work to continuous time. Furthermore, we generalised the theory of stopping times in relation to σ-integrable martingales and were, thus, able to adapt the idea of finite optional stopping to our setting. Everything we have proven will help us in the following chapters in order to improve Doob's classical L^p-inequality for submartingales and $p > 1$ as stated in Chapter 3.

Chapter 3

Doob's Classical L^p-Inequality for Submartingales on σ-Finite Measure Spaces

Now it is time to introduce and prove the central theorem of this thesis: Doob's L^p-inequality for submartingales and $p > 1$, since – as mentioned before – we hope to prove stricter versions for a more general setting of said inequality in Chapter 5. However, first we need to take note of other important inequalities introduced by Doob, known as Doob's maximum inequalities. [12, Lemma 19.11] proves these inequalities for his definition of martingales (as can be found in Remark 2.14), but only treats the case $T = \mathbb{N}$. Therefore, we will adapt his version to our setting where $T \subset \overline{\mathbb{R}}$. For our proof we use [13, Theorem 4.65] and proceed in a similar manner.

Let it be mentioned that there are various different approaches to proving Doob's inequality for the traditional definition of martingales (w.r.t. a random variable and the probability measure $\mathbb{P}$). A rather new and straightforward approach relies on deterministic inequalities, as shown in [1]. We will use this idea to improve Doob's maximum as well as his L^p inequalities later on.

Theorem 3.1. Doob's maximum inequalities
Let $(\Omega, \mathcal{F}, \mathbb{F}, \mu)$ be a σ-finite filtered measure space, $f := (f_t)_{t \in T}$ a submartingale according to Definition 2.13 and $T \subset \overline{\mathbb{R}}$ be non-empty such that

- *T is countable or*
- *T is a non-degenerate interval and f is left- or right-continuous,*

and $T \subset [s, v]$ for $s \le v$ in T. Then $\sup_{t \in T} f_t$ and $\inf_{t \in T} f_t$ are $\overline{\mathbb{R}}$-valued and

$\mathcal{F}$-measurable. Moreover, for every $\lambda \in \mathbb{R}$ we have

$$\begin{aligned}\lambda\,\mu\Big(\Big\{\sup_{t\in T} f_t \geq \lambda\Big\}\Big) &\leq \mathbb{E}_\mu[f_v \mathbb{1}_{\{\sup_{t\in T} f_t \geq \lambda\}}] - \mathbb{E}_\mu[(f_s - \lambda)^+] \\ &\leq \mathbb{E}_\mu[f_v \mathbb{1}_{\{\sup_{t\in T} f_t \geq \lambda\}}] \leq \mathbb{E}_\mu[f_v^+]\end{aligned} \tag{3.1}$$

(where the upper bounds not involving s are valid without their existence), and

$$\lambda\,\mu\Big(\Big\{\inf_{t\in T} f_t \leq -\lambda\Big\}\Big) \leq \mathbb{E}_\mu[f_v \mathbb{1}_{\{\inf_{t\in T} f_t > -\lambda\}}] - \mathbb{E}_\mu[f_s \vee (-\lambda)] \leq \mathbb{E}_\mu[f_v^+] - \mathbb{E}_\mu[f_s]. \tag{3.2}$$

Furthermore,

$$\lambda\,\mu\Big(\Big\{\sup_{t\in T} |f_t| \geq \lambda\Big\}\Big) \leq 2\,\mathbb{E}_\mu[f_v^+] - \mathbb{E}_\mu[f_s]. \tag{3.3}$$

Proof. First, note that $(f_t - \lambda)^+ \geq 0$ and $f_t \vee (-\lambda) \geq f_t$ for all $t \in T$. It follows that $\mathbb{E}_\mu[(f_s - \lambda)^+] \geq 0$ and $\mathbb{E}_\mu[f_s \vee (-\lambda)] \geq \mathbb{E}_\mu[f_s]$ by the monotonicity of the integral in Theorem A.12(3). Also, $f\mathbb{1}_F \leq f^+\mathbb{1}_F \leq f^+$ for $F \in \mathcal{F}$ and thus, $\mathbb{E}_\mu[f_v \mathbb{1}_{\{\sup_{t\in T} f_t \geq \lambda\}}] \leq \mathbb{E}_\mu[f_v^+]$ for all $u \in T$ by Theorem A.12(3). Hence, the second and third inequality in (3.1) and the second inequality in (3.2) follow immediately from what we have just observed and by combining these inequalities we can deduce (3.3). Therefore, it suffices to show the first inequality in (3.1) and (3.2).

We will start by proving (3.1) for non-empty finite sets $T = \{t_0, \ldots, t_n\}$ with $s = t_0 < t_1 < \ldots < t_n = v$ for $n \in \mathbb{N}$. Define

$$\tau = v \wedge \min\{t \in T : f_t \geq \lambda\},$$

where $\min \emptyset := t^*$. Then τ is a stopping time w.r.t. $\mathbb{F}$ by Theorem A.22(2) and Lemma 2.19. For

$$A := \Big\{\max_{t\in T} f_t \geq \lambda\Big\} = \bigcup_{t\in T} \{f_t \geq \lambda\}$$

it follows that $A \in \mathcal{F}_\tau$ (see (A.1) in Definition A.20), because $A \cap \{\tau \leq u\} = \bigcup_{t\in T, t\leq u} \{f_t \geq \lambda\} \in \mathcal{F}_u$ for all $u \in T$, where we used that f_t is $\mathcal{F}_t \subset \mathcal{F}_u$-measurable for $t \leq u$.

We claim that

$$\lambda \mathbb{1}_A \leq f_\tau \mathbb{1}_A - \underbrace{(f_s - \lambda)\mathbb{1}_{\{f_s \geq \lambda\}}}_{=(f_s-\lambda)^+}. \tag{3.4}$$

This can be proved by treating various cases.

- On A^c: $\max_{t\in T} f_t < \lambda \Rightarrow f_t < \lambda\ \forall t \in T \Rightarrow \mathbb{1}_{\{f_s\geq\lambda\}} = 0$. Thus, both sides of (3.4) are zero.
- On $\{f_s \geq \lambda\} \subset A$: $\tau = s \Rightarrow f_\tau = f_s$, because $s = t_0$ and thus, both sides of (3.4) equal λ.
- On $A \cap \{f_s < \lambda\}$: $\mathbb{1}_{\{f_s\geq\lambda\}} = 0$. Hence, (3.4) reduces to $f_\tau \geq \lambda$, which is true by definition of τ.

Therefore, by taking the expectation w.r.t. μ on both sides of (3.4) and applying finite stopping for submartingales we arrive at

$$\lambda\,\mu(A) \leq \mathbb{E}_\mu[f_\tau \mathbb{1}_A] - \mathbb{E}_\mu[(f_s-\lambda)^+] \overset{2.20}{\leq} \mathbb{E}_\mu[\mathbb{E}_\mu[f_v|\mathcal{F}_\tau]\mathbb{1}_A] - \mathbb{E}_\mu[(f_s-\lambda)^+].$$

By using that $\mathbb{1}_A$ is $\mathcal{F}_\tau$-measurable we can observe that

$$\lambda\,\mu(A) \overset{2.12(6)}{\leq} \mathbb{E}_\mu[\mathbb{E}_\mu[f_v\mathbb{1}_A|\mathcal{F}_\tau]] - \mathbb{E}_\mu[(f_s-\lambda)^+] \overset{2.12(2)}{=} \mathbb{E}_\mu[f_v\mathbb{1}_A] - \mathbb{E}_\mu[(f_s-\lambda)^+],$$

which concludes the proof of (3.1) for finite T.

Next, suppose T is countably infinite, then there exists an increasing sequence $(S_n)_{n\in\mathbb{N}}$ of non-empty finite sets such that $T = \bigcup_{n\in\mathbb{N}} S_n$. Additionally, let $(\lambda_m)_{m\in\mathbb{N}}$ be a sequence in $(-\infty,\lambda)$ such that $\lambda_m \nearrow \lambda$ as $m\to\infty$. Similarly to the first part of the proof let us make the following definitions:

$$A_{m,n} = \Big\{\max_{t\in S_n} f_t \geq \lambda_m\Big\},\ A_m = \bigcup_{n\in\mathbb{N}} A_{m,n} \text{ and } A = \Big\{\sup_{t\in T} f_t \geq \lambda\Big\}.$$

Then, $A = \bigcap_{m\in\mathbb{N}} A_m$. Note, that $(A_{m,n})_{n\in\mathbb{N}}$ is an increasing sequence for every $m\in\mathbb{N}$, whereas $(A_m)_{m\in\mathbb{N}}$ is decreasing. By applying the dominated convergence theorem as stated in Theorem A.17 and using what we have just proved for finite sets, it follows that

$$\begin{aligned}\lambda_m\,\mu(A_m) &= \lim_{n\to\infty} \lambda_m\,\mu(A_{m,n})\\ &\leq \lim_{n\to\infty} \mathbb{E}_\mu[f_v\mathbb{1}_{A_{m,n}}] - \mathbb{E}_\mu[(f_s-\lambda_m)^+]\\ &\overset{A.17}{=} \mathbb{E}_\mu[f_v\mathbb{1}_{A_m}] - \mathbb{E}_\mu[(f_s-\lambda_m)^+]\end{aligned}$$

for every $m\in\mathbb{N}$. In the same manner we may apply the dominated convergence theorem once more, which proves (3.1) for this case, because

$$\begin{aligned}\lambda\,\mu(A) &= \lim_{m\to\infty} \lambda_m\,\mu(A_m)\\ &\leq \lim_{m\to\infty} \mathbb{E}_\mu[f_v\mathbb{1}_{A_m}] - \mathbb{E}_\mu[(f_s-\lambda_m)^+]\\ &\overset{A.17}{=} \mathbb{E}_\mu[f_v\mathbb{1}_A] - \mathbb{E}_\mu[(f_s-\lambda)^+].\end{aligned}$$

Finally, let T be a non-degenerate interval and f left- or right-continuous and define $t^\circ = \inf T$ and $t^* = \sup T$ (note that $t^\circ, t^* \in \overline{\mathbb{R}}$). Then the set $\tilde{T} := T \cap (\mathbb{Q} \cup \{t^\circ, t^*\})$ is countable and $\sup_{t\in T} f_t = \sup_{t\in\tilde{T}} f_t$ by the left- or right-continuity of f. Hence, (3.1) follows and we are done with this part.

In the same manner as before, we will start with the finite non-empty set T defined at the beginning of this proof and $s := \min T$ and $v := \max T$ to prove (3.2). Define the stopping time $\sigma = v \wedge \min\{t \in T : f_t \leq -\lambda\}$ (again, see Theorem A.22(2) and Lemma 2.19) and $B = \{\min_{t\in T} f_t \leq -\lambda\}$, which is again an element of $\mathcal{F}_\sigma$ (see (A.1) in Definition A.20).

This time, we claim that

$$\lambda \mathbb{1}_B \leq f_s - f_\sigma \mathbb{1}_B - (f_s \vee (-\lambda)). \tag{3.5}$$

This can be proved by treating various cases.

- On B^{c}: $\min_{t\in T} f_t > -\lambda \Rightarrow f_s > -\lambda$. Thus, both sides of (3.5) are zero.
- On $\{f_s \leq -\lambda\} = \{\sigma = s\} \subset B$: $f_\sigma = f_s \Rightarrow f_s \vee (-\lambda) = -\lambda$. Thus, both sides of (3.5) equal λ.
- On $B \cap \{f_s > -\lambda\}$: $f_\sigma \leq -\lambda$, which is true by definition of σ.

Therefore, by taking the expectation w.r.t. μ on both sides of (3.5) we arrive at

$$\lambda\, \mu(B) \leq \mathbb{E}_\mu[f_s] - \mathbb{E}_\mu[f_\sigma \mathbb{1}_B] - \mathbb{E}_\mu[f_s \vee (-\lambda)]. \tag{3.6}$$

By applying finite stopping for submartingales and by using that $\mathbb{1}_B$ is $\mathcal{F}_\sigma$-measurable we can observe that

$$\mathbb{E}_\mu[f_s] \overset{2.20}{\leq} \mathbb{E}_\mu[\mathbb{E}_\mu[f_\sigma|\mathcal{F}_s]] \overset{2.12(2)}{=} \mathbb{E}_\mu[f_\sigma] = \mathbb{E}_\mu[f_\sigma \mathbb{1}_B] + \mathbb{E}_\mu[f_\sigma \mathbb{1}_{B^{\mathrm{c}}}]$$

Since

$$\mathbb{E}_\mu[f_\sigma \mathbb{1}_{B^{\mathrm{c}}}] \overset{2.20}{\leq} \mathbb{E}_\mu[\mathbb{E}_\mu[f_v|\mathcal{F}_\sigma]\mathbb{1}_{B^{\mathrm{c}}}] \overset{2.12(6)}{=} \mathbb{E}_\mu[\mathbb{E}_\mu[f_v \mathbb{1}_{B^{\mathrm{c}}}|\mathcal{F}_\sigma]] \overset{2.12(2)}{=} \mathbb{E}_\mu[f_v \mathbb{1}_{B^{\mathrm{c}}}]$$

by the $\mathcal{F}_\sigma$-measurability of $\mathbb{1}_{B^{\mathrm{c}}}$, we can rewrite (3.6) by using the inequalities above, which gives us

$$\lambda\, \mu(B) \leq \mathbb{E}_\mu[f_v \mathbb{1}_{B^{\mathrm{c}}}] - \mathbb{E}_\mu[f_s \vee (-\lambda)].$$

This concludes the proof of (3.2) for finite T.

Just like before, in the next step suppose T is countably infinite, then there exists an increasing sequence $(S_n)_{n\in\mathbb{N}}$ of non-empty finite sets such that $T = \bigcup_{n\in\mathbb{N}} S_n$. Again, let $(\lambda_m)_{m\in\mathbb{N}}$ be a sequence in $(-\infty, \lambda)$ such that $\lambda_m \nearrow \lambda$ as $m \to \infty$ and define

$$B_{m,n} = \Big\{\min_{t\in S_n} f_t \le -\lambda_m\Big\},\ B_m = \bigcup_{n\in\mathbb{N}} B_{m,n} \text{ and } B = \Big\{\inf_{t\in T} f_t \le -\lambda\Big\}.$$

Then, $B = \bigcap_{m\in\mathbb{N}} B_m$. This time $(B_{m,n})_{n\in\mathbb{N}}$ is a decreasing sequence for every $m \in \mathbb{N}$, whereas $(B_m)_{m\in\mathbb{N}}$ is increasing. By applying the dominated convergence theorem as stated in Theorem A.17 twice and what we have just proved for finite sets, it follows that

$$\begin{aligned}
\lambda\,\mu(B) &= \lim_{m\to\infty} \lambda_m\,\mu(B_m)\\
&= \lim_{m\to\infty}\Big(\lim_{n\to\infty} \lambda_m\,\mu(B_{m,n})\Big)\\
&\le \lim_{m\to\infty}\Big(\lim_{n\to\infty} \mathbb{E}_\mu[f_v \mathbb{1}_{B^c_{m,n}}] - \mathbb{E}_\mu[f_s \vee (-\lambda_m)]\Big)\\
&\overset{A.17}{=} \lim_{m\to\infty}(\mathbb{E}_\mu[f_v \mathbb{1}_{B^c_m}] - \mathbb{E}_\mu[f_s \vee (-\lambda_m)])\\
&\overset{A.17}{=} \mathbb{E}_\mu[f_v \mathbb{1}_B] - \mathbb{E}_\mu[f_s \vee (-\lambda)],
\end{aligned}$$

which proves (3.2) for this case.

Lastly we will treat the case of a non-degenerate interval T and a left- or right-continuous submartingale f. Once more define $t^\circ = \inf T$ and $t^* = \sup T$ as the endpoints in $\overline{\mathbb{R}}$ and note that the set $\tilde{T} := T \cap (\mathbb{Q} \cup \{t^\circ, t^*\})$ is countable. In this case we have $\inf_{t\in T} f_t = \inf_{t\in\tilde{T}} f_t$ by the left- or right-continuity of f and hence, (3.2) follows. □

Maximum inequalities play a vital role in studying fluctuations of random processes. Theorem 3.1 is the key to proving Doob's classical L^p-inequality for submartingales, as we will see shortly.

Theorem 3.2. Doob's classical L^p-inequality for submartingales and $p > 1$
Let $(\Omega, \mathcal{F}, \mathbb{F}, \mu)$ be a σ-finite filtered measure space and $f := (f_t)_{t\in T}$ be a positive submartingale according to Definition 2.13. Define $f^* = \sup_{t\in T} f_t$ and $f^*_u = \sup_{t\in T, t\le u} f_t$ for every $u \in T$.

- *If T is countable or*
- *if T is a non-degenerate interval and f is left- or right-continuous,*

we have for $u \in T$ and every $1 < p < \infty$

$$\mathbb{E}_\mu[(f^*_u)^p] \le \Big(\frac{p}{p-1}\Big)^p \mathbb{E}_\mu[f^p_u]. \tag{3.7}$$

Furthermore,

$$\mathbb{E}_\mu[(f^*)^p] \leq \left(\frac{p}{p-1}\right)^p \sup_{t\in T} \mathbb{E}_\mu[f_t^p]. \tag{3.8}$$

In order to prove Doob's inequality in the above version, we first need to take note of a few necessary lemmata which will help us later on. Please, refer to [12, Theorem 13.8 and Corollary 13.13] for the proofs of Lemma 3.3 and Lemma 3.4.

Lemma 3.3. TONELLI'S THEOREM
Let $(\Sigma, \mathcal{F}, \mu)$ and $(\Omega, \mathcal{G}, \nu)$ be σ-finite measure spaces and let $f : \Sigma \times \Omega \to [0, \infty]$ be a $\mathcal{F} \otimes \mathcal{G}$-measurable function. Then

(1) *$\sigma \mapsto f(\sigma, \omega)$ is $\mathcal{G}$-measurable for all $\omega \in \Omega$ and $\omega \mapsto f(\sigma, \omega)$ is $\mathcal{F}$-measurable for all $\sigma \in \Sigma$;*

(2) *$\sigma \mapsto \int_\Omega f(\sigma, \omega)\, \nu(d\omega)$ is $\mathcal{G}$-measurable and $\omega \mapsto \int_\Sigma f(\sigma, \omega)\, \mu(d\sigma)$ is $\mathcal{F}$-measurable;*

(3) *and*

$$\int_{\Sigma\times\Omega} f\, d(\mu\otimes\nu) = \int_\Sigma \int_\Omega f(\sigma, \omega)\, \mu(d\sigma)\, \nu(d\omega) = \int_\Omega \int_\Sigma f(\sigma, \omega)\, \nu(d\omega)\, \mu(d\sigma), \tag{3.9}$$

which is $[0, \infty]$-valued.

Lemma 3.4. *As usual, let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and let $\varphi : [0, \infty) \to [0, \infty)$ be an increasing and continuously differentiable function with $\varphi(0) = 0$. Then for all non-negative, real-valued and $\mathcal{F}$-measurable functions f we have*

$$\int \varphi \circ f\, d\mu = \int_0^\infty \varphi'(x)\mu(\{f \geq x\})\, dx. \tag{3.10}$$

Particularly, for $\varphi(x) := x^p$, $p \geq 1$, we have

$$\int f^p\, d\mu = \int_0^\infty px^{p-1}\mu(\{f \geq x\})\, dx. \tag{3.11}$$

Now we have all the tools we need to prove Doob's inequality as stated in Theorem 3.2. [12, Theorem 19.12] proves the theorem for his definition of martingales (see Remark 2.14) in σ-finite measure spaces and time $\mathbb{N}$. We will use his approach and expand it to our definition of submartingales and the conditional expectation. We use [13, Theorem 4.77] for our proof and proceed in a similar manner.

Proof. Theorem 3.2
Fix $u \in T$ and assume $\mathbb{E}_\mu[f_u^p] < \infty$ since (3.7) and (3.8) are trivial otherwise. Note that the function $\mathbb{R}^+ \ni x \mapsto x^p$ is increasing and convex. Thus, $(f_t^p)_{t\in T}$ is again a submartingale according to Remark 2.16(6), where the integrability of f_t^p follows for all $t \in T$ with $t \leq u$, because $\mathbb{E}_\mu[f_t^p] \leq \mathbb{E}_\mu[\mathbb{E}_\mu[f_u^p|\mathcal{F}_t]] = \mathbb{E}_\mu[f_u^p] < \infty$ by the submartingale property in (2.15) and the law of total expectation in Theorem 2.12(2).

Now, let $S \subset T \cap [-\infty, u]$ be finite and define $f_S^* = \max_{s\in S} f_s$. Again, it follows that $(f_S^*)^p$ is μ-integrable, because

$$0 \leq \mathbb{E}_\mu[(f_S^*)^p] \overset{\text{A.12(3)}}{\leq} \mathbb{E}_\mu\left[\sum_{s\in S} f_s^p\right] \overset{\text{A.12(2)}}{=} \sum_{s\in S} \underbrace{\mathbb{E}_\mu[f_s^p]}_{<\infty} < \infty,$$

since S is finite.

We further assume that $\mathbb{E}_\mu[f_S^p] > 0$, since otherwise (3.7) holds for f_S^* in place of f_u^* otherwise. With this and by using one of Doob's maximal inequalities (see Theorem 3.1) we have

$$\begin{aligned}
\mathbb{E}_\mu[(f_S^*)^p] &= \int (f_S^*)^p \, d\mu \\
&\overset{(3.11)}{=} \int_0^\infty p x^{p-1} \mu(\{f_S^* \geq x\}) \, dx \\
&\overset{(3.1)}{\leq} p \int_0^\infty x^{p-1} \left(\frac{1}{x} \int_{\{f_S^* \geq x\}} f_u \, d\mu \right) dx.
\end{aligned}$$

Now we can use Tonelli's theorem and integrate, which yields

$$\begin{aligned}
p \int_0^\infty \int_{\{f_S^* \geq x\}} x^{p-2} f_u \, d\mu \, dx &\overset{3.3}{=} p \int f_u \left(\int_0^{f_S^*} x^{p-2} \, dx \right) d\mu \\
&= \frac{p}{p-1} \int f_u (f_S^*)^{p-1} \, d\mu.
\end{aligned}$$

By using Hölder's inequality in Theorem A.15 we arrive at

$$\frac{p}{p-1} \int f_u (f_S^*)^{p-1} \, d\mu \leq \frac{p}{p-1} \left(\int f_u^p \, d\mu \right)^{1/p} \left(\int (f_S^*)^p \, d\mu \right)^{1-1/p}.$$

Dividing by $\left(\int (f_S^*)^p \, d\mu\right)^{1-1/p}$ and raising the resulting inequality to the p-th power finally gives us

$$\mathbb{E}_\mu[(f_S^*)^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}_\mu[f_u^p], \tag{3.12}$$

which, of course, is (3.7) for f_S^* in place of f_u^*.

Now, let $S \subset T \cap [-\infty, u]$ be countably infinite, then there exists an increasing sequence $(S_n)_{n\in\mathbb{N}}$ of finite subsets of S such that $S = \bigcup_{n\in\mathbb{N}} S_n$. For these subsets we define $f_{S_n}^* = \max_{s\in S_n} f_s$. Then $f_{S_n}^* \nearrow f_S^* := \sup_{s\in S} f_s$ as $n \to \infty$. By applying the monotone convergence theorem as stated in Theorem A.18 we have

$$\mathbb{E}_\mu[(f_S^*)^p] = \mathbb{E}_\mu\left[\lim_{n\to\infty}(f_{S_n}^*)^p\right] \overset{\text{A.18}}{=} \lim_{n\to\infty}\mathbb{E}_\mu[(f_{S_n}^*)^p] \overset{(3.12)}{\leq} \left(\frac{p}{p-1}\right)^p \mathbb{E}_\mu[f_u^p],$$

which, again, is (3.7) for f_S^* in place of f_u^*.

In case T is an interval, define $t^\circ = \inf T$. Then $S := \{t \in T | t \leq u\} \cap (\mathbb{Q} \cup \{t^\circ, u\})$ is countable and we can apply our former results to this case since $f_u^* = \sup_{s\in S} f_s$ due to the left- or right-continuity of f.

Now, (3.8) remains to be proven. First, note that (3.8) is in fact an upper bound to (3.7). If $t^* = \sup T$ is an element of T itself then we have (3.8) for t^* in place of u. Otherwise, let $(u_n)_{n\in\mathbb{N}}$ in T be an increasing sequence such that $u_n \nearrow t^*$ as $n \to \infty$. Then $f_{u_n}^* \nearrow f^*$ pointwise on Ω as $n \to \infty$. Thus, we may apply the monotone convergence theorem once again and it follows

$$\mathbb{E}_\mu[(f^*)^p] \overset{\text{A.18}}{=} \lim_{n\to\infty}\mathbb{E}_\mu[(f_{u_n}^*)^p] \leq \left(\frac{p}{p-1}\right)^p \sup_{t\in T}\mathbb{E}_\mu[f_t^p],$$

which concludes the proof. □

At the beginning of this chapter we generalised Doob's maximum inequalities to our new definition of martingales on σ-finite measure spaces. Furthermore, we proved that Doob's classical L^p-inequalities hold true as well for σ-finite measure spaces and submartingales as defined in Definition 2.13. We chose to approach the proofs rather indirectly in order to use some of the observations we have already introduced in the course of this thesis to extend the martingale theory. As mentioned before, there is a more modern way to go about proving Doob's classical L^p-inequality and also Doob's maximum inequalities by relying on purely deterministic inequalities. We will make use of this approach in the following chapter and show that Doob's inequalities can be improved and generalised further by considering σ-integrable submartingales.

Chapter 4

Improved Versions of Doob's Maximum Inequalities

Doob's maximum inequalities can also be proven by relying on deterministic inequalities. The proof of the classical maximum inequalities in Theorem 3.1 relies on a theoretical approach using theory we have developed for σ-integrable submartingales. However, we wish to present further improvements – or rather generalisations – of the inequalities in Theorem 3.1, by using a simple deterministic inequality proved by [13].

Let it be noted that the proof of the following lemma is not of the author's making but will be presented here for the sake of completeness. Please refer to [13, Proposition 4.70] for the original proposition and its proof if interested.

Lemma 4.1. *Define* $\overline{x}_k = \max\{x_0, \dots, x_k\}$ *and* $\underline{x}_k = \min\{x_0, \dots, x_k\}$ *for* $x_0, \dots, x_n \in \mathbb{R}$ *and* $k \in \{0, \dots, n\}$ *and* $\Delta x_{k+1} = x_{k+1} - x_k$ *for* $k \in \{0, \dots, n-1\}$ *and* $n \in \mathbb{N}$. *Then we have for every* $\lambda \in \mathbb{R}$ *that*

$$\lambda \mathbb{1}_{\{\overline{x}_n \geq \lambda\}} \leq x_n \mathbb{1}_{\{\overline{x}_n \geq \lambda\}} - \sum_{k=0}^{n-1} \mathbb{1}_{\{\overline{x}_k \geq \lambda\}} \Delta x_{k+1} - (x_0 - \lambda) \mathbb{1}_{\{x_0 \geq \lambda\}}, \tag{4.1}$$

where (4.1) *holds with equality if, and only if,* $\overline{x}_n < \lambda$ *or* $x_0 \geq \lambda$ *or the smallest* $k \in \{1, \dots, n\}$ *such that* $x_k \geq \lambda$ *satisfies* $x_k = \lambda$.

Furthermore,

$$\lambda \mathbb{1}_{\{\underline{x}_n \leq -\lambda\}} \leq x_n \mathbb{1}_{\{\underline{x}_n > -\lambda\}} - \sum_{k=0}^{n-1} \mathbb{1}_{\{\underline{x}_k > -\lambda\}} \Delta x_{k+1} - (x_0 \vee (-\lambda)), \tag{4.2}$$

where (4.2) *holds with equality if, and only if,* $\underline{x}_n > -\lambda$ *or* $x_0 \leq -\lambda$ *or the smallest* $k \in \{1, \dots, n\}$ *such that* $x_k \leq -\lambda$ *satisfies* $x_k = -\lambda$.

Proof. This proof will be done case-by-case depending of the first time the numbers $x_0, \dots, x_n$ reach or cross a given threshold.

- $\overline{x}_n < \lambda$: This implies that $\overline{x}_k < \lambda$ for all $k \in \{0, \dots, n\}$ and both sides of (4.1) equal zero.
- $x_0 \geq \lambda$: This implies that $\overline{x}_0, \dots, \overline{x}_n \geq \lambda$ and the right-hand side of (4.1) is a telescopic sum, which equals λ.
- $x_0 < \lambda$ and $\overline{x}_n \geq \lambda$: This implies the existence of a minimal $k \in \{1, \dots, n\}$ with $\overline{x}_0, \dots, \overline{x}_{k-1} < \lambda$ and $\overline{x}_k, \dots, \overline{x}_n \geq \lambda$. Therefore, $x_k \geq \lambda$ and thus, the right-hand side of (4.1) reduces to x_k and the inequality holds true.

Inequality (4.2) follows in the same manner.

- $\underline{x}_n > -\lambda$: This implies that $\underline{x}_k > -\lambda$ for all $k \in \{0, \dots, n\}$ and both sides of (4.2) equal zero.
- $x_0 \leq -\lambda$: This implies that $\underline{x}_0, \dots, \underline{x}_n \leq -\lambda$ and both sides of (4.2) equal λ.
- $x_0 > -\lambda$ and $\underline{x}_n \leq -\lambda$: This implies the existence of a minimal $k \in \{1, \dots, n\}$ with $\underline{x}_0, \dots, \underline{x}_{k-1} > -\lambda$ and $\underline{x}_k, \dots, \underline{x}_n \leq -\lambda$. Therefore, $x_k \leq -\lambda$ and thus, the right-hand side of (4.2) reduces to $-x_k$ and the inequality holds true.

This concludes the proof. □

We will use the deterministic inequalities above to prove more refined versions of Doob's maximum inequalities. The inequalities below focus on σ-integrable submartingales, hence, Theorem 4.2 also generalises the claims of Theorem 3.1.

Theorem 4.2. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mu)$ be a σ-finite filtered measure space and $T \subset \overline{\mathbb{R}}$ with $s, v \in T$ such that $s \leq v$ and $T \subset [s, v]$. Assume that $\mu|_{\mathcal{F}_s}$ is σ-finite and let $(f_t)_{t \in T}$ be a $\mathbb{F}$-adapted sequence of functions such that f_t is σ-integrable w.r.t. $\mathcal{F}_s$ for all $t \in T$ and $f_t \leq \mathbb{E}_\mu[f_u|\mathcal{F}_t]$ μ-a.e. for all $t \leq u$ in T. Then for $f^*_{s,v} := \operatorname{ess\,sup}_{t \in T} f_t$ and $f^\circ_{s,v} := \operatorname{ess\,inf}_{t \in T} f_t$ and every $\lambda \in \mathbb{R}$ we have*

$$\begin{aligned} \lambda\, \mathbb{E}_\mu\big[\mathbb{1}_{\{f^*_{s,v} \geq \lambda\}}\big|\mathcal{F}_s\big] &\leq \mathbb{E}_\mu\big[f_v \mathbb{1}_{\{f^*_{s,v} \geq \lambda\}}\big|\mathcal{F}_s\big] - (f_s - \lambda)^+ \\ &\leq \mathbb{E}_\mu\big[f_v \mathbb{1}_{\{f^*_{s,v} \geq \lambda\}}\big|\mathcal{F}_s\big] \leq \mathbb{E}_\mu[f_v^+|\mathcal{F}_s] \quad \mu\text{-a.e.} \end{aligned} \tag{4.3}$$

and

$$\lambda\,\mathbb{E}_\mu\big[\mathbb{1}_{\{f^\circ_{s,v}\le-\lambda\}}\big|\mathcal{F}_s\big] \le \mathbb{E}_\mu\big[f_v\mathbb{1}_{\{f^\circ_{s,v}>-\lambda\}}\big|\mathcal{F}_s\big] - (f_s\vee(-\lambda)) \\ \le \mathbb{E}_\mu[f_v^+|\mathcal{F}_s] - f_s \quad \mu\text{-}a.e. \tag{4.4}$$

Define the function $\psi_\lambda(x) = (x-\lambda)^+ + x\vee(-\lambda)$ *for* $x\in\mathbb{R}$*. Then, by combining the above inequalities,*

$$\lambda\,\mathbb{E}_\mu\big[\mathbb{1}_{\{\operatorname{ess\,sup}_{t\in T}|f_t|\ge\lambda\}}\big|\mathcal{F}_s\big] \le \mathbb{E}_\mu\big[f_v\big(\mathbb{1}_{\{f^*_{s,v}\ge\lambda\}}+\mathbb{1}_{\{f^\circ_{s,v}\le-\lambda\}}\big)\big|\mathcal{F}_s\big] - \psi(f_s) \\ \le 2\,\mathbb{E}_\mu[f_v^+|\mathcal{F}_s] - f_s \quad \mu\text{-}a.e. \tag{4.5}$$

Proof. First, note that $(f_s-\lambda)^+\ge 0$ and $f_s\vee(-\lambda)\ge f_s$. Similarly, since $f_t\mathbb{1}_F \le f_t^+\mathbb{1}_F \le f_t^+$ for $F\in\mathcal{F}$ and $t\in T$, it follows that $\mathbb{E}_\mu[f_v\mathbb{1}_{\{f^*_{s,v}\ge\lambda\}}|\mathcal{F}_s] \le \mathbb{E}_\mu[f_v^+|\mathcal{F}_s]$. Hence, the second and third inequalities in (4.3) and (4.4) follow. Therefore, it suffices to show the first inequalities in (4.3) and (4.4).

We will start by proving (4.3) for non-empty finite $T=\{t_0,\dots,t_n\}$ with $s=t_0<\dots<t_n=v$. For this purpose let $j\in\{0,\dots,n\}$ and define $\bar g_j=\max\{f_{t_0},\dots,f_{t_n}\}$, then

$$\mathbb{E}_\mu\big[\mathbb{1}_{\{\bar g_j\ge\lambda\}}\Delta f_{t_{j+1}}\big|\mathcal{F}_{t_j}\big] \overset{2.12(6)}{=} \mathbb{1}_{\{\bar g_j\ge\lambda\}}\underbrace{\mathbb{E}_\mu[f_{t_{j+1}}-f_{t_j}|\mathcal{F}_{t_j}]}_{\ge 0\ \mu\text{-a.e.}} \ge 0 \quad \mu\text{-a.e.} \tag{4.6}$$

by the $\mathcal{F}_{t_j}$-measurability of $\mathbb{1}_{\{\bar g_j\ge\lambda\}}$. The first inequality in Lemma 4.1 now gives

$$\lambda\mathbb{1}_{\{\bar g_n\ge\lambda\}} \le f_{t_n}\mathbb{1}_{\{\bar g_n\ge\lambda\}} - \sum_{j=0}^{n-1}\mathbb{1}_{\{\bar g_j\ge\lambda\}}\Delta f_{t_{j+1}} - (f_{t_0}-\lambda)^+ \quad \mu\text{-a.e.} \tag{4.7}$$

By taking the conditional expectation w.r.t. $\mathcal{F}_{t_k}$ for $k=n-1$ of the inequality above we arrive at

$$\lambda\,\mathbb{E}_\mu\big[\mathbb{1}_{\{\bar g_n\ge\lambda\}}\big|\mathcal{F}_{t_k}\big] \le \mathbb{E}_\mu\big[f_{t_n}\mathbb{1}_{\{\bar g_n\ge\lambda\}}\big|\mathcal{F}_{t_k}\big] - \sum_{j=0}^{k-1}\mathbb{E}_\mu\big[\mathbb{1}_{\{\bar g_j\ge\lambda\}}\Delta f_{t_{j+1}}\big|\mathcal{F}_{t_k}\big] \\ - \mathbb{E}_\mu[(f_{t_0}-\lambda)^+|\mathcal{F}_{t_k}] \quad \mu\text{-a.e.},$$

where the last term of the sum for k is greater than zero by (4.6) and thus, we have a further upper bound for $\lambda\,\mathbb{E}_\mu[\mathbb{1}_{\{\bar g_n\ge\lambda\}}|\mathcal{F}_{t_k}]$ if we leave out the last term. Taking conditional expectations iteratively for $k=n-2,\dots,0$ and using the tower property in Theorem 2.12(7) yields

$$\lambda\,\mathbb{E}_\mu\big[\mathbb{1}_{\{\bar g_n\ge\lambda\}}\big|\mathcal{F}_s\big] \le \mathbb{E}_\mu\big[f_v\mathbb{1}_{\{\bar g_n\ge\lambda\}}\big|\mathcal{F}_s\big] - \mathbb{E}_\mu[(f_s-\lambda)^+|\mathcal{F}_s] \quad \mu\text{-a.e.}$$

Since $(f_s - \lambda)^+$ is $\mathcal{F}_s$-measurable, we may rid ourselves from the conditional expectation w.r.t. $\mathcal{F}_s$ in the last term of the right-hand side (see Theorem 2.12(1)). Hence,

$$\lambda\, \mathbb{E}_\mu\big[\mathbb{1}_{\{\bar{g}_n \geq \lambda\}}\big|\mathcal{F}_s\big] \leq \mathbb{E}_\mu\big[f_v \mathbb{1}_{\{\bar{g}_n \geq \lambda\}}\big|\mathcal{F}_s\big] - (f_s - \lambda)^+ \quad \mu\text{-a.e.},$$

which is (4.3) for $\max_{t\in T} f_t$ in place of $f^*_{s,v}$.

For the general case note that there exists a sequence $(S_n)_{n\in\mathbb{N}}$ of finite subsets of T with $s, v \in S_n$ for all $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} S_n = T$. For $\bar{f}_{S_n} := \max_{t\in S_n} f_t$ it follows that $\bar{f}_{S_n} \nearrow f^*_{s,v}$ as $n \to \infty$ by Theorem A.24. Furthermore, let $(\lambda_m)_{m\in\mathbb{N}}$ be a sequence in $(-\infty, \lambda)$ such that $\lambda_m \nearrow \lambda$ as $m \to \infty$. Similarly to the proof of Theorem 3.2 let us make the following definitions.

$$A_{m,n} = \{\bar{f}_{S_n} \geq \lambda_m\},\;\; A_m = \bigcup_{n\in\mathbb{N}} A_{m,n},\;\; A = \{f^*_{s,v} \geq \lambda\}. \tag{4.8}$$

The sequence $(A_{m,n})_{n\in\mathbb{N}}$ is increasing for ever $m \in \mathbb{N}$, whereas $(A_m)_{m\in\mathbb{N}}$ is decreasing. Moreover, $A = \bigcap_{m\in\mathbb{N}} A_m$. We now know that (4.3) holds for every finite set S_n, $n \in \mathbb{N}$ and every λ_m, $m \in \mathbb{N}$. Hence, we can apply the conditional monotone convergence theorem in Theorem 2.12(5) to the positive sequence $(\mathbb{1}_{A_{m,n}})_{n\in\mathbb{N}}$ and the conditional dominated convergence theorem in Theorem 2.12(10) to the bounded sequence[1] $(f_v \mathbb{1}_{A_{m,n}})_{n\in\mathbb{N}}$ and conclude

$$\begin{aligned}
\lambda_m \mathbb{E}_\mu[\mathbb{1}_{A_m}|\mathcal{F}_s] &= \lambda_m \lim_{n\to\infty} \mathbb{E}_\mu[\mathbb{1}_{A_{m,n}}|\mathcal{F}_s] \\
&\leq \lim_{n\to\infty} \mathbb{E}_\mu[f_v \mathbb{1}_{A_{m,n}}|\mathcal{F}_s] - (f_s - \lambda_m)^+ \\
&= \mathbb{E}_\mu[f_v \mathbb{1}_{A_m}|\mathcal{F}_s] - (f_s - \lambda_m)^+ \quad \mu\text{-a.e.}
\end{aligned}$$

for every $m \in \mathbb{N}$. By applying the conditional monotone convergence theorem and the conditional dominated convergence theorem once more we can conclude that

$$\begin{aligned}
\lambda\, \mathbb{E}_\mu[\mathbb{1}_A|\mathcal{F}_s] &= \lim_{m\to\infty} \lambda_m\, \mathbb{E}_\mu[\mathbb{1}_{A_m}|\mathcal{F}_s] \\
&\leq \lim_{m\to\infty} \mathbb{E}_\mu[f_v \mathbb{1}_{A_m}|\mathcal{F}_s] - (f_s - \lambda_m)^+ \\
&= \mathbb{E}_\mu[f_v \mathbb{1}_A|\mathcal{F}_s] - (f_s - \lambda)^+ \quad \mu\text{-a.e.}
\end{aligned}$$

Now we will show (4.4) in the same manner. Again let $T = \{t_0, \ldots, t_n\}$ with $s = t_0 < \ldots < t_n = v$ be finite and define $\underline{g}_j = \min\{f_{t_0}, \ldots, f_{t_n}\}$ for

[1] This is possible since f_v is σ-integrable w.r.t. $\mathcal{F}_s$, which is equivalent to $|f_v|$ being σ-integrable w.r.t. $\mathcal{F}_s$ and because $f_v \mathbb{1}_{A_{m,n}} \leq |f_v|$ for all $m, n \in \mathbb{N}$.

$j \in \{0, \dots, n\}$. Then

$$\mathbb{E}_\mu\big[\mathbb{1}_{\{\underline{g}_j > -\lambda\}} \Delta f_{t_{j+1}} \big| \mathcal{F}_{t_j}\big] \overset{2.12(6)}{=} \mathbb{1}_{\{\underline{g}_j > -\lambda\}} \underbrace{\mathbb{E}_\mu[f_{t_{j+1}} - f_{t_j} | \mathcal{F}_{t_j}]}_{\geq 0\, \mu\text{-a.e.}} \geq 0 \quad \mu\text{-a.e.} \tag{4.9}$$

by the $\mathcal{F}_{t_j}$-measurability of $\mathbb{1}_{\{\underline{g}_j \geq \lambda\}}$. The second inequality in Lemma 4.1 now gives

$$\lambda \mathbb{1}_{\{\underline{g}_n \leq -\lambda\}} \leq f_{t_n} \mathbb{1}_{\{\underline{g}_n > -\lambda\}} - \sum_{j=0}^{n-1} \mathbb{1}_{\{\underline{g}_j > -\lambda\}} \Delta f_{t_{j+1}} - (f_{t_0} \vee (-\lambda)) \quad \mu\text{-a.e.}$$

By taking the conditional expectation w.r.t. $\mathcal{F}_{t_k}$ for $k = n-1$ of the inequality above we arrive at

$$\begin{aligned}\lambda \mathbb{E}_\mu\big[\mathbb{1}_{\{\underline{g}_n \leq -\lambda\}} \big| \mathcal{F}_{t_k}\big] \leq \mathbb{E}_\mu\big[f_{t_n} \mathbb{1}_{\{\underline{g}_n > -\lambda\}} | \mathcal{F}_{t_k}\big] - \sum_{j=0}^{k-1} \mathbb{E}_\mu\big[\mathbb{1}_{\{\underline{g}_j > -\lambda\}} \Delta f_{t_{j+1}} \big| \mathcal{F}_{t_k}\big] \\ - \mathbb{E}_\mu[f_{t_0} \vee (-\lambda) | \mathcal{F}_{t_k}] \quad \mu\text{-a.e.},\end{aligned}$$

where – just like in the first part of the proof – leaving out the last term of the sum for k gives a further upper bound by (4.9). Taking conditional expectations iteratively for $k = n-2, \dots, 0$ and using the tower property in Theorem 2.12(7) yields

$$\lambda \mathbb{E}_\mu\big[\mathbb{1}_{\{\underline{g}_n \leq -\lambda\}} \big| \mathcal{F}_s\big] \leq \mathbb{E}_\mu\big[f_v \mathbb{1}_{\{\underline{g}_n > -\lambda\}} \big| \mathcal{F}_s\big] - \mathbb{E}_\mu[f_s \vee (-\lambda) | \mathcal{F}_s] \quad \mu\text{-a.e.}$$

Once more the $\mathcal{F}_s$-measurability of $f_s \vee (-\lambda)$ gives the desired result:

$$\lambda \mathbb{E}_\mu\big[\mathbb{1}_{\{\underline{g}_n \leq -\lambda\}} \big| \mathcal{F}_s\big] \leq \mathbb{E}_\mu\big[f_v \mathbb{1}_{\{\underline{g}_n > -\lambda\}} \big| \mathcal{F}_s\big] - f_s \vee (-\lambda) \quad \mu\text{-a.e.},$$

which is (4.4) for $\min_{t \in T} f_t$ in place of $f^\circ_{s,v}$.

For the general case note there exists a sequence $(S_n)_{n \in \mathbb{N}}$ of finite subsets of T with $s, v \in S_n$ for all $n \in \mathbb{N}$ such that $\bigcup_{n \in \mathbb{N}} S_n = T$. For $\underline{f}_{S_n} := \min_{t \in S_n} f_t$ it follows that $\underline{f}_{S_n} \nearrow f^\circ_{s,v}$ as $n \to \infty$ by Theorem A.24. Furthermore, let $(\lambda_m)_{m \in \mathbb{N}}$ be a sequence in $(-\infty, \lambda)$ such that $\lambda_m \nearrow \lambda$ as $m \to \infty$ (then $-\lambda_m \searrow -\lambda$ as $m \to \infty$). Similarly to the proof of Theorem 3.2 let us make the following definitions.

$$B_{m,n} = \{\underline{f}_{S_n} \leq -\lambda_m\}, \; B_m = \bigcup_{n \in \mathbb{N}} B_{m,n}, \; B = \{f^\circ_{s,v} \leq -\lambda\}. \tag{4.10}$$

The sequence $(B_{m,n})_{n \in \mathbb{N}}$ is decreasing for ever $m \in \mathbb{N}$, whereas $(B_m)_{m \in \mathbb{N}}$ is increasing. Moreover, $B = \bigcap_{m \in \mathbb{N}} B_m$. We now know that (4.3) holds for every finite set S_n, $n \in \mathbb{N}$ and every λ_m, $m \in \mathbb{N}$. Hence, we can apply the conditional monotone convergence theorem in Theorem 2.12(5) to the positive sequence $(\mathbb{1}_{B_{m,n}})_{n\in\mathbb{N}}$ and the conditional dominated convergence theorem in Theorem 2.12(10) to the bounded sequence[2] $(f_v \mathbb{1}_{B^c_{m,n}})_{n\in\mathbb{N}}$ and conclude

$$\begin{aligned}\lambda_m \mathbb{E}_\mu[\mathbb{1}_{B_m}|\mathcal{F}_s] &= \lambda_m \lim_{n\to\infty} \mathbb{E}_\mu[\mathbb{1}_{B_{m,n}}|\mathcal{F}_s] \\ &\leq \lim_{n\to\infty} \mathbb{E}_\mu[f_v \mathbb{1}_{B^c_{m,n}}|\mathcal{F}_s] - f_s \vee (-\lambda_m) \\ &= \mathbb{E}_\mu[f_v \mathbb{1}_{B^c_m}|\mathcal{F}_s] - f_s \vee (-\lambda_m) \quad \mu\text{-a.e.}\end{aligned}$$

for every $m \in \mathbb{N}$. By applying the conditional monotone convergence theorem and the conditional dominated convergence theorem once more we can conclude that

$$\begin{aligned}\lambda\, \mathbb{E}_\mu[\mathbb{1}_B|\mathcal{F}_s] &= \lim_{m\to\infty} \lambda_m\, \mathbb{E}_\mu[\mathbb{1}_{B_m}|\mathcal{F}_s] \\ &\leq \lim_{m\to\infty} \mathbb{E}_\mu[f_v \mathbb{1}_{B^c_m}|\mathcal{F}_s] - f_s \vee (-\lambda_m) \\ &= \mathbb{E}_\mu[f_v \mathbb{1}_{B^c}|\mathcal{F}_s] - f_s \vee (-\lambda) \quad \mu\text{-a.e.}\end{aligned}$$

This concludes the proof. □

Remark 4.3. The claims of Theorem 4.2 can also be derived by similar methods like in the proof of Theorem 3.1 on page 30. For this purpose let T satisfy one of the conditions in Theorem 3.1 and let $(f_t)_{t\in T}$, which fulfills all the assumptions in Theorem 4.2, be left- or right-continuous. Then (4.3), (4.4) and (4.5) follow.

Indeed: Consider the setting and assumptions in Theorem 4.2 and define[3] $\tau = v \wedge \min\{t \in T : f_t \geq \lambda\}$, where $\min \emptyset := t^*$. First, let T be a non-empty finite set. Similarly to the proof of Theorem 3.1 on page 30 let us take the conditional expectation of (3.4) w.r.t. $\mathcal{F}_s$, where[4] $A := \{\max_{t\in T} f_t \geq \lambda\}$. Then we immediately arrive at (4.3), when observing that

$$\mathbb{E}_\mu[f_\tau \mathbb{1}_A|\mathcal{F}_s] \leq \mathbb{E}_\mu[\mathbb{E}_\mu[f_v|\mathcal{F}_\tau]\mathbb{1}_A|\mathcal{F}_s] = \mathbb{E}_\mu[\mathbb{E}_\mu[f_v \mathbb{1}_A|\mathcal{F}_\tau]|\mathcal{F}_s] = \mathbb{E}_\mu[f_v \mathbb{1}_A|\mathcal{F}_s] \tag{4.11}$$

μ-a.e. by Lemma 2.20 and Theorem 2.12(6) and (7).

When we consider $B := \{\min_{t\in T} f_t \leq -\lambda\}$ and[5] $\sigma := v \wedge \min\{t \in T : f_t \leq -\lambda\}$ and look at the conditional expectation of (3.5) w.r.t. $\mathcal{F}_s$, we arrive at

[2]This is possible since f_v is σ-integrable w.r.t. $\mathcal{F}_s$, which is equivalent to $|f_v|$ being σ-integrable w.r.t. $\mathcal{F}_s$ and because $f_v \mathbb{1}_{B^c_{m,n}} \leq |f_v|$ for all $m, n \in \mathbb{N}$.

[3]Note that τ is a stopping time w.r.t. $\mathbb{F}$ by Theorem A.22(2) and Lemma 2.19.

[4]Note that $A \in \mathcal{F}_\tau$, since $A = \bigcup_{t\in T}\{f_t \geq \lambda\}$ and $A \cap \{\tau \leq u\} = \bigcup_{t\in T, t\leq u}\{f_t \geq \lambda\} \in \mathcal{F}_u$ for all $u \in T$.

[5]Note that σ is also a stopping time w.r.t. $\mathbb{F}$ by Theorem A.22(2) and Lemma 2.19.

(4.4) instead, since

$$f_s \leq \mathbb{E}_\mu[f_\sigma|\mathcal{F}_s] = \mathbb{E}_\mu[f_\sigma \mathbb{1}_B|\mathcal{F}_s] + \mathbb{E}_\mu[f_\sigma \mathbb{1}_{B^c}|\mathcal{F}_s] \quad \mu\text{-a.e.} \tag{4.12}$$

and[6]

$$\begin{aligned}\mathbb{E}_\mu[f_\sigma \mathbb{1}_{B^c}|\mathcal{F}_s] &\leq \mathbb{E}_\mu[\mathbb{E}_\mu[f_v|\mathcal{F}_\sigma]\mathbb{1}_{B^c}|\mathcal{F}_s] \overset{2.12(6)}{=} \mathbb{E}_\mu[\mathbb{E}_\mu[f_v \mathbb{1}_{B^c}|\mathcal{F}_\sigma]|\mathcal{F}_s] \\ &= \mathbb{E}_\mu[f_v \mathbb{1}_{B^c}|\mathcal{F}_s] \quad \mu\text{-a.e.}\end{aligned}$$

by Lemma 2.20 and the tower property.

The other cases of T follow in the same manner as in the proof of Theorem 4.2. Therefore, please refer to the proof above for the final steps.

Corollary 4.4. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mu)$ be a σ-finite filtered measure space and $T \subset \overline{\mathbb{R}}$ with $v \in T$ and define $t^\circ = \inf T$. Assume that $\mu|_{\mathcal{F}_{t^\circ}}$ is σ-finite with $\mathcal{F}_{t^\circ} := \bigcap_{t\in T} \mathcal{F}_t$. Let $(f_t)_{t\in T}$ be a $\mathbb{F}$-adapted sequence of functions such that f_v is σ-integrable w.r.t. $\mathcal{F}_{t^\circ}$ for all $t \in T$ and $f_t \leq \mathbb{E}_\mu[f_u|\mathcal{F}_t]$ μ-a.e. for all $t \leq u$ in T. Then for $f_v^* := \operatorname{ess\,sup}_{t\in T, t\leq v} f_t$ and $f_v^\circ := \operatorname{ess\,inf}_{t\in T, t\leq v} f_t$ the following inequalities hold[7].*

$$\begin{aligned}\lambda\, \mathbb{E}_\mu\big[\mathbb{1}_{\{f_v^*\geq\lambda\}}\big|\mathcal{F}_{t^\circ}\big] &\leq \mathbb{E}_\mu\big[f_v \mathbb{1}_{\{f_v^*\geq\lambda\}}\big|\mathcal{F}_{t^\circ}\big] - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[(f_t-\lambda)^+|\mathcal{F}_{t^\circ}] \\ &\leq \mathbb{E}_\mu\big[f_v \mathbb{1}_{\{f_v^*\geq\lambda\}}\big|\mathcal{F}_{t^\circ}\big] \leq \mathbb{E}_\mu[f_v^+|\mathcal{F}_{t^\circ}] \quad \mu\text{-a.e.}\end{aligned} \tag{4.13}$$

$$\begin{aligned}\lambda\, \mathbb{E}_\mu\big[\mathbb{1}_{\{f_v^\circ\leq-\lambda\}}\big|\mathcal{F}_{t^\circ}\big] &\leq \mathbb{E}_\mu\big[f_v \mathbb{1}_{\{f_v^\circ>-\lambda\}}\big|\mathcal{F}_{t^\circ}\big] - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[f_t \vee (-\lambda)|\mathcal{F}_{t^\circ}] \\ &\leq \mathbb{E}_\mu\big[f_v \mathbb{1}_{\{f_v^\circ>-\lambda\}}\big|\mathcal{F}_{t^\circ}\big] - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[f_t|\mathcal{F}_{t^\circ}] \\ &\leq \mathbb{E}_\mu[f_v^+|\mathcal{F}_{t^\circ}] - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[f_t|\mathcal{F}_{t^\circ}] \quad \mu\text{-a.e.}\end{aligned} \tag{4.14}$$

Furthermore, for the function ψ defined in Theorem 4.2 it follows that

$$\begin{aligned}\lambda\, \mathbb{E}_\mu\big[\mathbb{1}_{\{\operatorname{ess\,sup}_{t\in T}|f_t|\geq\lambda\}}\big|\mathcal{F}_{t^\circ}\big] \leq \mathbb{E}_\mu\big[f_v\big(\mathbb{1}_{\{f_v^*\geq\lambda\}} + \mathbb{1}_{\{f_v^\circ>-\lambda\}}\big)\big|\mathcal{F}_{t^\circ}\big] \\ - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[\psi(f_t)|\mathcal{F}_{t^\circ}] \quad \mu\text{-a.e.}\end{aligned} \tag{4.15}$$

[6]Note that $B^c \in \mathcal{F}_\sigma$, since $B = \bigcap_{t\in T}\{f_t > -\lambda\}$ and $B \cap \{\sigma \leq u\} = \bigcap_{t\in T, t\leq u}\{f_t > -\lambda\} \in \mathcal{F}_u$ for all $u \in T$.

[7]When dealing with probability spaces and martingales $(M_t)_{t\in T}$ (see (1.1) in the Introduction) Doob's backward convergence theorem as stated in [13, Theorem 7.49] gives conditions for the existence of $(M_{t^\circ} - \lambda)^+$, hence there is no need to concern oneself with the essential infimum. In particular, the theorem states that the term $\operatorname{ess\,inf}_{t\in T} \mathbb{E}[(M_t - \lambda)^+|\mathcal{F}_{t^\circ}]$ can be identified with the limit $\lim_{t\in F, t\searrow t^\circ}(M_t - \lambda)^+$ for each countable subset $F \subset T$ such that $t^\circ = \inf F$, if t° is not an element of T itself. Define $X_t = (M_t - \lambda)^+$ for $t \in T$, which is a submartingale by the conditional Jensen inequality. Then Doob's backward convergence theorem gives the existence of $\lim_{t\in F, t\searrow t^\circ} X_t$ a.s. and further claims that there exists an a.s. unique $\mathcal{F}_{t^\circ}$-measurable random variable X_{t° such that $X_{t^\circ} = \lim_{t\in F, t\searrow t^\circ} X_t$ and $X_{t^\circ} \leq \mathbb{E}[X_t|\mathcal{F}_{t^\circ}]$ a.s. for all $t \in T$. Similar observations can be made for $\operatorname{ess\,inf}_{t\in T} \mathbb{E}[M_t \vee (-\lambda)|\mathcal{F}_{t^\circ}]$ and $\operatorname{ess\,inf}_{t\in T} \mathbb{E}[\psi(M_t)|\mathcal{F}_{t^\circ}]$.

Proof. Again, it suffices to prove the first inequalities in (4.13) and (4.14). If $t^\circ \in T$, simply take $s := t^\circ$ and (4.13) and (4.14) follow immediately from (4.3) and (4.4), since $f^*_{t^\circ,v} = f^*_v$ and $f^\circ_{t^\circ,v} = f^\circ_v$. Otherwise, first take the conditional expectation of (4.3) w.r.t. $\mathcal{F}_{t^\circ}$ and apply the tower property. Now let $(s_n)_{n\in\mathbb{N}}$ in T be an increasing sequence such that $s_n \searrow t^\circ$ as $n \to \infty$. Then $f^*_{s_n,v} \to f^*_v$ and $f^\circ_{s_n,v} \to f^\circ_v$ μ-a.e. on Ω as $n \to \infty$. Furthermore, let $(\lambda_m)_{m\in\mathbb{N}}$ be a sequence in $(-\infty, \lambda)$ such that $\lambda_m \nearrow \lambda$ as $m \to \infty$. We will use the same trick as in the proof of Theorem 4.2 in order to show the proof for any $T \subset \overline{\mathbb{R}}$. For this purpose define $A_{m,n}$, A_m and A similarly to (4.8) on page 40 as well as $B_{m,n}$, B_m and B similarly to (4.10) on page 41, namely

$$A_{m,n} = \{f^*_{s_n,v} \geq \lambda_m\},\ A_m = \bigcup_{n\in\mathbb{N}} A_{m,n},\ A = \{f^*_v \geq \lambda\}$$

and

$$B_{m,n} = \{f^\circ_{s_n,v} \leq -\lambda_m\},\ B_m = \bigcup_{n\in\mathbb{N}} B_{m,n},\ B = \{f^\circ_v \leq -\lambda\}.$$

Note that $(A_{m,n})_{n\in\mathbb{N}}$ is an increasing sequence for every $m \in \mathbb{N}$, whereas $(A_m)_{m\in\mathbb{N}}$ is decreasing and $A = \bigcup_{m\in\mathbb{N}} A_m$. Moreover, $(B_{m,n})_{n\in\mathbb{N}}$ is a decreasing sequence for every $m \in \mathbb{N}$, whereas $(B_m)_{m\in\mathbb{N}}$ is increasing, and $B = \bigcap_{m\in\mathbb{N}} B_m$. We may now apply the conditional monotone convergence theorem in Theorem 2.12(5) and the conditional dominated convergence theorem in Theorem 2.12(10), which together with (4.3) (note that using the essential infimum just gives a further upper bound) yields

$$\begin{aligned}
\lambda\,\mathbb{E}_\mu[\mathbb{1}_A|\mathcal{F}_{t^\circ}] &= \lim_{m\to\infty}\Big(\lim_{n\to\infty} \lambda_m\,\mathbb{E}_\mu[\mathbb{1}_{A_{m,n}}|\mathcal{F}_{t^\circ}]\Big)\\
&\leq \lim_{m\to\infty}\Big(\lim_{n\to\infty} \mathbb{E}_\mu[f_v\mathbb{1}_{A_{m,n}}|\mathcal{F}_{t^\circ}]\Big) - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[(f_t-\lambda)^+|\mathcal{F}_{t^\circ}]\\
&= \lim_{m\to\infty}\big(\mathbb{E}_\mu[f_v\mathbb{1}_{A_m}|\mathcal{F}_{t^\circ}]\big) - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[(f_t-\lambda)^+|\mathcal{F}_{t^\circ}]\\
&= \mathbb{E}_\mu[f_v\mathbb{1}_A|\mathcal{F}_{t^\circ}] - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[(f_t-\lambda)^+|\mathcal{F}_{t^\circ}] \quad \mu\text{-a.e.}
\end{aligned}$$

The same arguments and (4.4) imply that

$$\begin{aligned}
\lambda\,\mathbb{E}_\mu[\mathbb{1}_B|\mathcal{F}_{t^\circ}] &= \lim_{m\to\infty}\Big(\lim_{n\to\infty} \lambda_m\,\mathbb{E}_\mu[\mathbb{1}_{B_{m,n}}|\mathcal{F}_{t^\circ}]\Big)\\
&\leq \lim_{m\to\infty}\Big(\lim_{n\to\infty} \mathbb{E}_\mu[f_v\mathbb{1}_{B^c_{m,n}}|\mathcal{F}_{t^\circ}]\Big) - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[f_t\vee(-\lambda)|\mathcal{F}_{t^\circ}]\\
&= \lim_{m\to\infty}\big(\mathbb{E}_\mu[f_v\mathbb{1}_{B^c_m}|\mathcal{F}_{t^\circ}]\big) - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[f_t\vee(-\lambda)|\mathcal{F}_{t^\circ}]\\
&= \mathbb{E}_\mu[f_v\mathbb{1}_{B^c}|\mathcal{F}_{t^\circ}] - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[f_t\vee(-\lambda)|\mathcal{F}_{t^\circ}] \quad \mu\text{-a.e.}
\end{aligned}$$

Now we only need to combine the two inequalities above to deduce (4.15). □

To make further improvements we would now like to free ourselves from any endpoints. This is discussed in the following corollary.

Corollary 4.5. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mu)$ be a σ-finite filtered measure space and $T \subset \overline{\mathbb{R}}$. Define $t^\circ = \inf T$ and $t^* = \sup T$ and assume that $\mu|_{\mathcal{F}_{t^\circ}}$ is σ-finite with $\mathcal{F}_{t^\circ} := \bigcap_{t \in T} \mathcal{F}_t$. Let $(f_t)_{t\in T}$ be a $\mathbb{F}$-adapted sequence of functions such that f_u is σ-integrable w.r.t. $\mathcal{F}_{t^\circ}$ and $f_t \le \mathbb{E}_\mu[f_u|\mathcal{F}_t]$ μ-a.e. for all $t \le u$ in T. Then for $f^* := \operatorname{ess\,sup}_{t\in T} f_t$ and $f^\circ := \operatorname{ess\,inf}_{t\in T} f_t$ the following inequalities hold*[8]*.*

$$\begin{aligned}\lambda\, \mathbb{E}_\mu\big[\mathbb{1}_{\{f^* \ge \lambda\}}\big|\mathcal{F}_{t^\circ}\big] &\le \operatorname*{ess\,sup}_{t\in T} \mathbb{E}_\mu\big[f_t \mathbb{1}_{\{f^*\ge\lambda\}}\big|\mathcal{F}_{t^\circ}\big] - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[(f_t-\lambda)^+|\mathcal{F}_{t^\circ}] \\ &\le \operatorname*{ess\,sup}_{t\in T} \mathbb{E}_\mu[f_t^+|\mathcal{F}_{t^\circ}] - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[(f_t-\lambda)^+|\mathcal{F}_{t^\circ}] \\ &\le \operatorname*{ess\,sup}_{t\in T} \mathbb{E}_\mu[f_t^+|\mathcal{F}_{t^\circ}] \quad \mu\text{-a.e.}\end{aligned} \tag{4.16}$$

$$\begin{aligned}\lambda\, \mathbb{E}_\mu\big[\mathbb{1}_{\{f^\circ \le -\lambda\}}\big|\mathcal{F}_{t^\circ}\big] &\le \operatorname*{ess\,sup}_{t\in T} \mathbb{E}_\mu\big[f_t \mathbb{1}_{\{f^\circ > -\lambda\}}\big|\mathcal{F}_{t^\circ}\big] - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[f_t \vee (-\lambda)|\mathcal{F}_{t^\circ}] \\ &\le \operatorname*{ess\,sup}_{t\in T} \mathbb{E}_\mu[f_t^+|\mathcal{F}_{t^\circ}] - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[f_t|\mathcal{F}_{t^\circ}] \quad \mu\text{-a.e.}\end{aligned} \tag{4.17}$$

Again, with ψ as defined in Theorem 4.2 the two inequalities above imply

$$\begin{aligned}\lambda\, \mathbb{E}_\mu\big[\mathbb{1}_{\{\operatorname{ess\,sup}_{t\in T}|f_t| \ge \lambda\}}\big|\mathcal{F}_{t^\circ}\big] \le \operatorname*{ess\,sup}_{t\in T} \mathbb{E}_\mu\big[f_t\big(\mathbb{1}_{\{f^*\ge\lambda\}} + \mathbb{1}_{\{f^\circ > -\lambda\}}\big)\big|\mathcal{F}_{t^\circ}\big] \\ - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[\psi(f_t)|\mathcal{F}_{t^\circ}] \quad \mu\text{-a.e.}\end{aligned} \tag{4.18}$$

Proof. Once more it suffices to show the first inequality in (4.16) and (4.17). If t^* and t° are elements of T themselves then we immediately arrive at (4.16) and (4.17), because $f^* = f^*_{t^\circ,t^*}$ and $f^\circ = f^\circ_{t^\circ,t^*}$. Otherwise, let $(v_n)_{n\in\mathbb{N}}$ in T be an increasing sequence such that $v_n \nearrow t^*$ as $n \to \infty$. Then $f^*_{v_n} \to f^*$ and $f^\circ_{v_n} \to f^\circ$ μ-a.e. on Ω as $n \to \infty$. Again, let $(\lambda_m)_{m\in\mathbb{N}}$ be a sequence in

[8] When dealing with probability spaces and submartingales $(M_t)_{t\in T}$ (see (1.2) in the Introduction) Doob's almost sure convergence theorem as stated in [13, Theorem 7.35] gives conditions for the existence of $M_{t^*}\mathbb{1}_{\{M^*\ge\lambda\}}$, hence there is no need to concern oneself with the essential supremum. In particular, the theorem states that the term $\operatorname{ess\,sup}_{t\in T} \mathbb{E}[M_{t^*}\mathbb{1}_{\{M^*\ge\lambda\}}|\mathcal{F}_{t^\circ}]$ can be identified with the limit $\lim_{t\in F, t\nearrow t^*} M_t 1_{\{M^*\ge\lambda\}}$ for each countable subset $F \subset T$ such that $t^* = \sup F$, if t^* is not an element of T itself. Define $X_t = M_t 1_{\{M^*\ge\lambda\}}$ for $t \in T$, which is again a submartingale. Then Doob's almost sure convergence theorem gives the existence of $\lim_{t\in F, t\searrow t^*} X_t$ a.s. and further claims that there exists an a.s. unique $\mathcal{F}_{t^*}$-measurable random variable X_{t^*} such that $X_{t^*} = \lim_{t\in F, t\nearrow t^*} X_t$ a.s. Similar observations can be made for $\operatorname{ess\,sup}_{t\in T} \mathbb{E}[M_t\mathbb{1}_{\{M^*\le -\lambda\}}|\mathcal{F}_{t^\circ}]$.

$(-\infty, \lambda)$ such that $\lambda_m \nearrow \lambda$ as $m \to \infty$. Once more, let us make the following definitions:

$$A_{m,n} = \{f^*_{v_n} \geq \lambda_m\},\ A_m = \bigcup_{n\in\mathbb{N}} A_{m,n},\ A = \{f^* \geq \lambda\};$$

$$B_{m,n} = \{f^\circ_{v_n} \leq -\lambda_m\},\ B_m = \bigcup_{n\in\mathbb{N}} B_{m,n},\ B = \{f^\circ \leq -\lambda\}.$$

Note that $(A_{m,n})_{n\in\mathbb{N}}$ is an increasing sequence for every $m \in \mathbb{N}$, whereas $(A_m)_{m\in\mathbb{N}}$ is decreasing and $A = \bigcup_{m\in\mathbb{N}} A_m$. Moreover, $(B_{m,n})_{n\in\mathbb{N}}$ is a decreasing sequence for every $m \in \mathbb{N}$, whereas $(B_m)_{m\in\mathbb{N}}$ is increasing, and $B = \bigcap_{m\in\mathbb{N}} B_m$. Again, using the conditional monotone convergence theorem and the conditional dominated convergence theorem it follows by (4.13) (note that using the essential supremum just gives a further upper bound) that

$$\begin{aligned}
\lambda\,\mathbb{E}_\mu[\mathbb{1}_A|\mathcal{F}_{t^\circ}] &= \lim_{m\to\infty}\Big(\lim_{n\to\infty} \lambda_m\,\mathbb{E}_\mu[\mathbb{1}_{A_{m,n}}|\mathcal{F}_{t^\circ}]\Big) \\
&\leq \operatorname*{ess\,sup}_{t\in T} \lim_{m\to\infty}\Big(\lim_{n\to\infty} \mathbb{E}_\mu[f_t\mathbb{1}_{A_{m,n}}|\mathcal{F}_{t^\circ}]\Big) - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[(f_t-\lambda)^+|\mathcal{F}_{t^\circ}] \\
&= \operatorname*{ess\,sup}_{t\in T} \lim_{m\to\infty}\big(\mathbb{E}_\mu[f_t\mathbb{1}_{A_m}|\mathcal{F}_{t^\circ}]\big) - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[(f_t-\lambda)^+|\mathcal{F}_{t^\circ}] \\
&= \operatorname*{ess\,sup}_{t\in T} \mathbb{E}_\mu[f_t\mathbb{1}_A|\mathcal{F}_{t^\circ}] - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[(f_t-\lambda)^+|\mathcal{F}_{t^\circ}] \quad \mu\text{-a.e.}
\end{aligned}$$

Using the same arguments we can deduce the following by using (4.14):

$$\begin{aligned}
\lambda\,\mathbb{E}_\mu[\mathbb{1}_B|\mathcal{F}_{t^\circ}] &= \lim_{m\to\infty}\Big(\lim_{n\to\infty} \lambda_m\,\mathbb{E}_\mu[\mathbb{1}_{B_{m,n}}|\mathcal{F}_{t^\circ}]\Big) \\
&\leq \operatorname*{ess\,sup}_{t\in T} \lim_{m\to\infty}\Big(\lim_{n\to\infty} \mathbb{E}_\mu[f_t\mathbb{1}_{B^c_{m,n}}|\mathcal{F}_{t^\circ}]\Big) - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[f_t \vee (-\lambda)|\mathcal{F}_{t^\circ}] \\
&= \operatorname*{ess\,sup}_{t\in T} \lim_{m\to\infty}\big(\mathbb{E}_\mu[f_t\mathbb{1}_{B^c_m}|\mathcal{F}_{t^\circ}]\big) - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[f_t \vee (-\lambda)|\mathcal{F}_{t^\circ}] \\
&= \operatorname*{ess\,sup}_{t\in T} \mathbb{E}_\mu[f_t\mathbb{1}_{B^c}|\mathcal{F}_{t^\circ}] - \operatorname*{ess\,inf}_{t\in T} \mathbb{E}_\mu[f_t \vee (-\lambda)|\mathcal{F}_{t^\circ}] \quad \mu\text{-a.e.}
\end{aligned}$$

□

This chapter proved a generalised versions of Doob's maximum inequalities by relying on purely deterministic inequalities. A more theoretical approach using measure theory was undertaken in Chapter 3. Furthermore, we showed that it is possible to omit the need for a given interval; particularly, we may rid ourselves from a starting and an endpoint by considering the infimum and the supremum of our time span T. We will proceed in a similar manner in the following chapter as we strive to improve Doob's well-known L^p-inequalities.

Chapter 5

Improved Versions of Doob's L^p-Inequality for σ-Finite Measure Spaces

In this chapter, which builds the core of this thesis, we will present three versions of Doob's classical L^p-inequality for $p > 1$, $p = 1$ and $p \in (0, 1)$ and prove that they hold true on a more general setting. In particular, there are various new approaches here.

Firstly, by using deterministic inequalities proved by [13] it is possible to find sharper upper bounds than those given by Doob. Secondly, the improved inequalities hold true when considering submartingales according to Definition 2.13 on σ-finite measure spaces (see Chapter 3). Thirdly, the need for integrability can be omitted when considering σ-integrable functions. Hence, the improved inequalities hold for σ-integrable submartingales. Finally, there is no need for assumptions concerning the time span $T \in \overline{\mathbb{R}}$ when working with the essential supremum.

As a first step we would like to refine and generalise (3.7) and (3.8) before proving another L^p-inequality for $p \in (0, 1)$. We work along the lines of [13, Theorem 4.81 and 4.86] as well as [10, Lemma 3.2(c) and Satz 3.3(b)] in order to prove our claims.

5.1 Inequalities for $p > 1$

We will start out with a simple deterministic inequality proved by [13]. Let it be noted that the proof of the following lemma is not of the author's making but will be presented here for the sake of completeness. Please refer to [13, Proposition 4.80] for the original proposition and its proof if interested.

Lemma 5.1. *Define* $\overline{x}_k = \max\{x_0, \ldots, x_k\}$ *for* $x_0, \ldots, x_n \in \mathbb{R}_+$ *and* $k \in \{0, \ldots, n\}$ *and* $\Delta x_{k+1} = x_{k+1} - x_k$ *for* $k \in \{0, \ldots, n-1\}$, $n \in \mathbb{N}$. *Let* $p, q \in (1, \infty)$ *such that* $1/p + 1/q = 1$ *and* $c > 1$. *Then*

$$\overline{x}_n^p \leq \frac{c}{c-1}\frac{q}{p}\Big(c^{p/q}x_n^p - p\sum_{k=0}^{n-1}\overline{x}_k^{p-1}\Delta x_{k+1} - x_0^p\Big). \tag{5.1}$$

Proof. It follows from the fundamental theorem of calculus (see Theorem A.35 in the Appendix) that

$$\overline{x}_n^p = p\int_0^\infty \lambda^{p-2}\lambda\mathbb{1}_{\{\overline{x}_n \geq \lambda\}}\, d\lambda.$$

We can now plug in the estimate in (4.1), which gives us

$$\overline{x}_n^p \leq p\int_0^\infty \lambda^{p-2}x_n\mathbb{1}_{\{\overline{x}_n\geq\lambda\}}\, d\lambda - p\sum_{k=0}^{n-1}\int_0^\infty \lambda^{p-2}\mathbb{1}_{\{\overline{x}_k\geq\lambda\}}\Delta x_{k+1}\, d\lambda \\ - p\int_0^\infty \lambda^{p-2}(x_0-\lambda)\mathbb{1}_{\{x_0\geq\lambda\}}\, d\lambda. \tag{5.2}$$

We assumed $p/(p-1) = q$. Hence, it follows by integration that

$$\overline{x}_n^p \leq qx_n\overline{x}_n^{p-1} - q\sum_{k=0}^{n-1}\overline{x}_k^{p-1}\Delta x_{k+1} - \underbrace{qx_0x_0^{p-1} + x_0^p}_{=(q+1)x_0^p}. \tag{5.3}$$

By using Young's inequality for products in Theorem A.36 and $(p-1)q = p$ we can see that

$$qx_n\overline{x}_n^{p-1} = qc^{1/q}x_n\frac{\overline{x}_n^{p-1}}{c^{1/q}} \leq \frac{q}{p}c^{p/q}x_n^p + \frac{\overline{x}_n^p}{c}. \tag{5.4}$$

Plugging the inequality above into (5.3) and solving the resulting inequality for $\overline{x}_n^p$ we arrive at the claimed result by using that $q-1 = q/p$. □

A proof of (5.1) for $c = q^{1/p}$ can be found in [1, Proposition 2.1]. With the help of the inequality above, it is now possible to improve Theorem 3.2 a little further as well as generalise it to hold even for σ-integrable submartingales on σ-finite measure spaces.

Theorem 5.2. IMPROVED VERSION OF DOOB'S CLASSICAL L^p-INEQUALITY FOR SUBMARTINGALES
Let $(\Omega, \mathcal{F}, \mathbb{F}, \mu)$ *be a* σ*-finite filtered measure space and* $T \subset \overline{\mathbb{R}}$ *with* $s, v \in T$

such that $s \le v$ and $T \subset [s,v]$. Assume that $\mu|_{\mathcal{F}_s}$ is σ-finite and let $(f_t)_{t\in T}$ be a $\mathbb{F}$-adapted sequence of positive functions such that f_t is σ-integrable w.r.t. $\mathcal{F}_s$ for all $t \in T$ and $f_t \le \mathbb{E}_\mu[f_u|\mathcal{F}_t]$ μ-a.e. for all $t \le u$ in T (i.e. $(f_t)_{t\in T}$ is a σ-integrable submartingale according to Definition 2.15). Then for $f^*_{s,v} := \operatorname{ess\,sup}_{t\in T} f_t$, $c > 1$ and $p, q \in (1,\infty)$ such that $1/p + 1/q = 1$ we have[1]

$$\mathbb{E}_\mu[(f^*_{s,v})^p|\mathcal{F}_s] \le \frac{c}{c-1}\frac{q}{p}\big(c^{p/q}\,\mathbb{E}_\mu[f^p_v|\mathcal{F}_s] - (\mathbb{E}_\mu[f_v|\mathcal{F}_s])^p\big) \quad \mu\text{-a.e.} \tag{5.5}$$

Remark 5.3. Since Definition 2.15 gives a generalisation of the submartingale property, Theorem 5.2 certainly holds for submartingales defined according to Definition 2.13. Furthermore, note that the right-hand side of (5.5) simplifies, if f is a σ-integrable martingale, because then $\mathbb{E}_\mu[f_v|\mathcal{F}_s] = f_s$ μ-a.e. By the submartingale property in w.r.t. σ-integrable functions in (2.21) it is possible to derive a further upper bound for (5.5) by subtracting f^p_s instead of $(\mathbb{E}_\mu[f_v|\mathcal{F}_s])^p$.

Proof. Theorem 5.2
We will start by proving (5.5) for finite T and the σ-integrable martingale[2] $(\tilde{f}_t)_{t\in T}$ with $\tilde{f}_t := \mathbb{E}_\mu[f_v|\mathcal{F}_t]$ for $t \in T$. Since $f^*_{s,v} \le \operatorname{ess\,sup}_{t\in T}\mathbb{E}_\mu[f_v|\mathcal{F}_t] =: \tilde{f}^*_{s,v}$ μ-a.e., the claim then follows.

It suffices to prove that (5.5) holds true on every $A_l := \{\mathbb{E}_\mu[\tilde{f}^p_v|\mathcal{F}_s] \le l\} \in \mathcal{F}_s$ for $l \in \mathbb{N}$, because it certainly does on $\Omega \setminus \bigcup_{l\in\mathbb{N}} A_l$. For this purpose, fix $l \in \mathbb{N}$ and define $g_t = \mathbb{1}_{A_l}\tilde{f}_t$ for $t \in T$. Then $(g_t)_{t\in T}$ is a sequence of positive functions. Consider a sequence $(\Omega_l)_{l\in\mathbb{N}}$ in $\mathcal{F}_s$ with $\Omega_l \nearrow \Omega$ as $l \to \infty$ such that $\mu(\Omega_l) < \infty$ for all $l \in \mathbb{N}$ (such a sequence exists, since $\mu|_{\mathcal{F}_s}$ is σ-finite by assumption). Then $\overline{\Omega}_l := A_l \cap \Omega_l$ is also in $\mathcal{F}_s$ for all $l \in \mathbb{N}$, $\overline{\Omega}_l \nearrow \Omega$ as $l \to \infty$ and $\mu(\overline{\Omega}_l) < \infty$ for all $l \in \mathbb{N}$. Furthermore, $g_t\mathbb{1}_{\overline{\Omega}_l} \in L^1(\Omega,\mathcal{F},\mu)$ for all $l \in \mathbb{N}$ by Lemma 2.4(3). Thus, g_t is σ-integrable w.r.t. $\mathcal{F}_s$ for all $t \in T$. Moreover,

$$\mathbb{E}_\mu[g_u - g_t|\mathcal{F}_t] \overset{2.12(6)}{=} \mathbb{1}_{A_l}\mathbb{E}_\mu[\tilde{f}_u - \tilde{f}_t|\mathcal{F}_t] = 0 \quad \mu\text{-a.e.} \tag{5.6}$$

for all $t \le u$ in T and

$$\mathbb{E}_\mu[g^p_v\mathbb{1}_{\Omega_l}] \overset{2.12(6)}{=} \mathbb{E}_\mu[\mathbb{1}_{A_l}\underbrace{\mathbb{E}_\mu[\tilde{f}^p_v|\mathcal{F}_s]}_{\le l \text{ on } A_l}\mathbb{1}_{\Omega_l}] \le l\mu(\overline{\Omega}_l) < \infty$$

[1] For the conditional expectation of $(f^*_{s,v})^p$ please refer to Remark 2.11.

[2] $(\tilde{f}_t)_{t\in T}$ is in fact a martingale defined according to Definition 2.15, because $\mathbb{E}_\mu[\mathbb{E}_\mu[f_v|\mathcal{F}_u]|\mathcal{F}_t] = \mathbb{E}_\mu[f_v|\mathcal{F}_t]$ μ-a.e. for all $t \le u$ in T by the tower property in Theorem 2.12(7).

by the $\mathcal{F}_s$-measurability of $\mathbb{1}_{A_l}$. Thus, g_v^p is σ-integrable w.r.t. $\mathcal{F}_s$.

Suppose $T = \{t_0, \ldots, t_n\}$ with $s = t_0 < t_1 < \ldots < t_n = v$ and define $\bar{g}_j = \max\{g_{t_0}, \ldots, g_{t_j}\}$ for $j \in \{0, \ldots, n\}$ and $n \in \mathbb{N}$. It follows that $\bar{g}_j^{p-1}(g_{t_{j+1}} - g_{t_j})$ is also σ-integrable w.r.t. $\mathcal{F}_{t_j}$ for all $j \in \{0, \ldots, n-1\}$ by Lemma 2.4(1) and (3). Hence,

$$\mathbb{E}_\mu[\bar{g}_j^{p-1}(g_{t_{j+1}} - g_{t_j})|\mathcal{F}_{t_j}] \overset{2.12(6)}{=} \bar{g}_j^{p-1}(\underbrace{\mathbb{E}_\mu[g_{t_{j+1}} - g_{t_j}|\mathcal{F}_{t_j}]}_{=0\,\mu\text{-a.e. by (5.6)}}) = 0 \quad \mu\text{-a.e.}, \tag{5.7}$$

where we used the $\mathcal{F}_{t_j}$-measurability of $\bar{g}_j$. Substituting $x_0, \ldots, x_n$ in (5.1) with $g_{t_0}, \ldots, g_{t_n}$ now gives us

$$\bar{g}_n^p \le \frac{c}{c-1}\frac{q}{p}\Big(c^{p/q} g_{t_n}^p - p\sum_{j=0}^{n-1} \bar{g}_j^{p-1}\Delta g_{t_{j+1}} - g_{t_0}^p\Big) \quad \mu\text{-a.e.}$$

Let us now take the conditional expectation w.r.t. $\mathcal{F}_{t_k}$ for $k = n-1$ of the inequality above. This gives

$$\mathbb{E}_\mu[\bar{g}_n^p|\mathcal{F}_{t_k}] \le \frac{c}{c-1}\frac{q}{p}\Big(c^{p/q}\,\mathbb{E}_\mu[g_{t_n}^p|\mathcal{F}_{t_k}] - p\sum_{j=0}^{k-1}\mathbb{E}_\mu[\bar{g}_j^{p-1}\Delta g_{t_{j+1}}|\mathcal{F}_{t_k}] - \mathbb{E}_\mu[g_{t_0}^p|\mathcal{F}_{t_k}]\Big)$$

μ-a.e., where the last term of the sum vanishes for k due to (5.7). Taking conditional expectations iteratively for $k = n-2, \ldots, 0$ and using the tower property in Theorem 2.12(7) yields

$$\mathbb{E}_\mu[\bar{g}_n^p|\mathcal{F}_s] \le \frac{c}{c-1}\frac{q}{p}(c^{p/q}\,\mathbb{E}_\mu[g_v^p|\mathcal{F}_s] - g_s^p) \quad \mu\text{-a.e.},$$

where we used that $\mathbb{E}_\mu[g_s^p|\mathcal{F}_s] = g_s^p$ μ-a.e. by assumption of $\mathbb{F}$-adaptedness for the final term. By Theorem 2.12(1) the inequality above is (5.5) on A_l with $\tilde{f}_T := \max_{t\in T}\tilde{f}_t$ in place of $\tilde{f}_{s,v}^*$, thus, we are done with this part.

For the general case note there exists a sequence $(S_n)_{n\in\mathbb{N}}$ of finite subsets of T with $s, v \in S_n$ for all $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} S_n = T$. For $f_{S_n} := \max_{t\in S_n} f_t$ it follows that $f_{S_n} \nearrow f_{s,v}^*$ as $n \to \infty$ by Theorem A.24. We then know that (5.5) holds on A_l for every finite set S_n and thus,

$$\begin{aligned}\mathbb{E}_\mu[(f_{s,v}^*)^p|\mathcal{F}_s] &= \lim_{n\to\infty}\mathbb{E}_\mu[(f_{S_n})^p|\mathcal{F}_s] \\ &\le \frac{c}{c-1}\frac{q}{p}\big(c^{p/q}\,\mathbb{E}_\mu[f_v^p|\mathcal{F}_s] - (\mathbb{E}_\mu[f_v|\mathcal{F}_s])^p\big) \quad \mu\text{-a.e.}\end{aligned}$$

by the conditional monotone convergence theorem in Theorem 2.12(5).

□

We can weaken the assumptions in Theorem 5.2 a little bit further by omitting the need for adaptedness. In the following corollary we will only look at the increments of a sequence of functions, where we ask those to be adapted instead of the sequence itself. In the spirit of Definition 2.15 this then gives a generalisation of the theorem above.

Corollary 5.4. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mu)$ be a σ-finite filtered measure space and $T \subset \overline{\mathbb{R}}$ with $s, v \in T$ such that $T \subset [s, v]$. Assume $\mu|_{\mathcal{F}_s}$ is σ-finite and let $(f_t)_{t\in T}$ be a sequence of functions such that $f_t - f_s$ is $\mathcal{F}_t$-measurable and σ-integrable w.r.t. $\mathcal{F}_s$ for all $t \in T$. If $\mathbb{E}_\mu[f_u - f_t|\mathcal{F}_t] = 0$ μ-a.e. for all $t \le u$ in T, it follows for $c > 1$ and $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$ that*[3]

$$\mathbb{E}_\mu\left[\left(\operatorname*{ess\,sup}_{t\in T} |f_t - f_s|\right)^p \middle| \mathcal{F}_s\right] \le \frac{c}{c-1}\frac{q}{p}\left(c^{p/q}\,\mathbb{E}_\mu[|f_v - f_s|^p|\mathcal{F}_s] - (\mathbb{E}_\mu[|f_v - f_s|\,|\mathcal{F}_s])^p\right)$$

holds true μ-a.e.

Proof. Note that $0 = \mathbb{E}_\mu[f_u - f_t|\mathcal{F}_t] = \mathbb{E}_\mu[f_u - f_s|\mathcal{F}_t] - (f_t - f_s)$ μ-a.e. for all $t \le u$ in T by the $\mathcal{F}_t$-measurability of the increment $(f_t - f_s)$. This is equivalent to $f_t - f_s = \mathbb{E}_\mu[f_u - f_s|\mathcal{F}_t]$ μ-a.e. Then by Theorem 2.12(9) it follows that $|f_t - f_s| = |\mathbb{E}_\mu[f_u - f_s|\mathcal{F}_t]| \le \mathbb{E}_\mu[|f_u - f_s|\,|\mathcal{F}_t]$ μ-a.e. Hence, we can apply the improved version of Doob's L^p-inequality to the $\mathbb{F}$-adapted sequence of positive functions $(|f_t - f_s|)_{t\in T}$, which immediately yields the claim. □

A starting point $s \in T$ might not always be readily available. Thus, it can be helpful to consider the infimum of our time span $T \subset \overline{\mathbb{R}}$ instead. The following corollary gives an estimate for our sequence of functions without the need of such a starting point.

Corollary 5.5. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mu)$ be a σ-finite filtered measure space and $T \subset \overline{\mathbb{R}}$ with $v \in T$ and define $t^\circ = \inf T$. Assume that $\mu|_{\mathcal{F}_{t^\circ}}$ is σ-finite with $\mathcal{F}_{t^\circ} := \bigcap_{t\in T} \mathcal{F}_t$. Let $(f_t)_{t\in T}$ be an $\mathbb{F}$-adapted sequence of positive functions such that f_v is σ-integrable w.r.t. $\mathcal{F}_{t^\circ}$ for all $t \in T$ and $f_t \le \mathbb{E}_\mu[f_u|\mathcal{F}_t]$ μ-a.e. for all $t \le u$ in T. Then for $f_v^* := \operatorname{ess\,sup}_{t\in T, t\le v} f_t$, $c > 1$ and $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$ it follows that*[4]

$$\mathbb{E}_\mu[(f_v^*)^p|\mathcal{F}_{t^\circ}] \le \frac{c}{c-1}\frac{q}{p}\left(c^{p/q}\,\mathbb{E}_\mu[f_v^p|\mathcal{F}_{t^\circ}] - (\mathbb{E}_\mu[f_v|\mathcal{F}_{t^\circ}])^p\right) \quad \mu\text{-a.e.}$$

[3]For the conditional expectation of $\operatorname{ess\,sup}_{t\in T} |f_t - f_s|^p$ please refer to Remark 2.11.
[4]For the conditional expectation of $(f_v^*)^p$ please refer to Remark 2.11.

Proof. If $t^\circ \in T$, simply take $s := t^\circ$ and the claim follows immediately from (5.5), since $f^*_{t^\circ,v} = f^*_v$. Otherwise, consider the martingale $(T \cup \{t^\circ\}) \cap [-\infty, v] \ni t \to \mathbb{E}_\mu[f_v|\mathcal{F}_t] =: \tilde{f}_t$ and define $T_{\leq v} = T \cap [-\infty, v]$. Similarly to the proof of Theorem 5.2 we can observe that

$$f^*_v = \operatorname*{ess\,sup}_{t \in T_{\leq v}} f_t \leq \operatorname*{ess\,sup}_{t \in T_{\leq v}} \tilde{f}_t \leq \operatorname*{ess\,sup}_{t \in T_{\leq v} \cup \{t^\circ\}} \tilde{f}_t \quad \mu\text{-a.e.} \tag{5.8}$$

Since $T_{\leq v} \cup \{t^\circ\} \subset [t^\circ, v]$, we can apply Theorem 5.2 to $\operatorname{ess\,sup}_{t \in T_{\leq v} \cup \{t^\circ\}} \tilde{f}_t$, which gives us

$$\mathbb{E}_\mu\left[\operatorname*{ess\,sup}_{t \in T_{\leq v} \cup \{t^\circ\}} \tilde{f}_t | \mathcal{F}_{t^\circ}\right] \leq \frac{c}{c-1}\frac{q}{p}\big(c^{p/q}\,\mathbb{E}_\mu[\tilde{f}^p_v|\mathcal{F}_{t^\circ}] - (\mathbb{E}_\mu[\tilde{f}_v|\mathcal{F}_{t^\circ}])^p\big) \quad \mu\text{-a.e.}$$

This implies the claim by (5.8) and Theorem 2.12(1). $\square$

Note that Corollary 5.5 is in fact an improvement to (3.7). Finally, we would like to prove that also (3.8) can be improved and adapted to our setting.

Corollary 5.6. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mu)$ be a σ-finite filtered measure space and $T \subset \overline{\mathbb{R}}$. Define $t^\circ = \inf T$ and assume that $\mu|_{\mathcal{F}_{t^\circ}}$ is σ-finite with $\mathcal{F}_{t^\circ} := \bigcap_{t \in T} \mathcal{F}_t$. Let $(f_t)_{t \in T}$ be an $\mathbb{F}$-adapted sequence of positive functions such that f_u is σ-integrable w.r.t. $\mathcal{F}_{t^\circ}$ and $f_t \leq \mathbb{E}_\mu[f_u|\mathcal{F}_t]$ μ-a.e. for all $t \leq u$ in T. Then for $f^* := \operatorname{ess\,sup}_{t \in T} f_t$, $c > 1$ and $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$ we have*[5]

$$\mathbb{E}_\mu[(f^*)^p|\mathcal{F}_{t^\circ}] \leq \frac{c}{c-1}\frac{q}{p} \operatorname*{ess\,sup}_{v \in T}\big(c^{p/q}\,\mathbb{E}_\mu[f^p_v|\mathcal{F}_{t^\circ}] - (\mathbb{E}_\mu[f_v|\mathcal{F}_{t^\circ}])^p\big) \quad \mu\text{-a.e.}$$

Proof. If $t^* = \sup T$ is an element of T itself then we immediately arrive at the claim, because $f^* = f^*_{t^*}$. Otherwise, let $(v_n)_{n \in \mathbb{N}}$ in T be an increasing sequence such that $v_n \nearrow t^*$ as $n \to \infty$. Then $f^*_{v_n} \nearrow f^*$ μ-a.e. on Ω as $n \to \infty$. Thus, we may apply the conditional monotone convergence theorem in Theorem 2.12(5) and it follows that

$$\begin{aligned}
\mathbb{E}_\mu[(f^*)^p|\mathcal{F}_{t^\circ}] &= \lim_{n \to \infty} \mathbb{E}_\mu[(f^*_{v_n})^p|\mathcal{F}_{t^\circ}] \\
&\overset{5.5}{\leq} \frac{c}{c-1}\frac{q}{p} \sup_{n \in \mathbb{N}}\big(c^{p/q}\,\mathbb{E}_\mu[f^p_{v_n}|\mathcal{F}_{t^\circ}] - (\mathbb{E}_\mu[f_{v_n}|\mathcal{F}_{t^\circ}])^p\big) \\
&\leq \frac{c}{c-1}\frac{q}{p} \operatorname*{ess\,sup}_{v \in T}\big(c^{p/q}\,\mathbb{E}_\mu[f^p_v|\mathcal{F}_{t^\circ}] - (\mathbb{E}_\mu[f_v|\mathcal{F}_{t^\circ}])^p\big) \quad \mu\text{-a.e.},
\end{aligned}$$

which concludes the proof. $\square$

[5] For the conditional expectation of $(f^*)^p$ and f^p_v please refer to Remark 2.11.

5.2 Inequalities for $p = 1$

In this section we will, once more, use a deterministic inequality to derive an improved version of Doob's L^1-inequality for submartingales. For the sake of completeness we provide the proof for said deterministic inequality as presented and proven in [13, Proposition 4.85]. Therefore, let it be noted that the proof for the following lemma is not of the author's making. For further reading and more information please refer to [13] if interested.

Lemma 5.7. *Define $\overline{x}_k = \max\{x_0, \dots, x_k\}$ for $k \in \{0, \dots, n\}$ and $x_0, \dots, x_n \in \mathbb{R}_+$ with $x_0 > 0$. Furthermore, $\Delta x_{k+1} := x_{k+1} - x_k$ for $k \in \{0, \dots, n-1\}$, $n \in \mathbb{N}$, and let $c > 1$. Then*

$$\overline{x}_n \leq \frac{c}{c-1}\Big(x_0 + x_n \log\frac{c}{\mathrm{e}} + x_n \log x_n - x_0 \log x_0 - \sum_{k=0}^{n-1} \Delta x_{k+1} \log \overline{x}_k\Big). \tag{5.9}$$

Proof. If $a \in \mathbb{R}_+$ and $b, c > 0$, then[6]

$$a \log b \leq a \log a + \frac{b}{c} + a \log\frac{c}{e}. \tag{5.10}$$

We will first prove (5.10) in order to derive the inequalities in Lemma 5.7. If $a = 0$, (5.10) is trivial. To prove the inequality for $a, b, c > 0$ note that $\log x \leq x/c + \log(c/e)$ for $x > 0$, because both sides and their first derivatives agree at $x = c$ and the left-hand side is concave while the right-hand side is linear on $(0, \infty)$. (5.10) follows directly now by plugging in $x = b/a$ with $a, b > 0$ and using the functional equation of the natural logarithm.

In order to prove (5.9) first note that $x_0 > 0$ implies $\overline{x}_0, \dots, \overline{x}_n > 0$. By using (4.1) and noting that the integral of $\mathbb{1}_{\{x_0 \geq \lambda\}}$ over $\lambda \in [x_0, \infty)$ vanishes, we arrive at

$$\begin{aligned} \overline{x}_n &= x_0 + \int_{x_0}^{\infty} \mathbb{1}_{\{\overline{x}_n \geq \lambda\}}\, d\lambda \\ &\leq x_0 + x_n \int_{x_0}^{\infty} \frac{1}{\lambda} \mathbb{1}_{\{\overline{x}_n \geq \lambda\}}\, d\lambda - \sum_{k=0}^{n-1} \Delta x_{k+1} \int_{x_0}^{\infty} \frac{1}{\lambda} \mathbb{1}_{\{\overline{x}_k \geq \lambda\}}\, d\lambda \\ &= x_0 + x_n \log \overline{x}_n \underbrace{- x_n \log x_0 + \sum_{k=0}^{n-1} \Delta x_{k+1} \log x_0}_{= x_0 \log x_0} - \sum_{k=0}^{n-1} \Delta x_{k+1} \log \overline{x}_k \end{aligned} \tag{5.11}$$

[6] Note that there is equality in (5.10) if, and only if, $a, b > 0$ and $c = b/a$.

Applying (5.10) to $x_n \log \overline{x}_n$ and solving the inequality above for $\overline{x}_n$ finally gives (5.9). □

A proof of (5.7) for $c = \mathrm{e}$ can be found in [1, Proposition 2.1(II)]. We are now ready to show that also a generalised and improved version of Doob's L^1-inequality can be formulated for our specific setting.

Theorem 5.8. Improved version of Doob's L^1-inequality for submartingales
Consider the setting and assumption in Theorem 5.2 and define

$$\varphi : \mathbb{R}_+ \to \mathbb{R}, \quad \varphi(x) = \begin{cases} x \log x & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{5.12}$$

Then for all $c > 1$ it follows[7]

$$\mathbb{E}_\mu[f^*_{s,v}|\mathcal{F}_s] \leq \frac{c}{c-1}\Big(\log(c)\,\mathbb{E}_\mu[f_v|\mathcal{F}_s] + \mathbb{E}_\mu[\varphi(f_v)|\mathcal{F}_s] - \varphi(\mathbb{E}_\mu[f_v|\mathcal{F}_s])\Big) \quad \mu\text{-a.e.} \tag{5.13}$$

Remark 5.9. Definition 2.15 generalises the submartingale property. Hence, Theorem 5.8 certainly holds for submartingales defined according to Definition 2.13. Furthermore, note that the right-hand side of (5.13) simplifies, if $(f_t)_{t\in T}$ is a σ-integrable martingale, because then $\mathbb{E}_\mu[f_v|\mathcal{F}_s] = f_s$ μ-a.e. However, replacing $\varphi(\mathbb{E}_\mu[f_v|\mathcal{F}_s])$ with $\varphi(f_s)$ for positive σ-integrable submartingales can lead to a wrong inequality. This can be seen in [6, p. 3] for $c := \mathrm{e}$ treating the case of random variables and probability spaces.

Proof. Theorem 5.8
Again, we will start by proving (5.13) for finite T and the σ-integrable martingale $(\tilde{f}_t)_{t\in T}$ with $\tilde{f}_t := \mathbb{E}_\mu[f_v|\mathcal{F}_t]$ for $t \in T$ (see Footnote number 2 on page 49). Since $f^*_{s,v} \leq \operatorname{ess\,sup}_{t\in T} \mathbb{E}_\mu[f_v|\mathcal{F}_t] =: \tilde{f}^*_{s,v}$ μ-a.e., the claim then follows.

Again, it suffices to prove (5.13) on every $A_l := \{\mathbb{E}_\mu[\varphi(\tilde{f}_v)|\mathcal{F}_s] \leq l\} \in \mathcal{F}_s$ for $l \in \mathbb{N}$, which suffices, because it certainly does on $\Omega \setminus \bigcup_{l\in\mathbb{N}} A_l$. For this purpose, fix $l \in \mathbb{N}$ and define $g_t = \mathbb{1}_{A_l}\tilde{f}_t$ for $t \in T$. Then $(g_t)_{t\in T}$ is a sequence of positive functions and by the same arguments as in the proof of Theorem 5.2 (see page 49) it follows that g_t is σ-integrable w.r.t. $\mathcal{F}_s$ for all $t \in T$. Simply consider a sequence $(\Omega_l)_{l\in\mathbb{N}}$ in $\mathcal{F}_s$ with $\Omega_l \nearrow \Omega$ as $l \to \infty$ such that $\mu(\Omega_l) < \infty$ for all $l \in \mathbb{N}$ (such a sequence exists, since $\mu|_{\mathcal{F}_s}$ is σ-finite by assumption). Then $\overline{\Omega}_l := A_l \cap \Omega_l$ is also in $\mathcal{F}_s$ for all $l \in \mathbb{N}$, $\overline{\Omega}_l \nearrow \Omega$ as

[7] For the conditional expectation of $f^*_{s,v}$ and $\varphi(f_u)$ please refer to Remark 2.11.

$l \to \infty$, $\mu(\overline{\Omega}_l) < \infty$ for all $l \in \mathbb{N}$ and $g_t \mathbb{1}_{\overline{\Omega}_l} \in L^1(\Omega, \mathcal{F}, \mu)$ for all $t \in T$. Moreover,

$$\mathbb{E}_\mu[g_u - g_t | \mathcal{F}_t] \overset{2.12(6)}{=} \mathbb{1}_{A_l} \mathbb{E}_\mu[\tilde{f}_u - \tilde{f}_t | \mathcal{F}_t] = 0 \quad \mu\text{-a.e.} \tag{5.14}$$

for all $t \leq u$ in T.

Note that φ as defined in (5.12) is a convex function and bounded from below with $\varphi(0) = 0$. Thus, $\varphi(g_v)$ is σ-integrable w.r.t. $\mathcal{F}_s$, because

$$\mathbb{E}_\mu[\varphi(g_v)\mathbb{1}_{\Omega_l}] \overset{2.12(6)}{=} \mathbb{E}_\mu[\mathbb{1}_{A_l} \underbrace{\mathbb{E}_\mu[\varphi(f_v)|\mathcal{F}_s]}_{\leq l \text{ on } A_l} \mathbb{1}_{\Omega_l}] \leq l\mu(\overline{\Omega}_l) < \infty$$

by the $\mathcal{F}_s$-measurability of $\mathbb{1}_{A_l}$.

Suppose $T = \{t_0, \ldots, t_n\}$ with $s = t_0 < t_1 < \ldots < t_n = v$ and define $g_T = \max_{t \in T} g_t$. We want to prove (5.13) on A_l with g_T in place of $f^*_{s,v}$. For this purpose, consider $g_{\rho,t} := 1 + \rho(g_t - 1)$ for $\rho \in (0,1)$ and all $t \in T$. Note that $(g_{\rho,t})_{t \in T}$ is σ-integrable by Lemma 2.4(1) and (2). Moreover,

$$\mathbb{E}_\mu[g_{\rho,u} - g_{\rho,t} | \mathcal{F}_t] \overset{2.12(4)}{=} \rho(\mathbb{E}_\mu[g_u - g_t|\mathcal{F}_t]) \overset{(5.14)}{=} 0 \quad \mu\text{-a.e.} \tag{5.15}$$

for all $t \leq u$ in T. Furthermore, $g_{\rho,t} \geq 1 - \rho > 0$ for all $t \in T$ and $\rho \in (0,1)$ and $g_{\rho,t} \to g_t$ for all $t \in T$ as $\rho \nearrow 1$. Since φ is continuous, bounded below and increasing on $[1, \infty)$ and $g_{\rho,t} \leq g_t$ for all $\rho \in (0,1)$ on $\{g_t \geq 1\}$, we can apply the conditional monotone convergence theorem in Theorem 2.12(5) and conclude[8]

$$\mathbb{E}_\mu[\varphi(g_{\rho,t})|\mathcal{F}_s] \to \mathbb{E}_\mu[\varphi(g_t)|\mathcal{F}_s] \quad \mu\text{-a.e. as } \rho \nearrow 1. \tag{5.16}$$

Define $\bar{g}_{\rho,j} = \max\{g_{\rho,t_0}, \ldots, g_{\rho,t_j}\}$ for $j \in \{0, \ldots, n\}$ and $n \in \mathbb{N}$. Since $\log(1-\rho) \leq \log(\bar{g}_{\rho,j}) \leq \log(\bar{g}_{\rho,n})$ by the monotonicity of the logarithm and $|g_{\rho,t_{j+1}} - g_{\rho,t_j}| \leq \max\{g_{\rho,t_{j+1}}, g_{\rho,t_j}\} \leq \bar{g}_{\rho,n}$, it follows that

$$\bar{g}_{\rho,n} \log(1-\rho) \leq \underbrace{|g_{\rho,t_{j+1}} - g_{\rho,t_j}|}_{\leq \bar{g}_{\rho,n}} \underbrace{\log(\bar{g}_{\rho,j})}_{\leq |\log(\bar{g}_{\rho,n})|} \leq |\varphi(\bar{g}_{\rho,n})| \quad \mu\text{-a.e.},$$

which implies that $(g_{\rho,t_{j+1}} - g_{\rho,t_j}) \log(\bar{g}_{\rho,j})$ is also σ-integrable w.r.t. $\mathcal{F}_{t_j}$ for all $j \in \{0, \ldots, n-1\}$. Thus,

$$\mathbb{E}_\mu[(g_{\rho,t_{j+1}} - g_{\rho,t_j}) \log(\bar{g}_{\rho,j}) | \mathcal{F}_{t_j}] \overset{2.12(6)}{=} \log(\bar{g}_{\rho,j}) (\underbrace{\mathbb{E}_\mu[g_{\rho,t_{j+1}} - g_{\rho,t_j} | \mathcal{F}_{t_j}]}_{=0\,\mu\text{-a.e. by (5.15)}}) = 0 \tag{5.17}$$

[8]In case $\varphi(g_{\rho,t})$ is not σ-integrable w.r.t. $\mathcal{F}_s$, please refer to Remark 2.11 for a generalised definition of the conditional expectation.

μ-a.e., where we used the $\mathcal{F}_{t_j}$-measurability of $\log(\bar{g}_{\rho,j})$. Substituting $x_0, \ldots, x_n$ in (5.9) with $g_{\rho,t_0}, \ldots, g_{\rho,t_n}$ (which we can do, since $g_{\rho,t_0} \geq 1 - \rho > 0$) now yields

$$\bar{g}_{\rho,t_n} \leq \frac{c}{c-1}\Big(g_{\rho,t_0} + g_{\rho,t_n} \log\Big(\frac{c}{\mathrm{e}}\Big) + g_{\rho,t_n} \log(g_{\rho,t_n}) - g_{\rho,t_0} \log(g_{\rho,t_0}) \\ - \sum_{k=0}^{n-1} \log(\bar{g}_{\rho,t_k}) \Delta g_{\rho,t_{k+1}}\Big) \quad \mu\text{-a.e.}$$

Let us now take the conditional expectation w.r.t. $\mathcal{F}_{t_k}$ for $k = n-1$ of the inequality above. This gives

$$\mathbb{E}_\mu[\bar{g}_{\rho,t_n}|\mathcal{F}_{t_k}] \leq \frac{c}{c-1}\Big(\mathbb{E}_\mu[g_{\rho,t_0}|\mathcal{F}_{t_k}] + \mathbb{E}_\mu[g_{\rho,t_n}|\mathcal{F}_{t_k}] \log\Big(\frac{c}{\mathrm{e}}\Big) + \mathbb{E}_\mu[\varphi(g_{\rho,t_n})|\mathcal{F}_{t_k}] \\ - \mathbb{E}_\mu[\varphi(g_{\rho,t_0})|\mathcal{F}_{t_k}] - \sum_{j=0}^{k-1} \mathbb{E}_\mu[\log(\bar{g}_{\rho,t_j}) \Delta g_{\rho,t_{j+1}}|\mathcal{F}_{t_k}]\Big)$$

μ-a.e., where the last term of the sum vanished for k due to (5.17). Taking conditional expectations iteratively for $k = n-2, \ldots, 0$ and using the tower property in Theorem 2.12(7) yields

$$\mathbb{E}_\mu[\bar{g}_{\rho,v}|\mathcal{F}_s] \leq \frac{c}{c-1}\big(\mathbb{E}_\mu[g_{\rho,s}|\mathcal{F}_s] + \mathbb{E}_\mu[g_{\rho,v}|\mathcal{F}_s] \log(c/\mathrm{e}) \\ + \mathbb{E}_\mu[\varphi(g_{\rho,v})|\mathcal{F}_s] - \mathbb{E}_\mu[\varphi(g_{\rho,s})|\mathcal{F}_s]\big) \quad \mu\text{-a.e.} \tag{5.18}$$

Since $\mathbb{E}_\mu[g_{\rho,s}|\mathcal{F}_s] = \mathbb{E}_\mu[g_{\rho,v}|\mathcal{F}_s]$ μ-a.e. by (5.15), we can deduce that

$$\mathbb{E}_\mu[g_{\rho,s}|\mathcal{F}_s] + \mathbb{E}_\mu[g_{\rho,v}|\mathcal{F}_s] \log(c/\mathrm{e}) = \log(c)\, \mathbb{E}_\mu[g_{\rho,v}|\mathcal{F}_s] \quad \mu\text{-a.e.}$$

and

$$\mathbb{E}_\mu[\varphi(g_{\rho,s})|\mathcal{F}_s] \overset{2.12(9)}{\geq} \varphi(\mathbb{E}_\mu[g_{\rho,s}|\mathcal{F}_s]) = \varphi(\mathbb{E}_\mu[g_{\rho,v}|\mathcal{F}_s]) \quad \mu\text{-a.e.}, \tag{5.19}$$

where we used the conditional version of Jensen's inequality. Therefore,

$$\mathbb{E}_\mu[\bar{g}_{\rho,v}|\mathcal{F}_s] \leq \frac{c}{c-1}\big(\log(c)\, \mathbb{E}_\mu[g_{\rho,v}|\mathcal{F}_s] + \mathbb{E}_\mu[\varphi(g_{\rho,v})|\mathcal{F}_s] - \varphi(\mathbb{E}_\mu[g_{\rho,v}|\mathcal{F}_s])\big) \quad \mu\text{-a.e.}$$

is another upper bound for (5.18). Finally, by sending $\rho \nearrow 1$ and using (5.16) it follows that

$$\mathbb{E}_\mu[g_T|\mathcal{F}_s] \leq \frac{c}{c-1}\big(\log(c)\, \mathbb{E}_\mu[g_v|\mathcal{F}_s] + \mathbb{E}_\mu[\varphi(g_v)|\mathcal{F}_s] - \varphi(\mathbb{E}_\mu[g_v|\mathcal{F}_s])\big) \quad \mu\text{-a.e.},$$

which is (5.13) on A_l with $\tilde{f}_T := \max_{t \in T} \tilde{f}_t$ in place of $\tilde{f}^*_{s,v}$ by Theorem 2.12(1).

The rest of the proof for (5.13) goes along the same lines as the proof of Theorem 5.2 and will be omitted at this point. □

Just like it was the case for the L^p-inequality in Chapter 5.1, we can weaken the assumptions in Theorem 5.8 some more by omitting the need for adaptedness. In the following corollary we will only look at the increments of a sequence of functions, where we ask those to be adapted instead of the sequence itself. In the spirit of Definition 2.15 this then gives a generalisation of the theorem above.

Corollary 5.10. *Consider the setting and assumptions in Corollary 5.4 and define the function φ as in (5.12). Then*[9]

$$\mathbb{E}_\mu\left[\operatorname*{ess\,sup}_{t\in T}|f_t - f_s|\middle|\mathcal{F}_s\right] \leq \frac{c}{c-1}\big(\log(c)\,\mathbb{E}_\mu[|f_v - f_s|\,|\mathcal{F}_s] + \mathbb{E}_\mu[\varphi(|f_v - f_s|)|\mathcal{F}_s] - \varphi(\mathbb{E}_\mu[|f_v - f_s|\,|\mathcal{F}_s])\big) \quad \mu\text{-a.e.}$$

Proof. The claim follows in exactly the same manner as in the proof of Corollary 5.4. Please refer to page 51 for more details. □

Again we would like to free ourselves of the need for a starting point in the time span $T \subset \overline{\mathbb{R}}$. This can be done as follows.

Corollary 5.11. *Consider the setting and assumptions in Corollary 5.5 and define the function φ as in (5.12). Then*[10]

$$\mathbb{E}_\mu[f_v^*|\mathcal{F}_{t^\circ}] \leq \frac{c}{c-1}\big(\log(c)\,\mathbb{E}_\mu[f_v|\mathcal{F}_{t^\circ}] + \mathbb{E}_\mu[\varphi(f_v)|\mathcal{F}_{t^\circ}] - \varphi(\mathbb{E}_\mu[f_v|\mathcal{F}_{t^\circ}])\big) \quad \mu\text{-a.e.}$$

Proof. With the same arguments as in the proof of Corollary 5.5 we can deduce the claim from (5.13): If $t^\circ \in T$, simply take $s := t^\circ$ and the claim follows immediately, because $f^*_{t^\circ,v} = f^*_v$. Otherwise, consider the martingale $(T \cup \{t^\circ\}) \cap [-\infty, v] \ni t \to \mathbb{E}_\mu[f_v|\mathcal{F}_t] =: \tilde{f}_t$ and define $T_{\leq v} = T \cap [-\infty, v]$. Again we can observe that

$$f_v^* = \operatorname*{ess\,sup}_{t\in T_{\leq v}} f_t \leq \operatorname*{ess\,sup}_{t\in T_{\leq v}} \tilde{f}_t \leq \operatorname*{ess\,sup}_{t\in T_{\leq v}\cup\{t^\circ\}} \tilde{f}_t \quad \mu\text{-a.e.} \tag{5.20}$$

Since $T_{\leq v} \cup \{t^\circ\} \subset [t^\circ, v]$, we can apply Theorem 5.8 to $\operatorname{ess\,sup}_{t\in T_{\leq v}\cup\{t^\circ\}} \tilde{f}_t$, which gives us

$$\mathbb{E}_\mu\left[\operatorname*{ess\,sup}_{t\in T_{\leq v}\cup\{t^\circ\}} \tilde{f}_t\middle|\mathcal{F}_{t^\circ}\right] \leq \frac{c}{c-1}\big(\log(c)\,\mathbb{E}_\mu[\tilde{f}_v|\mathcal{F}_{t^\circ}] + \mathbb{E}_\mu[\varphi(\tilde{f}_v)|\mathcal{F}_{t^\circ}] - \varphi(\mathbb{E}_\mu[\tilde{f}_v|\mathcal{F}_{t^\circ}])\big) \quad \mu\text{-a.e.}$$

By (5.20) and Theorem 2.12(1) we are done. □

[9] For the conditional expectation of $\operatorname{ess\,sup}_{t\in T}|f_t - f_s|$ and $\varphi(|f_v - f_s|)$ please refer to Remark 2.11.

[10] For the conditional expectation of f_v^* and $\varphi(f_v)$ please refer to Remark 2.11.

Corollary 5.12. *Consider the setting and assumptions in Corollary 5.6 and define the function φ as in* (5.12). *Then*

$$\mathbb{E}_\mu[f^*|\mathcal{F}_{t^\circ}] \le \frac{c}{c-1} \operatorname*{ess\,sup}_{v\in T}\big(\log(c)\,\mathbb{E}_\mu[f_v|\mathcal{F}_{t^\circ}] + \mathbb{E}_\mu[\varphi(f_v)|\mathcal{F}_{t^\circ}] - \varphi(\mathbb{E}_\mu[f_v|\mathcal{F}_{t^\circ}])\big) \quad \mu\text{-a.e.} \tag{5.21}$$

Proof. If $t^* = \sup T$ is an element of T itself then we immediately arrive at (5.21), because $f^* = f^*_{t^*}$. Otherwise, let $(v_n)_{n\in\mathbb{N}}$ in T be an increasing sequence such that $v_n \nearrow t^*$ as $n \to \infty$. Then $f^*_{v_n} \nearrow f^*$ pointwise on Ω as $n \to \infty$. Thus, we may apply the conditional monotone convergence theorem in Theorem 2.12(5) and it follows

$$\begin{aligned}
\mathbb{E}_\mu[(f^*)^p|\mathcal{F}_{t^\circ}] &= \lim_{n\to\infty} \mathbb{E}_\mu[(f^*_{v_n})^p|\mathcal{F}_{t^\circ}] \\
&\overset{5.11}{\le} \frac{c}{c-1} \sup_{n\in\mathbb{N}}\big(\log(c)\,\mathbb{E}_\mu[f_{v_n}|\mathcal{F}_{t^\circ}] + \mathbb{E}_\mu[\varphi(f_{v_n})|\mathcal{F}_{t^\circ}] \\
&\quad -\varphi(\mathbb{E}_\mu[f_{v_n}|\mathcal{F}_{t^\circ}])\big) \\
&\le \frac{c}{c-1} \operatorname*{ess\,sup}_{v\in T}\big(\log(c)\,\mathbb{E}_\mu[f_v|\mathcal{F}_{t^\circ}] + \mathbb{E}_\mu[\varphi(f_v)|\mathcal{F}_{t^\circ}] \\
&\quad -\varphi(\mathbb{E}_\mu[f_v|\mathcal{F}_{t^\circ}])\big),
\end{aligned}$$

which holds true μ-a.e. □

5.3 Inequalities for $p \in (0, 1)$

The following extended versions of Doob's L^p-inequality for $p \in (0, 1)$ are generalisations of the corresponding textbook presentation in [10, Lemma 3.2(c) and Satz 3.3(b)]. In order to further develop our theory on inequalities for σ-integrable submartingales we will need the following lemma.

Lemma 5.13. *Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ such that $\mu|_\mathcal{G}$ is σ-finite. Furthermore, let f be an $\mathcal{F}$-measurable and $\overline{\mathbb{R}}_+$-valued function and g a positive $\mathcal{G}$-measurable function. If*

$$\mathbb{E}_\mu\big[\mathbb{1}_{\{f\ge\lambda\}}\big|\mathcal{G}\big] \le \mathbb{1}_{\{g\ge\lambda\}} + \frac{1}{\lambda}\, g\, \mathbb{1}_{\{g<\lambda\}} \quad \mu\text{-a.e.} \tag{5.22}$$

for all $\lambda > 0$, then it follows for every $p \in (0, 1)$ that[11]

$$\mathbb{E}_\mu[f^p|\mathcal{G}] \le \frac{1}{1-p}\, g^p \quad \mu\text{-a.e.} \tag{5.23}$$

[11] In case f^p is not σ-integrable w.r.t. $\mathcal{G}$ refer to Remark 2.11 for the generalised definition of $\mathbb{E}_\mu[f^p|\mathcal{G}]$.

Proof. Similarly to the proofs of the properties of the conditional expectation, we will divide the proof into two parts. Then we can apply the generalised definition of the conditional expectation in Remark 2.11.

Step 1: Let $G \in \mathcal{R}_{g,\mathcal{G}}$ (see Definition 2.6) and define $f_n = f \wedge n$. Then f_n is σ-integrable w.r.t. $\mathcal{G}$ and so is f_n^p. Moreover,

$$\mu(f_n^p \mathbb{1}_G \geq \lambda) = \mu(\{f_n \geq \lambda^{1/p}\} \cap G) = \mathbb{E}_\mu[\mathbb{E}_\mu[\mathbb{1}_{\{f_n \geq \lambda^{1/p}\}}|\mathcal{G}]\mathbb{1}_G],$$

where

$$\mathbb{E}_\mu\big[\mathbb{1}_{\{f_n \geq \lambda^{1/p}\}}\big|\mathcal{G}\big] \leq \mathbb{E}_\mu\big[\mathbb{1}_{\{f \geq \lambda^{1/p}\}}\big|\mathcal{G}\big] \leq \mathbb{1}_{\{g \geq \lambda^{1/p}\}} + \frac{1}{\lambda^{1/p}}\, g\, \mathbb{1}_{\{g < \lambda^{1/p}\}} \quad \mu\text{-a.e.}$$

by (5.22). Therefore, it follows by Lemma 3.3 that

$$\begin{aligned}\mathbb{E}_\mu[\mathbb{E}_\mu[f_n^p|\mathcal{G}]\mathbb{1}_G] &\overset{2.7}{=} \mathbb{E}_\mu[f_n^p \mathbb{1}_G] = \int_{(0,\infty)} \mu(f_n^p \mathbb{1}_G > \lambda)\, d\lambda \\ &\leq \mathbb{E}_\mu\Big[\mathbb{1}_G \int_{(0,g^p]} 1\, d\lambda\Big] + \mathbb{E}_\mu\Big[\mathbb{1}_G\, g \int_{(g^p,\infty)} \frac{1}{\lambda^{1/p}}\, d\lambda\Big] \\ &= \mathbb{E}_\mu\Big[\mathbb{1}_G\Big(g^p + \frac{g\,p}{1-p} g^{p-1}\Big)\Big].\end{aligned}$$

The claim follows by calculating the two integrals, because the steps above yield

$$\mathbb{E}_\mu[f_n^p \mathbb{1}_G] = \mathbb{E}_\mu[\mathbb{E}_\mu[f_n^p|\mathcal{G}]\mathbb{1}_G] \leq \frac{1}{1-p}\mathbb{E}_\mu[g^p \mathbb{1}_G],$$

which implies (5.23) by Lemma 2.5(1).

Step 2: Now we can apply the conditional monotone convergence since $f_n^p \geq 0$ for all $n \in \mathbb{N}$ and $f_n^p \nearrow f^p$ for $n \to \infty$. This yields

$$\mathbb{E}_\mu[f^p|\mathcal{G}] = \lim_{n\to\infty} \mathbb{E}_\mu[f_n^p|\mathcal{G}] \leq \frac{g^p}{1-p} \quad \mu\text{-a.e.},$$

which is what we wanted to show □

Theorem 5.14. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mu)$ be a σ-finite filtered measure space and $T \subset \overline{\mathbb{R}}$ with $s, v \in T$ such that $s \leq v$ and $T \subset [s, v]$. Assume that $\mu|_{\mathcal{F}_s}$ is σ-finite and let $(f_t)_{t\in T}$ be an $\mathbb{F}$-adapted sequence of positive functions such that f_t is σ-integrable w.r.t. $\mathcal{F}_s$ for all $t \in T$ and $f_t \leq \mathbb{E}_\mu[f_u|\mathcal{F}_t]$ μ-a.e. for all $t \leq u$ in*

T. *Then for* $f^*_{s,v} := \operatorname{ess\,sup}_{t\in T} f_t$, $\lambda \in \mathbb{R}$ *and* $p \in (0,1)$ *we have*[12]

$$\begin{aligned}\mathbb{E}_\mu[(f^*_{s,v})^p|\mathcal{F}_s] &\le \frac{1}{1-p}\big(\mathbb{E}_\mu\big[f_v \mathbb{1}_{\{f^*_{s,v}>\lambda\}}\big|\mathcal{F}_s\big] - (f_s-\lambda)^+\big)^p \\ &\le \frac{1}{1-p}\big(\mathbb{E}_\mu\big[f_v \mathbb{1}_{\{f^*_{s,v}>\lambda\}}\big|\mathcal{F}_s\big]\big)^p \\ &\le \frac{1}{1-p}(\mathbb{E}_\mu[f_v|\mathcal{F}_s])^p \quad \mu\text{-a.e.}\end{aligned} \tag{5.24}$$

Proof. The second and the third inequality in (5.24) follow immediately because $(f_s-\lambda)^+ \ge 0$ and $f_v\mathbb{1}_F \le f_v$ for $F \in \mathcal{F}$. Theorem 4.2 implies that

$$\mathbb{E}_\mu\big[\mathbb{1}_{\{f^*_{s,v}\ge\lambda\}}\big|\mathcal{F}_s\big] \le \frac{1}{\lambda}\big(\mathbb{E}_\mu\big[f_v\mathbb{1}_{\{f^*_{s,v}\ge\lambda\}}\big|\mathcal{F}_s\big] - (f_s-\lambda)^+\big) \quad \mu\text{-a.e.} \tag{5.25}$$

Define $g = \mathbb{E}_\mu[f_v\mathbb{1}_{\{f^*_{s,v}\ge\lambda\}}|\mathcal{F}_s] - (f_s-\lambda)^+$ and note that g is $\mathcal{F}_s$-measurable due to the definition of the conditional expectation w.r.t. $\mathcal{F}_s$ and the adaptedness of f_s. Since we now have

$$\mathbb{E}_\mu\big[\mathbb{1}_{\{f^*_{s,v}\ge\lambda\}}\big|\mathcal{F}_s\big] \le \mathbb{1}_{\{g\ge\lambda\}} + \frac{1}{\lambda}\, g\, \mathbb{1}_{\{g<\lambda\}} \quad \mu\text{-a.e.}$$

by 5.25, we may use Lemma 5.13, which immediately yields the desired result. □

Again we may rid ourselves of the need for adaptedness by looking at the increments of our sequence of functions instead.

Corollary 5.15. *Consider the setting and assumptions in Corollary 5.4 but let* $p \in (0,1)$ *and define* $g_t = |f_t - f_s|$ *for* $t \in T$. *Then for* $g^*_{s,v} := \operatorname{ess\,sup}_{t\in T} |f_t - f_s|$ *and* $\lambda \in \mathbb{R}$ *it follows that*[13]

$$\begin{aligned}\mathbb{E}_\mu[(g^*_{s,v})^p|\mathcal{F}_s] &\le \frac{1}{1-p}\big(\mathbb{E}_\mu\big[g_v \mathbb{1}_{\{g^*_{s,v}>\lambda\}}\big|\mathcal{F}_s\big] - (-\lambda)^+\big)^p \\ &\le \frac{1}{1-p}\big(\mathbb{E}_\mu\big[g_v \mathbb{1}_{\{g^*_{s,v}>\lambda\}}\big|\mathcal{F}_s\big]\big)^p \\ &\le \frac{1}{1-p}(\mathbb{E}_\mu[g_v|\mathcal{F}_s])^p \quad \mu\text{-a.e.}\end{aligned}$$

Proof. Note that the last term in (5.24) reduces to $(-\lambda)^+$ by the definition of g_s. The rest of the proof works in the same manner as the proof of Corollary 5.4. Please refer to page 51 for more details. □

[12] For the conditional expectation of $(f^*_{s,v})^p$ please refer to Remark 2.11.
[13] For the conditional expectation of $(g^*_{s,v})^p$ please refer to Remark 2.11.

The following corollary gives a further improvement to Theorem 5.14 by omitting the need for a starting point in our time span $T \subset \overline{\mathbb{R}}$.

Corollary 5.16. *Consider the setting and assumptions in Corollary 5.5 but let $p \in (0,1)$. Then*[14]

$$\begin{aligned}\mathbb{E}_\mu[(f_v^*)^p|\mathcal{F}_{t^\circ}] &\leq \frac{1}{1-p}\left(\mathbb{E}_\mu\left[f_v \mathbb{1}_{\{f_v^*\geq\lambda\}}\middle|\mathcal{F}_{t^\circ}\right] - \operatorname*{ess\,inf}_{t\in T}\mathbb{E}_\mu[(f_t-\lambda)^+|\mathcal{F}_{t^\circ}]\right)^p \\ &\leq \frac{1}{1-p}\left(\mathbb{E}_\mu\left[f_v \mathbb{1}_{\{f_v^*\geq\lambda\}}\middle|\mathcal{F}_{t^\circ}\right]\right)^p \leq \frac{1}{1-p}(\mathbb{E}_\mu[f_v|\mathcal{F}_{t^\circ}])^p \quad \mu\text{-a.e.}\end{aligned}$$

Proof. Again, it suffices to prove the first inequality. If $t^\circ \in T$, simply take $s := t^\circ$ and the claim follows immediately, since $f_{t^\circ,v}^* = f_v^*$. Otherwise, note that by Corollary 4.4

$$\mathbb{E}_\mu\left[\mathbb{1}_{\{f_v^*\geq\lambda\}}\middle|\mathcal{F}_{t^\circ}\right] \leq \frac{1}{\lambda}\left(\mathbb{E}_\mu\left[f_v \mathbb{1}_{\{f_v^*\geq\lambda\}}\middle|\mathcal{F}_{t^\circ}\right] - \operatorname*{ess\,inf}_{t\in T}\mathbb{E}_\mu[(f_t-\lambda)^+|\mathcal{F}_{t^\circ}]\right) \quad \mu\text{-a.e.} \tag{5.26}$$

Define $g = \mathbb{E}_\mu[f_v \mathbb{1}_{\{f_v^*\geq\lambda\}}|\mathcal{F}_{t^\circ}] - \operatorname{ess\,inf}_{t\in T}\mathbb{E}_\mu[(f_t-\lambda)^+|\mathcal{F}_{t^\circ}]$ and note that g is $\mathcal{F}_{t^\circ}$-measurable due to the definition of the conditional expectation w.r.t. $\mathcal{F}_{t^\circ}$. Since we now have

$$\mathbb{E}_\mu\left[\mathbb{1}_{\{f_v^*\geq\lambda\}}\middle|\mathcal{F}_{t^\circ}\right] \leq \mathbb{1}_{\{g\geq\lambda\}} + \frac{1}{\lambda}\, g\, \mathbb{1}_{\{g<\lambda\}} \quad \mu\text{-a.e.}$$

by 5.26, we may use Lemma 5.13, which immediately yields the desired result. □

Corollary 5.17. *Consider the setting and assumptions in Corollary 5.6 but let $p \in (0,1)$. Then*[15]

$$\begin{aligned}\mathbb{E}_\mu[(f^*)^p|\mathcal{F}_{t^\circ}] &\leq \frac{1}{1-p}\left(\operatorname*{ess\,sup}_{t\in T}\mathbb{E}_\mu\left[f_t \mathbb{1}_{\{f^*\geq\lambda\}}\middle|\mathcal{F}_{t^\circ}\right] - \operatorname*{ess\,inf}_{t\in T}\mathbb{E}_\mu[(f_t-\lambda)^+|\mathcal{F}_{t^\circ}]\right)^p \\ &\leq \frac{1}{1-p}\left(\operatorname*{ess\,sup}_{t\in T}\mathbb{E}_\mu\left[f_t \mathbb{1}_{\{f^*\geq\lambda\}}\middle|\mathcal{F}_{t^\circ}\right]\right)^p \\ &\leq \frac{1}{1-p}\left(\operatorname*{ess\,sup}_{t\in T}\mathbb{E}_\mu[f_t|\mathcal{F}_{t^\circ}]\right)^p \quad \mu\text{-a.e.}\end{aligned}$$

[14] The footnote in Corollary 4.4 explains that in the world of probability spaces and random variables $\operatorname{ess\,inf}_{t\in T}\mathbb{E}_\mu[(f_t-\lambda)^+|\mathcal{F}_{t^\circ}])^p$ may be identified further using Doob's backward convergence theorem.

[15] The footnote in Corollary 4.5 explains that in the world of probability spaces and random variables $\operatorname{ess\,sup}_{t\in T}\mathbb{E}_\mu[f_t \mathbb{1}_{\{f^*\geq\lambda\}}|\mathcal{F}_{t^\circ}]$ may be further identified using Doob's almost sure convergence theorem.

Proof. Once more, it suffices to prove the first inequality. If $t^* = \sup T$ is an element of T itself then we immediately arrive at the claim, because $f^* = f^*_{t^*}$. Otherwise, note that by Corollary 4.5

$$\mathbb{E}_\mu\big[\mathbb{1}_{\{f^*\geq\lambda\}}\big|\mathcal{F}_{t^\circ}\big] \leq \frac{1}{\lambda}\Big(\operatorname*{ess\,sup}_{t\in T}\mathbb{E}_\mu\big[f_t\mathbb{1}_{\{f^*\geq\lambda\}}\big|\mathcal{F}_{t^\circ}\big] - \operatorname*{ess\,inf}_{t\in T}\mathbb{E}_\mu[(f_t-\lambda)^+|\mathcal{F}_{t^\circ}]\Big) \quad \mu\text{-a.e.} \tag{5.27}$$

Define $g = \operatorname{ess\,sup}_{t\in T}\mathbb{E}_\mu[f_t\mathbb{1}_{\{f^*\geq\lambda\}}|\mathcal{F}_{t^\circ}] - \operatorname{ess\,inf}_{t\in T}\mathbb{E}_\mu[(f_t-\lambda)^+|\mathcal{F}_{t^\circ}]$ and note that g is $\mathcal{F}_{t^\circ}$-measurable due to the definition of the conditional expectation w.r.t. $\mathcal{F}_{t^\circ}$. Since we now have

$$\mathbb{E}_\mu\big[\mathbb{1}_{\{f^*\geq\lambda\}}\big|\mathcal{F}_{t^\circ}\big] \leq \mathbb{1}_{\{g\geq\lambda\}} + \frac{1}{\lambda}\,g\,\mathbb{1}_{\{g<\lambda\}} \quad \mu\text{-a.e.}$$

by (5.27), we may use Lemma 5.13, which immediately yields the desired result. □

5.4 Examples For Equality and Sharp Inequalities

In the proofs of the improved and generalised versions of Doob's maximum and L^p-inequalities we rely on deterministic inequalities, which may hold with equality given certain conditions. Lemma 4.1, for example, discusses under what circumstances equality may hold in (4.1) and (4.2). As we discuss in the Appendix, we may achieve equality in Young's inequality as well if we make the necessary assumptions outlined in Theorem A.36. For this reason we decided to look into finding examples for processes and functions which may yield equality in our newly developed inequalities.

The following example shows that there are, in fact, processes that imply equality in (5.5). For this purpose let us return to the world of probability spaces and stochastic processes.

Example 5.18. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $T = \overline{\mathbb{R}}_+$. For a positive real-valued[16] random variable $\tau \sim \operatorname{Exp}(1)$ define the indicator process $Y = (Y_t)_{t\in T}$ by $Y_t = \mathbb{1}_{[\tau,\infty]}(t)$ and let $\mathbb{F} = (\mathcal{F}_t)_{t\in T}$ be the filtration generated by Y, i.e.[17] $\mathcal{F}_t = \sigma(\{Y_s : T \ni s \leq t\})$ for $t \in T$. Furthermore, define the process $X := (X_t)_{t\in T}$ by[18]

$$X_t = \Big(\mathbb{1}_{[0,\tau)}(t) + \frac{1}{c}\mathbb{1}_{[\tau,\infty]}(t)\Big)\exp\Big(\frac{c-1}{c}(t\wedge\tau)\Big), \quad t \in T. \tag{5.28}$$

[16]We want to ensure that $\mathbb{P}(\tau < \infty) = 1$.

[17]Note that $\mathbb{F}$ is, in fact, the smallest filtration such that τ is a stopping time w.r.t. $\mathbb{F}$.

[18]Note that this process satisfies the conditions in Proposition 5.21 below.

For ease of readability we will make us of the convention to denote the exponential function using Euler's number e. Define $\gamma = (c-1)/c$ and note that $c(\gamma - 1) = -1$. We will show that X is, in fact, a martingale[19], which we do by proving that the map $T \ni t \mapsto \mathbb{E}[X_t]$ is constant and that X is a submartingale. Example A.29 in the Appendix then implies, that X is a martingale. Therefore, let $t \in T$. Then

$$\mathbb{E}[X_t] = \mathbb{E}[\mathbb{1}_{\{\tau>t\}}\mathrm{e}^{\gamma t}] + \frac{1}{c}\mathbb{E}[\mathbb{1}_{\{\tau\le t\}}\mathrm{e}^{\gamma\tau}] = \underbrace{\mathbb{P}(\tau > t)}_{=\mathrm{e}^{-t}}\mathrm{e}^{\gamma t} + \frac{1}{c}\underbrace{\int_0^t \mathrm{e}^{s(\gamma-1)}\,ds}_{=(\mathrm{e}^{t(\gamma-1)}-1)/(\gamma-1)}$$

$$= \left(1 + \frac{1}{c(\gamma-1)}\right)\mathrm{e}^{t(\gamma-1)} - \frac{1}{c(\gamma-1)} = 1.$$

This also implies the integrability of X. Since $X_t \le \mathrm{e}^{\tau\gamma}/c$ for all $t \in T$ the process is also uniformly integrable[20] (see Definition A.19). Moreover, in order to show that X is a martingale we will use that $\mathcal{F}_t = \sigma(\{\tau \le s\} : s \in [0,t])$ for $t \in T$ equals the σ-algebra[21]

$$\mathcal{G}_t := \{F \in \mathcal{F}_t : \{\tau > t\} \subset F \text{ or } \{\tau > t\} \subset F^{\mathrm{c}}\}$$

for all $t \in T$. This is true, since $\{\tau > t\} \in \mathcal{F}_t$ for all $t \in T$, $\{\tau > t\} \subset \{\tau \le s\}^{\mathrm{c}}$ for all $s \in [0,t]$, which implies that for all $F \in \mathcal{F}_t$ we have either $\{\tau > t\} \subset F$ or $\{\tau > t\} \subset F^{\mathrm{c}}$, hence, $\mathcal{F}_t \subset \mathcal{G}_t$ (and clearly, $\mathcal{G}_t \subset \mathcal{F}_t$).

We will now prove the submartingale property by relying on the memorylessness of the exponential distribution[22]. For this purpose let $s \le t$ be elements of T.

$$\mathbb{E}[X_t|\mathcal{F}_s] \overset{\text{a.s.}}{=} \mathbb{E}[\mathbb{1}_{\{\tau>t\}}\mathrm{e}^{\gamma t}|\mathcal{F}_s] + \frac{1}{c}\mathbb{E}[\mathbb{1}_{\{\tau\le t\}}\mathrm{e}^{\gamma\tau}|\mathcal{F}_s]. \tag{5.29}$$

Let us treat each summand on the right-hand side separately. Since τ is exponentially distributed we can use its memorylessness and deduce that

$$\mathbb{E}[\mathbb{1}_{\{\tau>t\}}\mathrm{e}^{\gamma t}|\mathcal{F}_s] = \mathrm{e}^{\gamma t}\underbrace{\mathbb{P}(\tau > t|\mathcal{F}_s)}_{=\mathbb{P}(\tau>t-s)\text{ a.s.}} \overset{\text{a.s.}}{=} \mathrm{e}^{t(\gamma-1)+s}. \tag{5.30}$$

We can rewrite the second summand on the right-hand side of (5.29) using $\mathbb{1}_{\{\tau\le t\}} = \mathbb{1}_{\{\tau\le s\}} + \mathbb{1}_{\{s<\tau\le t\}}$ and

$$\mathbb{E}[\mathbb{1}_{\{\tau\le s\}}\mathrm{e}^{\gamma\tau}|\mathcal{F}_s] = \mathbb{E}[\mathbb{1}_{\{\tau\le s\}}\mathrm{e}^{\gamma(\tau\wedge s)}|\mathcal{F}_s] \overset{\text{a.s.}}{=} \mathbb{1}_{\{\tau\le s\}}\mathrm{e}^{\gamma(\tau\wedge s)} \tag{5.31}$$

[19] For the definition please refer to the corresponding paragraph in the Introduction on page 1.

[20] For more information of uniformly integrable functions please refer to [13, Chapter 4.2, p. 85–91].

[21] The fact that $\mathcal{G}_t$ is a σ-algebra is easy to prove and will be left to the reader.

[22] e.g. $\mathbb{P}(\tau \ge s+t|\tau \ge s) = \mathbb{P}(\tau \ge t)$ for all $s, t \ge 0$

by applying Theorem A.28(5) to the $\mathcal{F}_s$-measurable function $\mathbb{1}_{\{\tau\leq s\}}\mathrm{e}^{\gamma(\tau\wedge s)}$. We claim that

$$\mathbb{E}[\mathbb{1}_{\{s<\tau\leq t\}}\mathrm{e}^{\gamma\tau}|\mathcal{F}_s] \overset{\text{a.s.}}{=} \mathbb{1}_{\{s<\tau\}}\mathrm{e}^{\gamma s}\mathbb{E}[\mathbb{1}_{\{\tau\leq t-s\}}\mathrm{e}^{\gamma\tau}] \tag{5.32}$$

For $F\in\mathcal{F}_s$ we have

$$\begin{aligned}\int_F \mathbb{E}[\mathbb{1}_{\{s<\tau\leq t\}}\mathrm{e}^{\gamma\tau}|\mathcal{F}_s]\,d\mathbb{P} &= \int_F \mathbb{1}_{\{s<\tau\leq t\}}\mathrm{e}^{\gamma\tau}\,d\mathbb{P}\\ &= \int_{F\cap\{\tau\leq s\}} \mathbb{1}_{\{s<\tau\leq t\}}\mathrm{e}^{\gamma\tau}\,d\mathbb{P} + \int_{F\cap\{\tau>s\}} \mathbb{1}_{\{s<\tau\leq t\}}\mathrm{e}^{\gamma\tau}\,d\mathbb{P}.\end{aligned}$$

The first integral on the right-hand side reduces to zero because $\{\tau\leq s\}\cap\{s<\tau\leq t\}=\emptyset$. Furthermore, we have shown before, every $F\in\mathcal{F}_s$ is also an element of $\mathcal{G}_s$, hence, $F\cap\{\tau>s\}=\{\tau>s\}$. Thus, by using $\{s<\tau\leq t\}\subset\{\tau>s\}$ we arrive at

$$\begin{aligned}\int_F \mathbb{E}[\mathbb{1}_{\{s<\tau\leq t\}}\mathrm{e}^{\gamma\tau}|\mathcal{F}_s]\,d\mathbb{P} &= \int_{\{s<\tau\leq t\}} \mathrm{e}^{\gamma\tau}\,d\mathbb{P} = \int_s^t \mathrm{e}^{r(\gamma-1)}\,dr\\ &= \frac{1}{\gamma-1}\big(\mathrm{e}^{t(\gamma-1)}-\mathrm{e}^{s(\gamma-1)}\big).\end{aligned}$$

Secondly, since

$$\mathbb{E}[\mathbb{1}_{\{\tau\leq t-s\}}\mathrm{e}^{\gamma\tau}] = \int_0^{t-s}\mathrm{e}^{r(\gamma-1)}\,dr = \frac{1}{\gamma-1}\big(\mathrm{e}^{t(\gamma-1)}\mathrm{e}^{-s(\gamma-1)}-1\big)$$

and $\mathrm{e}^{\gamma s}\mathbb{P}(s<\tau)=\mathrm{e}^{s(\gamma-1)}$, it follows that

$$\int_F \mathbb{1}_{\{s<\tau\}}\mathrm{e}^{\gamma s}\mathbb{E}[\mathbb{1}_{\{s<\tau\leq t-s\}}\mathrm{e}^{\gamma\tau}]\,d\mathbb{P} = \frac{1}{\gamma-1}\big(\mathrm{e}^{t(\gamma-1)}-\mathrm{e}^{s(\gamma-1)}\big).$$

Hence, (5.32) holds true a.s. Using (5.30), (5.31) and (5.32) and plugging them into (5.29), we can deduce that

$$\begin{aligned}\mathbb{E}[X_t|\mathcal{F}_s] &\overset{\text{a.s.}}{=} \mathrm{e}^{t(\gamma-1)+s}+\frac{1}{c}\mathbb{1}_{\{\tau\leq s\}}\mathrm{e}^{\gamma(\tau\wedge s)}-\mathbb{1}_{\{\tau>s\}}\mathrm{e}^{\gamma s}\big(\mathrm{e}^{t(\gamma-1)}\mathrm{e}^{-s(\gamma-1)}-1\big)\\ &= \mathrm{e}^{t(\gamma-1)+s}+\frac{1}{c}\mathbb{1}_{\{\tau\leq s\}}\mathrm{e}^{\gamma(\tau\wedge s)}-\mathbb{1}_{\{\tau>s\}}\big(\mathrm{e}^{t(\gamma-1)+s}-\mathrm{e}^{\gamma s}\big)\\ &= \Big(\mathbb{1}_{\{\tau>s\}}+\frac{1}{c}\mathbb{1}_{\{\tau\leq s\}}\Big)\mathrm{e}^{\gamma(\tau\wedge s)}+\mathrm{e}^{t(\gamma-1)+s}(1-\mathbb{1}_{\{\tau>s\}}).\end{aligned}$$

Since $\mathrm{e}^{t(\gamma-1)+s}(1-\mathbb{1}_{\{\tau>s\}})\geq 0$, we can conclude that

$$\mathbb{E}[X_t|\mathcal{F}_s] \overset{\text{a.s.}}{\geq} \Big(\mathbb{1}_{\{\tau>s\}}+\frac{1}{c}\mathbb{1}_{\{\tau\leq s\}}\Big)\mathrm{e}^{\gamma(\tau\wedge s)} = X_s,$$

which means, X is a submartingale.

Finally, we would like to show that (5.5) holds with equality when applied to X. Define the supremum process $X_t = \sup_{s\in[0,t]} X_s$ for $t \in T$. As the interval in question we wish to consider $[s,v] := [0,\infty] = T$ and prove that

$$\mathbb{E}[(X_v^*)^p|\mathcal{F}_s] \overset{a.s.}{=} \frac{c}{c-1}\frac{q}{p}\big(c^{p/q}\mathbb{E}[X_v^p|\mathcal{F}_s] - (\mathbb{E}[X_v|\mathcal{F}_s])^p\big). \tag{5.33}$$

In order to do so, note that $\mathcal{F}_s = \mathcal{F}_0 = \{\emptyset, \mathbb{R}_+\}$, hence, we can omit all the conditions in the equality above. Calculating the left-hand side of (5.33) then yields

$$\mathbb{E}[(X_v^*)^p|\mathcal{F}_0] = \mathbb{E}[e^{p\gamma\tau}] = \frac{c}{c-pc+p} \tag{5.34}$$

by the representation of the moment-generating function of an exponentially distributed random variable. Furthermore,

$$\mathbb{E}[X_v^p|\mathcal{F}_s] = \frac{1}{c^p}\mathbb{E}[e^{p\gamma\tau}] = \frac{c}{c^p(c-pc+p)} \tag{5.35}$$

and

$$(\mathbb{E}[X_v|\mathcal{F}_s])^p = \frac{1}{c^p}(\mathbb{E}[e^{\gamma\tau}])^p = \frac{1}{c^p}\frac{1}{(1-\gamma)^p} = 1. \tag{5.36}$$

Now, inserting (5.34), (5.35) and (5.36) into (5.33) and solving the equation gives zero on both sides, so (5.33) yields a true result. Hence, (5.33) holds true.

We aim to find families of functions and processes for which Theorems 5.2 and 5.8 hold with equality. Lemma 4.1 already hints towards which kind of processes and functions could achieve the desired result: One of these three different conditions needs to be satisfied in order for (4.1) to hold with equality:

- the maximum of a sequence of real numbers $x_0, \dots, x_n$ for $n \in \mathbb{N}$ is strictly smaller than a certain threshold $\lambda \in \mathbb{R}$;
- x_0 is greater or equal to λ;
- or if for the smallest $k \in \{1, \dots, n\}$ such that the number x_k is greater or equal λ, it is already equal λ.

Furthermore, in the proof of (5.1) we use Young's inequality, for which we discuss conditions for equality in Theorem A.36. In particular, (5.4) holds with equality, if $c\,x_n = \overline{x}_n := \max\{x_0, \dots, x_n\}$ for $c > 1$. Therefore, we can make the following observation:

Proposition 5.19. *Let $\lambda \in \mathbb{R}$, $c > 1$ and $s \leq v$ in $T \subset \overline{\mathbb{R}}$ such that $T \subset [s, v]$. The improved version of Doob's L^p-inequality for $p > 1$ in* (5.5) *holds with equality if, and only if, for $f := (f_t)_{t \in T}$ fulfilling the assumptions in Theorem 5.2 we have that $c\, f_v = \operatorname{ess\,sup}_{t \in T} f_t$ and f satisfies one of the following conditions.*

(1) $\operatorname{ess\,sup}_{t \in T} f_t \leq \lambda$, *i.e. f is bounded from above by λ.*

(2) $f_s \geq \lambda$, *i.e. f is bounded from below at the start of the period.*

(3) *If for $t, u \in T$ with $t \leq u$ such that $f_t \leq \lambda$ and $f_u \geq \lambda$, there exists a $\tilde{u} \in T$ such that $f_{\tilde{u}} = \lambda$, i.e. if f crosses a certain threshold throughout the period, f has to take a value at the mentioned threshold.*

Another way to prove the improved version of Doob maximum inequalities relies on stopping times, as Remark 4.3 shows. For this purpose let us review (3.4), which is essential for deriving (3.1). In the first part of the proof of Theorem 3.1 we require T to be finite and define $\tau = v \wedge \min\{t \in T : f_t \geq \lambda\}$ and $A = \{\max_{t \in T} f_t \geq \lambda\}$ and explain that (3.4), which states

$$\lambda \mathbb{1}_A \leq f_\tau \mathbb{1}_A - (f_s - \lambda)^+,$$

holds with equality on

- A^{c} (meaning $\max_{t \in T} f_t < \lambda$), since both sides are zero;
- $\{f_s \geq \lambda\} \subset A$, because both sides equal λ.

Remark 4.3 explains that by taking the conditional expectation of (3.4) we can deduce (4.3) in Theorem 4.2 by applying Theorem 2.20. Furthermore, we attain equality in (4.11) if, and only if, our sequence of functions $(f_t)_{t \in T}$ is in fact a σ-integrable martingale.

Finally, it follows from the proof of Lemma 5.1, where we apply Lemma 4.1 to the maximum of the sequences of real-numbers, that the sequence $(f^*_{s,u})_{u \in T}$ with $f^*_{s,u} := \operatorname{ess\,sup}_{t \in [s,u]} f_t$ needs to be continuous in order for (5.2) applied to $(f^*_{s,u})_{u \in T}$ to hold with equality. Hence, we can expect the inequality in Theorem 5.2 to hold with equality for the following sequences of functions.

Conjecture 5.20. *Let $c > 1$ and $T \subset \overline{\mathbb{R}}$ be non-empty such that T is countable or a non-degenerate interval with $T \subset [s, v]$ for $s \leq v$ in T. If $f := (f_t)_{t \in T}$ is a positive, right-continuous σ-integrable martingale such that the supremum process $(f^*_{s,u})_{u \in T}$ with $f^*_{s,u} := \operatorname{ess\,sup}_{t \in [s,u]} f_t$ is continuous and $c\, f_v = f^*_{s,v}$, then the improved version of Doob's L^p-inequality for $p > 1$ as stated in Theorem 5.2 applied to f holds with equality.*

We would need Doob's optional stopping theorem adapted to σ-finite measure spaces and σ-integrable martingales in order to prove this proposition. So far we can prove equality in (3.4) only for finite T (see Theorem 2.20) but would need optional stopping for continuous time to ensure the martingale property remains true. This would exceed the scope of this thesis, hence, we cannot prove Conjecture 5.20 beyond a reasonable doubt at this point in time. However, if we return to probability spaces, we can make use of Doob's optional stopping theorem (as stated in Theorem A.30) to prove the following.

Proposition 5.21. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, $c > 1$ and $T \subset \overline{\mathbb{R}}$ be non-empty such that T is countable or a non-degenerate interval with $T \subset [s, v]$ for $s \le v$ in T. If $X := (X_t)_{t\in T}$ is a positive, right-continuous martingale and if there exists a continuous and $\mathbb{F}$-adapted process $(X^*_{s,u})_{u\in T}$ such that $X^*_{s,u} = \sup_{t\in[s,u]} X_t$ a.s. and $c\,X_v = X^*_{s,v}$, then the improved version of Doob's L^p-inequality for $p > 1$ as stated in Theorem 5.2 applied to X holds with equality.*

Proof. The right-continuity of X allows us to apply Doob's optional stopping theorem (see Theorem A.30) and deduce that (3.4) holds true for countable as well as non-degenerate intervals $T \subset \overline{\mathbb{R}}$ and $A := \{X^*_{s,v} \ge \lambda\}$. Furthermore, by Theorem A.30 it follows that

$$\mathbb{E}[X_\tau \mathbb{1}_A | \mathcal{F}_s] \overset{\text{a.s.}}{=} \mathbb{E}[\mathbb{E}[X_v|\mathcal{F}_\tau]\mathbb{1}_A|\mathcal{F}_s] \overset{\text{a.s.}}{=} \mathbb{E}[X_v \mathbb{1}_A|\mathcal{F}_s]$$

for an $\mathbb{F}$- or $\mathbb{F}^+$-stopping time τ by the $\mathcal{F}_s$-measurability of $\mathbb{1}_A$ and the integrability of martingales (see Theorem A.28(5) and (1)). As we have explained above, we attain equality in (5.2) if the supremum process $(X^*_{s,u})_{u\in T}$ is continuous. Since we assumed $c\,X_v = X^*_{s,v}$, equality follows for (5.3). □

Another example for a process that satisfies the conditions in Proposition 5.21 and therefore yields equality in the improved version of Doob's L^p-inequality can be found in [1, proof of Theorems 3.1 and 1.2, p. 11]. We would like to present this example here for further illustration.

Example 5.22. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $T = \mathbb{R}_+$ and $B := (B_t)_{t\in T}$ a Brownian motion starting at $B_0 = 1$. Consider the stopped process $B^{\tau_c} := (B_{t\wedge\tau_c})_{t\in T}$ for $c > 1$ together with the stopping time $\tau_c := \inf\{t > 0 : B_t \le B^*_t/c\}$, where $(B^*_t)_{t\in T}$ is given by $B^*_t := \sup_{s\in[0,t]} B_s$. Then B^{τ_c} is uniformly integrable[23]. Furthermore, consider the process $\tilde{B} := (\tilde{B}_t)_{t\in[0,v]}$ defined by

[23]For the proof please refer to [13, Lemma 7.10]. The lemma also proves that $\mathbb{P}(\tau_c < \infty) = 1$.

$\tilde{B}_t = B_{t/(v-t)\wedge\tau_c}$ for $t \leq v$ in T. Then $\tilde{B}$ is a non-negative right-continuous martingale, the supremum process $(\tilde{B}_t^*)_{t\in[0,v]}$ is continuous and $c\,\tilde{B}_v = \tilde{B}_v^*$ for $v \in T$. Hence, we may apply Proposition 5.21 and deduce that

$$\mathbb{E}[\tilde{B}_v^*|\mathcal{F}_s] \overset{a.s.}{=} \frac{c}{c-1}\frac{q}{p}\big(c^{p/q}\,\mathbb{E}[\tilde{B}_v^p|\mathcal{F}_s] - (\mathbb{E}[\tilde{B}_v|\mathcal{F}_s])^p\big)$$

for $p, q \in (1,\infty)$ such that $1/p + 1/q = 1$ and $0 \leq s \leq v$.

We can make similar observations for Theorem 5.8 as the ones above. Again, we need (5.10) in the proof of Lemma 5.7. This inequality holds with equality if, and only if, $a, b > 0$ and $c = b/a$ as the footnote at the beginning of the proof on page 53 mentions. We later apply (5.10) to $x_n \log \overline{x}_n$, therefore, we achieve equality in (5.1) if, and only if, $c = \log \overline{x}_n / x_n$. Moreover, an important step for deriving (5.13) is (5.19), where we use the conditional version of Jensen's inequality. We achieve equality in (5.19) if there exists an $\mathcal{F}$-measurable modification[24] of the process. In a deterministic setting this means that the sequence of functions has to be constant μ-a.e. Therefore, we need to assume that the submartingale attains a constant value at the starting point. Considering all the arguments brought forth we can deduce the following necessary assumptions for equality in the improved version of Doob's L^1-inequality in Theorem 5.8.

Proposition 5.23. *Let $\lambda \in \mathbb{R}$, $c > 1$ and $s \leq v$ in $T \subset \overline{\mathbb{R}}$ such that $T \subset [s, v]$. The improved version of Doob's L^1-inequality in* (5.13) *holds with equality if, and only if, for $f := (f_t)_{t\in T}$ fulfilling the assumptions in Theorem 5.8 we have that $f_s = k \in \mathbb{R}$, $c\,f_v = \log(\operatorname{ess\,sup}_{t\in T} f_t)$ and f satisfies one of the three conditions in Proposition 5.19.*

Since we also use Lemma 4.1 when proving Lemma 5.7 (in particular, (5.11) is the key aspect here), the sequence $(f_{s,u}^*)_{u\in T}$ with $f_{s,u}^* := \operatorname{ess\,sup}_{t\in[s,u]} f_t$ has to be continuous once again. Hence, we can expect (5.13) to hold with equality for the following sequences of functions.

Conjecture 5.24. *Let $c > 1$ and $T \subset \overline{\mathbb{R}}$ be non-empty such that T is countable or a non-degenerate interval with $T \subset [s, v]$ for $s \leq v$ in T. Let $f := (f_t)_{t\in T}$ be a positive, right-continuous σ-integrable martingale with $f_s = k \in \mathbb{R}$. If the supremum process $(f_{s,u}^*)_{u\in T}$, where $f_{s,u}^* := \operatorname{ess\,sup}_{t\in[s,u]} f_t$, is continuous and $c\,f_v = \log(f_{s,v}^*)$, then the improved version of Doob's L^1-inequality as stated in Theorem 5.8 applied to f holds with equality.*

[24] Let I be a non-empty index set and let $f := (f_i)_{i\in I}$ and $(g_i)_{i\in I}$ be sequences of functions on a σ-finite measure space $(\Omega, \mathcal{F}, \mu)$ with values in a measurable space $(S, \mathcal{S})$. We call f and g *modifications* of one another, if the set $\{f_i \neq g_i\}$ is contained in a μ-null set for every $i \in I$. (This definition was inspired by [13, Definition 2.83] focusing on probability spaces and stochastic processes.)

Again, without Doob's optional stopping theorem adapted to σ-finite measure spaces and σ-integrable martingales we cannot prove Conjecture 5.24 beyond a reasonable doubt at this point in time. However, if we return to probability spaces, we can make use of Doob's optional stopping theorem (as stated in Theorem A.30) to prove the following.

Proposition 5.25. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, $c > 1$ and $T \subset \overline{\mathbb{R}}$ be non-empty such that T is countable or a non-degenerate interval with $T \subset [s, v]$ for $s \leq v$ in T. Furthermore, let $X := (X_t)_{t \in T}$ be a positive, right-continuous martingale such that $X_s = k \in \mathbb{R}$. If there exists a continuous and $\mathbb{F}$-adapted process $(X^*_{s,u})_{u \in T}$ such that $X^*_{s,u} = \sup_{t \in [s,u]} X_t$ a.s. and $c\, X_v = \log(X^*_{s,v})$, then the improved version of Doob's L^1-inequality as stated in Theorem 5.8 applied to X holds with equality.*

Proof. The right-continuity of X allows us to apply Doob's optional stopping theorem (see Theorem A.30) and deduce that (3.4) holds true for countable as well as non-degenerate intervals $T \subset \overline{\mathbb{R}}$ and $A := \{X^*_{s,v} \geq \lambda\}$. Furthermore, by Theorem A.30 it follows that

$$\mathbb{E}[X_\tau \mathbb{1}_A | \mathcal{F}_s] \overset{\text{a.s.}}{=} \mathbb{E}[\mathbb{E}[X_v | \mathcal{F}_\tau] \mathbb{1}_A | \mathcal{F}_s] \overset{\text{a.s.}}{=} \mathbb{E}[X_v \mathbb{1}_A | \mathcal{F}_s]$$

for an $\mathbb{F}$- or $\mathbb{F}^+$-stopping time τ by the $\mathcal{F}_s$-measurability of $\mathbb{1}_A$ and the integrability of martingales (see Theorem A.28(5) and (1)). As we have explained above, we attain equality in (5.2) if the supremum process $(X^*_{s,u})_{u \in T}$ is continuous. Since we assumed $c\, X_v = \log(X^*_{s,v})$ and $X_s = k \in \mathbb{R}$, equality follows in (5.13). □

Chapter 5 showed various generalisations and improvements to Doob's L^p-inequalities for submartingales, as they hold true (slightly adapted) even when dealing with σ-integrable functions rather than random variables. Furthermore, the need for adaptedness and integrability can be weakened such that the inequalities hold for more general assumptions. As we have shown, these inequalities may even hold with equality given certain conditions. All this can be of great help in actuarial practice, as will be outlined in the next chapter.

Chapter 6

Practical Applications in Intertemporal Risk Control

In this chapter we wish to give a short overview of possible applications of our newly developed theory on (sub-)martingales within the fields of financial and actuarial mathematics. We will see that especially the improved versions of Doob's L^p-inequalities for $p \geq 1$ can provide practitioners with helpful means to derive upper bounds for various key figures within their fields of expertise.

6.1 Mathematical Finance

A rather obvious application of our newly developed (sub-)martingale inequalities can be found in mathematical finance. According to the fundamental theorem of asset pricing in finite discrete time, the market does not allow for arbitrage if, and only if, there exists a probability measure $\mathbb{Q}$ equivalent to the original probability measure $\mathbb{P}$ such that the discounted price process is a martingale under $\mathbb{Q}$. The measure $\mathbb{Q}$ is then called a *martingale measure*. This motivates to apply the findings of Chapter 5 to the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T}, \mathbb{Q})$ and the discounted price process denoted by $X^{(i)} := S^{(i)}/S^{(0)}$ given a model for a financial market with $d \in \mathbb{N}$ assets $(S^{(1)}, \ldots, S^{(d)})$, a numeraire $S^{(0)}$ and a time span $T \subset \overline{\mathbb{R}}$ such that $S_t^{(i)}$ denoting the price of the i-th asset at time $t \in T$ is a non-negative random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.

Theorem 5.8 gives an upper bound for the conditional expectation of the essential supremum of the discounted asset, whereas Theorem 5.2 yields similar results for its second moment. Both might not be easily calculated, thus, sharp inequalities may be good alternatives. As a possible scenario think of X as the discounted price of a plain vanilla option in a short position.

The asset then turns into a liability fraught with risk. Using our newly developed inequalities it is possible to predict the worst case scenario, i.e. the highest possible payment obligation, *at any time* during a given period. Here lies the significance for intertemporal risk control. Let $T \subset \overline{\mathbb{R}}$ with $s, v \in T$ such that $s \leq v$ and $T \subset [s, v]$ and $c > 1$. Then for $p = q = 2$ (5.5) reduces to

$$\mathbb{E}_{\mathbb{Q}}\Big[\Big(\operatorname*{ess\,sup}_{t\in T} X_t^{(i)}\Big)^2\Big|\mathcal{F}_s\Big] \overset{\text{a.s.}}{\leq} \frac{c}{c-1}\big(c\,\mathbb{E}_{\mathbb{Q}}[(X_v^{(i)})^2|\mathcal{F}_s] - (X_s^{(i)})^2\big) \tag{6.1}$$

for $i = 1, \ldots, d$ by the martingale property of $X^{(i)}$. If we assume that the discounted price process is square-integrable, then taking the expectation w.r.t. the probability measure $\mathbb{Q}$ in (6.1) yields

$$\mathbb{E}_{\mathbb{Q}}\Big[\Big(\operatorname*{ess\,sup}_{t\in T} X_t^{(i)}\Big)^2\Big] \leq \frac{c}{c-1}\big(c\,\mathbb{E}_{\mathbb{Q}}[(X_v^{(i)})^2] - \mathbb{E}_{\mathbb{Q}}[(X_s^{(i)})^2]\big) \tag{6.2}$$

Since $\mathbb{R}_+ \ni x \mapsto x^2$ is convex and X is positive then a further upper bound that does not need the starting point $s \in T$ can be derived for (6.2) using Jensen's inequality (see Theorem A.16). This yields $\mathbb{E}_{\mathbb{Q}}[(X_s^{(i)})^2] \geq (\mathbb{E}_{\mathbb{Q}}[X_s^{(i)}])^2 = (\mathbb{E}_{\mathbb{Q}}[X_v^{(i)}])^2$, hence,

$$\mathbb{E}_{\mathbb{Q}}\Big[\Big(\operatorname*{ess\,sup}_{t\in T} X_t^{(i)}\Big)^2\Big] \leq \frac{c^2}{c-1}\mathbb{V}_{\mathbb{Q}}[X_v^{(i)}] + c\,(\mathbb{E}_{\mathbb{Q}}[X_v^{(i)}])^2. \tag{6.3}$$

Minimising $c^2/(c-1)\,\mathbb{V}_{\mathbb{Q}}[X_v^{(i)}] + c\,(\mathbb{E}_{\mathbb{Q}}[X_v^{(i)}])^2$ yields that (6.3) is minimal for

$$\hat{c} = 1 + \frac{\sqrt{(\mathbb{V}_{\mathbb{Q}}[X_v^{(i)}])^2 - \mathbb{V}_{\mathbb{Q}}[X_v^{(i)}](\mathbb{E}_{\mathbb{Q}}[X_v^{(i)}])^2}}{\mathbb{E}_{\mathbb{Q}}[(X_v^{(i)})^2]}. \tag{6.4}$$

Thus, we need to make further assumptions for our discounted price process in order to ensure the existence of $\hat{c}$:

- $\mathbb{E}_{\mathbb{Q}}[(X_v^{(i)})^2] > 0$;
- $(\mathbb{V}_{\mathbb{Q}}[X_v^{(i)}])^2 > \mathbb{V}_{\mathbb{Q}}[X_v^{(i)}](\mathbb{E}_{\mathbb{Q}}[X_v^{(i)}])^2$.

Given these conditions it is possible to find a minimal upper bound for the essential supremum of the discounted price process. As mentioned before, [1] proves Theorem 5.2 for $c = \mathrm{e}$. By (6.4) it becomes clear now, that our generalised version of Doob's L^p-inequality for $p > 1$ is indeed an improvement to both [1, Proposition 2.1]and Theorem 3.2.

On the other hand, (5.13) and the martingale property give us

$$\mathbb{E}_{\mathbb{Q}}\left[\operatorname*{ess\,sup}_{t\in T} X_t^{(i)}\middle|\mathcal{F}_s\right] \overset{\text{a.s.}}{\leq} \frac{c}{c-1}\left(\log(c)X_s^{(i)} + \mathbb{E}_{\mathbb{Q}}[\varphi(X_v^{(i)})|\mathcal{F}_s] - \varphi(X_s^{(i)})\right) \quad (6.5)$$

for $i = 1, \ldots, d$, where the function φ is defined as in (5.12). Let us assume that $\varphi(X_t^{(i)})$ is integrable for all $t \in T$. Then taking the expected value in (6.5) yields

$$\mathbb{E}_{\mathbb{Q}}\left[\operatorname*{ess\,sup}_{t\in T} X_t^{(i)}\right] \leq \frac{c}{c-1}\left(\log(c)\,\mathbb{E}_{\mathbb{Q}}[X_s^{(i)}] + \mathbb{E}_{\mathbb{Q}}[\varphi(X_v^{(i)})] - \mathbb{E}_{\mathbb{Q}}[\varphi(X_s^{(i)})]\right). \quad (6.6)$$

By Jensen's inequality (see Theorem A.16) it follows that $\mathbb{E}_{\mathbb{Q}}[\varphi(X_v^{(i)})] \geq \mathbb{E}_{\mathbb{Q}}[\varphi(X_s^{(i)})]$ for all $s \leq v$ in T, because $(\varphi(X_t^{(i)}))_{t\in T}$ is a submartingale. Again, we may look at minimal upper bounds. Minimising the right-hand side of (6.6) for $c > 1$, yields that the inequality is minimal for

$$\hat{c} - \log(\hat{c}) = 1 + \frac{\mathbb{E}_{\mathbb{Q}}[\varphi(X_v^{(i)})] - \mathbb{E}_{\mathbb{Q}}[\varphi(X_s^{(i)})]}{\mathbb{E}_{\mathbb{Q}}[X_s^{(i)}]}. \quad (6.7)$$

Again, we detect the need for further assumptions in order to ensure the existence of a solution $\hat{c}$ in (6.7):

- $\mathbb{E}_{\mathbb{Q}}[X_s^{(i)}] > 0$;
- $\mathbb{E}_{\mathbb{Q}}[\varphi(X_v^{(i)})] > \mathbb{E}_{\mathbb{Q}}[\varphi(X_s^{(i)})]$.

Due to the intermediate value theorem (see Theorem A.41) it is always possible to solve (6.7) given the newly recognised assumptions. The solution is given by the so-called *Lambert W function*[1], which is defined as the inverse function of $f(W) := W\mathrm{e}^W$. As we have shown, we are now able to estimate the expected value of the essential supremum of the discounted price process as well. This may give practitioners an edge on what maximal financial obligation can be expected at any time during a certain period.

Finally, Theorem 5.8 can also provide us with an upper bound for the maximal fluctuation of a financial instrument. The positive submartingale $Z := (Z_t)_{t\in T}$ with $T \subset \mathbb{R}$ and $Z_t := (Y_t - \mathbb{E}[Y_t])^2$, where $Y := (Y_t)_{t\in T}$ does not necessarily need to be a positive process (e.g. think of a swap contract), satisfies the assumptions of Theorem 5.8. Hence, for $s \leq v$ in T with $T \subset [s, v]$ and $c > 1$ it follows that

$$\mathbb{E}_{\mathbb{Q}}\left[\operatorname*{ess\,sup}_{t\in T} Z_t\middle|\mathcal{F}_s\right] \overset{\text{a.s.}}{\leq} \frac{c}{c-1}\left(\log(c)\,Z_s + \mathbb{E}_{\mathbb{Q}}[\varphi(Z_v)|\mathcal{F}_s] - \varphi(Z_s)\right)$$

[1]This function is implemented in various technical computing systems. E.g. in Wolfram Mathematica the function is called by `ProductLog[z]`.

and

$$\mathbb{E}_{\mathbb{Q}}\Big[\operatorname*{ess\,sup}_{t\in T} Z_t\Big] \leq \frac{c}{c-1}\big(\log(c)\,\mathbb{E}_{\mathbb{Q}}[Z_s] + \mathbb{E}_{\mathbb{Q}}[\varphi(Z_v)] - \mathbb{E}_{\mathbb{Q}}[\varphi(Z_s)]\big)$$

by assuming the integrability of Z and $\varphi(Z)$. We have just proven in the paragraph above under which circumstances the upper bound can be minimal for $c > 1$. Since $\mathbb{E}_{\mathbb{Q}}[\operatorname{ess\,sup}_{t\in T}(Y_t - \mathbb{E}[Y_t])^2]$ can be seen as a generalisation of the variance of Y, the inequalities above provide information on the maximal fluctuation of the process Y at any time during a certain period of time. Of course, the applicability and usefulness of such an upper bound cannot be overlooked.

In conclusion, the generalised versions of Doob's L^2-inequality and his L^1-inequality provide practitioners in the field of mathematical finance with an outlook on the expectation of a financial risk (e.g. a payment obligation) and gives upper bounds thereof. Particularly, the newly developed theory on Doob's L^p-inequalities may aid in controlling intertemporal risks not just at the beginning or the end of a period but *at any point in time* during a given period. Here lie the novelty and the importance of the findings of this thesis, as they provide practitioners with intertemporal risk constraints.

6.2 Actuarial Science

Besides the applications in financial mathematics, our improved versions of Doob's L^p-inequalities provide estimates for the loss random variable over the lifetime of an actuarial reserve. Hattendorff's theorem, developed 1868 by K. Hattendorff, demonstrates that the variance in the present value of the loss of an issued insurance policy can be allocated to the future years during which the insured is still alive. This, in turn, facilitates the management of risk prevalent in such insurance contracts over short periods of time. In particular, the theorem states that the loss random variables for different time periods are uncorrelated and the expected value is zero.

At the time of its development Hattendorff's theorem was viewed as quite controversial and came as a great surprise to many researchers and practitioners in the field in actuarial science. Today it is an important part of the standard curriculum for actuarial science. More than a century after the initial development, Bühlmann connected Hattendorff's theorem to the theory of martingales in 1976. In 1992 Norberg generalised Hattendorff's theorem further and connected it to modern martingale theory and showed that the variance of the loss random variable can be properly calculated, if we assume the state-space to satisfy the Markov property and take deterministic actuarial payment functions into consideration. A stochastic process has the Markov property if the conditional probability distribution of future states of the process (conditional on both past and present states) depends only upon the present state, not on the sequence of events that preceded it. If we cannot (or do not want to) make such assumptions, inequalities can be of great help. In particular, the improved versions of Doob's L^1- and L^2-inequalities for martingales will help us in deriving estimates for the expectation as well as the second moment of the essential supremum of the loss random variable. For more information on the history behind Hattendorff's theorem the interested reader is referred to [11, p. 489–491]

First, we would like to introduce Hattendorff's theorem as formulated in [8, p. 85–87][2]. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. Consider a cash flow modeling the benefits for the insured by a stochastic process B, which is discounted by another stochastic process v. The present value at time $t \geq 0$ is then given by $V_t = \int_t^\infty v(r)\,dB(r)$, where we assume the Lebesgue–Stieltjes integral is well-defined and V_t has finite expectation for all $t \geq 0$. Assume that v and B are both adapted to the filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$. Then the prospective actuarial reserve $V_{\mathbb{F}}^+(t)$ at time $t \geq 0$ is given by

$$V_{\mathbb{F}}^+(t) \overset{\text{a.s.}}{=} \frac{1}{v(t)}\,\mathbb{E}\Big[\int_t^\infty v(r)\,dB(r)\Big|\mathcal{F}_t\Big].$$

The *loss of the insurer* in the time interval $(s,t]$ with $s \leq t$ discounted to the time 0 of a cash flow B is now defined by

$$L(s,t) = \int_s^t v(r)\,dB(r) + v(t)V_{\mathbb{F}}^+(t) - v(s)V_{\mathbb{F}}^+(s) \overset{\text{a.s.}}{=} M(t) - M(s), \quad (6.8)$$

where for $t \geq 0$ we define

$$M(t) = \mathbb{E}[V_0|\mathcal{F}_t] \overset{\text{a.s.}}{=} \int_0^t v(r)\,dB(r) + v(t)V_{\mathbb{F}}^+(t).$$

Based on (6.8) one can see that the loss is composed of payments in the interval $(s,t]$ (represented by $\int_s^t v(r)\,dB(r)$), the value of the policy at the end of the period (denoted by $v(t)V_{\mathbb{F}}^+(t)$) as well as the value of the policy at the beginning of the period (denoted by $v(s)V_{\mathbb{F}}^+(s)$). Hattendorff's theorem now states the following.

[2]To be exact, [8] uses a decreasing sequence of stopping times for the definition. However, since the constant time is also a stopping time by A.22(1) it was decided to formulate the theorem without the use of proper stopping times, because we work with constant times when applying the improved versions of Doob's L^p-inequalities.

Theorem 6.1. *If $V \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, then*

(1) $\mathbb{E}[L(s,t)] = 0$ *and* $\operatorname{Cov}(L(s,t), L(u,v)) = 0$ *for all* $0 \le s \le t$ *and* $0 \le u \le v$ *such that* $(s,t] \cap (u,v] = \emptyset$*;*

(2) $\mathbb{E}[L(s,t)|\mathcal{F}_r] = 0$ *and* $\operatorname{Cov}(L(s,t), L(u,v)|\mathcal{F}_r) = 0$ *a.s. for all* $r \ge 0$ *and* $0 \le s \le t$ *and* $0 \le u \le v$ *such that* $(s,t] \cap (u,v] = \emptyset$*.*

The proof heavily relies on the fact that the stochastic process $(M(t))_{t\ge 0}$ is an $\mathbb{F}$-martingale; in particular, it is used that the increments of martingales have mean zero. We omit the proof, but interested readers may refer to [8, Theorem 7.2.5] for the case of a countable infinite time span $T \subset \mathbb{R}_+$ or to [11, Satz 9.24] for the general setting.

As mentioned before, the improved versions of Doob's L^p-inequalities in Chapter 5 now allow us to determine estimates for the conditional expectation of the essential supremum. Let $T \subset \overline{\mathbb{R}}$ with $s, v \in T$ such that $s \le v$ and $T \subset [s, v]$. Then $(L(s,t)^+)_{t\in T}$ (which describes the losses of an insurer) satisfies the conditions in Theorem 5.2, thus, we may conclude that for $c > 1$ and $p = q = 2$ that

$$\mathbb{E}\Big[\Big(\operatorname*{ess\,sup}_{t\in T} L(s,t)^+\Big)^2\Big|\mathcal{F}_s\Big] \overset{\text{a.s.}}{\le} \frac{c}{c-1}\big(c\,\mathbb{E}[(L(s,v)^+)^2|\mathcal{F}_s] - (\mathbb{E}[L(s,v)^+|\mathcal{F}_s])^2\big). \tag{6.9}$$

Furthermore, we can use Theorem 5.8 and conclude that

$$\begin{aligned}\mathbb{E}\Big[\operatorname*{ess\,sup}_{t\in T} L(s,t)^+\Big|\mathcal{F}_s\Big] \overset{\text{a.s.}}{\le} \frac{c}{c-1}\big(&\log(c)\,\mathbb{E}[L(s,v)^+|\mathcal{F}_s] + \mathbb{E}[\varphi(L(s,v)^+)|\mathcal{F}_s] \\ &- \varphi(\mathbb{E}[L(s,v)^+|\mathcal{F}_s])\big),\end{aligned} \tag{6.10}$$

where the function φ is defined as in (5.12).

If we assume $(L(s,t)^+)_{t\in T}$ to be square-integrable and $(\varphi(L(s,t)^+))_{t\in T}$ to be integrable, we can make similar observations as those on the expected value and its second moment of the discounted price process in the previous chapter. Taking the expected value in (6.9) yields

$$\mathbb{E}\Big[\Big(\operatorname*{ess\,sup}_{t\in T} L(s,t)^+\Big)^2\Big] \le \frac{c}{c-1}\big(c\,\mathbb{E}[(L(s,v)^+)^2] - (\mathbb{E}[L(s,v)^+)^2\big).$$

Furthermore, we can deduce

$$\begin{aligned}\mathbb{E}\Big[\operatorname*{ess\,sup}_{t\in T} L(s,t)^+\Big] \le \frac{c}{c-1}\big(&\log(c)\,\mathbb{E}[L(s,v)^+] + \mathbb{E}[\varphi(L(s,v)^+)] \\ &- \varphi(\mathbb{E}[L(s,v)^+])\big)\end{aligned}$$

by taking the expected value in (6.10). We could consider minimising the inequalities above again, in order to come up with minimal bounds. As we have discussed this at length in the previous chapter, we leave this to the reader and will omit repetition at this point. However, once more, the importance and applicability to real life cases becomes apparent.

All we have discussed in this chapter shows that an insurer can gain control over intertemporal financial risks (e.g. the possible losses stemming from an insurance contract within a given period). The the upper bounds for the expectation and the second moment of the loss random variable provide information on whether or not the insurer remains solvent w.r.t. a (life) insurance policy *at all times* within a certain period. The novelty here is that not only is the insurer able to give an outlook on their solvency at the end of the period but the bounds allow them to assess their solvency at any moment within the period. Hence, Theorem 5.2 and 5.8 provide the insurer with means to control intertemporal risks.

6.3 Utility Maximasation

Theorem 5.14 may find applications in the fields of financial and actuarial mathematics as well, because we can connect its findings to utility theory. Define the function

$$u(x) = \frac{x^p}{1-p}, \quad p \in (0,1), x > 0.$$

Then u is a utility function[3], because $u'(x) = px^{p-1}/(1-p) > 0$ and $u''(x) = -px^{p-2} < 0$ for all $x > 0$ and $p \in (0,1)$. By the definition of u, we can rewrite the last inequality[4] in (5.24). Consider $s \leq v$ in T, where $T \subset \mathbb{R}$, such that $T \subset [s,v]$. Then

$$\mathbb{E}_\mu[(f^*_{s,v})^p|\mathcal{F}_s] \leq u(\mathbb{E}_\mu[f_v|\mathcal{F}_s]) \quad \mu\text{-a.e.}$$

Furthermore,

$$\mathbb{E}_\mu[(f^*_{s,v})^p|\mathcal{F}_s] = (1-p)\,\mathbb{E}_\mu[u(f^*_{s,v})|\mathcal{F}_s] \leq u(\mathbb{E}_\mu[f_v|\mathcal{F}_s]) \quad \mu\text{-a.e.}$$

Since $u(f^*_{s,v}) = \operatorname{ess\,sup}_{t\in T} u(f_t)$ we may deduce

$$\mathbb{E}_\mu[\operatorname*{ess\,sup}_{t\in T} u(f_t)|\mathcal{F}_s] \leq \frac{1}{1-p} u(\mathbb{E}_\mu[f_v|\mathcal{F}_s]) \quad \mu\text{-a.e.} \tag{6.11}$$

[3] i.e. $u'(x) > 0$ and $u''(x) < 0$ for all $x > 0$

[4] Of course, we could make similar statements as the ones below for the sharper bounds, but since the last bound is easier calculated and more intuitive it was chosen as the focus of this discussion.

Utility theory tells us that for some risks it might make sense to focus on the expected utility of a gamble rather than its expectation. Theorem 5.14 gives us an upper bound for the conditional expected utility of the essential supremum of a sequence of functions.

If we go back to a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a stochastic process $X := (X_t)_{t\in T}$ describing a short position on a financial instrument, (6.11) gives us the means to estimate the maximal utility of this contract. Hence, we can weigh the expectation of the maximal value of such an instrument, for which we have an upper bound with (6.6), against its expected utility with an upper bound thereof given by (6.11). Again, (6.11) yields an upper bound that gives information on the utility of the maximal amount *at any given time* during a certain period. This may aid practitioners in deciding, whether or not entering into such a contract would be worth the investment and the possible risk stemming from an uncertain payment obligation.

As we have learned, the improved versions of Doob's L^p-inequalities can be put to great use in the fields of both financial and actuarial mathematics. Doob's classical L^p-inequalities for submartingales are certainly already of value in those fields. However, as the upper bounds can be improved even further, sharper inequalities can be derived. This may help practitioners around the globe to derive rather sharp estimates for the essential supremum of the loss random variable in case of application in life insurance mathematics or the essential supremum of the discounted price process when dealing with financial markets. Both can be of great value in the applied world of mathematics.

Chapter 7

Conclusion

The goal of this thesis was to expand the theory of martingales by adapting it to σ-finite measure spaces and proving refined and generalised versions of Doob's maximum and L^p-inequalities. Chapter 2 focused on expanding the definition of the conditional expectation to σ-integrable functions and σ-finite measure spaces. Proving the existence of a conditional expectation of a σ-integrable function under a σ-finite measure heavily relied on the Radon–Nikodým theorem. Theorem 2.12 was a fundamental finding within this chapter, because it shows that many well-known properties of the conditional expectation w.r.t. probability measures hold for σ-finite measures as well. Moreover, some of these properties may even hold for functions that have the right measurability but are not σ-integrable. We proved this by introducing a further generalisation of the conditional expectation in Remark 2.11.

There are two different definitions for (sub-/super-)martingales on σ-finite measure spaces in this thesis. The first one (see Definition 2.13) is quite similar to the definition of martingales in probability spaces, because we assume integrability and adaptedness, and we proved Doob's maximum inequalities and his classical L^p-inequality for submartingales and $p > 1$ for this setting. In this section we relied on a theoretical approach using measure theory. However, we could prove that the main theorems in this thesis do not need integrability and may even make do without adaptedness. For this reason we introduced the term *σ-integrable (sub-/super-)martingale* (see Definition 2.15), which lies at the core of this thesis and its findings.

The main focus of this thesis was to prove various generalisations and improvements of Doob's maximum and L^p-inequalities for σ-integrable submartingales. This can be found in Chapter 4 and 5. The proofs rely on rather basic deterministic inequalities, which helped us to find sharper upper bounds. In both chapters we start out with a given interval with a starting and an endpoint. However, by considering the infimum and the supremum of our time span we proved that we may omit the need for such points. Hence, we were able to expand the martingale theory by introducing (and proving) sharper and more general versions of Doob's maximum inequalities (see Theorem 4.2), Doob's inequalities for $p > 1$ (see Theorem 5.2) and $p = 1$ (see Theorem 5.8) as well as $p \in (0, 1)$ (see Theorem 5.14).

The deterministic inequalities we used to prove our refined inequalities may hold with equality given certain conditions. For this reason an investigation into families of processes, where equality in the newly developed theorems of Chapter 4 and 5 follows, was carried out. For this part we chose to return to the world of probability spaces and stochastic processes and gave examples to support our claims. Proposition 5.21 (resp. Proposition 5.25) shows what kind of processes imply equality in Theorem 5.2 (resp. Theorem 5.8).

The final chapter gave an outlook on how our improved versions of Doob's L^p-inequalities can be of help to practitioners in the fields of financial and actuarial mathematics. For example, the findings in our thesis may provide upper bounds for the expectation of the essential supremum of the discounted price process. Another possible application of the newly developed inequalities in this thesis can be found in the field of actuarial mathematics, since they can be used to find upper bounds for the expectation of the essential supremum of the loss random variable. In both cases, it is possible to determine minimal upper bounds. Furthermore, the improved version of Doob's L^p-inequality for $p \in (0, 1)$ finds application in utility maximisation.

In conclusion, this thesis achieved its goal to expand the theory of martingales. The main findings are refined versions of Doob's maximum and L^p-inequalities for σ-integrable submartingales and σ-finite measure spaces. Other interesting topics within this setting would be Doob's optional stopping theorem and its implications as well as a generalised definition of local martingales and their connection to stochastic integrals. A complete discussion of these topics would go beyond the scope of this thesis, however, this leaves much room for further research into the matter. The author hopes that the findings in this thesis ease the path for further projects as mathematicians strive to delve further into the vast and open world of mathematics.

Appendix

The following pages list various established results presented in well-known literature concerning measure theory and the conditional expectation. Except for Remark A.2, Lemma A.11, Remark A.21, Theorem A.22, Example A.26, Example A.29, Lemma A.31, the alternative proof of Theorem 2.7 on page 92 and Theorem A.36 non of the results presented here are of the author's conception. Futhermore, we do not claim to have been the firsts to prove the findings listed above (with the exception of the alternative proof for Theorem 2.7). We proved them due to the lack of time to research corresponding literature or due to the simple nature of the proofs. All definitions and results below have been marked with the corresponding source material.

A.1 Some Measure Theory

Definition A.1. σ-FINITE MEASURES
Let $(\Omega, \mathcal{F})$ be a measurable space and μ a measure on it. We call μ *σ-finite*, if one of following three properties holds.

(1) [1] Ω can be covered with at most countably many measurable sets of finite measure, i.e. there exist $\Omega_1, \Omega_2, \ldots \in \mathcal{F}$ with $\mu(\Omega_n) < \infty$ for all $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$.

(2) [2] Ω can be covered with at most countably many disjoint sets of finite measure, i.e. there exist $\Omega_1, \Omega_2, \ldots \in \mathcal{F}$ with $\mu(\Omega_n) < \infty$ for all $n \in \mathbb{N}$ and $\Omega_m \cap \Omega_n = \emptyset$ for all $m, n \in \mathbb{N}$ with $m \neq n$ such that $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$.

(3) [3] Ω can be covered with a monotone sequence of measurable sets of finite measure, i.e. there exist $\Omega_1, \Omega_2, \ldots \in \mathcal{F}$ with $\Omega_1 \subset \Omega_2 \subset \cdots$ and $\mu(\Omega_n) < \infty$ for all $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$.

[1] See [9, Definition 3.9].
[2] See `https://en.wikipedia.org/wiki/%CE%A3-finite_measure`.
[3] See [12, Definition 4.2].

Remark A.2. The three properties in Definition A.1 are equivalent.

Indeed: For (1) $\Rightarrow$ (2) consider the sequence $(\bar{\Omega}_n)_{n\in\mathbb{N}}$ with $\bar{\Omega}_1 = \Omega_1$ and $\bar{\Omega}_{n+1} = \Omega_{n+1} \cap (\bigcap_{i=1}^n \Omega_i^c)$, where the $(\Omega_n)_{n\in\mathbb{N}}$ satisfies (1). Then $(\bar{\Omega}_n)_{n\in\mathbb{N}}$ satisfies (2).

Of course, (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1). This leaves us to show (1) $\Rightarrow$ (3). This follows, because the sequence $(\tilde{\Omega}_n)_{n\in\mathbb{N}}$ defined by $\tilde{\Omega}_n = \bigcup_{i=1}^n \Omega_i$, where the $(\Omega_n)_{n\in\mathbb{N}}$ satisfies (1), satisfies (3).

Definition A.3. SIMPLE FUNCTION[4]
Let Ω be an arbitrary set. We call $f : \Omega \to \mathbb{R}$ *simple function*, if there exist $\gamma_1, \ldots, \gamma_m \in \mathbb{R}$ and a finite partition $\Omega_1, \ldots, \Omega_m$ of Ω such that $f(\omega) = \sum_{k=1}^m \gamma_k \mathbb{1}_{\Omega_k}(\omega)$ for all $\omega \in \Omega$.

Lemma A.4. *Let $(\Omega, \mathcal{F})$ be a measurable space. For every non-negative $\mathcal{F}$-measurable functions f there exists a sequence of non-negative monotonously increasing simple functions $(f_n)_{n\in\mathbb{N}}$ such that $f = \lim_{n\to\infty} f_n$.*

Proof. For the proof please refer to [9, Satz 7.30]. □

Theorem A.5. *Let $(\Omega, \mathcal{F})$ be a measurable space. If $f_n : \Omega \to \overline{\mathbb{R}}$ are $\mathcal{F}$-measurable functions for all $n \in \mathbb{N}$, then so are*

$$\sup_{n\in\mathbb{N}} f_n, \quad \inf_{n\in\mathbb{N}} f_n, \quad \limsup_{n\to\infty} f_n, \quad \liminf_{n\to\infty} f_n.$$

The same holds for $\lim_{n\to\infty} f_n$ whenever it exists pointwise.

Proof. For the proof please refer to [12, Corollary 8.9]. □

Definition A.6. δ-RING[5]
Let S be a set and $\mathcal{R} \subset \mathcal{P}(S)$ a non-empty collection of subsets of S. We call $\mathcal{R}$ a *δ-ring*, if

(1) $A, B \in \mathcal{R} \Rightarrow A \cup B \in \mathcal{R}$,

(2) $A, B \in \mathcal{R} \Rightarrow A \setminus B \in \mathcal{R}$, and

(3) $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{R}$ for every countable collection $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{R}$.

Definition A.7. SIGNED MEASURE[6]
Let $\mathcal{R}$ be a δ-ring on a set S. An $\mathbb{R}^d$-valued (or $\mathbb{C}^d$-valued) measure on $\mathcal{R}$ is a map $\mu : \mathcal{R} \to \mathbb{R}^d$ (or $\mu : \mathcal{R} \to \mathbb{C}^d$) such that $\sum_{n\in\mathbb{N}} \mu(A_n) = \mu(\bigcup_{n\in\mathbb{N}} A_n)$ for

[4] See [9, Definition 7.25].
[5] See [13, Definition 13.88].
[6] See [13, Definition 13.93].

every sequence $(A_n)_{n\in\mathbb{N}}$ of disjoint sets in $\mathcal{R}$ such that the union $\bigcup_{n\in\mathbb{N}} A_n$ is also in $\mathcal{R}$. The convergence of the series $\sum_{n\in\mathbb{N}} \mu(A_n)$ is part of the requirement. An $\mathbb{R}$-valued (or $\mathbb{C}$-valued) measure on $\mathcal{R}$ is called *signed (or complex) measure* on $\mathcal{R}$.

Definition A.8. Singular measure[7]
Let $\mathcal{R}$ be a δ-ring on a set S. We call two $\overline{\mathbb{R}}_+$-valued (or signed or complex) measures μ and ν on $\mathcal{R}$ *singular* on $\mathcal{R}$, if for every $A \in \mathcal{R}$ there exists a partition $B, C \in \mathcal{R}$ such that $\mu(B) = 0$ and $\nu(C) = 0$. This is denoted by $\mu \perp \nu$.

Theorem A.9. Jordan decomposition
Let $\mathcal{R}$ be a δ-ring on a set S. For every signed measure ν on $\mathcal{R}$ there exists a unique decomposition with two $\mathbb{R}_+$-valued measures ν^+ and ν^- that are singular on $\mathcal{R}$ such that $\nu = \nu^+ - \nu^-$. For every $A \in \mathcal{R}$ we call these two measures the positive *and* negative variation *of ν. They are given by $\nu^+(A) = \nu(A^+)$ and $\nu^-(A) = -\nu(A^-)$ (hence, they satisfy $\nu^+(A^-) = \nu^-(A^+) = 0$), where (A^+, A^-) denotes any Hahn decomposition of A w.r.t. ν on $\mathcal{R}$.*

Proof. For the proof please refer to [13, Theorem 13.101]. For the definition and the proof of existence of a Hahn decomposition please refer to [13, Theorem 13.98]. □

Definition A.10. Absolutely continuous and equivalent measures[8]
Let μ and ν be two (positive measures) on a measurable space $(\Omega, \mathcal{F})$.

(1) We call μ *absolutely continuous* w.r.t. ν on $\mathcal{F}$, if $\mu(F) = 0$ for all $F \in \mathcal{F}$ with $\nu(F) = 0$. This is denoted by $\mu \ll \nu$.

(2) If $\mu \ll \nu$ and $\nu \ll \mu$ on $\mathcal{F}$, then we call μ and ν *equivalent* on $\mathcal{F}$ and write $\mu \sim \nu$.

Lemma A.11. *Any non-zero σ-finite measure μ on a measurable space $(\Omega, \mathcal{F})$ is equivalent to a probability measure on $(\Omega, \mathcal{F})$.*

Proof. Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of disjoint sets in $\mathcal{F}$ such that $0 < \mu(\Omega_n) < \infty$ and $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$ (which exists, because we require μ to be σ-finite) and let $(w_n)_{n\in\mathbb{N}}$ be a sequence of strictly positive numbers (weights) such that $\sum_{n=1}^{\infty} w_n = 1$. Then the measure ν defined by

$$\nu(F) = \sum_{n=1}^{\infty} w_n \frac{\mu(F \cap \Omega_n)}{\mu(\Omega_n)}, \quad F \in \mathcal{F},$$

[7] See [13, Definition 13.100].
[8] See [13, Definition 7.15(a),(b)].

is a probability measure on Ω: Evidently, $\nu(F) \geq 0$ for all $F \in \mathcal{F}$, $\nu(\emptyset) = 0$ and $\nu(\Omega) = 1$. The σ-additivity of ν follows by

$$\nu\Big(\bigcup_{n\in\mathbb{N}} F_n\Big) = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} w_n \frac{\mu(F_i \cap \Omega_n)}{\mu(\Omega_n)} = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} w_n \frac{\mu(F_i \cap \Omega_n)}{\mu(\Omega_n)} = \sum_{i=1}^{\infty} \nu(F_i)$$

for a disjoint sequence $(F_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ because all summands are positive, which allows us to exchange the sums[9]. Since $\nu(F) = 0$ if, and only if, $\mu(F) = 0$ for $F \in \mathcal{F}$ it follows that $\mu \sim \nu$. □

Theorem A.12. Basic properties of integrals
Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, let $f, g \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ and $\alpha \in \mathbb{R}$. Then the following properties hold.

(1) Homogeneity: $\alpha f \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ *and* $\int \alpha f \, d\mu = \alpha \int f \, d\mu$.

(2) Additivity: $f + g \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ *and* $\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$.

(3) Monotonicity: $f \leq g \Rightarrow \int f \, d\mu \leq \int g \, d\mu$.

(4) $|\int f \, d\mu| \leq \int |f| \, d\mu$.

Proof. For the proofs please refer to [12, Theorem 10.4]. □

Lemma A.13. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and f a measurable function. Then $f \in L^1(\Omega, \mathcal{F}, \mu)$ if, and only if, $|f| \in L^1(\Omega, \mathcal{F}, \mu)$.*

Proof. For the proof please refer to [9, Folgerung 9.25]. □

Theorem A.14. Radon–Nikodým theorem
Let μ and ν be two measures of the measurable space $(\Omega, \mathcal{F})$. If μ is σ-finite, then the following two statements are equivalent:

(1) $\nu \ll \mu$;

(2) $\nu(F) = \int_F f \, d\mu$, *for all $F \in \mathcal{F}$ and some a.e. unique $f \in L^0_+(\Omega, \mathcal{F}, \mu)$.*

Furthermore, f is real-valued μ-a.e. if, and only if, ν is σ-finite. We call a function with property (2) *a* density of ν w.r.t. μ *and denote it by $f = \frac{d\nu}{d\mu}$.*

Proof. For the proof please refer to [9, Satz 11.19]. □

[9] See [9, Satz A.16] for the proof here.

Theorem A.15. Hölder's inequality
Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$. Then we have for all measurable real- or complex-valued functions $f \in L^p(\Omega, \mathcal{F}, \mu)$ and $g \in L^q(\Omega, \mathcal{F}, \mu)$

$$\left(\int |fg| \, d\mu \right) \leq \left(\int |f|^p \, d\mu \right)^{1/p} \left(\int |g|^q \, d\mu \right)^{1/q}.$$

Proof. For the proof please refer to [12, Theorem 12.2]. □

Theorem A.16. Jensen's inequality
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X an integrable random variable with values in an interval $C \subset \mathbb{R}$. If $\varphi : C \to \mathbb{R}$ is a convex function, then $\varphi \circ X$ is integrable and $\varphi(\mathbb{E}[X]) \leq \mathbb{E}[\varphi \circ X]$.

Proof. For the proof please refer to [9, Satz 13.1]. In the theorem C is defined as an interval (a, b), which may lead to the conclusion that the theorem only holds for intervals (a, b) with $a, b \in \mathbb{R}$. In the proof, however, the cases where $a = -\infty$ and $b = \infty$ are treated as well. □

Theorem A.17. Dominated convergence theorem
Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f : \Omega \to \mathbb{R}$ a measurable function. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable functions with $f_n : \Omega \to \mathbb{R}$ such that $f_n \to f$ as $n \to \infty$ and $|f_n| \leq g$ for all $n \in \mathbb{N}$ and some $g \in L^1(\Omega, \mathcal{F}, \mu)$. Then

$$\int_\Omega f \, d\mu = \lim_{n \to \infty} \int_\Omega f_n \, d\mu.$$

Proof. For the proof please refer to [14, Section 5.9, p. 54–55]. □

Theorem A.18. Monotone convergence theorem
Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f : \Omega \to [0, \infty]$ a measurable function. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of non-negative, measurable functions with $f_n : \Omega \to [0, \infty]$ such that $f_n \nearrow f$ as $n \to \infty$. Then

$$\int_\Omega f \, d\mu = \lim_{n \to \infty} \int_\Omega f_n \, d\mu.$$

Proof. For the proof please refer to [14, Section 5.3, p. 51–52, and Appendix A5.4, p. 213]. □

Definition A.19. Uniform integrability[10]
Let Φ be a non-empty set of measurable $\mathbb{K}^d$-valued functions on a measure

[10] See [13, Definition 4.25].

space $(\Omega, \mathcal{F}, \mu)$. We call Φ *uniformly integrable*, if for every $\epsilon > 0$ there exists a μ-integrable function $\omega_\epsilon : \Omega \to \mathbb{R}_+$ such that

$$\sup_{\varphi \in \Phi} \int_{\{\|\varphi\| > \omega_\epsilon\}} \|\varphi\| \, d\mu < \epsilon.$$

From now on let $T \subset \overline{\mathbb{R}}$. Furthermore, define $t^* = \sup T$ and $\mathcal{F}_{t^*} = \sigma(\bigcup_{t \in T} \mathcal{F}_t)$.

Definition A.20. STOPPING TIME[11]
Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathbb{F} := (\mathcal{F}_t)_{t \in T}$ a filtration. A map $\tau : \Omega \to \overline{T}$ is called *stopping time* w.r.t. $\mathbb{F}$, if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in T$. The associated σ-algebra is given by

$$\mathcal{F}_\tau := \{F \in \mathcal{F}_{t^*} \mid F \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \in T\}. \tag{A.1}$$

Remark A.21. It can be easily shown that $\mathcal{F}_\tau$ as defined in (A.1) is indeed a σ-algebra. For this purpose let $t \in T$.

- $\Omega \in \mathcal{F}_\tau$: This follows directly because $\Omega \cap \{\tau \leq t\} = \{\tau \leq t\} \in \mathcal{F}_t$, since τ is a stopping time w.r.t. $\mathbb{F}$.

- $F \in \mathcal{F}_\tau \Rightarrow F^c \in \mathcal{F}_\tau$: The properties of the compliment in set theory imply that for all $t \in T$ we have

$$F^c \cap \{\tau \leq t\} = \underbrace{(F \cap \{\tau \leq t\})^c}_{\in \mathcal{F}_t} \cap \underbrace{\{\tau \leq t\}}_{\in \mathcal{F}_t} \in \mathcal{F}_t,$$

  where we used that τ is a stopping time w.r.t. $\mathbb{F}$.

- $(F_n)_{n \in \mathbb{N}}$ with $F_n \in \mathcal{F}_\tau \ \forall n \in \mathbb{N} \Rightarrow \bigcup_{n \in \mathbb{N}} F_n \in \mathcal{F}_\tau$: This follows by the properties of σ-algebras and stopping times, because

$$\left(\bigcup_{n \in \mathbb{N}} F_n\right) \cap \{\tau \leq t\} = \bigcup_{n \in \mathbb{N}} \underbrace{(F_n \cap \{\tau \leq t\})}_{\in \mathcal{F}_t} \in \mathcal{F}_t,$$

  for all $t \in T$.

Theorem A.22. LIST OF PROPERTIES OF STOPPING TIMES
For all $t \in T$ and stopping times σ and τ w.r.t. $\mathbb{F}$, we have

(1) *the constant time $\eta : \Omega \to T : \omega \mapsto t$ is a stopping time w.r.t. $\mathbb{F}$ and $\mathcal{F}_\eta = \mathcal{F}_t$;*

[11] See [13, Definition 3.7 and Definition 3.10].

(2) *the pointwise maximum* $\sigma \vee \tau$ *and the pointwise minimum* $\sigma \wedge \tau$ *are stopping times w.r.t.* $\mathbb{F}$*;*

(3) τ *is* $\mathcal{F}_\tau$*-measurable;*

(4) *if* $\sigma \leq \tau$ *(pointwise), then* $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$*;*

(5) $\mathbb{F}^\tau := (\mathcal{F}_{\tau \wedge t})_{t \in T}$ *is a filtration;*

(6) $\mathcal{F}_{\sigma \wedge \tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$*;*

(7) $F \cap \{\sigma \leq \tau\} \in \mathcal{F}_{\sigma \wedge \tau}$ *for all* $F \in \mathcal{F}_\sigma$*;*

(8) $F \cap \{\sigma = \tau\} \in \mathcal{F}_{\sigma \wedge \tau}$ *for all* $F \in \mathcal{F}_\sigma$*;*

(9) *if* $(\tau_n)_{n \in \mathbb{N}}$ *is a sequence of stopping times and* $\tau := \sup_{n \in \mathbb{N}} \tau_n$ *takes values in* $\overline{T}$*, then* τ *is a stopping time, too.*

Proof. (1) It is easy to see that η is a stopping time, since for all $s \in T$ we have that

$$\mathcal{F}_s \ni \{\eta \leq s\} = \begin{cases} \emptyset & t > s, \\ \Omega & t \leq s. \end{cases}$$

Now, let us consider $A \in \mathcal{F}_\eta$. By (A.1) $A \cap \{\eta \leq s\} \in \mathcal{F}_s$ for all $s \in T$. In particular, for the constant time we have $A \cap \{\eta \leq t\} = A \cap \Omega = A \in \mathcal{F}_t$, which implies $\mathcal{F}_\eta \subset \mathcal{F}_t$.

Finally, let $B \in \mathcal{F}_t$. Then for $t > s \in T$ we have $B \cap \{\eta \leq s\} = \emptyset \in \mathcal{F}_s$. Conversely, for $t \leq s$ it follows that $B \cap \{\eta \leq s\} = B \in \mathcal{F}_t \subset \mathcal{F}_s$, since $\mathbb{F}$ is a filtration. Therefore, $B \in \mathcal{F}_\eta$ and thus, $\mathcal{F}_t \subset \mathcal{F}_\eta$.

(2) Let $t \in T$. The claim follows directly due to the properties of σ-algebras, because

$$\{\sigma \vee \tau \leq t\} = \underbrace{\{\sigma \leq t\}}_{\in \mathcal{F}_t} \cap \underbrace{\{\tau \leq t\}}_{\in \mathcal{F}_t} \in \mathcal{F}_t,$$
$$\{\sigma \wedge \tau \leq t\} = \underbrace{\{\sigma \leq t\}}_{\in \mathcal{F}_t} \cup \underbrace{\{\tau \leq t\}}_{\in \mathcal{F}_t} \in \mathcal{F}_t.$$

(3) Define $B = (a, b]$ for $a \leq b \in \overline{T}$. Then it suffices to show that $\tau^{-1}(B) \in \mathcal{F}_\tau$, because intervals like B generate the Borel σ-algebra $\mathcal{B}_{\overline{T}}$. For this purpose we will use the convention $\{\tau \in B\} := \tau^{-1}(B)$. Then

$$\{\tau \in B\} \cap \{\tau \leq t\} = \{\tau \leq a\}^{\mathrm{c}} \cap \{\tau \leq b\} \cap \{\tau \leq t\} = \{\tau \leq a\}^{\mathrm{c}} \cap \{\tau \leq b \wedge t\}.$$

The set $\{\tau \le b \wedge t\}$ is in $\mathcal{F}_t$ for all $t \in T$, because we have for $b \le t$ that $\{\tau \le b \wedge t\} \in \mathcal{F}_b \subset \mathcal{F}_t$ and $\{\tau \le b \wedge t\} \in \mathcal{F}_t$ for $b > t$ by the definition of stopping times and filtrations.

Furthermore, if $a < b \wedge t$, then $\{\tau \le a\}^c \in \mathcal{F}_a \subset \mathcal{F}_{b\wedge t} \subset \mathcal{F}_t$. Otherwise $\{\tau > a\} \cap \{\tau \le b \wedge t\} = \emptyset \in F_t$, because $\{\tau > b \wedge t\} \cap \{\tau \le b \wedge t\} = \emptyset$. Thus, $\{\tau \in B\} \cap \{\tau \le t\} \in F_t$ for all $t \in T$, which means $\tau^{-1}(B) \in \mathcal{F}_\tau$.

(4) For $F \in \mathcal{F}_\sigma$ we have $F \cap \{\sigma \le t\} \in \mathcal{F}_t$ for all $t \in T$ by (A.1). Since $\sigma \le \tau$ pointwise, it follows that $\{\tau \le t\} \subset \{\sigma \le t\}$ for all $t \in T$ which implies

$$F \cap \{\tau \le t\} = \underbrace{F \cap \{\sigma \le t\}}_{\in \mathcal{F}_t} \cap \underbrace{\{\tau \le t\}}_{\in \mathcal{F}_t} \in \mathcal{F}_t,$$

by the properties of σ-algebras. Thus, $F \in \mathcal{F}_\tau$.

(5) Because of (1) and (2) we know that $\tau \wedge t$ is a stopping time w.r.t. $\mathbb{F}$. Using Remark A.21 we can conclude that $\mathcal{F}_{\tau\wedge t}$ is a σ-algebra for all $t \in T$. The fact that $(\mathcal{F}_{\tau\wedge t})_{t\in T}$ is an increasing sequence of sub-σ-algebras in t then follows directly from (4), because $\mathcal{F}_{\tau\wedge t} \subset \mathcal{F}_{\tau\wedge s}$ for all $t \le s \in T$.

(6) Let us start with $F \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau$, which implies $F \cap \{\sigma \le t\} \in F_t$ and $F \cap \{\tau \le t\} \in \mathcal{F}_t$ for all $t \in T$ by (A.1). Then it follows immediately that

$$\begin{aligned} F \cap \{\sigma \wedge \tau \le t\} &= F \cap (\{\sigma \le t\} \cup \{\tau \le t\}) \\ &= \underbrace{(F \cap \{\sigma \le t\})}_{\in \mathcal{F}_t} \cup \underbrace{(F \cap \{\tau \le t\})}_{\in \mathcal{F}_t} \in \mathcal{F}_t \end{aligned}$$

for all $t \in T$. Thus, $F \in \mathcal{F}_{\sigma\wedge\tau}$ and $\mathcal{F}_\sigma \cap \mathcal{F}_\tau \subset \mathcal{F}_{\sigma\wedge\tau}$.

The other inclusion follows immediately from (4), since $\sigma \wedge \tau \le \sigma$ and $\sigma \wedge \tau \le \tau$ which implies $\mathcal{F}_{\sigma\wedge\tau} \subset \mathcal{F}_\sigma$ and $\mathcal{F}_{\sigma\wedge\tau} \subset \mathcal{F}_\tau$. Therefore, for $F \in \mathcal{F}_{\sigma\wedge\tau}$ we have $F \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau$ and thus, $\mathcal{F}_{\sigma\wedge\tau} \subset \mathcal{F}_\sigma \cap \mathcal{F}_\tau$.

(7)[12] Note that, for every $t \in T$, $\{\sigma \le \tau\} \cap \{\sigma \wedge \tau \le t\} \subset \{\sigma \le t\}$. This implies,

$$(F \cap \{\sigma \le \tau\}) \cap \{\sigma \wedge \tau \le t\} = \underbrace{(F \cap \{\sigma \le t\})}_{\in \mathcal{F}_t} \cap \underbrace{\{\sigma \wedge t \le \tau\}}_{=\{\sigma\wedge t \le \tau\wedge t\}} \cap \underbrace{\{\sigma \wedge \tau \le t\}}_{\in \mathcal{F}_t} \in \mathcal{F}_t$$

for every $F \in \mathcal{F}_\sigma$ and $t \in T$. The above holds true, because $\sigma \wedge t$ is a stopping time according to (1) and (2) and it is measurable w.r.t. $\mathcal{F}_{\sigma\wedge t} \subset \mathcal{F}_t$ by (3) and (4); similarly $\tau \wedge t$ is $\mathcal{F}_t$-measurable and thus, $\{\sigma \wedge t \le \tau \wedge t\} \in \mathcal{F}_t$ for all $t \in T$.

[12] The proof for this point can be found in [13, Lemma 3.11(g)], which we would like to present here for the sake of completeness. Let it be noted, however, that all other proofs are of the author's making.

(8) Note that $\{\sigma = \tau\} = \{\sigma \leq \tau\} \setminus \{\sigma < \tau\}$. We already know from (7) that $F \cap \{\sigma \leq \tau\} \subset \mathcal{F}_{\sigma \wedge t}$ for all $F \in \mathcal{F}_\sigma$. Since

$$\{\sigma < \tau\} \cap \{\tau \leq t\} = \bigcup_{n=0}^{t} \bigcup_{k=0}^{n-1} \{\tau = n\} \cap \{\sigma = k\}$$

for all $t \in T$, we have that $\{\sigma < \tau\} \in \mathcal{F}_\tau$, because $\{\tau = n\} = \{\tau \leq n\} \cap \{\tau < n\}^{\mathrm{c}} = \{\tau \leq n\} \cap \{\tau \leq n-1\}^{\mathrm{c}} \in \mathcal{F}_n \subset F_t$ since $n \leq t$ and similarly $\{\sigma = k\} \in \mathcal{F}_t$. Thus, $\{\sigma < \tau\} \cap \{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in T$. $\{\sigma < \tau\} \cap \{\sigma \leq t\} \in \mathcal{F}_t$ for all $t \in T$ follows in the same manner. Hence $\{\sigma < \tau\} \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau = \mathcal{F}_{\sigma \wedge \tau}$ by (6).

(9) This follows quite quickly due to the properties of σ-algebras. Simply bear in mind that for a disjoint sequence $(F_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$, the countable intersection $\bigcap_{n\in\mathbb{N}} F_n$ is also in $\mathcal{F}$, since $\bigcap_{n\in\mathbb{N}} F_n = (\bigcup_{n\in\mathbb{N}} F_n^{\mathrm{c}})^{\mathrm{c}} \in \mathcal{F}$.

Thus, for our case it follows that

$$\{\tau \leq t\} = \left\{\sup_{n\in\mathbb{N}} \tau_n \leq t\right\} = \bigcap_{n\in\mathbb{N}} \{\tau_n \leq t\} \in \mathcal{F}_t$$

for all $t \in T$. □

The following lemma is similar to Lemma 2.19 but focuses on probability spaces and stochastic processes. Moreover, the second part is a generalisation of the first.

Lemma A.23. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and (S, ρ) a metric space. Furthermore, let $X : T \times \Omega \to S$ be an $\mathbb{F}$-adapted process and $\sigma : \Omega \to \overline{T}$ a stopping time w.r.t. $\mathbb{F}$. For $A \in \mathcal{B}_S$ define the* first hitting time of A after σ *(also called* first entrance time*) by*

$$\tau = \inf\{t \in T : \sigma \leq t, X_t \in A\},$$

where we define $\inf \emptyset = t^*$. Then τ is an $\mathbb{F}$-stopping time under each of these conditions:

(1) *T is countable and every non-empty subset, which is bounded below, contains its infimum (think of T as finite, $T \subset \mathbb{Z}$ or $T = \{k - \frac{1}{n} : k \in \mathbb{Z}, n \in \mathbb{N}\}$);*

(2) *T is an interval in $\overline{\mathbb{R}}$, the set A is closed and X is continuous.*

Proof. For the proof please refer to [13, Lemma 3.51(a) and (b)]. □

Theorem A.24. Existence of the essential supremum
Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and let Φ be a collection of measurable functions $\varphi : \Omega \to \overline{\mathbb{R}}$.

(1) *Then there exists a function $\varphi^* \in \Phi$ such that $\varphi^* \geq \varphi$ μ-a.e. for all $\varphi \in \Phi$ and $\varphi^* \leq \psi$ μ-a.e. for every function $\psi \in \Phi$ with $\psi \geq \varphi$ μ-a.e. for all $\varphi \in \Phi$.*

(2) *Additionally, assume that for $\varphi, \tilde{\varphi} \in \Phi$ there exists a $\psi \in \Phi$ such that $\psi \geq \varphi \vee \tilde{\varphi}$. Then there exists an increasing sequence $(\varphi_n)_{n\in\mathbb{N}}$ such that $\varphi^* = \lim_{n\to\infty} \varphi_n$ μ-a.e.*

Proof. The claims are trivial for[13] $\Phi = \emptyset$ and a zero measure, therefore, we may assume $\Phi \neq \emptyset$ and $\mu \neq 0$ for the proof. Note that μ only appears in the theorem above through its null sets, which do not change when passing to an equivalent measure. Since any non-zero σ-finite measure is equivalent to a probability measure (see Theorem A.11), it suffices to prove the claim for probability spaces and random variables. This can be found in [5, Theorem A.33]. □

Definition A.25. Essential supremum[14]
We call the function φ^* in Theorem A.24 the *essential supremum* of Φ and denote it by

$$\operatorname{ess\,sup} \Phi = \operatorname*{ess\,sup}_{\varphi\in\Phi} \varphi := \varphi^*.$$

The *essential infimum* of Φ is defined by

$$\operatorname{ess\,inf} \Phi = \operatorname*{ess\,inf}_{\varphi\in\Phi} \varphi := -\operatorname*{ess\,sup}_{\varphi\in\Phi}(-\varphi).$$

Example A.26. The difference between the supremum and the essential supremum can be made apparent by an easy example regarding the measure space $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \lambda)$, where λ denotes the Borel–Lebesgue measure. Simply note that

$$\sup_{\substack{N\in\mathcal{B}_\mathbb{R}\\ \lambda(N)=0}} \mathbb{1}_N = 1,$$

whereas

$$\operatorname*{ess\,sup}_{\substack{N\in\mathcal{B}_\mathbb{R}\\ \lambda(N)=0}} \mathbb{1}_N = 0.$$

[13]See [13, Remark 13.48(b)].
[14]See [5, Definition A.34].

A.2 Some Theory on Conditional Expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Theorem A.27. Existence of conditional expectation
Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$ and $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{K}^d$ for $d \in \mathbb{N}$. Then there exists an almost surely unique $\mathbb{K}^d$-valued $\mathcal{G}$-measurable random vector Y with $\mathbb{E}[|Y|] < \infty$ such that

$$\mathbb{E}[X \mathbb{1}_G] = \mathbb{E}[Y \mathbb{1}_G]$$

for all $G \in \mathcal{G}$. We call $\mathbb{E}[X|\mathcal{G}] := Y$ the conditional expectation of X w.r.t. $\mathcal{G}$.

Proof. For the proof please refer to [14, Theorem 9.2] □

Theorem A.28. List of properties
Let $\mathcal{G}, \mathcal{H}$ be sub-σ-algebras of $\mathcal{F}$ and $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{K}^d$ for $d \in \mathbb{N}$. Then the following properties hold:

(1) $\mathbb{E}\left[\mathbb{E}[X|\mathcal{G}]\right] = \mathbb{E}[X]$.

(2) *If X is $\mathcal{G}$-measurable, then $\mathbb{E}[X|\mathcal{G}] \stackrel{a.s.}{=} X$.*

(3) Monotonicity: *If $X \leq Y$ a.s., then $\mathbb{E}[X|\mathcal{G}] \leq \mathbb{E}[Y|\mathcal{G}]$ a.s. In particular, if $X \stackrel{a.s.}{=} Y$ then $\mathbb{E}[X|\mathcal{G}] \stackrel{a.s.}{=} \mathbb{E}[Y|\mathcal{G}]$.*

(4) Linearity: *Let $a, b \in \mathbb{K}$ and X, Y be integrable $\mathbb{K}^d$-valued random vectors. Then $\mathbb{E}[aX + bY|\mathcal{G}] \stackrel{a.s.}{=} a\,\mathbb{E}[X|\mathcal{G}] + b\,\mathbb{E}[Y|\mathcal{G}]$.*

(5) Take out what is known: *Let Y be a $\mathcal{G}$-measurable. If XY is integrable, then $\mathbb{E}[XY|\mathcal{G}] \stackrel{a.s.}{=} Y\,\mathbb{E}[X|G]$.*

(6) Tower property: *If $\mathcal{H} \subset \mathcal{G}$, then $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] \stackrel{a.s.}{=} \mathbb{E}[X|\mathcal{H}]$.*

(7) Conditional version of Jensen's inequality: *Let $C \subset \mathbb{R}$ be an interval or $C \subset \mathbb{K}^d$ an open convex set. Assume X take values in C and that $\varphi : C \to \mathbb{R}$ is a convex function such that $\varphi(X) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then $\mathbb{E}[\varphi(X)|\mathcal{G}] \leq \varphi(\mathbb{E}[X|\mathcal{G}])$ a.s.*

(8) Conditional dominated convergence theorem: *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{K}^d$-valued random vectors in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $|X_n| \leq X$ for all $n \in \mathbb{N}$ and $X_n \to X$ a.s. as $n \to \infty$. Then $\mathbb{E}[X_n|\mathcal{G}] \to \mathbb{E}[X|\mathcal{G}]$ a.s. as $n \to \infty$.*

(9) Conditional monotone convergence theorem: *Again, let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of* $\mathbb{R}$*-valued random vectors in* $L^1(\Omega,\mathcal{F},\mathbb{P})$ *such that* $X_n \geq 0$ *for all* $n\in\mathbb{N}$ *and* $X_n \nearrow X$ *a.s. as* $n\to\infty$*. Then* $\mathbb{E}[X_n|\mathcal{G}] \nearrow \mathbb{E}[X|\mathcal{G}]$ *a.s. as* $n\to\infty$*.*

Proof. For the proof of the conditional version of Jensen's inequality in (7) in the vector valued case, see [3, Subsection 10.2.7, p. 349]. The remaining properties follow from the one-dimensional real case treated in [14, Section 9.7 and 9.8, p. 88–90] by considering components and real and imaginary part. □

Example A.29. Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a filtered measure space and $X := (X_t)_{t\in T}$ be a sub- or a supermartingale[15]. If the map $T\ni t\mapsto \mathbb{E}[X_t]$ is constant, then X is a martingale.

Indeed: Let X be a supermartingale and define $M_s = X_s - \mathbb{E}[X_t|\mathcal{F}_s]$ for $s\leq t$ in T and note that this defines a non-negative process. Then $\mathbb{E}[M_s]=0$ by assumption and the law of total expectation. Since a non-negative random variable Y with expectation 0 is 0 almost everywhere (this follows because $\mathbb{P}(Y\geq 2^{-n}) \leq 2^n\mathbb{E}[Y] = 0$ for all $n\in\mathbb{N}$), $M_s = 0$ a.s., which implies the claim.

In case X is a submartingale simply consider the non-negative process $\mathbb{E}[X_t|\mathcal{F}_s] - X_s$ for $s\leq t$ in T and the claim follows in the same manner as before.

Theorem A.30. Doob's optional stopping theorem
Let $X := (X_t)_{t\in T}$ *be a submartingale w.r.t.* $\mathbb{F}$ *and* $\sigma,\tau:\Omega\to\overline{T}$*, where*

- T *is countable and* σ,τ *are stopping times w.r.t.* $\mathbb{F}$*, or*
- T *is a non-degenerate interval,* X *is right-continuous,* τ *is an*[16] $\mathbb{F}^+$*-stopping time and* σ *a stopping time w.r.t. either* $\mathbb{F}^+$ *or* $\mathbb{F}$*.*

If there exists an $a\in T$ *such that* $a\leq\sigma\wedge\tau$*, then the following holds:*

(1) *For every* $u\in t$ *the random variables* $X_{\tau\wedge u}$ *and* $X_{\sigma\wedge\tau\wedge u}$ *are integrable and*
$$X_{\sigma\wedge\tau\wedge u} \overset{a.s.}{\leq} \mathbb{E}[X_{\tau\wedge u}|\mathcal{F}_{\sigma(+)}].$$

(2) *If* $t^*\notin T$ *and if* $\{X^+_{\tau\wedge u}\}$ *is uniformly integrable, then* X_τ *and* $X_{\sigma\wedge\tau}$ *are a.s. well defined and integrable, and*
$$X_{\sigma\wedge\tau} \overset{a.s.}{\leq} \mathbb{E}[X_\tau|\mathcal{F}_{\sigma(+)}].$$

[15] For the definition please refer to page 1 in the Introduction.

[16] $\mathbb{F}^+$ is the filtration defined by $\mathcal{F}_{t(+)} = \bigcap_{u\in T,u>t}\mathcal{F}_u$ for $t\in T\setminus\{t^*\}$ and $\mathcal{F}_{t(+)} = \mathcal{F}_{t^*}$ if $t^*\in T$.

Proof. For the proof please refer to [13, Theorem 4.89 and Remark 7.43]. □

As mentioned before, there is a different way to prove the existence of the conditional expectation for σ-integrable functions (see Theorem 2.7), which will be listed here for further reading. For the proof we need the Bayes' formula, where we rely on several properties of the conditional expectation w.r.t. probability measures and random variables. A list of these properties can be found above (see Theorem A.28).

Lemma A.31. Bayes' formula
Let η and ν be probability measures on $(\Omega, \mathcal{F})$ with $\eta \ll \nu$ on $\mathcal{F}$, $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra and $f \in L^1(\Omega, \mathcal{F}, \eta)$. Define $\rho = \frac{d\eta}{d\nu}$ (this density exists according to the Radon–Nikodým theorem in A.14). Then

$$\mathbb{E}_\eta[f|\mathcal{G}]\ \mathbb{E}_\nu[\rho|\mathcal{G}] = \mathbb{E}_\nu[f\rho|\mathcal{G}] \quad \nu\text{-a.e.} \tag{A.2}$$

Proof. First, let $\eta \ll \nu$ on $\mathcal{F}$. $f \in L^1(\Omega, \mathcal{F}, \eta)$ implies that $f\rho \in L^1(\Omega, \mathcal{F}, \nu)$ and that $\mathbb{E}_\nu[f\rho]$ is well-defined since $\mathbb{E}_\nu[|f\rho|] = \mathbb{E}_\eta[|f|] < \infty$.

Now, for all $G \in \mathcal{G}$ the following holds true.

$$\mathbb{E}_\nu[f\rho\mathbb{1}_G] = \mathbb{E}_\eta[f\mathbb{1}_G] = \mathbb{E}_\eta[\mathbb{1}_G\mathbb{E}_\eta[f|\mathcal{G}]] = \mathbb{E}_\nu[\rho\mathbb{1}_G\mathbb{E}_\eta[f|\mathcal{G}]],$$

where quietly we used the existence of the conditional expectation in Theorem A.27 for the second equality. We can use the same tricks again and deduce

$$\mathbb{E}_\nu[\rho\mathbb{1}_G\mathbb{E}_\eta[f|\mathcal{G}]] = \mathbb{E}_\nu[\mathbb{1}_G\mathbb{E}_\nu[\rho\mathbb{E}_\eta[f|\mathcal{G}]|\mathcal{G}]] \overset{\text{A.28(5)}}{=} \mathbb{E}_\nu[\mathbb{1}_G\mathbb{E}_\nu[\rho|\mathcal{G}]\mathbb{E}_\eta[f|\mathcal{G}]]$$

by the $\mathcal{G}$-measurability of $\mathbb{E}_\eta[f|\mathcal{G}]$. Since

$$\mathbb{E}_\nu[f\rho\mathbb{1}_G] = \mathbb{E}_\nu[\mathbb{1}_G\mathbb{E}_\nu[f\rho|\mathcal{G}]],$$

we can conclude

$$\mathbb{E}_\nu[\mathbb{1}_G\mathbb{E}_\nu[f\rho|\mathcal{G}]] = \mathbb{E}_\nu[\mathbb{1}_G\mathbb{E}_\nu[\rho|\mathcal{G}]\mathbb{E}_\eta[f|\mathcal{G}]],$$

which implies (A.2). □

Proof. Alternative proof of Theorem 2.7
To start off, let f be σ-finite w.r.t. $\mathcal{G}$ with values in $\mathbb{R}$ and let $f \geq 0$ w.l.o.g. (otherwise consider $f = f^+ - f^-$). Define

$$\tilde{\mu}_n(\cdot) = \begin{cases} \frac{\mu(\cdot\ \cap\ \Omega_n)}{\mu(\Omega_n)} & \text{if } \mu(\Omega_n) > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{A.3}$$

and

$$g_n = \begin{cases} \mathbb{E}_{\tilde{\mu}_n}[f\mathbb{1}_{\Omega_n}|\mathcal{G}] & \text{on } \Omega_n, \\ 0 & \text{otherwise.} \end{cases}$$

With these definitions one has $g_n \geq 0$ $\tilde{\mu}_n$-a.e. for all $n \in \mathbb{N}$.
Now, let $m \leq n$. We want to show

$$\mathbb{E}_{\tilde{\mu}_n}[f\mathbb{1}_{\Omega_n}|\mathcal{G}]\mathbb{1}_{\Omega_m} = \mathbb{E}_{\tilde{\mu}_m}[f\mathbb{1}_{\Omega_m}|\mathcal{G}] \quad \tilde{\mu}_m\text{-a.e.},$$

which is equivalent to

$$g_n\mathbb{1}_{\Omega_m} = g_m \quad \tilde{\mu}_m\text{-a.e.}$$

Due to the definition of $\tilde{\mu}_n$ in (A.3) we know for $m \leq n$ that $\tilde{\mu}_m \ll \tilde{\mu}_n$ on $\mathcal{G}$ and that a corresponding density is given by

$$\bar{\rho}_{m,n} := \frac{\mu(\Omega_n)}{\mu(\Omega_m)}\mathbb{1}_{\Omega_m}. \tag{A.4}$$

Using that $\bar{\rho}_{m,n}$ is $\mathcal{G}$-measurable, it follows from Bayes' formula (see Lemma A.31) that

$$\mathbb{E}_{\tilde{\mu}_n}[\bar{\rho}_{m,n}|\mathcal{G}]\, \mathbb{E}_{\tilde{\mu}_m}[f\mathbb{1}_{\Omega_m}|\mathcal{G}] = \mathbb{E}_{\tilde{\mu}_n}[f\mathbb{1}_{\Omega_m}\bar{\rho}_{m,n}|\mathcal{G}] \quad \tilde{\mu}_m\text{-a.e.}$$

We can rewrite this equation by using that $\mathbb{1}_{\Omega_m}\bar{\rho}_{m,n} = \mathbb{1}_{\Omega_n}\mathbb{1}_{\Omega_m}\bar{\rho}_{m,n}$ and that $\mathbb{E}_{\tilde{\mu}_n}[\bar{\rho}_{m,n}|\mathcal{G}] = \bar{\rho}_{m,n}$ $\tilde{\mu}_n$-a.e. Since $\mathbb{1}_{\Omega_m}\bar{\rho}_{m,n}$ is $\mathcal{G}$-measurable, we now have

$$\bar{\rho}_{m,n}\mathbb{E}_{\tilde{\mu}_m}[f\mathbb{1}_{\Omega_m}|\mathcal{G}] \overset{\text{A.28(5)}}{=} \bar{\rho}_{m,n}\mathbb{1}_{\Omega_m}\mathbb{E}_{\tilde{\mu}_n}[f\mathbb{1}_{\Omega_n}|\mathcal{G}] \quad \tilde{\mu}_m\text{-a.e.},$$

which is equivalent to

$$\mathbb{1}_{\Omega_m}g_m = \mathbb{1}_{\Omega_m}g_n \quad \tilde{\mu}_m\text{-a.e.}$$

This proves what we wanted to show as stated in (A.2) on Ω_m. Since g_m is 0 elsewhere we are done with this part. Therefore, we can conclude that $g_n \nearrow g \in L^0(\Omega, \mathcal{G}, \mu)$ for $n \to \infty$ and $g\mathbb{1}_{\Omega_n} = g_n$ μ-a.e. for all $n \in \mathbb{N}$.

Now, note that with (A.3) we have $\tilde{\mu}_n \ll \mu$ on $\mathcal{G}$. A corresponding density is given by

$$\rho_n := \frac{1}{\mu(\Omega_n)}\mathbb{1}_{\Omega_n}.$$

Let $G \in \mathcal{R}_{f,\mathcal{G}}$, then the following holds true by (A.2).

$$\begin{aligned} \mathbb{E}_\mu[f\mathbb{1}_{\Omega_n}\mathbb{1}_G] &= \mu(\Omega_n)\mathbb{E}_\mu\left[\frac{f\mathbb{1}_{\Omega_n}\mathbb{1}_G}{\mu(\Omega_n)}\right] = \mu(\Omega_n)\mathbb{E}_\mu[f\rho_n\mathbb{1}_G] \\ &= \mu(\Omega_n)\mathbb{E}_{\tilde{\mu}_n}[f\mathbb{1}_{\Omega_n}\mathbb{1}_G] \overset{\text{A.27}}{=} \mu(\Omega_n)\mathbb{E}_{\tilde{\mu}_n}[g_n\mathbb{1}_G] \\ &= \mu(\Omega_n)\mathbb{E}_\mu\left[\frac{g_n\mathbb{1}_{\Omega_n}\mathbb{1}_G}{\mu(\Omega_n)}\right] = \mathbb{E}_\mu[\underbrace{g_n\mathbb{1}_{\Omega_n}}_{=g\mathbb{1}_{\Omega_n}}\mathbb{1}_G], \end{aligned} \tag{A.5}$$

where we used the existence of the conditional expectation for probability measures (see Theorem A.27). By using the monotone convergence theorem in Theorem A.18 we can conclude that

$$\mathbb{E}_\mu[f\mathbb{1}_G] \overset{\text{A.18}}{=} \lim_{n\to\infty} \mathbb{E}_\mu[f\mathbb{1}_{\Omega_n}\mathbb{1}_G] \overset{\text{(A.5)}}{=} \lim_{n\to\infty} \mathbb{E}_\mu[g\mathbb{1}_{\Omega_n}\mathbb{1}_G] \overset{\text{A.18}}{=} \mathbb{E}_\mu[g\mathbb{1}_G],$$

which proves (2.3).

It remains to show that g is a.e. unique. In order to do so, let h have the same properties as g and consider an $\epsilon > 0$ such that for $B := \{g \geq h + \epsilon\}$ we have $\mu(B) > 0$ w.l.o.g. Then together with the definition $B \cap \Omega_n = B_n$ (note that $B_n \in \mathcal{R}_{f,\mathcal{G}}$) for $n \in \mathbb{N}$ we have

$$\begin{aligned}\mu(B) &\leq \sum_{n\in\mathbb{N}} \mu(B\cap\Omega_n) = \sum_{n\in\mathbb{N}} \mathbb{E}_\mu[\mathbb{1}_{B\cap\Omega_n}] \leq \frac{1}{\epsilon}\sum_{n\in\mathbb{N}} \mathbb{E}_\mu[(g-h)\mathbb{1}_{B_n}] \\ &= \frac{1}{\epsilon}\sum_{n\in\mathbb{N}} \underbrace{\mathbb{E}_\mu[g\mathbb{1}_{B_n}]}_{=\mathbb{E}_\mu[f\mathbb{1}_{B_n}]} - \underbrace{\mathbb{E}_\mu[h\mathbb{1}_{B_n}]}_{=\mathbb{E}_\mu[f\mathbb{1}_{B_n}]} = 0.\end{aligned}$$

This implies $g = h$ μ-a.e., thus the claim follows.

The general case of f taking values in $\mathbb{K}^d$ follows by considering the components separately (and splitting f into real and the imaginary part for $\mathbb{C}$-valued functions). □

A.3 Miscellaneous

Lemma A.32. *Let $\varphi : [a,b] \to \mathbb{R}$ be a convex and $\psi : [a,b] \to \mathbb{R}$ a concave function. Then φ and ψ have a finite right-hand derivative φ'_+ and ψ'_+ at every point in the open interval (a,b) which fulfill*

$$\varphi(x) \geq \varphi'_+(y)(x-y) + \varphi(y), \quad x, y \in (a,b),$$

and

$$\psi(x) \leq \psi'_+(y)(x-y) + \psi(y), \quad x, y \in (a,b).$$

In particular, φ is the upper envelope of all linear functions below its graph, i.e.

$$\varphi(x) = \sup\{l(x) : l(z) = \alpha z + \beta \leq \varphi(z), \forall z \in (a,b), \alpha, \beta \in \mathbb{R}\}, \quad x \in (a,b),$$

and ψ is the lower envelope of all linear functions above its graph, i.e.

$$\psi(x) = \inf\{l(x) : l(z) = \alpha z + \beta \geq \psi(z), \forall z \in (a,b), \alpha, \beta \in \mathbb{R}\}, \quad x \in (a,b).$$

Proof. For the proof please refer to [12, Lemma 12.13]. □

Lemma A.33. *Let $(y_n^{(m)} : m \in \mathbb{N}, n \in \mathbb{N})$ be an array of numbers in $\mathbb{R} \cup \{+\infty\}$, such that $(y_n^{(m)})_{m\in\mathbb{N}}$ is an increasing sequence for all $n \in \mathbb{N}$ and $(y_n^{(m)})_{n\in\mathbb{N}}$ is an increasing sequence for all $m \in \mathbb{N}$. Suppose that for fixed $n \in \mathbb{N}$ $y_n^{(m)} \nearrow y_n := \lim_{m\to\infty} y_n^{(m)}$ and $y_n^{(m)} \nearrow y^{(m)} := \lim_{n\to\infty} y_n^{(m)}$ for fixed $m \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \lim_{m\to\infty} y_n^{(m)} = \lim_{m\to\infty} \lim_{n\to\infty} y_n^{(m)} = \lim_{m\to\infty} y^{(m)}.$$

Proof. For the proof please refer to [14, Section A5.1, p. 211]. □

Theorem A.34. Cauchy–Schwarz inequality
Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space. Then

$$|\langle x, y\rangle| \leq \|x\| \, \|y\|$$

for all $x, y \in V$, where $\|\cdot\|$ is the norm induced by the inner product.

Proof. For the proof please refer to [15, Proposition 3.1.2(i)]. □

Theorem A.35. Fundamental theorem of calculus
Let $a < b$ in $\mathbb{R}$. Then the following two statements hold true.

(1) *If $f : [a, b] \to \mathbb{R}$ is continuous for all $x \in [a, b]$, then $F(x) := \int_a^x f(t)\,dt$ is differentiable and $F' = f$.*

(2) *If $F : [a, b] \to \mathbb{R}$ is a continuously differentiable function, then $\int_a^b F'(t)\,dt = F(b) - F(a)$.*

Proof. For the proof please refer to [4, Satz 7.24]. □

Theorem A.36. Young's inequality for products
Let $a, b \in \mathbb{R}_+$ and $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

with equality if, and only if, $a^p = b^q$.

Proof. For $a = b = 0$ the claim is certainly true, since both sides equal zero. Furthermore, for $a = 0$ and $b > 0$ the inequality holds true as well (of course, the same goes for $a > 0$ and $b = 0$). Therefore, assume $a, b > 0$. Since the logarithm is concave it follows for $t := 1/p$ that

$$\ln(ta^p + (1-t)b^q) \geq t\ln(a^p) + (1-t)\ln(b^q) = \ln(a) + \ln(b) = \ln(ab)$$

with equality if, and only if, $a^p = b^q$. The claim now follows immediately by exponentiating. □

For the following definitions and theorem let E be an arbitrary Euclidean space over the real numbers $\mathbb{R}$ equipped with the inner product $\langle \cdot, \cdot \rangle$. We could also consider $\mathbb{R}^d$ w.l.o.g., $d \in \mathbb{N}$, of real (column) d-dimensional vectors together with its standard inner product.

Definition A.37. FENCHEL CONJUGATE[17]
Let $f : E \to \overline{\mathbb{R}}$ a function. Then the *Fenchel conjugate* $f^* : E \to \overline{\mathbb{R}}$ is defined by

$$f^*(\varphi) = \sup_{x \in E} \{\langle \varphi, x \rangle - f(x)\}, \quad \varphi \in E.$$

We call the Fenchel conjugate of f^* the *biconjugate* and denote it by f^{**}.

Remark A.38. The map $E \ni \varphi \mapsto \varphi(x) - f(x) \in \mathbb{R}$ is affine and continuous.

Definition A.39. CLOSED FUNCTIONS[18]
We call a function $f : E \to \overline{\mathbb{R}}$ *closed* if its epigraph, i.e. the set $\{(x, \alpha) \in E \times \mathbb{R} \mid f(x) \leq \alpha\}$, is a closed set.

Theorem A.40. FENCHEL–MOREAU THEOREM
Let $f : E \to \overline{\mathbb{R}}$ a function. Then the following three properties are equivalent.

1. *f is closed and convex.*
2. *$f = f^{**}$*
3. *For all $x \in E$ we have that*[19]

$$f(x) = \sup\{\alpha(x) | \alpha \textit{ is an affine minorant of } f\}.$$

Proof. For the proof please refer to [2, Theorem 4.2.1]. □

Theorem A.41. INTERMEDIATE VALUE THEOREM
Let $f : [a, b] \to \mathbb{R}$ be a continuous real-valued function and $a < b \in \mathbb{R}$. Then for all $u \in [f(a), f(b)]$ if $f(a) \leq f(b)$ (respectively $u \in [f(b), f(a)]$ if $f(b) < f(a)$) there exists a $c \in [a, b]$ such that $f(c) = u$.

Proof. For the proof please refer to [4, Satz 5.47]. □

[17] See [2, Chapter 3.3, p. 49].
[18] See [2, Chapter 4.2, p. 76].
[19] Hence, the conjugacy operation induces a bijection between proper (i.e. $f(x) < +\infty$ for at least one $x \in E$ and $f(x) > -\infty$ for every $x \in E$) closed convex functions.

Abbreviations, Conventions and Notation

List of Abbreviation

a.e.	almost everywhere (with respect to a measure)
a.s.	almost surely (with respect to a probability measure)
e.g.	for example (lat. *exempli gratia*)
i.e.	that is to say (lat. *id est*)
resp.	respectively
w.l.o.g.	without loss of generality
w.r.t.	with respect to

Conventions

- We use *increasing*, *decreasing*, *larger* and *smaller* in the weak sense and use *strictly increasing*, *strictly decreasing*, *strictly larger* and *strictly smaller*, if we want to exclude equality.
- We may use *positive* in place of *non-negative*, and say *strictly positive*, when we mean it.
- The subset relation $\subset$ does not exclude equality of the two sets.

Symbols

$\ll$	absolute continuity (between two measures)
$\lvert \cdot \rvert$	absolute value on $\mathbb{R}$ and $\mathbb{C}$, the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{C}^d$ (for $d \in \mathbb{N}$)
$\lVert \cdot \rVert_p := (\int \lvert \cdot \rvert^p \, d\mu)^{1/p}$	L^p-norm
$(\cdot)^{\mathrm{c}}$	complement of a set

$\to$	converges to
$\nearrow$	monotone increasing and converging to
$\searrow$	monotone decreasing and converging to
$:=$	defining equality
$\emptyset$	empty set
$\exists$	there exists, there is
$\forall$	for all, for every
$\vee$	maximum of two functions
$\wedge$	minimum of two functions
$(\cdot)^-$	negative part of a function of measure
$(\cdot)^+$	positive part of a function of measure
$\langle \cdot, \cdot \rangle$	inner product defined on $\mathbb{K}^d$
$\sim$	distribution of random variable, equality between measures
$\circ$	composition of maps
$\times$	product of sets
$\otimes$	product of σ-algebras
$\perp$	singularity (between two measures)

Notation

$\mathcal{B}_k$	σ-algebra of the k-dimensional Borel sets
$\mathcal{B}_S$	Borel σ-algebra of the topological space S
$\mathbb{C}$	the field of complex numbers
$Cov(X,Y)$	covariance matrix $\mathbb{E}[(X - \mathbb{E}[x])(Y - \mathbb{E}[Y])^\perp]$
e	Euler's number 2,71828..., also used for the exponential function
$\operatorname{ess\,sup}$	essential supremum
$\operatorname{ess\,inf}$	essential infimum
$\mathrm{Exp}(\alpha)$	exponential distribution with parameter α
$\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P}$	expected value of random variable
$\mathbb{E}[X \vert \mathcal{F}]$	conditional expectation of random variable under the σ-algebra $\mathcal{F}$
$\mathbb{E}_\mu[f]$	$\int_\Omega f \, d\mu$
$\mathbb{E}_\mu[f \vert \mathcal{F}]$	conditional expectation of a function f under the σ-algebra $\mathcal{F}$
$\mathcal{F}$	σ-algebra of the sample space Ω
$\mathcal{F}_\tau$	σ-algebra associated with the stopping time τ
$\mathbb{F}$	filtration of a σ-algebra $\mathcal{F}$
$\mathcal{G}$	mostly a sub-σ-algebra of $\mathcal{F}$

$\mathbb{1}_F$	indicator function of the set F
$\mathbb{K}^d$	d-dimensional vector space of real or of complex numbers (for $d \in \mathbb{N}$)
$\mathcal{L}^p(\Omega, \mathcal{F}, \mu)$	set of all measurable functions f who's p-th power is integrable (for $p \geq 1$)
$L^p(\Omega, \mathcal{F}, \mu)$	quotient space of $\mathcal{L}^p$ such that $\|\cdot\|_p$ defines a norm on it (for $p \geq 1$)
$L^0(\Omega, \mathcal{F}, \mu)$	vector space of measurable functions w.r.t. $\mathcal{F}$
λ	the Borel–Lebesgue measure
μ	mostly a σ-finite measure
$\mathbb{N} = \{1, 2, 3, ...\}$	natural numbers (without zero)
$\mathcal{P}(\Omega)$	power set, i.e. the set of all subsets of Ω
$\mathbb{P}$	probability measure on the measure space $(\Omega, \mathcal{F})$
$\mathbb{Q}$	the field of rational numbers or a probability measure
$\mathbb{R}$	the field of real numbers
$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$	set of extended real numbers
$\mathbb{R}_+ = [0, \infty)$	set of positive real numbers
$\overline{\mathbb{R}}_+ = [0, \infty]$	set of extended positive real numbers
$\mathcal{R}_{f,\mathcal{G}}$	delta-ring of all sets in $\mathcal{G}$ such that f is integrable
σ	can refer to a stopping time w.r.t. a filtration when defined as such
T	a subset of $\overline{\mathbb{R}}$
$\overline{T}$	the set T with its supremum t^* added
t°	the infimum of T in $\overline{\mathbb{R}}$
t^*	the supremum of T in $\overline{\mathbb{R}}$
τ	a stopping time w.r.t. a filtration
u	a utility function
u', u''	first and second derivative of u
$\mathbb{V}_{\mathbb{Q}}$	variance of a random variable w.r.t a probability measure $\mathbb{Q}$
$X : \Omega \to \mathbb{R}$	a random variable
$X : T \times \Omega \to \mathbb{R}$	a stochastic process
$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$	the commutative ring of integers
Ω	sample space
$(\Omega, \mathcal{F}, \mu)$	mostly a σ-finite measure space (unless stated otherwise)
$(\Omega, \mathcal{F}, \mathbb{F}, \mu)$	a filtered σ-finite measure space
$(\Omega, \mathcal{F}, \mathbb{P})$	a probability space
$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$	a filtered σ-finite measure space

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