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Attempts on Holographic Renormalization of Conformal Chern-Simons Gravity in 3D Flat Space

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Abstract

Two methods for calculating the 2-point function of the holographic dual CFT to Chern-Simons gravity on an 3-dimensional Lorentzian flat space background are presented with partial success. A promising procedure to calculate the full set of 2-point functions is suggested.

Acknowledgments

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Contents

1 Introduction

1.1 The Holographic Principle

In 1974, Stephen Hawking predicted the existence of the now so called Hawking radiation of black holes by considering effects of quantum field theory in the presence of a black hole event horizon. As the vacuum expectation value for spontaneous creation and annihilation of particle-antiparticle pairs is nonzero, Hawking concluded that these events must also take place near the surface of black holes, which ultimately leads to a black body radiation thereof. This enables us to assign a concrete entropy to a black hole by the Bekenstein-Hawking formula

$$S = \frac{A}{4G} \tag{1}$$

with G being Newton's gravitational constant.

Surprisingly, the entropy of a black hole is linearly depending on the surface area A, and not, as one may expect, the volume. As one can always trigger the creation of a black hole by subsequently adding mass to the system of consideration, the Bekenstein-Hawking formula gives an upper bound for the entropy of a system in a given spacetime region, or else the second law of thermodynamics would be violated. From this observation we can conclude that the degrees of freedom of a theory of quantum gravity can not be proportional to the volume of a spacetime region, like for classical fields and their quantum field counterpart with a short range cut off, but have to be proportional to the surface [1]. This leads ultimately to the formulation of the holographic principle: It must be possible to store the information of given system in a spacetime region on the surface of this spacetime region, just like the 3 dimensional image of a hologram can be stored on a 2 dimensional holographic medium [2].

Concrete realizations of the holographic principle are known as gauge/gravity dualities, which state that there is is a one-to-one correspondence between the theory of gravity and a gauge theory on the boundary of the manifold of the gravity theory. The first correspondence of this kind has been found in 1998 by Maldacena in his seminal work [3], in which he related a string theory in anti-de Sitter space (spacetime of negative constant curvature) to a supersymmetric Yang-Mills theory with conformal symmetry, known widely in short as AdS/CFT (Anti-de Sitter/Conformal Field Theory) correspondence.

Recently, more and more such gauge/gravity duals have been found, for example for conformal Chern-Simons gravity on an AdS background in 3D [4] or Einstein gravity in 3D flat space [5], both dual to a specific 2D CFT. These findings motivate the idea that holographic principle could be a fundamental principle that holds true in general and paves the way to a theory of quantum gravity.

1.2 Chern-Simons Gravity in 3D

Before getting started, I would like to address some questions that may occur at this point: What is, and why study conformal Chern-Simons gravity? Why study gauge/gravity duality in 2+1 dimensions? How does the relation between this gravity theory and its dual CFT look like?

First of all, let us define the action functional S_{CSG} of conformal Chern-Simons gravity in 3D in vacuum:

$$S_{CSG}[g] = \frac{k}{4\pi} \int_M dx^3 \, CS[\Gamma] \tag{2}$$

$$CS[\Gamma] = \epsilon^{\lambda\mu\nu} \, \Gamma^\sigma{}_{\lambda\rho} \left(\partial_\mu \Gamma^\rho{}_{\nu\sigma} + \frac{2}{3} \, \Gamma^\rho{}_{\mu\tau} \, \Gamma^\tau{}_{\nu\sigma} \right) \tag{3}$$

$$\Gamma^\rho{}_{\nu\sigma} = \frac{1}{2} \, g^{\rho\alpha} \left(\partial_\nu g_{\sigma\alpha} + \partial_\sigma g_{\nu\alpha} - \partial_\alpha g_{\nu\sigma} \right) \tag{4}$$

where k is a coupling constant, $CS[\Gamma]$ the gravitational Chern-Simons term, $\epsilon^{\lambda\mu\nu}$ the Levi-Civita symbol, $\Gamma^\rho{}_{\nu\sigma}$ the Christoffel symbol and $g_{\mu\nu}$ the metric ($g^{\mu\nu}$ the inverse metric). In this thesis we will choose following metric signature:

$$sign(g) = (-, +, +). \tag{5}$$

Although $CS[\Gamma]$ is defined in terms of the Christoffel symbol (which is not a tensor and therefore inherently frame dependent), the Chern-Simons term is a purely topological term which exist only for manifolds of dimension 4n-1 ($n \in \mathbb{N}$). This means that the action S_{CSG} is actually depending only on the topological properties of the spacetime manifold, which makes conformal Chern-Simons gravity a topological theory. We can derive the equation of motion (EOM) of conformal Chern-Simons gravity by varying the action with respect to the inverse metric and setting it to zero

$$\delta S_{CSG}[g] = \frac{k}{2\pi} \int_M dx^3 \sqrt{-g} \, C_{\mu\nu} \, \delta g^{\mu\nu} = 0 \tag{6}$$

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where

$$C_{\mu\nu} = C_{\nu\mu} = \frac{1}{2}\left(\varepsilon_\mu{}^{\kappa\lambda}\nabla_\kappa R_{\lambda\nu} + \varepsilon_\nu{}^{\kappa\lambda}\nabla_\kappa R_{\lambda\mu}\right) = \varepsilon_\mu{}^{\kappa\lambda}\nabla_\kappa\left(R_{\lambda\nu} - \frac{1}{4}g_{\lambda\nu}R\right) \quad (7)$$

is the Cotton tensor, and

$$\varepsilon_\mu{}^{\kappa\lambda} = g_{\mu\rho}\varepsilon^{\rho\kappa\lambda} \quad (8)$$

$$\varepsilon^{\rho\kappa\lambda} = \frac{1}{\sqrt{-g}}\epsilon^{\rho\kappa\lambda} \quad (9)$$

the covariant tensor related to the Levi-Civita symbol,

$$g = det[g_{\mu\nu}] \quad (10)$$

the determinant of the metric $g_{\mu\nu}$,

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho{}_{\nu\sigma} - \partial_\nu\Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\tau}\Gamma^\tau{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\tau}\Gamma^\tau{}_{\mu\sigma} \quad (11)$$

is the Riemann curvature tensor,

$$R_{\sigma\nu} = R^\rho{}_{\sigma\rho\nu} \quad (12)$$

is the Ricci tensor, and

$$R = g^{\sigma\nu}R_{\sigma\nu} \quad (13)$$

the Ricci scalar or scalar curvature. The covariant derivative ∇_κ of a tensor of type(r,s) $T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}$ is defined as usual

$$\nabla_\kappa T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s} = \partial_\kappa T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s} + \Gamma^{\mu_1}{}_{\lambda\kappa}T^{\lambda\dots\mu_r}{}_{\nu_1\dots\nu_s} + \dots + \Gamma^{\mu_r}{}_{\lambda\kappa}T^{\mu_1\dots\lambda}{}_{\nu_1\dots\nu_s}$$
$$- \Gamma^\lambda{}_{\nu_1\kappa}T^{\mu_1\dots\mu_r}{}_{\lambda\dots\nu_s} - \dots - \Gamma^\lambda{}_{\nu_s\kappa}T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\lambda} \quad (14)$$

The vacuum solutions to the EOM

$$C_{\mu\nu} = 0 \quad (15)$$

are conformally flat spacetime metrics which locally can be written in the form

$$g_{\mu\nu}(x^\kappa) = e^{2\phi(x^\kappa)}\eta_{\mu\nu} \quad (16)$$

with $\eta_{\mu\nu}$ being the Minkowski metric and $\phi(x^\kappa)$ a smooth scalar function on the manifold.

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If we take a closer look at the EOM, we can see that we have an additional gauge freedom, as the EOM stay unchanged under the transformation

$$g_{\mu\nu}(x^\kappa) \to \tilde{g}_{\mu\nu}(x^\kappa) = e^{2\,\omega(x^\kappa)}\,g_{\mu\nu}(x^\kappa). \tag{17}$$

As we can always choose

$$\omega(x^\kappa) = -\phi(x^\kappa) \tag{18}$$

we can restrict without loss of generality $g_{\mu\nu}$ locally to the Minkowski metric, but not necessarily globally.

Now that we have some idea about what conformal Chern-Simons gravity is, we can start answering the questions from the beginning of this section. First of all, the most obvious reason for studying conformal Chern-Simons gravity in 3D is that there is just no Chern-Simons term in 3+1 spacetime dimension. But this does not mean that our efforts here are futile, since in 11 dimensions, as required by M-theory, there is a well defined Chern-Simons term. Further more, we can see gravity theories in reduced spacetime dimensions as toy models, as the reduction of dimensions drastically simplifies calculations. This helps us to gain a deep understanding of the mechanisms of these theories, which can often be generalized to higher dimensions.

There is another good reason for studying gauge/gravity duality in 3D: the symmetries of conformal field theories in 2D strongly restrict the dynamics of the CFTs as the algebra of infinitesimal conformal transformations in 2D is infinite dimensional [6]. For example, conformal symmetry of a CFT on an euclidean plane fixes the vacuum expectation value (vev) for the energy-momentum tensor to be zero

$$\langle 0|\,T(z)\,|0\rangle = 0 \tag{19}$$

and the vev for the two point function to be

$$\langle 0|\,T(z)T(0)\,|0\rangle = \frac{c}{2\,z^4} \tag{20}$$

with the constant c being the central charge of the symmetry algebra (see section 4.1).

This finally brings us to the last question: in which way is conformal Chern-Simons gravity connected to its dual CFT? Or in other words, what is the dictionary that translates between calculations made on the "gravity side" and on the corresponding "CFT side"? The proposed and in the past successfully applied answer for the vev for the 1- and 2-point functions of the stress tensor $T_{\mu\nu}$ of the CFT dual to a gravity theory governed by the action $\Gamma[g]$ is

$$\frac{\delta\Gamma[g]}{\delta g^{\mu\nu}(x^\kappa)} \propto \langle 0| \, T_{\mu\nu}(x^\kappa) \, |0\rangle \tag{21}$$

$$\frac{\delta^2\Gamma[g]}{\delta g^{\mu\nu}(x^\kappa) \, \delta g^{\rho\sigma}(y^\kappa)} \propto \langle 0| \, T_{\mu\nu}(x^\kappa) \, T_{\rho\sigma}(y^\kappa) \, |0\rangle \,. \tag{22}$$

The Symbol $\Gamma[g]$ refers to the bulk action $S[g]$ supplemented additionally with suitable boundary terms. Only the boundary terms give nonzero contribution to the vev of the 1- and 2-point functions if we impose that the metric (variation of the metric) obey the equation of motion (obey the linearized equation of motion).

In the next two chapters we will try to calculate with two different methods the vev of the two-point function (22) on the gravity side.

2 First Attempt: The Elegant Trick

In this section we will try to calculate the 2-point function for conformal Chern-Simons gravity in flat space by reducing the calculation to already known results for Einstein gravity in euclidean flat space [5]. This procedure has been inspired by the successful application of this elegant trick for Chern-Simons gravity in AdS space [4].

2.1 The Second Variation of the Bulk Action

We want to calculate the second variation of the bulk action $S_{CSG}[g]$

$$\delta^2 S_{CSG}[g] = \frac{k}{2\pi} \int_M dx^3 \left(\sqrt{-g}\, C_{\mu\nu}\, \delta^2 g^{\mu\nu} + \delta\sqrt{-g}\, C_{\mu\nu}\, \delta g^{\mu\nu} + \sqrt{-g}\, \delta C_{\mu\nu}\, \delta g^{\mu\nu} \right).$$

(23)

The first two terms vanish on-shell (EOM $C_{\mu\nu} = 0$ are fulfilled)

$$\delta^2 S_{CSG}[g]\big|_{EOM} = \frac{k}{2\pi} \int_M dx^3 \sqrt{-g}\, \delta C_{\mu\nu}\, \delta g^{\mu\nu}.$$

(24)

For the bulk term $\delta^2 S_{CSG}[g]$ to vanish, the variations of the metric $\delta g_{\mu\nu}$ have to fulfill the linearized equations of motion (LEOM)

$$\delta C_{\mu\nu} = 0.$$

(25)

Let us take a look on the variation of the Cotton tensor

$$\delta C_{\mu\nu} = \delta \left(\varepsilon_\mu{}^{\kappa\lambda} \nabla_\kappa \left(R_{\lambda\nu} - \frac{1}{4}\, g_{\lambda\nu}\, R \right) \right)$$

(26)

We are interested in Chern-Simons gravity on a flat space background

$$R^\rho{}_{\sigma\mu\nu} = 0,\, R_{\sigma\nu} = 0,\, R = 0$$

(27)

which implies the vanishing commutator of covariant derivatives when acting on tensors

$$[\nabla_\mu, \nabla_\nu] = 0.$$

(28)

The variation of the Cotton tensor on a flat space (FS) background reads

$$\delta C_{\mu\nu}\big|_{FS} = \varepsilon_\mu{}^{\kappa\lambda} \nabla_\kappa \left(\delta R_{\lambda\nu} - \frac{1}{4}\, g_{\lambda\nu}\, \delta R \right).$$

(29)

We can now define a differential operator

$$\mathcal{D}_\mu{}^\lambda = \varepsilon_\mu{}^{\kappa\lambda}\,\nabla_\kappa \tag{30}$$

and rewrite (24) on-shell in FS

$$\delta^2 S_{CSG}[g]\big|_{EOM,FS} = \frac{k}{2\pi}\int_M dx^3\sqrt{-g}\,\mathcal{D}_\mu{}^\lambda\left(\delta R_{\lambda\nu} - \frac{1}{4}\,g_{\lambda\nu}\,\delta R\right)\delta g^{\mu\nu}. \tag{31}$$

2.2 Chern-Simons VS. Einstein-Hilbert Action

Let us now take a look on the first variation of the Einstein-Hilbert action functional $S_{EH}[g]$

$$\delta S_{EH}[g] = \frac{1}{8\pi G} \int_M dx^3 \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu} \tag{32}$$

and the second variation on-shell in FS gives us

$$\delta^2 S_{EH}[g] \big|_{EOM,FS} = \frac{1}{8\pi G} \int_M dx^3 \sqrt{-g} \left(\delta R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \delta R \right) \delta g^{\mu\nu}. \tag{33}$$

As we demand that the metric variations obey the LEOM, $\delta R_{\mu\nu}$ is restricted to

$$\delta R_{\mu\nu} = 0, \tag{34}$$

which implies in FS

$$\delta R \big|_{FS} = \delta \left(g^{\rho\sigma} R_{\rho\sigma} \right) = g^{\rho\sigma} \delta R_{\rho\sigma} = 0. \tag{35}$$

Practically, this limits the variation of the metric to mappings of flat space to flat spaces. With this restriction we can now compare the second variations of $S_{CSG}[g]$ and $S_{EH}[g]$ to each other

$$\delta^2 S_{CSG}[g] \big|_{EOM,FS} = \frac{k}{2\pi} \int_M dx^3 \sqrt{-g} \, \mathcal{D}_\mu{}^\lambda \, \delta R_{\lambda\nu} \, \delta g^{\mu\nu} \tag{36}$$

$$\delta^2 S_{EH}[g] \big|_{EOM,FS} = \frac{1}{8\pi G} \int_M dx^3 \sqrt{-g} \, \delta R_{\mu\nu} \, \delta g^{\mu\nu}. \tag{37}$$

The variation of the Riemann tensor can be written as

$$\delta R^\rho{}_{\sigma\mu\nu} = \nabla_\mu \, \delta \Gamma^\rho{}_{\sigma\nu} - \nabla_\nu \, \delta \Gamma^\rho{}_{\sigma\mu} \tag{38}$$

hence

$$\delta R_{\mu\nu} = \nabla_\rho \, \delta \Gamma^\rho{}_{\mu\nu} - \nabla_\nu \, \delta \Gamma^\rho{}_{\mu\rho} \tag{39}$$

and the variation of the Christoffel symbol[1] as

$$\delta \Gamma^\rho{}_{\lambda\sigma} = \frac{1}{2} g^{\rho\kappa} \left(\nabla_\lambda \, \delta g_{\sigma\kappa} + \nabla_\sigma \, \delta g_{\lambda\kappa} - \nabla_\kappa \, \delta g_{\lambda\sigma} \right). \tag{40}$$

---

[1]Note that the variation of the Christoffel *is* a tensor, as it can be seen as the infinitesimal difference between two different Christoffel symbols, which transforms as a tensor.

Inserting (40) into (39) gives

$$\delta R_{\mu\nu} = \frac{1}{2} \left(\nabla^\rho \nabla_\mu \, \delta g_{\rho\nu} + \nabla^\rho \nabla_\nu \, \delta g_{\mu\rho} - \nabla^\rho \nabla_\rho \, \delta g_{\mu\nu} - \nabla_\mu \nabla_\nu \, g^{\rho\sigma} \, \delta g_{\rho\sigma} \right) \quad (41)$$

which can be reformulated as

$$\delta R_{\mu\nu} = \frac{1}{2} \left(\nabla^\rho \nabla_\mu \, \delta_\nu^\sigma + \nabla^\sigma \nabla_\nu \, \delta_\mu^\rho - \nabla^2 \, \delta_\mu^\rho \, \delta_\nu^\sigma - \nabla_\mu \nabla_\nu \, g^{\rho\sigma} \right) \delta g_{\rho\sigma} \quad (42)$$

We define another differential operator

$$\mathcal{R}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2} \left(\nabla^\rho \nabla_\mu \, \delta_\nu^\sigma + \nabla^\sigma \nabla_\nu \, \delta_\mu^\rho - \nabla^2 \, \delta_\mu^\rho \, \delta_\nu^\sigma - \nabla_\mu \nabla_\nu \, g^{\rho\sigma} \right) \quad (43)$$

which enables us to put (36) and (37) into the form

$$\delta^2 S_{CSG}[g]\big|_{EOM,FS} = \frac{k}{2\pi} \int_M dx^3 \sqrt{-g} \, (\mathcal{D} \, \mathcal{R} \, \delta g)_{\mu\nu} \, \delta g^{\mu\nu} \quad (44)$$

$$\delta^2 S_{EH}[g]\big|_{EOM,FS} = \frac{1}{8\pi G} \int_M dx^3 \sqrt{-g} \, (\mathcal{R} \, \delta g)_{\mu\nu} \, \delta g^{\mu\nu}. \quad (45)$$

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2.3 Formulation of the Trick

We can finally express following statement: If the two differential operators $\mathcal{D}$ and $\mathcal{R}$ commute

$$[\mathcal{D}, \mathcal{R}] \overset{?}{=} 0 \tag{46}$$

and the action of $\mathcal{D}$ on the non-normalizable metric variations[2] $\delta g_{EH\mu\nu}^{NN(i)}$ of flat space Einstein gravity [5] generates linear combinations thereof

$$\left(\mathcal{D}\, \delta g_{EH}^{NN(i)}\right)_{\mu\nu} \overset{?}{=} \sum_{j=1}^{2} a_{ij}\, \delta g_{EH\mu\nu}^{NN(j)} \tag{47}$$

with a_{ij} being constants, then, and only then, we can reduce the vev of the 2-point functions of Chern-Simons gravity to a linear combination with the constants b_{ij}^{kl} of the vev of the 2-point functions of Einstein gravity

$$\langle 0|\, T_{(i)}\, T_{(j)}\, |0\rangle_{CSG} \overset{?}{\propto} \sum_{m,n=1}^{2} b_{ij}^{kl}\, \langle 0|\, T_{(k)}\, T_{(l)}\, |0\rangle_{EH}\,. \tag{48}$$

---

[2]The non-normalizable metric variations satisfy the LEOM as the usual normalizable metric variations, but give non-trivial contributions to the boundary terms of the action variations.

2.4 Check-Up on the Trick

Let us first test if (46) is true for flat space

$$
\begin{aligned}
(\mathcal{D}\,\mathcal{R}\,\delta g)_{\mu\nu} &= \mathcal{D}_\mu{}^\lambda\,\mathcal{R}_{\lambda\nu}{}^{\rho\sigma}\,\delta g_{\rho\sigma} \\
&= \varepsilon_\mu{}^{\kappa\lambda}\,\nabla_\kappa\,\frac{1}{2}\left(\nabla^\rho\nabla_\lambda\,\delta_\nu^\sigma + \nabla^\sigma\nabla_\nu\,\delta_\lambda^\rho - \nabla^2\,\delta_\lambda^\rho\,\delta_\nu^\sigma - \nabla_\lambda\nabla_\nu\,g^{\rho\sigma}\right)\delta g_{\rho\sigma} \\
&= \varepsilon_\mu{}^{\lambda\kappa}\,\nabla_\kappa\,\frac{1}{2}\left(\nabla^\sigma\,\nabla_\nu\,\delta g_{\lambda\sigma} - \nabla^2\,\delta g_{\lambda\nu}\right)
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
(\mathcal{R}\,\mathcal{D}\,\delta g)_{\mu\nu} &= \mathcal{R}_{\mu\nu}{}^{\rho\sigma}\,\mathcal{D}_\rho{}^\lambda\,\delta g_{\lambda\sigma} \\
&= \frac{1}{2}\left(\nabla^\rho\nabla_\mu\,\delta_\nu^\sigma + \nabla^\sigma\nabla_\nu\,\delta_\mu^\rho - \nabla^2\,\delta_\mu^\rho\,\delta_\nu^\sigma - \nabla_\mu\nabla_\nu\,g^{\rho\sigma}\right)\varepsilon_\rho{}^{\kappa\lambda}\,\nabla_\kappa\,\delta g_{\lambda\sigma} \\
&= \frac{1}{2}\left(\varepsilon_\rho{}^{\kappa\lambda}\nabla_\kappa\nabla^\rho\nabla_\mu\delta g_{\lambda\nu} + \varepsilon_\mu{}^{\kappa\lambda}\nabla_\kappa\nabla^\sigma\nabla_\nu\delta g_{\lambda\sigma} \right. \\
&\qquad \left. -\varepsilon_\mu{}^{\kappa\lambda}\nabla_\kappa\nabla^2\delta g_{\lambda\nu} - \varepsilon^{\sigma\kappa\lambda}\nabla_\kappa\nabla_\mu\nabla_\nu\delta g_{\lambda\sigma}\right) \\
&= \frac{1}{2}\left(\varepsilon_\mu{}^{\kappa\lambda}\nabla_\kappa\nabla^\sigma\nabla_\nu\delta g_{\lambda\sigma} - \varepsilon_\mu{}^{\kappa\lambda}\nabla_\kappa\nabla^2\delta g_{\lambda\nu}\right) \\
&= \varepsilon_\mu{}^{\lambda\kappa}\,\nabla_\kappa\,\frac{1}{2}\left(\nabla^\sigma\,\nabla_\nu\,\delta g_{\lambda\sigma} - \nabla^2\,\delta g_{\lambda\nu}\right)
\end{aligned}
\tag{50}
$$

and we find that the commutator really vanishes in flat space

$$
[\mathcal{D}, \mathcal{R}] = 0.
\tag{51}
$$

This is good news as the first condition (46) to reduce the Chern-Simons 2-point functions to the ones of Einstein gravity is fulfilled. But here comes the bad news: As can be shown by a lengthy but simple calculation, the application of $\mathcal{D}$ on non-normalizable metric variations found in [5] for flat space Einstein gravity does not create linear combinations of these, so

$$
\left(\mathcal{D}\,\delta g_{EH}^{NN(i)}\right)_{\mu\nu} \neq \sum_{j=1}^2 a_{ij}\,\delta g_{EH\mu\nu}^{NN(j)}.
\tag{52}
$$

This leads us to the conclusion, that the elegant trick used successfully in [4] for conformal Chern-Simons gravity in AdS space can not be used for conformal Chern-Simons in flat space and we have to begin from scratch, as we will do in the next section.

3 Second Attempt: Brute Force

We will start in this section from scratch and calculate the second variation of the full action $\delta^2\Gamma_{CSG}[g]$ of conformal Chern-Simons gravity and take the boundary terms carefully into account.

3.1 First Variation

Defining the full action of conformal Chern-Simons gravity

$$
\Gamma_{CSG}[g] = \frac{k}{4\pi}\int_M dx^3\, CS[\Gamma] = \frac{k}{4\pi}\int_M dx^3\, \epsilon^{\lambda\mu\nu}\, \Gamma^\sigma{}_{\lambda\rho}\left(\partial_\mu\Gamma^\rho{}_{\nu\sigma} + \frac{2}{3}\Gamma^\rho{}_{\mu\tau}\,\Gamma^\tau{}_{\nu\sigma}\right)
$$
$$
= \frac{k}{4\pi}\int_M dx^3\, \epsilon^{\lambda\mu\nu}\left(\Gamma^\sigma{}_{\lambda\rho}\,\partial_\mu\Gamma^\rho{}_{\nu\sigma} + \frac{2}{3}\Gamma^\sigma{}_{\lambda\rho}\,\Gamma^\rho{}_{\mu\tau}\,\Gamma^\tau{}_{\nu\sigma}\right),
$$
(53)

the first variation reads

$$
\delta\Gamma_{CSG}[g] = \frac{k}{4\pi}\int_M dx^3\epsilon^{\lambda\mu\nu}\left(\delta\left(\Gamma^\sigma{}_{\lambda\rho}\,\partial_\mu\Gamma^\rho{}_{\nu\sigma}\right) + \frac{2}{3}\delta\left(\Gamma^\sigma{}_{\lambda\rho}\,\Gamma^\rho{}_{\mu\tau}\,\Gamma^\tau{}_{\nu\sigma}\right)\right). \quad (54)
$$

In the following calculations we will rely heavily on the symmetries of the expressions in use (see appendix A). The first term of (54) can be rewritten as

$$
\begin{aligned}
\epsilon^{\lambda\mu\nu}\delta\left(\Gamma^\sigma{}_{\lambda\rho}\,\partial_\mu\Gamma^\rho{}_{\nu\sigma}\right) &= \epsilon^{\lambda\mu\nu}\left(\delta\Gamma^\sigma{}_{\lambda\rho}\,\partial_\mu\Gamma^\rho{}_{\nu\sigma} + \Gamma^\sigma{}_{\lambda\rho}\,\partial_\mu\delta\Gamma^\rho{}_{\nu\sigma}\right) \\
&= \epsilon^{\lambda\mu\nu}\left(\delta\Gamma^\sigma{}_{\lambda\rho}\,\partial_\mu\Gamma^\rho{}_{\nu\sigma} + \partial_\mu\left(\Gamma^\sigma{}_{\lambda\rho}\,\delta\Gamma^\rho{}_{\nu\sigma}\right) - \partial_\mu\Gamma^\sigma{}_{\lambda\rho}\,\delta\Gamma^\rho{}_{\nu\sigma}\right) \\
&= \epsilon^{\lambda\mu\nu}\left(\delta\Gamma^\sigma{}_{\lambda\rho}\,\partial_\mu\Gamma^\rho{}_{\nu\sigma} + \partial_\mu\left(\Gamma^\sigma{}_{\lambda\rho}\,\delta\Gamma^\rho{}_{\nu\sigma}\right) + \partial_\mu\Gamma^\sigma{}_{\nu\rho}\,\delta\Gamma^\rho{}_{\lambda\sigma}\right) \\
&= \epsilon^{\lambda\mu\nu}\left(\partial_\mu\left(\Gamma^\sigma{}_{\lambda\rho}\,\delta\Gamma^\rho{}_{\nu\sigma}\right) + 2\,\delta\Gamma^\sigma{}_{\lambda\rho}\,\partial_\mu\Gamma^\rho{}_{\nu\sigma}\right)
\end{aligned}
$$
(55)

and the second term of (54) reads

$$
\begin{aligned}
\epsilon^{\lambda\mu\nu}\delta\left(\Gamma^\sigma{}_{\lambda\rho}\,\Gamma^\rho{}_{\mu\tau}\,\Gamma^\tau{}_{\nu\sigma}\right) &= \epsilon^{\lambda\mu\nu}\left(\delta\Gamma^\sigma{}_{\lambda\rho}\,\Gamma^\rho{}_{\mu\tau}\,\Gamma^\tau{}_{\nu\sigma} + \Gamma^\sigma{}_{\lambda\rho}\,\delta\Gamma^\rho{}_{\mu\tau}\,\Gamma^\tau{}_{\nu\sigma}\right. \\
&\qquad\qquad \left. + \Gamma^\sigma{}_{\lambda\rho}\,\Gamma^\rho{}_{\mu\tau}\,\delta\Gamma^\tau{}_{\nu\sigma}\right) \\
&= \epsilon^{\lambda\mu\nu}\left(\delta\Gamma^\sigma{}_{\lambda\rho}\,\Gamma^\rho{}_{\mu\tau}\,\Gamma^\tau{}_{\nu\sigma} + \Gamma^\sigma{}_{\nu\rho}\,\delta\Gamma^\rho{}_{\lambda\tau}\,\Gamma^\tau{}_{\mu\sigma}\right. \qquad (56) \\
&\qquad\qquad \left. + \Gamma^\sigma{}_{\mu\rho}\,\Gamma^\rho{}_{\nu\tau}\,\delta\Gamma^\tau{}_{\lambda\sigma}\right) \\
&= \epsilon^{\lambda\mu\nu}\,3\,\delta\Gamma^\sigma{}_{\lambda\rho}\,\Gamma^\rho{}_{\mu\tau}\,\Gamma^\tau{}_{\nu\sigma}.
\end{aligned}
$$

By insertion into (54) we get

$$\delta\Gamma_{CSG}[g] =$$

$$= \frac{k}{4\pi} \int_M dx^3 \epsilon^{\lambda\mu\nu} \left(\partial_\mu \left(\Gamma^\sigma{}_{\lambda\rho} \, \delta\Gamma^\rho{}_{\nu\sigma} \right) + 2\, \delta\Gamma^\sigma{}_{\lambda\rho} \, \partial_\mu \Gamma^\rho{}_{\nu\sigma} + 2\, \delta\Gamma^\sigma{}_{\lambda\rho} \, \Gamma^\rho{}_{\mu\tau} \, \Gamma^\tau{}_{\nu\sigma} \right)$$

$$= \frac{k}{4\pi} \int_M dx^3 \epsilon^{\lambda\mu\nu} \left(\partial_\mu \left(\Gamma^\sigma{}_{\lambda\rho} \, \delta\Gamma^\rho{}_{\nu\sigma} \right) + \delta\Gamma^\sigma{}_{\lambda\rho} \, 2 \left(\partial_\mu \Gamma^\rho{}_{\nu\sigma} + \Gamma^\rho{}_{\mu\tau} \, \Gamma^\tau{}_{\nu\sigma} \right) \right)$$

$$= \frac{k}{4\pi} \int_M dx^3 \epsilon^{\lambda\mu\nu} \left(\partial_\mu \left(\Gamma^\sigma{}_{\lambda\rho} \, \delta\Gamma^\rho{}_{\nu\sigma} \right) \right.$$

$$\left. + \delta\Gamma^\sigma{}_{\lambda\rho} \left(\partial_\mu \Gamma^\rho{}_{\nu\sigma} + \Gamma^\rho{}_{\mu\tau} \, \Gamma^\tau{}_{\nu\sigma} + \partial_\mu \Gamma^\rho{}_{\nu\sigma} + \Gamma^\rho{}_{\mu\tau} \, \Gamma^\tau{}_{\nu\sigma} \right) \right) \quad (57)$$

$$= \frac{k}{4\pi} \int_M dx^3 \epsilon^{\lambda\mu\nu} \left(\partial_\mu \left(\Gamma^\sigma{}_{\lambda\rho} \, \delta\Gamma^\rho{}_{\nu\sigma} \right) \right.$$

$$\left. + \delta\Gamma^\sigma{}_{\lambda\rho} \left(\partial_\mu \Gamma^\rho{}_{\nu\sigma} + \Gamma^\rho{}_{\mu\tau} \, \Gamma^\tau{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} - \Gamma^\rho{}_{\nu\tau} \, \Gamma^\tau{}_{\mu\sigma} \right) \right)$$

$$= \frac{k}{4\pi} \int_M dx^3 \epsilon^{\lambda\mu\nu} \left(\partial_\mu \left(\Gamma^\sigma{}_{\lambda\rho} \, \delta\Gamma^\rho{}_{\nu\sigma} \right) + \delta\Gamma^\sigma{}_{\lambda\rho} \, R^\rho{}_{\sigma\mu\nu} \right).$$

Let us examine the second term in (57)

$$\delta\Gamma^\sigma{}_{\lambda\rho} \, R^\rho{}_{\sigma\mu\nu} = \frac{1}{2} g^{\sigma\kappa} \left(\nabla_\lambda \, \delta g_{\rho\kappa} + \nabla_\rho \, \delta g_{\lambda\kappa} - \nabla_\kappa \, \delta g_{\lambda\rho} \right) R^\rho{}_{\sigma\mu\nu}$$

$$= \frac{1}{2} \left(\nabla_\lambda \left(R^{\rho\kappa}{}_{\mu\nu} \, \delta g_{\rho\kappa} \right) + \nabla_\rho \left(R^{\rho\kappa}{}_{\mu\nu} \, \delta g_{\lambda\kappa} \right) - \nabla_\kappa \left(R^{\rho\kappa}{}_{\mu\nu} \, \delta g_\rho \right) \right.$$

$$\left. - \nabla_\lambda R^{\rho\kappa}{}_{\mu\nu} \, \delta g_{\rho\kappa} - \nabla_\rho R^{\rho\kappa}{}_{\mu\nu} \, \delta g_{\lambda\kappa} + \nabla_\kappa R^{\rho\kappa}{}_{\mu\nu} \, \delta g_\rho \right) \quad (58)$$

$$= \frac{1}{2} \left(2 \, \nabla_\rho \left(R^{\rho\kappa}{}_{\mu\nu} \, \delta g_{\lambda\kappa} \right) - 2 \, \nabla_\rho R^{\rho\kappa}{}_{\mu\nu} \, \delta g_{\lambda\kappa} \right)$$

$$= \nabla_\rho \left(R^{\rho\kappa}{}_{\mu\nu} \, \delta g_{\lambda\kappa} \right) - \nabla_\rho R^{\rho\kappa}{}_{\mu\nu} \, \delta g_{\lambda\kappa}.$$

Cycling this back into (57) we find

$$\delta\Gamma_{CSG}[g] =$$

$$= \frac{k}{4\pi} \int_M dx^3 \epsilon^{\lambda\mu\nu} \left(\partial_\mu \left(\Gamma^\sigma{}_{\lambda\rho} \, \delta\Gamma^\rho{}_{\nu\sigma} \right) + \nabla_\rho \left(R^{\rho\kappa}{}_{\mu\nu} \, \delta g_{\lambda\kappa} \right) - \nabla_\rho R^{\rho\kappa}{}_{\mu\nu} \, \delta g_{\lambda\kappa} \right). \quad (59)$$

The last term in (59) can be written as

$$
\begin{aligned}
\epsilon^{\lambda\mu\nu} \nabla_\rho R^{\rho\kappa}{}_{\mu\nu} \, \delta g_{\lambda\kappa} &= \epsilon^{\lambda\mu\nu} \left(\nabla_\mu R^\kappa{}_\nu - \nabla_\nu R^\kappa{}_\mu \right) \delta g_{\lambda\kappa} \\
&= \left(\epsilon^{\lambda\mu\nu} \nabla_\mu R^\kappa{}_\nu + \epsilon^{\lambda\nu\mu} \nabla_\nu R^\kappa{}_\mu \right) \delta g_{\lambda\kappa} \\
&= 2\sqrt{-g}\, C^{\lambda\kappa} \, \delta g_{\lambda\kappa} = -2\sqrt{-g}\, C_{\lambda\kappa} \, \delta g^{\lambda\kappa}
\end{aligned}
\tag{60}
$$

We can put the first variation of the action into its final form

$$
\begin{aligned}
\delta\Gamma_{CSG}[g] &= \\
&= \frac{k}{4\pi} \int_M dx^3 \epsilon^{\lambda\mu\nu} \left(\partial_\mu \left(\Gamma^\sigma{}_{\lambda\rho} \, \delta\Gamma^\rho{}_{\nu\sigma} \right) + \nabla_\rho \left(R^{\rho\kappa}{}_{\mu\nu} \, \delta g_{\lambda\kappa} \right) \right) - 2\sqrt{-g}\, C^{\lambda\kappa} \, \delta g_{\lambda\kappa} \\
&= \delta S_{CSG}[g] + \frac{k}{4\pi} \int_M dx^3 \epsilon^{\lambda\mu\nu} \left(\partial_\mu \left(\Gamma^\sigma{}_{\lambda\rho} \, \delta\Gamma^\rho{}_{\nu\sigma} \right) + \nabla_\rho \left(R^{\rho\kappa}{}_{\mu\nu} \, \delta g_{\lambda\kappa} \right) \right).
\end{aligned}
\tag{61}
$$

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3.2 Second Variation and Boundary Terms

Finally we can calculate the full second variation on-shell for flat space

$$
\delta^2 \Gamma_{CSG}[g]\big|_{EOM,FS} =
$$
$$
= \frac{k}{4\pi} \int\limits_M dx^3 \epsilon^{\lambda\mu\nu} \left(\partial_\mu \left(\Gamma^\sigma{}_{\lambda\rho} \delta^2 \Gamma^\rho{}_{\nu\sigma} \right) + \nabla_\rho \left(\delta R^{\rho\kappa}{}_{\mu\nu} \delta g_{\lambda\kappa} \right) \right) - 2\sqrt{-g}\, \delta C^{\lambda\kappa} \delta g_{\lambda\kappa}
$$
$$
= \delta^2 S_{CSG}[g]\big|_{EOM,FS} + \frac{k}{4\pi} \int\limits_M dx^3 \epsilon^{\lambda\mu\nu} \left(\partial_\mu \left(\Gamma^\sigma{}_{\lambda\rho} \delta^2 \Gamma^\rho{}_{\nu\sigma} \right) + \nabla_\rho \left(\delta R^{\rho\kappa}{}_{\mu\nu} \delta g_{\lambda\kappa} \right) \right)
$$

$$(62)$$

and find two distinct boundary terms. The first one

$$
\frac{k}{4\pi} \int\limits_M dx^3 \epsilon^{\lambda\mu\nu}\, \partial_\mu \left(\Gamma^\sigma{}_{\lambda\rho} \delta^2 \Gamma^\rho{}_{\nu\sigma} \right)
\tag{63}
$$

is a non-covariant term which is inherently frame dependent. We want this term to vanish as we want physics to be frame independent. The second term is covariant

$$
\frac{k}{4\pi} \int\limits_M dx^3 \sqrt{-g}\, \nabla_\rho \left(\varepsilon^{\lambda\mu\nu} \delta R^{\rho\kappa}{}_{\mu\nu} \delta g_{\lambda\kappa} \right) = \frac{k}{4\pi} \oint\limits_{\partial M} dx^2 \sqrt{\sigma\gamma}\, n_\rho \left(\varepsilon^{\lambda\mu\nu} \delta R^{\rho\kappa}{}_{\mu\nu} \delta g_{\lambda\kappa} \right)
\tag{64}
$$

and should give us the desired contribution to the 2-point functions, whereas

$$
g_{\mu\nu} = \sigma\, n_\mu\, n_\nu + \gamma_{\mu\nu}
\tag{65}
$$

n_μ is the unit normal vector to, and $\gamma_{\mu\nu}$ the metric on the boundary surface ∂M, and

$$
\sigma = n^\mu\, n_\mu = \pm 1.
\tag{66}
$$

3.3 Non-Normalizable Metric Variations

Up to his point, we have successfully calculated the full second variation of flat space Chern-Simons gravity and identified the boundary term which we expect to give us the desired contribution to the 2-point functions for the dual CFT. Our next step is to find a set of Lorentzian metric variations that satisfy the LEOM ($\delta^2 S_{CSG}[g] = 0$), let the non-covariant term (63) vanish and result into a non-trivial contribution of (64). We will use in this section a promising set of metric variations from [7]. First of all we will choose Eddington-Finkelstein gauge in which we will perform our calculation. In this gauge the flat space background metric in the coordinates (u, r, ϕ) reads

$$g_{\mu\nu} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & r^2 \end{pmatrix} \tag{67}$$

$$dx^\mu = \begin{pmatrix} du \\ dr \\ d\phi \end{pmatrix} \tag{68}$$

$$ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu = -du^2 - 2 \, du \, dr + r^2 \, d\phi \tag{69}$$

Next we consider following set of metrics

$$\tilde{g}_{\mu\nu}(\mu_L(u,\phi), \mu_M(u,\phi), r) \, dx^\mu \, dx^\nu =$$
$$\left(r^2 \mu_L^2 + 2r \left(\mu_L' \left(1 + \mu_M \right) - \mu_L \, \mu_M' \right) - \left(1 + \mu_M \right)^2 - 2 \left(1 + \mu_M \right) \mu_M'' + \mu_M'^2 \right) du^2$$
$$- \left(1 + \mu_M \right) 2 \, du \, dr$$
$$+ \left(r^2 \, \mu_L - r \, \mu_M' \right) 2 \, du \, d\phi$$
$$+ r^2 \, d\phi^2 \tag{70}$$

which have the form of

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} \tilde{g}_{uu} & \tilde{g}_{ur} & \tilde{g}_{u\phi} \\ \tilde{g}_{ur} & 0 & 0 \\ \tilde{g}_{u\phi} & 0 & \tilde{g}_{\phi\phi} \end{pmatrix} \tag{71}$$

and

$$\mu_{L,M}' = \partial_\phi \mu_{L,M}$$
$$\dot{\mu}_{L,M} = \partial_u \mu_{L,M}. \tag{72}$$

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In [7] the functions $\mu_L(u, \phi)$ and $\mu_M(u, \phi)$ are chemical potentials that act as sources for our gravity theory. For our purpose μ_L and μ_M are considered as free functions of the variables u and ϕ, which shall generate non-normalizable metric variations. We can expand the metrics $\tilde{g}$ in orders of μ_L and μ_M

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}\left(\mu_L, \mu_M\right) + O\left(\mu_L^2, \mu_M^2, \mu_L\,\mu_M\right) \tag{73}$$

$$h_{\mu\nu} = \begin{pmatrix} 2\left(r\,\mu_L' - \mu_M'' - \mu_M\right) & -\mu_M & r^2\,\mu_L - r\,\mu_M' \\ -\mu_M & 0 & 0 \\ r^2\,\mu_L - r\,\mu_M' & 0 & 0 \end{pmatrix}. \tag{74}$$

We choose the metric variations to be

$$\delta g_{\mu\nu} = h_{\mu\nu} \tag{75}$$

and require that they satisfy the LEOM (25). As the LEOM include covariant derivatives of the metric variations up to third order, which lead to a fast amount of components to calculate, the computation of the LEOM has been done in *Mathematica*. The used *Mathematica* script is appended in appendix B. We find following non-vanishing components from this computation which we want to be zero

$$-\frac{1}{r^2}\left(\mu_M' + 2\,\mu_M''' + \mu_M''''' + r\left(-\mu_L'' - \mu_L'''' + \dot{\mu}_M' + \dot{\mu}_M'''\right)\right) = 0$$

$$-\frac{1}{r^2}\left(\mu_M' + \mu_M'''\right) = 0 \tag{76}$$

$$\frac{1}{r}\left(-r\left(\mu_L' + \mu_L'''\right) + 2\left(\mu_M'' + \mu_M''''\right)\right) = 0$$

$$-2\left(\mu_M' + \mu_M'''\right) = 0$$

which constrain the free functions to

$$\mu_L - \mu_L'' = 0$$
$$\mu_M - \mu_M'' = 0. \tag{77}$$

Furthermore, if we want the non-covariant term (63) to vanish, the free functions have to fulfill

$$3\,\mu_L\,\mu_L' + \mu_L\,\dot{\mu}_M = 0$$
$$\mu_L'\,\mu_M' + \mu_M'\,\dot{\mu}_M + \mu_M\,\dot{\mu}_M' - \mu_L\,\mu_M = 0. \tag{78}$$

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3.4 Resulting Boundary Term Contribution

Now something interesting happens: even if we loose the constraints on μ_L and μ_M and ignore (78), the covariant boundary term (64) vanishes!

$$0 = \frac{k}{4\pi} \oint_{\partial M} dx^2 \sqrt{\sigma\gamma} \, n_\rho \left(\varepsilon^{\lambda\mu\nu} \, \delta R^{\rho\kappa}{}_{\mu\nu} \, \delta g_{\lambda\kappa} \right)\Bigg|_{EOM, FS, LEOM} \tag{79}$$

What has happened? Did we do something wrong? Did we fail? To answer this questions we will take a closer look on the dual CFT in the next section.

4   Conclusion and Outlook

This section is dedicated to understand the outcome of our calculations in section 3. We will analyze in more detail the CFT dual to Chern-Simons gravity to achieve this. In the second subsection we will present a promising proposal for further investigations.

4.1   Examining the Dual CFT

Let us first have a look on the properties of a general CFT. As already mentioned, CFTs in 2D are highly restricted by their conformal symmetries. The symmetry algebra of a CFT in 2 dimensions consists of two copies of an infinite dimensional Virasoro algebra, which is a central extensions of the the Witt algebra, with the generators L_n and $\bar{L}_n$

$$
\begin{aligned}
[L_n,\, L_m] &= (n-m)\, L_{n+m} + \frac{c_L}{12}\left(n^3 - n\right)\delta_{n+m,\,0} \\
[\bar{L}_n,\, \bar{L}_m] &= (n-m)\, \bar{L}_{n+m} + \frac{c_{\bar{L}}}{12}\left(n^3 - n\right)\delta_{n+m,\,0} \\
[L_n,\, \bar{L}_m] &= 0
\end{aligned}
\tag{80}
$$

The constants c_L and $c_{\bar{L}}$ are called central charges and play an important role, as the spectrum of specific operators, like the stress tensor, can be related to the values of the central charges [6].

Now, coming from the gravity side, it can be shown that the asymptotic symmetry group for asymptotically flat spacetimes, generated by all (non-trivial) diffeomorphisms preserving the asymptotic flat space boundary conditions, is the Bondi-Metzner-Sachs (BMS) group. In 3 dimensions, the associated symmetry algebra is the centrally extend BMS algebra, generated by Virasoro generators L_n and supertranslations M_n, and reads

$$
\begin{aligned}
[M_n,\, M_m] &= 0 \\
[L_n,\, M_m] &= (n-m)\, M_{n+m} + \frac{c_M}{12}\left(n^3 - n\right)\delta_{n+m,\,0} \\
[L_n,\, L_m] &= (n-m)\, L_{n+m} + \frac{c_L}{12}\left(n^3 - n\right)\delta_{n+m,\,0}.
\end{aligned}
\tag{81}
$$

In [8] it was shown that for 3D flat space Chern-Simons gravity the central charges c_L and c_M take the values

$$
c_L = 24\, k, \; c_M = 0.
\tag{82}
$$

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This means that the non-trivial part of (81) reduces to one copy of the Virasoro algebra, which leads to the conlusion that the holographic dual CFT to 3D Chern-Simons gravity is the chiral half of a standard 2D CFT.

During the final phase in the development of this thesis there has been found a surprisingly elegant and, in comparison to our brute force calculation, easy method [9] to calculate any n-point function of the stress tensor of our dual CFT with a recursive a formula from the (n-1)-point functions.

The 2-point functions for the components of the stress tensor[3] L and M are found to be

$$
\begin{aligned}
\langle 0| M(u, \phi) M(0,0) |0\rangle &= 0 \\
\langle 0| M(u, \phi) L(0,0) |0\rangle &= 0 \\
\langle 0| L(u, \phi) L(0,0) |0\rangle &= \frac{c_L}{2 \sin^4 \frac{\phi}{2}} \quad .
\end{aligned}
\tag{83}
$$

---

[3]The components of the stress tensor are called L and M because they can be mode expanded in terms of the generators L_n and M_n.

4.2   Proposal for Onward Procedure

If we compare now our findings in (79) to (83) we see that we got like $\frac{2}{3}$ of it right. But why we do not get the last non-vanishing vev with our calculations? We can find an insightful clue in [4]. In order to obtain a well-defined Dirichlet boundary value problem for 3D Chern-Simons gravity in AdS space one has to add to the bulk action an additional boundary term

$$\frac{k}{2\pi} \oint_{\partial M} dx^2 \sqrt{\sigma\gamma}\, K^+_{\mu\nu}\, K^{-\,\mu\nu} \tag{84}$$

$$K^{\pm}_{\mu\nu} = \left(\delta^{\lambda}_{\mu} \pm \epsilon_{\mu}{}^{\lambda}\right) K_{\lambda\nu} \tag{85}$$

$$K_{\mu\nu} = \gamma_{\mu}{}^{\lambda}\, \nabla_{\lambda}\, n_{\nu} \tag{86}$$

where $K_{\mu\nu}$ is the extrinsic curvature of ∂M and $\epsilon^{\mu\nu}$ the 2-dimensional Levi-Civita symbol. This additional boundary term does not change the EOM but gives rise to an additional non-vanishing 2-point function.

We propose here for future investigations in metric formulation of the 2-point functions of the CFT dual to Chern-Simons gravity in 3D flat space to redefine the action as

$\Gamma_{CSG}[g] =$

$$= \frac{k}{4\pi} \left(\int_M dx^3 \epsilon^{\lambda\mu\nu}\, \Gamma^{\sigma}{}_{\lambda\rho} \left(\partial_{\mu}\Gamma^{\rho}{}_{\nu\sigma} + \frac{2}{3} \Gamma^{\rho}{}_{\mu\tau}\, \Gamma^{\tau}{}_{\nu\sigma} \right) + 2 \oint_{\partial M} dx^2 \sqrt{\sigma\gamma}\, K^+_{\mu\nu}\, K^{-\,\mu\nu} \right) \tag{87}$$

and to take this additional boundary term into account for the calculations of the vev for the stress tensor of the CFT.

4.3 Final Conclusion

In this thesis, two different approaches to calculate the 2-point functions of the stress tensor of the holographic dual 2D CFT to 3D conformal Chern-Simons gravity have been presented.

With our first method we tried to reduce the calculations to known results for euclidean flat space Einstein gravity. Although we successfully showed that the differential operators $\mathcal{D}$ and $\mathcal{R}$ commute for FS, it turned out that $\mathcal{D}$ applied on the non-normalizable metric variations of euclidean flat space Einstein gravity does not generate linear combinations thereof, which ultimately lead to the conclusion that this method is not applicable in this case.

In our second approach we started from scratch and calculated analytically the second variation of the action. We could identify a covariant boundary term and expected that this term would give us the desired contribution to the 2-point functions. However, for non-normalizable metric variations which obey the LEOM this boundary term vanishes.

In the final section of this thesis we examined the holographic dual CFT and compared our results to the findings in [9], in which an elegant mathematical trick has been presented to calculate any n-point function from the (n-1)-point functions. It turns out that two out of three 2-point functions actually vanish, in accordance with our results. Inspired by the outcome in [4] for Chern-Simons gravity in AdS space, we proposed to supplement the action of 3D conformal Chern-Simons gravity in FS with an additional boundary term (87). This additional boundary term does not change the EOM and we expect that this term should give us the desired last non-vanishing 2-point function. Although the verification of our outlined proposal remains for now an open issue, it gives us a clear pathway for future investigations.

A   Useful Identities

$$g_{\mu\nu} = g_{\nu\mu} \tag{88}$$

$$\Gamma^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu} \tag{89}$$

$$g_{\kappa\rho} R^{\rho}{}_{\sigma\mu\nu} = R_{\kappa\sigma\mu\nu} = -R_{\sigma\kappa\mu\nu} = -R_{\kappa\sigma\nu\mu} = R_{\mu\nu\kappa\sigma} \tag{90}$$

$$R^{\rho}{}_{\sigma\mu\nu} + R^{\rho}{}_{\nu\sigma\mu} + R^{\rho}{}_{\mu\nu\sigma} = 0 \tag{91}$$

$$R_{\mu\nu} = R_{\nu\mu} \tag{92}$$

$$\varepsilon^{\lambda\mu\nu} = \varepsilon^{\nu\lambda\mu} = \varepsilon^{\mu\nu\lambda} = -\varepsilon^{\lambda\nu\mu} = -\varepsilon^{\nu\mu\lambda}. \tag{93}$$

The variations of these object inherit their symmetries. Note that from

$$\delta\left(g^{\mu\lambda} g_{\lambda\nu}\right) = \delta\left(\delta^{\mu}_{\nu}\right) = 0 \tag{94}$$

follows

$$\delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma}. \tag{95}$$

B Mathematica Script

```
(* Defining the list of variables. *)
var = {u, r, phi};

(* Defining the background metric and
   its inverse in Eddington-Finkelstein gauge.*)
g[u_, r_, phi_] = {{-1, -1, 0}, {-1, 0, 0}, {0, 0, r^2}};
g[u, r, phi] // MatrixForm
gInv[u_, r_, phi_] = FullSimplify[Inverse[g[u, r, phi]]];
gInv[u, r, phi] // MatrixForm
```

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & r^2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \frac{1}{r^2} \end{pmatrix}$$

```
(* Defining the variation of the metric in
 terms of the free functions muL and muM. *)
h[u_, r_, phi_] = FullSimplify[
  {{2 (r D[muL[u, phi], phi] - D[D[muM[u, phi], phi], phi] -
       muM[u, phi]), -muM[u, phi],
    r^2 muL[u, phi] - r D[muM[u, phi], phi]}, {-muM[u, phi], 0, 0},
   {r^2 muL[u, phi] - r D[muM[u, phi], phi], 0, 0}}]
```

$$\left\{ \left\{ -2\left(muM[u, phi] - r\, muL^{(0,1)}[u, phi] + muM^{(0,2)}[u, phi]\right), \right.\right.$$
$$\left. -muM[u, phi], r\left(r\, muL[u, phi] - muM^{(0,1)}[u, phi]\right)\right\},$$
$$\left. \{-muM[u, phi], 0, 0\}, \left\{r\left(r\, muL[u, phi] - muM^{(0,1)}[u, phi]\right), 0, 0\right\}\right\}$$

```
(* Defining the Christoffel symbol. *)
gamma[u_, r_, phi_] = FullSimplify[
    Table[Table[Table[1 / 2 Sum[gInv[u, r, phi][[i, l]]
            (D[g[u, r, phi][[k, l]], var[[j]]] + D[g[u, r, phi][[j,
                 l]], var[[k]]] - D[g[u, r, phi][[k, j]], var[[l]]]),
          {l, 1, 3}], {k, 1, 3}], {j, 1, 3}], {i, 1, 3}]];
gamma[u, r, phi] // MatrixForm
```

$$
\left(
\begin{array}{ccc}
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r} \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{r} \\ 0 \end{pmatrix}
\end{array}
\right)
$$

```
(* Computing the first covariant
 dervative of the metric variation. *)
dh[u_, r_, phi_] = FullSimplify[
  Table[Table[Table[D[h[u, r, phi][[j, k]], var[[i]]] -
      Sum[gamma[u, r, phi][[l, i, j]] h[u, r, phi][[l, k]] +
        gamma[u, r, phi][[l, i, k]] h[u, r, phi][[l, j]],
       {l, 1, 3}], {k, 1, 3}], {j, 1, 3}], {i, 1, 3}]]
```

$\big\{\big\{\big\{-2\,\big(\text{muM}^{(1,0)}[\text{u, phi}] - r\,\text{muL}^{(1,1)}[\text{u, phi}] + \text{muM}^{(1,2)}[\text{u, phi}]\big),$

$\quad -\text{muM}^{(1,0)}[\text{u, phi}], r\,\big(r\,\text{muL}^{(1,0)}[\text{u, phi}] - \text{muM}^{(1,1)}[\text{u, phi}]\big)\big\},$

$\quad \big\{-\text{muM}^{(1,0)}[\text{u, phi}], 0, 0\big\},$

$\quad \big\{r\,\big(r\,\text{muL}^{(1,0)}[\text{u, phi}] - \text{muM}^{(1,1)}[\text{u, phi}]\big), 0, 0\big\}\big\},$

$\big\{\big\{2\,\text{muL}^{(0,1)}[\text{u, phi}], 0, r\,\text{muL}[\text{u, phi}]\big\},$

$\quad \{0, 0, 0\}, \{r\,\text{muL}[\text{u, phi}], 0, 0\}\big\},$

$\big\{\big\{-2\,\big(\text{muM}^{(0,1)}[\text{u, phi}] - r\,\text{muL}^{(0,2)}[\text{u, phi}] + \text{muM}^{(0,3)}[\text{u, phi}]\big),$

$\quad -r\,\text{muL}[\text{u, phi}],$

$\quad r\,\big(\text{muM}[\text{u, phi}] - r\,\text{muL}^{(0,1)}[\text{u, phi}] + \text{muM}^{(0,2)}[\text{u, phi}]\big)\big\},$

$\quad \{-r\,\text{muL}[\text{u, phi}], 0, r\,\text{muM}[\text{u, phi}]\},$

$\quad \big\{r\,\big(\text{muM}[\text{u, phi}] - r\,\text{muL}^{(0,1)}[\text{u, phi}] + \text{muM}^{(0,2)}[\text{u, phi}]\big),$

$\quad r\,\text{muM}[\text{u, phi}], 2\,r^2\,\big(-r\,\text{muL}[\text{u, phi}] + \text{muM}^{(0,1)}[\text{u, phi}]\big)\big\}\big\}\big\}$

```
(* Computing the second covariant dervative of the
 metric variation. *)ddh[u, r, phi] = FullSimplify[
  Table[Table[Table[Table[D[dh[u, r, phi][[i, j, k]], var[[m]]] -
      Sum[gamma[u, r, phi][[l, m, i]] dh[u, r, phi][[l, j, k]] +
        gamma[u, r, phi][[l, m, j]] dh[u, r, phi][[i, l, k]] +
        gamma[u, r, phi][[l, m, k]] dh[u, r, phi][[i, l, j]],
      {l, 1, 3}], {k, 1, 3}], {j, 1, 3}], {i, 1, 3}], {m, 1, 3}]]
```

$\{\{\{-2\,(\text{muM}^{(2,0)}[u,\text{phi}] - r\,\text{muL}^{(2,1)}[u,\text{phi}] + \text{muM}^{(2,2)}[u,\text{phi}]),$

$-\text{muM}^{(2,0)}[u,\text{phi}], r\,(r\,\text{muL}^{(2,0)}[u,\text{phi}] - \text{muM}^{(2,1)}[u,\text{phi}])\},$

$\{-\text{muM}^{(2,0)}[u,\text{phi}], 0, 0\},$

$\{r\,(r\,\text{muL}^{(2,0)}[u,\text{phi}] - \text{muM}^{(2,1)}[u,\text{phi}]), 0, 0\}\},$

$\{\{2\,\text{muL}^{(1,1)}[u,\text{phi}], 0, r\,\text{muL}^{(1,0)}[u,\text{phi}]\},$

$\{0, 0, 0\}, \{r\,\text{muL}^{(1,0)}[u,\text{phi}], 0, 0\}\},$

$\{\{-2\,(\text{muM}^{(1,1)}[u,\text{phi}] - r\,\text{muL}^{(1,2)}[u,\text{phi}] + \text{muM}^{(1,3)}[u,\text{phi}]),$

$-r\,\text{muL}^{(1,0)}[u,\text{phi}],$

$r\,(\text{muM}^{(1,0)}[u,\text{phi}] - r\,\text{muL}^{(1,1)}[u,\text{phi}] + \text{muM}^{(1,2)}[u,\text{phi}])\},$

$\{-r\,\text{muL}^{(1,0)}[u,\text{phi}], 0, r\,\text{muM}^{(1,0)}[u,\text{phi}]\},$

$\{r\,(\text{muM}^{(1,0)}[u,\text{phi}] - r\,\text{muL}^{(1,1)}[u,\text{phi}] + \text{muM}^{(1,2)}[u,\text{phi}]),$

$r\,\text{muM}^{(1,0)}[u,\text{phi}],$

$2\,r^2\,(-r\,\text{muL}^{(1,0)}[u,\text{phi}] + \text{muM}^{(1,1)}[u,\text{phi}])\}\}\},$

$\{\{\{2\,\text{muL}^{(1,1)}[u,\text{phi}], 0, r\,\text{muL}^{(1,0)}[u,\text{phi}]\}, \{0, 0, 0\},$

$\{r\,\text{muL}^{(1,0)}[u,\text{phi}], 0, 0\}\}, \{\{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}\},$

$\left\{\left\{\dfrac{2\,(\text{muM}^{(0,1)}[u,\text{phi}] + \text{muM}^{(0,3)}[u,\text{phi}])}{r},\right.\right.$

$0, -\text{muM}[u,\text{phi}] - \text{muM}^{(0,2)}[u,\text{phi}]\Big\},$

$\{0, 0, -\text{muM}[u,\text{phi}]\}, \{-\text{muM}[u,\text{phi}] - \text{muM}^{(0,2)}[u,\text{phi}],$

$-\text{muM}[u,\text{phi}], -2\,r\,\text{muM}^{(0,1)}[u,\text{phi}]\}\}\},$

$\{\{\{-2\,(\text{muM}^{(1,1)}[u,\text{phi}] - r\,\text{muL}^{(1,2)}[u,\text{phi}] + \text{muM}^{(1,3)}[u,\text{phi}]),$

$-r\,\text{muL}^{(1,0)}[u,\text{phi}],$

$r\,(\text{muM}^{(1,0)}[u,\text{phi}] - r\,\text{muL}^{(1,1)}[u,\text{phi}] + \text{muM}^{(1,2)}[u,\text{phi}])\},$

$\{-r\,\text{muL}^{(1,0)}[u,\text{phi}], 0, r\,\text{muM}^{(1,0)}[u,\text{phi}]\},$

$\{r\,(\text{muM}^{(1,0)}[u,\text{phi}] - r\,\text{muL}^{(1,1)}[u,\text{phi}] + \text{muM}^{(1,2)}[u,\text{phi}]),$

$r\,\text{muM}^{(1,0)}[u,\text{phi}], 2\,r^2\,(-r\,\text{muL}^{(1,0)}[u,\text{phi}] + \text{muM}^{(1,1)}[u,\text{phi}])\}\},$

$\left\{\left\{\dfrac{2\,(\text{muM}^{(0,1)}[u,\text{phi}] + \text{muM}^{(0,3)}[u,\text{phi}])}{r}, 0,\right.\right.$

$-\text{muM}[u,\text{phi}] - \text{muM}^{(0,2)}[u,\text{phi}]\Big\},$

$\{0, 0, -\text{muM}[u,\text{phi}]\}, \{-\text{muM}[u,\text{phi}] - \text{muM}^{(0,2)}[u,\text{phi}],$

$$-\text{muM}[\text{u, phi}], -2\,\text{r}\,\text{muM}^{(0,1)}[\text{u, phi}]\}\Big\},$$

$$\Big\{\Big\{-2\left(\text{muM}^{(0,2)}[\text{u, phi}] + \text{muM}^{(0,4)}[\text{u, phi}]\right) +$$

$$2\,\text{r}\left(\text{muL}^{(0,1)}[\text{u, phi}] + \text{muL}^{(0,3)}[\text{u, phi}] + \text{muM}^{(1,0)}[\text{u, phi}] -\right.$$

$$\left.\text{r}\,\text{muL}^{(1,1)}[\text{u, phi}] + \text{muM}^{(1,2)}[\text{u, phi}]\right),$$

$$-\text{muM}[\text{u, phi}] - \text{muM}^{(0,2)}[\text{u, phi}] + \text{r}\,\text{muM}^{(1,0)}[\text{u, phi}],$$

$$\text{r}\left(3\,\text{muM}^{(0,1)}[\text{u, phi}] + 3\,\text{muM}^{(0,3)}[\text{u, phi}] + \text{r}\left(-3\,\text{muL}^{(0,2)}[\text{u, phi}] -\right.\right.$$

$$\left.\left.\text{r}\,\text{muL}^{(1,0)}[\text{u, phi}] + \text{muM}^{(1,1)}[\text{u, phi}]\right)\right)\Big\},$$

$$\Big\{-\text{muM}[\text{u, phi}] - \text{muM}^{(0,2)}[\text{u, phi}] + \text{r}\,\text{muM}^{(1,0)}[\text{u, phi}],$$

$$-2\,\text{muM}[\text{u, phi}], \text{r}\left(3\,\text{r}\,\text{muL}[\text{u, phi}] - \text{muM}^{(0,1)}[\text{u, phi}]\right)\Big\},$$

$$\Big\{\text{r}\left(3\,\text{muM}^{(0,1)}[\text{u, phi}] + 3\,\text{muM}^{(0,3)}[\text{u, phi}] +\right.$$

$$\left.\text{r}\left(-3\,\text{muL}^{(0,2)}[\text{u, phi}] - \text{r}\,\text{muL}^{(1,0)}[\text{u, phi}] + \text{muM}^{(1,1)}[\text{u, phi}]\right)\right),$$

$$\text{r}\left(3\,\text{r}\,\text{muL}[\text{u, phi}] - \text{muM}^{(0,1)}[\text{u, phi}]\right), 0\}\Big\}\Big\}\Big\}$$

```
(* Computing the third covariant
 dervative of the metric variation. *)
dddh[u, r, phi] = FullSimplify[
  Table[D[ddh[u, r, phi], var[[c]]] - Table[
     Table[Table[Table[Sum[gamma[u, r, phi][[k, c, b]] ddh[u, r,
              phi]][[k, a, i, j]] + gamma[u, r, phi][[k, c, a]]
           ddh[u, r, phi][[b, k, i, j]] + gamma[u, r, phi][[
            k, c, i]] ddh[u, r, phi][[b, a, k, j]] +
          gamma[u, r, phi][[k, c, j]] ddh[u, r, phi][[b, a, i, k]],
        {k, 3}], {j, 3}], {i, 3}], {a, 3}], {b, 3}], {c, 3}]]
```

$$\Big\{\Big\{\Big\{\Big\{\Big\{-2\left(\text{muM}^{(3,0)}[\text{u, phi}] - \text{r}\,\text{muL}^{(3,1)}[\text{u, phi}] + \text{muM}^{(3,2)}[\text{u, phi}]\right),$$

$$-\text{muM}^{(3,0)}[\text{u, phi}], \text{r}\left(\text{r}\,\text{muL}^{(3,0)}[\text{u, phi}] - \text{muM}^{(3,1)}[\text{u, phi}]\right)\Big\},$$

$$\Big\{-\text{muM}^{(3,0)}[\text{u, phi}], 0, 0\Big\},$$

$$\Big\{\text{r}\left(\text{r}\,\text{muL}^{(3,0)}[\text{u, phi}] - \text{muM}^{(3,1)}[\text{u, phi}]\right), 0, 0\Big\}\Big\},$$

$$\Big\{\Big\{2\,\text{muL}^{(2,1)}[\text{u, phi}], 0, \text{r}\,\text{muL}^{(2,0)}[\text{u, phi}]\Big\},$$

$$\{0, 0, 0\}, \Big\{\text{r}\,\text{muL}^{(2,0)}[\text{u, phi}], 0, 0\Big\}\Big\},$$

$$\Big\{\Big\{-2\left(\text{muM}^{(2,1)}[\text{u, phi}] - \text{r}\,\text{muL}^{(2,2)}[\text{u, phi}] + \text{muM}^{(2,3)}[\text{u, phi}]\right),$$

$$-\text{r}\,\text{muL}^{(2,0)}[\text{u, phi}],$$

$$\text{r}\left(\text{muM}^{(2,0)}[\text{u, phi}] - \text{r}\,\text{muL}^{(2,1)}[\text{u, phi}] + \text{muM}^{(2,2)}[\text{u, phi}]\right)\Big\},$$

$$\Big\{-\text{r}\,\text{muL}^{(2,0)}[\text{u, phi}], 0, \text{r}\,\text{muM}^{(2,0)}[\text{u, phi}]\Big\},$$

$$\Big\{\text{r}\left(\text{muM}^{(2,0)}[\text{u, phi}] - \text{r}\,\text{muL}^{(2,1)}[\text{u, phi}] + \text{muM}^{(2,2)}[\text{u, phi}]\right),$$

$$\text{r}\,\text{muM}^{(2,0)}[\text{u, phi}],$$

$$2\,\text{r}^2\left(-\text{r}\,\text{muL}^{(2,0)}[\text{u, phi}] + \text{muM}^{(2,1)}[\text{u, phi}]\right)\Big\}\Big\}\Big\},$$

$$\Big\{\Big\{\Big\{2\,\text{muL}^{(2,1)}[\text{u, phi}], 0, \text{r}\,\text{muL}^{(2,0)}[\text{u, phi}]\Big\}, \{0, 0, 0\},$$

$$\Big\{\text{r}\,\text{muL}^{(2,0)}[\text{u, phi}], 0, 0\Big\}\Big\}, \{\{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}\},$$

$$\left\{\left\{\frac{1}{r}2\left(\text{muM}^{(1,1)}[\text{u, phi}]+\text{muM}^{(1,3)}[\text{u, phi}]\right),\, 0,\right.\right.$$

$$\left.-\text{muM}^{(1,0)}[\text{u, phi}]-\text{muM}^{(1,2)}[\text{u, phi}]\right\},$$

$$\left\{0,\, 0,\, -\text{muM}^{(1,0)}[\text{u, phi}]\right\},\, \left\{-\text{muM}^{(1,0)}[\text{u, phi}]-\text{muM}^{(1,2)}[\text{u, phi}],\right.$$

$$\left.\left.-\text{muM}^{(1,0)}[\text{u, phi}],\, -2\, r\, \text{muM}^{(1,1)}[\text{u, phi}]\right\}\right\}\right\},$$

$$\left\{\left\{\left\{-2\left(\text{muM}^{(2,1)}[\text{u, phi}]-r\, \text{muL}^{(2,2)}[\text{u, phi}]+\text{muM}^{(2,3)}[\text{u, phi}]\right),\right.\right.\right.$$

$$-r\, \text{muL}^{(2,0)}[\text{u, phi}],$$

$$\left.r\left(\text{muM}^{(2,0)}[\text{u, phi}]-r\, \text{muL}^{(2,1)}[\text{u, phi}]+\text{muM}^{(2,2)}[\text{u, phi}]\right)\right\},$$

$$\left\{-r\, \text{muL}^{(2,0)}[\text{u, phi}],\, 0,\, r\, \text{muM}^{(2,0)}[\text{u, phi}]\right\},$$

$$\left\{r\left(\text{muM}^{(2,0)}[\text{u, phi}]-r\, \text{muL}^{(2,1)}[\text{u, phi}]+\text{muM}^{(2,2)}[\text{u, phi}]\right),\right.$$

$$r\, \text{muM}^{(2,0)}[\text{u, phi}],$$

$$\left.\left.2\, r^2\left(-r\, \text{muL}^{(2,0)}[\text{u, phi}]+\text{muM}^{(2,1)}[\text{u, phi}]\right)\right\}\right\},$$

$$\left\{\left\{\frac{1}{r}2\left(\text{muM}^{(1,1)}[\text{u, phi}]+\text{muM}^{(1,3)}[\text{u, phi}]\right),\, 0,\right.\right.$$

$$\left.-\text{muM}^{(1,0)}[\text{u, phi}]-\text{muM}^{(1,2)}[\text{u, phi}]\right\},$$

$$\left\{0,\, 0,\, -\text{muM}^{(1,0)}[\text{u, phi}]\right\},\, \left\{-\text{muM}^{(1,0)}[\text{u, phi}]-\text{muM}^{(1,2)}[\text{u, phi}],\right.$$

$$\left.\left.-\text{muM}^{(1,0)}[\text{u, phi}],\, -2\, r\, \text{muM}^{(1,1)}[\text{u, phi}]\right\}\right\},$$

$$\left\{\left\{-2\left(\text{muM}^{(1,2)}[\text{u, phi}]+\text{muM}^{(1,4)}[\text{u, phi}]\right)+\right.\right.$$

$$2\, r\left(\text{muL}^{(1,1)}[\text{u, phi}]+\text{muL}^{(1,3)}[\text{u, phi}]+\text{muM}^{(2,0)}[\text{u, phi}]-\right.$$

$$\left.r\, \text{muL}^{(2,1)}[\text{u, phi}]+\text{muM}^{(2,2)}[\text{u, phi}]\right),$$

$$-\text{muM}^{(1,0)}[\text{u, phi}]-\text{muM}^{(1,2)}[\text{u, phi}]+r\, \text{muM}^{(2,0)}[\text{u, phi}],$$

$$r\left(3\, \text{muM}^{(1,1)}[\text{u, phi}]+3\, \text{muM}^{(1,3)}[\text{u, phi}]+r\left(-3\, \text{muL}^{(1,2)}[\text{u, phi}]-\right.\right.$$

$$\left.\left.\left.r\, \text{muL}^{(2,0)}[\text{u, phi}]+\text{muM}^{(2,1)}[\text{u, phi}]\right)\right)\right\},$$

$$\left\{-\text{muM}^{(1,0)}[\text{u, phi}]-\text{muM}^{(1,2)}[\text{u, phi}]+r\, \text{muM}^{(2,0)}[\text{u, phi}],\right.$$

$$\left.-2\, \text{muM}^{(1,0)}[\text{u, phi}],\, r\left(3\, r\, \text{muL}^{(1,0)}[\text{u, phi}]-\text{muM}^{(1,1)}[\text{u, phi}]\right)\right\},$$

$$\left\{r\left(3\, \text{muM}^{(1,1)}[\text{u, phi}]+3\, \text{muM}^{(1,3)}[\text{u, phi}]+r\left(-3\, \text{muL}^{(1,2)}[\text{u, phi}]-\right.\right.\right.$$

$$\left.\left.r\, \text{muL}^{(2,0)}[\text{u, phi}]+\text{muM}^{(2,1)}[\text{u, phi}]\right)\right),$$

$$\left.\left.\left.\left.r\left(3\, r\, \text{muL}^{(1,0)}[\text{u, phi}]-\text{muM}^{(1,1)}[\text{u, phi}]\right),\, 0\right\}\right\}\right\}\right\},$$

$$\left\{\left\{\left\{\left\{2\, \text{muL}^{(2,1)}[\text{u, phi}],\, 0,\, r\, \text{muL}^{(2,0)}[\text{u, phi}]\right\},\right.\right.\right.$$

$$\left\{0,\, 0,\, 0\right\},\, \left\{r\, \text{muL}^{(2,0)}[\text{u, phi}],\, 0,\, 0\right\}\right\},$$

$$\left\{\{0,\, 0,\, 0\},\, \{0,\, 0,\, 0\},\, \{0,\, 0,\, 0\}\right\},$$

$$\left\{\left\{\frac{1}{r}2\left(\text{muM}^{(1,1)}[\text{u, phi}]+\text{muM}^{(1,3)}[\text{u, phi}]\right),\right.\right.$$

$$\left.0,\, -\text{muM}^{(1,0)}[\text{u, phi}]-\text{muM}^{(1,2)}[\text{u, phi}]\right\},$$

$$\left\{0,\, 0,\, -\text{muM}^{(1,0)}[\text{u, phi}]\right\},\, \left\{-\text{muM}^{(1,0)}[\text{u, phi}]-\text{muM}^{(1,2)}[\text{u, phi}],\right.$$

$$\left.\left.\left.-\text{muM}^{(1,0)}[\text{u, phi}],\, -2\, r\, \text{muM}^{(1,1)}[\text{u, phi}]\right\}\right\}\right\},$$

$$\Big\{\{\{0,\, 0,\, 0\},\, \{0,\, 0,\, 0\},\, \{0,\, 0,\, 0\}\},$$

$$\{\{0,\, 0,\, 0\},\, \{0,\, 0,\, 0\},\, \{0,\, 0,\, 0\}\},$$

$$\Big\{\Big\{-\frac{1}{r^2}4\,\big(\text{muM}^{(0,1)}[\text{u},\,\text{phi}] + \text{muM}^{(0,3)}[\text{u},\,\text{phi}]\big),$$

$$0,\, \frac{1}{r}2\,\big(\text{muM}[\text{u},\,\text{phi}] + \text{muM}^{(0,2)}[\text{u},\,\text{phi}]\big)\Big\},$$

$$\Big\{0,\, 0,\, \frac{2\,\text{muM}[\text{u},\,\text{phi}]}{r}\Big\},\, \Big\{\frac{1}{r}2\,\big(\text{muM}[\text{u},\,\text{phi}] + \text{muM}^{(0,2)}[\text{u},\,\text{phi}]\big),$$

$$\frac{2\,\text{muM}[\text{u},\,\text{phi}]}{r},\, 4\,\text{muM}^{(0,1)}[\text{u},\,\text{phi}]\Big\}\Big\}\Big\},$$

$$\Big\{\Big\{\Big\{\frac{1}{r}2\,\big(\text{muM}^{(1,1)}[\text{u},\,\text{phi}] + \text{muM}^{(1,3)}[\text{u},\,\text{phi}]\big),\, 0,$$

$$-\text{muM}^{(1,0)}[\text{u},\,\text{phi}] - \text{muM}^{(1,2)}[\text{u},\,\text{phi}]\Big\},$$

$$\Big\{0,\, 0,\, -\text{muM}^{(1,0)}[\text{u},\,\text{phi}]\Big\},\, \Big\{-\text{muM}^{(1,0)}[\text{u},\,\text{phi}] - \text{muM}^{(1,2)}[\text{u},\,\text{phi}],$$

$$-\text{muM}^{(1,0)}[\text{u},\,\text{phi}],\, -2\,r\,\text{muM}^{(1,1)}[\text{u},\,\text{phi}]\Big\}\Big\},$$

$$\Big\{\Big\{-\frac{1}{r^2}4\,\big(\text{muM}^{(0,1)}[\text{u},\,\text{phi}] + \text{muM}^{(0,3)}[\text{u},\,\text{phi}]\big),\, 0,\, \frac{1}{r}$$

$$2\,\big(\text{muM}[\text{u},\,\text{phi}] + \text{muM}^{(0,2)}[\text{u},\,\text{phi}]\big)\Big\},$$

$$\Big\{0,\, 0,\, \frac{2\,\text{muM}[\text{u},\,\text{phi}]}{r}\Big\},\, \Big\{\frac{1}{r}2\,\big(\text{muM}[\text{u},\,\text{phi}] + \text{muM}^{(0,2)}[\text{u},\,\text{phi}]\big),$$

$$\frac{2\,\text{muM}[\text{u},\,\text{phi}]}{r},\, 4\,\text{muM}^{(0,1)}[\text{u},\,\text{phi}]\Big\}\Big\},\, \Big\{\Big\{-\frac{1}{r}2$$

$$\big(-2\,\big(\text{muM}^{(0,2)}[\text{u},\,\text{phi}] + \text{muM}^{(0,4)}[\text{u},\,\text{phi}]\big) + r\,\big(\text{muL}^{(0,1)}[\text{u},\,\text{phi}] +$$

$$\text{muL}^{(0,3)}[\text{u},\,\text{phi}] + \text{muM}^{(1,0)}[\text{u},\,\text{phi}] + \text{muM}^{(1,2)}[\text{u},\,\text{phi}]\big)\big),$$

$$\frac{1}{r}2\,\big(\text{muM}[\text{u},\,\text{phi}] + \text{muM}^{(0,2)}[\text{u},\,\text{phi}]\big) - \text{muM}^{(1,0)}[\text{u},\,\text{phi}],$$

$$-6\,\text{muM}^{(0,1)}[\text{u},\,\text{phi}] + 3\,r\,\text{muL}^{(0,2)}[\text{u},\,\text{phi}] -$$

$$6\,\text{muM}^{(0,3)}[\text{u},\,\text{phi}] - r\,\text{muM}^{(1,1)}[\text{u},\,\text{phi}]\Big\},$$

$$\Big\{\frac{1}{r}2\,\big(\text{muM}[\text{u},\,\text{phi}] + \text{muM}^{(0,2)}[\text{u},\,\text{phi}]\big) - \text{muM}^{(1,0)}[\text{u},\,\text{phi}],$$

$$\frac{4\,\text{muM}[\text{u},\,\text{phi}]}{r},\, -3\,r\,\text{muL}[\text{u},\,\text{phi}] + 2\,\text{muM}^{(0,1)}[\text{u},\,\text{phi}]\Big\},$$

$$\Big\{-6\,\text{muM}^{(0,1)}[\text{u},\,\text{phi}] + 3\,r\,\text{muL}^{(0,2)}[\text{u},\,\text{phi}] - 6\,\text{muM}^{(0,3)}[\text{u},\,\text{phi}] -$$

$$r\,\text{muM}^{(1,1)}[\text{u},\,\text{phi}],\, -3\,r\,\text{muL}[\text{u},\,\text{phi}] + 2\,\text{muM}^{(0,1)}[\text{u},\,\text{phi}],\, 0\Big\}\Big\}\Big\}\Big\},$$

$$\Big\{\Big\{\Big\{\Big\{-2\,\big(\text{muM}^{(2,1)}[\text{u},\,\text{phi}] - r\,\text{muL}^{(2,2)}[\text{u},\,\text{phi}] + \text{muM}^{(2,3)}[\text{u},\,\text{phi}]\big),$$

$$-r\,\text{muL}^{(2,0)}[\text{u},\,\text{phi}],$$

$$r\,\big(\text{muM}^{(2,0)}[\text{u},\,\text{phi}] - r\,\text{muL}^{(2,1)}[\text{u},\,\text{phi}] + \text{muM}^{(2,2)}[\text{u},\,\text{phi}]\big)\Big\},$$

$$\Big\{-r\,\text{muL}^{(2,0)}[\text{u},\,\text{phi}],\, 0,\, r\,\text{muM}^{(2,0)}[\text{u},\,\text{phi}]\Big\},$$

$\{r \left(\text{muM}^{(2,0)}[u, phi] - r\,\text{muL}^{(2,1)}[u, phi] + \text{muM}^{(2,2)}[u, phi]\right),$

$\quad r\,\text{muM}^{(2,0)}[u, phi],$

$\quad 2\,r^2 \left(-r\,\text{muL}^{(2,0)}[u, phi] + \text{muM}^{(2,1)}[u, phi]\right)\}\},$

$\left\{\left\{\dfrac{1}{r}2 \left(\text{muM}^{(1,1)}[u, phi] + \text{muM}^{(1,3)}[u, phi]\right), 0,\right.\right.$

$\quad -\text{muM}^{(1,0)}[u, phi] - \text{muM}^{(1,2)}[u, phi]\},$

$\quad \{0, 0, -\text{muM}^{(1,0)}[u, phi]\}, \{-\text{muM}^{(1,0)}[u, phi] - \text{muM}^{(1,2)}[u, phi],$

$\quad -\text{muM}^{(1,0)}[u, phi], -2\,r\,\text{muM}^{(1,1)}[u, phi]\}\},$

$\left\{\left\{-2 \left(\text{muM}^{(1,2)}[u, phi] + \text{muM}^{(1,4)}[u, phi]\right) + \right.\right.$

$\quad\quad 2\,r \left(\text{muL}^{(1,1)}[u, phi] + \text{muL}^{(1,3)}[u, phi] + \text{muM}^{(2,0)}[u, phi] - \right.$

$\quad\quad\quad r\,\text{muL}^{(2,1)}[u, phi] + \text{muM}^{(2,2)}[u, phi]\big),$

$\quad -\text{muM}^{(1,0)}[u, phi] - \text{muM}^{(1,2)}[u, phi] + r\,\text{muM}^{(2,0)}[u, phi],$

$\quad r \left(3\,\text{muM}^{(1,1)}[u, phi] + 3\,\text{muM}^{(1,3)}[u, phi] + r \left(-3\,\text{muL}^{(1,2)}[u, phi] - \right.\right.$

$\quad\quad\quad r\,\text{muL}^{(2,0)}[u, phi] + \text{muM}^{(2,1)}[u, phi]\big)\big)\},$

$\quad \{-\text{muM}^{(1,0)}[u, phi] - \text{muM}^{(1,2)}[u, phi] + r\,\text{muM}^{(2,0)}[u, phi],$

$\quad -2\,\text{muM}^{(1,0)}[u, phi], r \left(3\,r\,\text{muL}^{(1,0)}[u, phi] - \text{muM}^{(1,1)}[u, phi]\right)\},$

$\quad \{r \left(3\,\text{muM}^{(1,1)}[u, phi] + 3\,\text{muM}^{(1,3)}[u, phi] + r \left(-3\,\text{muL}^{(1,2)}[u, phi] - \right.\right.$

$\quad\quad\quad r\,\text{muL}^{(2,0)}[u, phi] + \text{muM}^{(2,1)}[u, phi]\big)\big),$

$\quad r \left(3\,r\,\text{muL}^{(1,0)}[u, phi] - \text{muM}^{(1,1)}[u, phi]\right), 0\}\}\},$

$\left\{\left\{\left\{\dfrac{1}{r}2 \left(\text{muM}^{(1,1)}[u, phi] + \text{muM}^{(1,3)}[u, phi]\right), 0,\right.\right.\right.$

$\quad -\text{muM}^{(1,0)}[u, phi] - \text{muM}^{(1,2)}[u, phi]\},$

$\quad \{0, 0, -\text{muM}^{(1,0)}[u, phi]\}, \{-\text{muM}^{(1,0)}[u, phi] - \text{muM}^{(1,2)}[u, phi],$

$\quad -\text{muM}^{(1,0)}[u, phi], -2\,r\,\text{muM}^{(1,1)}[u, phi]\}\},$

$\left\{\left\{-\dfrac{1}{r^2}4 \left(\text{muM}^{(0,1)}[u, phi] + \text{muM}^{(0,3)}[u, phi]\right), 0, \dfrac{1}{r}\right.\right.$

$\quad 2 \left(\text{muM}[u, phi] + \text{muM}^{(0,2)}[u, phi]\right)\},$

$\quad \left\{0, 0, \dfrac{2\,\text{muM}[u, phi]}{r}\right\}, \left\{\dfrac{1}{r}2 \left(\text{muM}[u, phi] + \text{muM}^{(0,2)}[u, phi]\right),\right.$

$\quad \dfrac{2\,\text{muM}[u, phi]}{r}, 4\,\text{muM}^{(0,1)}[u, phi]\}\}, \left\{\left\{-\dfrac{1}{r}2\right.\right.$

$\quad\quad \left(-2 \left(\text{muM}^{(0,2)}[u, phi] + \text{muM}^{(0,4)}[u, phi]\right) + r \left(\text{muL}^{(0,1)}[u, phi] + \right.\right.$

$\quad\quad\quad \text{muL}^{(0,3)}[u, phi] + \text{muM}^{(1,0)}[u, phi] + \text{muM}^{(1,2)}[u, phi]\big)\big),$

$\quad \dfrac{1}{r}2 \left(\text{muM}[u, phi] + \text{muM}^{(0,2)}[u, phi]\right) - \text{muM}^{(1,0)}[u, phi],$

$\quad -6\,\text{muM}^{(0,1)}[u, phi] + 3\,r\,\text{muL}^{(0,2)}[u, phi] - $

$\quad\quad 6\,\text{muM}^{(0,3)}[u, phi] - r\,\text{muM}^{(1,1)}[u, phi]\},$

$$\left\{\frac{1}{r}2\left(\text{muM}[u,\,phi]+\text{muM}^{(0,2)}[u,\,phi]\right)-\text{muM}^{(1,0)}[u,\,phi],\right.$$

$$\frac{4\,\text{muM}[u,\,phi]}{r},\,-3\,r\,\text{muL}[u,\,phi]+2\,\text{muM}^{(0,1)}[u,\,phi]\Big\},$$

$$\left\{-6\,\text{muM}^{(0,1)}[u,\,phi]+3\,r\,\text{muL}^{(0,2)}[u,\,phi]-6\,\text{muM}^{(0,3)}[u,\,phi]-\right.$$

$$\left.r\,\text{muM}^{(1,1)}[u,\,phi],\,-3\,r\,\text{muL}[u,\,phi]+2\,\text{muM}^{(0,1)}[u,\,phi],\,0\Big\}\Big\}\right\},$$

$$\left\{\left\{\left\{-2\left(\text{muM}^{(1,2)}[u,\,phi]+\text{muM}^{(1,4)}[u,\,phi]\right)+\right.\right.\right.$$

$$2\,r\left(\text{muL}^{(1,1)}[u,\,phi]+\text{muL}^{(1,3)}[u,\,phi]+\text{muM}^{(2,0)}[u,\,phi]-\right.$$

$$\left.r\,\text{muL}^{(2,1)}[u,\,phi]+\text{muM}^{(2,2)}[u,\,phi]\right),$$

$$-\text{muM}^{(1,0)}[u,\,phi]-\text{muM}^{(1,2)}[u,\,phi]+r\,\text{muM}^{(2,0)}[u,\,phi],$$

$$r\left(3\,\text{muM}^{(1,1)}[u,\,phi]+3\,\text{muM}^{(1,3)}[u,\,phi]+r\left(-3\,\text{muL}^{(1,2)}[u,\,phi]-\right.\right.$$

$$\left.\left.r\,\text{muL}^{(2,0)}[u,\,phi]+\text{muM}^{(2,1)}[u,\,phi]\right)\right)\Big\},$$

$$\left\{-\text{muM}^{(1,0)}[u,\,phi]-\text{muM}^{(1,2)}[u,\,phi]+r\,\text{muM}^{(2,0)}[u,\,phi],\right.$$

$$-2\,\text{muM}^{(1,0)}[u,\,phi],\,r\left(3\,r\,\text{muL}^{(1,0)}[u,\,phi]-\text{muM}^{(1,1)}[u,\,phi]\right)\Big\},$$

$$\left\{r\left(3\,\text{muM}^{(1,1)}[u,\,phi]+3\,\text{muM}^{(1,3)}[u,\,phi]+r\left(-3\,\text{muL}^{(1,2)}[u,\,phi]-\right.\right.\right.$$

$$\left.\left.r\,\text{muL}^{(2,0)}[u,\,phi]+\text{muM}^{(2,1)}[u,\,phi]\right)\right),$$

$$r\left(3\,r\,\text{muL}^{(1,0)}[u,\,phi]-\text{muM}^{(1,1)}[u,\,phi]\right),\,0\Big\}\Big\},\,\left\{\left\{-\frac{1}{r}2\right.\right.$$

$$\left(-2\left(\text{muM}^{(0,2)}[u,\,phi]+\text{muM}^{(0,4)}[u,\,phi]\right)+r\left(\text{muL}^{(0,1)}[u,\,phi]+\right.\right.$$

$$\left.\left.\text{muL}^{(0,3)}[u,\,phi]+\text{muM}^{(1,0)}[u,\,phi]+\text{muM}^{(1,2)}[u,\,phi]\right)\right),$$

$$\frac{1}{r}2\left(\text{muM}[u,\,phi]+\text{muM}^{(0,2)}[u,\,phi]\right)-\text{muM}^{(1,0)}[u,\,phi],$$

$$-6\,\text{muM}^{(0,1)}[u,\,phi]+3\,r\,\text{muL}^{(0,2)}[u,\,phi]-$$

$$6\,\text{muM}^{(0,3)}[u,\,phi]-r\,\text{muM}^{(1,1)}[u,\,phi]\Big\},$$

$$\left\{\frac{1}{r}2\left(\text{muM}[u,\,phi]+\text{muM}^{(0,2)}[u,\,phi]\right)-\text{muM}^{(1,0)}[u,\,phi],\right.$$

$$\frac{4\,\text{muM}[u,\,phi]}{r},\,-3\,r\,\text{muL}[u,\,phi]+2\,\text{muM}^{(0,1)}[u,\,phi]\Big\},$$

$$\left\{-6\,\text{muM}^{(0,1)}[u,\,phi]+3\,r\,\text{muL}^{(0,2)}[u,\,phi]-6\,\text{muM}^{(0,3)}[u,\,phi]-\right.$$

$$\left.r\,\text{muM}^{(1,1)}[u,\,phi],\,-3\,r\,\text{muL}[u,\,phi]+2\,\text{muM}^{(0,1)}[u,\,phi],\,0\Big\}\right\},$$

$$\left\{\left\{2\left(2\,\text{muM}^{(0,1)}[u,\,phi]+\text{muM}^{(0,3)}[u,\,phi]-\text{muM}^{(0,5)}[u,\,phi]+\right.\right.\right.$$

$$r\left(\text{muL}^{(0,2)}[u,\,phi]+\text{muL}^{(0,4)}[u,\,phi]+3\left(\text{muM}^{(1,1)}[u,\,phi]-\right.\right.$$

$$\left.\left.\left.r\,\text{muL}^{(1,2)}[u,\,phi]+\text{muM}^{(1,3)}[u,\,phi]\right)\right)\right),$$

$$-4\,\text{muM}^{(0,1)}[u,\,phi]-4\,\text{muM}^{(0,3)}[u,\,phi]+$$

$$3\,r\left(\text{muL}^{(0,2)}[u,\,phi]+r\,\text{muL}^{(1,0)}[u,\,phi]\right),$$

$$r\left(-3\,\text{muM}[u,\,phi]+2\,\text{muM}^{(0,2)}[u,\,phi]+5\,\text{muM}^{(0,4)}[u,\,phi]+\right.$$

$$r\left(-2\,\text{muL}^{(0,1)}[u,\,phi]-5\,\text{muL}^{(0,3)}[u,\,phi]-3\left(\text{muM}^{(1,0)}[u,\,\right.\right.$$

$$\left.\left.\left.phi]-r\,\text{muL}^{(1,1)}[u,\,phi]+\text{muM}^{(1,2)}[u,\,phi]\right)\right)\right)\Big\},$$

$$\Big\{-4\,\text{muM}^{(0,1)}[u,\,phi]-4\,\text{muM}^{(0,3)}[u,\,phi]+$$
$$3\,r\,\Big(\text{muL}^{(0,2)}[u,\,phi]+r\,\text{muL}^{(1,0)}[u,\,phi]\Big),\,-6\,r\,\text{muL}[u,\,phi],$$
$$3\,r\,\Big(-\text{muM}[u,\,phi]+r\,\Big(\text{muL}^{(0,1)}[u,\,phi]-\text{muM}^{(1,0)}[u,\,phi]\Big)\Big)\Big\},$$
$$\Big\{r\,\Big(-3\,\text{muM}[u,\,phi]+2\,\text{muM}^{(0,2)}[u,\,phi]+5\,\text{muM}^{(0,4)}[u,\,phi]+$$
$$r\,\Big(-2\,\text{muL}^{(0,1)}[u,\,phi]-5\,\text{muL}^{(0,3)}[u,\,phi]-3\,\Big(\text{muM}^{(1,0)}[u,$$
$$phi]-r\,\text{muL}^{(1,1)}[u,\,phi]+\text{muM}^{(1,2)}[u,\,phi]\Big)\Big)\Big),$$
$$3\,r\,\Big(-\text{muM}[u,\,phi]+r\,\Big(\text{muL}^{(0,1)}[u,\,phi]-\text{muM}^{(1,0)}[u,\,phi]\Big)\Big),$$
$$6\,r^2\,\Big(-2\,\text{muM}^{(0,1)}[u,\,phi]-\text{muM}^{(0,3)}[u,\,phi]+$$
$$r\,\Big(\text{muL}[u,\,phi]+\text{muL}^{(0,2)}[u,\,phi]+$$
$$r\,\text{muL}^{(1,0)}[u,\,phi]-\text{muM}^{(1,1)}[u,\,phi]\Big)\Big)\Big\}\Big\}\Big\}\Big\}$$

```
(* Defining the LEOM. *)
leom[u_, r_, phi_] = FullSimplify[
  1 / 4 Table[Table[Sum[Sum[Sum[LeviCivitaTensor[3][[a, i, j]]
        Sum[Sum[gInv[u, r, phi][[b, c]]
          (g[u, r, phi][[m, a]] (dddh[u, r, phi][[n, b, i, j,
            c]] - dddh[u, r, phi][[c, b, i, j, n]]) + g[u, r,
            phi][[n, a]] (dddh[u, r, phi][[m, b, i, j, c]] -
            dddh[u, r, phi][[c, b, i, j, m]])), {c, 3}],
        {b, 3}], {j, 3}], {i, 3}], {a, 3}], {n, 3}], {m, 3}]]
```

$$\Big\{\Big\{-\frac{1}{r^2}\Big(\text{muM}^{(0,1)}[u,\,phi]+2\,\text{muM}^{(0,3)}[u,\,phi]+$$
$$\text{muM}^{(0,5)}[u,\,phi]+r\,\Big(-\text{muL}^{(0,2)}[u,\,phi]-$$
$$\text{muL}^{(0,4)}[u,\,phi]+\text{muM}^{(1,1)}[u,\,phi]+\text{muM}^{(1,3)}[u,\,phi]\Big)\Big),$$
$$-\frac{1}{r^2}\Big(\text{muM}^{(0,1)}[u,\,phi]+\text{muM}^{(0,3)}[u,\,phi]\Big),$$
$$\frac{1}{r}\Big(-r\,\Big(\text{muL}^{(0,1)}[u,\,phi]+\text{muL}^{(0,3)}[u,\,phi]\Big)+$$
$$2\,\Big(\text{muM}^{(0,2)}[u,\,phi]+\text{muM}^{(0,4)}[u,\,phi]\Big)\Big)\Big\},$$
$$\Big\{-\frac{1}{r^2}\Big(\text{muM}^{(0,1)}[u,\,phi]+\text{muM}^{(0,3)}[u,\,phi]\Big),\,0,\,0\Big\},$$
$$\Big\{\frac{1}{r}\Big(-r\,\Big(\text{muL}^{(0,1)}[u,\,phi]+\text{muL}^{(0,3)}[u,\,phi]\Big)+$$
$$2\,\Big(\text{muM}^{(0,2)}[u,\,phi]+\text{muM}^{(0,4)}[u,\,phi]\Big)\Big),$$
$$0,\,-2\,\Big(\text{muM}^{(0,1)}[u,\,phi]+\text{muM}^{(0,3)}[u,\,phi]\Big)\Big\}\Big\}$$

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```
(* Defining the list of rules so that the LEOM vanish. *)
Lleom = {muM^(0,2)[u, phi] → -muM[u, phi],
    muM^(0,3)[u, phi] → -muM^(0,1)[u, phi],
    muM^(0,4)[u, phi] → muM[u, phi], muM^(0,5)[u, phi] →
     muM^(0,1)[u, phi], muM^(1,3)[u, phi] → -muM^(1,1)[u, phi],
    muL^(0,2)[u, phi] → -muL[u, phi], muL^(0,3)[u, phi] →
     -muL^(0,1)[u, phi], muL^(0,4)[u, phi] → muL[u, phi]};


(* Test if the LEOM really vanish for
 the previously defined list of rules. *)
leom[u, r, phi] /. Lleom // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
(* Defining the covariant boundary term without
 contraction with the unit normal vector n. *)
btWOn[u_, r_, phi_] =
 FullSimplify[Table[Table[Table[Sum[Sum[Sum[Sum[
          LeviCivitaTensor[3][[a, i, j]] gInv[u, r, phi][[b, c]]
           (g[u, r, phi][[a, m]] (ddh[u, r, phi][[b, j, n, c]] +
                 ddh[u, r, phi][[b, n, j, c]] - ddh[u, r, phi][[
                  b, c, j, n]] - ddh[u, r, phi][[j, n, b, c]]) +
             g[u, r, phi][[a, n]] (ddh[u, r, phi][[b, j, m, c]] +
                 ddh[u, r, phi][[b, m, j, c]] - ddh[u, r, phi][[
                  b, c, j, m]] - ddh[u, r, phi][[j, m, b, c]])),
          {j, 3}], {c, 3}], {b, 3}], {a, 3}],
      {n, 3}], {m, 3}], {i, 3}]]
```

$$\left\{\left\{\left\{-\frac{1}{r}4\ \left(\mathrm{muM}^{(0,1)}[u,\,phi]+\mathrm{muM}^{(0,3)}[u,\,phi]\right),\,0,\,0\right\},\,\{0,\,0,\,0\},\right.\right.$$

$$\{0,\,0,\,0\}\Big\},\,\left\{\left\{\frac{1}{r}4\ \left(\mathrm{muM}^{(0,1)}[u,\,phi]+\mathrm{muM}^{(0,3)}[u,\,phi]\right),\,\frac{1}{r}\right.\right.$$

$$2\ \left(\mathrm{muM}^{(0,1)}[u,\,phi]+\mathrm{muM}^{(0,3)}[u,\,phi]\right),\,-2\ \left(\mathrm{muM}^{(0,2)}[u,\,phi]-\right.$$

$$\left.r\ \left(\mathrm{muL}^{(0,1)}[u,\,phi]+\mathrm{muL}^{(0,3)}[u,\,phi]\right)+\mathrm{muM}^{(0,4)}[u,\,phi]\right)\Big\},$$

$$\left\{\frac{1}{r}2\ \left(\mathrm{muM}^{(0,1)}[u,\,phi]+\mathrm{muM}^{(0,3)}[u,\,phi]\right),\,0,\,0\right\},$$

$$\left\{-2\ \left(\mathrm{muM}^{(0,2)}[u,\,phi]-\right.\right.$$

$$\left.r\ \left(\mathrm{muL}^{(0,1)}[u,\,phi]+\mathrm{muL}^{(0,3)}[u,\,phi]\right)+\mathrm{muM}^{(0,4)}[u,\,phi]\right),$$

$$\left.0,\,4\ r\ \left(\mathrm{muM}^{(0,1)}[u,\,phi]+\mathrm{muM}^{(0,3)}[u,\,phi]\right)\right\}\Big\},$$

$$\left\{\left\{-\frac{1}{r^2}4\ \left(\mathrm{muM}^{(0,2)}[u,\,phi]-r\ \left(\mathrm{muL}^{(0,1)}[u,\,phi]+\mathrm{muL}^{(0,3)}[u,\,phi]\right)+\right.\right.\right.$$

$$\left.\mathrm{muM}^{(0,4)}[u,\,phi]\right),\,0,\,\frac{1}{r}$$

$$2\ \left(\mathrm{muM}^{(0,1)}[u,\,phi]+\mathrm{muM}^{(0,3)}[u,\,phi]\right)\Big\},\,\{0,\,0,\,0\},$$

$$\left\{\frac{1}{r}2\ \left(\mathrm{muM}^{(0,1)}[u,\,phi]+\mathrm{muM}^{(0,3)}[u,\,phi]\right),\,0,\,0\right\}\Big\}\Big\}$$

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```
(* Calculating the boundary term
 while the LEOM being satisfied. *)
btWOn[u, r, phi] /. Lleom // MatrixForm
```

$$
\left(
\begin{array}{ccc}
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\end{array}
\right)
$$

```
(* Defining the non-covariant boundary term without
   contraction with the partial derivatives. *)
noncovWOd[u_, r_, phi_] = FullSimplify[
  Table[Sum[Sum[LeviCivitaTensor[3][[i, j, k]]
     Sum[Sum[gamma[u, r, phi][[m, j, n]]
       Sum[Sum[Sum[gInv[u, r, phi][[n, p]] gInv[u, r, phi][[
              q, a]] h[u, r, phi][[p, q]], {q, 3}], {p, 3}]
          (dh[u, r, phi][[k, m, a]] + dh[u, r, phi][[m, k, a]] -
            dh[u, r, phi][[a, m, k]]), {a, 3}],
       {n, 3}], {m, 3}], {k, 3}], {j, 3}], {i, 3}]]
```

$$
\left\{ \frac{1}{r}2\,\mathrm{muM}[u,\,\mathrm{phi}]\,\left(-r\,\mathrm{muL}[u,\,\mathrm{phi}] + \mathrm{muM}^{(0,1)}[u,\,\mathrm{phi}]\right), \right.
$$

$$
\frac{1}{r}2\,\left(\mathrm{muM}[u,\,\mathrm{phi}]\,\left(\mathrm{muM}^{(0,1)}[u,\,\mathrm{phi}] - \right.\right.
$$

$$
\left. r\,\mathrm{muL}^{(0,2)}[u,\,\mathrm{phi}] + \mathrm{muM}^{(0,3)}[u,\,\mathrm{phi}]\right) - r\,\mathrm{muM}^{(0,1)}[u,\,\mathrm{phi}]
$$

$$
\left(\mathrm{muL}^{(0,1)}[u,\,\mathrm{phi}] + \mathrm{muM}^{(1,0)}[u,\,\mathrm{phi}]\right) + r\,\mathrm{muL}[u,\,\mathrm{phi}]
$$

$$
\left.\left(-r\,\mathrm{muL}^{(0,1)}[u,\,\mathrm{phi}] + 2\,\mathrm{muM}^{(0,2)}[u,\,\mathrm{phi}] + r\,\mathrm{muM}^{(1,0)}[u,\,\mathrm{phi}]\right)\right),
$$

$$
\left. \frac{1}{r}2\,\mathrm{muL}[u,\,\mathrm{phi}]\,\left(-r\,\mathrm{muL}[u,\,\mathrm{phi}] + \mathrm{muM}^{(0,1)}[u,\,\mathrm{phi}]\right)\right\}
$$

```
(* Calculating the contraction of the previously defined term
   with the partial derivatives which gives the full non-
 covariant boundary term. *)
noncovWd[u_, r_, phi_] =
 Sum[D[noncovWOd[u, r, phi][[i]], var[[i]]], {i, 3}] /. Lleom //
   FullSimplify
```

$\frac{1}{r}2 \left(-\text{muL}[u, phi] \left(\text{muM}[u, phi] + 3\,r\,\text{muL}^{(0,1)}[u, phi]\right) + \right.$

$\quad \text{muM}^{(0,1)}[u, phi] \left(\text{muL}^{(0,1)}[u, phi] + \text{muM}^{(1,0)}[u, phi]\right) +$

$\quad \left. \text{muM}[u, phi] \left(-r\,\text{muL}^{(1,0)}[u, phi] + \text{muM}^{(1,1)}[u, phi]\right)\right)$

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