TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

DIPLOMARBEIT

The shape of death

A brief discussion on the RRR transform and multidimensional methods of pricing mortality derivatives

ausgeführt am Institut für

Wirtschaftsmathematik
Forschungsgruppe Finanz- und Versicherungsmathematik

unter der Anleitung von

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Wien, 9. Februar 2016 ______________________________

Affidavit

I hereby declare that this master thesis has been written only by the undersigned and without any assistance from third parties.

Furthermore, I confirm that no sources have been used in the preparation of this thesis other than those indicated in the thesis itself.

Vienna, January 29, 2016 ______________________________

Kurzfassung

Derivative Finanzinstrumente existieren in den unterschiedlichsten Märkten, zum Beispiel in Aktienmärkte, Energiemärkte sowie Öl- und Gasmärkte. Deren Underlying ist schon seit langem kein gewöhnlicher Stock mehr. Dennoch gibt es für Mortalitätsderivate bis heute noch keinen liquiden Markt.
Diese Arbeit befasst sich mit der Modellierung der Sterblichkeitsintensität, welche sowohl Korellationen über die Zeit als auch zwischen Altersgruppen zulässt. Dieser Ansatz führt zu einer neuen Art von sterblichkeitsbezogenen Finanzinstrumenten, welche mit diesem Modell bepreist werden können. Ein Beispiel eines Mortalitätsderivates wird vorgestellt und eine geschlossene Preisformel wird hergeleitet. Weiters wird das Einbeziehen eines Sicherheitspolsters in die Modellierung der Sterblichkeitsintensität diskutiert, sowie eine Transformation, welche diesen Ansatz approximiert, vorgestellt.

Schlagwörter: *Cox-Ingersoll-Ross Prozess, CIR-field, Correlation Bond, Mortality Swap, Gompertz-Makeham Modell, Second Order Shift, Esscher Transformation, RRR Transformation*

Abstract

Derivative financial instruments occur in different types of markets, like stock markets, energy markets and oil and gas markets. Their underlying does not have to be an ordinary stock any more. However, a liquid market for mortality derivatives does not exist yet.
This thesis is about modelling the force of mortality by including correlations over time and between age classes. This approach leads to a new kind of mortality linked financial instrument which can be priced within this model. An example of a mortality derivative will be given and a closed pricing formula will be established. Furthermore, this thesis discusses including a safety margin for modelling the force of mortality and introduces a transformation which aims to approximate this approach.

Keywords: *Cox-Ingersoll-Ross process, CIR-field, Correlation bond, Mortality swap, Gompertz-Makeham model, Second order shift, Esscher transform, RRR transform*

Acknowledgements

I would like to thank Alex, Alex, Arpad, Axel, Bernhart, Bianca, Brigitte, Caroline, Cetin, Christiane, Christina, Daniel, Georg, Jacqueline, Jadranka, Joel, Jonas, Julia, July, Laura, Ludwig, Marie, Nicole, Niki, Paul, Philip, Roscoe, Sandra, Tilmann and Thomas as well as Andy, Marlene and Martin and, of course, Anna.

Most of all, I would like to thank Prof. Thorsten Rheinländer, who was the supervisor of this thesis. He provided a friendly environment where we could discuss problems on an equal footing. Nevertheless, he set high requirements and enabled me to deliver new ideas.

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CHAPTER 1

Introduction

A mortality derivative is a financial instrument whose underlying is related to the mortality of a certain cohort. Consider a derivative depending on interest rates. A mortality derivative is a similar product. For example, a so called longevity swap is a contract between two parties. The payment of one party depends on the mortality of a cohort, e.g. the uncertain number of survivors at the maturity. This party pays the floating leg against a predefined fixed leg. The underlying is the force of mortality, which can be compared to ordinary swaps where the interest rate is the underlying index. The upper example is a so called bespoken over the counter swap. There are already some papers focusing on these products, for example Biffis et al. [BBPS14]. However, a liquid market for mortality linked derivatives does not exist yet. There are organisations such as the *Life & Longevity Market Association* who want to establish a liquid market. For further information please visit their homepage `http://www.llma.org`.

The aim of this thesis is to provide a pricing framework for mortality derivatives. In contrast to existing frameworks this model includes correlations of the mortality rate over time and between age classes. This is accomplished by shifted Cox-Ingersoll-Ross processes which model the deviation of the mortality rate from a given deterministic function. Since the correlation between age classes is taken into account, it is possible to define new kinds of mortality linked instruments which depend on these correlations. Furthermore, this model is able to provide a closed pricing formula for such a product.

The structure of this thesis is as follows:
Chapter 2 gives an example of a mortality linked financial instrument which reflects the correlation between different cohorts. This product, called correlation bond, motivates the final model and can be priced within this model.

Chapter 3 introduces a deterministic model for the mortality rate, namely the Gompertz-Makeham model. Furthermore, it discusses the so called *second order shifts*. Insurance and reinsurance undertakings use second order shifts in order to gain higher profits or to be more prudent within the calculation of best estimates of future cash flows. These cash flows will only occur, if the insured person is alive at a certain time. Therefore, the present value of this contract is linked to the probability

of death of this person. The second order shift simply shifts the persons age, e.g. for two years; see figure 1.
This thesis tries to receive a similar output using a different approach, namely the *RRR transform*. Consider the probability of a person with age x dying within the next t years, i.e. ${}_tq_x$. These probabilities refer to the distribution function of a random variable representing the remaining lifetime of a person with age x. This paper will introduce a transformation of the random variable's density function.
The transformation looks similar to the Esscher transform. However, the Esscher transform is not the right choice to solve this problem, which will be discussed within this chapter.

Chapter 4 adds a simple stochastic movement, i.e. the Brownian motion, to the model. Some basic theorems of stochastic analysis will be stated. They are required in order to derive the price of the correlation bond of chapter 2. This model includes correlations over time, but has not implied correlations between age classes yet.

Chapter 5 is about the main model of this thesis. The shifted Cox-Ingersoll-Ross process will be introduced and analysed which finally leads to the CIR-field, a model for the force of mortality. It consists of a basic force of mortality, which does not include stochastic movements and serves as trend function for the force of mortality. For example, the Gompertz model with Makeham extension from chapter 3 can be used for this part. The randomness is modelled by two shifted Cox-Ingersoll-Ross processes which are driven by independent Brownian motions. One process models the deviations over time and the other process reflects deviations over the person's age. The usage of a shifted CIR-process ensures a variation of the trend function, which is bounded from below. Therefore, modelling a negative mortality rate can be avoided. Furthermore, the correlation of the mortality between two age classes is a decreasing function with respect to age differences. A rigorous proof of this statement will be accomplished within this chapter.

Chapter 6 applies the prior defined model to financial products such as mortality swaps and the correlation bond mentioned in chapter 2. The derivation of the price requires results from term structure modelling. Moreover, the moment generating function of the non-central chi-square distribution occurs. Finally a closed pricing formula will be stated.

CHAPTER 2

Motivation

2.1. Setting

The financial product presented in this short chapter reflects correlations of the mortality between certain cohorts and motivates the model, which will be introduced in chapter 5. Clearly, this product is very simple. However, having a look at more complex models shows that the calculation of such products leads to challenging but solvable problems.

Consider a reference society, e.g. a large city, or a whole country, which is assumed to be closed. The following uses a notation similar to Biffis et al. [BM].
Let $\mathcal{I} = \{x_1, \dots, x_n\}$ denote the society, where each x_i refers to a cohort, e.g. people of a certain age and gender. An individual j of cohort $x \in \mathcal{I}$ is linked to a random variable $\tau^{x,j}$ which describes the residual lifetime.

Fix $T > 0$. The price at time $t < T$ of a pure endowment contract, paying S_T to each individual who is alive at time T, is given by

$$P_t^{\mathcal{B}} := \int_{\mathcal{B}} \mathbb{E}\left[\exp\left(-\int_t^T (r_s + \mu_{s,x})\,\mathrm{d}s\right) S_T\right] \Psi_t(\mathrm{d}x) \qquad (1)$$

where $\mathcal{B}$ is a subset of $\mathcal{I}$, r denotes the interest rate, $\mu_{\cdot,x}$ is the force of mortality of the cohort $x \in \mathcal{I}$ and

$$\Psi_t(\mathcal{B}) := \int_{\mathcal{B}} \sum_{j=1}^{|x|} \mathrm{I}_{\{\tau^{x,j} > t\}}\,\mathrm{d}x$$

describes the numbers of survivors at time t over all cohorts of interest.

2.2. Correlation bond - pricing formula

Since the force of mortality $(\mu_{s,x})_{(s,x)\in[0,\infty)\times\mathcal{I}}$ is a mathematical surface, correlations between the different cohorts can be implied. Let $\tilde{x} \in \mathcal{I}$ be a specific cohort. Assume a party named A offers derivative financial instruments whose underlying asset is $P_t^B / P_t^{\tilde{x}}$. The corresponding prices are calculated by (1) with payment S_T equal to 1. Furthermore, let $B \subset \mathcal{I}$ be a subset of $\mathcal{I}$ containing only the specific cohort $\tilde{x}$ and one other cohort $y \in \mathcal{I}$. This leads to the definition of the correlation bond.

Definition 2.1. Let $\tilde{x}, y \in \mathcal{I}$ denote cohorts. The *Correlation bond* is defined via

$$\tilde{P}_t^y := \frac{P_t^{\tilde{x},y}}{P_t^{\tilde{x}}} \tag{2}$$

where $P_t^{\tilde{x},y}$ and $P_t^{\tilde{x}}$ is defined by (1).

Remark: Assume all cohorts to have the same cardinality. If the mortality of the cohort $\tilde{x}$ is perfectly positive correlated to the mortality of the cohort y, the value of $\tilde{P}_t^y$ will be (omitting interest rates) constant. A pension fund trying to hedge this kind of correlation would invest in an instrument, benefiting from a constant development of (2)*, e.g. a short position in a barrier option. Therefore, $\tilde{P}_t^y$ serves as some kind of measure for these correlations.*

Assuming the term structure is independent of the force of mortality, the time zero price of this product is equal to

$$\tilde{P}_0^y = \frac{B(0,T)\int_{\{\tilde{x},y\}} \mathbb{E}\left[\exp\left(-\int_0^T \mu_{s,x}\,\mathrm{d}s\right)\right]\Psi_0(\,\mathrm{d}x)}{B(0,T)\int_{\{\tilde{x}\}} \mathbb{E}\left[\exp\left(-\int_0^T \mu_{s,x}\,\mathrm{d}s\right)\right]\Psi_0(\mathrm{d}x)} \tag{3}$$

$$= \frac{|\tilde{x}|\;\mathbb{E}\left[\exp\left(-\int_0^T \mu_{s,\tilde{x}}\,\mathrm{d}s\right)\right] + |y|\;\mathbb{E}\left[\exp\left(-\int_0^T \mu_{s,y}\,\mathrm{d}s\right)\right]}{|\tilde{x}|\;\mathbb{E}\left[\exp\left(-\int_0^T \mu_{s,\tilde{x}}\,\mathrm{d}s\right)\right]} \tag{4}$$

where $B(t,T) = \mathbb{E}\left[e^{-\int_t^T r_s\,\mathrm{d}s}|\mathcal{F}_t\right]$ denotes the time t-price of a zero coupon bond using a filtration $(\mathcal{F}_t)_{t\geq 0}$ for this market. For further information on term structures see Filipović [Fil09].

This serves as an example of a mortality linked financial product. The following sections will provide different frameworks in order to compute the terms in expression (4). Since this product is able to reflect the correlation between cohorts, the last model introduced in chapter 5 will be the most suitable for pricing the correlation bond.

CHAPTER 3

Gompertz-Makeham law of mortality

This chapter considers the Gompertz model and the Gompertz model with Makeham extension, *short:* the Gompertz-Makeham model, for the force of mortality. The mortality is modelled by a deterministic function. Therefore, formula (4) can be computed straightforward. However, this chapter discusses an important topic for insurance and reinsurance undertakings, namely *second order shifts* which will be described in section 3.1. How this method affects formula (4) is described below. Furthermore, the results can be applied in extended stochastic models of the following sections. First, focus on the Gompertz-Makeham model.

Applying the parametrisation similar to Wüthrich et al. [WM13] and Carriere [Car], the force of mortality, following the Gompertz law of mortality, is given by

$$\mu(x) = \frac{e^{\frac{x-m}{\zeta}}}{\zeta}$$

and the Gompertz-Makeham law of mortality is given by

$$\mu(x) = \lambda + \frac{e^{\frac{x-m}{\zeta}}}{\zeta} \tag{5}$$

with location parameter $m > 0$, dispersion parameter $\zeta > 0$ and an additional parameter $\lambda > 0$.
Wüthrich and Merz [WM13] use the Gompertz law of mortality which applies in some examples of section 3.3.1. The Makeham extension of the Gomperz model allows to include effects which are independent of a person's age. For further information, see Marshall et al. [MO07].
Since λ equal to zero leads to the Gompertz model, the following uses the Makeham extension.

Consider a person with age x. The remaining lifetime is described by a positive random variable τ_x. Under the Gompertz-Makeham law of mortality, a person with age x is going to survive the next t years with probability

$$\begin{aligned} {}_tp_x = \mathbb{P}\left[\tau_x > t\right] &= \exp\left(-\int_0^t \mu(x+s)\,\mathrm{d}s\right) \\ &= \exp\left\{-\lambda t - e^{\frac{x-m}{\zeta}}\left(e^{\frac{t}{\zeta}} - 1\right)\right\} \end{aligned}$$

Furthermore, the probability of death occurring within t years follows

$$ {}_tq_x = 1 - {}_tp_x = 1 - \exp\left\{-\lambda t - \xi\left(e^{\frac{t}{\zeta}} - 1\right)\right\} \tag{6} $$

with

$$ \xi := \xi(x, m, \zeta) := e^{\frac{x-m}{\zeta}} $$

If the input parameters of ξ are different from the model parameters x, m and ζ, the more explicit notation $\xi(\cdot, \cdot, \cdot)$ will be used. This only occurs in section 3.1. Otherwise the short notation ξ is sufficient.

3.1. Second order probability

The second order shift is a parallel shift of the probabilities of death. Figure 1 illustrates this approach with a shift of two years, i.e. instead of a person with age x, the probabilities refer to a person with age $x - 2$. In general, the level of this shift is denoted by $k \in \mathbb{N}$.
As mentioned in chapter 1, insurance and reinsurance undertakings might be using this approach to guarantee a prudent estimation for their future cash flows. For example, consider an insurance contract paying a certain amount if the insured person is alive at time T in the future, i.e. a pure endowment contract. This cash flow needs to be discounted and weighted by probability of survival to obtain the contract's expected present value. Shifting the mortality by two years leads to a reduction of the probability of death and therefore increases the present value of the future payment. As a consequence this results in increasing technical provisions, which insurance and reinsurance undertakings have to calculate in order to guarantee and fulfil all contractual obligations.
Alternatively, technical provisions of a life insurance contracts will drop, if the mortality drops. This occurs due to the fact that the event of paying money to the insured person at a future date becomes less likely.

Remark: Choosing a value for k might look arbitrary. However, it is bounded by two borders. On the one hand a company wants to maximise their profit and for this purpose the company does not choose a very prudent second order shift. On the other hand, due to solvency requirements, the company is committed to a certain level of prudence.

This chapter studies parallel shifts, which reduce the mortality by reducing the persons age x. Considering formula (6), the choice of level k for the second order probability is equal to replacing parameter m by $m^+ = m + k > m$ for $k \in \mathbb{N}$. This leads to the following definition.

Definition 3.1. Let $k \in \mathbb{N}$. The second order probabilities with level k for the Gompertz-Makeham model are given by

$$\begin{aligned} {}_tq_x^{(k)} &= 1 - \exp\left\{-\lambda t - \xi(x, m+k, \zeta)\left(e^{\frac{t}{\zeta}} - 1\right)\right\} \\ &= 1 - \exp\left\{-\lambda t - \xi e^{\frac{-k}{\zeta}}\left(e^{\frac{t}{\zeta}} - 1\right)\right\} \end{aligned} \tag{7}$$

The notation used in the upper formula (7) is based on Wüthrich et al. [WM13].

Remark: $\frac{\partial\, {}_tq_x^{(k)}}{\partial t} = \frac{\xi}{\zeta} \exp\left\{\frac{t-k}{\zeta} - \xi e^{\frac{-k}{\zeta}}\left(e^{\frac{t}{\zeta}} - 1\right)\right\}$

The next section introduces an alternative transformation in order to approximate the second order probabilities.

3.2. RRR transform

The RRR transform is a variation of the Esscher transform, see Gerber and Shiu [GS94], in order to generate transformed probabilities of death. Similar to the Esscher transform the RRR transformation is defined on density functions, i.e.

Definition 3.2. Let $h \in \mathbb{C}$. The RRR transform of a density f is defined by

$$f(t; h) = \frac{1}{\mathcal{L}\{f\}(h)} e^{-ht} f(t) \tag{8}$$

where $\mathcal{L}\{f\}(h)$ is the Laplace transform of a random variable with density function f, see Marshall et al. [MO07]

$$\mathcal{L}\{f\}(h) = \int_0^\infty e^{-hs} f(s)\, \mathrm{d}s \tag{9}$$

In order to apply this transformation to the probabilities of death, their density is required.
The probability ${}_tq_x$ to die within t years is a distribution function in $t \in \mathbb{R}_+$. The corresponding density function, denoted by $f(t)$, equals the derivation

$$\begin{aligned} f(t) := \frac{\partial\, {}_tq_x}{\partial t} &= \frac{\partial}{\partial t}\left[1 - \exp\left\{-\lambda t - \xi\left(e^{\frac{t}{\zeta}} - 1\right)\right\}\right] \\ &= \left(\lambda + \frac{\xi}{\zeta} e^{\frac{t}{\zeta}}\right) \exp\left\{-\lambda t - \xi\left(e^{\frac{t}{\zeta}} - 1\right)\right\} \end{aligned} \tag{10}$$

To receive the transformed probability of death, the density function $f(t; h)$ in definition (8) needs to be integrated, i.e.

$$ {}_tq_x(h) = \int_0^t f(s; h)\, \mathrm{d}s \tag{11}$$

Remark: Since the Laplace transform is defined for complex-valued parameters h such that the integral in (9) *exists, the value of h is allowed to be negative as long as the integral exists.*

Using the Gompertz-Makeham law of mortality, the Laplace transform of (10) is given by

$$\begin{aligned}\mathcal{L}\{f\}(h) &= \int_0^\infty e^{-hs} f(s)\,\mathrm{d}s \\ &= \int_0^\infty e^{-hs}\left(\lambda + \frac{\xi}{\zeta}e^{\frac{s}{\zeta}}\right)\exp\left\{-\lambda s - \xi\left(e^{\frac{s}{\zeta}} - 1\right)\right\}\,\mathrm{d}s \\ &= e^{\xi}\xi^{\zeta(\lambda+h)}\Big[\lambda\zeta\Gamma\left(-\zeta\left(\lambda+h\right),\xi\right) + \Gamma\left(-\zeta\left(\lambda+h\right)+1,\xi\right)\Big] \end{aligned} \tag{12}$$

The function $\Gamma(\cdot,\cdot)$ denotes the upper incomplete gamma function, i.e.

$$\Gamma(a,x) := \int_x^\infty u^{a-1}e^{-u}\,\mathrm{d}u \tag{13}$$

where a is allowed to be negative as long as x is greater than zero.

According to definition (11), the transformed density of (10) follows

$$\begin{aligned}{}_tq_x(h) &= \int_0^t f(s;h)\,\mathrm{d}s \\ &= \frac{1}{\mathcal{L}\{f\}(h)}\int_0^t e^{-hs}\left(\lambda + \frac{\xi}{\zeta}e^{\frac{s}{\zeta}}\right)\exp\left\{-\lambda s - \xi\left(e^{\frac{s}{\zeta}} - 1\right)\right\}\,\mathrm{d}s \\ &= \frac{\lambda\zeta\Gamma\left(-\zeta\left(\lambda+h\right),\xi,\xi e^{\frac{t}{\zeta}}\right) + \Gamma\left(-\zeta\left(\lambda+h\right)+1,\xi,\xi e^{\frac{t}{\zeta}}\right)}{\lambda\zeta\Gamma\left(-\zeta\left(\lambda+h\right),\xi\right) + \Gamma\left(-\zeta\left(\lambda+h\right)+1,\xi\right)}\end{aligned} \tag{14}$$

where in addition to the upper incomplete gamma function (13), the generalised incomplete gamma function

$$\Gamma(a,x,y) := \int_x^y u^{a-1}e^{-u}\,\mathrm{d}u \tag{15}$$

appears. Note that the disparity in the notation between the incomplete and the generalised incomplete gamma function is the number of input variables.
A rigorous computation of equations (12) and (14) is accomplished in appendix A.

Remark: Using the Gompertz law of mortality, which means to choose λ of definition (5) *equal to zero, simplifies the upper results. The Laplace transform in equation* (12) *reduces to*

$$\mathcal{L}\{f\}(h) = e^{\xi}\xi^{\zeta h}\Gamma\left(-\zeta h + 1,\xi\right) \tag{16}$$

and the transformed probability in equation (14) *follows*

$${}_tq_x(h) = \frac{\Gamma\left(-\zeta h + 1,\xi,\xi e^{\frac{t}{\zeta}}\right)}{\Gamma\left(-\zeta h + 1,\xi\right)} \tag{17}$$

3.3. Examples

The examples presented in this section shall visualise the application of the upper results. The corresponding graphics can be found in appendix B. Numerical results and graphics have been computed by programs which are stated and described in appendix C.

3.3.1. Using the Gompertz law of mortality.

Female Population:
In the first example the fit using the RRR transform compared to the second order probabilities is illustrated. For this purpose the parameter-values, similar to example 8.19 of Marshall et al. [MO07], are taken from Carriere [Car]. These values have been estimated to achieve the best fit for a certain female population.

In figure 3 the second order probabilities, with shift k equal to 2, are plotted. Furthermore, the parameter h has been chosen to achieve the least minimum square difference between the considered curves, which is received for h equal to -0.025. This can be seen in figure 2. As mentioned in section 3.2, since the appearing integrals exist, a negative value of h is appropriate. The program, which computes this result, is illustrated in appendix C code C.2.

Male Population:
Similar to the previous example the parameter-values are given by Carriere [Car] for a certain male population. Figure 4 shows that the least square difference for the considered curves are given by h equal to -0.02. This leads to RRR transformed probabilities plotted in figure 5.

3.3.2. Using the Gompertz Makeham law of mortality.

Danish population:
For the parametrisation of the Gompertz-Makeham model, a Danish dataset has been used. The parameter-values are taken from Norberg [Nor02], section 3.2. which correspond to the *G82M mortality table*. Figure 6 shows the 3D-plot. The level of h archiving the least square difference is marked by the green plane. Figure 7 explicitly shows the transformed probabilities with this specific value for h.

3.4. A brief discussion on the Esscher transform

Previous studies tried to accomplish similar results, using the Esscher transform in order to achieve transformed probabilities. This section investigates the approach using Esscher transformed probabilities. As a result, this choice of transformation is not admissible.
Since Gerber and Shiu [GS94] used the Esscher transform for the price of options,

it is an interesting tool in financial and actuarial mathematics. Similar to their notation, the Esscher transform is given by

$$f(t;h) = \frac{1}{M_f(h)} e^{ht} f(t) \tag{18}$$

where $M_f(h)$ is defined as the moment generating function of a random variable with density function f, i.e.

$$M_f(h) = \int_{-\infty}^{\infty} e^{ht} f(t)\,\mathrm{d}t \tag{19}$$

The appearing random variable is, for this purpose, the remaining lifetime with distribution function ${}_tq_x$. The probability ${}_tq_x$ in the Gompertz-Makeham model is given by formula (6), which is

$${}_tq_x = 1 - \exp\left\{-\lambda t - \xi\left(e^{\frac{t}{\zeta}} - 1\right)\right\}$$

This holds for t greater than zero.
Since ${}_tq_x$ equals zero for $t \leq 0$, the function defined for $t \in \mathbb{R}$ is given by

$${}_tq_x = \begin{cases} 0 & ,t \leq 0 \\ 1 - \exp\left\{-\lambda t - \xi\left(e^{\frac{t}{\zeta}} - 1\right)\right\} & ,t > 0 \end{cases} \tag{20}$$

This function is non-decreasing, $\lim_{t\to-\infty} {}_tq_x$ equals zero, $\lim_{t\to\infty} {}_tq_x$ equals one and since

$$\begin{aligned} \lim_{t\to 0+} {}_tq_x &= \lim_{t\to 0+} \left[1 - \exp\left\{-\lambda t - \xi\left(e^{\frac{t}{\zeta}} - 1\right)\right\}\right] \\ &= 0 = \lim_{t\to 0-} {}_tq_x \end{aligned}$$

the function is continuous in $\mathbb{R}$. Therefore, ${}_tq_x$ is a distribution function.
Trying to calculate the derivative of this function, which is necessary to calculate the moment generating function (19), leads to

$$f(t) = \begin{cases} 0 & ,t \leq 0 \\ \left(\lambda + \frac{\xi}{\zeta} e^{\frac{t}{\zeta}}\right) \exp\left\{-\lambda t - \xi\left(e^{\frac{t}{\zeta}} - 1\right)\right\} & ,t > 0 \end{cases} \tag{21}$$

where the result for $t > 0$ has already been calculated in section 3.2. However, since

$$\begin{aligned} \lim_{t\to 0+} f(t) &= \lim_{t\to 0+} \left[\left(\lambda + \frac{\xi}{\zeta} e^{\frac{t}{\zeta}}\right) \exp\left\{-\lambda t - \xi\left(e^{\frac{t}{\zeta}} - 1\right)\right\}\right] \\ &= \lambda + \frac{\xi}{\zeta} \\ &\neq 0 = \lim_{t\to 0-} f(t) \end{aligned} \tag{22}$$

the density function is not continuous.
The Esscher transformation (18) requires a density function defined on $\mathbb{R}$. Since in the Gompertz-Makeham model the derivation of ${}_tq_x$ does not exist in t equal to zero, the Esscher transform can not be applied. This also concerns the Gompertz model, because (22) is still non-zero even if λ equals zero .

Note that the RRR transform (8) is defined on the positive half-line and therefore this problem has been avoided.

3.5. Correlation bond - pricing formula

Second order shifts can be applied within the computation of the correlation bond's price of section 2.2. Since this financial instrument is provided by an insurance or reinsurance undertaking, the price needs to be adjusted by a certain risk margin. This assumption is realistic from an economical point of view, since a financial institution, selling this product, expects to make profit. Pricing this contract with its true value has no financial incentive. Furthermore, a company's business, that's only purpose is to sell these types of products, is going bankrupt with probability equal to one. Therefore, second order shifts or the RRR tranform are appropriate tools within pricing methods.

Consider the pricing formula (4) of section 2.2. Applying the second order shift and the fact, that in this model everything is assumed to be deterministic, the corresponding price is equal to

$$\tilde{P}_0^y = \frac{|\tilde{x}| \ \exp\left(-\int_0^T \mu_{s,\tilde{x}} \, \mathrm{d}s\right) + |y| \ \exp\left(-\int_0^T \mu_{s,y} \, \mathrm{d}s\right)}{|\tilde{x}| \ \exp\left(-\int_0^T \mu_{s,\tilde{x}} \, \mathrm{d}s\right)}$$

with

$$\exp\left(-\int_0^T \mu_{s,x} \, \mathrm{d}s\right) = \exp\left\{-\lambda T - \xi(x, m+k, \zeta)\left(e^{\frac{T}{\zeta}} - 1\right)\right\} \tag{23}$$

where $x \in \{\tilde{x}, y\}$.

Note that the formula consists of cardinalities of the initial cohorts weighted by the probability of surviving the next T years. Thus, the expected numbers of survivors.

A similar result can be achieved by using survival probabilities corresponding to formula (14).

The deterministic case might not be useful. Still the Gompertz-Makeham model is used in practice, therefore the introduced methods of producing a safety margin for such pricing formulas can be applied. Furthermore, the following stochastic mortality models will include a deterministic part. This deterministic function will describe the trend of the mortality rate. Results of this section can be applied, on the assumption that the mortality follows the Gompertz-Makeham law but has further stochastic components.

CHAPTER 4

Added Brownian motion

In the previous chapter a deterministic way of modelling the force of mortality and pricing the correlation bond of section 2.2 was introduced. This chapter focusses on the Brownian motion in order to add a stochastic movement to the model. This will happen in a very simple setting. Chapter 5 introduces a more complex model. Still, this first approach of including randomness requires some results from Stochastic Analysis like Nivikov's condition for stochastic exponentials. Furthermore, Ito's integration by parts formula will be stated, which will also find application in the following chapters. The source of randomness is reflected by a Brownian motion.

A *standard Brownian motion*, similar to Revuz et al. [RY99], is defined as a continuous stochastic process $(W_t)_{t\geq 0}$ with the following properties:

(1) $W_0 = 0$, almost sure
(2) the process is almost surely continuous
(3) W_t has independent increments
(4) the increments are normally distributed, i.e. $W_t - W_s \sim N(0, t-s)$ for $s \leq t$

The force of mortality shall consist of a deterministic part δ and a Brownian motion:

$$\mu_{s,x} = \delta(s,x) + W_s \tag{24}$$

For example, the deterministic function $\delta(s,x)$ can be equal to the force of mortality under the Gompertz-Makeham law.
Formula (4) of section 2.2 contains expressions of the form $\mathbb{E}\left[\exp\left(-\int_0^T \mu_{s,x}\,\mathrm{d}s\right)\right]$. In order to compute this expectation, several results of the theory on time continuous stochastic integration are necessary. For details see Revuz et al. [RY99], Kazamaki [Kaz94] and Øksendal [Øks03].

4.1. Stochastic exponential

Suppose $(M_t)_{t\geq 0}$ is a continuous local martingale, then the *stochastic exponential* is defined as the solution of

$$\mathrm{d}Y_t = Y_t\,\mathrm{d}M_t$$

with initial condition $Y_0 = 1$.
Applying Itō's formula with $f(x) = e^x$ provides the expression

$$\mathcal{E}(M)_t = \exp\left(M_t - \frac{1}{2}[M]_t\right)$$

The stochastic exponential is again a continuous local martingale, since it is defined as a stochastic integral. A sufficient condition to guarantee the martingale property is the *Novikov condition*:
If

$$\mathbb{E}\left[\exp\left(\frac{1}{2}[M]_t\right)\right] < \infty \qquad \text{for all } 0 \le t \le T \tag{25}$$

holds, then the stochastic exponential $(\mathcal{E}(M)_t)_{0\le t\le T}$ is a true martingale.

4.2. Integration by parts

Referring to Øksendal [Øks03], let $(X_t)_{t\ge 0}$ and $(Y_t)_{t\ge 0}$ be two continuous semimartingales. The *integration by parts formula* states

$$X_tY_t = X_0Y_0 + \int_0^t X_s\,\mathrm{d}Y_s + \int_0^t Y_s\,\mathrm{d}X_s + [X,Y]_t \tag{26}$$

4.3. Correlation bond - pricing formula

The problem of computing expressions of the form $\mathbb{E}\left[\exp\left(-\int_0^T \mu_{s,x}\,\mathrm{d}s\right)\right]$ can be solved by applying the upper results. Using (24) leads to

$$\begin{aligned}
\mathbb{E}\left[e^{-\int_0^T \mu_{s,x}\,\mathrm{d}s}\right] &= e^{-\int_0^T \delta(s,x)\,\mathrm{d}s}\mathbb{E}\left[e^{-\int_0^T W_s\,\mathrm{d}s}\right] \\
&= e^{-\int_0^T \delta(s,x)\,\mathrm{d}s}\mathbb{E}\left[e^{\int_0^T s\,\mathrm{d}W_s - T\cdot W_T}\right] \qquad (27)\\
&= e^{-\int_0^T \delta(s,x)\,\mathrm{d}s}\mathbb{E}\left[e^{\int_0^T (s-T)\,\mathrm{d}W_s}\right] \\
&= e^{-\int_0^T \delta(s,x)\,\mathrm{d}s}\mathbb{E}\left[e^{\int_0^T (s-T)\,\mathrm{d}W_s - \frac{1}{2}\int_0^T (s-T)^2\,\mathrm{d}s}\right] e^{\frac{1}{2}\int_0^T (s-T)^2\,\mathrm{d}s} \\
&= e^{\frac{T^3}{6} - \int_0^T \delta(s,x)\,\mathrm{d}s}\mathbb{E}\left[\mathcal{E}\left(\int_0^\bullet (s-T)\,\mathrm{d}W_s\right)_T\right] \\
&= \exp\left(\frac{T^3}{6} - \int_0^T \delta(s,x)\,\mathrm{d}s\right) \qquad (28)
\end{aligned}$$

The integration by parts formula (26) yields to equality (27). Note that the covariation term vanishes, since one argument has finite variation. For equation (28), the Novikov condition (25) has been used. This step needs to be executed carefully since T is part of the integrand as well as the upper bound.
Let $T \ge 0$ be fixed. Consider a process $(M_t)_{0\le t\le T}$ defined by

$$M_t := \int_0^t (s-T)\,\mathrm{d}W_s \qquad \text{for all } 0 \le t \le T$$

This process is a continuous martingale and by the *Itō isometry*, its variation process is given by

$$[M]_t = \int_0^t (s-T)^2 \, \mathrm{d}s = \frac{t^3}{3} - t^2 T + tT^2$$

Therefore, the Novikov condition (25) is fulfilled and $\left(\mathcal{E}\left(\int_0^\bullet (s-T)\,\mathrm{d}W_s\right)_t\right)_{0\le t\le T}$ is a martingale. This leads to a constant expectation and since M_0 equals zero

$$\mathbb{E}\left[\mathcal{E}\left(\int_0^\bullet (s-T)\,\mathrm{d}W_s\right)_t\right] = 1 \qquad \text{for all } 0 \le t \le T$$

especially for t equal to T.
Applying (28) to formula (4) of section 2.2 leads to

$$\begin{aligned}
\tilde{P}_0^y &= \frac{|\tilde{x}| \ \mathbb{E}\left[\exp\left(-\int_0^T \mu_{s,\tilde{x}}\,\mathrm{d}s\right)\right] + |y| \ \mathbb{E}\left[\exp\left(-\int_0^T \mu_{s,y}\,\mathrm{d}s\right)\right]}{|\tilde{x}| \ \mathbb{E}\left[\exp\left(-\int_0^T \mu_{s,\tilde{x}}\,\mathrm{d}s\right)\right]} \\
&= \frac{|\tilde{x}| \ \exp\left(\frac{T^3}{6} - \int_0^T \delta(s,\tilde{x})\,\mathrm{d}s\right) + |y| \ \exp\left(\frac{T^3}{6} - \int_0^T \delta(s,y)\,\mathrm{d}s\right) +}{|\tilde{x}| \ \exp\left(\frac{T^3}{6} - \int_0^T \delta(s,\tilde{x})\,\mathrm{d}s\right)}
\end{aligned}$$

The upper expression $\tilde{P}_0^y$ is just an example of a financial product. Important in this setting was the calculation of the occurring terms $\mathbb{E}\left[\exp\left(-\int_0^T \mu_{s,x}\,\mathrm{d}s\right)\right]$. This is the expected survival probability, which can be computed explicitly once the deterministic function δ is chosen.
It is necessary to require properties for δ in order to achieve these results, e.g. integrability such that (28) exists.

CHAPTER 5

Dependencies between cohorts

This chapter introduces the *Cox-Ingersoll-Ross-process*, which will be denoted by the short notation *CIR-process*. The model in the previous section does not include correlations between cohorts. The source of randomness, i.e. the Brownian motion, was indexed by t, therefore only dependencies between different points in time have been captured.

The following approach includes dependencies between cohorts. The idea simply consists of using two CIR-processes driven by independent Brownian motions. One process represents dependency between time, and the second one is capturing the dependency between age classes. For this purpose the set $\mathcal{I}$ of cohorts is going to be an interval of the form $[0, w)$, where w denotes the final age.

In anticipation of the final model, the application of CIR-processes for the force of mortality follows.

Consider the mortality modelled by:

$$\mu_{t,x} = \delta(t, x) + Y_t + Z_x \tag{29}$$

where $(Y_t)_{t\geq 0}$ and $(Z_x)_{x\in\mathcal{I}}$ are both shifted CIR-processes.

Details on the modelling follow below. First, the CIR-process and its properties will be introduced.

5.1. Cox-Ingersoll-Ross-process

For details on CIR-processes see Cox et al. [CIR85]. This section uses the notation similar to Deelstra et al. [DP]. Furthermore, the idea of a shifted CIR-process has been taken from Deelstra et al.

Definition 5.1. A Cox-Ingersoll-Ross-process $(Y_t^*)_{t\geq 0}$ follows the stochastic differential equation

$$\mathrm{d}Y_t^* = \kappa(\theta - Y_t^*)\,\mathrm{d}t + \sigma\sqrt{Y_t^*}\,\mathrm{d}W_t$$

with initial condition $Y_0^* = y_0^* > 0$ and $\kappa, \theta > 0$, σ being real parameters.

A basic property of the CIR-process is its mean reversion. Furthermore, the condition $2\kappa\theta \geq \sigma^2$ ensures that the process is strictly positive.

For the purpose of definition (29) the CIR-process shall be shifted by its mean θ, i.e. $Y_t := Y_t^* - \theta$. This leads to the stochastic differential equation

$$\mathrm{d}Y_t = -\kappa Y_t \,\mathrm{d}t + \sigma\sqrt{Y_t + \theta}\,\mathrm{d}W_t$$

Since the condition $2\kappa\theta \geq \sigma^2$ is taken for granted, the process $(Y_t^*)_{t\geq 0}$ is positive. Therefore, the process $(Y_t)_{t\geq 0}$ can be negative, but its lower bound is equal to $-\theta$ and its mean reverting level is zero. That means, if the process is negative the trend will be positive such that the process will get positive again. Once the process is positive, the trend-function will become negative in order to decrease the process. Therefore, the process will fluctuate around zero.
These are appropriate properties, since the CIR-processes in (29) are intended to reflect disturbance of the trend $\delta(t,x)$. The mean trend of this variation shall be equal to zero and since a negative force of mortality should be avoided, a lower bound for the shifted CIR-processes is suitable.
The solution of this SDE can be derived by applying the integration by parts formula (26):

$$\begin{aligned}
\mathrm{d}\left(e^{\kappa t}Y_t\right) &= e^{\kappa t}\,\mathrm{d}Y_t + \kappa e^{\kappa t}Y_t\,\mathrm{d}t + \underbrace{\mathrm{d}\left[e^{\kappa \bullet}, Y\right]_t}_{=0} \\
&= -\kappa e^{\kappa t}Y_t\,\mathrm{d}t + e^{\kappa t}\sigma\sqrt{Y_t+\theta}\,\mathrm{d}W_t + \kappa e^{\kappa t}Y_t\,\mathrm{d}t \\
&= e^{\kappa t}\sigma\sqrt{Y_t+\theta}\,\mathrm{d}W_t
\end{aligned}$$

which leads to

$$\begin{aligned}
e^{\kappa t}Y_t - Y_0 = \sigma\int_0^t e^{\kappa v}\sqrt{Y_v+\theta}\,\mathrm{d}W_v \qquad &\Longleftrightarrow \\
Y_t = e^{-\kappa t}y_0 + e^{-\kappa t}\sigma\int_0^t e^{\kappa v}\sqrt{Y_v+\theta}\,\mathrm{d}W_v &
\end{aligned}$$

For further considerations the covariance of this process is needed.
Let $s \leq t$,

$$\begin{aligned}
\mathrm{Cov}\left[Y_s\,, Y_t\right] &= \mathbb{E}\left[\left(Y_s - \mathbb{E}Y_s\right)\left(Y_t - \mathbb{E}Y_t\right)\right] \\
&= \mathbb{E}\left[\left(e^{-\kappa s}\sigma\int_0^s e^{\kappa u}\sqrt{Y_u+\theta}\,\mathrm{d}W_u\right)\left(e^{-\kappa t}\sigma\int_0^t e^{\kappa v}\sqrt{Y_v+\theta}\,\mathrm{d}W_v\right)\right] \\
&= \sigma^2 e^{-\kappa(s+t)}\mathbb{E}\left[\left(\int_0^s e^{\kappa u}\sqrt{Y_u+\theta}\,\mathrm{d}W_u\right)\left(\int_0^s e^{\kappa v}\sqrt{Y_v+\theta}\,\mathrm{d}W_v\right)\right. \\
&\quad \left. + \left(\int_0^s e^{\kappa u}\sqrt{Y_u+\theta}\,\mathrm{d}W_u\right)\left(\int_s^t e^{\kappa v}\sqrt{Y_v+\theta}\,\mathrm{d}W_v\right)\right]
\end{aligned}$$

By Itō-isometry this equation reduces to

$$\begin{aligned}\mathrm{Cov}\left[Y_s\,,Y_t\right] &= \sigma^2 e^{-\kappa(s+t)}\mathbb{E}\left[\int_0^s e^{2\kappa u}\left(Y_u+\theta\right)\,\mathrm{d}u\right] \\ &= \sigma^2 e^{-\kappa(s+t)}\int_0^s e^{2\kappa u}\left(\mathbb{E}Y_u+\theta\right)\,\mathrm{d}u \\ &= \sigma^2 e^{-\kappa(s+t)}\left[y_0\int_0^s e^{\kappa u}\,\mathrm{d}u+\theta\int_0^s e^{2\kappa u}\,\mathrm{d}u\right] \\ &= \frac{\sigma^2 e^{-\kappa(s+t)}}{2\kappa}\left[2y_0\left(e^{\kappa s}-1\right)+\theta\left(e^{2\kappa s}-1\right)\right] \end{aligned} \tag{30}$$

This expression will be used in the study of the correlation between cohorts using the force of mortality (29). The purpose of this section was an introduction to CIR-processes. Since the current mortality model (29) requires two CIR-processes, the parameter-notation will be adapted.

5.2. CIR-field-model

The introduction of this chapter briefly mentioned a mortality model using two CIR-processes. A rigorous definition follows.

Definition 5.2. For $s \in [0,\infty)$ and $x \in \mathcal{I}$ the force of mortality, called *CIR-field*, is given by

$$\mu_{t,x} = \delta(t,x) + Y_t + Z_x$$

where

- $\delta\colon \mathbb{R}^+ \times \mathcal{I} \to \mathbb{R}^+$ is a deterministic function
- $(Y_t)_{t\geq 0}$ is a shifted CIR process, following the SDE
$$\mathrm{d}Y_t = -\tilde{\kappa} Y_t\,\mathrm{d}t + \tilde{\sigma}\sqrt{Y_t+\tilde{\theta}}\,\mathrm{d}\tilde{W}_t$$
with initial condition $Y_0 = y_0$ and positive constants $\tilde{\kappa}, \tilde{\theta}$ and $\tilde{\sigma}$.
- $(Z_x)_{x\in\mathcal{I}}$ is a shifted CIR process, following the SDE
$$\mathrm{d}Z_x = -\kappa Z_x\,\mathrm{d}x + \sigma\sqrt{Z_x+\theta}\,\mathrm{d}W_x$$
with initial condition $Z_0 = z_0$ and positive constants κ, θ and σ.
- The appearing standard Brownian motions $(\tilde{W}_t)_{t\geq 0}$ and $(W_x)_{x\in\mathcal{I}}$ are assumed to be independent.

Remark: This model is named CIR-field-model, because it can be regarded as a stochastic field $(\mu_{t,x})_{(t,x)\in[0,\infty)\times\mathcal{I}}$.

What now has been achieved is a mortality consisting of a trend function with the ability of including disturbance over time and age classes. The function δ in definition 5.2 describes the mean trend of the mortality. For example the Gompertz-Makeham law of mortality might be a choice for δ, or the RRR transform can be chosen. However, the describing function, which the mortality should follow, can be

chosen separately. It is not necessary to choose a stochastic model and then fit the trend part in order to describe mortality.
In this model the stochastic disturbance, i.e. the deviation from the function δ, is modelled by Cox-Ingersoll-Ross processes. As described in the previous section, an attribute of CIR-processes is the possibility of being strictly positive. This finds application in term structure modelling. The modelling of mortality does not require a strict positive disturbance function, since also negative deviations are possible and need to be included. However, this property still can be used properly. Performing a certain shift of the CIR-process, allows to gain negative values. Note that these negative values correspond to the disturbance functions, not the final mortality process. Nevertheless, due to the non-negativity of the non-shifted process, the shifted one is bounded from below. This is an important property for mortality models in order to avoid negative values for the mortality process. For this purpose the shift levels θ and $\tilde{\theta}$ have to be chosen properly such that the deterministic function δ is able to catch the negative deviation. Otherwise a negative force of mortality is possible.
The shift-level is exactly the same as the mean trend of the non-shifted process. This leads to a new disturbing process, whose mean trend is zero, meaning that this process will develop around zero. Since the trend of the mortality is fully captured by the function δ, a mean trend equal to zero is appropriate. However, if a modeller wants to include a certain trend in the variation of the mortality function, simply a shift smaller or bigger than θ has to be performed. This might be the case if a more prudent modelling is required and the modeller is expecting a general deviation. Another way of including such prudent modelling assumptions has been discussed in section 3.2.
The model uses shifted CIR-processes for the deviation over time as well as for deviations over age classes. For this purpose independent Brownian motions have been used, because disturbance over time is assumed to be independent of the disturbance over age classes. Modelling the mortality with the capability of including correlations between different ages leads to the following requirement. From a natural point of view, the mortality of two ages with a small difference should be higher correlated than very different ones. This means, that the correlation between two age classes should be monotonically decreasing, as the age-difference becomes greater. The following section provides a detailed proof of this property.

5.3. Proof of decreasing correlation

A goal in modelling dependencies between age classes is the following. The mortality of people with similar ages should be stronger correlated compared to people with a great difference in their ages. That means, that the correlation between two persons with age x and $x + \Delta$ shall be a monotonically decreasing function in Δ.
This section proves, that the CIR-field-model fulfils this requirement. Some expressions can be very large and hence will be evaluated separately.

Definition 5.3. Let the correlation between $\mu_{s,x}$ and $\mu_{t,y}$ be denoted by

$$\rho\left((s,x),(t,y)\right) := \frac{\mathrm{Cov}\left[\mu_{s,x},\mu_{t,y}\right]}{\sqrt{\mathbb{V}\mu_{s,x}\mathbb{V}\mu_{t,y}}}$$

The requirement mentioned above means, in terms of this notation, that $\rho((t,x),(t,x+\Delta))$ is a monotonically decreasing function in Δ.

The following theorem delivers a condition with which this can be achieved. This condition is only necessary, since the framework does not want to exclude a starting value of the CIR-process different from zero. If the process starts in zero, the condition will always be fulfilled. Nevertheless, if the modeller prefers a different starting value, the following inequality has to be satisfied in order to achieve a decreasing correlation.

Theorem 5.1. *The correlation $\rho((t,x),(t,x+\Delta))$ is a decreasing function in Δ if either the initial value z_0 equals zero, or, in the case of $z_0 \neq 0$, if the condition*

$$\kappa < \frac{\ln\left(2+\theta/z_0\right)}{x+\Delta} \tag{31}$$

holds.

For the proof of this statement some groundwork is necessary.

First, an expression for the correlation in terms of variance and covariance of the CIR-processes will be derived. Therefore, the property of the covariance being bilinear will be used. Due to the independence of the Brownian motions the following transformations hold.

$$\begin{aligned}\rho((t,x),(t,x+\Delta)) &= \frac{\mathrm{Cov}\left[\delta(t,x)+Y_t+Z_x;\delta(t,x+\Delta)+Y_t+Z_{x+\Delta}\right]}{\left(\mathbb{V}\left(\delta(t,x)+Y_t+Z_x\right)\mathbb{V}\left(\delta(t,x+\Delta)+Y_t+Z_{x+\Delta}\right)\right)^{1/2}}\\ &= \frac{\mathbb{V}Y_t+\mathrm{Cov}[Z_x;Z_{x+\Delta}]}{\left((\mathbb{V}Y_t+\mathbb{V}Z_x)(\mathbb{V}Y_t+\mathbb{V}Z_{x+\Delta})\right)^{1/2}}\end{aligned} \tag{32}$$

In order to compute the deviation with respect to Δ, $\frac{\partial}{\partial\Delta}\mathrm{Cov}[Z_x;Z_{x+\Delta}]$ and $\frac{\partial}{\partial\Delta}\mathbb{V}Z_{x+\Delta}$ are needed. These expressions have already been calculated in the previous section and are given by formula (30). Adapting the parameters leads to:

$$\mathrm{Cov}[Z_x;Z_{x+\Delta}] = \frac{1}{2\kappa}\sigma^2 e^{-\kappa(2x+\Delta)}\left[2z_0(e^{\kappa x}-1)+\theta(e^{2\kappa x}-1)\right] \tag{33}$$

$$\mathbb{V}Z_{x+\Delta} = \frac{1}{2\kappa}\sigma^2 e^{-2\kappa(x+\Delta)}\left[2z_0(e^{\kappa(x+\Delta)}-1)+\theta(e^{2\kappa(x+\Delta)}-1)\right] \tag{34}$$

and therefore the deviations with respect to Δ can be computed. Straight forward calculations lead to:

$$\frac{\partial}{\partial\Delta}\mathrm{Cov}[Z_x; Z_{x+\Delta}] = -\frac{1}{2}\sigma^2 e^{-\kappa(2x+\Delta)}\left[2z_0(e^{\kappa x}-1)+\theta(e^{2\kappa x}-1)\right] \tag{35}$$

$$\begin{aligned}\frac{\partial}{\partial\Delta}\mathbb{V}Z_{x+\Delta} &= -\sigma^2 e^{-2\kappa(x+\Delta)}\left[2z_0(e^{\kappa(x+\Delta)}-1)+\theta(e^{2\kappa(x+\Delta)}-1)\right]\\ &\quad+\sigma^2 e^{-2\kappa(x+\Delta)}\left[z_0 e^{\kappa(x+\Delta)}+\theta e^{2\kappa(x+\Delta)}\right]\\ &= \sigma^2 e^{-2\kappa(x+\Delta)}\left[\theta - z_0 e^{\kappa(x+\Delta)}+2z_0\right]\end{aligned} \tag{36}$$

The final proof of theorem 5.1 includes a downward estimation provided by the corollary below. This inequality shall be separately verified.

Corollary 5.1.1. *The covariation* (33) *and variation* (34) *fulfil*

$$\mathrm{Cov}\left[Z_x; Z_{x+\Delta}\right] < \mathbb{V}Z_{x+\Delta}$$

and

$$\frac{\mathrm{Cov}\left[Z_x; Z_{x+\Delta}\right]}{\mathbb{V}Z_{x+\Delta}} < \frac{\mathbb{V}Y_t+\mathrm{Cov}\left[Z_x; Z_{x+\Delta}\right]}{\mathbb{V}Y_t+\mathbb{V}Z_{x+\Delta}} \tag{37}$$

PROOF. Since $\mathbb{V}Y_t$ and $\mathbb{V}Z_{x+\Delta}$ are greater than zero, the second statement (37) follows directly from the first inequality. This can be seen by the following.
Let $a, b, c \in \mathbb{R}$ and assume b and c are positive. Then the following transformations hold:

$$\begin{aligned}\frac{a}{b} &< \frac{a+c}{b+c} && \Longleftrightarrow\\ ab+ac &< ab+bc && \Longleftrightarrow\\ a &< b\end{aligned}$$

The proof of the first inequality will be accomplished by straightforward computation. Inserting formulas (33) and (34) into $\mathrm{Cov}[Z_x; Z_{x+\Delta}] < \mathbb{V}Z_{x+\Delta}$ leads to an equivalent inequality.

$$\begin{aligned}\mathrm{Cov}[Z_x; Z_{x+\Delta}] &< \mathbb{V}Z_{x+\Delta} && \Longleftrightarrow\\ 2z_0(e^{\kappa x}-1)+\theta(e^{2\kappa x}-1) &< e^{-\kappa\Delta}\left[2z_0(e^{\kappa(x+\Delta)}-1)+\theta(e^{2\kappa(x+\Delta)}-1)\right] && \Longleftrightarrow\\ 0 &< 2z_0(\underbrace{1-e^{-\kappa\Delta}}_{>0})+\theta(\underbrace{e^{\kappa(2x+\Delta)}-e^{2\kappa x}}_{>0}+\underbrace{1-e^{-\kappa\Delta}}_{>0})\end{aligned}$$

□

In order to apply this downward estimation, it is necessary that $\frac{\partial}{\partial\Delta}\mathbb{V}Z_{x+\Delta}$ is positive. This required positivity is provided by the following corollary.

Corollary 5.1.2. *Condition* (31) *is equivalent to* $\frac{\partial}{\partial\Delta}\mathbb{V}Z_{x+\Delta}$ *being positive.*

PROOF. $\frac{\partial}{\partial\Delta}\mathbb{V}Z_{x+\Delta}$ being positive is equal to

$$\begin{aligned} 0 &< \theta - z_0 e^{\kappa(x+\Delta)} + 2z_0 && \Longleftrightarrow \\ e^{\kappa(x+\Delta)} &< 2 + \theta/z_0 && \Longleftrightarrow \\ \kappa &< \frac{\ln(2+\theta/z_0)}{x+\Delta} \end{aligned}$$

□

Note that the first inequality in the upper proof holds, if $z_0 = 0$. Therefore $z_0 = 0$ is sufficient for $\frac{\partial}{\partial\Delta}\mathbb{V}Z_{x+\Delta}$ being positive.
Merging these results finally leads to the proof of theorem 5.1.

PROOF OF THEOREM 5.1. Equation (32) provides the following form of the correlation's derivation with respect to Δ

$$\begin{aligned} \frac{\partial}{\partial\Delta}\rho((t,x),(t,x+\Delta)) = D^{-1}\Big[& D^{1/2}\frac{\partial}{\partial\Delta}\mathrm{Cov}[Z_x;Z_{x+\Delta}] \\ & -\frac{1}{2}(\mathbb{V}Y_t + \mathrm{Cov}[Z_x;Z_{x+\Delta}])D^{-1/2}(\mathbb{V}Y_t+\mathbb{V}Z_x)\frac{\partial}{\partial\Delta}\mathbb{V}Z_{x+\Delta}\Big] \end{aligned}$$

where $D := (\mathbb{V}Y_t + \mathbb{V}Z_x)(\mathbb{V}Y_t + \mathbb{V}Z_{x+\Delta})$ denotes the squared denominator of the correlation.
The statement, that needs to be proved is

$$\frac{\partial}{\partial\Delta}\rho((t,x),(t,x+\Delta)) < 0$$

Multiplying this inequality by $D^{1/2}$ leads to

$$\begin{aligned} \frac{\partial}{\partial\Delta}\mathrm{Cov}[Z_x;Z_{x+\Delta}] &< \frac{1}{2}(\mathbb{V}Y_t + \mathrm{Cov}[Z_x;Z_{x+\Delta}])D^{-1}(\mathbb{V}Y_t+\mathbb{V}Z_x)\frac{\partial}{\partial\Delta}\mathbb{V}Z_{x+\Delta} \\ &< \frac{1}{2}(\mathbb{V}Y_t + \mathrm{Cov}[Z_x;Z_{x+\Delta}])(\mathbb{V}Y_t+\mathbb{V}Z_{x+\Delta})^{-1}\frac{\partial}{\partial\Delta}\mathbb{V}Z_{x+\Delta} \end{aligned}$$

Due to corollary 5.1.2, $\frac{\partial}{\partial\Delta}\mathbb{V}Z_{x+\Delta}$ is positive if either the condition (31) or $z_0 = 0$ is fulfilled. Therefore, applying corollary 5.1.1 is a downward estimation for the right hand side. This leads to a stronger inequality

$$\frac{\partial}{\partial\Delta}\mathrm{Cov}[Z_x;Z_{x+\Delta}] < \frac{1}{2}\mathrm{Cov}[Z_x;Z_{x+\Delta}](\mathbb{V}Z_{x+\Delta})^{-1}\frac{\partial}{\partial\Delta}\mathbb{V}Z_{x+\Delta}$$

which implies the upper expression.
Furthermore, inserting (33), (34), (35) and (36) leads to

$$\begin{aligned} \frac{\partial}{\partial\Delta}\mathrm{Cov}[Z_x;Z_{x+\Delta}] &< \frac{1}{2}\mathrm{Cov}[Z_x;Z_{x+\Delta}](\mathbb{V}Z_{x+\Delta})^{-1}\frac{\partial}{\partial\Delta}\mathbb{V}Z_{x+\Delta} \qquad \Longleftrightarrow \\ -\frac{1}{2}\left[2z_0(e^{\kappa x}-1)+\theta(e^{2\kappa x}-1)\right] &< \left[2z_0(e^{\kappa x}-1)+\theta(e^{2\kappa x}-1)\right] \\ &\quad \cdot e^{\kappa\Delta}\left[2z_0(e^{\kappa(x+\Delta)}-1)+\theta(e^{2\kappa(x+\Delta)}-1)\right]^{-1} \\ &\quad \cdot e^{-\kappa\Delta}\left[\theta - z_0 e^{\kappa(x+\Delta)} + 2z_0\right] \end{aligned}$$

which is equivalent to

$$
\begin{aligned}
-\frac{1}{2}\left[2z_0(e^{\kappa(x+\Delta)}-1)+\theta(e^{2\kappa(x+\Delta)}-1)\right] &< \theta - z_0 e^{\kappa(x+\Delta)} + 2z_0 &\iff\\
z_0 - \frac{\theta}{2}\left(e^{2\kappa(x+\Delta)}-1\right) &< \theta + 2z_0 &\iff\\
-z_0 - \frac{\theta}{2}\left(e^{2\kappa(x+\Delta)}+1\right) &< 0
\end{aligned}
$$

which obviously holds and finishes the proof.

□

What has been shown, is that under condition (31) the goal of decreasing correlations is achieved. This can also be guaranteed , if z_0 is chosen equal to zero, which is a proper assumption since there is no reason to start with a non-zero value of disturbance. However, theorem 5.1 has been proved on a more general setting in order to capture modelling preferences.

The most general setting for modelling mortality, which is provided in this paper, has been accomplished. The following chapter discusses several applications of this model in financial markets. Furthermore, the survival probability, which is necessary for the price of the correlation bond, will be calculated.

CHAPTER 6

Application in financial markets

The mortality model of the previous section allows to discuss certain financial products which can be computed now. Since the CIR-field model is capturing the correlation between time and age classes, new kinds of financial products are possible. This section will introduce simple products and their price calculation. Since CIR-processes are used, some known results from interest rate modelling will find application in this context.
Considering a zero coupon bond, which pays 1 at the maturity, and using continuous interest rates allows, that the expected survival probability can be seen as the price of a zero coupon bond. The *Faynman-Kac formula* will occur and the usage of an exponential affine approach will lead to *Riccati differential equations*. This allows to derive an explicit formula for the expected survival probability, which is highly important to price several products.

6.1. Correlation future

Assuming a financial institution, denoted by *party A*, wants to reduce its risk of an uncertain development of the correlation between two age classes. Party A would like to hold a derivative financial product, which trades their uncertain correlation of two ages with a predefined value, provided by *party B*.
This product equals a future contract, where the underlying stock is the correlation between these ages and the strike is a predefined value. Similar to basic derivative markets, the price of such an derivative is linked to the strike. Assuming that the price equals zero, the fair strike equals the correlation given by the model.
Formula (30) provides all necessary terms of the correlation $\rho((t,t),(x,y))$. For illustration purposes the starting value of the CIR-processes, describing disturbance over time and age classes, y_0 and z_0 are set equal to zero.
The pay-off of this future contract, expiring at maturity T, is

$$P(\mu_{T,x},\mu_{T,y})-K$$

where $P(\mu_{T,x},\mu_{T,y})$ denotes the correlation of $\mu_{t,x}$ and $\mu_{t,y}$ at time $t=T$, which is not known before, and K denotes the strike, i.e. the constant correlation, which will be traded against the floating leg.
As mentioned above, the fair strike, such that the contract has value zero at time

$t = 0$, is given by formula (32)

$$\rho((T,x),(T,y)) = \frac{\mathbb{V}Y_T + \mathrm{Cov}[Z_x; Z_y]}{((\mathbb{V}Y_T + \mathbb{V}Z_x)(\mathbb{V}Y_T + \mathbb{V}Z_y))^{1/2}}$$

The occurring expressions can be computed using equation (30):

$$\begin{aligned}
\mathbb{V}Y_T &= \frac{\tilde{\sigma}^2\tilde{\theta}\exp(-2\tilde{\kappa}T)}{2\tilde{\kappa}}\left(e^{2\tilde{\kappa}T} - 1\right) \\
&= \frac{\tilde{\sigma}^2\tilde{\theta}}{2\tilde{\kappa}}\left(1 - e^{-2\tilde{\kappa}T}\right) \\
\mathbb{V}Z_x &= \frac{\sigma^2\theta}{2\kappa}\left(1 - e^{-2\kappa x}\right) \\
\mathbb{V}Z_y &= \frac{\sigma^2\theta}{2\kappa}\left(1 - e^{-2\kappa y}\right) \\
\mathrm{Cov}[Z_x, Z_y] &= \frac{\sigma^2\theta\exp(-\kappa(x+y))}{2\kappa}\left(e^{2\kappa y} - 1\right)
\end{aligned}$$

This is not a highly exotic contract. However, this example shall illustrate that this model provides computational background for new kinds of correlation linked mortality derivatives. In general all well known derivatives like vanilla, barrier, look-back, American or Asian options can be defined using the force of mortality as underlying. Furthermore, the correlation between age classes is taken into account and offers new possible products.

Another example of combining well-known products with these new possibilities is the following.

6.2. Mortality swap and correlation bonds

Considering a mortality swap or, more precisely, a trade on the effective number of survivors at time T against a predefined number. Assuming a group consists of N individuals at time 0 and the whole group belongs to age class x.

The pay-off's expectation of this swap equals:

$$N \cdot \mathbb{E}\left[e^{-\int_0^T \mu_{s,x}\,\mathrm{d}s}\right] - N^*$$

where N^* denotes the predefined number of survivors, i.e. the fixed leg. Similar to the pricing formula (4) of section 2 an expression of the form $\mathbb{E}\left[\exp\left(-\int_0^T \mu_{s,x}\,\mathrm{d}s\right)\right]$ needs to be computed. Therefore, this section delivers the price of a mortality swap as well as the price for the product of section 2.

Due to

$$\mathbb{E}\left[e^{-\int_0^T \mu_{s,x}\,\mathrm{d}s}\right] = \underbrace{e^{-\int_0^T \delta(s,x)\,\mathrm{d}s}}_{①}\underbrace{\mathbb{E}\left[e^{-\int_0^T Y_s\,\mathrm{d}s}\right]}_{②}\underbrace{\mathbb{E}\left[e^{-TZ_x}\right]}_{③} \tag{38}$$

three parts fall into account. Expression ① depends on the chosen trend function δ. For example, in the case of the Gompertz-Makeham law of mortality, ① has already been computed in chapter 3 by equation (23) with k equal to zero.

Expression ③ is the moment generating function of the Cox-Ingersoll-Ross process. Since the distribution of this process (conditionally on its initial value) follows a

non-central chi-square distribution, its moment generating function can be derived and evaluated.
The calculation of the term ② requires results from term structure modelling, which occur since the CIR-process finds application in modelling term structures. The reason for this is its non-negativity and the possibility to derive explicit formulas for bond prices. These properties will be used in modelling mortality.
The following calculations are mainly based on [Fil09].

6.2.1. Calculation via bond prices.

In order to compute $\mathbb{E}\left[\exp\left(-\int_0^T Y_s\,\mathrm{d}s\right)\right]$ some calculations are required.
Let $T > 0$ be fixed and $t \leq T$. Define

$$V(t, Y_t) := \mathbb{E}\left[e^{-\int_t^T Y_s\,\mathrm{d}s} | \mathcal{F}_t\right]$$

where the filtration $(\mathcal{F}_t)_{t\geq 0}$ is generated by a Brownian motion $(\tilde{W}_t)_{t\geq 0}$.
Consider

$$e^{-\int_0^t Y_s\,\mathrm{d}s} V(t, Y_t) = \mathbb{E}\left[e^{-\int_0^T Y_s\,\mathrm{d}s} | \mathcal{F}_t\right] \tag{39}$$

which obviously holds because $e^{-\int_0^t Y_s\,\mathrm{d}s} V(t, Y_t)$ is $\mathcal{F}_t$ - measurably. Furthermore, due to

$$\mathbb{E}\left[e^{-\int_0^T Y_s\,\mathrm{d}s}\right] \leq e^{\tilde{\theta}T} < \infty \tag{40}$$

the process

$$\left(e^{-\int_0^t Y_s\,\mathrm{d}s} V(t, Y_t)\right)_{t\geq 0}$$

is integrable and by (39) it is a martingale.
The following notation will be used:

$$\begin{aligned} V &:= V(t, Y_t) \\ V^{(t)} &:= \frac{\partial}{\partial t} V(t, Y_t) \\ V^{(y)} &:= \frac{\partial}{\partial y} V(t, Y_t) \\ V^{(yy)} &:= \frac{\partial^2}{\partial y^2} V(t, Y_t) \end{aligned}$$

Assume V is continuously differentiable, i.e. $V \in C^{1,2}$. Applying the integration by parts and the Itō formula leads to

$$\begin{aligned} \mathrm{d}\left(e^{-\int_0^t Y_s \,\mathrm{d}s} V(t,Y_t)\right) &= e^{-\int_0^t Y_s V \,\mathrm{d}s}\left(-Y_t \,\mathrm{d}t + \mathrm{d}V(t,Y_t)\right) \\ &= e^{-\int_0^t Y_s \,\mathrm{d}s}\left(-Y_t V \,\mathrm{d}t + V^{(t)}\,\mathrm{d}t + V^{(y)}\mathrm{d}Y_t + \frac{1}{2}V^{(yy)}\mathrm{d}[Y]_t\right) \\ &= e^{-\int_0^t Y_s \,\mathrm{d}s}\left(\tilde{\sigma}\sqrt{Y_t+\tilde{\theta}}V^{(y)}\,\mathrm{d}\tilde{W}_t\right. \\ &\qquad \left. + \left(-Y_t V + V^{(t)} - \tilde{\kappa} Y_t V^{(y)} + \frac{1}{2}\tilde{\sigma}^2 (Y_t+\tilde{\theta})V^{(yy)}\right)\mathrm{d}t\right) \end{aligned}$$

Since $\left(\exp\left(-\int_0^t Y_s \,\mathrm{d}s\right) V(t,Y_t)\right)_{t\geq 0}$ is a martingale, the dt term in the upper expression has to be equal to zero, which leads to the *Feynman-Kac formula*:

$$-Y_t V + V^{(t)} - \tilde{\kappa} Y_t V^{(y)} + \frac{1}{2}\tilde{\sigma}^2 (Y_t+\tilde{\theta})V^{(yy)} = 0 \tag{41}$$

The next step relies on the theory of affine term structures. Assume V has some exponential affine structure, i.e.

$$V(t,y) = \exp\left(-A(t,T) - B(t,T)y\right)$$

Therefore, the partial derivations are

$$\begin{aligned} V^{(t)} &= \big(-\underbrace{\frac{\partial}{\partial t}A(t,T)}_{=:A^{(t)}(t,T)} - \underbrace{\frac{\partial}{\partial t}B(t,T)}_{=:B^{(t)}(t,T)}\, y\big)V \\ V^{(y)} &= -B(t,T)V \\ V^{(yy)} &= B(t,T)^2 V \end{aligned}$$

Thus, equation (41) delivers

$$V\left(-y - A^{(t)}(t,T) - B^{(t)}(t,T)y + \tilde{\kappa}B(t,T)y + \frac{1}{2}\tilde{\sigma}^2(y+\tilde{\theta})B(t,T)^2\right) = 0$$

Applying affine matching, i.e. comparing terms of y and y^0 leads to the system of differential equations:

$$\begin{aligned} B^{(t)}(t,T) &= \frac{1}{2}\tilde{\sigma}^2 B(t,T)^2 + \tilde{\kappa}B(t,T) - 1 \qquad (42) \\ B(T,T) &= 0 \\ A^{(t)}(t,T) &= \frac{1}{2}\tilde{\sigma}^2\tilde{\theta}B(t,T)^2 \qquad (43) \\ A(T,T) &= 0 \end{aligned}$$

These partial differential equations are called *Riccati differential equations* and can be explicitly solved. The solution is given, for example by adapting the result from [Fil09]:

Theorem 6.1. *The solution of* (42) *is given by*

$$B(t,T) = \frac{2\left(e^{\gamma(T-t)}-1\right)}{(\gamma+\tilde{\kappa})\left(e^{\gamma(T-t)}-1\right)+2\gamma} \tag{44}$$

where $\gamma := \sqrt{\tilde{\kappa}^2 + 2\tilde{\sigma}^2}$.

The formal derivation of the Riccati differential equations is not focus of this thesis. Nevertheless, the solution shall be verified.

Proof. In order to proof the statement, the explicit solution (44) will be inserted in its differential equation (42). For this purpose let

$$N := (\gamma+\tilde{\kappa})\left(e^{\gamma(T-t)}-1\right)+2\gamma$$

denote the denominator of $B(t,T)$. The left hand side of (42), i.e. the partial derivation with respect to t, follows

$$\begin{aligned} B^{(t)}(t,T) &= \frac{\partial}{\partial t}\left(\frac{2\left(e^{\gamma(T-t)}-1\right)}{(\gamma+\tilde{\kappa})\left(e^{\gamma(T-t)}-1\right)+2\gamma}\right) \\ &= \frac{1}{N^2}\Big(-2\gamma e^{\gamma(T-t)}\left((\gamma+\tilde{\kappa})\left(e^{\gamma(T-t)}-1\right)+2\gamma\right) \\ &\qquad + 2\gamma(\gamma+\tilde{\kappa})e^{\gamma(T-t)}\left(e^{\gamma(T-t)}-1\right)\Big) \\ &= \frac{-4\gamma^2 e^{\gamma(T-t)}}{N^2} \end{aligned}$$

On the other hand, the right hand side of (42) follows

$$\begin{aligned} \frac{1}{2}\tilde{\sigma}^2 B(t,T)^2 + &\tilde{\kappa}B(t,T) - 1 = \\ &= \frac{1}{N^2}\left(2\tilde{\sigma}^2\left(e^{\gamma(T-t)}-1\right)^2 + 2\tilde{\kappa}N\left(e^{\gamma(T-t)}-1\right) - N^2\right) \\ &= \frac{1}{N^2}\Bigg(\left(e^{\gamma(T-t)}-1\right)^2 \underbrace{\left(2\tilde{\sigma}^2 + 2\tilde{\kappa}(\gamma+\tilde{\kappa}) - (\gamma+\tilde{\kappa})^2\right)}_{=\,2\tilde{\sigma}^2+(\gamma+\tilde{\kappa})(\tilde{\kappa}-\gamma)\,=\,2\tilde{\sigma}^2+\tilde{\kappa}^2-\gamma^2\,=\,0} \\ &\qquad + 4\tilde{\kappa}\gamma\left(e^{\gamma(T-t)}-1\right) - 4\gamma(\gamma+\tilde{\kappa})\left(e^{\gamma(T-t)}-1\right) - 4\gamma^2\Bigg) \\ &= \frac{-4\gamma^2\left(e^{\gamma(T-t)}-1\right) - 4\gamma^2}{N^2} \\ &= \frac{-4\gamma^2 e^{\gamma(T-t)}}{N^2} \end{aligned}$$

Furthermore, the terminal condition $B(T,T) = 0$ is fulfilled and therefore (44) is a solution of (42). □

In the next step the solution of equation (43) $A(t,T)$ is derived. For some Riccati differential equations straight forward integration of (43) is possible. In this case, due to the term $B(t,T)^2$, this might be more difficult. The following expression for

$A(t,T)$ has been derived by the adapted solution in the case of a non-shifted CIR processes. To proof the correctness, this solution is verified.

Theorem 6.2. *The solution of* (43) *is given by*

$$A(t,T) = -\frac{2\tilde{\theta}\tilde{\kappa}}{\tilde{\sigma}^2} \ln\left(\frac{2\gamma e^{(\gamma+\tilde{\kappa})(T-t)/2}}{N}\right) + \tilde{\theta}B(t,T) - \tilde{\theta}(T-t) \tag{45}$$

where again $N = (\gamma+\tilde{\kappa})\left(e^{\gamma(T-t)} - 1\right) + 2\gamma$ *denotes the denominator of* $B(t,T)$, $\gamma := \sqrt{\tilde{\kappa}^2 + 2\tilde{\sigma}^2}$ *and* $B(t,T)$ *is given by* (44).

Proof. In order to verify, that (45) is a solution of (43) the upper expression is inserted into the differential equation. Using the differential equation (42), which is satisfied for $B(t,T)$, equation (43) equals:

$$\begin{aligned} A^{(t)}(t,T) &= -\frac{2\tilde{\theta}\tilde{\kappa}}{\tilde{\sigma}^2}\frac{\partial}{\partial t}\left(\ln(2\gamma) + (\gamma+\tilde{\kappa})\frac{T-t}{2} - \ln(N)\right) + \tilde{\theta}B^{(t)}(t,T) + \tilde{\theta} \\ &= -\frac{2\tilde{\theta}\tilde{\kappa}}{\tilde{\sigma}^2}\left(-\frac{1}{2}(\gamma+\tilde{\kappa}) - \frac{1}{N}\frac{\partial}{\partial t}N\right) + \underbrace{\frac{1}{2}\tilde{\sigma}^2\tilde{\theta}B(t,T)^2}_{=A^{(t)}(t,T)} + \tilde{\theta}\tilde{\kappa}B(t,T) \end{aligned}$$

Therefore, it is sufficient to show

$$\begin{aligned} \tilde{\theta}\tilde{\kappa}B(t,T) &= \frac{2\tilde{\theta}\tilde{\kappa}}{\tilde{\sigma}^2}\left(-\frac{1}{2}(\gamma+\tilde{\kappa}) - \frac{1}{N}\frac{\partial}{\partial t}N\right) \\ &= \frac{\tilde{\theta}\tilde{\kappa}}{\tilde{\sigma}^2 N}\left(-N(\gamma+\tilde{\kappa}) + 2\gamma(\gamma+\tilde{\kappa})e^{\gamma(T-t)}\right) \\ &= \frac{\tilde{\theta}\tilde{\kappa}(\gamma+\tilde{\kappa})}{\tilde{\sigma}^2 N}\left(-(\gamma+\tilde{\kappa})\left(e^{\gamma(T-t)} - 1\right) - 2\gamma + 2\gamma e^{\gamma(T-t)}\right) \\ &= \frac{\tilde{\theta}\tilde{\kappa}}{\tilde{\sigma}^2 N}\underbrace{(\gamma+\tilde{\kappa})(\gamma-\tilde{\kappa})}_{=2\tilde{\sigma}^2}\left(e^{\gamma(T-t)} - 1\right) \\ &= \tilde{\theta}\tilde{\kappa}\frac{2\left(e^{\gamma(T-t)} - 1\right)}{N} \\ &= \tilde{\theta}\tilde{\kappa}B(t,T) \end{aligned}$$

Furthermore, the terminal condition $A(T,T) = 0$ is fulfilled, which completes this proof. □

The upper calculations are very basic in the context of term structure modelling. Applying a very common process of this theory, i.e. the CIR-process, enables to use these results for the model.

As a final step in the calculation of (38), term ③, i.e. the moment generating function of the Cox-Ingersoll-Ross process, has to be derived.

6.2.2. Calculation via the non-central chi-square distribution.

The last step in pricing mortality swaps, mentioned at the beginning of this section, consists in calculating the moment generating function of the CIR-process. Thus, ③ in expression (38) can be computed, which is

$$\mathbb{E}\left[e^{-TZ_x}\right] \tag{46}$$

The distribution of a CIR process conditioned on its initial value follows a non-central chi-square distribution. The following shall briefly prove this statement and derive the proper parameters of the non-central chi-square distribution. The derivation of the distribution of the CIR-process is adapted from [Cai04].

First, the *non-central chi-square distribution* is introduced. This definition is taken from [MO07], where the notation has been adapted to this setting.
Let d denote the degrees of freedom and $X_1, ..., X_d$ independent normally distributed random variables with mean μ_i and variance 1. Define

$$R_t := \sum_{i=1}^{d} X_i(t)^2$$

R_t has a non-central chi-square distribution with d degrees of freedom and *non-centrality parameter* $\lambda = \sum_{i=1}^{d} \mu_i^2$.

Remark: It can be shown, that the distribution only depends on the squared sum λ and not on the particular μ_i. Therefore, the distribution is given by two parameters, the degrees of freedom d and by the non-centrality parameter λ.

Before the explicit representation of (46) can be computed, some prior calculations are required. Remember that Z denotes the shifted CIR-process. Therefore, the basic CIR-process $\tilde{Z}$ will be studied. This process is given by the differential equation

$$\mathrm{d}\tilde{Z}_x = \kappa\left(\theta - \tilde{Z}_x\right)\mathrm{d}x + \sigma\sqrt{\tilde{Z}_x}\,\mathrm{d}W_x \tag{47}$$

with initial value $\tilde{Z}_0 = \tilde{z}_0$.

Theorem 6.3. *Assume that $\frac{4\theta\kappa}{\sigma^2}$ is an integer. The distribution of*

$$\frac{4\kappa}{\sigma^2}\frac{1}{1-e^{-\kappa x}}\tilde{Z}_x$$

conditionally on its initial value follows a non-central chi-square distribution with parameters

$$d = \frac{4\theta\kappa}{\sigma^2} \qquad \text{degrees of freedom}$$
$$\lambda = \frac{4\kappa}{\sigma^2}\frac{e^{-\kappa x}}{1-e^{-\kappa x}}\tilde{z}_0 \qquad \text{non-centrality parameter}$$

PROOF. Let d be an integer and consider $X_1, ..., X_d$ where for each $i \in \{1, ..., d\}$ X_i is given by the differential equation

$$\mathrm{d}X_i(t) = -\frac{1}{2}\kappa X_i(t)\,\mathrm{d}t + \sqrt{\kappa}\,\mathrm{d}W_i(t)$$

with initial value $X_i(0) = x_i(0)$ and W_i being independent Brownian motions.
Note that this process is a special case of an *Ornstein Uhlenbeck process.* For further information see Ornstein et al. [OU]. This differential equation can be solved explicitly by the integration by parts formula introduced in chapter 4. Consider

$$\begin{aligned}\mathrm{d}\left(e^{\frac{1}{2}\kappa t}X_i(t)\right) &= e^{\frac{1}{2}\kappa t}\left(\frac{1}{2}\kappa X_i(t)\,\mathrm{d}t + \mathrm{d}X_i(t)\right)\\ &= e^{\frac{1}{2}\kappa t}\sqrt{\kappa}\,\mathrm{d}W_i(t)\end{aligned}$$

Therefore

$$\begin{aligned}e^{\frac{1}{2}\kappa t}X_i(t) - X_i(0) &= \int_0^t e^{\frac{1}{2}\kappa s}\sqrt{\kappa}\,\mathrm{d}W_i(s) \qquad \Longleftrightarrow\\ X_i(t) &= e^{-\frac{1}{2}\kappa t}x_i(0) + \int_0^t e^{-\frac{1}{2}\kappa(t-s)}\sqrt{\kappa}\,\mathrm{d}W_i(s)\end{aligned}$$

Observe that $X_i(t)$ is normally distributed with mean

$$\mathbb{E}\left[X_i(t)\right] = e^{-\frac{1}{2}\kappa t}x_i(0)$$

The variance can be computed using the Itō-isometry

$$\begin{aligned}\mathbb{V}\left[X_i(t)\right] &= \mathbb{E}\left[\left(X_i(t) - \mathbb{E}\left[X_i(t)\right]\right)^2\right]\\ &= \mathbb{E}\left[\left(\int_0^t e^{-\frac{1}{2}\kappa(t-s)}\sqrt{\kappa}\,\mathrm{d}W_i(s)\right)^2\right]\\ &= \mathbb{E}\left[\int_0^t e^{-\kappa(t-s)}\kappa\,\mathrm{d}s\right] = \int_0^t e^{-\kappa(t-s)}\kappa\,\mathrm{d}s\\ &= 1 - e^{-\kappa t}\end{aligned}$$

Note that all X_i have the same variance.
Let R be defined by

$$R_t := \sum_{i=1}^d X_i(t)^2$$

Therefore, $\frac{R_t}{1-e^{-\kappa t}}$ is non-central chi-square distributed with d degrees of freedom and non-centrality parameter

$$\lambda = \frac{1}{1-e^{-\kappa t}} \sum_{i=1}^{d} e^{-\kappa t} x_i^2(0) = \frac{e^{-\kappa t}}{1-e^{-\kappa t}} R(0)$$

Furthermore, applying Itō's formula leads to the differential equation for R:

$$\begin{aligned} \mathrm{d}R_t &= 2\sum_{i=1}^{d} X_i(t)\,\mathrm{d}X_i(t) + \sum_{i=1}^{d} \mathrm{d}\left[X_i\right]_t \\ &= -\sum_{i=1}^{d} \kappa X_i^2(t)\,\mathrm{d}t + 2\sum_{i=1}^{d} \sqrt{\kappa} X_i(t)\,\mathrm{d}W_i(t) + d\kappa\,\mathrm{d}t \\ &= \kappa\left(d - R_t\right)\,\mathrm{d}t + 2\sqrt{\kappa}\sum_{i=1}^{d} X_i(t)\,\mathrm{d}W_i(t) \end{aligned}$$

A new one-dimensional Brownian motion, defined by

$$\mathrm{d}W(t) := \frac{1}{\sqrt{\sum_{i=1}^{d} X_i^2(t)}} \sum_{i=1}^{d} X_i(t)\,\mathrm{d}W_i(t)$$

leads to

$$\mathrm{d}R_t = \kappa\left(d - R_t\right)\,\mathrm{d}t + 2\sqrt{\kappa}\sqrt{R_t}\,\mathrm{d}W(t)$$

Remark: Note that R is a weak solution of the upper differential equation.

Replacing the index t by x and defining $\tilde{Z}_x := \frac{\sigma^2}{4\kappa} R_x$ provides the differential equation for $\tilde{Z}$:

$$\begin{aligned} \mathrm{d}Z_x &= \kappa \frac{\sigma^2}{4\kappa}\left(d - R_x\right)\,\mathrm{d}x + \underbrace{\frac{\sigma^2}{2\sqrt{\kappa}}\sqrt{R_x}}_{=\sigma\sqrt{\frac{\sigma^2}{4\kappa}R_x}}\,\mathrm{d}W(x) \\ &= \kappa\left(\frac{d\sigma^2}{4\kappa} - \tilde{Z}_x\right)\,\mathrm{d}x + \sigma\sqrt{\tilde{Z}_x}\,\mathrm{d}W(x) \end{aligned}$$

This differential equation is equal to equation (47) if

$$\frac{d\sigma^2}{4\kappa} = \theta \qquad \Longleftrightarrow$$

$$d = \frac{4\theta\kappa}{\sigma^2}$$

holds.

Furthermore, the non-centrality parameter λ equals

$$\lambda = \frac{e^{-\kappa x}}{1-e^{-\kappa x}} R(0) = \frac{4\kappa}{\sigma^2} \frac{e^{-\kappa x}}{1-e^{-\kappa x}} \tilde{z}_0$$

which completes the proof. □

Remark: The only additional restriction given by this theorem is the requirement of $d = 4\theta\kappa/\sigma^2$ being an integer. However, the definition of the non-central chi-square distribution can be extended to non-integer values of d. Therefore, it is not necessary, that $4\theta\kappa/\sigma^2$ is an integer.

Remark: Remember that the condition $2\kappa\theta \geq \sigma^2$ is taken for granted in order to receive a positive non-shifted CIR-process. This leads to the circumstance that the degrees of freedom are greater or equal to two.

Returning to the main task of this subsection, it is now possible to calculate $\mathbb{E}\left[e^{-TZ_x}\right]$. This can be accomplished by changing to the non-shifted process and extend it in such a way that the upper theorem can be applied.

$$\begin{aligned}\mathbb{E}\left[e^{-TZ_x}\right] &= e^{\theta T}\mathbb{E}\left[e^{-T\tilde{Z}_x}\right]\\ &= e^{\theta T}\mathbb{E}\Bigg[\exp\Bigg(-T\frac{\sigma^2}{4\kappa}\left(1-e^{-\kappa x}\right)\underbrace{\left(\frac{4\kappa}{\sigma^2}\frac{1}{1-e^{-\kappa x}}\tilde{Z}_x\right)}_{\sim\chi^2(\lambda,d)}\Bigg)\Bigg]\\ &= e^{\theta T}M\left(-T\frac{\sigma^2}{4\kappa}\left(1-e^{-\kappa x}\right)\right)\end{aligned}$$

where $M(\cdot)$ denotes the moment generating function of the non-central chi-square distribution with parameters λ and d. Referring to Ravishanker et al. [RD01] the function is given by

$$M(t) = (1-2t)^{-\frac{d}{2}}\exp\left(\frac{\lambda t}{1-2t}\right)$$

which is well defined for $t \leq 2$. This condition is not violated in this framework since the argument of the moment generating function is always negative. Putting all results together finally leads to an explicit expression for $\mathbb{E}\left[e^{-TZ_x}\right]$.

Therefore, it is now possible to compute the expected probability of survival (38). A closed expression is given in the following section.

6.2.3. The probability of survival in the CIR-field.

The results of the last sections enable to state a closed formula of the probability of survival (38). Summing up these results leads to

$$\begin{aligned}\mathbb{E}\left[e^{-\int_0^T \mu_{s,x}\,\mathrm{d}s}\right] &= e^{-\int_0^T \delta(s,x)\,\mathrm{d}s}\mathbb{E}\left[e^{-\int_0^T Y_s\,\mathrm{d}s}\right]\mathbb{E}\left[e^{-TZ_x}\right] \\ &= \exp\Bigg(-\int_0^T \delta(s,x)\,\mathrm{d}s - A(0,T) - B(0,T)y_0 \\ &\qquad +\theta T + \frac{-\lambda T\frac{\sigma^2}{4\kappa}\left(1-e^{-\kappa x}\right)}{1+2T\frac{\sigma^2}{4\kappa}\left(1-e^{-\kappa x}\right)}\Bigg)\left(1+2T\frac{\sigma^2}{4\kappa}\left(1-e^{-\kappa x}\right)\right)^{-d/2}\end{aligned}$$

with

$$\begin{aligned}\gamma &= \sqrt{\tilde{\kappa}^2+2\tilde{\sigma}^2} \\ N &= (\gamma+\tilde{\kappa})\left(e^{\gamma(T-t)}-1\right)+2\gamma \\ B(t,T) &= \frac{2\left(e^{\gamma(T-t)}-1\right)}{(\gamma+\tilde{\kappa})\left(e^{\gamma(T-t)}-1\right)+2\gamma} \\ A(t,T) &= -\frac{2\tilde{\theta}\tilde{\kappa}}{\tilde{\sigma}^2}\ln\left(\frac{2\gamma e^{(\gamma+\tilde{\kappa})(T-t)/2}}{N}\right)+\tilde{\theta}B(t,T)-\tilde{\theta}(T-t) \\ d &= \frac{4\theta\kappa}{\sigma^2} \\ \lambda &= \frac{4\kappa}{\sigma^2}\frac{e^{-\kappa x}}{1-e^{-\kappa x}}\tilde{z}_0\end{aligned}$$

One term depends on the predefined deterministic function δ which describes the mortality without taking into account some disturbances over time and age classes. For example, the Gompertz-Makeham model with second order shift is a possible choice for δ. The expression $\exp\left(-\int_0^T \mu_{s,x}\,\mathrm{d}s\right)$ is provided by equation (23).
The second term has been calculated using results similar to bond prices related to term structure models. The third term can be transformed in order to receive the moment generating function of a non-central chi-square distribution. Additionally, the needed parameters have been derived.
The examples of mortality derivatives require the expected probability of survival to state the price of the product. Multiplied by the number of individuals of a cohort, who are alive at time $t = 0$ delivers the expected number of survivors at time T. The possibility to receive an explicit form of such expressions allows to work with some mortality derivatives and is necessary for a theory that wants to provide a sophisticated setting for pricing such derivatives.

CHAPTER 7

Conclusion

In this thesis the goal of developing a sophisticated framework for pricing mortality derivatives has been accomplished. The final model, namely the *CIR-field*, for the force of mortality allows to include correlations between time and age classes. Moreover, resulting properties of this model are suitable for practical use such as a bounded disturbance from the trend function of the mortality in order to avoid a negative mortality. The correlation itself is a decreasing function with respect to the difference between the ages, which also reflects the reality and makes this model more applicable for practical use. Additionally, the model allows to derive explicit expressions for some mortality derivatives such as mortality swaps. A financial product, which can be used to hedge the correlation between the mortality of different cohorts, has been introduced. The model provides a closed pricing formula for this product, which has been derived rigorously.
Due to the relation to term structures and the moment generating function of the non-central chi squared distribution the model allows application and provides closed pricing formulas for further derivatives.
In addition to this stochastic model, the Gompertz-Makeham model has been discussed. An alternative of second order shift, the *RRR transform*, has been introduced and evaluated. These transformations lead to a more prudent consideration of the mortality and might be obligatory due to regulatory requirements. Furthermore, the stochastic CIR-field includes a deterministic function, for which the transformed Gompertz-Makeham model can be chosen.

Further research could be a study on the occurring parameters of the CIR-field. Detailed historical data could allow to fit this model to observed fluctuations on the mortality. Moreover, numerical examples for prices under this model could be evaluated. A study on the sensitivity of the price of a certain financial product with respect to parameters might lead to further interesting properties of this model.
Another topic of interest is the application of this model to new kind of derivatives which are linked to the correlation of age classes. The new way of modelling the force of mortality might lead to financial products which have not been considered yet.

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APPENDIX A

Details on the computations

The following computation summarises some calculations from previous chapters.

$$\begin{aligned}
&\int_0^t e^{-hs}\left(\lambda+\frac{\xi}{\zeta}e^{\frac{s}{\zeta}}\right)\exp\left\{-\lambda s-\xi\left(e^{\frac{s}{\zeta}}-1\right)\right\}\,\mathrm{d}s\\
&=e^{\xi}\int_0^t\left(\lambda+\frac{\xi}{\zeta}e^{\frac{s}{\zeta}}\right)\exp\left\{-s\left(\lambda+h\right)-\xi e^{\frac{s}{\zeta}}\right\}\,\mathrm{d}s=\left|\begin{matrix}\xi e^{\frac{s}{\zeta}}=u\\ \mathrm{d}s=\zeta u^{-1}\,\mathrm{d}u\end{matrix}\right|\\
&=e^{\xi}\zeta\int_{\xi}^{\xi e^{t/\zeta}}\left(\lambda+\frac{u}{\zeta}\right)\exp\left\{-\left(\lambda+h\right)\zeta\ln\left(\frac{u}{\xi}\right)-u\right\}u^{-1}\,\mathrm{d}u\\
&=e^{\xi}\zeta\xi^{\zeta(\lambda+h)}\int_{\xi}^{\xi e^{t/\zeta}}\left(\lambda+\frac{u}{\zeta}\right)u^{-\zeta(\lambda+h)-1}e^{-u}\,\mathrm{d}u\\
&=e^{\xi}\xi^{\zeta(\lambda+h)}\left[\lambda\zeta\Gamma\left(-\zeta\left(\lambda+h\right),\xi,\xi e^{\frac{t}{\zeta}}\right)+\Gamma\left(-\zeta\left(\lambda+h\right)+1,\xi,\xi e^{\frac{t}{\zeta}}\right)\right]
\end{aligned}$$

Since the incomplete gamma function appears, the following notation has been used:

$$\Gamma\left(a,x,y\right):=\int_x^y u^{a-1}e^{-u}\,\mathrm{d}u$$

$$\Gamma\left(a,x\right):=\int_x^\infty u^{a-1}e^{-u}\,\mathrm{d}u$$

with $y\geq x>0$. Note that a might be negative. However, this will not cause a problem for the incomplete gamma function with $x>0$.

This computation has been used in previous calculations. For example, choosing λ equal to zero leads to the Gompertz model. Additional, t approaching infinity will deliver the corresponding result for the calculation of the Laplace transform in equation (16). The final RRR transformed probability in the Gompertz model, see equation (17), is provided by the upper expression (with $\lambda=0$) divided by the Laplace transform, i.e.:

$$\begin{aligned}
{}_tq_x(h)&=\frac{e^{\xi}\xi^{\zeta h}\Gamma\left(-\zeta h+1,\xi,\xi e^{\frac{t}{\zeta}}\right)}{e^{\xi}\xi^{\zeta h}\Gamma\left(-\zeta h+1,\xi\right)}\\
&=\frac{\Gamma\left(-\zeta h+1,\xi,\xi e^{\frac{t}{\zeta}}\right)}{\Gamma\left(-\zeta h+1,\xi\right)}
\end{aligned}$$

APPENDIX B

Graphics

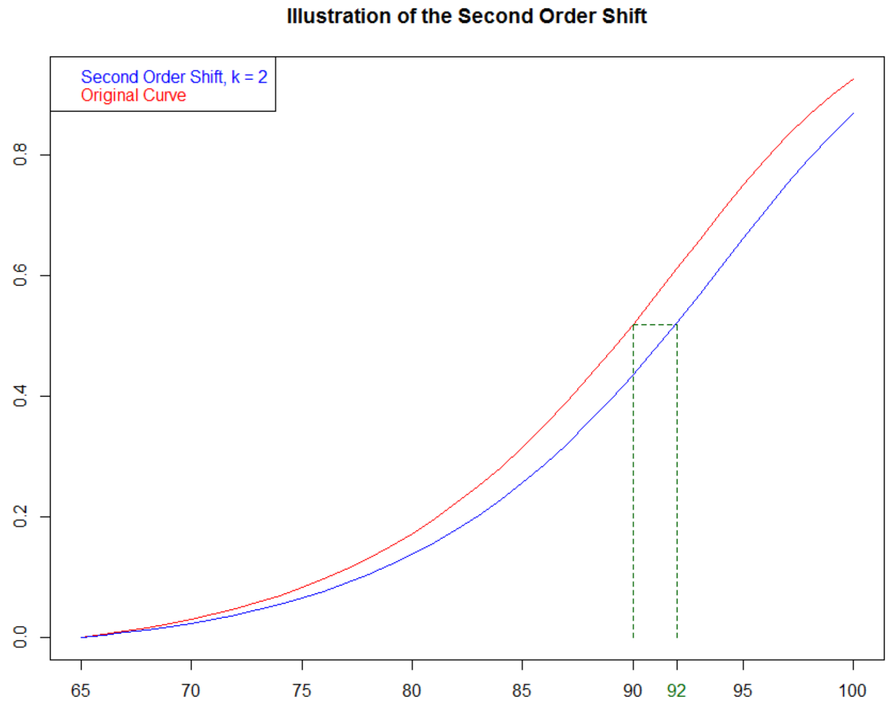

FIGURE 1. Probabilities of a person with age 65 to die within year t for $t \in \{0, ..., 100 - 65\}$, using Gompertz law of mortality with $m = 92.16$ and $\zeta = 8.11$ for a female population, compared to the second order shift of the same curve.

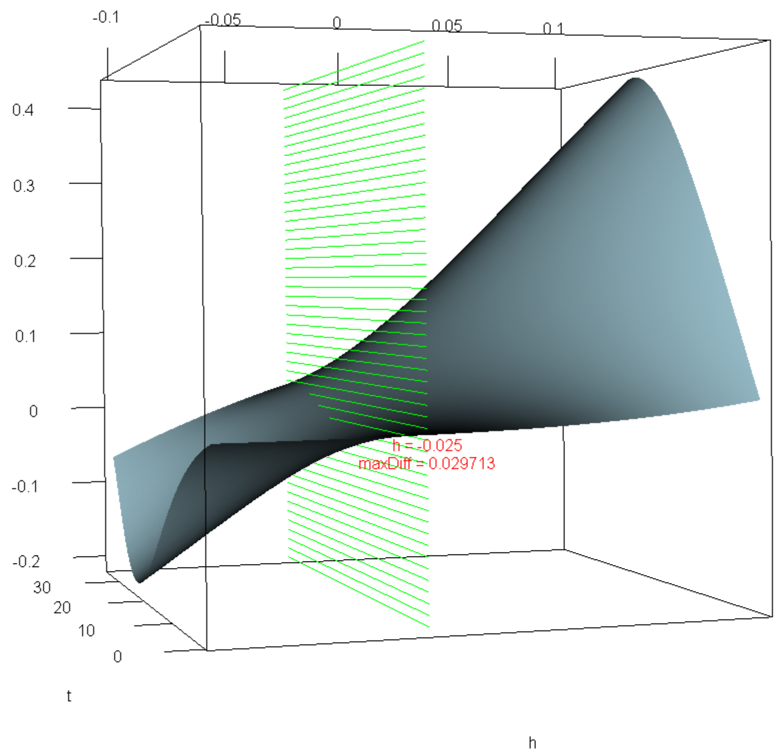

FIGURE 2. The difference of the second order probabilities and the RRR transformed probabilities using Gompertz law of mortality with age $x = 65$, $m = 92.16$ and $\zeta = 8.11$ for a female population. The x-axis represents the values of h and the y-axis the time $t \in \{0, ..., 100 - 65\}$. Furthermore, the green plain describes the level for h where the least square differences are achieved. This level, as well as the absolute value of the greatest difference over the time, for this particular h are displayed in the graphic.

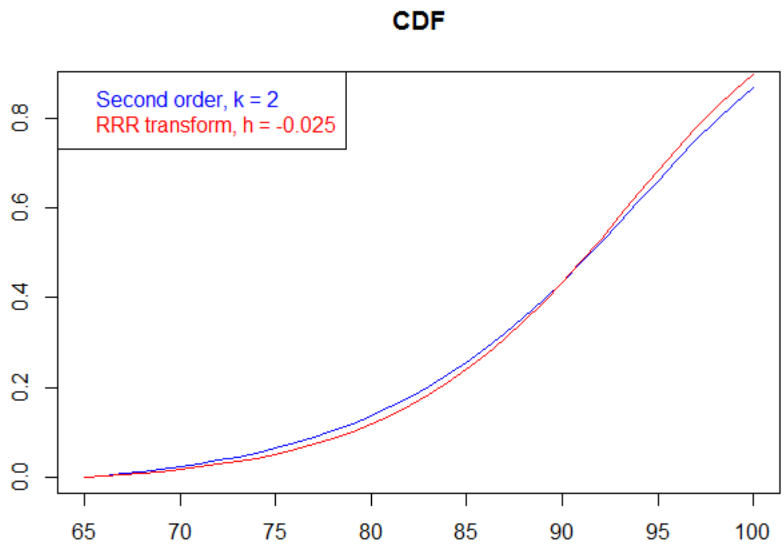

FIGURE 3. Probabilities of a person with age 65 to die within year t for $t \in \{0, ..., 100 - 65\}$, using Gompertz law of mortality with $m = 92.16$ and $\zeta = 8.11$ for a female population.

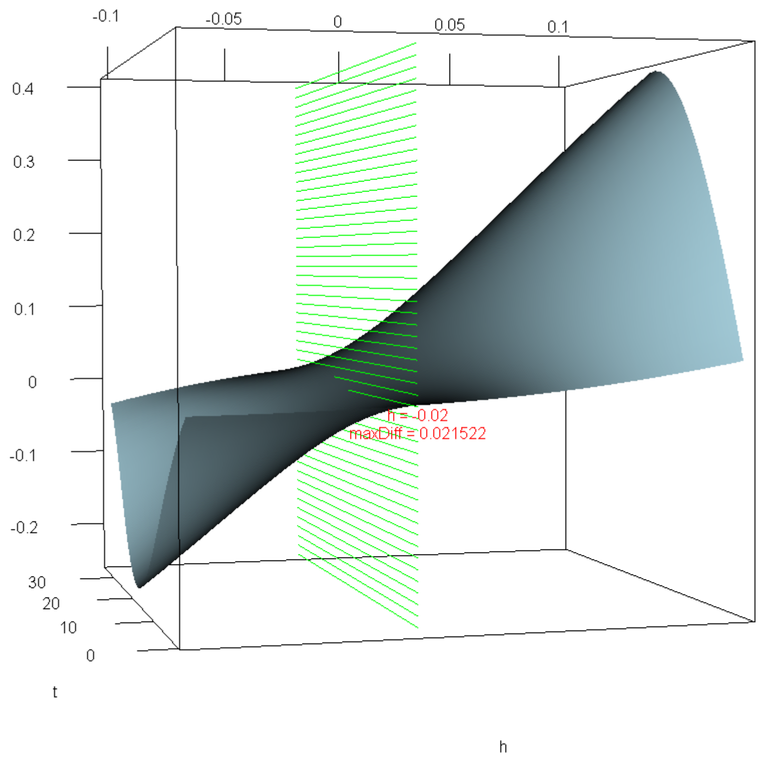

FIGURE 4. The difference of the second order probabilities and the RRR transformed probabilities using Gompertz law of mortality with age $x = 65$, $m = 86.37$ and $\zeta = 9.83$ for a male population. The x-axis represents the values of h and the y-axis the time $t \in \{0, ..., 100 - 65\}$. Furthermore, the green plain describes the level for h where the least square differences are achieved. This level, as well as the absolute value of the greatest difference over the time, for this particular h are displayed in the graphic.

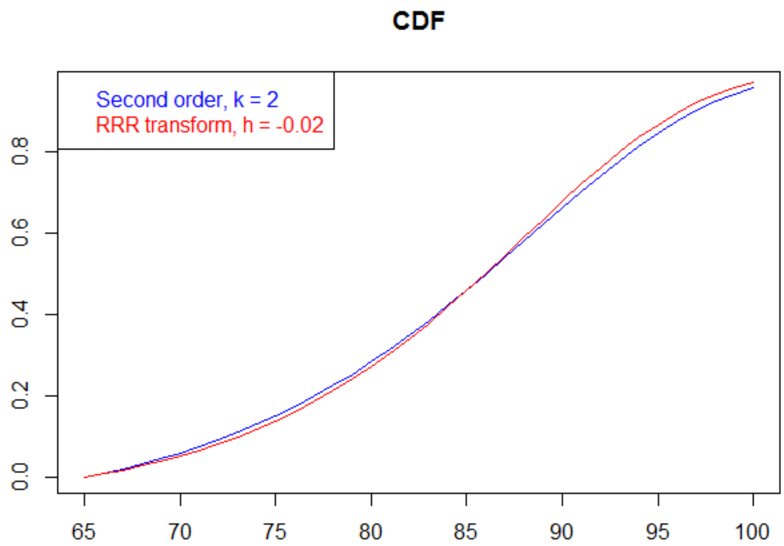

Figure 5. Probabilities of a person with age 65 to die within year t for $t \in \{0, ..., 100-65\}$, using Gompertz law of mortality with $m = 86.37$ and $\zeta = 9.83$ for a male population.

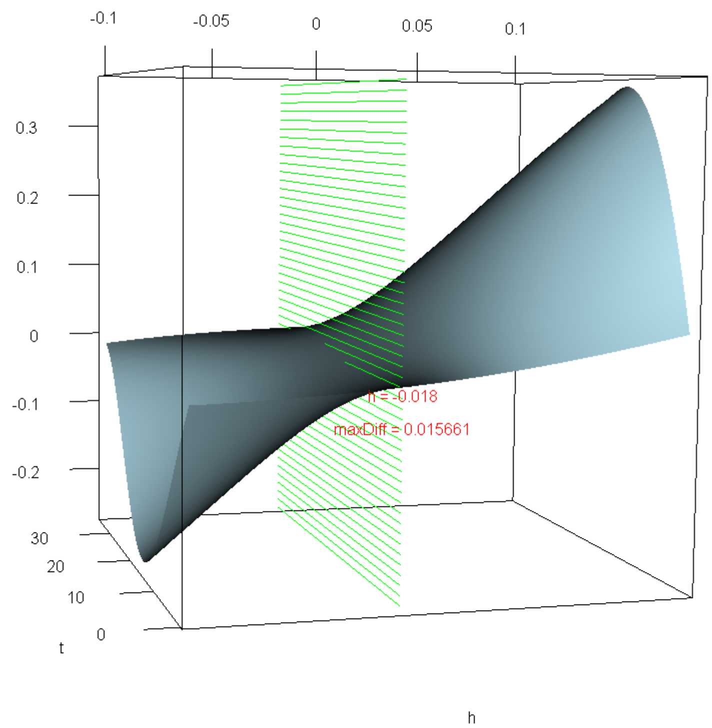

FIGURE 6. The difference of the second order probabilities and the RRR transformed probabilities using Gompertz-Makeham law of mortality with age $x = 65$, $m = 80.58$, $\zeta = 11.42$ and $\lambda = 5 * 10^{-4}$ for a male population. The x-axis represents the values of h and the y-axis the time $t \in \{0, ..., 100 - 65\}$. Furthermore, the green plain describes the level for h where the least square differences are achieved. This level, as well as the absolute value of the greatest difference over the time, for this particular h are displayed in the graphic.

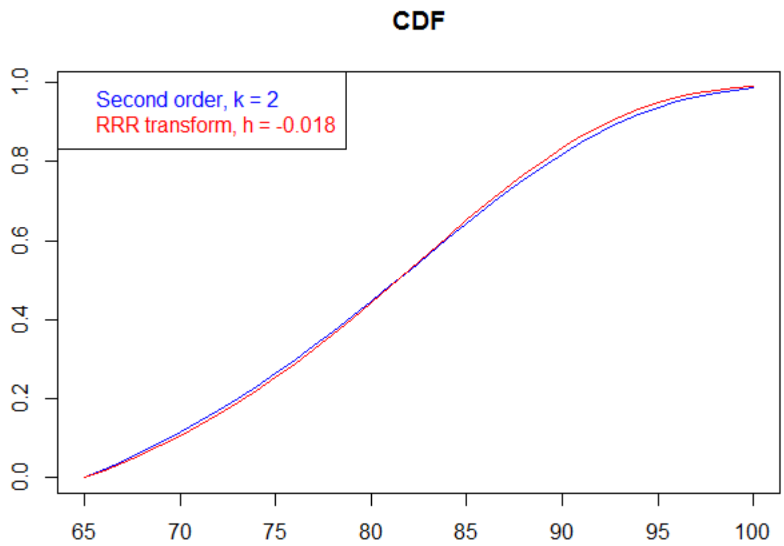

FIGURE 7. Probabilities of a person with age 65 to die within year t for $t \in \{0, ..., 100 - 65\}$, using Gompertz-Makeham law of mortality with $m = 80.58$, $\zeta = 11.42$ and $\lambda = 5 \cdot 10^{-4}$ for a male population.

APPENDIX C

Description of *R*-codes

This section introduces the used codes in order to compute the graphics of appendix B and therefore the results of the examples of section 3.3. For this purpose the software *R*, see [R C13], has been used.

```
1  GcdfPlot <- function(x,m,zeta,lam,k,h,p){
2    xi<-exp((x-m)/zeta)
3    t<-seq(0,100-x,1)
4    q1<- 1-exp(- lam*t-xi*exp((-k)/zeta)*(exp(t/zeta)-1))
5    q2<-(lam*zeta*(gamma_inc(-zeta*(lam+h), xi) -gamma_inc(-zeta*(lam+h)
         ,xi*exp(t/zeta)))+(gamma_inc(-zeta*(lam+h)+1, xi) -gamma_inc(-
         zeta*(lam+h)+1, xi*exp(t/zeta))))/(lam*zeta*gamma_inc(-zeta*(lam
         +h), xi)+gamma_inc(-zeta*(lam+h)+1, xi))
6    if (p==1){
7      plot(seq(x,100,1),q1,type="l",col="blue",xlab="",ylab="")
8      lines(seq(x,100,1),q2,type="l",col="red")
9      legend("topleft",c(paste("Second␣order,␣k␣=",k), paste("RRR␣
           transform,␣h␣=",h)),text.col=c("blue","red"))
10     title("CDF")}}
```

Listing C.1. *R*-code for generating figures 3, 5 and 7.

The upper code computes the second order probabilities of section 3.1 and the RRR transform defined in section 3.2. Furthermore, the code creates a plot of these two probability functions, which has been applied to create figures 3, 5 and 7.
The input parameters are the current age x, the model parameters for the Gompertz-Makeham model m, ζ and λ. The parameter k denotes the applied shift for the second order probabilities and h is needed for the RRR transform, see definition (8). The parameter p has been used for further studies and is not important for this setting.
In line 2 ξ is calculated which will simplify the further calculation and has also been used in the corresponding formulas of chapter 3. The vector t, which is declared in line 3, denotes the remaining years. Note that the calculations and graphics are generated till the person's age reaches 100. In line 4 the second order probabilities are computed. This code-line corresponds to formula (7). On the other hand the RRR transform according to formula (14) is implied in line 5. In this formula the generalised incomplete gamma function, see (15) appears. In order to imply this function, the upper incomplete gamma function has been used. That is why the implementation of this formula contains differences, e.g.

`gamma_inc(-zeta*(lam+h), xi) - gamma_inc(-zeta*(lam+h),xi*exp(t/zeta))`

which equals

$$\begin{aligned}\Gamma\Big(-\zeta(\lambda+h),\xi\Big)-\Gamma\left(-\zeta(\lambda+h),\xi e^{\frac{t}{\zeta}}\right) &= \int_{\xi}^{\infty} u^{-\zeta(\lambda+h)-1}e^{-u}\,\mathrm{d}u \\ &\quad -\int_{\xi e^{t/\zeta}}^{\infty} u^{-\zeta(\lambda+h)-1}e^{-u}\,\mathrm{d}u \\ &= \int_{\xi}^{\xi e^{t/\zeta}} u^{-\zeta(\lambda+h)-1}e^{-u}\,\mathrm{d}u\end{aligned}$$

The further code generates the plot, which includes the computed probabilities for the remaining years.

```
ApproxHdiff <- function(x,m,zeta,k,lam) {
  xi<-exp((x-m)/zeta)
  h<-seq(-0.1,0.1,0.001)
  t<-seq(0,100-x,1)
  transf<-matrix(0,nrow=length(h),ncol=length(t))
  ls<-matrix(0,nrow=length(h),1)
  SecOrder<-matrix(0,nrow=length(h),ncol=length(t))

  for (i in 1:length(t)) {temp<-(lam*zeta*(gamma_inc(-zeta*(lam+h), xi) -gamma_inc(-zeta*(lam+h),xi*exp(i/zeta)))+(gamma_inc(-zeta*(lam+h)+1, xi) -gamma_inc(-zeta*(lam+h)+1, xi*exp(i/zeta))))/(lam*zeta*gamma_inc(-zeta*(lam+h), xi)+gamma_inc(-zeta*(lam+h)+1, xi))
                           transf[,i]<- temp
                           SecOrder[,i]<- 1-exp(- lam*i-xi*exp((-k)/zeta)*(exp(i/zeta)-1)) }
  uv<-max(transf-SecOrder)
  lv<-min(transf-SecOrder)

  persp3d(h,t,transf-SecOrder,col = "lightblue",zlab="")->res
  for (i in 1:length(h)) {ls[i]<-sum((transf[i,]-SecOrder[i,])^2) }

  h_min<-h[which(ls==min(ls))]
  text3d(h_min,0,-0.01,c(paste("h =",h_min)),col="red")
  text3d(h[which(ls==min(ls))],0,-0.05,c(paste("maxDiff =",round(max(abs(transf[which(h==h_min),]-SecOrder[which(h==h_min),])),6))),col="red")

  for (i in seq(lv,uv,(uv-lv)/50)){lines3d(h_min,t,i,pmat=res,col="green")}
  return(h_min) }
```

LISTING C.2. *R*-code for generating

The aim of the upper code is to calculate the parameter h of the RRR transform (8) in order to fit the second order probabilities (7). This will be accomplished by least square fitting. This particular h is the return of the code. Furthermore, a 3D-plot will be generated which shows the difference between the second order probabilities and the RRR transform, i.e. the error using the RRR transform. Additionally the level h of the least square difference is highlighted. The code has been used within the examples of section 3.3. Moreover, figures 2, 4 and 6 have been generated by this code.

The input parameters are the persons age x and the model parameters for the Gompertz-Makeham model m, ζ and λ. Furthermore, k describes the shift for the second order probabilities (7). Line 2 computes ξ which has been used in the formulas of chapter 3 and only serves as simplification. The vector h in line 3 describes the range of values which will be tested in order to fit the RRR transform to the second order probabilities. For the examples in this paper the occurring range was sufficient. Otherwise the range has to be adapted.

Line 4 computes the vector t of calculated years. The resulting graphic as well as the considered years for the least square fit are only going to include the years till the person's age reaches 100. The lines 5 to 7 are generating empty matrices which will be used below. The matrix in line 5 will contain the RRR transformed probabilities (14), where the dimensions of the matrix are the lengths of the vectors h and t. Thus, the code computes for each value in h the transformed probabilities. The matrix in line 6, which is only a vector, will be used to save the squared differences between the second order probabilities and the RRR transform in order to evaluate the least square difference. Analogous to the RRR transform, line 7 declares an empty matrix for the second order probabilities.

The loop in line 9 computes the RRR transform for each value in t. The occurring formula (14) is similar implied as in code C.1 and explained by an example. In the next line the computed vector is added to the matrix. In line 11 the corresponding vector of second order probabilities is calculated and saved. The values calculated in lines 12 and 13 will be needed for the graphic below. These values describe the range of the z-axis. In line 15 the basic graphic is generated. The x-axis is the vector h, i.e. the range of tested parameters for the RRR transform. The y-axis is the vector t of considered years and the z-axis is the difference between the RRR transform and the second order probabilities. In order to receive the least squared difference, line 16 computes the added up differences for each value of h and line 18 evaluates their minimum. The next two lines are inserting some text to the graphic, i.e. the values of the particular h where the least squared difference can be achieved and the least squared difference at this level. Furthermore, line 22 adds a surface to the graphic, which is indicated by lines. This surface shall highlight the level of the least squared difference and therefore makes the goodness of fit by using the RRR transformed more visible. Finally, this value for h is the only output of this code, which will be returned in line 23.