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Dissertation

Consistency of Option Prices under Bid-Ask Spreads and Implied Volatility Slope Asymptotics

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Kurzfassung der Dissertation

Im ersten Teil leiten wir zunächst eine Verallgemeinerung des Strassen Theorems her. Dieses Theorem besagt, dass es zu jedem stochastischen Prozess, dessen Randverteilungen in konvexer Ordnung wachsen, ein Martingal mit denselben Randverteilungen gibt. Solche Prozesse bzw. Folgen von Verteilungen werden in der Literatur üblicherweise als Peacock bezeichnet. Wir wollen dieses Resultat erweitern: anstatt Martingalen mit fest vorgegebenen Randverteilungen zu betrachten, wollen wir wissen, unter welchen Bedingungen es Martingale gibt, deren Randverteilungen eine vorgegebene Distanz zu gegebenen Verteilungen nicht überschreiten. Entfernungen werden mit Metriken auf dem Raum der Wahrscheinlichkeitsmaße mit endlichem Erwartungswert gemessen. In unserem Hauptresultat ist die Metrik die Unendlich-Wasserstein-Distanz. Wir werden notwendige und hinreichende Bedingungen für die Existenz von Peacocks in vorgegebener Unendlich-Wasserstein-Distanz formulieren und beweisen. Wir betrachten dabei zunächst abzählbare und danach überabzählbare Indexmengen. Anschließend befassen wir uns noch mit dem gleichen Problem für die Stop-Loss-Distanz, die Lévy-Distanz und die Prokhorov-Distanz.

Die Resultate bezüglich der Unendlich-Wasserstein-Distanz haben eine finanzmathematische Anwendung. Angenommen, wir können die Kauf- und Verkaufspreise europäischer Call-Optionen auf ein Underlying beobachten. Diese Call-Optionen unterscheiden sich nur hinsichtlich des Ausübungspreises und des Ausübungszeitpunktes. Wir versuchen folgende Frage zu beantworten: Unter welchen Voraussetzungen gibt es ein mathematisches, arbitragefreies Modell eines Finanzmarktes, welches diese Preise erzeugt? Anders als in der bisherigen Literatur, wollen wir dabei auch Modelle berücksichtigen, in denen der zukünftige Bid-Ask-Spread auf das Underlying positive Werte annehmen kann. Wir werden beweisen, dass es ausreicht, Konsistenzbedinguen für jeden Ausübungszeitpunkt einzeln anzugeben, wenn dieser Bid-Ask-Spread unbeschränkt ist.

Im Weiteren fokussieren wir uns daher auf Modelle, in denen der Bid-Ask Spread durch eine vorgegebene Konstante beschränkt ist. Wir werden notwendige und hinreichende Bedingungen für die Existenz geeigneter Modelle formulieren und beweisen. Außerdem geben wir Arbitragestrategien an, für den Fall, dass die notwendigen Bedingungen nicht erfüllt sind. Wir unterscheiden dabei zwischen modellunabhängigen Arbitragestrategien und sogenannten schwachen Arbitragestrategien, die nur von den Nullmengen des ausgewählten Modells abhängen. Wir geben eine vollständige Lösung für einzelne Laufzeiten an und einige Teillösungen für mehrere Laufzeiten. Die theoretischen Resultate des ersten Teils dieser Dissertation wurden bereits eingereicht ([47]) und ein Paper über die Anwendungen ist in Arbeit ([46]).

Im zweiten Teil dieser Dissertation untersuchen wir die erste Ableitung nach dem Strike

der impliziten Volatilität in exponentiellen Lévy Modellen. Genauer interessieren wir uns für die Asymptotik der at-the-money Steigung der impliziten Volatilität für kurze Laufzeiten. Zunächst stellen wir einen Zusammenhang zwischen dieser Asymptotik und dem Preis einer zugehörigen Digitaloption her. Im Hauptresultat betrachten wir dann Modelle mit unendlicher Aktivität, die auch eine Brown'sche Komponente haben. Als technisches Hilfsmittel verwenden wir die Mellin-Transformation und leiten damit eine asymptotische Reihenentwicklung für die gesuchte Steigung her. Als Nebenprodukt bekommen wir außerdem eine asymptotische Reihenentwicklung für den at-the-money Preis von Digitaloptionen mit kurzer Laufzeit. Letztendlich besprechen wir noch den Zusammenhang der hergeleiteten Asymptotik mit der gesamten Form der impliziten Volatilität mit Hilfe von Lees Momenten Formel. Wir zeigen anhand einiger Modelle, dass die at-the-money Steigung der impliziten Volatilität im Zusammenhang mit den Enden selbiger steht. Die Resultate des zweiten Teils stehen auch in [48] (under revision) zur Verfügung.

Letztendlich beschäftigen wir uns noch mit einem Thema aus dem Gebiet der Portfolio-Optimierung. Wir analysieren dabei Strategien, bei denen der Investor das Portfolio nur dann umschichtet, wenn der Unterschied zwischen dem aktuellen Anteil des riskanten Assets – gemessen am Gesamtvermögen – und des Merton-Anteils zu groß wird. Wir beschränken uns dabei auf das Black-Scholes-Modell ohne Transaktionskosten und leiten eine asymptotische Darstellung der Wachstumsrate her.

Abstract

The first part of this thesis deals with a generalisation of Strassen's theorem and its applications to check option prices for consistency in markets with positive bid-ask spreads. Strassen's theorem asserts that a stochastic process is increasing in convex order if and only if there is a martingale with the same marginal distributions. Such processes, or families of measures, are nowadays known as peacocks. We extend this classical result in a novel direction, relaxing the requirement on the martingale. Instead of equal marginal laws, we just require them to be within closed balls, defined by some metric on the space of probability measures. In our main result, the metric is the infinity Wasserstein distance. Existence of a peacock within a prescribed distance is reduced to a countable collection of rather explicit conditions. We also solve this problem when the underlying metric is the stop-loss distance, the Lévy distance and the Prokhorov distance.

The result for the infinity Wasserstein distance has a financial application, as it allows to check European call option quotes for consistency. To be more precise, given a set of European call option prices with different maturities and strikes on one underlying, we want to know when there is a model which is consistent with these prices. In contrast to previous studies, we allow models where the underlying trades at a bid-ask spread. The main question then is how large (in terms of a deterministic bound) this spread must be to explain the given prices. We fully solve this problem in the case of a single maturity, and give several partial results for multiple maturities.

We will prove that in case the bid-ask spread is not bounded there is no interplay between the current price of the underlying and and the option prices. Therefore we focus on models where the bid-ask spread is bounded by a predefined constant. We fully solve this problem in the case of a single maturity, and give several partial results for multiple maturities.

The theoretical results of this part of the thesis are already submitted ([47]) and there is a working paper about the financial applications ([46]).

In the second part of this thesis we will derive asymptotics for the at-the-money strike derivative of implied volatility in Lévy models as maturity tends to zero. Our main results quantify the behavior of the slope for infinite activity exponential Lévy models including a Brownian component. As auxiliary results, we obtain asymptotic expansions of short maturity at-the-money digital call options, using Mellin transform asymptotics. Finally, we discuss when the at-the-money slope is consistent with the steepness of the smile wings, as given by Lee's moment formula. The results of the second part can be found in [48] (under revision).

Finally, we briefly deal with a topic from portfolio optimisation. In the classical Black-Scholes framework without transaction costs we will analyse the following trading strategy: the investor leaves her portfolio unchanged until the almost surely finite time when the distance between the risky fraction of the portfolio and the Merton proportion exceeds a certain threshold β. The portfolio is then rebalanced such that the new risky fraction is equal to the Merton proportion. This evolution is repeated indefinitely. We derive an asymptotic expansion of the growth rate for small β.

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Son olarak, sevgili annem ve babam benden sevginizi ve desteğinizi hiç esirgemediğiniz, bu günlere gelmemde bana yardımcı olduğunuz için çok teşekkür ederim. Sizi çok seviyorum.

Contents

I.

A Variant of Strassen's Theorem with an Application to the Consistency of Option Prices

1

Introduction and problem formulation

A celebrated result, first proved by Strassen in 1965,[1] states that, for a given sequence of probability measures $(\mu_n)_{n\in\mathbb{N}}$, there exists a martingale $M = (M_n)_{n\in\mathbb{N}}$ such that the law of M_n is μ_n for all n, if and only if all μ_n have finite mean and $(\mu_n)_{n\in\mathbb{N}}$ is increasing in convex order (see Definition 2.1). Such sequences, and their continuous time counterparts, are nowadays referred to as peacocks, a pun on the French acronym PCOC, for "Processus Croissant pour l'Ordre Convexe" [51]. For further references on Strassen's theorem and its predecessors, see the appendix of [21], p.380 of Dellacherie and Meyer [27], and [5].

The theorem gave rise to plenty of generalisations, one of the most famous being Kellerer's theorem [58, 59]. It states that, for a peacock $(\mu_t)_{t\geq 0}$ with index set $\mathbb{R}^+$, there is a *Markov* martingale $M = (M_t)_{t\geq 0}$ such that $M_t \sim \mu_t$ for all $t \geq 0$. Several proofs and ramifications of Kellerer's theorem can be found in the literature. Hirsch and Roynette [52] construct martingales as solutions of stochastic differential equations and use an approximation argument. Lowther [67, 68] shows that under some regularity assumptions there exists an ACD martingale with marginals $(\mu_t)_{t\geq 0}$. Here, ACD stands for "almost-continuous diffusion", a condition implying the strong Markov property and stochastic continuity. Beiglböck, Huesmann and Stebegg [6] use a certain solution of the Skorokhod problem, which is Lipschitz-Markov, to construct a martingale which is Markov. The recent book by Hirsch, Profeta, Roynette, and Yor [51] contains a wealth of constructions of peacocks and associated martingales.

While there are many works that aim at producing martingales with additional properties, we extend Strassen's theorem in a different direction. The main question that we consider in Chapter 2 is the following: given $\epsilon > 0$, a metric d on $\mathcal{M}$ – the set of all probability measures on $\mathbb{R}$ with finite mean – and a sequence of measures $(\mu_t)_{t\in T}$ in $\mathcal{M}$, when does a sequence $(\nu_t)_{t\in T}$ in $\mathcal{M}$ exist, such that $d(\mu_t, \nu_t) \leq \epsilon$ and such that the sequence $(\nu_t)_{t\in T}$ is a peacock? Here T is either $\mathbb{N}$ or the interval $[0, 1]$. Once we have constructed a peacock, we know, from the results mentioned above, that there is a martingale (with certain properties) with these marginals. We thus want to find out when there is a martingale M such that the law of M_t is close to μ_t for all t. We will state necessary and sufficient conditions when d is the infinity Wasserstein distance, the stop-loss distance, the Prokhorov distance, and the Lévy distance.

[1] See Theorem 8 in [83]. (Another result from that paper, relative to the usual stochastic order instead of the convex order, is also sometimes referred to as Strassen's theorem; see [65].)

The infinity Wasserstein distance is a natural analogue of the well-known p-Wasserstein distance. It seems to have made only a few appearances in the literature, one being [18], where the authors study it in an optimal transport setting. It also has applications in graph theory, where it is referred to as the bottleneck distance (see p. 216 of [31]). We will give an alternative representation of the infinity Wasserstein distance, which shows some similarity to the better known Lévy distance. The stop-loss distance was introduced by Gerber in [44] and has been studied in actuarial science (see for instance [26, 55]).

For both of these metrics, we translate existence of a peacock within ϵ-distance into a more tractable condition: There has to exist a real number (with the interpretation of the desired peacock's mean) that satisfies a countable collection of finite-dimensional conditions, each explicitly expressed in terms of the call functions $x \mapsto \int (y-x)^+ \mu_t(dy)$ of the given sequence of measures. For the infinity Wasserstein distance, the existence proof is not constructive, as it uses Zorn's lemma. For the stop-loss distance, the problem is much simpler, and our proof is short and constructive.

Our proof approach is similar for both metrics: we will construct minimal and maximal elements (with respect to convex ordering) in closed balls, and then use these elements to derive our conditions. In the case of the infinity Wasserstein distance, we will make use of the lattice structure of certain subsets of closed balls.

The Lévy distance was first introduced by Lévy in 1925 (see [64]). Its importance is partially due to the fact that d^{L} metrizes weak convergence of measures on $\mathbb{R}$. The Prokhorov distance, first introduced in [75], is a metric on measures on an arbitrary separable metric space, and is often referred to as a generalisation of the Lévy metric, since d^{P} metrizes weak convergence on any separable metric space. For these two metrics, peacocks within ϵ-distance *always* exist, and can be explicitly constructed.

The structure of the second chapter is as follows. Section 2.1 specifies our notation and introduces the most important definitions. In Section 2.2 contains our main results, concerning the described variant of Strassen's theorem for the infinity Wasserstein distance. A continuous time version of this can be found in Section 2.2.3. In Section 2.3 we will treat the stop-loss distance. After collecting some facts on the Lévy and Prokhorov distances in Section 2.4, we will prove a variant of Strassen's theorem for these metrics in Sections 2.4.1-2.4.2.

In Chapter 3 we will apply the results for the infinity Wasserstein distance, as we calibrate a model to given call option prices. Calibrating martingales to reproduce given option prices is a central topic of mathematical finance, and it is thus a natural question which sets of option prices admit such a fit, and which do not. Note that we are not interested in *approximate* model calibration, but in the consistency of option prices, and thus in arbitrage-free models that fit the given prices *exactly*. Moreover, we do not consider continuous call prices surfaces, but restrict to the (practically more relevant) case of finitely many strikes and maturities. Therefore, consider a financial asset with finitely many European call options written on it. Carr and Madan [16] assume that interest rates, dividends and bid-ask spreads are zero, and derive necessary and sufficient conditions for the existence of arbitrage free models. Essentially, the given call prices must not admit calendar or butterfly arbitrage. Davis and Hobson [21] include interest rates and dividends and give similar results. They also describe explicit arbitrage strategies, whenever arbitrage exists. Concurrent related work has been done by Buehler [14]. More recently, Tavin [85] considers options on multiple assets and studies the existence of arbitrage strategies in this setting.

As with virtually any result in mathematical finance, robustness with respect to market frictions is an important issue in assessing the practical appeal of these findings. Somewhat surprisingly, not much seems to be known in this direction, the single exception being a paper by Cousot [20]. He allows positive bid-ask spreads on the options, but not on the underlying, and finds conditions on the prices that determine the existence of an arbitrage-free model explaining them.

The novelty of our setting is that we allow a bid-ask spread on the underlying. Without any further assumptions on the size of this spread, it turns out that there is no connection between the quoted price of the underlying and those of the calls: Any strategy trying to exploit unreasonable prices can be made impossible by a sufficiently large bid-ask spread; see Example 3.3 and Proposition 3.14. In this respect, the problem is *not* robust w.r.t. the introduction of a spread on the underlying. However, an arbitrarily large spread seems questionable, given that spreads are usually tight for liquid underlyings. We thus enunciate that the appropriate question is not "when are the given prices consistent", but rather "how large a bid-ask spread on the underlying is needed to explain them?" We thus put a bound $\epsilon \geq 0$ on the (discounted) spread of the underlying and want to determine the smallest such ϵ that leads to a model explaining the given prices. We then refer to the call prices as ϵ-consistent (with the absence of arbitrage).

We assume discrete trading times and finite probability spaces throughout; no gain in tractability or realism is to be expected by not doing so. The main technical tool used in the papers [16, 20, 21] mentioned above to construct arbitrage-free models is Strassen's theorem [83], or modifications thereof. In this context that theorem essentially states that option prices have to increase with maturity, but this property breaks down if a spread on the underlying is allowed. We will therefore need to work with a generalisation of Strassen's theorem which will be exactly Theorem 2.11.

The structure of Chapter 3 is as follows. In Section 3.1 we will describe our setting and give a precise formulation of our problem. Then in Sections 3.2 an 3.3 we will formulate conditions for the existence of arbitrage free models with bounded bid-ask spreads for single maturities resp. multiple maturities. In Section 3.4 we will discuss the case where models without spread bounds are allowed.

2

A variant of Strassen's Theorem

2.1. Notation and preliminaries

Let $\mathcal{M}$ denote the set of all probability measures on $\mathbb{R}$ with finite mean. We start with the definition of convex order.

Definition 2.1. Let μ, ν be two measures in $\mathcal{M}$. Then we say that μ is smaller in convex order than ν, in symbols $\mu \leq_{\mathrm{c}} \nu$, if for every convex function $\phi : \mathbb{R} \to \mathbb{R}$ we have that $\int \phi \, d\mu \leq \int \phi \, d\nu$, whenever both integrals are finite.[1] A family of measures $(\mu_t)_{t\in\mathcal{T}}$ in $\mathcal{M}$, where $\mathcal{T} \subseteq [0,\infty)$, is called peacock, if $\mu_s \leq_{\mathrm{c}} \mu_t$ for all $s \leq t$ in $\mathcal{T}$ (see Definition 1.3 in [51]).

Intuitively, $\mu \leq_{\mathrm{c}} \nu$ means that ν is more dispersed than μ, as convex integrands tend to emphasize the tails. By choosing $\phi(x) = x$ resp. $\phi(x) = -x$, we see that $\mu \leq_{\mathrm{c}} \nu$ implies that μ and ν have the same mean. As mentioned in the introduction, Strassen's theorem asserts the following:

Theorem 2.2. *(Strassen [83]) For any peacock, there is a martingale whose family of one-dimensional marginal laws coincides with it.*

The converse implication is of course true as well, as a trivial consequence of Jensen's inequality. For $\mu \in \mathcal{M}$ and $x \in \mathbb{R}$ we define

$$R_\mu(x) = \int_{\mathbb{R}} (y-x)^+ \mu(dy) \quad \text{and} \quad F_\mu(x) = \mu\big((-\infty, x]\big).$$

We call R_μ the call function of μ, as in financial terms it is the (undiscounted) price of a call option with strike x, written on an underlying with law μ at maturity. (It is also

[1] The apparently stronger requirement that the inequality $\int \phi \, d\mu \leq \int \phi \, d\nu$ holds for convex ϕ whenever it makes sense, i.e., as long as both sides exist in $[-\infty, \infty]$, leads to an equivalent definition. This can be seen by the following argument, similar to Remark 1.1 in [51]: Assume that the inequality holds if both sides are finite, and let ϕ (convex) be such that $\int \phi \, d\mu = \infty$. We have to show that then $\int \phi \, d\nu = \infty$. Since ϕ is the envelope of the affine functions it dominates, we can find convex ϕ_n with $\phi_n \uparrow \phi$ pointwise, and such that each ϕ_n is C^2 and ϕ_n'' has compact support. By monotone convergence, we then have $\int \phi \, d\nu = \lim \int \phi_n \, d\nu \geq \lim \int \phi_n \, d\mu = \int \phi \, d\mu = \infty$. With similar arguments we can deal with the case where $\int \phi \, d\nu = -\infty$.

known as integrated survival function [72].) The mean of a measure μ will be denoted by $\mathbb{E}\mu = \int y\, \mu(dy)$. The following proposition summarizes important properties of call functions.

Proposition 2.3. *Let μ, ν be two measures in $\mathcal{M}$. Then:*

(i) *R_μ is convex, decreasing and strictly decreasing on $\{R_\mu > 0\}$. Hence the right derivative of R_μ always exists and is denoted with R'_μ.*

(ii) *$\lim_{x\to\infty} R_\mu(x) = 0$ and $\lim_{x\to-\infty}(R_\mu(x) + x) = \mathbb{E}\mu$. In particular, if $\mu([a,\infty)) = 1$ for $a > -\infty$, then $\mathbb{E}\mu = R_\mu(a) + a$.*

(iii) *$R'_\mu(x) = -1 + F_\mu(x)$ and $R_\mu(x) = \int_x^\infty (1 - F_\mu(y))\, dy$, for all $x \in \mathbb{R}$.*

(iv) *$\mu \leq_{\mathrm{c}} \nu$ holds if and only if $\mathbb{E}\mu = \mathbb{E}\nu$ and $R_\mu(x) \leq R_\nu(x)$ for all $x \in \mathbb{R}$.*

(v) *For $x_1 \leq x_2 \in \mathbb{R}$, we have $R_\mu(x_2) - R_\mu(x_1) = \int_{x_1}^{x_2} R'_\mu(y)\, dy$.*

Conversely, if a function $R : \mathbb{R} \to \mathbb{R}$ satisfies (i) and (ii), then there exists a probability measure $\mu \in \mathcal{M}$ with finite mean such that $R_\mu = R$.

Proof. As for (v), note that R'_μ is increasing, thus integrable, and that the fundamental theorem of calculus holds for right derivatives. See [11] for a short proof. The other assertions are proved in [52], Proposition 2.1, and [51], Exercise 1.7. ❑

For a metric d on $\mathcal{M}$, denote with $B^d(\mu, \epsilon)$ the closed ball with respect to d, with center μ and diameter ϵ. Then our main question is:

Problem 2.4. Given $\epsilon > 0$, a metric d on $\mathcal{M}$, and a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, when does there exist a peacock $(\nu_n)_{n\in\mathbb{N}}$ with $\nu_n \in B^d(\mu_n, \epsilon)$ for all n?

Note that this can also be phrased as

$$d_\infty((\mu_n)_{n\in\mathbb{N}}, (\nu_n)_{n\in\mathbb{N}}) \leq \epsilon,$$

where

$$d_\infty((\mu_n)_{n\in\mathbb{N}}, (\nu_n)_{n\in\mathbb{N}}) = \sup_{n\in\mathbb{N}} d(\mu_n, \nu_n)$$

defines a metric on $\mathcal{M}^{\mathbb{N}}$ (with possible value infinity; see the remark before Proposition 2.6 below). For some results on this kind of infinite product metric, we refer to [10]. Clearly, a solution to Problem 2.4 settles the case of finite sequences $(\mu_n)_{n=1,\ldots,n_0}$, too, by extending the sequence with $\mu_n := \mu_{n_0}$ for $n > n_0$.

To fix ideas, consider the case where the given sequence $(\mu_n)_{n=1,2}$ has only two elements. We want to find measures $\nu_n \in B^d(\mu_n, \epsilon)$, $n = 1, 2$, such that $\nu_1 \leq_{\mathrm{c}} \nu_2$. Intuitively, we want ν_1 to be as small as possible and ν_2 to be as large as possible, in the convex order. Recall that a peacock has constant mean, which is fixed as soon as ν_1 is chosen. We will denote the set of probability measures on $\mathbb{R}$ with mean $m \in \mathbb{R}$ by $\mathcal{M}_m$. These considerations lead us to the following problem.

Problem 2.5. Suppose that a metric d on $\mathcal{M}$, a measure $\mu \in \mathcal{M}$ and two positive numbers ϵ, m are given. When are there two measures $\mu^{\min}, \mu^{\max} \in B^d(\mu, \epsilon) \cap \mathcal{M}_m$ such that

$$\mu^{\min} \leq_{\mathrm{c}} \nu \leq_{\mathrm{c}} \mu^{\max}, \quad \text{for all } \nu \in B^d(\mu, \epsilon) \cap \mathcal{M}_m\ ?$$

The following proposition defines the infinity Wasserstein distance[2] W^∞, and explains its connection to call functions. For various other probability metrics and their relations, see [50]. We will use the words "metric" and "distance" for mappings $\mathcal{M} \times \mathcal{M} \to [0, \infty]$ in a loose sense. Since all our results concern *concrete* metrics, there is no need to give a general definition (as, e.g., Definition 1 in Zolotarev [88]). For the sake of completeness, we include a proof that W^∞ satisfies the classical properties of a metric. Note also that allowing metrics to take the value ∞, as we do, leaves much of the theory of metric spaces unchanged; see, e.g., [15].

Proposition 2.6. *The mapping $W^\infty : \mathcal{M} \times \mathcal{M} \to [0, \infty]$, defined by*

$$W^\infty(\mu, \nu) = \inf \|X - Y\|_\infty ,$$

satisfies the metric axioms. The infimum is taken over all probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and random pairs (X, Y) with marginals given by μ and ν. This metric has the following representation in terms of call functions, which is more useful for our purposes:

$$W^\infty(\mu, \nu) = \inf\Big\{h > 0 : R'_\mu(x - h) \le R'_\nu(x) \le R'_\mu(x + h), \ \forall x \in \mathbb{R}\Big\}. \tag{2.1}$$

Proof. For the equivalence of the two representations see [66], p. 127. Clearly, W^∞ is symmetric and $W^\infty(\mu, \mu) = 0$. If we assume that $W^\infty(\mu, \nu) = 0$, then we have for each $n \in \mathbb{N}$ and $x \in \mathbb{R}$

$$R'_\mu\Big(x - \frac{1}{n}\Big) \le R'_\nu(x) \le R'_\mu\Big(x + \frac{1}{n}\Big),$$

and hence $R'_\nu(x) \le R'_\mu(x)$. By symmetry, we get $R'_\mu(x) \le R'_\nu(x)$, which implies that $R_\mu = R_\nu$ and hence $\mu = \nu$.

Given three measures $\mu_1, \mu_2, \mu_3 \in \mathcal{M}$ such that $W^\infty(\mu_1, \mu_2) = \epsilon_1 < \infty$ and $W^\infty(\mu_2, \mu_3) = \epsilon_2 < \infty$ we obtain that

$$\begin{aligned} R'_{\mu_1}\Big(x - (\epsilon_1 + \epsilon_2 + \frac{2}{n})\Big) &\le R'_{\mu_2}\Big(x - (\epsilon_2 + \frac{1}{n})\Big) \\ &\le R'_{\mu_3}(x) \le R'_{\mu_2}\Big(x + (\epsilon_2 + \frac{1}{n})\Big) \\ &\le R'_{\mu_1}\Big(x + (\epsilon_1 + \epsilon_2 + \frac{2}{n})\Big). \end{aligned}$$

Thus

$$W^\infty(\mu_1, \mu_3) \le \epsilon_1 + \epsilon_2 = W^\infty(\mu_1, \mu_2) + W^\infty(\mu_2, \mu_3).$$

Note that the triangle-inequality trivially holds if $\max\{\epsilon_1, \epsilon_2\} = \infty$. ❑

By (2.1) and Proposition 2.3 *(iii)*, W^∞ can also be written as

$$W^\infty(\mu, \nu) = \inf\Big\{\epsilon > 0 : F_\mu(x - \epsilon) \le F_\nu(x) \le F_\mu(x + \epsilon), \ \forall x \in \mathbb{R}\Big\}.$$

We will see below (Proposition 2.8) that, when d is the infinity Wasserstein distance, Problem 2.5 has a solution $(\mu^{\min}, \mu^{\max})$ if and only if $|m - \mathbb{E}\mu| \le \epsilon$. As an easy consequence,

[2]The name "*infinite* Wasserstein distance" is also in use, but "*infinity* Wasserstein distance" seems to make more sense (cf. "*infinity* norm").

given $(\mu_n)_{n=1,2}$, the desired "close" peacock $(\nu_n)_{n=1,2}$ exists if and only if there is an m with $|m-\mathbb{E}\mu_1| \leq \epsilon$, $|m-\mathbb{E}\mu_2| \leq \epsilon$ such that the corresponding measures $\mu_1^{\min}, \mu_2^{\max}$ satisfy $\mu_1^{\min} \leq_{\mathrm{c}} \mu_2^{\max}$. Then, $(\nu_1, \nu_2) = (\mu_1^{\min}, \mu_2^{\max})$ is a possible choice.

Besides the infinity Wasserstein distance, we will solve Problems 2.4 and 2.5 also for the stop-loss distance (Proposition 2.23), for index sets $\mathbb{N}$ and $[0,1]$ (see Theorems 2.11, 2.22, 2.25, and 2.27). For the Lévy distance and the Prokhorov distance we will use different techniques and solve Problem 2.4 for index set $\mathbb{N}$ (see Corollary 2.36 and Theorem 2.37).

2.2. Strassen's Theorem for the infinity Wasserstein distance

We now start to investigate the interplay between the infinity Wasserstein distance and the convex order. It is a well known fact that the ordered set $(\mathcal{M}_m, \leq_{\mathrm{c}})$ is a lattice for all $m \in \mathbb{R}$, with least element δ_m (Dirac delta). See for instance [60, 73]; recall that $\mathcal{M}_m$ denotes the set of probability measures on $\mathbb{R}$ with mean m. The lattice property means that, given any two measures $\mu, \nu \in \mathcal{M}_m$, there is a unique supremum, denoted with $\mu \vee \nu$, and a unique infimum, denoted with $\mu \wedge \nu$, with respect to convex order. It is easy to prove that $R_{\mu\vee\nu} = R_\mu \vee R_\nu$ and $R_{\mu\wedge\nu} = \mathrm{conv}(R_\mu, R_\nu)$. Here and in the following $\mathrm{conv}(R_\mu, R_\nu)$ denotes the convex hull of R_μ and R_ν, i.e., the largest convex function that is majorized by $R_\mu \wedge R_\nu$.

In the following we will denote balls with respect to W^∞ with B^∞. The next lemma shows that $(B^\infty(\mu,\epsilon) \cap \mathcal{M}_m, \leq_{\mathrm{c}})$ is a sublattice of $(\mathcal{M}_m, \leq_{\mathrm{c}})$, which will be important afterwards. Recall that two measures can be comparable w.r.t. convex order only if their means agree. This accounts for the relevance of sublattices of the form $(B^\infty(\mu,\epsilon)\cap\mathcal{M}_m, \leq_{\mathrm{c}})$ for our problem: If a peacock $(\nu_n)_{n\in\mathbb{N}}$ satisfying $\nu_n \in B^\infty(\mu_n, \epsilon)$ for all $n \in \mathbb{N}$ exists, then we necessarily have $\nu_n \in B^\infty(\mu_n,\epsilon) \cap \mathcal{M}_m$, $n \in \mathbb{N}$, with $\mathbb{E}\nu_1 = \mathbb{E}\nu_2 = \cdots = m$.

Lemma 2.7. *Let $\mu \in \mathcal{M}$, $\epsilon > 0$ and $m \in \mathbb{R}$. Then for $\nu_1, \nu_2 \in B^\infty(\mu,\epsilon) \cap \mathcal{M}_m$ we have*

$$\nu_1 \vee \nu_2 \in B^\infty(\mu,\epsilon) \cap \mathcal{M}_m \quad \textit{and} \quad \nu_1 \wedge \nu_2 \in B^\infty(\mu,\epsilon) \cap \mathcal{M}_m.$$

Proof. Denote the call functions of ν_1 and ν_2 with R_1 and R_2. We start with $\nu_1 \vee \nu_2$. It is easy to check that $R : x \mapsto R_1(x) \vee R_2(x)$ is a call function such that $R'(x) \in \{R_1'(x), R_2'(x)\}$ for all $x \in \mathbb{R}$. By Proposition 2.3 *(ii)*, it is also clear that $\nu_1 \vee \nu_2 \in \mathcal{M}_m$. This proves the assertion.

As for the infimum, we will first assume that there exists $x_0 \in \mathbb{R}$ such that $R_1(x) \leq R_2(x)$ for $x \leq x_0$ and $R_2(x) \leq R_1(x)$ for $x \geq x_0$. Then there exist $x_1 \leq x_0$ and $x_2 \geq x_0$ such that the convex hull of R_1 and R_2 can be written as (see [74])

$$\mathrm{conv}(R_1,R_2)(x) = \begin{cases} R_1(x), & x \leq x_1, \\ R_1(x_1) + \frac{R_2(x_2)-R_1(x_1)}{x_2-x_1}(x-x_1), & x \in [x_1,x_2], \\ R_2(x), & x \geq x_2. \end{cases}$$

Now observe that for all $x \in [x_1, x_2)$

$$\begin{aligned} R'_\mu(x-\epsilon) &\leq R'_2(x) \leq R'_2(x_2-) \\ &\leq \frac{R_2(x_2) - R_1(x_1)}{x_2 - x_1} \\ &\leq R'_1(x_1) \leq R'_1(x) \leq R'_\mu(x+\epsilon), \end{aligned}$$

and hence $\operatorname{conv}(R_1, R_2)'(x) \in [R'_\mu(x-\epsilon), R'_\mu(x+\epsilon)]$. Therefore $\nu_1 \wedge \nu_2 \in B^\infty(\mu, \epsilon) \cap \mathcal{M}_m$.

For the general case note that for all $x \in \mathbb{R}$ we have by [74] that either $\operatorname{conv}(R_1, R_2)(x) = R_\mu(x) \wedge R_\nu(x)$ or that x lies in an interval I such that $\operatorname{conv}(R_1, R_2)$ is affine on I. If the latter condition is the case then we can derive bounds for the right-derivative $\operatorname{conv}(R_1, R_2)'(x), x \in I$, exactly as before. The situation is clear if we have that either $\operatorname{conv}(R_1, R_2)(x) = R_1(x)$ or $\operatorname{conv}(R_1, R_2)(x) = R_2(x)$. ❑

We now show that the sublattice $(B^\infty(\mu, \epsilon) \cap \mathcal{M}_m, \leq_c)$ contains a least and a greatest element with respect to convex order. This is the subject of the following proposition, and is also the solution to Problem 2.5 for the infinity Wasserstein distance. As for the assumption $m \in [\mathbb{E}\mu - \epsilon, \mathbb{E}\mu + \epsilon]$ in Proposition 2.8, it is necessary to ensure that $B^\infty(\mu, \epsilon) \cap \mathcal{M}_m$ is not empty. Indeed, if $W^\infty(\mu_1, \mu_2) \leq \epsilon$ for some $\mu_1, \mu_2 \in \mathcal{M}$, then by (2.1), Proposition 2.3 *(ii)*, *(v)*, and the continuity of call functions, we obtain

$$R_{\mu_1}(x+\epsilon) \leq R_{\mu_2}(x) \leq R_{\mu_1}(x-\epsilon), \quad x \in \mathbb{R}. \tag{2.2}$$

By part *(ii)* of Proposition 2.3, it follows that $|\mathbb{E}\mu_1 - \mathbb{E}\mu_2| \leq \epsilon$.

Proposition 2.8. *Given $\epsilon > 0$, a measure $\mu \in \mathcal{M}$ and $m \in [\mathbb{E}\mu - \epsilon, \mathbb{E}\mu + \epsilon]$, there exist unique measures $S(\mu), T(\mu) \in B^\infty(\mu, \epsilon) \cap \mathcal{M}_m$ such that*

$$S(\mu) \leq_c \nu \leq_c T(\mu) \quad \textit{for all } \nu \in B^\infty(\mu, \epsilon) \cap \mathcal{M}_m.$$

The call functions of $S(\mu)$ and $T(\mu)$ are given by

$$R^{\min}_\mu(x) = R_{S(\mu)}(x) = \Big(m + R_\mu(x-\epsilon) - (\mathbb{E}\mu + \epsilon)\Big) \vee R_\mu(x+\epsilon), \tag{2.3}$$

$$R^{\max}_\mu(x) = R_{T(\mu)}(x) = \operatorname{conv}\Big(m + R_\mu(\cdot + \epsilon) - (\mathbb{E}\mu - \epsilon), R_\mu(\cdot - \epsilon)\Big)(x). \tag{2.4}$$

To highlight the dependence on ϵ and m we will sometimes write $S(\mu; m, \epsilon)$ and $R^{\min}_\mu(\cdot; m, \epsilon)$, respectively $T(\mu; m, \epsilon)$ and $R^{\max}_\mu(\,\cdot\,; m, \epsilon)$.

Proof. We define $R^{\min}_\mu$ and $R^{\max}_\mu$ by the right hand sides of (2.3) resp. (2.4), and argue that the associated measures $S(\mu)$ and $T(\mu)$ have the stated property. Clearly $R^{\min}_\mu$ is a call function, and we have that

$$\begin{aligned} \mathbb{E}R^{\min}_\mu &= \lim_{x \to -\infty} \Big(m + R_\mu(x-\epsilon) - (\mathbb{E}\mu + \epsilon) + x\Big) \vee \Big(R_\mu(x+\epsilon) + x\Big) \\ &= m \vee (\mathbb{E}\mu - \epsilon) = m. \end{aligned}$$

From the convexity of R_μ we can deduce the existence of $v \in \mathbb{R} \cup \{\pm\infty\}$ such that

$$R^{\min}_\mu(x) = \begin{cases} m + R_\mu(x-\epsilon) - (\mathbb{E}\mu + \epsilon), & x \leq v, \\ R_\mu(x+\epsilon) & x \geq v. \end{cases}$$

Hence we get that $(R_\mu^{\min})'(x) \in [R'_\mu(x-\epsilon), R'_\mu(x+\epsilon)]$ for all x. According to Proposition 2.6, the measure associated with $R_\mu^{\min}$ lies in $B^\infty(\mu,\epsilon)\cap\mathcal{M}_m$. To the left of v, $R_\mu^{\min}$ is as steep as possible (where steepness refers to the absolute value of the right derivative), and to the right of v it is as flat as possible. From this and convexity, it is easy to see that $S(\mu)$ is the least element.

Similarly we can show that $\mathbb{E}R_\mu^{\max} = m$, and thus it suffices to show that

$$(R_\mu^{\max})'(x) \in [R'_\mu(x-\epsilon), R'_\mu(x+\epsilon)].$$

But this can be done exactly as in Lemma 2.7. ❑

Remark 2.9. It is not hard to show that

$$R_\mu^{\max}(x) = \begin{cases} m + R_\mu(x+\epsilon) - (\mathbb{E}\mu - \epsilon), & x \le x_1, \\ R_\mu(x_1+\epsilon) + \frac{(\mathbb{E}\mu-\epsilon)-m}{2\epsilon}(x - x_1 - 2\epsilon), & x \in [x_1, x_1+2\epsilon], \\ R_\mu(x-\epsilon), & x \ge x_1 + 2\epsilon, \end{cases}$$

where

$$x_1 = \inf\left\{ x \in \mathbb{R} : R'_\mu(x+\epsilon) \ge -\frac{m - (\mathbb{E}\mu-\epsilon)}{2\epsilon} \right\}.$$

Before formulating our main theorem, we recall that in Definition 2.1 we defined a peacock to be a sequence of probability measures with finite mean and increasing w.r.t. convex order. We now give a simple reformulation of this property. For a given sequence of call functions $(R_n)_{n\in\mathbb{N}}$, define, for $N \in \mathbb{N}$ and $x_1,\dots,x_N \in \mathbb{R}$,

$$\Phi_N(x_1,\dots,x_N) = R_1(x_1) + \sum_{n=2}^{N} \Big(R_n(x_n) - R_n(x_{n-1}) \Big) - R_{N+1}(x_N). \tag{2.5}$$

Proposition 2.10. *A sequence of call functions $(R_n)_{n\in\mathbb{N}}$ with constant mean defines a peacock if and only if $\Phi_N(x_1,\dots,x_N) \le 0$ for all $N \in \mathbb{N}$ and $x_1,\dots,x_N \in \mathbb{R}$.*

Proof. According to Proposition 2.3 *(iv)*, we need to check whether the sequence of call functions increases. Let $n \in \mathbb{N}$ be arbitrary. If we set the n-th component of $(x_1,\dots,x_{n+1})$ to an arbitrary $x \in \mathbb{R}$ and let all others tend to ∞, we get

$$\Phi_{n+1}(\infty,\dots,\infty,x,\infty) = R_n(x) - R_{n+1}(x).$$

The sequence of call functions thus increases, if Φ is always non-positive. Conversely, assume that $(R_n)_{n\in\mathbb{N}}$ increases. Then, for $N \in \mathbb{N}$ and $x_1,\dots,x_N \in \mathbb{R}$,

$$\begin{aligned} \Phi_N(x_1,\dots,x_N) &\le R_1(x_1) + \sum_{n=2}^{N} R_{n+1}(x_n) - \sum_{n=2}^{N} R_n(x_{n-1}) - R_{N+1}(x_N) \\ &= R_1(x_1) + \sum_{n=3}^{N+1} R_n(x_{n-1}) - \sum_{n=2}^{N} R_n(x_{n-1}) - R_{N+1}(x_N) \\ &= R_1(x_1) - R_2(x_1) \le 0. \end{aligned}$$

❑

We now extend the definition of Φ_N for $x_1, \dots, x_N \in \mathbb{R}$, $m \in \mathbb{R}$, and $\epsilon > 0$ as follows, using the notation from Proposition 2.8:

$$\Phi_N(x_1, \dots, x_N; m, \epsilon) = R_1^{\min}(x_1; m, \epsilon) \\ + \sum_{n=2}^{N} \Big(R_n(x_n + \epsilon\sigma_n) - R_n(x_{n-1} + \epsilon\sigma_n) \Big) - R_{N+1}^{\max}(x_N; m, \epsilon). \quad (2.6)$$

Here, $R_1^{\min}$ is the call function of $S(\mu_1; m, \epsilon)$, $R_{N+1}^{\max}$ is the call function of $T(\mu_{N+1}; m, \epsilon)$, and

$$\sigma_n = \operatorname{sgn}(x_{n-1} - x_n) \quad (2.7)$$

depends on x_{n-1} and x_n. Clearly, for $\epsilon = 0$ and $\mathbb{E}\mu_1 = \mathbb{E}\mu_2 = \cdots = m$, we recover (2.5):

$$\Phi_N(x_1, \dots, x_N; m, 0) = \Phi_N(x_1, \dots, x_N), \quad N \in \mathbb{N},\ x_1, \dots, x_N \in \mathbb{R}. \quad (2.8)$$

The following theorem gives an equivalent condition for the existence of a peacock within W^∞-distance ϵ of a given sequence of measures, thus solving Problem 2.4 for the infinity Wasserstein distance, and is the main result of this chapter. By Proposition 2.10 and (2.8), it is consistent with Strassen's theorem (Theorem 2.2), i.e., the case $\epsilon = 0$. Also, note that the functions Φ_N defined in (2.6) have explicit expressions in terms of the given call functions, as $R^{\min}$ and $R^{\max}$ are explicitly given by (2.3) and (2.4). The existence criterion we obtain is thus rather explicit; the existence *proof* is not constructive, though, as mentioned in the introduction. (For a constructive proof of Strassen's theorem, and references to earlier constructive proofs, see Müller and Rüschendorf [72].) Moreover, note that we use Strassen's theorem in the proof; for $\epsilon = 0$, the proof reduces to a triviality, and not to a proof of Strassen's theorem.

Theorem 2.11. *Let $\epsilon > 0$ and $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$ such that*

$$I := \bigcap_{n\in\mathbb{N}} [\mathbb{E}\mu_n - \epsilon, \mathbb{E}\mu_n + \epsilon]$$

is not empty. Denote by $(R_n)_{n\in\mathbb{N}}$ the corresponding call functions, and define Φ_N by (2.6). Then there exists a peacock $(\nu_n)_{n\in\mathbb{N}}$ such that

$$W^\infty(\mu_n, \nu_n) \le \epsilon, \quad \textit{for all } n \in \mathbb{N}, \quad (2.9)$$

if and only if for some $m \in I$ and for all $N \in \mathbb{N}$, $x_1, \dots, x_N \in \mathbb{R}$, we have

$$\Phi_N(x_1, \dots, x_N; m, \epsilon) \le 0. \quad (2.10)$$

In this case it is possible to choose $\mathbb{E}\nu_1 = \mathbb{E}\nu_2 = \cdots = m$.

The proof of Theorem 2.11 is given towards the end of the present section, building on Theorem 2.16 and Corollary 2.17 below.

For $\epsilon = 0$, condition (2.10) is equivalent to the sequence of call functions (R_n) being increasing, see Proposition 2.10. For $\epsilon > 0$, analogously to the proof of Proposition 2.10, we see that (2.10) implies

$$R_n(x + \epsilon) \le R_{n+1}(x - \epsilon), \quad x \in \mathbb{R}, n \in \mathbb{N}. \quad (2.11)$$

It is clear that (2.11) is necessary for the existence of the peacock $(\nu_n)_{n\in\mathbb{N}}$, since, by (2.2) and Proposition 2.3 *(iv)*,

$$R_n(x+\epsilon) \le R_{\nu_n}(x) \le R_{\nu_{n+1}}(x) \le R_{n+1}(x-\epsilon), \quad x \in \mathbb{R}, n \in \mathbb{N}.$$

On the other hand, it is easy to show that (2.11) is not sufficient for (2.10):

Example 2.12. Fix $m > 1$ and $\epsilon = 1$ and define two measures

$$\mu_1 = \frac{2}{m+1}\delta_0 + \frac{m-1}{m+1}\delta_{m+1}, \quad \mu_2 = \delta_{m+1},$$

where δ denotes the Dirac delta. It is simple to check that (2.11) is satisfied, i.e.

$$R_{\mu_1}(x+\epsilon) \le R_{\mu_2}(x-\epsilon), \quad x \in \mathbb{R}.$$

Now assume that we want to construct a peacock $(\nu_n)_{n=1,2}$ such that $W^\infty(\mu_n, \nu_n) \le 1$.

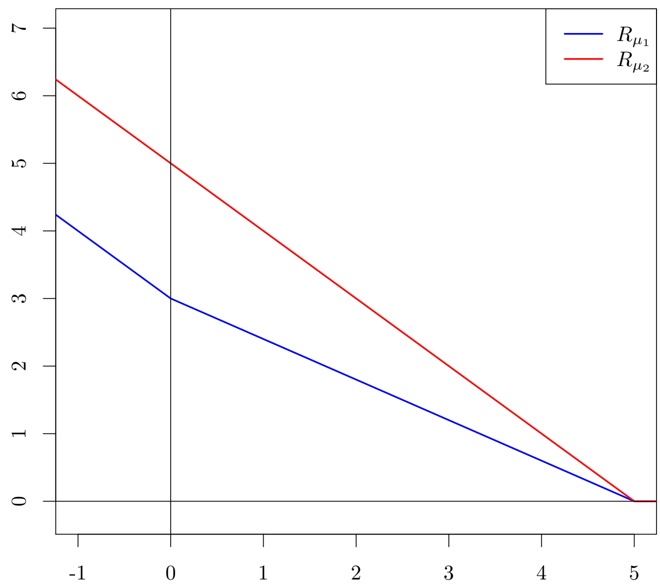

Figure 2.1.: The call functions of μ_1 (lower curve) and μ_2 (upper curve) from Example 2.12, for $m = 4$ and $\epsilon = 1$. The call function of ν_1 is the call function of μ_1 shifted to the right by one. Similarly, shifting the call function of μ_2 by one to the left yields the call function of ν_2.

Then the only possible mean for this peacock is m, which easily follows from $\mathbb{E}\mu_1 = m-1$ and $\mathbb{E}\mu_2 = m+1$ (see the remark before Proposition 2.8). Therefore the peacock has to satisfy $\nu_n \in B^\infty(\mu_n, 1) \cap \mathcal{M}_m$, $n = 1, 2$, and the only possible choice is

$$\nu_1 = \frac{2}{m+1}\delta_1 + \frac{m-1}{m+1}\delta_{m+2}, \quad \nu_2 = \delta_m.$$

But since $R_{\nu_1}(x) > R_{\nu_2}(x)$ for $x \in (1, m+2)$, $(\nu_n)_{n=1,2}$ is not a peacock; see Figure 2.1.

If the sequence $(\mu_n)_{n=1,2}$ has just two elements, then it suffices to require (2.10) only for $N = 1$. It then simply states that there is an $m \in I$ such that $R_1^{\min}(x; m, \epsilon) \le R_2^{\max}(x; m, \epsilon)$ for all x, which is clearly necessary and sufficient for the existence of $(\nu_n)_{n=1,2}$. However, if the sequence $(\mu_n)_{n\in\mathbb{N}}$ has more than two elements, then

$$R_k^{\min}(x; m, \epsilon) \le R_n^{\max}(x; m, \epsilon), \quad k \le n \tag{2.12}$$

is only necessary but not sufficient for the existence of a suitable peacock, as is shown in the next example.

Example 2.13. Let $\epsilon = 1$ and consider three measures

$$\begin{aligned}
\mu_1 &= \frac{1}{2}\delta_1 + \frac{1}{3}\delta_6 + \frac{1}{6}\delta_9, \quad \mathbb{E}\mu_1 = 4, \\
\mu_2 &= \frac{1}{2}\delta_1 + \frac{1}{2}\delta_9, \quad \mathbb{E}\mu_2 = 5, \\
\mu_3 &= \frac{1}{2}\delta_3 + \frac{1}{3}\delta_9 + \frac{1}{6}\delta_{11}, \quad \mathbb{E}\mu_3 = 6.
\end{aligned}$$

If a peacock (ν_1, ν_2, ν_3) in W^∞-neighborhood of (μ_1, μ_2, μ_3) exists its mean has to be $\mathbb{E}\nu_1 = 5$. A simple calculation reveals that

$$\begin{aligned}
S(\mu_1; 5, 1) = T(\mu_1; 5, 1) &= \frac{1}{2}\delta_2 + \frac{1}{3}\delta_7 + \frac{1}{6}\delta_{10}, \\
S(\mu_2; 5, 1) &= \frac{1}{2}\delta_2 + \frac{1}{2}\delta_8, \\
T(\mu_2; 5, 1) &= \frac{1}{2}\delta_0 + \frac{1}{2}\delta_{10}, \\
S(\mu_3; 5, 1) = T(\mu_3; 5, 1) &= \frac{1}{2}\delta_2 + \frac{1}{3}\delta_7 + \frac{1}{6}\delta_{10}.
\end{aligned}$$

It is now plain to check that (2.12) holds for $k, n \in \{1, 2, 3\}$. In particular, we have that $S(\mu_1; 5, 1) = T(\mu_3; 5, 1)$. Therefore, a peacock satisfying (2.9) exists if and only if $S(\mu_1; 5, 1) \in B^\infty(\mu_2, 1)$. But this is not the case, since (2.1) does not hold for $x = 7$:

$$-\frac{1}{6} = R'_{S(\mu_1)}(x) \notin [R'_{\mu_2}(x-1), R'_{\mu_2}(x+1)] = \left\{-\frac{1}{2}\right\}.$$

Thus, we have shown that (2.12) is not sufficient for the existence of suitable peacocks in W^∞-neighborhoods.

Unsurprisingly, the peacock from Theorem 2.11 is in general not unique:

Example 2.14. Let $\epsilon > 0$ and consider the constant sequences $R_n(x) = (-x)^+$, $n \in \mathbb{N}$, and

$$P_n(x, c) = \begin{cases} -x, & x \le -\epsilon, \\ \epsilon - \frac{\epsilon(x+\epsilon)}{c+\epsilon}, & -\epsilon \le x \le c, \\ 0, & x \ge c. \end{cases}$$

Then, for any $c \in [0, \epsilon]$, it is easy to verify that the sequence of call functions $P_n(\cdot, c)$ defines a peacock satisfying (2.9).

2.2.1. Illustrations

In this section we want to illustrate the minimal and maximal measures from Proposition 2.8 with some examples.

Example 2.15. (*i*) Suppose that $\mu = \delta_m$ for some $m \in \mathbb{R}$, then clearly $S(\mu; m, \epsilon) = \mu$ for all $\epsilon \geq 0$. A simple calculation shows that $T(\mu, m, \epsilon) = \frac{1}{2}\delta_{m-\epsilon} + \frac{1}{2}\delta_{m+\epsilon}$.

(*ii*) Suppose that $\mu \in \mathcal{M}$ is symmetric, i.e. $F_\mu(x) = 1 - F_\mu(-x)$ for all $x \in \mathbb{R}$ and assume that μ has a density f_μ with respect to the Lebesgue measure, such that $F_\mu(x) = \int_{-\infty}^{x} f_\mu(z)\ dz$. For fixed $\epsilon \geq 0$ we will determine $S(\mu) := S(\mu; 0, \epsilon)$ and $T(\mu) := T(\mu; 0, \epsilon)$. A simple calculation reveals that the unique solution of

$$R_\mu(x - \epsilon) - \epsilon = R_\mu(x + \epsilon),$$

is given by $x = 0$. Therefore the distribution function of $S(\mu)$ is given by

$$F_{S(\mu)}(x) = \begin{cases} F_\mu(x - \epsilon), & x < 0, \\ F_\mu(x + \epsilon), & x \geq 0. \end{cases}$$

Note that in general $S(\mu)$ has an atom at 0: $S(\mu)(\{0\}) = F_\mu(\epsilon) - F_\mu(-\epsilon)$=:p. We can now decompose $S(\mu)$ as follows:

$$S(\mu) = p\delta_0 + (1 - p)\nu,$$

where δ_0 denotes Dirac delta function at 0 and ν is a symmetric measure which has a density f_ν given by

$$f_\nu(x) = \begin{cases} f_\mu(x - \epsilon), & x \leq 0, \\ f_\mu(x + \epsilon), & x \geq 0. \end{cases}$$

Similarly we can construct $T(\mu)$. By Remark 2.9, we get that the call function of $T(\mu)$ can be written as

$$R_{T(\mu)}(x) = \begin{cases} R_\mu(x + \epsilon) + \epsilon, & x \leq -\epsilon, \\ \frac{1}{2}(\epsilon - x) + R_\mu(0), & x \in [-\epsilon, \epsilon], \\ R_\mu(x - \epsilon), & x \geq \epsilon. \end{cases}$$

Note that this implies that the distribution function of $T(\mu)$ is flat in $[-\epsilon, \epsilon]$, i.e. for all Borel sets $A \subseteq [-\epsilon, \epsilon]$ we have that $T(\mu)(A) = 0$.

Figure 2.2 illustrates the densities of $S(\mu)$ and $T(\mu)$ when μ is the standard normal distribution and $\epsilon = \frac{1}{2}$.

This highlights the fact that $S(\mu)$ has more mass in the center than μ and less mass in the tails. Conversely, $T(\mu)$ has no mass in the center, but more mass in the tails than μ.

(*iii*) If in (*ii*) μ is the uniform distribution on $[-n, n]$ for $n > \epsilon$, then $T(\mu)$ is the uniform distribution on $[-\epsilon - n, -\epsilon] \cup [\epsilon, \epsilon + n]$ and $S(\mu)$ is the convex combination of the uniform distribution on $[-n+\epsilon, n-\epsilon]$ and the dirac measure on zero. In this case $S(\mu)$ is a simple fusion of μ (see [32] for definitions): $S(\mu)$ can be obtained from μ by fusing all the mass from $[-n, -n + \epsilon] \cup [n - \epsilon, n]$ into zero.

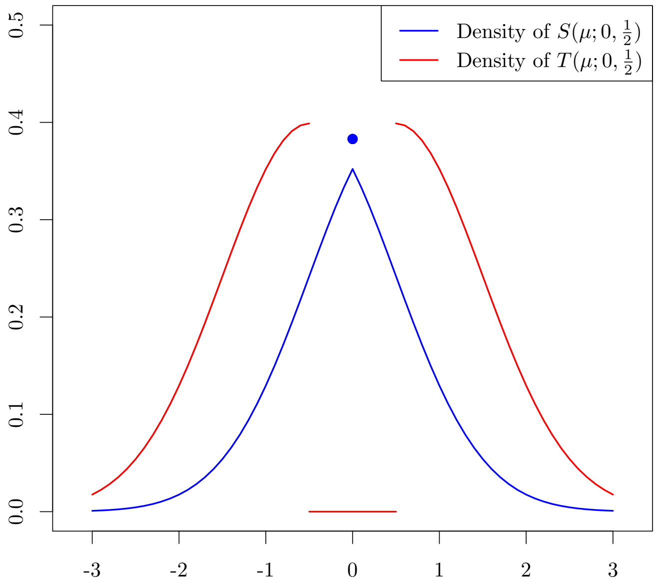

Figure 2.2.: Densities of $S(\mu)$ and $T(\mu)$ from *(ii)* of Example 2.15, when μ is the standard normal distribution. The blue circle indicates that $S(\mu)$ has an atom at zero.

2.2.2. Proof of Theorem 2.11

The following theorem furnishes the main step for the induction proof of Theorem 2.11, given at the end of the present section. In each induction step, the next element of the desired peacock should be contained in a certain ball, it should be larger in convex order than the previous element (ν in Theorem 2.16), and it should be as small as possible in order not to hamper the existence of the subsequent elements. This leads us to search for a least element of the set (2.13). The conditions defining this least element translate into inequalities on the corresponding call function. Part *(ii)* of Theorem 2.16 states that, at each point of the real line, at least one of the latter conditions becomes an equality.

Theorem 2.16. *Let μ, ν be two measures in $\mathcal{M}$ such that the set*

$$A_\mu^\nu := \left\{\theta \in B^\infty(\mu;\epsilon) : \ \nu \leq_{\mathrm{c}} \theta\right\} \tag{2.13}$$

is not empty.

(i) The set A_μ^ν contains a least element $S_\nu(\mu)$ with respect to $\leq_{\mathrm{c}}$, i.e. for every $\theta \in A_\mu^\nu$ we have that

$$\nu \leq_{\mathrm{c}} S_\nu(\mu) \leq_{\mathrm{c}} \theta.$$

Equivalently, if

$$R_\nu(x) \leq R_{T(\mu)}(x; \mathbb{E}\nu, \epsilon), \quad x \in \mathbb{R},$$

there exists a pointwise smallest call function R^* which is greater than R_ν and satisfies $(R^*)'(x) \in [R'_\mu(x-\epsilon), R'_\mu(x+\epsilon)]$ for all $x \in \mathbb{R}$.

(ii) The call function R^* is a solution of the following variational type inequality:

$$\min\Big\{R^*(x) - R_\nu(x), (R^*)'(x) - R'_\mu(x-\epsilon), R'_\mu(x+\epsilon) - (R^*)'(x)\Big\} = 0, \quad x \in \mathbb{R}. \tag{2.14}$$

Proof. The equivalence in *(i)* follows from Proposition 2.3 *(iv)*. We now argue that $S_\nu(\mu)$ exists. An easy application of Zorn's lemma shows that there exist minimal elements in A^ν_μ. If θ_1 and θ_2 are two minimal elements of A^ν_μ then according to Lemma 2.7 the measure $\theta_1 \wedge \theta_2$ lies in $B^\infty(\nu, \epsilon) \cap \mathcal{M}_{\mathbb{E}\nu}$. Moreover, the convex function R_ν nowhere exceeds R_{θ_1} and R_{θ_2}, and hence we have that $R_\nu \leq \mathrm{conv}(R_{\theta_1} \wedge R_{\theta_2}) = R_{\theta_1 \wedge \theta_2}$. Therefore $\theta_1 \wedge \theta_2$ lies in A^ν_μ. Now clearly $\theta_1 \wedge \theta_2 \leq_{\mathrm{c}} \theta_1$ and $\theta_1 \wedge \theta_2 \leq_{\mathrm{c}} \theta_2$, and from the minimality we can conclude that $\theta_1 \wedge \theta_2 = \theta_1 = \theta_2$.

Now let θ^* be the unique minimal element and let $\theta \in A^\nu_\mu$ be arbitrary. Exactly as before we can show that $\theta^* \wedge \theta$ lies in A^ν_μ. Moreover $\theta^* = \theta^* \wedge \theta \leq_{\mathrm{c}} \theta$ and therefore θ^* is the least element of A^ν_μ.

It remains to show *(ii)*. We set

$$R^*(x) = \inf\{R_\theta(x) : \ \theta \in A^\nu_\mu\}. \tag{2.15}$$

Clearly R^* is a decreasing function with $\lim_{x\to\infty} R^*(x) = 0$ and $\lim_{x\to-\infty} R^*(x) + x = \mathbb{E}\nu$. We will show that R^* is convex, which is equivalent to the convexity of the epigraph $\mathcal{E}$ of R^*. Pick two points $(x_1, y_1), (x_2, y_2) \in \mathcal{E}$. Then there exist measures $\theta_1, \theta_2 \in A^\nu_\mu$ such that $R_{\theta_1}(x_1) \leq y_1$ and $R_{\theta_2}(x_2) \leq y_2$. Using Lemma 2.7 once more, we get that $\theta := \theta_1 \wedge \theta_2 \in A^\nu_\mu$ and $R_\theta(x_i) \leq y_i, i = 1, 2$. Therefore, the whole segment with endpoints (x_1, y_1) and (x_2, y_2) lies in the epigraph of R_θ and hence in $\mathcal{E}$. This implies that R^* is a call function, and the associated measure has to be $S_\nu(\mu)$. Also, we can therefore conclude that the infimum in (2.15) is attained for all x.

Now assume that (2.14) is wrong. Since all functions appearing in (2.14) are right-continuous, there must then exist an open interval (a, b) where (2.14) does not hold, i.e. $R^*(x) > R_\nu(x)$ and $(R^*)'(x) \in (R'_\mu(x-\epsilon), R'_\mu(x+\epsilon))$ for all $x \in (a, b)$.

Case 1: There exists an open interval $I \subseteq (a, b)$ where R^* is strictly convex. Then we can pick $x_1 \in I$ and $h_1 > 0$ such that $x_1 + h_1 \in I$ and such that the tangent

$$P_1(x) := R^*(x_1) + (R^*)'(x_1)(x - x_1), \quad x \in [x_1, x_1 + h_1]$$

satisfies $R_\nu(x) < P_1(x) < R^*(x)$ for $x \in (x_1, x_1 + h_1]$. Also, since $(R^*)'(x_1) > R'_\mu(x_1 - \epsilon)$ and since R'_μ is right-continuous, we can choose h_1 small enough to guarantee $(R^*)'(x_1) \geq R'_\mu(x_1 + h_1 - \epsilon)$. Next pick $x_2 \in (x_1, x_1 + h_1)$, such that $R'_\mu(\cdot + \epsilon)$ is continuous at x_2 and set

$$P_2(x) := R^*(x_2) + (R^*)'(x_2)(x - x_2), \quad x \in [x_2 - h_2, x_2].$$

We can choose h_2 small enough to ensure that $R_\nu(x) < P_2(x) < R^*(x)$ and $(R^*)'(x_2) \leq R'_\mu(x_2 - h_2 + \epsilon)$. Also, if x_1 and x_2 are close enough together, then there is an intersection of P_1 and P_2 in (x_1, x_2). Now the function

$$\widetilde{R}(x) := \begin{cases} P_1(x) \vee P_2(x), & x \in [x_1, x_2], \\ R^*(x), & \text{otherwise}, \end{cases}$$

is a call function which is strictly smaller than R^* and satisfies $\widetilde{R}'(x) \in [R'_\mu(x-\epsilon), R'_\mu(x+\epsilon)]$ for all $x \in \mathbb{R}$. This is a contradiction to (2.15). See Figure 2.3 for an illustration.

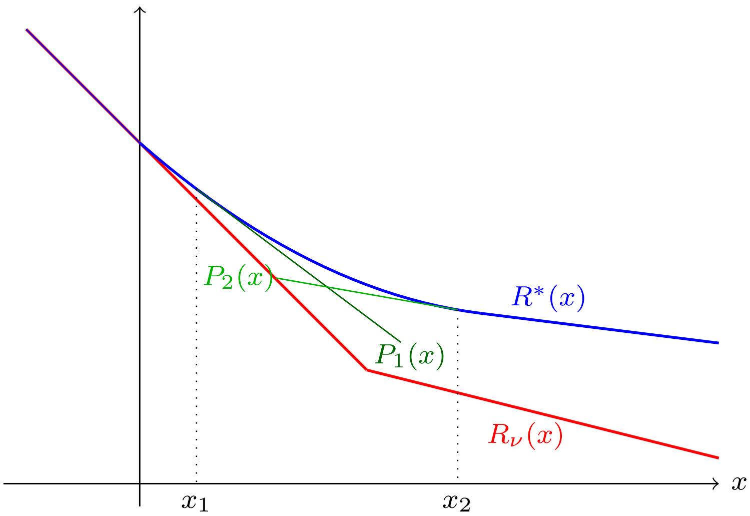

Figure 2.3.: Case 1 of the proof of Theorem 2.16. If R^* is strictly convex, then we can deform it using two appropriate tangents, contradicting minimality of the associated measure.

Case 2: If there is no open interval in (a, b) where R^* is strictly convex, then R^* has to be affine on some closed interval $I \subseteq (a, b)$ (see p. 7 in [76]). Therefore, there exist k, d in $\mathbb{R}$ such that

$$R^*(x) = kx + d, \quad x \in I.$$

By Proposition 2.3 *(ii)*, the slope k has to lie in the open interval $(-1, 0)$, since R^* is greater than R_ν on I. We set

$$a_1 := \sup\{x \in \mathbb{R} : (R^*)'(x) < k\} > -\infty,$$
$$b_1 := \inf\{x \in \mathbb{R} : (R^*)'(x) > k\} < \infty;$$

the finiteness of these quantities follows from Proposition 2.3 *(ii)*. From the convexity of R_ν and the fact that $R_\nu \leq R^*$, we get that $R^*(x) > R_\nu(x)$ for all $x \in (a_1, b_1)$, as well as $(R^*)'(x) > R'_\mu(x-\epsilon)$ for all $x \in (a_1, b)$ and $(R^*)'(x) < R'_\mu(x+\epsilon)$ for all $x \in (a, b_1)$. We now define lines P_1 and P_2, with analogous roles as in Case 1. Their definitions depend on the behavior of $(R^*)'$ at a_1 and b_1.

If $(R^*)'(a_1-) < k$, then we set $x_1 = a_1$ and $P_1(x) = R^*(x_1) + k_1(x - x_1)$ for $x \geq x_1$, with an arbitrary $k_1 \in ((R^*)'(x_1-), k)$; see Figure 2.4.

If, on the other hand, $(R^*)'(a_1-) = k$, then we can find $x_1 < a_1$ such that $R^*(x_1) > R_\nu(x_1)$ and $(R^*)'(x_1) > R'_\mu(x_1 - \epsilon)$. In this case we define

$$P_1(x) := R^*(x_1) + (R^*)'(x_1)(x - x_1), \quad x \geq x_1.$$

Similarly, if $(R^*)'(b_1) > k$, then we define $x_2 = b_1$ and $P_2(x) = R^*(x_2) + k_2(x - x_2)$ for $x \le x_2$ and for $k_2 \in (k, (R^*)'(b_1))$, and otherwise we can find $x_2 > b_1$ such that $R^*(x_2) > R_\nu(x_2)$ and $(R^*)'(x_2) < R'_\mu(x_2 + \epsilon)$. We then set

$$P_2(x) := R^*(x_2) + (R^*)'(x_2)(x - x_2), \quad x \le x_2.$$

We can choose $h_1, h_2 > 0$, $\widetilde{d} < d$ and k_1, k_2 such that the function

$$\widetilde{R}(x) := \begin{cases} P_1(x), & x \in [x_1, x_1 + h_1], \\ kx + \widetilde{d}, & x \in [x_1 + h_1, x_2 - h_2], \\ P_2(x), & x \in [x_2 - h_2, x_2], \\ R^*(x), & \text{otherwise}, \end{cases}$$

is a call function which is strictly smaller than R^* but not smaller than R_ν. Also, if h_1 and h_2 are small enough we have that $\widetilde{R}'(x) \in [R'_\mu(x - \epsilon), R'_\mu(x + \epsilon)]$ for all $x \in \mathbb{R}$, which is a contradiction to (2.15). ❑

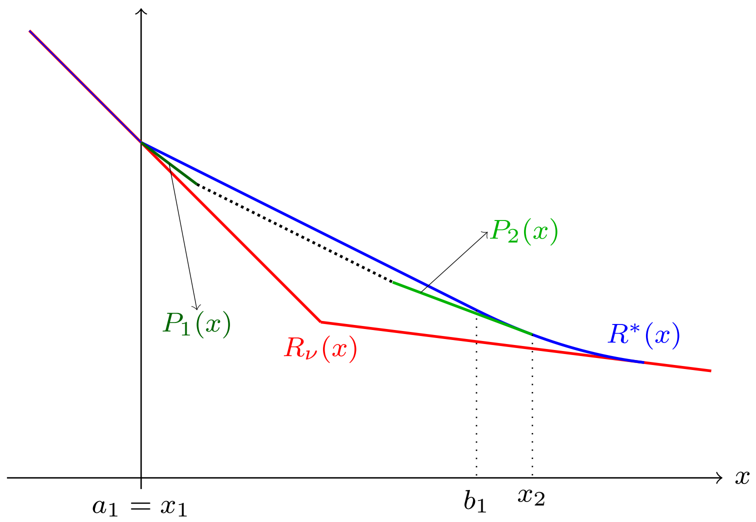

Figure 2.4.: Case 2 of the proof of Theorem 2.16, with $(R^*)'(a_1-) < k$ and $(R^*)'(b_1) = k$.

In part *(i)* of Theorem 2.16, we showed that A^ν_μ has a least element. The weaker statement that it has an infimum follows from [60], p. 162; there it is shown that *any* subset of the lattice $(\mathcal{M}_m, \le_c)$ has an infimum. (The stated requirement that the set be bounded from below is always satisfied, as the Dirac delta δ_m is the least element of $(\mathcal{M}_m, \le_c)$.) This infimum is, of course, given by the least element $S_\nu(\mu)$ that we found.

If $\nu = \delta_m$, then $S_\nu(\mu) = S(\mu)$, the least element from Proposition 2.8. In this case we have that

$$(R^*)'(x) = \begin{cases} R'_\mu(x - \epsilon), & x < x^*, \\ R'_\mu(x + \epsilon), & x \ge x^*, \end{cases}$$

where x^* is the unique solution of

$$m + R_\mu(x - \epsilon) - (\mathbb{E}\mu + \epsilon) = R_\mu(x + \epsilon).$$

The following corollary establishes an alternative representation of the inequality (2.14), which we will use to prove Theorem 2.11. Note that, in general, (2.14) has more than one solution, not all of which are call functions. However, R^* is always a solution.

Corollary 2.17. *Assume that the conditions from Theorem 2.16 hold and denote the call function of $S_\nu(\mu)$ by R^*. Then for all $x \in \mathbb{R}$ there exists $y \in \mathbb{R} \cup \{\pm\infty\}$ such that*

$$R^*(x) = R_\nu(y) - R_\mu(y + \epsilon\sigma) + R_\mu(x + \epsilon\sigma),$$

where $\sigma = \operatorname{sgn}(y - x)$. Here and in the following we set $R(\infty) = 0$ for all call functions R and

$$R_1(-\infty \pm \epsilon) - R_2(-\infty \pm \epsilon) := \lim_{x \to -\infty} (R_1(x \pm \epsilon) - R_2(x \pm \epsilon)),$$

for call functions R_1 and R_2.

Proof. By Theorem 2.16 we know that R^* is a solution of (2.14). Let x be an arbitrary real number. If $R^*(x) = R_\nu(x)$, then the above relation clearly holds for $y = x$. Otherwise, we have $R^*(x) > R_\nu(x)$, and one of the other two expressions on the left hand side of (2.14) must vanish at x. First we assume that $(R^*)'(x) = R'_\mu(x + \epsilon)$. Define

$$y := \inf\{z \geq x : \ (R^*)'(z) < R'_\mu(z + \epsilon)\}.$$

If $y < \infty$, then by definition $(R^*)'(y) < R'_\mu(y + \epsilon)$. By (2.14), we have $R^*(y) = R_\nu(y)$. It follows that

$$R^*(z) = R_\nu(y) - R_\mu(y + \epsilon) + R_\mu(z + \epsilon), \quad \text{for all } z \in [x, y].$$

If $y = \infty$, then this equation, i.e. $R^*(z) = R_\mu(z + \epsilon)$, $z \geq x$, also holds.

If, on the other hand, $(R^*)'(x) = R'_\mu(x - \epsilon)$, then we similarly define

$$y := \sup\{z \leq x : \ (R^*)'(z) > R'_\mu(z - \epsilon)\}.$$

If $y > -\infty$ then $(R^*)'(y-) > R'_\mu((y - \epsilon)-)$ and hence $R^*(y) = R_\nu(y)$ by (2.14). Therefore we can write

$$R^*(z) = R_\nu(y) - R_\mu(y - \epsilon) + R_\mu(z - \epsilon), \quad \text{for all } z \in [y, x].$$

If $y = -\infty$ then $(R^*)'(z) = R'_\mu(z - \epsilon)$ for all $z \leq x$. The above equation holds if we take the limit $y \to -\infty$ on the right hand side. ❑

Corollary 2.18. *Using Proposition 2.8 and Theorem 2.16, for a given sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, we inductively define the measures*

$$\theta_1 = S(\mu_1; m, \epsilon), \quad \theta_k = S_{\theta_{k-1}}(\mu_k), \quad k \geq 2,$$

if the sets

$$\{\nu \in B^\infty(\mu_k, \epsilon) : \ \theta_{k-1} \leq_c \nu\}$$

are not empty. Then the following relation holds:

$$R_{\theta_n}(x) = R_{\theta_{n-1}}(y) - R_{\mu_n}(y + \epsilon\sigma) + R_{\mu_n}(x + \epsilon\sigma),$$

where $n \geq 2$, $y \in \mathbb{R} \cup \{\pm\infty\}$ depends on x and $\sigma = \operatorname{sgn}(y - x)$.

Proof. The result follows by simply applying Theorem 2.16 and Corollary 2.17 with $\nu = \theta_{n-1}$ and $\mu = \mu_n$. ❑

We can now prove Theorem 2.11, the main result of this chapter. As in Strassen's theorem (Theorem 2.2), the "if" direction is the more difficult one.

Proof of Theorem 2.11. Suppose that (2.10) holds for some $m \in I$ and all $x_1, \ldots, x_N \in \mathbb{R}$, $N \in \mathbb{N}$. We will inductively construct a sequence $(P_n)_{n\in\mathbb{N}}$ of call functions, which will correspond to the measures $(\nu_n)_{n\in\mathbb{N}}$. Define $P_1 = R_1^{\min}(\,\cdot\,; m, \epsilon)$. For $N = 1$, (2.10) yields that $R_1^{\min}(x) \leq R_2^{\max}(x)$. Note that the continuity of the R_n guarantee that (2.10) also holds for $x_n \in \{\pm\infty\}$, if we set $\operatorname{sgn}(\infty - \infty) = \operatorname{sgn}(-\infty + \infty) = 0$. We can now use Theorem 2.16 together with Corollary 2.17, with $R_\nu = R_1^{\min}$ and $R_\mu = R_2$, to construct a call function P_2, which satisfies

$$P_2(x) = R_1^{\min}(x_1) + R_2(x + \epsilon\sigma) - R_2(x_1 + \epsilon\sigma), \quad x \in \mathbb{R},$$

where $\sigma = \operatorname{sgn}(x_1 - x)$, and x_1 depends on x. If we use (2.10) we get that

$$R_1^{\min}(x_1) + R_2(x + \epsilon\sigma_2) - R_2(x_1 + \epsilon\sigma_2) \leq R_n^{\max}(x; s, \epsilon), \quad n \geq 3,\ x_1, x \in \mathbb{R}.$$

Hence $P_2(x) \leq R_n^{\max}(x)$ for all $x \in \mathbb{R}$ and for all $n \geq 3$. Now suppose that we have already constructed a finite sequence $(P_1, \ldots, P_N)$ such that $P_n \leq P_{n+1}$, $1 \leq n < N$, and such that $P_N \leq R_n^{\max}$ for all $x \in \mathbb{R}$ and for all $n \geq N + 1$. Then by induction we know that for all $x \in \mathbb{R}$ there exists $(x_1, \ldots, x_{N-1})$ such that

$$P_N(x) = R_1^{\min}(x_1) + \sum_{n=2}^{N-1} \Big(R_n(x_n + \epsilon\sigma_n) - R_n(x_{n-1} + \epsilon\sigma_n)\Big) + R_N(x + \epsilon\sigma_N) - R_N(x_{N-1} + \epsilon\sigma_N),$$

with $\sigma_N = \operatorname{sgn}(x_{N-1} - x)$. In particular, we have that $P_N \leq R_{N+1}^{\max}$. We can therefore again use Corollary 2.17, with $R_\mu = R_{N+1}$ and $R_\nu = P_N$, to construct a call function P_{N+1}, such that

$$P_{N+1}(x) = R_1^{\min}(x_1) + \sum_{n=2}^{N} \Big(R_n(x_n + \epsilon\sigma_n) - R_n(x_{n-1} + \epsilon\sigma_n)\Big) \\ + R_{N+1}(x + \epsilon\sigma_{N+1}) - R_{N+1}(x_N + \epsilon\sigma_{N+1}),$$

where $\sigma_{N+1} = \operatorname{sgn}(x_N - x)$ and $(x_1, \ldots, x_N)$ depend on x. Assumption (2.10) guarantees that $P_{N+1} \leq R_n^{\max}$ for all $n \geq N + 1$.

We have now constructed a sequence of call functions, such that $P_n \leq P_{n+1}$. Their associated measures, which we will denote by ν_n, satisfy $W^\infty(\mu_n, \nu_n) \leq \epsilon$ and $\nu_n \leq_c \nu_{n+1}$. Thus we have constructed a peacock with mean m.

Conversely, assume that $(\nu_n)_{n\in\mathbb{N}}$ is a peacock such that $W^\infty(\mu_n, \nu_n) \leq \epsilon$ and set $m = \mathbb{E}\nu_1$. Denote the call function of ν_n by P_n. We will show by induction that (2.10) holds. For $N = 1$ we have that

$$R_1^{\min}(x; m, \epsilon) \leq P_1(x) \leq P_2(x) \leq R_2^{\max}(x; m, \epsilon), \quad x \in \mathbb{R},$$

by Proposition 2.8.

For $N = 2$ and $x_1 \le x_2$ we can use (v) of Proposition 2.3 to obtain

$$\begin{aligned} R_1^{\min}(x_1; m, \epsilon) + R_2(x_2 - \epsilon) - R_2(x_1 - \epsilon) &\le P_2(x_1) + \int_{x_1}^{x_2} R_2'(z - \epsilon)\, dz \\ &\le P_2(x_1) + \int_{x_1}^{x_2} P_2'(z)\, dz \\ &= P_2(x_2) \le P_3(x_2) \le R_3^{\max}(x_2; m, \epsilon). \end{aligned}$$

Similarly, if $x_2 \le x_1$,

$$\begin{aligned} R_1^{\min}(x_1; m, \epsilon) + R_2(x_2 + \epsilon) - R_2(x_1 + \epsilon) &\le P_2(x_1) - \int_{x_2}^{x_1} R_2'(z + \epsilon)\, dz \\ &\le P_2(x_1) - \int_{x_2}^{x_1} P_2'(z)\, dz \\ &= P_2(x_2) \le P_3(x_2) \le R_3^{\max}(x_2; m, \epsilon). \end{aligned}$$

If (2.10) holds for $N - 1$ and $x_{N-1} \le x_N$, then

$$\begin{aligned} &R_1^{\min}(x_1; m, \epsilon) + \sum_{n=2}^{N} \left(R_n(x_n + \epsilon\sigma_n) - R_n(x_{n-1} + \epsilon\sigma_n)\right) \\ &\quad\le P_{N-1}(x_{N-1}) + R_N(x_N - \epsilon) - R_N(x_{N-1} - \epsilon) \\ &\quad\le P_N(x_{N-1}) + \int_{x_{N-1}}^{x_N} P_N'(z)\, dz \\ &\quad\le P_{N+1}(x_N) \le R_{N+1}^{\max}(x_N; m, \epsilon). \end{aligned}$$

The case where $x_{N-1} \ge x_N$ can be dealt with similarly. ❑

Remark 2.19. In Theorem 2.11, it is actually not necessary that the balls centered at the measures μ_n are all of the same size. The theorem easily generalises to the following result. For $m \in \mathbb{R}$, a sequence of non-negative numbers $(\epsilon_n)_{n\in\mathbb{N}}$, and a sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, define

$$\begin{aligned} &\Phi_N(x_1, \ldots, x_N; m, \epsilon_1, \ldots, \epsilon_{N+1}) = R_1^{\min}(x_1; m, \epsilon_1) \\ &\qquad + \sum_{n=2}^{N} \Big(R_n(x_n + \epsilon_n\sigma_n) - R_n(x_{n-1} + \epsilon_n\sigma_n)\Big) - R_{N+1}^{\max}(x_N; m, \epsilon_{N+1}), \\ &\qquad\qquad N \in \mathbb{N},\ x_1, \ldots, x_N \in \mathbb{R}, \end{aligned} \tag{2.16}$$

with σ_n defined in (2.7), and assume that

$$I := \bigcap_{n\in\mathbb{N}} [\mathbb{E}\mu_n - \epsilon_n, \mathbb{E}\mu_n + \epsilon_n]$$

is not empty. Then there exists a peacock $(\nu_n)_{n\in\mathbb{N}}$ such that

$$W^\infty(\mu_n, \nu_n) \le \epsilon_n, \quad \text{for all } n \in \mathbb{N},$$

if and only if for some $m \in I$ and for all $N \in \mathbb{N}$, $x_1, \ldots, x_N \in \mathbb{R}$, we have

$$\Phi_N(x_1, \ldots, x_N; m, \epsilon_1, \ldots, \epsilon_{N+1}) \le 0.$$

To prove this result, one simply has to replace ϵ by ϵ_n in the preceding proof.

Remark 2.20. It is easy to see that (2.10) is equivalent to the following condition which is slightly easier to check:

$$R_1^{\min}(x_1; m, \epsilon) + \sum_{n=2}^{N} \Big(R_n(x_n + \epsilon\sigma_n) - R_n(x_{n-1} + \epsilon\sigma_n) \Big) \\ \leq \Big(m + R_{N+1}(x_N + \epsilon) - (\mathbb{E}\mu_{N+1} - \epsilon) \Big) \wedge R_{N+1}(x_N - \epsilon). \quad (2.17)$$

Note that the convex envelope of the right hand side of (2.17) is exactly $R_{N+1}^{\max}(x_N; m, \epsilon)$ and therefore (2.10) implies (2.17). On the other hand if we go through the proof Theorem 2.11 on more time we see that the call functions P_N constructed there satisfy

$$P_N(x) \leq \Big(m + R_{N+1}(x + \epsilon) - (\mathbb{E}\mu_{N+1} - \epsilon) \Big) \wedge R_{N+1}(x - \epsilon), \quad x \in \mathbb{R}.$$

Then, from the convexity of P_N we can deduce that $P_N(x) \leq R_{N+1}^{\max}(x; m, \epsilon)$ and thus (2.17) implies (2.10).

Remark 2.21. If a probability metric is comparable with the infinity Wasserstein distance, then Theorem 2.11 implies a corresponding result about that metric (but, of course, not an "if and only if" condition).

For instance, denote by W^p the p-Wasserstein distance ($p \geq 1$), defined by

$$W^p(\mu, \nu) = \inf \Big(\mathbb{E}[|X - Y|^p] \Big)^{1/p}, \quad \mu, \nu \in \mathcal{M}.$$

The infimum is taken over all probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and random pairs (X, Y) with marginals given by μ and ν. Clearly, we have that for all $\mu, \nu \in \mathcal{M}$ and $p \geq 1$

$$W^\infty(\mu, \nu) \geq W^p(\mu, \nu).$$

Hence, given a sequence $(\mu_n)_{n \in \mathbb{N}}$, (2.10) is a sufficient condition for the existence of a peacock $(\nu_n)_{n \in \mathbb{N}}$, such that $W^p(\mu_n, \nu_n) \leq \epsilon$ for all $n \in \mathbb{N}$. But since the balls with respect to W^p are in general strictly larger than the balls with respect to W^∞, we cannot expect (2.10) to be necessary.

2.2.3. Strassen's theorem for the infinity Wasserstein distance: continuous time

In this section we will formulate a version of Theorem 2.11 for continuous index sets. We generalise the definition of Φ_N from (2.6) as follows. For finite sets $\mathcal{S} = \{t_1, \dots, t_{N+1}\} \subseteq [0, 1]$ with $t_1 < t_2 < \cdots < t_{N+1}$, we set

$$\Phi_{\mathcal{S}}(x_1, \dots, x_N; m, \epsilon) = R_{t_1}^{\min}(x_1; m, \epsilon) \\ + \sum_{n=2}^{N} \Big(R_{t_n}(x_n + \epsilon\sigma_n) - R_{t_{n-1}}(x_n + \epsilon\sigma_n) \Big) - R_{t_{N+1}}^{\max}(x_N; m, \epsilon). \quad (2.18)$$

Here, $R_{t_1}^{\min}$ is the call function of $S(\mu_{t_1}; m, \epsilon)$, $R_{t_{N+1}}^{\max}$ is the call function of $T(\mu_{t_{N+1}}; m, \epsilon)$, and $\sigma_n = \operatorname{sgn}(x_{n-1} - x_n)$ depends on x_{n-1} and x_n. Using $\Phi_{\mathcal{S}}$, we can now formulate a necessary and sufficient condition for the existence of a peacock within ϵ-distance. The continuity assumption (2.19) occurs in the proof in a natural way; we do not know to which extent it can be relaxed.

Theorem 2.22. *Assume that $(\mu_t)_{t\in[0,1]}$ is a family of measures in $\mathcal{M}$ such that*

$$I := \bigcap_{t\in[0,1]} [\mathbb{E}\mu_t - \epsilon, \mathbb{E}\mu_t + \epsilon]$$

is not empty and such that

$$\lim_{s\uparrow t} F_{\mu_s} = F_{\mu_t}, \quad t \in [0,1], \tag{2.19}$$

pointwise on $\mathbb{R}$. Then there exists a peacock $(\nu_t)_{t\in[0,1]}$ with

$$W^\infty(\mu_t, \nu_t) \leq \epsilon, \quad \textit{for all } t \in [0,1],$$

if and only if there exists $m \in I$ such that for all finite sets $\mathcal{S} = \{t_1, \dots, t_{N+1}\} \subset \mathbb{Q} \cap [0,1]$ with $t_1 < t_2 < \dots < t_{N+1}$, and for all $x_1, \dots, x_N \in \mathbb{R}$ we have that

$$\Phi_{\mathcal{S}}(x_1, \dots, x_N; m, \epsilon) \leq 0. \tag{2.20}$$

In this case it is possible to choose $\mathbb{E}\nu_t = m$ for all $t \in [0,1]$.

Proof. By Theorem 2.11, condition (2.20) is clearly necessary for the existence of such a peacock. In order to show that it is sufficient, we will first construct ν_q for $q \in \mathbb{Q} \cap [0,1]$. Therefore fix $m \in I$ such that (2.20) holds and fix $q = \frac{s}{r} \in \mathbb{Q} \cap [0,1]$. We will define a sequence of measures $(\nu_q^{(n)})_{n\in\mathbb{N}}$ as follows (recall the notation from Theorem 2.16): for fixed $n \in \mathbb{N}$ set $\theta_1^{(n)} = S_{\mu_0}(\mu_{\frac{1}{sn}})$ and $\theta_k^{(n)} = S_{\theta_{k-1}^{(n)}}(\mu_{\frac{k}{sn}})$, where $k = 2, \dots rn$. Then

$$\nu_q^{(n)} := \theta_{rn}^{(n)}.$$

Condition (2.20) guarantees that $\nu_q^{(n)}$ exists. Denote the call function of $\nu_q^{(n)}$ by R_n. Then we have that

$$R_{S(\mu_q;m,\epsilon)} \leq R_n \leq R_{n+1} \leq R_{T(\mu_q;m,\epsilon)}, \quad n \in \mathbb{N}, \tag{2.21}$$

and thus the bounded and increasing sequence (R_n) converges pointwise to a function R. As a limit of decreasing convex functions R is also decreasing and convex and together with (2.21) we see that R is a call function with $\lim_{x\to-\infty} = R(x) + x = m$. Therefore R can be associated to a measure $\nu_q \in \mathcal{M}_m$.

Next, we will show that $\nu_q \in B^\infty(\mu_q, \epsilon)$. From the convexity of the R_n we get that

$$\begin{aligned} R'(x) &= \lim_{h\downarrow 0} \lim_{n\to\infty} \frac{R_n(x+h) - R_n(x)}{h} \\ &\geq \lim_{h\downarrow 0} \lim_{n\to\infty} R_n'(x+h) \\ &\geq \lim_{h\downarrow 0} \lim_{n\to\infty} R_{\mu_q}'(x+h-\epsilon) = R_{\mu_q}'(x-\epsilon), \end{aligned}$$

and similarly

$$\begin{aligned} R'(x) &= \lim_{h\downarrow 0} \lim_{n\to\infty} \frac{R_n(x+h) - R_n(x)}{h} \\ &\leq \lim_{h\downarrow 0} \lim_{n\to\infty} R_n'(x) \\ &\leq \lim_{n\to\infty} R_{\mu_q}'(x+\epsilon) = R_{\mu_q}'(x+\epsilon), \end{aligned}$$

thus $W^\infty(\nu_q, \mu_q) \le \epsilon$.

Now for $p, q \in \mathbb{Q} \cap [0,1]$ we want to show that $\nu_p \le_c \nu_q$. We first illustrate the idea for $p = \frac{1}{3}$ and $q = \frac{1}{2}$. Recall that ν_p was defined via an approximating sequence $(\nu_p^{(n)})_{n\in\mathbb{N}}$. An easy observation reveals that $\nu_p^{(2)} \le_c \nu_q^{(3)}$: Indeed, $\nu_p^{(2)}$ is defined to be the smallest element in $B^\infty(\mu_p, \epsilon) \cap \mathcal{M}_m$ which dominates $S_{\mu_0}(\mu_{\frac{1}{6}})$, and $\nu_q^{(3)}$ is defined to be the smallest element in $B^\infty(\mu_q, \epsilon) \cap \mathcal{M}_m$ which dominates $\nu_p^{(2)}$. With similar arguments we can show that $\nu_p^{(4)} \le_c \nu_q^{(6)}$, or more generally $\nu_p^{(2^k)} \le_c \nu_q^{(3\cdot 2^{k-1})}$, for all $k \in \mathbb{N}$.

For general $p, q \in \mathbb{Q} \cap [0,1]$ with $p = \frac{r_1}{s_1}$ and $q = \frac{r_2}{s_2}$, we pick subsequences $(l_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ such that $l_k s_1 = n_k s_2$ for all $k \in \mathbb{N}$. Then clearly $\nu_p^{(l_k)} \le_c \nu_q^{(n_k)} \le_c \nu_q$ for all $k \in \mathbb{N}$ and therefore $\nu_p \le_c \nu_q$. We have shown that $(\nu_t)_{t\in\mathbb{Q}\cap[0,1]}$ is a peacock.

The next step is to define measures ν_t for $t \notin \mathbb{Q} \cap [0,1]$. Therefore pick such a t and an increasing sequence $q_n \in \mathbb{Q} \cap [0,1]$ which converges to t. Similar reasoning as before shows that the sequence $(R_{\nu_{q_n}})_{n\in\mathbb{N}}$ converges pointwise to a call function, and we define ν_t to be the associated measure. Then clearly $\mathbb{E}\nu_t = m$. Furthermore, using the continuity of the distribution functions, we get that

$$\begin{aligned} R'_{\nu_t}(x) &= \lim_{h\downarrow 0} \lim_{n\to\infty} \frac{R_{\nu_{q_n}}(x+h) - R_{\nu_{q_n}}(x)}{h} \\ &\ge \lim_{h\downarrow 0} \lim_{n\to\infty} R'_{\nu_{q_n}}(x+h) \\ &\ge \lim_{h\downarrow 0} \lim_{n\to\infty} R'_{\mu_{q_n}}(x+h-\epsilon) \\ &= \lim_{h\downarrow 0} R'_{\mu_t}(x+h-\epsilon) = R'_{\mu_t}(x-\epsilon), \end{aligned}$$

and similarly we see that $R'_{\nu_t}(x) \le R'_{\mu_t}(x+\epsilon)$. We have shown that $\nu_t \in B^\infty(\mu_t, \epsilon)$ for all $t \in [0,1]$.

From the definition of ν_t we have that $\nu_q \le_c \nu_t$ for $q < t, q \in \mathbb{Q} \cap [0,1]$ and $\nu_t \le_c \nu_p$ for $p > t, p \in \mathbb{Q} \cap [0,1]$. This implies $\nu_s \le_c \nu_t$ for all $0 \le s \le t \le 1$, thus $(\nu_t)_{t\in[0,1]}$ is a peacock with mean m. ❑

2.3. Strassen's theorem for the stop-loss distance

For two measures $\mu, \nu \in \mathcal{M}$ we define the stop-loss distance as

$$d^{\mathrm{SL}}(\mu, \nu) = \sup_{x\in\mathbb{R}} |R_\mu(x) - R_\nu(x)|.$$

We will denote closed balls with respect to d^{SL} by B^{SL}. In the following proposition, we use the same notation for least elements as in the case of the infinity Wasserstein distance; no confusion should arise.

Proposition 2.23. *Given $\epsilon > 0$, a measure $\mu \in \mathcal{M}$ and $m \in [\mathbb{E}\mu - \epsilon, \mathbb{E}\mu + \epsilon]$, there exists a unique measure $S(\mu) \in B^{\mathrm{SL}}(\mu, \epsilon) \cap \mathcal{M}_m$, such that*

$$S(\mu) \le_c \nu, \quad \text{for all } \nu \in B^{\mathrm{SL}}(\mu, \epsilon) \cap \mathcal{M}_m.$$

The call function of $S(\mu)$ is given by

$$R_\mu^{\min}(x) = R_{S(\mu)}(x) = (m-x)^+ \vee (R_\mu(x) - \epsilon). \tag{2.22}$$

To highlight the dependence on ϵ and m we will sometimes write $S(\mu; m, \epsilon)$ or $R_\mu^{\min}(\cdot; m, \epsilon)$.

Proof. It is easy to check that $R_{S(\mu)}$ defines a call function, and by *(ii)* of Proposition 2.3 we have that

$$\begin{aligned}
\mathbb{E}R_{S(\mu)} &= \lim_{x\to-\infty} R_{S(\mu)}(x) + x \\
&= \lim_{x\to-\infty} (m \vee (R_\mu(x) + x - \epsilon)) \\
&= m \vee (\mathbb{E}\mu - \epsilon) = m.
\end{aligned}$$

The rest is clear. ❑

Remark 2.24. The set $B^{\mathrm{SL}}(\mu,\epsilon)\cap\mathcal{M}_m$ does not contain a greatest element. To see this, take an arbitrary $\nu \in B^{\mathrm{SL}}(\mu,\epsilon)\cap\mathcal{M}_m$ and define $x_0 \in \mathbb{R}$ as the unique solution of $R_\nu(x) = \frac{1}{2}\epsilon$. Then for $n \in \mathbb{N}$ define new call functions

$$R_n(x) = \begin{cases} (x-x_0)\frac{R_\nu(x_0+n)-R_\nu(x_0)}{n} + R_\nu(x_0), & x \in [x_0, x_0+n], \\ R_\nu(x), & \text{otherwise.} \end{cases}$$

It is easy to check that R_n is indeed a call function and the associated measures θ_n lie in $B^{\mathrm{SL}}(\mu,\epsilon)\cap\mathcal{M}_m$. Furthermore, from the convexity of R_ν we can deduce that $R_\nu \le R_n \le R_{n+1}$, and hence $\nu \le_{\mathrm{c}} \theta_n \le_{\mathrm{c}} \theta_{n+1}$. The call functions R_n converge to a function R which is not a call function since $R(x) = R_\nu(x_0) = \frac{\epsilon}{2}$ for all $x \ge x_0$. Therefore no greatest element can exist.

However it is true that a measure ν is in $B^{\mathrm{SL}}(\mu,\epsilon)$ if and only if

$$R_\mu^{\min}(\,.\,;\mathbb{E}\nu,\epsilon) \le R_\nu \le R_\mu + \epsilon.$$

Theorem 2.25. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$ such that*

$$I := \bigcap_{n\in\mathbb{N}} [\mathbb{E}\mu_n - \epsilon, \mathbb{E}\mu_n + \epsilon],$$

is not empty. Denote by $(R_n)_{n\in\mathbb{N}}$ the corresponding call functions. Then there exists a peacock $(\nu_n)_{n\in\mathbb{N}}$ such that

$$d^{\mathrm{SL}}(\mu_n, \nu_n) \le \epsilon, \quad n \in \mathbb{N}, \tag{2.23}$$

if and only if for some $m \in I$

$$R_k^{\min}(x; m, \epsilon) \le R_n(x) + \epsilon, \quad \text{for all } k \le n \text{ and } x \in \mathbb{R}. \tag{2.24}$$

Here $R_k^{\min}$ denotes the call function of $S(\mu_k; m, \epsilon)$. In this case it is possible to choose $\mathbb{E}\nu_1 = m$.

Proof. Suppose (2.24) holds for $m \in I$. We will define the ν_n via their call functions P_n. Therefore we set $P_1(x) = R_1^{\min}(x; m, \epsilon)$ and

$$P_n(x) = \max\{P_{n-1}(x), R_n^{\min}(x; m, \epsilon)\}, \quad n \geq 2. \tag{2.25}$$

It is easily verified that P_n is a call function and satisfies

$$R_n^{\min}(x) \leq P_n(x) \leq R_n(x) + \epsilon, \quad x \in \mathbb{R}, \tag{2.26}$$

and therefore ν_n, the measure associated to P_n, satisfies $\nu_n \in B^{\mathrm{SL}}(\mu_n, \epsilon)$. Furthermore $P_n \leq P_{n+1}$, and thus $(\nu_n)_{n\in\mathbb{N}}$ is a peacock with mean m.

Now assume that $(\nu_n)_{n\in\mathbb{N}}$ is a peacock such that $d^{\mathrm{SL}}(\mu_n, \nu_n) \leq \epsilon$. We will denote the call function of ν_n by P_n and set $m = \mathbb{E}\nu_1 \in I$. Then for $k \leq n$ and $x \in \mathbb{R}$ we get with Proposition 2.23

$$R_k^{\min}(x; m, \epsilon) \leq P_k(x) \leq P_n(x) \leq R_n(x) + \epsilon.$$

❑

Note that (2.24) trivially holds for $k = n$. Moreover, unwinding the recursive definition (2.25) and using (2.22), we see that P_n has the explicit expression

$$P_n(x) = \max\{(m - x)^+, R_1(x) - \epsilon, \ldots, R_n(x) - \epsilon\}, \quad x \in \mathbb{R}, n \in \mathbb{N}.$$

The following proposition shows that the peacock from Theorem 2.25 is never unique.

Proposition 2.26. *In the setting of Theorem 2.25, suppose that (2.24) holds. Then there are infinitely many peacocks satisfying (2.23).*

Proof. Define P_n as in the proof of Theorem 2.25, and fix $x_0 \in \mathbb{R}$ with $P_1(x_0) < \epsilon$. For arbitrary $c \in (0, 1)$, we define

$$G(x) = \begin{cases} P_1(x_0), & x \leq x_0, \\ P_1(x_0) + cP_1'(x_0)(x - x_0), & x \geq x_0. \end{cases}$$

Thus, in a right neighborhood of x_0, the graph of G is a line that lies above P_1. We then put $\tilde{P}_n = P_n \vee G$, for $n \in \mathbb{N}$. It is easy to see that $(\tilde{P}_n)_{n\in\mathbb{N}}$ is an increasing sequence of call functions with mean m, and thus defines a peacock. Moreover, we have

$$\tilde{P}_n \leq (R_n + \epsilon) \vee G \leq R_n + \epsilon,$$

by (2.26) and the fact that $G \leq \epsilon$. The lower estimate $\tilde{P}_n \geq P_n \geq R_n - \epsilon$ is also obvious. ❑

Theorem 2.25 easily extends to continuous index sets.

Theorem 2.27. *Assume that $(\mu_t)_{t\in[0,1]}$ is a family of measures in $\mathcal{M}$ such that*

$$I := \bigcap_{t\in[0,1]} [\mathbb{E}\mu_t - \epsilon, \mathbb{E}\mu_t + \epsilon]$$

is not empty. Denote the call function of μ_t by R_t. Then there exists a peacock $(\nu_t)_{t\in[0,1]}$ with

$$d^{\mathrm{SL}}(\mu_t, \nu_t) \leq \epsilon, \quad \textit{for all } t \in [0, 1],$$

if and only if there exists $m \in I$ such that for all $0 \leq s < t \leq 1$ we have that

$$R_s^{\min}(x; m, \epsilon) \leq R_t(x) + \epsilon, \quad \textit{for all } x \in \mathbb{R}. \tag{2.27}$$

Here $R_s^{\min}$ denotes the call function of $S(\mu_s; m, \epsilon)$. In this case it is possible to choose $\mathbb{E}\nu_1 = m$.

Proof. If (2.27) holds for $m \in I$ we set

$$P_t(x) = \sup_{s \leq t} R_s^{\min}(x; m, \epsilon), \quad t \in [0, 1].$$

Then P_t is a call function which satisfies $R_t^{\min}(x; m, \epsilon) \leq P_t(x) \leq R_t(x) + \epsilon$ for $x \in \mathbb{R}$. The rest can be done as in the proof of Theorem 2.25. ❑

2.4. Lévy distance and Prokhorov distance

We will begin with the definition the Lévy distance and the Prokhorov distance. For further information concerning these metrics, their properties and their relations to other metrics, we refer the reader to [53] (p.27 ff). The Lévy distance is a metric on the set of all measures on $\mathbb{R}$, defined as

$$d^{\mathrm{L}}(\mu, \nu) = \inf\Big\{h > 0 : F_\mu(x - h) - h \leq F_\nu(x) \leq F_\mu(x + h) + h, \ \forall x \in \mathbb{R}\Big\}.$$

Its importance is partially due to the fact that d^{L} metrizes weak convergence of measures on $\mathbb{R}$. The Prokhorov distance is a metric on measures on an arbitrary separable metric space (S, ρ). For measures μ, ν on S it can be written as

$$d^{\mathrm{P}}(\mu, \nu) = \inf\Big\{h > 0 : \nu(A) \leq \mu(A^h) + h, \ \text{for all closed sets } A \subseteq S\Big\},$$

where $A^h = \{x \in S : \inf_{a \in A} \rho(x, a) \leq h\}$. We will denote closed balls with respect to d^{L} resp. d^{P} by B^{L} resp. B^{P}.

The Prokhorov distance is often referred to as a generalisation of the Lévy metric, since d^{P} metrizes weak convergence on any separable metric space. Note, though, that d^{L} and d^{P} do not coincide when $(S, \rho) = (\mathbb{R}, |\,.\,|)$, as shown in the following example.

Example 2.28. Let $\epsilon = \frac{1}{8}$, μ be the uniform distribution on $[0, 1]$, and ν be the uniform distribution on $[2\epsilon, 1 - 2\epsilon]$. Then it is easy to check that $d^{\mathrm{L}}(\mu, \nu) \leq \epsilon$. Also we have that

$$F_\mu\Big(\frac{1}{4} - \epsilon\Big) - \epsilon = F_\nu\Big(\frac{1}{4}\Big),$$

hence $d^{\mathrm{L}}(\mu, \nu) = \epsilon$. Next, we will show that the Prokhorov distance of μ and ν is larger than $\frac{1}{6}$, and hence not equal to the Lévy distance. Consider the closed set $A = [2\epsilon, 1 - 2\epsilon]$. Then $\nu(A) = 1$, and the inequality

$$\begin{aligned} 1 &\leq \mu(A^h) + h \\ &= \mu([2\epsilon - h, 1 - 2\epsilon + h]) + h = 1 - 4\epsilon + 3h \end{aligned}$$

is true for all $h \geq \frac{1}{6}$, and therefore $d^{\mathrm{P}}(\mu, \nu) \geq \frac{1}{6}$.

It is easy to see that the Prokhorov distance of two measures on $\mathbb{R}$ is an upper bound for the Lévy distance. See [53] p.36; we include the simple proof for completeness.

Lemma 2.29. *Let μ and ν be two probability measures on $\mathbb{R}$. Then*

$$d^{\mathrm{L}}(\mu,\nu) \le d^{\mathrm{P}}(\mu,\nu).$$

Proof. We set $\epsilon = d^{\mathrm{P}}(\mu,\nu)$. Then for any $x \in \mathbb{R}$ and all $n \in \mathbb{N}$ we have that

$$\begin{aligned} F_\nu(x) = \nu((-\infty,x]) &\le \mu\Big(\Big(-\infty, x+\epsilon+\frac{1}{n}\Big]\Big) + \epsilon + \frac{1}{n} \\ &= F_\mu\Big(x+\epsilon+\frac{1}{n}\Big) + \epsilon + \frac{1}{n}, \end{aligned}$$

and by the symmetry of d^{P} the above relation also holds with μ and ν interchanged. This implies that $d^{\mathrm{L}}(\mu,\nu) \le \epsilon$. ❑

2.4.1. Modified Lévy distance and Prokhorov distance

We will first define slightly different distances d_p^{L} and d_p^{P} on the set of probability measures on $\mathbb{R}$, which in general are not metrics in the classical sense (recall the remark before Proposition 2.6). These distances are useful for two reasons: First, it will turn out that balls with respect to d^{L} and d^{P} can always be written as balls w.r.t. d_p^{L} and d_p^{P}, see Lemma 2.30. Second, the function d_p^{P} has a direct link to minimal distance couplings which are especially useful for applications, see Proposition 2.32 and Theorem 3.15.

For $p \in [0,1]$ we define

$$d_p^{\mathrm{L}}(\mu,\nu) := \inf\Big\{h > 0 :\ F_\mu(x-h) - p \le F_\nu(x) \le F_\mu(x+h) + p, \forall x \in \mathbb{R}\Big\} \tag{2.28}$$

and

$$d_p^{\mathrm{P}}(\mu,\nu) := \inf\Big\{h > 0 : \nu(A) \le \mu(A^h) + p,\ \text{for all closed sets } A \subseteq S\Big\}. \tag{2.29}$$

It is easy to show that $d_p^{\mathrm{P}}(\mu,\nu) = d_p^{\mathrm{P}}(\nu,\mu)$ (see e.g. Propositon 1 in [29]). Note that $d_p(\mu,\nu) = 0$ does not imply that $\mu = \nu$. We will refer to d_p^{L} as the modified Lévy distance, and to d_p^{P} as the modified Prokhorov distance. The corresponding closed balls are denoted by B_p^{L} resp. B_p^{P}.

The following Lemma explains the connection between the Lévy distance d^{L} and the modified Lévy distance d_p^{L}, resp. the Prokhorov distance d^{P} and the modified Prokhorov distance d_p^{P}.

Lemma 2.30. *Let $\mu \in \mathcal{M}$. Then for every $\epsilon \in [0,1]$ we have that*

$$B^{\mathrm{L}}(\mu,\epsilon) = B_\epsilon^{\mathrm{L}}(\mu,\epsilon), \quad \text{and} \quad B^{\mathrm{P}}(\mu,\epsilon) = B_\epsilon^{\mathrm{P}}(\mu,\epsilon).$$

Proof. For $\nu \in \mathcal{M}$, the assertion $\nu \in B^{\mathrm{P}}(\mu,\epsilon)$ is equivalent to

$$\mu(A) \le \nu(A^{\epsilon+\delta}) + \epsilon + \delta, \quad \delta > 0,\ A \subseteq \mathbb{R} \text{ closed}, \tag{2.30}$$

whereas $\nu \in B_\epsilon^{\mathrm{P}}(\mu,\epsilon)$ means that

$$\mu(A) \le \nu(A^{\epsilon+\delta}) + \epsilon, \quad \delta > 0,\ A \subseteq \mathbb{R} \text{ closed.} \tag{2.31}$$

Obviously, (2.31) implies (2.30). Now suppose that (2.30) holds, and let $\delta \downarrow 0$. Notice that $A^{\epsilon+\delta_1} \subseteq A^{\epsilon+\delta_2}$ for $\delta_1 \le \delta_2$. The continuity of ν then gives

$$\mu(A) \le \nu(A^\epsilon) + \epsilon \le \nu(A^{\epsilon+\delta}) + \epsilon \quad \delta > 0,\ A \subseteq \mathbb{R} \text{ closed,}$$

and thus $B^{\mathrm{P}}(\mu,\epsilon) = B_\epsilon^{\mathrm{P}}(\mu,\epsilon)$.

Replacing A by intervals $(-\infty, x]$ for $x \in \mathbb{R}$ in (2.30) and (2.31) proves that $B^{\mathrm{L}}(\mu,\epsilon) = B_\epsilon^{\mathrm{L}}(\mu,\epsilon)$. ❏

Similarly to Lemma 2.29 we can show that the modified Lévy distance of two measures never exceeds the modified Prokhorov distance.

Lemma 2.31. *Let μ and ν be two probability measures on $\mathbb{R}$ and let $p \in [0,1]$. Then*

$$d_p^{\mathrm{L}}(\mu,\nu) \le d_p^{\mathrm{P}}(\mu,\nu).$$

Proof. We set $\epsilon = d_p^{\mathrm{P}}(\mu,\nu)$. Then for any $x \in \mathbb{R}$ and all $n \in \mathbb{N}$ we have that

$$\begin{aligned} F_\nu(x) = \nu((-\infty, x]) &\le \mu\Big(\Big(-\infty, x + \epsilon + \frac{1}{n}\Big]\Big) + p \\ &= F_\mu\Big(x + \epsilon + \frac{1}{n}\Big) + p, \end{aligned}$$

and by the symmetry of d^{P} the above relation also holds with μ and ν interchanged. This implies that $d_p^{\mathrm{L}}(\mu,\nu) \le \epsilon$. ❏

The following result was first proved by Strassen and was then extended by Dudley [29, 83]. It explains the connection of d_p^{P} to minimal distance couplings.

Proposition 2.32. *Given measures μ,ν on $\mathbb{R}$, $p \in [0,1]$, and $\epsilon > 0$ there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with random variables $X \sim \mu$ and $Y \sim \nu$ such that*

$$\mathbb{P}(|X - Y| > \epsilon) \le p, \tag{2.32}$$

if and only if

$$d_p^{\mathrm{P}}(\mu,\nu) \le \epsilon. \tag{2.33}$$

2.4.2. Strassen's theorem for Prokhorov distance and Lévy distance

In this section we will prove variants of Strassen's theorem, first for the modified Prokhorov distance and later on for the modified Lévy distance, the Prokhorov distance, and the Lévy distance. It turns out that Problem 2.4 always has a solution for these distances, regardless of the size of ϵ. In the following we denote the quantile function of a measure $\mu \in \mathcal{M}$ by G_μ, i.e.

$$G_\mu(p) = \inf\{x \in \mathbb{R} :\ F_\mu(x) \ge p\}, \quad p \in [0,1].$$

Proposition 2.33. *Let $\mu \in \mathcal{M}$, $p \in (0,1]$, and $m \in \mathbb{R}$. Then the set*

$$B_p^{\mathrm{P}}(\mu, 0) \cap \mathcal{M}_m$$

is not empty. Moreover, this set contains at least one measure with bounded support.

Proof. The statement is clear for $p = 1$, and so so we focus on $p \in (0,1)$. Given a measure μ we set $I = [G_\mu(\frac{p}{4}), G_\mu(1 - \frac{p}{4}))$. We will first define a measure η with bounded support which lies in $B_p^{\mathrm{P}}(\mu, 0)$, and then we will modify it to obtain a measure θ with mean m. We set

$$F_\eta(x) := \begin{cases} 0, & x < G_\mu(\frac{p}{4}), \\ F_\mu(x), & x \in I, \\ 1, & x \geq G_\mu(1 - \frac{p}{4}), \end{cases}$$

which is clearly a distribution function of a measure η. Note that η has finite support, so in particular η has finite mean. Next we define

$$\theta = \Big(1 - \frac{p}{2}\Big)\eta + \frac{p}{2}\delta_w,$$

where w is chosen such that $\mathbb{E}\theta = m$. Since η has bounded support, we can deduce that θ also has bounded support. Now for every closed set $A \subseteq \mathbb{R}$ we have that

$$\begin{aligned} \theta(A) &\leq (1 - \frac{p}{2})\eta(A) + \frac{p}{2} \\ &\leq (1 - \frac{p}{2})\eta(A \cap \mathrm{int}(I)) + p \\ &\leq \mu(A) + p, \end{aligned}$$

where $\mathrm{int}(I)$ denotes the interior of I. For the last inequality, note that μ and η are equal on $\mathrm{int}(I)$. The last equation implies that $\theta \in B_p^{\mathrm{P}}(\mu, 0) \cap \mathcal{M}_m$. ❑

Note that in Proposition 2.33 it is not important that μ has finite mean. The statement is true for all measures on $\mathbb{R}$. The same is true for all subsequent results.

Proposition 2.34. *Let $\nu \in \mathcal{M}$ be a measure with bounded support and $p \in (0,1)$. Then for all measures $\mu \in \mathcal{M}$ there exists a measure $\theta \in B_p^{\mathrm{P}}(\mu, 0)$ with bounded support such that $\nu \leq_{\mathrm{c}} \theta$.*

Proof. Fix $\mu, \nu \in \mathcal{M}$ and $p \in (0,1)$, and set $m = \mathbb{E}\nu$. Then, by Proposition 2.33, there is a measure $\theta_0 \in B_{p/2}^{\mathrm{P}}(\mu, 0) \cap \mathcal{M}_m$ which has bounded support. For $n \in \mathbb{N}$ we define

$$\theta_n = (1 - \frac{p}{2})\theta_0 + \frac{p}{4}\delta_{m-n} + \frac{p}{4}\delta_{m+n}.$$

These measures have bounded support and mean m. Furthermore, for $A \subseteq \mathbb{R}$ closed, we have

$$\begin{aligned} \theta_n(A) &\leq (1 - \frac{p}{2})\theta_0(A) + \frac{p}{2} \\ &\leq \theta_0(A) + \frac{p}{2} \\ &\leq \mu(A) + p, \quad n \in \mathbb{N}, \end{aligned}$$

and hence $\nu_n \in B_p^{\mathrm{P}}(\mu, 0)$ for all $n \in \mathbb{N}$. Now observe that for all $n \in \mathbb{N}$ and $x \in (m - n, m+n)$ we have

$$R_{\theta_n}(x) = (1 - \frac{p}{2})R_{\theta_0}(x) + \frac{p}{4}(m + n - x), \tag{2.34}$$

which tends to infinity as n tends to infinity. Therefore there has to exist $n_0 \in \mathbb{N}$ such that $\nu \leq_{\mathrm{c}} \theta_{n_0}$. ❑

In Proposition 2.34 it is important that $p > 0$. For $p = 0$ the limit in 2.34 is finite.

Theorem 2.35. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$, $\epsilon > 0$. and $p \in (0, 1]$. Then, for all $m \in \mathbb{R}$ there exists a peacock $(\nu_n)_{n\in\mathbb{N}}$ with mean m such that*

$$d_p^{\mathrm{P}}(\mu_n, \nu_n) \leq \epsilon.$$

Proof. If $p = 1$ then $B_p^{\mathrm{P}}(\mu, 0)$ contains all probability measures on $\mathbb{R}$, which is easily seen from the definition of d_p^{P}, and the result is trivial. So we consider the case $p < 1$. Since $B_p^{\mathrm{P}}(\mu, 0) \subseteq B_p^{\mathrm{P}}(\mu, \epsilon)$, it suffices to prove the statement for $\epsilon = 0$. By Proposition 2.33, there exists a measure $\nu_1 \in B_p^{\mathrm{P}}(\mu_1, 0) \cap \mathcal{M}_m$ with bounded support. By Proposition 2.34 there exists a measure $\nu_2 \in B_p^{\mathrm{P}}(\mu_2, 0)$ such that $\nu_1 \leq_{\mathrm{c}} \nu_2$. Since ν_2 has again finite support, we can proceed inductively to finish the proof. ❑

Setting $\epsilon = p \in (0, 1]$ in the previous result, we obtain the following corollary.

Corollary 2.36. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$ and $\epsilon > 0$. Then, for all $m \in \mathbb{R}$ there exists a peacock $(\nu_n)_{n\in\mathbb{N}}$ with mean m such that*

$$d^{\mathrm{P}}(\mu_n, \nu_n) \leq \epsilon.$$

Proof. By Lemma 2.30 we have that $B^{\mathrm{P}}(\mu, \epsilon) = B_\epsilon^{\mathrm{P}}(\mu, \epsilon)$ for all $\mu \in \mathcal{M}$ and $\epsilon \in [0, 1]$. The result now easily follows from Theorem 2.35. ❑

Since balls with respect to the modified Prokhorov metric are larger than balls with respect to the Lévy metric, we get the following corollary.

Theorem 2.37. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$, $\epsilon > 0$, and $p \in (0, 1]$. Then, for all $m \in \mathbb{R}$ there exists a peacock $(\nu_n)_{n\in\mathbb{N}}$ with mean m such that*

$$d_p^{\mathrm{L}}(\mu_n, \nu_n) \leq \epsilon.$$

In particular, there exists a peacock $(\nu_n)_{n\in\mathbb{N}}$ with mean m such that

$$d^{\mathrm{L}}(\mu_n, \nu_n) \leq \epsilon.$$

Proof. Fix $\epsilon > 0$ and $p \in (0, 1]$, and let $(\nu_n)_{n\in\mathbb{N}}$ be the peacock from Theorem 2.35 resp. Corollary 2.36. Then by Lemma 2.31 resp. Lemma 2.29, we have that $\nu_n \in B_p^{\mathrm{L}}(\mu_n, \epsilon)$ resp. $\nu_n \in B^{\mathrm{L}}(\mu_n, \epsilon)$ for all $n \in \mathbb{N}$. ❑

2.4.3. Least element for the modified Lévy distance

In this subsection we will show that sets of the $B_p^{\mathrm{L}}(\mu,\epsilon)\cap\mathcal{M}_m$ contain a least element. It is described in the Proposition 2.39.

Before we formulate it we make some preparations.

Lemma 2.38. *Given a measure $\mu\in\mathcal{M}$, $p\in(0,1)$, $\epsilon>0$ and $m\in\mathbb{R}$, set*

$$a = G_\mu(p)+\epsilon, \quad b = G_\mu(1-p)-\epsilon,$$

as well as

$$\begin{aligned} P_\mu(x) &= \Big(R_\mu(x-\epsilon) - px - (R_\mu(a-\epsilon) - pa) + (m-a)^+\Big)\mathbb{1}_{[a,\infty)}(x), \\ Q_\mu(x) &= \Big(R_\mu(x+\epsilon) + px - (R_\mu(b+\epsilon)+pb)\Big)\mathbb{1}_{(-\infty,b]}(x), \end{aligned}$$

for $x\in\mathbb{R}$. Then the following statements hold.

(i) P_μ is strictly decreasing and convex on $[a,\infty)$ and satisfies $P_\mu(a)=(m-a)^+$ and $\lim_{x\to\infty} P_\mu(x)=-\infty$.

(ii) Q_μ is decreasing and convex on $\mathbb{R}$ and satisfies $Q_\mu(b)$=0 and $Q_\mu(x)<(m-x)^+$ for all sufficiently small x.

(iii) The function

$$R_{S(\mu)} = \max\Big\{P_\mu(x), Q_\mu(x), (m-x)^+\Big\}, \quad x\in\mathbb{R}, \tag{2.35}$$

is convex and decreasing.

Proof. The assertions in *(i)* and *(ii)* are clear. For *(iii)* note that the function $R_{S(\mu)}$ is convex and decreasing on $[a,\infty)$. For $x<a$ we have that

$$R_{S(\mu)}(x) = \max\{Q_\mu(x),(m-x)^+\}$$

and hence $R_{S(\mu)}$ is convex and decreasing on $(-\infty,a)$. We will now argue that $R'_{S(\mu)}(a-)\le R'_{S(\mu)}(a+)$, which will establish the convexity of $R_{S(\mu)}$ on $\mathbb{R}$. If $R_{S(\mu)}(a)>P_\mu(a)$ then there has to exist an interval I containing a, such that $R_{S(\mu)}(x)=Q_\mu(x)$ for all $x\in I$, which implies the convexity $R_{S(\mu)}$ in this case. On the other hand, if $R_{S(\mu)}(a)=P_\mu(a)$ then $R'_{S(\mu)}(a-)=-1\le R'_{S(\mu)}(a+)$. From the convexity of $R_{S(\mu)}$ we can deduce its continuity and therefore $R(S(\mu))$ is decreasing on $\mathbb{R}$. ❑

Proposition 2.39. *Given a measure $\mu\in\mathcal{M}$, $p\in(0,1)$, $\epsilon>0$ and $m\in\mathbb{R}$, the set $B_p^{\mathrm{L}}(\mu,\epsilon)\cap\mathcal{M}_m$ contains a least element with respect to $\le_{\mathrm{c}}$, i.e. there exists a measure $S(\mu)$ such that*

$$S(\mu)\le_{\mathrm{c}}\theta, \quad \text{for all } \theta\in B_p^{\mathrm{L}}(\mu,\epsilon)\cap\mathcal{M}_m.$$

The call function of $S(\mu)$ is given by (2.35). Furthermore $S(\mu)$ has bounded support.

Proof. From Lemma 2.38 we know that $R_{S(\mu)}$ is convex and decreasing and from *(i)* and *(ii)* we can deduce that

$$\lim_{x\to-\infty} R_{S(\mu)}(x) + x = \lim_{x\to-\infty} (m-x)^+ + x = m,$$

and

$$\lim_{x\to\infty} R_{S(\mu)}(x) = \lim_{x\to\infty} (m-x)^+ = 0.$$

Therefore $R_{S(\mu)}$ is a call function and the associated measure has bounded support.

The minimality of $S(\mu)$ can be shown similar to the case where $p = 0$: to the right of a, P_μ describes a function that is as steep as possible. To the left of a there is no lower bound to the right-derivative of call functions (except of course the trivial bound -1). Similarly, to the left of b, Q_μ describes a function that is as flat as possible, to the right of b the upper bound for the right derivative is 0. ❑

3

Consistency of Option prices in markets with bid-ask spreads

In this chapter we will apply the results for the infinity Wasserstein distance and the modified Prokhorov distance. Given a finite set of European call option prices on a single underlying, we want to know when there is a market model which is consistent with these prices. In contrast to previous studies, we allow models where the underlying trades at a bid-ask spread. The next section explain our notation.

3.1. Notation and Preliminaries

Our time index set will be $\mathcal{T} = \{0, \dots, T\}$ throughout, where $1 \leq T \in \mathbb{N}$. Whenever we talk about "the given prices" or similarly, we mean the following data:

$$\text{A positive deterministic bank account } (B(t))_{t \in \mathcal{T}} \text{ with } B(0) = 1, \tag{3.1}$$

$$\text{strikes} \quad 0 < K_{t,1} < K_{t,2} < \cdots < K_{t,N_t}, \quad N_t \geq 1,\ t \in \mathcal{T}, \tag{3.2}$$

corresponding call option bid and ask prices (at time zero)

$$0 \leq \underline{r}_{t,i} \quad \text{resp.} \quad 0 \leq \overline{r}_{t,i}, \quad \text{such that} \quad \underline{r}_{t,i} \leq \overline{r}_{t,i} \qquad 1 \leq i \leq N_t,\ t \in \mathcal{T}, \tag{3.3}$$

$$\text{and the current bid and ask price of the underlying} \quad 0 < \underline{S}_0 \leq \overline{S}_0. \tag{3.4}$$

We write $D(t) = B(t)^{-1}$ for the time zero price of a zero-coupon bond maturing at t. The discounted strikes will be denoted by $k_{t,i} = D(t)K_{t,i}$.

In the presence of a bid-ask spread on the underlying, it is not obvious how to define the payoff of an option; this issue seems to have been somewhat neglected in the transaction costs literature. Indeed, suppose that an agent holds a call option with strike \$100, and that at maturity $T = 1$ bid and ask are $\underline{S}_1 = \$90$ resp. $\overline{S}_1 = \$110$. Then, the agent might wish to exercise the option to obtain a security that would cost him \$10 more in the market, or he may forfeit the option on the grounds that spending \$100 would earn him a position whose liquidation value is only \$90. Thus, the exercise decision cannot be nailed down without making any further assumptions.

In the literature on option pricing under transaction costs , it is usually assumed that bid and ask of the underlying are *constant* multiples of a mid-price (often assumed to be geometric Brownian motion). This mid-price is then used as trigger to decide whether an option should be exercised, followed by physical delivery [9, 23, 86]. The assumption that such a constant-proportion mid-price triggers exercise seems to be rather ad-hoc, though. To deal with this problem in a parsimonious way, we assume that call options are cash-settled, using a reference price process S^C. This process evolves within the bid-ask spread. It is not a traded asset by itself, but just serves to fix the call option payoff $(S_t^C - K)^+$ for strike K and maturity t. This payoff is immediately transferred to the bank account without any costs.

Definition 3.1. A *model* consists of a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a discrete filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$ and three adapted stochastic processes $\underline{S}$, $\overline{S}$, and S^C, satisfying

$$0 < \underline{S}_t \leq S_t^C \leq \overline{S}_t, \quad t \in \mathcal{T}.$$

We now give a definition for consistency of option prices, allowing for bid-ask spreads on both the underlying and the options.

Definition 3.2. The prices (3.1)-(3.4) are *consistent with the absence of arbitrage,* if there is a model such that

- $\mathbb{E}[(D(t)S_t^C - k_{t,i})^+] \in [\underline{r}_{t,i}, \overline{r}_{t,i}], \quad 1 \leq i \leq N_t,\ t \in \mathcal{T}.$
- There is a consistent price system for the underlying, i.e., a process S^* such that $\underline{S}_t \leq S_t^* \leq \overline{S}_t$ for $t \in \mathcal{T}$ and such that $(D(t)S_t^*)_{t\in\mathcal{T}}$ is a $\mathbb{P}$-martingale.

The process S^* is also called a shadow price. According to Kabanov and Stricker [56], these requirements yield an arbitrage free model comprising bid and ask price processes for the underlying and each call option. Indeed, for the call with maturity t and strike $K_{t,i}$, one may take $\big(\underline{r}_{t,i}\mathbb{1}_{\{s=0\}} + B(s)\mathbb{E}[(D(t)S_t^C - k_{t,i})^+|\mathcal{F}_s]\mathbb{1}_{\{s>0\}}\big)_{s\in\mathcal{T}}$ as bid price process (and similarly for the ask price), and $\big(B(s)\mathbb{E}[(D(t)S_t^C - k_{t,i})^+|\mathcal{F}_s]\big)_{s\in\mathcal{T}}$ as consistent price system.

As mentioned in the introduction, if consistency is defined according to Definition 3.2, then there is no interplay between the current prices of the underlying and the options, which seems to make little sense. The following example shows how frictionless arbitrage strategies may fail in the presence of a sufficiently large spread, whereas a general result is given in Section 3.4 below.

Example 3.3. Let $c > 0$ be arbitrary. We set $k := k_{1,1} = k_{2,1} = 1$ and assume

$$B(1) = B(2) = 1, \quad \underline{S}_0 = \overline{S}_0 = 2, \quad r_1 := \underline{r}_{1,1} = \overline{r}_{1,1} = c + 1, \quad r_2 := \underline{r}_{2,1} = \overline{r}_{2,1} = 1.$$

Thus $C_1(k)$ is "too expensive", and without frictions, buying $C_2(k) - C_1(k)$ would be an arbitrage opportunity (upon selling one unit of stock if $C_1(k)$ expires in the money). In particular, the first condition from Corollary 4.2 in [21] and equation (5) in [20], are violated: they both state that $r_1 \leq r_2$ is necessary for the absence of arbitrage strategies.

But with spreads we can choose c as large as we want and still the above prices would be consistent with no-arbitrage. Indeed, we can define a deterministic model as follows:

$$\underline{S}_1 = \underline{S}_2 = 2, \quad \overline{S}_1 = 2c + 2, \quad \overline{S}_2 = 2, \quad S^C = \frac{1}{2}(\underline{S} + \overline{S}).$$

Note that
$$(S_2^C - k)^+ = 1 \quad \text{and} \quad (S_1^C - k)^+ = c + 1.$$
This model is clearly free of arbitrage (see also Proposition 3.14). In particular, consider the portfolio $C_2(k) - C_1(k)$: the short call $-C_1(k)$ finishes in the money with payoff $-(c+1)$. This cannot be compensated by going short in the stock, because its bid price stays at 2. The payoff at time $t = 2$ of this strategy, with shorting the stock at time $t = 1$, is
$$(S_2^C - k)^+ - (S_1^C - k)^+ - (\overline{S}_2 - \underline{S}_1) = -c < 0.$$

Our focus thus will be on a stronger notion of consistency, where the discounted spread on the underlying is bounded. Hence, our goal becomes to determine how large a spread is needed to explain given option prices.

Definition 3.4. Let $\epsilon \geq 0$. Then the prices (3.1)-(3.4) are *ϵ-consistent with the absence of arbitrage,* or simply *ϵ-consistent,* if they are consistent (Definition 3.2) and the following conditions hold:
$$\overline{S}_t - \underline{S}_t \leq \epsilon B(t), \quad t \in \mathcal{T}, \tag{3.5}$$
$$S_t^C \geq \epsilon B(t), \quad t \in \mathcal{T}. \tag{3.6}$$

The bound (3.6) is a mild assumption made for tractability, and makes sense given the actual size of market prices and spreads (recall that $\underline{S} \leq S^C$). Also, when checking for ϵ-consistency we will always assume that all strikes are larger than ϵ, which can be justified by the same reasoning.

Definition 3.5. By a semi-static portfolio, we mean a (self-financing) portfolio where the positions in the options are fixed at time zero, and the position in the underlying asset can only be modified at trading times in $\mathcal{T}$. In the following let $\epsilon \geq 0$.

(i) The prices (3.1)-(3.4) admit *model-independent arbitrage* with respect to spread bound ϵ if we can form a semi-static portfolio in the underlying asset and the options such that the initial portfolio value is negative and, for any model satisfying 3.5 and 3.6, all subsequent cash flows are non-negative.

(ii) There is a *weak arbitrage opportunity* with respect to spread bound ϵ if there is no model-independent arbitrage strategy (with respect to spread bound ϵ), but for any model for any model for any model satisfying 3.5 and 3.6, there is a semi-static portfolio such that the initial portfolio value is non-positive, but all subsequent cashflows are non-negative and the probability of a positive cashflow is positive.

Most of the time we will fix $\epsilon \geq 0$ and only write model-independent arbitrage meaning model-independent arbitrage with respect to spread bound ϵ.

The notion of weak, i.e. model-dependent, arbitrage was first used in [21], where the authors give examples to highlight the differences between weak arbitrage and model-independent arbitrage. Note that the process $(D(t)S_t^C)_{t\in\mathcal{T}}$ does not have to be a martingale, since S^C is not traded on the market. The option prices give us some information about the marginals of the process S^C, though. On the other hand, the process

$(D(t)S_t^*)_{t\in\mathcal{T}}$ has to be a martingale, but we have no information about its marginals, except that $|S_t^* - S_t^C| \le \epsilon B(t)$. This is equivalent to

$$W^\infty\Big(\mathcal{L}(D(t)S_t^C), \mathcal{L}(D(t)S_t^*)\Big) \le \epsilon, \tag{3.7}$$

where W^∞ denotes the infinity Wasserstein distance (see Proposition 2.6), and $\mathcal{L}$ the law of a random variable. For the next proposition we will use the notation from Chapter 2. It explains the connection between ϵ-consistent prices, and Theorem 2.11.

Proposition 3.6. *For $\epsilon \ge 0$ the prices (3.1)-(3.4) are ϵ-consistent with the absence of arbitrage, if there are sequences of measures $(\mu_t)_{t\in\mathcal{T}}$ and $(\nu_t)_{t\in\mathcal{T}}$ such that:*

(a) $R_{\mu_t}(k_{t,i}) \in [\underline{r}_{t,i}, \overline{r}_{t,i}]$ for all $t \in \mathcal{T}$ and $i \in \{1,\dots,N_t\}$ and $\mu_t([\epsilon,\infty)) = 1$,

(b) $(\nu_t)_{t\in\mathcal{T}}$ is a peacock and its mean satisfies $\mathbb{E}\nu \in [\underline{S}_0, \overline{S}_0]$, and

(c) $W^\infty(\mu_t,\nu_t) \le \epsilon$ for all $t \in \mathcal{T}$.

Proof. Let $(\mu_t)_{t\in\mathcal{T}}$ and $(\nu_t)_{t\in\mathcal{T}}$ be as above. Then by Strassen's theorem (Theorem 2.2) and the definition of the infinity Wasserstein distance, there exists a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a stochastic process $(S_t^C)_{t\in\mathcal{T}}$ and a martingale $(\widetilde{S}_t)_{t\in\mathcal{T}}$ such that for all $t \in \mathcal{T}$ we have $D(t)S_t^C \sim \mu_t$, $\widetilde{S}_t \sim \nu_t$ and

$$\mathbb{P}(|D(t)S_t^C - \widetilde{S}_t| \le \epsilon) = 1.$$

We then simply set

$$S_t^* := B(t)\widetilde{S}_t, \quad \underline{S}_t := S_t^C \wedge S_t^*, \quad \overline{S}_t := S_t^C \vee S_t^*,$$

for $t \ge 1$ and have an arbitrage free model. ❑

3.2. Single maturity: ϵ-consistency

The consistency conditions for a single maturity are similar to those derived in Theorem 3.1 of [21] and Proposition 3 of [20]. In addition to the conditions given there, we have to assume that the mean of S_t^C is "close enough" to S_0.

In the following we fix $t = 1 \in \mathcal{T}$ and often drop the time index for notational convenience, i.e. we write $\overline{r}_i$ instead of $\overline{r}_{1,i}$ etc.

In the frictionless case the underlying can be identified with an option with strike $k = 0$. Here we will do something similar. In the formulation of next theorem we set $k_0 = \epsilon$, as if we would introduce an option with strike $\epsilon B(1)$, but we think of $C(\epsilon B(1))$ as the underlying. The choices for $\underline{r}_0$ and $\overline{r}_0$ are most easily understood from the following considerations: In every model which is ϵ-consistent with the absence of arbitrage, (3.6) implies that the discounted expected payoff of an option with strike $\epsilon B(1)$ has to satisfy

$$D(1)\mathbb{E}[(S_1^C - \epsilon B(1))^+] = D(1)\mathbb{E}[S_1^C] - \epsilon.$$

Furthermore, to guarantee the existence of a consistent price system, $D(1)\mathbb{E}[S_1^C]$ has to lie in the closed interval $[\underline{S}_0 - \epsilon, \overline{S}_0 + \epsilon]$, which implies that the price of an option with strike $B(1)\epsilon$ has to lie in the interval $[\underline{S}_0 - 2\epsilon, \overline{S}_0]$.

So in the next theorem we will use the symbol $C_t(\epsilon B(t))$ as a reference to the underlying and $-C_t(\epsilon B(t))$ is a reference to a short position in the underlying plus an additional deposit of 2ϵ at the bank account. Recall that we assume that $k_1 > \epsilon$.

Theorem 3.7. *Let $\epsilon \geq 0$ and consider prices as at the beginning of Section 3.1. Moreover, for ease of notation we set $k_0 = \epsilon$, $\underline{r}_0 = \underline{S}_0 - 2\epsilon$, and $\overline{r}_0 = \overline{S}_0$. Then the prices are ϵ-consistent (see Definition 3.4) if and only if the following conditions hold:*

(i)

$$\frac{\overline{r}_l - \underline{r}_j}{k_l - k_j} \geq \frac{\underline{r}_j - \overline{r}_i}{k_j - k_i}, \quad 0 \leq i < j < l \leq N, \tag{3.8}$$

(ii)

$$\frac{\overline{r}_l - \underline{r}_i}{k_l - k_i} \geq -1, \quad 0 \leq i < l \leq N, \tag{3.9}$$

(iii)

$$\underline{r}_j \leq \overline{r}_i, \quad 0 \leq i < j \leq N, \tag{3.10}$$

(iv)

$$\underline{r}_j = \overline{r}_i \;\Rightarrow\; \underline{r}_j = 0, \quad 0 \leq i < j \leq N. \tag{3.11}$$

Moreover, there is a model-independent arbitrage, as soon as any of the conditions (i)-(iii) is not satisfied.

Proof. We first show that the conditions are necessary. Throughout the proof we will denote the option $C_1(K_{1,i})$ by C^i to ease notation.

(i) Suppose that $1 \leq i < j < l$ are such that (3.8) does not hold. We buy a so called butterfly spread, which is the contract

$$BF^{i,j,l} = \frac{K_l - K_j}{K_l - K_i} C^i + \frac{K_j - K_i}{K_l - K_i} C^l - C^j$$

and get an initial payment. Its payoff at maturity is positive if S_1^C expires in the interval (K_i, K_l) and zero otherwise, so we have model-independent arbitrage.

If (3.8) fails for $i = 0$ we buy the contract

$$BF^{0,j,l} = \frac{K_l - K_j}{K_l - B(1)\epsilon} S + \frac{K_j - B(1)\epsilon}{K_l - B(1)\epsilon} C^l - C^j$$

and make an initial profit. At maturity the cash value of the contract is given by

$$\frac{K_l - K_j}{K_l - B(1)\epsilon} \underline{S}_1 + \frac{K_j - B(1)\epsilon}{K_l - B(1)\epsilon} (S_1^C - K_l)^+ - (S_1^C - K_j)^+$$

which is always non-negative.

(ii) Suppose (3.9) fails for $1 \leq i < l$. Then we buy a call spread $C^l - C^i$ and invest $k_l - k_i$ in the bank account. This earns an initial profit and at maturity the cashflow is non-negative.

Now we consider the case where $i = 0$. Note that in this case (3.9) is equivalent to

$$\frac{\overline{r}_l - \underline{S}_0}{k_l + \epsilon} \geq -1.$$

If this fails we buy C^l, sell one unit of the underlying and invest $k_l + \epsilon$ in the bank account. Again we earn an initial profit and at maturity the cashflow is non-negative.

(iii) If (3.10) fails for $0 < i < j$, then we buy the call spread $C^i - C^j$ and get an initial payment. Its payoff at maturity is always non-negative.

If (3.10) fails for $i = 0$, then we sell C^j and buy one unit of the stock which also yields model-independent arbitrage.

(iv) We will show that we cannot find an arbitrage-free model for the given prices, if (3.11) fails. In Proposition 3.8 we will argue that there is a weak arbitrage opportunity in this case (which entails, according to Definition 3.5, that there is no model-independent arbitrage). In any model where $\mathbb{P}(S_1^C > K_j) = 0$ we could sell C^j. Since this option is never exercised, this yields arbitrage. If on the other hand $\mathbb{P}(S_1^C > K_j) > 0$ and $i > 0$, then we buy the call spread $C^i - C^j$ at zero cost. At maturity the probability of a positive cashflow is positive. If $i = 0$, then we buy the contract $S - C^j$ instead, and at maturity the cashflow is given by $\underline{S}_1 - (S_1^C - K_j)$ which is positive with positive probability.

Now we show that the stated conditions are sufficient for ϵ-consistency. We first argue that we may w.l.o.g. assume that $\overline{r}_N = \underline{r}_N = 0$. Indeed, we could choose

$$k_{N+1} \geq \max\left\{\frac{\overline{r}_i k_j - \underline{r}_j k_i}{\overline{r}_i - \underline{r}_j} :\ 0 \leq i < j \leq N,\ \overline{r}_i - \underline{r}_j > 0\right\} \vee \max\{k_j + \underline{r}_j : 0 \leq j \leq N\}$$

and set $\overline{r}_{N+1} = \underline{r}_{N+1} = 0$. Then all conditions from Theorem 3.7 would still hold, if we included an additional option with strike k_{N+1} and bid and ask price equal to zero. So from now on we assume that $\overline{r}_N = \underline{r}_N = 0$.

We will first show, that for $s \in \{0, \dots, N\}$ we can find $e_s \in [\underline{r}_s, \overline{r}_s]$ such that the linear interpolation L of the points $(k_s, e_s), s \in \{0, \dots, N\}$ is convex, decreasing, and such that the right derivative of L satisfies $L'(k_0) \geq -1$. Then we will extend L to a call function, and its associated measure will be the law of $D(1)S_1^C$. The sequence $(e_s)_{s\in\{1,\dots,N\}}$ can then be interpreted as shadow prices of the options with strikes $(k_s)_{s\in\{1,\dots,N\}}$.

Before we start we will introduce some notation. For $j, l \in \{1, \dots, N\}, j < l$ we denote the line connecting $(k_j, \underline{r}_j)$ and $(k_l, \overline{r}_l)$ by $f_{j,l}$, i.e.

$$f_{j,l}(x) = \underline{r}_j + \frac{\overline{r}_l - \underline{r}_j}{k_l - k_j} \cdot (x - k_j).$$

If e_s is known for some $s \in \{0 \dots, N\}$, then we denote the line connecting (k_s, e_s) and $(k_i, \overline{r}_i), i \in \{s+1, \dots, N\}$ by $g_{s,i}$, i.e.

$$g_{s,i}(x) = e_s + \frac{\overline{r}_i - e_s}{k_i - k_s} \cdot (x - k_s).$$

The linear interpolation of (k_s, e_s) and $(k_j, \underline{r}_j), j \in \{s+1, \dots, N\}$ will be denoted by $h_{s,j}$:

$$h_{s,j}(x) = e_s + \frac{\underline{r}_j - e_s}{k_j - k_s} \cdot (x - k_s).$$

We will refer to the slopes of these lines as $f'_{j,l}, g'_{s,i}$ and $h'_{s,j}$ respectively.

First we will construct e_0. In order to get all desired properties – this will become clear towards the end of the proof – e_0 has to satisfy

$$e_0 \geq \max_{0 \leq j < l \leq N} f_{j,l}(k_0), \tag{3.12}$$

and

$$e_0 \leq \min_{0 \leq i \leq N} (k_i + \overline{r}_i - k_0). \tag{3.13}$$

We will argue that we can pick such an e_0 by showing that

$$f_{j,l}(k_0) \leq k_i + \overline{r}_i - k_0, \quad i, j, l \in \{0, \dots, N\}, j \leq l. \tag{3.14}$$

Using (3.9) twice we can immediately see that (3.14) holds for $i \geq j$:

$$f_{j,l}(k_0) \leq \underline{r}_j + k_j - k_0 \leq \overline{r}_i + k_i - k_0.$$

If on the other hand $i < j$ we rewrite the right hand side of (3.14) to $h_i(k_0)$, where $h_i(x) = -x + \overline{r}_i + k_i$. Then from (3.8) we get that

$$f_{j,l}(k_i) \leq \overline{r}_i = h_i(k_i),$$

and since $f'_{j,l} \geq -1 = h'_i$ the inequality follows.

The above reasoning shows that existence of an e_0 such that (3.12) and (3.13) hold. Next we want to construct e_1 for given e_0. It has to satisfy the requirements

$$e_1 \geq \max_{1 \leq j < l \leq N} f_{j,l}(k_1) \vee (e_0 + k_0 - k_1) \tag{3.15}$$

and

$$e_1 \leq \min_{1 \leq i \leq N} g_{0,i}(k_1). \tag{3.16}$$

Again we will argue that we can pick such an e_1 by considering the corresponding inequalities. First note that the inequality

$$e_0 + k_0 - k_1 \leq g_{0,i}(k_1), \quad i \in \{1, \dots, N\},$$

follows directly from (3.12). Next we want to prove that

$$f_{j,l}(k_1) \leq g_{0,i}(k_1), \quad i, j, l \in \{1, \dots, N\}, j < l. \tag{3.17}$$

Therefore observe that

$$f_{j,l}(k_0) \leq e_0 = g_{0,i}(k_0).$$

If $i < j$ (3.17) follows from (3.8), since $f_{j,l}(k_i) \leq \overline{r}_i = g_{0,i}(k_i)$. For $i = j$ we may simply use the fact that $\underline{r}_i \leq \overline{r}_i$ and hence we get that $f_{j,l}(k_i) \leq \overline{r}_i = g_{0,i}(k_i)$. For $i > j$ we may use $f_{j,l}(k_0) \leq e_0 = h_{0,j}(k_0)$ to get

$$f_{j,l}(k_1) \leq h_{0,j}(k_1) \leq g_{0,i}(k_1),$$

where the last inequality follows from the fact that $h_{0,j}(k_0) = g_{0,i}(k_0) = e_0$ and that

$$h'_{0,j} = \frac{\underline{r}_j - e_0}{k_j - k_0} \leq \frac{\overline{r}_i - e_0}{k_i - k_0} = g'_{0,i}.$$

In the last step we used that $e_0 \geq f_{j,i}(k_0)$.

Now suppose we have already constructed $e_1, \dots e_{s-1}, s \in \{1, \dots, N\}$. Then for $r \in \{1, \dots s-1\}$ we have that

$$e_r \geq \left(e_{r-1} + \frac{e_{r-1} - e_{r-2}}{k_{r-1} - k_{r-2}} \cdot (k_r - k_{r-1})\right) \vee \max_{r \leq j < l \leq N} f_{j,l}(k_r), \tag{3.18}$$

and

$$e_r \leq \min_{r \leq i \leq N} g_{r-1,i}(k_r). \tag{3.19}$$

Note that for $r = 1$ we need an appropriate e_{-1} and k_{-1} in order for 3.18 to hold. For instance, we can set $k_{-1} = -1$ and $e_{-1} = e_0 - (k_0 + 1) \cdot (e_1 - e_0)/(k_1 - k_0)$.

We want to show that we can choose e_s such that (3.18) and (3.19) hold for $r = s$. First, the inequality

$$e_{s-1} + \frac{e_{s-1} - e_{s-2}}{k_{s-1} - k_{s-2}} \cdot (k_s - k_{s-1}) \leq g_{s-1,i}(k_s), \quad i \in \{s, \dots, N\},$$

is equivalent to

$$\frac{e_{s-1} - e_{s-2}}{k_{s-1} - k_{s-2}} \leq \frac{\overline{r}_i - e_{s-1}}{k_i - k_{s-1}}$$

which is again equivalent to

$$e_{s-1} \leq g_{s-2,i}(k_{s-1})$$

and holds by (3.19).

The inequality

$$f_{j,l}(k_s) \leq g_{s-1,i}(k_s), \quad i, j, l \in \{s, \dots, N\}, j < l,$$

can be shown using the same arguments as before: first we note that $f_{j,l}(k_{s-1}) \leq e_{s-1} = g_{s-1,i}(k_s)$ and then we distinguish between $i < j$, $i = j$ and $i > j$.

We have now constructed a finite sequence $(e_s)_{s \in \{0,\dots,N\}}$. Observe that for all $s \in \{0, \dots, N\}$ the bounds on e_s from above, namely (3.12) and (3.13) for $s = 0$, (3.15) and (3.16) for $s = 1$ and (3.18) and (3.19) for $s > 1$, ensure that $e_s \in [\underline{r}_s, \overline{r}_s]$. Denote by $L : [k_0, k_N] \to \mathbb{R}$ the linear interpolation of the points $(k_s, e_s), s \in \{0, \dots, N\}$. Then L is convex, which is easily seen from

$$e_s \geq e_{s-1} + \frac{e_{s-1} - e_{s-2}}{k_{s-1} - k_{s-2}} \cdot (k_s - k_{s-1}), \quad s \geq 2.$$

Furthermore, by (3.15)

$$L'(k_0) = \frac{e_1 - e_0}{k_1 - k_0} \geq -1.$$

Finally, L is strictly decreasing on $\{L > 0\}$ which is most easily seen from $e_s \leq g_{s-1,N}(k_s)$. Therefore L can be extended to a call-function R as follows:

$$R(x) = \begin{cases} L(k_0) + k_0 - x, & x \leq k_0, \\ L(x), & x \in [k_0, k_N], \\ 0, & x \geq k_N. \end{cases}$$

Let μ be the associated measure. Then $\mathbb{E}\mu = R(0) = L(k_0) + k_0 \in [\underline{S}_0 - \epsilon, \overline{S}_0 + \epsilon]$. If $\mathbb{E}\mu < \underline{S}_0$ we define a measure ν by setting $\nu(A) = \mu(A - \epsilon)$ for Borel sets A. The set $A - \epsilon$ is defined as $\{a - \epsilon : a \in A\}$. Then $\mathbb{E}\nu = \mathbb{E}\mu + \epsilon \in [\underline{S}_0, \overline{S}_0]$. Similarly, if $\mathbb{E}\mu > \overline{S}_0$ we define $\nu(A) = \mu(A + \epsilon)$ for Borel sets A, and if $\mathbb{E}\mu \in [\underline{S}_0, \overline{S}_0]$ then we simply set $\nu = \mu$. Furthermore for $x < k_0$ we have that $R'(x) = -1$, therfore μ has support $[\epsilon, \infty)$. By Proposition 3.6 the prices are ϵ-consistent with the absence of arbitrage. ❑

Supplementing Theorem 3.7, we now show that there is only weak arbitrage if (3.11) fails, i.e., no model-independent arbitrage. This is the content of the following proposition; its proof is a slight modification of the last part of the proof of Theorem 3.1 in [21].

Proposition 3.8. *Under the assumptions of Theorem 3.7, there is a weak arbitrage opportunity if* (3.8), (3.9) *and* (3.10) *hold, but* (3.11) *fails, i.e. there exist $i < j$ such that* $\overline{r}_i = \underline{r}_j > 0$.

Proof. As we have seen in part *(iv)* of the necessity proof of Theorem 3.7, there is an arbitrage opportunity that depends on the null sets of the model. We will show that there is no model-independent arbitrage strategy. Suppose, on the contrary, that there is one. Then we can construct a portfolio $C_\gamma = \sum_{l=-1}^{N} \gamma_l C(K_l)$, where $\gamma_l \in \mathbb{R}$ such that its initial cost is negative, i.e.

$$r_\gamma := \gamma_{-1} + (\gamma_0^+ \overline{S}_0 - \gamma_0^- \underline{S}_0) + \sum_{l=1}^{N} (\gamma_l^+ \overline{r}_l - \gamma_l^- \underline{r}_l) < 0$$

and such that the cashflow at maturity is non-negative, i.e.

$$\gamma_{-1} B(1) + (\gamma_0^+ \overline{S}_1 - \gamma_0^- \underline{S}_t) + \sum_{l=1}^{N} \gamma_l (S_1^C - K_l)^+ \geq 0.$$

Here $C(K_{-1})$ denotes the bank account and $C(K_0)$ denotes the underlying as usual. Without loss of generality we can assume that $\sum_{l=-1}^{N} |\gamma_l| = 1$.

Next we construct $e_0, \ldots, e_N$ as in the sufficiency proof of Theorem 3.7. Clearly we then have $\overline{r}_i = e_i = e_{i+1} = \cdots = e_N$. The idea is to consider a market with slightly different shadow prices $\widetilde{e}_l$, which can be obtained from the original shadow prices e_l by shifting them down. More precisely, we set $l_0 = \max_{l \in \{0,\ldots,N\}} (e_l + k_l = e_0 + k_0)$ and define

$$z = \min\left\{ -\frac{r_\gamma}{2},\ (e_{l_0+1} + k_{l_0+1} - e_{l_0} - k_{l_0}) \cdot \frac{\sum\limits_{s=l_0}^{N} (k_s - k_{l_0})}{k_{l_0+1} - k_{l_0}},\ e_N \cdot \frac{\sum\limits_{s=l_0}^{N} (k_s - k_{l_0})}{k_N - k_{l_0}} \right\}.$$

Then we set $\widetilde{e}_l = e_l$ for $l \leq l_0$ and for $l > l_0$

$$\widetilde{e}_l = e_l - z \frac{k_l - k_{l_0}}{\sum\limits_{s=l_0}^{N} (k_s - k_{l_0})}.$$

Now consider a modified set of prices, where bid and ask price of the l-th call, $0 \leq l \leq N$, are both defined by $\widetilde{e}_l$. It is easy to check that these prices satisfy all conditions from

Theorem 3.7, and hence do not admit any arbitrage opportunities. Indeed, the second expression in the definition of z guarantees that $\widetilde{e}_{l_0+1}$ is not too small, i.e.

$$\frac{\widetilde{e}_{l_0+1} - \widetilde{e}_{l_0}}{k_{l_0+1} - k_{l_0}} \geq -1,$$

and the third expression ensures that $\widetilde{e}_N$ is not too small, i.e. $\widetilde{e}_N \geq 0$. A simple calculation shows that

$$\begin{aligned}
\gamma_{-1} + (\gamma_0^+ \overline{S}_0 - \gamma_0^- \underline{S}_0) + \sum_{l=1}^{N} \gamma_l \widetilde{e}_l &= \gamma_{-1} + (\gamma_0^+ \overline{S}_0 - \gamma_0^- \underline{S}_0) + \sum_{l=1}^{N} \gamma_l e_l - \sum_{l=l_0+1}^{N} \gamma_l (e_l - \widetilde{e}_l) \\
&\leq r_\gamma - \sum_{l=l_0+1}^{N} \gamma_l (e_l - \widetilde{e}_l) \\
&\leq r_\gamma + z \sum_{l=l_0+1}^{N} |\gamma_l| \frac{k_l - k_{l_0}}{\sum\limits_{s=l_0}^{N} (k_s - k_{l_0})} \\
&\leq r_\gamma + z \leq \frac{r_\gamma}{2} < 0,
\end{aligned}$$

and so the portfolio $C_\gamma = \sum_{l=-1}^{N} \gamma_l C(K_l)$ in the modified market has negative cost. But its cashflow at maturity is unchanged and hence non-negative, and we have thus constructed a model-independent arbitrage strategy for the modified set of prices, which is a contradiction. ❑

For $\epsilon = 0$ and $\underline{r}_i = \overline{r}_i = r_i$ the conditions from Theorem 3.7 simplify to

$$0 \geq \frac{r_{i+1} - r_i}{k_{i+1} - k_i} \geq \frac{r_i - r_{i-1}}{k_i - k_{i-1}} \geq -1, \quad \text{for } i \in \{1, \ldots, N-1\},$$

and

$$r_i = r_{i-1} \text{ implies } r_i = 0, \quad \text{for } i \in \{1, \ldots, N\}.$$

These are exactly the conditions required in Theorem 3.1 of [21].

We close this section with the following remark.

Remark 3.9. Note that in contrast to the frictionless case, we do not have to require that bid or ask prices are decreasing as the strike increases, in order to get models which are ϵ-consistent with the absence of arbitrage. i.e. we do not have to require that $\underline{r}_i \geq \underline{r}_j$ or $\overline{r}_i \geq \overline{r}_j$ for $i < j$. This is shown in the following example.

Consider the case with two options, where $\epsilon = 0$, and for $i = 1, 2$ the prices are given by

$$B(1) = 1, \quad \underline{S}_0 = \overline{S}_0 = 4, \quad \overline{r}_i = i + 5, \quad \underline{r}_i = 1 + \frac{i}{2}, \quad k_i = i.$$

The prices and a possible choice of shadow prices are shown in Figure 3.1. Clearly all conditions from Theorem 3.7 are satisfied, and therefore there exists an arbitrage free model. For example we can choose $\mu = \delta_5$, where δ denotes the Dirac delta. This example shows that in our setting, prices which are plausible from a no-arbitrage point of view, are not necessarily rational prices in an economic sense: since at maturity the cashflow of $C(K_2)$ never exceeds the cashflow of $C(K_1)$ the utility indifference ask-price of $C(K_2)$ should not be higher than the utility indifference ask-price ask-price of $C(K_1)$.

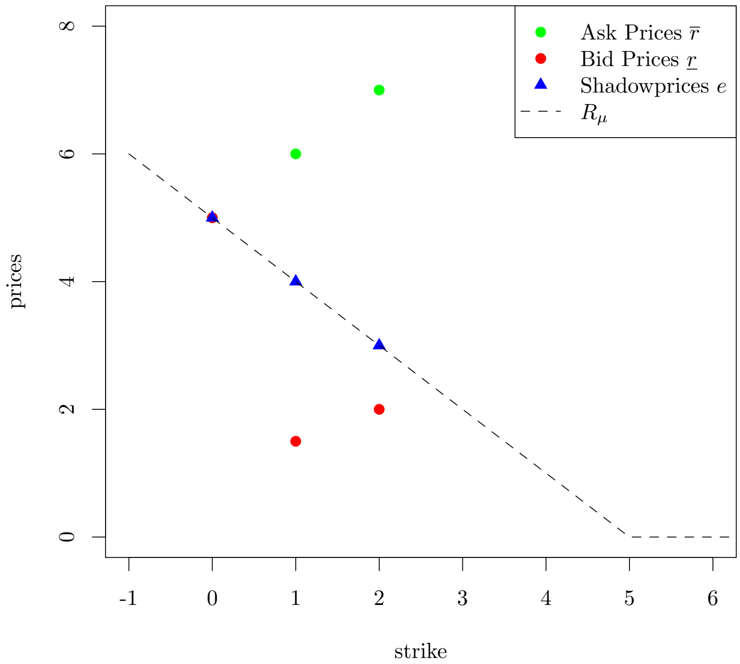

Figure 3.1.: This examples shows that it is not necessary that the ask-prices resp. bid-prices decrease with the strike. The line represents the call function of δ_5.

3.3. Multiple maturities: ϵ-consistency

The ϵ-consistency conditions for single maturities (Theorem 3.7) are a straightforward generalisation of the results of [20, 21]. They guarantee that for each single maturity $t \in \mathcal{T}$ the option prices can be associated to a measure μ_t, such that $\mathbb{E}\mu_t \in [\underline{S}_0, \overline{S}_0]$. In this section we want to state necessary conditions for multiple periods. In the case where there is no market friction on the underlying, it suffices to compare prices with only three or two different maturities (see equations (4), (5) and (6) in [20] and Corollary 4.2 in [21]) to obtain suitable consistency conditions. These conditions ensure that the family of measures $(\mu_t)_{t\in\mathcal{T}}$ is a peacock.

If we want to check for ϵ-consistency ($\epsilon > 0$) it is clear from (2.10) that we need conditions which involve all maturities simultaneously (see also Example 2.13). This is why we will introduce calendar vertical baskets (CVB), a portfolio which consists of various long and short positions in the call options. We first give a proper definition of CVBs. Then, in Proposition 3.11 we will study a certain trading strategy involving a short position in a CVB: this strategy will then serve as a base for the conditions in Theorem 3.12. Note that our definition of a CVB depends on $\epsilon \geq 0$: the contract defined in Definition 3.10 only provides necessary conditions in markets where the bid-ask spread is bounded by $\epsilon \geq 0$.

Definition 3.10. Fix $u \in \mathcal{T}$ and $\epsilon \geq 0$ and assume that vectors $\sigma = (\sigma_1, \dots, \sigma_u)$, $x = (x_1, \dots, x_u)$, $I = (i_1, \dots, i_{u-1})$ and $J = (j_1, \dots, j_u)$ are given, such that (*i*) $x_t \in \mathbb{R}$ for all $t \in \{1, \dots, u\}$,

(*ii*) $\sigma_1 \in \{-1, 1\}$ and $\sigma_t = \operatorname{sgn}(x_{t-1} - x_t)$ for all $t \in \{2, \dots, u\}$,

(*iii*) $i_t \in \{0, \dots, N_{t+1}\}$ and $k_{t+1,i_t} \leq x_t + \epsilon\sigma_{t+1}$ for all $t \in \{1, \dots, u-1\}$,

(*iv*) $j_t \in \{0, \dots, N_t\}$ and $k_{t,j_t} = x_t + \epsilon\sigma_t$ for all $t \in \{1, \dots, u\}$.

Then we define a calendar vertical basket with these parameters as the contract

$$CVB_u(\sigma, x, I, J) = C_1(K_{1,j_1}) + \sum_{t=2}^{u}\Big(C_t(K_{t,j_t}) - C_t(K_{t,i_{t-1}})\Big) - 2\epsilon\mathbb{1}_{\{\sigma_1 = -1\}}. \tag{3.20}$$

The market ask resp. bid-price of $CVB_u(\sigma, x, I, J)$ are given by

$$\overline{r}_u^{CVB}(\sigma, x, I, J) = \overline{r}_{1,j_1} + \sum_{t=2}^{u}\big(\overline{r}_{t,j_t} - \underline{r}_{t,i_{t-1}}\big) - 2\epsilon\mathbb{1}_{\{\sigma_1=-1\}},$$
$$\underline{r}_u^{CVB}(\sigma, x, I, J) = \underline{r}_{1,j_1} + \sum_{t=2}^{u}\big(\underline{r}_{t,j_t} - \overline{r}_{t,i_{t-1}}\big) + 2\epsilon\mathbb{1}_{\{\sigma_1=-1\}}.$$

We will refer to u as the maturity of the CVB.

Proposition 3.11. *Fix $\epsilon \geq 0$ and assume that 3.5 and 3.6 hold. Then for all parameters u, σ, x, I, J as in Definition 3.10, there is a semi-static portfolio whose initial value is given by $-\underline{r}_u^{CVB}(\sigma, x, I, J)$ and such that the cash-flow at each time $t \in \mathcal{T}$ is either zero or given by $-(S_t^C - B(t)(k_{t,j} - \epsilon\sigma_t + \epsilon))$.*

Proof. Assume that we buy the contract

$$-CVB_u(\sigma, x, I, J) = -C_1(K_{1,j_1}) + \sum_{t=2}^{u}\Big(C_t(K_{t,i_{t-1}}) - C_t(K_{t,j_t})\Big) + 2\epsilon\mathbb{1}_{\{\sigma_1=-1\}}, \tag{3.21}$$

thus we are getting an initial payment of $\underline{r}_u^{CVB}(\sigma, x, I, J)$. We have to keep in mind that if $i_t = 0$ for some $t \in \{1, \dots, u-1\}$ then the corresponding expression in (3.21) denotes a long position in the underlying, and if $j_t = 0$ for some $t \in \{1, \dots, u\}$ the expression $-C_t(K_{t,j_t})$ in (3.21) denotes a short position in the underlying plus an additional deposit of 2ϵ at the bank account at time-0. To ease notation we will write $K_{t,i}$ instead of $K_{t,i_{t-1}}$ and $K_{t,j}$ instead of K_{t,j_t}. Also we will write ϵ_t for $\epsilon B(t)$.

We will show inductively that at the end of each period $t \in \{1, \dots, u\}$ we can end up in one of two scenarios: either the value of our bank account is non-negative, we will call this scenario A, or the value of our bank account is at least $-(S_t^C - K_{t,j} - \epsilon_t\sigma_t + \epsilon_t)$, we will refer to this as scenario B. If at any period we end up in scenario B, we go short in one unit of the underlying, such that the value of the bank account becomes $K_{t,j} - \epsilon_t\sigma_t$, thus we start the next period with a short position in the underlying.

We will first deal with the case where $\sigma_1 = -1$ and afterwards with the case $\sigma_1 = 1$. We start with $t = 1$ and first assume that $j_1 > 0$. If $C_1(K_{1,j})$ expires out of the money, the cashflow at time t is given by $2\epsilon_1 \geq 0$, so we are in scenario A. Otherwise we sell one unit of the underlying and the resulting cash flow is

$$2\epsilon_1 + K_{1,j} - S_1^C + \underline{S}_1 \geq K_{1,j} + \epsilon_1 = K_{1,j} - \sigma_1\epsilon_1,$$

thus we have scenario B. If $j_1 = 0$ then $K_{1,j} = \epsilon_1$. We do not close the short position in this case and the value of our bank account becomes $4\epsilon_1 \geq K_{1,j} - \sigma_1\epsilon_1$, so we also get to scenario B.

For the induction step we split up the proof into two parts. In part A we will assume that at the beginning of time t we are in scenario A, and in part B we will assume that at the beginning of time t we are in scenario B.

Part A: We will show that at the end of time t we can end up either in situation A or B. First we assume that $j_t, i_{t-1} > 1$ such that both expressions in (3.21) with maturity t denote options. Under this assumptions the cashflow at the beginning of time t is given by

$$CF_t \geq \left(S_t^C - K_{t,i}\right)^+ - \left(S_t^C - K_{t,j}\right)^+.$$

Clearly, if $K_{t,i} \leq K_{t,j}$ or if both options expire out of the money then $CF_t \geq 0$, and we are in situation A at the end of t. So suppose that $K_{t,i} > K_{t,j}$ and that $S_t^C > K_{t,j}$. This also implies that $\sigma_t = 1$. If this is the case, we go short in one unit of the underlying and at the end of time t and get

$$\begin{aligned} CF_t &\geq \left(S_t^C - K_{t,i}\right)^+ - \left(S_t^C - K_{t,j}\right) + \underline{S}_t \\ &\geq K_{t,j} - \epsilon_t\sigma_t. \end{aligned}$$

This corresponds to situation B. Next assume that $j_t = 0$ and $i_t > 0$. Then we have that $K_{t,j} = \epsilon_t$. At the and of the t-th period we end up in scenario B:

$$CF_t \geq \left(S_t^C - K_{t,j}\right)^+ + 2\epsilon_t \geq K_{t,j} - \epsilon_t\sigma_t.$$

We proceed with the case that $j_t > 0$ and $i_{t-1} = 0$. Then since $K_{t,j} > \epsilon_t$ we can close the long position in the underlying and end up in scenario A after time t:

$$CF_t \geq \underline{S}_t - \left(S_t^C - K_{t,j}\right)^+ \geq 0.$$

The case where $j_t = i_{t-1} = 0$ is easily handled since the long and the short position simply cancel out. We are done with part A.

Part B: Assume that at the beginning of time t we are in scenario B, thus the value of our bank account is given by $k_{t-1,j}B(t) - \epsilon_t\sigma_{t-1}$. First we will consider the case where $j_t, i_{t-1} > 1$. If at time t the option with strike $K_{t,j}$ expires in the money, we do not close the short position and have

$$\begin{aligned} CF_t &\geq k_{t-1,j}B(t) - \epsilon_t\sigma_{t-1} + \left(S_t^C - K_{t,i}\right)^+ - \left(S_t^C - K_{t,j}\right) \\ &= k_{t-1,j}B(t) - \epsilon_t\sigma_{t-1} + K_{t,j} - K_{t,i} \\ &\geq K_{t,j} - \epsilon_t\sigma_t. \end{aligned}$$

which means that we end up in scenario B. Now we distinguish two cases according to $x_{t-1} \leq x_t$ and $x_{t-1} > x_t$ and always assume that $C_t(K_{t,j})$ expires out of the money. If $x_{t-1} \leq x_t$ then we also have that $k_{t,i} \leq k_{t,j}$ and that $\sigma_t = -1$. We close the short position to end up in scenario A:

$$\begin{aligned} CF_t \geq k_{t-1,j}B(t) - \epsilon_t\sigma_{t-1} + \left(S_t^C - K_{t,i}\right)^+ - \overline{S}_t \\ K_{t,i} - \epsilon_t\sigma_t - K_{t,i} - \epsilon \geq 0. \end{aligned}$$

If on the other hand $x_{t-1} > x_t$ and $\sigma_t = 1$, we do not trade at time t to stay in scenario B:

$$\begin{aligned} CF_t &\geq k_{t-1,j}B(t) - \epsilon_t\sigma_{t-1} + \left(S_t^C - K_{t,i}\right)^+ \\ &> K_{t,j} - \epsilon_t\sigma_t. \end{aligned}$$

We proceed with the case where $j_t = \ 0$ and $i_t > 0$. As before, we have that $K_{t,j} = \epsilon_t$ and we can close one short position to stay in scenario B:

$$\begin{aligned} CF_t &\geq k_{t-1,j}B(t) - \epsilon_t\sigma_{t-1} + \left(S_t^C - K_{t,i}\right)^+ + 2\epsilon_t - \overline{S}_t \\ &\geq k_{t-1,j}B(t) - \epsilon_t\sigma_{t-1} - K_{t,i} + \epsilon_t \\ &\geq \epsilon_t - \epsilon_t\sigma_t = K_{t,j} - \epsilon_t\sigma_t. \end{aligned}$$

If $j_t > 0$ and $i_{t-1} = 0$ then we distinguish two cases: either $C_t(K_{t,j})$ expires out of the money in which case we cancel out the long and short position in the underlying and have:

$$CF_t \geq k_{t-1,j}B(t) - \epsilon_t\sigma_{t-1} \geq 0,$$

which corresponds to scenario A. Or, $C_t(K_{t,j})$ expires in the money. Then we sell one unit of the underlying such that we end up in scenario B:

$$\begin{aligned} CF_t &\geq k_{t-1,j}B(t) - \epsilon_t\sigma_{t-1} - S_t^C + K_{t,j} + \underline{S}_t \\ &\geq k_{t-1,j}B(t) - \epsilon_t\sigma_{t-1} + K_{t,j} - \epsilon_t \\ &\geq K_{t,j} - \epsilon_t\sigma_t. \end{aligned}$$

In the last inequality we have used that $k_{t-1,j}B(t) - \epsilon_t\sigma_{t-1} = B(t)x_{t-1} \geq K_{t,i} - \epsilon_t\sigma_t$, and that $K_{t,i} = \epsilon_t$.

The case where $j_t = i_{t-1} = 0$ is again easy to handle since the long and the short position cancel out and we are in scenario B at the end of the t-th period.

Thus at the end of time u we are either in scenario A or scenario B, which proves the assertion if $\sigma_1 = -1$.

The proof for $\sigma_1 = 1$ is similar. We will show that at the end of time-1 we can either be in scenario A or scenario B and the statement of the proposition then follows by induction exactly as in the case $\sigma_1 = -1$.

First we assume that $j_1 > 0$. Then if the option $C_1(K_{1,j})$ expires out of the money we are in scenario A, otherwise we go short in the underlying and have

$$CF_1 \geq -S_1^C + K_{1,j} + \underline{S}_1 \geq K_{1,j} - \epsilon,$$

which corresponds to scenario B. If $j_1 = 0$ then we also have that $K_{j,1} = \epsilon_1$ and hence we are in scenario B. ❑

According to Proposition 3.11, there is a trading strategy for the buyer of the contract $-CVB_u(\sigma, x, I, J)$, defined in (3.21), such that cashflow of this contract at maturity u only depends on $\sigma_u, k_{u,j}$. In the following we will use this strategy and only write $-CVB_u(\sigma_u, k_{u,j})$ resp. $\underline{r}_u^{CVB}(\sigma_u, k_{u,j})$ instead of $-CVB_u(\sigma, x, I, J)$ resp. $\underline{r}_u^{CVB}(\sigma, x, I, J)$.

If the cashflow of $-CVB_u(\sigma_u, k_{u,j})$ at time u is zero we will say that the calendar vertical basket expires out of the money, otherwise we will say that it expires in the money.

The next theorem states necessary conditions for the absence of arbitrage in markets with spread bound $\epsilon \geq 0$.

Theorem 3.12. *Let $\epsilon \geq 0$, $s, t, u \in \mathcal{T}$ such that $s < t$ and $s < u$ and $i \in \{0, \ldots, N_t\}$, $j \in \{0, \ldots, N_s\}$, $l \in \{0, \ldots, N_u\}$. Then for all calendar vertical basket with maturity $s \in \mathcal{T}$ and parameters $k_{s,j}$ and σ_s the following conditions are necessary for the existence of ϵ-consistent models:*

(i)

$$\frac{\underline{r}_s^{CVB}(\sigma_s, k_{s,j}) - \overline{r}_{t,i}}{(k_{s,j} - \epsilon\sigma_s) - (k_{t,i} + \epsilon)} \leq \frac{\overline{r}_{u,l} - \underline{r}_s^{CVB}(\sigma_s, k_{s,j})}{k_{u,l} + \epsilon - (k_{s,j} - \epsilon\sigma_s)}, \quad \text{if} \quad k_{t,i} + \epsilon < k_{s,j} - \epsilon\sigma_s < k_{u,l} + \epsilon, \tag{3.22}$$

(ii)

$$\frac{\overline{r}_{u,l} - \underline{r}_s^{CVB}(\sigma_s, k_{s,j})}{k_{u,l} + \epsilon - (k_{s,j} - \epsilon\sigma_s)} \geq -1, \quad \text{if} \quad k_{s,j} - \epsilon\sigma_s < k_{u,l} + \epsilon, \tag{3.23}$$

(iii)

$$\underline{r}_s^{CVB}(\sigma_s, k_{s,j}) - \overline{r}_{t,i} \leq 0, \quad \text{if} \quad k_{s,j} - \epsilon\sigma_s \geq k_{t,i} + \epsilon, \tag{3.24}$$

(iv)

$$\underline{r}_s^{CVB}(\sigma_s, k_{s,j}) - \overline{r}_{t,i} = 0 \;\Rightarrow\; \overline{r}_{t,i} = 0, \quad \text{if} \quad k_{s,j} - \epsilon\sigma_s > k_{t,i} + \epsilon. \tag{3.25}$$

Proof. We will assume that $s < t \leq u$ and that $i, l > 0$. The other cases can be dealt with similarly. In all three cases we will assume that until time s we will follow the trading strategy described in Proposition 3.11.

(i) If (3.22) fails we set

$$\theta = \frac{k_{u,l} + \epsilon - (k_{s,j} - \epsilon\sigma_s)}{k_{u,l} - k_{t,i}} \in (0, 1)$$

and buy $\theta C_t(K_{t,i}) + (1-\theta)C_u(K_{u,l}) - CVB_s(\sigma_s, K_{s,j})$, making an initial profit. If the calendar vertical basket $CVB_s(\sigma_s, K_{s,j})$ expires out-of-the-money we have model-independent arbitrage. Otherwise we sell one unit of the underlying at time s. In order to close the short position we buy θ units of the underlying at time t and we buy $1 - \theta$ units of the underlying at time u. At time u the cash value of this strategy is non-negative:

$$\begin{aligned}
&(k_{s,j} - \epsilon\sigma_s + \epsilon)B(u) + \theta(S_t^C - K_{t,i})^+ \frac{B(u)}{B(t)} + (1-\theta)(S_u^C - K_{u,l})^+ \\
&\qquad\qquad + (\underline{S}_s - S_s^C)\frac{B(u)}{B(s)} - \theta \overline{S}_t \frac{B(u)}{B(t)} - (1-\theta)\overline{S}_u \\
&\qquad \geq (k_{s,j} - \epsilon\sigma_s)B(u) + \theta\frac{B(u)}{B(t)}\Big(S_t^C - K_{t,i} - \overline{S}_t\Big) + (1-\theta)\Big(S_u^C - K_{u,l} - \overline{S}_u\Big) \\
&\qquad \geq \Big(k_{s,j} - \epsilon\sigma_s - \theta k_{t,i} - (1-\theta)k_{u,l} - \epsilon\Big)B(u) = 0.
\end{aligned}$$

(ii) Next, assume that (3.23) fails. Then buying the contract

$$C_u(K_{u,l}) - CVB_s(\sigma_s, K_{s,j}) + k_{u,l} + \epsilon - (k_{s,j} - \epsilon\sigma_s)$$

earns an initial profit. If CVB_s expires out of the money all remaining cashflows are non-negative. Otherwise we enter a short position at time s and close it at time u:

$$\begin{aligned}(k_{s,j} - \epsilon\sigma_s + \epsilon)B(u) + (\underline{S}_s - S^C_s)\frac{B(u)}{B(s)} + (S^C_u - K_{u,l})^+ - \overline{S}_u \\ + \Big(k_{u,l} + \epsilon - (k_{s,j} - \epsilon\sigma_s)\Big)B(u) \geq 0.\end{aligned}$$

(iii) If (3.24) fails we buy the contract $C_t(K_{t,i}) - CVB_s(\sigma_s, k_{s,j})$ for negative cost. If $CVB_s(\sigma_s, k_{s,j})$ expires in the money we have model-independent arbitrage. Otherwise we sell one unit of the underlying at time s and close the short position at time t. The resulting cashflow then satisfies:

$$(k_{s,j} - \epsilon\sigma_s + \epsilon)B(t) + (\underline{S}_s - S^C_s)\frac{B(t)}{B(s)} + (S^C_t - K_{t,j})^+ - \overline{S}_t \geq 0.$$

(iv) We will show that there cannot exist an ϵ-consistent model, if (3.25) fails. In every model where the probability that $CVB_s(\sigma_s, k_{s,j})$ expires in the money is zero, we could simply sell $CVB_s(\sigma_s, k_{s,j})$ and follow the trading strategy from Proposition 3.11. realising (model-dependent) arbitrage. On the other hand, if $CVB_s(\sigma_s, k_{s,j})$ expires in the money with positive probability, then we can use the same strategy as in the proof of *(iii)*. At time t the probability of a positive cashflow is positive. ❑

Note that if $\epsilon = 0$ then $CVB_s(\sigma_s, k_{s,j})$ has the same payoff as $-C_s(K_{s,j})$. Keeping this in mind, it is easy to verify that the conditions from Theorem 3.12 are a generalisation of equations (4), (5) and (6) in [20].

3.3.1. Sufficient conditions under simplified assumptions

It remains open whether the conditions from Theorem 3.12 together with the conditions for single maturities are sufficient for the existence of ϵ-consistent models or not.

In Proposition 3.13, we will state conditions which guarantee the existence of ϵ-consistent models under simplified assumptions. This can be regarded as a first step towards solving the puzzle of finding sufficient conditions for the original problem.

In this section we will modify the settings described in (3.2) and (3.3) as follows:

(i) For all maturities $t \in \mathcal{T}$ options with all strikes $k \in \mathbb{R}$ are traded.

(ii) The bid price and the ask price of all options are equal. We will write $R_t(k)$ for the price of an option with strike $B(t)k$ and maturity t.

(iii) For all $t \in \mathcal{T}$ the function $k \mapsto R_t(k)$ is a call function and the associated measure μ_t has finite support which is a subset of $[\epsilon, \infty)$.

(iv) The initial bid-ask spread on the underlying is zero, i.e. $\underline{S}_0 = \overline{S}_0 = S_0$ and $\mathbb{E}\mu_t \in [S_0 - \epsilon, S_0 + \epsilon]$.

These assumptions allow us to circumvent many problems, as we only have to check whether the family $(\mu_t)_{t\in\mathcal{T}}$ satisfies the condition from Theorem 2.11. Recall that if $k \mapsto R_t(k)$ is not a call-function the prices cannot be consistent with the absence of arbitrage (see Theorem 3.7). The conditions in Proposition 3.13 can be directly derived from (2.10). Note that in our case $m = S_0$ and $\mathbb{E}\mu_t = R_t(\epsilon) + \epsilon$.

Proposition 3.13. *Let $\epsilon \geq 0$. Then for all $u \in \{2, \dots, T\}$ and for all $k_1, \dots, k_{u-1}$ the following conditions are necessary and sufficient for the the existence of ϵ-consistent models.*

(i)

$$\sum_{t=1}^{u-1} \Big(R_{t+1}(k_t + \epsilon\sigma_{t+1}) - R_t(k_t + \epsilon\sigma_t) \Big) + R_1(\epsilon) - R_u(\epsilon) + 2\epsilon \geq 0, \tag{3.26}$$

where $\sigma_1 = -1$ and $\sigma_u = 1$.

(ii)

$$\sum_{t=1}^{u-1} \Big(R_{t+1}(k_t + \epsilon\sigma_{t+1}) - R_t(k_t + \epsilon\sigma_t) \Big) + R_1(\epsilon) - S_0 + 2\epsilon \geq 0, \tag{3.27}$$

where $\sigma_1 = -1$ and $\sigma_u = -1$.

(iii)

$$\sum_{t=1}^{u-1} \Big(R_{t+1}(k_t + \epsilon\sigma_{t+1}) - R_t(k_t + \epsilon\sigma_t) \Big) + S_0 - R_u(\epsilon) \geq 0, \tag{3.28}$$

where $\sigma_1 = 1$ and $\sigma_u = 1$.

(iv)

$$\sum_{t=1}^{u-1} \Big(R_{t+1}(k_t + \epsilon\sigma_{t+1}) - R_t(k_t + \epsilon\sigma_t) \Big) \geq 0, \tag{3.29}$$

where $\sigma_1 = 1$ and $\sigma_u = -1$.

Here we set $\sigma_t = \operatorname{sgn}(k_{t-1} - k_t)$.

Proof. We will first show that there is model-independent arbitrage with respect to spread bound ϵ if any of the above conditions fail. We will assume that $u = T$. Throughout the first part of the proof we fix $k_1, \dots k_{T-1} \in \mathbb{R}$ and set $K_t = B(t)k_t$ and $\epsilon_t = B(t)\epsilon$, for $t \in \mathcal{T}$.

(i) If (3.26) fails, we buy the contract

$$2\epsilon + \sum_{t=1}^{T} \left(C_t(k_{t-1}B(t) + \epsilon_t\sigma_t) - C_t(K_t + \epsilon_t\sigma_t) \right),$$

where $k_0 = k_T = 0$ and $\sigma_t = \operatorname{sgn}(k_{t-1} - k_t)$, making an initial profit. We will show inductively that at the end of each period $t \in \{1, \dots, T\}$ we can either have at least K_t in our bank account – which we will refer to as scenario A′ – or we can have long position the underlying asset S and a non-negative value in our bank account, to which we will refer as scenario B′. We start with $t = 1$. If both options with maturity $t = 1$ expire in the money, the cashflow is given by

$$(S_1^C - \epsilon_1) - (S_1^C - K_1 + \epsilon_1) + 2\epsilon_1 = K_1,$$

which correspond to scenario A′. If on the other hand $S_1^C \leq K_1 - \epsilon_1$ we get the amount $S_1^C + \epsilon_1$ transferred to our bank account, which is enough to buy the underlying asset for $\overline{S}_1 \leq S_1^C + \epsilon_1$.

Now suppose that at the beginning of the t-th period we are in scenario A′. If $k_{t-1} \leq k_t$ the cashflow at time t is given by

$$CF_t = B(t)k_{t-1} + \left(S_t^C - B(t)k_{t-1} + \epsilon_t\right)^+ - \left(S_t^C - K_t + \epsilon_t\right)^+.$$

If both options are in the money the value of our bank account becomes K_t and we finish the t-th period in scenario A′, otherwise $S_t^C \leq K_t - \epsilon_t$ in which case we have enough money to buy the underlying for $\overline{S}_t$ and we end up in scenario B′.

If $k_{t-1} \geq k_t$ the cashflow at time $t+1$ is given by

$$CF_t = B(t)k_{t-1} + \left(S_t^C - B(t)k_{t-1} + \epsilon_t\right)^+ - \left(S_t^C - K_t - \epsilon_t\right)^+ \geq K_t$$

Now assume that at the beginning of the t-th period we are in situation B′. If $k_{t-1} \leq k_t$ the cashflow at time t is non-negative and we stay at scenario B′. If on the other hand $k_{t-1} > k_t$ we sell the underlying for $\underline{S}_t$ and the resulting cashflow is always greater than K_t, thus we end up in scenario B′. This completes the first part of the proof.

(ii) If (3.27) is violated, then buying the contract

$$2\epsilon + \sum_{t=1}^{T} C_t(k_{t-1}B(t) + \epsilon_t\sigma_t) - C_t(K_t + \epsilon_t\sigma_t) + C_{T+1}(k_T B(T+1) - \epsilon_{T+1}) - S,$$

where $k_0 = 0$, earns an initial profit. Following the same strategy as in *(i)* we can either have K_T in our bank account or one unit of the underlying and a non-negative value in our bank account, at the beginning of the T-th period. In both cases the terminal value is non-negative.

(iii) If (3.28) fails, we buy the contract

$$S - C_1(K_1 + \epsilon_1) + \sum_{t=2}^{T} C_t(k_{t-1}B(t) + \epsilon_t\sigma_t) - C_t(K_t + \epsilon_t\sigma_t),$$

where $k_T = 0$, making an initial profit. Whenever $C_1(K_1 + \epsilon_1)$ expires out of the money we still have the underlying, whereas if $S_1^C \geq K_1 + \epsilon_1$ we sell one unit of the underlying and the value of our bank account is at least K_1. The rest can be done by induction, as before.

(iv) If (3.29) fails, then buying the contract

$$-C_1(K_1 + \epsilon_1) + \sum_{t=2}^{T-1} C_t(k_{t-1}B(t) + \epsilon_t\sigma_t) - C_t(K_t + \epsilon_t\sigma_t) + C_T(k_{T-1}B(T) - \epsilon_T),$$

earns an initial profit. We will show inductively that at the end of the t-th period, $t \in \{1, \dots, T\}$, it is possible to have either a non-negative amount in our bank account, or having more than K_t in our bank account and being short one unit of the underlying asset. These two scenarios are exactly scenario A resp. scenario B from the proof of Proposition 3.11. We start with $t = 1$. If $C_1(K_1 + \epsilon_1)$ expires out of the money the former condition is satisfied, otherwise we short-sell one unit of the asset. The value of our bank account is then given by $\underline{S}_1 - (S_1^C - K_1 - \epsilon_1) \geq K_1$.

Now suppose that at the beginning of the t-th period the value of our bank account is non-negative. If $k_t \geq k_{t-1}$ the cashflow at time t is non-negative, so the value of the bank account stays non-negative. If on the other hand $k_t \leq k_{t-1}$, we sell one unit of the underlying at time t. The value of the bank account is then given by

$$CF_t = \left(S_t^C - B(t)k_{t-1} - \epsilon_t\right)^+ - \left(S_t^C - K_t - \epsilon_t\right)^+ + \underline{S}_t \geq K_t.$$

Now assume that at the beginning of the t-th period we are in situation B, meaning we have a short position in the underlying and have at least $k_{t-1}B(t)$ at our bank account. Then if both options with maturity t are in the money, the new value of the bank account is K_t. Otherwise we close the short position, which leaves us with a non-negative value in our bank account. At time T the cashflow in both cases is non-negative.

If all four conditions hold then by Theorem 2.11 and Remark 2.20 there exists a peacock $(\nu_t)_{t \in \mathcal{T}}$ with mean S_0 such that

$$W^\infty(\mu_t, \nu_t) \leq \epsilon, \quad t \in \mathcal{T}.$$

By Proposition 3.6 the prices are ϵ-consistent with the absence of arbitrage. ❑

3.4. Multiple maturities: consistency

As mentioned in the introduction, our main goal is to find the least bound on the underlying's bid-ask spread that allows to reproduce given option prices. The following result clarifies the situation if *no* such bound is imposed (see also Example 3.3). By enlarging the class of models, the no-arbitrage conditions become looser. In particular, we do not have any intertemporal conditions. Recall the notation used in Theorem 3.7, where $i = 0$ is allowed in (3.8)-(3.11), inducing a dependence of these conditions on $\underline{S}_0$ and $\overline{S}_0$. In the following proposition, on the other hand, we require $i, j, l \geq 1$, meaning that the current bid and ask prices of the underlying are irrelevant. This result is the main motivation for introducing the concept of ϵ-consistency.

Proposition 3.14. *The prices (3.1)-(3.4) are consistent with the absence of arbitrage (see Definition 3.2) if and only if, for all $t \in \mathcal{T}$, the conditions (3.8)-(3.11) from Theorem 3.7 hold for $i, j, l \in \{1, \dots, N_t\}$.*

Proof. By Theorem 3.7 these conditions are necessary. Now fix $t \in \mathcal{T}$ and assume that all conditions hold. Exactly as in the sufficiency proof of Theorem 3.7, we can construct $e_{t,1}, e_{t,2}, \dots, e_{t,N_t}$ such that $e_{t,i} \in [\underline{r}_{t,i}, \overline{r}_{t,i}]$. The linear interpolation L_t of the points $(k_{t,i}, e_{t,i})_{i \in \{1,\dots,N_t\}}$ can then be extended to a call function of a measure μ_t (see the final part of the sufficiency proof of Theorem 3.7).

We define random variables S_t^C such that the law of $D(t)S_t^C$ is given by μ_t. Then we have that

$$D(t)\mathbb{E}[(S_t^C - K_{t,i})^+] = e_{t,i} \in [\underline{r}_{t,i}, \overline{r}_{t,i}], \quad i \in \{1, \dots N_t\}.$$

Furthermore we pick $s \in [\underline{S}_0, \overline{S}_0]$ and set $\nu_t = \delta_s$ (Dirac delta) for all $t \in \mathcal{T}$. Clearly $(\nu_t)_{t\in\mathcal{T}}$ is a peacock and we set $S_t^* = B(t)s$ such that $D(t)S_t^* \sim \nu_t$. Finally we define $\underline{S}_t = S_t^* \wedge S_t^C$ and $\overline{S}_t = S_t^* \vee S_t^C$ for $t \in \mathcal{T}$ and have thus constructed an arbitrage free model. ❏

It turns out the the conditions of Theorem 3.7 are implied by an even weaker notion of no-arbitrage, where the spread bound has to hold only with a certain probability:

Theorem 3.15. *Let $p \in (0, 1]$ and $\epsilon \geq 0$. For prices (3.1)-(3.4) the following are equivalent:*

- *The prices satisfy Definition 3.4 (ϵ-consistency), but with (3.5) replaced by the weaker condition*
$$\mathbb{P}\Big(\overline{S}_t - \underline{S}_t \geq \epsilon B(t)\Big) \leq p, \quad t \in \mathcal{T}.$$
- *For all $t \in \mathcal{T}$ conditions (3.8)-(3.11) from Theorem 3.7 hold for $i, j, l \in \{1, \dots, N_t\}$.*

Proof. By Theorem 3.7, the second assertion implies the first one. To show the other implication, we define probability measures $(\mu_t)_{t\in\mathcal{T}}$ as in the proof of Proposition 3.14, such that $R_{\mu_t}(k_{t,i}) \in [\underline{r}_{t,i}, \overline{r}_{t,i}]$, for $i \in \{1, \dots N_t\}$ and $t \in \mathcal{T}$. Now we pick $s \in [\underline{S}_0, \overline{S}_0]$. Then by Theorem 2.35 there exists a peacock $(\nu_t)_{t\in\mathcal{T}}$ with mean s such that $d_p^{\mathrm{P}}(\mu_t, \nu_t) \leq \epsilon$ for all $t \in \mathcal{T}$. We can then use Proposition 2.32 to conclude that there exist stochastic processes $(\widetilde{S^C}_t)_{t\in\mathcal{T}}$ and $(\widetilde{S_t^*})_{t\in\mathcal{T}}$ whose marginal distributions are given by μ_t resp. ν_t, such that $(\widetilde{S_t^*})_{t\in\mathcal{T}}$ is a martingale and such that

$$\mathbb{P}\Big(|\widetilde{S_t^*} - \widetilde{S_t^C}| \geq \epsilon\Big) \leq p, \quad t \in \mathcal{T}.$$

For $t \in \mathcal{T}$ we then simply put

$$S_t^* = B(t)\widetilde{S_t^*}, \quad S_t^C = B(t)\widetilde{S_t^C}, \quad \underline{S}_t = S_t^* \wedge S_t^C, \quad \text{and } \overline{S}_t = S_t^* \vee S_t^C.$$

❏

4

Conclusion

Given a family of measures $(\mu_t)_{t\in T}$ on $\mathbb{R}$ with finite means, we derived conditions for the existence of a peacock $(\nu_t)_{t\in T}$ within a certain distance to the given family. We formulated necessary and sufficient conditions for the infinity Wasserstein distance and the stop-loss distance, and showed that such a peacock always exists if we measure distance with the Lévy metric or the Prokhorov metric. In particular, we get the following result, which is a simple corollary of Theorem 2.11, Theorem 2.35 and Corollary 2.36. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}$ with finite means, $\epsilon > 0$, and $p \in (0, 1]$. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a stochastic process $(X_n)_{n\in\mathbb{N}}$ whose marginal laws are given by $(\mu_n)_{n\in\mathbb{N}}$ and a martingale $(M_n)_{n\in\mathbb{N}}$ such that:

(*i*) $\mathbb{P}(|X_n - M_n| > \epsilon) \leq p$ for all $n \in \mathbb{N}$,

(*ii*) $\mathbb{P}(|X_n - M_n| > \epsilon) \leq \epsilon$ for all $n \in \mathbb{N}$,

(*iii*) $\mathbb{P}(|X_n - M_n| > \epsilon) = 0$ for all $n \in \mathbb{N}$ if and only if condition 2.10 holds,

(*iv*) $\mathbb{P}(X_n = M_n) = 1$ for all $n \in \mathbb{N}$ if and only if $(\mu_n)_{n\in\mathbb{N}}$ is a peacock.

Notice that (*iv*) is simply Strassen's theorem. In future work we hope to prove similar statements for other metrics, e.g. the p-Wasserstein distance W^p for $p \geq 1$ (see also Remark 2.21).

Furthermore we used these theoretical results to calibrate models to a given set of European option prices. We allowed models where the future bid-ask spread on the underlying can take positive values and formulated necessary conditions for the existence of consistent models distinguishing whether the bid-ask spread is bounded by a constant or not. We argued that in case it is not bounded by a constant there are no intertemporal conditions (see Proposition 3.14 and Theorem 3.15). In case the bid-ask spread is bounded by a predefined constant, we stated necessary and sufficient conditions for single maturities (Theorem 3.7): these conditions guarantee that for each maturity $t \in \mathcal{T}$ there exists a measure μ_t which explains the associated prices.

For multiple periods we stated necessary conditions which we derived from Theorem 2.11, see Theorem 3.12. Unfortunately, it remains open whether these conditions are sufficient for the existence of ϵ-consistent models or not. In future work, we would like to further study these settings and – if necessary – complete the conditions from Theorem 3.12 to make them also sufficient.

II.

Small-maturity asymptotics for the at-the-money implied volatility slope in Lévy models

1

Introduction

Recent years have seen an explosion of the literature on asymptotics of option prices and implied volatilities (see, e.g., [4, 42] for many references). Such results are of practical relevance for fast model calibration, qualitative model assessment, and parametrisation design. The small-time behavior of the *level* of implied volatility in Lévy models (and generalisations) has been investigated in great detail [12, 35, 36, 37, 78, 84]. We, on the other hand, focus on the at-the-money *slope* of implied volatility, i.e., the strike derivative, and investigate its behavior as maturity becomes small. For diffusion models, there typically exists a limiting smile as the maturity tends to zero, and the limit slope is just the slope of this limit smile (e.g., for the Heston model, this follows from [30, Section 5]). Our focus is, however, on exponential Lévy models. There is no limit smile here that one could differentiate, as the implied volatility blows up off-the-money [84]. In fact, this is a desirable feature, since in this way Lévy models are better suited to capture the steep short maturity smiles observed in the market. But it also implies that the limiting slope cannot be deduced directly from the behavior of implied volatility itself, and requires a separate analysis. (Note that a limiting smile does exist if maturity and log-moneyness tend to zero jointly in an appropriate way [71].)

It turns out that the presence of a Brownian component has a decisive influence: Without it, the ATM (at-the-money) slope explodes (under mild conditions). The blowup is of order $T^{-1/2}$ for many models, but may also be slower (CGMY model with $Y \in (1, 2)$, e.g.; see Example 2.10). Our main results are on Lévy models *with* a Brownian component, though. We provide a result (Corollary 2.6 in Section 2.4) that translates the asymptotic behavior of the moment generating function to that of the ATM slope. When applied to concrete models, we see that the slope may converge to a finite limit (Normal Inverse Gaussian, Meixner, CGMY models), or explode at a rate slower than $T^{-1/2}$ (generalised tempered stable model). Note that several studies [1, 2, 17] highlight the importance of a Brownian component when fitting to historical data or option prices. In particular, in many pure jump Lévy models ATM implied volatility converges to zero as $T \downarrow 0$ (see Proposition 5 in [84] for a precise statement), which seems undesirable.

From a practical point of view, the asymptotic slope is a useful ingredient for model calibration: E.g., if the market slope is negative, then a simple constraint on the model parameters forces the (asymptotic) model slope to be negative, too. Our numerical tests show that the sign of the slope is reliably identified by a first order asymptotic approximation, even if the maturity is not short at all. With our formulas, the asymptotic slope (and, of course, its sign) can be easily determined from the model parameters. For instance, the slope of the NIG (Normal Inverse Gaussian) model is positive if and only if the skewness parameter satisfies $\beta > -\frac{1}{2}$.

To obtain these results, we investigate the asymptotics of ATM digital calls; their relation to the implied volatility slope is well known. While, for Lévy processes X, the small-time behavior of the transition probabilities $\mathbb{P}[X_T \geq x]$ (in finance terms, digital call prices) has been well studied for $x \neq X_0$ (see, e.g., [38] and the references therein), not so much is known for $x = X_0$. Still, first order asymptotics of $\mathbb{P}[X_T \geq X_0]$ are available, and this suffices if there is no Brownian component. If the Lévy process has a Brownian component, then it is well known that $\lim_{T\to 0} \mathbb{P}[X_T \geq X_0] = \frac{1}{2}$. In this case, it turns out that the second order term of $\mathbb{P}[X_T \geq X_0]$ is required to obtain slope asymptotics. For this, we use a novel approach involving the Mellin transform (w.r.t. time) of the transition probability (Sections 2.3 and 2.4). We believe that this method is of wide applicability to other problems involving time asymptotics of Lévy processes, and hope to elaborate on it in future work.

Finally, we consider the question whether a positive at-the-money slope requires the right smile wing to be the steeper one, and vice versa. Wing steepness refers to large-strike asymptotics here. It turns out that this is indeed the case for several of the infinite activity models we consider. This results in a qualitative limitation on the smile shape that these models can produce.

One of the few other works dealing with small-time Lévy slope asymptotics is the comprehensive recent paper by Andersen and Lipton [4]. Besides many other problems on various models and asymptotic regimes, they study the small-maturity ATM digital price and volatility slope for the tempered stable model (Propositions 8.4 and 8.5 in [4]). This includes the CGMY model as a special case (see Example 2.10 for details). Their proof method is entirely different from ours, exploiting the explicit form of the characteristic function of the tempered stable model. Using mainly the dominated convergence theorem, they also analyse the convexity. We, on the other hand, assume a certain asymptotic behavior of the characteristic function, and use its explicit expression only when calculating concrete examples. Our approach covers, e.g., the ATM slope of the generalised tempered stable, NIG, and Meixner models without additional effort.

The recent preprint [39] is also closely related to our work. There, the Brownian component is generalised to stochastic volatility. On the other hand, the assumptions on the Lévy measure exclude, e.g., the NIG and Meixner models. Section 2.5 has additional comments on how our results compare to those of [4] and [39]. Alòs et al. [3] also study the small time implied volatility slope under stochastic volatility and jumps, but the latter are assumed to have finite activity, which is not our focus. Results on the *large* time slope can be found in [41]; see also [43], p. 63f.

2

Digital call prices and slope asymptotics

2.1. Digital call prices

We denote the underlying by $S = e^X$, normalised to $S_0 = 1$, and the pricing measure by $\mathbb{P}$. W.l.o.g. the interest rate is set to zero, and so S is a $\mathbb{P}$-martingale. Suppose that the log-underlying $X = (X_t)_{t\geq 0}$ is a Lévy process with characteristic triplet (b, σ^2, ν) and $X_0 = 0$. The moment generating function (mgf) of X_T is

$$M(z, T) = \mathbb{E}[e^{zX_T}] = \exp\left(T\psi(z)\right),$$

where

$$\psi(z) = \tfrac{1}{2}\sigma^2 z^2 + bz + \int_{\mathbb{R}} \left(e^{zx} - 1 - zx\right) \nu(dx). \tag{2.1}$$

This representation is valid if the Lévy process has a finite first moment, which we of course assume, as even $S_t = e^{X_t}$ should be integrable. If, in addition, X has paths of finite variation, then $\int_{\mathbb{R}} |x|\, \nu(dx) < \infty$, and

$$\psi(z) = \tfrac{1}{2}\sigma^2 z^2 + b_0 z + \int_{\mathbb{R}} \left(e^{zx} - 1\right) \nu(dx),$$

where the drift b_0 is defined by

$$b_0 = b - \int_{\mathbb{R}} x\, \nu(dx).$$

The following theorem collects some results about the small-time behavior of $\mathbb{P}[X_T \geq 0]$. All of them are known, or easily obtained from known results. We are mainly interested in the case where $S = e^X$ is a martingale, and so $\mathbb{P}[X_T \geq 0]$ has the interpretation of an at-the-money digital call price. Still, we mention that this assumption is not necessary for parts *(i)*-*(iv)*. In part *(iv)*, the following condition from [80] is used:

(**H**-α) The Lévy measure ν has a density $g(x)/|x|^{1+\alpha}$, where g is a non-negative measurable function admitting left and right limits at zero:

$$c_+ := \lim_{x\downarrow 0} g(x), \quad c_- := \lim_{x\uparrow 0} g(x), \quad \text{with } c_+ + c_- > 0.$$

Theorem 2.1. *Let X be a Lévy process with characteristic triplet (b, σ^2, ν) and $X_0 = 0$.*

(*i*) *If X has finite variation, and $b_0 \neq 0$, then*

$$\lim_{T \downarrow 0} \mathbb{P}[X_T \geq 0] = \begin{cases} 1, & b_0 > 0 \\ 0, & b_0 < 0. \end{cases}$$

(*ii*) *If $\sigma > 0$, then $\lim_{T\downarrow 0} \mathbb{P}[X_T \geq 0] = \frac{1}{2}$.*

(*iii*) *If X is a Lévy jump diffusion, i.e., it has finite activity jumps and $\sigma > 0$, then*

$$\mathbb{P}[X_T \geq 0] = \frac{1}{2} + \frac{b_0}{\sigma\sqrt{2\pi}}\sqrt{T} + O(T), \quad T \downarrow 0.$$

(*iv*) *Suppose that $\sigma = 0$ and that ($\mathbf{H}$-α) holds for some $\alpha \in [1, 2)$. If $\alpha = 1$, we additionally assume $c_- = c_+ =: c$ and $\int_0^1 x^{-1}|g(x) - g(-x)|\, dx < \infty$. Then*

$$\lim_{T \downarrow 0} \mathbb{P}[X_T \geq 0] = \begin{cases} \frac{1}{2} + \frac{1}{\pi} \arctan \frac{b^*}{\pi c} & \textit{if} \quad \alpha = 1, \\ \frac{1}{2} + \frac{\alpha}{\pi} \arctan(\beta \tan(\frac{\alpha\pi}{2})) & \textit{if} \quad \alpha \neq 1, \end{cases}$$

where $b^* = b - \int_0^\infty (g(x) - g(-x))/x\, dx$ and $\beta = (c_+ - c_-)/(c_+ + c_-)$.

(*v*) *If e^X is a martingale and the Lévy measure satisfies $\nu(dx) = e^{-x/2}\nu_0(dx)$, where ν_0 is a symmetric measure, then*

$$\mathbb{P}[X_T \geq 0] = \Phi(-\sigma_{\text{imp}}(1, T)\sqrt{T}/2),$$

where Φ denotes the standard Gaussian cdf.

Proof. (*i*) We have $\mathbb{P}[X_T \geq 0] = \mathbb{P}[T^{-1}X_T \geq 0]$, but $T^{-1}X_T$ converges a.s. to b_0, by Theorem 43.20 in [81].

(*ii*) If $\sigma > 0$, then $T^{-1/2}X_T$ converges in distribution to a centered Gaussian random variable with variance σ^2 (see [81]). For further CLT-type results in this vein, see [28, 49].

(*iii*) Conditioning on the first jump time τ, which has an exponential distribution, we find

$$\begin{aligned} \mathbb{P}[X_T \geq 0] &= \mathbb{P}[X_T \geq 0 | \tau \leq T] \cdot \mathbb{P}[\tau \leq T] + \mathbb{P}[X_T \geq 0 | \tau > T] \cdot \mathbb{P}[\tau > T] \\ &= O(T) + \mathbb{P}[\sigma W_T + b_0 T \geq 0](1 + O(T)) \\ &= \mathbb{P}[\sigma W_T + b_0 T \geq 0] + O(T) \\ &= \Phi(b_0\sqrt{T}/\sigma) + O(T). \end{aligned} \tag{2.2}$$

Now apply the expansion

$$\Phi(x) = \frac{1}{2} + \frac{x}{\sqrt{2\pi}} + O(x^3), \quad x \to 0. \tag{2.3}$$

(*iv*) By Proposition 1 in [80], the rescaled process $X_t^{\varepsilon,\alpha} := \varepsilon^{-1}X_{\varepsilon^\alpha t}$ converges in law to a strictly α-stable process $X_t^{*,\alpha}$ as $\varepsilon \downarrow 0$. Therefore

$$\lim_{T \downarrow 0} \mathbb{P}[X_T \geq 0] = \lim_{\varepsilon \downarrow 0} \mathbb{P}[\varepsilon^{-1}X_{\varepsilon^\alpha} \geq 0] = \mathbb{P}[X_1^{*,\alpha} \geq 0],$$

and it suffices to evaluate the latter probability. For $\alpha = 1$, $X_1^{*,1}$ has a Cauchy distribution with characteristic exponent

$$\log \mathbb{E}[\exp(iuX_1^{*,1})] = ib^* u - \pi c|u|,$$

hence $\mathbb{P}[X_1^{*,1} \geq 0] = \frac{1}{\pi} \arctan \frac{b^*}{\pi c}$. (Our b^* is denoted γ^* in [80].)

If $1 < \alpha < 2$, then $X_1^{*,\alpha}$ has a strictly stable distribution with characteristic exponent

$$\log \mathbb{E}[\exp(iuX_1^{*,\alpha})] = -|du|^\alpha \Big(1 - i\beta\, sgn(u) \tan(\frac{\alpha\pi}{2})\Big),$$

where

$$d_\pm^\alpha = -\Gamma(-\alpha)\cos(\frac{\alpha\pi}{2})c_\pm \geq 0, \quad d^\alpha = d_+^\alpha + d_-^\alpha, \quad \beta = \frac{d_+^\alpha - d_-^\alpha}{d^\alpha} \in (-1,1).$$

The desired expression for $\mathbb{P}[X_1^{*,\alpha} \geq 0]$ then follows from [24]. See [35] for further related references.

(v) Under this assumption, the market model is symmetric in the sense of [33, 34]. The statement is Theorem 3.1 in [33]. ❑

The variance gamma model and the CGMY model with $0 < Y < 1$ are examples of finite variation models (of course, only when $\sigma = 0$), and so part *(i)* of Theorem 2.1 is applicable. Part *(iii)* is applicable, clearly, to the well-known jump diffusion models by Merton and Kou. In Section 2.5, we will discuss two examples for part *(iv)* (NIG and Meixner).

2.2. Implied Volatility Slope and Digital Options with Small Maturity

The (Black-Scholes) implied volatility is the volatility that makes the Black-Scholes call price equal the call price with underlying S:

$$C_{\mathrm{BS}}(K,T,\sigma_{\mathrm{imp}}(K,T)) = C(K,T) := \mathbb{E}[(S_T - K)^+].$$

Since no explicit expression is known for $\sigma_{\mathrm{imp}}(K,T)$ (see [45]), many authors have investigated approximations (see, e.g., the references in the introduction). The following relation between implied volatility slope and digital calls is well known [43]; we give a proof for completeness. (Note that absolute continuity of S_T holds in all Lévy models of interest, see Theorem 27.4 in [81], and will be assumed throughout.)

Lemma 2.2. *Suppose that the law of S_T is absolutely continuous for each $T > 0$, and that*

$$\lim_{T \downarrow 0} C(K,T) = (S_0 - K)^+, \quad K > 0. \tag{2.4}$$

Then, for $T \downarrow 0$,

$$\partial_K \sigma_{\mathrm{imp}}(K,T)|_{K=1} \sim \sqrt{\frac{2\pi}{T}} \Big(\frac{1}{2} - \mathbb{P}[S_T \geq 1] - \frac{\sigma_{\mathrm{imp}}(1,T)\sqrt{T}}{2\sqrt{2\pi}} + O\big((\sigma_{\mathrm{imp}}(1,T)\sqrt{T})^2\big)\Big). \tag{2.5}$$

Proof. By the implicit function theorem, the implied volatility slope has the representation

$$\partial_K \sigma_{\text{imp}}(K,T) = \frac{\partial_K C(K,T) - \partial_K C_{\text{BS}}(K,T,\sigma_{\text{imp}}(K,T))}{\partial_\sigma C_{\text{BS}}(K,T,\sigma_{\text{imp}}(K,T))}.$$

Since the law of S_T is absolutely continuous, the call price $C(K,T)$ is continuously differentiable w.r.t. K, and $\partial_K C(K,T) = -\mathbb{P}[S_T \geq K]$. Inserting the explicit formulas for the Black-Scholes Vega and digital price, and specialising to the ATM case $K = S_0 = 1$, we get

$$\partial_K \sigma_{\text{imp}}(K,T)|_{K=1} = \frac{\Phi(-\sigma_{\text{imp}}(1,T)\sqrt{T}/2) - \mathbb{P}[S_T \geq 1]}{\sqrt{T}\varphi(\sigma_{\text{imp}}(1,T)\sqrt{T}/2)},$$

where Φ and φ denote the standard Gaussian cdf and density, respectively. By Proposition 4.1 in [79], our assumption (2.4) implies that the annualised implied volatility $\sigma_{\text{imp}}(1,T)\sqrt{T}$ tends to zero as $T \downarrow 0$. (The second assumption used in [79] are the no-arbitrage bounds $(S_0 - K)^+ \leq C(K,T) \leq S_0$, for $K, T > 0$, but these are satisfied here because our call prices are generated by the martingale S.) Using the expansion (2.3) and $\varphi(x) = \frac{1}{\sqrt{2\pi}} + O(x^2)$, we thus obtain (2.5). ❑

The asymptotic relation (2.5) is, of course, consistent with the small-moneyness expansion presented in [25], where $\sqrt{2\pi/T}\left(\frac{1}{2} - \mathbb{P}[S_T \geq K]\right)$ appears as second order term (i.e., first derivative) of implied volatility.

Lemma 2.2 shows that, in order to obtain first order asymptotics for the at-the-money (ATM) slope, we need first order asymptotics for the ATM digital call price $\mathbb{P}[S_T \geq 1]$. (Recall that $S_0 = 1$.) For models where $\lim_{T\downarrow 0} \mathbb{P}[S_T \geq 1] = \frac{1}{2}$, we need the second order term of the digital call as well, and the first order term of $\sigma_{\text{imp}}(1,T)\sqrt{T}$. The limiting value $1/2$ for the ATM digital call is typical for diffusion models (see [49]), and Lévy processes that contain a Brownian motion. For infinite activity models without diffusion component, $\mathbb{P}[S_T \geq 1]$ may converge to $1/2$ as well (e.g., in the CGMY model with $Y \in (1,2)$), but other limiting values are also possible. See the examples in Section 2.5.

From part *(i)* of Theorem 2.1 and Lemma 2.2 we can immediately conclude the following result. Note that we assume throughout that X is such that $S = e^X$ is a martingale with $S_0 = 1$.

Proposition 2.3. *Suppose that the Lévy process X has finite variation (and thus, necessarily, that $\sigma = 0$), and that $b_0 \neq 0$. Then the ATM implied volatility slope satisfies*

$$\partial_K \sigma_{\text{imp}}(K,T)|_{K=1} \sim -\sqrt{\pi/2}\,\text{sgn}(b_0) \cdot T^{-1/2}, \quad T \downarrow 0.$$

Note that $T^{-1/2}$ is the fastest possible growth order for the slope, in any model (see Lee [63]).

If X is a Lévy jump diffusion with $\sigma > 0$, then by part *(iii)* of Theorem 2.1, (2.5), and the fact that $\sigma_{\text{imp}} \to \sigma$ (implied volatility converges to spot volatility), we obtain the finite limit

$$\lim_{T\downarrow 0} \partial_K \sigma_{\text{imp}}(K,T)|_{K=1} = -\frac{b_0}{\sigma} - \frac{\sigma}{2}. \tag{2.6}$$

(It is understood that the substitution $K = 1$ is to be performed before the limit $T \downarrow 0$.) Notice that the expression on the right hand side of (2.6) does depend on the jump parameters, because the drift b_0, fixed by the condition $\mathbb{E}[\exp(X_1)] = 1$, depends on them. Moreover, (2.6) is consistent with the formal calculation of the variance slope

$$\lim_{T\downarrow 0} \partial_K \sigma^2_{\mathrm{imp}}(K,T)|_{K=1} = -2b_0 - \sigma^2$$

on p. 61f in [43]. In fact (2.6) is well known for jump diffusions, see [3, 87].

2.3. General remarks on Mellin transform asymptotics

As mentioned after Lemma 2.2, we need the second order term for the ATM digital call if we want to find the limiting slope in Lévy models with a Brownian component. While this is easy for finite activity models (see the end of the preceding section), it is more difficult in the case of infinite activity jumps. We will find this second order term using Mellin transform asymptotics. For further details and references on this technique, see e.g. [40]. The Mellin transform of a function H, locally integrable on $(0, \infty)$, is defined by

$$(\mathcal{M}H)(s) = \int_0^\infty T^{s-1} H(T)\, dT.$$

Under appropriate growth conditions on H at zero and infinity, this integral defines an analytic function in an open vertical strip of the complex plane. The function H can be recovered from its transform by Mellin inversion (see formula (7) in [40]):

$$H(T) = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} (\mathcal{M}H)(s) T^{-s}\, ds, \tag{2.7}$$

where κ is a real number in the strip of analyticity of $\mathcal{M}H$. For the validity of (2.7), it suffices that H is continuous and that $y \mapsto (\mathcal{M}H)(\kappa + iy)$ is integrable. Denote by $s_0 \in \mathbb{R}$ the real part of the left boundary of the strip of analyticity. A typical situation in applications is that $\mathcal{M}H$ has a pole at s_0, and admits a meromorphic extension to a left half-plane, with further poles at $s_0 > s_1 > s_2 > \dots$ Suppose also that the meromorphic continuation satisfies growth estimates at $\pm i\infty$ which allow to shift the integration path in (2.7) to the left. We then collect the contribution of each pole by the residue theorem, and arrive at an expansion (see formula (8) in [40])

$$H(T) = \mathrm{Res}_{s=s_0} (\mathcal{M}H)(s) T^{-s} + \mathrm{Res}_{s=s_1} (\mathcal{M}H)(s) T^{-s} + \dots$$

Thus, the basic principle is that singularities s_i of the transform are mapped to terms T^{-s_i} in the asymptotic expansion of H at zero. Simple poles of $\mathcal{M}H$ yield powers of T, whereas double poles produce an additional logarithmic factor $\log T$, as seen from the expansion $T^{-s} = T^{-s_i}(1 - (\log T)(s - s_i) + O((s - s_i)^2))$.

2.4. Main results: digital call prices and slope asymptotics

The mgf $M(z,T)$ of X_T is analytic in a strip $z_- < \mathrm{Re}(z) < z_+$, given by the critical moments

$$z_+ = \sup\{z \in \mathbb{R} : \mathbb{E}[e^{zX_T}] < \infty\}$$

and

$$z_- = \inf\{z \in \mathbb{R} : \mathbb{E}[e^{zX_T}] < \infty\}.$$

Since X is a Lévy process, the critical moments do not depend on T. We will obtain asymptotic information on the transition probabilities (i.e., digital call prices) from the Fourier representation [62]

$$\begin{aligned}
\mathbb{P}[S_T \geq 1] &= \mathbb{P}[X_T \geq 0] \\
&= \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} \frac{M(z,T)}{z}\, dz \\
&= \frac{1}{\pi} \mathrm{Re} \int_0^\infty \frac{M(a+iy,T)}{a+iy}\, dy, \qquad (2.8)
\end{aligned}$$

where the real part of the vertical integration contour satisfies $a \in (0,1) \subseteq (z_-, z_+)$, and convergence of the integral is assumed throughout. We are going to analyse the asymptotic behavior of this integral, for $T \downarrow 0$, by computing its Mellin transform. Asymptotics of the probability (digital price) $\mathbb{P}[X_T \geq 0]$ are then evident from (2.8). The linearity of $\log M$ as a function of T enables us to evaluate the Mellin transform in semi-explicit form.

Lemma 2.4. *Suppose that $S = e^X$ is a martingale, and that $\sigma > 0$. Then, for any $a \in (0,1)$, the Mellin transform of the function*

$$H(T) := \int_0^\infty \frac{e^{T\psi(a+iy)}}{a+iy}\, dy, \quad T > 0, \qquad (2.9)$$

is given by

$$(\mathcal{M}H)(s) = \Gamma(s)F(s), \quad 0 < \mathrm{Re}(s) < \tfrac{1}{2}, \qquad (2.10)$$

where

$$F(s) = \int_0^\infty \frac{(-\psi(a+iy))^{-s}}{a+iy}\, dy, \quad 0 < \mathrm{Re}(s) < \tfrac{1}{2}. \qquad (2.11)$$

Moreover, $|(\mathcal{M}H)(s)|$ decays exponentially, if $\mathrm{Re}(s) \in (0, \frac{1}{2})$ is fixed and $|\mathrm{Im}(s)| \to \infty$.

See section 2.7 for the proof of Lemma 2.4. With the Mellin transform in hand, we now proceed to convert an expansion of the mgf at $i\infty$ to an expansion of $\mathbb{P}[X_T \geq 0]$ for $T \downarrow 0$. The following result covers, e.g., the NIG and Meixner models, and the generalised tempered stable model, all with $\sigma > 0$. See Section 2.5 for details.

Theorem 2.5. *Suppose that $S = e^X$ is a martingale, and that $\sigma > 0$. Assume further that there are constants $a \in (0,1)$, $c \in \mathbb{C}$, $\nu \in [1,2)$ and $\varepsilon > 0$ such that the Laplace exponent satisfies*

$$\psi(z) = \frac{1}{2}\sigma^2 z^2 + cz^\nu + O(z^{\nu-\varepsilon}), \quad \mathrm{Re}(z) = a,\ \mathrm{Im}(z) \to \infty. \qquad (2.12)$$

Then the ATM digital call price satisfies

$$\mathbb{P}[X_T \geq 0] = \frac{1}{2} + C_{\tilde{\nu}} T^{\tilde{\nu}} + o(T^{\tilde{\nu}}), \quad T \downarrow 0, \tag{2.13}$$

where $C_{\tilde{\nu}} = \frac{\tilde{\nu}}{2\pi}\left(\frac{1}{2}\sigma^2\right)^{\tilde{\nu}-1} \mathrm{Im}(e^{-i\pi\tilde{\nu}}c)\Gamma(-\tilde{\nu})$ *with* $\tilde{\nu} = (2-\nu)/2 \in (0, \frac{1}{2}]$*. For* $\nu = 1$*, this simplifies to*

$$\mathbb{P}[X_T \geq 0] = \frac{1}{2} + \frac{\mathrm{Re}(c)}{\sigma\sqrt{2\pi}}\sqrt{T} + o(\sqrt{T}), \quad T \downarrow 0.$$

Together with Lemma 2.2, this theorem implies the following corollary, which is our main result on the implied volatility slope as $T \downarrow 0$.

Corollary 2.6. *Under the assumptions of Theorem 2.5, the ATM implied volatility slope behaves as follows:*

(*i*) *If* $\nu = 1$*, then*

$$\lim_{T\downarrow 0} \partial_K \sigma_{\mathrm{imp}}(K,T)|_{K=1} = -\frac{\mathrm{Re}(c)}{\sigma} - \frac{\sigma}{2},$$

with c *from* (2.12).

(*ii*) *If* $1 < \nu < 2$ *and* $C_{\tilde{\nu}} \neq 0$*, then*

$$\partial_K \sigma_{\mathrm{imp}}(K,T)|_{K=1} \sim -\sqrt{2\pi} C_{\tilde{\nu}} T^{\tilde{\nu}-1/2}, \quad T \downarrow 0.$$

Proof of Theorem 2.5. From (2.8) and (2.9) we know that

$$\mathbb{P}[X_T \geq 0] = \frac{1}{\pi} \mathrm{Re}\, H(T). \tag{2.14}$$

We now express $H(T)$ by the Mellin inversion formula (2.7), with $\kappa \in (0, \frac{1}{2})$. This is justified by Lemma 2.4, which yields the exponential decay of the transform $\mathcal{M}H$ along vertical rays. (Continuity of H, which is also needed for the inverse transform, is clear.) Therefore, we have

$$H(T) = \frac{1}{2\pi i} \int_{1/4-i\infty}^{1/4+i\infty} \Gamma(s)F(s)T^{-s}\, ds, \quad T \geq 0. \tag{2.15}$$

As outlined in Section 2.3, we now show that $\Gamma(s)F(s)$ has a meromorphic continuation, then shift the integration path in (2.15) to the left, and collect residues. It is well known that Γ is meromorphic with poles at the non-positive integers, so it suffices to discuss the continuation of F, defined in (2.11). As in the proof of Lemma 2.4, we put $h(y) := -\psi(a+iy)$, $y \geq 0$. To prove exponential decay of the desired meromorphic continuation, it is convenient to split the integral:

$$\begin{aligned} F(s) &= \int_0^{y_0} \frac{h(y)^{-s}}{a+iy}\, dy + \int_{y_0}^{\infty} \frac{h(y)^{-s}}{a+iy}\, dy \\ &=: A_0(s) + \tilde{F}(s), \quad 0 < \mathrm{Re}(s) < \tfrac{1}{2}. \end{aligned} \tag{2.16}$$

The constant $y_0 \geq 0$ will be specified later. It is easy to see that A_0 is analytic in the half-plane $\mathrm{Re}(s) < \frac{1}{2}$, and so $\tilde{F}$ captures all poles of F in that half-plane. By (2.12), the function h has the expansion (with a possibly decreased ε, to be precise)

$$h(y) = \tfrac{1}{2}\sigma^2 y^2 + \tilde{c} y^\nu + O(y^{\nu-\varepsilon}), \quad y \to \infty, \tag{2.17}$$

where

$$\tilde{c} := \begin{cases} -ci^\nu & \nu > 1, \\ -(c+\sigma^2 a)i & \nu = 1. \end{cases}$$

The reason why F (or $\tilde{F}$) is not analytic at $s = 0$ is that the second integral in (2.16) fails to converge for y large. We thus subtract the following convergence-inducing integral from $\tilde{F}$:

$$\begin{aligned} \tilde{G}_1(s) &:= \int_{y_0}^{\infty} \frac{(\frac{1}{2}\sigma^2 y^2)^{-s}}{a+iy}\, dy \\ &= -\pi i (\tfrac{1}{2}a^2\sigma^2)^{-s} \frac{e^{i\pi s}}{\sin 2\pi s} - \int_0^{y_0} \frac{(\frac{1}{2}\sigma^2 y^2)^{-s}}{a+iy}\, dy \\ &=: G_1(s) + A_1(s). \end{aligned} \tag{2.18}$$

Note that G_1 is meromorphic, and that A_1 is analytic for $\mathrm{Re}(s) < \frac{1}{2}$. From the expansion

$$h(y)^{-s} = (\tfrac{1}{2}\sigma^2 y^2)^{-s} - \frac{2\tilde{c}s}{\sigma^2}\left(\frac{\sigma^2}{2}\right)^{-s} y^{\nu-2s-2} + O(y^{\nu-2\,\mathrm{Re}(s)-2-\varepsilon}), \quad y \to \infty, \tag{2.19}$$

for s fixed, we see that the function

$$\tilde{F}_1(s) := \int_{y_0}^{\infty} \frac{1}{a+iy}\left(h(y)^{-s} - (\tfrac{1}{2}\sigma^2 y^2)^{-s}\right)\, dy \tag{2.20}$$

is analytic for $-\tilde{\nu} < \mathrm{Re}(s) < \frac{1}{2}$, and, clearly, for $0 < \mathrm{Re}(s) < \frac{1}{2}$ we have

$$\tilde{F}(s) = \tilde{F}_1(s) + \tilde{G}_1(s). \tag{2.21}$$

We have thus established the meromorphic continuation of $\tilde{F}$ to the strip $-\tilde{\nu} < \mathrm{Re}(s) < \frac{1}{2}$. To continue $\tilde{F}$ even further, we look at the second term in (2.19) and define

$$\begin{aligned} \tilde{G}_2(s) &:= -\frac{2\tilde{c}s}{\sigma^2}\left(\frac{\sigma^2}{2}\right)^{-s} \int_{y_0}^{\infty} \frac{y^{\nu-2s-2}}{a+iy}\, dy \\ &= -\frac{2\tilde{c}\pi}{\sigma^2}\left(\frac{\sigma^2}{2}\right)^{-s} s a^{\nu-2s-2} \frac{e^{(2s-\nu+3)\pi i/2}}{\sin \pi(\nu-2s)} + \frac{2\tilde{c}s}{\sigma^2}\left(\frac{\sigma^2}{2}\right)^{-s} \int_0^{y_0} \frac{y^{\nu-2s-2}}{a+iy}\, dy \\ &=: G_2(s) + A_2(s) \end{aligned}$$

and the compensated function

$$\tilde{F}_2(s) := \int_{y_0}^{\infty} \frac{1}{a+iy}\left(h(y)^{-s} - (\tfrac{1}{2}\sigma^2 y^2)^{-s} + \frac{2\tilde{c}s}{\sigma^2}\left(\frac{\sigma^2}{2}\right)^{-s} y^{\nu-2s-2}\right)\, dy.$$

By (2.19), the function $\tilde{F}_2$ is analytic for $\mathrm{Re}(s) \in (-\tilde{\nu} - \varepsilon/2, (\nu-1)/2)$. Moreover, by definition we have

$$\tilde{F}_1(s) = \tilde{F}_2(s) + \tilde{G}_2(s), \quad -\tilde{\nu} < \mathrm{Re}(s) < \tfrac{\nu-1}{2},$$

and so the meromorphic continuation of $\tilde{F}$ to the region $-\tilde{\nu} - \varepsilon/2 < \mathrm{Re}(s) < \frac{1}{2}$ is established.

In order to shift the integration path in (2.15) to the left, we have to ensure that the integral converges. This is the content of Lemma 2.7 below, which also yields the existence of an appropriate $y_0 \geq 0$, to be used in the definition of $\tilde{F}$ in (2.16). By the residue theorem, we obtain

$$\begin{aligned} H(T) &= \mathrm{Res}_{s=0}(\mathcal{M}H)(s)T^{-s} + \mathrm{Res}_{s=-\tilde{\nu}}(\mathcal{M}H)(s)T^{-s} \\ &\qquad + \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} (\mathcal{M}H)(s)T^{-s}\, ds, \quad T \geq 0, \end{aligned} \tag{2.22}$$

where $\kappa = -\tilde{\nu} - \varepsilon/4$, and $\mathcal{M}H$ now of course denotes the meromorphic continuation of the Mellin transform. We then compute the residues. According to (2.16) and (2.21), the continuation of $\mathcal{M}H$ in a neighborhood of $s = 0$ is given by $\Gamma(s)(A_0(s) + \tilde{F}_1(s) + \tilde{G}_1(s))$. Therefore,

$$\begin{aligned} \mathrm{Res}_{s=0}(\mathcal{M}H)(s)T^{-s} &= A_0(0) + \tilde{F}_1(0) + A_1(0) + \mathrm{Res}_{s=0}\Gamma(s)G_1(s)T^{-s} \\ &= \mathrm{Res}_{s=0}\Gamma(s)G_1(s)T^{-s} \\ &= \tfrac{1}{2}\pi + i(\tfrac{1}{2}\gamma - \log(a\sigma/\sqrt{2}) + \tfrac{1}{2}\log T), \end{aligned} \tag{2.23}$$

where γ is Euler's constant. Note that $A_0(0) = -A_1(0)$ and $\tilde{F}_1(0) = 0$ by definition. The remaining residue (2.23) is straightforward to compute from (2.18) (with a computer algebra system, e.g.) and has real part $\frac{1}{2}\pi$. Notice that the logarithmic term $\log T$, resulting from the *double* pole at zero (see the end of Section 2.3), appears only in the imaginary part. Recalling (2.14), we see that the first term on the right-hand side of (2.22) thus yields the first term of (2.13).

Similarly, we compute for $\nu > 1$

$$\begin{aligned} \mathrm{Res}_{s=-\tilde{\nu}}(\mathcal{M}H)(s)T^{-s} &= \mathrm{Res}_{s=-\tilde{\nu}}\Gamma(s)G_2(s)T^{-s} \\ &= \frac{\Gamma(-\tilde{\nu})}{2\pi}\left[\frac{2\tilde{c}s}{\sigma^2}\left(\frac{\sigma^2}{2}\right)^{-s}\pi a^{\nu-2s-2}e^{(2s-\nu+3)\pi i/2}T^{-s}\right]_{s=-\tilde{\nu}}. \end{aligned}$$

In the case $\nu = 1$, the function G_1 also has a pole at $-\tilde{\nu} = -\frac{1}{2}$, and we obtain

$$\begin{aligned} \mathrm{Res}_{s=-\tilde{\nu}}(\mathcal{M}H)(s)T^{-s} &= \mathrm{Res}_{s=-1/2}\Gamma(s)(G_1(s)+G_2(s))T^{-s} \\ &= \sqrt{\frac{\pi}{2}}\left(\frac{i\tilde{c}}{\sigma} - a\sigma\right)\sqrt{T}. \end{aligned}$$

A straightforward computation shows that the stated formula for $C_{\tilde{\nu}}$ is correct in both cases. The integral on the right-hand side of (2.22) is clearly $O(T^{-\kappa}) = o(T^{\tilde{\nu}})$, and so the proof is complete. ❑

Lemma 2.7. *There is $y_0 \geq 0$ such that the meromorphic continuation of $\mathcal{M}H$ constructed in the proof of Theorem 2.5, which depends on y_0 via the definition of $\tilde{F}$ in (2.16), decays exponentially as $|\mathrm{Im}(s)| \to \infty$.*

Lemma 2.7 is proved in section 2.7.

2.5. Examples

We now apply our main results (Theorem 2.5 and Corollary 2.6) to several concrete models.

Example 2.8. The NIG (Normal Inverse Gaussian) model has Laplace exponent

$$\psi(z) = \tfrac{1}{2}\sigma^2 z^2 + \mu z + \delta\big(\sqrt{\hat{\alpha}^2 - \beta^2} - \sqrt{\hat{\alpha}^2 - (\beta+z)^2}\big),$$

where $\delta > 0$, $\hat{\alpha} > \max\{\beta + 1, -\beta\}$. (The notation $\hat{\alpha}$ should avoid confusion with α from Theorem 2.1.) Since S is a martingale, we must have

$$\mu = -\tfrac{1}{2}\sigma^2 + \delta\big(\sqrt{\hat{\alpha}^2 - (\beta+1)^2} - \sqrt{\hat{\alpha}^2 - \beta^2}\big).$$

The relation between μ and b from (2.1) is $\mu + \beta\delta/\sqrt{\hat{\alpha}^2 - \beta^2} = b$, as seen from the derivative of the Laplace exponent ψ at $z = 0$. The Lévy density is

$$\frac{\nu(dx)}{dx} = \frac{\delta\hat{\alpha}}{\pi|x|} e^{\beta x} K_1(\hat{\alpha}|x|),$$

where K_1 is the modified Bessel function of second order and index 1.

First assume $\sigma = 0$. Since $K_1(x) \sim 1/x$ for $x \downarrow 0$, condition ($\mathbf{H}$-α) is satisfied with $\alpha = 1$, with $c_+ = c_- = \delta/\pi$. The integrability condition in part *(iv)* of Theorem 2.1 is easily checked, and we conclude

$$\lim_{T \downarrow 0} \mathbb{P}[X_T \geq 0] = \frac{1}{2} + \frac{1}{\pi}\arctan\Big(\frac{\mu}{\delta}\Big), \quad \sigma = 0.$$

Note that $b^* = \mu = b - \frac{\delta\hat{\alpha}}{\pi}\int_0^\infty K_1(\hat{\alpha}x)(e^{\beta x} - e^{-\beta x})\,dx$. By Lemma 2.2, the implied volatility slope of the NIG model thus satisfies

$$\partial_K \sigma_{\mathrm{imp}}(K,T)|_{K=1} \sim -\sqrt{2/\pi}\arctan(\mu/\delta)\cdot T^{-1/2}, \quad T \downarrow 0, \quad \sigma = 0,\ \mu \neq 0.$$

Now assume that $\sigma > 0$. Since $\sqrt{\hat{\alpha}^2 - (\beta+z)^2} = -iz + O(1)$ as $\mathrm{Im}(z) \to \infty$, the expansion (2.12) becomes

$$\psi(z) = \tfrac{1}{2}\sigma^2 z^2 + (\mu + i)z + O(1), \quad \mathrm{Re}(z) = a,\ \mathrm{Im}(z) \to \infty.$$

We can thus apply Theorem 2.5 to conclude that the ATM digital price satisfies

$$\mathbb{P}[X_T \geq 0] = \frac{1}{2} + \frac{\mu}{\sigma\sqrt{2\pi}}\sqrt{T} + o(\sqrt{T}), \quad T \downarrow 0, \quad \sigma > 0.$$

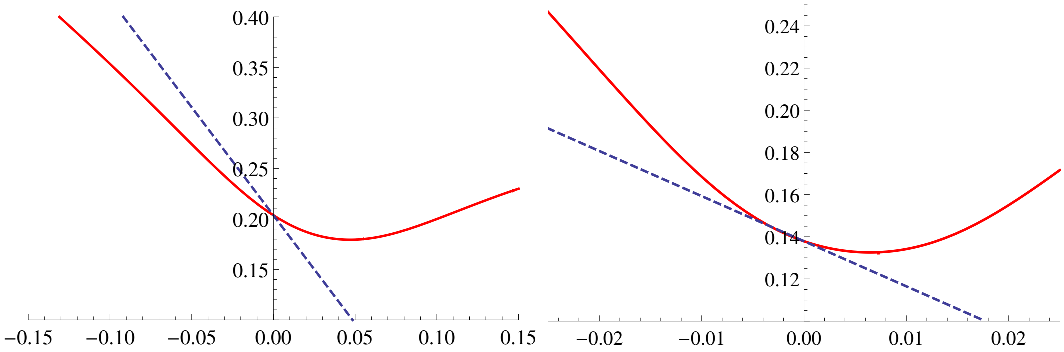

Figure 2.1.: The volatility smile, as a function of log-strike, of the NIG model with parameters $\sigma = 0.085$, $\hat{\alpha} = 4.237$, $\beta = -3.55$, $\delta = 0.167$, and maturity $T = 0.1$ (left panel) respectively $T = 0.01$ (right panel). The parameters were calibrated to S&P 500 call prices from Appendix A of [13]. The dashed line is the slope approximation (2.24). We did the calibration and the plots with Mathematica, using the Fourier representation of the call price.

By part *(i)* of Corollary 2.6, the limit of the implied volatility slope is given by

$$\begin{aligned}\lim_{T\downarrow 0} \partial_K \sigma_{\text{imp}}(K,T)|_{K=1} &= -\frac{\mu}{\sigma} - \frac{\sigma}{2} \\ &= \frac{\delta}{\sigma}\left(\sqrt{\hat{\alpha}^2 - \beta^2} - \sqrt{\hat{\alpha}^2 - (\beta+1)^2}\right), \quad \sigma > 0. \end{aligned} \tag{2.24}$$

This limit is positive if and only if $\beta > -\frac{1}{2}$.

See Figure 2.1 for a numerical example. Let us stress again that we identify the correct *sign* of the slope, while we find that explicit asymptotics do not approximate the *value* of the slope very accurately. Still, in the right panel of Figure 2.1 we have zoomed in at very short maturity to show that our approximation gives the asymptotically correct tangent in this example.

Example 2.9. The Laplace exponent of the Meixner model is

$$\psi(z) = \tfrac{1}{2}\sigma^2 z^2 + \mu z + 2\hat{d}\log\frac{\cos(\hat{b}/2)}{\cosh\frac{1}{2}(-\hat{a}iz - i\hat{b})},$$

where $\hat{d} > 0$, $\hat{b} \in (-\pi, \pi)$, and $0 < \hat{a} < \pi - \hat{b}$. (We follow the notation of Schoutens [82], except that we write μ instead of m, and $\hat{a}, \hat{b}, \hat{d}$ instead of a, b, d.) The Lévy density is

$$\frac{\nu(dx)}{dx} = \hat{d}\frac{\exp(\hat{b}x/\hat{a})}{x\sinh(\pi x/\hat{a})}.$$

We can proceed analogously to Example 2.8. For $\sigma = 0$ we again apply part *(iv)* of Theorem 2.1, with $\alpha = 1$, where now $c_+ = c_- = \hat{d}\hat{a}/\pi$. Consequently,

$$\lim_{T\downarrow 0} \mathbb{P}[X_T \geq 0] = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{\mu}{\hat{a}\hat{d}}\right), \quad \sigma = 0,$$

and

$$\partial_K \sigma_{\text{imp}}(K,T)|_{K=1} \sim -\sqrt{2/\pi}\arctan\left(\frac{\mu}{\hat{a}\hat{d}}\right) \cdot T^{-1/2}, \quad T \downarrow 0, \quad \sigma = 0,\ \mu \neq 0.$$

Now assume $\sigma > 0$. The expansion of the Laplace exponent is

$$\psi(z) = \tfrac{1}{2}\sigma^2 z^2 + (\mu + \hat{a}\hat{d}i)z + O(1), \quad \operatorname{Re}(z) = a,\ \operatorname{Im}(z) \to \infty.$$

By Theorem 2.5, the ATM digital price in the Meixner model thus satisfies

$$\mathbb{P}[X_T \geq 0] = \frac{1}{2} + \frac{\mu}{\sigma\sqrt{2\pi}}\sqrt{T} + o(\sqrt{T}), \quad T \downarrow 0.$$

The limit of the implied volatility slope is given by

$$\begin{aligned}\lim_{T\downarrow 0} \partial_K \sigma_{\text{imp}}(K,T)|_{K=1} &= -\frac{\mu}{\sigma} - \frac{\sigma}{2} \\ &= \frac{2\hat{d}}{\sigma}\log\left(\frac{\cos(\hat{b}/2)}{\cosh\frac{1}{2}(-(\hat{a}+\hat{b})i)}\right), \quad \sigma > 0.\end{aligned}$$

Example 2.10. The Laplace exponent of the CGMY model is

$$\psi(z) = \tfrac{1}{2}\sigma^2 z^2 + \mu z + C\Gamma(-Y)((M-z)^Y - M^Y + (G+z)^Y - G^Y), \tag{2.25}$$

where we assume $C > 0$, $G > 0$, $M > 1$, $0 < Y < 2$, and $Y \neq 1$.

The case $\sigma = 0$ and $Y \in (0,1)$ need not be discussed, as it is a special case of Proposition 8.5 in [4]. Our Proposition 2.3 could also be applied, as the CGMY process has finite variation in this case.

If $\sigma = 0$ and $Y \in (1,2)$, then the ATM digital call price converges to $\frac{1}{2}$, and the slope explodes, of order $T^{1/2-1/Y}$. This is a special case of Corollary 3.3 in [39]. Note that Proposition 8.5 in [4] is not applicable here, because the constant $C_{\mathfrak{M}}$ from this proposition vanishes for the CGMY model, and so the leading term of the slope is not obtained. Theorem 2.1 *(iv)* from Section 2 is not useful, either; it gives the correct digital call limit price $\frac{1}{2}$, but does not provide the second order term necessary to get slope asymptotics.

We now proceed to the case $\sigma > 0$, which is our main focus. The expansion of ψ at $i\infty$ is

$$\psi(z) = \tfrac{1}{2}\sigma^2 z^2 + c_Y z^Y + \mu z + O(z^{Y-1}), \quad \operatorname{Re}(z) = a,\ \operatorname{Im}(z) \to \infty,$$

with the complex constant $c_Y := C\Gamma(-Y)(1 + e^{-i\pi Y})$. First assume $0 < Y < 1$. Then we proceed analogously to the preceding examples, applying Theorem 2.5 and Corollary 2.6. The ATM digital price thus satisfies

$$\mathbb{P}[X_T \geq 0] = \frac{1}{2} + \frac{\mu}{\sigma\sqrt{2\pi}}\sqrt{T} + o(\sqrt{T}), \quad T \downarrow 0, \tag{2.26}$$

and the limit of the implied volatility slope is given by

$$\begin{aligned}\lim_{T\downarrow 0} \partial_K \sigma_{\text{imp}}(K,T)|_{K=1} &= -\frac{\mu}{\sigma} - \frac{\sigma}{2} \\ &= \frac{1}{\sigma}C\Gamma(-Y)((M-1)^Y - M^Y + (G+1)^Y - G^Y).\end{aligned} \tag{2.27}$$

Now assume $1 < Y < 2$. In principle, Theorem 2.5 is applicable, with $\nu = Y$; however, the constant $C_{\tilde{\nu}}$ in (2.13) is zero, and so we do not get the second term of the expansion immediately. What happens is that the Mellin transform of H (see the proof of Theorem 2.5) may have further poles in $-\frac{1}{2} < \mathrm{Re}(s) < 0$, but none of them gives a contribution, since the corresponding residues have zero real part. Therefore, (2.26) and (2.27) are true also for $1 < Y < 2$. Note that (2.26) and (2.27) also follow from concurrent work by Figueroa-López and Ólafsson [39]. For $0 < Y < 1$, they also follow from Proposition 8.5 in [4], but not for $1 < Y < 2$, because then the constant $C_{\mathfrak{M}}$ from that proposition vanishes when specialising it to the CGMY model.

In the following example, we discuss the generalised tempered stable model. The tempered stable model, which is investigated in [4], is obtained by setting $\alpha_- = \alpha_+$.

Example 2.11. The generalised tempered stable process [19] is a generalisation of the CGMY model, with Lévy density

$$\frac{\nu(dx)}{dx} = \frac{C_-}{|x|^{1+\alpha_-}} e^{-\lambda_-|x|} 1_{(-\infty,0)}(x) + \frac{C_+}{|x|^{1+\alpha_+}} e^{-\lambda_+|x|} 1_{(0,\infty)}(x),$$

where $\alpha_\pm < 2$ and $C_\pm, \lambda_\pm > 0$. For $\alpha_\pm \notin \{0,1\}$ the Laplace exponent of the generalised tempered stable process is

$$\psi(z) = \tfrac{1}{2}\sigma^2 z^2 + \mu z + \Gamma(-\alpha_+)C_+\Big((\lambda_+ - z)^{\alpha_+} - \lambda_+^{\alpha_+}\Big) \\ + \Gamma(-\alpha_-)C_-\Big((\lambda_- + z)^{\alpha_-} - \lambda_-^{\alpha_-}\Big).$$

For $\sigma > 0$, $\alpha_+ \in (1,2)$, and $\alpha_- < \alpha_+$ we have the following expansion:

$$\psi(z) = \tfrac{1}{2}\sigma^2 z^2 + \Gamma(-\alpha_+)C_+ e^{-i\pi\alpha_+} z^{\alpha_+} + O(z^{\max\{1,\alpha_-\}}), \quad \mathrm{Re}(z) = a,\ \mathrm{Im}(z) \to \infty.$$

We now apply Theorem 2.5 with $\nu = \alpha_+$, and find that the second order expansion of the ATM digital call is

$$\mathbb{P}[X_T \geq 0] = \frac{1}{2} + C_{\tilde{\nu}} T^{\tilde{\nu}} + o(T^{\tilde{\nu}}), \quad T \downarrow 0,$$

with $\tilde{\nu} = 1 - \alpha_+/2 \in (0, \frac{1}{2})$ and the real constant

$$\begin{aligned} C_{\tilde{\nu}} &= \frac{\tilde{\nu}}{2\pi} \left(\tfrac{1}{2}\sigma^2\right)^{\tilde{\nu}-1} \Gamma(-\alpha_+)C_+ \,\mathrm{Im}(e^{-i\pi\tilde{\nu}} e^{-i\pi\alpha_+})\Gamma(-\tilde{\nu}) \\ &= \frac{\tilde{\nu}}{2\pi} \left(\tfrac{1}{2}\sigma^2\right)^{\tilde{\nu}-1} \Gamma(-\alpha_+)C_+ \sin(-\pi(1+\alpha_+/2))\Gamma(-\tilde{\nu}). \end{aligned}$$

By Corollary 2.6 *(ii)*, the ATM implied volatility slope explodes, but slower than $T^{-1/2}$:

$$\partial_K \sigma_{\mathrm{imp}}(K,T)|_{K=1} \sim -\sqrt{2\pi} C_{\tilde{\nu}} T^{\tilde{\nu}-1/2}, \quad T \downarrow 0.$$

Note that these results also follow from the concurrent paper [39], which treats tempered stable-like models.

If $\sigma > 0$ and $\alpha_+ < 1$, then part *(i)* of Corollary 2.6 is applicable, and formulas analogous to (2.26) and (2.27) hold.

2.6. Robustness of Lee's Moment Formula

As we have already mentioned, our first order slope approximations give limited accuracy for the size of the slope, but usually succeed at identifying its sign, i.e., whether the smile increases or decreases at the money. It is a natural question whether this sign gives information on the smile as a whole: If the slope is positive, does it follow that the right wing is steeper than the left one, and vice versa? To deal with this issue, recall Lee's moment formula [61]. Under the assumption that the critical moments z_+ and z_-, defined in (2.4) and (2.4), are finite, Lee's formula states that

$$\limsup_{k\to\infty} \frac{\sigma_{\mathrm{imp}}(K,T)}{\sqrt{k}} = \sqrt{\frac{\Psi(z_+ - 1)}{T}} \tag{2.28}$$

and

$$\limsup_{k\to-\infty} \frac{\sigma_{\mathrm{imp}}(K,T)}{\sqrt{-k}} = \sqrt{\frac{\Psi(-z_-)}{T}}, \tag{2.29}$$

where $T > 0$ is fixed, $k = \log K$, and $\Psi(x) := 2 - 4(\sqrt{x^2 + x} - x)$. According to Lee's formula, the slopes of the wings depend on the size of the critical moments. In Lévy models, the critical moments do not depend on T. The compatibility property we seek now becomes:

$$\lim_{k\to\infty} \frac{\sigma_{\mathrm{imp}}(K,T)}{\sqrt{k}} > \lim_{k\to-\infty} \frac{\sigma_{\mathrm{imp}}(K,T)}{\sqrt{-k}} \quad \text{for all } T > 0 \tag{2.30}$$

if and only if

$$\partial_K \sigma_{\mathrm{imp}}(K,T)|_{K=1} > 0 \qquad \text{for all sufficiently small } T. \tag{2.31}$$

That is, the right wing of the smile is steeper than the left wing deep *out-of-the-money* if and only if the small-maturity *at-the-money* slope is positive. We now show that this is true for several infinite activity Lévy models. By our methods, this can certainly be extended to other infinite activity models. It does not hold, though, for the Merton and Kou jump diffusion models. The parameter ranges in the following theorem are the same as in the examples in Section 2.5.

Theorem 2.12. *Conditions* (2.30) *and* (2.31) *are equivalent for the following models. For the latter three, we assume that* $\sigma > 0$ *or* $\mu \neq 0$.

- *Variance gamma with* $\sigma = 0$, $b_0 \neq 0$
- *NIG*
- *Meixner*
- *CGMY*

Put differently, these models are *not* capable (at short maturity) of producing a smile that has, say, its minimum to the left of $\log K = k = 0$, and thus a positive ATM slope, but whose left wing is steeper than the right one.

Proof. The critical moments are clearly finite for all of these models. Moreover, it is well known that the lim sup in (2.28) and (2.29) can typically be replaced by a genuine limit, for instance using the criteria given by Benaim and Friz [8]. Their conditions on the mgf are easily verified for all our models; in fact Benaim and Friz [8] explicitly treat the variance gamma model with $b_0 = 0$ and the NIG model. We thus have to show that (2.31) is equivalent to $\Psi(z_+ - 1) > \Psi(-z_-)$. Since Ψ is strictly decreasing on $(0, \infty)$, the latter condition is equivalent to $z_+ - 1 < -z_-$. It remains to check the equivalence

$$z_+ - 1 < -z_- \quad \Longleftrightarrow \quad (2.31). \tag{2.32}$$

The mgf of the variance gamma model is (see [69])

$$M(z,T) = e^{Tb_0 z}(1 - \theta\nu z - \tfrac{1}{2}\hat{\sigma}^2\nu z^2)^{-T/\nu},$$

where $\hat{\sigma}, \nu > 0$ and $\theta \in \mathbb{R}$. Its paths have finite variation, and so Proposition 2.3 shows that (2.31) is equivalent to $b_0 < 0$. The critical moments are

$$z_\pm = -\frac{\nu\theta \pm \sqrt{2\nu\hat{\sigma}^2 + \nu^2\theta^2}}{\nu\hat{\sigma}^2},$$

and we have $-z_- + 1 - z_+ = 1 + 2\theta/\hat{\sigma}^2$. This is positive if and only if

$$b_0 = \nu^{-1}\log(1 - \theta\nu - \tfrac{1}{2}\hat{\sigma}^2\nu) < 0,$$

which yields (2.32).

As for the other three models, first suppose that $\sigma > 0$. The examples in Section 2.5 show that (2.31) is equivalent to $\mu < -\frac{1}{2}\sigma^2$. The critical moments of the NIG model are $z_+ = \hat{\alpha} - \beta$ and $z_- = -\hat{\alpha} - \beta$. Therefore, $z_+ - 1 < -z_-$ if and only if $\beta > -\frac{1}{2}$, and this is indeed equivalent to

$$\mu + \tfrac{1}{2}\sigma^2 = \delta\left(\sqrt{\hat{\alpha}^2 - (\beta+1)^2} - \sqrt{\hat{\alpha}^2 - \beta^2}\right) < 0.$$

For the Meixner model, we have $z_\pm = (\pm\pi - \hat{b})/\hat{a}$, which yields $-z_- + 1 - z_+ = 1 + 2\hat{b}/\hat{a}$. On the other hand,

$$\mu + \tfrac{1}{2}\sigma^2 = -2\hat{d}\log\frac{\cos(\hat{b}/2)}{\cos((\hat{a}+\hat{b})/2)},$$

which is negative if and only if $\cos(\hat{b}/2) > \cos((\hat{a}+\hat{b})/2)$, and this is equivalent to $\hat{a}+2\hat{b} > 0$.

Finally, in case of the CGMY model, we have

$$\mu + \tfrac{1}{2}\sigma^2 = -C\Gamma(-Y)\big((M-1)^Y - M^Y + (G+1)^Y - G^Y\big).$$

Since, for $Y \in (0,1)$, $\Gamma(-Y) < 0$ and the function $x \mapsto x^Y - (x+1)^Y$ is strictly increasing on $(0, \infty)$, we see that $\mu + \frac{1}{2}\sigma^2 < 0$ if and only if $M - 1 < G$. This is the desired condition, since the explicit expression (2.25) shows that $z_+ = M$ and $z_- = -G$. The case $Y \in (1,2)$ is analogous.

It remains to treat the case $\sigma = 0$. First, note that the critical moments do not depend on σ. Furthermore, from the examples in Section 2.5, we see that (2.31) holds if and only if $\mu < 0$. Now observe that adding a Brownian motion σW_t to a Lévy model adds $-\frac{1}{2}\sigma^2$ to the drift, if the martingale property is to be preserved. Therefore, the assertion follows from what we have already proved about $\sigma > 0$. ❑

2.7. Proofs of Lemmas 2.4 and 2.7

Proof of Lemma 2.4. Since $S = e^X$ is a martingale, we have $\psi'(0) = \mathbb{E}[X_1] < 0$. Then $\psi(0) = 0$ implies that $\psi(a) < 0$ for all sufficiently small $a > 0$. In fact, it easily follows from $\psi(1) = 0$ and the concavity of ψ that all $a \in (0,1)$ satisfy $\psi(a) < 0$. Let us fix such an a. From

$$\mathrm{Re}(-\psi(a+iy)) = -\psi(a) + \frac{1}{2}\sigma^2 y^2 + \int_{\mathbb{R}} e^{ax}(1-\cos(yx))\,\nu(dx)$$

we obtain that the function $h(y) := -\psi(a+iy)$, $y \geq 0$, satisfies

$$\mathrm{Re}\, h(y) > \tfrac{1}{2}\sigma^2 y^2 \geq 0, \quad y \geq 0. \tag{2.33}$$

For $0 < \mathrm{Re}(s) < \frac{1}{2}$ define the function

$$g(T) = T^{\mathrm{Re}(s)-1} \int_0^\infty \frac{e^{-T\,\mathrm{Re}(h(y))}}{|a+iy|}\,dy, \quad T > 0.$$

Using Fubini's theorem and substituting $T\,\mathrm{Re}(h(y)) = u$, we then calculate for $\mathrm{Re}(s) > 0$

$$\begin{aligned}
\int_0^\infty g(T)\,dT &= \int_0^\infty \frac{1}{|a+iy|} \int_0^\infty e^{-T\,\mathrm{Re}(h(y))} T^{\mathrm{Re}(s)-1}\,dT dy \\
&= \int_0^\infty \frac{\mathrm{Re}(h(y))^{-\mathrm{Re}(s)}}{|a+iy|} \left(\int_0^\infty e^{-u} u^{\mathrm{Re}(s)-1}\,du \right) dy \\
&= \Gamma(\mathrm{Re}(s)) \int_0^\infty \frac{\mathrm{Re}(h(y))^{-\mathrm{Re}(s)}}{|a+iy|}\,dy.
\end{aligned}$$

From (2.33), we get

$$\int_0^\infty \frac{\mathrm{Re}(h(y))^{-\mathrm{Re}(s)}}{|a+iy|}\,dy \leq (\tfrac{1}{2}\sigma^2)^{-\mathrm{Re}(s)} \int_0^\infty \frac{y^{-2\,\mathrm{Re}(s)}}{|a+iy|}\,dy.$$

The restriction $\mathrm{Re}(s) < \frac{1}{2}$ ensures that the last integral is finite and thus the integrability of g. Using the dominated convergence theorem and Fubini's theorem, the Mellin transform of H can now be calculated as

$$\int_0^\infty H(T)T^{s-1}\,dT = \int_0^\infty \frac{1}{a+iy} \int_0^\infty e^{-Th(y)} T^{s-1}\,dT dy.$$

The substitution $Th(y) = u$ gives us the result. Note that $h(y)$ is in general non-real; it is easy to see, though, that Euler's integral

$$\Gamma(s) = \int_0^\infty u^{s-1} e^{-u}\,du, \quad \mathrm{Re}(s) > 0,$$

still represents the gamma function if the integration is performed along any complex ray emanating from zero, as long as the ray stays in the right half-plane. The latter holds, since $\mathrm{Re}(h(y)) > 0$.

It remains to prove the exponential decay of the Mellin transform $\mathcal{M}H(s) = \Gamma(s)F(s)$ for large $|\operatorname{Im}(s)|$. First, note that

$$\begin{aligned}\operatorname{Im}\psi(a+iy) &= by + \sigma^2 ay + \int_{\mathbb{R}} (e^{ax}\sin xy + xy)\,\nu(dx)\\ &= O(y), \quad y\to\infty,\end{aligned}$$

which together with (2.33) yields the existence of an $\varepsilon > 0$ such that $|\arg h(y)| \le \frac{1}{2}\pi - \varepsilon$ for all $y \ge 0$. We then estimate, with $\mathrm{Re}(s) \in (0, \frac{1}{2})$ fixed,

$$\begin{aligned}|F(s)| &\le \int_0^\infty \frac{e^{-\mathrm{Re}(s\log h(y))}}{|a+iy|}\,dy\\ &= \int_0^\infty \frac{e^{-\mathrm{Re}(s)\log|h(y)| + \operatorname{Im}(s)\arg h(y)}}{|a+iy|}\,dy\\ &\le e^{(\pi/2-\varepsilon)|\operatorname{Im}(s)|}\int_0^\infty \frac{(\frac{1}{2}\sigma^2 y^2)^{-\mathrm{Re}(s)}}{|a+iy|}\,dy.\end{aligned}$$

The integral converges, and thus this estimate is good enough, since Stirling's formula yields $|\Gamma(s)| = \exp\big(-\frac{1}{2}\pi|\operatorname{Im}(s)|(1+o(1))\big)$. ❑

Proof of Lemma 2.7. Recall that, in the proof of Theorem 2.5, we defined the following meromorphic continuation of $F(s)$, to the strip $-\tilde{\nu} - \frac{1}{2}\varepsilon < \mathrm{Re}(s) < \frac{1}{2}$:

$$\begin{aligned}&A_0(s) + \tilde{G}_1(s) + \tilde{F}_1(s), \quad -\tilde{\nu} < \mathrm{Re}(s) < \tfrac{1}{2},\\ &A_0(s) + \tilde{G}_1(s) + \tilde{G}_2(s) + \tilde{F}_2(s), \quad -\tilde{\nu} - \tfrac{1}{2}\varepsilon < \mathrm{Re}(s) < \tfrac{1}{2}(\nu - 1).\end{aligned}$$

As noted at the end of the proof of Lemma 2.4, Stirling's formula implies $|\Gamma(s)| = \exp\big(-\frac{1}{2}\pi|\operatorname{Im}(s)|(1+o(1))\big)$. By (2.10), it thus suffices to argue that the continuation of $F(s)$ is $O(\exp((\frac{1}{2}\pi - \varepsilon)|\operatorname{Im}(s)|))$ for some $\varepsilon > 0$. The functions $\tilde{G}_1$ and $\tilde{G}_2$ are clearly $O(1)$. As for A_0, defined in (2.16), we have

$$\begin{aligned}|A_0(s)| &\le \int_0^{y_0} \frac{e^{-\mathrm{Re}(s\log h(y))}}{|a+iy|}\,dy\\ &= \int_0^{y_0} \frac{|h(y)|^{-\mathrm{Re}(s)} e^{\operatorname{Im}(s)\arg h(y)}}{|a+iy|}\,dy.\end{aligned}$$

Now note that

$$|h(y)|^{-\mathrm{Re}(s)} \le \begin{cases}(\frac{1}{2}\sigma^2 y^2)^{-\mathrm{Re}(s)} & 0 < \mathrm{Re}(s) < \frac{1}{2},\\ (\max_{0\le y\le y_0}|h(y)|)^{-\mathrm{Re}(s)} & \mathrm{Re}(s) \le 0,\end{cases}$$

and that

$$\exp(\operatorname{Im}(s)\arg h(y)) \le \exp((\tfrac{\pi}{2}-\varepsilon)|\operatorname{Im}(s)|)$$

for some $\varepsilon > 0$, as argued in the proof of Lemma 2.4.

It remains to establish a bound for $\tilde{F}_1$, defined in (2.20). (The bound for $\tilde{F}_2$ is completely analogous, and we omit the details.) In what follows, we assume that $-\tilde{\nu} < \operatorname{Re}(s) < \frac{1}{2}$. By (2.17), we have (where the O is uniform w.r.t. s, and $y_0 \ge 0$ is still arbitrary):

$$\begin{aligned}\tilde{F}_1(s) &= \int_{y_0}^{\infty} \frac{1}{a+iy}\Big((\tfrac{1}{2}\sigma^2 y^2)^{-s}(1+O(y^{\nu-2}))^{-s} - (\tfrac{1}{2}\sigma^2 y^2)^{-s}\Big)\,dy \\ &= \int_{y_0}^{\infty} \frac{1}{a+iy}(\tfrac{1}{2}\sigma^2 y^2)^{-s}\Big((1+O(y^{\nu-2}))^{-s} - 1\Big)\ dy. \end{aligned} \tag{2.34}$$

We now choose y_0 such that, for some constant $C_0 > 0$,

$$\begin{aligned}|\log|1+O(y^{\nu-2})|| &\le \tfrac{1}{4}\pi, \\ |\arg(1+O(y^{\nu-2}))| &\le \tfrac{1}{4}\pi, \\ |\log(1+O(y^{\nu-2}))| &\le C_0 y^{\nu-2},\end{aligned}$$

hold for all $y \ge y_0$. (By a slight abuse of notation, here $O(y^{\nu-2})$ of course denotes the function hiding behind the $O(y^{\nu-2})$ in (2.34).) For all $w \in \mathbb{C}$ we have the estimate

$$|e^w - 1| \le |w| e^{|\operatorname{Re}(w)|}.$$

Using this in (2.34), we find

$$\begin{aligned}\Big|(1+O(y^{\nu-2}))^{-s} - 1\Big| &= \Big|\exp(-s\log(1+O(y^{\nu-2}))) - 1\Big| \\ &\le |s\log(1+O(y^{\nu-2}))| \cdot \exp(|\operatorname{Re}(s\log(1+O(y^{\nu-2}))|) \\ &\le C_1|s|y^{\nu-2}\exp(\tfrac{1}{4}\pi|\operatorname{Im}(s)|),\end{aligned}$$

where $C_1 = C_0\exp(\frac{1}{4}\pi\sup_s|\operatorname{Re}(s)|)$, and thus

$$\begin{aligned}|\tilde{F}_1(s)| &\le C_2|s|e^{\frac{1}{4}\pi|\operatorname{Im}(s)|}\int_{y_0}^{\infty} y^{-2\operatorname{Re}(s)+\nu-3}\,dy \\ &= \exp\big(\tfrac{1}{4}\pi|\operatorname{Im}(s)|(1+o(1))\big).\end{aligned}$$

❑

3 Conclusion

The main result of this part of the thesis (Corollary 2.6) translates asymptotics of the log-underlying's moment generating function to first-order asymptotics for the at-the-money implied volatility slope. As a byproduct we obtain asymptotics for ATM digital call prices (Theorem 2.5). Checking the requirements of Theorem 2.5 resp. Corollary 2.6 only requires Taylor expansion of the moment generating function, which has an explicit expression in many models of practical interest: we illustrated this in section 2.5 with several examples.

Higher order expansions can be obtained by the same proof technique, if desired. They will follow in a relatively straightforward way from higher order expansions of the moment generating function, by collecting further residues of the Mellin transform. In future work, we hope to connect our assumptions on the moment generating function with properties of the Lévy triplet, which should give additional insight on how the slope depends on model characteristics.

III.

Portfolio Optimisation using adaptive strategies

1

Introduction

The problem of portfolio optimisation with respect to given utility function goes back to a result by Merton in 1971 ([70]): he showed that in order to maximise expected utility in the Black-Scholes model the investor has to keep a constant fraction π^* invested in the risky asset. This quantity π^* has an easy representation in terms of the parameters of the Black-Scholes model and is often referred to as Merton Proportion. Realising this strategy requires the investor to trade continuously, which is not applicable in practice for many reasons. One of these reasons is that in presence of transaction costs, continuous trading leads to immediate bankruptcy.

In 1990 Davis and Norman ([22]) described a optimal strategy in the Black-Scholes model with proportional transaction costs: instead of keeping the risky fraction, i.e. the proportion of wealth held in the risky asset, constant at π^*, the investor is only supposed to trade, if the actual risky fraction π_t leaves a certain interval I containing π^*. The optimal strategy is then to instantaneously rebalance the portfolio to assure that π_t stays in the no-trade region I. Mathematically, this leads to theory of reflected diffusions: if the investor follows the optimising strategy, her risky fraction is a reflected diffusion with domain I and trading only occurs infinitesimally to ensure that the risky fraction stays in the no-trade region.

We want to analyse the following trading strategy: while the risky fraction π_t is in the interior of an interval I containing the Merton proportion π^* we do not trade. Once the risky fraction reaches the boundary of I we rebalance the portfolio such that the risky fractions jumps to π^*. This strategy – although not optimal – is better applicable in practice since there are only finitely many trading times compared with the optimal strategy derived by Davis and Norman. Therefore it makes sense to study this strategy also in a frictionless setting which is what we will do.

Similar strategies have been studied by by Rogers in 2001 ([77]) and by Irle and Sass in 2006 ([54]). In [77] the author studies strategies where the investor in a frictionless market is only allowed to rebalance her portfolio in times which are a multiple of $h > 0$. He shows that the loss a *relaxed* investor faces compared to an optimal investor are relatively small. The authors of [54] study so called constant boundary (CB) strategies in markets with proportional and fixed transaction costs. These are strategies, where the investor rebalances her portfolio only when the risky-fraction is about to leave a certain domain D. They show that there exists a CB-strategy which is optimal in the class of all CB-strategies and they provide numerical examples on how to determine this strategy.

The strategy we are interested in is also a CB-strategy: in our case D is a small interval $[\pi^* - \alpha, \pi^* + \beta]$ containing the Merton proportion π^*. Our aim is to study the growth rate in dependence of the parameters α and β. We restrict ourselves to the Black-Scholes model.

On a technical level we will work with diffusions with jump boundaries: these are diffusions starting in a domain $D \subseteq \mathbb{R}$ and which jump to interior of D once they hit the boundary ∂D of D. This evolution is then repeated indefinitely. In [7] the authors prove that diffusions with jump boundaries are ergodic and have an invariant distribution. They also give an representation of its probability density in terms of the Greens's function.

2

Asymptotics of the Growth rate in the Black-Scholes model

2.1. Problem Formulation

Consider a financial market consisting of a risky asset S and a risk-free bank account B. Their dynamics are given by

$$\begin{aligned} dB_t &= rB_t dt, \quad B_0 = 1, \\ dS_t &= \mu S_t dt + \sigma S_t dW_t, \quad S_0 = 1, \end{aligned}$$

where W is a one-dimensional Brownian motion. Denote by ϕ_t^0 and ϕ_t the number of bonds and stocks hold by the investor at time t. Then the total wealth of the investor at time t is given by $V_t = \phi_t^0 B_t + \phi_t S_t$ and

$$\pi_t = \frac{\phi_t S_t}{V_t}$$

is the fraction of total wealth held in stocks. A simple calculation shows that the V can be written as

$$V_t = v \exp\Big(rt + \sigma \int_0^t \pi_s \, dW_s + \int_0^t (\mu - r)\pi_s - \frac{1}{2}\sigma^2 \pi_s^2 \, ds\Big),$$

where $v > 0$ denotes the investor's starting capital.

We want to study the growth rate R

$$R = \lim_{T\to\infty} \frac{1}{T} \mathbb{E}[\log V_T]$$

for the following strategy: we choose ϕ_0^0 and ϕ_0 such that $V_0 = v$ and $\pi_0 = \pi^*$, where π^* denotes the Merton proportion,

$$\pi^* = \frac{\mu - r}{\sigma^2}.$$

The portfolio is left unchanged until the almost surely finite time τ when π leaves a given interval $(\pi^* - \alpha, \pi^* + \beta)$ around the Merton proportion. Afterwards the portfolio is rebalanced such that π jumps back to π^* at time τ. This evolution is repeated indefinitely.

2.2. Main results

We consider the case where $\pi^* \notin \{0,1\}$. The logarithm of the stock price at time t will be denoted with $X_t > 0$. By replacing μ with $\mu - r$ and adding r to R we can assume that $r = 0$, which makes notation lighter. For given $v > 0$, $\alpha \in [0, \pi^*)$ and $\beta \in [0, 1-\pi^*)$ we choose

$$\phi_t^0 = v(1-\pi^*), \quad \phi_t = v\pi^*, \quad t \in [0, \tau],$$

where τ is defined as

$$\tau = \inf\Big\{t \geq 0 : \pi_t - \pi^* \notin (-\alpha, \beta)\Big\} = \inf\Big\{t \geq 0 : X_t \notin (a, b)\Big\}. \tag{2.1}$$

It is easy to see that $a, b \in \mathbb{R}$ can be chosen such that the above relation holds, if α, β are sufficiently small, e.g. if $\pi^* \in (0,1)$ we have

$$a = \log\left(\frac{1}{\pi^*}\frac{-\alpha}{1+\alpha-\pi^*} + 1\right), \quad b = \log\left(\frac{1}{\pi^*}\frac{\beta}{1-\beta-\pi^*} + 1\right).$$

Note that in any case $a < 0 < b$ such that $\mathbb{E}[\tau] < \infty$.

The portfolio is left unchanged until time τ where the portfolio is rebalanced such that the new risky fraction equals the Merton proportion π^*. This evolution is repeated indefinitely.

More precisely we set $\tau_0 = 0$, $\tau_1 = \tau$ and inductively we define

$$\phi_t^0 = V_{\tau_n}(1-\pi^*), \quad \phi_t = \frac{V_{\tau_n}\pi^*}{S_{\tau_n}}, \quad t \in (\tau_n, \tau_{n+1}],$$

$$\tau_{n+1} = \inf\{t > \tau_n : \pi_t \notin (\pi^* - \alpha, \pi^* + \beta)\}.$$

Then for $t \in (\tau_n, \tau_{n+1}]$,

$$\pi_t = \frac{\pi^* S_t/S_{\tau_n}}{1 - \pi^* + \pi^* S_t/S_{\tau_n}}.$$

Since $(S_t/S_{\tau_n})_{t \geq \tau_n}$ is again a geometric Brownian Motion starting at 1 at time τ_n and independent from $\mathcal{F}_{\tau_n}$, we can use the theory of Brownian motion with jump boundary (see [7]) to give a more elegant formulation of our problem.

In the following we set $D = [a, b]$. Suppose that $\{X^n : n \in \mathbb{N}_0\}$ is a sequence of processes which all have the same law as X and which are all killed at the boundary of D, denoted with ∂D. Furthermore we set $\sigma_1 := \tau_1$ and

$$\sigma_{n+1} = \inf\{t > 0 : X_t^n \in \partial D\}.$$

Note that $\tau_n \overset{d}{=} \sum_{k=1}^n \sigma_n$. A Brownian motion with drift with jumps from the boundary is defined as

$$Z_t := \sum_{n=0}^{\infty} \mathbf{1}_{[\tau_n, \tau_{n+1})}(t) X_{t-\tau_n}^n.$$

The process π can now be expressed in terms of Z as follows:

$$\pi_t = \frac{\pi^* \exp(Z_t)}{1 - \pi^* + \pi^* \exp(Z_t)} =: g(Z_t).$$

The main ingredient to determine the growth rate is the following Proposition. For the following we set $\gamma = \pi^* - \frac{1}{2}$. Strictly speaking the following result is only true for $\gamma \neq 0$, the case where $\gamma = 0$ is discussed later.

Proposition 2.1. *The process Z has an invariant distribution ν which is absolutely continuous with respect to the Lebesgue measure on D. Its density is given by*

$$f_\nu(y) = \frac{2\exp(\gamma y)\sinh\Big(\gamma\big(b-(0\vee y)\big)\Big)\sinh\Big(\gamma\big((0\wedge y)-a\big)\Big)}{ae^{a\gamma}\sinh(b\gamma)-be^{b\gamma}\sinh(a\gamma)}, \quad y \in D. \tag{2.2}$$

Proof. According to [7], Proposition 1, Z has an invariant distribution with density

$$f_\nu(y) = \frac{G(0,y)}{\int_D G(0,z)\,dz},$$

where $G(x,y)$ is the Green's function of the process X killed at the boundary of D. An explicit expression for G is well known in the literature (see [57], p.198):

$$G(x,y) = \frac{2\big(P(b)-P(x\vee y)\big)\big(P(x\wedge y)-P(a)\big)}{\sigma^2\big(P(b)-P(a)\big)P'(y)}, \quad x,y \in (a,b),$$

and

$$P(x) = \int_0^x \exp(-2\gamma u)\,du,$$

is the so-called scale function of X. Putting everything together we see that

$$G(x,y) = \frac{2\exp(\gamma(y-x))\sinh\Big(\gamma\big(b-(x\vee y)\big)\Big)\sinh\Big(\gamma\big((x\wedge y)-a\big)\Big)}{\sigma^2\gamma\sinh\big(\gamma(b-a)\big)}, \quad x,y \in D. \tag{2.3}$$

Furthermore

$$\int_D G(0,z)\,dz = \frac{ae^{a\gamma}\sinh(b\gamma)-be^{b\gamma}\sinh(a\gamma)}{\sigma^2\gamma\sinh\big((b-a)\gamma\big)},$$

which yields (2.2). ❑

The growth rate can now easily be determined. Denote by h the function

$$h(y) := \mu g(y) - \frac{1}{2}\sigma^2 g(y)^2$$

Then we have that

$$\lim_{T\to\infty}\frac{1}{T}\,\mathbb{E}[\log V_T] = \log(v) + \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbb{E}[\mu\pi_t - \frac{1}{2}\sigma^2\pi_t^2]\,dt$$

$$= \log(v) + \int_a^b h(y)f_\nu(y)\,dy.$$

It is possible to derive an explicit but lengthy expression in terms of the incomplete beta function. More importantly we can now prove the following.

Theorem 2.2. *For $a = b$ the growth rate R satisfies*

$$R = \log(v) + \frac{1}{2}(\pi^*\sigma)^2 - \frac{1}{12}\Big((1-\pi^*)\pi^*\sigma^2\Big)^2 b^2 + O(b^4),$$

for $b \to 0$.

Proof. Since the function h does not depend on b we can replace h by its second order Taylor polynomial T_h^2 in the definition of the growth rate to get an approximation. We note that

$$T_h^2(y) = \frac{1}{2}(\pi^*\sigma)^2 - \frac{1}{2}\Big((1-\pi^*)\pi^*\sigma^2\Big)^2 y^2,$$

and that

$$\int_{-b}^{b} y^3 f_\nu(y)\, dy = O(b^4).$$

Hence we get the following

$$\begin{aligned} R - \log(v) &= \int_{-b}^{b} h(y) f_\nu(y)\, dy \\ &= \int_{-b}^{b} T_h^2(y) f_\nu(y)\, dy + O(b^4) \\ &= \frac{1}{2}(\pi^*\sigma)^2 - \frac{1}{12}\Big((1-\pi^*)\pi^*\sigma^2\Big)^2 b^2 + O(b^4) \\ &= \frac{1}{2}(\pi^*\sigma)^2 - \frac{1}{12}\sigma^2\beta^2 + O(\beta^3). \end{aligned}$$

❑

This result shows that the growth rate R, regarded as a function in β has a local maximum at $\beta = 0$, which illustrates the optimality of π^*.

Finally, we want to summarise the results for $\gamma = 0$. They can be obtained from the general results simply by taking the limit, but since there is a very simple expression for the growth rate in this case we want to state it explicitly.

Corollary 2.3. *Assume that $\gamma = 0$. Then for $\alpha = \beta$*

$$R = \log(v) - \sigma^2 \frac{\log(1-4\beta^2)}{2\Big(\log(1+2\beta) - \log(1-2\beta)\Big)^2}.$$

Proof. Let $\gamma = 0$. Then the Green's function of Z is given by

$$G(x,y) = \frac{2}{\sigma^2(b-a)}\big(b - (x \vee y)\big)\big((x \wedge y) - a\big), \quad x, y \in D,$$

which can be seen by either taking the limit in (2.3) or by using $P(x) = x$. As before, P denotes the scale function. Therefore the density of the invariant distribution is given by

$$f_\nu(y) = -\frac{G(0,y)}{ab}.$$

The growth rate can then be written as

$$\begin{aligned}
R - \log(v) &= \int_a^b h(y) f_\nu(y)\, dy \\
&= \frac{1}{b-a}\left(\frac{1}{a}\log\Big(\frac{2}{1+\exp(-a)}\Big) + \frac{1}{b}\log\Big(\frac{1+\exp(b+)}{2}\Big) - \sigma\right) \\
&= -\sigma^2 \frac{\log(1-4\beta^2)}{2\Big(\log(1+2\beta) - \log(1-2\beta)\Big)^2} \\
&= \frac{\sigma^2}{8} - \frac{\sigma^2\beta^2}{12} + O(\beta^4),
\end{aligned}$$

where the second equality holds only for $\beta = \alpha$. ❑

Bibliography

[1] Y. Aït-Sahalia, *Telling from discrete data whether the underlying continuous-time model is a diffusion*, Journal of Finance, 57 (2002), pp. 2075–2113.

[2] Y. Aït-Sahalia and J. Jacod, *Is Brownian motion necessary to model high-frequency data?*, Ann. Statist., 38 (2010), pp. 3093–3128.

[3] E. Alòs, J. A. León, and J. Vives, *On the short-time behavior of the implied volatility for jump-diffusion models with stochastic volatility*, Finance Stoch., 11 (2007), pp. 571–589.

[4] L. Andersen and A. Lipton, *Asymptotics for exponential Lévy processes and their volatility smile: survey and new results*, Int. J. Theor. Appl. Finance, 16 (2013). Paper no. 1350001, 98 pages.

[5] B. Armbruster, *A short proof of Strassen's theorem using convex analysis.* Preprint, available at http://users.iems.northwestern.edu/~armbruster/, 2013.

[6] M. Beiglböck, M. Huesmann, and F. Stebegg, *Root to Kellerer*, arXiv preprint arXiv:1507.07690, (2015).

[7] I. Ben-Ari and R. G. Pinsky, *Ergodic behavior of diffusions with random jumps from the boundary*, Stochastic Process. Appl., 119 (2009), pp. 864–881.

[8] S. Benaim and P. Friz, *Smile asymptotics II: Models with known moment generating functions*, J. Appl. Probab., 45 (2008), pp. 16–32.

[9] M. Bichuch, *Pricing a contingent claim liability with transaction costs using asymptotic analysis for optimal investment*, Finance Stoch., 18 (2014), pp. 651–694.

[10] C. R. Borges, *The sup metric on infinite products*, Bull. Austral. Math. Soc., 44 (1991), pp. 461–466.

[11] M. W. Botsko and R. A. Gosser, *Stronger versions of the fundamental theorem of calculus*, The American Mathematical Monthly, 93 (1986), pp. 294–296.

[12] S. I. Boyarchenko and S. Z. Levendorskiĭ, *Non-Gaussian Merton-Black-Scholes theory*, vol. 9 of Advanced Series on Statistical Science & Applied Probability, World Scientific Publishing Co., Inc. River Edge, NJ, 2002.

[13] Y. Bu, *Option pricing using Lévy processes*, master's thesis, Chalmers University of Technology, Göteborg, 2007.

[14] H. Buehler, *Expensive martingales*, Quant. Finance, 6 (2006), pp. 207–218.

[15] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, vol. 33 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2001.

[16] P. Carr and D. B. Madan, *A note on sufficient conditions for no arbitrage*, Finance Research Letters, 2 (2005), pp. 125–130.

[17] P. Carr and L. Wu, *What type of process underlies options? A simple robust test*, The Journal of Finance, 58 (2003), pp. 2581–2610.

[18] T. Champion, L. De Pascale, and P. Juutinen, *The ∞-Wasserstein distance: Local solutions and existence of optimal transport maps*, SIAM Journal on Mathematical Analysis, 40 (2008), pp. 1–20.

[19] R. Cont and P. Tankov, *Financial modelling with jump processes*, Chapman & Hall/CRC Financial Mathematics Series, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[20] L. Cousot, *Conditions on option prices for absence of arbitrage and exact calibration*, Journal of Banking and Finance, 31 (2007), pp. 3377–3397.

[21] M. H. A. Davis and D. G. Hobson, *The range of traded option prices*, Math. Finance, 17 (2007), pp. 1–14.

[22] M. H. A. Davis and A. R. Norman, *Portfolio selection with transaction costs*, Math. Oper. Res., 15 (1990), pp. 676–713.

[23] M. H. A. Davis, V. G. Panas, and T. Zariphopoulou, *European option pricing with transaction costs*, SIAM J. Control Optim., 31 (1993), pp. 470–493.

[24] Y. A. Davydov and I. A. Ibragimov, *On asymptotic behavior of some functionals of processes with independent increments*, Theory Probab. Appl., 16 (1971), pp. 162–167.

[25] L. De Leo, V. Vargas, S. Ciliberti, and J.-P. Bouchaud, *We've walked a million miles for one of these smiles*. Preprint, available at http://arxiv.org/abs/1203.5703, 2012.

[26] N. De Pril and J. Dhaene, *Error bounds for compound Poisson approximations of the individual risk model*, Astin Bulletin, 22 (1992), pp. 135–148.

[27] C. Dellacherie and P.-A. Meyer, *Probabilities and potential. C*, vol. 151 of North-Holland Mathematics Studies, North-Holland Publishing Co., Amsterdam, 1988. Potential theory for discrete and continuous semigroups, Translated from the French by J. Norris.

[28] R. A. Doney and R. A. Maller, *Stability and attraction to normality for Lévy processes at zero and at infinity*, J. Theoret. Probab., 15 (2002), pp. 751–792.

[29] R. M. Dudley, *Distances of probability measures and random variables*, Ann. Math. Statist, 39 (1968), pp. 1563–1572.

[30] V. Durrleman, *From implied to spot volatilities*, Finance Stoch., 14 (2010), pp. 157–177.

[31] H. Edelsbrunner and J. L. Harer, *Computational topology*, American Mathematical Society, Providence, RI, 2010. An introduction.

[32] D. Èlton and T. P. Khill, *Fusions of a probability distribution: preliminary announcement*, Teor. Veroyatnost. i Primenen., 37 (1992), pp. 402–406.

[33] J. Fajardo, *Barrier style contracts under Lévy processes: an alternative approach*. Preprint, available on SSRN, 2014.

[34] J. Fajardo and E. Mordecki, *Symmetry and duality in Lévy markets*, Quant. Finance, 6 (2006), pp. 219–227.

[35] J. E. Figueroa-López and M. Forde, *The small-maturity smile for exponential Lévy models*, SIAM J. Financial Math., 3 (2012), pp. 33–65.

[36] J. E. Figueroa-López, R. Gong, and C. Houdré, *Small-time expansions of the distributions, densities, and option prices of stochastic volatility models with Lévy jumps*, Stochastic Process. Appl., 122 (2012), pp. 1808–1839.

[37] J. E. Figueroa-López, R. Gong, and C. Houdré, *High-order short-time expansions for ATM option prices of exponential Lévy models*. To appear in Mathematical Finance, 2014.

[38] J. E. Figueroa-López and C. Houdré, *Small-time expansions for the transition distributions of Lévy processes*, Stochastic Process. Appl., 119 (2009), pp. 3862–3889.

[39] J. E. Figueroa-López and S. Ólafsson, *Short-time asymptotics for the implied volatility skew under a stochastic volatility model with Lévy jumps*. Preprint, available at http://arxiv.org/abs/1502.02595, 2015.

[40] P. Flajolet, X. Gourdon, and P. Dumas, *Mellin transforms and asymptotics: harmonic sums*, Theoret. Comput. Sci., 144 (1995), pp. 3–58. Special volume on mathematical analysis of algorithms.

[41] M. Forde, A. Jacquier, and J. E. Figueroa-López, *The large-time smile and skew for exponential Lévy models*. Preprint, 2011.

[42] P. Friz, S. Gerhold, A. Gulisashvili, and S. Sturm, *On refined volatility smile expansion in the Heston model*, Quantitative Finance, 11 (2011), pp. 1151–1164.

[43] J. Gatheral, *The Volatility Surface, A Practitioner's Guide*, Wiley, 2006.

[44] H. U. Gerber, *An introduction to mathematical risk theory*, vol. 8, SS Huebner Foundation for Insurance Education, Wharton School, University of Pennsylvania Philadelphia, 1979.

[45] S. Gerhold, *Can there be an explicit formula for implied volatility?*, Appl. Math. E-Notes, 13 (2013), pp. 17–24.

[46] S. Gerhold and I. C. Gülüm, *Consistency of option prices under bid-ask spreads*. Working paper, 2016.

[47] ——, *A variant of strassen's theorem: Existence of martingales within a prescribed distance*. Submitted, 2016.

[48] S. Gerhold, I. C. Gülüm, and A. Pinter, *The small maturity implied volatility slope for lévy models*. In Revision, 2016.

[49] S. Gerhold, M. Kleinert, P. Porkert, and M. Shkolnikov, *Small time central limit theorems for semimartingales with applications*, Stochastics, 87 (2015), pp. 723–746.

[50] A. L. Gibbs and F. E. Su, *On choosing and bounding probability metrics*, International Statistical Review, 70 (2002), pp. 419–435.

[51] F. Hirsch, C. Profeta, B. Roynette, and M. Yor, *Peacocks and associated martingales, with explicit constructions*, vol. 3 of Bocconi & Springer Series, Springer, Milan; Bocconi University Press, Milan, 2011.

[52] F. Hirsch and B. Roynette, *A new proof of Kellerer's theorem*, ESAIM Probab. Stat., 16 (2012), pp. 48–60.

[53] P. J. Huber and E. M. Ronchetti, *Robust statistics*, Wiley Series in Probability and Statistics, John Wiley & Sons, Inc., Hoboken, NJ, second ed., 2009.

[54] A. Irle and J. Sass, *Good portfolio strategies under transaction costs: a renewal theoretic approach*, in Stochastic finance, Springer, New York, 2006, pp. 321–341.

[55] R. Kaas, A. Van Heerwaarden, and M. Goovaerts, *On stop-loss premiums for the individual model*, Astin Bulletin, 18 (1988), pp. 91–97.

[56] Y. M. Kabanov and C. Stricker, *The Harrison-Pliska arbitrage pricing theorem under transaction costs*, J. Math. Econom., 35 (2001), pp. 185–196. Arbitrage and control problems in finance.

[57] S. Karlin and H. M. Taylor, *A second course in stochastic processes*, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981.

[58] H. G. Kellerer, *Markov-Komposition und eine Anwendung auf Martingale*, Math. Ann., 198 (1972), pp. 99–122.

[59] ——, *Integraldarstellung von Dilationen*, in Transactions of the Sixth Prague Conference on Information Theory, Statistical Decision Functions, Random Processes (Tech. Univ., Prague, 1971; dedicated to the memory of Antonín Špaček), Academia, Prague, 1973, pp. 341–374.

[60] R. P. Kertz and U. Rösler, *Complete lattices of probability measures with applications to martingale theory*, in Game theory, optimal stopping, probability and statistics, vol. 35 of IMS Lecture Notes Monogr. Ser., Inst. Math. Statist., Beachwood, OH, 2000, pp. 153–177.

[61] R. W. Lee, *The moment formula for implied volatility at extreme strikes*, Math. Finance, 14 (2004), pp. 469–480.

[62] ——, *Option pricing by transform methods: Extensions, unification, and error control*, Journal of Computational Finance, 7 (2004), pp. 51–86.

[63] ——, *Implied volatility: statics, dynamics, and probabilistic interpretation*, in Recent advances in applied probability, Springer, New York, 2005, pp. 241–268.

[64] P. Lévy, *Calcul des probabilités*, vol. 9, Gauthier-Villars Paris, 1925.

[65] T. Lindvall, *On Strassen's theorem on stochastic domination*, Electron. Comm. Probab., 4 (1999), pp. 51–59 (electronic). Review, pointing out and correcting an

error, available at http://www.ams.org/mathscinet-getitem?mr=1711599.

[66] ——, *Lectures on the coupling method*, Dover Publications, Inc, 2002. Corrected reprint of the 1992 original.

[67] G. Lowther, *Fitting martingales to given marginals*, arXiv preprint arXiv:0808.2319, (2008).

[68] ——, *Limits of one-dimensional diffusions*, Ann. Probab., 37 (2009), pp. 78–106.

[69] D. Madan, P. Carr, and E. Chang, *The variance gamma process and option pricing*, European Finance Review, 2 (1998), pp. 79–105.

[70] R. C. Merton, *Optimum consumption and portfolio rules in a continuous-time model*, J. Econom. Theory, 3 (1971), pp. 373–413.

[71] A. Mijatović and P. Tankov, *A new look at short-term implied volatility in asset price models with jumps*. Preprint, available at http://arxiv.org/abs/1207.0843, 2012.

[72] A. Müller and L. Rüschendorf, *On the optimal stopping values induced by general dependence structures*, J. Appl. Probab., 38 (2001), pp. 672–684.

[73] A. Müller and M. Scarsini, *Stochastic order relations and lattices of probability measures*, SIAM J. Optim., 16 (2006), pp. 1024–1043 (electronic).

[74] A. M. Oberman, *The convex envelope is the solution of a nonlinear obstacle problem*, Proc. Amer. Math. Soc., 135 (2007), pp. 1689–1694 (electronic).

[75] Y. V. Prokhorov, *Convergence of random processes and limit theorems in probability theory*, Theory of Probability & Its Applications, 1 (1956), pp. 157–214.

[76] A. W. Roberts and D. E. Varberg, *Convex functions*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1973. Pure and Applied Mathematics, Vol. 57.

[77] L. C. G. Rogers, *The relaxed investor and parameter uncertainty*, Finance Stoch., 5 (2001), pp. 131–154.

[78] M. Roper, *Implied volatility explosions: European calls and implied volatilities close to expiry in exponential Lévy models*. Preprint, available at http://arxiv.org/abs/0809.3305, 2008.

[79] M. Roper and M. Rutkowski, *On the relationship between the call price surface and the implied volatility surface close to expiry*, Int. J. Theor. Appl. Finance, 12 (2009), pp. 427–441.

[80] M. Rosenbaum and P. Tankov, *Asymptotic results for time-changed Lévy processes sampled at hitting times*, Stochastic Process. Appl., 121 (2011), pp. 1607–1632.

[81] K.-i. Sato, *Lévy processes and infinitely divisible distributions*, vol. 68 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1999.

[82] W. Schoutens, *Meixner processes: Theory and applications in finance*, EURANDOM Report 2002-004, EURANDOM, Eindhoven, 2002.

[83] V. Strassen, *The existence of probability measures with given marginals*, Ann. Math.

Statist., 36 (1965), pp. 423–439.

[84] P. Tankov, *Pricing and hedging in exponential Lévy models: review of recent results*, in Paris-Princeton Lectures on Mathematical Finance 2010, vol. 2003 of Lecture Notes in Math., Springer, Berlin, 2011, pp. 319–359.

[85] B. Tavin, *Detection of arbitrage in a market with multi-asset derivatives and known risk-neutral marginals*, Journal of Banking & Finance, 53 (2015), pp. 158–178.

[86] A. E. Whalley and P. Wilmott, *An asymptotic analysis of an optimal hedging model for option pricing with transaction costs*, Math. Finance, 7 (1997), pp. 307–324.

[87] S. Yan, *Jump risk, stock returns, and slope of implied volatility smile*, Journal of Financial Economics, 99 (2011), pp. 216–233.

[88] V. M. Zolotarev, *Metric distances in spaces of random variables and of their distributions*, Mat. Sb. (N.S.), 101(143) (1976), pp. 416–454, 456.

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Curriculum Vitæ

General information

Born — Istanbul, Turkey, August 11, 1987

Nationality — Austria

Education

2011-Present — **Ph.D. in Financial and Actuarial Mathematics** at Vienna University of Technology

Supervisor: Associate Prof. Stefan Gerhold

2009–2011 — **Master of Science in Financial and Actuarial Mathematics** at Vienna University of Technology

Supervisor: Prof. Peter Grandits

2006–2009 — **Bachelor of Science in Financial and Actuarial Mathematics** at Vienna University of Technology

Supervisor: Dr. Martin Predota

Academic and work experience

Assistant Researcher

since 2011 — at the Institute of Mathematical Methods in Economics, for the Research Unit Financial and Actuarial Mathematics

Employed in the FWF project 24880-N25, Asymptotics of volatility surfaces and option prices

Portfolio Management Programm (PMP)

2011-2013 — Analyst/Portfolio Manager, ZZ Group, Vienna

The PMP is a program for students in Vienna sponsored by Peter Pühringer, hedge fund manager and founder of the ZZ Vermögensverwaltung GmbH.

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Academic Honors

2011 — "AVÖ"-prize by the Austrian Actuarial Society for master thesis

2011 — Graduation to Master of Science with distinction in minimum time

2009 — Graduation to Bachelor of Science with distinction in minimum time

2008–2010 — Scholarship of Vienna University of Technology

Publications and working papers

- S. Gerhold, I. C. Gülüm, A. Pinter, Small-maturity asymptotics for the at-the-money implied volatility slope in Lévy models, 2016, Submitted.
- S. Gerhold, I. C. Gülüm, A variant of Strassen's theorem: Existence of martingales within a prescribed distance, 2016, Submitted.
- S. Gerhold, I. C. Gülüm, Consistency of option prices under bid-ask spreads, 2016, Work in Progress.
- S. Gerhold, I. C. Gülüm, Portfolio Optimisation using adaptive strategies, 2016, Work in Progress
- I. C. Gülüm, U. Schmock, On the existence of an equivalent martingale measure in the Dalang-Morton-Willinger theorem, which preserves the dependence structure, 2016, Work in Progress.

Teaching experience at Vienna University of Technology

2015 Exercise class Life Insurance Mathematics

2014 Exercise class Stochastic Analysis in Financial and Actuarial Mathematics 2

2011 Exercise class Mathematical Finance 1: Discrete-Time Models

2010 Assistant for the lecture Risk Management in Finance and Insurance

2010 Assistant for the lecture Mathematical Finance 1: Discrete-Time Models

Other Teaching Experience

July 2013 Lecturer at a one-week **summer school on Quantitative Methods in Risk Management** in Canazei/Italy, together with Prof. Friedrich Hubalek

Main subjects: Standard methods for Value-at-Risk and Expected Shortfall determination, Time series analysis, Copulas in theory and practice, Dependence measures, Extreme value theory

2012 Assistant for the lecture Mathematical Finance 1: Discrete-Time Models

Assistant of Prof. Schmock at the University of Salzburg, Austria

Conferences and Seminars

2016 **12$^{\text{th}}$ German Probability and Statistics Days 2016 - Bochumer Stochastik-Tage**, Bochum, Gemany
Talk: Consistency of option prices under bid-ask spreads

2015 **Vienna Seminar in Mathematical Finance and Probability**, Vienna, Austria
Talk: A variant of Strassen's theorem: Existence of martingales within a prescribed distance

2015 **14$^{\text{th}}$ Winter school on Mathematical Finance - Special Topics: Nonlinear Pricing, Dependence and Model Risk**, Lunteren, Netherlands.

2013 **One-day workshop on Portfolio Risk Management (PRisMa Day)**, Vienna, Austria
Talk: On the Existence of an equivalent martingale measure which preserves the Dependence Structure

2013 **Endowment and Asset Management Conference**, Vienna, Austria