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Effects of Age-Specific Immigration: An Optimal Control Approach

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Zusammenfassung

Zahlreiche Industrieländer sind von fallenden Sterbe- als auch Geburtenraten betroffen, welche in weiterer Folge zu einer Überalterung der Bevölkerung führen. Diese demographischen Veränderungen haben unter anderem Auswirkungen auf die Sozialsysteme in den jeweiligen Ländern, allen voran auf die öffentlichen Pensionssysteme, welche von einer immer kleiner werdenden arbeitenden Bevölkerungsgruppe finanziert werden müssen. In diesem Zusammenhang wird oft Immigration als Schlüssel zur Eindämmung des Alterungsprozesses genannt. Aufgrund von höherer Mobilität, politischen Turbulenzen und ökonomischen Ungleichgewichten auf der Welt gab es einen massiven Anstieg der Nettomigration in hoch entwickelte Länder. Das führte dazu, dass sich Immigration im Laufe der letzten Jahrzehnte zu einem immer wichtigeren Thema entwickelt hat. Ziel der vorliegenden Arbeit ist es, das reale Phänomen der Immigration aus Sicht des Gastlandes durch Abstrahierung mittels mathematischer Modelle nachzubilden, um so ein besseres Verständnis für die zugrunde liegenden Mechanismen zu bekommen. Das beinhaltet naturgemäß auch die analytische sowie numerische Lösung der beschreibenden Gleichungen.

Das Hauptaugenmerk liegt auf der Untersuchung von ökonomischen und demographischen Effekten von Immigration durch Methoden der dynamischen Optimierung. Im Speziellen sollen qualitative und quantitative (altersspezifische) Effekte der Immigration im Gastland bestimmt werden. Die Anwendung von dynamischer Optimierung für die Untersuchung dieser spezifischen, demographischen Fragestellungen, führt insbesondere zu einer Weiterentwicklung der verwendenten, mathematischen Methoden und zeigt auch die Vielfalt ihrer Einsatzbereiche.

Es werden mehrere mathematische Modelle erstellt und gelöst, um die oben genannten Untersuchungen durchzuführen. Die ersten beiden Modelle beschränken sich rein auf die demographischen Auswirkungen von Immigration und befassen sich mit dem Einfluss dieser auf zukünftige Bevölkerungsgrößen und -strukturen und in weiterer Folge mit deren Einfluss auf demographische Indikatoren. Die Frage nach der optimalen altersspezifischen Immigrationspolitik, welche den Abhängigkeitsquotienten der resultierenden Bevölkerung langfristig minimiert, wird mittels eines Kontrollmodells untersucht. Der Abhängigkeitsquotient bezeichnet das Verhältnis der wirtschaftlich abhängigen Altersgruppen (Personen, die noch nicht bzw. nicht mehr im erwerbsfähigen Alter sind) zur Bevölkerung im erwerbsfähigen Alter. Dabei wird eine stationäre Bevölkerung betrachtet. Das resultierende optimale Kontrollmodell ist linear und umfasst zusätzliche Beschränkungen an die Zustandsgröße oder alternativ an die Kontrollvaria-

ble. Ein sehr allgemeines Maximumsprinizip muss daher zur Bestimmung der optimalen Lösung angewendet werden. Es werden zwei verschiedene Modellformulierungen betrachtet. Die erste Formulierung beinhaltet die zusätzliche Forderung, dass die Anzahl der Immigranten, die jährlich ins Land strömen, fest gewählt ist. Während diese Forderung durchaus realistisch ist, zeigt das mathematische Modell, dass es in diesem Fall optimal ist, dass Immigranten nahe dem maximalen Lebensalter ins Land einwandern, da diese auf natürliche Weise den Abhängigkeitsquotienten nur gering belasten. Daher wird in einem nächsten Schritt ein Modell gewählt, welches anstelle der Anzahl der Immigranten die zusätzliche Nebenbedingung einer fixen Bevölkerungsgröße beinhaltet. Für dieses Modell wird gezeigt, dass unter der Bedingung, dass die altersspezifischen Schranken stark gelockert werden, es optimal ist, dass Migranten mit Mitte 30 einwandern.

In einem nächsten Schritt wird dann die Annahme einer stationären Bevölkerung fallen gelassen, und das über die Zeit variable Altersprofil der Immigranten in eine Bevölkerung konstanter Größe bestimmt, welches die Zahl der Arbeiter in einer Bevölkerung maximiert. Das erfordert die Formulierung eines spezifischen Kontrollmodells, in dem die Dynamiken mittels Differentialgleichungen mit verteilten Parametern beschrieben werden, untersucht werden. Aus mathematischer Sicht ist dieses Problem aus den folgenden Gründen interessant: (i) das Modell beinhaltet ein Problem mit verteilten Parametern und einer speziellen Zustandsnebenbedingung (ii) das Problem ist auf unendlichem Zeithorizont gestellt, und (iii) es handelt sich um ein Maximierungsproblem mit einem nicht-konkaven Funktional als Zielfunktion, sodass Wohlgestelltheit des Problems und insbesondere Existenz der Lösung nicht automatisch folgen.

Es kann gezeigt werden, dass unter einer generischen Bedingung und der Annahme, dass die Bevölkerung konstante Mortalität und Fertilität aufweist, das optimale altersspezifische Migrationsprofil unabhängig von den Anfangsdaten ist und konstant über die Zeit. Daher kann die Lösung durch das bereits untersuchte stationäre Problem charakterisiert werden.

Da Immigration nicht nur demographische Auswirkungen hat, werden in einem weiteren Schritt ökonomische Modelle betrachtet, um auch die wirtschaftlichen Auswirkungen von Immigration zu untersuchen. Daher werden bestehende überlappende Generationenmodelle dahingehend erweitert, dass auch Immigranten darin abgebildet werden können. Dabei liegen die Herausforderungen in der ökonomisch und mathematisch sauberen und konsistenten makro- und mikroökonomischen Modellierung. Es werden zeitkontinuierliche überlappende Generationenmodelle verwendet, welche auch die Formulierung und Lösung von partiellen Differentialgleichungen beinhalten. In einem ersten Modell werden die ökonomischen Auswirkungen eines Immigrationsschocks auf die verschiedenen Generationen in der Ökonomie betrachtet.

In einem letzten Modell wird untersucht, welche Rolle das Alter der Immigranten für das Pensionssystem des Gastlandes spielt. Die Auswirkungen der altersspezifischen Einwanderung auf die Höhe der Sozialversicherungssteuer und die Pensionszahlungen im Allgemeinen werden untersucht. Dabei wird ein Pay-as-you-go-Rentensystem, in dem die Höhe der Renten fest gewählt ist, betrachtet. Für das betrachtete numerische Experiment wird gezeigt, dass die Sozialversicherungssteuer sinkt, wenn die Einwanderer in einem höhere Alter kommen, obwohl der Altersquotient in der Bevölkerung erheblich steigt. Das beruht auf der Tatsache, dass die Zuwanderer Pensionsansprüche im Gastland aufweisen. Darüber hinaus kann gezeigt werden, dass unter den gewählten Voraussetzungen, Immigranten, über alle Altersgruppen hinweg, Nettozahler des Rentensystems sind. Daher können sie zumindest zu einem gewissen Teil die finanzielle Lücke, die durch die Überalterung der einheimischen Bevölkerung verursacht wird, schließen. Allerdings sieht man auch, dass Zuwanderung allein auf lange Sicht die fiskalen Herausforderungen nicht lösen kann. Dafür wäre eine unrealistische Erhöhung der Sozialversicherungssteuer notwendig, um einen ausgeglichenen Haushalt gewährleisten zu können. Daher müssen auch andere Maßnahmen, wie eine Erhöhung des gesetzlichen Rentenalters und Veränderungen in den Parametern des Rentensystems, betrachtet werden.

Abstract

Declining mortality rates combined with decreasing fertility rates have led to the prevailing situation of population aging in many developed countries. Social insurance, pension schemes etc., of a higher share of elderly people will have to be financed by a lower share of the younger and middle-aged population constituting the labor force. Therefore, immigration is very often named as a remedy to counteract these demographic changes. Moreover, higher mobility, political instabilities, and economic imbalances have led to an increase in net immigration to high-income countries over the last decades. Consequently, immigration has become an important and interesting topic.

In this thesis, economic and demographic effects of immigration are investigated by developing suitable mathematical models. In particular, it is an attempt to tackle demographic and economic questions by applying optimal control theory. Qualitative and quantitative effects of (age-specific) immigration patterns on the receiving country are determined. Hence, the ideas used in this thesis may be fruitful for the study of immigration policies.

In this thesis, models for the study of (age-specific) effects of immigration in the host country are developed. The first two models only cover demographic effects of immigration, dealing with the impact of immigrants on future population size and structure and consequently on demographic indicators. The question of the optimal age-specific immigration policy that minimizes the dependency in a population in the long-run is posed. A stationary problem is considered which consists of the investigation of a rather specific linear optimal control model including a state constraint which makes it necessary to apply a very general maximum principle. Two alternative policies are considered. In the first one, the total number of immigrants is prescribed. In the second one, the total population size is fixed while the rest of the model remains the same. It turns out that the solution exhibits a bang-bang behavior, which depends on the sign of the so-called switching function. In the model with a fixed total number of immigrants, it is shown that in the optimal solution there are ages in the vicinity of the maximum attainable age where immigration occurs. When the total population size of the receiving country is fixed, the optimal solution is such that immigration happens at not more than two separate age intervals and always in ages younger than the retirement age. When relaxing the age-specific bounds for immigration in the model, it turns out that the optimal age of immigration would be in the mid-thirties in the case study of Austria.

More generally, in a later step the time-varying age-specific immigration pattern to a population of fixed size that maximizes the number of workers in a population is investigated. This leads to the formulation of a very specific, distributed parameter model. From a mathematical point of view the considered problem is challenging for three reasons: (i) it has the form of a distributed optimal control problem with state constraints (although rather specific); (ii) the time horizon is infinite and a theory for infinite-horizon optimal control problems for age-structured systems is missing; (iii) it is a maximization problem for a non-concave functional, where the existence of a solution and the well-posedness are problematic. It turns out that under an additional generic well-posedness condition for a population with time-invariant mortality and fertility the optimal age-density of the migration is time-invariant and independent of the initial data. Hence, the solution can be found by solving the associated steady-state problem, as it has been studied before.

Since immigration is never solely a demographic issue, in a next step economic models are considered to investigate also the economic effects of immigration. Existing overlapping generations models have to be extended in order to be able to deal with immigration. Here, the challenges consist of an economic and mathematical sound adaptation of the macro- and microeconomic modeling. Continuous time overlapping generations models are used which also include the formulation and solving of partial differential equations. In a first model, the welfare effects of immigration on the various cohorts of the host population are investigated.

In a second model, the focus is on the description of the life-cycle behavior of immigrants entering at various ages of their life to determine their impact on the pension schemes of a country. The impact of age-specific immigration on the social security rate and the pension expenditure rate in a benefit-defined pay-as-you-go pension scheme are presented. Moreover, scaled pension expenditures and tax payments for the two groups, natives and immigrants, are given. For the presented numerical experiment the social security rate decreases with the age of the arriving immigrants although the old-age dependency ratio increases substantially. This is because of the fact that immigrants qualify for fewer pensions in the host country. Moreover, across all age groups immigrants are net payers of the pension system. Hence, they are at least to a small extend able to close the financial gap caused by the aging of the native population. However, one also sees that immigration alone cannot solve the fiscal problems arising with the demographic change because an unbearably high increase of the social security rate would be necessary to guarantee a balanced budget. Hence, also other measures such as an increase in the statutory retirement age and changes in the parameters of the pension system would additionally be necessary.

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'J'ai décidé d'être heureux parce que c'est bon pour la santé.'
Voltaire

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Motivation

This thesis deals with mathematical models and methods for the analysis of demographic and economic consequences of immigration. Consequently, this work focuses on the development and the solution of age-structured optimal control problems. This first chapter of the thesis explains why immigration is an important and interesting topic to be studied by mathematical methods.

Immigration has become a more and more central topic in the last decades up until now. One reason for this development is the aging of populations in most developed countries caused by low birth rates and an increasing life expectancy. This aging has substantial impact on the society through changes of the size and the structure of the labor force and the non-productive part of the population. Another reason is the increase of mobility and immigration in the world. Especially, fertility rates well below the replacement level, as faced by many European countries, have given rise to the question of whether young workers from outside the country may help to counteract the negative consequences of the demographic changes.

Immigration is studied in various fields of research due to its impact on several dimensions of social life, including economy, demography, politics, etc. This makes it thoroughly an interdisciplinary topic. Subsequently, we will build up mathematical models which solely focus on the demographic and economic dimension of immigration.

Immigration through the eyes of a demographer

From a demographic point of view, it is interesting to investigate how immigration affects population dynamics in a below-replacement fertility context, that is, when the population would shrink without migration. The future population size and age structure of a country depend on three demographic variables: fertility, mortality and net migration. Therefore, in order to influence future population structures policy makers may either set up measures to encourage young couples to get children and therefore increase fertility levels, or to steer immigration in order to off-set population aging. While the former measure may or may not work and its effectiveness is not as straight forward, immigration directly affects the age structure of the host country.

In United Nations (2001), it is analyzed whether migration is a suitable approach to avoid population aging and decline in various low-fertility countries. It is concluded that, in general, it is already too late to fully stop the aging process in the countries under investigation. With respect to migration, it is concluded that in the short and middle-term it would be advantageous in comparison to other measures, such as the increase of the retirement age, which would have to be set unrealistically high. In United Nations (2001) it is also stated that very high numbers of immigrants would be needed in order to seriously affect the ongoing aging. However, most of the studies the United Nations build their results on, assumed that the age profile of inflowing migrants would remain the same in the future. In Arthur and Espenshade (1988), on the other hand, it was shown how sensitively the ultimate population size and age structure depends on variations in immigrants' ages. Hence, in the chapters below, we show by using the optimal control approach how an age-specific immigration policy can be used to reduce the number of dependent persons in the population. We use optimal control theory to study the aforementioned question of an optimal age-specific immigration policy. While there are certainly other aspects that have to be taken into account when formulating an immigration policy, age definitely plays an important role as already reflected in immigration policies of countries such as Australia and Canada.

Economic effects of immigration

Migration is never a solely demographic issue. It also affects the economy of both sending and receiving country as well as the economic well-being of their inhabitants. Investigating the mutual effects of migration on both, sending and receiving country, economists speak of brain drain, and correspondingly, brain gain, see for example Stark et al. (1997), that the affected countries may observe. Papers dealing with the effect of emigration usually investigate the role of remittances for the economic development of the sending country, see Rapoport and Docquier (2005). In this thesis, however, the focus was solely put on the role of immigration for the economy of the receiving country and its inhabitants.

A large part of the research on the impact of immigration on the receiving countries consists of empirical papers investigating the effect of immigration on wages. Most of these studies focus on US data, cf. Borjas (1994, 2003, 2005). In general, the research has shown small and very often insignificant effects of immigration on the wages of native workers. However, recent contributions such as by Borjas (2003) have found a significant negative effect of immigration on the wages of natives on national level having no high school diploma. Hence, the importance of differentiation with respect to education and other characteristics seems to matter in the estimation of wage effects of immigration, as has also been summarized and further developed in Ottaviano and Peri (2012). We incorporate this reasoning in the model elaborated in Chapter 5 and carefully model the heterogeneous economic behavior of immigrants and natives.

Other economic papers focus on the effect of immigration or in particular immigration shocks on the native's welfare, see for example Razin and Sadka (2000); Fehr et al. (2003); Boldrin and Montes (2008); Jinno (2013). The focus is on how the various generations are affected differently. The general economic mechanism behind this consideration is that, under the assumption that the country does not have good access to international capital markets and the assumption of no capital mobility, those who are the holder of capital win due the rise in the price for capital, because of a decrease in the capital-per-capita. Simultaneously, due to the additional labor economy wide wage rates fall. In line with this is the first model in Chapter 5.

Another main line of economic research investigates the effect of immigration on the host country's public finances and in particular the social security system, cf. Storesletten (2000, 2003); Mayr (2005), where they empirically and theoretically determine the fiscal impact of immigrants. In line with this, in the last part of this thesis it is investigated how immigrants affect the pension system of the host country and whether they can act as a rejuvenating and fiscal balancing force. To model the economic interaction of people of different ages, overlapping generations models (OLG) are the typical work horse representing also the common tool to investigate the economic consequences of the ongoing demographic transition, cf. Sánchez-Romero et al. (2013).

Immigration and mathematical modeling

The described importance for society has made immigration an interesting topic in mathematical modeling. There are several ways to mathematically model migration processes, including integral equations, projection matrices and difference equations. Here, we will focus on ordinary and partial differential equations. As its importance has already been pointed out by the celebrated demographer and mathematician Nathan Keyfitz in a work together with his daughter, Keyfitz and Keyfitz (1997), the famous *Mc Kendrick - von Foerster* equation is a perfect tool to study population dynamics due to its easy adaption to study migration effects. The quite recent development of new mathematical methods and tools, such as the invention of new maximum principles in the area of optimal control of population dynamics, see Feichtinger et al. (2003); Veliov (2008), and the invention of the vintage capital theory, has made it possible to study the optimization of complex demographic processes including immigration. For example, Feichtinger et al. (2004) introduce intertemporal and age-dependent features to a theory of population policy at the macro-level, and combine a Lotka-type renewal model of population dynamics with a Solow/Ramsey economy. They characterize meaningful qualitative results for the optimal migration path and the optimal saving rate. Another interesting question is posed in Larramona et al. (2007), where the authors build a three-stage optimal control model in order to investigate the optimal timing of immigration over the life-cycle. They show that for individuals with a higher education it is optimal to immigrate earlier as for those with a lower education.

However, not only populations as a whole but also their sub-populations, such as a country's labor force or electorate as well as a company's employee underlie an aging process. Hence, the question about the sequence of recruitment numbers that generates a given stock trajectory plays an important role not only in demography, but also in manpower planning. In a recent line of research the aging of learned societies was investigated. Learned societies are an example of an organization of fixed size where annual intake is strictly determined by the number of deaths and quits or by the number of members reaching a statutory retirement age.

Feichtinger and Veliov (2007) investigated the optimal recruitment processes to a learned society with similar optimal control techniques as presented in this thesis. Recruitment processes may be viewed as a specific form of immigration and therefore the mathematical modeling is rather similar. A related problem was also treated in Feichtinger et al. (2007). The interesting results in these articles on aging learned societies, made it natural to adapt the methods in-use and investigate immigration processes to an aging society. Hence, the results established in Feichtinger and Veliov (2007); Feichtinger et al. (2007); Dawid et al. (2009), representing parts of previous research projects, have built the starting point of the thesis in hand. However, the models investigated in Chapter 3 and 4 not only brought a new area of application in focus, but also represented a further development from a mathematical point of view. A new complexity was introduced in form of a non-local boundary condition being a result of the fact that the number of births in a population depends on the current population. This made the mathematical treatment much more cumbersome.

1.2 Structure of the thesis

In this thesis, four mathematical models are presented which deal with demographic and fiscal effects of immigration to a population with below-replacement fertility. Three different types of mathematical modeling tools are used below: optimal control models including ordinary differential equations, distributed parameter models and overlapping generations models. The commonality of the models below, besides of the demographic context, is the use of optimal control theory. Since optimal control theory is not a common tool in demography, dealing with demographic questions required the adaptation of the methods in use. This led to interesting mathematical challenges such as the formulation of a specific, new Maximum Principle.

Chapter 2 gives a comprehensive overview of the mathematical concepts used in the models below. Hence, the basic notions and concepts of population dynamics and the control of such are introduced.

In Chapter 3 the first model investigating the optimal immigration age profile to a stationary population, where the optimality criterion is given by the total dependency ratio of the resulting population, is presented. The content of Chapter 3 is heavily based on a joint work with Gustav Feichtinger and Anton Belyakov and was published in Simon et al. (2012).

In Chapter 4 a distributed parameter control model is set up to determine intertemporal optimal immigration profiles to a fixed size population. The content of Chapter 4 was a joint work with Vladimir Veliov and Bernhard Skritek and was published in Simon et al. (2013).

Finally, in Chapter 5 continuous time overlapping generations models are used to investigate, first, the effect of an immigration shock on the welfare of the host country's inhabtiants, and second, the impact of immigration on the sustainability of the pension system of a country with below-replacement fertility.

In Chapter 6 the main results of the thesis are discussed.

Chapter 2

Brief review of the relevant theory

This chapter is devoted to the mathematical analysis of populations with age structure and serves as a short guide to give an overview of the mathematical concepts used in the rest of the thesis below. We introduce linear age-dependent population dynamics and methods for the optimization of such. In particular, in Section 2.2 we also briefly discuss distributed parameter control and in Section 2.3 we describe overlapping generations models.

2.1 Linear age-dependent population dynamics

Among the different ways to model population dynamics, including integral equations, difference equations, and projection matrices, we will focus here on the most elegant one, namely (partial) differential equations. The presentation of the mathematical concepts in this section follows closely Webb (1985) and Anita (2000), which provide a good overview on the mathematical formulation of population dynamics via partial differential equations and the control of such.

The study of population dynamics has already a long history. In 1798 Malthus proposed his famous model of population dynamics where the rate of population growth is proportional to the size of the population. The solution to such a model is an exponentially growing population. Obviously, the Malthusian model is by far an appropriate way to study the dynamics of the populations in the 21st century. Nowadays, populations are characterized by low fertility levels and high numbers of immigrants. Hence, as it has already been pointed out in Keyfitz and Keyfitz (1997), it is of great importance to choose a modeling method which allows for an explicit modeling of the immigration term as it is possible by extending the famous *McKendrick-von Foerster Equation*.

In the following, we will denote by $N(t, a)$ the (non-probabilistic) density of females of age $a \in [0, \omega]$, $\omega \in (0, \infty]$, at time $t \geq 0$ in a one-sex population. Here, ω denotes the maximal attainable age.

The measurable, non-negative and bounded functions $f(a)$ and $\mu(a)$ denote the age-specific fertility and mortality rates of the population. Then, $l(a) := e^{-\int_0^a \mu(s)\,\mathrm{d}s}$ is the probability of surviving from 0 to age a.

The boundedness assumption for the mortality rate needs some explanation. There is no empirical evidence about boundedness or unboundedness of $\mu(a)$. We can assume equally well that the mortality rate is unbounded close to some maximal age $a = \omega$ in such a way that all the population dies till age ω. An alternative (not less plausible, in our opinion) is that $\mu(a)$ is bounded and large enough after a certain age, say 110 years, so that individuals above this age exist only mathematically.

With $M(t, a) \geq 0$ we denote the 'number' of immigrants of age a entering the population at time t.

Then, the population dynamics for the total population $N(t, a)$ is given by the equation

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right) N(t, a) = -\mu(a)\, N(t, a) + M(t, a), \quad t,\, a \geq 0, \tag{2.1}$$

where $\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right)$ means the directional derivative

$$\mathcal{D}\, N(t, a) = \lim_{h \to 0} \frac{N(t+h, a+h) - N(t, a)}{h}. \tag{2.2}$$

The symbol $\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right)$ is used instead of $\mathcal{D}\, N$ due to historical reasons. However, it is important to interpret this symbol as above since $\mathcal{D}\, N$ may exist in cases where the partial derivatives $\frac{\partial}{\partial a}$ and $\frac{\partial}{\partial t}$ are not existing, and this is typically the case in the controlled population models.

More specifically, in Chapter 4, it will be assumed that $M(t, a) = R(t)u(t, a)$, where $R(t)$ is the number of immigrants at time t and $u(t, a)$ is the age-density which satisfies

$$0 \leq \underline{u}(a) \leq u(t, a) \leq \bar{u}(a), \qquad \int_0^\omega u(t, a)\, \mathrm{d}a = 1.$$

For $M(t, a) = 0$, equation (2.1) reduces to the famous *McKendrick-von Foerster Equation*, cf. Keyfitz and Keyfitz (1997), which describes the dynamics of a population without migration. The partial differential equation in (2.1) is linear and of first-order. It belongs to the class of transport equations, which are typically solved via the method of characteristics, cf. Evans (2010). However, there is a crucial difference between the classical transport equation and the Mc Kendrick-von Foerster equation due to the endogenous and non-local boundary condition involved in the latter, see (2.4).

The initial condition for (2.1) is given by the initial population density

$$N(0, a) = N_0(a), \quad a \in (0, \omega), \tag{2.3}$$

where $N_0 : [0, \infty) \to \mathbf{R}$ is non-negative and bounded, and the boundary conditions are given by the number of births in the population for every time t:

$$N(t, 0) = \int_0^\omega f(a)\, N(t, a)\, \mathrm{d}a, \quad t > 0. \tag{2.4}$$

Note, that this condition depends on the solution $N(t, a)$ itself and is therefore an endogenous and *non-local boundary condition*.

In order to let (2.1) be meaningful, one has to specify the mathematical setting of the above problem.

Therefore, for any positive number T we abbreviate

$$D_T := [0, T] \times [0, \infty), \quad D := [0, \infty) \times [0, \infty), \tag{2.5}$$

and we define the space $\mathcal{N}$ that consists of all functions $N : D \to \mathbf{R}$ which are
(i) measurable, and the function $t \mapsto \int_0^\infty |N(t, a)| \, \mathrm{d}a < \infty$ is finite and locally bounded;
(ii) locally absolutely continuous on almost every line $t - a =$ const intersected with D (these are the characteristic lines of the differential operator in (2.1)).

In Figure 2.1 some characteristic lines are plotted. In the context of population dynamics one also calls them *life lines*, because the life of a cohort born lets say at time $t - a$ evolves along these lines until the final age ω.

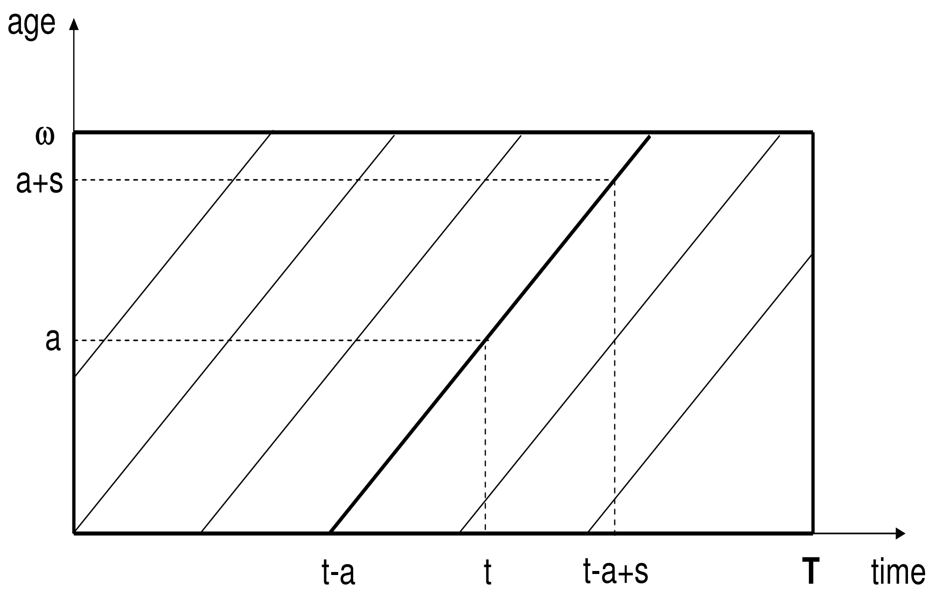

Figure 2.1: Lexis Diagram

Then for any $N \in \mathcal{N}$ the directional derivative $\mathcal{D}\, N(t, a)$ is well-defined for a.e. $(t, a) \in D$. Moreover, $M : D \to \mathbb{R}^+$ is measurable and locally bounded.

Let us first define a *stationary solution* $N(a)$ of problem (2.1) - (2.4) as a solution $N(t, a) = N(a)$ independent of time. By definition, this stationary solution is absolutely continuous and satisfies

$$\frac{dN(a)}{da} = -\mu(a)N(a) + M(a), \quad a \in (0, \omega) \tag{2.6}$$

$$N(0) = \int_0^\omega f(a)N(a) \, \mathrm{d}a. \tag{2.7}$$

For the study of the existence of a stationary solution, we have to introduce an important concept in population dynamics, namely the *net reproduction rate* (NRR),

$$NRR := \int_0^\omega f(a)l(a)\,\mathrm{d}a. \tag{2.8}$$

In the one-sex model we consider, the NRR gives the number of children expected to be born to a single individual during her life under the assumption of constant age-specific mortality and birth rates.

Then, we easily see that for the existence of a solution of the stationary, homogenous problem it holds:

Proposition 2.1. *Let $M = 0$ and μ and f are fixed as above. Then equations* (2.6) – (2.7) *have a non-negative and non-trivial solution if and only if $NRR = 1$, where NRR is given as in* (2.8).

For positive immigration $M(a)$ the following holds:

Proposition 2.2. *Let f and μ be fixed as above and let $M(a) \geq 0$ not be identically equal to zero. Then equations* (2.6), (2.7) *have a non-negative solution if and only if $NRR < 1$. In this case the solution is unique and given by the formula*

$$N(a) = \frac{1}{1 - NRR} \int_0^\omega \left(f(\tau) \int_0^\tau \frac{l(\tau)}{l(s)} M(s)\,\mathrm{d}s \right) \mathrm{d}\tau\; l(a).$$

In Chapter 3, we will focus on the stationary problem formulation (2.6), (2.7) and determine the optimal choice of $M(a)$ under certain additional constraints. The optimality criterion will be given by a specific demographic indicator.

Let us now return to the original problem (2.1)–(2.4) and discuss the existence, non-negativity and uniqueness of a solution. Note, that with a solution of (2.1)–(2.4) we mean a function $N \in \mathcal{N}$ if the equations are satisfied almost everywhere with $\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right) N$ interpreted as $\mathcal{D}N$.

Theorem 2.1. *Problem* (2.1)–(2.4) *has a unique solution $N \in \mathcal{N}$ and N is (essentially) bounded on every subset $D_T \subset D$, $0 < T < \infty$. The solution is non-negative.*

We omit the proof because it is similar to the proof of Lemma 4.2 in Chapter 4.

In Chapter 4 we will further restrict $N(t, a)$ by requiring that

$$\int_0^\omega N(t, a)\,\mathrm{d}a = \bar{N},$$

where $0 < \bar{N} < \infty$ is fixed and the requirement holds for almost every t. This requirement will lead us to the study of a two-dimensional system of integral equations when investigating existence and uniqueness of a solution. . For more general existence and uniqueness results we refer to Brokate (1985); Webb (1985); Feichtinger et al. (2003).

2.2 Optimal control of age-dependent population dynamics

This section serves for a better understanding of the optimization techniques for population models as above. The most important notions and concepts of optimal control theory are explained. In no way this section should be considered as a complete overview of the theory, but more as a short guide to get the basic tools to be equipped for the next chapters. For a more complete overview of optimal control theory, the reader may consult books such as Pontryagin et al. (1962); Feichtinger and Hartl (1986); Grass et al. (2008); Léonard and Long (1992).

2.2.1 Basic concepts in optimal control theory

Optimal control theory aims to identify optimal ways to control a dynamic system. Many different questions can be studied via an optimal control problem. Applications range from macroeconomic models and population dynamics, to mechanical motions, forestry and even drug control. In this section, we present an optimal control model which is general enough to be used for studying age-structured problems with it.

In general, to determine an optimal control means that the control input to the system, named *control*, which has to lie in a given control region, is chosen such that the *state* is steered optimally, where optimality is measured by an objective functional. The trajectory of the state follows the law of a dynamic system, which is influenced by the choice of the control variable, and is typically subjected to additional state constraints. In age-structured systems, the state is typically the age-density of a population or of a sub-population. Depending on the application, the control might be an age-specific migration policy, the harvesting effort of farmed populations such as fish, or age-specific drug prevention techniques to reduce drug abuse.

The control region is defined via constraints on the control variable. Sometimes mixed constraints, involving both the state and the control, are present. As mentioned, the optimality criterion is given by the so-called *objective function*.

Let us denote by $u \in \mathbb{R}^m$ the control variable and by $x \in \mathbb{R}^n$ the state variable of a system. Then, a general optimal control problem can be formulated as follows:

$$\begin{aligned} \mathscr{L}_0(x(\cdot), u(\cdot)) &= \int_{a_0}^{\omega} L(a, x(a), u(a))\,\mathrm{d}a \\ &\quad + l_0(x(a_0), x(\omega)) \to \inf, \qquad (2.9) \\ \frac{\mathrm{d}x}{\mathrm{d}a} &= f(a, x(a), u(a)), \quad u \in \mathcal{U}([0,\omega]), \qquad (2.10) \\ \mathscr{L}_i(x(\cdot), u(\cdot)) &= \int_{a_0}^{\omega} f_i(a, x(a), u(a))\,\mathrm{d}a \\ &\quad + l_i(x(a_0), x(\omega)) \lesseqgtr 0, \qquad (2.11) \end{aligned}$$

where $i = 1, 2, \ldots, m$.

Here, (2.9) defines the objective function $\mathscr{L}_0(x, u)$, where ω denotes the final age and a_0 is the initial age, typically $a_0 = 0$. It may hold that $\omega = \infty$ as it is also the case in the model presented in Chapter 4. The value $l_0(x(a_0), x(\omega))$ is called the *scrap value*.

Equation (2.10) describes the state dynamics in form of an ODE. In (2.11) constraints on the state or mixed-constraints may be described. Here, $\mathcal{U}([0,\omega])$ denotes the set of admissible controls, which is defined as a class of functions, $u : [0,\omega] \mapsto U$, typically being measurable or piecewise continuous.

Note also that in many economic applications, where the independent variable is time and not age, it is common that the decision maker discounts time with a time preference rate ρ, reflecting the fact, that the future state of the system is less important than its current one. Then, the integrand in the objective function takes the form $e^{-\rho a}L(a, x(a), u(a))$. Typically, this is the case in so-called life-cycle models, see e.g. Heijdra and Romp (2009), where agents aim to optimally choose the consumption over their life-cycle. Such models are essentially involved in Chapter 5.

Now assume that for $u(\cdot) \in \mathcal{U}([0,\omega])$, $\mathscr{L}(u) := \mathscr{L}_0(x(u), u)$ exists and that (2.10) has a unique solution $x(\cdot)$. Then a solution of problem (2.9)–(2.11) may be defined in the following way:

Definition 2.1. *: A solution to problem* (2.9) – (2.11) *is a control $u^*(\cdot) \in \mathcal{U}([0,\omega])$ and the corresponding state $x^*(\cdot)$, which fulfill* (2.11), *such that $\forall u(\cdot) \in \mathcal{U}([0,\omega])$ either* (2.11) *does not hold or $\mathscr{L}(u(\cdot)) \leq \mathscr{L}(u^*(\cdot))$.*

This is the standard definition of optimality and is of key importance in the analysis of optimal control models.

2.2.2 Pontryagin's Maximum Principle

The solution of an optimal control problem of the above type can be characterized by necessary optimality conditions of Pontryagin type called Pontryagin's Maximum Principle. The maximum principle, loosely speaking, consists of a maximization condition on the so-called Hamiltonian function, a law of motion which has to be fulfilled by the so-called adjoint variable and a transversality condition.

Note, that although for (2.9)–(2.11) it would be more accurate to formulate a minimum principle, we still hold on to the maximum formulation because it is more common in the literature.

The so-called Pontryagin function is defined as

$$H(a, x, u, \xi) = \xi(a)\, f(a, x, u) - \sum_{i=0}^{m} \lambda_i f_i(a, x, u), \quad \xi \in \mathbb{R}^n$$

where function $\xi(a)$ is called the adjoint variable and λ_i are the Lagrange multipliers. In the following we formulate Pontryagin's Maximum Principle in a similar form stated in Alekseev et al. (1987) on p.218:

Theorem 2.2 (Pontryagin Maximum Principle)**.** *Let G be an open set in the space $\mathbf{R} \times \mathbf{R}^n$, let W be an open set in the space $\mathbf{R} \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$ and let $\mathcal{U}$ be an arbitrary topological space. Let the functions f_i: $G \times \mathcal{U} \to \mathbf{R}$, $i = 0, 1, \ldots, m$, f: $G \times \mathcal{U} \to \mathbf{R}^n$, and their partial derivatives with respect to x be continuous in $G \times \mathcal{U}$, and let the functions l_i, $i = 1, \ldots, m$ be continuously differentiable in W.*

If $(x^*(\cdot), u^*(\cdot))$ *is an optimal solution for the problem* (2.9)–(2.11), *then there are Lagrange multipliers*

$$\lambda_0 \geq 0, \quad \lambda = (\lambda_1, \ldots, \lambda_m),$$

not all zero, and an adjoint variable $\xi(\cdot)$ *such that:*

a) *the adjoint equation*

$$\begin{aligned}\frac{d\xi}{da} &= -\xi(a)\frac{\partial f}{\partial x}(a, x^*(a), u^*(a)) + \sum_{i=0}^{m} \lambda_i \frac{\partial f_i}{\partial x}(a, x^*(a), u^*(a)) \\ &= -\frac{\partial H}{\partial x}(a, x^*(a), u^*(a), \xi(a)),\end{aligned}$$

along with the transversality conditions

$$\begin{aligned}\xi(\omega) &= -\sum_{i=0}^{m} \lambda_i \frac{\partial l_i}{\partial x_\omega}(x_0^*, x_\omega^*) \\ \xi(a_0^*) &= \sum_{i=0}^{m} \lambda_i \frac{\partial l_i}{\partial x_0}(x_0^*, x_\omega^*)\end{aligned}$$

the maximum principle in Hamiltonian (Pontryagin) form

$$H^*(a) \equiv H(a, x^*(a), u^*(a), \xi(a)) \equiv \max_{v \in \mathcal{U}} H(a, x^*(a), v, \xi(a))$$

the function $H^*(a)$ *being continuous on the closed interval* $[a_0, \omega]$.

b) *the conditions of concordance of signs hold:*

$$\lambda_i \gtreqless 0, \tag{2.12}$$

c) *the conditions of complementary slackness hold:*

$$\lambda_i \mathscr{L}_i(x^*(\cdot), u^*(\cdot)) = 0, \quad i = 1, 2, \ldots, m$$

(inequalities (2.12) *mean that* $\lambda_i \geq 0$ *if* $\mathscr{L}_i \leq 0$ *in condition* (2.11), $\lambda_i \leq 0$ *if* $\mathscr{L}_i \geq 0$, *and* λ_i *may have an arbitrary sign if* $\mathscr{L}_i = 0$*).*

The proof of the Maximum Principle involves many important concepts of optimal control theory. Some of them are also used in a later chapter for the proof of necessary optimality conditions of a distributed control system.

2.2.3 Distributed parameter control

The models in Chapter 3 and Chapter 4 represent two examples for the optimal control of age-structured control systems, where the area of application is demography. Similarly, such models can also be found in the research of the control of epidemiological processes, drug initiation, cf. Almeder et al. (2004), or harvesting and birth control, Brokate (1985). More extensive bibliography can be found in Feichtinger et al. (2003). In all these fields of applications the age of the individual plays an important role.

While the control of the stationary dynamics in Chapter 3 can be studied by rather standard techniques, the problem in Chapter 4 represents a very specific form of a so-called *distributed parameter model* and required the extension of existing techniques for the characterization of the solution of such problems.

Distributed parameter models are the proper tool for studying the control of heterogeneous, dynamic systems. Heterogeneous, dynamic systems are characterized by state variables which exhibit an explicit dependence on a certain parameter, which for example might be age. The dynamics is typically given by partial differential equations. Very often the solution of a distributed parameter model is characterized by the formulation of a maximum principle of Pontryagin's type. Rather recently new maximum principles were developed such as the ones presented in Feichtinger et al. (2003); Veliov (2008), where the authors formulate maximum principles for fairly general distributed parameter systems. In Veliov (2008) the analysis is not only directed to age-dependent dynamics. The author considers a very broad form of heterogeneity which is represented by a parameter taking values in an abstract measurable space, so that continuous and discrete heterogeneities, as well as probabilistic heterogeneities, may be included in the problem formulation.

Whereas, as already mentioned, distributed parameter control may also arise in other applications than age-structured systems, for example size-structured systems in forestry control, we will put here a focus on the control of age-dependent population dynamics. A general maximum principle for nonlinear population dynamics was obtained in Brokate (1985). In recent years, a vast literature dealing with age-structured models and various extensions of the McKendrick-type system treated in there arose where the existing optimality conditions were not applicable and hence required the development of new theoretical results. In Feichtinger et al. (2003) the analysis targeted age-structured optimal control models, similar to the problem considered in Chapter 4. However, the results obtained in there were not sufficient for the study of the model presented in this thesis and hence required a further development of existing results.

This section is aimed to present the state of the present theory. In Feichtinger et al. (2003), the authors considered the following general distributed optimal control model:

$$\min_{u,v,w} J(u,v,w) := \int_0^\omega l(a, N(T,a))\,\mathrm{d}a$$
$$+ \int_0^T \int_0^\omega L(t,a,N(t,a),P(t,a),B(t),u(t,a),v(t),w(a))\,\mathrm{d}a\,\mathrm{d}t, \tag{2.13}$$

subject to the equations

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right) N(t,a) = f(t,N(t,a),P(t,a),B(t),u(t,a)), \tag{2.14}$$

$$P(t,a) = \int_0^\omega g(t,a,a',N(t,a'),u(t,a'))\,\mathrm{d}a', \tag{2.15}$$

$$B(t) = \int_0^\omega h(t,a,N(t,a),P(t,a),B(t),u(t,a))\,\mathrm{d}a, \tag{2.16}$$

the initial condition

$$N(0,a) = N_0(a, w(a)), \tag{2.17}$$

the boundary condition

$$N(t,0) = \varphi(t, B(t), v(t)), \tag{2.18}$$

and the control constraints

$$u(t,a) \in U,\ v(t) \in V,\ w(a) \in W. \tag{2.19}$$

Here, $t \in [0,T]$ and $a \in [0,\omega]$ denote time and age, respectively. Then, $N(t,a) \in \mathbb{R}^m$, $P(t,a) \in \mathbb{R}^n$ and $B(t) \in \mathbb{R}^r$ are the state variables of the system. The variables $u(t,a)$, $v(t)$ and $w(a)$ are the distributed, boundary and initial controls. We will omit here the strict mathematical definition of the problem and refer the reader to Feichtinger et al. (2003).

Now assume that for admissible controls $u(t,a)$, $v(t)$ and $w(a)$, $J(u,v,w)$ exists and that problem (2.14)–(2.18) has a unique solution (N,P,B). Then a solution of problem (2.13)–(2.19) may be defined in the following way:

Definition 2.2. *: A solution to problem* (2.13)–(2.19) *are controls* $u^*(t,a) \in U$, $v^*(t) \in V$, $w^*(a) \in W$ *and the corresponding states* $(N^*(t,a), P^*(t,a), B^*(t))$ *such that* $\forall(u,v,w)$ *that fulfill* (2.19) *it holds that* $J(u,v,w) \geq J(u^*,w^*,v^*)$ *.*

The above system might be interpreted as follows. For $m = 1$ the distributed state variable $N(t,a)$ may give the density of the population or in case of $m > 1$ it may describe the density of various sub-populations of a society, for instance skilled and unskilled workers as typically investigated in a demo-economic model, being of age a at time t . The term $N(t,a)$ states that each individual is characterized by its age a at any time t. Hence, the age a is the distributed parameter in this model. The term $B(t)$ is an aggregated quantity such as the size of the population or of a sub-population. The term $P(t,a)$ may model the effect that cohorts have on the dynamics of each other. Function $u(t,a)$ is an age-specific control such as an age-specific training rate, when we think again of the model of skilled and unskilled workers. The boundary control $v(t)$ may reflect a birth control measure or in a vintage capital model, the number of purchased new machines.

For the characterization of a solution of the above problem, we assume sufficient smoothness of the involved functions. Below ∇_x denotes the differentiation with respect to variable x. We define the following initial, boundary and distributed Hamiltonians

$$
\begin{aligned}
&H_0(a,w) := \xi(0,a)N_0(a,w) + \int_0^T L(s,a,w)\,\mathrm{d}s,\\
&H_b(t,v) := \xi(t,0)\varphi(t,v) + \int_0^\omega L(t,b,v)\,\mathrm{d}b,\\
&H(t,a,u) := L(t,a,u) + \xi(t,a)f(t,a,u)\\
&+ \int_0^\omega \eta(t,a')g(t,a',a,u)\,\mathrm{d}a' + \zeta(t)h(t,a,u).
\end{aligned}
$$

and the corresponding adjoint system, whose existence may be assumed:

$$\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial t}\right)\xi(t,a)=\xi(t,a)\nabla_N f(t,a)+\zeta(t)\nabla_N h(t,a)+\nabla_N L(t,a) \tag{2.20}$$

$$\xi(t,\omega)=0,\quad \xi(T,a)=\nabla_y l(a,N^(T,a)), \tag{2.21}$$

$$\eta(t,a)=\nabla_P L(t,a)+\xi(t,a)\nabla_Q f(t,a)+\zeta(t)\nabla_P h(t,a), \tag{2.22}$$

$$\zeta(t)=\xi(t,0)\nabla_Q\varphi(t)+\int_0^\omega\left[\xi(t,a)\nabla_Q f(t,a)+\nabla_Q L(t,a)+\zeta(t)\nabla_Q h(t,a)\right]\mathrm{d}a. \tag{2.23}$$

Then, the following necessary optimality conditions for a solution of (2.13) – (2.19) hold.

Theorem 2.3. *Let* $(N^*,P^*,B^*,u^*,v^*,w^*)$ *be an optimal solution of problem* (2.13) – (2.19). *Then the adjoint system (2.20 - 2.23) has a unique solution* (ξ,η,ζ)*in* $L_\infty([0,T]\times[0,\omega])$ *and then for a.e.* $t_0\in[0,T]$, $a_0\in[0,\omega]$ *and* $(t,a)\in D_T$ *the following holds:*

$$\begin{aligned}
&\frac{\partial H_0}{\partial w}(a_0,w^*(a_0))(w-w^*(a_0))\geq 0,\quad \forall w\in W,\\
&\frac{\partial H_b}{\partial v}(v-v^*(t_0))\geq 0,\quad \forall v\in V,\\
&H(t,a,u)-H(t,a,u^*(t,a))\geq 0,\quad \forall u\in U.
\end{aligned}$$

This newly established maximum principle could be applied to a wide range of problems, cf. Feichtinger et al. (2004, 2006); Prskawetz and Veliov (2007); Wrzaczek et al. (2010). There are several reasons for the wide applicability of the new optimality conditions for age-structured systems. For example, they can deal with integral quantities such as $P(t,a)$ that exhibit an age-dependence. Another extension to existing maximum principles is the involvement of boundary controls $v(t)$. Another crucial feature is that the distributed control may appear in both dynamics and the boundary condition.

We may now give a specification to problem (2.13) – (2.19) and turn to the model of Chapter 4. In the model considered in Chapter 4 the following specifications hold true. The objective function is given by

$$\max_{u(t,a),R(t)}\int_0^T e^{-rT}\left\{qR(t))+\int_0^\omega p(a)N(t,a)\,\mathrm{d}a\right\}\mathrm{d}t \tag{2.24}$$

The objective function is subjected to

$$\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial t}\right)N(t,a)=-\mu(a)N(t,a)+R(t)u(t,a), \tag{2.25}$$

$$B(t)=\int_0^\omega f(a)N(t,a)\,\mathrm{d}a, \tag{2.26}$$

with initial and boundary conditions

$$N(t,0)=B(t),\quad N(0,a)=N_0(a), \tag{2.27}$$

and the control constraints

$$\underline{u}(a) \le u(t,a) \le \bar{u}(a), \quad \int_0^\omega u(t,a)\,\mathrm{d}a = 1, \tag{2.28}$$

$$R(t) \ge 0. \tag{2.29}$$

The other variables and functions are missing in this specific model formulation. Hence, so far, on a finite domain $[0,\omega] \times [0,T]$ the solution of the model could be characterized by the above maximum principle. However, the problem posed in Chapter 4 represents a substantial extension to the above problem setting. First, it is posed on an infinite horizon, i.e. $T = \omega = \infty$. A general maximum principle for infinite horizon age-structured problems does not exist yet. Second, the problem introduced in Chapter 4 exhibits another difficulty because it is required that the size of the total population is held constant, namely $\int_0^\omega N_0(a)\,\mathrm{d}a = M$, i.e.

$$\int_0^\omega N(t,a)\,\mathrm{d}a = M, \quad \text{for almost every t,} \tag{2.30}$$

which is a sort of a state constraint and such are not included in Brokate (1985); Feichtinger et al. (2003); Veliov (2008). In Chapter 4, we showed that the requirement (2.30) can always be fulfilled when $R(t)$ is chosen in the feedback form

$$R(t) := \int_0^\infty (\mu(a) - f(a))\,N(t,a)\,\mathrm{d}a.$$

Hence, in Chapter 4 it was necessary to establish a new, specific maximum principle in order to characterize the solution of Problem (2.24) – (2.30).

2.3 Overlapping generations models in continuous time

In Chapter 5 we make use of another class of economic-mathematical models, namely *overlapping generations models* (OLG). Similar to the above mentioned distributed parameter control models, OLG models are a work horse for studying the effects caused by the age-heterogeneity of agents. OLG models are commonly a tool for macro economists. They follow the neoclassical reasoning and give micro-funded models of the whole economy, where the various players in the economy are brought together by a general equilibrium mechanism. This general equilibrium mechanism typically requires to solve a system of fixed point equations. Moreover, they make it possible to study phenomena where life-cycle aspects are of importance and no representative agent hypothesis is suitable. In general, OLG models are able to replicate several key mechanisms of economic activity:

- transfers in an economy happen between different generations,
- agents face a finite but uncertain life time of which they know of and hence behave accordingly,
- the main sectors of economic activity, being firms, households and government, can be modeled,

- the economic effects of the demographic changes in the industrialized countries.

In the basic version, individuals determine their optimal consumption and hence saving over the life cycle. However, many extensions exist where the life cycle choices of individuals do not only include the optimal consumption over the life horizon but also other important life-time decisions such as labor endowment, the education period, see Boucekkine et al. (2002), the retirement age, see Heijdra and Romp (2009), health investments, see Kuhn et al. (2011), and so on.

The first pillar of an OLG model are the households that inhabit the economy and solve an optimization problem, where they optimally determine several life time decisions. In most OLG models, with exceptions for example in Evans et al. (2009), individuals have perfect foresight meaning that they are able to perfectly forecast future prices. The households rent their labor and assets to firms. These firms produce goods that are in return bought by the households. And last but not least, there is the government which collects taxes from the economic agents in order to finance inter- and intragenerational transfers, such as pensions, health care and so on.

All these different economic agents meet at the corresponding markets, which are assumed to be cleared in each time step. Hence, there is also no unemployment, meaning that the firms demand exactly as much labor as it is supplied by the households. OLG models contain the Ramsey (1928) model as a special case.

One distinguishes between discrete time and continuous time OLG models. Many discrete OLG models only have two or three overlapping generations and are suitable for the study of fundamental economic mechanisms. For example, in three period OLG models, the implicit length of such a generation is roughly 20–30 years, adding up to a life-time of roughly 80–90 years, which does not allow for a realistic modeling of the demography. In an attempt to investigate economies of specific countries, Auerbach and Kotlikoff (1987) invented the so-called *computational general equilibrium models* (CGE) which exhibited a high number of co-existing agents. These CGE models are not analytically tractable and can only be solved by numerical methods.

However, analytic tractability is desirable because it makes it possible to derive general conclusions. Hence, continuous time OLG models come into play. In Blanchard (1985), the author developed the first continuous time OLG model based on previous ideas by Yaari (1965). The Blanchard-Yaari model is also known as perpetual youth model due to the fact that the agents always face a constant probability to die.

Hence, a natural extension of the Blanchard-Yaari model is the introduction of a finite life time and a probability density function which models the changing probability of survival over the life horizon.

In a continuous time OLG model the individual behavior can be modeled by an optimal control model. Hence, OLG models exhibit two important features that are of great interest for this work: dynamic optimization and heterogeneous modeling.

Continuous time OLG models make it possible to realistically model the demographic developments in a society. Hence, they gained in importance when people started to investigate the consequences of aging societies. There are several papers involving OLG models that deal with the economic consequences of a mortality shock or changes in the fertility, see for example Heijdra and Lighart (2006).

The model studied in Chapter 5 is also a continuous time OLG model with finite horizon. We extend the standard model by including another heterogeneity, namely the agents nativity. We model immigrants and natives who interact in an economy. In contrast to other papers, where the influence of fertility or mortality on economic parameters was investigated such as the growth rate or the per-capita capital, we focus in here on the immigration profile.

Chapter 3

Minimizing the dependency ratio in a population with below-replacement fertility through immigration

Here, we present an optimal control model to determine the optimal age-specific immigration profile to a stationary population. We consider two alternative problem formulations. First, in Section 3.3, we fix the total number of people who annually immigrate to a country. Then, in Section 3.4, we prescribe the (stationary) total size of the receiving country's population.

3.1 Introduction

In many industrialized countries fertility rates are below-replacement level. Additionally, these countries face a mortality decline, in particular at ages after retirement. Since fertility decline is very often the dominating effect, the population of these countries would decline without immigration. Moreover, the age structure of these countries' population is changing, showing a growth in the number of elderly people and a declining number of young people.

One important indicator of age structures is the so-called dependency ratio, which is the ratio of persons of nonworking age to persons of working age, usually the 20 to 65-year-olds. A low dependency ratio is desirable because it indicates that there are proportionally more adults of working age who can support the young and the elderly of the population. This in turn is advantageous for the countries' health-care system and pension schemes. A downfall of the relative number of working people in a population also has negative impacts on the growth path of the economy. A possible way to counter the risks of these demographic changes is to step up immigration.

Similar to Arthur and Espenshade (1988); Mitra (1990); Schmertmann (1992); Wu and Li (2003), in this work we consider a population where we assume that immigration, fertility, and mortality rates are constant and fertility is below-replacement level. These studies already have shown that such populations eventually converge to stationary populations. Following Schmertmann (1992) from now on we will denote this kind of population as SI, meaning *stationary through immigration*. Below-replacement level fertility and mortality rates indicate that without immigration the population would converge to zero. In our model we assume that the age-specific fertility rates of immigrants equal those of the natives. Following Schmertmann (2012) we do not account for emigration.

In this work, we aim to find the age-specific immigration profile that minimizes the dependency ratio in a stationary population. We do so by applying optimal control theory which is a rather new methodology in demographic research, see for example Feichtinger and Veliov (2007). We formulate an optimal control problem where the age-specific immigration profile is the control variable and the age structure of the population is the state variable. By deriving a maximum principle, we characterize the optimal solution to the posed optimal control problem.

A similar question to the one posed here is asked in United Nations (2001) where the authors determine whether migration of a country can be used to hinder a decline or aging of its population. They refer to this as replacement migration. They examine the situation of eight industrialized countries during the time period from 1995 to 2050.

In Schmertmann (2012) the question is raised how age-targeted immigration policy can be used to increase the relative number of working people in a population. There, the total number of annual immigrants is fixed and the problem is reduced to a static optimization problem. What is shown is that the highest relative number of workers can be achieved if all immigrants arrive at one single age under the assumption that at each age an arbitrarily high number of immigrants can be recruited. Schmertmann's paper leaves the question open what the optimal age-specific immigration profile would look like if not all immigrants are admitted at one single age. This issue, among others things, is tackled below.

From a mathematical point of view, a similar linear optimal control problem to the one proposed here is considered in Dawid et al. (2009). The authors determine the optimal recruitment policy of a stationary learned society, i.e. a professional and hierarchical organization, that minimizes the average age of the organization for a fixed number of recruits.
Feichtinger and Veliov (2007) extended their study to the transitory case. Remarkably, the optimal recruitment is the same as in the stationary case. That is why we also start with the stationary case. In Chapter 4 we extend the model of this chapter in various directions. One major difference is that the model in Chapter 4 is a so-called distributed parameter model, dealing with the transitory dynamics.

In the following, we consider two alternative policies in order to investigate their impact on the optimal immigration profile.

Policy 1: We fix the total number of people who annually immigrate to a country.

Policy 2: We prescribe the (stationary) total size of the receiving country's population.

Moreover, we assume that there are age-specific upper bounds for immigration. This is a reasonable assumption because it takes into account that the present immigration cannot be changed dramatically.

We find that the optimal immigration profile for both policies exhibits a *bang-bang* pattern, meaning that the solution jumps from one age-specific bound to the other and takes no values in between. We prove that for the optimal profile under Policy 1 that besides immigration at young and middle ages, immigration takes place also in the vicinity of the maximum attainable age. Such counter-intuitive old-age immigration does not happen under Policy 2. We show that under reasonable assumptions about the vital rates and the age-specific immigration bounds, the optimal immigration profile under Policy 2 is such that it is optimal to allow maximum immigration on not more than two separate age intervals before the retirement age.

The optimal control approach enables us to determine the marginal value of an immigrant at a certain age in terms of the dependency ratio,
cf. Wrzaczek et al. (2010), by interpretation of the so-called adjoint variable, cf. Grass et al. (2008), whose clear meaning will be defined in Section 3.3. As a consequence we are able to decide what age-specific immigration profile is optimal for minimizing the dependency ratio. Moreover, the impact of an a-year-old immigrant on the dependency ratio can be represented as a sum of two components. The first component, which is referred to as the *direct* effect, accounts for a woman's expected life time inside and outside the work force. The second component, known as the *indirect* effect accounts for the effect on the dependency ratio contributed by her expected number of descendants. Clearly, when an immigrant arrives towards the end of childbearing age she will have less children than a younger woman which indicates that she will be less of a burden for the dependency ratio of the resulting stationary population. However, the expected remaining time in the working population is then also reduced, meaning that she will be dependent for a relatively longer time.

We give numerical illustrations of our findings for the case study of the Austrian population based on demographic data from 2008 and find that in the case of a fixed population size together with very loose age-specific bounds for immigration, the optimal age of immigration lies in the mid-thirties.

3.2 Model description and preliminary statements

In the following, α and β denote the lower and upper age limits determining the working age population and ω is the maximum attainable age of an individual. We aim to minimize the dependency ratio given as

$$D(M(\cdot)) := \frac{\int_0^\alpha N(a)\,\mathrm{d}a + \int_\beta^\omega N(a)\,\mathrm{d}a}{\int_\alpha^\beta N(a)\,\mathrm{d}a}, \quad 0 < \alpha < \beta < \omega,$$

by choosing the age distribution of immigrants $M(\cdot)$. With $D(M(\cdot))$ we mean the dependency ratio that results when realizing the immigration profile $M(\cdot)$ and $N(a)$ denotes the number of resulting females in the population of age a. Here, the age a is considered as a continuous variable and subsequently $\dot{N}(a)$ denotes the derivative of $N(a)$ with respect to a.

We come up with the following optimal control problem:

$$\min_{M(\cdot)} D(M(\cdot)), \tag{3.1}$$

subject to

$$\dot{N}(a) = -\mu(a)N(a) + M(a), \tag{3.2}$$

$$N(0) = \int_0^\omega f(a)N(a)\,\mathrm{d}a, \tag{3.3}$$

$$0 \le M(a) \le \bar{M}(a),\ 0 < a < \omega. \tag{3.4}$$

Additionally, we prescribe one of the two alternative constraints, corresponding to Policy 1 and Policy 2 mentioned in Section 3.1:

Problem 1. This is problem (3.1) – (3.4) with the additional constraint that the total number of immigrants, M_{tot} is fixed:

$$M_{\text{tot}} = \int_0^\omega M(a)\,\mathrm{d}a. \tag{3.5}$$

Problem 2. This is problem (3.1) – (3.4) with the additional constraint that the size of the stationary population, N_{tot} is fixed:

$$N_{\text{tot}} = \int_0^\omega N(a)\,\mathrm{d}a. \tag{3.6}$$

From a mathematical point of view, these two problems differ significantly, since (3.5) is an integral control constraint while (3.6) a state constraint.

The immigration age profile $M(\cdot)$ is referred to as *control*, since it is the decision variable in the optimization problem. The population structure is determined by the so-called state variable of the problem, $N(\cdot)$. In contrast to the control, the state variable cannot instantaneously be influenced since it has its own dynamics, see (3.2).

By $f(a)$ and $\mu(a)$ we denote age-specific fertility and mortality rates which do not change with time and are continuous functions of a. Additionally, we assume that $\int_0^\omega \mu(a)\,\mathrm{d}a = +\infty$, cf. Anita (2000), which ensures that $N(\omega) = 0$ holds. Subsequently, we will with $l(a) := e^{-\int_0^a \mu(s)\,\mathrm{d}s}$ denote the age-specific probability of surviving until age a.

The support of $f(\cdot)$ is a subset $[a_{\min}, a_{\max}] \subset [0, \omega]$, where $a_{\min}$ and $a_{\max}$ denote the youngest and the oldest age of childbearing, respectively. Subsequently, for both problems we assume that fertility $f(\cdot)$ is below-replacement, i.e. that the following assumption holds:

Assumption 3.1. *Age-specific fertility $f(\cdot)$ and mortality $\mu(\cdot)$ are such that*

$$NRR := \int_0^\omega f(a)l(a)\,\mathrm{d}a < 1$$

holds.

Below-replacement fertility means that a population cannot reproduce itself.

With $\bar{M}(a)$ we denote the age-specific immigration bounds which are assumed to be continuous. From a mathematical point of view the reason for imposing these age-specific bounds is the applicability of Pontryagin's Maximum Principle. However, more practically spoken these bounds are justifiable because they may reflect the fact that age is not the only factor that should be taken into account when determining the optimal immigration policy and also the number of potential immigrants of a certain age is limited.

Notice that the control $M(\cdot)$ enters linearly the problem. This property of the optimal control problem is responsible for the bang-bang behavior of the solution obtained below. The dynamics (3.2) describing the age structure of the population only holds for a stationary population. See also Chapter 2 for a detailed discussion of dynamic population models.

Subsequently, for a given age interval $[\alpha, \beta] \subseteq [0, \omega]$ the function $\mathbb{I}_{[\alpha,\beta]}(\cdot)$ is defined as the characteristic function

$$\mathbb{I}_{[\alpha,\beta]}(a) = \begin{cases} 1 & \text{if } a \in [\alpha, \beta], \\ 0 & \text{otherwise.} \end{cases}$$

We also recall the notion of the *reproductive value* of an a-year-old female, introduced in Fisher (1930) (see also Keyfitz (1977)), which is the expected number of future daughters of an individual from her current age onward, given that she has survived to this age:

$$v(a) = \int_a^\omega \frac{l(x)}{l(a)} f(x)\,\mathrm{d}x. \tag{3.7}$$

Notice that $v(0) = NRR$. Then, we determine the solution of (3.2) by using the Cauchy formula:

$$\begin{aligned} N(a) =& e^{-\int_0^a \mu(s)\,\mathrm{d}s} N(0) + \int_0^a e^{\int_s^a \mu(\tau)\,\mathrm{d}\tau} M(s)\,\mathrm{d}s \\ =& l(a)N(0) + \int_0^a \frac{l(a)}{l(s)} M(s)\,\mathrm{d}s. \end{aligned}$$

From (3.3) it follows that

$$\begin{aligned} N(0) =& \frac{1}{1-NRR} \int_0^\omega f(a) \int_0^a \frac{l(a)}{l(s)} M(s)\,\mathrm{d}s\,\mathrm{d}a \\ =& \frac{1}{1-NRR} \int_0^\omega f(a)l(a) \int_0^a \frac{M(s)}{l(s)}\,\mathrm{d}s\,\mathrm{d}a \\ =& \frac{1}{1-NRR} \int_0^\omega \frac{M(s)}{l(s)} \int_s^\omega f(a)l(a)\,\mathrm{d}a\,\mathrm{d}s \\ =& \frac{1}{1-NRR} \int_0^\omega M(s)v(s)\,\mathrm{d}s. \end{aligned}$$

With this presentation of the solution one can obtain an alternative representation of the objective function (3.1): Consequently, for the total population the following relations hold:

$$\begin{aligned}\int_0^{\omega} N(a)\,\mathrm{d}a &= \int_0^{\omega} \frac{l(a)}{1-NRR}\,\mathrm{d}a \int_0^{\omega} M(s)v(s)\,\mathrm{d}s + \int_0^{\omega}\int_0^{a} \frac{l(a)}{l(s)} M(s)\,\mathrm{d}s\,\mathrm{d}a \\ &= \frac{e_{[0,\omega]}(0)}{1-NRR}\int_0^{\omega} M(s)v(s)\,\mathrm{d}s + \int_0^{\omega} e_{[0,\omega]}(s)M(s)\,\mathrm{d}s \\ &= \int_0^{\omega}\left(\frac{e_{[0,\omega]}(0)}{1-NRR}v(s) + e_{[0,\omega]}(s)\right)M(s)\,\mathrm{d}s \\ &= \int_0^{\omega} G(s)M(s)\,\mathrm{d}s,\end{aligned}$$

where

$$G(s) := \frac{e_{[0,\omega]}(0)}{1-NRR}v(s) + e_{[0,\omega]}(s). \tag{3.8}$$

Analogously, we obtain

$$\int_{\alpha}^{\beta} N(a)\,\mathrm{d}a = \int_0^{\omega} F(s)M(s)\,\mathrm{d}s,$$

where

$$F(s) := \frac{e_{[\alpha,\beta]}(0)}{1-NRR}v(s) + e_{[\alpha,\beta]}(s), \tag{3.9}$$

and hence the following relation holds:

$$D(M(\cdot)) = 1 - \frac{\int_0^{\omega} F(a)M(a)\,\mathrm{d}a}{\int_0^{\omega} G(a)M(a)\,\mathrm{d}a}. \tag{3.10}$$

Here, $e_{[0,\omega]}(a) = \int_a^{\omega} \frac{l(x)}{l(a)}\,\mathrm{d}x$ is the life expectancy in $[0,\omega]$ at age a. Similarly, $e_{[\alpha,\beta]}(a) = \int_a^{\omega} \frac{l(x)}{l(a)} \mathbb{I}_{[\alpha,\beta]}(x)\,\mathrm{d}x$ is the working life expectancy of an a-year-old, reflecting the expected number of years an a-year-old would spend working. Clearly, $e_{[\alpha,\beta]}(a) = 0$ for $a \geq \beta$. We will need the alternative representation (3.10) of the objective function in a later step, see Lemma 3.1 and Theorem 3.2.

3.3 The optimal immigration profile for a fixed number of immigrants

In this section, we analyze Problem 1, given by (3.1)–(3.5), by making use of optimal control theory.

In order to determine an optimal immigration profile, $M^*(\cdot)$, we derive necessary conditions to characterize the optimal solution.

Now, we restate the problem so that the maximum principle, see Theorem 2.2 in Chapter 2, is applicable. Therefore, in addition to $N(a)$ we introduce the auxiliary state variables $X(a)$, $Y(a)$, being absolutely continuous functions of a. The corresponding state equations read as

$$\dot{X}(a) = N(a), \quad X(0) = 0,$$
$$\dot{Y}(a) = \mathbb{I}_{[\alpha,\beta]}(a)N(a), \quad Y(0) = 0.$$

Equivalently, it holds that

$$X(a) = \int_0^a N(\tau)\,\mathrm{d}\tau \quad \text{and} \quad Y(a) = \int_\alpha^a \mathbb{I}_{[\alpha,\beta]}(\tau)N(\tau)\,\mathrm{d}\tau.$$

In this way we can express the objective function (3.1) by evaluating the functions $X(\cdot)$ and $Y(\cdot)$ at the terminal value ω. Therefore, Problem 1 is equivalent to the problem

$$\min_{M(a)} \frac{X(\omega)}{Y(\omega)}, \tag{3.11}$$

subject to

$$\dot{N}(a) = -\mu(a)N(a) + M(a), \tag{3.12}$$
$$\dot{X}(a) = N(a), \quad X(0) = 0, \tag{3.13}$$
$$\dot{Y}(a) = \mathbb{I}_{[\alpha,\beta]}(a)N(a), \quad Y(0) = 0, \tag{3.14}$$
$$N(0) = \int_0^\omega f(a)N(a)\,\mathrm{d}a, \tag{3.15}$$
$$0 \le M(a) \le \bar{M}(a), \tag{3.16}$$
$$M_{\text{tot}} = \int_0^\omega M(a)\,\mathrm{d}a. \tag{3.17}$$

Introducing scalars λ_1, λ_2, further referred to as Lagrange multipliers, and the so-called adjoint variables $\xi(\cdot)$, $\zeta(\cdot)$ and $\eta(\cdot)$ we define the Hamiltonian of (3.11) - (3.17) as

$$\begin{aligned}
&H(a, N, X, Y, M, \xi, \zeta, \eta) = \\
&\xi(a)\,(-\mu(a)N(a) + M(a)) + \zeta(a)N(a) + \eta(a)\mathbb{I}_{[\alpha,\beta]}(a)N(a) \\
&- \lambda_0 \frac{X(\omega)}{Y(\omega)} - \lambda_1 f(a)N(a) - \lambda_2 M(a).
\end{aligned}$$

The next theorem provides necessary optimality conditions and can be summarized as a maximization condition for the Hamiltonian

$$\max_{0 \le M \le \bar{M}(a)} H$$

with appropriate adjoint variables.

Theorem 3.1. *Let* $(N^*(\cdot), M^*(\cdot))$ *be an optimal solution of problem* (3.1)–(3.5). *Then there exist Lagrange multipliers* $\lambda_1, \lambda_2 \in \mathbb{R}$ *and an absolutely continuous function* $\xi(\cdot)$*, such that*

(i) $\xi(\cdot)$ *satisfies on* $[0, \omega]$ *the equations*

$$\begin{aligned}
&\dot{\xi}(a) = \mu(a)\xi(a) - \lambda_1 f(a) - \frac{(D(M^*(\cdot)) + 1)^2}{N_{\text{tot}}(M^*(\cdot))}\mathbb{I}_{[\alpha,\beta]}(a) + \frac{(D(M^*(\cdot)) + 1)}{N_{\text{tot}}(M^*(\cdot))}, \\
&\xi(0) = \lambda_1, \quad \xi(\omega) = 0,
\end{aligned} \tag{3.18}$$

(ii) and the following maximum principle holds for almost every $a \in (0, \omega)$:

$$(\xi(a) - \lambda_2)M^*(a) = \max_{0 \leq M \leq \bar{M}(a)} (\xi(a) - \lambda_2)M. \tag{3.19}$$

Proof. The theorem follows from the application of Pontryagin's maximum principle as stated in Theorem 2.2. Given the Hamiltonian

$$H(a, N, X, Y, M, \xi, \zeta, \eta) =$$
$$\xi\left(-\mu(a)N + M\right) + \zeta N + \eta \mathbb{I}_{[\alpha,\beta]}(a)N - \lambda_0 \frac{X(\omega)}{Y(\omega)} - \lambda_1 f(a)N - \lambda_2 M.$$

We have

$$\begin{aligned}
\dot{\xi}(a) &= -\frac{\partial H}{\partial N}, \ \xi(0) = \lambda_1, \quad \xi(\omega) = 0,\\
\dot{\eta}(a) &= -\frac{\partial H}{\partial Y}, \quad \eta(\omega) = \frac{X(\omega)}{Y^2(\omega)},\\
\dot{\zeta}(a) &= -\frac{\partial H}{\partial X}, \quad \zeta(\omega) = \frac{1}{Y(\omega)},
\end{aligned}$$

where $\eta = \frac{X(\omega)}{Y^2(\omega)} = \frac{(D+1)^2}{N_{\text{tot}}}$ and $\zeta = \frac{1}{Y(\omega)} = \frac{D+1}{N_{\text{tot}}}$. Hence,

$$\begin{aligned}
\dot{\xi}(a) &= \mu(a)\xi(a) - \lambda_1 f(a) - \frac{X(\omega)}{Y^2(\omega)} \mathbb{I}_{[\alpha,\beta]}(a) + \frac{1}{Y(\omega)},\\
&= \mu(a)\xi(a) - \lambda_1 f(a) - \frac{(D(M^*(\cdot)) + 1)^2}{N_{\text{tot}}(M^*(\cdot))} \mathbb{I}_{[\alpha,\beta]}(a) + \frac{(D(M^*(\cdot)) + 1)}{N_{\text{tot}}(M^*(\cdot))}.
\end{aligned}$$

Q.E.D.

From (3.19) the following can be concluded:

$$M^*(a) = \begin{cases} \bar{M}(a) & \text{if } \xi(a) > \lambda_2, \\ \text{undetermined} & \text{if } \xi(a) = \lambda_2, \\ 0 & \text{if } \xi(a) < \lambda_2. \end{cases} \tag{3.20}$$

Hence, we introduce the following assumption on mortality $\mu(a)$ and fertility $f(a)$:

Assumption 3.2. *For all real numbers d_0, d_1 and d_2 it holds that*

$$meas\{a \in \Omega : d_0\mu(a) - d_1 f(a) = d_2\} = 0.$$

Proposition 3.1. *Let Assumption 3.2 be fulfilled, then the optimal control $M^*(a)$ exhibits no singular arc, that is, the indeterminacy in* (3.20) *may happen only on a set of measure zero.*

Proof. Assume that there exists a non void interval $[\underline{a}, \bar{a}] \subset [0, \omega]$ where the optimal solution $M^*(a)$ exhibits a singular arc. Then, $\xi(a) = \lambda_2$ and simultaneously $\dot{\xi}(a) = 0$ on $[\underline{a}, \bar{a}]$. From (3.18) we have that

$$\dot{\xi}(a) = \mu(a)\lambda_2 - f(a)\lambda_1 - c_1 \mathbb{I}_{[\alpha,\beta]}(a) + c_2 = 0 \quad \forall a \in [\underline{a}, \bar{a}].$$

where

$$c_1 = \frac{(D(M^*(\cdot)) + 1)^2}{N_{\text{tot}}(M^*(\cdot))}$$

and

$$c_2 = \frac{(D(M^*(\cdot)) + 1)}{N_{\text{tot}}(M^*(\cdot))}$$

. This contradicts Assumption 3.2 and hence the result follows. Q.E.D.

Assumption 3.2 means that equality $\xi(a) = \lambda_2$ happens only at isolated points, so that the values $M^*(a)$ at these points have no effect on the dependency ratio. Assumption 3.2 holds for fertility and morality rates that are not linearly related on a set of positive measure.

It can immediately be concluded that the optimal control is of *bang-bang* type, jumping from one boundary to the other. Therefore, function $\xi(\cdot) - \lambda_2$ is usually referred to as *switching function* because the change of its sign determines the ages a at which the optimal control switches from one boundary to the other.

Hence, to obtain the optimal immigration profile it is necessary to determine $\xi(\cdot)$ and λ_2. We note, that the right hand side of the differential equation (3.18) is discontinuous at ages $a = \alpha$ and $a = \beta$ and therefore the solution ξ has two kinks at each of these ages. Additionally, we see that (3.18) is a boundary value problem for a linear differential equation. By using the Cauchy formula for (3.18) we obtain the solution

$$\xi(a) = \lambda_1 v(a) + \frac{D(M^*(\cdot)) + 1}{N_{\text{tot}}(M^*(\cdot))}\big((D(M^*(\cdot)) + 1)e_{[\alpha,\beta]}(a) - e_{[0,\omega]}(a)\big). \tag{3.21}$$

Using the boundary condition $\xi(0) = \lambda_1$ and taking into account that $NRR = v(0)$ we obtain that

$$\lambda_1 = \frac{\frac{D(M^*(\cdot))+1}{N_{\text{tot}}(M^*(\cdot))}\Big((D(M^*(\cdot)) + 1)e_{[\alpha,\beta]}(0) - e_{[0,\omega]}(0)\Big)}{1 - NRR}. \tag{3.22}$$

With (3.20) and expressions (3.21), (3.22) we are now able to obtain the optimal immigration profile $M^*(\cdot)$, where the Lagrange multiplier λ_2 has to be determined in such a way, that (3.5) holds for the resulting solution.

The above introduced scalar λ_1 is the Lagrange multiplier corresponding to the initial condition (3.3) and it reflects the marginal worth of an increase in the annual flow of newborns. The constant λ_2 corresponds to the control constraint (3.5) and gives the marginal change in the dependency ratio when adding an additional immigrant.

In economic applications of optimal control theory, the adjoint variable is interpreted as shadow price of the state variable. The term shadow price is commonly used in capital theory, cf. Dorfman (1969); Léonard and Long (1992). There, it is interpreted as the highest hypothetical price a rational decision-maker would be willing to pay for owning an additional unit of the corresponding state variable at time a measured by the discounted (extra) future profit. Notice that the shadow price is not a real market price and therefore can also have a negative value. See also for example Grass et al. (2008), where an overview of recent developments of optimal control theory is given, for a more detailed discussion of the economic interpretation of the maximum principle. In line with this interpretation, $\xi(a)$ gives the shadow price of an individual of age a. As it can be seen below, for this particular optimal control problem considered here the shadow price is a part of the effect of adding an additional immigrant of age a.

Problem 1 has a rather peculiar property which in fact questions its credibility, although, in general, prescribing the number of immigrants to a population seems to be a quite reasonable policy. Namely, the optimal immigration policy for Problem 1 always involves very old immigrants

In the theorem below, it is shown that the optimal immigration profile $M^*(\cdot)$ is such that arbitrarily close to the maximum age ω there are ages where immigration is optimal. An individual's contribution consists of her own expected years lived in the host country and the analogous contribution of all her future descendants. Since we aim to minimize the relative number of dependent people in the population, the fact that immigration at the end of the life horizon is optimal seems to be counter-intuitive. However, in the next theorem it is shown that this is not only a numerical effect as found in the case study below, but can be proven theoretically.

Furthermore, we assume that the following assumption holds:

Regularity Assumption 1. *For any $c > 0$ it holds that*

$$F(a) \neq c\, G(a),$$

almost everywhere in $[0, \omega]$.

Functions G and F are given as in (3.8) and (3.9). This assumption means that an immigrant's effect on the working population is not proportional to its effect on the overall population.

Moreover, for the proof of Theorem 3.2 below we need the following Lemma:

Lemma 3.1. *For any immigration profile $M(\cdot)$ satisfying* (3.16), (3.17), *there exists a set $\Gamma \subset [0, \omega]$, meas$(\Gamma) > 0$ such that $M(a) > 0$ for $a \in \Gamma$ and*

$$\frac{F(a)}{G(a)} < J(M(\cdot)), \quad \forall a \in \Gamma,$$

holds.

Proof. Because of the regularity assumption the strict inequality

$$F(a) \int_0^\omega G(a) M(a)\, \mathrm{d}a > G(a) \int_0^\omega F(a) M(a)\, \mathrm{d}a,$$

holds on a subset $\Gamma \subset \Gamma^0$ of positive measure. Multiplying both sides by $M(a)$ and integrating on $[0, \omega]$ we obtain

$$\int_0^\omega F(a)M(a)\,\mathrm{d}a \int_0^\omega G(a)M(a)\,\mathrm{d}a > \int_0^\omega G(a)M(a)\,\mathrm{d}a \int_0^\omega F(a)M(a)\,\mathrm{d}a,$$

which gives a contradiction. Q.E.D.

Theorem 3.2. *Let $M(\cdot)$ be an arbitrary immigration profile which fulfills* (3.4), (3.5) *and additionally $M(a) < \bar{M}(a)$ for $a \in [\omega - \delta, \omega]$ and some $\delta > 0$. Then there is an immigration profile $\tilde{M}(\cdot)$ which satisfies* (3.4),(3.5) *such that*

$$D(\tilde{M}) < D(M).$$

Proof. For the proof we consider the equivalent maximization problem

$$\max_{M(a)} J(M(\cdot)),$$

subject to

$$M_{\text{tot}} = \int_0^\omega M(a)\,\mathrm{d}a,$$
$$0 \leq M(a) \leq \bar{M}(a).$$

This problem is equivalent to the minimization problem (3.1)–(3.5). Note, that $J(M(\cdot)) = 1 - D(M(\cdot))$.

Let Γ be the set from Lemma 3.1, and let $b \in \Gamma$ be a Lebesgue point. Recall that almost every point of Γ is such. Let us define an immigration profile $\tilde{M}(\cdot)$

$$\tilde{M}(a) := \begin{cases} M(a) & a \notin [b-\delta, b] \cup [\omega - \delta, \omega], \\ M(a) - h & a \in [b - \delta, b], \\ M(a) + h & a \in [\omega - \delta, \omega], \end{cases}$$

where $M(a) > 0$ and $0 < h \leq \bar{M}(a) - M(a)$ holds. The corresponding objective value reads as

$$J(\tilde{M}(\cdot)) = \frac{\int_0^\omega F(a)M(a)\,\mathrm{d}a - h\int_{b-\delta}^b F(a)\,\mathrm{d}a + h\int_{\omega-\delta}^\omega F(a)\,\mathrm{d}a}{\int_0^\omega G(a)M(a)\,\mathrm{d}a - h\int_{b-\delta}^b G(a)\,\mathrm{d}a + h\int_{\omega-\delta}^\omega G(a)\,\mathrm{d}a}.$$

We define

$$H(\delta) := h\int_{x-\delta}^x F(a)\,\mathrm{d}a, \quad x = b, \omega,$$

where, by transformation of the independent variable, $H(\delta) = h\int_0^\delta F(x-t)\,\mathrm{d}t$ holds. By Taylor expansion around 0 we obtain

$$\begin{aligned} H(\delta; x) &= h(H(0;x) + \delta H'(0;x) + \delta^2 H''(0;x) + o(\delta^2)), \\ &= h\delta F(x) + h\delta^2 F'(x) + ho(\delta^2). \end{aligned}$$

As usual, $o(\epsilon)$ means a function such that $O(\epsilon)/\epsilon \to 0$ with $\epsilon \to 0$. The same approach is used for G. Therefore, by neglecting all terms but the linear one in δ,

$$J(\tilde{M}(\cdot)) = \frac{\int_0^\omega F(a)M(a)\,\mathrm{d}a - \delta h F(b) + \delta h F(\omega)}{\int_0^\omega G(a)M(a)\,\mathrm{d}a - \delta h G(b) + \delta h G(\omega)}.$$

Note, that $G(\omega) = F(\omega) = 0$ and therefore it holds that

$$\begin{aligned}
& & J(\tilde{M}(\cdot)) - J(M(\cdot)) > 0 \\
\Leftrightarrow & & \frac{\int_0^\omega F(a)M(a)\,\mathrm{d}a - \delta h F(b)}{\int_0^\omega G(a)M(a)\,\mathrm{d}a - \delta h G(b)} > \frac{\int_0^\omega F(a)M(a)\,\mathrm{d}a}{\int_0^\omega G(a)M(a)\,\mathrm{d}a} \\
\Leftrightarrow & & -F(b)\int_0^\omega G(a)M(a)\,\mathrm{d}a > -G(b)\int_0^\omega F(a)M(a)\,\mathrm{d}a \\
\Leftrightarrow & & \frac{F(b)}{G(b)} < \frac{\int_0^\omega F(a)M(a)\,\mathrm{d}a}{\int_0^\omega G(a)M(a)\,\mathrm{d}a},
\end{aligned}$$

which is fulfilled by the choice of $b \in \Gamma$ as was proven in Lemma 3.1. Since problem (3.1)–(3.5) and problem (3.11)–(3.16) and therefore $J(\tilde{M}(\cdot)) > J(M(\cdot))$ and $D(\tilde{M}(\cdot)) < D(M(\cdot))$ are equivalent we have thus proven Theorem 3.2. Q.E.D.

This counter-intuitive property of the optimal solution is due to the age-specific immigration bounds, (3.4), that are introduced in this model. If they are removed, as done in a static setup in Schmertmann (2012), this effect probably cannot be observed anymore. We also overcome this counter-intuitive result in Section 3.4 by considering Problem 2, where we fix the size of the stationary population instead of the size of the immigration.

3.3.1 A case study: the Austrian case

The numerical results for the optimal immigration profile and the dependency ratio obtained in this section are based on the analytical derivations above. In the following, we will assume that $\alpha = 20$, $\beta = 65$, and $\omega = 110$. For the computations we initialize the age structure of demographic variables referring to Austrian data as of 2008, cf. Figure 3.1, and interpolate these data piecewise linearly to obtain continuous representations of the fertility and mortality rate. The actual age-specific immigration numbers of 2008 are denoted by $M_{\text{act}}(a)$. In 2010 the total dependency ratio for Austria was 62.2% and under the current fertility, mortality and immigration rates it would rise up to a level of 78%.

In the following, we investigate two different Scenarios for Problem 1:

Scenario 1. We set

$$\bar{M}(a) = 2M_{\text{act}}(a), \quad \forall a \in [0, \omega],$$

which corresponds to a possible doubling of the number of immigrants at any age compared with the 2008 level. Since it is argued that a rise in immigration would help

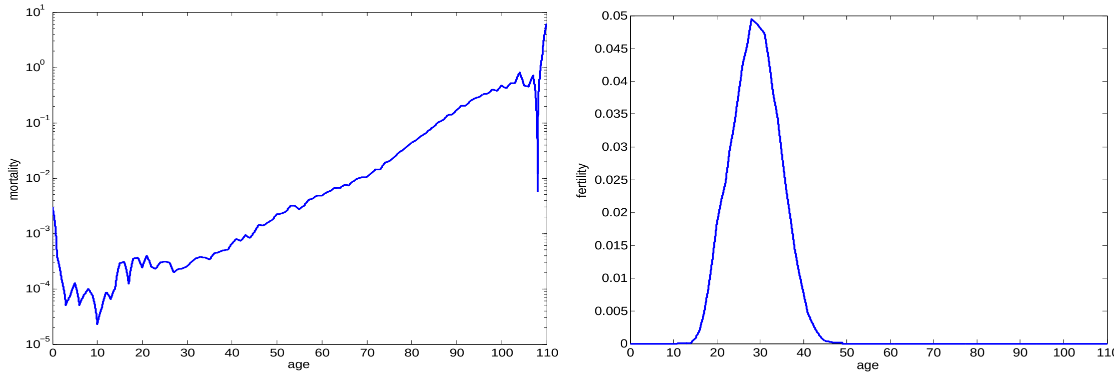

Figure 3.1: Female mortality rate $\mu(a)$ (left, logarithmic scale) and fertility rate $f(a)$ (right); Austria 2008

to reduce the aging process, we prescribe a total volume of $M_{\text{tot}} = 80000$ females in order to simulate an increase in the number of annual immigrants.
The resulting age profile that fulfills the maximization condition (3.19) is

$$M^*(a) = \begin{cases} \bar{M}(a) & \text{if } a \in [11, 49] \cup [82, 110], \\ 0 & \text{otherwise.} \end{cases}$$

This can be concluded from the values of the adjoint variable $\xi(a)$ at ages a depicted in Figure 3.2. The solid line in Figure 3.2 corresponds to the λ_2-level. Consequently, for ages where $\xi(a)$ has values larger than λ_2 immigration is at its upper bound and for ages where $\xi(a)$ is smaller than λ_2 the optimal immigration profile is zero. The adjoint variable $\xi(\cdot)$ exhibits two kinks at ages $\alpha = 20$ and $\beta = 65$, due to the discontinuity of the right hand side of the differential equation (3.18). For a detailed explanation of the shape of the adjoint variable as a function of a see Section 3.5.

The resulting optimal immigration profile $M^*(a)$ is illustrated in Figure 3.4 and is such that in particular workers close to retirement and young retirees are excluded from immigration. Since the lower age-specific immigration bounds are assumed to be zero for all ages, due to its bang-bang characteristic the $M^*(a)$ is either zero or takes its maximal value.

In Figure 3.6, the age structure of the optimal stationary through immigration population is depicted. As typical for a closed stationary population, the age structure of a stationary through immigration population exhibits a flat line at young ages due to the low mortality at these ages. The resulting total SI (stationary through immigration) population size is 13.0 million females. This large number for the population size results from the fact that immigrants enter at young ages and hence get more children while they live in the host country.

The resulting minimal dependency ratio is 75.14%, which corresponds to about 75 dependents per 100 workers.

Scenario 2. We also performed the calculations with $M_{\text{tot}} = 50000$ which is close to the actual total number of (female) immigrants for Austria in 2008. The age-specific upper bound was set to $\bar{M}(a) = 20M_{\text{act}}(a)$ which corresponds to a high supply of immigrants at all ages. This stylized scenario is considered to see what the optimal immigration profile is if the age-specific immigration rates are relaxed – so to be relatively free to choose from all ages – while the number of annual immigrants is kept on the current level.

From the switching function depicted in Figure 3.3, we can conclude that the optimal immigration profile reads

$$M^*(a) = \begin{cases} \bar{M}(a) & \text{if } a \in [33,36] \cup [109,110], \\ 0 & \text{otherwise.} \end{cases}$$

See also Figure 3.5. Clearly, relaxing the age-specific immigration bounds shortens the optimal age interval for immigration drastically. Scenario 2 reflects the case where there is a huge supply of immigrants in all ages. In this case only immigrants in the mid-thirties are appreciated because they spend a long time working in the host country and also have a lower net reproduction rate.

The resulting minimal dependency ratio is 72.24%. The resulting total size of the female SI population is 4.1 million females which would mean that the size of the Austrian population would be close to the current one. Hence, with such an age-targeted immigration policy the dependency ratio could be reduced by more than 5% compared to a scenario where the fertility, immigration and mortality rates would remain on current levels. However, a rise of 10% from the 2010 level would still remain.

Figure 3.7 represents the age structure of the optimal SI population. The steep increase in the mid thirties comes from the very restrictive immigration policy which focuses on immigrants at these ages. In comparison to Scenario 1, it is optimal that immigrants enter at older ages and hence a smaller stationary population results.

In both scenarios it is optimal to let people immigrate at the end of the life time, although they are part of the economically dependent population. This can be explained by the fact that (3.5) has to be fulfilled and the age-specific bounds hold. Although, in practice, a restriction of the number of immigrants is clearly a meaningful immigration policy, this rather peculiar result, as already explained analytically above, makes us move to a new problem formulation, see Problem 2, to overcome this result. Alternatively, one could also think of a scenario where instead of prescribing the total number of immigrants, the government fixes an upper bound for the total immigration influx,

$$\int_0^{\omega} M(a) \leq M_{tot}, \ M_{tot} > 0,$$

For this policy, however, the problem becomes incorrect in the sense that the optimal solution converges to zero. Hence, subsequently, in Problem 2, we fix the total population size instead.

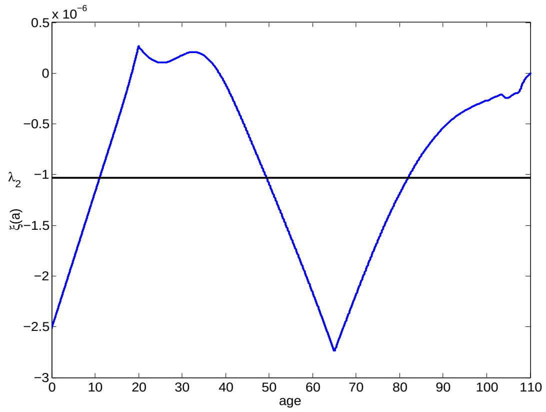

Figure 3.2: Scenario 1: The adjoint variable $\xi(\cdot)$ for Problem 1

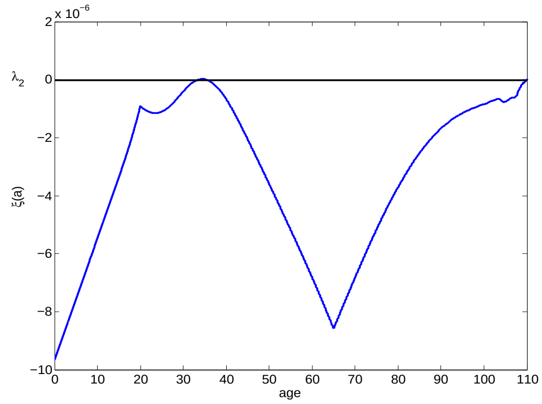

Figure 3.3: Scenario 2: The adjoint variable $\xi(\cdot)$ for Problem 1

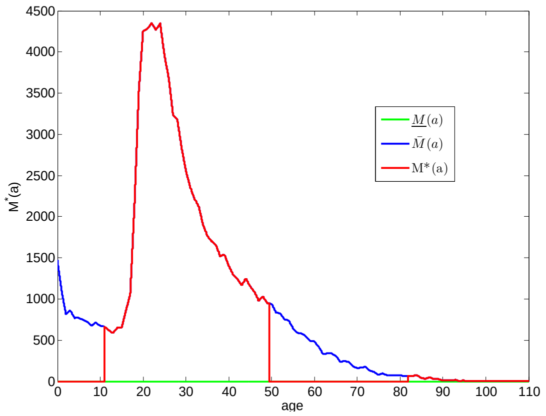

Figure 3.4: Scenario 1: The optimal immigration profile $M^*(\cdot)$ for Problem 1

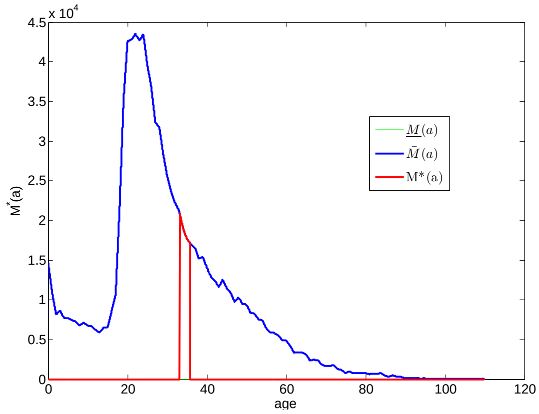

Figure 3.5: Scenario 2: The optimal immigration profile $M^*(\cdot)$ for Problem 1

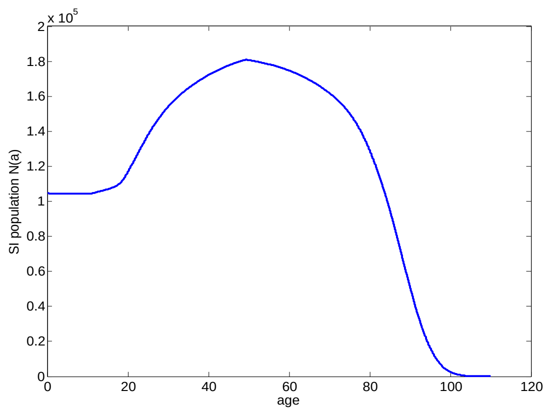

Figure 3.6: Scenario 1: The age structure of the SI population $N^*(\cdot)$ for Problem 1

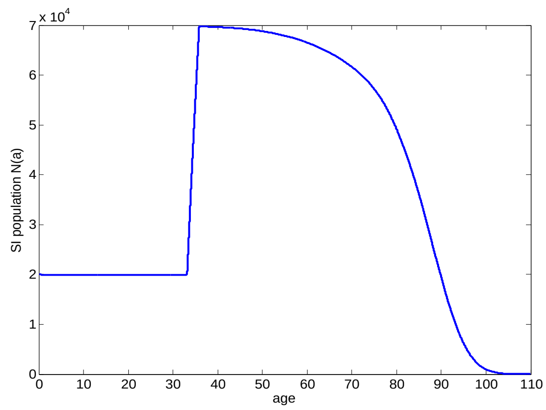

Figure 3.7: Scenario 2: The age structure of the SI population $N^*(\cdot)$ for Problem 1

3.4 The optimal immigration profile for a fixed population size

We slightly change the model and instead of fixing the volume of immigrants (Policy 1), we require that the number of people in the population equals a prescribed value (Policy 2), i.e. we consider problem (3.1)–(3.4) with the additional constraint (3.6). Theorem 3.2 below states necessary conditions for the optimal solution.

Theorem 3.3. *If* $(N^*(\cdot), M^*(\cdot))$ *is an optimal solution of problem* (3.1)–(3.4) *and* (3.6)*, then there are Lagrange multipliers* λ_1, λ_2*, and an absolutely continuous function* $\xi : [0, \omega] \to \mathbb{R}$ *such that:*

i) the function $\xi(\cdot)$ *satisfies*

$$\begin{aligned} &\dot{\xi}(a) = \mu(a)\xi(a) - \lambda_1 f(a) - \mathbb{I}_{[\alpha,\beta]}(a) + \lambda_2,\ a \in [0, \omega], \\ &\xi(0) = \lambda_1, \quad \xi(\omega) = 0, \end{aligned} \tag{3.23}$$

ii) and the maximum principle holds for almost every $a \in (0, \omega)$*:*

$$\xi(a)M^*(a) = \max_{0 \le M \le \bar{M}(a)} \xi(a)M.$$

Proof. Minimization of the dependency ratio D in a population with fixed size is equivalent to maximization of the number of working people:

$$\max_{M(a)} \int_0^\omega \mathbb{I}_{[\alpha,\beta]}(a)N(a)\,\mathrm{d}a.$$

Again, we define the Pontryagin function as

$$\begin{aligned} &H(a, N, X, Y, M, \xi) = \\ &\xi(-\mu(a)N + M) + \mathbb{I}_{[\alpha,\beta]}(a)N - \lambda_1 f(a)N - \lambda_2 N, \end{aligned}$$

and apply Pontryagin's maximum principle presented in Theorem 2.2 in Chapter 2. The optimality conditions for (N^*, M^*) can be formulated by the following expressions:

$$\xi(a)M^*(a) = \max_{0 \le M \le \bar{M}(a)} \xi(a)M(a), \tag{3.24}$$

$$\dot{\xi}(a) = \mu(a)\xi(a) - \lambda_1 f(a) + \mathbb{I}_{[\alpha,\beta]}(a) + \lambda_2, \quad \xi(0) = \lambda_1, \quad \xi(\omega) = 0,$$

where λ_1 should be calculated in such a way that (3.6) is satisfied for the resulting maximizer in (3.24). Q.E.D.

Again we assume that Assumption 3.2 holds. Then, the optimal immigration profile is also of bang-bang type,

$$M^*(a) = \begin{cases} \bar{M}(a) & \text{if } \xi(a) > 0, \\ \text{singular} & \text{if } \xi(a) = 0, \\ 0 & \text{if } \xi(a) < 0, \end{cases} \tag{3.25}$$

and it remains to determine $\xi(\cdot)$. Similar calculations as in Section 3.3 give

$$\xi(a) = \lambda_1 v(a) + e_{[\alpha,\beta]}(a) - \lambda_2 e_{[0,\omega]}(a). \tag{3.26}$$

where $v(a)$ is given by (3.7). Using the boundary condition $\xi(0) = \lambda_1$, we obtain

$$\lambda_1 = \frac{e_{[\alpha,\beta]}(0) - \lambda_2 e_{[0,\omega]}(0)}{1 - NRR}. \tag{3.27}$$

In order to determine the optimal solution $(N^*(\cdot), M^*(\cdot))$, the Lagrange multiplier λ_2 in (3.23) has to be chosen in such a way that condition (3.6) is fulfilled. Therefore, the value of λ_2 depends on the choice of the prescribed value N_{tot}. Note, that $\xi(\cdot)$ is independent of the optimal solution $(N^*(\cdot), M^*(\cdot))$ and can therefore be calculated separately for each λ_2.

3.4.1 Analytical study of the optimal immigration profile

In the following, we derive general results for the optimal immigration profile for given age-specific fertility $f(a)$ and mortality $\mu(a)$ rates. This section is driven by the question of whether there is always one optimal age interval where $M^*(a)$ is on the upper bound $\bar{M}(a)$, or if the optimal immigration policy could possibly consist of several separated age intervals of maximal immigration. This reasoning is similar to Feichtinger and Veliov (2007) where it was shown that the optimal recruitment policy is bi-polar meaning that there are two distinct age intervals where recruitment is optimal. Here, it is also shown that the optimal immigration profile attains its upper bound $\bar{M}(a)$ on no more than two separate intervals.

Since the change of the sign of the adjoint variable $\xi(a)$ determines the switches of the optimal solution from one limit to the other, we count how many times the switching function (3.26) can cross its switching level $\xi(a) = 0$. To estimate this number from above we count how many times the derivative in (3.23) can change its sign at level $\xi(a) = 0$ from positive to negative

$$\dot{\xi}(a)\big|_{\xi=0} = -\lambda_1 f(a) - \mathbb{I}_{[\alpha,\beta]}(a) + \lambda_2 = 0. \tag{3.28}$$

Assumption 3.3. *The upper limit $\bar{M}(a)$ is such that if $M(a) \equiv \bar{M}(a)$, then the corresponding solution $N(\cdot)$ of* (3.2), (3.3) *satisfies*

$$\int_0^\omega N(a)\,\mathrm{d}a > N_{\text{tot}}.$$

Corollary 3.1. *There is at least one interval with $\xi(a) > 0$.*

Otherwise the optimality condition (3.25) requires $M(a) = 0$ for almost every a. This, however, leads to a contradiction between Assumption 3.1 and $N_{\text{tot}} > 0$ in (3.6). Q.E.D.

Proposition 3.2. *For the Lagrange multiplier λ_2 it holds that $\lambda_2 \in [0, 1]$.*

Proof. From equation (3.27) we conclude that if $\lambda_2 < 0$ leads to $\lambda_1 > 0$ and both lead to $\dot{\xi}(a)\big|_{\xi=0} < 0$ in equation (3.28) $\forall a \in [0, \omega)$. Hence, $\xi(a) > 0$ on $a \in [0, \omega)$ and $M^*(a) = \bar{M}(a)$ which contradicts Assumption 3.3.

If $\lambda_2 > 1$ then $\lambda_1 < 0$ because $e_{[\alpha,\beta]}(0) < e_{[0,\omega]}(0)$, thus the derivative in (3.28) has the following property $\dot{\xi}(a)\big|_{\xi=0} = -\lambda_1 f(a) - \mathbb{I}_{[\alpha,\beta]}(a) + \lambda_2 > -1 + \lambda_2 > 0$ for all $a \in [0, \omega]$, since $\min\{f(a)\} = 0$. But to satisfy the terminal condition $\xi(\omega) = 0$ for the adjoint variable it should hold, that $\xi(a) < 0$ for $a \in [0, \omega)$. That contradicts Assumption 3.1 and $N_{\text{tot}} > 0$ in (3.6). Q.E.D.

Proposition 3.3. *The following relations hold true:*

a) $\xi(a) < 0$ *if* $\lambda_2 > 0$ *for all* $a \in [\beta, \omega)$*,*

b) $\xi(a) = 0$ *if* $\lambda_2 = 0$ *for all* $a \in [\beta, \omega]$*.*

Proof. Since $e_{[\alpha,\beta]}(a) = 0$ and $v(a) = 0$ holds for all $a \in [\beta, \omega]$ it follows from (3.26) and Proposition 3.2 that $\xi(a) = -\lambda_2 e_{[0,\omega]}(a) \leq 0$, $a \in [\beta, \omega]$. Thus, **b)** is obvious and **a)** follows from the inequality $e_{[0,\omega]}(a) > 0$ for all $a \in [0, \omega)$. Q.E.D.

Assumption 3.4. *The fertility* $f(a)$ *is single peaked with support to the left from* β *and to the right from* 0*, i.e.* $a_{min} < \alpha < a_{max} \leq \beta$*.*

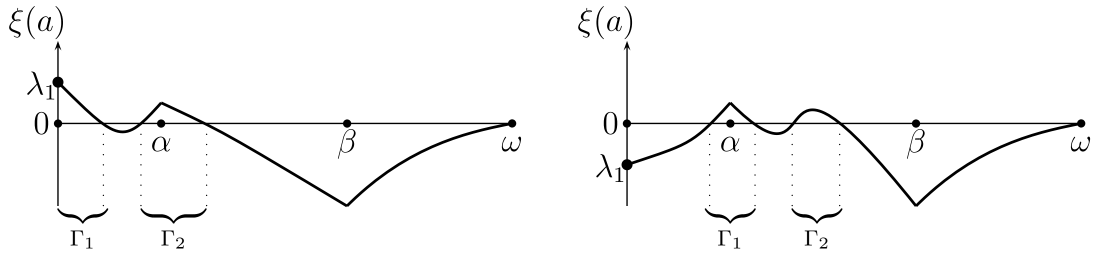

Figure 3.8: The adjoint variable $\xi(\cdot)$ determining the optimal immigration for two separate age intervals Γ_1 and Γ_2 in two cases: **a)** $\lambda_1 > 0$ (left) and **b)** $\lambda_1 < 0$ (right)

Let us denote the age of maximal fertility by $a_{fmax} = \arg\max(f(a))$.

Proposition 3.4. *The set of* $\{a : \xi(a) > 0\}$ *consists of at most two disconnected intervals* Γ_1 *and* Γ_2 *(see Fig. 3.8). Moreover,*

a) *if* $a_{fmax} < \alpha$ *then* $\alpha \in \Gamma_2$*,*

b) *if* $a_{fmax} > \alpha$ *then* $\alpha \in \Gamma_1$*.*

Proof. It follows from Proposition 3.3 that $\Gamma_1, \Gamma_2 \subset [0, \beta]$.

The derivative (3.28) can change its sign from plus to minus only at $a = \alpha$ because of the jump of the function $\mathbb{I}_{[\alpha,\beta]}(a)$ or/and at $a = a_0$, where a_0 is such a root of the equation $\dot{\xi}(a_0)\big|_{\xi=0} = 0$ that $\ddot{\xi}(a_0)\big|_{\xi=0} = -\lambda_1 \dot{f}(a_0) < 0$. It follows from Proposition 3.2 and Assumption 3.4 that equation $\dot{\xi}(a_0)\big|_{\xi=0} = 0$ cannot have more than two roots all located either in $[0, \alpha)$ or in $[\alpha, \beta]$ depending on the sign of λ_1.

If $\lambda_1 > 0$ then equation $\dot{\xi}(a_0)\big|_{\xi=0} = 0$ can only have roots in $[0, \alpha)$, where a_0 is the first root, if any, of the equation $-\lambda_1 f(a) + \lambda_2 = 0$.

If $\lambda_1 < 0$ then equation $\dot{\xi}(a_0)\big|_{\xi=0} = 0$ can have roots only in $[\alpha, \beta]$ so a_0 is the second root, if any, of the equation $-\lambda_1 f(a) - 1 + \lambda_2 = 0$, which can happen only when $a_{fmax} > \alpha$.

Thus, it follows from the continuity of the function $\xi(a)$ that it can be positive on not more than two separate intervals. It is also easy to see graphically in Fig. 3.8 that if the function $\xi(a)$ is positive on two separate intervals $\Gamma_1, \Gamma_2 \subset [0, \beta]$, these intervals must contain both points a_0 and α where derivative (3.28))changes its sign, so that $a_0 \in \Gamma_1$, $\alpha \in \Gamma_2$ when $\lambda_1 > 0$ and $\alpha \in \Gamma_1$, $a_0 \in \Gamma_2$ when $\lambda_1 < 0$. Q.E.D.

The above proposition gives us some information on the shape of the optimal immigration profile. It tells us that it is optimal that immigrants come in not more than two separate age groups. Hence, for example, depending on the parameters of the problem, immigrant's children may be valued positively or not, see also the Figures for the Austrian case study below.

3.4.2 A case study: the Austrian case

For the calculations we initialize again the fertility and mortality profiles with Austrian data as of 2008, cf. Figure 3.1.

Similar to Problem 2, in the numerical procedure to obtain the solution (N^*, M^*), first, the boundary value problem consisting of the adjoint equation and the corresponding initial and boundary value, see (3.18), has to be solved. Here, the adjoint equation does not explicitly depend on the optimal solution (N^*, M^*) itself but on the free parameter λ_2. Hence, one has to guess an initial value, λ_2^0 and plug it into (3.26)–(3.27) . We approximate the occurring integrals in the solution presentation, see (3.26), by using the trapezoidal rule. Subsequently, the new candidate for a solution $M^*_{new}(\cdot)$ is determined according to (3.25). The procedure is stopped if the absolute value of the difference of the resulting population size and the prescribed value N_{tot} in (3.6) is sufficiently small. Otherwise a new guess for λ_2 has to be provided and the procedure is repeated.

For the total population size we prescribe the resulting sizes from Section 3.3, i.e. $N_{\text{tot}} = 13.0$ million and $N_{\text{tot}} = 4.1$ million, respectively.

Scenario 1. Therefore, by setting $N_{\text{tot}} = 13.0$ million and $\bar{M}(a) = 2M_{act}(a)$, we achieve a corresponding dependency ratio $D = 74.73\%$ which is slightly smaller than the one we obtain above and the resulting volume of immigrants is 72000. The corresponding optimal immigration profile reads as

$$M^*(a) = \begin{cases} \bar{M}(a) & \text{if } a \in [9, 41], \\ 0 & \text{otherwise,} \end{cases}$$

which is determined according to (3.25) and shown in Figure 3.10. Figure 3.9 shows the corresponding adjoint variable $\xi(\cdot)$ and Figure 3.11 the optimal immigration profile. Notice that the adjoint variable in Figure 3.9 exhibits only one inner maximum. The optimal age structure is depicted in Figure 3.13.

Scenario 2. We also calculate the optimal immigration profile for $N_{\text{tot}} = 4.1$ million females and $\bar{M}(a) = 20M_{act}(a)$,

$$M^*(a) = \begin{cases} \bar{M}(a) & \text{if } a \in [33, 36], \\ 0 & \text{otherwise.} \end{cases}$$

Figure 3.10 shows the adjoint variable $\xi(\cdot)$ and Figure 3.12 the optimal immigration profile. The resulting optimal population age structure is shown in Figure 3.14. For these parameter values we achieve a corresponding dependency ratio $D = 72.24\%$ and the resulting volume of immigrants is 50000.

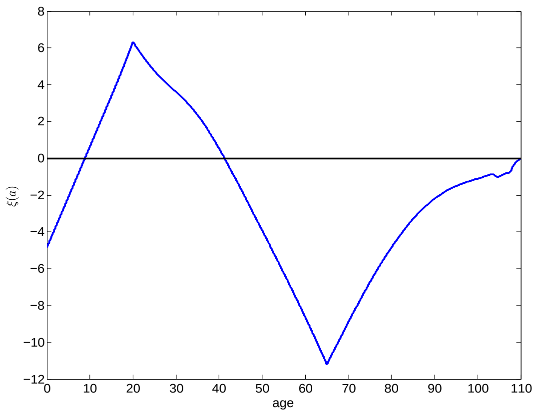

Figure 3.9: Scenario 1: The adjoint variable $\xi(\cdot)$ for Problem 2

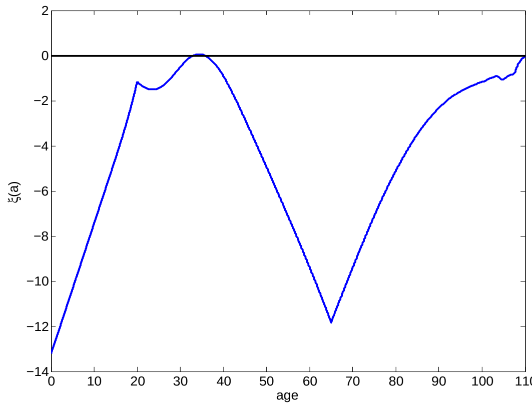

Figure 3.10: Scenario 2: The adjoint variable $\xi(\cdot)$ for Problem 2

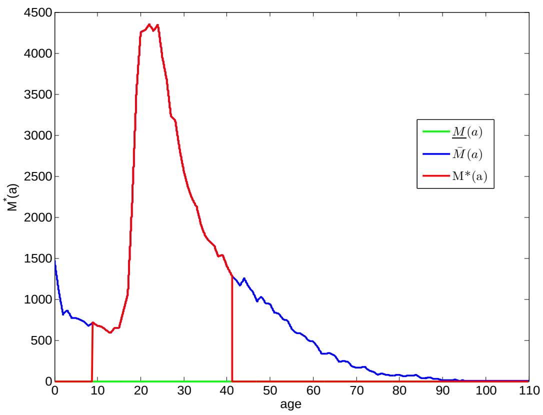

Figure 3.11: Scenario 1: The optimal immigration profile $M^*(\cdot)$ for Problem 2

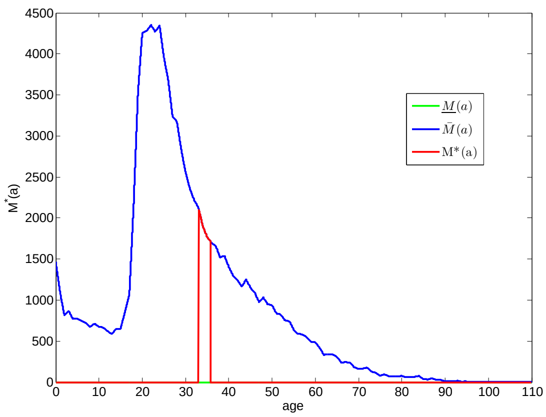

Figure 3.12: Scenario 2: The optimal immigration profile $M^*(\cdot)$ for Problem 2

Notice that the optimal solution (N^*, M^*) depends on the data N_{tot} and $\bar{M}(a)$ and hence the adjoint function changes with these exogenous parameters.

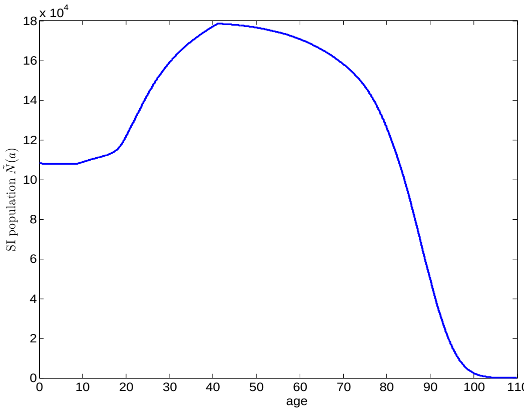

Figure 3.13: Scenario 1: The age structure of the SI population $N^*(\cdot)$ for Problem 2

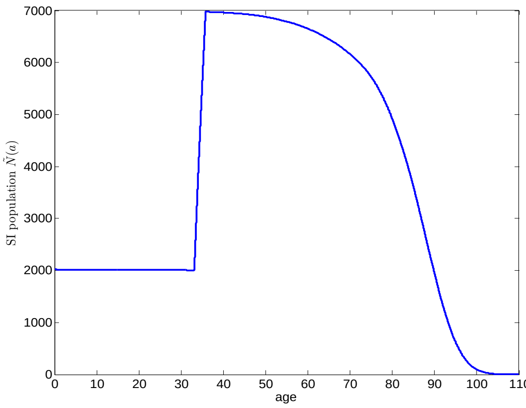

Figure 3.14: Scenario 2: The age structure of the SI population $N^*(\cdot)$ for Problem 2

In consequence of the below figures, we may conclude that if we target a relatively large population, and if the supply of acceptable immigrants is relatively low, namely two-times the actual figures, then immigrants with children (older than 9 years old) are appreciated. If a relatively small population is targeted, and if the supply of immigrants in all ages is high, i.e. $20M_{act}(a)$, then optimal immigration is concentrated in the mid thirties and immigrants with children are not appreciated.

Moreover, we find that by considering Problem 2 the peculiar result of old-age immigration does not appear anymore.

3.5 Direct and indirect effect of an additional individual

The adjoint variable in Problem 1 and Problem 2, $\xi(a)$, may also be interpreted as shadow price of the corresponding state variable, $N(a)$. This means that it reflects the decrease of the dependency ratio, when the optimal age structure of the population is marginally increased at age a, roughly speaking, when the population is increased by one a-year-old. A positive value of the adjoint variable means a decrease in the dependency ratio. In this dynamic set up, a change of the (optimal) age structure at one particular age, also affects the age structure at other ages.

This shadow price, see equation (3.21) and (3.26), consists of two parts

$$\xi(a) = \xi^d(a) + \lambda_1 v(a).$$

The *direct* effect, $\xi^d(a)$, represents the marginal value of an individual of age a given by her participation in the labor force: For Problem 1

$$\xi^d(a) = \frac{(D(M^*(\cdot)) + 1)}{N_{\text{tot}}(M^*(\cdot))}\big((D(M^*(\cdot)) + 1)e_{[\alpha,\beta]}(a) - e_{[0,\omega]}(a)\big),$$

holds and correspondingly for Problem 2

$$\xi^d(a) = e_{[\alpha,\beta]}(a) - \lambda_2 e_{[0,\omega]}(a),$$

holds. The direct effect accounts positively for her expected remaining years in $[\alpha, \beta]$ and negatively for her remaining life expectancy in $[0, \alpha]$ (for $a \leq \alpha$) and $[\beta, \omega]$.

The *indirect* effect of an a-year-old, $\lambda_1 v(a)$, is her reproductive value, i.e. the number of expected future daughters, weighted by the shadow price of newborns, λ_1, since $\xi(0) = \lambda_1$. Therefore, the indirect effect can be interpreted as the value of expected future births of an a-year-old in units of the dependency ratio. This is a generalization of the interpretation of the reproductive value, cf. Fisher (1930); Wrzaczek et al. (2010). Note, that the indirect effect can also be negative, namely when an additional newborn is negatively valued for the population.

For Problem 1, the Lagrange multiplier λ_2 may also be interpreted as the marginal effect on the dependency ratio when changing the total number of immigrants M_{tot}. Similarly, for Problem 2, λ_2 measures the effect of a marginal change of the prescribed population size N_{tot} on the dependency ratio.

In Figure 3.15 we plot the direct and indirect effect of an additional a-year-old separately. The figure corresponds again to the Austrian case for Problem 1, (3.1)–(3.5), where we set $M_{\text{tot}} = 50000$ and $\bar{M}(a) = 20M_{act}(a)$. The dotted line gives the weighted reproductive value, representing the indirect effect. The dashed line gives the direct effect. The sum of these two lines, by definition, exhibits $\xi(\cdot)$, which is depicted by the solid line.
As it can be seen in Figure 3.15, the indirect effect reduces the absolute value of the adjoint variable $\xi(\cdot)$ in early ages, preventing these ages to be optimal. Furthermore, this effect is zero for ages older than the maximum age of childbearing. Therefore, after this age the direct effect and the adjoint variable coincide. We also see from equation (3.21) that the direct effect always increases until age 20. This is due to the fact, that the remaining life expectancy decreases, implying a higher value of this individual in units of the dependency ratio and also because the ratio between the number of person-years lived in the working ages, $\int_\alpha^\beta l(x)\,\mathrm{d}x$, and the individual's probability to survive until age a, $l(a)$, increases with a.

Moreover, we see in Figure 3.15, that the direct effect reaches its maximum at age 20, since these individuals spend their whole working life in the receiving country, and then falls monotonically until age 65. However, the sharp increase in the indirect effect between ages $[20, 40]$ shifts the optimal age away from 20 and further to the right.

The increase of the direct effect after age 65 is due to the fact that the remaining life expectancy in $[0, \omega]$, which is the only term left in equation (3.21), is decreasing with age, and therefore the burden induced by these females on the dependency ratio is reduced.

Moreover, for the particular optimal control problem considered here it holds that for Problem 1, $\xi(a) - \lambda_2$, and for Problem 2, $\xi(a)$, give the decrease in the dependency ratio when changing the optimal age structure of immigrant inflows. So, under Policy 2 the shadow price is only a part of the total effect of an additional immigrant.

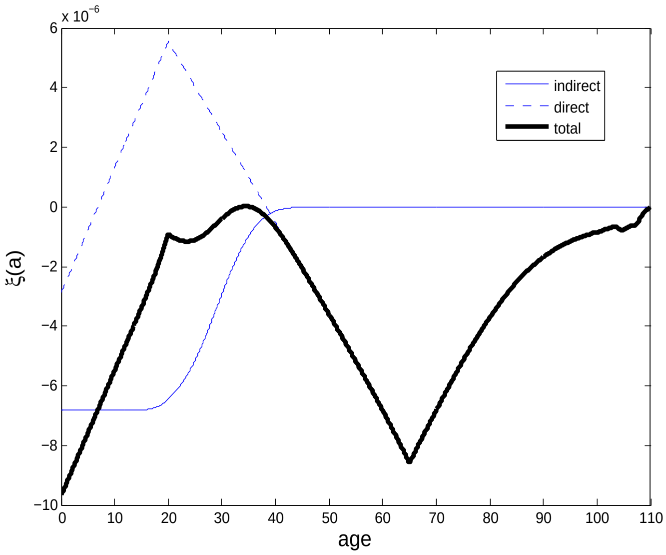

Figure 3.15: The direct (dashed line), indirect (solid blue line) and total (solid black line) effect of an additional a-year-old

3.6 Discussion

The aim of the present study was to determine the age-specific immigration policy that minimizes the dependency ratio in a population with below-replacement fertility assuming that the vital rates remain constant over time. We apply optimal control theory which is a rather new approach in demographic research. We assume that there are age-specific bounds that constrain the immigration profile from above. This is a reasonable assumption since it takes into account the present immigration profile that is hard to change drastically.

Two alternative policies are considered. In the first one, the total number of immigrants is prescribed. In the second one, the total population size is fixed while the rest of the model remains the same. It turns out that the solution exhibits a bang-bang behavior, which depends on the sign of the so-called switching function. The shape of the switching function with varying age a is determined by the adjoint variable.

In the model with a fixed total number of immigrants, it is shown that in the optimal solution there are ages in the vicinity of the maximum attainable age where immigration occurs. When we fix the total population size of the receiving country, the optimal solution is that immigration happens at not more than two separate age intervals and in ages younger than the retirement age. We present numerical results for a case study of the Austrian population based on demographic data from 2008 which underline our theoretical findings.

Moreover, by analyzing the shape of the switching function or, equivalently, the adjoint variable, and interpreting it as a shadow price, we determine the marginal value of an a-year-old individual in terms of the objective function.

3.7 Extensions

A straightforward extension is the study of the transitory case as it is investigated in Chapter 4. Similar as in Feichtinger and Veliov (2007), the resulting problem is a distributed control problem, which is formalized on infinite horizon. The state dynamics is a first order partial differential equation, which is of McKendrick-type Keyfitz (1977); Keyfitz and Keyfitz (1997). Although, the similarity in the structure of the problem indicates that as in Feichtinger and Veliov (2007) it holds that for stationary data, i.e. fertility and mortality rates, the optimal solution is also stationary, this result does not follow immediately and needs some deeper mathematical involvement as shown in Chapter 4. Therefore, optimality conditions for this distributed parameter control model have to be derived in order to carry out the relevant analysis.

Chapter 4

Optimal immigration in a fixed size population: A distributed control model

In this chapter the question of time-varying optimal choice of the immigration age-profile to a fixed size population within certain bounds is investigated. In Section 4.4 we formulate necessary optimality conditions of Pontryagin type for the resulting non-standard age-structured control system posed on an infinite horizon. In Section 4.5 we investigate stationarity, uniqueness and the structure of the optimal solution and finally in Section 4.6 we give numerical examples.

This chapter presents a joint work with Vladimir Veliov and Bernhard Skritek, published in Simon et al. (2013).

4.1 Introduction

In what follows, we consider a human population where immigration is allowed, although subjected to restrictions. Many countries face low fertility levels combined with an increase in life expectancy especially in older ages. These demographic developments influence the populations' well-being in many ways and lead for example to severe challenges for their social security systems. One possible way to counteract these developments is to steer immigration in an appropriate way.

It is assumed that the intensity of the migration inflow and, to a certain extend, the age-structure of the migrants can be used as control (policy) instruments. The problem we consider is to keep the size of the population constant by choosing immigration appropriately which, in addition, optimizes a certain objective function.

Of course, the problem is meaningful only if the population would steadily decrease without immigration. This means we consider a population with *below-replacement fertility*.

The subsequent model is related to the problem considered in Chapter 3, where a stationary population was considered and the optimal age-specific immigration profile that minimizes the dependency ratio while fixing either the population size or the immigration quota was determined. However, the model considered in this chapter is substantially more difficult. Here we move to non-stationary populations which led to the formulation and study of a distributed control problem on an infinite horizon. Hence, the population dynamics in this case can be modeled by an extension of the McKendrick-von Foerster equation, see Keyfitz and Keyfitz (1997).

From the mathematical point of view the considered problem is challenging for three reasons: (i) it has the form of a distributed optimal control problem with state constraints (although rather specific) given by the fixed size condition; (ii) the time horizon is infinite and a theory for infinite-horizon optimal control problems for age-structured systems is missing (Chan and Zhu (1990) and Feichtinger and Veliov (2007) are exceptions, as well as a few non-sound papers); (iii) we deal with a maximization problem for a non-concave functional, where the existence of a solution and the well-posedness are problematic.

As an application we consider the Austrian female population in 2009 and determine from the available data the age-specific mortality and fertility rates and the initial age structure of the population and of the immigration. Then we consider the age-profile of the immigration as a control (policy) variable, allowing for modifications of the age-profile from 2009. Using the analytical results which we will establish in this chapter below, we then determine numerically the immigration policy that maximizes the aggregate number of workers over time. We considered two scenarios. While in the first scenario we do not account for adaptation costs of immigrants, in the second scenario we do. It turns out that for both scenarios the optimal immigration intensity is at its upper bound on a single age-interval and on its lower bound at all other ages. In the first scenario, the optimal immigration age-pattern is such that it is optimal that immigrants from the age 20 to the mid-thirties migrate. For the scenario with immigration costs we observe that the optimal immigration age-pattern is moved to younger ages and even to ages before the youngest working age which was set to be 20.

For the presentation of the problem and the proofs below ideas from Feichtinger and Veliov (2007) are used, where the authors investigate the recruitment problem of organizations of fixed size. Like here, a distributed control problem is involved. However, the present problem is substantially more complicated due to the involvement of births in the boundary condition. As mentioned earlier, see for example Section 2.1 in Chapter 2, the considered problem exhibits a non-local boundary condition. This leads to the study of a system of integral equations when investigating the existence and uniqueness of a solution.

4.2 Population dynamics and preliminary statements

Subsequently, a particular linear age-dependent population model is posed and investigated. The optimization problem of which is then formulated in Section 4.3. Hence, the age-structured population dynamics, including initial and boundary conditions, and the fixed size condition are introduced and then existence and uniqueness are proven as well as some qualitative results of the solution are stated.

Note, that if not stated differently we follow the notations of Chapter 2. Below $t \geq 0$ denotes time, $a \geq 0$ denotes age and $a \mapsto N(t,a) \geq 0$ is the (non-probabilistic) age-density of a population. The mortality and the fertility rate at age a are denoted by $\mu(a)$ and $f(a)$, respectively. The immigration flux (number of immigrants) at time t will be denoted by $R(t) \geq 0$, and the immigration age-density by $u(t,a)$, $a \in [0,\infty)$ and $t \in [0,\infty)$. That is, u satisfies

$$u(t,a) \geq 0, \quad \int_0^\infty u(t,a)\,\mathrm{d}a = 1. \tag{4.1}$$

Hence, $R(t)u(t,a)$ is the flow of immigrants of age a at time t. Then the evolution of the population is described as

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right) N(t,a) = -\mu(a)\, N(t,a) + R(t)\, u(t,a), \quad t,\, a \geq 0, \tag{4.2}$$

$$N(0,a) = N_0(a), \quad a \geq 0, \tag{4.3}$$

$$N(t,0) = B(t), \quad t \geq 0, \tag{4.4}$$

where $N_0(\cdot)$ is the initial population density and

$$B(t) := \int_0^\infty f(a)\, N(t,a)\,\mathrm{d}a. \tag{4.5}$$

are the births in the population. The formal meaning of these equations will be given below.

Now we pass to a strict formulation of the previous consideration, starting with some basic assumptions (BA) on the data.

(BA) The functions μ, f, $N_0 : [0,\infty) \to \mathbf{R}$ are assumed to be Lipschitz continuous and otherwise given as stated in Section 2.1 of Chapter 2. Mortality μ satisfies $\mu(a) \geq \mu_0 > 0$ for all sufficiently large a; $f(a)$ and $N_0(a)$ are equal to zero for all sufficiently large a; there is $a_0 \geq 0$ such that $f(a_0) > 0$ and $N_0(a) > 0$ for $a \in [0,a_0]$; N_0 satisfies $\int_0^\infty N_0(a)\,\mathrm{d}a = \bar{N}$ with some positive $\bar{N} < \infty$.

Remark 4.1. The Lipschitz continuity assumption is made just for technical convenience and can be relaxed. The remaining assumptions about the fertility $f(a)$, the present age-density of the population, $N_0(a)$, and that the population is non-void until some fertile age a_0 are factual. The boundedness assumption for the mortality is justified according to Section 2.1 in Chapter 2.

Recall the definition of the domains D and D_T in (2.5) of Section 2.1. Then, let $u : D \to \mathbf{R}$ be an immigration age-profile, that is u is measurable and locally bounded and satisfies (4.1). Moreover, let $R : [0,\infty) \to \mathbf{R}$ be also measurable and locally bounded.

Function N is assumed to belong to the function space $\mathcal{N}$ as defined in Section 2.1. Then, by definition $N \in \mathcal{N}$ is a solution of (4.2)–(4.4) if the equations are satisfied almost everywhere with $\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right) N$ interpreted as $\mathcal{D}N$. Notice that for functions from $\mathcal{N}$ the right-hand side of (4.4) makes sense, and that the traces $N(0,\cdot)$ and $N(\cdot,0)$ are a.e. well-defined and measurable (see Feichtinger et al. (2003) for more details). The above definition of a solution is equivalent to the ones commonly used in the literature, e.g. Anita (2000); Webb (1985).

After defining a solution we are now able to discuss its uniqueness. Hence, the following lemma states the uniqueness of a solution of the above problem:

Lemma 4.1. *Let u and R be fixed as above. Then system (4.2)–(4.4) has a unique solution $N \in \mathcal{N}$ and $N \in L_\infty(D_T)$ for every $T > 0$. The function B is locally bounded.*

The proof of Lemma 4.1 is omitted as it is essentially the same as that of Lemma 4.2, but easier, since R is given and we deal with only one Volterra equation – that for B. Hence, see the proof of Lemma 4.2 for more details.

Given an immigration profile, $u(t,a)$, one can always keep the size of the population constant (equal to $\bar{N} := \int_0^\infty N_0(a)\,\mathrm{d}a$) by an appropriate choice of the immigration intensity, namely by choosing $R(t)$ in the feedback form

$$R(t) := \int_0^\infty (\mu(a) - f(a))\, N(t,a)\,\mathrm{d}a. \tag{4.6}$$

Hence, in parallel we may consider the system

$$\mathcal{D}N(t,a) = -\mu(a)\,N(t,a) + \int_0^\infty (\mu(s) - f(s))\,N(t,s)\,\mathrm{d}s\,u(t,a) \tag{4.7}$$

with side conditions (4.3) and (4.4). The meaning of a solution $N \in \mathcal{N}$ is the same as for (4.2)–(4.4), regarding the fact that the integral on the right-hand side of (4.7) is well-defined and finite due to the fact that N belongs to $\mathcal{N}$. Then, the following holds:

Lemma 4.2. *Let u be fixed as above. Then equation (4.7) with side conditions (4.3)–(4.4) has a unique solution $N \in \mathcal{N}$ and N is (essentially) bounded on every subset $D_T \subset D$, $0 < T < \infty$. Moreover, the functions R and B, defined by (4.6) and (4.5), are locally Lipschitz continuous.*

Proof of Lemma 4.2. Let us start with the uniqueness. Let $N \in \mathcal{N}$ be a solution of (4.3), (4.4), (4.7). Let $R(t)$ and $B(t)$ be defined by (4.5) and (4.6), respectively. Both are measurable and locally bounded, according to the properties $\mathcal{N}$.

The function N has the following representation, resulting from solving (4.2) along the characteristic lines:

$$N(t,a) := \begin{cases} e^{-\int_0^a \mu(\tau)\,\mathrm{d}\tau} B(t-a) + \int_{t-a}^t e^{-\int_{a-t+s}^a \mu(\tau)\,\mathrm{d}\tau} R(s)\,u(s,a-t+s)\,\mathrm{d}s & \text{if } a < t, \\ e^{-\int_{a-t}^a \mu(\tau)\,\mathrm{d}\tau} N_0(a-t) + \int_0^t e^{-\int_{a-t+s}^a \mu(\tau)\,\mathrm{d}\tau} R(s)\,u(s,a-t+s)\,\mathrm{d}s & \text{if } a \geq t, \end{cases} \tag{4.8}$$

for $t \in [0,\infty)$. Inserting this expression for N in (4.5) and (4.6) and changing the order of integration in the double integrals we obtain the following system of Volterra equations of the second kind for B and R:

$$\begin{aligned} B(t) &= \int_0^t R(s) \int_0^\infty e^{-\int_a^{a+t-s} \mu(\tau)\,\mathrm{d}\tau} f(a+t-s)\, u(s,a)\,\mathrm{d}a\,\mathrm{d}s \\ &\quad + \int_0^t B(s)\, e^{-\int_0^{t-s} \mu(\tau)\,\mathrm{d}\tau} f(t-s)\,\mathrm{d}s + \int_0^\infty e^{-\int_s^{t+s} \mu(\tau)\,\mathrm{d}\tau} f(s+t)\, N_0(s)\,\mathrm{d}s, \\ R(t) &= \int_0^t R(s) \int_0^\infty e^{-\int_a^{a+t-s} \mu(\tau)\,\mathrm{d}\tau} \nu(a+t-s)\, u(s,a)\,\mathrm{d}a\,\mathrm{d}s \\ &\quad + \int_0^t B(s)\, e^{-\int_0^{t-s} \mu(\tau)\,\mathrm{d}\tau} \nu(t-s)\,\mathrm{d}s + \int_0^\infty e^{-\int_s^{t+s} \mu(\tau)\,\mathrm{d}\tau} \nu(s+t)\, N_0(s)\,\mathrm{d}s, \end{aligned} \tag{4.9}$$

where $\nu(a) := \mu(a) - f(a)$. The system can be written as

$$x(t) = \int_0^t k(t,s)x(s)\,\mathrm{d}s + \Phi(t),$$

where $x = (B, R)$ and the kernel of this system $k(t,s) = \big(k_{i,j}(t,s)\big)$ is given by

$$\begin{aligned} k_{1,1}(t,s) &= e^{-\int_0^{t-s} \mu(\tau)\,\mathrm{d}\tau} f(t-s), \\ k_{1,2}(t,s) &= \int_0^\infty u(s,a)\, e^{-\int_a^{a+t-s} \rho(\tau)\,\mathrm{d}\tau} f(a+t-s)\,\mathrm{d}a, \\ k_{2,1}(t,s) &= \int_0^\infty u(s,a)\, e^{-\int_a^{a+t-s} \rho(\tau)\,\mathrm{d}\tau} \nu(a+t-s)\,\mathrm{d}a, \\ k_{2,2}(t,s) &= e^{-\int_0^{t-s} \rho(\tau)\,\mathrm{d}\tau} \nu(t-s), \end{aligned}$$

and

$$\Phi(t) = \begin{pmatrix} \int_t^\infty e^{-\int_0^{s-t} \mu(\tau)\,\mathrm{d}\tau} f(s) N_0(s-t)\,\mathrm{d}s \\ \int_t^\infty e^{-\int_0^{s-t} \mu(\tau)\,\mathrm{d}\tau} \nu(s) N_0(s-t)\,\mathrm{d}s \end{pmatrix}.$$

Notice that all the four components of $k(t,s)$ are bounded due to the properties of u and the data. Indeed, take for example the most complicated component

$$\begin{aligned} &\left| \int_0^\infty e^{-\int_a^{a+t-s} \mu(\tau)\,\mathrm{d}\tau} \nu(a+t-s)\, u(s,a)\,\mathrm{d}a \right| \\ &\le \sup_{a\ge 0} \left\{ e^{-\int_a^{a+t-s} \mu(\tau)\,\mathrm{d}\tau} |\nu(a+t-s)| \right\} \int_0^\infty u(s,a)\,\mathrm{d}a \\ &\le \bar{\nu}, \quad 0 \le s \le t < \infty, \end{aligned}$$

where $\bar{\nu} = \sup_{a\ge 0} |\nu(a)| < \infty$.

According to Theorems 5.4 and 5.5 in Chapter 9 of Gripenberg et al. (1990) this system has a unique locally bounded solution $(B,\, R)$, so that B and R are uniquely determined, hence N is also uniquely determined by (4.8).

On the other hand, from the existence of the locally bounded solution $(B,\, R)$ we obtain a function N from (4.8). Due to the local boundedness of B and R and due to $\int_0^\infty N_0(a)\,\mathrm{d}a = \bar{N}$, we have that $N \in \mathcal{N}$. It is straightforward to check that N satisfies (4.3), (4.4), (4.7) which proves the existence.

It remains to prove that the functions R and B are locally Lipschitz continuous. We have

$$\begin{aligned} B(t+h) - B(t) &= \int_0^\infty f(a)(N(t+h,a) - N(t+h,a+h))\,\mathrm{d}a \qquad (4.10) \\ &+ \int_0^\infty f(a)(N(t+h,a+h) - N(t,a))\,\mathrm{d}a. \end{aligned}$$

Notice that both N and $\mathcal{D}N$ are bounded on every set D_T. Then the second integral is proportional to h because of the absolute continuity of N along the characteristics. For the first integral it holds that

$$\begin{aligned} &\int_0^\infty f(a)(N(t+h,a) - N(t+h,a+h))\,\mathrm{d}a \\ &= \int_0^h f(a)\,N(t+h,a)\,\mathrm{d}a + \int_h^\infty f(a)\,N(t+h,a)\,\mathrm{d}a - \int_0^\infty f(a)\,N(t+h,a+h)\,\mathrm{d}a \\ &= \int_0^h f(a)\,N(t+h,a)\,\mathrm{d}a + \int_0^\infty (f(a+h) - f(a))\,N(t+h,a+h)\,\mathrm{d}a. \end{aligned}$$

The last integral is proportional to h due to the Lipschitz continuity of f and the boundedness of N. Therefore, the Lipschitz continuity follows. The proof for R is analogous. Q.E.D.

The fact that R satisfies (4.6) has an obvious demographic meaning and can be established by integration of equation (4.2) with respect to a, provided that N is a differentiable function and the equation is satisfied in the classical sense:

$$\int_0^\infty \frac{\partial}{\partial a} N(t,a)\,\mathrm{d}a + \int_0^\infty \frac{\partial}{\partial t} N(t,a)\,\mathrm{d}a = -\int_0^\infty \mu(a)N(t,a)\,\mathrm{d}a + R(t)\int_0^\infty u(t,a)\,\mathrm{d}a.$$

Since $\int_0^\infty u(t,a)\,\mathrm{d}a = 1$ holds and additionally the size of the population is constant over time, namely, $\int_0^\infty N(t,a)\,\mathrm{d}a = \bar{N}$, we obtain

$$\lim_{a\to\infty} N(t,a) - N(t,0) = -\int_0^\infty \mu(a)N(t,a)\,\mathrm{d}a + R(t),$$

and finally $N(t,0) = \int_0^\infty f(a)N(t,a)\,\mathrm{d}a$ gives

$$R(t) = \int_0^\infty (\mu(a) - f(a))N(t,a)\,\mathrm{d}a.$$

However, differentiability is not necessarily the case which might be caused by an inconsistency of the initial and the boundary conditions (4.3), (4.4), or by discontinuities of u. It turns out that the optimal control for the problem described in the next section is indeed discontinuous indeed. Therefore, we give a strict proof of the fact that$R$ satisfies (4.6) is equivalent to the fixed size condition in the subsequent Lemma:

Lemma 4.3. *Let u and R be as above and let N be the unique solution of (4.2)–(4.4). Then the population N has a fixed size (that is, $\int_0^\infty N(t,a)\,\mathrm{d}a = \bar{N}$) if and only if the function R satisfies (4.6). In this case N coincides with the unique solution of (4.7) with side conditions (4.3)–(4.4).*

Proof of Lemma 4.3. First, let us show that the function $t \mapsto \bar{N}(t) := \int_0^\infty N(t,a)\,\mathrm{d}a$ is locally Lipschitz and thus almost everywhere differentiable. Hence, consider the following equation

$$\begin{aligned}
&\int_0^\infty N(t+h,a)\,\mathrm{d}a - \int_0^\infty N(t,a)\,\mathrm{d}a = \\
&= \int_0^\infty [N(t+h,a),a+h) - N(t+h,a)]\,\mathrm{d}a + \int_0^\infty [N(t+h,a) - N(t+h,a+h)]\,\mathrm{d}a.
\end{aligned} \tag{4.11}$$

Then, the first integral is Lipschitz because of the absolute continuity of N along the characteristic lines. For the second integral it holds that

$$\begin{aligned}
&\int_0^\infty [N(t+h,a) - N(t+h,a+h)]\,\mathrm{d}a = \int_0^\infty N(t+h,a)\,\mathrm{d}a - \int_h^\infty N(t+h,a)\,\mathrm{d}a \\
&= \int_0^h N(t+h,a)\,\mathrm{d}a,
\end{aligned}$$

and hence Lipschitz continuity follows from the boundedness of N.

The above function being almost everywhere differentiable, gives that the fixed size property is equivalent to $\frac{\mathrm{d}}{\mathrm{d}t}\bar{N}(t) = 0$ for a.e. t. The latter is equivalent to having the weak derivative of $\bar{N}(\cdot)$ equal to zero. That is, having

$$\int_0^\infty \Psi(t)\frac{\mathrm{d}}{\mathrm{d}t}\bar{N}(t)\,\mathrm{d}t = 0$$

for every $\Psi(t) \in C_0^\infty(0,\infty)$ (the space of all infinitely differentiable function with compact support and $\Psi(0) = 0$).

We have

$$\int_0^\infty \Psi(t)\frac{\mathrm{d}}{\mathrm{d}t}\bar{N}(t)\,\mathrm{d}t = [\Psi(t)\bar{N}(t)]_{t=0}^\infty - \int_0^\infty \bar{N}(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t)\,\mathrm{d}t.$$

The first term is zero because of the properties of $\Psi(t)$. The second we can rewrite along the characteristic lines of (4.2) as follows:

$$\begin{aligned}
&\int_0^\infty \Psi(t)\frac{\mathrm{d}}{\mathrm{d}t}\bar{N}(t)\,\mathrm{d}t = -\int_0^\infty\int_0^\infty N(t,a)\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{d}a\,\Psi(t)\,\mathrm{d}t \\
&= -\int_0^\infty\int_0^\infty N(s,\tau+s)\frac{\mathrm{d}}{\mathrm{d}s}\Psi(s)\,\mathrm{d}s\,\mathrm{d}\tau - \int_0^\infty\int_0^\infty N(\tau+s,s)\frac{\mathrm{d}}{\mathrm{d}s}\Psi(\tau+s)\,\mathrm{d}s\,\mathrm{d}\tau.
\end{aligned}$$

Integrating again by parts and using (4.4) we obtain

$$\begin{aligned}
\int_0^\infty \Psi(t)\frac{\mathrm{d}}{\mathrm{d}t}\bar{N}(t)\,\mathrm{d}t \quad = \quad & -\left[\int_0^\infty N(s,\tau+s)\Psi(s)\right]_0^\infty \mathrm{d}\tau \\
& + \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty N(s,\tau+s)\,\Psi(s)\,\mathrm{d}s\,\mathrm{d}\tau \\
& - \left[\int_0^\infty N(\tau+s,s)\Psi(\tau+s)\right]_0^\infty \mathrm{d}\tau \\
& + \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty N(\tau+s,s)\,\Psi(\tau+s)\,\mathrm{d}s\,\mathrm{d}\tau.
\end{aligned} \tag{4.12}$$

The fact that $N(\tau,0) = \int_0^\infty f(a)N(\tau,a)\,\mathrm{d}a$ gives

$$\begin{aligned}\int_0^\infty \Psi(t)\frac{\mathrm{d}}{\mathrm{d}t}\bar{N}(t)\,\mathrm{d}t &= \int_0^\infty\int_0^\infty f(a)N(\tau,a)\,\mathrm{d}a\,\Psi(\tau)\,\mathrm{d}\tau \\ &+ \int_0^\infty\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}N(s,\tau+s)\,\Psi(s)\,\mathrm{d}s\,\mathrm{d}\tau \\ &+ \int_0^\infty\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}N(\tau+s,s)\,\Psi(\tau+s)\,\mathrm{d}s\,\mathrm{d}\tau.\end{aligned}$$

Using (4.2) and rewriting the integral again in the (t,a)-plane we obtain

$$\int_0^\infty \Psi(t)\frac{\mathrm{d}}{\mathrm{d}t}\bar{N}(t)\,\mathrm{d}t = \int_0^\infty\int_0^\infty \Psi(t)[-\mu(a)N(t,a) + f(a)N(t,a) + R(t)u(t,a)]\,\mathrm{d}a\,\mathrm{d}t.$$

The fact $\int_0^\infty u(t,a)\,\mathrm{d}a = 1$ and the arbitrary choice of $\Psi \in C_0^\infty(0,\infty)$ imply that the left hand side is zero if and only if $R(t) = \int_0^\infty [\mu(a) - f(a)]N(t,a)\,\mathrm{d}a$.

The last claim of the lemma is evident. Q.E.D.

Note, that since we consider a human population, negative values for R and N make no sense. However, so far it is not clear whether the solution is non-negative. In the following lemma, boundedness and non-negativity of the involved functions N, B and R are investigated:

Lemma 4.4. *Let u be fixed as above. Assume that for the unique solution $N \in \mathcal{N}$ of (4.7) with side conditions (4.3)–(4.4) it holds that $R(t) \geq 0$ for every $t \geq 0$, where R is defined by (4.6). Then the functions N, B and R are non-negative and bounded, uniformly with respect to u as above for which the assumption $R(t) \geq 0$ is fulfilled.*

Proof of Lemma 4.4. First we shall prove that under the conditions in Lemma 4.4 we have $B(t) \geq 0$ for all $t \geq 0$. We recall that B is a continuous function due to Lemma 4.2. Moreover, from assumption (BA) we have $B(0) = \int_0^\infty f(s)N_0(s)\,\mathrm{d}s > 0$. Denote

$$\theta = \sup\{t \geq 0 :\ B(s) > 0 \text{ on } [0,t)\}.$$

Assume that θ is finite (otherwise we are done). Then $B(\theta) = 0$, where $B(\theta) = \int_0^\infty f(s)N(\theta,s)\,\mathrm{d}s$. On the other hand we have from the presentation of the solution $N(t,a)$ in (4.8) that

$$\begin{aligned}B(\theta) &= \int_0^\theta f(s)B(\theta-s)e^{-\int_0^s \mu(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s \\ &+ \int_0^\theta\int_{\theta-s}^\theta e^{\int_{s-\theta+a}^s \mu(\tau)\,\mathrm{d}\tau}R(a)u(a,s-\theta+a)\,\mathrm{d}a\,\mathrm{d}s \\ &+ \int_0^\infty N_0(\theta-s)e^{-\int_{s-\theta}^s \mu(\tau)\,\mathrm{d}\tau}f(s)\,\mathrm{d}s \\ &+ \int_\theta^\infty\int_0^\theta e^{-\int_{s-\theta+a}^s \mu(\tau)\,\mathrm{d}\tau}R(a)u(a,s-\theta+a)\,\mathrm{d}a\,\mathrm{d}s\end{aligned}$$

and hence, we observe that

$$0 = B(\theta) \geq \int_0^\theta B(s)\,e^{-\int_0^{\theta-s}\mu(\tau)\,\mathrm{d}\tau}f(\theta-s)\,\mathrm{d}s + \int_0^\infty e^{-\int_s^{\theta+s}\mu(\tau)\,\mathrm{d}\tau}f(s+\theta)\,N_0(s)\,\mathrm{d}s, \quad (4.13)$$

where we use the assumption $R(t) \geq 0$. Obviously both terms are non-negative. We shall show that at least one of them is strictly positive, which contradicts (4.13). If $f(a) > 0$ for some $a \in [0, \theta)$, then the first integral in (4.13) is strictly positive since $B(s) > 0$ on $[0, \theta)$. Alternatively, let $f(a) = 0$ for all $a \in [0, \theta)$. Then $a_0 \geq \theta$ (see assumption (BA)). Take $s = a_0 - \theta$. Then the integrand $e^{-\int_s^{\theta+s} \mu(\tau)\, d\tau} f(s+\theta)\, N_0(s)$ is strictly positive (see (BA)), hence the second integral in (4.13) is strictly positive, too. The obtained contradiction proves that $B(t) \geq 0$ for all $t \geq 0$. From (4.8) it follows also that $N(t, a) \geq 0$.

Then the boundedness follows:

$$
\begin{aligned}
R(t) &\leq \int_0^\infty |\mu(a) - f(a)|\, |N(t,a)|\, da \leq (\bar{\mu} + \bar{f}) \int_0^\infty |N(t,a)|\, da \\
&= (\bar{\mu} + \bar{f}) \int_0^\infty N(t,a)\, da = (\bar{\mu} + \bar{f})\, \bar{N},
\end{aligned}
$$

where $\bar{f}$ is the upper bound for f. The same argument proves also boundedness of B. Then the boundedness of $N(t, a)$ follows from (4.8) and the fact that $u(t, a) = 0$ for all sufficiently large a. Q.E.D.

We mention that the requirement $R(t) \geq 0$ is related, but not necessarily implied by the standard below-replacement condition, see the discussion in the beginning of the next section.

4.3 The optimization problem

The main aim in this chapter is to determine optimal age patterns of immigrants in a population of fixed size. The specific optimization problem that we introduce below arises only for populations that need a positive immigration in order to sustain their size, as it is the case for most European countries. Many of these countries face below-replacement fertility,

$$
\int_0^\infty f(a)\, e^{-\int_0^a \mu(\theta)\, d\theta}\, da < 1, \tag{4.14}
$$

for a long period. Below replacement fertility implies extinction of the population without immigration and thus immigration is needed to sustain the size. However, it does not imply that the population, i.e. N, will decrease in the short run, therefore negative immigration, i.e. emigration, $R(t) < 0$, may be needed for some t (see Keyfitz (1971)). In Figure 1-3 we provide an illustrative example which shows that the immigration rate $R(t)$ determined by (4.6) takes negative values for some t several generations after the initial time, although fertility and mortality satisfy condition (4.14). Figure 1 presents the fertility $f(a)$, mortality $\mu(a)$ and the immigration profile $u(a)$ for this stylized example. The initial population $N_0(a)$ and the population $N(T, a)$ at time $T = 90$ are depicted in Figure 2.

Hence, Figure 4.3 shows that below-replacement fertility is not enough to guarantee positive immigration, i.e. $R(t) > 0$. Since in the present model we use immigration as a

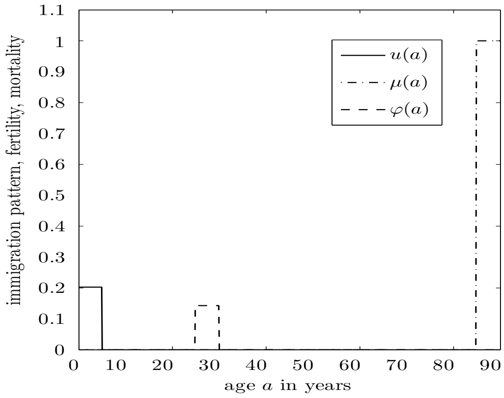

Figure 4.1: Functions $f(a)$ (dashed line), $\mu(a)$ (dashed-dotted line) and $u(a)$ (solid line)

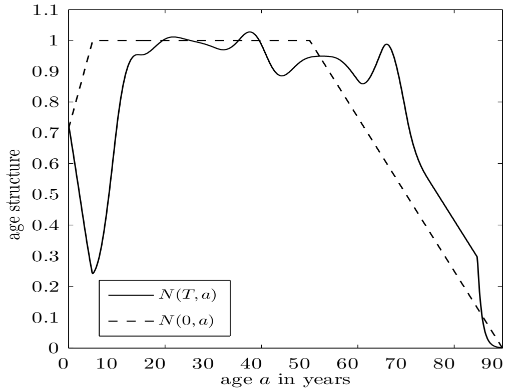

Figure 4.2: The initial $N_0(a)$ (dashed line) and the age structure $N(T, a)$ (solid line) at $T = 90$

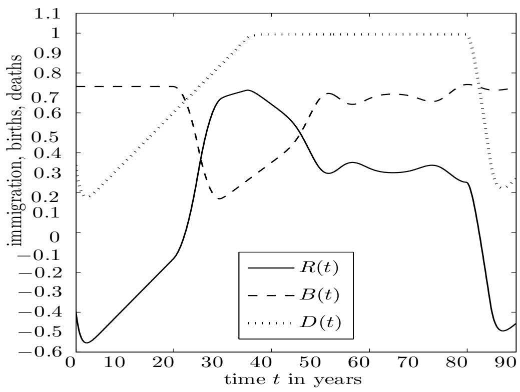

Figure 4.3: The number of births $B(t)$ (dashed line), the number of deaths $D(t)$ (dotted line) and the number of immigrants $R(t)$ (solid line) over time

policy instrument and negative immigration is not admissible, we have to eliminate this possibility by introducing assumption (A2) below, which is stronger than the below-replacement fertility condition.

Clearly, in practice, discrimination of immigrants will not happen based on age only. Therefore, we introduced certain bounds on the immigration profile. Nevertheless, the age of applicants for a visa is taken into account, for example, by the Australian authorities[1]. There, in the skilled point test, 60 points are needed for a working permit and 30 of those can be gained by being a member of the age group ranging from 25 to 29, while for age 45+ zero points are awarded.

Let $m(a)$ be the present immigration profile, that is, at time $t = 0$, which is historically determined by habits, policies or other factors. Then the present normalized

[1] www.visabureau.com/australia

age-density of immigration is given by

$$\hat{u}(a) := \frac{m(a)}{\int_0^\infty m(a)\,\mathrm{d}a}. \tag{4.15}$$

Therefore, when using the age-density of the immigration as a control (policy) variable we can implement only slight changes in $\hat{u}(a)$. For this reason we consider control constraints of the form $\underline{u}(a) \le u(t,a) \le \bar{u}(a)$, where the lower and the upper bounds are not much different from the present values $\hat{u}(a)$, say $\underline{u}(a) = (1-\varepsilon)\hat{u}(a)$ and $\bar{u}(a) = (1+\varepsilon)\hat{u}(a)$ with some small $\varepsilon > 0$.

The optimization problem we consider in what follows is:

$$\max_{R,u} \int_0^\infty e^{-rt} \left[\int_0^\infty p(a)\, N(t,a)\,\mathrm{d}a - q\, R(t) \right] \mathrm{d}t, \tag{4.16}$$

subject to

$$\mathcal{D}N(t,a) \;=\; -\mu(a)N(t,a) + R(t)\,u(t,a), \quad (t,a) \in D, \tag{4.17}$$

$$N(0,a) = N_0(a), \quad a \ge 0, \tag{4.18}$$

$$N(t,0) = \int_0^\infty f(a)\, N(t,a)\,\mathrm{d}a, \quad t \ge 0, \tag{4.19}$$

$$\int_0^\infty N(t,a)\,\mathrm{d}a = \bar{N}, \tag{4.20}$$

$$\underline{u}(a) \le u(t,a) \le \bar{u}(a), \quad \int_0^\infty u(t,a)\,\mathrm{d}a = 1, \tag{4.21}$$

$$R(t) \ge 0. \tag{4.22}$$

Hence, we end up with a distributed control problem where the age-specific immmigration $u(t,a)$ is the control variable and the population density $N(t,a)$ is the state variable. See also Chapter 2 for a short discussion on distributed control models. Here, function $p(a)$ is a weight function that is higher if people of a certain age are more valuable from point of view of the policy maker solving this optimization problem. For example, $p(a)$ could be the function taking the value 1 for ages $a \in [20,65]$, representing the working ages, and 0 otherwise. The second term penalizes the size of immigration (if $q > 0$). Equation (4.20) represents a state constraint. The intertemporal discount rate is r. The constant q represents the benefits or costs of immigration arising, for example, from possible integration or education expenditures. If $q = 0$ then maximizing the performance value is related to minimizing the dependency ratio of the population (considered in a steady state in Chapter 3), that is the fraction of non-workers to workers in a population, which is a measure of how solvent a social security system is.

Additionally to (BA) we make the following assumptions.

(**A1**) The discount rate r is strictly positive, the function $p : [0,\infty) \to \mathbf{R}$ is measurable and bounded; q is a real number; the functions $\underline{u}$, $\bar{u} : [0,\infty) \to \mathbf{R}$ are measurable and bounded, and satisfy the relations $0 \leq \underline{u}(a) \leq \bar{u}(a)$ for all $a \geq 0$, $\bar{u}(a) = 0$ for all sufficiently large a; moreover $\int_0^\infty \underline{u}(a)\, \mathrm{d}a < 1$ and $\int_0^\infty \bar{u}(a)\, \mathrm{d}a > 1$.

According to Lemma 4.3, we can reformulate problem (4.16)–(4.22) in the following way:

$$\max_{u \in \mathcal{U}} J(u) := \int_0^\infty e^{-rt} \left[\int_0^\infty [p(a) - q(\mu(a) - f(a))]\, N(t,a)\, \mathrm{d}a \right] \mathrm{d}t, \tag{4.23}$$

$$\mathcal{D}N(t,a) = -\mu(a)\, N(t,a) + u(t,a) \int_0^\infty (\mu(s) - f(s))\, N(t,s)\, \mathrm{d}s, \quad (t,a) \in D, \tag{4.24}$$

$$N(0,a) = N_0(a), \quad a \geq 0, \tag{4.25}$$

$$N(t,0) = \int_0^\infty f(a)\, N(t,a)\, \mathrm{d}a, \quad t \geq 0. \tag{4.26}$$

Here, the set of admissible controls $\mathcal{U}$ is defined as

$$\mathcal{U} := \left\{ u : D \to \mathbf{R} : \underline{u}(a) \leq u(t,a) \leq \bar{u}(a), \quad \int_0^\infty u(t,a)\, \mathrm{d}a = 1 \right\}. \tag{4.27}$$

Due to the last requirement in (A1) the set of admissible controls, $\mathcal{U}$, is nonempty.

The condition for a non-negative immigration rate, $R(t) \geq 0$, is disregarded in the above reformulation. It will be stipulated by the following additional assumption.

(**A2**) For any $u \in \mathcal{U}$ the immigration intensity R, defined by (4.6) for the corresponding solution of (4.24)–(4.26), is strictly positive for all t.

Even with below-replacement fertility, see (4.14), it could happen that $R(t) < 0$ for some t. We assume that for the present immigration pattern $\hat{u}(a)$ the resulting immigration size satisfies $\hat{R}(t) \geq R_0 > 0$. Implicitly this property requires that the initial density $N_0(a)$ results from a population which has experienced below-replacement fertility for quite a while before the present time $t = 0$. This is the situation in most of the European countries in the 21st century, for example, as in our case study in Section 4.6. We assume a bit more, namely that $R(t) > 0$ for any admissible control u, having in mind that all admissible controls are close to $\hat{u}$.

Existence of a solution of the optimization problem

Since $R(t) > 0$, Lemma 4.4 together with $r > 0$ and the boundedness of p, μ and f, imply that $J(u)$ is finite for every $u \in \mathcal{U}$ and that $\sup_{u \in \mathcal{U}} J(u)$ is finite. Thanks to this we can use the standard definition of optimality: $u \in \mathcal{U}$ is optimal if $J(u) \geq J(v)$ for every $v \in \mathcal{U}$. We mention that the proof of existence is not routine since we deal with a problem of maximization of a non-concave functional. Indeed, the mapping $\mathcal{U} \ni u \longrightarrow$ "objective value $J(u)$" is not concave, as argued in Feichtinger and Veliov (2007) even in the substantially simpler case $f = 0$.

Proposition 4.1. *Let assumptions (BA), (A1), (A2) be fulfilled. Then the optimal control problem* (4.23)–(4.26) *has a solution.*

Proof of Proposition 4.1. The proof is a modification of that in Feichtinger and Veliov (2007) while the underlying idea stams from Anita et al. (1998). Denote by $J(u)$ the objective value for an admissible control u. Due to Lemma 4.4 and $r > 0$, the value $J(u)$ is finite and uniformly bounded with respect to $u \in \mathcal{U}$. Then $J^* = \sup_{u \in \mathcal{U}} J(u)$ is also finite. Pick a maximizing sequence $\{u_k\}$ of admissible controls for which $J(u_k) \geq J^* - \frac{1}{k}$. Denote by N_k the corresponding solution of (4.24)–(4.27), and let R_k be defined as in (4.6). According to Assumption (A2) and Lemma 4.4, there is a constant C such that $0 \leq N_k(t,a) \leq C$ and $0 < R_k(t) \leq C$ almost everywhere and for all k.

The sequence $\{e^{-rt}N_k\}$ of elements of $L_1(D)$ is weakly relatively compact due to the Dunford-Pettis criterion.

Therefore, there exists a subsequence, which will also be denoted by N_k, such that $e^{-rt}N_k$ converges $L_1(D)$-weakly to some $e^{-rt}N^*$, and N^* is obviously bounded by the same constant C. According to Mazur's lemma there exist a sequence

$$e^{-rt}\tilde{N}_k := \sum_{i=k}^{n_k} p_i^k e^{-rt} N_i, \;\; p_i^k \geq 0, \;\; \sum_{i=k}^{n_k} p_i^k = 1,$$

that (strongly) converges to $e^{-rt}N^*$ in $L_1(D)$. Obviously for every $T > 0$ the sequence $\tilde{N}_k$ converges to N^* in $L_1(D_T)$. With the same weights p_i^k we define

$$\tilde{R}_k(t) := \sum_{i=k}^{n_k} p_i^k R_i(t) = \int_0^\infty (\mu(a) - f(a))\tilde{N}_k(t,a)\,\mathrm{d}a. \tag{4.28}$$

Since $R_k(t) > 0$ holds for all $k > 0$ and $t > 0$, this also holds for $\tilde{R}_k$ and we can define

$$\tilde{u}_k(t,a) := \frac{1}{\tilde{R}_k(t)} \sum_{i=k}^{n_k} p_i^k R_i(t) u_i(t,a).$$

Obviously $\tilde{u}_k$ is also an admissible control. Moreover we have that

$$\mathcal{D}\tilde{N}_k = \sum_{i=k}^{n_k} p_i^k(-\mu N_i + R_i u_i) = -\mu\tilde{N}_k + \tilde{R}_k\tilde{u}_k, \tag{4.29}$$

$$\tilde{N}_k(t,0) = \sum_{i=k}^{n_k} p_i^k N_i(t,0) = \sum_{i=k}^{n_k} p_i^k \int_0^\infty f(a)N_i(t,a)\,\mathrm{d}a = \int_0^\infty f(a)\tilde{N}_k(t,a)\,\mathrm{d}a, \tag{4.30}$$

which means that $(\tilde{u}_k, \tilde{N}_k)$ is an admissible control-trajectory pair for problem (4.23)–(4.27).

Since $\tilde{N}_k$ converges to N^* in $L_1(D_T)$, we may pass to an almost everywhere converging subsequence that we denote again by $\tilde{N}_k$. Moreover, we may assume (passing again to a subsequence) that $e^{-rt}\tilde{u}_k$ converges to some $e^{-rt}u^*$ weakly in $L_1(D)$. Now we will show that u^* is an admissible control. For every measurable and bounded set $\Gamma \subset [0,\infty)$ it holds that

$$\int_\Gamma \int_0^{\bar{a}} \tilde{u}_k(t,a)\,\mathrm{d}a\,\mathrm{d}t \to \int_\Gamma \int_0^{\bar{a}} u^*(t,a)\,\mathrm{d}a\,\mathrm{d}t$$

where $\bar{a}$ is such that $\bar{u}(a) = 0$ for $a \geq \bar{a}$, hence also $u^*(t,a) = 0$ (see (A1)). Since $\tilde{u}_k$ are admissible controls the left hand side is equal to meas(Γ), and thus also the right hand side. Since this holds for any measurable and bounded set Γ, this implies that $\bar{u}$ satisfies the integral constraint in (4.27). The inequality constraints are obviously also satisfied. Therefore, u^* is an admissible control. In the next paragraph we shall prove that the pair N^* solves (4.24)–(4.26) with $u = u^*$.

Let us define

$$R^*(t) = \int_0^\infty (\mu(a) - f(a))N^*(t,a)\,\mathrm{d}a.$$

Due to the pointwise convergence of $\tilde{N}_k$ in D_T we obtain by passing to a limit in (4.28) that $\tilde{R}_k(t) \longrightarrow R^*(t)$ for a.e. $t \in [0,T)$, and since T is arbitrary this holds for a.e. $t \geq 0$. Moreover, for a.e. t the mapping $[0, T-t] \ni s \longrightarrow \tilde{N}_k(t+s,s)$ is uniformly Lipschitz continuous. From here it easily follows (see Feichtinger et al. (2003) for more details) that $N^*(t,0)$ is well defined for a.e. t and $\tilde{N}_k(t,0) \longrightarrow N^*(t,0)$. Then by passing to a limit in (4.30) we obtain that N^* satisfies the boundary condition (4.26) for a.e. $t \in [0,T]$, hence for all a.e. $t \geq 0$. In a similar way one can prove that N^* satisfies the initial condition (4.25). In order to show that (4.24) is also satisfied we take an arbitrary measurable set $\Gamma \subset [0,T]$ integrate with respect to $a \in \Gamma$ the representation (4.8) of the solution $\tilde{N}_k$ for $u = \tilde{u}_k$. Due to the established properties we can pass on to the limit. Since Γ is arbitrary, we obtain that (u^*, N^*, R^*) satisfy (4.8) on D_T, hence N^* is a solution of (4.24) on D for $u = u^*$.
Thus (u^*, N^*) is an admissible control-trajectory pair.

Now let us show that $J(u^*) \geq J^*$. We have

$$\begin{aligned}
J(\tilde{u}_k) &= \int_0^\infty e^{-rt}\left[\int_0^\infty p(a)\tilde{N}_k(t,a)\,\mathrm{d}a - q\tilde{R}_k(t)\right]\mathrm{d}t \\
&= \int_0^\infty e^{-rt}\left[\int_0^\infty p(a)\sum_{i=k}^{n_k} p_i^k N_i(t,a)\,\mathrm{d}a - q\sum_{i=k}^{n_k} p_i^k R_i(t)\right]\mathrm{d}t \\
&= \sum_{i=k}^{n_k} p_i^k J(u_i) = \sum_{i=k}^{n_k} p_i^k\left(J^* - \frac{1}{i}\right) \geq J^* - \frac{1}{k}.
\end{aligned}$$

Using this we obtain

$$\begin{aligned}
J^* &\leq \limsup_k \left(J(\tilde{u}_k) + \frac{1}{k}\right) = \limsup_k J(\tilde{u}_k) \\
&= \limsup_k \int_0^\infty e^{-rt}\left[\int_0^\infty p(a)\tilde{N}_k(t,a)\,\mathrm{d}a - q\tilde{R}_k(t)\right]\mathrm{d}t \\
&= \int_0^\infty e^{-rt}\left[\int_0^\infty p(a)N^*(t,a)\,\mathrm{d}a - qR^*(t)\right]\mathrm{d}t \\
&= J(u^*).
\end{aligned}$$

Q.E.D.

4.4 Necessary optimality conditions

In this section we formulate and prove necessary optimality conditions of Pontryagin's type for problem (4.23)-(4.27). The problem at hand has a similar structure as those studied in Brokate (1985), Feichtinger et al. (2003), Veliov (2008), with the substantial difference that here the time-horizon is infinite. There are no results in the literature that provide necessary optimality conditions for age-structured optimal problems on infinite horizon, except the ones mentioned in the introduction, which are not applicable for problem (4.23)-(4.26).

The subsequent Pontryagin-type necessary optimality conditions involve (i) an appropriate *adjoint equation*; (ii) an appropriate *transversality condition* that uniquely determines a solution of the adjoint equation; (iii) a *maximization condition* for each t separately. The word "appropriate" in (i) and (ii) means that the maximization condition in (iii) holds true with the "appropriate" adjoint function.

The appropriate adjoint equation associated with our problem will be shown to have the form

$$\begin{aligned} \mathcal{D}\,\xi(t,a) \;&=\; (r+\mu(a))\,\xi(t,a) \;-\; f(a)\,\xi(t,0) \\ &\quad -(\mu(a)-f(a))\int_0^\infty \xi(t,\alpha)\,u(t,\alpha)\,\mathrm{d}\alpha \;-\;p(a)\;+\;q(\mu(a)-f(a)). \end{aligned} \tag{4.31}$$

A main challenge is to define an appropriate transversality condition. Following ideas originating in Aubin and Clarke (1979) and developed in Aseev and Veliov (2012) for ordinary differential systems and also in Feichtinger and Veliov (2007) for a problem which is similar but substantially simpler than (4.23)-(4.26), we introduce the "transversality" condition $\|\xi\|_\infty < \infty$. This is justified by Proposition 4.2 and Theorem 4.1 below.

Subsequently, we first show that the adjoint equation associated with our problem and assumed to be given by (4.31) has a unique bounded solution and then formulate the main result, i.e. the necessary optimality condition, in Theorem 4.1.

The adjoint equation has a unique bounded solution

We start with some preliminary results and an additional assumption.

In the following (u, N) denotes an optimal solution of problem (4.23)–(4.26) and R and B are the number of immigrants and births given by (4.6) and (4.5).

We introduce the notations

$$\rho(a) := r+\mu(a), \qquad \nu(a) := \mu(a)-f(a), \qquad \phi(a) := p(a)-q\,(\mu(a)-f(a))$$

and the auxiliary variables

$$\lambda(t) := \xi(t,0), \qquad \eta(t) := \int_0^\infty \xi(t,a)\,u(t,a)\,\mathrm{d}a. \tag{4.32}$$

Then the adjoint equation becomes

$$\mathcal{D}\,\xi(t,a) = \rho(a)\,\xi(t,a) - f(a)\,\lambda(t) - \nu(a)\,\eta(t) - \phi(a). \tag{4.33}$$

In the next lemma, we first show for arbitrary bounded functions $\lambda(t)$ and $\eta(t)$ that a unique solution of the adjoint equation exists.

Lemma 4.5. *Let assumptions (BA), (A1), (A2) be fulfilled. Then, for any given functions* λ, $\eta \in L_\infty(0,\infty)$ *equation (4.33) has a unique bounded solution on D and it is given by the formula*

$$\xi(t,a) = \int_a^\infty e^{-\int_a^s \rho(\theta)\,\mathrm{d}\theta}\,[f(s)\,\lambda(s+t-a) + \nu(s)\,\eta(s+t-a) + \phi(s)]\,\mathrm{d}s, \tag{4.34}$$

where the integral is absolutely convergent.

Proof. The integral in (4.34) is absolutely convergent and the function ξ is bounded due to the boundedness of the term in brackets and the inequality

$$\int_a^\infty e^{-\int_a^s \rho(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s \le \int_a^\infty e^{-r(s-a)}\,\mathrm{d}s = \frac{1}{r}.$$

One can verify by substitution that ξ defined by (4.34) satisfies (4.33).

To prove the uniqueness assertion we consider the difference $\Delta\xi(t,a)$ between two bounded solutions which is also bounded. It satisfies the equation

$$\mathcal{D}\,\Delta\xi(t,a) = \rho(a)\,\Delta\xi(t,a).$$

If $\Delta\xi(t,a) \neq 0$ for some t and a, then the function $x(s) := \Delta\xi(t+s, a+s)$, $s \geq 0$, satisfies $\dot{x}(s) = \rho(a+s)\,x(s)$ with $x(0) = \Delta\xi(t,a)$. Due to $\rho(a+s) \geq r > 0$, $x(s)$ is unbounded since $x(0) = \Delta\xi(t,a) \neq 0$. This contradiction completes the proof. Q.E.D.

By substituting (4.34) in (4.32) one obtains that ξ is a bounded solution of (4.31) if and only if it is the unique bounded solution of (4.34) with the functions λ and η determined as bounded solutions of the resulting system of equations after substitution. This system is given by equations (4.35) – (4.36):

$$\begin{aligned}
\lambda(t) &= \xi(t,0), \\
&= \int_0^\infty e^{-\int_0^s \rho(\theta)\,\mathrm{d}\theta}[f(s)\lambda(s+t) + \nu(s)\eta(s+t) + \phi(s)]\,\mathrm{d}s, \\
&= \int_t^\infty e^{-\int_0^{s-t} \rho(\theta)\,\mathrm{d}\theta}[f(s-t)\lambda(s) + \nu(s-t) + \eta(s) + \phi(s-t)]\,\mathrm{d}s. \qquad (4.35)
\end{aligned}$$

And similarly for η we obtain,

$$\begin{aligned}
\eta(t) \;=\; \int_t^\infty \int_0^\infty u(t,a)\,e^{-\int_a^{a+s-t} \rho(\tau)\,\mathrm{d}\tau}\,&[f(a+s-t)\,\lambda(s) \qquad (4.36) \\
&+\nu(a+s-t)\,\eta(s) + \phi(a+s-t)]\,\mathrm{d}a\;\mathrm{d}s.
\end{aligned}$$

Hence, one has to show that λ and η, solving (4.35)–(4.36), are unique.

In the following, we reformulate the above system of equation as

$$x(t) = \int_t^\infty k(t,s)x(s)\,\mathrm{d}s + \Phi(t), \tag{4.37}$$

where $x = (\lambda, \eta)$, $k(t,s) = \big(k_{i,j}(t,s)\big)$ is the matrix

$$
\begin{aligned}
k_{1,1}(t,s) &= e^{-\int_0^{s-t}\rho(\tau)\,\mathrm{d}\tau} f(s-t) && (4.38)\\
k_{1,2}(t,s) &= e^{-\int_0^{s-t}\rho(\tau)\,\mathrm{d}\tau} \nu(s-t) && (4.39)\\
k_{2,1}(t,s) &= \int_0^\infty u(t,a)\, e^{-\int_a^{a+s-t}\rho(\tau)\,\mathrm{d}\tau} f(a+s-t)\,\mathrm{d}a && (4.40)\\
k_{2,2}(t,s) &= \int_0^\infty u(t,a)\, e^{-\int_a^{a+s-t}\rho(\tau)\,\mathrm{d}\tau} \nu(a+s-t)\,\mathrm{d}a, && (4.41)
\end{aligned}
$$

and the inhomogeneity is

$$
\Phi(t) = \begin{pmatrix} \int_t^\infty e^{-\int_0^{s-t}\rho(\tau)\,\mathrm{d}\tau} \phi(s-t)\,\mathrm{d}s \\ \int_t^\infty \int_0^\infty u(t,a)\, e^{-\int_a^{a+s-t}\rho(\tau)\,\mathrm{d}\tau} \phi(a+s-t)\,\mathrm{d}a\ \mathrm{d}s \end{pmatrix}.
$$

Then, a key point in the subsequent analysis is to show that integral equation (4.37) has a unique bounded solution. (The precise statement is formulated in Lemma 4.6 below.)

This, however requires an additional assumption about the kernel $k(t,s)$, which is formulated in terms of the below numbers κ_{ij}:

$$
\kappa_{11} := \int_0^\infty e^{-\int_0^a \rho(\theta)\,\mathrm{d}\theta} f(a)\,\mathrm{d}a, \qquad \kappa_{12} := \int_0^\infty e^{-\int_0^a \rho(\theta)\,\mathrm{d}\theta} |\nu(a)|\,\mathrm{d}a, \tag{4.42}
$$

$$
\kappa_{21} := \max_{a\ge 0} \int_0^\infty e^{-\int_a^{a+\tau} \rho(\theta)\,\mathrm{d}\theta} f(a+\tau)\,\mathrm{d}\tau, \qquad \kappa_{22} := \max_{a\ge 0} \int_0^\infty e^{-\int_a^{a+\tau} \rho(\theta)\,\mathrm{d}\theta} |\nu(a+\tau)|\,\mathrm{d}\tau.
$$

In the proof of Lemma 4.6 it is shown that the following condition (A3) ensures that the integral operator in (4.37) is contractive in an appropriate norm:

(A3) The following inequality is fulfilled

$$
\frac{1}{2}\left[\kappa_{11} + \kappa_{22} + \sqrt{(\kappa_{11}-\kappa_{22})^2 + 4\,\kappa_{12}\,\kappa_{21}}\right] < 1. \tag{4.43}
$$

In the proof of Lemma 4.6 it is shown that the appropriate norm mentioned above is given by $\|(y_1,y_2)\| := \max\{\|y_1\|_{L_\infty}, \alpha\|y_2\|_{L_\infty}\}$ for an appropriate $\alpha > 0$. The proof also reveals the reason for assuming (A3).

Note, that **(A3)** imposes a condition solely on the data of the problem. This condition is not only used in the proof of the optimality condition below, but also for the characterization of the optimal solution in the next section (see the proof of Lemma 4.7). The results in these sections, however, are also valid under an alternative condition that involves the set of admissible controls and the numbers

$$
\begin{aligned}
\bar\kappa_{21} &:= \int_0^\infty \hat u(a) \int_0^\infty e^{-\int_a^{a+\tau}\rho(\theta)\,\mathrm{d}\theta} f(a+\tau)\,\mathrm{d}\tau\,\mathrm{d}a,\\
\bar\kappa_{22} &:= \int_0^\infty \hat u(a) \int_0^\infty e^{-\int_a^{a+\tau}\rho(\theta)\,\mathrm{d}\theta} |\nu(a+\tau)|\,\mathrm{d}\tau\,\mathrm{d}a,
\end{aligned}
$$

where $\hat u$ is a reference time-invariant control (see (4.15) and the explanations there around).

(A3′) For some $\varepsilon > 0$ it holds that

$$\bar{u}(a) \leq (1+\varepsilon)\hat{u}(a), \quad a \geq 0,$$

and the following inequality is fulfilled:

$$\frac{1}{2}\left[\kappa_{11} + (1+\varepsilon)\bar{\kappa}_{22} + \sqrt{(\kappa_{11} - (1+\varepsilon)\bar{\kappa}_{22})^2 + 4\,(1+\varepsilon)\kappa_{12}\,\bar{\kappa}_{21}}\,\right] < 1. \tag{4.44}$$

Remark 4.2. Assumptions (A3) and (A3′) implicitly require that $\kappa_{11} < 1$ which is equivalent to the below-replacement fertility condition if $r = 0$. For $r > 0$ the below-replacement fertility condition is stronger than $\kappa_{11} < 1$. We mention also that $\bar{\kappa}_{21} \leq \kappa_{21}$ and $\bar{\kappa}_{22} \leq \kappa_{22}$, so that (A3′) may be a weaker assumption than (A3) if ε is sufficiently small which is the reason for considering them both in the below analysis.

Having formulated Assumption (A3) and (A3′) we are now able to state the following key lemma:

Lemma 4.6. *Under (BA) and (A1)–(A3), system (4.35), (4.36) has a unique solution in $L_\infty(0,\infty)$. The same is true also under (BA), (A1), (A2), (A3′).*

Proof of Lemma 4.6. Consider the kernel $k(t,s)$ of the integral equation (4.37) defined in (4.38). It defines an operator $K : (L_\infty(0,\infty))^2 \to (L_\infty(0,\infty))^2$. The operator depends on u, so we need the existence of a solution of the integral equation for every admissible u. If the operator norm is smaller than one, a resolvent $L : (L_\infty(0,\infty))^2 \to (L_\infty(0,\infty))^2$ with kernel $l(t,s)$ exists according to Corollary 3.10 and Theorem 3.6 in Gripenberg et al. (1990) and can be written as

$$x(t) = \Phi(t) - \int_0^\infty l(t,s)\Phi(s)\,\mathrm{d}s.$$

To show that the norm is smaller than one, we define for $\alpha > 0$ a new norm in $(L_\infty(0,\infty))^2$.

$$\|(x_1,x_2)\| = \max\{\|x_1\|_{L_\infty}, \alpha\|x_2\|_{L_\infty}\}.$$

Take $x \in (L_\infty(0,\infty))^2$ with $\|x\| = 1$, and estimate the norm of $y = Kx$:

$$\begin{aligned}
\|y\| &= \max\{y_1, \alpha y_2\} \\
&= \max\Big\{\sup_{t\geq 0}\int_t^\infty [k_{11}(t,s)x_1(s) + k_{12}(t,s)x_2(s)]\,\mathrm{d}s, \\
&\qquad \alpha\sup_{t\geq 0}\int_t^\infty [k_{21}(t,s)x_1(s) + k_{22}(t,s)x_2(s)]\,\mathrm{d}s\Big\} \\
&\leq \max\Big\{\sup_{t\geq 0}\int_t^\infty |k_{11}(t,s)|\,\mathrm{d}s\,\|x_1\|_\infty + \sup_{t\geq 0}\int_0^\infty \frac{1}{\alpha}|k_{12}(t,s)|\,\mathrm{d}s\,\alpha\|x_2\|_\infty, \qquad (4.45) \\
&\qquad \sup_{t\geq 0}\int_t^\infty \alpha|k_{21}(t,s)|\,\mathrm{d}s\,\|x_1\|_\infty + \sup_{t\geq 0}\int_0^\infty |k_{22}(t,s)|\,\mathrm{d}s\,\alpha\|x_2\|_\infty\Big\}.
\end{aligned}$$

Since $\int_0^\infty u(t,a)\,\mathrm{d}a = 1$, with κ_{ij} defined in (4.42), it holds that:

$$\sup_{t\geq 0} \int_0^\infty |k_{ij}(t,s)|\,\mathrm{d}s \leq \kappa_{ij}, \qquad i,j \in \{1,2\}.$$

With this, (4.45) and $\|x\| = 1$ it can be concluded that

$$\|y\| \leq \max\{\kappa_{11} + \frac{1}{\alpha}\kappa_{12}, \alpha\kappa_{21} + \kappa_{22}\}.$$

Thus, the right hand side is an estimation for the operator norm of K. The operator norm being smaller than unity is implied by the existence of $\theta_0 < 1$ and $\alpha > 0$ such that

$$\kappa_{11} + \frac{1}{\alpha}\kappa_{12} \leq \theta_0, \tag{4.46}$$

$$\alpha\kappa_{21} + \kappa_{22} \leq \theta_0. \tag{4.47}$$

Since the first line is monotonously decreasing and the second is increasing in α, for the optimal α, which allows for the smallest possible θ_0, both equations are fulfilled as equality. Therefore, we solve the equation

$$\kappa_{11} + \frac{1}{\alpha}\kappa_{12} = \alpha\kappa_{21} + \kappa_{22}$$

for α and obtain

$$\alpha_{1,2} = \frac{1}{2\kappa_{21}}\left[\kappa_{11} - \kappa_{22} \pm \sqrt{(\kappa_{11} - \kappa_{22})^2 + 4\kappa_{12}\kappa_{21}}\right].$$

We insert the positive solution for α into the second line of (4.47) to obtain θ_0:

$$\theta_0 = \frac{1}{2}\left[\kappa_{11} + \kappa_{22} + \sqrt{(\kappa_{11} - \kappa_{22})^2 + 4\kappa_{12}\kappa_{21}}\right].$$

The requirement $\theta_0 < 1$ is exactly inequality (4.44) in condition (A3).

The sufficiency of (A3') follows because the assumption implies that for all admissible u it holds that $\sup_{t\geq 0} \int_0^\infty k_{ij}(t,s)\,\mathrm{d}s \leq (1+\varepsilon)\bar{\kappa}_{2j}$ for $j = 1,2$. System (4.46) – (4.47) then reads as

$$\kappa_{11} + \frac{1}{\alpha}\kappa_{12} \leq \theta_0, \tag{4.48}$$

$$(1+\varepsilon)(\alpha\bar{\kappa}_{21} + \bar{\kappa}_{22}) \leq \theta_0.$$

By following the same steps as above, we obtain that (4.43) is sufficient for the operator norm of K to be smaller than one.
Q.E.D.

As a consequence of the above two lemmas in combination, we obtain the following proposition which states the uniqueness of a bounded solution of the adjoint variable.

Proposition 4.2. *Under (BA) and (A1)–(A3), the adjoint equation (4.31) has a unique solution in $L_\infty(D)$. The same is true also under (BA), (A1), (A2), (A3').*

Proof. According to Lemma 4.6 system (4.35), (4.36) has a bounded solution. Then according to Lemma 4.5 equation (4.31) also has a bounded solution, obtained by substitution of the solution (λ, η) of (4.35), (4.36) in (4.33).

For any bounded solution ξ of (4.31) the functions λ and η defined by (4.32) are bounded and satisfy (4.35), (4.36). Therefore λ and η are uniquely determined (Lemma 4.6), hence ξ is unique (Lemma 4.5). Q.E.D.

Necessary optimality condition of Pontryagin type

The next theorem gives a necessary optimality condition of the type of the Pontryagin maximum principle for problem (4.23)-(4.26).

Theorem 4.1. *Let assumptions (BA), (A1)–(A3) (or alternatively (BA), (A1), (A2), (A3')) be fulfilled and let (u, N) be an optimal solution of problem* (4.23)-(4.26). *Let ξ be the unique solution in $L_\infty(D)$ of the adjoint equation (4.31). Then for a.e. $t \geq 0$ the optimal control $u(t, \cdot)$ maximizes the integral*

$$\int_0^\infty \xi(t, a) v(a) \, \mathrm{d}a, \tag{4.49}$$

on the set of measurable functions $v(\cdot)$ satisfying

$$\underline{u}(a) \leq v(a) \leq \bar{u}(a), \quad \int_0^\infty v(a) \, \mathrm{d}a = 1. \tag{4.50}$$

Proof. Subsequently, we show that if u is an optimal control and hence $\Delta J \leq 0$, due to optimality, then $v(a) = u(t, a)$ maximizes the integral in (4.49). Let J be the optimal objective value and let ξ be the unique bounded solution of the adjoint equation (4.31) on D (see Proposition 4.2). Let us fix an arbitrary $\theta > 0$, let $h > 0$ be arbitrary (and presumably small) and $T > 0$ be such that $\theta - h \geq 0$ and $\theta + h \leq T$. Denote $\Theta_h := [\theta - h, \theta + h] \times [0, \infty) \subset D$ and define a "disturbed" control

$$\tilde{u}(t, a) := \begin{cases} u(t, a) & \text{for } (t, a) \notin \Theta_h, \\ v(a) & \text{for } (t, a) \in \Theta_h, \end{cases} \tag{4.51}$$

where v is any measurable function satisfying (4.50).

Then $\tilde{u}$ satisfies the control constraints. Let $\tilde{N}$ be the corresponding solution of (4.24)–(4.26) and $\tilde{R}$, $\tilde{B}$ be corresponding functions (immigration and birth flows) defined by (4.6) and (4.5), while R and B correspond to N. Denote $\Delta J = J(\tilde{u}) - J(u)$, $\Delta u = \tilde{u} - u$, $\Delta N = \tilde{N} - N$, $\Delta R = \tilde{R} - R$, $\Delta B = \tilde{B} - B$, all depending on the chosen h and v. According to (A2), $\tilde{R}$ is non-negative, and all the functions introduced above are bounded (see Lemma 4.4).

Clearly,

$$\Delta J = \int_0^T e^{-rt} \int_0^\infty \phi(a) \, \Delta N(t, a) \, \mathrm{d}a \, \mathrm{d}t. \tag{4.52}$$

In order to obtain an expression for ΔN we multiply the equation

$$\begin{aligned} \mathcal{D}\Delta N(t,a) &= -\mu(a)\,\Delta N(t,a) \\ &\quad + \int_0^\infty \nu(\alpha)\,[\Delta N(t,s)\,u(t,a) + N(t,s)\,\Delta u(t,a) + \Delta N(t,s)\Delta u(t,a)]\,\mathrm{d}s \end{aligned} \tag{4.53}$$

resulting from (4.24) by $e^{-rt}\xi(t,a)$ and integrate on D. Since $D = \{(s, x+s) : s, x \geq 0\} \cup \{(x+s, s) : s, x \geq 0\}$ and the two sets on the right intersect only on a set of measure zero, we may represent

$$\begin{aligned} \int_0^\infty \int_0^\infty \mathcal{D}\Delta N(t,a) e^{-rt}\xi(t,a)\,\mathrm{d}t\,\mathrm{d}a &= \int_0^\infty \int_0^\infty e^{-rs}\xi(s, x+s)\frac{\mathrm{d}}{\mathrm{d}s}\Delta N(s, x+s)\,\mathrm{d}s\,\mathrm{d}x \\ &\quad + \int_0^\infty \int_0^\infty e^{-r(x+s)}\xi(x+s, s)\frac{\mathrm{d}}{\mathrm{d}s}\Delta N(x+s, s)\,\mathrm{d}s\,\mathrm{d}x. \end{aligned} \tag{4.54}$$

By integration by parts (for the inner interval) the first term on the right-hand side gives

$$\begin{aligned} &\int_0^\infty \Big[e^{-rs}\xi(s, x+s)\Delta N(s, x+s)|_{s=0}^{\infty} \\ &- \int_0^\infty \Delta N(s, x+s) e^{-rs}(-r\xi(s, x+s) + \frac{d}{ds}\xi(s, x+s))\,\mathrm{d}s \Big]\,\mathrm{d}x. \end{aligned} \tag{4.55}$$

The term with $s \to \infty$ is zero because both ξ and N (and therefore ΔN) are bounded, and the term with $s = 0$ is zero because $\Delta N(0, a) = 0$.

The second term on the right-hand side of (4.54) is treated in the same way and combining the two terms we obtain that the right-hand side of (4.54) is equal to

$$\begin{aligned} &- \int_0^\infty e^{-rt}\xi(t,0)\Delta N(t,0)\,\mathrm{d}t \\ &- \int_0^\infty \int_0^\infty e^{-rt}(\mathcal{D}\xi(t,a) - r\xi(t,a))\Delta N(t,a)\,\mathrm{d}a\,\mathrm{d}t, \end{aligned} \tag{4.56}$$

where we changed again to the (t, a)-plane. To obtain equation (4.58) below, first, we take into account that $N(t, 0) = \int_0^\infty f(a)N(t, a)\,\mathrm{d}a$ and observe that (4.56) equals the right-hand side of equation (4.54). Then, we observe that

$$\begin{aligned} D\Delta N(t,a) &= -\mu(a)\Delta N(t,a) + \int_0^\infty (\mu(s) - f(s))\Delta N(t,s)\,\mathrm{d}s\; u(t,a), \\ &= -\mu(a)\Delta N(t,a) + \int_0^\infty \nu(s)\Delta N(t,s)\,\mathrm{d}s\; u(t,a), \end{aligned}$$

and insert this relation into the left-hand side of equation (4.54). Hence, equation (4.54) becomes

$$\begin{aligned} &\int_0^\infty \int_0^\infty e^{-rt}\xi(t,a)\Big[\mu(a)\Delta N(t,a) + \int_0^\infty \nu(s)\Delta N(t,s)\,\mathrm{d}s\Big]\,\mathrm{d}a\,\mathrm{d}t \\ &= -\int_0^\infty \int_0^\infty e^{-rt}\xi(t,0) f(a)\Delta N(t,a)\,\mathrm{d}a\,\mathrm{d}t \\ &\quad - \int_0^\infty \int_0^\infty e^{-rt}(\mathcal{D}\xi(t,a) - r\xi(t,a))\Delta N(t,a)\,\mathrm{d}a\,\mathrm{d}t. \end{aligned} \tag{4.57}$$

Reordering of the terms gives

$$0 = \int_0^\infty \int_0^\infty e^{-rt}\Big[(\xi(t,0)f(a) + \mathcal{D}\xi(t,a) - r\xi(t,a) - \mu(a)\xi(t,a))\Delta N(t,a) \qquad (4.58)$$
$$+ \int_0^\infty \nu(s)\xi(t,a)\left(\Delta N(t,s)u(t,a) + N(t,s)\Delta u(t,a) + \Delta N(t,s)\Delta u(t,a)\right)\,\mathrm{d}s\Big]\,\mathrm{d}a\,\mathrm{d}t.$$

Using the adjoint equation (4.31) and change the order of integration in the first term of the right-hand side of equation (4.58), we obtain that

$$0 = \int_0^\infty \int_0^\infty e^{-rt}\Big[-\phi(a)\Delta N(t,a)$$
$$+ \int_0^\infty \left(\nu(s)\xi(t,a)N(t,s)\Delta u(t,s) + \nu(s)\xi(t,a)\Delta N(t,s)\Delta u(t,a)\right)\,\mathrm{d}s\Big]\,\mathrm{d}a\,\mathrm{d}t.$$

Adding this to (4.52) we get

$$\Delta J = \int_0^\infty \int_0^\infty \int_0^\infty e^{-rt}\nu(s)\xi(t,a)\Big[N(t,s)\Delta u(t,a) + \Delta N(t,s)\Delta u(t,a)\Big]\,\mathrm{d}s\,\mathrm{d}a\,\mathrm{d}t$$
$$= \int_0^\infty \int_0^\infty e^{-rt}R(t)\xi(t,a)\Delta u(t,a)\,\mathrm{d}a\,\mathrm{d}t \qquad (4.59)$$
$$+ \int_0^\infty \int_0^\infty \int_0^\infty e^{-rt}\nu(s)\xi(t,a)\Delta N(t,s)\Delta u(t,a)\,\mathrm{d}s\,\mathrm{d}a\,\mathrm{d}t.$$

Next we shall show that the second term on the right-hand side above is of second order with respect to h (cf. (4.51)). Note, that $\Delta u(t,a) = 0$ for $t \notin [\theta - h, \theta + h]$, thus we need an estimation of ΔN only on the time-horizon $[0,T]$ with some $T > \theta$, say $T = \theta + 1$.

By solving equation (4.24) along the characteristic lines we obtain the representation

$$N(t,a) = B(t-a)e^{-\int_{0\vee(a-t)}^{a}\mu(\tau)\,\mathrm{d}\tau} + \int_{0\wedge(t-a)}^{t} e^{-\int_{a-t+s}^{a}\mu(\tau)\,\mathrm{d}\tau}R(s)u(s,a-t+s)\,\mathrm{d}s,$$

where B is extended as $B(t) = N_0(-t)$ for $t < 0$ and $0 \vee \alpha := \max\{0,\alpha\}$. Due to assumption (BA) we can estimate $e^{-\int_\alpha^a \mu(\tau)\,\mathrm{d}\tau} \le 1$. A similar equality holds for $\tilde{N}(t,a)$ corresponding to the control $\tilde{u}$. Subtracting the two expressions and estimating, we obtain that

$$|\Delta N(t,a)| \le |\Delta B(t-a)| + \int_{0\vee(t-a)}^{t} \left|\Delta R(s)\tilde{u}(s,a-t+s) + R(s)\Delta u(s,a-t+s)\right|\,\mathrm{d}s.$$

From (4.6), (4.5) we can estimate

$$|\Delta R(t)| \le c_1 \int_0^\infty |\Delta N(t,a)|\,\mathrm{d}a = c_1\|\Delta N(t,\cdot)\|_{L_1},$$

where c_1 is a constant depending only on f and μ, and analogously we can estimate $|\Delta B(t)|$. Then it is a matter of routine estimations (taking into account that Δu is non-zero on a set of measure proportional to h) to obtain the inequality

$$\|\Delta N(t,\cdot)\|_{L_1} \le \int_0^t c_2\|\Delta N(x,\cdot)\|_{L_1}e^{-(t-x)\mu_0}\,\mathrm{d}x + c_3 h, \quad t \in [0,\theta+1],$$

where c_2 and c_3 are independent of h (although may depend on θ and the data of the problem).

Since $T = \theta + 1$ is finite, Gronwall's lemma gives

$$\|\Delta N(t,\cdot)\|_{L_1} \le Ch, \;\; t \le T.$$

Now it is straightforward to estimate the last term in (4.59) by

$$\int_{\theta-h}^{\theta+h} e^{-rt} C\, h\, \|\nu\|_{L_\infty} \|\xi\|_{L_\infty} \int_0^{\bar{a}} |\Delta u(t,\alpha)|\, \mathrm{d}\alpha\, \mathrm{d}t \le C\, h^2.$$

Using this in the estimation (4.59) and having in mind the definition of $\tilde{u}(t,a)$ in (4.51) and the fact that $\Delta J \le 0$ due to the optimality of u we obtain that

$$\frac{1}{2h}\int_{\theta-h}^{\theta+h} e^{-rt} \int_0^\infty R(t)\, \xi(t,a)\, u(t,a)\, \mathrm{d}a\, \mathrm{d}t \ge \frac{1}{2h}\int_{\theta-h}^{\theta+h} e^{-rt} \int_0^\infty R(t)\, \xi(t,a)\, v(a)\, \mathrm{d}a\, \mathrm{d}t - \frac{C}{2}\, h.$$

Almost every s is a Lebesgue point of the function $t \to \int_0^\infty R(t)\xi(t,a)u(t,a)\,\mathrm{d}a$, and $R(t) > 0$. Therefore, we can conclude the proof of the theorem by taking the limit $h \to 0$. Q.E.D.

4.5 Uniqueness, stationarity, and structure of the optimal immigration pattern

In this section, we use Theorem 1 to obtain some qualitative properties of the optimal solution of problem (4.23)-(4.26). The most interesting one is that the optimal control u (that is, the optimal immigration profile) is unique and time-invariant: $u(t,a) = u(a)$. This fact is not evident. Its proof is based on stability condition (A3) and on an additional well-posedness condition. The latter also implies a bang-bang structure of the optimal control $u(a)$.

To prove uniqueness and stationarity of the optimal solution we rewrite the adjoint equation in a feedback form. To do this we introduce the functional $\sigma(\cdot)$:

$$\sigma(g) = \max_{v \in \mathcal{V}} \int_0^\infty g(a)v(a)\,\mathrm{d}a, \quad g \in L_\infty(0,\infty), \tag{4.60}$$

where $\mathcal{V}$ is the set of functions $v : [0,\infty) \to \mathbf{R}$ satisfying (4.50). Then using the optimization condition in Theorem 4.1 we can rewrite the adjoint equation (4.31) in the feedback form

$$\mathcal{D}\xi(t,a) = (r + \mu(a))\xi(t,a) - f(a)\xi(t,0) - (\mu(a) - f(a))\sigma(\xi(t,\cdot)) - \phi(a). \tag{4.61}$$

The existence of a solution in $L_\infty(D)$ to this equation follows from the necessity of the maximum principle.

Lemma 4.7. *If assumption (A2) is fulfilled, then equation* (4.61) *has a unique bounded solution.*

Proof of Lemma 4.7. The proof is similar to the one of Lemma 4.6. Let us take two bounded solutions, ξ_1 and ξ_2 and denote by $\Delta\xi(t,a)$ the difference between the two. The solutions ξ_i can be written as (cf. (4.34))

$$\xi_i(t,a) = \int_a^\infty e^{-\int_a^s \rho(\theta)\,\mathrm{d}\theta}[f(s)\xi_i(s+t-a,0) + \nu(s)\sigma(\xi_i(s+t-a,\cdot)) + \phi(s)]\,\mathrm{d}s, \;\; i = 1,2.$$

Let $\Delta\lambda(t) := \xi_1(t,0) - \xi_2(t,0)$ and $\Delta\sigma(t) := \sigma(\xi_1(t,\cdot)) - \sigma(\xi_2(t,\cdot))$, then we obtain the homogeneous system of integral equations

$$\begin{aligned}
\Delta\xi(t,a) &= \int_0^\infty e^{-\int_a^{s+a}\rho(\theta)\,\mathrm{d}\theta}(f(s+a)\Delta\lambda(s+t) + \nu(s+a)\Delta\sigma(\xi(s+t,\cdot)))\,\mathrm{d}s,\\
\Delta\lambda(t) &= \int_0^\infty e^{-\int_0^s \rho(\theta)\,\mathrm{d}\theta}[f(s)\Delta\lambda(s+t) + \nu(s)\Delta\sigma(s+t)]\,\mathrm{d}s.
\end{aligned}$$

Denote by $K : (L_\infty(0,\infty))^2 \to (L_\infty(0,\infty))^2$ the integral operator representing the system of integral equations above. The existence of a unique solution of this system is guaranteed if $\|K\| < 1$. To show this, take the norm in the system of equations and use that σ is Lipschitz with Lipschitz constant 1 (cf. Lemma 4.1 in Feichtinger and Veliov (2007)),

$$\begin{aligned}
\|\Delta\xi\|_\infty &\le \kappa_{21}\|\Delta\lambda\|_\infty + \kappa_{22}\|\Delta\xi\|_\infty\\
\|\Delta\lambda\|_\infty &\le \kappa_{11}\|\Delta\lambda\|_\infty + \kappa_{12}\|\Delta\xi\|_\infty.
\end{aligned}$$

As in the proof of Lemma 4.6 we define a norm $\|(\Delta\xi,\Delta\lambda)\| := \max\{\|\Delta\xi\|_\infty, a\|\Delta\lambda\|_\infty\}$ for $\alpha > 0$. We choose again an appropriate $\alpha > 0$ such that the norm of the operator K is minimized. The minimum is exactly the left hand side of (4.43) and Assumption (A3) guarantees that it is smaller than one. Therefore, a unique solution exists to the homogeneous system, which is obviously $\Delta\xi = 0$. Q.E.D.

Now we introduce a *regularity assumption* that ensures that the maximization condition in Theorem 4.1 determines a unique control. As shown in Feichtinger and Veliov (2007) in a simpler version of the problem considered here, without a certain regularity assumption the uniqueness fails and this is due to non-concavity of the problem. On the other hand the regularity assumption is in a reasonable sense generic and easy to check.

(**A4**) For all real numbers d_0, d_1 and d_2 it holds that

$$\text{meas}\{a \in [0,\infty] : d_0 + d_1\mu(a) + d_2 f(a) - p(a) = 0\} = 0.$$

This assumption requires that μ, f and p must not be linearly related on a set of positive measure.

Theorem 4.2. *Let assumptions (BA), (A1)–(A4) be fulfilled. Then optimal control problem* (4.23) – (4.27) *has a unique optimal control u and it is time-invariant: $u(t,a) = u(a)$.*

Proof. First we shall prove that (4.61) has a stationary bounded solution $\hat{\xi}(t,a) = \hat{\xi}(a)$. To do this we show that the equation

$$\xi'(a) = (r + \mu(a))\xi(a) - f(a)\xi(0) - (\mu(a) - f(a))\sigma(\xi(\cdot)) - \phi(a) \tag{4.62}$$

has a bounded solution. Denote $\lambda = \xi(0)$ and $\eta = \sigma(\xi(\cdot))$, then we can write the solution to the differential equation as

$$\xi(a) = \int_a^\infty e^{-\int_a^s \rho(\theta)\,\mathrm{d}\theta}[f(s)\lambda + \nu(s)\eta + \phi(s)]\,\mathrm{d}s.$$

Using the definition of $\sigma(\xi(\cdot))$, (4.60), the equations for λ and η are

$$\lambda = \int_0^\infty e^{-\int_0^s \rho(\tau)\,\mathrm{d}\tau}[f(s)\lambda + \nu(s)\eta + \phi(s)]\,\mathrm{d}s$$
$$\eta = \max_{v\in\mathcal{V}} \int_0^\infty v(a) \int_a^\infty e^{-\int_a^s \rho(\tau)\,\mathrm{d}\tau}[f(s)\lambda + \nu(s)\eta + \phi(s)]\,\mathrm{d}s\,\mathrm{d}a.$$

Denoting the terms independent from λ and η by (b_1, b_2) we can write the equations above as

$$(I - K)\begin{pmatrix}\lambda\\ \eta\end{pmatrix} = \begin{pmatrix}b_1\\ b_2\end{pmatrix}, \tag{4.63}$$

where I is the 2×2 identity matrix and K is the matrix defined by the right hand side. As in the proof of Lemma 4.6 and 4.7, define a norm on $\mathbf{R}^2$ as $\|(x,y)\| = \max\{|x|, \alpha|y|\}$, $\alpha > 0$. Estimating the operator norm of K in the same way as in the proof of Lemma 4.6 gives that the norm is smaller or equal to the left hand side in (4.44). Then assumption (A3) states that the norm is smaller than one, thus $(I - K)$ is invertible and therefore a unique solution to (4.63) exists. Thus, a bounded solution $\hat{\xi}(a)$ of (4.62) exists and it is obviously a stationary bounded solution of (4.61).

According to Lemma 4.7 the stationary function $\hat{\xi}(a)$ is the unique bounded solution of (4.61).

On the other hand, Theorem 4.1 claims that for every optimal control u, the adjoint equation (4.31) has a unique bounded solution $\xi(t,a)$ and for a.e. $t \geq 0$

$$\int_0^\infty \xi(t,a)u(t,a)da = \max_{v\in\mathcal{V}} \int_0^\infty \xi(t,a)v(a)\,\mathrm{d}a = \sigma(\xi(t,\cdot)).$$

Then ξ is a bounded solution also of (4.61), which implies that $\xi = \hat{\xi}$. The above maximization condition reads now as

$$\int_0^\infty \hat{\xi}(a)u(t,a)da = \max_{v\in\mathcal{V}} \int_0^\infty \hat{\xi}(a)v(a)\,\mathrm{d}a, \tag{4.64}$$

where $\hat{\xi}$ is the unique bounded solution of (4.62).

Assumption (A4) obviously implies that the solution $\hat{\xi}$ of (4.62) cannot be constant on a set of positive measure. Then similarly as in Corollary 5.1, in Feichtinger and Veliov (2007) one can prove that (4.64) uniquely determines (modulo a set of measure zero) a control $u \in \mathcal{U}$, it is time-invariant and has the following structure: there is a real number l such that

$$u(t,a) = \begin{cases} \underline{u}(a) & \text{if } \hat{\xi}(a) \leq l, \\ \bar{u}(a) & \text{if } \hat{\xi}(a) > l. \end{cases} \tag{4.65}$$

Q.E.D.

We formulate the last finding in the proof of the above theorem as a corollary.

Corollary 4.1. *Let $\hat{\xi}$ be the unique bounded solution of (4.62). Then, there is $l \in \mathbf{R}$ such that the unique optimal control $u(t,a) = \hat{u}(a)$ is determined by (4.65). This number l is the only one for which the resulting $\hat{u}$ satisfies $\int_0^\infty \hat{u}(a)\,\mathrm{d}a = 1$.*

Thus the optimal solution is of bang-bang type. A related result is obtained in Chapter 3 for a static counterpart of the problem considered in this chapter. Since $\xi(a)$ can be interpreted marginally as "shadow price" of an a-year-old individual, the above corollary asserts that there is a critical value l such that it is optimal to encourage as much as possible migration in ages for which the shadow price is higher than l ($u(a) = \bar{u}(a)$) and restrict as much as possible migration in ages for which the shadow price is smaller than l. The remarkable fact here is that the shadow price is independent of the initial age-distribution of the population and time-invariant.

4.6 A case study: the Austrian Population

In this section, we numerically determine the optimal immigration policy given by (4.23)–(4.26) for the case study of the Austrian population. The numerical results for the optimal time-invariant immigration profile and the population's age structure obtained in this section are based on the analytical results above. In all the numerical calculations below, we specify $p(a)$ in (4.23) as the characteristic function of the age interval $[20, 65]$. If we additionally set $q = 0$ the objective function (4.23) is the discounted and aggregated number of workers over time. It is related to the so-called dependency ratio, which is the ratio of nonworking age population to the working age population. The dependency ratio is an important demographic indicator for the solvency of the social security system of a population. The case of $q > 0$, which is also discussed below, accounts for possible costs for the integration of immigrants.

For the computations, we initialize the age structure of demographic variables referring to Austrian data as of 2009 and interpolate these data piecewise linearly to obtain continuous representations of the vital rates, $f(a)$, $\mu(a)$. As already mentioned in Remark 4.1, we assume that $\mu(a) = \mu(95)$ for $a \geq 95$. These demographic data together with an intertemporal discount rate of $r = 0.04$ satisfy assumption (A3) with $\kappa_{11} = 0.0737$, $\kappa_{12} = 0.0774$, $\kappa_{21} = 0.1480$, $\kappa_{22} = 0.1506$. For these values the quantity in the left hand side of (4.43) equals 0.2259 and is therefore well below 1. For the initial age structure $N_0(a)$ we take the annual average numbers of the Austrian female population in 2009, see Figure 4.5 (solid line). The normalized immigration age-density of

2009 is denoted by $\hat{u}(a)$, see Figure 4.4. We set the lower and upper age-specific limits for immigration to

$$\underline{u}(a) = 0 \text{ and } \overline{u}(a) = 2\hat{u}(a).$$

In the following, we analyze three scenarios: in the uncontrolled case the immigration age density remains the same in the future $u(t,a) = \hat{u}(a)$; then we assume that q in (4.23) takes the value zero and the immigration age density is chosen optimally $u(t,a) = u^*(a)$; and additionally we set $q = 200$ where again $u(t,a) = u^*(a)$ is chosen optimally. With the last scenario we analyze the effect of immigration costs on the optimal immigration age-pattern, see Figure 4.4.

The optimal solution $u(t,a) = u^*(a)$ in the above scenarios is depicted in Figure 4.4. As it can be seen in this figure, the optimal age profile of immigrants is at its upper bound from slightly before the lowest working age of $a = 20$ until the mid thirties. It is on its lower bound at any other ages. Notice also that increasing the costs of immigration shifts the optimal age pattern to the left as indicated by the dashed line in Figure 4.4.

In Figure 4.5 we compare the age structure of the initial population with the stationary population at $t = 400$ which results in the uncontrolled case and when applying the optimal $u^*(a)$ for $q = 200$. The sharp increase of the optimal population $N^*(400, a)$ at the low working ages is due to the annual inflow of immigrants at these ages. Hence, one can follow that the age structure in the controlled case is more favorable, i.e. younger, than in the uncontrolled case. In particular, there are more young and middle-aged workers.

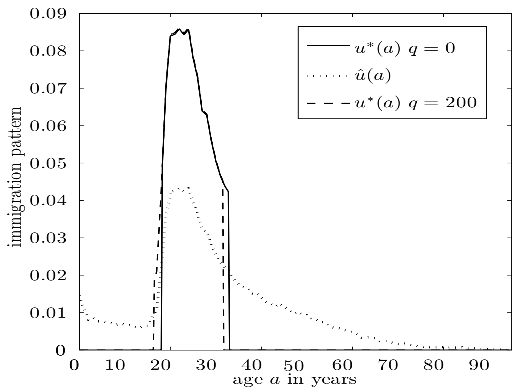

Figure 4.4: The actual age density $\hat{u}(a)$ (dotted line) and the optimal immigration density $u^*(a)$ for $q = 0$ (solid) and $q = 200$ (dashed line)

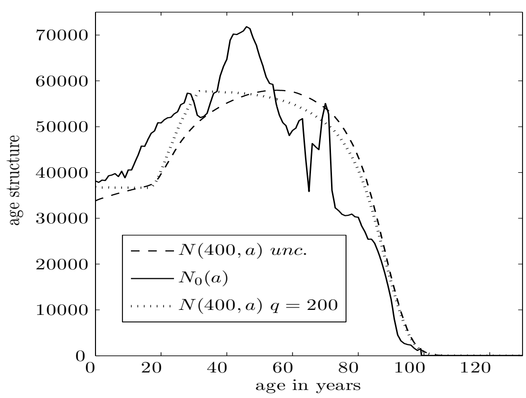

Figure 4.5: The initial age structure $N_0(a)$ (solid line) and $N(400, a)$ for the uncontrolled case (dashed) and for the optimal control with $q = 200$ (dotted)

In Figure 4.6 we plot the evolution of the number of newborns $B(t)$, the number of deaths $D(t)$ and the recruitment rate $R(t)$ on the time horizon $[0, 400]$, where $D(t) = R(t) + B(t)$. Notice that for the uncontrolled as well as for the controlled immigration, there is a huge increase in the number of immigrants $R(t)$ at the beginning, caused by the high number of deaths which would be a result of the baby boom, that occurred in Austria in the 50s and 60s.

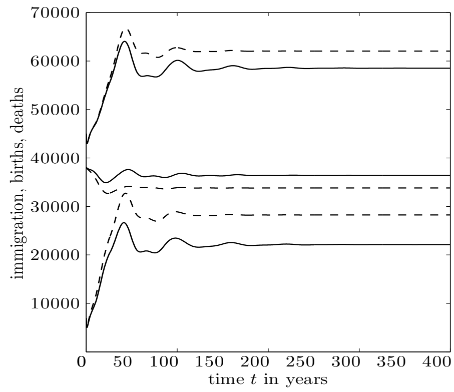

Figure 4.6: The evolution of the number of deaths $D(t)$ (upper solid line), the number of births $B(t)$ (middle solid line) and the number of immigrants $R(t)$ (lower solid line) over time for the optimal control and $q = 0$ compared with the uncontrolled case (corresponding dashed lines)

In Figure 4.7 the change of the number of workers and in Figure 4.8 the dependency ratio over time are shown. We compare the scenario with $q = 0$ to the case where current age-specific immigration rates would remain the same in the future. Clearly, we can sustain a higher number of workers and simultaneously a lower dependency ratio when applying the optimal immigration pattern $u^*(a)$.

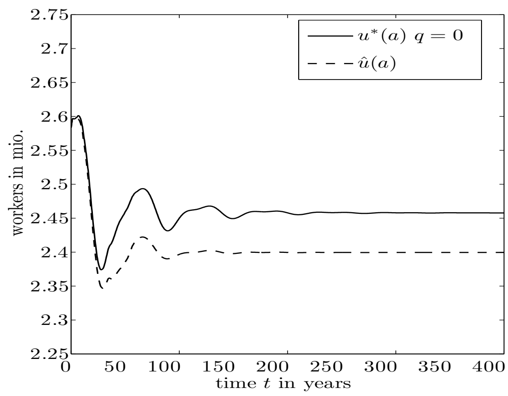

Figure 4.7: The evolution of the number of workers over time for $q = 0$ (solid line) and the uncontrolled case (dashed line)

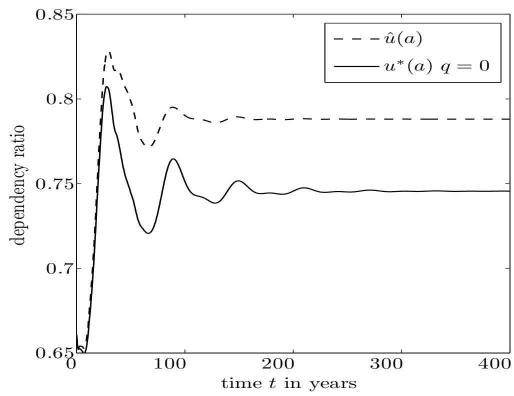

Figure 4.8: The evolution of the so-called dependency ratio over time for $q = 0$ (solid line) and the uncontrolled case (dashed line).

4.7 Discussion

The main contributions of the above investigations are as follows. We obtain a new Pontryagin type maximum principle with a transversality condition in the form of boundedness of the adjoint variable. This is done under suitable stability assumptions which are fulfilled for populations with sufficiently low fertility. Existence of an optimal solution is also proven.

The most striking result is that under an additional generic well-posedness condition for a population with time-invariant mortality and fertility the optimal age-density of the migration turns out to be time-invariant and independent of the initial data. This makes it possible to find it by solving the associated steady-state problem, which is an optimal control problem for an ordinary differential equation and was studied in details in Chapter 3 and published in Simon et al. (2012). Thanks to this property also qualitative results for the optimal policy are obtained.

However, two of the key assumptions in the above model pose questions: the stability conditions (A3) (or (A3′)) and the assumption that the discount rate r is strictly positive.

We were not able to prove the main results—the optimality conditions and the stationarity of the optimal immigration age-profile—only assuming below-replacement fertility (4.14), which has a clear demographic meaning. A challenging question is whether the “stability" provided by (4.14) is not enough for the validity of the results. Apparently this question requires more profound analysis of the stability of the involved systems of integral equations.

The question whether the optimality conditions (especially the “transversality" condition for the associated adjoint equation) and the stationarity result can be obtained in the case of no discount (that is, for $r = 0$) seems to be important since discounting is not a common practice in the “evaluation" of demographic processes.

Chapter 5

Overlapping generations models with immigration

In this chapter we develop overlapping generations models (OLG) which explicitly include immigration. Taking account of immigration in standard economic models requires adaptations in various respects.

In the following, we propose two general equilibrium models. In Section 5.1 we investigate the effect of a sudden variation in the number of immigrants on the host country to shed light on the welfare effects of immigration for the various generations of the host country's population. Subsequently, in Section 5.2, we vary the age structure of the inflowing migrants and determine the impact of immigration on the pension system and capital accumulation.

5.1 Modeling an immigration shock with a continuous time OLG model

Immigration is a complex process. People who immigrate to a country differ, among other characteristics, in ethnicity, religious believes, age, skill level and their economic situation. In this work we will focus only on the last three of these features. The number of annual immigrants as well as their skill level and age distribution impact the population structure and the productivity of the work force. The economic situation of immigrants, in here, reflected by their capital endowment when entering the country, changes the capital labor ratio in the presumably closed economy. As a consequence, all these characteristics impact the economic activities in the country and henceforth the welfare of its inhabitants.

OLG models are typically used to investigate how different generations interact with each other in an economy. Here, we extend this model structure by adding a new heterogeneity, namely by explicitly modeling natives and immigrants.

Here, we aim to determine the welfare consequences of an exogenous immigration shock to a closed economy. This means a rapid change in the number of immigrants for a small period of time due to policy changes or possibly also unstable conditions such as wars or economic crises in other countries.

In order to identify the welfare effects for different cohorts of the native population, the model must replicate the age structure of the population and the life cycle choices for the agents of the different vintages.

In Ben-Gad (2003, 2006) the population is modeled in form of overlapping dynasties where arriving immigrants are the founders of new dynasties. This provides a first step to a realistic description of the population structure. However, immigrants have all the same age when entering the host country. Moreover, these dynasties live infinitely long. In contrast to these articles, here we consider an age pattern of inflowing immigrants and finite but uncertain life times.

In Boldrin and Montes (2008), like here, it is investigated how an immigration shock affects the welfare of different cohorts. There, they use the framework of a three period discrete time overlapping generations model. The age structure of the population is determined by assigning the individuals to these three periods. As a result there are only three coexisting generations at each point of time. In order to depict a realistic finite life time, the length of these periods is approximately 25-30 years.

Unlike the other model treated in this chapter, where the focus was on the long term effects, i.e. steady state changes, here we investigate the intertemporal changes in the macroeconomic variables and therefore also consider a temporal shock which lasts only for a couple of years.

In the following, we assume that immigrants enter without any assets. The life cycle of both, natives and immigrants, is divided into a schooling, a working and a retirement period. Cohorts living in different periods are linked by intergenerational transfers, i.e. labor taxes are redistributed to the old in form of pension payments. While receiving their education, native agents accumulate debts, due to own consumption and the lack of labor income. They pay back and start saving during their working period.

It is assumed that immigrants enter the country after finishing their education. This is in accordance with Austrian data, where more than 85 % of all immigrants between ages 16-24 enter after finishing schooling. This holds for even 95 % of all immigrants over the age of 35. We assume that human capital is solely accumulated by education. There are also no intergenerational knowledge spillovers.

We distinguish two different cases. First, we assume that the number of schooling years, this means the length of education, is exogenously given and may vary between natives and immigrants. Since education determines the efficiency of labor, we aim to investigate how differences in education between natives and immigrants impact the welfare of the native population. This impact on the welfare varies for different generations. As a consequence, some generations win and others lose in terms of life cycle utility.

Later, we endogenize the education decision of natives and let them decide over their optimal number of schooling years.

While natives accumulate capital through saving, immigrants consume immediately what they earn during working life and receive pensions after retirement. Therefore, immigrants do not hold any assets in course of their life cycle. Here, we follow Ruist (2011). He argues that if it is assumed that immigrants are close to the bottom of the income structure, they have little incentives to save and invest in the host country, because saving incentives are correlated with income. Moreover, most immigrants remit much of their savings to their country of origin and some of them even intend to go back after some time.

It is assumed that immigrants and natives feature the same fertility and mortality rates. Immigrants' offspring is considered as native and therefore also acts as saver and capital owner.

5.1.1 Model

Age structured populations are studied in economics through overlapping generations models. These models allow for a realistic determination of life-cycle behaviors. Here, an age-structured population with immigration is considered. The household side of the closed economy is modeled by an overlapping generations framework. The firm sector is assumed to consist of one representative firm that uses aggregate capital and labor for the production of a single good.

Population Structure

The economy is populated by different age cohorts whose lifespan is uncertain but bounded. In the following, time is denoted by $t \geq 0$, where $t = 0$ is the starting time of the consideration of the economy, and a cohort's birth date is τ. The age of death is a random variable over $[0, \omega]$, where $\omega < \infty$ is the maximal reachable age. As Chapter 2 and Chapter 3 the probability of surviving of an individual born at time τ until age $a = t - \tau \in [0, \omega]$ is again denoted by $l(a) \in C^1[0, \omega]$ and does not change with time. Therefore, $-l'(a)$ is again the unconditional probability of dying at age a. Accordingly,

$$\mu(a) = -\frac{l'(a)}{l(a)},$$

is equal to the conditional probability of dying at age a, given that the individual survives until this age. Therefore, $\mu(a)$ is the density function of the random variable describing the age of death. For the probabilistic density function $\mu(a)$ it holds $\int_0^\omega \mu(a)\,\mathrm{d}a = l(0) = 1$.

Let $N(\tau, t)$ denote the number of natives [1] and $M(\tau, t)$ the number of individuals born outside the host country at time τ and still being alive at time $t > \tau$. Then $N(\tau, 0)$ and $M(\tau, 0)$ for $\tau \in [-\omega, 0]$ represent the age structure of natives and immigrants at the starting point of the economy. The native cohort with birth date τ changes over time according to

$$\frac{dN(\tau, t)}{dt} = -N(\tau, t)\mu(t - \tau). \tag{5.1}$$

[1]The term "number of people"' is strictly speaking not correct. To be more correct, one would have to speak of $N(\cdot, t)$ as a density representing the distribution of individuals along cohorts.

The births are given as a boundary condition for this equation at $\tau = t$ as

$$N(\tau, \tau) = \int_{\tau-\omega}^{\tau} f(\tau - s)\,(N(s, \tau) + M(s, \tau))\,\mathrm{d}s, \tag{5.2}$$

where $f(\cdot) \in C[0, \omega]$ denotes the age-specific fertility rate. Again we assume, see Chapter 2, that

$$NRR < 1.$$

Additionally, we assume that the children of immigrants are part of the native population and that immigrants and natives have the same age-specific fertility and mortality rates. The dynamics of the population $M(\tau, t)$ reads as,

$$\frac{dM(\tau,t)}{dt} = -M(\tau,t)\mu(t-\tau) + m(\tau,t), \quad M(\tau,\tau) = 0. \tag{5.3}$$

Here, $m(\tau, t)$ denotes the age-specific and possibly time-varying immigration profile. Then, the number of natives in the population at time t is given by

$$N(t) = \int_{t-\omega}^{\omega} N(t, \tau)\, d\tau,$$

and the number of immigrants is

$$M(t) = \int_{t-\omega}^{\omega} M(t, \tau)\, d\tau.$$

Individual Optimal Behavior

Let the utility from consumption $c > 0$ of any individual be denoted by $u(c)$ and consider a CRRA-utility function

$$u(c) = \begin{cases} \frac{c^{1-\sigma}}{1-\sigma} & \text{if } \sigma \in (0,1) \cup (1, +\infty), \\ \ln(c) & \text{if } \sigma = 1. \end{cases}$$

where σ is the risk aversion coefficient and $1/\sigma$ is the intertemporal elasticity of substitution between consumption over time. The higher $1/\sigma$, i.e. the lower the risk aversion coefficient, the more willing is the household to substitute consumption over time. Function $u(c)$ belongs to the family of constant relative risk aversion utilities (CRRA).

Since in our considerations the age of dying $a \in [0, \omega]$ is not a fixed number but a random variable, we adopt the expected utility hypothesis. fa an individual of cohort τ chooses a consumption profile $c(\tau, \cdot)$ such that her expected life-time discounted utility $\mathrm{E}[u]$ is maximized. The subjective discount rate is denoted by ρ. It defines how the preference for consumption decreases over the life time and is assumed to be constant.

Agents have perfect foresight meaning that agents perfectly forecast the rates of return on capital, $r(t)$, and labor, $w(t)$. Consequently, they supply labor such that its actual return, in form of wage, meets their expectations and the same holds for the saving decision determining the supplied capital and the expected return on capital. A typical life cycle consists of a schooling period, h, a work period and retirement after the fixed age R. We assume that people born at time τ are identical from the economic point of view.

An agent of the cohort born at time τ takes the real return on assets $r(t)$ and the wage rate $w(t)$ as given and chooses her consumption in order to maximize her expected utility

$$\int_{\tau}^{\tau+\omega} e^{-\int_{\tau}^{t}(\rho+\mu(\eta-\tau))d\eta} \frac{c^{\sigma}(\tau,t)}{\sigma} \, \mathrm{d}t \tag{5.4}$$

subject to the flow dynamics

$$\begin{aligned} \frac{da(\tau,t)}{dt} &= (r(t)+\mu(t-\tau))a(\tau,t) + (1-\theta)w(t)e(h,t-\tau) - c(\tau,t) \\ &\quad + \mathbb{I}_{[R,\omega]}(t-\tau)p(t), \; t \in (\tau, \tau+\omega). \end{aligned} \tag{5.5}$$

Here, $a(\tau,t)$ denote the financial assets of an agent born in τ at time t. We assume that each agent depending on her age $t-\tau$ is endowed with *efficient units of labor*, $e(h,t-\tau) : [0,R] \times [0,\omega] \to [0,\infty)$, i.e. for a given age $t-\tau$ function $e(h,\cdot)$ determines her productivity in the production process. Consequently, labor income equals $w(t)e(h,t-\tau)$.

For a fixed number of schooling years h we define

$$e(h,t-\tau) = \begin{cases} e^{g(h,t-\tau)} & \text{if } h \le t-\tau \le R, \\ 0 & \text{otherwise,} \end{cases}$$

where $g(h,t-\tau) := \delta_1 h + \delta_2(t-\tau-h) + \delta_3(t-\tau-h)^2$ and $\delta_1, \delta_2 > 0$ and $\delta_3 < 0$.

Therefore, the productivity, and consequently, the wage of an agent is a concave function in work experience measured in working years $t-\tau-h$. This representation of $e(h,t-\tau)$ follows Mincer (1974), where the logarithm of wages is modeled as the sum of a linear function of years of education h and a quadratic function of years of experience. During schooling and after retirement there is no supply of labor. A share θ of the labor income must be paid into a Pay-As-You-Go (PAYG) pension system. They benefit from the payments of the working cohorts when they retire in form of pension payments $p(t)$, $t \in [\tau+R, \tau+\omega]$.

Agents hold all their assets in form of annuities, cf. Yaari (1965). Since the life-insurance company redistributes the wealth of the agents who died to those who survived in the same age cohort, the real rate of return $r(t)$ is augmented by the age-specific mortality rate $\mu(t-\tau)$.

Agents have no assets when they enter the economy except of those who are alive at time $t = 0$:

$$a(\tau,0) \text{ given, if } \tau \in (-\omega, 0), \tag{5.6}$$

$$a(\tau,\tau) = 0, \text{ if } \tau \ge 0. \tag{5.7}$$

Moreover, one cannot die indebted[2]:

$$a(\tau,\tau+\omega) = 0. \tag{5.8}$$

[2]In fact we should require that, $a \ge 0$, but at the optimum equality holds.

Since the individual utility maximizing problem (5.4)–(5.8) constitutes a dynamic optimization problem, we apply Pontryagin's Maximum Principle to obtain the optimal consumption profile. The corresponding present value Hamiltonian reads as

$$\begin{aligned}\bar{H}(t,\tau,c,a,\lambda) &= e^{-\rho(t-\tau)} l(t-\tau)\frac{c^{1-\sigma}(\tau,t)}{1-\sigma} \\ &+ \bar{\lambda}(\tau,t)((r(t)+\mu(t-\tau))a(\tau,t) + (1-\theta)w(t)e(h,t-\tau) \\ &- c(\tau,t) + \mathbb{I}_{[R,\omega]}(t-\tau)p(t)).\end{aligned}$$

The first order necessary optimality conditions, see Theorem 2.2, are:

$$\frac{\partial \bar{\lambda}(\tau,t)}{\partial t} = -\bar{\lambda}(\tau,t)(r(t)+\mu(t-\tau)), \tag{5.9}$$

$$\frac{\partial \bar{H}(\tau,t)}{\partial c} = e^{-\rho(t-\tau)} l(t-\tau)c^{-\sigma}(\tau,t) - \bar{\lambda}(\tau,t) = 0. \tag{5.10}$$

From (5.10) we obtain the following expression for the optimal consumption profile

$$c(\tau,t) = e^{\frac{-\rho(t-\tau)}{\sigma}} \left(\frac{\bar{\lambda}(\tau,t)}{l(t-\tau)}\right)^{\frac{-1}{\sigma}}. \tag{5.11}$$

By introducing

$$\lambda(\tau,t) := \bar{\lambda}(\tau,t)e^{\rho(t-\tau)}\frac{1}{l(t-\tau)} \tag{5.12}$$

we derive a differential equation for the shadow price that is independent of the mortality function:

$$\frac{\partial \lambda(\tau,t)}{\partial t} = \frac{\partial \bar{\lambda}(\tau,t)}{\partial t}\frac{e^{\rho(t-\tau)}}{l(t-\tau)} + \rho e^{\rho(t-\tau)}\frac{\bar{\lambda}(\tau,t)}{l(t-\tau)} - e^{\rho(t-\tau)}\bar{\lambda}(\tau,t)\frac{l'(t-\tau)}{l^2(t-\tau)}$$

And by using the optimality condition (5.9) and relation (5.12) we find that

$$\frac{\partial \lambda(\tau,t)}{\partial t} = (-r(t)+\rho)\lambda(\tau,t)$$

holds. Therefore, the equation

$$\lambda(\tau,t) = e^{\int_\tau^t (-r(\eta)+\rho)\,\mathrm{d}\eta}\lambda_0(\tau) \tag{5.13}$$

holds. Inserting (5.13) into expression (5.11) and defining

$$\tilde{\lambda}(\tau) := \lambda_0^{\frac{-1}{\sigma}}(\tau),$$

yields an expression for $c(\tau,t)$ that is solely represented by exogenous variables except for the initial value $\lambda_0(\tau)$ which yet has to be determined

$$c(\tau,t) = e^{\frac{-1}{\sigma}\int_\tau^t (r(\eta)-\rho)\,\mathrm{d}\eta}\tilde{\lambda}(\tau). \tag{5.14}$$

We substitute consumption (5.14) into the budget constraint (5.5) and determine $\tilde{\lambda}(\tau)$ in such a way that the boundary conditions (5.6), (5.8) or alternatively (5.7)–(5.8) are fulfilled.

More precisely, for $(\tau,t) \in \{(\tau,t) : \tau \in (0,\infty), t \in [\tau,\tau+\omega)\}$ the optimal consumption is given by

$$c(\tau,t) = \tilde{\lambda}(\tau) e^{\frac{1}{\sigma}\int_\tau^t (r(\eta)-\rho)\,\mathrm{d}\eta},$$

with

$$\tilde{\lambda}(\tau) = \frac{\int_\tau^{\tau+\omega} e^{-\int_\tau^t (r(\eta)+\mu(\eta-\tau))\,\mathrm{d}\eta} \Big((1-\theta)w(t)e(h,t-\tau) + \mathbb{I}_{[R,\omega]}(t-\tau)p(t)\Big)\,\mathrm{d}t}{\int_\tau^{\tau+\omega} e^{\int_\tau^t (\frac{1}{\sigma}((1-\sigma)r(\eta)-\rho)+\mu(\eta-\tau))\,\mathrm{d}\eta}\,\mathrm{d}t},$$

and for $(\tau,t) \in \{(\tau,t) : \tau \in (-\omega,0), t \in (0,\tau+\omega)\}$

$$c(\tau,t) = \tilde{\lambda}(\tau) e^{\frac{1}{\sigma}\int_\tau^t (r(\eta)-\rho)\,\mathrm{d}\eta},$$

$$\tilde{\lambda}(\tau) = \frac{a(\tau,0)}{\int_0^{\tau+\omega} e^{\int_\tau^t (\frac{1}{\sigma}((1-\sigma)r(\eta)-\rho)+\mu(\eta-\tau))\,\mathrm{d}\eta}\,\mathrm{d}t} +$$
$$\frac{\int_0^{\tau+\omega} e^{-\int_\tau^t (r(\eta)+\mu(\eta-\tau))d\eta} \Big((1-\theta)w(t)e(h,t-\tau) + \mathbb{I}_{[R,\omega]}(t-\tau)p(t)\Big)\,\mathrm{d}t}{\int_0^{\tau+\omega} e^{\int_\tau^t (\frac{1}{\sigma}((1-\sigma)r(\eta)-\rho)+\mu(\eta-\tau))\,\mathrm{d}\eta}\,\mathrm{d}t}.$$

The case of $c(\tau,t) < 0$ for some t can be ruled out because it could only happen for $\tilde{\lambda}(\tau) < 0$ which would imply, see Equation (5.14), that the shadow price would be negative for all $t \in [\tau,\tau+\omega]$, thus $c(\tau,t)$ would be negative for all t, which contradicts the optimality of $c(\tau,\cdot)$.

Endogenous education decision

It is well-known that education plays an important role when it comes to economic performance of a country in general, and hence also when one aims to determine economic effects of immigration because immigrants, among other things, change the skill composition of the labor force. Whereas many models, cf. Lacomba and Lagos (2010); Razin and Sadka (2000), consider different skill groups to account of the educational heterogeneity in the population, here we explicitly model the accumulation of human capital of the agent which determines her efficiency in the production process and therefore is related to her skill level. While it is assumed that immigrants have an exogenous, fixed education level when they enter the country, a native agent endogenously chooses her optimal period of education. Children of immigrants can become higher educated than their parents when they choose to be so. In general, human capital can be accumulated through education and/or learning-by-doing. Here, we only allow for an education period at the beginning of the life time. In the beginning of the life-cycle agents dedicate their time to education and during that time they do not work. An increase of education leads to an increase in the efficiency units of labor.

Subsequently, we model the agent's decision on her optimal length of schooling. Doing so, a rational agent compares the future income stream of an additional schooling year with the potential income of quitting schooling now.

We assume that the decision of quitting school is once and for all. We obtain a necessary condition for the optimal number of schooling years by using the Lagrange method. Since the only heterogeneity here is the vintage of the cohort represented by τ, all members of a cohort receive the same education. Therefore, the schooling period is a function of the vintage τ, $h(\tau)$. Wherever we consider a fixed cohort τ we suppress the dependence on τ and simply write h.

The schooling problem of those belonging to the cohort τ reads as

$$\max_{h\in[0,R]} \int_0^\omega e^{-\int_0^s (\rho+\mu(\eta))d\eta} u(c(\tau,\tau+s))\,\mathrm{d}s,$$

subject to the budget constraint

$$\begin{aligned}&\int_0^\omega e^{\int_s^\omega (r(\tau+\eta)+\mu(\eta))d\eta}((1-\theta)w(\tau+s)e(h,s)+\mathbb{I}_{[R,\omega]}(s)p(\tau+s))\,\mathrm{d}s\\ &=\int_0^\omega e^{\int_s^\omega (r(\tau+\eta)+\mu(\eta))d\eta} c(\tau,\tau+s)\,\mathrm{d}s.\end{aligned} \tag{5.15}$$

Equation (5.15) is obtained by using the Cauchy formula for the linear differential equation in (5.5).

The corresponding Lagrangian $\mathcal{L}$ reads as

$$\begin{aligned}\mathcal{L}(h,\mu)=&\int_0^\omega (e^{-\int_0^s (\rho+\mu(\eta))\,\mathrm{d}\eta} u(c(\tau,\tau+s))\\ &+\mu(e^{\int_s^\omega (r(\tau+\eta)+\mu(\eta))\,\mathrm{d}\eta}((1-\theta)w(\tau+s)e(h,s)\\ &+\mathbb{I}_{[R,\omega]}(s)p(\tau+s)-c(\tau,\tau+s))))\,\mathrm{d}s.\end{aligned}$$

Hence by using the necessary condition $\frac{\partial\mathcal{L}}{\partial h}=0$, we obtain the optimal number of schooling years h^*,

$$\begin{aligned}&\int_h^R e^{\int_s^\omega (r(\tau+\eta)+\mu(\tau+\eta))\,\mathrm{d}\eta}(1-\theta)w(\tau+s)\frac{\partial e(h,s)}{\partial h}\,\mathrm{d}s\\ &=e^{\int_h^\omega (r(\tau+\eta)+\mu(\eta))\,\mathrm{d}\eta}(1-\theta)w(\tau+h)e(h,h).\end{aligned} \tag{5.16}$$

Observe that

$$\frac{\partial e(h,s)}{\partial h}=\begin{cases} 0 & \text{if } s<h, s>R\\ e^{g(h,s)}(\delta_1-\delta_2-2\delta_3 s+2\delta_3 h) & \text{if } s\in[h,R].\end{cases}$$

The right hand side of Equation (5.16) is the expected forgone income when not realizing h as the number of years to be spent at school and the left hand side determines the expected gain during the remaining working years from postponing the working entry age.

The term $\frac{\partial e(h,s)}{\partial h}$ in the left hand side of Equation (5.16) determines the resulting marginal increase in productivity for age s.

The optimal schooling time is not easy to determine explicitly for non-constant $r(t)$ and $w(t)$ and general, non rectangular survival laws. In our model $w(t)$ and $r(t)$ are determined endogenously through profit maximizing of the representative firm at every instant of time. In general, (5.16) can only be solved numerically and only provides a necessary condition for the optimal $h \equiv h^*(\tau)$. Moreover, existence of an optimal number of school years is not granted.

To simplify the implicit relation (5.16) for h we make the assumption of $\delta_3 = 0$, which reflects a linear increase of efficiency in experience. Then we obtain

$$(\delta_1 - \delta_2) \int_h^R e^{-\int_0^s r(\tau+\eta)\,\mathrm{d}\eta} e^{\delta_2(s-h)} l(s)(1-\theta) w(\tau+s)\,\mathrm{d}s$$
$$= e^{-\int_0^h r(\tau+\eta)\,\mathrm{d}\eta} l(h)(1-\theta) w(\tau+h). \tag{5.17}$$

For constant r and w and a rectangular survival function, (5.17) reduces to

$$(\delta_1 - \delta_2) \int_h^R e^{-rs} e^{\delta_2(s-h)}\,\mathrm{d}s = e^{-rh}.$$

Hence,

$$(\delta_1 - \delta_2) e^{-\delta_2 h} \frac{-1}{r-\delta_2} \left(e^{R(\delta_2 - r)} - e^{(\delta_2 - r)h} \right) = e^{-rh}.$$

Then, the explicit expression for the optimal solution h reads as

$$h^* = R + \frac{1}{r-\delta_2} \ln \left(1 - \frac{r-\delta_2}{\delta_1 - \delta_2} \right).$$

We see that h^* is independent of the constant wage rate w.

5.1.2 Government

Each time t the government collects taxes θ on labor to finance the implemented PAYG pension system. It is required that at any time t the government must have a balanced budget:

$$\theta w(t) \int_{t-R}^{t} \left(e(h^*(\tau), t-\tau) N(\tau,t) + e(h^M, t-\tau) M(\tau,t) \right)\,\mathrm{d}\tau$$
$$= p(t) \int_{t-\omega}^{t-R} \left(N(\tau,t) + M(\tau,t) \right)\,\mathrm{d}\tau.$$

5.1.3 Firms

In our model economy agents interact with firms. We apply the representative firm hypothesis. The firm produces output $Y(t)$ with labor $L(t)$ and capital $K(t)$ as input factors. The firm pays wages for labor input and borrows the services of capital from households and also pays for these services. The production function is of neoclassical type,

$$Y(t) = F(K(t), L(t)) = K^\alpha(t) L^{1-\alpha}(t),$$

where

$$K(t) = \int_{t-\omega}^{t} N(\tau,t)a(\tau,t)\,\mathrm{d}\tau,$$

$$L(t) = \int_{t-R}^{t} \Big(e(h^*(\tau), t-\tau)N(\tau,t) + e(h^M, t-\tau)M(\tau,t)\Big)\,\mathrm{d}\tau.$$

We assume here that immigrants and natives are perfect substitutes. Output can either be used for consumption or for increasing the capital stock. Firms maximize their profits by choosing capital $K(t)$ and labor $L(t)$ in an optimal way. The firm's problem reads as

$$\max_{K,L}\{Y(t) - R(t)K(t) - w(t)L(t)\}.$$

Factors receive their marginal products,

$$R(t) = F_K(K(t), L(t)),$$
$$w(t) = F_L(K(t), L(t)).$$

Let us denote by $k(t) = \frac{K(t)}{L(t)}$ the capital-(effective) labor ratio and let $f(k) := k^\alpha$. Therefore the factor returns can be obtained by

$$R(t) = f'(k(t)),$$
$$w(t) = f(k(t)) - f'(k(t))k(t).$$

5.1.4 Numerical Experiments

Subsequently, we consider a benchmark case where at the moment of shock the economy as well as the population are in a steady state.

Demography

In Arthur and Espenshade (1988) it was shown that any population with below-replacement fertility and a constant number of annual immigrants with a fixed age distribution as well as constant age-specific mortality rates, eventually converge to a stationary population.

Here, for each $a = t_0 - \tau$ we calibrate $N(a, t_0)$ with the number of members of cohort τ in the native female population of Austria in 2001 and $M(a, t_0)$ is the number of individuals born outside the country of the corresponding cohort τ [3]. We simulate equations (5.1) – (5.3) with constant age-specific fertility rates $f(a)$, where again $a = t - \tau$, and constant age-specific mortality rates $\mu(a)$ and a constant inflow of immigrants $m(a)$ until a stationary population is reached, see Figure 5.1. In the following, time is measured in years.

For the fertility rates $f(a)$ and the immigration rates $m(a)$ we took linearly interpolated Austrian data of 2008. For the numerical examples below we follow Boucekkine et al. (2002) and consider a survival function of the form

$$l(a) = \frac{e^{-a\mu_0} - \epsilon}{1 - \epsilon},$$

[3]No later data could be found for $M(a, t_0)$.

with $\epsilon > 1, \mu_0 < 0$. This survival law fulfills $l(0) = 1$ and ω is determined such that $l(\omega) = 0$ holds,

$$\omega = -\frac{\ln(\epsilon)}{\mu_0}.$$

Therefore, $\lim_{a\to\omega} \mu(a) = +\infty$. We fully specify $l(a)$ by setting $\mu_0 = 0.068$ and $\omega = 80$. For these specifications, the net reproduction rate (NRR) is approximately 0.7, which is below replacement level.

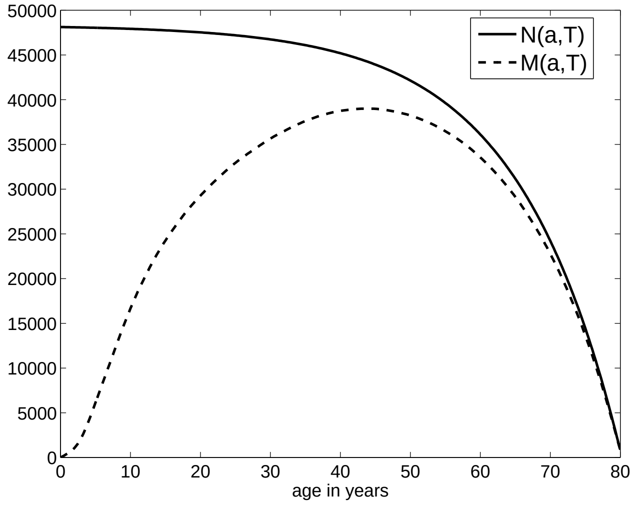

Figure 5.1: Steady state population age structure at time of the shock

Figure 5.1 shows the steady state age structure of the two sub-populations. Notice that in the stationary population the share of immigrants is about 35%.

Economic Parameters

We set the initial assets profile $a(\tau, 0)$ to the steady state solution,

$$a(\tau, 0) = a(t - \tau),\ \tau < t,$$

before the immigration shock. Table 5.1 summarizes the important parameters the calculations we present in this section. In the economic model age 0 corresponds to the real age of 16, because this is the age when compulsory schooling typically ends. The period of the life-cycle before age 16 is not modeled explicitly. The consumption of these agents is assumed to be part of the parents consumption.

During the additional education time agents accumulate debts due to their lack of labor income.

Immigration shock

We normalize time such that the time when the immigration shock happens is $t = 0$. The immigration shock is modeled as a doubling of the number of immigrants from

Parameters	*Core Model*
education migr. h_i^m	0,6,11
retirement age R	49 (65-16)
μ_0	0.068
life span ω	80 (96-16)
tax θ	0.12
CES σ	1
capital share α	1/3
δ_1	0.041
δ_2	0
δ_3	0
time pref. rate ρ	0
depreciation rate δ	0
shock period	$t \in [100, 105]$

Table 5.1: Parameter calibration

a pre-shock value of 35000 annual immigrants and lasts for 5 years. During this time twice as many immigrants enter the country while the age structure is held constant.

Such a scenario could be compared to the years 1989 -1993, where due to the war in former Yugoslavia, the numbers of net migrants to Austria where in some years even three times as high. Figure 5.2 shows how the number of natives $N(t)$ and immigrants $M(t)$ change over time as a consequence of the shock. The immigration shock leads in later consequence to a higher number of natives, since the immigrants children are assumed to integrate themselves fully in the host country.

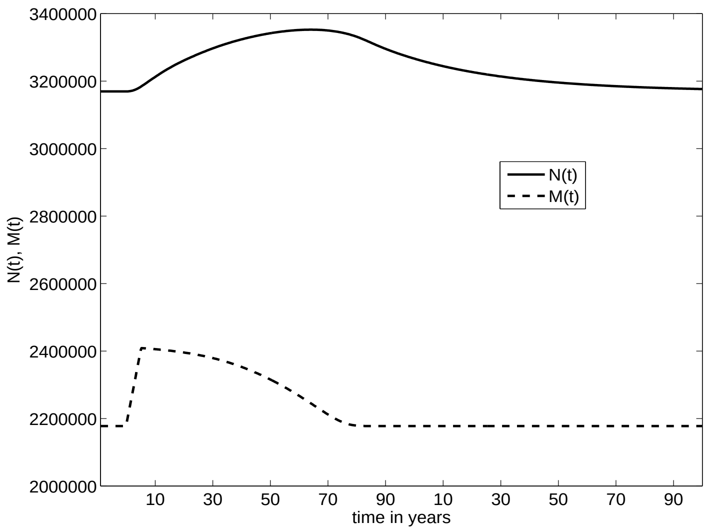

Figure 5.2: Number of natives and immigrants over time: natives (solid); immigrants (dashed)

Numerical Results

We first analyze the case where also the education of the natives is exogenously given and consider three scenarios with respect to the educational achievements of immigrants. First, we assume that immigrants who enter have only the basic education, $h_1^m = 0$, then we consider that immigrants obtain the same number of schooling years as natives, $h_2^m = h = 6$, and the last case reflects an inflow of immigrants with a high education i.e. $h_3^m = 11$ [4]. We then compare the utility changes of the different cohorts for all three scenarios.

Due to the increase in the number of immigrants at the beginning of the shock, i.e. at time $t = 0$, the capital (effective) labor ratio $k(t)$ decreases. Consequently, the interest rate goes up and the wage rate goes down as can be seen in Figure 5.3. This favors those who are owners of capital and affects adversely workers.

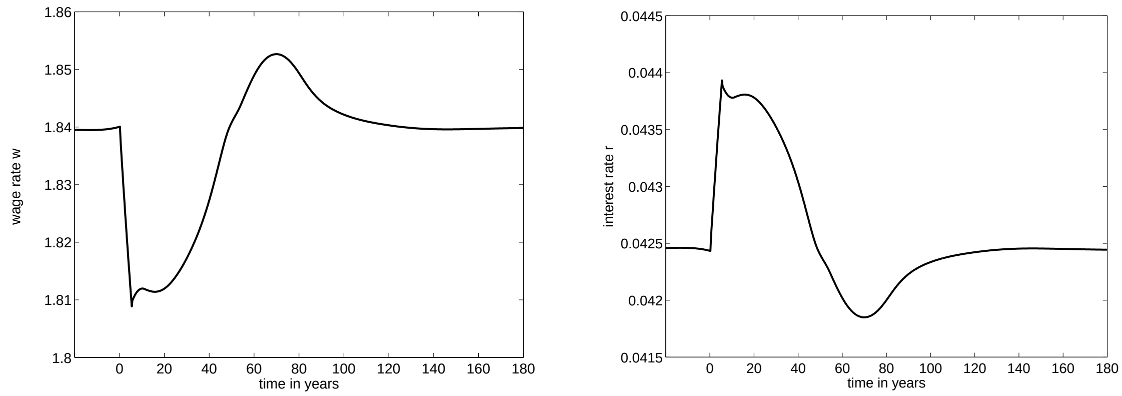

Figure 5.3: Wage rate (left) and interest rate (right) over time for h_1^m

Figure 5.4 represents the immigration shock effects for the welfare of different cohorts. It shows the relative change in life time utility over time for the various cohorts compared to the steady state value for the various h_i^m.

The figure shows that the cohorts who are middle aged and old at the time of the shock benefit. This is because they are the owners of capital at that time. The welfare is the highest for those who are currently 46 (30 + 16) years old at the time of the shock. The second peak in Figure 5.4 is due to the flattening out of the number of immigrants and the fact that equipped offspring of the immigrants enter the economy. This causes a peak in the capital labor ratio. As a consequence the interest rate r decreases and the wage rate w increases. However, this peak is already damped as compared to the initial one.

The cohorts which have the severest drawbacks of the immigration shock are those who enter the economy in the decade after the shock. This is because they face very high interest rates at the beginning of their lifetime, when they actually accumulate

[4] However, this last scenario $h_3^m = 11$ does not really suit our setting, because high educated immigrants, might as well (similar to high educated natives) accumulate savings and therefore would also contribute to the capital stock.

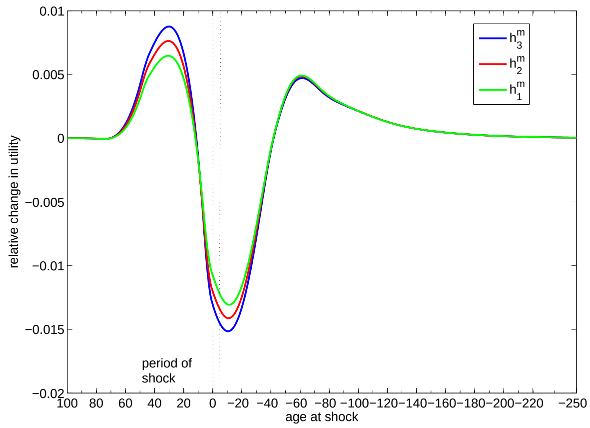

Figure 5.4: Relative change in welfare for different cohorts: $h_1^m = 0$ (green), $h_2^m = 6$ (blue), $h_3^m = 11$ (red) for $h = 6$

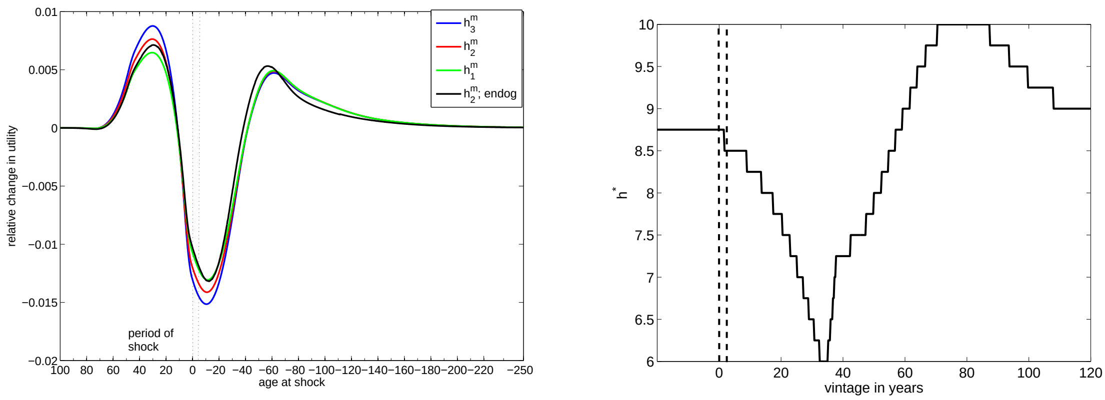

Figure 5.5: Relative change in welfare for different cohorts: $h_1^m = 0$ (green), $h_2^m = 6$ (blue), $h_3^m = 11$ (red), $h_2^m = 0$ with endogenous $h^*(\tau)$ (left). Optimal schooling time (right)

debts because they are still educating themselves. Moreover, they face a very low wage rate during their working life and decreasing interest rates. With respect to the different education of the immigrants, one can say that the higher the education of the immigrants, the more severe are the effects on the utility.

In a next step, we endogenize the education decision of the native individuals in order to see how the increase of labor effects the skill composition in the country. In Figure 5.5(a) the utility changes for various cohorts in case of an endogenous determination of the schooling period by the native agents is depicted. It shows that endogenous education slightly decreases the loss of future generations and the gain of old generations. Therefore, by choosing their education optimally young natives can damp the negative effect of the immigration shock on their life time utility.

In Figure 5.5(b) the change in the length of the schooling period of the various cohorts of vintages τ younger than the shock is depicted. One observes that at the shock the schooling period goes down. This fall is then followed by a period where cohorts go even longer to school then before the shock.

5.1.5 Conclusions

The model presented in this section focuses on the welfare effects induced by an immigration shock. The shock is modeled by an increase of the number of immigrants for a short period of time. By developing a continuous time overlapping generations model for a closed economy, we determine the resulting changes in life-time utility of different age cohorts. Numerical results for Austrian data are provided.

We conclude that the immigration shock is welfare improving for those cohorts being at the end of their working life or already retired. They benefit from the increased interest rate. Moreover, retirees may have an additional benefit from the increased tax payments of the higher number of workers due to the incoming immigrants. However, since the wage rate goes down, pension payments might also go down. So the effect of the immigration shock on pensions of retirees is not unambiguous.

The shock leads to the highest decrease in life cycle utility of those cohorts born during or after the shock. This result is interesting in terms of immigration policies. It implies that young cohorts would prefer closed borders whereas older cohorts would not. Moreover, we may conclude that the increased number of immigrants during the shock leads to an increase in the work force and therefore the wage rate goes down due to increased competition. This is accompanied by an increase in the interest rate because of the induced reduction in the capital-labor ratio.

5.1.6 Outlook

The present model can be extended in various ways. So far it only represents the first attempt to depict possible effects of an inflow of individuals who change the demographic and economic situation in the host country.

So far it is assumed that immigrants' offspring make the same life-cycle decisions as the natives of the same cohort do and are therefore considered as natives. In further investigations one could relax this assumption and investigate how the results change if a share of the children still behave as their parents did.

Moreover, so far we haven't accounted of emigration.

By considering a different pension system rather than the contribution defined PAYG system one may get more insight in what is driving the results and how they depend on this modeling assumption.

It would be of particular interest how the results depend on the steady-state age structure of the population. Since changes in the two factor returns, interest rate r and wage rate w, affect different cohorts, one may expect that the intermediate-term effects induced by the shock depend critically on the age structure.

5.2 Long-run impact of age-specific immigration

The subsequent model investigates the long-run impact of immigration by explicitly modeling the life-cycle of the immigrants in the host country.

The aging of the populations of Western countries goes hand in hand with the aging of their labor forces. This has severe effects on the sustainability of the social security system and especially the pension schemes. In many countries the pension system is a so-called pay-as-you-go (PAYG) system, where the currently working population pays for those in retirement. Hence, a growing number of older persons in comparison to a shrinking labor force, caused by low mortality rates at older ages and additionally low fertility rates, implies a shrinking money flow into the system. In what follows, the Austrian PAYG pension system will be mimed.

One remedy may be to step up immigration. It is a common belief that since the age-structure of immigrants is younger than the one of the native population, immigration could help to reduce the fiscal imbalance caused by the aging process. However, clearly these immigrants would also grow older and hence many people argue that in the long-run there would be no positive effect of immigration with respect to the fiscal balance.

In Storesletten (2000) it was shown that for a calibration with US data immigration is slightly beneficial for the government finances. Jinno (2013) investigated how an immigration policy which consists of the admission of unskilled immigrants, whose children incur assimilation costs in order to become skilled workers, positively influences the net pension benefits for native residents and immigrants under a defined-benefit pension system. They find that native residents do not always become net beneficiaries, even if the government admits an unlimited number of immigrants. Jinno (2013) also shows that this result does not hold in a defined contribution system.

Empirical studies on the effect of immigration on the Austrian economy have already been made, for example, by Winterebmer and Zweimüller (1996) and Mayr (2005). In Winterebmer and Zweimüller (1996), it was investigated how an increase in immigration affects wages of young native blue collar workers in Austria. It is found that in regions, industries, or firms with a higher share of foreign workers, natives earn higher wages.

In an empirical paper, Mayr (2005) used the general accounting method to study the intertemporal fiscal impact of immigration to Austria. It is concluded that under the assumption that future immigrants resemble those of the current immigration the total fiscal effect of immigration is positive. The reasons for the positive effect of immigration are (i) the young age structure, and (ii) lower per capita net transfer payments during retirement compensating for lower per capita net tax payments during working age. We try to replicate these empirical findings in a theoretical model where we explicitly take into account the age structure of immigrants and the fact that immigrants qualify for lower pensions.

Hence, in contrast to the aforementioned theoretical papers, where typically two- or three-period OLG models are considered and immigrants are assumed to arrive either in period one or period two, which roughly distinguishes between immigrants who arrive as children or in adulthood, we explicitly model how the age-structure of the immigrants affects the host country. With the subsequent model one may tackle the following questions:

- x What is the long-term effect of immigration on the sustainability of the pension system measured in terms of the social secruity rate and the pensions-to-output ratio?
- x Are immigrants net beneficiaries or net payers of the pension system?
- x Since the age of the immigrants has a strong impact on the age structure and size of a population the obvious question is whether the age structure of the immigrants matters for the pension system, and how?

While in Jinno (2013) a two-period OLG model is considered, we consider a continuous time OLG model, where a continuum of overlapping generations coexist at the same time. This allows a more accurate modeling of the demography.

Clearly, under the assumption of preserving below-replacement fertility, immigration is needed in order to avoid a major shrinking and aging of the population. Hence, we assume that fertility would remain on low levels, and investigate how age-specific immigration rates would be able to compensate for this. Moreover, we assume that immigrants have higher fertility than natives.

We explicitly model a pension system which realistically resembles current practice in many European countries and consider a pay-as-you-go pension system with defined benefits such as it is the case in Austria. In this work we focus on the steady state, and it is assumed that the government budget is balanced. Hence, there are no debts.

From an economic point of view, immigrants and natives differ significantly. Hence, in contrast to other macroeconomic models such as Fehr et al. (2003), where the focus was solely on the macroeconomic aspect of immigration, we explicitly distinguish between natives and immigrants in the model. As a matter of fact, they participate quite differently in the pension system. While natives spend their majority or even the whole working life in their home country and consequently earn high pensions, many immigrants arrive in the middle of their productive period and hence qualify for lower pensions in the host country. In what follows, we investigate how this difference is reflected in terms of the pension system. Moreover, immigrants and natives also differ during their productive period, which affects their contributions to the pension system. We model this difference by allowing that immigrants and natives do not act as perfect substitutes in the production process and they may have different productivity profiles. This leads to different wages for immigrants and natives.

In the numerical example we again focus on the Austrian case. We conclude that immigrants are net contributors to the pension system. In contrast to the native population the immigrant population pays more into the pension system then it earns and hence immigration contributes for the closing of the financial gap caused by the aging of the population.

In a stylized scenario, we find that although immigrants who enter in their mid-thirties spend a shorter time working in the host country and lead to a sharp increase of the aging ratio they still lead to a smaller social contribution rate and a lower pensions-to-output ratio in comparison to a scenario where all immigrants enter in their early twenties. However, we also find that with immigration alone it is not possible to keep the social contribution rate on current levels. Hence, additional measures have to be taken for a balanced pension system.

5.2.1 Population dynamics

In this section, we describe the demographic side of the model which is exogenous to the economic model. We consider a bench mark demographic scenario, where under the assumption of a constant annual inflow of immigrants, we denote by $M(a, a^*)$ the number of immigrants at age a who have arrived in the country at age $a^* \leq a$. Once immigrants have migrated, they stay in the host country for the rest of their lives.

The age-specific immigration density $m(\cdot)$ fulfills

$$\int_{a_{min}}^{a_{max}} m(a^*)\,\mathrm{d}a^* = 1, \quad m(a^*) \geq 0.$$

Moreover, we assume that age-specific immigration patterns $m(\cdot)$ as well as fertility $f(\cdot)$ and mortality $\mu(\cdot)$ are time-invariant. Natives' fertility and mortality are such that fertility is again under the replacement level, i.e. $NRR < 1$ holds, see Chapter 2 for the definition of NRR. Moreover, we assume that age-specific mortality rates of natives and immigrants are the same.

The resulting population is stationary through immigration, cf. Schmertmann (1992), and consists of natives $N(a)$ and immigrants $M(a) = \int_{a_{min}}^{a_{max}} M(a; a^*)\,\mathrm{d}a^*$. The parameters a_{min} and a_{max} are the minimal and maximal age of immigration, where $a \wedge b := \min\{a, b\}$ and $0 < a_{min} < a_{max} < R$ holds. Here, $\omega = 110$ denotes the maximal attainable age. The number of annual intakes is given by the exogenous parameter I which determines together with $m(\cdot)$ the size of the steady-state population. Hence, the changes in age structure of the immigrants follows the subsequent dynamic law

$$M'(a; a^*) = -\mu(a)M(a; a^*),\ a^* < a < \omega, \tag{5.18}$$

$$M(a^*; a^*) = m(a^*)I,\ a^* \in [a_{min}, a_{max}]. \tag{5.19}$$

Here, $M'(a; a^*)$ denotes the derivative with respect to age a. The age structure of the native population fulfills

$$N'(a) = -\mu(a)N(a),\ 0 < a < \omega, \tag{5.20}$$

$$N(0) = \int_0^{\omega} \left(f(a)N(a) + f_M(a) \int_{a_{min}}^{a_{max} \wedge a} M(a; a^*)\,\mathrm{d}a^* \right) \mathrm{d}a, \tag{5.21}$$

where $N(a)$ gives the number of natives of age a and $f_M(\cdot)$ is the age-specific fertility of immigrants. Equation (5.21) gives the number of births in the population, where it is assumed that the children of immigrants are considered as natives. This means that while immigrants of the first generation have a higher fertility than the average native, $f_M \geq f$, their children, i.e. immigrants of the second generation, show already no significant difference in their child bearing behavior compared with natives. According to Sobotka (2008), the fertility of immigrants converges to the fertility levels of the host country. We assume that this assimilation happens within one generation which is also indicated by some data as mentioned in Sobotka (2008).

According to the Cauchy formula, the solution of (5.18) – (5.19) reads as follows:

$$M(a,a^*) = Im(a^*)l(a),\ a^* < a < \omega, \tag{5.22}$$

and hence

$$\begin{aligned} M(a) &= \int_{a_{min}}^{a_{max}} M(a;a^*)\,\mathrm{d}a^*, \\ &= \int_{a_{min}}^{a_{max}\wedge a} \frac{l(a)}{l(a^*)} m(a^*) I\,\mathrm{d}a^*,\ a_{min} < a < \omega. \end{aligned}$$

As before $l(a) = e^{-\int_0^a \mu(s)\,\mathrm{d}s}$. For the solution of (5.20) it holds that

$$\begin{aligned} N(a) &= e^{-\int_0^a \mu(s)\,\mathrm{d}s} N(0) \\ &= l(a)N(0). \end{aligned}$$

Inserting this into (5.21) gives

$$\begin{aligned} N(0) &= N(0)\int_0^{\omega} f(a)l(a)\,\mathrm{d}a + \int_0^{\omega} f_M(a)M(a)\,\mathrm{d}a, \\ &= \frac{\int_0^{\omega} f_M(s)M(s)\,\mathrm{d}s}{1-\int_0^{\omega} f(s)l(s)\,\mathrm{d}s}, \end{aligned}$$

and hence

$$N(a) = \frac{\int_0^{\omega} f_M(s)M(s)\,\mathrm{d}s}{1-\int_0^{\omega} f(s)l(s)\,\mathrm{d}s} l(a),\ 0 < a < \omega. \tag{5.23}$$

5.2.2 The pension system

The literature distinguishes between two different prototypical social security systems: the pay-as-you-go (*PAYG*) system and the fully-funded system. In the fully-funded system, the contributions of the individuals earn the market interest. They accumulate over their working period and are paid out after retirement. In the PAYG system the currently working people finance the pensions of the retired. Due to this, the PAYG system leads to a crowding out of capital. There are two variants of the PAYG system: benefit defined (BD) and contribution defined (CD). In the BD version the pensions are fixed and the corresponding social security tax rate is determined by the general equilibrium mechanism. In the CD system the opposite holds true meaning that the pension benefits are calculated such that the government's financial goals are reached.

Subsequently, we aim to mime the Austrian pension system. The Austrian pension system consists of three pillars, where the first and dominant pillar is a PAYG. There were three major reforms: 2000, 2003, 2004, which lead to changes in N_B and p_1, see Knell et al. (2006). The following notions are of importance:

x N_B is the *assessment period*, i.e. it equals the number of an individual's working years used for calculating the pension entitlements,

x the so-called *assessment base* is derived from the average earnings over the assessment period N_B,

x p_1 is the annual *accrual rate* of the pension, which is the percentage of the annual wage payed out as pension. The accrual rate annually adds up over the whole working period to a maximum of 80%. Currently, $p_1 = 1.78\%$

Hence, natives' pensions are given by

$$p_N = p_1 \frac{R}{N_B} \int_{R-N_B}^{R} e_N(a) w_N \, \mathrm{d}a,$$

where $e_N(a)$ are the efficiency units of labor and w_N is the wage rate. Immigrants' pension payments are determined as follows. In Austria, immigrants who arrive later than 15 years before the statutory retirement age R, i.e. $a^* \in (R-15, a_{max}]$ get a minimal pension. Otherwise public pension payments p depend on the age of arrival a^*:

$$p(a^*) := \begin{cases} p_1 \frac{R-a^*}{N_B} \int_{R-N_B}^{R} w_M e_M(a;a^*) \, \mathrm{d}a & \text{if } 0 \leq a^* \leq R - N_B, \\ p_1 \int_{a^*}^{R} w_M e_M(a;a^*) \, \mathrm{d}a & \text{if } R - N_B < a^* \leq R - 15, \\ p_1 \int_{R-15}^{R} w_M e_M(a;a^*) \, \mathrm{d}a & \text{if } R - 15 < a^* < a_{max}, \end{cases} \tag{5.24}$$

where again $e_M(a;a^*)$ are the efficiency units of labor and w_M is the wage rate. This follows the set up of the third pillar of the Austrian pension system. It holds that $p_N = p(0)$. Here, for the sake of simplicity, we neglect pension portability from the home country to the host country. Pension portability would lead to a higher income of immigrants during their retirement but it would not affect the government budget since this part of the immigrant's pension would be financed by the sending country. Hence, we would have to deal with an open economy framework. For a discussion of pension portability see, for example, Jousten (2012). As a consequence of these assumptions, the pensions received by an average immigrant are considerably smaller and depend on her age of arrival in the host country.

5.2.3 Utility maximization of natives

Native households maximize their life-time utility by choosing the age-dependent consumption profile. Households are comprised of one adult and dependent children, and the number of households of a certain age is determined by the population structure. The number of new households entering the economy is determined by the country's fertility, mortality and immigration rates. It is assumed that children become independent, enter into the labor market and start a new household at age $a_0 = 18$. This is in accordance with empirical findings, cf. Sambt and Prskawetz (2011).

Let the utility from consumption $c_N > 0$ of any individual be denoted by $u(c_N)$. In the following we choose the specific utility function

$$u(c_N) = \begin{cases} \frac{c_N^{1-\sigma}}{1-\sigma} & \text{if } \sigma \in (0,1) \cup (1,+\infty), \\ \ln(c_N) & \text{if } \sigma = 1, \end{cases}$$

where σ is the risk aversion coefficient. For this particular utility function it is related to the intertemporal elasticity of substitution which is simply $1/\sigma$. The intertemporal elasticity of substitution gives the change in marginal consumption growth with respect to marginal utility growth. The higher $1/\sigma$, i.e. the lower the risk aversion coefficient, the more willing is the household to substitute consumption over time. When $\sigma \to \infty$, this is the case of infinite risk aversion. Function $u(c)$ belongs to the family of constant relative risk aversion utilities (CRRA). It is assumed that the dependence of the children is not directly reflected in the utility function as for example considered in Sánchez-Romero et al. (2013).

Since in our considerations the age of dying $a \in [0, \omega]$ is uncertain, we adopt the expected utility hypothesis. Therefore, an individual chooses a consumption profile $c(\cdot)$ such that her expected life-time discounted utility $\mathrm{E}[u]$ is maximized. The subjective discount rate is denoted by ρ, and is assumed to be constant. It gives the impatience of households for consumption. A high impatience means that households weigh later time points in life less. This leads to a higher consumption at earlier ages compared to a scenario with a low value of ρ. Hence, ρ defines how the preference for consumption decreases over the life time.

Again we recall that $l'(a)$ is the unconditional probability of dying at age a. We denote by $U_N(c_N)$ the expected, discounted and aggregated utility from consumption over the whole life horizon:

$$\begin{aligned} U_N(c_N) &= \int_0^{\omega} -l'(s) \int_0^{s} e^{-\rho a} u(c_N(a)) \,\mathrm{d}a \,\mathrm{d}s, && (5.25) \\ &= -\int_0^{\omega} e^{-\rho a} u(c_N(a)) \int_a^{\omega} l'(s) \,\mathrm{d}s \,\mathrm{d}a, && (5.26) \\ &= \int_0^{\omega} e^{-\rho a} u(c_N(a)) l(a) \,\mathrm{d}a, && (5.27) \\ &= \int_0^{\omega} e^{-\int_0^a (\rho+\mu(\tau))d\tau} u(c_N(a)) \,\mathrm{d}a. && (5.28) \end{aligned}$$

In Equation (5.26) we changed the order of integration. Moreover, it holds that $l(a) = -\int_a^{\omega} l'(s)\, ds$.

Households have perfect foresight and perfectly forecast the rates of return on capital, r, and labor, w. Consequently, they make their saving decisions in such a way that they meet their expectations on the return of capital. Individuals start working with 18 and retire at the fixed age R. They take the real return on assets r and the wage rate w as given and choose their consumption in order to maximize their expected utility:

$$\int_0^{\omega} e^{-\int_0^a (\rho+\mu(s))ds} \frac{c_N^{1-\sigma}(a)}{1-\sigma} \,\mathrm{d}a, \tag{5.29}$$

subject to the flow dynamics

$$k'_N((a) = ((r+\mu(a))k_N(a) + (1-\theta)w_N(a)e_N(a) - c_N(a) + \mathbb{I}_{[R,\omega]}(a)p_N, \ a \in (0,\omega) \tag{5.30}$$

Individuals have no assets when they enter the economy:

$$k_N(0) = 0. \tag{5.31}$$

Moreover, individuals cannot die indebted[5]:

$$k_N(\omega) = 0. \tag{5.32}$$

In (5.30), r denotes the rate of return of capital. We assume that each individual depending on her age a is endowed with $e_N(a)$ *efficient units of labor*, i.e. for a given age a function $e_N(\cdot)$ determines her productivity in the production process. Consequently, gross labor income equals $y_N(a) = w_N e_N(a)$. During schooling and after retirement the individual does not supply labor.

Individuals hold all their assets in form of annuities, cf. Yaari (1965). Due to these life-insurances, the wealth of the individuals who died are redistributed to those who survived in the same age cohort. Hence, the real rate of return r is augmented by the age-specific mortality rate $\mu(a)$.

During working life individuals pay a share θ of their labor income into a contributions defined PAYG pension system. They benefit from the payments of the working cohorts when they retire in form of pension payments p_N. Using the Cauchy formula for the life cycle profile of the financial assets in Equation (5.30), we obtain that

$$k(a) = \int_0^a e^{\int_s^a (r+\mu(\eta))\,\mathrm{d}\eta} \Big((1-\theta)w_N e_N(s) - c_N(s) + p_N \mathbb{I}_{[R,\omega]}(s) \Big)\,\mathrm{d}s. \tag{5.33}$$

holds.

Optimal consumption profile

The corresponding present-value Hamiltonian of problem (5.29)–(5.32) reads as

$$\begin{aligned} H_N = e^{-\rho a}u(c_N(a)) + \lambda_N(a)((r+\mu(a))k_N(a) \\ + (1-\theta)w_N e_N(a) - c_N(a) + \mathbb{I}_{[R,\omega]}(a)p_N). \end{aligned}$$

Again we apply Pontryagin's maximum principle, see Theorem 2.2 in Chapter 2, and obtain the first order necessary optimality conditions:

$$\begin{aligned} \lambda'_N(a) &= -\lambda_N(a)(r+\mu(a)), \\ \frac{\partial H_N}{\partial c_N} &= e^{-\rho a} l(a) c_N^{-\sigma}(a) - \lambda_N(a) = 0. \end{aligned} \tag{5.34}$$

Hence, we obtain

$$\lambda_N(a) = l(a)e^{-ra}\lambda_0.$$

[5]In fact we should require that $k_N \geq 0$ but at the optimum equality holds.

From (5.34) we obtain the following expression for the optimal consumption profile

$$c_N(a) = \left(e^{\rho a}\frac{\lambda_N(a)}{l(a)}\right)^{-\frac{1}{\sigma}}. \tag{5.35}$$

Hence, it holds that

$$c_N(a) = e^{\frac{(r-\rho)a}{\sigma}} c_0, \tag{5.36}$$

where $c_0 := \lambda_0^{-\frac{1}{\sigma}}$. In order to determine c_0 we substitute consumption (5.35) into the dynamic budget constraint (5.30). Then c_0 should be determined in such a way that boundary conditions (5.31)–(5.32) are fulfilled. To this end we use (5.33) with $a = \omega$ and express

$$c_0 = \frac{\int_0^\omega e^{-ra} l(a)((1-\theta)w_N e_N(a) - p_N \mathbb{I}_{[R,\omega]}(a))\,\mathrm{d}a}{\int_0^\omega l(a) e^{(r(1+\frac{1}{\sigma})-\frac{\rho}{\sigma})a}\,\mathrm{d}a}.$$

With the so determined c_0, formula (5.36) gives an explicit representation of the optimal consumption of the native population.

5.2.4 Remaining Life Time Utility Maximization of Immigrants

Now let us turn to the immigrant's perspective. We assume that immigrants, once they have migrated to a new country, remain their for the rest of their lives. It is assumed that they arrive without any assets[6] and maximize their rest of life time utility out of consumption. This means, that we do not model the immigrants' life time in the home country. This is consistent because of our assumption of a closed economy and therefore do not know the economic characteristics of the rest of the world. In contrast to many other models, where it is assumed that immigrants only enter at the beginning of the life-cylce, see e.g. Fehr et al. (2003), we assume that immigrants enter the country at ages $a^* \in [a_{min}, a_{max}]$. They arrive without any assets and after their arrival they choose optimally their consumption level $c_M(\cdot; a^*)$ over the remaining life cycle. Similarly as for natives, the life-time utility of consumption $c_M(\cdot)$ is

$$\begin{aligned}
U_M(c_M) &= \int_{a^*}^{\omega} -l'(s) \int_{a^*}^{s} e^{-\rho a} u(c_M(a))\,\mathrm{d}a\,\mathrm{d}s, \\
&= -\int_{a^*}^{\omega} e^{-\rho a} u(c_M(a)) \int_{a}^{\omega} l'(s)\,\mathrm{d}s\,\mathrm{d}a, \\
&= \int_{a^*}^{\omega} e^{-\rho a} u(c_M(a)) l(a)\,\mathrm{d}a.
\end{aligned}$$

Hence, the utility maximizing problem reads as,

$$\max_{c_M} \int_{a^*}^{\omega} e^{-\int_0^a (\rho+\mu(\eta))\,\mathrm{d}\eta} \frac{c_M^{1-\sigma}(a)}{1-\sigma}\,\mathrm{d}a, \tag{5.37}$$

[6] This corresponds to the fact that immigrants use their assets for the journey to the host country or leave the assets for their dependents in the home country

subject to

$$\begin{aligned} k'_M(a;a^*) &= (r+\mu(a))k_M(a;a^*) + (1-\theta)w_M e_M(a;a^*) \\ &\quad - c_M(a;a^*) + \mathbb{I}_{[R,\omega]}(a)p(a^*), \end{aligned} \tag{5.38}$$

$$k_M(a^*;a^*) = 0 \quad k_M(\omega;a^*) = 0. \tag{5.39}$$

We assume that each individual depending on her age a is endowed with $e_M(a;a^*)$ *efficient units of labor*, i.e. for a given age a function $e_M(\cdot;a^*)$ determines her productivity in the production process. Consequently, gross labor income equals $y_M(a;a^*) = w_M e_M(a;a^*)$. In general, we allow for dependence of productivity on a^* as empirically found in Storesletten (2000) for the US. The constant θ denotes the wage tax and p are the pensions as explained above. Using again the Cauchy formula for the life cycle profile of the financial assets of immigrants in Equation (5.38) we obtain that

$$k_M(a^*;a^*) = \int_{a^*}^{a} e^{\int_s^a (r+\mu(\eta))\,\mathrm{d}\eta} \Big((1-\theta)w_M e_M(s;a^*) - c_M(s;a^*) + p_M(a^*)\mathbb{I}_{[R,\omega]}(s)\Big)\,\mathrm{d}s. \tag{5.40}$$

Optimal consumption profile

The corresponding present-value Hamiltonian reads as

$$\begin{aligned} H_M &= e^{-\rho a}u(c_M(a;a^*)) + \lambda_M(a;a^*)((r+\mu(a))k_M(a;a^*) \\ &\quad + (1-\theta)w_M e_M(a;a^*) - c_M(a;a^*) + \mathbb{I}_{[R,\omega]}(a)p_M(a^*)). \end{aligned}$$

The first order necessary optimality conditions, see Theorem 2.2 in Chapter 2, are:

$$\begin{aligned} \lambda'_M(a;a^*) &= -\lambda_M(a;a^*)(r+\mu(a)), \\ \frac{\partial H_M}{\partial c_M} &= e^{-\rho a}l(a)c_M^{-\sigma}(a;a^*) - \lambda_M(a;a^*) = 0. \end{aligned} \tag{5.41}$$

Hence, we obtain

$$\lambda_M(a;a^*) = l(a)e^{-ra}\lambda_{a^*}.$$

From (5.41) we obtain the following expression for the optimal consumption profile

$$c_M(a;a^*) = \left(e^{\rho a}\frac{\lambda_M(a;a^*)}{l(a)}\right)^{-\frac{1}{\sigma}}.$$

Then

$$c_M(a;a^*) = e^{\frac{(r-\rho)}{\sigma}a}c_{a^*}, \tag{5.42}$$

where $c_{a^*} := \lambda_{a^*}^{-\frac{1}{\sigma}}$. We determine c_{a^*} by inserting the above expression in (5.40) :

$$c_{a^*} = \frac{\int_0^\omega e^{-ra}l(a)((1-\theta)w_N e_N(a) - p_M(a^*)\mathbb{I}_{[R,\omega]}(a))\,\mathrm{d}a}{\int_0^\omega l(a)e^{(r(1+\frac{1}{\sigma})-\frac{\rho}{\sigma})a}\,\mathrm{d}a}.$$

With the so determined c_{a^*} formula (5.42) gives an explicit representation of the optimal consumption of the immigrant population.

5.2.5 The government budget

In the following we give formulas for the aggregate values of the pension expenditures and the tax payments of the two sub-populations.

The pension expenditures for the immigrant population are, see (5.24),

$$PE_M := \int_R^{\omega} \int_{a_{min}}^{a_{max} \wedge a} p(a^*) M(a; a^*) \, \mathrm{d}a^* \, \mathrm{d}a.$$

The pension expenditures for the native population are

$$PE_N := p(0) \int_R^{\omega} N(a) \, \mathrm{d}a.$$

Hence, total pension expenditures depend on the age structure of the population, the parameters of the pension system, N_B and p_1, as-well as the wage rates of natives and immigrants w_M and w_N, respectively.

The tax payments of the immigrant population are

$$tax_M := \theta \int_0^{\omega} \int_{a_{min}}^{a_{max} \wedge a} w_M e_M(a; a^*) M(a; a^*) \, \mathrm{d}a^* \, \mathrm{d}a,$$

and finally, tax payments of the native population are

$$tax_N := \theta \int_0^{\omega} w_N e_N(a) N(a) \, \mathrm{d}a.$$

Hence, the aggregate values are given by $PE_{tot} = PE_M + PE_N$ and accordingly $tax_{tot} = tax_M + tax_N$.

Then, the *social security system* is balanced if

$$tax_{tot} = PE_{tot}. \tag{5.43}$$

The Austrian pension system is a PAYG defined benefits. Therefore, θ has to be adjusted such that (5.43) holds. In this work we focus on the steady state. It is assumed that the government budget is always balanced and there are no debts. Hence, the sustainability of the pension system is reflected by changes in the contribution rate θ. Higher benefits are counteracted by an increase in the contribution rate. The contribution rate can be viewed as a generalization of the demographic old-age dependency ratio because it relates the aggregate expenses for the pensions in a population to the total contributions of the working people. Similarly, the old- age dependency ratio relates the number of non-working people in a population to those who are working. A lower contribution rate means that less taxes have to be used to close the gap of the pension system caused by the demographic change. Hence, the additional contributions could be used for other pillars of the social security system such as health insurance.

5.2.6 Firm's problem

The production sector of the economy is modeled by a representative firm which uses capital and labor to produce a single consumption good. The consumption good can either be saved or consumed. To which extend the product is consumed or saved is decided by the individuals who inhabit the economy.

The production function is given by

$$Y = K^{\alpha}(AL)^{1-\alpha},$$

where Y is the output, L is the effective aggregate labor input and K is the capital stock. The constant α is the capital share and A is the labor-augmenting technological level. It is assumed that immigrants and natives are imperfect substitutes. Therefore, the aggregate effective labor L is taken to be a so-called CES (constant elasticity of substitution) aggregator which combines the two different kinds of labor:

$$L = \left(\gamma L_M^{\frac{\beta-1}{\beta}} + (1-\gamma)L_N^{\frac{\beta-1}{\beta}}\right)^{\frac{\beta}{\beta-1}},$$

where L_M and L_N are the effective labor input of immigrants and natives, respectively,

$$L_M = \int_0^{\omega} \int_{a_{min}}^{a_{max}\wedge a} e_M(a;a^*)M(a;a^*)\,\mathrm{d}a^*\,\mathrm{d}a, \tag{5.44}$$

$$L_N = \int_0^{\omega} e_N(a)N(a)\,\mathrm{d}a. \tag{5.45}$$

The weights γ and $1-\gamma$ are associated with the two different forms of labor in the labor force. The constant β, $0 < \beta < \infty$, is the *elasticity of substitution* between native labor and immigrant labor. If $\beta > 1$, then the two types of labor are substitutes, meaning that a reduction in the supply of one type increases the demand for the other. For $\beta < 1$, the two types of labor are compliments and therefore a reduction of the supply of one does not increase the demand for the other. If $\beta \to 1$, the CES aggregator reduces to a Cobb-Douglas function. The limit of $\beta \to \infty$ describes the case of perfect substitutes and $\beta \to 0$ means that immigrant labor and native labor are perfect compliments. The aggregate capital stock is given by:

$$K = \int_0^{\omega} \int_{a_{min}}^{a_{max}\wedge a} k^M(a;a^*)M(a;a^*)\,\mathrm{d}a^*\,\mathrm{d}a + \int_0^{\omega} k^N(a)N(a)\,\mathrm{d}a. \tag{5.46}$$

The representative firm maximizes profit by hiring labor L and renting capital K from households. Therefore, prices for workers and capital equal the corresponding marginal product

$$R = A^{1-\alpha}\alpha\hat{k}^{\alpha-1}, \tag{5.47}$$

$$\log(w_M) = \log(A^{1-\alpha}(1-\alpha)\hat{k}^{\alpha}) + \frac{1}{\beta}\log(L) + \log(\gamma) - \frac{1}{\beta}\log(L_M), \tag{5.48}$$

$$\log(w_N) = \log(A^{1-\alpha}(1-\alpha)\hat{k}^{\alpha}) + \frac{1}{\beta}\log(L) + \log(1-\gamma) - \frac{1}{\beta}\log(L_N), \tag{5.49}$$

where $\hat{k} := K/L$ is the capita per effective labor.

5.2.7 Definition of steady-state equilibrium

A *steady-state competitive equilibrium* is defined as the policy functions of individuals ($c_N(\cdot)$ and $c_M(\cdot, a^*)$), labor and capital demand of firms (K and L), factor prices (w_M, w_N and r), social contribution rate (θ) and the value of pensions (p_N and $p(a^*)$), that fulfill the following conditions:

x The functions $c_N(a)$ and $c_M(a, a^*)$ are optimal in terms of the optimization problems given by (5.29)–(5.32) and (5.37)–(5.39).

x Factor prices are equal to marginal products given by (5.47)–(5.49).

x The goods market clears.

x The budget of the pension system is balanced, i.e. Equation (5.43) holds.

In Table 5.2 we summarized the equations that have to be fulfilled in equilibrium. The determination of a steady-state equilibrium turns out to be a fixed-point problem in $\hat{k}$

$$\hat{k} = \phi(\hat{k}), \tag{5.50}$$

where ϕ is a non-linear function in $\hat{k}$. For more details on the solution of (5.50) see the description of the numerical algorithm in section 5.2.8 below.

5.2.8 Numerical Experiments

Calibration

The above model is now calibrated with Austrian data.

Demography For the computations we initialize the age structure of demographic variables, $f(a)$, $f_M(a)$, $\mu(a)$, $m(a)$, referring to Austrian data as of 2008 provided by, and interpolate these data piecewise linearly to obtain continuous representations of the vital rates. For the influx of migrants we take the mean value of net migration to Austria over the past 10 years, $I = 35000$. We assume a maximal attainable age of $\omega = 110$.

Households To construct age-specific efficiency profiles for immigrants and natives, we used the 2008, 2009 and 2010 Income, Social Inclusion and Living Conditions (EU-SILC) survey data for Austria. Due to a lack of data, we assumed that $e_M(a; a^*) = e_M(a)$, i.e. we did not account of the differences in wages depending on the age of arrival of the immigrant. In Figure 5.6 we plotted the estimated efficiency profiles. Notice that while the efficiency of the natives is always increasing with age, that of the immigrants is slightly bending backwards in the ages before retirement.

We set the subjective discount factor $\rho = 0$, meaning that the only source of discounting future preferences is the survival probability and the relative risk aversion $\sigma = 1.6$ which is in line with Sánchez-Romero et al. (2013).

$$c_N(a) = e^{\frac{(r-\rho)}{\sigma} a} \frac{\int_0^\omega e^{-ra} l(a)((1-\theta) w_N e_N(a) - p_N \mathbb{I}_{[R,\omega]}(a)) \, \mathrm{d}a}{\int_0^\omega l(a) e^{(r(1+\frac{1}{\sigma}) - \frac{\rho}{\sigma})a} \, \mathrm{d}a},$$

$$c_M(a;a^*) = e^{\frac{(r-\rho)}{\sigma} a} \frac{\int_0^\omega e^{-ra} l(a)((1-\theta) w_N e_N(a) - p_M(a^*) \mathbb{I}_{[R,\omega]}(a)) \, \mathrm{d}a}{\int_0^\omega l(a) e^{(r(1+\frac{1}{\sigma}) - \frac{\rho}{\sigma})a} \, \mathrm{d}a},$$

$$k_N(a) = \int_0^a e^{\int_s^a (r+\mu(\eta)) \, \mathrm{d}\eta} \Big((1-\theta) w_N e_N(s) - c_N(s) + p_N \mathbb{I}_{[R,\omega]}(s) \Big) \, \mathrm{d}s,$$

$$k_M(a;a^*) = \int_{a^*}^a e^{\int_s^a (r+\mu(\eta)) \, \mathrm{d}\eta} \Big((1-\theta) w_M e_M(s;a^*) - c_M(s;a^*) + p_M(a^*) \mathbb{I}_{[R,\omega]}(s) \Big) \, \mathrm{d}s,$$

$$L_N = \int_0^\omega e_N(a) N(a) \, \mathrm{d}a,$$

$$L_M = \int_0^\omega \int_{a_{min}}^{a_{max} \wedge a} e_M(a;a^*) M(a;a^*) \, \mathrm{d}a^* \, \mathrm{d}a,$$

$$K = \int_0^\omega \int_{a_{min}}^{a_{max} \wedge a} k^M(a;a^*) M(a;a^*) \, \mathrm{d}a^* \, \mathrm{d}a + \int_0^\omega k^N(a) N(a) \, \mathrm{d}a,$$

$$R = A^{1-\alpha} \alpha \hat{k}^{\alpha-1},$$

$$\log(w_M) = \log(A^{1-\alpha}(1-\alpha)\hat{k}^\alpha) + \frac{1}{\beta} \log(L_M + L_N) + \log(\gamma) - \frac{1}{\beta} \log(L_M),$$

$$\log(w_N) = \log(A^{1-\alpha}(1-\alpha)\hat{k}^\alpha) + \frac{1}{\beta} \log(L_M + L_N) + \log(1-\gamma) - \frac{1}{\beta} \log(L_N),$$

$$\theta \int_0^\omega \Big(\int_{a_{min}}^{a_{max} \wedge a} w_M e_M(a;a^*) M(a;a^*) \, \mathrm{d}a^* + w_N e_N(a) N(a) \, \mathrm{d}a \Big) \, \mathrm{d}a$$
$$= p(0) \int_R^\omega \Big(N(a) \int_{a_{min}}^{a_{max} \wedge a} p(a^*) M(a;a^*) \, \mathrm{d}a^* \Big) \, \mathrm{d}a,$$

$$M(a,a^*) = I m(a^*) l(a),$$

$$M(a) = \int_{a_{min}}^{a_{max} \wedge a} \frac{l(a)}{l(a^*)} m(a^*) I \, \mathrm{d}a^*,$$

$$N(a) = \frac{\int_0^\omega f_M(s) M(s) \, \mathrm{d}s}{1 - \int_0^\omega f(s) l(s) \, \mathrm{d}s} l(a).$$

Table 5.2: System of equations to determine the endogenous variables: microeconomic relations (first block); macroeconomic relations (second block); demography (third block);

Firm To properly estimate the weight γ, we assume that the differences in wages of immigrants and natives is solely given by their efficiency of labor

$$\frac{y_M(a)}{y_N(a)} = \frac{e_M(a)}{e_N(a)}.$$

Hence, $\frac{w_M}{w_N} = 1$, holds and consequently we can estimate γ by

$$\gamma = \frac{\left(\frac{L_M}{L_N}\right)^{1/\beta}}{1 + \left(\frac{L_M}{L_N}\right)^{1/\beta}}.$$

Note the dependence of γ on β.

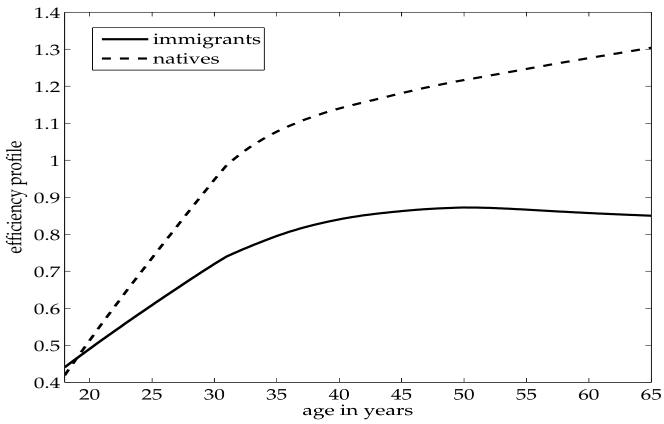

Figure 5.6: The efficiency profiles of immigrants and natives

We follow Sánchez-Romero et al. (2013) where the author's also dealt with the Austrian economy and chose the capital share, $\alpha = 0.31$ and the rate of capital depreciation, $\delta = 0.04$. The labor-augmenting productivity factor $A = 4.2489 \cdot 10^4$ is chosen such that aggregate output Y approximates the value of Austria's GDP.

Pension system We set the parameters of the pension system to those of the current Austrian pension system, where $N_B = 25$, $p_1 = 1.78\%$ and $R = 62.5$ holds.

Solution algorithm for the numerical solution

Below we numerically solve the system of equations of Table 5.2 determining the general equilibrium in the economy. Similar systems of equations have already been solved in other economic papers dealing with general equilibrium models. The general equilibrium mechanism is the most famous nonlinear equation problem in economics. A general solution algorithm is, for example, proposed in Judd (1999).

Since finding the equilibrium can be summarized to solving a non-linear fixed point equation in $\hat{k}$, in the following we apply a fixed-point iteration method. Subsequently, we assume that there exists a unique solution of equation (5.50). We take an initial guess $\hat{k}_0$ and insert it into the equations determining the marginal products R, w_M and w_N and with them we calculate the tax rate θ. Then, we compute per-capita consumption $c_N(a)$ and $c_M(a, a^*)$ and per-capita capital $k_N(a)$ and $k_M(a; a^*)$.

Here, in particular, we follow the below algorithm to find an equilibrium solution $\hat{k}^*$:

Step 1: First we choose an adjustment factor $\eta \geq 0$ and a tolerance $\epsilon > 0$ and small. The adjustment factor is chosen to guarantee stable convergence. The tolerance ϵ determines a stopping criterion for the solution algorithm. We initially compute the age densities of immigrants and natives, $M(a; a^*)$ and $N(a)$ according to (5.22), (5.23).Then, we make an initial guess $\hat{k}_0$.

Step 2: Given the initial guess $\hat{k}_0$ we compute the marginal products of capital and labor, R, w_M and w_N, according to equations (5.47)–(5.49).

Step 3: We subsequently determine the social security rate θ by solving (5.43).

Step 4: In the next step we compute the household problem for natives (5.29)–(5.32) and immigrants (5.37)–(5.39) meaning that we compute the age-specific consumption profiles $c_N(a)$ and $c_M(a, a^*)$ and subsequently also $k_N(a)$ and $k_M(a, a^*)$ with the previous determined values of R, w_M, w_N and θ. Everything else in the equations is exogenously given. Notice that for the immigrants the household problem depends on the arrival age a^* and hence has to be calculated for all $a_{min} \leq a^* \leq a_{max}$ separately.

Step 4: Subsequently, we compute the aggregate variables, K, L_M and L_N where K is determined as in (5.46) and L_M and L_N are given as in (5.44)–(5.45).

Step 5: Then, we compute a new guess $\hat{k}_{i+1} = K/(L_M + L_N)$.

Step 6: The procedure is stopped if $\|\eta\hat{k}_{i+1} + (1 - \eta)\hat{k}_i\| < \epsilon$. Otherwise we go back to Step 2 and set $i = i + 1$.

In the numerical example below it was necessary to set $0 < \eta < 1$ because for $\eta = 0$ unstable iterations appeared.

Numerical Results

Demography In Table 5.3 we summarize the various demographic scenarios, where we assumed that all immigrants arrive in a specific 5–year long sub-interval between $a_{min} = 18$ and $a_{max} = 40$.

In Figures 5.7 – 5.10, the resulting stationary through immigration populations are plotted. Notice that the younger the immigrants, the bigger is the resulting population. This is caused by a higher fertility of younger immigrants.

Since for the sustainability of the pension system not the total dependency of a population matters but instead the ratio of working to retired people, we calculated the resulting old-age ratios (OADs), see Table 5.3. For the calculation of the various OADs we used age groups $18 - 61$ and $62+$. We find that unlike in Chapter 3 where the optimal age to minimize the dependency was in the mid-thirties the OAD clearly rises with the age of immigration a^*. This is because for the dependency ratio, a high number of children, caused by a high fertility rate of young immigrants, is not beneficial since they increase the dependent population.

Demography				
$a^* \in$	[18,25]	[25,30]	[30,35]	[35,40]
$N_{tot} + M_{tot}$	9.1 m	7.2 m	4.2 m	2.2 m
$\frac{M_{tot}}{N_{tot}+M_{tot}}$	0.21	0.26	0.41	0.70
$\frac{M_{tot}}{N_{tot}}$	0.26	0.35	0.68	2.38
OAD	0.38	0.40	0.45	0.57

Table 5.3: Demographic results

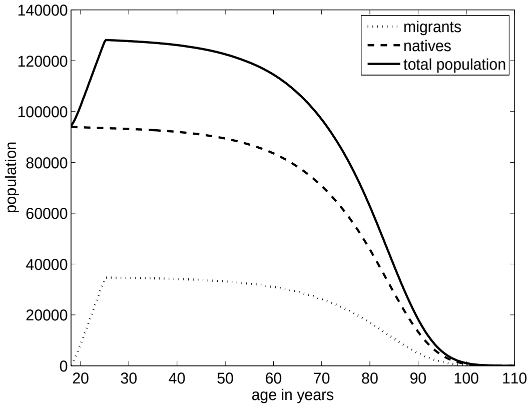

Figure 5.7: Long run population structure for immigration in ages $a^* \in [18, 25]$

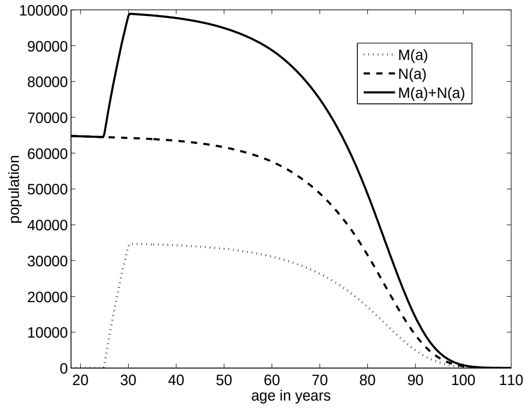

Figure 5.8: Long run population structure for immigration in ages $a^* \in [25, 30]$.

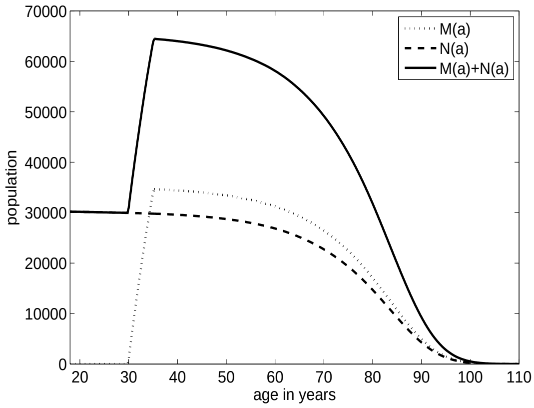

Figure 5.9: Long run population structure for immigration in ages $a^* \in [30, 35]$

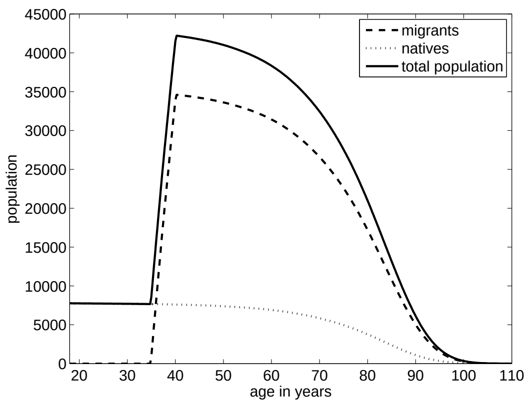

Figure 5.10: Long run population structure for immigration in ages $a^* \in [35, 40]$.

Effects on the Pension System In Table 5.4 the impact of age-specific immigration on the social security rate θ and the pension expenditure rate PE_{tot}/Y are presented. Moreover, scaled pension expenditures and tax payments for the two groups, natives and immigrants, are given. One can see that for the given scenario the social security rate decreases with the age of the arriving immigrants although the OAD increases substantially. This is because of the fact that immigrants qualify for fewer pensions in

$N_B = 25$, R=62.5, $e_N(a) \neq e_M(a)$, $\beta = 10000$				
$a^* \in$	[18,25]	[25,30]	[30,35]	[35,40]
θ	0.34	0.33	0.32	0.31
PE_M/w	$3.4 \cdot 10^5$	$3.0 \cdot 10^5$	$2.6 \cdot 10^5$	$2.2 \cdot 10^5$
PE_N/w	$14.5 \cdot 10^5$	$10.0 \cdot 10^5$	$4.6 \cdot 10^5$	$1.2 \cdot 10^5$
tax_N/w	$14.2 \cdot 10^5$	$9.7 \cdot 10^5$	$4.4 \cdot 10^5$	$1.2 \cdot 10^5$
tax_M/w	$3.7 \cdot 10^5$	$3.3 \cdot 10^5$	$2.8 \cdot 10^5$	$2.3 \cdot 10^5$
net transfers migrants/PE_{tot}	1.3 %	2.0 %	3.0 %	3.0%
PE_{tot}/Y in %	23.3 %	23.0 %	22.6%	21.4 %

Table 5.4: Pension expenditures and contribution rates

the host country. Moreover, in Table 5.4 we see that across all age groups immigrants are net payers of the pension system. Hence, they are at least to a small extend able to close the financial gap caused by the aging of the native population. However, one also sees that immigration alone cannot solve the fiscal problems arising with the demographic change because an increase of the social security rate to $\theta \in [0.31, 0.34]$ would be necessary to guarantee a balanced budget. Also pension expenditure rates would have to rise from currently 12.8%, cf. OECD (2012) to values between 21% and 23%. Hence, also other measures such as an increase in the statutory retirement age and changes in the parameters of the pension system would additionally be necessary.

Impact of immigration on economic variables and life cycle behavior Subsequently, we investigate the life cycle behavior of consumption and asset accumulation of natives and immigrants.

$N_B = 25$, R=62.5, $e_N(a) \neq e_M(a)$, $\beta = 10000$				
$a^* \in$	[18,25]	[25,30]	[30,35]	[35,40]
r	0.020	0.019	0.017	0.012
w	30840	31080	31560	32790

Table 5.5: Economic parameters

In Figures 5.11–5.12 the native's life-cycle profile of consumption and financial assets for the various entry scenarios of the immigrants are plotted. We note that unlike natives, immigrants even if they arrive at relatively young ages, they do not become net borrowers, see Figures 5.13 and 5.15. This is because they do not earn as high pensions as natives do and hence immediately start accumulating assets. There is also a clear dependence of the native's capital accumulation on the age of arrival of the immigrants. If immigrants arrive in early ages natives accumulate more capital. This is caused by a higher interest rate on capital, see Table 5.5, although there is a reverse effect caused by an increased θ. A higher θ is usually responsible for a crowding out of capital. Hence, the higher interest rate R compensates the crowding out effect of an increased θ. In Figures 5.12, 5.14 and 5.16 the life-cycle consumption profiles of immigrants and natives are plotted. Figure 5.12 shows that the earlier the immigrants enter the country the lower is the initial consumption level and native individuals borrow more at the beginning of the life cycle.

Moreover, we find that if immigrants enter the host country later in life they accumulate even more assets than a native individual because they have to anticipate the missing pension payments at the end of their life.

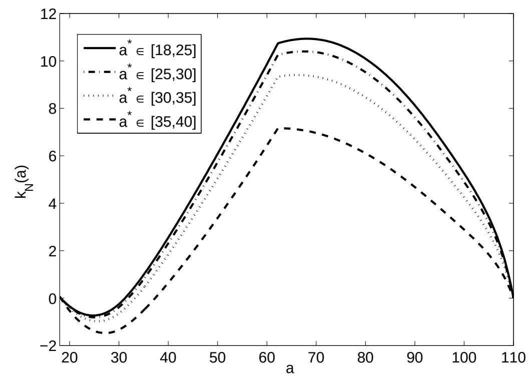

Figure 5.11: Scaled assets of natives over the life-cycle for $\beta = 10000$

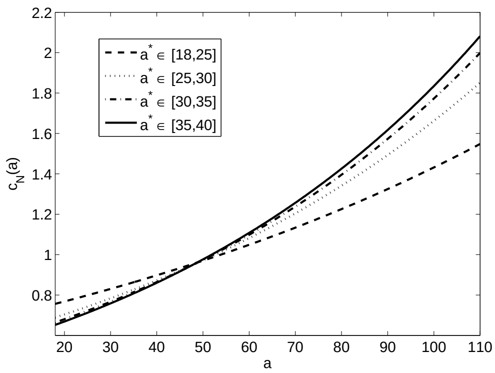

Figure 5.12: Scaled consumption of natives over the life-cycle for $\beta = 10000$

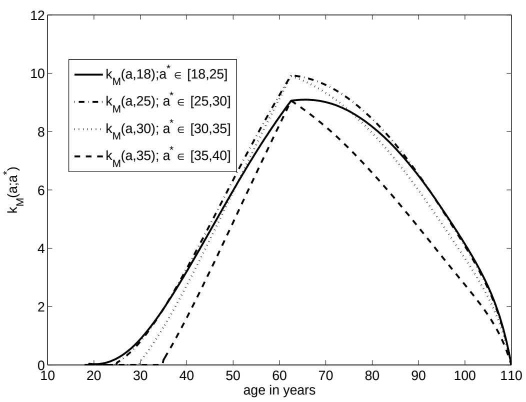

Figure 5.13: Scaled assets of immigrants over the life-cylce for $\beta = 10000$

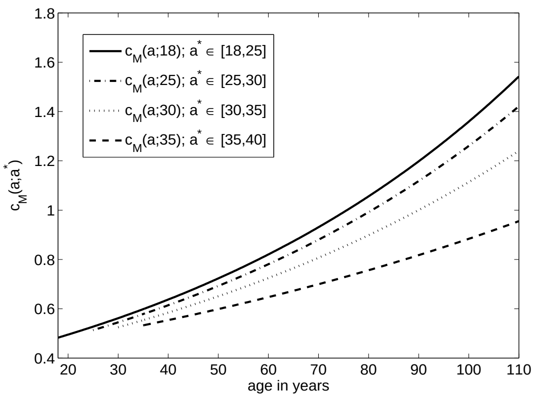

Figure 5.14: Scaled consumption of immigrants over the life-cylce for $\beta = 10000$

5.2.9 Outlook

In this work, the focus was on the steady state effects of immigrants' age structure regarding the sustainability of the pension system. Hence, in further investigations an extension to the transitory dynamics would make it possible to study the short term effects which would shed light on more recent developments. We found that since

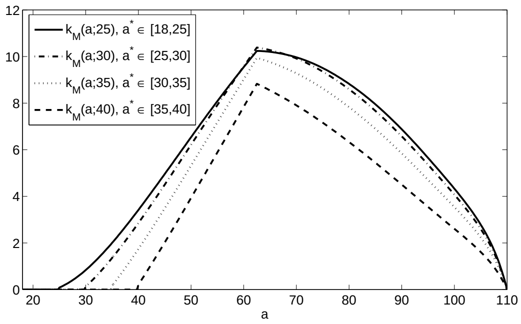

Figure 5.15: Scaled assets of immigrants over the life-cylce for $\beta = 10000$

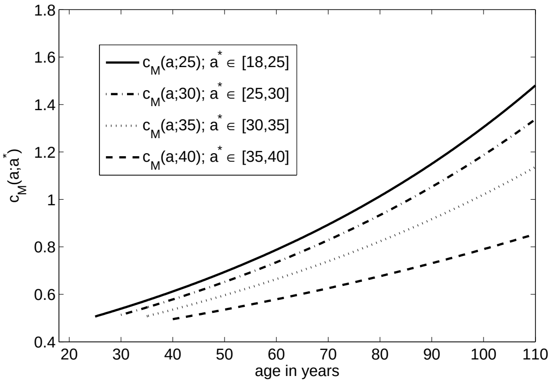

Figure 5.16: Scaled consumption of immigrants over the life-cylce for $\beta = 10000$

immigrants are heterogeneous with respect to their age of arrival they also earn different pensions after retirement. This heterogeneity may lead to different insentives for retirement when the age of retirement is not exogenous anymore. Hence, an interesting extension of this model could be to investigate how the results would change in case of an endogenous retirement decision. Moreover, one could also drop the strong assumption of no capital movement and pass over to an open economy framework.

Chapter 6

Conclusions and Possible Extensions

In this thesis, I tried to summarize and combine the work I have done during the last three years as a research assistant, partly in close cooperation with my colleagues and co-authors, A. Belyakov, G. Feichtinger, B. Skritek, and V. Veliov. Clearly, not all of it could be included. This thesis should be understood as a compilation of mathematical models dealing with immigration. It was an attempt to tackle demographic and economic questions by applying optimal control theory. In particular, it was aimed to determine qualitative and quantitative effects of (age-specific) immigration patterns on the receiving country to study demographic and economic consequences of immigration. Hence, the ideas used in this thesis may be fruitful for the study of immigration policies.

The theoretical approach used within this thesis is rather specific as it uses a method, i.e. optimal control theory, which is typically used in engineering or the natural sciences as it enables to solve models consisting of equations describing natural processes or physical laws. However, with the triumph of neoclassical economics, optimal control theory became also a very popular modeling method in economics. Consequently, the idea of using optimal control theory for answering purely demographic questions in this thesis also seemed not too far-fetched, although it is not as commonly used as in economics. Further development of the models and techniques used in the thesis could lead to new approaches to solve complex policy problems.

From a mathematical point of view the challenges to be tackled in this thesis lied in the formulation of valid models and the application of optimal control theory to age-structured systems as-well as the presentation of their analytical and numerical solution. The technique used in Chapter 4 for obtaining transversality conditions for a specific infinite-horizon age-structured optimal control problem is of independent interest and can be applied for various problems of this type.

In Chapter 3 and Chapter 4 purely demographic models were formulated.

In Chapter 3 the question of the optimal age-specific immigration policy that minimizes the dependency in a population in the long-run was posed. A stationary problem was considered which consisted of the investigation of a rather specific linear optimal control model including a state constraint which made it necessary to apply a very general maximum principle.

Subsequently, in Chapter 4 more generally, the time-varying age-specific immigration pattern to a population of fixed size that maximizes the number of workers in a population was investigated. This lead to the formulation of a very specific, distributed parameter model. From a mathematical point of view the considered problem was challenging for three reasons: (i) it has the form of a distributed optimal control problem with state constraints (although rather specific); (ii) the time horizon is infinite and a theory for infinite-horizon optimal control problems for age-structured systems is missing; (iii) it is a maximization problem for a non-concave functional, where the existence of a solution and the well-posedness are problematic. It turned out that under an additional generic well-posedness condition for a population with time-invariant mortality and fertility the optimal age-density of the migration is time-invariant and independent of the initial data. Hence, the solution could be found by solving the associated steady-state problem, as it had been studied in Chapter 3.

Since immigration is never solely a demographic issue, in a next step economic models were considered to investigate also the economic effects of immigration. Hence, in Chapter 5 existing overlapping generations models had to be extended in order to be able to deal with immigration. Here, the challenges consisted of an economic and mathematical sound adaptation of the macro- and microeconomic modeling. We used continuous time overlapping generations models which also included the formulation and solving of partial differential equations. In a first model, the welfare effects of immigration on the various cohorts of the host population were investigated. In a second model, the focus was on the description of the life-cycle behavior of immigrants entering at various ages of their life to determine their impact on the pension schemes of a country. The considered models could clearly be extended in various ways, depending on the question one aims to pose. Hence, for example, the rather specific objective function in Chapter 3 could be replaced by the labor participation rate or by a more promising measure of the future dependency in a population as presented in Sanderson and Scherbov (2010). More generally, the concepts used in the demographic models of Chapter 3 and Chapter 4 could be used for the study of resource allocation problems when considering biological populations instead of human populations. In Chapter 5 when investigating the effects of immigrants on the finances of the host country, one may pass over to a transitory problem in order to be able to capture more recent developments as the consequences of the aging process are faced in recent years. Moreover, pension portability and the consideration of an open economy would lead to a more realistic consideration.

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2. 07/2009: Talk on *Numerical Solution of Singular Eigenvalue Problems for Systems of Ordinary Differential Equations*, Johannes Kepler Universität Linz