Technische Universität Wien
Karlsplatz 13, 1040 Wien

Dissertation

Creatures and Cardinals

ausgeführt zum Zwecke der Erlangung des akademischen Grades
eines Doktors der technischen Wissenschaften

an der Fakultät für Mathematik und Geoinformation
der Technischen Universität Wien

Betreuung:
Ao. Univ.-Prof. Dipl.-Ing. Dr. Martin Goldstern
E104 Institut für Diskrete Mathematik und Geometrie
✉: martin.goldstern@tuwien.ac.at

Verfasser:
Dipl.-Ing. Lukas Daniel Klausner, BSc (E00625127)
786 860 Doktoratsstudium der technischen Wissenschaften – Technische Mathematik
⌂: Steinbachstraße 34–36
3001 Mauerbach
✉: lukas.d.klausner@tuwien.ac.at
klausner.lukas@gmail.com

This thesis is dedicated to all those
who did not dedicate their theses to themselves.

ABSTRACT

This thesis collects several related results on cardinal characteristics of the continuum, all of which employ the method of creature forcing.

In chapter A, we use a countable support product of lim sup creature forcing posets to show that consistently, for uncountably many different functions the associated Yorioka ideals' uniformity numbers can be pairwise different. In addition we show that, in the same forcing extension, for two other types of simple cardinal characteristics parametrised by reals (localisation and anti-localisation cardinals), for uncountably many parameters the corresponding cardinals are pairwise different. The proofs are based on standard creature forcing methods, relational systems and Tukey connections.

In chapter B, we disassemble, recombine and reimplement the creature forcing construction used by Fischer et al. [FGKS17] to separate Cichoń's diagram into five cardinals as a countable support product with more easily understandable internal structure. Using the fact that it is of countable support, we augment the construction by adding uncountably many additional cardinal characteristics, namely, localisation cardinals. The proofs use both creature forcing and combinatorial methods.

In chapter C, we introduce several cardinal characteristics related to the splitting number $\mathfrak{s}$, the reaping number $\mathfrak{r}$ and the independence number $\mathfrak{i}$ and prove bounds and consistency results to show that several of these cardinal characteristics are, in fact, new. Most proofs are of a combinatorial nature; one of the more sophisticated proofs utilises a creature forcing poset already introduced in chapter B.

All three chapters are self-contained except for minor cross-references.

CONTENTS

Abstract **i**

Acknowledgements **v**

Preface **vii**

A Yorioka Ideals **1**

A1 Introduction 1

A2 Yorioka Ideals and Cardinal Characteristics 5

A3 Motivational Example 9

A4 Lots and Lots of Auxiliary Functions 17

A5 The Grand Finale 21

A6 Open Questions 28

B Modular Framework for Creature Forcing **31**

B1 Introduction 31

B2 Motivational Prologue 34

B3 Defining the Forcing Factors 36

B4 Putting the Parts Together 45

B5 Bigness 55

B6 Continuous and Rapid Reading 58

B7 Unhalving and More Continuous Reading 62

B8 $\mathrm{cof}(\mathcal{N}) \leq \kappa_{\mathrm{cn}}$ 70

B9 $\mathrm{non}(\mathcal{M}) = \kappa_{\mathrm{nm}}$ . . . 72
B10 $\mathfrak{c}_{f_\xi, g_\xi} = \kappa_\xi$ . . . 75
B11 $\mathrm{non}(\mathcal{N}) \geq \kappa_{\mathrm{nn}}$ and $\mathrm{cof}(\mathcal{N}) \geq \kappa_{\mathrm{cn}}$ . . . 82
B12 $\mathrm{non}(\mathcal{N}) \leq \kappa_{\mathrm{nn}}$ . . . 87
B13 Failed Attempts and Open Questions . . . 93

C Halfway New Cardinal Characteristics **97**
C1 Introduction . . . 97
C2 Characteristics Related to $\mathfrak{s}$ . . . 98
C3 Separating $\mathfrak{s}_{1/2}$ and $\mathrm{non}(\mathcal{N})$ . . . 102
C4 Characteristics Related to $\mathfrak{r}$ and $\mathfrak{i}$ . . . 108
C5 Open Questions . . . 117

List of Figures **119**

Bibliography **121**

ACKNOWLEDGEMENTS

There are a great number of people in my life without whose support this thesis would not have been possible.

Most importantly, I want to thank my advisor, Martin Goldstern, for his advice and guidance in my graduate studies and in the creation of this thesis. I also owe thanks to all of my coauthors, especially to the wonderful people with whom I worked on the papers which constitute this thesis: Jörg Brendle, Lorenz J. Halbeisen, Marc Lischka, Diego Alejandro Mejía, Saharon Shelah, and, once more, Martin Goldstern.

Both my undergraduate and my graduate studies would not have been half as much fun without the friendship and company of my many amazing fellow students. You are all awesome.

In addition to my friends from university, my family and friends, in particular my marvellous wife Sarah Wantoch, offered the support, stability and outside perspective without which no long-term project would be achievable.

Finally, I thank the selfless and foolhardy people who daringly agreed to proofread this thesis and offered many helpful suggestions and corrections: Caroline Amsüss, Christian Amsüss, Irina Barnay, Astrid Berg, Thomas Klausner, Florian Libisch, and Sebastian Zivota. Without their help, this thesis would contain many more typos and mistakes. Any imperfections, errors or omissions that remain are due to me.

PREFACE

Set theory began as a mathematical subject when Georg Cantor discovered the notion of infinite cardinality and proved that the cardinality of the set of real numbers (the *continuum* $2^{\aleph_0}$) is different from the cardinality of the set of natural numbers $\aleph_0$. The question of "how different?" immediately became a focal point of the new subject and has kept its central place for more than a century. Even before Paul Cohen proved that Cantor's well-known continuum hypothesis cannot be refuted, i. e. that there can consistently be infinite sets of reals of intermediate cardinality, several cardinal numbers of potentially "intermediate" size (so-called cardinal characteristics, such as the unbounding number $\mathfrak{b}$ and the dominating number $\mathfrak{d}$, and of course $\aleph_1$) were known, and the inability of mathematicians to prove equalities between them already hinted at the vast range of unprovability results that emerged as Cohen's forcing method was developed and refined.

For a general overview of cardinal characteristics, see [Bla10], [Hal17, chapter 9] and [Vau90] as well as [BJ95]. We will give brief definitions of the most important standard concepts and terms from the study of cardinal characteristics which will appear in this thesis.

Given an ideal $\mathcal{I}$ on some base set X, we can define four cardinal characteristics:

- the additivity number $\operatorname{add}(\mathcal{I}) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} \notin \mathcal{I}\}$,
- the covering number $\operatorname{cov}(\mathcal{I}) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} = X\}$,
- the uniformity number $\operatorname{non}(\mathcal{I}) := \min\{|Y| \mid Y \subseteq X \text{ and } Y \notin \mathcal{I}\}$, and
- the cofinality $\operatorname{cof}(\mathcal{I}) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I} \text{ and } \forall\, B \in \mathcal{I}\ \exists\, A \in \mathcal{A}\colon B \subseteq A\}$.

In particular, we will refer to these cardinal characteristics for

- the ideal $\mathcal{N} := \{A \subseteq 2^\omega \mid \lambda(A) = 0\}$ of *Lebesgue null sets* and
- the ideal $\mathcal{M} := \{A \subseteq \omega^\omega \mid A = \bigcup_{n<\omega} A_n \text{ and } \forall\, n < \omega\colon A_n \text{ nowhere dense}\}$ of *meagre sets*.

In addition to these, we will refer to several other cardinal characteristics (where $f \leq^* g$ stands for $\forall^\infty n < \omega\colon f(n) \leq g(n)$):

- $\mathfrak{b} := \min\{|A| \mid A \subseteq \omega^\omega \text{ and } \forall\, g \in \omega^\omega\ \exists\, f \in A\colon f \not\leq^* g\}$ (the unbounding number),
- $\mathfrak{d} := \min\{|A| \mid A \subseteq \omega^\omega \text{ and } \forall\, g \in \omega^\omega\ \exists\, f \in A\colon g \leq^* f\}$ (the dominating number),
- $\mathfrak{s} := \min\{|\mathcal{S}| \mid \mathcal{S} \subseteq [\omega]^\omega \text{ and } \forall\, X \in [\omega]^\omega\ \exists\, S \in \mathcal{S}\colon |X \cap S| = |X \smallsetminus S| = \aleph_0\}$ (the splitting number),
- $\mathfrak{r} := \min\{|\mathcal{R}| \mid \mathcal{R} \subseteq [\omega]^\omega \text{ and } \nexists\, X \in [\omega]^\omega\ \forall\, R \in \mathcal{R}\colon |R \cap X| = |R \smallsetminus X| = \aleph_0\}$ (the reaping number), and
- $\mathfrak{i} := \min\{|\mathcal{I}| \mid \mathcal{I} \subseteq [\omega]^\omega \text{ and } \forall\, \mathcal{A} \cup \mathcal{B} \subseteq \mathcal{I}\colon \left|\bigcap_{A\in\mathcal{A}} A \cap \bigcap_{B\in\mathcal{B}} (\omega \smallsetminus B)\right| = \aleph_0\}$ (the independence number),

The common methods predominantly used in all three chapters are *countable support products* and *creature forcing*. The standard reference work for the latter is by Rosłanowski and Shelah [RS99], but this thesis is self-contained and, hopefully, more easily digestible.

Finally, we remark that we use the standard notation for forcing conditions where $q \leq p$ means that q is a stronger condition than p, and we also strive to follow Goldstern's alphabetic convention, i. e. stronger conditions are denoted by lexicographically bigger symbols.

CHAPTER A

YORIOKA IDEALS

This chapter is based on [KM18], which is joint work with Diego Alejandro Mejía.

A1 Introduction

This research forms part of the study of cardinal characteristics of the continuum which are parametrised by reals and of the forcing techniques required to separate many of them. The main motivation of this chapter is to produce a forcing model where several (even uncountably many) uniformity numbers of *Yorioka ideals* are pairwise different; in doing so, we included additional types of parametrised cardinal characteristics which we refer to as *localisation* and *anti-localisation cardinals*. We first review the definition of a Yorioka ideal:

Notation A1.1. We fix the following terminology.

(1) For $\sigma \in (2^{<\omega})^\omega$, let $[\sigma]_\infty := \bigcap_{n<\omega} \bigcup_{i\geq n}[\sigma(i)]$ and let $\mathrm{ht}_\sigma \in \omega^\omega$ be the function defined by $\mathrm{ht}_\sigma(i) := |\sigma(i)|$.
(2) For $f, g \in \omega^\omega$, we write $f \ll g$ to state "for every $m < \omega$, $f \circ \mathrm{pow}_m \leq^* g$", where $\mathrm{pow}_m \colon \omega \to \omega \colon i \mapsto i^m$.

Definition A1.2 (Yorioka [Yor02])**.** We define Yorioka ideals in two steps:

(1) For $g \in \omega^\omega$, define $\mathcal{J}_g := \{X \subseteq 2^\omega \mid \exists\, \sigma \in (2^{<\omega})^\omega \colon X \subseteq [\sigma]_\infty \text{ and } \mathrm{ht}_\sigma = g\}$.
(2) For $f \in \omega^\omega$ increasing, define $\mathcal{I}_f := \bigcup_{g\gg f} \mathcal{J}_g$. Any family of this form is called a *Yorioka ideal*.

Yorioka ideals are partial approximations of the σ-ideal $\mathcal{SN}$ of strong measure zero subsets of 2^ω. They were introduced by Yorioka [Yor02] to show that no inequality between $\mathrm{cof}(\mathcal{SN})$ and $\mathfrak{c} := 2^{\aleph_0}$ can be proved in ZFC. Though it is very easy to show that $\mathcal{SN} = \bigcap_{g\in\omega^\omega} \mathcal{J}_g$, the families of the form $\mathcal{J}_g$ are not ideals in general

(Kamo and Osuga [KO08]); however, $\mathcal{I}_f$ is a σ-ideal whenever f is increasing, and $\mathcal{SN} = \bigcap\{\mathcal{I}_f \mid f \in \omega^\omega \text{ increasing}\}$ characterises $\mathcal{SN}$, as well. Also note that $\mathcal{I}_f \subseteq \mathcal{N}$, where $\mathcal{N}$ denotes the ideal of Lebesgue measure zero subsets of 2^ω.

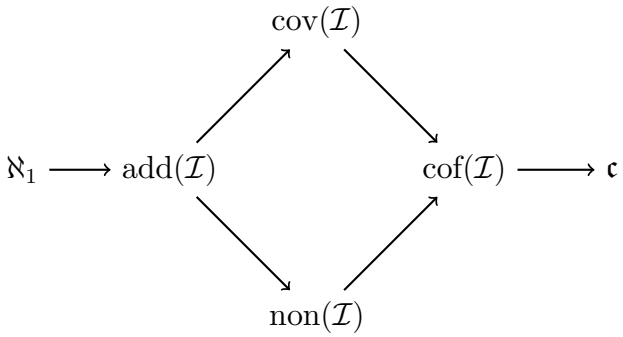

Figure 1: The standard inequalities for the cardinal characteristics associated with a σ-ideal $\mathcal{I}$ which contains all finite subsets and has a Borel basis.

The cardinal invariants associated with Yorioka ideals (that is, $\mathrm{add}(\mathcal{I}_f)$, $\mathrm{cov}(\mathcal{I}_f)$, $\mathrm{non}(\mathcal{I}_f)$ and $\mathrm{cof}(\mathcal{I}_f)$) have been studied for quite some time. Kamo and Osuga [KO08] showed that for any fixed increasing f, no other inequality consistent with the standard known inequalities (see Figure 1) can be proved in ZFC. This was improved by Cardona and Mejía [CM17] by constructing a ccc poset forcing that, for some f_0, the four cardinals associated with $\mathcal{I}_f$ are pairwise different for all $f \geq^* f_0$. On the other hand, Kamo and Osuga also showed that $\mathrm{add}(\mathcal{I}_f) \leq \mathfrak{b}$ and $\mathfrak{d} \leq \mathrm{cof}(\mathcal{I}_f)$. It is also known that $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{I}_f)$ and $\mathrm{cof}(\mathcal{I}_f) \leq \mathrm{cof}(\mathcal{N})$ in ZFC (attributed to Kamo; see also [CM17, section 3] for a proof), but equality cannot be proved (Cardona and Mejía [CM17]).

Later, Kamo and Osuga [KO14] forced, using a ccc poset, that infinitely (even uncountably) many covering numbers of Yorioka ideals are pairwise different; moreover, continuum many pairwise different covering numbers can be forced under the assumption that a weakly inaccessible cardinal exists. The dual of this result is the main question of this chapter.

Question A. *Is it consistent with* ZFC *that infinitely many cardinals of the form* $\mathrm{non}(\mathcal{I}_f)$ *are pairwise different?*

A key feature in Kamo and Osuga's model for infinitely many covering numbers is a relation they discovered between the covering numbers of Yorioka ideals and anti-localisation cardinals of the form $\mathfrak{c}^\exists_{c,h}$. The duals $\mathfrak{v}^\exists_{c,h}$ of these cardinals, as well as the localisation cardinals themselves, will also play an important role in solving the question above. They are defined as follows:

Notation A1.3. We fix some basic notation and terminology.

(1) Given $A \subseteq \omega$ and a formula φ, we write $\forall^\infty i \in A\colon \varphi$ for $\exists\, n < \omega\ (\forall\, i \geq n, i \in A)\colon \varphi$ (i. e. all but finitely many members of A satisfy φ), and $\exists^\infty i \in A\colon \varphi$ for $\forall\, n < \omega\ (\exists\, i \geq n, i \in A)\colon \varphi$ (i. e. infinitely many members of A satisfy φ). We often write $\forall^\infty i$ instead of $\forall^\infty i < \omega$, and likewise for $\exists^\infty i$.

(2) Let $c = \langle c(i) \mid i < \omega \rangle$ be a sequence of non-empty sets. We write $\prod c := \prod_{i<\omega} c(i)$ and $\mathrm{seq}_{<\omega}(c) := \bigcup_{n<\omega} \prod_{i<n} c(i)$. For $t \in \mathrm{seq}_{<\omega}(c)$, let $[t] = [t]_c := \{x \in \prod c \mid t \subseteq x\}$.
(3) In addition, if $h \in \omega^\omega$, let $\mathcal{S}(c,h) := \prod_{i<\omega} [c(i)]^{\leq h(i)}$, the set of all *h-slaloms* (sequences of subsets of size at most $h(i)$) in $\prod c$.
(4) When x and y are functions with domain ω, we write

$$\begin{aligned} y \textit{ localises } x, \text{ denoted by } x \in^* y, &\quad \text{iff } \forall^\infty i\colon x(i) \in y(i);\\ y \textit{ anti-localises } x, \text{ denoted by } x \notin^* y, &\quad \text{iff } \forall^\infty i\colon x(i) \notin y(i). \end{aligned}$$

(5) When, in addition, x and y go into the ordinal numbers, we write $x \leq y$ for $\forall\, i < \omega\colon x(i) \leq y(i)$ and $x \leq^* y$ for $\forall^\infty i < \omega\colon x(i) \leq y(i)$. We likewise use the notation $x < y$ and $x <^* y$.

Definition A1.4. Let $c = \langle c(i) \mid i < \omega \rangle$ be a sequence of non-empty sets and $h \in \omega^\omega$. We define the following cardinal characteristics:

$$\begin{aligned} \mathfrak{v}^\forall_{c,h} &:= \min\{|F| \mid F \subseteq \textstyle\prod c \text{ and } \neg\exists\, \varphi \in \mathcal{S}(c,h)\ \forall\, x \in F\colon x \in^* \varphi\}\\ \mathfrak{c}^\forall_{c,h} &:= \min\{|S| \mid S \subseteq \mathcal{S}(c,h) \text{ and } \forall\, x \in \textstyle\prod c\ \exists\, \varphi \in S\colon x \in^* \varphi\}\\ \mathfrak{v}^\exists_{c,h} &:= \min\{|E| \mid E \subseteq \textstyle\prod c \text{ and } \forall\, \varphi \in \mathcal{S}(c,h)\ \exists\, y \in E\colon y \notin^* \varphi\}\\ \mathfrak{c}^\exists_{c,h} &:= \min\{|R| \mid R \subseteq \mathcal{S}(c,h) \text{ and } \neg\exists\, y \in \textstyle\prod c\ \forall\, \varphi \in R\colon y \notin^* \varphi\} \end{aligned}$$

The first two types of cardinals are referred to as *localisation cardinals*, while the latter two are referred to as *anti-localisation cardinals*.[1]

The localisation and anti-localisation cardinals are a generalisation of the cardinals used in Bartoszyński's characterisations $\mathrm{add}(\mathcal{N}) = \mathfrak{v}^\forall_{\omega,h}$ and $\mathrm{cof}(\mathcal{N}) = \mathfrak{c}^\forall_{\omega,h}$ (where ω is to be interpreted as the constant sequence ω) when h goes to infinity, and $\mathrm{non}(\mathcal{M}) = \mathfrak{c}^\exists_{\omega,h}$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{v}^\exists_{\omega,h}$ when $h \geq^* 1$, where $\mathcal{M}$ denotes the ideal of meagre subsets of 2^ω (see e. g. [BJ95, Theorem 2.3.9, Lemmata 2.4.2 and 2.4.8]). Moreover, when c takes infinitely many infinite values, the localisation and anti-localisation cardinals are already characterised by other well-known cardinals (see [CM17, section 3]), so these cardinals are more interesting when $c \in \omega^\omega$ and $c >^* h$. Even then, we have some trivial values: $\mathfrak{v}^\forall_{c,h}$ is finite and $\mathfrak{c}^\forall_{c,h} = \mathfrak{c}$ when h does not go to infinity (Goldstern and Shelah [GS93]); also, $\mathfrak{c}^\exists_{c,h}$ is finite and $\mathfrak{v}^\exists_{c,h} = \mathfrak{c}$ when the quotient $\frac{h(k)}{c(k)}$ does not converge to 0 (see [CM17, section 3]). See Figure 2 for a summary of the ZFC-provable inequalities relating localisation and anti-localisation cardinals to other cardinal characteristics. The additional relation between the anti-localisation cardinals and the covering and uniformity numbers of $\mathcal{N}$ was previously hinted at for the case of $h = 1$ in [GS93]; we prove it in general in Lemma A2.3.

[1] A brief note on notation: $\mathfrak{c}^\forall_{c,h}$ and $\mathfrak{c}^\exists_{c,h}$ were used in [GS93, Kel08, KS09, KS12, KO14] with the meaning of *covering* $\prod c$ with slaloms, where the relations for covering are $\in^*$ and $\exists^\infty i\colon x(i) \in \varphi(i)$, respectively. The notation $\mathfrak{v}^\forall_{c,h}$ and $\mathfrak{v}^\exists_{c,h}$ is intended to be read as *avoiding* or *evading* such coverings.

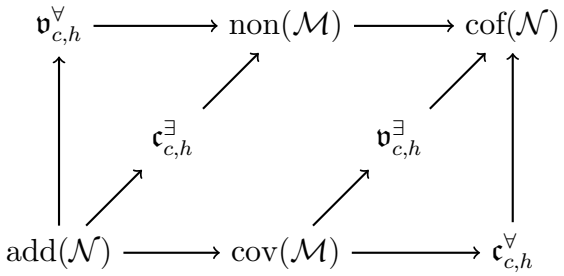

Figure 2: The ZFC-provable inequalities between localisation and anti-localisation cardinals and other well-known cardinal characteristics in Cichoń's diagram. Additionally, if $\sum_{i<\omega} \frac{h(i)}{c(i)} < \infty$, then $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{c}^{\exists}_{c,h}$ and $\mathfrak{v}^{\exists}_{c,h} \leq \mathrm{non}(\mathcal{N})$, and conversely, if $\sum_{i<\omega} \frac{h(i)}{c(i)} = \infty$, then $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{v}^{\exists}_{c,h}$ and $\mathfrak{c}^{\exists}_{c,h} \leq \mathrm{non}(\mathcal{N})$ (see Lemma A2.3).

One of the earliest appearance of these cardinals is in Miller's [Mil81] characterisations $\mathrm{non}(\mathcal{SN}) = \min\{\mathfrak{v}^{\exists}_{c,h} \mid c \in \omega^\omega\}$ $(h \geq^* 1)$ and $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{non}(\mathcal{SN})\}$.[2] Some time later, Goldstern and Shelah [GS93] proved that uncountably many cardinals of the form $\mathfrak{c}^{\forall}_{c,h}$ can be pairwise different. Kellner [Kel08] then improved this result by showing the consistency of continuum many pairwise different cardinals of the same type, and Kellner and Shelah [KS09, KS12] included similar consistency results for the type $\mathfrak{c}^{\exists}_{c,h}$; they also showed that there may be continuum many of both types of cardinals (in the same model). All of these consistency results were proved by using proper ω^ω-bounding forcing constructions with *normed creatures*.

By ccc forcing techniques, Brendle and Mejía [BM14] showed that uncountably many cardinals of the form $\mathfrak{v}^{\forall}_{c,h}$ can be pairwise different – and even continuum many, under the assumption that there exists a weakly inaccessible cardinal. However, the corresponding consistency result for $\mathfrak{v}^{\exists}_{c,h}$ remained unknown.

Question B. *Is it consistent with* ZFC *that there are infinitely many pairwise different cardinals of the form* $\mathfrak{v}^{\exists}_{c,h}$*?*

In this chapter we answer Question A and Question B in the positive. Assuming CH, we construct a forcing model where uncountably many cardinals of the form $\mathrm{non}(\mathcal{I}_f)$, $\mathfrak{v}^{\exists}_{c,h}$ and $\mathfrak{c}^{\forall}_{c,h}$ are pairwise different.

Main Theorem. *Assume* CH. *If* $\langle \kappa_\alpha \mid \alpha \in A\rangle$ *is a sequence of infinite cardinals such that* $|A| \leq \aleph_1$ *and* $\kappa_\alpha^{\aleph_0} = \kappa_\alpha$ *for every* $\alpha \in A$*, then there is a family* $\langle (a_\alpha, d_\alpha, f_\alpha, c_\alpha, h_\alpha) \mid \alpha \in A\rangle$ *of tuples of increasing functions in* ω^ω *and a proper* ω^ω*-bounding* $\aleph_2$*-cc poset* $\mathbb{Q}$ *that forces* $\mathfrak{v}^{\exists}_{c_\alpha,h_\alpha} = \mathrm{non}(\mathcal{I}_{f_\alpha}) = \mathfrak{c}^{\forall}_{a_\alpha,d_\alpha} = \kappa_\alpha$ *for every* $\alpha \in A$*.*

The family of increasing functions is constructed in such a way that for each $\alpha \in A$, $\mathfrak{v}^{\exists}_{c_\alpha,h_\alpha} \leq \mathrm{non}(\mathcal{I}_{f_\alpha}) \leq \mathfrak{c}^{\forall}_{a_\alpha,d_\alpha}$ is provable in ZFC. The poset $\mathbb{Q}$ is constructed by a

[2] By similar methods, it can also be proved that $\mathrm{cof}(\mathcal{M}) = \sup(\{\mathfrak{d}\} \cup \{\mathfrak{c}^{\exists}_{c,h} \mid c \in \omega^\omega\})$ and, whenever h goes to infinity, $\mathrm{add}(\mathcal{N}) = \min(\{\mathfrak{b}\} \cup \{\mathfrak{v}^{\exists}_{c,h} \mid c \in \omega^\omega\})$ and $\mathrm{cof}(\mathcal{N}) = \sup(\{\mathfrak{d}\} \cup \{\mathfrak{c}^{\forall}_{c,h} \mid c \in \omega^\omega\})$.

CS (countable support) product of proper ω^ω-bounding posets, as in Goldstern and Shelah [GS93], but instead of tree-like posets we use Silver-like posets in the product. These posets are used to add generic reals that increase each $\mathfrak{v}^\exists_{c_\alpha,h_\alpha}$; in fact, forcing $\kappa_\alpha \leq \mathfrak{v}^\exists_{c_\alpha,h_\alpha}$ is not difficult. On the other hand, guaranteeing that $\mathfrak{c}^\forall_{a_\alpha,d_\alpha} \leq \kappa_\alpha$ holds in the final forcing extension requires careful definition of the increasing functions and strong combinatorics for each forcing in the product. To be more precise, each poset $\mathbb{Q}^{d_\alpha}_{c_\alpha,h_\alpha}$ (which is used to increase $\mathfrak{v}^\exists_{c_\alpha,h_\alpha}$) is obtained by a lim sup creature construction and depends on d_α in the sense that this function determines the *bigness* needed to not increase the cardinals $\mathfrak{c}^\forall_{a_\beta,d_\beta}$ for $\beta \neq \alpha$.

This chapter is structured as follows.

- In section A2, we strengthen the connections Kamo and Osuga found between anti-localisation cardinals and Yorioka ideals' covering and uniformity numbers; besides, we find a (simple) connection between localisation and anti-localisation cardinals. These ensure that, in the Main Theorem, the family of increasing functions satifies $\mathfrak{v}^\exists_{c_\alpha,h_\alpha} \leq \mathrm{non}(\mathcal{I}_\alpha) \leq \mathfrak{c}^\forall_{a_\alpha,d_\alpha}$ for every $\alpha \in A$.
- In section A3, for $c, h \in \omega^\omega$, we define the Silver-like poset we use to increase $\mathfrak{v}^\exists_{c,h}$ and, by incorporating a norm with sufficiently large bigness, we present sufficient conditions on functions a, e which will guarantee that such a poset does not increase $\mathfrak{c}^\forall_{a,e}$.
- In section A4, we use the results of the previous sections to construct a family of functions as required in the Main Theorem, which is finally proved in section A5.
- The final section A6 is dedicated to discussions and open questions.

A2 Yorioka Ideals and Cardinal Characteristics

For notational simplicity, we describe the cardinal characteristics we are interested in through relational systems as below.

Definition A2.1. A *relational system* is a triplet $\mathbf{R} := \langle X, Y, \sqsubset \rangle$ where $\sqsubset$ is a relation contained in $X \times Y$. The *cardinal characteristics associated with* $\mathbf{R}$ are

$$\mathfrak{b}(\mathbf{R}) := \min\{|B| \mid B \subseteq X \text{ and } \neg\exists\, y \in Y\ \forall\, x \in B\colon x \sqsubset y\},$$
$$\mathfrak{d}(\mathbf{R}) := \min\{|D| \mid D \subseteq Y \text{ and } \forall\, x \in X\ \exists\, y \in D\colon x \sqsubset y\}.$$

The *dual of* $\mathbf{R}$ is the relational system $\mathbf{R}^\perp := \langle Y, X, \not\sqsupset \rangle$.

Let $\mathbf{R}' := \langle X', Y', \sqsubset' \rangle$ be another relational system. A pair (F, G) is a *Tukey connection from* $\mathbf{R}$ *to* $\mathbf{R}'$ if $F\colon X \to X'$, $G\colon Y' \to Y$ and for any $x \in X$ and $y' \in Y'$, $F(x) \sqsubset' y'$ implies $x \sqsubset G(y')$. When there is a Tukey connection from $\mathbf{R}$ to $\mathbf{R}'$, we say that $\mathbf{R}$ *is Tukey-below* $\mathbf{R}'$, which is denoted by $\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$. We say that $\mathbf{R}$ and $\mathbf{R}'$ are *Tukey-equivalent*, denoted by $\mathbf{R} \cong_{\mathrm{T}} \mathbf{R}'$, when $\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$ and $\mathbf{R}' \preceq_{\mathrm{T}} \mathbf{R}$.

Recall that $\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$ implies that $\mathfrak{d}(\mathbf{R}) \leq \mathfrak{d}(\mathbf{R}')$ and $\mathfrak{b}(\mathbf{R}') \leq \mathfrak{b}(\mathbf{R})$. Also, $\mathfrak{b}(\mathbf{R}^\perp) =$

$\mathfrak{d}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R}^\perp) = \mathfrak{b}(\mathbf{R})$. In this section, we will use such Tukey connections to prove inequalities between cardinal invariants.

Example A2.2. We give two examples of relational systems.

(1) Let $\mathcal{I}$ be a family of subsets of a set X that satisfies
   (i) $[X]^{<\aleph_0} \subseteq \mathcal{I}$,
   (ii) $X \notin \mathcal{I}$, and
   (iii) whenever $Y \in \mathcal{I}$ and $X \subseteq Y$, $X \in \mathcal{I}$.
   Consider the relational systems $\mathcal{I} := \langle \mathcal{I}, \mathcal{I}, \subseteq \rangle$ and $\mathbf{Cv}(\mathcal{I}) := \langle X, \mathcal{I}, \in \rangle$. Note that

$$\mathfrak{b}(\mathcal{I}) = \mathrm{add}(\mathcal{I}), \qquad \mathfrak{d}(\mathcal{I}) = \mathrm{cof}(\mathcal{I}),$$
$$\mathfrak{b}(\mathbf{Cv}(\mathcal{I})) = \mathrm{non}(\mathcal{I}), \qquad \mathfrak{d}(\mathbf{Cv}(\mathcal{I})) = \mathrm{cov}(\mathcal{I}),$$

   which are the *cardinal invariants associated with* $\mathcal{I}$. Note that $\mathbf{Cv}(\mathcal{I}) \preceq_{\mathrm{T}} \mathcal{I}$ and $\mathbf{Cv}(\mathcal{I})^\perp \preceq_{\mathrm{T}} \mathcal{I}$, so the well-known fact that $\mathrm{add}(\mathcal{I})$ is below both $\mathrm{cov}(\mathcal{I})$ and $\mathrm{non}(\mathcal{I})$ and that $\mathrm{cof}(\mathcal{I})$ is above those three is easily proved through the relational systems.
   If $\mathcal{J}$ is another family of subsets of X that satisfies (i)–(iii) above and $\mathcal{I} \subseteq \mathcal{J}$, then $\mathbf{Cv}(\mathcal{J}) \preceq_{\mathrm{T}} \mathbf{Cv}(\mathcal{I})$, so $\mathrm{cov}(\mathcal{J}) \leq \mathrm{cov}(\mathcal{I})$ and $\mathrm{non}(\mathcal{I}) \leq \mathrm{non}(\mathcal{J})$.

(2) Let $c = \langle c(i) \mid i < \omega \rangle$ be a sequence of non-empty sets and $h \in \omega^\omega$. Define the relational systems $\mathbf{Lc}(c,h) := \langle \prod c, \mathcal{S}(c,h), \in^* \rangle$ and $\mathbf{aLc}(c,h) := \langle \mathcal{S}(c,h), \prod c, \not\ni^* \rangle$. Note that

$$\mathfrak{b}(\mathbf{Lc}(c,h)) = \mathfrak{v}^\forall_{c,h}, \qquad \mathfrak{d}(\mathbf{Lc}(c,h)) = \mathfrak{c}^\forall_{c,h},$$
$$\mathfrak{b}(\mathbf{aLc}(c,h)) = \mathfrak{c}^\exists_{c,h}, \qquad \mathfrak{d}(\mathbf{aLc}(c,h)) = \mathfrak{v}^\exists_{c,h}.$$

   If, in addition, $c' = \langle c'(i) \mid i < \omega \rangle$ is a sequence of non-empty sets, $h' \in \omega^\omega$ and $|c(i)| \leq |c'(i)|$ and $h'(i) \leq h(i)$ for all but finitely many $i < \omega$, then $\mathbf{Lc}(c,h) \preceq_{\mathrm{T}} \mathbf{Lc}(c',h')$ and $\mathbf{aLc}(c',h') \preceq_{\mathrm{T}} \mathbf{aLc}(c,h)$. Hence, $\mathfrak{v}^\forall_{c',h'} \leq \mathfrak{v}^\forall_{c,h}$, $\mathfrak{c}^\forall_{c,h} \leq \mathfrak{c}^\forall_{c',h'}$, $\mathfrak{c}^\exists_{c,h} \leq \mathfrak{c}^\exists_{c',h'}$ and $\mathfrak{v}^\exists_{c',h'} \leq \mathfrak{v}^\exists_{c,h}$.

Lemma A2.3. *Let $c, h \in \omega^\omega$ and assume that $c \geq 1$ and $h \geq^* 1$.*

(a) If $\sum_{i<\omega} \frac{h(i)}{c(i)} < \infty$, then $\mathbf{Cv}(\mathcal{N}) \preceq_{\mathrm{T}} \mathbf{aLc}(c,h)^\perp$. In particular, $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{c}^\exists_{c,h}$ and $\mathfrak{v}^\exists_{c,h} \leq \mathrm{non}(\mathcal{N})$.

(b) If $\sum_{i<\omega} \frac{h(i)}{c(i)} = \infty$, then $\mathbf{Cv}(\mathcal{N}) \preceq_{\mathrm{T}} \mathbf{aLc}(c,h)$. In particular, $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{v}^\exists_{c,h}$ and $\mathfrak{c}^\exists_{c,h} \leq \mathrm{non}(\mathcal{N})$.

Before we engage in the proof, recall that whenever X is an uncountable Polish space and μ is a continuous (i. e. every singleton has measure zero) probability measure on the Borel σ-algebra of X, there is a Borel isomorphism $f\colon X \to 2^\omega$ that preserves the measures, i. e. $\mu(A)$ is equal to the Lebesgue measure of $f[A]$ for any Borel set $A \subseteq X$ (see e. g. [Kec95, Theorem 17.41]). Therefore, $\mathcal{N}(X) \cong_{\mathrm{T}} \mathcal{N}$ and $\mathbf{Cv}(\mathcal{N}(X)) \cong_{\mathrm{T}} \mathbf{Cv}(\mathcal{N})$. For the following proof, when $c \in \omega^\omega$, $c > 0$, and $c \neq^* 1$, $\prod c$ is the Polish space endowed with the product topology of the discrete spaces $c(i)$ (for $i < \omega$), and μ_c denotes the product measure of $\langle \mu_{c(i)} \mid i < \omega \rangle$, where

$\mu_{c(i)}$ is the measure on the power set of $c(i)$ such that each singleton has measure $1/c(i)$. It is clear that μ_c is a continuous probability measure on the Borel σ-algebra of $\prod c$.

Proof. To see (a), note that $F\colon \prod c \to \prod c$, defined as the identity map, and $G\colon \mathcal{S}(c,h) \to \mathcal{N}(\prod c)$, defined as $G(S) := \{x \in \prod c \mid \exists^\infty i\colon x(i) \in S(i)\}$, form the corresponding Tukey connection.

To see (b), first note that, in general, $\mathbf{aLc}(c,1) \cong_\mathrm{T} \mathbf{Ed}(c) := \langle \prod c, \prod c, \neq^* \rangle$, where $x \neq^* y$ means $\forall^\infty i\colon x(i) \neq y(i)$. We now first prove the following fact:

Claim. *If $h \geq 1$, then $\mathbf{Ed}(d) \preceq_\mathrm{T} \mathbf{aLc}(c,h)$, where $d(i) := \left\lceil \frac{c(i)}{h(i)} \right\rceil$.*

Proof. For each $i < \omega$, we can partition $c(i)$ into $d(i)$ many sets $\langle b_{i,j} \mid j < d(i) \rangle$ of size $\leq h(i)$. Define $F\colon \prod d \to \mathcal{S}(c,h)$ by $F(x)(i) := b_{i,x(i)}$, and define $G\colon \prod c \to \prod d$ such that for any $y \in \prod c$, $G(y)(i)$ is the unique $j \in d(i)$ such that $y(i) \in b_{i,j}$. It is clear that (F,G) is the required Tukey connection. □

Thanks to the preceding claim and the final part of Example A2.2 (2), it suffices to show (b) for $h = 1$, that is, $\mathbf{Cv}(\mathcal{N}) \preceq_\mathrm{T} \mathbf{Ed}(c)$. The case $c \leq^* 1$ is trivial because $\mathbf{Ed}(c) \cong_\mathrm{T} \mathbf{Ed}(1) \cong_\mathrm{T} \langle \{0\}, \{0\}, \neq \rangle$, so assume $c \not\leq^* 1$. For each $y \in \prod c$, define $G(y) := \{x \in \prod c \mid x(i) \neq^* y(i)\}$. Note that $\mu_c(G(y)) = \lim_{n\to\infty} \prod_{i \geq n} \left(1 - \frac{1}{c(i)}\right)$, and

$$\prod_{i \geq n} \left(1 - \frac{1}{c(i)}\right) \leq \prod_{i \geq n} e^{-\frac{1}{c(i)}} = e^{-\sum_{i \geq n} \frac{1}{c(i)}} = 0,$$

so $G(y) \in \mathcal{N}(\prod c)$. Defining $F\colon \prod c \to \prod c$ as the identity map, (F,G) hence witnesses $\mathbf{Cv}(\mathcal{N}) \preceq_\mathrm{T} \mathbf{Ed}(c)$. □

Kamo and Osuga [KO14] proved the following connections between the covering and uniformity numbers of Yorioka ideals and anti-localisation cardinals:

(1) Let $c \in \omega^\omega$ with $c \geq^* 2$. If $g \in \omega^\omega$ and $g(n) \geq \sum_{i \leq n} \log_2 c(i)$ for all but finitely many $n < \omega$, then $\mathbf{aLc}(c,1)^\perp \preceq_\mathrm{T} \mathbf{Cv}(\mathcal{J}_g)$. In particular, $\mathfrak{c}^\exists_{c,1} \leq \mathrm{cov}(\mathcal{J}_g)$ and $\mathrm{non}(\mathcal{J}_g) \leq \mathfrak{v}^\exists_{c,1}$.

(2) Let $c, h \in \omega^\omega$ and $g \in \omega^\omega$ monotonically increasing. If $1 \leq^* h \leq^* c$ and $c(n) \geq 2^{g(\sum_{i \leq n} h(i)-1)}$, then $\mathbf{Cv}(\mathcal{J}_g) \preceq_\mathrm{T} \mathbf{aLc}(c,h)^\perp$. In particular, $\mathrm{cov}(\mathcal{J}_g) \leq \mathfrak{c}^\exists_{c,h}$ and $\mathfrak{v}^\exists_{c,h} \leq \mathrm{non}(\mathcal{J}_g)$.

The two facts above are linked with Yorioka ideals in the sense that $\mathcal{J}_g \subseteq \mathcal{I}_f \subseteq \mathcal{J}_f$ whenever $f \ll g$ (so $\mathbf{Cv}(\mathcal{J}_f) \preceq_\mathrm{T} \mathbf{Cv}(\mathcal{I}_f) \preceq_\mathrm{T} \mathbf{Cv}(\mathcal{J}_g)$). For the purposes of this chapter, we improve Kamo and Osuga's results by showing a direct connection to the Yorioka ideals without passing through a family of the form $\mathcal{J}_g$.

Lemma A2.4. *Let $c, h \in \omega^\omega$, let $\langle I_n \mid n < \omega \rangle$ be the interval partition such that $|I_n| = h(n)$, and let $g_{c,h} \in \omega^\omega$ be defined by $g_{c,h}(k) := \lfloor \log_2 c(n) \rfloor$ whenever $k \in I_n$. If $c \geq^* 2$, $h \geq^* 1$, f is an increasing function and $g_{c,h} \gg f$, then $\mathbf{Cv}(\mathcal{I}_f) \preceq_\mathrm{T} \mathbf{aLc}(c,h)^\perp$. In particular, $\mathrm{cov}(\mathcal{I}_f) \leq \mathfrak{c}^\exists_{c,h}$ and $\mathfrak{v}^\exists_{c,h} \leq \mathrm{non}(\mathcal{I}_f)$.*

Proof. It suffices to find $F\colon 2^\omega \to \prod c$ and $G\colon \mathcal{S}(c,h) \to \mathcal{I}_f$ such that for any $y \in 2^\omega$ and $S \in \mathcal{S}(c,h)$, if $\exists^\infty n\colon F(y)(n) \in S(n)$, then $y \in G(S)$.

For each $n < \omega$, fix a one-to-one map $\iota_n\colon 2^{\lfloor \log_2 c(n) \rfloor} \to c(n)$. For $S \in \mathcal{S}(c,h)$, enumerate $S(n) =: \{m^S_{n,k} \mid k \in I_n\}$ and, whenever $k \in I_n$, define $\sigma_S(k) := \iota_n^{-1}(m^S_{n,k})$ when $m^S_{n,k} \in \operatorname{ran}\iota_n$; otherwise $\sigma_S(k)$ is allowed to be anything of length $\lfloor \log_2 c(n) \rfloor$.

Set $G(S) := [\sigma_S]_\infty$. Note that $|\sigma_S(k)| = g_{c,h}(k)$, so $\operatorname{ht}_{\sigma_s} \gg f$ and $G(S) \in \mathcal{I}_f$. On the other hand, for $y \in 2^\omega$, define $F(y)(n) := \iota_n(y{\restriction}_{\lfloor \log_2 c(n) \rfloor})$. If $F(y)(n) \in S(n)$ for infinitely many n, then for such n there is a $k \in I_n$ such that $F(y)(n) = m^S_{n,k}$, so $\sigma_S(k) = y{\restriction}_{\lfloor \log_2 c(n) \rfloor} \subseteq y$. As there are infinitely many such k, we conclude that $y \in G(S)$. □

Lemma A2.5. *Let $b, g \in \omega^\omega$ such that $b \geq^* 2$, $g \geq^* 1$, let $\langle J_n \mid n < \omega \rangle$ be the interval partition such that $|J_n| = g(n)$, and let $f_{b,g} \in \omega^\omega$ be defined by $f_{b,g}(k) := \sum_{\ell \leq n} \lceil \log_2 b(\ell) \rceil$ whenever $k \in J_n$. If f is an increasing function and there is some $1 \leq m < \omega$ such that $f_{b,g}(k) \leq f(k^m)$ for all but finitely many $k < \omega$, then $\mathbf{aLc}(b,g)^\perp \preceq_{\mathrm{T}} \mathbf{Cv}(\mathcal{I}_f)$. In particular, $\mathfrak{c}^\exists_{b,g} \leq \operatorname{cov}(\mathcal{I}_f)$ and $\operatorname{non}(\mathcal{I}_f) \leq \mathfrak{v}^\exists_{b,g}$.*

Proof. First note that for any function g', $\exists\, m > 0\ \forall^\infty k\colon g'(k) \leq f(k^m)$ iff $\forall\, f' \gg f\colon g' \leq^* f'$. It suffices to define functions $F\colon \prod b \to 2^\omega$ and $G\colon \mathcal{I}_f \to \mathcal{S}(b,g)$ such that for any $y \in \prod b$ and $X \in \mathcal{I}_f$, if $F(y) \in X$, then $\exists^\infty n\colon y(n) \in G(X)(n)$.

Consider the interval partition $\langle I_n \mid n < \omega \rangle$ of ω such that $|I_n| = \lceil \log_2 b(n) \rceil$. For each $n < \omega$, fix a one-to-one function $\iota_n\colon b(n) \to 2^{I_n}$. For $y \in \prod b$, define $F(y)$ as the concatenation of the binary sequences $\langle \iota_n(y(n)) \mid n < \omega \rangle$. On the other hand, for $X \in \mathcal{I}_f$, $X \subseteq [\sigma_X]_\infty$ for some $\sigma_X \in (2^{<\omega})^\omega$ such that $\operatorname{ht}_{\sigma_X} \gg f$, so we define

$$G(X)(n) := \{\iota_n^{-1}(\sigma_X(k){\restriction}_{I_n}) \mid \sigma_X(k){\restriction}_{I_n} \in \operatorname{ran}\iota_n,\ I_n \subseteq \operatorname{ht}_{\sigma_X}(k),\ k \in J_n\}$$

(since $f_{b,h} \leq^* \operatorname{ht}_{\sigma_X}$ and $\forall\, k \in J_n\colon I_n \subseteq \operatorname{ht}_{\sigma_X}$ holds for all but finitely many n).

If $F(y) \in X$, then there is an infinite $W \subseteq \omega$ such that for any $n \in W$, $\sigma_X(k) \subseteq F(y)$ for some $k \in J_n$; so $\sigma_X(k){\restriction}_{I_n} = F(y){\restriction}_{I_n} = \iota_n(y(n))$ and hence we have $y(n) \in G(X)(n)$. □

The following result shows a connection between localisation cardinals and anti-localisation cardinals. This is useful to include localisation cardinals of the type $\mathfrak{c}^\forall$ in our main result.

Lemma A2.6. *Let $c, h \in \omega^\omega$ with $c > 0$ and $h \geq^* 1$. If c' is a function with domain ω and $h' \in \omega^\omega$ such that for all but finitely many $i < \omega$, $c'(i) \geq \left|[c(i)]^{\leq h(i)} \smallsetminus \{\varnothing\}\right|$ and $h'(i) < {}^{c(i)}/_{h(i)}$, then $\mathbf{aLc}(c,h) \preceq_{\mathrm{T}} \mathbf{Lc}(c',h')$. In particular, $\mathfrak{v}^\forall_{c',h'} \leq \mathfrak{c}^\exists_{c,h}$ and $\mathfrak{v}^\exists_{c,h} \leq \mathfrak{c}^\forall_{c',h'}$.*

Proof. It suffices to show the result for the case when $h(i) \geq 1$, $c'(i) = [c(i)]^{\leq h(i)} \smallsetminus \{\varnothing\}$ and $h'(i) < {}^{c(i)}/_{h(i)}$ for all $i < \omega$.[3]

[3] Note that the c' we work with here and in the subsequent proof is such that $\prod c'$ already is a family of slaloms, which reduces the complexity of the Tukey connection (since we do not have to bijectively map sets to their cardinalities).

Define $F\colon \mathcal{S}(c,h) \to \prod c'$ such that $F(S)(i) := S(i)$ whenever $S(i) \neq \varnothing$; otherwise $F(\varphi)(i)$ is some arbitrary singleton. On the other hand, define $G\colon \mathcal{S}(c',h') \to \prod c$ such that $G(\varphi)(i) \in c(i) \setminus \bigcup \varphi(i)$ (which is fine because this union has size $\leq h(i) \cdot h'(i) < c(i)$). It is clear that (F,G) is the Tukey connection that we want. □

The next lemma is more or less the converse of the preceding one. We prove it only for completeness' sake, and it will not be used in the sequel.

Lemma A2.7. *Let $c, h \in \omega^\omega$ with $c > 0$ and $h \geq^* 1$. If $c', h' \in \omega^\omega$ with $c' > 0$ such that for all but finitely many $i < \omega$, $c'(i) \leq \left|[c(i)]^{\leq h(i)} \setminus \{\varnothing\}\right|$ and $h'(i) \geq \left|[c(i) - 1]^{\leq h(i)} \setminus \{\varnothing\}\right|$, then $\mathbf{Lc}(c',h') \preceq_{\mathrm{T}} \mathbf{aLc}(c,h)$.*

Proof. It suffices to show the result for the case when $h(i) \geq 1$, $c'(i) = [c(i)]^{\leq h(i)} \setminus \{\varnothing\}$ and $h'(i) = \left|[c(i) - 1]^{\leq h(i)} \setminus \{\varnothing\}\right|$ for all $i < \omega$.

Define $F\colon \prod c' \to \mathcal{S}(c,h)$ by $F(S) := S$ and $G\colon \prod c \to \mathcal{S}(c',h')$ by $G(y)(i) := [c(i) \setminus \{y(i)\}]^{\leq h(i)} \setminus \{\varnothing\}$. Assume that $S \in \prod c'$, $y \in \prod c$ and $y \notin^* F(S)$. This means that for all but finitely many $i < \omega$, $y(i) \notin S(i)$, so $S(i) \in G(y)(i)$, that is, $S \in^* G(y)$. □

By Lemma A2.6 and Lemma A2.7 applied to $c' = c$, $h = 1$ and $h' = c - 1$, it follows that $\mathbf{aLc}(c,1) \cong_{\mathrm{T}} \mathbf{Lc}(c, c-1)$ (although it is also quite simple to prove this directly). More properties of localisation and anti-localisation cardinals can be found in e. g. [GS93, CM17].

The following lemma shows how functions should be related to get a particular chain of inequalities between anti-localisation cardinals, localisation cardinals and uniformity numbers of Yorioka ideals, which will be useful for the main result of this chapter.

Lemma A2.8. *Assume that $a, d, b, g, f, c, h \in \omega^\omega$ such that*

(I1) $b \geq^* 2$, $g \geq^* 1$ and $\forall^\infty i < \omega\colon {}^{b(i)}\!/_{g(i)} > d(i)$,
(I2) $a(i) \geq \left|[b(i)]^{\leq g(i)} \setminus \{\varnothing\}\right|$ for all but finitely many i,
(I3) f is increasing and $f \geq^* f_{b,g}$ (see Lemma A2.5), and
(I4) $c \geq^* 2$, $h \geq^* 1$ and $g_{c,h} \gg f$ (see Lemma A2.4).

Then $\mathfrak{v}^{\exists}_{c,h} \leq \mathrm{non}(\mathcal{I}_f) \leq \mathfrak{v}^{\exists}_{b,g} \leq \mathfrak{c}^{\forall}_{a,d}$.

Proof. This is a direct consequence of Lemma A2.4, Lemma A2.5 and Lemma A2.6. □

A3 Motivational Example

We will now define a creature forcing poset whose countable support product we will use to increase the value of $\mathfrak{v}^{\exists}_{c,h}$ to be at least some prescribed cardinal κ. Through careful choice of the function parameters (and some additional auxiliary functions), we will show in section A5 that $\mathfrak{v}^{\exists}_{c,h}$ is indeed forced to be *exactly* κ, and

then use a countable support product of many such creature forcing posets with varying function parameters to force uncountably many Yorioka ideals' uniformity numbers to be different.

Definition A3.1. Let $c, h \in \omega^\omega$ with $c > h \geq^* 1$. We define a forcing poset $\mathbb{Q}_{c,h}$ as follows: A condition $p \in \mathbb{Q}_{c,h}$ is a sequence of *creatures* $p(n)$ such that each $p(n)$ is a non-empty subset of $[c(n)]^{\leq h(n)}$ and such that, letting the norm $\|\cdot\|_{c,h,n}$ be defined by

$$\|M\|_{c,h,n} := \max\{k \mid \forall Y \subseteq [c(n)]^{\leq k}\ \exists X \in M\colon Y \subseteq X\},$$

p fulfils $\limsup_{n\to\infty} \|p(n)\|_{c,h,n} = \infty$. The order is $q \leq p$ iff $q(n) \subseteq p(n)$ for all $n < \omega$ (i. e. stronger conditions consist of smaller subsets of $[c(n)]^{\leq h(n)}$). Note that $\mathbb{Q}_{c,h} \neq \varnothing$ iff $\limsup_{n\to\infty} h(n) = \infty$.

Given a condition p such as above, the finite initial segments in $p{\restriction}_{k+1}$ (for $k < \omega$) are sometimes referred to as *possibilities* and denoted by $\mathrm{poss}(p, {\leq}k) := \prod_{\ell\leq k}[p(\ell)]^1 = \{\langle\{z(\ell)\} \mid \ell \leq k\rangle \mid \forall \ell \leq k\colon z(\ell) \in p(\ell)\}$. We may also use the notation $\mathrm{poss}(p, {<}k) := \mathrm{poss}(p, {\leq}k-1)$. When $\eta \in \mathrm{poss}(p, {\leq}k)$, we write $p \wedge \eta$ to denote $\eta^\frown p{\restriction}_{[k+1,\omega)}$.[4]

We denote the indices of the non-trivial creatures by $\mathrm{split}(p) := \langle k < \omega \mid |p(k)| > 1\rangle$ and, for $n < \omega$, denote the n-th member of $\mathrm{split}(p)$ by $s_n(p)$. For $p, q \in \mathbb{Q}_{c,h}$, define $q \leq_n p$ as "$q \leq p$ and $q{\restriction}_{s_n(q)+1} = p{\restriction}_{s_n(q)+1}$", which means that q is stronger than p and they are identical up to (including) the n-th non-trivial creature.

Observation A3.2. In this section, we often work with the set $\mathbb{Q}^*_{c,h}$ of conditions $p \in \mathbb{Q}_{c,h}$ such that $\|p(s_n(p))\|_{c,h,s_n(p)} \geq n+1$ (i. e. the n-th non-trivial creature has norm at least $n+1$). It is clear that $\mathbb{Q}^*_{c,h}$ is dense in $\mathbb{Q}_{c,h}$.

Lemma A3.3. *The poset $\mathbb{Q}_{c,h}$ adds a generic slalom $\dot S \in \mathcal{S}(c,h)$ such that $\mathbb{Q}_{c,h}$ forces that any ground-model real in $\prod c$ is caught infinitely often by $\dot S$.*

Proof. As the set $D_n := \{p \in \mathbb{Q}_{c,h} \mid |p(n)| = 1\}$ is dense in $\mathbb{Q}_{c,h}$, we can define $\dot S(n)$ as the unique member of $\bigcap_{p\in\dot G} p(n)$ (where $\dot G$ is the generic filter), which clearly is a name for a member of $[c(n)]^{\leq h(n)}$.

Now assume that $x \in V \cap \prod c$ and fix a condition $p \in \mathbb{Q}_{c,h}$ and an $n_0 < \omega$. Pick some $k \geq n_0$ such that $\|p(k)\|_{c,h,k} \geq 1$ and strengthen p to q by setting $q(k) = \{t\}$ for some $t \in p(k)$ that contains $x(k)$ (which exists because, by the definition of the norm, $\|M\|_{c,h,n} \geq 1$ iff $\bigcup M = c(n)$). Hence $q \leq p$ and $q \Vdash x(k) \in \dot S(k)$.

The previous argument shows (by density) that for any $n_0 < \omega$, $\mathbb{Q}_{c,h}$ forces $\exists\, k \geq n_0\colon x(k) \in \dot S(k)$. □

[4] The usual creature forcing notation, which is used quite a lot in chapter B, defines the set of possibilities more abstractly as $\mathrm{poss}(p, {\leq}k) := \prod_{\ell\leq k} p(\ell)$ and defines $p \wedge \eta$ as a condition with an extended *trunk* (a concept which we did not deem necessary to introduce in this first chapter). Since working with possibilities η as sequences of singletons suffices for our proofs and is conceptually easier, we instead opted for the simpler definition in this first chapter.

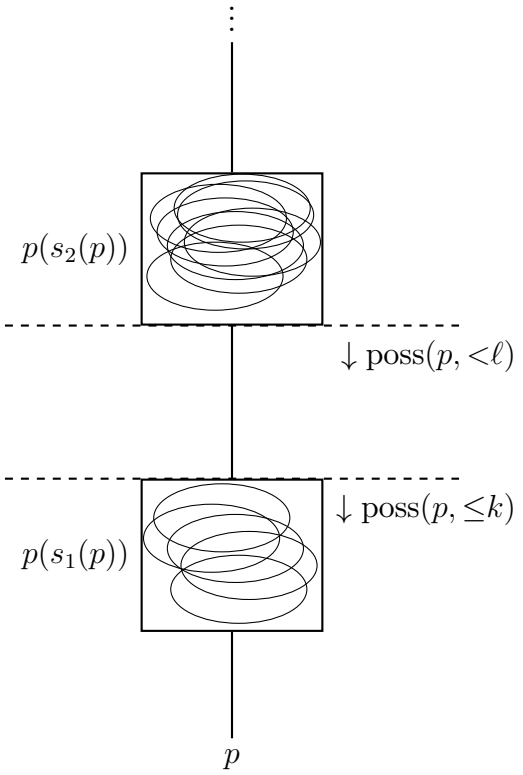

Figure 3: An example for the initial segment of a condition $p \in \mathbb{Q}_{c,h}$, with $k := s_1(p)$ and $\ell := s_2(p)$. Each ellipse represents one element of $[c(k)]^{\leq h(k)}$ and $[c(\ell)]^{\leq h(\ell)}$, respectively.

We now show that the poset $\mathbb{Q}_{c,h}$ is indeed a proper ω^ω-bounding poset and that, moreover, $\mathbb{Q}^*_{c,h}$ satisfies strong axiom A. It is clear that $|\mathbb{Q}_{c,h}| = \mathfrak{c}$ so, assuming CH, the standard Δ-system argument shows that this poset has $\aleph_2$-cc and hence preserves cardinalities and cofinalities.

Lemma A3.4. *If $n < \omega$, $p \in \mathbb{Q}_{c,h}$ and $D \subseteq \mathbb{Q}_{c,h}$ is open dense, then there is a condition $q \leq_n p$ in $\mathbb{Q}_{c,h}$ such that for any $\eta \in \mathrm{poss}(q, \leq s_n(q))$, $q \wedge \eta \in D$.*

Proof. Enumerate $\mathrm{poss}(p, \leq s_n(p)) =: \{\eta_k \mid k < m\}$. By recursion on $k \leq m$, define $p_0 := p$ and choose $p_{k+1} \leq \eta_k{}^\frown p_k{\restriction}_{[s_n(p)+1,\omega)}$ in D. Then

$$q := p{\restriction}_{s_n(p)+1}{}^\frown p_m{\restriction}_{[s_n(p)+1,\omega)},$$

is precisely the condition we are looking for. □

Lemma A3.5. *The poset $\mathbb{Q}^*_{c,h}$ satisfies strong axiom A. Concretely, it satisfies:*

(a) For any $p, q \in \mathbb{Q}^*_{c,h}$ and $m \leq n < \omega$, if $p \leq_n q$, then $p \leq_m q$ and $p \leq q$.
(b) Whenever $\langle p_n \mid n < \omega\rangle$ is a sequence in $\mathbb{Q}^*_{c,h}$ which satisfies $p_{n+1} \leq_n p_n$ for every $n < \omega$, there is some $q \in \mathbb{Q}^*_{c,h}$ such that $q \leq_n p_n$ for every $n < \omega$.
(c) If $A \subseteq \mathbb{Q}^*_{c,h}$ is an antichain, $p \in \mathbb{Q}^*_{c,h}$ and $n < \omega$, then there is a condition $q \leq_n p$ in $\mathbb{Q}^*_{c,h}$ such that only finitely many $r \in A$ are compatible with q.

In particular, $\mathbb{Q}_{c,h}$ is proper and ω^ω-bounding.

Proof. Property (a) follows immediately from the definition. To see property (b), first define $f\colon [-1,\omega) \to \omega$ such that $f(-1) := -1$ and, for $n \geq 0$, $f(n) = s_n(p_n)$. For each $n < \omega$ define $q(k) := p_n(k)$ for any $k \in (f(n-1), f(n)]$. It is clear that $s_n(q) = f(n)$, so $q \in \mathbb{Q}^*_{c,h}$. By the definition of q, $q \leq_n p_n$.

We now show property (c). Set D as the set of conditions $p \in \mathbb{Q}_{c,h}$ which are either stronger than some member of A or incompatible with every member of A. It is clear that D is an open dense set, so we can find $q \leq_n p$ in $\mathbb{Q}^*_{c,h}$ as in Lemma A3.4, that is, such that $q \wedge \eta \in D$ for any $\eta \in \mathrm{poss}(q, {\leq} s_n(q))$, which means that there is at most one $r_\eta \in A$ weaker than $q \wedge \eta$. Hence, if $r \in A$ is compatible with q, then it must be compatible with $q \wedge \eta$ for some $\eta \in \mathrm{poss}(q, {\leq} s_n(q))$ (of which there are only finitely many), so $r = r_\eta$ because A is an antichain. □

A sequence $\langle p_n \mid n < \omega\rangle$ as in property (b) is usually known as a *fusion sequence*, and the q obtained in its proof is known as the *fusion of* $\langle p_n \mid n < \omega\rangle$. As an application of this notion, we prove the properties of *continuous and timely reading of names* and *few possibilities* for the forcing $\mathbb{Q}_{c,h}$.

Definition A3.6. Let $p \in \mathbb{Q}_{c,h}$ and let $\dot\tau$ be a $\mathbb{Q}_{c,h}$-name for a function from ω into the ground model V.

(1) We say that p *reads* $\dot\tau$ *continously* iff for each $n < \omega$ there is some k_n such that for each $\eta \in \mathrm{poss}(p, {\leq} k_n)$, $p \wedge \eta$ decides $\dot\tau{\restriction}_n$.
(2) We say that p *reads* $\dot\tau$ *timely* iff for each $n \in \mathrm{split}(p)$ and each $\eta \in \mathrm{poss}(p, {\leq} n)$, $p \wedge \eta$ decides $\dot\tau{\restriction}_n$. (This is equivalent to continuous reading with $k_n = n$ for each $n \in \mathrm{split}(p)$.)

The term "continuous reading" refers to the fact that, in a sense, there is a continuous function from "branches" in p to V^ω, coded by a function from initial segments of p to initial segments of $\dot\tau$.[5] It is clear that timely reading of a name implies continuous reading thereof; we will only be using the stronger property of timely reading in the rest of this chapter.

Lemma A3.7 (timely reading of names)**.** *Let $p \in \mathbb{Q}_{c,h}$ and let $\dot\tau$ be a $\mathbb{Q}_{c,h}$-name for a function from ω into the ground model V. Then there is a condition $q \leq p$ in $\mathbb{Q}_{c,h}$ such that q reads $\dot\tau$ timely.*

Proof. Denote by D_n the set of conditions in $\mathbb{Q}_{c,h}$ which decide $\tau{\restriction}_n$, which is an open dense subset of $\mathbb{Q}_{c,h}$. By recursion on $\ell < \omega$, define an increasing function $f \in \omega^\omega$ and a sequence $\langle p_\ell \mid \ell < \omega\rangle$ of members of $\mathbb{Q}^*_{c,h}$ such that

(i) $p_0 \leq p$,
(ii) $f(\ell) = s_\ell(p_\ell)$,
(iii) $p_{\ell+1} \leq_\ell p_\ell$, and
(iv) for any $\eta \in \mathrm{poss}(p_{\ell+1}, {\leq} f(\ell))$, $p_{\ell+1} \wedge \eta \in D_{f(\ell)}$.

[5] See Observation B6.2 for a more detailed explanation.

For the construction, choose any $p_0 \leq p$ in $\mathbb{Q}^*_{c,h}$ and let $f(0) := s_0(p_0)$, which clearly satisfy (i) and (ii). Now assume that p_ℓ and $f(\ell)$ have been constructed and that they satisfy (ii). Find $p_{\ell+1} \in \mathbb{Q}^*_{c,h}$ by application of Lemma A3.4 to ℓ, p_ℓ and $D_{f(\ell)}$.

As in Lemma A3.5 (b), define $q = \langle q(k) \mid k < \omega\rangle$ such that $q{\restriction}_{(f(\ell)-1,f(\ell)]} := p_\ell{\restriction}_{(f(\ell-1),f(\ell)]}$ for any $\ell < \omega$ (letting $f(-1) := -1$). It is clear that $s_\ell(q) = f(\ell)$, even more, $\|q(f(\ell))\|_{c,h,f(\ell)} = \|p_\ell(f(\ell))\|_{c,h,f(\ell)} \geq \ell + 1$, so $q \in \mathbb{Q}^*_{c,h}$. Besides, $q \leq_\ell p_\ell$ for any $\ell < \omega$. On the other hand, by (iv), $q \wedge \eta \in D_{f(\ell)}$ for any $\eta \in \operatorname{poss}(q, {\leq}f(\ell))$. □

Lemma A3.8 (few possibilities)**.** *Let $g \in \omega^\omega$ be a function going to infinity. If $p \in \mathbb{Q}_{c,h}$, then there is a condition $q \leq p$ in $\mathbb{Q}^*_{c,h}$ such that $|\operatorname{poss}(q, {<}s_n(q))| < g(s_n(q))$ for all $n < \omega$.*

Proof. We construct $f \in \omega^\omega$ and a fusion sequence $\langle p_n \mid n < \omega\rangle$ in $\mathbb{Q}^*_{c,h}$ such that

(i) $p_0 \leq p$,
(ii) $f(n) = s_n(p_n)$,
(iii) $|\operatorname{poss}(p_n, {<}f(n))| < g(f(n))$, and
(iv) $p_{n+1} \leq_n p_n$.

Start with $p_{-1} := p$ and $f(-1) := -1$. For $n < \omega$ such that p_{n-1} has already been defined, choose $f(n) > f(n-1)$ in $\operatorname{split}(p_{n-1})$ such that $\|p_{n-1}(f(n))\|_{c,h,f(n)} \geq n+1$ and $|\operatorname{poss}(p_{n-1}, {\leq}f(n-1))| < g(f(n))$ (where $\operatorname{poss}(p, {\leq}{-1}) := \{\langle\,\rangle\}$). Define p_n such that $p_n(k) := p_{n-1}(k)$ for every $k \in [0, f(n-1)] \cup [f(n), \omega]$ and, for $k \in (f(n-1), f(n))$, $p_n(k)$ is a singleton contained in $p_{n-1}(k)$. It is clear that $p_n \leq_{n-1} p_{n-1}$ and that $|\operatorname{poss}(p_n, {<}f(n))| = |\operatorname{poss}(p_{n-1}, {\leq}f(n-1))| < g(f(n))$.

As in previous arguments, define the fusion $q \in \mathbb{Q}^*_{c,h}$ of the sequence $\langle p_n \mid n < \omega\rangle$ by $q(k) := p_n(k)$ whenever $k \in (f(n-1), f(n)]$. It is easy to see that q is as required. □

We are now interested to know for which functions $b, g \in \omega^\omega$ the poset $\mathbb{Q}_{c,h}$ will not increase the cardinal $\mathfrak{v}^\exists_{b,g}$ (i. e. any slalom in $\mathcal{S}(b,g)$ from the generic extension anti-localises some real in $\prod b$ from the ground model) or the cardinal $\mathfrak{c}^\forall_{b,g}$ (i. e. any real in $\prod b$ from the generic extension is localised by some slalom in $\mathcal{S}(b,g)$ from the ground model). This will be the key point to understand the relation between the functions so that the Main Theorem can be proved using a countable support product of our posets increasing cardinals of the form $\mathfrak{v}^\exists_{c,h}$. To ensure such *preservation*, it seems that our forcing requires the norms on the creatures to satisfy *bigness* (in the sense of [FGKS17] or section B5),[6] and for this we modify the original definition of $\mathbb{Q}_{c,h}$.

Definition A3.9. Let $c, h, d \in \omega^\omega$ with $c > h \geq^* 1$, $d \geq 2$, and

$$\limsup_{n\to\infty} \frac{1}{d(k)} \log_{d(k)} h(k) = \infty.$$

[6] Note that bigness is equivalent to the concept of *completeness* in the sense of [GS93], as is explained in detail in section B5.

Recalling Definition A3.1 of $\mathbb{Q}_{c,h}$, we now define $\mathbb{Q}^d_{c,h}$ as follows: A condition $p \in \mathbb{Q}^d_{c,h}$ is a sequence of creatures $p(n)$ such that each $p(n)$ is a subset of $[c(n)]^{\leq h(n)}$ and such that, replacing $\|\cdot\|_{c,h,n}$ with the norm $\|\cdot\|^d_{c,h,n}$ defined by

$$\|M\|^d_{c,h,n} := \frac{1}{d(n)} \log_{d(n)}(\|M\|_{c,h,n} + 1),$$

p fulfils $\limsup_{n\to\infty} \|p(n)\|^d_{c,h,n} = \infty$. The order is the same as on $\mathbb{Q}_{c,h}$.

Let $\mathbb{Q}^{*d}_{c,h}$ be the set of conditions in $\mathbb{Q}^d_{c,h}$ that satisfy $\|p(s_n(p))\|^d_{c,h,s_n(p)} \geq n+1$ for all $n < \omega$, which clearly is a dense subset of $\mathbb{Q}^d_{c,h}$.

The requirement $\limsup_{n\to\infty} {}^{1}/_{d(k)} \log_{d(k)} h(k) = \infty$ is what guarantees that the poset $\mathbb{Q}^d_{c,h}$ is non-empty. Note that $\mathbb{Q}^d_{c,h} \subseteq \mathbb{Q}_{c,h}$ and $\mathbb{Q}^{*d}_{c,h} \subseteq \mathbb{Q}^*_{c,h}$. Also note that the results proven so far remain equally valid for $\mathbb{Q}^d_{c,h}$ and $\mathbb{Q}^{*d}_{c,h}$ (the same proofs apply), in particular, that we can use the poset to increase $\mathfrak{v}^{\exists}_{c,h}$.

Lemma A3.10 (bigness)**.** *For any $n < \omega$, the norm $\|\cdot\|^d_{c,h,n}$ is strongly $d(n)$-big, i. e. whenever $M \subseteq [c(n)]^{\leq h(n)}$ and $f\colon M \to d(n)$, there is an $M^* \subseteq M$ such that $f{\restriction}_{M^*}$ is constant and $\|M^*\|^d_{c,h,n} \geq \|M\|^d_{c,h,n} - {}^{1}/_{d(n)}$.*

Proof. This is quite similar to [FGKS17, Lemma 8.1.2 (1)] or the case t = nn of Theorem B5.6 (i). For each $j < d(n)$, let $M_i := f^{-1}[\{j\}]$ and $m_j := \|M_i\|_{c,h,n}$. By Definition A3.9, there is some $a_j \subseteq c(n)$ of size $m_j + 1$ such that no member of M_j contains a_j, thus no member of M contains $a := \bigcup_{j<d(n)} a_j$, so $\|M\|_{c,h,n} \leq |a| - 1 \leq d(n) \cdot (m+1) - 1$ where $m := m_{j^*} = \max_{j<d(n)}\{m_j\}$. Therefore

$$\|M\|^d_{c,h,n} = \frac{1}{d(n)} \log_{d(n)}(\|M\|_{c,h,n} + 1) \leq \|M_{j^*}\|^d_{c,h,n} + \frac{1}{d(n)}$$

as required. □

Corollary A3.11. *Fix $n, m, k < \omega$. If ${}^{m}/_{k} \leq d(n)$, then, whenever $M \subseteq [c(n)]^{\leq h(n)}$ and $f\colon M \to m$, there is an $M^* \subseteq M$ such that $|f[M^*]| \leq k$ and $\|M^*\|^d_{c,h,n} \geq \|M\|^d_{c,h,n} - {}^{1}/_{d(n)}$.*

Proof. Partition m into sets $\{a_j \mid j < \ell\}$ of size at most k with $\ell \leq d(n)$. Define $f'\colon M \to \ell$ such that $f'(z) = j$ iff $f(z) \in a_j$. By Lemma A3.10 there is some $M^* \subseteq M$ such that $f'{\restriction}_{M^*}$ is constant with value j^* and $\|M^*\|^d_{c,h,n} \geq \|M\|^d_{c,h,n} - \frac{1}{d(n)}$. Hence $f[M^*] \subseteq a_{j^*}$, so $|f[M^*]| \leq k$. □

Thanks to bigness, timely reading of names can be strengthened in the following way:

Lemma A3.12 (early reading)**.** *Let $\langle A_n \mid n < \omega\rangle$ be a sequence of non-empty finite sets, and let $\dot\tau$ be a $\mathbb{Q}^d_{c,h}$-name for a member of $\prod_{n<\omega} A_n$. If d goes to infinity and $\left|\prod_{i<n} A_i\right| \leq d(n)$ for all $n < \omega$, then for any $p \in \mathbb{Q}^d_{c,h}$, there is some $q \leq p$ which reads $\dot\tau$ early, that is, for any $n < \omega$ and $\eta \in \mathrm{poss}(q, {<}n)$, $q \wedge \eta$ already decides $\dot\tau{\restriction}_n$.*

Proof. Without loss of generality, by Lemma A3.7 we may assume that $p \in \mathbb{Q}^{*d}_{c,h}$ reads $\dot\tau$ timely, that is, $p \wedge \eta$ decides $\dot\tau{\restriction}_n$ for any $\eta \in \operatorname{poss}(p, {\leq}n)$ and $n \in \operatorname{split}(p)$; by Lemma A3.8 we may also assume that $|\operatorname{poss}(p, {<}n)| < d(n)$ for any $n \in \operatorname{split}(p)$.

We construct $q(k)$ by recursion on $k < \omega$. When $k \notin \operatorname{split}(p)$, let $q(k) := p(k)$. Now assume that $k \in \operatorname{split}(p)$ and work with $p_k := q{\restriction}_k{}^\frown p{\restriction}_{[k,\omega)}$. Enumerate $\operatorname{poss}(p_k, {<}k) =: \{\eta_j \mid j < m\}$, where $m < d(k)$. By recursion on $j \leq m$, define $M_j \subseteq p(k)$ such that

(i) $M_0 = p(k)$,
(ii) $M_{j+1} \subseteq M_j$,
(iii) $\|M_{j+1}\|^d_{c,h,k} \geq \|M_j\|^d_{c,h,k} - {}^1\!/_{d(k)}$, and
(iv) there is some $r_j \in \prod_{i<k} A_i$ such that for any $t \in M_{j+1}$, $p_k \wedge (\eta_j{}^\frown\{t\})$ forces $\dot\tau{\restriction}_k = r_j$.

Assume we already have M_j. Define the function $h_j \colon M_j \to \prod_{i<k} A_i$ such that for any $t \in M_{j+1}$, $p_k \wedge (\eta_j{}^\frown\{t\})$ forces $\dot\tau{\restriction}_k = h_j(t)$. Hence, by Lemma A3.10, there is some $M_{j+1} \subseteq M_j$ as in (iii) such that $h_j{\restriction}_{M_{j+1}}$ is constant with value r_j.

Define $q(k) := M_m$. By property (iii),

$$\|q(k)\|^d_{c,h,k} \geq \|p(k)\|^d_{c,h,k} - \frac{m}{d(k)} > \|p(k)\|^d_{c,h,k} - 1$$

and, by property (iv), $p_k \wedge (\eta_j{}^\frown\{t\})$ forces $\dot\tau{\restriction}_k = r_j$ for any $t \in q(k)$ and $j < m$, which means that $q{\restriction}_{k+1}{}^\frown p{\restriction}_{[k+1,\omega)} \wedge \eta_j$ forces $\dot\tau{\restriction}_k = r_j$.

By the construction, when $k \in \operatorname{split}(p)$, $\|q(k)\|^d_{c,h,k} \geq \|p(k)\|^d_{c,h,k} - 1$, so $q \in \mathbb{Q}^d_{c,h}$. If $k \in \operatorname{split}(p)$ and $\eta \in \operatorname{poss}(q, {<}k)$, then $q \wedge \eta$ decides $\dot\tau{\restriction}_k$. Now, if $k \in \omega \smallsetminus \operatorname{split}(p)$ and $\eta \in \operatorname{poss}(q, {<}k)$, then there is a unique $\eta' \in \operatorname{poss}(q, {<}k')$ extending η, where k' is the smallest member of $\operatorname{split}(p)$ above k, so $q \wedge \eta = q \wedge \eta'$. As this decides $\dot\tau{\restriction}_{k'}$, it is clear that it also decides $\dot\tau{\restriction}_k$. □

The following result gives sufficient conditions on functions a, e to guarantee that $\mathbb{Q}^d_{c,h}$ does not increase $\mathfrak{c}^\forall_{a,e}$. Hereafter, we fix the notation $c^{\nabla h}(k) := |[c(k)]^{\leq h(k)}|$.

Lemma A3.13. *Assume that $c, d, h \in \omega^\omega$ are as in Definition A3.9 with d going to infinity, $a, e \in \omega^\omega$ with $a > 0$ and e going to infinity, and that they satisfy*

(L1) $\prod_{k<n} a(k) \leq d(n)$ *and* $\prod_{k<n} c^{\nabla h}(k) \leq e(n)$ *for all but finitely many n, and*
(L2) $\lim_{k\to\infty} \min\left\{\frac{c^{\nabla h}(k)}{e(k)}, \frac{a(k)}{d(k)}\right\} = 0$.

Then $\mathbb{Q}^d_{c,h}$ forces that any real in $\prod a$ is localised by some member of $\mathcal{S}(a,e) \cap V$.

Proof. We show that, whenever $p \in \mathbb{Q}^d_{c,h}$ and $\dot x$ is a $\mathbb{Q}^d_{c,h}$-name for a real in $\prod a$, then there is some $\varphi \in \mathcal{S}(a,e) \cap V$ and some $q \leq p$ in $\mathbb{Q}^d_{c,h}$ forcing $\dot x(k) \in \varphi(k)$ for all but finitely many k. Without loss of generality, we assume that

(i) $p \in \mathbb{Q}^{*d}_{c,h}$ reads $\dot x$ early (by (L1) and Lemma A3.12),
(ii) for any $k \in \operatorname{split}(p)$, $|\operatorname{poss}(p, {<}k)| \leq d(k)$ and

$$\min\left\{\frac{c^{\nabla h}(k)}{e(k)}, \frac{a(k)}{d(k)}\right\} \leq \frac{1}{2\,|\operatorname{poss}(p, {<}k)|}$$

(by (L2) and Lemma A3.8), and

(iii) $|\operatorname{poss}(p, {<}k)| \leq e(k)$ for every $k < \omega$ (by (L1)).

By recursion on k, we construct $q(k)$ and $\varphi(k)$ according to the following case distinction: First, let $p_k := q{\restriction_k}^\frown p{\restriction_{[k,\omega)}}$.

When $k \notin \operatorname{split}(q)$. As $\dot{x}(k)$ is decided by the possibilities in $\operatorname{poss}(p_k, {<}k)$, the set $\varphi(k) := \{\ell \in a(k) \mid \exists\, \eta \in \operatorname{poss}(p_k, {<}k)\colon p_k \wedge \eta \Vdash \dot{x}(k) = \ell\}$ has size $\leq e(k)$ by (iii), so $\varphi(k) \in [a(k)]^{\leq e(k)}$ and p_k forces $\dot{x}(k) \in \varphi(k)$. Set $q(k) := p(k)$.

When $k \in \operatorname{split}(q)$. According to (ii), we split into two subcases. If

$$|\operatorname{poss}(p_k, {<}k)| \leq \frac{e(k)}{2c^{\nabla h}(k)},$$

then $\operatorname{poss}(p_k, {\leq}k)$ has size $< e(k)$, so the set of possible values $\varphi(k)$ for $\dot{x}(k)$ has size $< e(k)$. Hence p_k forces $\dot{x}(k) \in \varphi(k)$ and $\varphi(k) \in [a(k)]^{\leq e(k)}$. Set $q(k) := p(k)$.

Now consider the case when $|\operatorname{poss}(p_k, {<}k)| \leq \frac{d(k)}{2a(k)}$. Note that

$$\frac{a(k)}{\left\lfloor \frac{e(k)}{|\operatorname{poss}(p_k,<k)|} \right\rfloor} \leq \frac{2a(k)|\operatorname{poss}(p_k, {<}k)|}{e(k)} \leq \frac{d(k)}{e(k)} \leq d(k).$$

We show how to find $q(k) \subseteq p(k)$ with $\|q(k)\|^{d(k)}_{c,h,k} \geq \|p(k)\|^{d(k)}_{c,h,k} - 1$ and $\varphi(k) \in [a(k)]^{\leq e(k)}$ such that $q{\restriction_{(k+1)}}^\frown p{\restriction_{[k+1,\omega)}}$ forces $\dot{x}(k) \in \varphi(k)$. Start by enumerating $\operatorname{poss}(p_k, {<}k) =: \{\eta_k \mid k < m\}$, where $m := |\operatorname{poss}(p_k, {<}k)| < d(k)$. By recursion on $j \leq m$, define $M_j \subseteq p(k)$ such that

(a) $M_0 = p(k)$,
(b) $M_{j+1} \subseteq M_j$,
(c) $\|M_{j+1}\|^d_{c,h,k} \geq \|M_j\|^d_{c,h,k} - {}^1\!/_{d(k)}$, and
(d) there is some $s_j \subseteq a(k)$ of size $\leq \left\lfloor \frac{e(k)}{m} \right\rfloor$ such that for any $t \in M_{j+1}$, $p_k \wedge (\eta_j{}^\frown\{t\})$ forces $\dot{x}(k) \in s_j$.

Assume we already have M_j. By (i), $p_k \wedge (\eta_j{}^\frown\{t\})$ decides $\dot{x}(k)$ for every $t \in M_j$, which is a value in $a(k)$. As ${}^{a(k)}\!/_{\lfloor \frac{e(k)}{m} \rfloor} \leq d(k)$, by Corollary A3.11 there are M_{j+1} and s_j as in (b)–(d). Once we have M_m, set $q(k) := M_m$ and $\varphi(k) := \bigcup_{j<m} s_j$.

At the end of the construction, it is clear that q forces $\dot{x}(k) \in \dot{\varphi}(k)$ for any sufficiently large k, and that $\varphi(k) \in [a(k)]^{\leq e(k)}$. □

Observation A3.14. The function $c^{\nabla h}$ is used as an upper bound of $\langle |p(k)| \mid k < \omega\rangle$ for any $p \in \mathbb{Q}^d_{c,h}$, since $p(k) \subseteq [c(k)]^{\leq h(k)}$. In the same way, we could use c^h (pointwise exponentiation) instead (as long as $h \geq^* 2$), because $|[m]^{\leq k}| \leq m^k$ whenever $m, k < \omega$ and $k \neq 1$ (since $|[m]^{\leq 1}| = m + 1$). In fact, in the following sections, every instance of $c^{\nabla h}$ could be replaced by c^h without affecting the results and the proofs.

Thanks to the Tukey connection constructed in Lemma A2.6, we can now easily deduce sufficient conditions on functions b, g to guarantee that $\mathfrak{v}^\exists_{b,g}$ is not increased by $\mathbb{Q}^d_{c,h}$.

Corollary A3.15. *Assume that $c, d, h \in \omega^\omega$ are as in Definition A3.9 with d going to infinity, $b, g, d \in \omega^\omega$ with $b, g > 0$, $e(k) := \lceil \frac{b(k)}{g(k)} \rceil - 1$ going to infinity, and that they satisfy*

(AL1) $\prod_{k<n} b^{\nabla g}(k) \leq d(n)$ *and* $\prod_{k<n} c^{\nabla h}(k) \leq e(n)$ *for all but finitely many n, and*
(AL2) $\lim_{k\to\infty} \min\left\{\frac{c^{\nabla h}(k)}{e(k)}, \frac{b^{\nabla g}(k)}{d(k)}\right\} = 0$.

Then $\mathbb{Q}^d_{c,h}$ forces that any slalom in $\mathcal{S}(b, g)$ is anti-localised by some member of $\prod b \cap V$.

Proof. Set $a(k) := b^{\nabla g}(k)$. By Lemma A2.6, there is a definable Tukey connection (F, G) which witnesses $\mathbf{aLc}(b, g) \preceq_{\mathrm{T}} \mathbf{Lc}(a, e)$ (even in forcing-generic extensions). As a and e satisfy the assumptions in Lemma A3.13, in any $\mathbb{Q}^d_{c,h}$-generic extension, any real in $\prod a$ is localised by some slalom in $\mathcal{S}(a, e) \cap V$. Hence any slalom in $\mathcal{S}(b, g)$ is anti-localised by some member of $\prod b \cap V$. □

A4 Lots and Lots of Auxiliary Functions

To show that we can separate uncountably many different Yorioka ideals' uniformity numbers, we could in principle define two sequences of integers acting as universal bounds on our function parameters and auxiliary functions (similar to [GS93]). However, for the sake of clarity we will give the definition of the sequences as part of an inductive definition in the construction of the auxiliary functions. We stress that there also is an a priori definition, which would, however, make the text less readable.

Either way, we fix two sequences n^-_k, n^+_k of natural numbers ≥ 2 such that

(i) $n^-_k \cdot n^+_k < n^-_{k+1}$ for any $k < \omega$, and
(ii) $\lim_{n\to\infty} \log_{n^-_k} n^+_k = \infty$.

Given Lemma A3.13 and Corollary A3.15 above, we now make the following definition.

Definition A4.1. Given the bounding sequences n^-_k, n^+_k, we call a family $\mathcal{F} = \langle (a_\alpha, d_\alpha, b_\alpha, g_\alpha, f_\alpha, c_\alpha, h_\alpha) \mid \alpha \in A\rangle$ of tuples of increasing functions in ω^ω *suitable* if it fulfils the following properties for any $\alpha \in A$:

(S1) The functions a_α, d_α, b_α, g_α, $b^{\nabla g_\alpha}_\alpha$, ${}^{b_\alpha}\!/_{g_\alpha}$, h_α and $c^{\nabla h_\alpha}_\alpha$ are bounded from below by n^-_k and bounded from above by n^+_k, i. e. for any $k < \omega$, we have

$$a_\alpha(k), d_\alpha(k), b_\alpha(k), g_\alpha(k), b^{\nabla g_\alpha}_\alpha(k), \frac{b_\alpha(k)}{g_\alpha(k)}, h_\alpha(k), c^{\nabla h_\alpha}_\alpha(k) \in [n^-_k, n^+_k].$$

(S2) $h_\alpha < c_\alpha$ and $\limsup_{k\to\infty} \frac{1}{d_\alpha(k)} \log_{d_\alpha(k)}(h_\alpha(k) + 1) = \infty$.
(S3) ${}^{b_\alpha}\!/_{g_\alpha} > d_\alpha$.
(S4) $a_\alpha \geq b^{\nabla g_\alpha}_\alpha$.

(S5) There is some $\ell > 0$ such that $f_{b_\alpha,g_\alpha} \leq^* f_\alpha \circ \mathrm{pow}_\ell$ for f_{b_α,g_α} as in Lemma A2.5.[7]
(S6) $f_\alpha \ll g_{c_\alpha,h_\alpha}$ for g_{c_α,h_α} as in Lemma A2.4.
(S7) For any $\beta \in A$, if $\beta \neq \alpha$, then

$$\lim_{k\to\infty} \min\left\{ \frac{c_\beta^{\nabla h_\beta}(k)}{d_\alpha(k)}, \frac{a_\alpha(k)}{d_\beta(k)} \right\} = 0.$$

Properties (S3)–(S6) ensure that, by Lemma A2.8, $\mathfrak{v}^\exists_{c_\alpha,h_\alpha} \leq \mathrm{non}(\mathcal{I}_{f_\alpha}) \leq \mathfrak{v}^\exists_{b_\alpha,g_\alpha} \leq \mathfrak{c}^\forall_{a_\alpha,d_\alpha}$. When we prove the Main Theorem in the following section, we will aim to force all these cardinals to be equal to some predetermined κ_α for each α. We therefore increase the cardinals $\mathfrak{v}^\exists_{c_\alpha,h_\alpha}$ by using a countable support product of posets of the form $\mathbb{Q}^{d_\alpha}_{c_\alpha,h_\alpha}$, while at the same time ensuring that $\mathfrak{c}^\forall_{a_\alpha,d_\alpha}$ does not exceed the desired value. Property (S2) guarantees that each poset $\mathbb{Q}^{d_\alpha}_{c_\alpha,h_\alpha}$ is non-empty and, thanks to Lemma A3.13, property (S7) will ensure that $\mathbb{Q}^{d_\beta}_{c_\beta,h_\beta}$ will not increase $\mathfrak{c}^\forall_{a_\alpha,d_\alpha}$ for $\beta \neq \alpha$. To this end, it is also necessary that $\prod_{k<n} a(k) \leq d(n)$ for all but finitely many n (property (L1) in Lemma A3.13), which is a consequence of property (S1). In fact, property (S1) is what allows us to explicitly control the functions and the number of possibilities of the creatures in the poset.

Theorem A4.2. *There are bounding sequences n_k^-, n_k^+ such that there is an uncountable suitable family $\mathcal{F} = \langle (a_\alpha, d_\alpha, b_\alpha, g_\alpha, f_\alpha, c_\alpha, h_\alpha) \mid \alpha \in 2^\omega \rangle$.*

Proof. For clarity, we first explain how to construct a suitable family for one single α, that is, a suitable tuple of the form (a, d, b, g, f, c, h) (as in Definition A4.1 for the case $|A| = 1$, for which (S7) is vacuous). For motivational purposes, assume we have already defined n_k^- and $d(k) \geq n_k^-$, and the values of all the functions at $\ell < k$ (with the exception of f, which depends on some interval partition as indicated in the following argument).

To have property (S2), it suffices to define $h(k)$ such that $\frac{1}{d(k)} \log_{d(k)}(h(k)+1) \geq k+1$, so we let $h(k) := d(k)^{(k+1)\cdot d(k)}$. (Later, when we define $c(k)$, it will be clear that $c(k) > h(k)$.) The value of $g(k)$ is determined such that property (S6) is satisfied, but for now we will just assume that we have already defined it and will later explain how to define it according to our needs. (The value of $g(k)$ will only depend on $h(k)$ as well as $h(\ell)$ and $g(\ell)$ for $\ell < k$.)

Now, to fulfil (S3), we let $b(k) := 2^{g(k)+d(k)}$ (such that $\log_2 b(k)$ will be an integer, making the definition of $f_{b,g}$ a bit nicer). Now recall that $f_{b,g}$ is defined, along the interval partition $\langle I_n \mid n < \omega \rangle$ of ω which satisfies $|I_n| = g(n)$, as $f_{b,g}(j) = \sum_{\ell \leq n} \log_2 b(\ell)$ for any $j \in I_n$. In our construction, we may assume that we have already defined $g(\ell)$ and I_ℓ for any $\ell \leq k$, so we explain how to define $f{\restriction}_{I_k}$ such that property (S5) is satisfied – even more, such that $f_{b,g}(j) \leq f(j)$ for any $j \in I_k$.

[7] We can simplify property (S5) by restricting it to $\ell = 1$: Assume the statement holds for some $\ell' > 1$. Let $f'_\alpha = f_\alpha \circ \mathrm{pow}_{\ell'}$. Then $f_{b_\alpha,g_\alpha} \leq^* f'_\alpha$ and $f'_\alpha \ll g_{c_\alpha,h_\alpha}$ still holds (by the definition of $\ll$). This is why we can work with f'_α instead of f_α, in which case property (S5) is already satisfied for $\ell = 1$ and property (S6) remains true for f'_α.

We also require that f is increasing, so we define

$$f(j) := \sum_{\ell \le k} \log_2 b(\ell) + j - \min(I_k)$$

for any $j \in I_k$. This definition allows f to be increasing when attached to the previous intervals because

$$\begin{aligned} f(\max(I_{k-1})) &= \sum_{\ell \le k-1} \log_2 b(\ell) + \max(I_{k-1}) - \min(I_{k-1}) \\ &= \sum_{\ell \le k-1} \log_2 b(\ell) + g(k-1) - 1 < \sum_{\ell \le k} \log_2 b(\ell) = f(\min(I_k)). \end{aligned}$$

Recall that $g_{c,h}$ is defined, along the interval partition $\langle J_n \mid n < \omega \rangle$ of ω which satisfies $|J_n| = h(n)$, as $g_{c,h}(j) = \lfloor \log_2 c(n) \rfloor$ for all $j \in J_n$. In order to have property (S6), it suffices to define $c(k)$ such that $f(j^{k+2}) \le \log_2 c(k)$ for any $j \in J_k$. If we can ensure that $j^{k+2} \in I_k$ for any $j \in J_k$, then

$$\begin{aligned} f(j^{k+2}) &= \sum_{\ell \le k} \log_2 b(\ell) + j^{k+2} - \min(I_k) \\ &\le \sum_{\ell \le k} \log_2 b(\ell) + \max(I_k) - \min(I_k) < \sum_{\ell \le k} \log_2 b(\ell) + g(k), \end{aligned}$$

so it suffices to define

$$c(k) := 2^{\sum_{\ell \le k} \log_2 b(l) + g(k)} = 2^{g(k)} \cdot \prod_{\ell \le k} b(\ell).$$

Now, to ensure that $j^{k+2} \in I_k$ for any $j \in J_k$, it suffices to have that $\min(J_\ell)^{\ell+1} = \min(I_\ell)$ for any $\ell \le k+1$ (as $\min(J_0) = 0$ and $\min(J_{\ell+1}) = \min(J_\ell) + h(\ell)$ as well as $\min(I_{\ell+1}) = \min(I_\ell) + g(\ell)$, the values of $\min(J_{k+1})$ and $\min(I_{k+1})$ are already known). So we assume that we had already ensured $\min(J_k)^{k+1} = \min(I_k)$ from the beginning (so, before n_k^- is even determined; also note that this is true for $k = 0$) and, after defining $h(k)$, we let $g(k) := \min(J_{k+1})^{k+2} - \min(I_k)$. This implies that $\min(I_{k+1}) = \min(I_k) + g(k) = \min(J_{k+1})^{k+2}$ as required.

We can choose $a(k)$ above $c^{\nabla h}(k)$ and $b^{\nabla g}(k)$, and define $n_k^+ := a(k)$, which guarantees property (S4). Note that by our definitions, we have

$$n_k^- < d(k) < h(k) < g(k) < b(k) < c(k) < c^{\nabla h}(k) < a(k) = n_k^+$$

as well as

$$n_k^- < d(k) < \frac{b(k)}{g(k)} < b(k) < b^{\nabla g}(k) < a(k) = n_k^+,$$

which together ensure property (S1).

We can then choose n_{k+1}^- above $n_k^- \cdot n_k^+$ (note that $\log_{n_k^-} n_k^+ \ge \log_{d(k)}(h(k)+1) > k$) and define $d(k+1)$ to be larger than n_{k+1}^-, and then the iterative construction continues for $k+1$.

Now, we define both the bounding sequences and the suitable family $\mathcal{F}$ of size continuum inductively. More or less the same strategy as above will suffice to define the values at a fixed k, but some extra work is needed in order to guarantee property (S7). We construct tuples of functions $\langle (a_t, d_t, b_t, g_t, f_t, c_t, h_t) \mid t \in 2^{<\omega} \rangle$ such that

(i) $|a_t| = |d_t| = |b_t| = |g_t| = |c_t| = |h_t| = |t|$ and $|f_t| = \sum_{k<|t|} g_t(n)$, and
(ii) if $t \subseteq t'$ in $2^{<\omega}$, then $a_t \subseteq a_{t'}$, and likewise for the other functions.

Using these, we define $a_\alpha := \bigcup_{n<\omega} a_{\alpha\restriction_n}$ for $\alpha \in 2^\omega$, and likewise for the other functions, as well as $n_k^- := d_{\bar{0}}(k) - 1$ and $n_k^+ := a_{\bar{1}}(k)$ (where $\bar{e} = \langle e, e, e, \ldots \rangle$ for $e \in \{0, 1\}$) and claim that these are as desired in the theorem.

Denote by $\lhd$ the lexicographic order in both 2^ω and 2^n for any $n < \omega$. We construct $\langle (a_t, d_t, b_t, g_t, f_t, c_t, h_t) \mid t \in 2^n \rangle$ by recursion on n. When $n = 0$, it is clear that $a_{\langle\,\rangle}$, $d_{\langle\,\rangle}$, etc. are defined as the empty sequence. Assuming we have reached step n of the construction, we show how to obtain the functions for $t \in 2^{n+1}$. We do this in the following two steps:

1. Assuming we have already defined $d_t(n)$, we define $a_t(n)$, $b_t(n)$ and so on in the same way as in the definitions for the simple case $|A| = 1$.
2. By recursion on $\langle 2^{n+1}, \lhd \rangle$, we construct $d_t(n)$ for $t \in 2^{n+1}$.

We first show step 1. At this point, we have sequences $\langle I_\ell^{t\restriction_{\ell+1}} \mid \ell < n \rangle$ and $\langle J_\ell^{t\restriction_{\ell+1}} \mid \ell < n \rangle$ of consecutive intervals covering an initial segment of ω such that $|I_\ell^{t\restriction_{\ell+1}}| = g_t(\ell)$ and $|J_\ell^{t\restriction_{\ell+1}}| = h_t(\ell)$. In fact, $I_\ell^{t\restriction_{\ell+1}} = [|f_{t\restriction_\ell}|, |f_{t\restriction_\ell}| + g_t(\ell))$ for any $\ell < n$. Note that we can already define $\min(I_n^t)$ and $\min(J_n^t)$ at this stage (considering that we are constructing interval partitions of ω). Further assume that $\min(I_n^t) = \min(J_n^t)^{n+1}$, which is trivially true for $n = 0$.

Define $h_t(n) := d_t(n)^{(n+1)\cdot d_t(n)}$, $g_t(n) := (\max(J_n^t) + 1)^{n+2} - \min(I_n^t)$, where $J_n^t := [\min(J_n^t), \min(J_n^t) + h_t(n))$, and $b_t(n) := 2^{g_t(n)+d_t(n)}$. Let $I_n^t := [|f_{t\restriction_n}|, |f_{t\restriction_n}| + g_t(n))$ and, for $k \in I_n^t$, define

$$f_t(k) := \sum_{\ell \le n} \log_2 b_t(\ell) + k - |f_{t\restriction_n}|.$$

By definition, $|f_t| = |f_{t\restriction_n}| + g_t(n) = \max(I_n^t) + 1$. Finally, let

$$c_t(n) := 2^{g_t(n)} \cdot \prod_{\ell \le n} b_t(\ell)$$

and choose $a_t(n)$ above $c_t^{\nabla h_t}(n)$ and $b_t^{\nabla g_t}(n)$. As in the case of $|A| = 1$, $d_t(n) < h_t(n) < g_t(n) < b_t(n) < c_t(n) < a_t(n)$ (it will be clear from the next step that d_t will always take values ≥ 2), the other three functions to be bounded lie somewhere in between, and f_t is increasing.

To see step 2, start by choosing $d_{\bar{0}\restriction_{n+1}}(n) > d_{\bar{0}\restriction_n}(n-1) \cdot a_{\bar{1}\restriction_n}(n-1) + 2$ (in case $n = 0$, just choose $d_{\bar{0}\restriction_1}(0)$ to be any number above 2). Now assume that we have $d_t(n)$ and that $t^+ \in 2^{n+1}$ is the immediate successor of t with respect to $\lhd$. Let $d_{t^+}(n) := (n+1) \cdot a_t(n)$. This completes the construction. Note that, whenever

$t, t' \in 2^{n+1}$ and $t \lhd t'$, then $\frac{a_t(n)}{d_{t'}(n)} \leq \frac{1}{n+1}$, which is what ultimately guarantees property (S7).

Now we finally show that $\mathcal{F} = \langle (a_\alpha, d_\alpha, b_\alpha, g_\alpha, f_\alpha, c_\alpha, h_\alpha) \mid \alpha \in 2^\omega \rangle$ is a suitable family for the bounding sequences n_k^- and n_k^+, that is, it satisfies the properties in Definition A4.1. Properties (S1)–(S6) are immediate by construction, as in the case $|A| = 1$. Note that $I_n^\alpha := I_n^{\alpha\restriction_{n+1}}$ and $J_n^\alpha := J_n^{\alpha\restriction_{n+1}}$ (for $n < \omega$) define interval partitions of ω such that $|I_n^\alpha| = g_\alpha(n)$ and $|J_n^\alpha| = h_\alpha(n)$, so properties (S5) and (S6) can be proved in the same way as before.

To prove property (S7), since $a_\alpha > c_\alpha^{\nabla h_\alpha}$ for any α, it suffices to show that, whenever $\alpha \lhd \beta$ in 2^ω, $\lim_{k\to\infty} \frac{a_\alpha(k)}{d_\beta(k)} = 0$. Let n be the minimal number such that $\alpha(n) < \beta(n)$; by the definition of d_β, $d_\beta(k) \geq (k+1) \cdot a_\alpha(k)$ for any $k \geq n$, which implies that the sequence of the $\frac{a_\alpha(k)}{d_\beta(k)}$ converges to 0. □

A5 The Grand Finale

Now that we have defined suitable families, we can put everything together to prove the Main Theorem:

Theorem A5.1. *Assume* CH *and let $\langle \kappa_\alpha \mid \alpha \in A \rangle$ be a sequence of infinite cardinals such that $|A| \leq \aleph_1$ and $\kappa_\alpha^\omega = \kappa_\alpha$ for every $\alpha \in A$. Given bounding sequences n_k^-, n_k^+ and a suitable family $\mathcal{F} = \langle (a_\alpha, d_\alpha, b_\alpha, g_\alpha, f_\alpha, c_\alpha, h_\alpha) \mid \alpha \in A \rangle$, the poset*

$$\mathbb{Q} := \prod_{\alpha\in A} (\mathbb{Q}^{d_\alpha}_{c_\alpha,h_\alpha})^{\kappa_\alpha}$$

(where the product and all powers have countable support) forces

$$\mathfrak{v}^\exists_{c_\alpha,h_\alpha} = \mathrm{non}(\mathcal{I}_{f_\alpha}) = \mathfrak{v}^\exists_{b_\alpha,g_\alpha} = \mathfrak{c}^\forall_{a_\alpha,d_\alpha} = \kappa_\alpha$$

for every $\alpha \in A$.

(Recall Definition A4.1 for the definition of a suitable family and Definition A3.9 for the definition of the poset $\mathbb{Q}^{d_\alpha}_{c_\alpha,h_\alpha}$.)

This section is dedicated to proving the theorem above. Fix a disjoint family $\langle K_\alpha \mid \alpha \in A \rangle$ such that $|K_\alpha| = \kappa_\alpha$ and let $K := \bigcup_{\alpha\in A} K_\alpha$. Hence, we can express $\mathbb{Q}$ as the countable support product of $\langle \mathbb{Q}_\xi \mid \xi \in K \rangle$, where $\mathbb{Q}_\xi := \mathbb{Q}^{d_\alpha}_{c_\alpha,h_\alpha}$ whenever $\xi \in K_\alpha$. For any $p \in \mathbb{Q}$, $\xi \in \mathrm{supp}(p)$ and $k < \omega$ we write $p(\xi, k) := p(\xi)(k)$.

By the following lemma, without loss of generality we may assume that any $p \in \mathbb{Q}$ is *modest* (unless we explicitly say the opposite), that is, for every $\ell < \omega$ there is at most one $\xi \in \mathrm{supp}(p)$ such that $\ell \in \mathrm{split}(p(\xi))$.

Lemma A5.2 (modesty)**.** *The set of modest conditions in $\mathbb{Q}$ is dense.*

Proof. Fix $p \in \mathbb{Q}$. By induction on $n < \omega$ it is possible to define a function $f = (f_0, f_1) \colon \omega \to \mathrm{supp}(p) \times \omega$ such that

(i) f_1 is strictly increasing,
(ii) $f_1(n) \in \mathrm{split}(p(f_0(n)))$ and the norm of $p(f_0(n))$ (with respect to $\mathbb{Q}_{f_0(n)}$) is larger than $n+1$, and
(iii) for any $\xi \in \mathrm{supp}(p)$, the set $f_0^{-1}[\{\xi\}]$ is infinite.

Define q with $\mathrm{supp}(q) = \mathrm{supp}(p)$ such that $q(\xi,k) = p(\xi,k)$ whenever $(\xi,k) \in \mathrm{ran}\, f$, otherwise $q(\xi,k)$ is a singleton contained in $p(\xi,k)$. By the definition of f it is clear that $q \in \mathbb{Q}$ and $q \leq p$. Moreover, by (i), q is modest. □

Thanks to modesty, $\mathbb{Q}$ behaves very much like a single forcing as in section A3. Before we show Theorem A5.1, we revisit the notation and results of section A3 and revise them for the context of $\mathbb{Q}$.

Notation A5.3. In the context of $\mathbb{Q}$, for any $p \in \mathbb{Q}$ and $k, n < \omega$:

(1) $\mathrm{split}(p) := \bigcup_{\xi \in \mathrm{supp}(\xi)} \mathrm{split}(p_\xi)$, which is a disjoint union by modesty.
(2) $s_n(p)$ denotes the n-th member of $\mathrm{split}(p)$.
(3) For $k \in \mathrm{split}(p)$, let $\xi^p(k)$ denote the unique $\xi \in \mathrm{supp}(p)$ such that $k \in \mathrm{split}(p(\xi))$.
(4) $p{\restriction}^w := \langle p(\xi){\restriction}_w \mid \xi \in \mathrm{supp}(p)\rangle$ when w is a subset of ω (usually an interval).
(5) $\mathrm{poss}(p, {\leq}k) := \prod_{\xi \in \mathrm{supp}(p)} \mathrm{poss}(p(\xi), {\leq}\ell)$. By modesty, this set is finite with size $\leq \prod_{\ell \leq k} n^+_\ell < n^-_{k+1}$. Define $\mathrm{poss}(p, < k)$ similarly.
(6) For $\eta \in \mathrm{poss}(p, {\leq}k)$, $p \wedge \eta$ denotes the condition $\langle p(\xi) \wedge \eta(\xi) \mid \xi \in \mathrm{supp}(p)\rangle$.
(7) For $\xi \in K$, $\dot{S}_\xi$ denotes the slalom added by $\mathbb{Q}_\xi$. Note that if $\xi \in K_\alpha$, then $\dot{S}_\xi$ is a $\mathbb{Q}_\xi$-name for a slalom in $\mathcal{S}(c_\alpha, h_\alpha)$.
(8) If $q \in \mathbb{Q}$, $q \leq_n p$ means that $q \leq p$, $\xi^q(s_\ell(q)) \in \mathrm{supp}(p)$ for every $\ell \leq n$, and $(q{\restriction}^{s_n(q)+1}){\restriction}_{\mathrm{supp}(p)} = p{\restriction}^{s_n(q)+1}$.
(9) If $q \in \mathbb{Q}$ and $F \subseteq K$ is finite, $q \leq_{n,F} p$ means that $q \leq p$, $F \subseteq \mathrm{supp}(p)$ and $q(\xi) \leq_n p(\xi)$ for any $\xi \in F$.

Lemma A5.4. *If $n < \omega$, $p \in \mathbb{Q}$, and $D \subseteq \mathbb{Q}$ is open dense, then there is some $q \leq_n p$ in $\mathbb{Q}$ such that for any $\eta \in \mathrm{poss}(q, {\leq}s_n(q))$, $q \wedge \eta \in D$.*

Proof. The same argument as in the proof of Lemma A3.4 works here. Concretely, enumerate $\mathrm{poss}(p, {\leq}s_n(p)) =: \{\eta_k \mid k < m\}$ and, by recursion on $k \leq m$, choose $p_{k+1} \leq \eta'_k {}^\frown p_k{\restriction}^{[s_n(p)+1,\omega)}$ ('pointwise' concatenation) in D such that $s_0(p_{k+1}(\xi)) > s_n(p)$ for any $\xi \in \mathrm{supp}(p_{k+1}) \setminus \mathrm{supp}(p)$ and η'_k is the unique member of $\mathrm{poss}(p_k, {\leq}s_n(p))$ containing η_k . Let $q := r {}^\frown p_m{\restriction}^{[s_n(p)+1,\omega)}$, where r has domain $\mathrm{supp}(p_m) \times (s_n(p)+1)$, $r(\xi,\ell) := p(\xi,\ell)$ whenever $\xi \in \mathrm{supp}(p)$ and $r(\xi,\ell) := p_m(\xi,\ell)$ otherwise (which is a singleton). □

Define $\mathbb{Q}^*$ as the set of conditions in $p \in \mathbb{Q}$ such that for any $n < \omega$, the norm of $p(\xi^p(s_n(p)), s_n(p))$ is above $n+1$. Note that the condition q found in Lemma A5.2 is actually in $\mathbb{Q}^*$, so this set is dense in $\mathbb{Q}$. Also, if $p \in \mathbb{Q}^*$, $\alpha \in A$, and $\xi \in K_\alpha \cap \mathrm{supp}(p)$, then $p(\xi) \in \mathbb{Q}^*_\xi := \mathbb{Q}^{*d_\alpha}_{c_\alpha, h_\alpha}$.

Lemma A5.5 (fusion)**.** *Let $\langle p_n, F_n \mid n < \omega\rangle$ be a sequence such that*

(i) $p_n \in \mathbb{Q}^*$ and $F_n \subseteq K$ is non-empty finite,

(ii) $p_{n+1} \leq_{n,F_n} p_n$,
(iii) $F_n \subseteq F_{n+1}$ *and* $W := \bigcup_{n<\omega} F_n$ *is equal to* $\bigcup_{n<\omega} \mathrm{supp}(p_n)$.

Then there is a condition $q \in \mathbb{Q}$ *with* $\mathrm{supp}(q) = W$ *such that* $q \leq_{n,F_n} p$ *for every* $n < \omega$.

Proof. For each $\xi \in W$, let $n_\xi := \min\{n < \omega \mid \xi \in F_n\}$. It is clear that $\langle p_n(\xi) \mid n \geq n_\xi\rangle$ is a sequence in $\mathbb{Q}^*_\xi$ and that $p_{n+1}(\xi) \leq_n p_n(\xi)$. Thus, there is a fusion $q(\xi) \in \mathbb{Q}^*_\xi$ of that sequence as in the proof of Lemma A3.5. Note that $s_k(q(\xi)) = s_k(p_n(\xi))$ for any $n \geq n_\xi$ and $k \leq n$.

Let $q := \langle q(\xi) \mid \xi \in W\rangle$. If $\xi, \zeta \in W$ are different and $k, k' < \omega$, then, for some sufficiently large n, we have that $s_k(q(\xi)) = s_k(p_n(\xi)) \neq s_{k'}(p_n(\zeta)) = s_{k'}(q(\zeta))$ (by modesty). Hence q is modest, so $q \in \mathbb{Q}$. □

Lemma A5.6. *Let* χ *be a sufficiently large regular cardinal and let* $N \preceq H_\chi$ *be countable such that* $\mathbb{Q} \in N$. *Let* $\langle D_n \mid n < \omega\rangle$ *be a sequence of open dense subsets of* $\mathbb{Q}$ *such that each* $D_n \in N$. *If* $p \in \mathbb{Q} \cap N$, *then there is a condition* $q \leq p$ *in* $\mathbb{Q}$ *such that for any* $n < \omega$ *and* $\eta \in \mathrm{poss}(q, s_n(q))$, $q \wedge \eta \in D_n$.

Proof. Enumerate $K \cap N =: \{\xi_k \mid k < \omega\}$ and let $F_n := \{\xi_k \mid k \leq n\}$ for every $n < \omega$. By recursion on $n < \omega$, construct a sequence $\langle p_n \mid n < \omega\rangle$ such that

(i) $p_0 \leq p$,
(ii) $p_n \in \mathbb{Q}^* \cap N$ and $F_n \subseteq \mathrm{supp}(p_n)$,
(iii) $L_n := \langle s_k(p_n(\xi)) \mid k \leq n, \xi \in F_n\rangle$ is an initial segment of $\mathrm{split}(p_n)$,
(iv) for any $k < |L_n|$ and $\eta \in \mathrm{poss}(p_n, {\leq}s_k(p_n))$, $p_n \wedge \eta \in D_k$, and
(v) $p_{n+1} \leq_{n,F_n} p_n$.

Each step of this construction takes place inside N, though it is very likely that the final sequence is outside N.

Figure 4 illustrates the idea of the following construction. Choose some $p' \leq p$ in $\mathbb{Q}^*$ that satisfies (ii) and (iii), that is, $\xi_0 \in \mathrm{supp}(p')$ and $s_0(p') = s_0(p'(\xi_0))$, and find $p_0 \leq_0 p'$ by application of Lemma A5.4 to D_0. Assume that we have constructed p_n. By recursion on $k < 2(n+2)$, construct $p_{n,k}$ as follows:

(1) First note that $\max(L_n) = s_{m_n}(p_n)$ where $m_n := |L_n| - 1 = (n+1)^2 - 1$. Let $p_{n,0} := p_n$.
(2) Having defined $p_{n,k}$ for $k \leq n$, choose an $\ell \in \mathrm{split}(p_{n,k}(\xi_k))$ larger than $s_{m_n+k}(p_{n,k})$, choose some $p'_{n,k} \leq_{m_n+k} p_{n,k}$ such that $p'_{n,k}(\xi_k, \ell) = p_{n,k}(\xi_k, \ell)$ and $\ell = s_{m_n+k+1}(p'_{n,k})$, and find $p_{n,k+1} \leq_{m_n+k+1} p'_{n,k}$ by application of Lemma A5.4 to D_{m_n+k+1}.
(3) Having defined $p_{n,n+1}$, choose some $p'_{n,n+1} \leq_{m_n+n+1} p_{n,n+1}$ such that $\xi_{n+1} \in \mathrm{supp}(p'_{n,n+1})$ and $s_0(p'_{n,n+1}(\xi_{n+1})) = s_{m_n+n+2}(p'_{n,n+1})$, and afterwards find $p_{n,n+2} \leq_{m_n+n+2} p'_{n,n+1}$ by application of Lemma A5.4 to D_{m_n+n+2}.
(4) Having defined $p_{n,k}$ at $n+2 \leq k < 2(n+2)-1$, choose an $\ell \in \mathrm{split}(p_{n,k}(\xi_{n+1}))$ larger than $s_{m_n+k}(p_{n,k})$, choose some $p'_{n,k} \leq_{m_n+k} p_{n,k}$ such that $p'_{n,k}(\xi_{n+1}, \ell) = p_{n,k}(\xi_{n+1}, \ell)$ and $\ell = s_{m_n+k+1}(p'_{n,k})$, and find $p_{n,k+1} \leq_{m_n+k+1} p'_{n,k}$ by application of Lemma A5.4 to D_{m_n+k+1}.

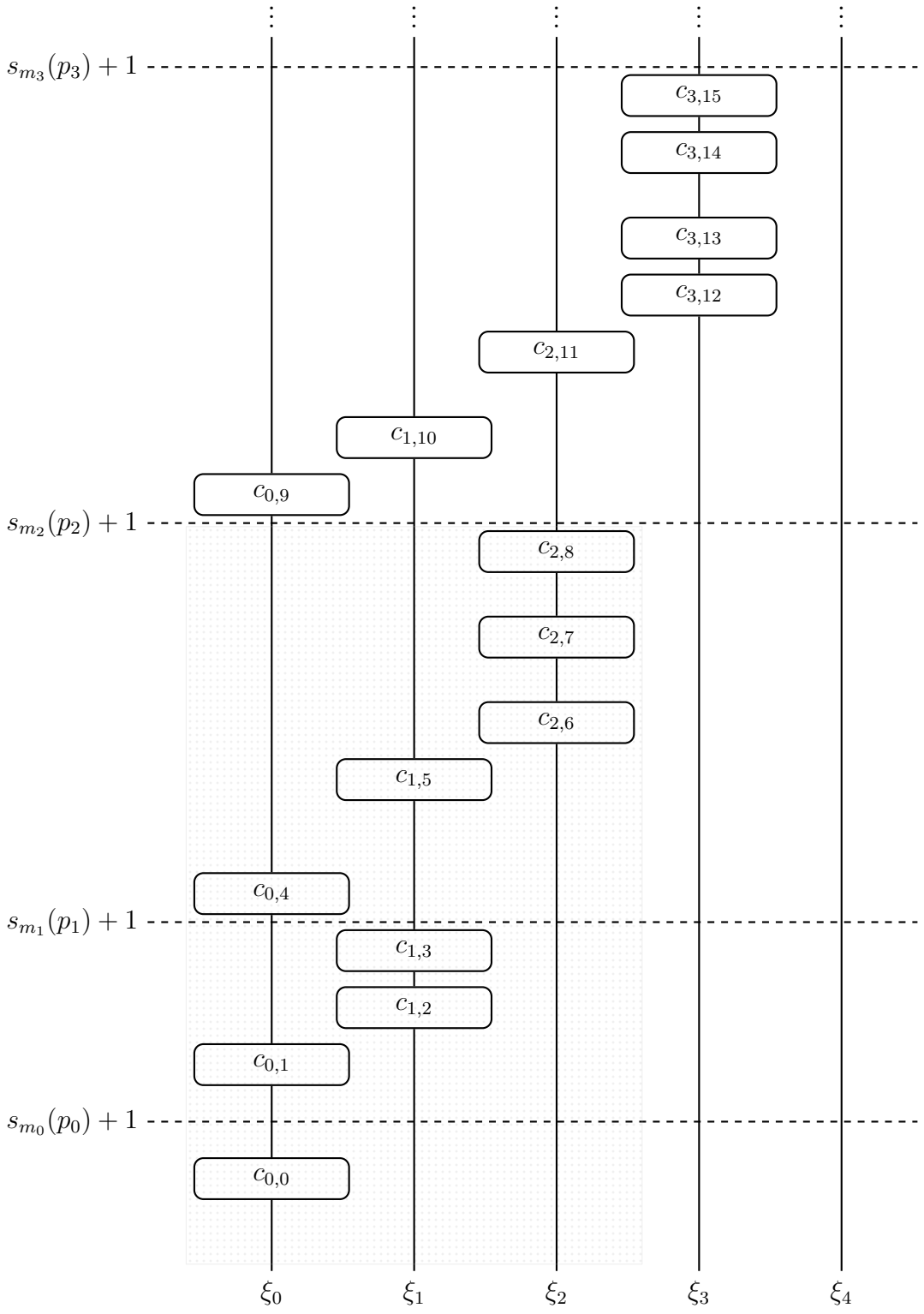

Figure 4: The proof of Lemma A5.6 up to the construction of p_3. The term "$c_{k,\ell}$" is short for the creature $q(\xi_k, s_\ell(q))$. The shaded area shows the part of p_2 that is not modified in the construction of p_3, and will also remain constant in any further steps of the construction. Note that $m_0 = 0$, $m_1 = 3$, $m_2 = 8$ and $m_3 = 15$.

(5) At the end, $p_{n+1} := p_{n,2(n+2)-1}$ is as desired.

By Lemma A5.5, we can find a fusion q of $\langle p_n \mid n < \omega\rangle$ with $\mathrm{supp}(q) = K \cap N$ such that $q \leq_{n,F_n} p_n$ for any $n < \omega$. By (iii), $q \leq_{|L_n|-1} p_n$ for each $n < \omega$, so (iv) implies that q is as desired. □

Corollary A5.7. *$\mathbb{Q}$ is proper and ω^ω-bounding.*

Proof. Properness is immediate by applying Lemma A5.6 to the enumeration of all the open dense subsets in N.

To show that $\mathbb{Q}$ is ω^ω-bounding, let $\dot{x}$ be a $\mathbb{Q}$-name for a member of ω^ω and let $p \in \mathbb{Q}$. Let χ be a sufficiently large regular cardinal and let $N \preceq H_\chi$ be countable such that $\mathbb{Q}, p, \dot{x} \in N$. Let D_n be the set of conditions of $\mathbb{Q}$ that decide $\dot{x}(n)$, which is an open dense set that belongs to N. Find $q \leq p$ by application of Lemma A5.6 to $\langle D_n \mid n < \omega\rangle$, which implies that $q \Vdash \dot{x} \in B_n$, where $B_n := \{k < \omega \mid \exists \eta \in \mathrm{poss}(q, s_n(q)) \colon q \wedge \eta \Vdash \dot{x}(n) = k\}$ is a finite set. Hence, q forces that $\dot{x}$ is bounded by $f \in V$, where $f(n) := \max(B_n)$. □

Corollary A5.8 (timely reading of names)**.** *Let $p \in \mathbb{Q}$ and let $\dot{\tau}$ be a $\mathbb{Q}$-name for a function from ω into the ground model V. Then there is a condition $q \leq p$ in $\mathbb{Q}$ such that q reads $\dot{\tau}$ timely.*

Proof. A fusion argument as in the proof of Lemma A5.6 works, but here the dense sets need to be defined within the construction of the fusion sequence. Concretely, start with a sufficiently large regular cardinal χ and a countable model $N \preceq H_\chi$ such that $\mathbb{Q}, p, \dot{\tau} \in N$, and construct the fusion sequence $\langle p_n \mid n < \omega\rangle$ satisfying conditions (i)–(iii) and (v) in the proof of Lemma A5.6 – but instead of (iv) demand

(iv') for any $k < |L_n|$ and $\eta \in \mathrm{poss}(p_n, s_k(p_n))$, $p_n \wedge \eta \in D_k$, where D_k is the (open dense) set of conditions in $\mathbb{Q}^*$ that decide $\dot{\tau}{\restriction}_{s_k(p_n)}$.

The construction is the same as in Lemma A5.6 except that the dense sets D_k are defined just before each application of Lemma A5.4. □

Lemma A5.9. *The poset $\mathbb{Q}$ is $\aleph_2$-cc.*

Proof. This follows by a typical Δ-system argument under CH. □

Lemma A5.10 (few possibilities)**.** *Let $\langle g_\xi \mid \xi \in \mathrm{supp}(p)\rangle$ be a sequence of functions from ω into ω that go to infinity. If $p \in \mathbb{Q}$, then there is a condition $q \leq p$ in $\mathbb{Q}^*$ such that $\mathrm{supp}(q) = \mathrm{supp}(p)$ and for any $\xi \in \mathrm{supp}(q)$ and $\ell \in \mathrm{split}(q(\xi))$, $|\mathrm{poss}(q, {<}\ell)| < g_\xi(\ell)$.*

Proof. Enumerate $\mathrm{supp}(p) =: \{\xi_n \mid n < \omega\}$ and let $F_n := \{\xi_k \mid k \leq n\}$ for every $n < \omega$. By recursion on $n < \omega$, construct a sequence $\langle p_n \mid n < \omega\rangle$ such that

(i) $p_0 \leq p$,
(ii) $p_n \in \mathbb{Q}^*$ and $\mathrm{supp}(p_n) = \mathrm{supp}(p)$,
(iii) $L_n := \langle s_k(p_n(\xi)) \mid k \leq n, \xi \in F_n\rangle$ is an initial segment of $\mathrm{split}(p_n)$,
(iv) for any $\ell \in L_n$, $|\mathrm{poss}(p_n, {<}\ell)| < g_{\xi^{p_n}(\ell)}(\ell)$, and

(v) $p_{n+1} \leq_{n,F_n} p_n$.

Choose $\ell_0 \in \mathrm{split}(p(\xi_0))$ such that both $g_{\xi_0}(\ell_0)$ and the norm of $p(\xi_0, \ell_0)$ are larger than 1, and choose $p_0 \leq p$ in $\mathbb{Q}^*$ with $\mathrm{supp}(p_0) = \mathrm{supp}(p)$ such that $p_0(\xi, k)$ is a singleton for all $k < \ell_0$ and $p_0(\xi_0, \ell_0) = p(\xi_0, \ell_0)$ (so $s_0(p_0) = \ell_0$). Having constructed p_n, we can define by recursion an increasing sequence $\langle \ell_{n,k} \mid k < 2n+3 \rangle$ of natural numbers such that

(I) $\ell_{n,0} > \max(L_n)$,
(II) $\ell_{n,k} \in \mathrm{split}(p_n(\xi_{k_*}))$ with $k_* := \min\{k, n+1\}$, and
(III) $g_{\xi_{k_*}}(\ell_{n,k})$ is larger than $\mathrm{poss}(p_n, {\leq}\max(L_n)) \cdot \prod_{i<k} |p_n(\xi_{i_*}, \ell_{n,i})|$.

Define p_{n+1} with $\mathrm{supp}(p_{n+1}) = \mathrm{supp}(p_n)$ such that $p_{n+1}(\xi, k) = p_n(\xi, k)$ when either $\xi \in \mathrm{supp}(p_n)$ and $k \in [0, \max(L_n)] \cup [\ell_{n,2n+2}, \omega)$, or $\xi = \xi_{i_*}$ and $k = \ell_{n,i}$ for some $i < 2(n+2)$, and otherwise such that $p_{n+1}(\xi, k)$ is a singleton contained in $p_n(\xi, k)$. It is clear that p_{n+1} is as required.

Define q as in Lemma A5.5. □

Lemma A5.11 (early reading)**.** *Let $\langle X_k \mid k < \omega \rangle$ be a sequence of non-empty sets with $|X_k| \leq n_k^+$, and let $\dot\tau$ be a $\mathbb{Q}$-name for a member of $\prod_{k<\omega} X_k$. Then for any $p \in \mathbb{Q}$, there is a condition $q \leq p$ in $\mathbb{Q}$ which reads $\dot\tau$ early, that is, for any $n < \omega$ and $\eta \in \mathrm{poss}(q, {<}n)$, $q \wedge \eta$ already decides $\dot\tau{\restriction}_n$.*

Proof. Without loss of generality, by Corollary A5.8 assume that p reads $\dot\tau$ timely. By recursion on $k < \omega$ we define $q(\xi, k)$ for all $\xi \in \mathrm{supp}(p)$ (so at the end, $\mathrm{supp}(q) = \mathrm{supp}(p)$). When $k \notin \mathrm{split}(p)$, let $q(\xi, k) := p(\xi, k)$. Assume $k \in \mathrm{split}(p)$ and let $p_k := q{\restriction}^{k}{}^\frown p{\restriction}^{[k,\omega)}$ ('pointwise' concatenation). Exactly as in the proof of Lemma A3.12 (bigness can be used because $|\prod_{i<k} X_i| \leq \prod_{i<k} n_i^+ < n_k^- < d_\alpha(k)$ for any $\alpha \in A$), we can find $q(\xi^p(k), k) \subseteq p(\xi^p(k), k)$ such that the difference between their norms is less that 1 and, for every $\eta \in \mathrm{poss}(p_k, {<}k)$, there is an r_η such that $p_k \wedge \eta' \Vdash \dot\tau{\restriction}_k = r_\eta$ for any $\eta' \in \mathrm{poss}(p_k, {\leq}k)$ that extends η with $\eta'(\xi^p(k), k) \subseteq q(\xi^p(k), k)$. For $\xi \neq \xi^p(k)$, just define $q(\xi, k) := p(\xi, k)$. □

Proof of Theorem A5.1. First, we remark that since $\mathbb{Q}$ has $\aleph_2$-cc, it preserves cardinalities and cofinalities. Now note that since $\mathcal{F}$ is suitable, the assumptions in Lemma A2.8 are fulfilled for any $\alpha \in A$: (S1) and (S3) imply (I1); (S4) implies (I2); (S1) and (S5) imply (I3); and (S1) and (S6) imply (I4). Hence $\mathfrak{v}^\exists_{c_\alpha,h_\alpha} \leq \mathrm{non}(\mathcal{I}_{f_\alpha}) \leq \mathfrak{v}^\exists_{b_\alpha,g_\alpha} \leq \mathfrak{c}^\forall_{a_\alpha,d_\alpha}$, so it suffices to show $\mathbb{Q} \Vdash \mathfrak{v}^\exists_{c_\alpha,h_\alpha} \geq \kappa_\alpha$ and $\mathbb{Q} \Vdash \mathfrak{c}^\forall_{a_\alpha,d_\alpha} \leq \kappa_\alpha$ for any $\alpha \in A$.

The lower bound. Fix some $\alpha \in A$. We prove $\mathbb{Q} \Vdash \mathfrak{v}^\exists_{c_\alpha,h_\alpha} \geq \kappa_\alpha$ when $\kappa_\alpha > \aleph_1$ (otherwise it is trivial) as follows (analogously to the proof of Lemma A3.3): Let $\dot F$ be a $\mathbb{Q}$-name for a subset of $\prod c_\alpha$ of size $< \kappa_\alpha$. By $\aleph_2$-cc, there is some $B \in V$ with $|B| < \kappa_\alpha$ such that $\dot F$ is a $\mathbb{Q}{\restriction}_B$-name. Hence there is some $\xi \in K_\alpha \setminus B$.

It suffices to show that for any $\mathbb{Q}{\restriction}_B$-name $\dot x$ for a member of $\prod c_\alpha$, $\mathbb{Q}$ forces that $\dot x(k) \in \dot S_\xi(k)$ for infinitely many $k < \omega$. Only for this argument, we can briefly forget about modesty. Fix $p \in \mathbb{Q}$ and some $n_0 < \omega$ and, without loss of generality, also assume $\xi \in \mathrm{supp}(p)$. Pick a $k \geq n_0$ such that the norm of $p(\xi, k)$ is at

least 1; afterwards, find an $r \leq p{\restriction}_B$ which decides $\dot{x}(k) = z$. Define $p' \in \mathbb{Q}$ such that $\mathrm{supp}(p') = \mathrm{supp}(r) \cup \mathrm{supp}(p)$, $p'(\gamma) := r(\gamma)$ for $\gamma \in B$ and $p'(\gamma) := p(\gamma)$ otherwise. Finally, strengthen p' to q with the same support by first setting $q{\restriction}_{K \setminus \{\xi\}} := p'{\restriction}_{K \setminus \{\xi\}}$, and then setting $q(\xi, k) := \{t\}$ for some $t \in p'(\xi, k)$ which contains z (which exists by the definition of the norm, as explained in Lemma A3.3) and $q(\xi, n) := p'(\xi, n)$ when $n \neq k$. Hence $q \leq p'$ and $q \Vdash \dot{x}(k) \in \dot{S}_\xi(k)$.

This argument shows that given any family of size less than κ_α, this set cannot be localising in the sense of $\mathfrak{v}^{\exists}_{c_\alpha, h_\alpha}$, and hence $\mathbb{Q} \Vdash \mathfrak{v}^{\exists}_{c_\alpha, h_\alpha} \geq \kappa_\alpha$.

The upper bound. This argument is very similar to Lemma A3.13. Fix $\alpha \in A$ and let $C^\alpha := \bigcup \{K_\beta \mid \kappa_\beta \leq \kappa_\alpha,\ \beta \in A\}$. Note that $|C^\alpha| = \kappa_\alpha$ and $\mathbb{Q}{\restriction}_{C^\alpha}$ forces that $\mathfrak{c} = \kappa_\alpha$. To show $\mathbb{Q} \Vdash \mathfrak{c}^{\forall}_{a_\alpha, d_\alpha} \leq \kappa_\alpha$ it suffices to prove that for any $p \in \mathbb{Q}$ and any $\mathbb{Q}$-name $\dot{x}$ for a member of $\prod a_\alpha$, there is some $q \leq p$ in $\mathbb{Q}$ and some $\mathbb{Q}{\restriction}_{C^\alpha}$-name $\dot{\varphi}$ such that $q \Vdash$ "$\dot{\varphi} \in \mathcal{S}(a_\alpha, d_\alpha)$ and $\dot{x} \in \dot{\varphi}$". (This means that whenever G is $\mathbb{Q}$-generic over V, any member of $\prod a_\alpha$ is localised in $V[G]$ by some slalom in $\mathcal{S}(a_\alpha, d_\alpha) \cap V[G \cap \mathbb{Q}{\restriction}_{C^\alpha}]$, where the latter set has size κ_α.) Without loss of generality, we may assume that

(i) $p \in \mathbb{Q}^*$ reads $\dot{x}$ early (by Lemma A5.11 because $a_\alpha(k) \leq n_k^+$), and
(ii) for any $\xi \in \mathrm{supp}(p) \setminus C^\alpha$, if $\beta \in A$ and $\xi \in K_\beta$, then for any $k \in \mathrm{split}(p(\xi))$,

$$\min\left\{\frac{c_\beta^{\nabla h_\beta}(k)}{d_\alpha(k)}, \frac{a_\alpha(k)}{d_\beta(k)}\right\} \leq \frac{1}{2\,|\,\mathrm{poss}(p, {<}k)|}$$

(by (S7) and Lemma A5.10 applied to the function g_ξ defined by

$$g_\xi(k) := \left(2 \cdot \min\left\{\frac{c_\beta^{\nabla h_\beta}(k)}{d_\alpha(k)}, \frac{a_\alpha(k)}{d_\beta(k)}\right\}\right)^{-1}$$

when ξ is as above, or the identity on ω otherwise).

By recursion on k, we construct $\dot{\varphi}(k)$ and $q(\xi, k)$ for any $\xi \in \mathrm{supp}(p)$ (so at the end, $\mathrm{supp}(q) = \mathrm{supp}(p)$). Let $p_k := q{\restriction}^{k}{}^\frown p{\restriction}^{[k,\omega)}$. We distinguish two cases.

<u>When $k \notin \mathrm{split}(p)$.</u> In this case, for any $\eta \in \mathrm{poss}(p_k, {<}k)$, $p_k \wedge \eta$ already decides $\dot{x}(k)$ to be some $r_\eta \in a_\alpha(k)$, so we define $\dot{\varphi}(k)$ (in the ground model) as the set of those r_η. Note that $|\dot{\varphi}(k)| \leq |\,\mathrm{poss}(p_k, {<}k)| < n_k^- < d_\alpha(k)$. Let $q(\xi, k) := p(\xi, k)$ for any $\xi \in \mathrm{supp}(p)$.

<u>When $k \in \mathrm{split}(p)$.</u> First, assume that $\xi^p(k) \in C^\alpha$. For any $\eta \in \mathrm{poss}(p_k, {<}k)$ and $s \in p(\xi^p(k), k)$, there is a unique $\eta' \in \mathrm{poss}(p_k, {\leq}k)$ that extends η and such that $\eta'(\xi^p(k), k) = \{s\}$, so by (i) there is some $r_{\eta,s} \in a_\alpha(k)$ such that $p_k \wedge \eta' \Vdash \dot{x}(k) = r_{\eta,s}$. Define $\dot{r}_\eta$ as the $\mathbb{Q}{\restriction}_{C^\alpha}$-name for a member of $a_\alpha(k)$ such that any condition $p' \in \mathbb{Q}{\restriction}_{C^\alpha}$ with $p'(\xi^p(k), k) = \{s\}$ for some $s \in p(\xi^p(k), k)$ forces that $\dot{r}_\eta = r_{\eta,s}$. Let $\dot{\varphi}(k) := \{\dot{r}_\eta \mid \eta \in \mathrm{poss}(p_k, {<}k)\}$, which is clearly a $\mathbb{Q}{\restriction}_{C^\alpha}$-name for a set of size $< d_\alpha(k)$. Also, for any $\eta \in \mathrm{poss}(p_k, {<}k)$, $p_k \wedge \eta$ forces $\dot{x} \in \dot{\varphi}(k)$. Let $q(\xi, k) := p(\xi, k)$ for all $\xi \in \mathrm{supp}(p)$.

Now assume that $\xi^p(k) \notin C^\alpha$ and let $\beta \in A$ be such that $\xi^p(k) \in K_\beta$. According

to (ii), we distinguish two subcases. If

$$|\operatorname{poss}(p_k, {<}k)| \leq \frac{d_\alpha(k)}{2c_\beta^{\nabla h_\beta}(k)},$$

then $|\operatorname{poss}(p_k, {\leq}k)| \leq |\operatorname{poss}(p_k, {<}k)| \cdot c_\beta^{\nabla h_\beta}(k) < d_\alpha(k)$. Let $\dot\varphi(k)$ (in the ground model) be the set of objects in $a_\alpha(k)$ that are decided to be $\dot x(k)$ by $p_k \wedge \eta$ for some $\eta \in \operatorname{poss}(p_k, {\leq}k)$. It is clear that this set has size $< d_\alpha(k)$, so we can define $q(\xi, k) := p(\xi, k)$ for all $\xi \in \operatorname{supp}(p)$.

The other subcase is when $|\operatorname{poss}(p_k, {<}k)| \leq \frac{d_\beta(k)}{2a_\alpha(k)}$. Exactly as in the proof of Lemma A3.13 (with $a = a_\alpha$, $d = d_\beta$ and $e = d_\alpha$), we can find $q(\xi^p(k), k) \subseteq p(\xi^p(k), k)$ such that the difference between their norms is less than 1, as well as a $\dot\varphi(k) \in [a_\alpha(k)]^{\leq d_\alpha(k)}$ (in the ground model) such that $p_{k+1} := q{\restriction}^{k+1}{}^\frown p{\restriction}^{[k+1,\omega)}$ forces that $\dot x \in \dot\varphi(k)$. For any $\xi \in \operatorname{supp}(p) \setminus \{\xi^p(k)\}$, let $q(\xi, k) := p(\xi, k)$.

In the end, both q and φ are as required. □

A6 Open Questions

Concerning the consistency of infinitely many pairwise different cardinals associated with Yorioka ideals, the following summarises the current open questions.

Question C. *Is each of the statements below consistent with* ZFC*?*

(Q1) *There are continuum many pairwise different cardinal invariants of the form* $\operatorname{cov}(\mathcal{I}_f)$.

(Q2) *There are continuum many pairwise different cardinal invariants of the form* $\operatorname{non}(\mathcal{I}_f)$.

(Q3) *There are infinitely many pairwise different cardinal invariants of the form* $\operatorname{add}(\mathcal{I}_f)$.

(Q4) *There are infinitely many pairwise different cardinal invariants of the form* $\operatorname{cof}(\mathcal{I}_f)$.

It is also interesting to consider the consistency of the conjunction of the statements above.

As mentioned in the introduction, Kamo and Osuga [KO14] used the existence of a weakly inaccessible cardinal to force the statement in (Q1), but its consistency is still unknown assuming only ZFC. We believe that extending the ideas in our construction with lim inf techniques as in [KS12] would work to solve this problem (even simultaneously with (Q2)).

Very little is known about the additivity and cofinality of Yorioka ideals. Even the following questions are still open (see also [CM17, section 6]).

Question D. *Is it consistent with* ZFC *that there are two Yorioka ideals with different additivity numbers (or cofinality numbers)?*

Any idea to solve this question in the positive could be used to prove the consistency of (Q4) using a lim sup creature construction as in this chapter. However, as the additivity numbers of Yorioka ideals are below $\mathfrak{b}$, the typical ω^ω-bounding creature forcing notions will not work to prove the consistency of (Q3).

As for the localisation and anti-localisation cardinals, the remaining open problems are the following.

Question E. *Is each of the statements below consistent with* ZFC*?*

(Q5) *There are continuum many pairwise different cardinal invariants of the form* $\mathfrak{v}^{\forall}_{c,h}$.

(Q6) *There are continuum many pairwise different cardinal invariants of the form* $\mathfrak{v}^{\exists}_{c,h}$.

Brendle and Mejía [BM14] used a weakly inaccessible cardinal to force (Q5), but its consistency with respect to ZFC alone is still open.

CHAPTER B

MODULAR FRAMEWORK FOR CREATURE FORCING

This chapter is based on [GK18], which is joint work with my advisor, Martin Goldstern.

B1 Introduction

Much like the first chapter, this research forms part of the study of cardinal characteristics of the continuum and the forcing techniques required to separate many of them.

Some of the most popular cardinal characteristics are collected in Cichoń's diagram. The paper [FGKS17] is one in a series of progressively more difficult results showing that more and more of the cardinals from Cichoń's diagram can in fact simultaneously be different, in suitably constructed models of set theory. In that particular paper it was shown that those cardinals in Cichoń's diagram which are neither $\mathrm{cov}(\mathcal{N})$ nor provably below $\mathfrak{d}$ (specifically: $\mathrm{non}(\mathcal{M})$, $\mathrm{non}(\mathcal{N})$, $\mathrm{cof}(\mathcal{N})$, and $2^{\aleph_0}$) can have quite arbitrary values (subject to the known inequalities which the diagram expresses).

The older paper [GS93] presented a consistency result about infinitely many pairwise different cardinal characteristics of the continuum with particularly simple definitions, answering a question of Blass [Bla93, p. 78]. Specifically, they showed that uncountably many so-called *localisation cardinals*

$$\mathfrak{c}_{f,g} := \min\left\{ |\mathcal{S}| \;\middle|\; \mathcal{S} \subseteq \prod_{k<\omega} [f(k)]^{\leq g(k)}, \forall\, x \in \prod_{k<\omega} f(k)\ \exists\, S \in \mathcal{S} \colon x \in^* S \right\}$$

can be pairwise different in a suitably constructed set-theoretic universe.[8]

The common method used in both of these papers is *creature forcing*. While the method of [GS93] was a rather straightforward countable support product of natural tree-like forcing posets, the elements of the forcing poset in [FGKS17] were sequences of so-called *compound creatures*, and the forcing poset was not obviously decomposable as a product of simpler forcing posets. The apparent complexity of that construction may have deterred some readers from taking a closer look at this method.

In this chapter, we will revisit the construction of [FGKS17], but in a more modular way. Using (mostly) a countable support product of lim sup creature forcing posets, together with a lim inf creature forcing poset, we construct a ZFC universe in which the cardinal characteristics $\aleph_1$, $\mathrm{non}(\mathcal{M})$, $\mathrm{non}(\mathcal{N})$, $\mathrm{cof}(\mathcal{N})$ and $2^{\aleph_0}$ are all distinct, and moreover distinct from uncountably many localisation cardinals.

We give a brief outline of the construction. The original forcing construction from [FGKS17] can be decomposed and modified to become a product consisting of three factors: a countable support product of lim sup creature forcing posets, a lim sup creature forcing poset that is not further decomposable, and a lim inf creature forcing poset. The latter two are still simpler than the parts of the original creature forcing construction corresponding to them; we believe they cannot be replaced by countable support products of creature forcing posets. This new representation allows describing the methods and proofs used in a more modular way, which can then more easily be combined with other lim sup creature forcing posets. As a motivating example, we show how to add a variant of the lim sup creature forcing posets used to separate the localisation cardinals $\mathfrak{c}_{f,g}$ from [GS93] to this construction.

The main result is the following:

Theorem B1.1. *Let*

- $\mathrm{types} := \{\mathrm{nm}, \mathrm{nn}, \mathrm{cn}, \mathrm{ct}\} \cup \bigcup_{\xi<\omega_1}\{\xi\}$,
- $\mathrm{types}_{\limsup} := \mathrm{types} \setminus \{\mathrm{nm}\}$,
- $\mathrm{types}_{\mathrm{modular}} := \mathrm{types}_{\limsup} \setminus \{\mathrm{ct}\}$, *and*
- $\mathrm{types}_{\mathrm{slalom}} := \bigcup_{\xi<\omega_1}\{\xi\}$.

Assume CH *in the ground model. Assume we are given cardinals* $\kappa_{\mathrm{nm}} \leq \kappa_{\mathrm{nn}} \leq \kappa_{\mathrm{cn}} \leq \kappa_{\mathrm{ct}}$ *as well as a sequence of cardinals* $\langle \kappa_\xi \mid \xi < \omega_1\rangle$ *with* $\kappa_\xi \leq \kappa_{\mathrm{cn}}$ *for all* $\xi < \omega_1$ *such that for each* $\mathrm{t} \in \mathrm{types}$, $\kappa_{\mathrm{t}}^{\aleph_0} = \kappa_{\mathrm{t}}$. *Further assume we are given a congenial sequence*[9] *of function pairs* $\langle f_\xi, g_\xi \mid \xi < \omega_1\rangle$.

Then there are natural lim sup *creature forcing posets* $\mathbb{Q}_{\mathrm{t}}$ *for each* $\mathrm{t} \in \mathrm{types}_{\mathrm{modular}}$, *a* lim sup *creature forcing poset* $\mathbb{Q}_{\mathrm{ct},\,\kappa_{\mathrm{ct}}}$ *and a* lim inf *creature forcing poset* $\mathbb{Q}_{\mathrm{nm},\,\kappa_{\mathrm{nm}}}$

[8] For more detail on $\mathfrak{c}_{f,g}$, see section B10; for a more general treatment of localisation and anti-localisation cardinals, see Definition A1.4.

[9] We will define this term in Definition B3.2.

such that

$$\mathbb{Q} := \left(\prod_{\mathrm{t} \in \mathrm{types}_{\mathrm{modular}}} \mathbb{Q}_{\mathrm{t}}^{\kappa_{\mathrm{t}}} \right) \times \mathbb{Q}_{\mathrm{ct},\,\kappa_{\mathrm{ct}}} \times \mathbb{Q}_{\mathrm{nm},\,\kappa_{\mathrm{nm}}}$$

(where all products and powers have countable support) forces:

(M1) $\mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \aleph_1$,
(M2) $\mathrm{non}(\mathcal{M}) = \kappa_{\mathrm{nm}}$,
(M3) $\mathrm{non}(\mathcal{N}) = \kappa_{\mathrm{nn}}$,
(M4) $\mathrm{cof}(\mathcal{N}) = \kappa_{\mathrm{cn}}$,
(M5) $\mathfrak{c}_{f_\xi, g_\xi} = \kappa_\xi$ *for all* $\xi < \omega_1$, *and*
(M6) $2^{\aleph_0} = \kappa_{\mathrm{ct}}$.

Moreover, $\mathbb{Q}$ *preserves all cardinals and cofinalities.*

See Figure 5 for a graphical representation of our results.

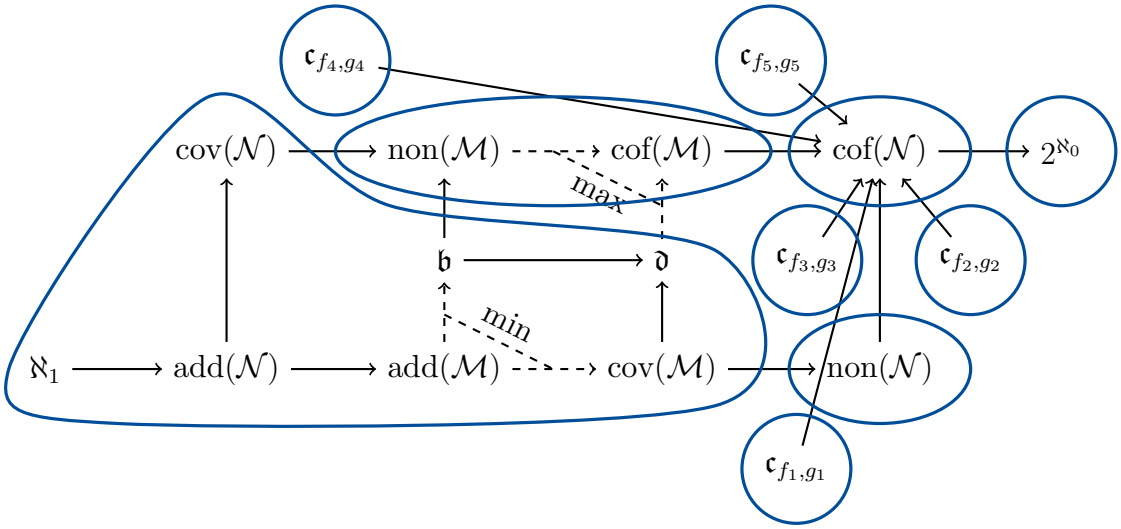

Figure 5: Cichoń's diagram with some exemplary $\mathfrak{c}_{f,g}$ added to it; cardinals which are forced to be equal are grouped together, and each such group can be forced to be different from the others subject to the usual constraints.

We give a brief outline of this chapter.

- In section B3, we define all the constituent parts of the forcing construction, and in section B4, we show how to put them together and prove a few fundamental properties.
- We then introduce and prove the main properties of the forcing construction which will be used throughout the chapter – bigness in section B5 and continuous and rapid reading in section B6 and section B7. The latter section also contains proofs of properness and ω^ω-bounding, as well as the "easy" parts of the main theorem ((M1) and (M6)).
- The following sections contain the proofs to the remaining parts of the main theorem:

  - section B8 and section B11 prove (M4),
  - section B9 proves (M2),
  - section B10 proves (M5), and
  - section B11 and section B12 prove (M3).
- Finally, in section B13, we give a brief account of the limitations of the method (and some of our failed attempts to add factors to the construction) and open questions.

B2 Motivational Prologue

We now define the basic framework of the forcing poset. We will not be defining each and every cog of the machinery right from the start; we will instead fill in the blanks one by one, to reduce the complexity and allow for more easily digestible reading.

At the most elementary level, our forcing poset is a product of four parts, each of which employs creature forcing constructions. In such a creature forcing construction, conditions are sequences of so-called creatures holding some finite amount of information on the generic real. For technical reasons, we will separate these forcing posets into different sets of levels – the (compound) creatures in the lim inf forcing poset $\mathbb{Q}_{\mathrm{nm},\,\kappa_{\mathrm{nm}}}$ will be enumerated by integers of the form $4k$, the creatures in the modular lim sup forcing posets $\mathbb{Q}_{\mathrm{nn}}$ and $\mathbb{Q}_{\mathrm{cn}}$ will be enumerated by integers of the form $4k+1$, the creatures in the modular lim sup forcing posets $\mathbb{Q}_{\xi}$ will be enumerated by integers of the form $4k+2$ and the creatures in the Sacks-like lim sup forcing poset $\mathbb{Q}_{\mathrm{ct},\,\kappa_{\mathrm{ct}}}$ will be enumerated by integers of the form $4k+3$.

The modular lim sup forcing posets are not too complicated, having just a creature C_ℓ at each level ℓ for each index in the support. Each such C_ℓ is a subset of some finite set of so-called possibilities $\mathrm{POSS}_{t,\ell}$.

The lim sup forcing poset $\mathbb{Q}_{\mathrm{ct},\,\kappa_{\mathrm{ct}}}$ cannot be separated into a countable support product of factors. (To be precise, *we* cannot separate it into a countable support product of factors or replace it by such a forcing poset.) We are quite certain that this is due to fundamental structural reasons, namely that in order to prove Lemma B8.1, we have to group the levels (and hence the associated creatures) in this forcing poset together in a certain way, and these partitions need to be compatible, i. e. there must be a single level partition shared by all indices in the support of $\mathbb{Q}_{\mathrm{ct},\,\kappa_{\mathrm{ct}}}$.

Each element of the lim inf forcing poset $\mathbb{Q}_{\mathrm{nm},\,\kappa_{\mathrm{nm}}}$ consists of a sequence of grids of creatures C_ℓ. Each such grid has a finite support $S_\ell \subseteq A_{\mathrm{nm}}$ (where A_{nm} is some index set of size κ_{nm}). For each ℓ, there is a finite set J_ℓ (i. e. some natural number), and the grid consists of a $|S_\ell|$-tuple $\langle C_{\ell,\alpha} \mid \alpha \in S_\ell\rangle$ of stacked creatures $C_{\ell,\alpha}$; each stacked creature $C_{\ell,\alpha}$, in turn, is a finite sequence of creatures $C_{(\ell,0),\alpha}, \ldots, C_{(\ell,J_\ell-1),\alpha}$. We will also refer to such (ℓ, i) as "sublevels". Additionally, each lim inf level ℓ also has a so-called "halving parameter" $d(\ell)$, a natural number.[10]

[10] See Figure 8 for a graphical representation of this structure.

For easier reading, we will be using the term "height" to mean "level" for the lim sup forcing posets or "sublevel" for the lim inf forcing poset. A height $L \in \text{heights}$ is thus either a level $\ell = 4k+1$, $\ell = 4k+2$ or $\ell = 4k+3$ or a sublevel (ℓ, i) with $\ell = 4k$ and $i \in J_\ell$. (For $\mathbb{Q}_{\mathrm{ct},\,\kappa_{\mathrm{ct}}}$, we will consider all creatures within the same class of the level partition as a unit, which complicates induction on the heights a little bit.)

The descriptions of these forcing posets as "lim inf forcing" or "lim sup forcing" refers to the kind of requirements we demand of the sequences of creatures. Each of these creature forcing posets has a norm (a sequence of functions from $2^{\mathrm{POSS}_{\mathrm{t},L}}$ to the non-negative reals) associated with it. As one would expect from the nomenclature, we will demand that for any given condition $p \in \mathbb{Q}$, for each lim sup forcing poset $\mathbb{Q}_{\mathrm{t}}$ we have $\limsup_{\ell\to\infty} \|p(\alpha,\ell)\|_{\mathrm{t},\ell} = \infty$ for each $\alpha \in \mathrm{supp}(p)$ (again, for $\mathbb{Q}_{\mathrm{ct},\,\kappa_{\mathrm{ct}}}$, this will look a tiny bit different) as well as that $\liminf_{\ell\to\infty} \|p(\ell)\|_{\mathrm{nm}} = \infty$. (Note that in this statement, we are deliberately referring to levels and not to heights, and the limits are to be understood as limits in terms of the tg-appropriate levels.)

The forcing posets involved will depend on certain parameters, which we will define iteratively by induction on the heights. For each height L, we will also inductively define natural numbers $n^P_{<L} < n^R_{<L} < n^B_L < n^S_L$.

We want to briefly explain the purpose of these sequences:

- $n^P_{<L}$ will be an upper bound on the *number of possibilities* below L (corresponding to maxposs from [FGKS17] and n^- from [GS93]). By this we mean that $n^P_{<L}$ will bound the number of different possible maximal strengthenings[11] of a condition p below L and hence e. g. the number of iterations we have to go through whenever we want to consider all possible such strengthenings.
- n^B_L will be a lower bound on the *bigness* of a creature at height L (corresponding to b from [FGKS17] and also n^- from [GS93]), which we will be defining a bit later. For now, think of this as follows: Whenever we partition a creature C_L into at most n^B_L many sets (e. g. according to which value they force some name to have), there is always one set such that strengthening C_L to a subcreature corresponding to that set will only very slightly decrease the norm.
- n^S_L will be an upper bound on the *size* of $\mathrm{POSS}_{\mathrm{t},L}$ for all $\mathrm{t} \in \text{types}$ (corresponding to M from [FGKS17] and n^+ from [GS93]).
- $n^R_{<L}$ will be used to control how quickly a condition p decides finite initial segments of reals (corresponding to H from [FGKS17]), i. e. its *rapidity*. This decision of initial segments will be referred to as "reading" in the sequel.

[11] This will be explained in more detail in the following. By "strengthening of p below L", we mean conditions $q \leq p$ such that p and q are identical at all heights $K \geq L$, and by "maximal strengthening" we mean that there is no stronger q with this property.

B3 Defining the Forcing Factors

Let us now begin to define the framework of the forcing construction.

Definition B3.1. Assume we are given cardinals $\kappa_{\mathrm{nm}} \leq \kappa_{\mathrm{nn}} \leq \kappa_{\mathrm{cn}} \leq \kappa_{\mathrm{ct}}$ and a sequence of cardinals $\langle \kappa_\xi \mid \xi < \omega_1 \rangle$ with $\kappa_\xi \leq \kappa_{\mathrm{cn}}$ for all $\xi < \omega_1$ such that for each $\mathrm{t} \in \mathrm{types}$, $\kappa_{\mathrm{t}}^{\aleph_0} = \kappa_{\mathrm{t}}$.[12] Choose disjoint index sets A_{t} of size κ_{t} for each $\mathrm{t} \in \mathrm{types}$. We will use the shorthand notations

- $A_{\mathrm{slalom}} := \bigcup_{\xi<\omega_1} A_\xi$,
- $A_{*\mathrm{n}} := A_{\mathrm{cn}} \cup A_{\mathrm{nn}}$, and
- $A := A_{\mathrm{nm}} \cup A_{*\mathrm{n}} \cup A_{\mathrm{slalom}} \cup A_{\mathrm{ct}}$,

as well as the notations

- $\mathrm{typegroups} := \{\mathrm{nm}, *\mathrm{n}, \mathrm{slalom}, \mathrm{ct}\}$, and
- $\mathrm{typegroups}_{\limsup} := \mathrm{typegroups} \setminus \{\mathrm{nm}\}$.

For each $\ell = 4k$, we will fix some J_ℓ with $0 < J_\ell < \omega$. We will refer to the set of *heights*

$$\mathrm{heights} := \bigcup_{k<\omega} \left(\{(4k, i) \mid i \in J_{4k}\} \cup \{4k+1, 4k+2, 4k+3\} \right)$$

as well as its subsets

- $\mathrm{heights}_{\mathrm{nm}} := \bigcup_{k<\omega} \{(4k, i) \mid i \in J_{4k}\}$,
- $\mathrm{heights}_{*\mathrm{n}} := \{4k+1 \mid k < \omega\}$,
- $\mathrm{heights}_{\mathrm{slalom}} := \{4k+2 \mid k < \omega\}$, and
- $\mathrm{heights}_{\mathrm{ct}} := \{4k+3 \mid k < \omega\}$.

The heights will be ordered in the obvious way, that is:

$$\ldots < 4k-1 < (4k,0) < (4k,1) < \ldots < (4k, J_{4k}-1) < 4k+1 < 4k+2 < 4k+3 < (4k+4, 0) < \ldots$$

We will also use L^+ and L^- to refer to the successor and predecessor of a height L in this order.

The creatures of our forcing poset $\mathbb{Q}$ will "live" on (some subset of)

$$\mathrm{DOMAIN} := A_{\mathrm{nm}} \times \mathrm{heights}_{\mathrm{nm}} \cup A_{*\mathrm{n}} \times \mathrm{heights}_{*\mathrm{n}} \cup A_{\mathrm{slalom}} \times \mathrm{heights}_{\mathrm{slalom}} \cup A_{\mathrm{ct}} \times \mathrm{heights}_{\mathrm{ct}},$$

that is, each $p \in \mathbb{Q}$ will have creatures for each α in a countably infinite $\mathrm{supp}(p) \subseteq \mathrm{DOMAIN}$ (though for each height, only finitely many will be non-trivial). For each

$$(\alpha, L) \in \bigcup_{\mathrm{tg} \in \mathrm{typegroups}} (A_{\mathrm{tg}} \cap \mathrm{supp}(p)) \times \mathrm{heights}_{\mathrm{tg}},$$

there will be a finite set $\mathrm{POSS}_{\alpha,L}$, and the creatures $C_{\alpha,L}$ will be some non-empty subsets of these. (See Figure 6 for a schematic representation of the structure of $\mathbb{Q}$.)

[12] Recall that $\mathrm{types} = \{\mathrm{nm}, \mathrm{nn}, \mathrm{cn}, \mathrm{ct}\} \cup \bigcup_{\xi<\omega_1} \{\xi\}$.

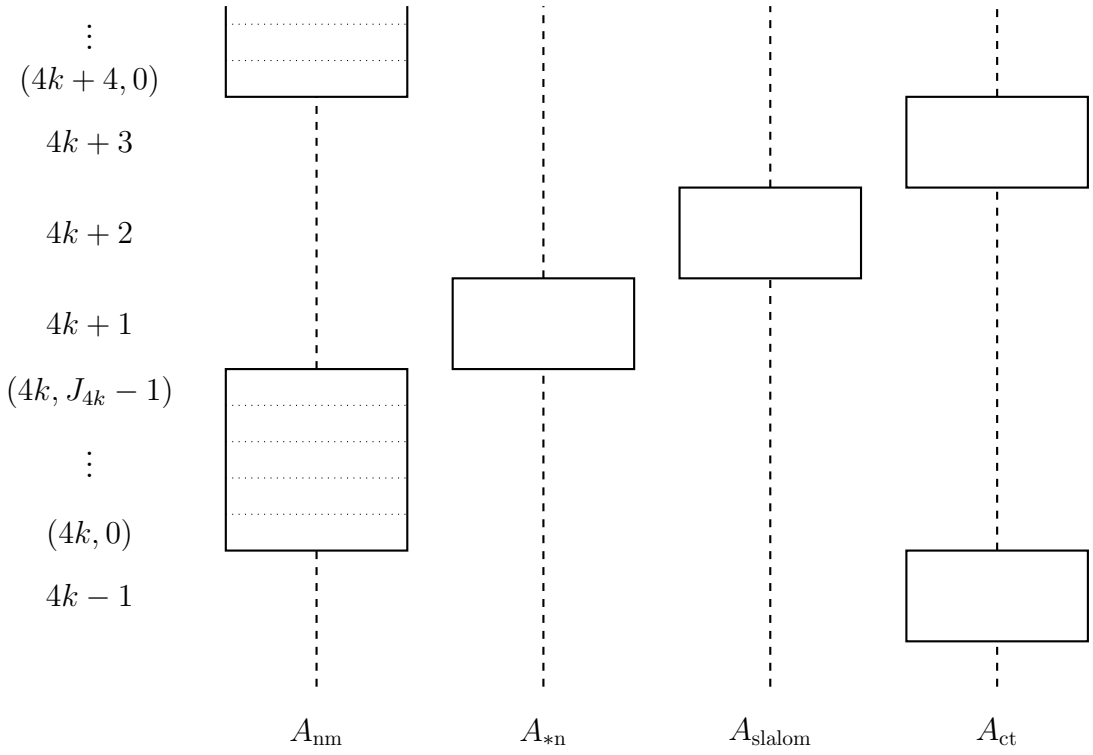

Figure 6: A diagram of the basic structure of the forcing poset $\mathbb{Q}$.

Given some index $\alpha \in A$ respectively some height $L \in \mathrm{heights}$, we will use $\mathrm{tg}(\alpha)$ respectively $\mathrm{tg}(L)$ to denote the appropriate group of types, i. e. the tg such that $\alpha \in A_{\mathrm{tg}}$ respectively the tg such that $L \in \mathrm{heights}_{\mathrm{tg}}$.

We will now first define the forcing posets themselves. However, the inductive definitions of the forcing posets and those of the auxiliary sequences $n^P_{<L}, n^B_L, n^S_L, n^R_{<L}$ mentioned above are actually intertwined. We will be using the auxiliary functions as parameters here and very diligently make sure in section B4 that when inductively defining them, we will not be using anything not previously defined up to that step of the induction process (mostly, this means taking care not to commit off-by-one errors). For now, think of these four sequences as growing very, very quickly and fulfilling $n^P_{<L} \ll n^R_{<L} \ll n^B_L \ll n^S_L \ll n^P_{<L^+}$.[13]

Keep the following in mind: To define creature forcing posets, we mainly have to define the sets of *possibilities* $\mathrm{POSS}_{t,L}$ and the associated norms. The reasons for the specific choices of the norms will only become clear later in section B5, when we define the concept of bigness.

We will start with $\mathbb{Q}_{\mathrm{slalom}}$.

Definition B3.2. Given the sequences $n^P_{<L}, n^B_L, n^S_L$, we call a sequence of function pairs (f_ξ, g_ξ) in ω^ω *congenial* if:

[13] As a matter of fact, in the inductive definitions of these sequences, we will only be demanding that they be far larger than some other term, and we define them in some appropriate way to ensure that; making them larger still would not pose any problems.

(i) for each ξ and for all $k < \omega$, $n^B_{4k+2} \leq g_\xi(k) < f_\xi(k) \leq n^S_{4k+2}$,
(ii) for each ξ, $\lim_{k\to\infty} \frac{\log f_\xi(k)}{n^P_{<4k+2}\cdot \log g_\xi(k)} = \infty$, and
(iii) for all ξ, ζ with $\xi \neq \zeta$, either $\lim_{k\to\infty} \frac{f_\zeta(k)^2}{g_\xi(k)} = 0$ or $\lim_{k\to\infty} \frac{f_\xi(k)^2}{g_\zeta(k)} = 0$.[14]

When referring to a single pair of functions in a congenial sequence, we will call this a *congenial pair of functions*.

The choice of $n^B_L \ll n^S_L$ will ensure that there are sufficiently many different such function pairs.

Definition B3.3. Given the sequences $n^P_{<L}, n^B_L, n^S_L, n^R_{<L}$ and a congenial pair of functions (f_ξ, g_ξ), the forcing factor $\mathbb{Q}_\xi$ is defined as the set of all conditions p fulfilling the following:

(i) p consists of a sequence of creatures $\langle p(L) \mid L \in \text{heights}_{\text{slalom}}\rangle$. Each such L is of the form $4k+2$.
(ii) The sets of possibilities are given by the subsets of $\text{POSS}_{\xi,L} := f_\xi((L-2)/4) = f_\xi(k)$. This means that for each such L, $p(L) = p(4k+2) \subseteq f_\xi(k) = f_\xi((L-2)/4)$ (and $p(L) \neq \varnothing$).
(iii) The norm $\|\cdot\|_{\xi,L}$ is given by $\|M\|_{\xi,L} := \frac{\log|M|}{n^P_{<L}\cdot\log g_\xi((L-2)/4)} = \frac{\log|M|}{n^P_{<4k+2}\cdot\log g_\xi(k)}$.[15]
(iv) There is an increasing sequence of $L_i \in \text{heights}_{\text{slalom}}$ such that $\|p(L_i)\|_{\xi,L_i} \geq i$. Equivalently, $\limsup_{L\to\infty} \|p(L)\|_{\xi,L} = \infty$. This means that for these L_i, $|p(L_i)|$ is much larger than $g_\xi((L_i-2)/4)$ (in more legible notation: for these k_i, i. e. such that $L_i =: 4k_i+2$, we have that $|p(4k_i+2)|$ is much larger than $g_\xi(k_i)$).

A condition q is stronger than a condition p if $q(L) \subseteq p(L)$ holds for each $L \in \text{heights}_{\text{slalom}}$.

Note that Definition B3.2 (ii) ensures that $\mathbb{Q}_\xi$ is non-empty.

Next, we define $\mathbb{Q}_{\text{nn}}$.

Definition B3.4. Given the sequences $n^P_{<L}, n^B_L, n^S_L, n^R_{<L}$, the forcing factor $\mathbb{Q}_{\text{nn}}$ is defined as the set of all conditions p fulfilling the following:

(i) p consists of a sequence of creatures $\langle p(L) \mid L \in \text{heights}_{*\text{n}}\rangle$.
(ii) For each such L, fix a finite interval $I_L \subseteq \omega$ (for notational simplicity, disjoint from all I_K for $K < L$) such that with the definitions given below, $\|\,\text{POSS}_{\text{nn},L}\,\|_{\text{nn},L} > n^B_L$.

[14] Property (iii) here corresponds to the assumption in [GS93, Theorem 3.1], but is more specific. While [GS93] only demands (if we ignore the distinction between g and h) that for each $k < \omega$, either $f_\zeta(k)$ is much smaller than $g_\xi(k)$ or $g_\zeta(k)$ is much bigger than $f_\xi(k)$, but the order could be inverted for $k+1$, we actually demand that the functions are eventually ordered the same way. We could just as well work with the more general property, but we believe our restriction makes the proofs somewhat easier to digest.

[15] As stated previously, the specific definition of the norm here is not really important, and other definitions might work equally well; we mainly require the norms to have a property called "bigness" (defined in section B5), which will be proved in Theorem B5.6.

(iii) The sets of possibilities are given by

$$\mathrm{POSS}_{\mathrm{nn},L} := \left\{ X \subseteq 2^{I_L} \;\middle|\; |X| = \left(1 - \frac{1}{2^{n_L^B}}\right) \cdot |2^{I_L}| \right\},$$

that is, all subsets X of 2^{I_L} of relative size $1 - 2^{-n_L^B}$.

(iv) The norm $\|\cdot\|_{\mathrm{nn},L}$ on subsets of $\mathrm{POSS}_{\mathrm{nn},L}$ is given by

$$\|M\|_{\mathrm{nn},L} := \frac{\log \|M\|_L^{\mathrm{intersect}}}{n_L^B \log n_L^B}$$

with $\|M\|_L^{\mathrm{intersect}} := \min\{|Y| \mid Y \subseteq 2^{I_L}, \forall\, X \in M\colon X \cap Y \neq \varnothing\}$.

(v) There is an increasing sequence of $L_i \in \mathrm{heights}_{*\mathrm{n}}$ such that $\|p(L_i)\|_{\mathrm{nn},L_i} \geq i$. Equivalently, $\limsup_{L\to\infty} \|p(L)\|_{\mathrm{nn},L} = \infty$.

Note that the minimum in the definition above is equal to ${}^{2^{|I_L|}}\!/_{2^{n_L^B}}$ (up to rounding errors) for $M = \mathrm{POSS}_{\mathrm{nn},L}$. Therefore, fulfilling $\|\mathrm{POSS}_{\mathrm{nn},L}\|_{\mathrm{nn},L} > n_L^B$ (and the lim sup condition on the norms for conditions) is possible and $\mathbb{Q}_{\mathrm{nn}}$ is non-empty.

A condition q is stronger than a condition p if $q(L) \subseteq p(L)$ holds for each $L \in \mathrm{heights}_{*\mathrm{n}}$.

Next, we define $\mathbb{Q}_{\mathrm{cn}}$. The norm we give here is technically different from (and hopefully simpler than) the one given in [FGKS17], but fulfils the same purpose.

Definition B3.5. Given the sequences $n^P_{<L}, n^B_L, n^S_L, n^R_{<L}$, the forcing factor $\mathbb{Q}_{\mathrm{cn}}$ is defined as the set of all conditions p fulfilling the following:

(i) p consists of a sequence of creatures $\langle p(L) \mid L \in \mathrm{heights}_{*\mathrm{n}}\rangle$.

(ii) For each such L, fix a finite interval $I_L \subseteq \omega$ (for notational simplicity, disjoint from all I_K for $K < L$) such that with the definitions given below, $\|\mathrm{POSS}_{\mathrm{cn},L}\|_{\mathrm{cn},L} > n_L^B$.

(iii) The sets of possibilities are given by

$$\mathrm{POSS}_{\mathrm{cn},L} := \left\{ X \subseteq 2^{I_L} \;\middle|\; |X| = \left(1 - \frac{1}{2^{n_L^B}}\right) \cdot |2^{I_L}| \right\},$$

that is, all subsets X of 2^{I_L} of relative size $1 - 2^{-n_L^B}$. This is the same kind of possibility set as for $\mathbb{Q}_{\mathrm{nn}}$, but the norm is different:

(iv) The norm $\|\cdot\|_{\mathrm{cn},L}$ on subsets of $\mathrm{POSS}_{\mathrm{cn},L}$ is given by

$$\|M\|_{\mathrm{cn},L} := \frac{\log |M| - \log \binom{2^{|I_L|-1}}{2^{n_L^B}-1}}{2^{\min I_L} \cdot (n_L^B)^2 \cdot \log 3n_L^B}.$$

(v) There is an increasing sequence of $L_i \in \mathrm{heights}_{*\mathrm{n}}$ such that $\|p(L_i)\|_{\mathrm{cn},L_i} \geq i$. Equivalently, $\limsup_{L\to\infty} \|p(L)\|_{\mathrm{cn},L} = \infty$.

A condition q is stronger than a condition p if $q(L) \subseteq p(L)$ holds for each $L \in \mathrm{heights}_{*\mathrm{n}}$.

Note that if the I_L are chosen as above, then $\mathbb{Q}_{\mathrm{cn}}$ is non-empty; see the observation below on why such a choice of I_L is possible.

This is the only forcing poset which we have substantially modified as compared to [FGKS17], so let us briefly explain what we have done and why that is fine. (We will omit the rounding to integers in the following calculations.)

Observation B3.6. The construction in [FGKS17] combines two different norms which provide properties required for the proofs, $\mathrm{nor}_b^\cap$ and $\mathrm{nor}^\div_{I,b}$. One can easily see that $\mathrm{nor}_b^\cap(x) = \left\lfloor \frac{\log x}{\log 3b} \right\rfloor$; this is not explicitly stated in [FGKS17], but is is straightforward from the definitions (setting $M(\delta,\ell) := 3^{\ell/\delta}$).

On the other hand, $\mathrm{nor}^\div_{I,b}(x) = {}^{x}/_{\binom{2^{|I|-1}}{2^{b-1}}}$. Consider $\mathrm{nor}^\div_{I,b}(\mathrm{POSS}_{\mathrm{cn},L})$ for $b := n^B_L$; for appropriately large (with respect to n^B_L) choices of I, this can become arbitrarily large. But then the same holds for $\log \mathrm{nor}^\div_{I,b}(x)$ and also for $\frac{\log \mathrm{nor}^\div_{I,b}(x)}{\log 3b}$, and we have only decreased the norm by modifying it this way. This norm now is almost the same as $\mathrm{nor}_b^\cap$ except for the subtrahend, but we have already established that this norm still goes to infinity. Hence, if we replace the norm in [FGKS17, Definition 10.1.1 (3)] (which defines $\mathbb{Q}_{\mathrm{cn}}$) by this instead, all relevant properties are preserved and we have used a slightly nicer, closed form instead.

Next, we define $\mathbb{Q}_{\mathrm{ct},\,\kappa_{\mathrm{ct}}}$. As mentioned before, this forcing poset is a lim sup forcing poset, but not decomposable into factors. We first define an auxiliary norm.

Definition B3.7. We define the *split norm* $\|\cdot\|^{\mathrm{split}}$ of a finite tree T by

$$\|T\|^{\mathrm{split}} := \max\{k \mid \exists\, S \subseteq T\colon S \cong 2^{\leq k}\},$$

that is, the maximal k such that the complete binary tree $2^{\leq k}$ of height k order-embeds into T.

Given a finite interval $I \subseteq \omega$ and a non-empty $X \subseteq 2^I$, we identify X with the tree $T_X := X \cup \{\eta{\restriction}_{I\cap n} \mid \eta \in X, n \in I\}$ and write $\|X\|^{\mathrm{split}}$ for $\|T_X\|^{\mathrm{split}}$.[16]

Definitions 2.3.1 and 2.3.4 in [FGKS17] define a norm $\mathrm{nor}^{B,\,m}_{\mathrm{Sacks}}(X)$ as

$$\mathrm{nor}^{B,\,m}_{\mathrm{Sacks}}(X) := \max\left(\{i \mid F^B_m(i) \leq \|X\|^{\mathrm{split}}\} \cup \{0\}\right)$$

for some function $F^B_m(i)$; we will use this norm without repeating the technical arguments, instead briefly referring to the relevant properties the norm is proved to have in [FGKS17, Lemma 2.3.6].

Definition B3.8. Given a cardinal κ_{ct} with $\kappa_{\mathrm{ct}}^\omega = \kappa_{\mathrm{ct}}$, an index set A_{ct} of size κ_{ct} and the sequences $n^P_{<L}, n^B_L, n^S_L, n^R_{<L}$, the forcing poset $\mathbb{Q}_{\mathrm{ct},\,\kappa_{\mathrm{ct}}}$ is defined as the set of all conditions p with countable $\mathrm{supp}(p) \subseteq A_{\mathrm{ct}}$ fulfilling the following:

(i) There is a partition of heights$_{\mathrm{ct}}$ into a sequence of consecutive intervals, which we will call a *frame*. To avoid confusion with the intervals I_L, we will refer to the intervals (i. e. partition classes) of the frame as *segments*.

[16] We were tempted to simply write $\eta{\restriction}_n$ in the definition of T_X here, but we think that $\eta{\restriction}_{I\cap n}$ more explicitly shows that we actually mean $\eta{\restriction}_{\{k\in I\mid k<n\}}$.

(ii) We formalise the frame as a function $\mathrm{segm}\colon \mathrm{heights}_{\mathrm{ct}} \to \mathrm{heights}_{\mathrm{ct}}^{<\omega}$ mapping each height to the finite tuple of heights constituting the segment it belongs to. We will also use $F(L)$ to refer to $\min(\mathrm{segm}(L))$, so

$$\mathrm{segm}(L) = [F(L), F(L^*) - 4] \cap \mathrm{heights}_{\mathrm{ct}},$$

where L^* is the minimal $L' \in \mathrm{heights}_{\mathrm{ct}}$ above L such that $\mathrm{segm}(L') \neq \mathrm{segm}(L)$. (See Figure 7 for the structure of a frame.)

(iii) For each $\alpha \in \mathrm{supp}(p)$, $p(\alpha)$ consists of a sequence of creatures $\langle p(\alpha, L) \mid L \in \mathrm{heights}_{\mathrm{ct}} \rangle$

(iv) Given a segment $\bar{M} := \langle M_1, \ldots, M_m \rangle$, we will use the abbreviated notation $p(\alpha, \bar{M})$ to denote $\langle p(\alpha, M_1), \ldots, p(\alpha, M_m) \rangle$. We will call $p(\alpha, \bar{M})$ a *creature segment*.

(v) For each $L \in \mathrm{heights}_{\mathrm{ct}}$, fix a finite interval $I_L \subseteq \omega$ (for notational simplicity, disjoint from all I_K for $K < L$) such that with the definitions given below, $\| \mathrm{POSS}_{\mathrm{ct},L} \|_{\mathrm{ct},L} > n_L^B$. (This ensures that even for the trivial frame consisting of only singleton segments, there are valid conditions.)

(vi) The sets of possibilities are given by $\mathrm{POSS}_{\mathrm{ct},L} := 2^{I_L}$. This means that for each such L and $\alpha \in \mathrm{supp}(p)$, $p(\alpha, L) \subseteq 2^{I_L}$ (and $p(L) \neq \varnothing$).

(vii) We will treat each creature segment as a unit and define the norm of a condition in $\mathbb{Q}_{\mathrm{ct},\,\kappa_{\mathrm{ct}}}$ on creature segments. Let $\bar{X} := \langle X_1, \ldots, X_m \rangle$ be a creature segment of $p(\alpha)$ (for some $\alpha \in \mathrm{supp}(p)$) associated with the segment $\bar{K} := \langle K_1, \ldots, K_m \rangle$. This means that for some $i < \omega$, we have $K_j = 4(i+j)+3$ and $X_j \subseteq \mathrm{POSS}_{\mathrm{ct},K_j}$ for $j \in \{1, \ldots, m\}$.

(viii) The norm $\| \cdot \|_{\mathrm{ct},L}$ on a creature segment $\bar{X}$ is given by

$$\|\bar{X}\|_{\mathrm{ct},K_1} := \max_{j \in \{1,\ldots,m\}} \mathrm{nor}_{\mathrm{Sacks}}^{n^B_{K_1},\,k}(X_j),$$

where k is such that $K_1 = 4k + 3$.

(ix) For each $\alpha \in \mathrm{supp}(p)$, there is an increasing sequence of $L_i \in \mathrm{heights}_{\mathrm{ct}}$, each of which is the initial height in a segment, such that $\|p(\alpha, \mathrm{segm}(L_i))\|_{\mathrm{ct},L_i} \geq i$. Equivalently, $\limsup_{L \to \infty} \|p(\alpha, \mathrm{segm}(L))\|_{\mathrm{ct},F(L)} = \infty$.

A condition q is stronger than a condition p if $\mathrm{supp}(q) \supseteq \mathrm{supp}(p)$, $q(\alpha, L) \subseteq p(\alpha, L)$ holds for each $\alpha \in \mathrm{supp}(p)$ and each $L \in \mathrm{heights}_{\mathrm{ct}}$, and the frame of q is coarser than the frame of p.

Note that the choice of the I_L above ensures that $\mathbb{Q}_{\mathrm{ct},\,\kappa_{\mathrm{ct}}}$ is non-empty. Also note that we will sometimes for brevity write $\|X\|_{\mathrm{ct},K_1}$ to mean $\mathrm{nor}_{\mathrm{Sacks}}^{n^B_{K_1},\,k}(X)$, to avoid having to single out the ct case when it is not strictly necessary.

We remark that if $q \leq p$ only differs from p in that its frame is coarser, then the norms on the creature segments in q are greater or equal to the norms on the corresponding creature segments in p.

Observation B3.9. One could decompose the forcing poset $\mathbb{Q}_{\mathrm{ct},\,\kappa_{\mathrm{ct}}}$ as the composition of a σ-complete forcing poset $\mathbb{F}$ defining a frame partition $\mathcal{F}$ and a

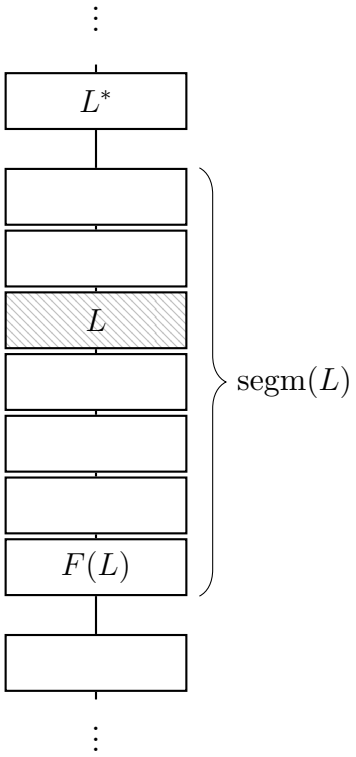

Figure 7: A visualisation of the segment segm(L) of the height L in a frame, as well as the last and first heights of the preceding and succeeding segment, respectively.

parametrised version $\mathbb{Q}^{\mathcal{F}}_{\mathrm{ct},\,\kappa_{\mathrm{ct}}}$ with a fixed frame, analogous to the well-known decomposition of Mathias forcing $\mathbb{R}$ into a forcing poset $\mathbb{U}$ adding an ultrafilter $\mathcal{U}$ and the parametrised Mathias forcing $\mathbb{R}_{\mathcal{U}}$ (cf. [Hal17, Lemma 26.10]). However, this decomposition of $\mathbb{Q}_{\mathrm{ct}\,\kappa_{\mathrm{ct}}}$ neither simplifies nor generalises our constructions, so we will not use it.

Finally, we define $\mathbb{Q}_{\mathrm{nm},\,\kappa_{\mathrm{nm}}}$ (the only lim inf forcing poset), which we will define *en bloc* instead of as a countable support product.

Definition B3.10. Given a cardinal κ_{nm} with $\kappa_{\mathrm{nm}}^{\omega} = \kappa_{\mathrm{nm}}$, an index set A_{nm} of size κ_{nm} and the sequences $n^P_{<L}, n^B_L, n^S_L, n^R_{<L}$, the forcing poset $\mathbb{Q}_{\mathrm{nm},\,\kappa_{\mathrm{nm}}}$ is defined as the set of all conditions p fulfilling the following:

(i) p consists of a sequence $\langle p(4k) \mid k < \omega\rangle$ of *compound creatures*, each of which has a finite support $S_{4k} \subseteq \mathrm{supp}(p) \subseteq A_{\mathrm{nm}}$, together with a sequence of reals $d(4k)$ which are called *halving parameters*. The supports S_{4k} are non-decreasing.

(ii) We define the finite sets of sublevels to be

$$J_{4k} := 3^{(4k+1)\cdot 2^{4k\cdot n^P_{<(4k,0)}}}.$$

(The reason for this definition will become clear in (xi) below.)

(iii) For each $\alpha \in S_{4k}$, $C_{\alpha,4k}$ is a stacked creature consisting of $|J_{4k}|$ many creatures $C_{\alpha,(4k,i)}$, $i \in J_{4k}$. So the compound creature $p(4k)$ is indexed by $S_{4k} \times \{(4k,i) \mid i \in J_{4k}\}$, and each $p(\alpha,4k) = C_{\alpha,4k}$ is a stacked creature. (See Figure 8 for an example of a compound creature.)

(iv) In the following, $L = (4k,i)$ will refer to some sublevel height of $p(4k)$.

(v) Define the *cell norm* $\|\cdot\|^{\mathrm{cell}}_L$ by $\|M\|^{\mathrm{cell}}_L := \frac{\log|M|}{n^B_L \log n^B_L}$.

(vi) For each $L = (4k,i)$, fix a finite interval $I_L \subseteq \omega$ (for notational simplicity, disjoint from all I_K for $K < L$) such that $\|2^{I_L}\|^{\mathrm{cell}}_L > n^B_L$.

(vii) $\mathrm{POSS}_{\mathrm{nm},L} := 2^{I_L}$; this means that for each such L and all $\alpha \in S_{4k}$, $p(\alpha,L) \subseteq 2^{I_L}$.

(viii) Call the minimal $4k < \omega$ such that there is an $\alpha \in \mathrm{supp}(p)$ and a $K = (4k,i) \in \mathrm{heights}_{\mathrm{nm}}$ with $|p(\alpha,K)| > 1$ the *trunk length* of p, denoted by $\mathrm{trklgth}(p)$. We call the part of p below $\mathrm{trklgth}(p)$ the *trunk* and denote it by $\mathrm{trunk}(p)$; the trunk of p consists of singletons $p(\alpha,L)$ in $\mathrm{POSS}_{\mathrm{nm},L}$ for each $\alpha \in \mathrm{supp}(p)$ and each $L = (4j,i)$ with $j < k$ and $i \in J_{4j}$. By definition, we let $S_{4j} = \varnothing$ for $j < k$.

(ix) Each sublevel fulfils a condition called *modesty*,[17] which means that for each $L = (4k,i)$, there is at most one index $\alpha \in S_{4k}$ such that $p(\alpha,L)$ is non-trivial, i. e. $|p(\alpha,L)| > 1$.

(x) Define the *stack norm* $\|\cdot\|^{\mathrm{stack}}_{4k}$ on stacked creatures as follows: $\|p(\alpha,4k)\|^{\mathrm{stack}}_{4k}$ is the maximal r such that there is an $X \subseteq J_{4k}$ with $\mu_{4k}(X) := \frac{\log_3|X|}{4k+1} \geq r$ and such that $\|p(\alpha,(4k,x))\|^{\mathrm{cell}}_{(4k,x)} \geq r$ for all $x \in X$.

Note that $\mu_{4k}(J_{4k}) = 2^{4k \cdot n^P_{<(4k,0)}}$. Consequently, the stack norm of a maximal stacked creature having the full 2^{I_L} at each height also is $2^{4k \cdot n^P_{<(4k,0)}}$, as $n^B_L > 2^{4k \cdot n^P_{<(4k,0)}}$ for $L > (4k,0)$.

(xi) Define $\|\cdot\|_{\mathrm{nm},4k}$ on compound creatures by

$$\|p(4k)\|_{\mathrm{nm},4k} := \frac{\log_2\left(\min\{\|p(\alpha,4k)\|^{\mathrm{stack}}_{4k} \mid \alpha \in S_{4k}\} - d(4k)\right)}{n^P_{<(4k,0)}}$$

or 0 if the above is ill-defined (for instance, because the minimal stacked creature norm is smaller than the halving parameter $d(4k)$).

Note that for the trunk, applying this norm to any subset of $\mathrm{supp}(p)$ and any level $4j < \mathrm{trklgth}(p)$ also just yields 0. Also note that the norm of the maximal compound creature consisting of the maximal stacked creatures thus is

$$\frac{\log_2(2^{4k \cdot n^P_{<(4k,0)}} - d(4k))}{n^P_{<(4k,0)}},$$

which for $d(4k) = 0$ is exactly $4k$.

[17] In [FGKS17], the set of modest conditions was introduced as a dense subset of the conditions instead; while we will do this similarly in the following section in Lemma B4.4, for sake of easier presentation, we prefer to define the lim inf conditions as modest right from the start. Note that if we drop modesty from the definition, applying Lemma 2.2.2 from [FGKS17] to an arbitrary condition p easily yields a stronger modest condition q.

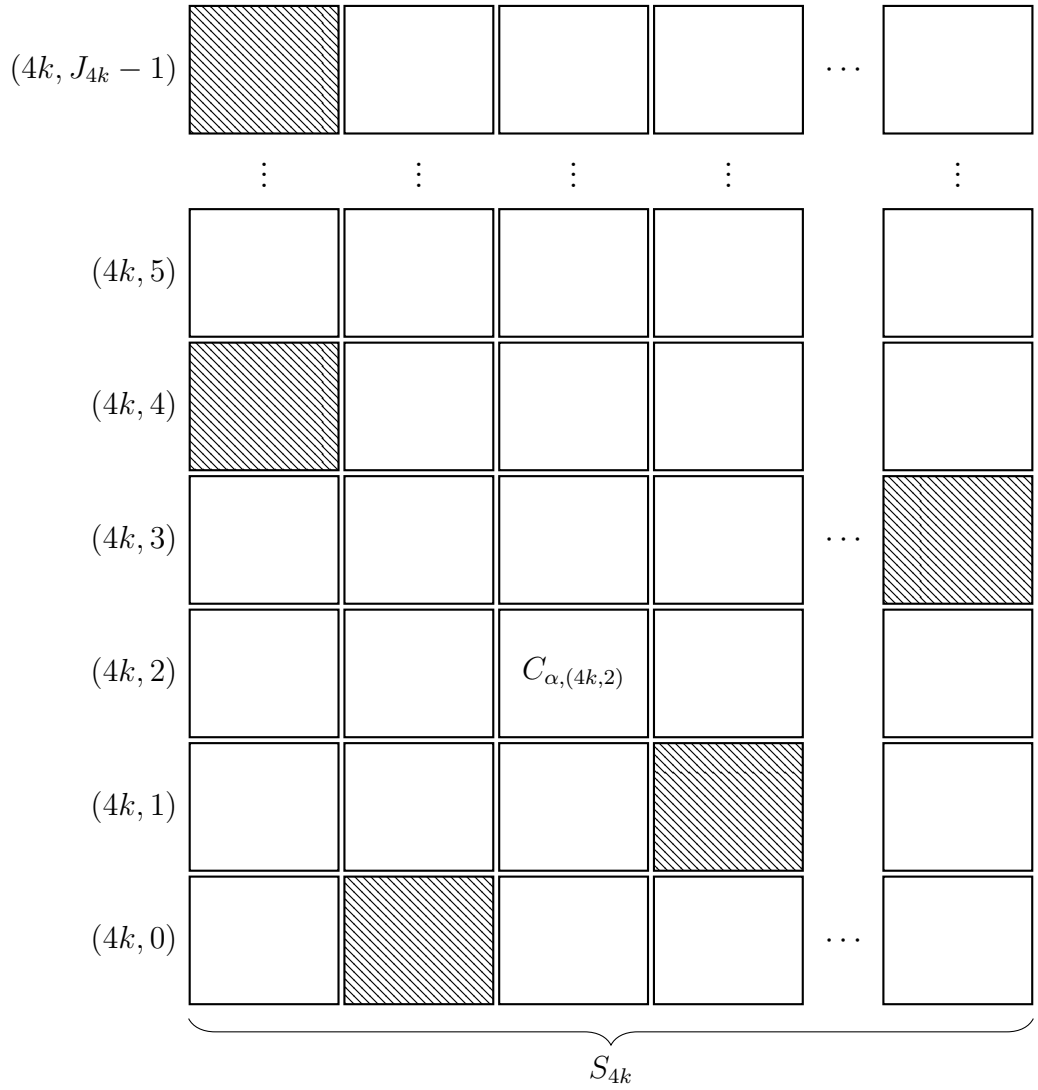

Figure 8: An example of a compound creature $C := p(4k)$ of a condition $p \in \mathbb{Q}_{\mathrm{nm},\,\kappa_{\mathrm{nm}}}$. A possible pattern of cells containing non-trivial creatures is hatched.

(xii) There is an increasing sequence of $k_i < \omega$ such that $\|p(4\ell)\|_{\mathrm{nm},4\ell} \geq i$ for all $\ell \geq k_i$. Equivalently, $\liminf_{k\to\infty} \|p(4k)\|_{\mathrm{nm},4k} = \infty$.

(xiii) The relative widths (i. e. the width-to-height ratio) of the compound creatures converge to 0, i. e. $\lim_{k\to\infty} \frac{|S_{4k}|}{4k+1} = 0$.

A condition q is stronger than a condition p if

- $\mathrm{trklgth}(q) \geq \mathrm{trklgth}(p)$ (the trunk may grow),
- $S_{4k}(q) \supseteq S_{4k}(p)$ for each $4k \geq \mathrm{trklgth}(q)$ (above the trunk, the supports do not shrink),
- for each $k < \omega$, for each $\alpha \in S_{4k}(p)$ and for each $i \in J_{4k}$, $q(\alpha, (4k, i)) \subseteq p(\alpha, (4k, i))$, and
- $d(q)(4k) \geq d(p)(4k)$ (the halving parameters do not decrease).

Note that for reasonably small halving parameters (namely, such that for some

$k_0 < \omega$ and some $\varepsilon > 0$

$$d(4k) < 2^{4k \cdot n^P_{<(4k,0)}} \cdot (1 - \varepsilon)$$

holds for all $k > k_0$), the choice of the I_L above ensures that $\mathbb{Q}_{\mathrm{nm},\,\kappa_{\mathrm{nm}}}$ is non-empty.

We want to briefly remark on the terminology: Our compound creatures are the smallest possible kind of compound creatures in [FGKS17], since there compound creatures could span multiple levels. Our cells and stacks are the subatoms and atoms of [FGKS17].

B4 Putting the Parts Together

We remark that we still have not shown that the definitions we make are possible, as we require the sequences $n^P_{<L}, n^B_L, n^S_L, n^R_{<L}$ to make the definitions. Before we rectify that omission, we define the full forcing poset.

Definition B4.1. Let

- $\mathrm{types} := \{\mathrm{nm}, \mathrm{nn}, \mathrm{cn}, \mathrm{ct}\} \cup \bigcup_{\xi<\omega_1} \{\xi\}$,
- $\mathrm{types}_{\,\limsup} := \mathrm{types} \setminus \{\mathrm{nm}\}$, and
- $\mathrm{types}_{\,\mathrm{modular}} := \mathrm{types}_{\,\limsup} \setminus \{\mathrm{ct}\}$.

Assume we are given cardinals $\kappa_{\mathrm{nm}} \leq \kappa_{\mathrm{nn}} \leq \kappa_{\mathrm{cn}} \leq \kappa_{\mathrm{ct}}$ as well as a sequence of cardinals $\langle \kappa_\xi \mid \xi < \omega_1 \rangle$ with $\kappa_{\mathrm{nm}} \leq \kappa_\xi \leq \kappa_{\mathrm{nn}}$ such that for each $\mathrm{t} \in \mathrm{types}$, $\kappa_{\mathrm{t}}^{\aleph_0} = \kappa_{\mathrm{t}}$. Then our forcing poset is defined as follows:

$$\mathbb{Q} := \prod_{\mathrm{t} \in \mathrm{types}_{\,\mathrm{modular}}} \mathbb{Q}_{\mathrm{t}}^{\kappa_{\mathrm{t}}} \times \mathbb{Q}_{\mathrm{ct},\,\kappa_{\mathrm{ct}}} \times \mathbb{Q}_{\mathrm{nm},\,\kappa_{\mathrm{nm}}},$$

where all products and powers have countable support.

Since $\mathbb{Q}$ is a product, a condition q is stronger than a condition p if each factor of q is stronger than the corresponding factor of p. See Fact B4.10 for a detailed description of all the properties subsumed by the statement "$q \leq p$".

Definition B4.2. Given $p \in \mathbb{Q}$, $\mathrm{tg} \in \mathrm{typegroups}_{\,\limsup}$ and $L \in \mathrm{heights}_{\,\mathrm{tg}}$, define $\mathrm{supp}(p, \mathrm{tg}, L)$ to be the set of all $\alpha \in A_{\mathrm{tg}}$ such that for some $K \leq L$ in $\mathrm{heights}_{\,\mathrm{tg}}$, $|p(\alpha, K)| > 1$. This means that the tg-specific support of a condition at some height L is the set of all indices of that group of types such that p has already had a non-trivial creature at that index up to L.

For $\mathrm{tg} = \mathrm{ct}$, we usually will refer to the support of a segment $\bar{K} = \langle K_1, \ldots, K_m \rangle$ (since we treat each creature segment as a whole) and mean $\mathrm{supp}(p, \mathrm{ct}, \bar{K}) = \mathrm{supp}(p, \mathrm{ct}, K_m)$.

For $\mathrm{tg} = \mathrm{nm}$, we define $\mathrm{supp}(p, \mathrm{nm}, 4k) := S_{4k}(p)$ and $\mathrm{supp}(p, \mathrm{nm}, (4k, i)) := \mathrm{supp}(p, \mathrm{nm}, 4k)$ for all $i \in J_{4k}$.

We define $\mathrm{supp}(p, L)$ to be the union of all appropriate $\mathrm{supp}(p, \mathrm{tg}, K)$ with $K \leq L$, and $\mathrm{supp}(p)$ to be the union of all $\mathrm{supp}(p, L)$ with $L \in \mathrm{heights}$.

We immediately remark that we will instead work with a dense subset of $\mathbb{Q}$:

Definition B4.3. We call a condition $p \in \mathbb{Q}$ *modest* if

(i) for each $\mathrm{tg} \in \mathrm{typegroups}_{\limsup}$, $\mathrm{supp}(p, \mathrm{tg}, \ell) = \varnothing$ for all $\ell < \mathrm{trklgth}(p)$,[18]
(ii) for each $L \in \mathrm{heights}$, there is at most one index $\alpha \in A$ such that $p(L, \alpha)$ is non-trivial, i. e. $|p(L, \alpha)| > 1$,
(iii) the segments of $p(\mathrm{ct})$ are such that for each segment $\bar{L} = \langle L_1, \ldots, L_m \rangle$ with $L_1 = 4k + 3$, for all $\alpha \in \mathrm{supp}(p, \mathrm{ct}, \bar{L})$ we have $\|p(\alpha, \bar{L})\|_{\mathrm{ct}, L_1} \geq k$ as well as $|\mathrm{supp}(p, \mathrm{ct}, \bar{L})| = |\mathrm{supp}(p, \mathrm{ct}, L_m)| < k$, and
(iv) for each segment $\bar{L} = \langle L_1, \ldots, L_m \rangle$ of the frame of $p(\mathrm{ct})$ (with $L_1 = 4k + 3$) and $\alpha \in A_{\mathrm{ct}}$ such that $p(\alpha, \bar{L})$ is non-trivial, there is exactly one $L^* \in \bar{L}$ such that $p(\alpha, L^*)$ is non-trivial, and furthermore $\|p(\alpha, \bar{L})\|_{\mathrm{ct}, L_1} = k$. Letting $c = F_k^{n^B_{L_1}}(k)$ (i. e. precisely the split norm necessary to achieve this ct norm), we furthermore demand that each such $p(\alpha, L^*)$ is already minimal; in particular, this means that there are exactly 2^c many possibilities in $p(\alpha, L^*)$.

Lemma B4.4. *The set of modest conditions is dense in $\mathbb{Q}$; moreover, for any $p \in \mathbb{Q}$ there is even a modest $q \leq p$ with the same support.*

Proof. Given an arbitrary $p \in \mathbb{Q}$, we have to find a modest $q \leq p$. We first pick arbitrary singletons in each non-trivial creature below $\mathrm{trklgth}(p)$ to fulfil (i). Then, we define q piecewise for each $\mathrm{tg} \in \mathrm{typegroups}$:

- For $\mathrm{tg} = \mathrm{nm}$, we have already defined the compound creatures such that they fulfil (ii).
- For $\mathrm{tg} \in \mathrm{typegroups}_{\mathrm{modular}}$, finding $q(\mathrm{tg})$ is just a matter of diagonalisation and bookkeeping (picking arbitrary singletons within creatures as required to fulfil (ii)).
- To achieve (iii), we coarsen the frame to encompass sufficiently large $p(\alpha, K)$ into the creature segments and/or strengthen to arbitrary singletons whenever necessary (plus bookkeeping, again).
- Property (iv) is fulfilled by choosing, for each $\alpha \in \mathrm{supp}(p, \bar{L})$, a single $L^* \in \bar{L}$ such that $\|p(\alpha, L^*)\|^{\mathrm{split}}$ is large enough, shrinking $p(\alpha, L^*)$ such that it is minimal and replacing all other $p(\alpha, L')$ by arbitrary singletons; by definition, all of this leaves the ct norms of such segments at least k and the resulting $q(\mathrm{ct})$ is still a valid (part of a) condition.

It is clear that $\mathrm{supp}(q) = \mathrm{supp}(p)$. □

Note that for any modest $p \in \mathbb{Q}$, property (ii) immediately implies that $\mathrm{supp}(p, L)$ is finite for any $L \in \mathrm{heights}$.

We will extend the meaning of the word "trunk" to refer to the entire single possibility of a modest condition p below the trunk length of p.

We will only ever work with modest conditions; whenever we speak of conditions, the qualifier "modest" is implied. Though the results of some constructions may

[18] In the preceding section (in Definition B3.10), $\mathrm{trklgth}(p)$ only took the nm part of p into account, but now we want to make sure that p has no non-trivial creatures below $\mathrm{trklgth}(p)$ at all.

not be modest conditions themselves, we can find stronger conditions with the same support by the preceding lemma; and moreover, if a condition is already partially modest (i. e. modest up to a certain height), we can keep that part when making it modest.

We remark that modesty properties (iii) and (iv) roughly correspond to the concept of "Sacks pruning" in [FGKS17, subsection 3.4] and [FGKS17, Lemma 2.3.6].

Modesty properties (ii)–(iv) are of vital importance to the entire construction. Without them, it would not be possible to define the sequences $n^P_{<L}, n^B_L, n^S_L, n^R_{<L}$ in a sensible manner, which we are now finally able to do. Before we do so, we have to introduce the "maximal strengthenings of a condition p below a height L" mentioned in the introductory remarks.

Definition B4.5. Given a condition $p \in \mathbb{Q}$, we call a height L *relevant* if either

- $L \in \text{heights}_{\text{ct}}$ is the minimum L_1 of a segment $\bar{L} = \langle L_1, \dots, L_m \rangle$ of the frame of $p(\text{ct})$; or
- $L \in \text{heights} \setminus \text{heights}_{\text{ct}}$ and there is an $\alpha \in \text{supp}(p)$ such that $p(\alpha, L)$ is non-trivial, i. e. such that $|p(\alpha, L)| > 1$.

We will use this terminology to simplify the structure of proofs in which we iterate over the heights and modify a condition at each height; naturally, we will only need to do this at the relevant heights, and in such a way that we treat the ct part of the condition at the lower boundaries of segments of the frame.

Definition B4.6. We define the *possibilities of a condition* $p \in \mathbb{Q}$ up to some height L as follows:

- For each sensible choice of $\alpha \in \text{supp}(p)$ and $K \in \text{heights}$ (i. e. such that $\text{tg}(\alpha) = \text{tg}(K)$), let $\text{poss}(p, \alpha, K) := p(\alpha, K)$.
- For each $\alpha \in \text{supp}(p) \setminus A_{\text{ct}}$ and each $L \in \text{heights}$, let

$$\text{poss}(p, \alpha, {<}L) := \prod_{\substack{K<L \\ \text{tg}(K)=\text{tg}(\alpha)\neq\text{ct}}} \text{poss}(p, \alpha, K).$$

- For each $\alpha \in \text{supp}(p) \cap A_{\text{ct}}$ and each segment $\bar{L} = \langle L_1, \dots, L_m \rangle$ of the frame of $p(\text{ct})$, let

$$\text{poss}(p, \alpha, {<}L_1) := \prod_{K<L_1} \text{poss}(p, \alpha, K),$$
$$\text{poss}(p, \alpha, {<}L_i) := \prod_{K\leq L_m} \text{poss}(p, \alpha, K)$$

  for all $i \in \{2, \dots, m\}$. This means that when talking about possibilities of $p(\text{ct})$ below some L_i, we have to take the whole segment of the frame into account unless we are at the lower boundary L_1 of such a segment.
- (For easier notation, consider $\text{poss}(p, \alpha, {<}L)$ for $L \in \text{heights} \setminus \text{heights}_{\text{ct}}$ and $\alpha \in \text{supp}(p) \cap A_{\text{ct}}$ to mean $\text{poss}(p, \alpha, {<}L^*)$ with $L^* := \min\{K \in \text{heights}_{\text{ct}} \mid L < K\}$.)

- For each $L \in \text{heights}$, let

$$\text{poss}(p, {<}L) := \prod_{\alpha \in \text{supp}(p) \setminus A_{\text{ct}}} \text{poss}(p, \alpha, {<}L) \times \prod_{\alpha \in \text{supp}(p) \cap A_{\text{ct}}} \text{poss}(p, \alpha, {<}L).$$

Note that while technically, the last product above is infinite, thanks to modesty only finitely (even boundedly) many of the factors will be non-trivial. The fact that for each p and L iterating over all $\eta \in \text{poss}(p, {<}L)$ only takes boundedly many steps (with the bound depending only on L) will be very important in many of the following proofs. Also note that for $L \leq \text{trklgth}(p)$, $|\,\text{poss}(p, {<}L)| = 1$.[19]

Definition B4.7. Given $p \in \mathbb{Q}$, $L \in \text{heights}$ and $\eta \in \text{poss}(p, {<}L)$, we define $p \wedge \eta =: q$ as the condition resulting from replacing all creatures below L as well as those above L in the current segment of the frame of $p(\text{ct})$ with the singletons from η. Formally, q is defined by

- $q(\alpha, K) := \{\eta(\alpha, K)\}$ for all $K < L$ and $q(\alpha, M) := p(\alpha, K)$ for all $K \geq L$ and all $\alpha \in \text{supp}(p) \setminus A_{\text{ct}}$, and
- $q(\alpha, K) := \{\eta(\alpha, K)\}$ for all $K < L^*$ and $q(\alpha, K) := p(\alpha, K)$ for all $K \geq L^*$ and all $\alpha \in \text{supp}(p) \cap A_{\text{ct}}$, where $L^* := \min\{M \in \text{heights}_{\text{ct}} \mid M \geq L \text{ and } M \text{ is the minimum of a segment of } p(\text{ct})\}$.

In some proofs, we will use the notation $p^{<L}$ or $q^{\geq L}$ to denote partial initial or terminal (pseudo-)conditions in the obvious sense of $p^{<L} := \langle p(K) \mid K < L\rangle$ and $q^{\geq L} := \langle q(M) \mid M \geq L\rangle$. We will denote the join of such partial conditions by $p^{<L} {}^\frown q^{\geq L}$; we will at those times take special care to make sure what we are writing down actually ends up being a proper condition.

We will now finally show that the definition of the sequences $n^P_{<L}, n^B_L, n^S_L, n^R_{<L}$ is possible in a consistent way. What we are actually doing is the following: We define the base sets in each level/height L of the forcing posets $\mathbb{Q}_{\text{t}}$, $\text{t} \in \text{types}_{\limsup}$, respectively in each sublevel/height L of $\mathbb{Q}_{\text{nm},\,\kappa_{\text{nm}}}$ iteratively by induction on the levels and also define the four sequences for that L in that step, assuming we already know the four sequences for $K < L$. The order of definitions is as follows:

1. $n^P_{<L}$,
2. $n^R_{<L}$,
3. n^B_L,
4. the section of the forcing poset for the height L, and finally
5. n^S_L.

Definition B4.8. Recall that L^- and L^+ denote the predecessor and successor of a height L, respectively. We define the sequences $n^P_{<L}, n^R_{<L}, n^B_L, n^S_L$ as follows:

$\boldsymbol{n^P_{<L}}$: We recall that $n^P_{<L}$ is meant to be an upper bound on the number of possibilities below the height L, hence $n^P_{<L^-} \cdot n^S_{L^-} < n^P_{<L}$ must hold. (Note that as an

[19] A further note: While as in [FGKS17], the "shapes" of possibilities are not really "nice", this is less of a conceptual problem in this chapter, as due to the compartmentalisation of the creatures to different heights depending on the factor they belong to, possibilities are by necessity tiered; the fact that the ct possibilities may be further "down" in the heights structure is less of a conceptual stretch now.

immediate consequence, we also get $\prod_{K<L} n^S_K < n^P_{<L}$.) For the initial step, simply let $n^P_{<(0,0)} := 1$. (The interpretation of this number still makes sense, as there is exactly one trivial – empty – possibility "below the first height".) For any height L not in heights$_{\text{ct}}$, let $n^P_{<L^+}$ be the minimal integer fulfilling the inequality

$$n^P_{<L} \cdot n^S_L < n^P_{<L^+}.$$

For a height $L = 4k + 3 \in$ heights$_{\text{ct}}$, let $n^P_{<L^+}$ be the minimal integer fulfilling the inequality

$$n^P_{<L} \cdot (n^S_L)^{k-1} < n^P_{<L^+}.$$

That this is indeed a sufficient bound even for the ct case is a result of modesty properties (iii) and (iv) – by shrinking the ct creatures down to a certain prescribed minimal size, we are certain to be bounded by $2^{I_{L_1}}$, even if the actual non-trivial creature turns out to be far above L; and there are at most $k - 1$ many of those creatures in the segment starting at $L = 4k + 3$.[20]

$\boldsymbol{n^R_{<L}}$: The definition of this sequence coding the rapidity of the reading depends on the forcing factor. (Technically, both $n^P_{<L}$ and $n^R_{<L}$ only require information about the previous height L^-, but it makes more sense to define them both at the beginning of the following height's definitions.) For technical reasons, we require $n^P_{<L} < n^R_{<L}$; apart from that, the definition's motivations should be clear once the concepts of rapid reading (Definition B6.1) respectively punctual reading (Definition B10.12) have been introduced.

As a general requirement, for any $L^- \in$ heights we demand the following: Let $\ell := 4k$ (if $L^- = (4k, i)$) or $\ell := L^-$ (otherwise). We then require $n^R_{<L}$ to be at least large enough that it fulfils the inequality

$$n^P_{<L} < n^R_{<L} < \frac{2^{n^R_{<L}}}{\ell}.$$

(While this is not strictly necessary, it makes the proof of Lemma B7.7 slightly nicer.) In most cases, this is easily fulfilled already, anyways, but in the case that for any lower heights the subsequent definitions are smaller than would be required by the above, we just pick $n^R_{<L}$ larger instead.

Depending on the specific typegroup,

- For $L^- \in$ heights$_{\text{nm}}$: Let $n^R_{<L} := n^P_{<L} + 2^{\max I_{L^-}+1}$.
- For $L^- \in$ heights$_{*\text{n}}$: Let $n^R_{<L} := n^P_{<L}+2 \nearrow (\max I_{L^-}+2)$, where $2 \nearrow x := 2^{2^x}$.
- For $L^- \in$ heights$_{\text{slalom}}$: In Lemma B10.13, we will define a function $z \in \omega^\omega$ in which the value of $z(k)$ only depends on the value of n^S_{4k+2}; we let $n^R_{<L} := n^P_{<L} + z(k)$.
- For $L^- \in$ heights$_{\text{ct}}$: We do not have any additional demands for this part of the sequence.

[20] In many cases (whenever the segments of the frame are not trivially short), we are actually way too generous here, but that does not matter.

$\boldsymbol{n^B_L}$: This is a straightforward definition; let

$$n^B_L := (n^B_{L^-})^{n^P_{<L}\cdot n^R_{<L}}$$

(with $n^B_{(0,0)^-} := 2$).[21]

Note that having defined these three numbers, all of the definitions of the various forcing factors can be made, though see the next paragraph regarding the slalom forcing posets.

$\boldsymbol{n^S_L}$: The definition of this sequence also depends on the specific forcing factor. We recall that this sequence is meant to be an upper bound on the size of the base sets at this height, i. e. the number of possibilities at this height.[22]

- For $L \in \text{heights}_{\text{nm}}$: The base sets for this factor is 2^{I_L} for some I_L, so let $n^S_L := 2^{|I_L|}$.
- For $L \in \text{heights}_{*\text{n}}$: For both cn and nn, the base set for these factors is the set of all subsets of 2^{I_L} of relative size $1 - 2^{-n^B_L}$; there are of course equally many of relative size $2^{-n^B_L}$, so let

$$n^S_L := \binom{2^{|I_L|}}{2^{|I_L|-n^B_L}}.$$

- For $L \in \text{heights}_{\text{slalom}}$: This is a bit different from the other cases. While for the other factors, the bound on the size is an a posteriori observation, for the slalom forcing factor, we actually define the bound n^S_L on the size a priori and then (in Lemma B10.10) define the congenial sequence of function pairs $\langle f_\xi, g_\xi \mid \xi < \omega_1\rangle$ such that they fit between n^B_L and n^S_L. For $L = 4k+2$, we hence pick

$$n^S_L := (n^B_L)^{e_k^{3^{2^{k+1}}}}$$

  (for some increasing sequence e_k strictly greater than n^B_L, also defined in Lemma B10.10).
- For $L \in \text{heights}_{\text{ct}}$: The base set for this factor is also 2^{I_L} for some I_L (though with very different requirements on the size of I_L), so let $n^S_L := 2^{|I_L|}$.

We immediately see that $n^P_{<L}$ and n^S_L work as intended:

Lemma B4.9. *For all $p \in \mathbb{Q}$ and $L \in$ heights, $|\operatorname{poss}(p, {<}L)| \leq n^P_{<L}$.*

Proof. For $L = (0,0)$, $\operatorname{poss}(p, {<}(0,0))$ is trivial and $n^P_{<(0,0)} = 1$. The rest follows by induction from $n^P_{<L}\cdot n^S_L < n^P_{<L^+}$ for $L \in \text{heights} \setminus \text{heights}_{\text{ct}}$ and $n^P_{<L}\cdot(n^S_L)^{k-1} < n^P_{<L^+}$ together with modesty for $L = 4k+3 \in \text{heights}_{\text{ct}}$. □

[21] Note that this definition implies $(n^B_{L^-})^{n^P_{<L}} < n^B_L$ as well as $n^B_{L^-}\cdot 2^{n^S_{L^-}+1} < n^B_L$ and $2^{n^P_{<L}\cdot n^R_{<L}} \leq (n^B_{L^-})^{n^P_{<L}\cdot n^R_{<L}}$.

[22] Also note that the definitions of the intervals I_L are such that $n^B_L < n^S_L$ holds for all L. However, in case the reader prefers not to verify this fact, she can just assume that the I_L are chosen even larger such that this inequality holds.

(The function of n^B_L and $n^R_{<L}$ will be shown in detail in section B5 and section B6, respectively.)

Having finally defined all parameters required for the forcing poset, we will now first remark on a few simple properties.

Fact B4.10. Since $\mathbb{Q}$ is a product, a condition q is stronger than a condition p if $q(\mathrm{tg})$ is stronger than $p(\mathrm{tg})$ for each $\mathrm{tg} \in \mathrm{typegroups}$; moreover, for each $\mathrm{t} \in \mathrm{types}_{\mathrm{modular}}$ (i. e. all but ct and nm), this statement can be broken down further to "$q(\alpha)$ is stronger than $p(\alpha)$ for each $\alpha \in A_{\mathrm{t}} \cap \mathrm{supp}(p)$".

To briefly summarise, "$q \leq p$" hence means that

- $\mathrm{trklgth}(q) \geq \mathrm{trklgth}(p)$ (the trunk may grow),
- $\mathrm{supp}(q) \supseteq \mathrm{supp}(p)$ (the support may grow),
- $\mathrm{supp}(q, \mathrm{nm}, 4k) \supseteq \mathrm{supp}(p, \mathrm{nm}, 4k)$ for each $k < \omega$ (above the trunk, the supports do not shrink for the lim inf factor),[23]
- the frame[24] of $q(\mathrm{ct})$ is coarser than the frame of $p(\mathrm{ct})$,
- for each $\alpha \in \mathrm{supp}(p)$ and each $L \in \mathrm{heights}$ with $\mathrm{tg}(\alpha) = \mathrm{tg}(L)$, $q(\alpha, L) \subseteq p(\alpha, L)$ (strengthening the creatures on the old support), and
- for each $k < \omega$, $d(q)(4k) \geq d(p)(4k)$ (the halving parameters do not decrease).

Lemma B4.11. *For any given countable set of indices $B \subseteq A$, there is a condition p such that* $\mathrm{supp}(p) = B$. *In particular, given any $\alpha \in A$, there is a condition p such that* $\mathrm{supp}(p) = \{\alpha\}$.

Proof. We prove the simple case first: Given any $\alpha \in A$, define p by letting $p(L)$ be equal to the full base set for each $L \in \mathrm{heights}_{\mathrm{tg}(\alpha)}$. (If $\alpha \in A_{\mathrm{nm}}$, let the halving parameter sequence be equal to the constant 0 sequence. If $\alpha \in A_{\mathrm{ct}}$, let the frame be the trivial partition of $\mathrm{heights}_{\mathrm{ct}}$ into singleton segments.)

Given an arbitrary countable $B \subseteq A$ (without loss of generality such that B has infinite intersection with A_{t} for each $\mathrm{t} \in \mathrm{types}$) instead, we first enumerate $B_{\mathrm{tg}} = A_{\mathrm{tg}} \cap B$ for each $\mathrm{tg} \in \mathrm{typegroups}_{\limsup}$ as $B_{\mathrm{tg}} =: \{\alpha_x, \alpha_{4+x}, \alpha_{2\cdot 4+x}, \alpha_{3\cdot 4+x}, \ldots\}$ (with the x depending on the tg, in such a way that the $4k + x$ correspond to the appropriate levels for this tg). Also enumerate $B_{\mathrm{nm}} = A_{\mathrm{nm}} \cap B$ as $B_{\mathrm{nm}} =: \{\beta_1, \beta_2, \ldots\}$.

We then define the condition p as follows:

- For $\mathrm{tg} \in \mathrm{typegroups}_{\limsup}$, we first let $p^*(\alpha_i, 4k + x)$ (again, x corresponding to tg) be equal to the full base sets for each $i \geq 4k+x$ and arbitrary singletons below that level. Let the frame of $p^*(\mathrm{ct})$ be the trivial partition of $\mathrm{heights}_{\mathrm{ct}}$ into singleton segments.

[23] In the forcing construction of [FGKS17], this was true in a more general sense, but we have restricted the concept of the trunk to the lim inf factor and defined the support at a height slightly differently. These changes mean that in fact, the support at a certain height may shrink in the lim sup factors in a stronger condition, because the non-trivial creatures witnessing that a certain index α was already in the support by height L may have been eliminated when extending the trunk, so α will then only enter the support at a later height. This conceptual change does not cause any problems, however.

[24] Recall Definition B3.8 for the definition of frames.

  - Now use some appropriate diagonalisation of B_{tg} to thin out $p^*(\mathrm{tg})$ in such a way that in the resulting $p(\mathrm{tg})$ fulfils modesty[25] (which only requires reducing creatures to singletons or to order-isomorphic copies of $2^{\leq k}$) while still fulfilling the requirements on the lim sup of the norms.
  - (It follows from the definitions of the forcing factors that the $p^*(\mathrm{tg})$ fulfil the lim sup conditions for each $\mathrm{tg} \in \mathrm{typegroups}_{\limsup}$, and so do the $p(\mathrm{tg})$ after diagonalisation.)
- For nm, we let $d(p)(4k) = 0$ for all $k < \omega$ and pick some increasing sequence s_{4k} (with $s_0 = 1$) such that $\lim_{k\to\infty} \frac{s_{4k}}{4k+1} = 0$. We will let $S_{4k}(p) := \{\beta_1, \dots, \beta_{s_{4k}}\}$, so $\lim_{k\to\infty} \frac{|S_{4k}(p)|}{4k+1} = 0$ is fulfilled. Note that without loss of generality $|S_{4k}(p)| = s_{4k}$ will be much smaller than $k+1$.
  - We will define p such that $p(\mathrm{nm}, 4k)$ has at least norm k. For each $\alpha \in S_{4k}(p)$, pick a set $X_\alpha \subseteq J_{4k}$ of size $3^{(4k+1)\cdot k}$ (which means $\mu_{4k}(X) = k$) disjoint from $X_{\alpha'}$ for each $\alpha' \in S_{4k}$ with $\alpha \neq \alpha'$. We let $p(\alpha, (4k, j))$ be equal to the full base set for each $j \in X_\alpha$ and some arbitrary singletons elsewhere. The full base sets have cell norms much larger than k, so the whole compound creature $p(\mathrm{nm}, 4k)$ has norm k and the lim inf condition is fulfilled.
  - The choice of these X_α is possible because we only require

$$s_{4k} \cdot 3^{(4k+1)\cdot k} < (k+1) \cdot 3^{(4k+1)\cdot k}$$

    many different sublevels to choose from to do that, and by our definition, $J_{4k} = 3^{(4k+1)\cdot 2^{4k\cdot n^P_{<(4k,0)}}}$ is much larger than that. □

Before proceeding, we recall the following combinatorial result from [FGKS17, Lemma 2.2.2].

Lemma B4.12. *Given $\ell \leq k$ and a family $\langle X_i \mid 1 \leq i \leq \ell\rangle$ of subsets of J_{4k}, there is a family $\langle X_i^* \mid 1 \leq i \leq \ell\rangle$ of pairwise disjoint sets such that for each $1 \leq i \leq \ell$, $X_i^* \subseteq X_i$ and $\mu_{4k}(X_i^*) \geq \mu_{4k}(X_i) - 1$.*

Lemma B4.13. *Given two conditions $p, q \in \mathbb{Q}$ with disjoint supports, identical (or compatible) frames and identical sequences of halving parameters, there is a condition r stronger than both.*

Proof. Since $\frac{|S_{4k}(p)|}{4k+1}$ and $\frac{|S_{4k}(q)|}{4k+1}$ must both converge to 0, there is some k_0 such that $\frac{|S_{4k}(p)|}{4k+1} \leq \frac{1}{2}$ and $\frac{|S_{4k}(q)|}{4k+1} \leq \frac{1}{2}$ for all $k \geq k_0$. Define $p' \leq p$ and $q' \leq q$ as the conditions resulting from extending the trunk to $4k_0$ (and choosing arbitrary singletons within all non-trivial creatures below).

We first define the pseudo-condition r^* as simply the union of p' and q' together with the finest frame coarser than the frames of $p(\mathrm{ct})$ and $q(\mathrm{ct})$. Of course, r^* might not fulfil modesty. For each $\mathrm{tg} \in \mathrm{typegroups}_{\limsup}$, we use diagonalisation to thin out $r^*(\mathrm{tg})$ and pick appropriately small subcreatures in $r^*(\mathrm{ct})$ in such a way that the resulting $r(\mathrm{tg})$ fulfils modesty.

[25] Recall Definition B4.3 for the definition of modesty.

As for each $\alpha \in \mathrm{supp}(r^*) \cap A_{\mathrm{ct}}$, the minimal elements of the segments – which are the reference points for the norms – can only have shrunk, it follows that $r^*(\mathrm{ct})$ is indeed a valid condition.

For $r^*(\mathrm{nm})$, we need to do a bit more. Assume without loss of generality that for all $k \geq k_0$, $\|p'(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} \geq 2$ and $\|q'(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} \geq 2$. We do the following procedure for each $k \geq k_0$:

- Let $S_{4k}(p') =: \{\alpha_1, \dots, \alpha_c\}$ and $S_{4k}(q') =: \{\beta_1, \dots, \beta_d\}$ and note that $c, d \leq 2k$ by our choice of k_0.
- Let $n_p := \|p^0(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k}$, $n_q := \|q^0(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k}$ and $n := \min(n_p, n_q)$. For each $1 \leq i \leq c$ and $1 \leq j \leq d$, there must be sets $A_i \subseteq J_{4k}$ respectively $B_j \subseteq J_{4k}$ such that they witness the stacked creature norm of $p'(\alpha_i, 4k)$ respectively $q'(\beta_j, 4k)$ being at least n. We remark that since n is at least 2, so are $\mu_{4k}(A_i)$ and $\mu_{4k}(B_j)$, and hence $|A_i|$ and $|B_j|$ are at least $3^{2\cdot(4k+1)}$.
- By applying Lemma B4.12 to the family $\langle A_1, \dots, A_c, B_1, \dots, B_d\rangle$, we get a family $\langle A_1^*, \dots, A_c^*, B_1^*, \dots, B_d^*\rangle$ of pairwise disjoint subsets of J_{4k} such that for each $1 \leq i \leq c$ and each $1 \leq j \leq d$,
  - $\mu_{4k}(A_i^*) \geq n - 1$ and $\mu_{4k}(B_j^*) \geq n - 1$,
  - for each $a \in A_i^*$, $\|p'(\alpha_i, (4k, a))\|^{\mathrm{cell}}_{(4k,a)} \geq n$, and
  - for each $b \in B_j^*$, $\|q'(\beta_j, (4k, b))\|^{\mathrm{cell}}_{(4k,b)} \geq n$.

  Define $r(\mathrm{nm}, 4k)$ by keeping the creatures in these sublevel index sets and replacing the others by arbitrary singletons.
- It follows that $\|r(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} \geq n - 1$.

Hence the resulting r is indeed a condition, and r is stronger than both p and q by construction. □

Corollary B4.14. *Given a condition $p \in \mathbb{Q}$ and any $\alpha \in A \setminus \mathrm{supp}(p)$, there is a $q \leq p$ with $\mathrm{supp}(q) = \mathrm{supp}(p) \cup \{\alpha\}$.*

Proof. By Lemma B4.11, there is a condition p_α with support $\{\alpha\}$. If $\alpha \in A_{\mathrm{nm}}$, we replace p_α by the condition with identical creatures, but the halving parameters of p instead; since p is a condition, the halving parameters must be small enough such that $\liminf_{k\to\infty} \|p(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} = \infty$, and hence the same must hold for p_α with the same halving parameters. Then we apply Lemma B4.13 to p and p_α; the resulting q is as required. □

We can now define the generic sequences added by the forcing.

Definition B4.15. Let G be a $\mathbb{Q}$-generic filter. For each $\mathrm{tg} \in$ typegroups and each $\alpha \in A_{\mathrm{t}} \subseteq A_{\mathrm{tg}}$, let $\dot{y}_\alpha$ be the name for

$$\{(L, z) \mid L \in \mathrm{heights}_{\mathrm{tg}}, \exists\, p \in G\colon\ \mathrm{trklgth}(p) > L \wedge p(\alpha, L) = \{z\}\}.$$

For $\mathrm{tg} \neq \mathrm{nm}$, there is the equivalent, clearer representation (with x the appropriate element of $\{1, 2, 3\}$)

$$\{(k, z) \mid 4k + x \in \mathrm{heights}_{\mathrm{tg}}, \exists\, p \in G\colon\ \mathrm{trklgth}(p) > 4k + x \wedge p(\alpha, 4k + x) = \{z\}\}.$$

We write $\dot{y}$ for $\langle \dot{y}_\alpha \mid \alpha \in A\rangle$. We remark that depending on the specific tg, the z in the definitions above are entirely different kinds of objects.

We note a few simple facts about generics and possibilities.

Fact B4.16. Let $p \in \mathbb{Q}$ and $L \in \text{heights}$.

- For $\eta \in \text{poss}(p, {<}L)$, $p \wedge \eta \leq p$.[26]
- $p \wedge \eta$ and $p \wedge \eta'$ are incompatible if $\eta, \eta' \in \text{poss}(p, {<}L)$ are distinct.
- $p \wedge \eta$ forces that $\dot{y}$ extends η, i. e. that $\dot{y}_\alpha$ extends $\eta(\alpha)$ for all $\alpha \in \text{supp}(p)$. In particular, p forces that $\dot{y}$ extends $p^{< \text{trklgth}(p)}$.
- $\eta \in \text{poss}(p, {<}L)$ iff p does not force that η is incompatible with $\dot{y}$.
- $\mathbb{Q}$ forces that $\dot{y}$ is defined everywhere. (This follows from Corollary B4.14.)

Lemma B4.17. *Given $q \leq p$ and $\eta \in \text{poss}(q, {<}L)$, there is a unique $\vartheta \in \text{poss}(p, {<}L)$ such that $q \wedge \eta \leq p \wedge \vartheta$.*

Proof. Recall Definition B4.6: Since the possibilities are structurally somewhat more complicated for ct, we need to take that into account when trimming η to get ϑ.

Set $L^* := \min\{M \in \text{heights}_{\text{ct}} \mid M \geq L, M \text{ is the minimum of a segment of } p(\text{ct})\}$. Let $\vartheta{\restriction}_{A \setminus A_{\text{ct}}} := \eta{\restriction}_{\text{supp}(p) \setminus A_{\text{ct}}}$ and $\vartheta{\restriction}_{A_{\text{ct}}} := \eta{\restriction}_{\text{supp}(p) \cap A_{\text{ct}}}({<}L^*)$ (this restriction is necessary for technical reasons, because $p(\text{ct})$ in general could have a finer frame than $q(\text{ct})$). Uniqueness follows from the incompatibility of $p \wedge \vartheta$ and $p \wedge \vartheta'$ for distinct $\vartheta, \vartheta' \in \text{poss}(p, {<}L)$. □

The first important fact about $\mathbb{Q}$ we will prove is the following:

Lemma B4.18. *Assuming* CH, *$\mathbb{Q}$ is $\aleph_2$-cc.*

Proof. Assume that $Z := \langle p_i \mid i < \omega_2 \rangle$ is a family of conditions. Using the Δ-system lemma for families of countable sets and CH, we can find $\Delta \subseteq A$ and thin out Z to a subset of the same size such that for any distinct $p, q \in Z$,

- $\Delta = \text{supp}(p) \cap \text{supp}(q)$,
- for all $k < \omega$, $d(p)(4k) = d(q)(4k)$,
- the frames of $p(\text{ct})$ and $q(\text{ct})$ are identical, and
- p and q are identical on Δ, i. e. for all $\alpha \in \Delta$ and all $L \in \text{heights}_{\text{tg}(\alpha)}$, $p(\alpha, L) = q(\alpha, L)$.

By Lemma B4.13 (applied to p and $q{\restriction}_{A \setminus \Delta}$), there is some $r \in \mathbb{Q}$ stronger than both p and q, hence Z is not an antichain. □

Lemma B4.19. *Assume that $B \subseteq A$ and either $A_{\text{nm}} \subseteq B$ or $A_{\text{nm}} \cap B = \varnothing$. Let $\mathbb{Q}_B \subseteq \mathbb{Q}$ consist of all $p \in \mathbb{Q}$ with $\text{supp}(p) \subseteq B$. Then $\mathbb{Q}_B$ is a complete subforcing poset of $\mathbb{Q}$. ($\mathbb{Q}_B$ has the same general properties as $\mathbb{Q}$, as it is essentially the same forcing poset, just with a smaller index set.)*

If we additionally assume that $A_{\text{ct}} \subseteq B$ or $A_{\text{ct}} \cap B = \varnothing$, then it is clear from the product structure of $\mathbb{Q} = \mathbb{Q}_B \times \mathbb{Q}_{A \setminus B}$ that $\mathbb{Q}_B$ is a complete subforcing poset of $\mathbb{Q}$.

[26] Recall Definition B4.7 for the definition of $p \wedge \eta$.

Proof. It is clear that the "stronger" relation and incompatibility work as required for a complete embedding. We have to show that given $q \in \mathbb{Q}$, there is some $\pi(q) =: p \in \mathbb{Q}_B$ such that any $p' \in \mathbb{Q}_B$ with $p' \leq p$ is compatible with q in $\mathbb{Q}$.

Let $\pi\colon \mathbb{Q} \to \mathbb{Q}_B$ be the projection mapping each $q \in \mathbb{Q}$ to $\pi(q) := q{\restriction}_{(\mathrm{supp}(p)\cap B)}$. Let $p := \pi(q)$ and fix an arbitrary $p' \in \mathbb{Q}_B$ stronger than p. Let $p^* := p{\restriction}_{A\setminus B}$ and apply Lemma B4.13 to p' and p^* (keeping in mind that their frames are necessarily compatible, in case that is relevant) to get an $r \in \mathbb{Q}$ stronger than p' and q. □

B5 Bigness

One key concept for many of the following proofs is the fact that by our construction, creatures at height L are much, much bigger than creatures at height L^- and much, much smaller than creatures at height L^+.[27] The exact nature of this size difference is encoded in the sequence n^B_L. While this concept is referred to as *completeness* in the older [GS93], we will be using the modern and more standard terminology of *bigness* from [FGKS17], while unifying the different concepts and generalising them even further in Definition B5.4.

Definition B5.1. Fix positive integers c and d.

(i) We say a non-empty set C and a norm

$$\|\cdot\|\colon (\mathcal{P}(C)\setminus\{\varnothing\}) \to \mathbb{R}_{\geq 0}$$

on the subsets of C are *c-big* (synonymously, *have c-bigness*) if the following holds: For each non-empty $X \subseteq C$ and each colouring $\chi\colon X \to c$ of X, there is a non-empty $Y \subseteq X$ such that $\chi{\restriction}_Y$ is constant and $\|Y\| \geq \|X\| - 1$. Equivalently, $(C, \|\cdot\|)$ is c-big if for each non-empty $X \subseteq C$ and each partition $X = X_1 \cup X_2 \cup \ldots \cup X_c$, there is some $i \in \{1, 2, \ldots, c\}$ such that $\|X_i\| \geq \|X\| - 1$.

(ii) We say $(C, \|\cdot\|)$ is *(c,d)-big* (synonymously, *has (c,d)-bigness*) if the following holds: For each non-empty $X \subseteq C$ and each colouring $\chi\colon X \to c$ of X, there is a non-empty $Y \subseteq X$ such that $|\operatorname{ran}\chi{\restriction}_Y| \leq d$ and $\|Y\| \geq \|X\| - 1$. Equivalently, $(C, \|\cdot\|)$ is (c,d)-big if for each non-empty $X \subseteq C$ and each partition $X = X_1 \cup X_2 \cup \ldots \cup X_c$, there is some d-tuple $\{i_1, i_2, \ldots, i_d\} \subseteq \{1, 2, \ldots, c\}$ such that $\|X_{i_1} \cup X_{i_2} \cup \ldots \cup X_{i_d}\| \geq \|X\| - 1$.[28]

(iii) We say $(C, \|\cdot\|)$ is *strongly c-big* (synonymously, *has strong c-bigness*) if in the above, even $\|Y\| \geq \|X\| - 1/c$ (respectively $\|X_i\| \geq \|X\| - 1/c$) holds.

Since the colouring and the partition formulations of the properties above are evidently equivalent, we will use whichever is more suited for that particular proof.

Fact B5.2. A few simple facts about bigness:

- If $(C, \|\cdot\|)$ has (strong) c-bigness, it also has (strong) c'-bigness for any $c' \leq c$.

[27] Recall that L^- and L^+ denote the predecessor and successor of a height L, respectively.

[28] This partition formulation of (c,d)-bigness is precisely the definition of (c,d)-completeness from [GS93, Definition 2.2].

- A simple example of a norm with c-bigness is $\log_c |\cdot|$.
- Modifying the norm to be ${}^{\log_c |\cdot|}/_c$ gives us strong c-bigness.
- An example of a (c,d)-big norm is $\log_{c/d} |\cdot|$.

The first fact can be generalised as follows:

Lemma B5.3. *If ${}^c/_d \leq b$ and $\|\cdot\|$ is b-big, then $\|\cdot\|$ is also (c,d)-big.*

Proof. Let $X = X_1 \cup X_2 \cup \ldots \cup X_c$; we have to find a d-tuple $\{i_1, i_2, \ldots, i_d\} \subseteq \{1, 2, \ldots, c\}$ such that $\|X_{i_1} \cup X_{i_2} \cup \ldots \cup X_{i_d}\| \geq \|X\| - 1$. Since ${}^c/_d \leq b$, we have $c \leq d \cdot b$. We regroup the partition of X as

$$X = (X_1 \cup X_2 \cup \ldots \cup X_b) \cup (X_{b+1} \cup \ldots \cup X_{2b}) \cup \ldots \cup (X_{(d-1)b+1} \cup \ldots \cup X_{db}),$$

where $X_k := \varnothing$ for $c < k \leq d \cdot b$. Define $Y_j := \bigcup_{1 \leq i \leq d} X_{ib+j}$ for $1 \leq j \leq b$. Then

$$X = Y_1 \cup Y_2 \cup \ldots \cup Y_b$$

and, since $\|\cdot\|$ is b-big, there is some j_0 such that $\|Y_{j_0}\| \geq \|X\| - 1$. Then $\{j_0, b+j_0, 2b+j_0, 3b+j_0, \ldots, (d-1)b+j_0\}$ is the d-tuple we had to provide. Possibly, some of these indices are not even necessary – namely if they point to empty X_k; in that case, pick arbitrary replacement indices pointing towards actually existing sets. □

Definition B5.4. We extend the definition of (c,d)-bigness and strong c-bigness in the following way:

(i) We say $(C, \|\cdot\|)$ is *e-strongly c-big* (synonymously, *has e-strong c-bigness*) if $X_i \subseteq X$ is even such that $\|X_i\| \geq \|X\| - {}^1/_e$.

(ii) We say $(C, \|\cdot\|)$ is *e-strongly (c,d)-big* (synonymously, *has e-strong (c,d)-bigness*) if the d-tuple is even such that $\|X_{i_1} \cup X_{i_2} \cup \ldots \cup X_{i_d}\| \geq \|X\| - {}^1/_e$.

(We omit the equivalent colouring formulations of the same definitions.)

Observation B5.5. Dividing a (c,d)-big or c-big norm by e yields an e-strongly (c,d)-big or e-strongly b-big norm, respectively.

Note that if ${}^c/_d \leq b$ and $\|\cdot\|$ has e-strong b-bigness, using the same method as in the preceding lemma gives us a d-tuple such that $\|X_{i_1} \cup X_{i_2} \cup \ldots \cup X_{i_d}\| \geq \|X\| - {}^1/_e$ and hence even e-strong (c,d)-bigness.

We have defined the norms of the various forcing factors in such a way that they have n^B_L-bigness at height L:

Theorem B5.6. *Recall the definitions of the norms in Definition B3.4 (for $\mathbb{Q}_{\mathrm{nn}}$), Definition B3.5 (for $\mathbb{Q}_{\mathrm{cn}}$), Definition B3.3 (for $\mathbb{Q}_{\mathrm{slalom}}$), Definition B3.8 (for $\mathbb{Q}_{\mathrm{ct},\,\kappa_{\mathrm{ct}}}$) and Definition B3.10 (for $\mathbb{Q}_{\mathrm{nm},\,\kappa_{\mathrm{nm}}}$).*

(i) For each $\mathrm{t} \in \mathrm{types}_{\,\mathrm{modular}}$ and each $L \in \mathrm{heights}_{\,\mathrm{tg(t)}}$, $(\mathrm{POSS}_{\mathrm{t},L}, \|\cdot\|_{\mathrm{t},L})$ has n^B_L-bigness. For $\mathrm{t} \in \{\mathrm{nn}, \mathrm{cn}\}$, we even have strong n^B_L-bigness. Letting $L := 4k+2$, for $\xi \in \mathrm{types}_{\,\mathrm{slalom}}$, we even have $n^P_{<4k+2}$-strong $g_\xi(k)$-bigness at height $4k+2$.

(ii) *Given a condition* $p(\mathrm{ct}) \in \mathbb{Q}_{\mathrm{ct},\,\kappa_{\mathrm{ct}}}$*, for each segment* $\bar{K} := \langle K_1, \dots, K_m \rangle$ *of its frame,* $(p(\mathrm{ct}, \bar{K}), \|\cdot\|_{\mathrm{ct},K_1})$ *has* $n^B_{K_1}$*-bigness.*
(iii) *For each* $L \in \mathrm{heights}_{\,\mathrm{nm}}$*,* $(\mathrm{POSS}_{\mathrm{nm},L}, \|\cdot\|^{\mathrm{cell}}_L)$ *has strong* n^B_L*-bigness.*

Proof. For $\mathrm{t} = \xi \in \mathrm{types}_{\,\mathrm{slalom}}$, the norm is the exemplary norm with $g_\xi((L-2)/4)$-bigness from Fact B5.2 divided by $n^P_{\leq 4k+2}$, so by Observation B5.5, we have $n^P_{\leq 4k+2}$-strong $g_\xi(k)$-bigness, and since $g_\xi((L-2)/4) \geq n^B_L$, we also have n^B_L-bigness.

For $\mathrm{t} = \mathrm{nn}$, let $X \subseteq \mathrm{POSS}_{\mathrm{nn},L}$ and fix a partition $X = X_1 \cup X_2 \cup \dots \cup X_{n^B_L}$. Consider $\|X_i\|^{\mathrm{intersect}}_L$ and let r be the maximal such intersect norm (letting i^* be some such index with $\|X_{i^*}\|^{\mathrm{intersect}}_L = r$); hence $\|X_i\|^{\mathrm{intersect}}_L \leq r$ for all $i \in n^B_L$, witnessed by sets Y_i. Then $Y := \bigcup Y_i$ witnesses that $\|X\|^{\mathrm{intersect}}_L \leq n^B_L \cdot r$; hence

$$\|X\|_{\mathrm{nn},L} \leq \frac{\log(n^B_L \cdot r)}{n^B_L \log n^B_L} \leq \frac{\log \max_{i<n^B_L} \|X_i\|^{\mathrm{intersect}}_L}{n^B_L \log n^B_L} + \frac{1}{n^B_L} = \|X_{i^*}\|_{\mathrm{nn},L} + \frac{1}{n^B_L}$$

and hence i^* is an index such that $\|X_{i^*}\|_{\mathrm{nn},L} \geq \|X\|_{\mathrm{nn},L} - 1/n^B_L$.

For $\mathrm{t} = \mathrm{cn}$, we first remark that

$$\|M\|_{\mathrm{cn},L} = \frac{\log |M|}{r \cdot n^B_L \log 3n^B_L} - s$$

for some positive r, s only depending on L. Given $X \subseteq \mathrm{POSS}_{\mathrm{cn},L}$ and a colouring $c\colon X \to n^B_L$, there is some c-homogeneous $Y \subseteq X$ with $|Y| \geq |X|/n^B_L$ and hence

$$\|Y\|_{\mathrm{cn},L} \geq \frac{\log |X| - \log n^B_L}{r \cdot n^B_L \log 3n^B_L} - s \geq \|X\|_{\mathrm{cn},L} - \frac{\log n^B_L}{r \cdot n^B_L \log 3n^B_L} \geq \|X\|_{\mathrm{cn},L} - \frac{1}{n^B_L}.$$

For ct, the claim is a direct translation of [FGKS17, Lemma 2.3.6 (6)], since (letting $K_1 =: 4k+3$) modesty ensures that $p(\mathrm{ct}, \bar{K})$ will contain at most a $(k-1)$-tuple of creature segments which are non-trivial (which correspond to the Sacks columns of [FGKS17]), all of which have a norm of at least k.[29]

For nm, the cell norm is exactly the exemplary norm with strong n^B_L-bigness from Fact B5.2. □

We remark that the n^B_L thus precisely describe the (strong) bigness properties at height L.

Corollary B5.7. *Let* $p \in \mathbb{Q}$*,* $\alpha \in \mathrm{supp}(p) \setminus A_{\mathrm{ct}}$ *and let* $L \in \mathrm{heights}_{\,\mathrm{tg}(\alpha)}$ *be a relevant height. Then for each colouring* $c\colon p(\alpha, L) \to n^B_L$*, there is a* c*-homogeneous* $q(\alpha, L) \subseteq p(\alpha, L)$ *such that* $\|q(\alpha, L)\|_{\mathrm{tg}(\alpha),L} \geq \|p(\alpha, L)\|_{\mathrm{tg}(\alpha),L} - 1$ *for* $\mathrm{tg}(\alpha) \neq \mathrm{nm}$ *and* $\|q(\alpha, L)\|^{\mathrm{cell}}_L \geq \|p(\alpha, L)\|^{\mathrm{cell}}_L - 1$ *for* $\mathrm{tg}(\alpha) = \mathrm{nm}$*.*

The same holds for $\mathrm{supp}(p) \cap A_{\mathrm{ct}}$*: Let* $\bar{L} = \langle L_1, \dots, L_m \rangle$ *be a segment of the frame of* $p(\mathrm{ct})$ *such that* $p(\mathrm{ct}, \bar{L})$ *is non-trivial. Then for each colouring* $c\colon p(\mathrm{ct}, \bar{L}) \to n^B_{L_1}$*, there is a* c*-homogeneous* $q(\mathrm{ct}, \bar{L}) \subseteq p(\alpha, \bar{L})$ *such that*

$$\|q(\alpha, \bar{L})\|_{\mathrm{ct},L_1} \geq \|p(\alpha, \bar{L})\|_{\mathrm{ct},L_1} - 1$$

for all $\alpha \in \mathrm{supp}(p, \mathrm{ct}, \bar{L})$*.*

[29] The demand imposed on their norms in the definition of modesty is only necessary to be able to apply the cited technical lemma without modifications.

Note that using the fact that n^B_L is big with respect to $n^P_{<L}$ and $n^R_{<L}$, this can be iterated downwards. (We will not use the following consideration directly, but a similar one will come up later on.) First note that a colouring $c\colon \mathrm{poss}(p, \leq L) \to n^R_{<L}$ can be reinterpreted as a colouring $d\colon p(\alpha, L) \to (n^R_{<L})^{\mathrm{poss}(p,<L)}$. Since $(n^R_{<L})^{\mathrm{poss}(p,<L)} \leq (n^R_{<L})^{n^P_L} \leq n^B_L$, we can use the preceding corollary to make the colouring independent of the possibilities at height L. (For ct, keep in mind we have to treat tuples of creature segments as units.) Iterating this downwards allows, for instance, the following:

- Given a colouring $c\colon \mathrm{poss}(p, <L) \to n^R_{<L'}$ for some relevant heights $L' < L$, we can strengthen $p(\alpha_K, K)$ to $q(\alpha_K, K)$ for all $L' \leq K < L$, decreasing the corresponding norms by at most 1, such that the colouring c restricted to $\mathrm{poss}(q, <L)$ only depends on $\mathrm{poss}(q, <L')$. (The number of colours here limits how far we can iterate this downwards.)
- In particular, if $c\colon \mathrm{poss}(p, <L) \to 2$ for some relevant height L, we can find $q \leq p$ such that $\mathrm{poss}(q, <L)$ is c-homogeneous.

Finally, we will require one similar specific consequence of strong bigness:

Lemma B5.8. *Let H be a finite subset of* $\mathrm{heights} \setminus \mathrm{heights}_{\mathrm{ct}}$ *and for each $L \in H$, assume we are given some type $\mathrm{t}_L \in \mathrm{tg}(L)$ and some $C_L \subseteq \mathrm{POSS}_{\mathrm{t}_L, L}$. Let K be the minimum of H and $F\colon \prod_{L\in H} C_L \to n^B_K$. Then there are $D_L \subseteq C_L$, with the norm of D_L decreasing by at most $1/n^B_L$ when compared to C_L, such that the value of F is constant on $\prod_{L\in H} D_L$.*

Proof. The case $|H| = 1$ is trivial, so assume $|H| \geq 2$ and let M be the maximum of H. We construct D_L by downwards induction on $L \in H$. Then F can be written as a function from C_M to $(n^B_K)^P$, where $P := \prod_{L\in H, L\neq M} C_L$. Since $(n^B_K)^{|P|} \leq (n^B_K)^{n^P_{<M}} \leq n^B_M$, we can use strong n^B_M-bigness to find D_L.

Continue the downwards induction with $H' := H \setminus \{M\}$. □

B6 Continuous and Rapid Reading

We now prove the main properties required to show that $\mathbb{Q}$ is proper and ω^ω-bounding.

Definition B6.1. Let $p \in \mathbb{Q}$ and let $\dot\tau$ be a $\mathbb{Q}$-name for an ordinal. We say that p *decides $\dot\tau$ below the height L* if $p \wedge \eta$ decides $\dot\tau$ for each $\eta \in \mathrm{poss}(p, <L)$; in other words, there is a function $T\colon \mathrm{poss}(p, <L) \to \mathrm{Ord}$ with $p \wedge \eta \Vdash \dot\tau = T(\eta)$ for each $\eta \in \mathrm{poss}(p, <L)$.

We say that p *essentially decides $\dot\tau$* if there is some height L such that p decides $\dot\tau$ below L. Let $\dot r$ be a $\mathbb{Q}$-name for a countable sequence of ordinals. We say that p *continuously reads $\dot r$* if p essentially decides each $\dot r(n)$.

Let $\dot s$ be a $\mathbb{Q}$-name for an element of 2^ω. We say that p *rapidly reads $\dot s$* if for each $L \in \mathrm{heights}$, $\dot s{\restriction}_{n^R_{<L}}$ is decided below L.

For $B \subseteq A$, we say that p continuously reads $\dot{r}$ *only using indices in* B if p continuously reads $\dot{r}$ and the value of $T(\eta)$ depends only on $\eta{\restriction_B}$. Analogously, we say that p continuously reads $\dot{r}$ *not using indices in* B if p continuously reads $\dot{r}$ only using indices in $A \setminus B$. (The same terminology will be used for "rapidly" instead of "continuously".)

Observation B6.2. The name "continuous reading" comes from the following consideration: For a fixed condition p, the possibilities form an infinite tree T_p; the set of branches $[T_p]$ carries a natural topology. A condition p continuously reads some $\dot{r}$ iff there is a function $f\colon T_p \to \mathrm{Ord}^{<\omega}$ in the ground model such that for the natural (continuous) extension $F\colon [T_p] \to \mathrm{Ord}^{\omega}$ of f, $p \Vdash \dot{r} = F(\dot{y})$, where $\dot{y}$ is the generic branch in $[T_p]$. In our case, the tree is finitely splitting and hence T_p is compact, so continuity and uniform continuity coincide.

Rapid reading then is equivalent to a kind of Lipschitz continuity. We remark that the $n^R_{<L}$ describe "how rapidly" p reads $\dot{s}$, i. e. they can be interpreted as corresponding to the Lipschitz constants.

Lemma B6.3. *If p continuously (or rapidly) reads $\dot{r}$ and $q \leq p$, then q continuously (or rapidly) reads $\dot{r}$. (The same holds if we add "only using indices in B" or "not using indices in B".)*

Proof. This follows immediately from Lemma B4.17. □

Lemma B6.4. *If $q \leq^* p$ and p essentially decides $\dot{\tau}$, then q also essentially decides $\dot{\tau}$.*

Proof. Since $q \leq^* p$, the frame of q must be coarser than the frame of p (because if not, then we could strengthen the frame of q in a way incompatible with the frame of p and get $r \leq q$ incompatible with p). p forces that $\dot{\tau}$ is decided below some height L; let $L^* \geq L$ be the minimum of the first segment of the frame of $q(\mathrm{ct})$ which is entirely above L. Clearly, p also forces that $\dot{\tau}$ is decided below L^*; so for each $\eta \in \mathrm{poss}(p, {<}L^*)$, we have $p \wedge \eta \Vdash \dot{\tau} = t$ for some $t \in \mathrm{Ord}$.

Since $q \leq^* p$ and since L^* is the minimum of segments in the frames of both p and q (which ensures that the possibilities of $p(\mathrm{ct})$ and $q(\mathrm{ct})$ below L^* have the same length), it is clear that $\mathrm{poss}(q{\restriction_{\mathrm{supp}(p)}}, {<}L^*) \subseteq \mathrm{poss}(p, {<}L^*)$ (because if not, then there would be an $r \leq q$ incompatible with p). Let $\vartheta \in \mathrm{poss}(q, {<}L^*)$. There is a unique $\eta \in \mathrm{poss}(q{\restriction_{\mathrm{supp}(p)}}, {<}L^*) \subseteq \mathrm{poss}(p, {<}L^*)$ such that $\vartheta = \eta{\restriction_{\mathrm{supp}(p)}}$. By $q \wedge \vartheta \leq^* p \wedge \eta$ it follows that $q \wedge \vartheta \Vdash \dot{\tau} = t$ must also hold. □

Lemma B6.5. *In the ground model, let $\kappa := \max(\aleph_0, |B|)^{\aleph_0}$ for some $B \subseteq A$. Then in the extension, there are at most κ many reals which are read continuously only using indices in B; more formally, letting G be a $\mathbb{Q}$-generic filter, there are at most κ many reals r such that there is a $p \in G$ and a name $\dot{s}$ such that p continuously reads $\dot{s}$ only using indices in B and such that $\dot{s}[G] = r$.*

Proof. The argument is a variation of the usual "nice names" consideration. Given p continuously reading some $\dot{s}$, we can define the canonical name $\dot{s}'$ continuously

read by $p' := p{\restriction}_B$ such that p forces $\dot{s} = \dot{s}'$. (We can do this by the following procedure: Let L_n be the height such that $\dot{s}(n)$ is decided below L_n. For each $\eta \in \mathrm{poss}(p, {<}L_n)$, we have $p \wedge \eta \Vdash \dot{s}(n) = x^\eta_n$ for some x^η_n. Define $\dot{s}'(n)$ as the name containing all pairs $\langle \check{x}^\eta_n, p \wedge \eta \rangle$.)

Hence it suffices to prove that there are at most κ many names of reals continuously read in this manner. There are at most κ many countable subsets of B and hence at most κ many conditions p' with $\mathrm{supp}(p') \subseteq B$, because

- there are countably many heights,
- for each such height $L \in \mathrm{heights}_{\mathrm{tg}}$, we have at most countably many indices in $B \cap A_{\mathrm{tg}}$, and
- for each such index α, we have to choose one of finitely many creatures (very often: singletons) to be $p'(\alpha, L)$.

Given any such p', there are only $2^{\aleph_0}$ many possible ways to continuously read a real $\dot{s}'$ with respect to p' (by picking the decision heights L_n and the values x^η_n for each of finitely many $\eta \in \mathrm{poss}(p', {<}L_n)$). □

We will now first prove that given a condition continuously reading some $\dot{r} \in 2^\omega$, we can find a stronger condition rapidly reading $\dot{r}$, and only afterwards prove that we can densely find conditions continuously reading any $\dot{\tau} \in \mathrm{Ord}^\omega$. (This sequence of proofs, the same as in [FGKS17], makes for an easier presentation.)

Theorem B6.6. *Given p continuously reading $\dot{r} \in 2^\omega$, there is a $q \leq p$ rapidly reading $\dot{r}$. (The same is true if we add "only using indices in B".)*

Proof. For each height L, we define:

$$K_{\mathrm{dec}}(L) \text{ is the maximal height such that } \dot{r}{\restriction}_{n^R_{<K_{\mathrm{dec}}(L)}} \text{ is decided below } L \text{ by } p. \quad (*_1)$$

The function K_{dec} is non-decreasing, and continuous reading already implies that K_{dec} is unbounded. (If it were bounded by K, that would mean that for any $K' \geq K$, $\dot{r}(n^B_{K'})$ were not essentially decided by p.) K_{dec} can, however, grow quite slowly. (p rapidly reading $\dot{r}$ translates to $K_{\mathrm{dec}}(L) \geq L$ for all L.)

For all heights $K \leq L$ we define

$$\dot{x}^L_K := \dot{r}{\restriction}_{n^R_{<\min(K, K_{\mathrm{dec}}(L))}} \text{ (which is, by definition, decided below } L).$$

There are at most $2^{n^R_{<K}}$ many possible values for $\dot{x}^L_K$, since $n^R_{<\min(K,K_{\mathrm{dec}}(L))} \leq n^R_{<K}$.

In the following, we will only consider relevant heights. Recall that relevant heights are those that are either in $\mathrm{heights}_{\mathrm{ct}}$ and the minimum of a segment of the frame of $p(\mathrm{ct})$, or are in $\mathrm{heights}_{\mathrm{tg}}$ for some $\mathrm{tg} \neq \mathrm{ct}$ and are such that there is an $\alpha_L \in \mathrm{supp}(p) \cap A_{\mathrm{tg}}$ with a non-trivial $p(\alpha_L, L)$. For a relevant height $L \notin \mathrm{heights}_{\mathrm{ct}}$, we will use α_L to refer to the corresponding index.

Step 1: Fix a relevant L. We will choose, by downwards induction on all relevant $L' \leq L$, objects $C^L_{L'}$ (which will be either creatures $C^L_{L'} \subseteq p(\alpha_{L'}, L')$ or tuples of creature segments $C^L_{L'} \subseteq p(\mathrm{ct}, \mathrm{segm}(L'))$) and functions $\psi^L_{L'}$.

Step 1a: To start the induction, for $L' = L$ we set $C^L_L := p(\alpha_L, L)$ respectively $C^L_L := p(\mathrm{ct}, \mathrm{segm}(L))$. We let ψ^L_L be the function with domain $\mathrm{poss}(p, {<}L)$ assigning to each $\eta \in \mathrm{poss}(p, {<}L)$ the corresponding value of $\dot{x}^L_L$. (This means that $p \wedge \eta \Vdash \dot{x}^L_L = \psi^L_L(\eta)$ for each $\eta \in \mathrm{poss}(p, {<}L)$.)

Step 1b: We continue the induction on L' and write $C' := C^L_{L'}$, $\psi' := \psi^L_{L'}$, $\dot{x}' := \dot{x}^L_{L'}$ for short.

Our plan is as follows:

- We will pick a creature C' stronger than $p(\alpha_{L'}, L')$ respectively a tuple of creature segments C' stronger than $p(\mathrm{ct}, L')$ such that the corresponding norm decreases by at most 1.
- ψ' will be a function with domain $\mathrm{poss}(p, {<}L')$ such that

  $$\text{modulo } \langle C^L_K \mid L' \leq K < L\rangle, \text{ each } \eta \in \mathrm{poss}(p, {<}L') \text{ decides } \dot{x}' \text{ to be } \psi'(\eta),$$

  or, put differently, that $p \wedge \eta$ forces $\dot{x}' = \psi'(\eta)$ if the generic $\dot{y}$ is compatible with C^L_K for all non-trivial heights K with $L' \leq K < L$.[30]

We will define C', ψ' as follows: Let L'' be the smallest relevant height above L'. By induction, we already have that $\psi'' := \psi^L_{L''}$ is a function with domain $\mathrm{poss}(p, {<}L'')$ such that modulo $\langle C^L_K \mid L'' \leq K < L\rangle$, each $\eta \in \mathrm{poss}(p, {<}L'')$ decides $\dot{x}'' := \dot{x}^L_{L''}$ to be $\psi''(\eta)$.

Let $\psi''_*(\eta)$ be the restriction of $\psi''(\eta)$ to $n^R_{<\min(L', K^{\mathrm{dec}}(L))}$. This means that ψ''_* maps each $\eta \in \mathrm{poss}(p, {<}L'')$ to a restriction of $\dot{x}''$ – a potential value for $\dot{x}'$.

We can refactor ψ''_* as a function $\psi''_* : X \times Y \to Z$, where $X := \mathrm{poss}(p, {<}L')$, $Y := p(\alpha_{L'}, L')$ respectively $Y := p(\mathrm{ct}, \mathrm{segm}(L'))$ and Z is the set of possible values of $\dot{x}'$, which has at most size $2^{n^R_{L'}}$. This implicitly defines a function from Y to Z^X; with $|Z^X| \leq 2^{n^P_{L'} \cdot n^R_{L'}}$, we can by Corollary B5.7 use bigness at height L' to find $C' \subseteq p(\alpha_{L'}, L')$ respectively $C' \subseteq p(\mathrm{ct}, \mathrm{segm}(L'))$ (with the norm decreasing by at most 1) such that ψ''_* does not depend on height L'. From this, we get a natural definition of ψ'.

Step 2: We perform a downwards induction as in step 1 (always in the original p) from each relevant height L, thus defining for each relevant $K < L$ the creatures/tuples of creature segments C^L_K and a function ψ^L_K fulfilling

$$\text{modulo } \langle C^L_{K'} \mid K \leq K' < L\rangle, \text{ each } \eta \in \mathrm{poss}(p, {<}K) \text{ decides } \dot{x}^L_K \text{ to be } \psi^L_K(\eta). \quad (*_2)$$

The corresponding norms of these creatures/tuples of creature segments decrease by at most 1.

Step 3: For a given K, there are only finitely many possibilities for both C^L_K and ψ^L_K. So by König's Lemma there necessarily exists a sequence $\langle C^*_K, \psi^*_K \mid K \text{ relevant}\rangle$ such that

$$\text{for each } L, \text{ there is } L^* > L \text{ such that for all } K \leq L,\ \langle C^{L^*}_K, \psi^{L^*}_K\rangle = \langle C^*_K, \psi^*_K\rangle. \quad (*_3)$$

[30] We could also introduce a term referring to "p, but replacing all $p(\alpha_K, K)$ by C^L_K" here, but for notational simplicity, we eschew this.

(These $\langle C^*_K, \psi^*_K \mid K \text{ relevant}\rangle$ thus form an infinite branch in the tree of all $\langle C^L_K, \psi^L_K\rangle$.)

Step 4: To define q, we replace all creatures and tuples of creature segments of p by $C^*_K \subseteq p(\alpha_K, K)$ respectively $C^*_K \subseteq p(\mathrm{ct}, \mathrm{segm}(K))$. Thus q has the same support as p, the same trunk, the same frame and the same halving parameters, and all corresponding norms decrease by at most 1, hence q actually is a condition. We now claim that q rapidly reads $\dot r$, i. e. we claim that each $\eta \in \mathrm{poss}(q, {<}K)$ decides $\dot r{\restriction}_{n^R_{<K}}$.

Step 5: To show this, we fix K and pick a $K' > K$ such that $K_{\mathrm{dec}}(K') \geq K$. According to its definition Eq. ($*_1$), this means that $\dot r{\restriction}_{n^R_{<K}}$ is decided below K'. Now pick $L^* > K'$ per Eq. ($*_3$) and note that per Eq. ($*_2$), $\dot x^{L^*}_K$ is decided below K by each $\eta \in \mathrm{poss}(p, {<}K)$ to be $\psi^{L^*}_K(\eta)$, modulo $\langle C^{L^*}_{K''} \mid K \leq K'' < L^*\rangle$. Since $K_{\mathrm{dec}}(K') \geq K$ and $L^* \geq K'$ (from which $K_{\mathrm{dec}}(L^*) \geq K_{\mathrm{dec}}(K')$ follows), we have $\min(K^{\mathrm{dec}}(L^*), K) = K$ and hence $\dot x^{L^*}_K = \dot r{\restriction}_{n^R_{<K}}$. As we had $K_{\mathrm{dec}}(K') \geq K$, $\dot x^{L^*}_K$ is already decided below K' by the original condition p. Hence, in "modulo $\langle C^{L^*}_{K''} \mid K \leq K'' < L^*\rangle$", we can actually disregard any $K'' > K'$.

However, by Eq. ($*_3$) we know that q has as its creatures and tuples of creature segments $C^{L^*}_L = C^*_L$ for all relevant $L < K'$. Hence q forces that the generic $\dot y$ be compatible with $C^{L^*}_L$ for all non-trivial $K \leq L < K'$. From that, we immediately have that $\psi^{L^*}_K = \psi^*_K$ correctly computes $\dot x^{L^*}_K = \dot r{\restriction}_{n^R_{<K}}$ modulo q, and hence q decides $\dot r{\restriction}_{n^R_{<K}}$ below K. As Step 5 holds for any K, q rapidly reads $\dot r$. □

B7 Unhalving and More Continuous Reading

This section will contain proofs constructing a fusion sequence of conditions in $\mathbb{Q}$. While the lemmata and theorems could be formulated more generally, this would not give any additional insight, as they are only of a technical character. Since the structure of the possibilities in the ct factor is a bit unpleasant to work with, we will anchor these fusion constructions at the easiest possible nm levels of a condition, which are those which lie exactly between the maximal height of one segment in the frame of the ct factor of the condition and the minimal height of the frame segments immediately succeeding it.

Definition B7.1. Given a condition $p \in Q$, we call a lim inf level $4k$ (respectively $4k+1 \in \mathrm{heights}_{*\mathrm{n}}$ respectively $4k+2 \in \mathrm{heights}_{\mathrm{slalom}}$) *$p$-agreeable* if the heights $4k-1$ and $4k+3$ in $\mathrm{heights}_{\mathrm{ct}}$ are such that $4k-1 = \max(\mathrm{segm}(4k-1))$ and $4k+3 = \min(\mathrm{segm}(4k+3))$.

Restricting our constructions to use these heights as the stepping stones makes the possibilities easier to think about.

This section will also be the only time we actually use the halving parameters, in the form of the following operation on conditions:

Definition B7.2. Given a condition $q \in \mathbb{Q}$ and $4h < \omega$, define $r := \mathrm{half}(q, \geq 4h)$ as the condition obtained by replacing the halving parameters $d(q)(4k)$ of q by

$$\begin{aligned} d(r)(4k) &:= d^*(q)(4k) \\ &:= d(q)(4k) + \frac{\min\{\|q(\alpha, 4k)\|^{\mathrm{stack}}_{4k} \mid \alpha \in \mathrm{supp}(q, \mathrm{nm}, 4k)\} - d(q)(4k)}{2} \end{aligned}$$

for all $4k \geq 4h$.

It is clear that for $r := \mathrm{half}(q, \geq 4h)$, the compound creature $r(\mathrm{nm}, 4k)$ is identical to $q(\mathrm{nm}, 4k)$ for each $4k < 4h$ and that for $4k \geq 4h$, the norm of the compound creature $r(\mathrm{nm}, 4k)$ has decreased by exactly $1/n^P_{<(4k,0)}$ compared to the norm of $q(\mathrm{nm}, 4k)$ (respectively, has remained 0 in case $4h \leq 4k < \mathrm{trklgth}(q)$).

The point of this is the following: Given $q \in \mathbb{Q}$ with relatively large nm norms and $r \leq \mathrm{half}(q, \geq 4h)$ such that some nm norms of r are rather small, we can find an "unhalved" version s of r such that $s \leq q$, s has relatively large nm norms and $s =^* r$. We will use this unhalving operation in the first part of the proof of continuous reading.

Lemma B7.3. *Fix $M \in \mathbb{R}$ and $h < \omega$. Given $q \in \mathbb{Q}$ such that $\|q(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} \geq M$ for all $4k \geq 4h$ as well as $r \leq \mathrm{half}(q, \geq 4h)$ such that $\mathrm{trklgth}(r) = 4h$ and $\|r(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} > 0$ for all $4k \geq 4h$, there are $s \in \mathbb{Q}$ and $h^* > h$ such that*

- *(i)* $s \leq q$,
- *(ii)* $\mathrm{trklgth}(s) = 4h$,
- *(iii)* $\|s(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} \geq M$ *for all* $4k \geq 4h^*$,
- *(iv)* s *is identical to* r *above* $(4h^*, 0)$, *which means:* $s(\alpha, L) = r(\alpha, L)$ *for each sensible choice of* $\alpha \in \mathrm{supp}(r) = \mathrm{supp}(s)$ *and* $L \in \mathrm{heights}$ *(and their halving parameters and frames are identical above* $(4h^*, 0)$*),*
- *(v)* $\|s(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} \geq M - 1/n^P_{<(4k,0)} \geq M - 1/n^P_{<(4h,0)}$ *for all* $4h \leq 4k < 4h^*$, *and*
- *(vi)* $\mathrm{poss}(s, {<}(4h^*, 0)) = \mathrm{poss}(r, {<}(4h^*, 0))$.

Taken together, (iv) and (vi) imply $s =^* r$ and hence by Lemma B6.4, if r essentially decides some $\dot{\tau}$, then so does s.

Proof. Let $h^\dagger \geq h$ such that $\|r(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} > M$ for all $4k \geq 4h^\dagger$. Set $h^* := h^\dagger + 1$. Define s to be identical to r except for the fact that for all $4h \leq 4k < 4h^*$, we replace

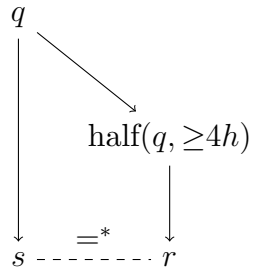

Figure 9: The construction in Lemma B7.3.

the halving parameters $d(r)(4k)$ by $d(q)(4k)$. (This means that for $4h \leq 4k < 4h^*$ we have $d(s)(4k) = d(q)(4k)$.)

It is clear that (i)–(iv) and (vi) are true; it remains to show that (v) holds. Fix k such that $4h \leq 4k < 4h^*$; we have to show that

$$\begin{aligned}\|s(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} &= \frac{\log_2\left(\min\{\|s(\alpha, 4k)\|^{\mathrm{stack}}_{4k} \mid \alpha \in \mathrm{supp}(s, \mathrm{nm}, 4k)\} - d(s)(4k)\right)}{n^P_{<(4k,0)}} \\ &\overset{!}{\geq} M - \frac{1}{n^P_{<(4k,0)}}.\end{aligned}$$

Recall the definition of d^* in the preceding definition; as $d^*(q)(4k)$ were the halving parameters of $\mathrm{half}(q, \geq 4h)$ and $r \leq \mathrm{half}(q, \geq 4h)$, we know that $d(r)(4k) \geq d^*(q)(4k)$.

Since we assumed $\|r(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} > 0$, we know that

$$\begin{aligned}0 &< \frac{\log_2\left(\min\{\|r(\alpha, 4k)\|^{\mathrm{stack}}_{4k} \mid \alpha \in \mathrm{supp}(r, \mathrm{nm}, 4k)\} - d(r)(4k)\right)}{n^P_{<(4k,0)}} \\ &= \frac{\log_2\left(\min\{\|s(\alpha, 4k)\|^{\mathrm{stack}}_{4k} \mid \alpha \in \mathrm{supp}(s, \mathrm{nm}, 4k)\} - d(r)(4k)\right)}{n^P_{<(4k,0)}}\end{aligned}$$

Fixing any $\beta \in \mathrm{supp}(s, \mathrm{nm}, 4k) = \mathrm{supp}(r, \mathrm{nm}, 4k)$, this shows

$$0 < \log_2\left(\|s(\beta, 4k)\|^{\mathrm{stack}}_{4k} - d(r)(4k)\right)$$

and thus

$$\begin{aligned}\|s(\beta, 4k)\|^{\mathrm{stack}}_{4k} &> d(r)(4k) \geq d^*(q)(4k) \\ &= d(q)(4k) + \frac{\min\{\|q(\alpha, 4k)\|^{\mathrm{stack}}_{4k} \mid \alpha \in \mathrm{supp}(q, \mathrm{nm}, 4k)\} - d(q)(4k)}{2}\end{aligned}$$

Hence (recalling $d(q)(4k) = d(s)(4k)$)

$$\|s(\beta, 4k)\|^{\mathrm{stack}}_{4k} - d(s)(4k) \geq \frac{\min\{\|q(\alpha, 4k)\|^{\mathrm{stack}}_{4k} \mid \alpha \in \mathrm{supp}(q, \mathrm{nm}, 4k)\} - d(q)(4k)}{2}$$

for any $\alpha \in \mathrm{supp}(s, \mathrm{nm}, 4k) = \mathrm{supp}(r, \mathrm{nm}, 4k)$. Taking $\log_2$ and then dividing by $n^P_{<(4k,0)}$ yields

$$\frac{\log_2\left(\|s(\beta, 4k)\|^{\mathrm{stack}}_{4k} - d(s)(4k)\right)}{n^P_{<(4k,0)}} \geq \|q(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} - \frac{1}{n^P_{<(4k,0)}}$$

and consequently (since this holds for any β)

$$\|s(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} \geq \|q(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} - \frac{1}{n^P_{<(4k,0)}} \geq M - \frac{1}{n^P_{<(4k,0)}},$$

proving (v). □

To prove that we can densely find conditions continuously reading a given name, we will first prove the following auxiliary lemma.

Lemma B7.4. *Let $\dot\tau$ be an arbitrary $\mathbb{Q}$-name and let $p^* \in \mathbb{Q}$ and $\ell^* < \omega$ and $M^* \geq 1$ be such that $4\ell^*$ is p^*-agreeable and $\|p^*(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} \geq M^* + 1$ holds for all $4k \geq 4\ell^*$. Then there is a condition q such that:*

- *$q \leq p^*$,*
- *q essentially decides $\dot\tau$,*
- *below $(4\ell^*, 0)$, q and p^* are identical on $\mathrm{supp}(p^*)$, and any $\alpha \in \mathrm{supp}(q) \setminus \mathrm{supp}(p^*)$ only enter the support above $(4\ell^*, 0)$ (as a consequence, $4\ell^*$ also is q-agreeable), and*
- *$\|q(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} \geq M^*$ for all $4k \geq 4\ell^*$.*

Proof. The proof consists of three parts.

Part 1: finding intermediate deciding conditions by applying the unhalving lemma

Suppose we are given $p \in \mathbb{Q}$, $\ell < \omega$ and $M \geq 1$ such that 4ℓ is p-agreeable and $\|p(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} \geq M + 1$ for all $4k \geq 4\ell$. We construct an extension $r(p, 4\ell, M)$ of p with certain properties:

First, enumerate $\mathrm{poss}(p, {<}(4\ell, 0))$ as $(\eta^1, \ldots, \eta^m)$ and note that $m \leq n^P_{<(4\ell,0)}$. Setting $p^0 := q^0 := p$, we now inductively construct conditions $p^1 \geq \ldots \geq p^m$ and auxiliary conditions $\tilde q^1, q^1, \ldots, \tilde q^m, q^m$ such that for each $n < m$, the following properties hold:

(1) $\tilde q^{n+1}$ is derived from p^n by replacing everything below $(4\ell, 0)$ (in $\mathrm{supp}(p)$) with η^{n+1}.
   - By (3) below, we will have $\mathrm{trklgth}(\tilde q^{n+1}) = 4\ell$.
   - For $n = 0$, $\tilde q^1$ is just $p^0 \wedge \eta^1$; but for $n \geq 1$, η^{n+1} will not actually be in $\mathrm{poss}(p^n, {<}4\ell)$, so we cannot formally use that notation.
   - Note that in general, $\mathrm{supp}(p^n)$ will be larger than $\mathrm{supp}(p)$, so we do not replace *everything* below 4ℓ with η^{n+1}, but only the part that is in $\mathrm{supp}(p)$.
   - We could also derive $\tilde q^{n+1}$ from q^n, since p^n and q^n only differ on the part being replaced to get $\tilde q^{n+1}$, anyway.

(2) $q^{n+1} \leq \tilde q^{n+1}$. (Note that, obviously, $q^{n+1} \not\leq q^n$, since their trunks are different and the conditions are hence incompatible.)

(3) $\mathrm{trklgth}(q^{n+1}) = 4\ell$. (This means that by strengthening $\tilde q^{n+1}$ to q^{n+1}, we do not increase the trunk lengths.)

(4) $\|q^{n+1}(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} \geq M + 1 - {}^{n+1}/_{n^P_{<(4\ell,0)}}$ for all $4k \geq 4\ell$.

(5) One of the following two cases holds:
   - q^{n+1} essentially decides $\dot\tau$.
   - $q^{n+1} = \mathrm{half}(\tilde q^{n+1}, {\geq}4\ell)$

   More explicitly: If the "decision" case is possible under the side conditions (2)–(4), then we use it (i. e. strengthen the condition to decide). If not, only then do we halve – and thereby certainly satisfy (2)–(4).

(6) We define p^{n+1} as follows: Below $(4\ell, 0)$, p^{n+1} is identical to p on $\mathrm{supp}(p)$; above (including) $(4\ell, 0)$ as well as outside $\mathrm{supp}(p)$, p^{n+1} is identical to q^{n+1}. In detail:
- For all $\alpha \in \mathrm{supp}(p)$, $p^{n+1}(\alpha, L) := p(\alpha, L)$ for all sensible $L < (4\ell, 0)$.
- For all $\alpha \in \mathrm{supp}(p)$, $p^{n+1}(\alpha, L) := q^{n+1}(\alpha, L)$ for all $L \geq (4\ell, 0)$.
- For all $\beta \in \mathrm{supp}(q) \setminus \mathrm{supp}(p)$, $p^{n+1}(\beta) := q^{n+1}(\beta)$.

(Note that as we required $\mathrm{trklgth}(q^{n+1})$ to remain 4ℓ, any newly added indices β can only start having non-trivial creatures starting with height $(4\ell, 0)$ by modesty.)

(7) $p^{n+1} \leq p^n$, so the $\langle p^n \mid n \in m+1 \rangle$ are a descending sequence of conditions.

Ultimately, we define $r(p, 4\ell, M) := p^m$ (the last of the p^n constructed above). $r := r(p, 4\ell, M)$ fulfils $r \leq p$ and $\|r(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} \geq M$ for all $4k \geq 4\ell$. As r differs from p only above $(4\ell, 0)$, it is also clear that 4ℓ is r-agreeable.

Furthermore, r has the following important decision property:

If $\eta \in \mathrm{poss}(r, {<}(4\ell, 0))$ and if there is an $s \leq r \wedge \eta$ such that s essentially decides $\dot\tau$, $\mathrm{trklgth}(s) = 4\ell$ and $\|s(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} > 0$ for all $4k \geq 4\ell$, then $r \wedge \eta$ already essentially decides $\dot\tau$. $(*_4)$

To prove Eq. $(*_4)$, note the following: η canonically corresponds to some $\eta^{n+1} \in \mathrm{poss}(p, {<}(4\ell, 0))$, therefore $s \leq r \wedge \eta \leq q^{n+1} \leq \tilde q^{n+1}$. We thus only have to show that q^{n+1} was constructed using the "decision" case. Assume, towards an indirect proof, that this was not the case; so q^{n+1} came about by halving $\tilde q^{n+1}$. Since s is stronger than $\mathrm{half}(\tilde q^{n+1}, {\geq}4\ell)$, we can use Lemma B7.3 and unhalve s to obtain some $s' \leq \tilde q^{n+1}$ with large norm such that $s' =^* s$. This means we could have used the "decision" case after all, which finishes this step of the proof.

Part 2: iterating the intermediate conditions to define q

Given p^*, ℓ^* and M^* as in the lemma's statement, we inductively construct conditions p_n and accompanying $\ell_n < \omega$ for each $n \geq 0$. Let $p_0 := p^*$ and $\ell_0 := \ell^*$. Given p_n and ℓ_n such that $4\ell_n$ is p_n-agreeable, define p_{n+1} and ℓ_{n+1} as follows:

- Choose $\ell_{n+1} > \ell_n$ such that:
  - $4\ell_{n+1}$ is p_n-agreeable,
  - $\|p_n(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} \geq M^* + n + 1$ for all $4k \geq 4\ell_{n+1}$, and
  - for each $\alpha \in \mathrm{supp}(p_n, (4\ell_n, 0)) \setminus A_{\mathrm{nm}}$ of type t, there is a height L with $(\ell_n, J_{4\ell_n} - 1) < L < (4\ell_{n+1}, 0)$ such that $\|p_n(\alpha, L)\|_{\mathrm{t},L} \geq M^* + n + 1$.
- Set $p_{n+1} := r(p_n, 4\ell_{n+1}, M^* + n + 1)$. (By the construction of r in the previous part, it follows that $4\ell_{n+1}$ then also is p_{n+1}-agreeable.)

Thus $\langle p_n \mid n < \omega \rangle$ is a descending sequence of conditions, which converges to a condition $q \in \mathbb{Q}$. To verify that q is indeed a condition, note the following: By construction, we have $\|q(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} \geq M^* + n$ for all $4k \geq 4\ell_{n+1}$. For all other types t and all indices $\alpha \in \mathrm{supp}(q) \cap A_{\mathrm{t}}$, we have assured the existence of a subsequence of creatures of strictly increasing norms of $q(\alpha)$, since below any $(4\ell_{n+1}, 0)$, q is equal to p_{n+1} (and also to p_n). Thus, q is indeed a condition. Clearly, $q \leq p^*$ also holds.

In the next and final part, we will show that q essentially decides $\dot{\tau}$ (proving the lemma). The following property will be central to the proof:

If $\eta \in \mathrm{poss}(q, {<}4(\ell_m, 0))$ for some m and if there is an $r \leq q \wedge \eta$ such that r essentially decides $\dot{\tau}$, $\mathrm{trklgth}(r) = 4\ell_m$ and $\|r(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} > 0$ for all $4k \geq 4\ell_m$, then $q \wedge \eta$ already essentially decides $\dot{\tau}$. $(*_5)$

To prove Eq. $(*_5)$, note that η canonically corresponds to some η^{n+1} which was already considered as a possible trunk when constructing the intermediate condition $p_m := r(p_{m-1}, 4\ell_m, M^* + m)$, so we can use Eq. $(*_4)$ to conclude Eq. $(*_5)$.

Part 3: using bigness to thin out q and prove its essential decision property

The final part of the proof is essentially a rerun of the proof of Theorem B6.6. This is the main reason we proved rapid reading before continuous reading, as the idea of the proof is easier to digest in the rather simpler Theorem B6.6, in our opinion. The difference is that this time, we do not homogenise with respect to the potential values for some names, but instead with respect to whether $q \wedge \eta$ essentially decides $\dot{\tau}$ or not.

Step 1: Fix a relevant height $L > (4\ell_0, 0)$. We will choose, by downwards induction on all relevant L' with $(4\ell_0, 0) \leq L' \leq L$, objects $C^L_{L'}$ (again, either creatures $C^L_{L'} \subseteq q(\alpha_{L'}, L')$ or tuples of creature segments $C^L_{L'} \subseteq q(\mathrm{ct}, \mathrm{segm}(L'))$) and subsets of possibilities $B^L_{L'}$.

Step 1a: To start the induction, for $L' = L$ we set $C^L_L := q(\alpha_L, L)$ respectively $C^L_L := q(\mathrm{ct}, \mathrm{segm}(L))$. We let B^L_L be the set of all $\eta \in \mathrm{poss}(q, {<}L)$ such that $q \wedge \eta$ essentially decides $\dot{\tau}$.

Step 1b: We continue the induction downwards on the relevant heights L' with $(4\ell_0, 0) \leq L' < L$. We construct $C^L_{L'}$ and $B^L_{L'}$ such that the following holds:

- $C^L_{L'}$ is a strengthening of $q(\alpha_{L'}, L')$ respectively $q(\mathrm{ct}, \mathrm{segm}(L'))$ such that the corresponding norm decreases by at most 1.
- $B^L_{L'}$ is a subset of $\mathrm{poss}(q, {<}L')$ such that for each $\eta \in B^L_{L'}$ and each $x \in C^L_{L'}$, we have $\eta^\frown x \in B^L_{L'^+}$, and analogously for each $\eta \in \mathrm{poss}(q, {<}L') \setminus B^L_{L'}$ and each $x \in C^L_{L'}$, we have $\eta^\frown x \notin B^L_{L'^+}$. (We will call this property "homogeneity".) Since we only concern ourselves with relevant heights, $B_{L'^+}$ might not be explicitly defined by this process – if not, just take the smallest relevant height L'' above L' and cut off the elements of $B^L_{L''}$ at height L'^+ to get $B^L_{L'^+}$.

Just as in the case of the proof of rapid reading in Theorem B6.6, we can find such objects using bigness:

- Define L'' to be the smallest relevant height above L'.
- By induction, there is a function F mapping each $\eta \in \mathrm{poss}(q, {<}L'')$ to $\{\in B^L_{L''}, \notin B^L_{L''}\}$.
- We thin out $q(\alpha_{L'}, L')$ to $C^L_{L'}$, decreasing the norm by at most 1, such that for each $\nu \in \mathrm{poss}(q, {<}L')$, each extension of ν compatible with $C^L_{L'}$ has the same F-value $F^*(\nu)$.
- This in turn defines $B^L_{L'}$.

Step 2: We perform a downwards induction as in step 1 (always in the original q) from each relevant height L above $(4\ell_0, 0)$. Given a relevant height K such that $(4\ell_0, 0) \leq K < L$ and $\eta \in \mathrm{poss}(q, {<}K)$, and given that $q \wedge \eta$ essentially decides $\dot\tau$ and that $\eta' \in \mathrm{poss}(q, {<}L)$ extends η, it is clear that $q \wedge \eta'$ also essentially decides $\dot\tau$. We thus have

$$\text{if } q \wedge \eta \text{ essentially decides } \dot\tau \text{ for } \eta \in \mathrm{poss}(q, {<}K), \text{ then } \forall\, L > K : \eta \in B^L_K. \tag{$*_6$}$$

Step 3: We now show the converse, namely:

$$\begin{aligned}&\text{Whenever } \eta \in B^L_{L'} \text{ for some relevant height } L \text{ with } L' = (4\ell_m, 0) \leq L \\ &\quad\text{(for some } m\text{), then } q \wedge \eta \text{ essentially decides } \dot\tau.\end{aligned} \tag{$*_7$}$$

To prove Eq. ($*_7$), derive a condition r from q by using η as the trunk and replacing creatures respectively tuples of creature segments at relevant heights K (with $L' \leq K \leq L$) with C^L_K. Now, since all $\eta' \in \mathrm{poss}(r, {<}L) \subseteq \mathrm{poss}(q, {<}L)$ are in B^L_L, all $q \wedge \eta' \geq^* r \wedge \eta'$ essentially decide $\dot\tau$, and consequently, so does r. Noting that $\|r(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} > 0$ for all $4k \geq 4\ell_m$, we can use Eq. ($*_5$) to get that $q \wedge \eta$ essentially decides $\dot\tau$.

Hence, to show that q essentially decides $\dot\tau$, by Eq. ($*_7$) it suffices to show that for all $\eta \in \mathrm{poss}(q, {<}(4\ell_0, 0))$ there is a height L such that $\eta \in B^L_{(4\ell_0,0)}$.

Step 4: As in Theorem B6.6, we choose an "infinite branch" $\langle C^*_K, B^*_K \mid K \text{ relevant}\rangle$. (Recall that this means that for each height L_0, there is some $L > L_0$ such that, for all $K \leq L_0$, $(C^L_K, B^L_K) = (C^*_K, B^*_K)$.) By replacing the creatures and tuples of creature segments of q at relevant heights K with C^*_K, we obtain a condition q^*.

Step 5: To show that q essentially decides $\dot\tau$, we thus have to show (as noted in Step 3) that $\eta \in B^*_{(4\ell_0,0)}$ for all $\eta \in \mathrm{poss}(q, {<}(4\ell_0, 0)) = \mathrm{poss}(q^*, {<}(4\ell_0, 0))$.

Fix any such η. Find an $r \leq q^* \wedge \eta$ deciding $\dot\tau$. Without loss of generality, for some m, $\mathrm{trklgth}(r) = 4\ell_m$ and $\|r(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} > 0$ for all $4k \geq 4\ell_m$. Let $\eta' > \eta$ be the trunk of r restricted to $\mathrm{supp}(q, (4\ell_m, 0))$, which ensures $\eta' \in \mathrm{poss}(q, {<}(4\ell_m, 0))$ and $r \leq q \wedge \eta'$. By Eq. ($*_5$), $q \wedge \eta'$ already essentially decides $\dot\tau$.

Now pick some relevant $L > (4\ell_m, 0)$ such that $(C^L_K, B^L_K) = (C^*_K, B^*_K)$ for all relevant $K \leq (4\ell_m, 0)$. According to Eq. ($*_6$), $\eta' \in B^*_K$ and by homogeneity $\eta \in B^*_{(4\ell_0,0)}$ (since η' is an extension of η). Hence by Eq. ($*_7$), $q \wedge \eta$ also essentially decides $\dot\tau$, which completes the proof. □

We can now use this lemma to prove continuous reading.

Theorem B7.5. *Let $\dot r$ be a $\mathbb{Q}$-name for an element of Ord^ω in V and $p \in \mathbb{Q}$. Then there is a $q \leq p$ continuously reading $\dot r$.*

Proof. We will iteratively construct conditions p_n in a similar way as in Part 2 of Lemma B7.4. Given $p^*, \ell^*, M^*, \dot\tau$ as in Lemma B7.4, we will denote the condition resulting from the application of that lemma by $s(p^*, \ell^*, M^*, \dot\tau)$.

Set $p_{-1} := p$. Let $p_0 := s(p_{-1}, \ell_{-1}, 1, \dot r(0))$, where ℓ_{-1} is the minimal ℓ such that

- 4ℓ is p_{-1}-agreeable,
- p_0 decides $\dot{r}(0)$ below $(4\ell, 0)$ (i.e., if L is the minimal height such that p_0 decides $\dot{r}(0)$ below L, then choose ℓ minimal such that $L \leq (4\ell, 0)$),
- $4\ell \geq \mathrm{trklgth}(p_{-1})$, and
- $\|p_{-1}(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} \geq 2$ for all $4k \geq 4\ell$.

Given p_n and ℓ_{n-1} such that $4\ell_{n-1}$ is p_n-agreeable, p_n essentially decides $\dot{r}{\restriction}_{\{0,\dots,n\}}$ and $\|p_n(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} \geq n+1$ for all $4k \geq 4\ell_{n-1}$ (which is evidently true for $n = 0$), we define p_{n+1} and ℓ_n as follows:

- Let $\ell_n > \ell_{n-1}$ be the minimal ℓ such that:
  - $4\ell_n$ is p_n-agreeable,
  - p_n decides $\dot{r}(n)$ (or, equivalently, $\dot{r}{\restriction}_{\{0,\dots,n\}}$) below $(4\ell, 0)$,
  - $\|p_n(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} \geq n+2$ for all $4k \geq 4\ell$, and
  - for each $\alpha \in \mathrm{supp}(p_n, (4\ell_{n-1}, 0)) \setminus A_{\mathrm{nm}}$ of type t, there is a height L with $(\ell_{n-1}, J_{4\ell_{n-1}} - 1) < L < (4\ell_n, 0)$ such that $\|p_n(\alpha, L)\|_{\mathrm{t},L} \geq n$.
- Let $p_{n+1} := s(p_n, \ell_n, n+1, \dot{r}(k+1))$.

Lemma B7.4 ensures that $p_{n+1} \leq p_n$, that p_{n+1} essentially decides $\dot{r}(n+1)$ (and thus $\dot{r}{\restriction}_{\{0,\dots,n+1\}}$) and that it fulfils $\|p_{n+1}(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} \geq n+1$ for all $4k \geq 4\ell_n$.

Similar to Part 2 of Lemma B7.4, $\langle p_n \mid n < \omega\rangle$ is a descending sequence of conditions converging to a condition $q \in \mathbb{Q}$. By construction, q continuously reads $\dot{r}$. □

Theorem B7.5 and Theorem B6.6 taken together show that for any $p \in \mathbb{Q}$ and any $\mathbb{Q}$-name $\dot{r}$ for a real, there is a $q \leq p$ rapidly reading $\dot{r}$. Even more important are the following consequences of the previous two sections, which prove the first, easier parts of this chapter's main theorem, Theorem B1.1:

Lemma B7.6. *$\mathbb{Q}$ satisfies the finite version of Baumgartner's axiom A and hence is proper and ω^ω-bounding. Assuming* CH *in the ground model, $\mathbb{Q}$ moreover preserves all cardinals and cofinalities.*

Proof. Define the relations $\leq_n$ by $\leq_0 := \leq$ and (for $n \geq 1$) $q \leq_n p$ if there is some $\ell \geq n$ such that 4ℓ is p-agreeable, such that p and q are identical below $(4\ell, 0)$ on $\mathrm{supp}(p)$ and such that $\|q(\mathrm{nm}, 4k)\|_{\mathrm{nm},4k} > 0$ for all $4k \geq 4\ell$. It is clear that any sequence $p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 \dots$ has a limit; and by Lemma B7.4, for any $p \in \mathbb{Q}$, $n < \omega$ and $\dot{\tau}$ a $\mathbb{Q}$-name for an ordinal, there is a $q \leq_n p$ essentially deciding $\dot{\tau}$, which means that it forces $\dot{\tau}$ to have one of finitely many values.

Lemma B4.18 shows that $\mathbb{Q}$ preserves all cardinals and cofinalities $\geq \aleph_2$, and since it is proper, it also preserves $\aleph_1$. This proves the "moreover" part of Theorem B1.1. □

Lemma B7.7. *Assuming* CH *in the ground model, in the extension $\mathfrak{d} = \aleph_1$ and $\mathrm{cov}(\mathcal{N}) = \aleph_1$.*

Proof. Since $\mathbb{Q}$ is ω^ω-bounding, it forces $\mathfrak{d}$ to be $\aleph_1$. To prove the second part of the statement, we show that each new real is forced to be contained in a ground

model null set, so the $\aleph_1$ many Borel null sets of the ground model cover the reals (in other words, $\mathbb{Q}$ adds no random reals) and hence $\mathrm{cov}(\mathcal{N})$ is forced to be $\aleph_1$.

Let $\dot{r}$ be a $\mathbb{Q}$-name for a real and $p \in \mathbb{Q}$. Let $q \leq p$ read $\dot{r}$ rapidly, which means that for each $L \in \text{heights}$, $\dot{r}{\restriction}_{n^R_{<L}}$ is determined by $\eta \in \mathrm{poss}(q, {<}L)$; let X^q_L be the set of possible values of $\dot{r}{\restriction}_{n^R_{<L}}$. For notational simplicity, consider only heights ℓ of the form $(4k, 0), 4k+1, 4k+2, 4k+3$ and identify $(4k, 0)$ with $4k$. Then it follows that $|X^q_\ell| \leq n^P_{<\ell} < n^R_{<\ell} < {}^{2^{n^R_{<\ell}}}/\ell$, where the last inequality holds by our general requirement on the $n^R_{<\ell}$. This means that the relative size of X^q_ℓ is bounded by $1/\ell$ and hence $\langle X^q_\ell \mid \ell < \omega\rangle$ can be used to define the ground model null set $N_q := \{s \in 2^\omega \mid \forall\, \ell < \omega \colon s{\restriction}_{n^R_\ell} \in X^q_\ell\}$. By definition, $q \Vdash \dot{r} \in N_q$. □

This proves (M1) of Theorem B1.1.

Lemma B7.8. *In the extension,* $2^{\aleph_0} = \kappa_{\mathrm{ct}}$.

Proof. If $\alpha, \beta \in A_{\mathrm{ct}}$ are distinct, then the reals $\dot{y}_\alpha$ and $\dot{y}_\beta$ are forced to be different, hence there are at least κ_{ct} many reals in the extension. But every real in the extension is read continuously by Theorem B7.5, hence by Lemma B6.5 there are at most $\kappa_{\mathrm{ct}}^{\aleph_0} = \kappa_{\mathrm{ct}}$ many reals in the extension. □

This proves (M6) of Theorem B1.1. It remains to prove points (M2)–(M5) of Theorem B1.1, which we will do in the following sections.

B8 $\mathrm{cof}(\mathcal{N}) \leq \kappa_{\mathrm{cn}}$

To show $\mathrm{cof}(\mathcal{N}) \leq \kappa_{\mathrm{cn}}$, we prove that $\mathbb{Q}$ has the Laver property over the intermediate forcing poset

$$\mathbb{Q}_{\text{non-ct}} := \left(\prod_{\mathrm{t} \in \text{types}_{\text{modular}}} \mathbb{Q}_{\mathrm{t}}^{\kappa_{\mathrm{t}}} \right) \times \mathbb{Q}_{\mathrm{nm},\, \kappa_{\mathrm{nm}}}$$

(and hence also the Sacks property, since it is ω^ω-bounding). We will use the same equivalent formulation as in [FGKS17, Lemmas 6.3.1–2], namely, we will prove:

Lemma B8.1. *Given a condition $p \in \mathbb{Q}$, a name $\dot{r} \in 2^\omega$ and a function $g \colon \omega \to \omega$ in V. Then there is a $q \leq p$ and a name $\dot{T} \subseteq 2^{<\omega}$ for a leafless tree such that:*

- *q reads $\dot{T}$ continuously not using any indices in A_{ct},*
- *$q \Vdash \dot{r} \in [\dot{T}]$, and*
- *$\left|\dot{T}{\restriction}_{2^{g(n)}}\right| < n+1$ for all $n < \omega$.*

Proof. We first note that we can increase g without loss of generality, since if $g_1(n) \leq g_2(n)$ for all n and $\dot{T}$ witnesses the lemma for g_2, then the same $\dot{T}$ also witnesses the lemma for g_1.

We can also assume without loss of generality that p is modest and rapidly reads $\dot{r}$, i. e. $\mathrm{poss}(p, {<}L)$ determines $\dot{r}{\restriction}_{n^R_{<L}}$ for all heights L. Considering this, we can find a strictly increasing sequence of segment-initial heights L_n (i. e. $\min(\mathrm{segm}(L_n)) = L_n$) such that $g(n) = n^R_{<L_n}$ for all $n < \omega$ (increasing g when necessary).

Hence, each $\eta \in \mathrm{poss}(p, {<}L_n)$ defines a value $\dot{R}^n(\eta)$ for $\dot{r}{\restriction}_{g(n)}$. We split each η into two components, η_{ct} and η_{rmdr} (i. e. the non-ct remainder). If we fix the η_{ct} component of η, then $\dot{R}^n(\cdot, \eta_{\mathrm{ct}})$ is a name not depending on the ct component, i. e. not using any indices in A_{ct}. (More formally: Given an η_{rmdr} compatible with the generic filter such that $(\eta_{\mathrm{rmdr}}, \eta_{\mathrm{ct}}) = \eta \in \mathrm{poss}(p, {<}L_n)$, $\dot{R}^n(\eta_{\mathrm{rmdr}}, \eta_{\mathrm{ct}})$ evaluates to $\dot{R}^n(\eta)$.)

We will now construct a stronger condition q and an increasing sequence $\langle i_n \mid n < \omega\rangle$ of natural numbers with the following properties: Given some i_{n+1}, let $i_n < m \leq i_{n+1}$ and $\eta \in \mathrm{poss}(q, {<}L_{i_{n+1}})$. Such an η extends a unique η^m in the set of possibilities $\mathrm{poss}(q, {<}L_m)$ cut off at height L_m, which we call $\mathrm{poss}^\dagger(q, {<}L_m)$. Restricting this η^m to the ct component yields $\eta^m_{\mathrm{ct}} := \eta^m{\restriction}_{A_{\mathrm{ct}}}$.[31] Then $q \wedge \eta$ forces the name $R^m(\cdot, \eta^m_{\mathrm{ct}})$ to be evaluated to $\dot{r}{\restriction}_{g(m)}$, and hence q forces $\dot{r}{\restriction}_{g(m)}$ to be an element of

$$\dot{T}^m := \{R^m(\cdot, \eta^m_{\mathrm{ct}}) \mid \eta \in \mathrm{poss}(q, {<}L_{i_{n+1}})\},$$

which is a name not using any indices in A_{ct}. It thus suffices to show that there are few such η^m_{ct}, i. e. that letting $P_m := \{\eta^m_{\mathrm{ct}} \mid \eta \in \mathrm{poss}(q, {<}L_{i_{n+1}})\}$, for all $m < \omega$ we have $|P_m| < m + 1$.

The condition q will have the same support as p. On $\mathrm{supp}(p) \setminus A_{\mathrm{ct}}$, we define q to be equal to p. Hence we now only have to define q on $\mathrm{supp}(p) \cap A_{\mathrm{ct}}$. We will inductively construct the sequence $\langle i_n\rangle$ and the new condition $q(\mathrm{ct})$ below L_{i_n}, and show that $|P_m| < m + 1$ holds for all $m \leq i_n$. To begin the induction, let $i_0 = 0$ and let $q(\mathrm{ct})$ below L_0 be identical to some arbitrary possibility in $\mathrm{poss}(p(\mathrm{ct}), {<}L_0)$, giving us $|P_{i_0}| = 1$.

By way of induction hypothesis, assume we already have i_n, q is defined up to L_{i_n} and $|P_m| < m + 1$ holds for all $m \leq i_n$. (By our choice of $i_0 = 0$, all this is fulfilled for $n = 0$.) Keep in mind that each L_i is the initial height in a segment of the frame of $p(\mathrm{ct})$.

Step 1: Let $\Sigma := \mathrm{supp}(p, \mathrm{ct}, L_{i_n}) \cap A_{\mathrm{ct}}$ and let m be such that $L_{i_n} = 4m + 3$. (Note that hence $|\Sigma| < m$, though this is not important to this proof.) Let c be minimal such that $\mathrm{nor}^{n^B_{L_{i_n}}, m}_{\mathrm{Sacks}}(2^c) = n$. Let $i' := (i_n + 1) \cdot 2^{c\cdot|\Sigma|}$. For each $\alpha \in \Sigma$, find $L^\alpha > L_{i'}$ (with $L^\alpha \neq L^\beta$ for $\alpha \neq \beta$) such that $\mathrm{nor}^{n^B_{L_{i_n}}, m}_{\mathrm{Sacks}}(p(\alpha, L^\alpha)) \geq n$. Finally, let $i_{n+1} > i'$ be minimal such that $L^\alpha < L_{i_{n+1}}$ for all $\alpha \in \Sigma$.

Step 2: We define $q(\mathrm{ct})$ from L_{i_n} up to (but excluding) $L_{i_{n+1}}$ as follows: For each

[31] Note that these η^m_{ct} are *not* possibilities in $\mathrm{poss}(q, \mathrm{ct}, {<}L_m)$, however, since such possibilities would actually go up to height $\max(\mathrm{segm}(L_{m-1}))$ instead of ending at height L_m. (They are, however, still possibilities in the "moral" sense, i. e. they are initial segments of the generic real $\dot{y}$.) Please excuse this minor abuse of notation. It makes sense when one considers that in the frame of p, L_m *used* to be a segment-initial height, even if it no longer is in the frame of q.

$\alpha \in \Sigma$, we take $p(\alpha, L^\alpha)$ and shrink it such that $\mathrm{nor}^{n^B_{L_{i_n}}, m}_{\mathrm{Sacks}}(q(\alpha, L^\alpha))$. For all other heights $L \in \mathrm{heights}_{\mathrm{ct}}$ with $L_{i_n} \leq L < L_{i_{n+1}}$, replace $p(\alpha, L)$ with an arbitrary singleton to get $q(\alpha, L)$. In particular, this means that for $L_{i_n} \leq L \leq L_{i'}$, $p(\alpha, \mathrm{ct}, L)$ is a singleton for each $\alpha \in \Sigma$.

For the frame of $q(\mathrm{ct})$, take the segments in the frame of $p(\mathrm{ct})$ starting at (the segment starting with) L_{i_n} and going up to, and including, the (segment ending with the) $\mathrm{heights}_{\mathrm{ct}}$-predecessor L' of $L_{i_{n+1}}$; merge all of them to form a single segment in the frame of $q(\mathrm{ct})$.

Step 3: For those indices α in $\mathrm{supp}(p, \mathrm{ct}, L')$ which are outside of Σ (i. e. those which enter the support of $p(\mathrm{ct})$ strictly above L_{i_n} and up to L'), also choose arbitrary singletons to get a trivial $q(\alpha, \mathrm{ct}, \langle L_{i_n}, \ldots, L' \rangle)$. (Such indices will be in the support of $q(\mathrm{ct})$ from $L_{i_{n+1}}$ onwards.)

Step 4: We now just have to prove that $|P_m|$ is sufficiently small up to (and including) i_{n+1}. First, let $i_n < m < i'$; for such m, we did not add any possibilities to q (as all new creature segments consist of singletons up to that height), so $|P_m| = |P_{i_n}| < i_n < m$. Now consider $i' \leq m \leq i_{n+1}$. For each $\alpha \in \Sigma$, the number of possibilities in $q(\alpha, \mathrm{ct}, \langle L_{i_n}, \ldots, L' \rangle)$ is exactly 2^c. By the induction hypothesis we already know that $|P_{i_n}| < i_n + 1$, and due to the choice of i', we altogether have

$$|P_m| \leq |P_{i_n}| \cdot 2^{c \cdot |\Sigma|} < (i_n + 1) \cdot 2^{c \cdot |\Sigma|} = i' < m + 1$$

and we are done with the induction. □

Having proved this, we now know that $\mathbb{Q}$ has the Sacks property over the intermediate forcing poset $\mathbb{Q}_{\text{non-ct}}$. By [BJ95, Theorem 2.3.12] (later restated as Theorem B10.3 in section B10, where we will use it a bit more extensively), this is equivalent to the fact that any null set in the model obtained by forcing with the entire $\mathbb{Q}$ is contained in a null set of the model obtained by forcing with $\mathbb{Q}_{\text{non-ct}}$, and hence we have shown that $\mathbb{Q} \Vdash \mathrm{cof}(\mathcal{N}) \leq \kappa_{\mathrm{cn}}$ by Lemma B6.5.

We will show $\mathbb{Q} \Vdash \mathrm{cof}(\mathcal{N}) \geq \kappa_{\mathrm{cn}}$ a bit later.

B9 $\mathrm{non}(\mathcal{M}) = \kappa_{\mathrm{nm}}$

The following proof does not use any specifics of the creatures and possibilities; it only requires that $\mathbb{Q}_{\mathrm{nm}, \kappa_{\mathrm{nm}}}$ is the only part of the forcing poset involving a lim inf construction.

Lemma B9.1. *The set of all reals that can be read continuously only using indices in A_{nm} is not meagre.*

Proof. Let $\dot{M}$ be a $\mathbb{Q}$-name for a meagre set. We can find $\mathbb{Q}$-names of nowhere dense trees $\dot{T}_n \subseteq 2^{<\omega}$ such that $\dot{M} = \bigcup_{n<\omega} \dot{T}_n$ is forced. We will show that there is a $\mathbb{Q}$-name for a real $\dot{r}$ which is continuously read only using indices in A_{nm} such that $\dot{r} \notin \dot{M}$; hence, the set of all such reals cannot be meagre.

First note that since $\mathbb{Q}$ is ω^ω-bounding and all $\dot{T}_n$ are nowhere dense, for each $n < \omega$, there is a ground model function $f_n \colon \omega \to \omega$ such that the following holds: For each $\rho \in 2^x$, there is a $\rho' \in 2^{f_n(x)}$ such that $\rho \subseteq \rho'$ and such that $\rho' \notin \dot{T}_n$ is forced.

We fix some $p \in \mathbb{Q}$ forcing the previously mentioned properties of $\dot{M}$ and $\langle \dot{T}_n \rangle$ and continuously reading all $\dot{T}_n$ (which is possible per Theorem B7.5). We will construct (in the ground model) $q \leq p$ and a real $\dot{r}$ continuously read by q only using indices in A_{nm} such that $q \Vdash \dot{r} \notin \dot{M}$.

We will define q inductively as the limit of a fusion sequence q_i. Assume we have already defined q in the form of a condition q_i up to some q_i-agreeable $4k_i$, and that we have an $x_i < \omega$ and a $\mathbb{Q}$-name $\dot{z}_i$ for an element of 2^{x_i} such that $\dot{z}_i$ is decided by $\mathrm{poss}(q_i{\restriction}_{A_{\mathrm{nm}}}, {<}(4k_i, 0))$. (The real $\dot{r}$ will be defined as the union of the $\dot{z}_i$.) Finally, assume that q_i already forces $\dot{z}_i \notin \dot{T}_0 \cup \dot{T}_1 \cup \ldots \cup \dot{T}_{i-1}$. The idea is to now extend $\dot{z}_i$ to a longer name $\dot{z}_{i+1}$ which is forced by q_{i+1} to avoid $\dot{T}_i$, as well.

To that end, enumerate $\mathrm{poss}(q_i, {<}(4k_i, 0))$ as $(\eta^0, \eta^1, \ldots, \eta^{m-1})$. Set $4k_i^0 := 4k_i$, $x_i^0 := x_i$, $\dot{z}_i^0 := \dot{z}_i$. By induction on j, $0 \leq j < m$, we deal with η^j: Assume we are given a name $\dot{z}_i^j$ for an element of 2^{x^j} that is decided by $\mathrm{poss}(q_i, {<}(4k^j, 0))$, and that we have already constructed a condition q_i^j such that

- $q_i^j \leq q_i$,
- q_i^j is identical to q_i below $4k_i$,
- $4k_i^j$ is q_i^j-agreeable, and
- between $4k_i^0 = 4k_i$ and $4k_i^j$, all creatures in $q_i^j{\restriction}_{A \setminus A_{\mathrm{nm}}}$ are singletons.

Let $x_i^{j+1} := f_n(x_i^j)$ and choose some q_i^j-agreeable $4k_i^{j+1}$ above $4k_i^j$ which is big enough for p to determine $\dot{X} := \dot{T}_i{\restriction}_{x_i^{j+1}}$, i. e. there is some function F from $\mathrm{poss}(p, {<}(4k_i^{j+1}, 0))$ to possible values of $\dot{X}$ (a consequence of continuous reading). We now define q_i^{j+1} as follows: Below $4k_i^j$ and above $4k_i^{j+1}$, q_i^{j+1} is identical to q_i^j. Between $4k_i^j$ and $4k_i^{j+1}$, we just leave $q_i^j(\mathrm{nm})$ as it is; and on $A \setminus A_{\mathrm{nm}}$, we just choose arbitrary singletons within q_i^j to get q_i^{j+1}.

We now briefly consider the name $\dot{X}$: A possibility $\nu \in \mathrm{poss}(p, {<}(4k_i^{j+1}, 0))$ consists of

- U, the part below $4k_i^j$,
- V, the part in $A \setminus A_{\mathrm{nm}}$ between $4k_i^j$ and $4k_i^{j+1}$, and
- W, the part in A_{nm} between $4k_i^j$ and $4k_i^{j+1}$.

So we can write $\dot{X} = F(U, V, W)$. Under the assumption that the generic follows η^j ($U = \eta^j$) and the singleton values of q_i^{j+1} on $A \setminus A_{\mathrm{nm}}$ ($V = V^j$ for some fixed V^j), then there is a name $\dot{X}' = F(\eta^j, V^j, \cdot)$ depending only on indices in A_{nm} such that $q_i^{j+1} \Vdash \dot{X} = \dot{X}'$.

Recall that $\dot{z}_i^j$ is already determined by the nm-part of η^j and that already p forces that there is some extension $z' \in 2^{x_i^{j+1}}$ of that value of $\dot{z}_i^j$ such that $z' \notin \dot{X}'$. By picking (in the ground model) for each possible choice of W some $z'(W) \in 2^{x_i^{j+1}} \setminus F(\eta_j, V^j, W)$ extending $\dot{z}_i^j$, we can define the name $\dot{z}_i^{j+1} := z'(\cdot)$ which depends only on indices in A_{nm} and is determined below $4k_i^{j+1}$. By construction, we have

that $q_i^{j+1} \Vdash \dot{z}_i^j \subseteq \dot{z}_i^{j+1}$ and $q_i^{j+1} \wedge \eta^j \Vdash \dot{z}_i^{j+1} \notin \dot{T}_i$.

Repeating this construction for all j, $0 \leq j < m$, finally define $\dot{z}_{i+1} := \dot{z}_i^m$ and $x_{i+1} := x_i^m$ and let $4k_{i+1}$ be such that

- $4k_{i+1}$ is above $4k_i^m$,
- $4k_{i+1}$ is q_i^m-agreeable, and
- for all $\alpha \in \mathrm{supp}(q_i^m, (4k_i^m, 0)) \setminus A_{\mathrm{nm}}$ of type t, there is a height $(4k_i^m, 0) < L < (4k_{i+1}, 0)$ such that $\|p(\alpha, L)\|_{\mathrm{t},L} \geq i$.

Define q_{i+1} to be equal to q_i^m below $4k_i^m$ and equal to p above $4k_i^m$. By our choice of $4k_{i+1}$, we have ensured that the lim sup part of the fusion condition $q := \bigcap_{i<\omega} q_i$ will actually be a condition (the lim inf part trivially is). By the construction of q_{i+1}, we have ensured that q_{i+1} forces that $\dot{r} := \bigcup_{i<\omega} \dot{z}_i$ avoids $\dot{T}_0 \cup \dot{T}_1 \cup \ldots \cup \dot{T}_i$, and hence q forces that $\dot{r}$ avoids $\dot{M}$. Finally, by the construction of the $\dot{z}_i$, they are continuously read by q only using indices in A_{nm}, and so is their union $\dot{r}$. □

Corollary B9.2. $\mathbb{Q}$ *forces* $\mathrm{non}(\mathcal{M}) \leq \kappa_{\mathrm{nm}}$.

Proof. By Lemma B6.5, the non-meagre set from Lemma B9.1 has size at most κ_{nm}, and hence we have $\mathbb{Q} \Vdash \mathrm{non}(\mathcal{M}) \leq \kappa_{\mathrm{nm}}$. □

To prove $\mathrm{non}(\mathcal{M}) \geq \kappa_{\mathrm{nm}}$, we first define some meagre sets in the extension. Recall that for $\alpha \in A_{\mathrm{nm}}$, the generic object $\dot{y}_\alpha$ is a $\mathrm{heights}_{\mathrm{nm}}$-sequence of objects in $\mathrm{POSS}_{\mathrm{nm},L} = 2^{I_L}$, or equivalently an ω-sequence of 0s and 1s. We define a name for a meagre set $\dot{M}_\alpha$ as follows: A real $r \in 2^\omega$ is in $\dot{M}_\alpha$ iff for all but finitely many $k < \omega$, there is an i_k such that $r{\restriction}_{I_{(4k,i_k)}} \neq \dot{y}_\alpha(4k, i_k)$, or equivalently

$$\dot{M}_\alpha := \bigcup_{n<\omega} \bigcap_{k \geq n} \{r \in 2^\omega \mid r{\restriction}_{I_{4k}} \neq \dot{y}_\alpha{\restriction}_{I_{4k}}\}$$

(abusing the notation by letting $I_{4k} := \bigcup_{i \in J_{4k}} I_{(4k,i)}$), whence it is clear that $\dot{M}_\alpha$ is indeed a meagre set.

By the choice of $n^R_{<L}$ for the nm case in Definition B4.8, if p rapidly reads $\dot{r}$, then for any $L \in \mathrm{heights}_{\mathrm{nm}}$, $\dot{r}{\restriction}_{I_L}$ is decided $\leq L$. Also note that if the cell norm $\|x\|^{\mathrm{cell}}_L$ of some creature x is at least 1, then it follows that $|x| > n^P_{<L}$.

Lemma B9.3. *Let $\dot{r} \in 2^\omega$ be a name for a real and let p rapidly read $\dot{r}$ not using the index $\alpha \in A_{\mathrm{nm}}$. Then $p \Vdash \dot{r} \in \dot{M}_\alpha$.*

Proof. We first remark that it suffices to prove that there is an $s \leq p$ such that $s \Vdash \dot{r} \in \dot{M}_\alpha$. Assume that we have shown this, and also assume that p does *not* force $\dot{r} \in \dot{M}_\alpha$; then there is a $q \leq p$ forcing the contrary, and q still rapidly reads $\dot{r}$ not using the index α. Since we can thus find an $s \leq q$ which *does* force $\dot{r} \in \dot{M}_\alpha$, we have arrived at the desired contradiction.

We only have to find $s \leq p$ forcing $\dot{r} \in \dot{M}_\alpha$. As a matter of fact, we will only have to modify p in very few places to arrive at the desired condition s. Without loss of generality, assume that $\alpha \in \mathrm{supp}(p)$. Recall that by the definition of $\mathbb{Q}_{\mathrm{nm},\,\kappa_{\mathrm{nm}}}$, there

is some k_1 such that for any $k \geq k_1$, $\|p(4k)\|_{\mathrm{nm},4k} \geq 1$; as a consequence, for each stacked creature $p(\alpha, 4k)$, there is at least one $i \in J_{4k}$ with $\|p(\alpha,(4k,i))\|^{\mathrm{cell}}_{(4k,i)} \geq 1$; for each $k \geq k_1$, we pick some such i_k.

Consider one of these $(4k, i_k) =: L$. We know that $\dot{r}\restriction_{I_L}$ is decided $\leq L$ by p – and actually even below L, since by modesty (ii), there can be at most one index β such that $p(L,\beta)$ is non-trivial, α already is such an index and p reading $\dot{r}$ does not depend on α. Since there are at most $n^P_{<L}$ many possibilities below L in p, there can be at most $n^P_{<L}$ many possible values for $\dot{r}\restriction_{I_L}$, and since $|p(\alpha, L)| > n^P_{<L}$, there must be some $x_k \in p(\alpha, L)$ different from all possible values of $\dot{r}\restriction_{I_L}$ under the reading by p.

We define the condition s by replacing each $p(\alpha,(4k,i_k))$ with the singleton $\{x_k\}$. It is clear that s is still a condition, as we have at most reduced each stacked creature's norm in $p(\alpha)$ by 1, which does not negatively affect the lim inf norm convergence. By definition, $s \Vdash \dot{r} \in \dot{M}_\alpha$ as required, since $\dot{r}\restriction_{I_{(4k,i_k)}}$ is different from $\dot{y}_\alpha$ for all $k \geq k_1$. □

Corollary B9.4. $\mathbb{Q}$ *forces* $\mathrm{non}(\mathcal{M}) \geq \kappa_{\mathrm{nm}}$.

Proof. Fix a condition p, some $\kappa < \kappa_{\mathrm{nm}}$ and a sequence of names of reals $\langle \dot{r}_i \mid i \in \kappa\rangle$. We find some $\alpha \in A_{\mathrm{nm}}$ such that $p \Vdash \{\dot{r}_i \mid i \in \kappa\} \subseteq \dot{M}_\alpha$.

For each $i \in \kappa$, fix a maximal antichain A_i below p such that each $a \in A_i$ rapidly reads $\dot{r}_i$. Recall that $\mathbb{Q}$ is $\aleph_2$-cc by Lemma B4.18. Since $\kappa_{\mathrm{nm}} > \kappa$ and without loss of generality $\kappa_{\mathrm{nm}} > \aleph_1$ (otherwise, there is nothing to prove), $S := \bigcup_{i\in\kappa}\bigcup_{a\in A_i} \mathrm{supp}(a)$ has size $\kappa < \kappa_{\mathrm{nm}}$ and we can find an index $\alpha \in A_{\mathrm{nm}} \setminus S$. Each $a \in A_i$ rapidly reads $\dot{r}_i$ not using the index α; so by the preceding lemma, for each i, each $a \in A_i$ forces $\dot{r}_i \in \dot{M}_\alpha$ and so does p (since A_i is predense below p), finishing the proof. □

This proves (M2) of Theorem B1.1.

B10 $\mathfrak{c}_{f_\xi,g_\xi} = \kappa_\xi$

Definition B10.1. Given $f, g \in \omega^\omega$ going to infinity such that $0 < g < f$, we call $S := \langle S_k \mid k < \omega\rangle \in ([\omega]^{<\omega})^\omega$ an (f,g)-*slalom* if $S_k \subseteq f(k)$ and $|S_k| \leq g(k)$ for all $k < \omega$, or in shorter notation, if $S \in \prod_{k<\omega}[f(k)]^{\leq g(k)}$.

We say a family of (f,g)-slaloms $\mathcal{S}$ is (f,g)-*covering* if for all $h \in \prod_{k<\omega} f(k)$ there is an $S \in \mathcal{S}$ such that $h \in^* S$ (i. e. $h(k) \in S_k$ for all but finitely many $k < \omega$).[32]

We then define the cardinal characteristic $\mathfrak{c}_{f,g}$, sometimes also denoted by $\mathfrak{c}^\forall_{f,g}$ and referred to as one of two kinds of *localisation cardinals*, as

$$\mathfrak{c}_{f,g} := \min\left\{ |\mathcal{S}| \;\middle|\; \mathcal{S} \subseteq \prod_{k<\omega}[f(k)]^{\leq g(k)}, \forall\, x \in \prod_{k<\omega} f(k)\ \exists\, S \in \mathcal{S}\colon x \in^* S \right\},$$

the minimal size of an (f,g)-covering family.

[32] Equivalently, $h \in S$ would lead to the same results.

A simple diagonalisation argument shows that under the assumptions above, $\mathfrak{c}_{f,g}$ is always uncountable. [GS93, section 1] contains a few simple properties following from the definition, but the only one we will be interested in here is monotonicity, in the following sense:

Fact B10.2. If $f \leq^* f'$ and $g \geq^* g'$, then $\mathfrak{c}_{f,g} \leq \mathfrak{c}_{f',g'}$.

For the following proof of $\mathfrak{c}_{f,g} \leq \mathrm{cof}(\mathcal{N})$, we recall a result from [Bar84] (as presented in [BJ95, Theorem 2.3.12]):

Theorem B10.3 (Bartoszyński)**.** *Let $M \subseteq N$ be transitive models of* ZFC**. The following are equivalent:*

(i) Every null set coded in N is covered by a Borel null set coded in M.
(ii) Every convergent series of positive reals in N is dominated by a convergent series in M.
(iii) For every function $h \in \omega^\omega \cap N$ there is a slalom $S \in \mathcal{C} \cap M$ such that $h(k) \in S(k)$ for almost all k.

In this theorem, $\mathcal{C}$ is defined as the set of all slaloms S such that

$$\sum_{k \geq 1} \frac{|S_k|}{k^2} < \infty,$$

which does not directly relate to our cardinal characteristics, but very nearly so:

Fact B10.4. Consider the following:

- We extend the definition of (f,g)-slalom and (f,g)-covering to allow $f \in (\omega+1)^\omega$. Write

$$\mathfrak{c}_{\omega,g} := \min\left\{ |\mathcal{S}| \;\middle|\; \mathcal{S} \subseteq \prod_{k<\omega} [\omega]^{\leq g(k)}, \forall\, x \in \omega^\omega\ \exists\, S \in \mathcal{S} \colon x \in^* S \right\},$$

  i. e. identify ω with the constant ω-valued function. By Fact B10.2, we then have that $\mathfrak{c}_{f,g} \leq \mathfrak{c}_{\omega,g}$.
- Note that $\mathfrak{c}_{\omega,g} \leq \kappa$ actually is a thinly veiled statement about the Sacks property.
- Finally, recall the well-known fact that the statement of the Sacks property is independent of the specific slalom size used (since for any two slalom size functions, the statements can be converted into each other by a simple coding argument). Hence it is clear that for any $g, g' \in \omega^\omega$ going to infinity with $0 < g, g'$, we have $\mathfrak{c}_{\omega,g} = \mathfrak{c}_{\omega,g'}$.

Definition B10.5. Let $\mathfrak{c}_{\omega,\mathcal{C}}$ be the minimal size of a family of slaloms in $\mathcal{C}$ covering all functions in ω^ω, i. e.

$$\mathfrak{c}_{\omega,\mathcal{C}} := \min\{|\mathcal{S}| \mid \mathcal{S} \subseteq \mathcal{C}, \forall\, x \in \omega^\omega\ \exists\, S \in \mathcal{S} \colon x \in^* S\}.$$

Lemma B10.6. *Let $g \in \omega^\omega$ be going to infinity with $0 < g$. Then $\mathfrak{c}_{\omega,g} = \mathfrak{c}_{\omega,\mathcal{C}}$.*

Proof. Letting $g^+(k) := k^2$ and $g^-(k) := \log k$, define

$$\mathcal{C}_+ := \left\{ S \;\middle|\; S \in^* \prod_{k<\omega} [\omega]^{\leq g^+(k)} \right\}$$

and

$$\mathcal{C}_- := \left\{ S \;\middle|\; S \in^* \prod_{k<\omega} [\omega]^{\leq g^-(k)} \right\}.$$

It is clear that $\mathcal{C}_+ \supseteq \mathcal{C} \supseteq \mathcal{C}_-$, which implies $\mathfrak{c}_{\omega,g^+} \leq \mathfrak{c}_{\omega,\mathcal{C}} \leq \mathfrak{c}_{\omega,g^-}$.

Now Fact B10.4 implies that $\mathfrak{c}_{\omega,g^+} = \mathfrak{c}_{\omega,g^-} = \mathfrak{c}_{\omega,g}$ for any $g \in \omega^\omega$ going to infinity with $0 < g$, and hence $\mathfrak{c}_{\omega,g^+} = \mathfrak{c}_{\omega,\mathcal{C}} = \mathfrak{c}_{\omega,g^-} = \mathfrak{c}_{\omega,g}$. □

Lemma B10.7. *For any $g \in \omega^\omega$ going to infinity with $0 < g$, $\mathfrak{c}_{\omega,g} = \mathrm{cof}(\mathcal{N})$.*

Proof. By Lemma B10.6, we instead show $\mathfrak{c}_{\omega,\mathcal{C}} = \mathrm{cof}(\mathcal{N})$. We use Theorem B10.3. To show $\mathfrak{c}_{\omega,g} \leq \mathrm{cof}(\mathcal{N})$, we use (i)⇒(iii):

Let $N := V$ and let M be a model of size κ such that every null set in N is covered by a Borel null set in M. Then Theorem B10.3 shows that $\{S \in N \mid S \in \mathcal{C}\}$ witnesses $\mathfrak{c}_{\omega,\mathcal{C}} \leq \kappa = \mathrm{cof}(\mathcal{N})$.

To show the converse result $\mathfrak{c}_{\omega,\mathcal{C}} \geq \mathrm{cof}(\mathcal{N})$, we use (iii)⇒(i) in the same vein. □

Theorem B10.8. *Given f, g as in Definition B10.1, $\mathfrak{c}_{f,g} \leq \mathrm{cof}(\mathcal{N})$.*

Proof. This follows immediately from Lemma B10.7 and Fact B10.4. □

Observation B10.9. Apart from this inequality, there are no limitations on the placement of the $\mathfrak{c}_{f_\xi,g_\xi}$ relative to the other cardinal characteristics in this chapter.

Recall that $\kappa_\xi^{\aleph_0} = \kappa_\xi$ for any $\xi < \omega_1$. As a consequence of the theorem, note that if we were to omit the $\mathrm{cof}(\mathcal{N})$ forcing factors entirely, we would then get the following result for $\mathrm{cof}(\mathcal{N})$ in $V[G]$: Let $\lambda := \sup_{\xi<\omega_1} \mathfrak{c}_{f_\xi,g_\xi}$. Then it is clear that $\lambda \leq \mathrm{cof}(\mathcal{N})$ by Lemma B10.7 and $\mathrm{cof}(\mathcal{N}) \leq \lambda^{\aleph_0}$ by the fact that there are only $\lambda^{\aleph_0}$ many reals after forcing with $\mathbb{Q}_{\text{non-ct}}$ (recall section B8). If $\mathrm{cof}\,\lambda \geq \omega_1$, then GCH in the ground model implies that $\lambda^{\aleph_0} = \lambda$ and hence $\mathrm{cof}(\mathcal{N}) = \lambda$.

Before we prove the cardinal characteristics' inequalities, we need to show that there indeed is a congenial ω_1-sequence of function pairs as defined in Definition B3.2. We can show even more:

Lemma B10.10. *There is a congenial sequence $\langle f_\xi, g_\xi \mid \xi < \mathfrak{c} \rangle$ of continuum many function pairs.*

Proof. (This proof is a modification and simplification of the construction in [GS93, Example 3.3].)

Recall that we need to show the following properties from Definition B3.2:

(i) For all ξ and for all $k < \omega$, $n^B_{4k+2} \leq g_\xi(k) < f_\xi(k) \leq n^S_{4k+2}$.
(ii) For all ξ, $\lim_{k\to\infty} \frac{\log f_\xi(k)}{n^P_{<4k+2}\cdot \log g_\xi(k)} = \infty$.
(iii) For all $\xi \neq \zeta$, either $\lim_{k\to\infty} \frac{f_\zeta(k)^2}{g_\xi(k)} = 0$ or $\lim_{k\to\infty} \frac{f_\xi(k)^2}{g_\zeta(k)} = 0$.

Also recall the definitions of n^B_{4k+2} and n^S_{4k+2} in Definition B4.8.

Let $\langle e_k \mid k < \omega\rangle$ be an increasing sequence such that $e_k > n^B_{4k+2} \geq 2$ for all $k < \omega$ (e. g. $e_k := n^B_{4k+2}+1$). Take the complete binary tree $T := 2^{<\omega}$ and enumerate $T\cap 2^k$ in lexicographic order as $\{s^1_k, \ldots, s^{2^k}_k\}$. We now define a pair of functions (f_ξ, g_ξ) for each branch $b_\xi \in [T]$ by the following rule: If $b_\xi\restriction_k = s^i_k$, then $f_\xi(k) := (n^B_{4k+2})^{e_k^{3^{2i}}}$ and $g_\xi(k) := (n^B_{4k+2})^{e_k^{3^{2i-1}}}$.

It is clear that by definition, $n^B_{4k+2} \leq g_\xi(k) < f_\xi(k)$, and recalling the fact that in Definition B4.8 we set

$$n^S_{4k+2} := (n^B_{4k+2})^{e_k^{3^{2^{k+1}}}},$$

we also have $f_\xi(k) \leq n^S_{4k+2}$. This proves property (i).

To show property (ii), we first remark that $g_\xi(k) \geq n^B_{4k+2} > 2^{n^P_{<4k+2}}$ and hence $\log g_\xi(k) > n^P_{<4k+2}$; it thus suffices to show

$$\lim_{k\to\infty} \frac{\log f_\xi(k)}{(\log g_\xi(k))^2} = \infty.$$

This is easy to see, as (for some $1 \leq i \leq 2^k$ depending on b_ξ and k)

$$\frac{\log f_\xi(k)}{(\log g_\xi(k))^2} = \frac{e_k^{3^{2i}}}{(e_k^{3^{2i-1}})^2 \cdot \log n^B_{4k+2}} = \frac{e_k^{3^{2i}-2\cdot 3^{2i-1}}}{\log n^B_{4k+2}} = \frac{e_k^{3^{2i-1}}}{\log n^B_{4k+2}} \geq \frac{e_k^3}{\log n^B_{4k+2}}$$

diverges to infinity as k goes to infinity.

Finally, consider $\xi \neq \zeta$, without loss of generality such that $b_\xi < b_\zeta$ in the natural lexicographic order on the branches of $[T]$. (If we have $b_\zeta < b_\xi$, we can just prove the other statement in property (iii) the same way.) Taking the first k such that $b_\xi(k-1) \neq b_\zeta(k-1)$, there are $1 \leq i < j \leq 2^k$ such that $b_\xi\restriction_k = s^i_k$ and $b_\zeta\restriction_k = s^j_k$ (and analogously for any larger k). Then it follows that

$$F(\xi,\zeta,k) := \frac{f_\xi(k)^2}{g_\zeta(k)} = (n^B_{4k+2})^{2\cdot e_k^{3^{2i}} - e_k^{3^{2j-1}}}$$

and since $e_k > 2$ and $2i < 2j-1$, we have that

$$2\cdot e_k^{3^{2i}} < e_k^{3^{2i+1}} \leq e_k^{3^{2j-1}}$$

and hence $F(\xi,\zeta,k) < 1/n^B_{4k+2}$, which goes to 0, as required to show property (iii). □

We point out once more that in order to make it easier to read, the construction above is actually slightly less general than the one in [GS93, Example 3.3]; in our case, the pairs of functions are not only pointwise "far apart", but instead even have the same ordering between them at each point. The reader can easily convince herself that the more general construction would also work in the same way.

Lemma B10.11. *$\mathbb{Q}$ forces that for all $\xi < \omega_1$, $\mathfrak{c}_{f_\xi,g_\xi} \geq \kappa_\xi$.*

Proof. Fix some $\xi < \omega_1$. Let G be $\mathbb{Q}$-generic and let $\mathcal{S}$ be some family of g_ξ-slaloms in $V[G]$ of size less than κ_ξ. Each $S \in \mathcal{S}$ is read continuously only using indices in some countable subset B_S of A and there are less than $\kappa_\xi^{\aleph_0} = \kappa_\xi = |A_\xi|$ many S, so letting $B := \bigcup_{S\in\mathcal{S}} B_S$, all of $\mathcal{S}$ is read continuously only using indices in B and there is some $\alpha \in A \setminus B$.

Now assume towards a contradiction that there were some g_ξ-slalom $S^* \in V[G{\restriction}_B]$ covering the generic $\dot{y}_\alpha$. Working in V, this means that there is a $\mathbb{Q}{\restriction}_B$-name $\dot{S}^*$ and a condition $p \in \mathbb{Q}$ such that $\Vdash_{\mathbb{Q}{\restriction}_B}$ "$\dot{S}^*$ is a g_ξ-slalom" and $p \Vdash_{\mathbb{Q}}$ "$\dot{S}^*$ covers $\dot{y}_\alpha$".

But then we can find some $k < \omega$ such that $|p(\alpha, 4k{+}2)| > g_\xi(k)$. Find $q \leq p$ by first strengthening $p{\restriction}_B$ to decide $\dot{S}^*_k = T$ and then finding some $x \in p(\alpha, 4k{+}2) \setminus T$ and replacing $p(\alpha, 4k{+}2)$ by $\{x\}$. The condition q then forces the desired contradiction, proving that fewer than κ_ξ many g_ξ-slaloms cannot suffice to cover all functions in $\prod_{k<\omega} f_\xi(k)$. □

To prove the converse, we first have to prepare just a few more technical tools.

Definition B10.12. Let $p \in \mathbb{Q}$ and let $\dot{t}$ be a $\mathbb{Q}$-name for a function in $\prod_{k<\omega} n^S_{4k+2}$. We say that p *punctually reads* $\dot{t}$ if for each $k < \omega$, $\dot{t}{\restriction}_{k+1}$ is decided below $4k + 2$.

Lemma B10.13. *Fix a coding function C which continuously maps functions in $\prod_{k<\omega} n^S_{4k+2}$ to 2^ω. Then there is a unique family of functions $\langle C_k \mid k < \omega\rangle$ with $C_k \colon 2^{<z(k)} \to \prod_{\ell\leq k} n^S_{4\ell+2}$ and a unique $z \in \omega^\omega$ such that the following holds: For any function $t \in \prod_{k<\omega} n^S_{4k+2}$, there is an $s \in 2^\omega$ such that for all $k < \omega$, we have $C_k(s{\restriction}_{z(k)}) = t{\restriction}_{k+1}$.*

Proof. Define $z(0) := \lceil \log_2(n^S_2)\rceil + 1$ and $z(k) := z(k-1) + \lceil \log_2(n^S_{4k+2})\rceil + 1$ for $k \geq 1$. The functions C_k are the natural restrictions of the inverse of the coding function C. □

Corollary B10.14. *Let $\dot{t}$ be a $\mathbb{Q}$-name for a function in $\prod_{k<\omega} n^S_{4k+2}$ and $p \in \mathbb{Q}$. Then there is a $q \leq p$ punctually reading $\dot{t}$.*

Proof. By the preceding lemma, there is a $\mathbb{Q}$-name $\dot{s}$ for an element of 2^ω such that for any $k < \omega$, $C_k(\dot{s}{\restriction}_{z(k)}) = \dot{t}{\restriction}_{k+1}$. Find $q \leq p$ rapidly reading $\dot{s}$; by the preceding lemma, q then punctually reads $\dot{t}$. □

Definition B10.15. For a modest $p \in \mathbb{Q}$, we call $4k + 2$ a *slalom-splitting level* if there is an $\alpha \in \mathrm{supp}(p)$ such that $|p(\alpha, 4k + 2)| > 1$. We refer to this unique index by α_k, and the corresponding type by $\zeta_k < \omega_1$.

Definition B10.16. Fix some $\xi < \omega_1$. We call a condition $p \in \mathbb{Q}$ *ξ-prepared* if for all $\zeta \neq \xi$ and all $k < \omega$, one of the following three statements holds:

- $4k + 2$ is not a slalom-splitting level of $p{\restriction}_{\mathrm{supp}(p)\setminus\{\xi\}}$,
- $f_\zeta(k)^2 < g_\xi(k)$, or
- $f_\xi(k)^2 < g_\zeta(k)$.

Lemma B10.17. *Fix some $\xi < \omega_1$ and let $p \in \mathbb{Q}$. Then there is a ξ-prepared $q \leq p$.*

Proof. We do the following steps for each $\zeta \neq \xi$.

First, note that per the proof of Lemma B10.10, there is some k_ζ such that for all $k < k_\zeta$, $f_\xi(k) = f_\zeta(k)$ and $g_\xi(k) = g_\zeta(k)$, but for all $k \geq k_\zeta$, the functions are different.

Second, per property (iii) in Definition B3.2, we know that either

$$\lim_{k\to\infty} \frac{f_\zeta(k)^2}{g_\xi(k)} = 0 \quad \text{or} \quad \lim_{k\to\infty} \frac{f_\xi(k)^2}{g_\zeta(k)} = 0$$

and hence there must be some $k^*_\zeta \geq k_\zeta$ such that $\frac{f_\zeta(k)^2}{g_\xi(k)} < 1$ for all $k \geq k^*_\zeta$ or $\frac{f_\xi(k)^2}{g_\zeta(k)} < 1$ for all $k \geq k^*_\zeta$. Now for $k < k^*_\zeta$, replace $p(\zeta, 4k+2)$ by arbitrary singletons to get $q(\zeta, 4k+2)$. The resulting q is then ξ-prepared. □

Lemma B10.18. *$\mathbb{Q}$ forces that for all $\xi < \omega_1$, $\mathfrak{c}_{f_\xi, g_\xi} \leq \kappa_\xi$.*

Proof. Fix some $\xi < \omega_1$ and let $A := \bigcup_{\kappa_\mathrm{t} \leq \kappa_\xi} A_\mathrm{t}$.[33] We will prove that the g_ξ-slaloms in $V^{\mathbb{Q}\restriction A}$ cover $\prod_{k<\omega} f_\xi(k)$; this suffices since by Lemma B6.5 (and by the fact that $\mathbb{Q}\restriction_A$ is a complete subforcing of $\mathbb{Q}$, see Lemma B4.19), $\Vdash_{\mathbb{Q}\restriction_A} 2^{\aleph_0} \leq \kappa_\xi$ and hence $\Vdash_{\mathbb{Q}} (2^{\aleph_0})^{V^{\mathbb{Q}\restriction A}} \leq \kappa_\xi$.

So let $\dot{t}$ be a $\mathbb{Q}$-name for a function in $\prod_{k<\omega} f_\xi(k)$ and let $p^* \in \mathbb{Q}$ be an arbitrary condition. Find $p \leq p^*$ such that p punctually reads $\dot{t}$ and is ξ-prepared.

We will find a condition $q \leq p$ and a $\mathbb{Q}\restriction_A$-name $\dot{S}$ for a g_ξ-slalom such that $q \Vdash$ "$\dot{S}$ covers $\dot{t}$".

To find q and define $\dot{S}$, we go through the levels of the form $4k+2$ and make the following case distinction. (We know that one of the following cases must hold since p is ξ-prepared.)

Case 0: $4k+2$ is not a slalom-splitting level of p.

In this case, we have that $|\operatorname{poss}(p, {<}4k+3)| = |\operatorname{poss}(p, {<}4k+2)|$ (since at level $4k+2$, there is only one possible extension for each possibility from below). Hence letting $q(\text{slalom}, 4k+2) := p(\text{slalom}, 4k+2)$ and defining

$$\dot{S}_k := \{x < \omega \mid \exists\, \eta \in \operatorname{poss}(q, {<}4k+3) \colon p \wedge \eta \Vdash \dot{t}(k) = x\},$$

we actually have a (ground model) set of size at most

$$|\operatorname{poss}(p, {<}4k+3)| = |\operatorname{poss}(p, {<}4k+2)| < n^P_{<4k+2} < n^B_{4k+2} \leq g_\xi(k),$$

and clearly $q \Vdash \dot{t}(k) \in \dot{S}_k$.

[33] Note, however, that the t $\notin$ slalom are not especially relevant here. The case distinction below only cares about the A_ζ with $\zeta < \omega_1$ – and whether $\kappa_\zeta \leq \kappa_\xi$ or $\kappa_\zeta > \kappa_\xi$ –, but the definition is just cleaner in this more general formulation.

Case 1: $4k+2$ is a slalom-splitting level of p, but ζ_k is such that $\kappa_{\zeta_k} \leq \kappa_\xi$.

In this case, $\alpha_k \in A$. We once more let $q(\text{slalom}, 4k+2) := p(\text{slalom}, 4k+2)$ and define $\dot{S}_k$ to be a $\mathbb{Q}{\restriction}_A$-name satisfying

$$\Vdash_{\mathbb{Q}{\restriction}_A} \dot{S}_k = \{x < \omega \mid \exists\, \eta \in \text{poss}(q, {<}4k+3)\colon p \wedge \eta \Vdash \text{“}\dot{t}(k) = x \text{ and } \eta(\alpha_k) = \dot{y}_{\alpha_k}\text{”}\},$$

which means we only allow those possibilities η which are compatible with the generic real $\dot{y}_{\alpha_k}$ added by the forcing factor $\mathbb{Q}_{\alpha_k}$. Similar to the previous case, this means that $\Vdash_{\mathbb{Q}{\restriction}_A} |\dot{S}_k| \leq |\,\text{poss}(p, {<}4k+2)|$, for the following reason: Let $\varepsilon := \dot{y}_{\alpha_k}$; then in the definition of $\dot{S}_k$ above, the only admissible possibilities $\eta \in \text{poss}(p, {<}4k+3)$ are those of the form $\eta = \nu^\frown\varepsilon$ for some $\nu \in \text{poss}(p, {<}4k+2)$. Hence $\Vdash_{\mathbb{Q}{\restriction}_A} |\dot{S}_k| < g_\xi(k)$, and by definition $q \Vdash \dot{t}(k) \in \dot{S}_k$.

Case 2: $4k+2$ is a slalom-splitting level of p, $\kappa_{\zeta_k} > \kappa_\xi$ and $f_\zeta(k)^2 < g_\xi(k)$.

From $f_\zeta(k)^2 < g_\xi(k)$, we get the following:

$$\begin{aligned}|\,\text{poss}(p, {<}4k+3)| &\leq |\,\text{poss}(p, {<}4k+2)| \cdot f_\zeta(k) \leq n^P_{<4k+2} \cdot f_\zeta(k)\\ &< n^B_{4k+2} \cdot f_\zeta(k) < f_\zeta(k)^2 < g_\xi(k)\end{aligned}$$

By once more letting $q(\text{slalom}, 4k+2) := p(\text{slalom}, 4k+2)$ and defining

$$\dot{S}_k := \{x < \omega \mid \exists\, \eta \in \text{poss}(q, {<}4k+3)\colon p \wedge \eta \Vdash \dot{t}(k) = x\},$$

we hence again have a (ground model) set of size at most $g_\xi(k)$, and by definition $q \Vdash \dot{t}(k) \in \dot{S}_k$.

Case 3: $4k+2$ is a slalom-splitting level of p, $\kappa_{\zeta_k} > \kappa_\xi$ and $f_\xi(k)^2 < g_\zeta(k)$.

This is the only case where we have to do any actual work to get q, as we cannot simply collect all potential values of $\dot{t}(k)$. Instead, we will first have to use the bigness properties of the norm to reduce the number of potential values. To begin, we remark that letting $m_k := |\,\text{poss}(p, {<}4k+2)|$, $c := f_\xi(k)$ and $d := {}^{g_\xi(k)}/_{m_k}$, we have

$$\frac{c}{d} = \frac{f_\xi(k)}{\frac{g_\xi(k)}{m_k}} = f_\xi(k) \cdot \frac{|\,\text{poss}(p, {<}4k+2)|}{g_\xi(k)} \leq f_\xi(k) \cdot \frac{n^P_{<4k+2}}{g_\xi(k)} < f_\xi(k) < g_\zeta(k)$$

and since $\|\cdot\|_{\zeta,4k+2}$ has $n^P_{<4k+2}$-strong $g_\zeta(k)$-bigness by Theorem B5.6, according to Observation B5.5 it also has $n^P_{<4k+2}$-strong (c,d)-bigness.

Enumerate $\text{poss}(p, {<}4k+2) =: \{\eta_1, \ldots, \eta_{m_k}\}$. We claim we can find a sequence of subsets $p(\alpha_k, 4k+2) = F^0_k \supseteq F^1_k \supseteq F^2_k \supseteq \ldots \supseteq F^{m_k}_k$ and a sequence of sets C_j with the following properties (for each $1 \leq j \leq m_k$):

(i) $\|F^j_k\|_{\zeta,4k+2} \geq \|F^{j-1}_k\|_{\zeta,4k+2} - {}^1/_{n^P_{<4k+2}}$
(ii) $|C_j| \leq d$
(iii) $p \wedge (\eta^j{}^\frown x) \Vdash \dot{t}(k) \in C_j$ holds for all $x \in F^j_k$.

We know that, given F^j_k, for each $x \in F^j_k$, we have that $p \wedge (\eta^j{}^\frown x)$ decides $\dot{t}(k)$ by punctual reading of $\dot{t}$ (noting that $\eta^j{}^\frown x \in \text{poss}(p, {<}4k+3)$). Since there are most c many possible values for $\dot{t}(k)$, we can use $n^P_{<4k+2}$-strong (c,d)-bigness of the

norm $\|\cdot\|_{\zeta,4k+2}$ to find $F_k^{j+1} \subseteq F_k^j$ and C_{j+1} with the desired properties, proving our claim.

Now, we define $F_k := F_k^{m_k}$. Since $m_k \leq n^P_{<4k+2}$, by (i) we have that

$$\|F_k\|_{\zeta,4k+2} \geq \|p(\alpha_k, 4k+2)\|_{\zeta,4k+2} - \frac{m_k}{n^P_{<4k+2}} \geq \|p(\alpha_k, 4k+2)\|_{\zeta,4k+2} - 1.$$

Hence, defining $q(\alpha_k, 4k+2) := F_k$ and $q(\text{slalom}, 4k+2) := p(\text{slalom}, 4k+2)$ elsewhere (i. e. on $A_{\text{slalom}} \setminus \{\alpha_k\}$) does not negatively affect the lim sup properties of the norm.

Finally, let $\dot{S}_k := \bigcup_{1\leq j\leq m_k} C_j$. Then by (ii), we have that $|\dot{S}_k| \leq d \cdot m_k = g_\xi(k)$, and (iii) implies $q \Vdash \dot{t}(k) \in \dot{S}_k$, finishing the proof. □

This proves (M5) of Theorem B1.1.

B11 $\text{non}(\mathcal{N}) \geq \kappa_{\text{nn}}$ and $\text{cof}(\mathcal{N}) \geq \kappa_{\text{cn}}$

The proofs in this section are more or less identical to those in [FGKS17], although we sincerely hope we have improved the presentation.

To prove $\text{non}(\mathcal{N}) \geq \kappa_{\text{nn}}$, we define some null sets in the extension, similar to the definition of $\dot{M}_\alpha$ in section B9. Recall that for $\alpha \in A_{\text{nn}}$, the generic object $\dot{y}_\alpha$ is a heights$_{*\text{n}}$-sequence of subsets $\dot{R}_{\alpha,L}$ of 2^{I_L} of relative size $1 - 2^{-n^B_L}$.[34] Since the sequence of N^B_L, $L \in \text{heights}_{*\text{n}}$, is strictly monotone, we have

$$\prod_{L\in\text{heights}_{*\text{n}}} \left(1 - \frac{1}{2^{n^B_L}}\right) > 0$$

and hence the set $\{r \in 2^\omega \mid \forall\, k < \omega\colon r{\restriction}_{I_{4k+1}} \in \dot{R}_{\alpha,4k+1}\}$ is positive. It follows that the set $\{r \in 2^\omega \mid \forall^\infty k < \omega\colon r{\restriction}_{I_{4k+1}} \in \dot{R}_{\alpha,4k+1}\}$ has measure one, and therefore its complement

$$\dot{N}_\alpha := \{r \in 2^\omega \mid \exists^\infty k < \omega\colon r{\restriction}_{I_{4k+1}} \notin \dot{R}_{\alpha,4k+1}\}$$

is a name for a null set.

Recall that (by Theorem B5.6) for each $L \in \text{heights}_{*\text{n}}$, $(\text{POSS}_{\text{nn},L}, \|\cdot\|_{\text{nn},L})$ has strong n^B_L-bigness. We show a similar, more specific property:

Lemma B11.1. *Let $L \in \text{heights}_{*\text{n}}$, $X \subseteq \text{POSS}_{\text{nn},L}$ and $E \subseteq 2^{I_L}$. Let $X' := \{H \in X \mid H \cap E = \varnothing\}$. Then $\|X'\|^{\text{intersect}}_L \geq \|X\|^{\text{intersect}}_L - |E|$.*

If additionally $|E| \leq n^P_{<L}$, then it follows that $\|X'\|_{\text{nn},L} \geq \|X\|_{\text{nn},L} - 1/\log n^B_L$.

Proof. For the first part, assume that some Y witnesses $\|X'\|^{\text{intersect}}_L$; then $Y \cup E$ certainly witnesses $\|X\|^{\text{intersect}}_L$.

[34] Recall that $\text{heights}_{*\text{n}} = \{4k+1 \mid k < \omega\}$.

For the second part, note that $n^P_{<L} \leq {}^{n^B_L}\!/_2$ and hence $|E| \leq {}^{(n^B_L)^{\|X\|_{\mathrm{nn},L}}}\!/_2$. Since $n^B_L \geq 2$ and assuming X is non-trivial (without loss of generality, assume $\|X\|_{\mathrm{nn},L} \geq 2$), a trivial inequality gives

$$\frac{(n^B_L)^{\|X\|_{\mathrm{nn},L}}}{2} = \frac{(\|X\|^{\mathrm{intersect}}_L)^{1/n^B_L}}{2} = \left(\frac{\|X\|^{\mathrm{intersect}}_L}{2^{n^B_L}}\right)^{1/n^B_L}$$
$$\leq \left(\left(1 - \frac{1}{2^{n^B_L}}\right)^{n^B_L} \cdot (\|X\|^{\mathrm{intersect}}_L)^{n^B_L}\right)^{1/n^B_L} = \left(1 - \frac{1}{2^{n^B_L}}\right) \cdot \|X\|^{\mathrm{intersect}}_L$$

and hence the first part implies

$$\|X'\|_{\mathrm{nn},L} = \frac{\log \|X'\|^{\mathrm{intersect}}_L}{n^B_L \log n^B_L} \geq \frac{\log(\|X\|^{\mathrm{intersect}}_L - |E|)}{n^B_L \log n^B_L} \geq \frac{\log({}^{\|X\|^{\mathrm{intersect}}_L}\!/_{2^{n^B_L}})}{n^B_L \log n^B_L}$$
$$= \frac{\log \|X\|^{\mathrm{intersect}}_L}{n^B_L \log n^B_L} - \frac{\log 2^{n^B_L}}{n^B_L \log n^B_L} = \|X\|_{\mathrm{nn},L} - \frac{1}{\log n^B_L}.$$ □

Lemma B11.2. *Let $\dot{r} \in 2^\omega$ be a name for a real and let p rapidly read $\dot{r}$ not using the index $\alpha \in A_{\mathrm{nn}}$. Then $p \Vdash \dot{r} \in \dot{N}_\alpha$.*

Proof. We first remark that as in Lemma B9.3, it suffices to prove that there is an $s \leq p$ such that $s \Vdash \dot{r} \in \dot{N}_\alpha$. Similar to that proof, we will only have to modify p in very few places to get the desired condition s. Without loss of generality, assume that $\alpha \in \mathrm{supp}(p)$.

We will only modify p at index α for infinitely many heights in heights$_{*\mathrm{n}}$. Assume we have already modified n many heights $L_0, \dots, L_{n-1}$; pick some $L_n := 4k_n + 1 \in$ heights$_{*\mathrm{n}}$ such that $p(\alpha, L_n)$ is non-trivial and has a norm of at least 2. By rapid reading, we know that $\dot{r}{\restriction}_{I_{L_n}}$ is decided $\leq L_n$ by p, and as in Lemma B9.3, by modesty and since $\dot{r}$ does not depend on the index α, it is even decided below L_n. Hence the set E_n of possible values for $\dot{r}{\restriction}_{I_{L_n}}$ has size at most $n^P_{<L}$; by the preceding lemma, replacing $p(\alpha, L_n)$ by $C_n := \{H \in p(\alpha, L_n) \mid H \cap E_n = \varnothing\}$ only decreases the norm by at most 1.

The condition s resulting from replacing each $p(\alpha, L_n)$ by C_n then fulfils $s \Vdash \dot{r} \in \dot{N}_\alpha$ by definition, as $s \Vdash \dot{r}{\restriction}_{I_{4k_n+1}} \notin \dot{R}_{\alpha,4k_n+1}$ holds for all $n < \omega$. □

Corollary B11.3. $\mathbb{Q}$ *forces* $\mathrm{non}(\mathcal{N}) \geq \kappa_{\mathrm{nn}}$.

Proof. The proof is identical to the proof of Corollary B9.4. □

To prove $\mathrm{cof}(\mathcal{N}) \geq \kappa_{\mathrm{cn}}$, we define null sets in the extension in the same way we did at the start of this section, namely

$$\dot{N}_\alpha := \{r \in 2^\omega \mid \exists^\infty k < \omega \colon r{\restriction}_{I_{4k+1}} \notin \dot{R}_{\alpha,4k+1}\}$$

for $\alpha \in A_{\mathrm{cn}}$. However, the purpose of these null sets will be quite different; rather than covering all reals in the extension which do not depend on α, they will avoid being covered by any null set not depending on α.

We wish to spare the reader the details of the combinatorial arguments from [FGKS17, section 9], and hence will only sketch the modifications necessary to see why the proofs in [FGKS17] still hold. The relevant result we will be using is [FGKS17, Lemma 10.2.1], in the following form:

Lemma B11.4. *Fix a height $L \in \text{heights}_{*n}$, an index $\alpha \in A_{\text{cn}}$ and a creature $C \subseteq \text{POSS}_{\alpha,L}$ such that $\|C\|_{\text{cn},L} \geq 2$.*

(i) Given $T \subseteq 2^{I_L}$ of relative size at least $1/2$, we can strengthen C to a creature D such that $T \not\subseteq X$ for all $X \in D$ and such that

$$\|D\|_{\text{cn},L} \geq \|C\|_{\text{cn},L} - \frac{1}{2^{\min I_L} \cdot n_L^B}.$$

(ii) Given a probability space Ω and a function $F\colon C \to \mathcal{P}(\Omega)$ mapping each $X \in C$ to some $F(X) \subseteq \Omega$ of measure at least $1/n_L^B$, we can strengthen C to a creature D such that $\bigcap_{X\in D} F(X)$ has measure at least $1/n_{L^+}^B$ and such that

$$\|D\|_{\text{cn},L} \geq \|C\|_{\text{cn},L} - \frac{1}{2^{\min I_L} \cdot n_L^B}.$$

Proof sketch. In [FGKS17], Lemma 10.2.1 is an immediate consequence of two properties – (i) follows from Lemma 9.2.2 and (ii) follows from Eq. (9.1.4).

The second part is straightforward: By the considerations in Observation B3.6, it is clear that (ii) still follows for our new norm as long as $n_{L^+}^B > n_L^B \cdot 2^{n_L^S+1}$, which is the case by the definition of $n_{L^+}^B$.

The first part (Lemma 9.2.2) requires a bit more thought (since our modification to take the logarithm of the $\text{nor}_{I,b}^{\div}$ complicates the direct argument). Define

$$\Delta_L := \binom{2^{|I_L|-1}}{2^{n_L^B-1}}.$$

The relevant statement in Lemma 9.2.2 then is: Given C and T as in (i), we can find D such that $T \not\subseteq X$ for all $X \in D$ and such that

$$|C \setminus D| \leq \Delta_L. \tag{$*_8$}$$

We first explain why Eq. ($*_8$) implies (i): Let $\delta := |C \setminus D|$. Then $\|C\|_{\text{cn},L} \geq 2$ implies $|C| \geq 2\Delta_L$, and $\Delta_L \geq \delta$ implies $|D| \geq |C|/2 \geq \delta$, so $\delta/|D| \leq 1$. By the well-known fact that

$$\frac{\log(z+\varepsilon) - \log z}{\varepsilon} \leq \frac{1}{z}$$

and the fact that $|C| = |D| + \delta$, we get

$$\log|C| - \log|D| \leq \frac{\delta}{|D|} \leq 1.$$

By all this, we know that having found such a D, the numerators of the fractions in $\|C\|_{\mathrm{cn},L}$ and $\|D\|_{\mathrm{cn},L}$ differ by at most 1, and hence the whole norms differ by at most $1/(2^{\min I_L} \cdot n_L^B)$ (actually, even less).

Finally, the reason why Eq. ($*_8$) holds is the same combinatorial consideration explained in [FGKS17, Lemma 9.2.2 (1)]. □

Fact B11.5. We require a few facts about the correspondence between trees of measure $1/2$ and null sets.

(i) Let $T \subseteq 2^{<\omega}$ be a leafless tree of measure $1/2$ (and recall that such trees bijectively correspond to closed sets of measure $1/2$). For $X \subseteq 2^\omega$, let $X + 2^{<\omega} := \bigcup\{X + r \mid r \in 2^{<\omega}\}$, the set of all rational translates of X (where $X + r := X + r^\frown\langle 000\ldots\rangle$). Then the set $N_T := 2^\omega \setminus ([T] + 2^{<\omega})$ is a null set closed under rational translations.

(ii) Conversely, given an arbitrary null set N, there is a leafless tree T of measure $1/2$ such that $N \subseteq N_T$, since the complement of $N + 2^{<\omega}$ must contain a closed set of size $1/2$.

(iii) Let $k < \omega$ and $s \in T \cap 2^k$. We define the relative measure of s in T as $2^k \cdot \lambda([T] \cap [s])$. Analogously, for finite trees $T \in 2^{\leq m}$ (such that there are no leaves below tree level m) we define the relative measure of $s \in T \cap 2^k$ for $k \leq m$ in the same way. (For $s \notin T$, the relative measure of s in T is 0, naturally.)

(iv) Given a leafless tree $T \subseteq 2^{<\omega}$, some $s \in T$ of positive relative measure and some $0 < \varepsilon < 1$, there is some extension t of s such that t has relative measure $> \varepsilon$. Moreover, it follows that for all tree levels above the tree level of t there is some extension u of t such that u has relative measure $>\varepsilon$. (These statements are a simple consequence of Lebesgue's density theorem.)

Since the measure of a tree T does not change if we remove any $s \in T$ of relative measure 0, we will be working with such trees instead:

Definition B11.6. We call $T \subseteq 2^{<\omega}$ a *sturdy tree* if it has measure $1/2$ and no $s \in T$ has relative measure 0 (in particular, this means T is leafless).

The considerations from Fact B11.5 also hold for sturdy trees, so we will be working with those instead.

Finally, we remark that 2^{2^k} is an upper bound for the cardinality of the set $2^{\leq k}$. We can thus code any name for a sturdy tree $\dot{T}$ by a real $\dot{t} \in 2^\omega$ such that $\dot{T} \cap 2^k$ is determined by $\dot{t}\restriction_{2^{2^k+1}}$, and by the definition of $n^R_{<L}$, if a condition p rapidly reads $\dot{t}$, then for each $\eta \in \mathrm{poss}(p, {\leq}L)$, $p \wedge \eta$ decides $\dot{T} \cap 2^{\max I_L}$; we abbreviate this fact by "p rapidly reads $\dot{T}$".

Lemma B11.7. *Let $\dot{T}$ be a name for a sturdy tree and let p rapidly read $\dot{T}$ not using the index $\alpha \in A_{\mathrm{cn}}$. Then $p \Vdash \dot{N}_\alpha \not\subseteq N_{\dot{T}}$, that is, p forces that there is an $s \in \dot{N}_\alpha \cap [\dot{T}]$.*

Proof. Once again, it suffices to find a $q \leq p$ and a $\mathbb{Q}$-name $\dot{s}$ for a real such that $q \Vdash \dot{s} \in \dot{N}_\alpha \cap [\dot{T}]$. Without loss of generality, assume that $\alpha \in \mathrm{supp}(p)$. To achieve

this, we will modify $p(\alpha)$ at infinitely many $*\mathrm{n}$ heights to get q and thereafter define the required real s inductively in the extension.

Let $L \in \mathrm{heights}_{*\mathrm{n}}$ be a height, above all the previously modified heights, such that $\|p(\alpha, L)\| \geq 3$. (The condition on the norm is necessary for us to be able to apply Lemma B11.4 (i) sufficiently often.) Let $\dot{T}^* := \dot{T} \cap 2^{\max I_L}$. By rapid reading, p decides $\dot{T}^*$ below L (since $\dot{T}$ does not depend on α and by modesty, there is no other index β such that $p(\beta, L)$ is non-trivial). In particular, this means that the set W of possible values of $\dot{T}^*$ has size at most $n^P_{<L}$.

We now enumerate all $U \in W$ and all $u \in U \cap 2^{\min I_L}$ with relative measure at least $1/2$ (measured in U). Clearly, there are at most $M := n^P_{<L} \cdot 2^{\min I_L}$ many such pairs (U, u). Starting with $C_0 := p(\alpha, L)$, we will iteratively apply Lemma B11.4 (i) to the creature C^n and the tree $u^\frown U{\restriction}_{2^{I_L}}$ to get a creature $C_{n+1} \subseteq C_n$ which then fulfils the following statement: For each $X \in C_{n+1}$, there is some $u' \in 2^{I_L} \setminus X$ such that $u^\frown u' \in U$, and

$$\|C_{n+1}\|_{\mathrm{cn},L} \geq \|C_n\|_{\mathrm{cn},L} - \frac{1}{2^{\min I_L} \cdot n^B_L}.$$

After going through all M many possible choices of (U, u), we arrive at $D := C_M$, which fulfils the following statement: For each $X \in D$ and each (U, u) as above, there is some $u' \in 2^{I_L} \setminus X$ such that $u^\frown u' \in U$, and $\|D\|_{\mathrm{cn},L} \geq \|p(\alpha, L)\|_{\mathrm{cn},L} - 1$, since $n^P_{<L} < n^B_L$ and hence

$$\frac{M}{2^{\min I_L} \cdot n^B_L} = \frac{n^P_{<L} \cdot 2^{\min I_L}}{n^B_L \cdot 2^{\min I_L}} \leq 1.$$

Denote the condition which emerges after repeating the process above for infinitely many heights by q (and note that $q \leq p$ and q only differs from p at index α). We will now work in the forcing extension $V[G]$ (for some generic filter G containing q) and construct some $s \in \dot{N}_\alpha \cap [\dot{T}]$. Recall that the requirements on s are that it is a branch of $[\dot{T}]$ and that for infinitely many $L \in \mathrm{heights}_{*\mathrm{n}}$ we have $s{\restriction}_{I_L} \notin \dot{R}_{\alpha,L}$.

Start with $s_0 := \varnothing$ and $k_0 := 0$. Assume we have already defined k_n and s_n (which will be equal to $s{\restriction}_{k_n}$) such that $s_n \in \dot{T}$. Since $\dot{T}$ is a sturdy tree and hence has no nodes of relative measure 0, by Fact B11.5 (iv) there is some $k' > k_n$ and a $t \in \dot{T} \cap 2^{k'}$ such that t extends s_n and has relative measure at least $1/2$. Pick a height $L \in \mathrm{heights}_{*\mathrm{n}}$ such that L was considered in the construction of q and such that $\min I_L =: k'' > k'$. Also by Fact B11.5 (iv), there is (still) a $u \in \dot{T} \cap 2^{k''}$ such that u extends s_n and has relative measure at least $1/2$. Let $U := \dot{T} \cap 2^{\max I_L}$ and note that in the construction of q, we dealt with the pair (U, u). Hence for all $X \in q(\alpha, L)$ (in particular, the $\dot{R}_{\alpha,L}$ chosen by the generic filter G), there is some $u' \in 2^{I_L} \setminus X$ such that $u^\frown u' \in U$. So we can set $s_{n+1} := u^\frown u'$ and $k_{n+1} := \max I_L$ and continue the induction; the resulting $s := \bigcup_{n<\omega} s_n$ is as required. □

Corollary B11.8. $\mathbb{Q}$ *forces* $\mathrm{cof}(\mathcal{N}) \geq \kappa_{\mathrm{cn}}$.

Proof. Fix a condition p, some $\kappa < \kappa_{\mathrm{cn}}$ and a sequence of names of null sets $\langle \dot{N}_i \mid i \in \kappa \rangle$ which p forces to be a basis of null sets. As described above, for each

$i \in \kappa$, we can assume that $\dot{N}_i = N_{\dot{T}_i}$ for some name for a sturdy tree $\dot{T}_i$. The rest of the proof is identical to the proof of Corollary B9.4. □

This proves (M4) of Theorem B1.1.

B12 $\mathrm{non}(\mathcal{N}) \leq \kappa_{\mathrm{nn}}$

For the final proofs, we will require two more lemmata. First, we show that the slalom part of the forcing construction has a property similar to Lemma B11.4 (ii).

Lemma B12.1. *Fix a height $L \in \mathrm{heights}_{\mathrm{slalom}}$, a slalom type $\xi \in \mathrm{types}_{\mathrm{slalom}}$, an index $\alpha \in A_\xi$, and a creature $C \subseteq \mathrm{POSS}_{\xi,L}$ such that $\|C\|_{\xi,L} \geq 2$.*

Given a probability space Ω and a function $F\colon C \to \mathcal{P}(\Omega)$ mapping each $X \in C$ to some $F(X) \subseteq \Omega$ of measure at least $1/n^B_L$, we can strengthen C to a creature D such that $\bigcap_{X\in D} F(X)$ has measure at least $1/n^B_{L^+}$ and such that

$$\|D\|_{\xi,L} \geq \|C\|_{\xi,L} - \frac{1}{2^{\min I_L} \cdot n^B_L}.$$

Proof. As in the proof of Lemma B11.4 (ii), we only require that a statement analogous to [FGKS17, Eq. (9.1.4)] holds (as $n^B_{L^+} > n^B_L \cdot 2^{n^S_L+1}$ is true for any L). We already know that Eq. (9.1.4) holds for a norm with the basic structure[35] of $\frac{\log x}{\log 3b}$; the slalom norms have the basic structure $\frac{\log x}{\log g_\xi(k)}$ (with $L = 4k+2$), and in our construction in Lemma B10.10, each $g_\xi(k)$ is defined as some e-th power of n^B_L; each such exponent e is assured to be at least 8 and even the smallest $n^B_{(0,0)} \geq 8$, hence $(n^B_L)^e \geq 3n^B_L$ and the same basic property holds for this norm structure, as well.[36] □

The other lemma is one more combinatorial statement about trees.

Lemma B12.2. *Given a tree $T \subseteq 2^{<\omega}$ of positive measure and an $\varepsilon > 0$, we call $s \in T \cap 2^k$ fat if*

$$\lambda([T] \cap [s]) \geq \frac{1-\varepsilon}{2^k}.$$

Then there is a $k^* < \omega$ such that for all $k \geq k^*$, there are at least $|T \cap 2^k| \cdot (1-\varepsilon)$ many fat nodes $s \in T \cap 2^k$.

Proof. (This is the same proof as the one of [FGKS17, Lemma 10.5.3].)

[35] "Basic structure" in the sense of "ignoring additional factors in the denominator or additive terms in the numerator, the norm is fundamentally of logarithmic character".

[36] Alternatively, we could simply amend Definition B3.2 (i) to demand $3n^B_{4k+2} \leq g_\xi(k)$, instead.

Let $\mu := \lambda([T])$. Since $|T \cap 2^k| \cdot 2^{-k}$ decreasingly converges to μ, there is some k^* such that for all $k \geq k^*$, we have

$$\frac{|T \cap 2^k|}{2^k} - \mu \cdot \varepsilon^2 \leq \mu. \tag{$*_9$}$$

Fix some $k \geq k^*$ and let f be the number of fat $s \in T \cap 2^k$ (and $\ell := |T \cap 2^k| - F$ the number of non-fat s). It suffices to show $f \geq \mu \cdot 2^k \cdot (1 - \varepsilon)$.

Note that

$$\mu < f \cdot \frac{1}{2^k} + \ell \cdot \frac{1-\varepsilon}{2^k} = \frac{|T \cap 2^k|}{2^k} - \frac{\ell \cdot \varepsilon}{2^k} \tag{$*_{10}$}$$

and hence Eq. ($*_9$) and Eq. ($*_{10}$) together imply $\mu \cdot 2^k \cdot \varepsilon \geq \ell$. Since $f + \ell = |T \cap 2^k| \geq \mu \cdot 2^k$, it follows that $f \geq \mu \cdot 2^k - \ell \geq \mu \cdot 2^k \cdot (1 - \varepsilon)$. □

Now recall that in section B8, we proved that $\mathbb{Q}$ had the Sacks property over the complete subforcing poset $\mathbb{Q}_{\text{non-ct}}$ (which consists of all conditions p with $\text{supp}(p) \cap A_{\text{ct}} = \varnothing$). In particular, this implied that any null set in the $\mathbb{Q}$-extension is already contained in some null set in the $\mathbb{Q}_{\text{non-ct}}$-extension.

We will now show that the set R of all reals read rapidly only using indices in $A_{\text{nm}} \cup A_{\text{nn}}$ is not null; by the consideration above, we can work entirely with $\mathbb{Q}_{\text{non-ct}}$ and show that it is not null there. As in the preceding section, we will work with sturdy trees instead of null sets.

Lemma B12.3. *Let $\dot{T}$ be a name for a sturdy tree and let $p \in \mathbb{Q}_{\text{non-ct}}$ rapidly read $\dot{T}$. Then there is a $q \leq p$ in $\mathbb{Q}_{\text{non-ct}}$ and a name $\dot{r}$ for a real such that q rapidly reads $\dot{r}$ only using indices in $A_{\text{nm}} \cup A_{\text{nn}}$ (i. e. not using any indices in $A_{\text{cn}} \cup A_{\text{slalom}}$) and such that $q \Vdash \dot{r} \in [\dot{T}]$.*

Proof. We will construct q and $\dot{r}$ by induction on $n < \omega$. For each n, we will define or show the following:

(i) We will define some $L_n := (4k_n, 0) \in \text{heights}_{\text{nm}}$.[37]

(ii) We will define conditions $q_n \leq p$ such that
- $\|q_n(4k)\|_{\text{nm},4k} \geq n + 3$ for all $k \geq k_n$,
- $q_{n+1} \leq q_n$,
- q_{n+1} and q_n are identical on $\text{supp}(q_n)$ below L_n and any new $\alpha \in \text{supp}(q_{n+1}) \setminus \text{supp}(q_n)$ only enter the support of q_{n+1} above L_n,
- $\|q_{n+1}(4k)\|_{\text{nm},4k} \geq n$ for all $k_n \leq k < k_{n+1}$, and
- for each $\alpha \in \text{supp}(q_{n+1}, L_n) \setminus A_{\text{nm}}$ of type t, there is a height L with $L_n < L < L_{n+1}$ such that $\|q_{n+1}(\alpha, L)\|_{\text{t},L} \geq n$.

Thus $\langle q_n \mid n < \omega \rangle$ will be a descending sequence of conditions converging to a condition q.

(iii) We will define some $i_n < \omega$ and a name $\dot{r}_n$ for an element of $\dot{T} \cap 2^{i_n}$ such that q_n decides $\dot{r}_n$ below L_n only using indices in $A_{\text{nm}} \cup A_{\text{nn}}$.

[37] Since we are working in $\mathbb{Q}_{\text{non-ct}}$, in contrast to section B7 we do not have to worry about the heights being p-agreeable.

(iv) We will require that i_n is not "too large" with respect to L_n in the sense that $2^{i_n+2} < n^B_{L_n}$. (Since n^B_L grows quickly and monotonously, it will suffice to show $2^{i_n+2} < 4k_n$.)[38]

(v) The i_n will be such that $i_{n+1} > i_n$.

(vi) The $\dot r_n$ will be such that $\dot r_{n+1}$ is forced (by q_{n+1}) to extend $\dot r_n$.

Thus q will force that $\dot r := \bigcup_{n<\omega} \dot r_n$ will the the desired branch in $[\dot T]$.

(vii) Finally, we will also construct a name $\dot T_n$ which q_n will force to be
- a subtree of $\dot T$ with stem $\dot r_n$ and relative measure greater than $1/2$ (i. e. $\lambda([\dot T_n]) > 1/2 \cdot 2^{-i_n}$)
- which is read continuously by q_n in such a way that below L_n, the reading only uses indices in $A_{\mathrm{nm}} \cup A_{\mathrm{nn}}$.

Step 0: To start the induction, define $i_0 := 0$, $\dot r_0 := \langle\,\rangle$ and $\dot T_0 := \dot T$. Choose $L_0 = (4k_0, 0)$ such that $\|p(4k')\|_{\mathrm{nm},4k'} \geq 3$ for all $k' \geq k_0 \geq 1$ ("≥ 1" to ensure property (iv)) and let q_0 be the condition resulting from extending the trunk of p to L_0. It is clear that properties (i)–(vi) are fulfilled by definition, and property (vii) holds since below L_0, there is only a single possibility, and hence the reading of $\dot T_0$ cannot depend on any indices in $A_{\mathrm{cn}} \cup A_{\mathrm{slalom}}$ below L_0.

In the following steps, assume we have constructed the required objects ($L_n = (4k_n, 0)$, q_n, i_n, $\dot r_n$ and $\dot T_n$) for some $n < \omega$; we will now proceed to construct them for $n+1$.

Step 1: Choose a height $L^* = (4k^*, 0)$ large enough such that for each $\alpha \in \mathrm{supp}(q_n, L_n) \setminus A_{\mathrm{nm}}$ of type t, there is a height L with $L_n < L < L^*$ such that $\|q_n(\alpha, L)\|_{\mathrm{t},L} \geq n+1$.

It is forced (by q_n) that Lemma B12.2 holds for $\dot T_n$ and $\varepsilon := 1/(n^P_{<L_n} \cdot n^P_{<L^*})$. Hence there is a name for a tree level $\dot m$ such that from $\dot m$ upwards, there are many fat nodes in $\dot T_n$. We can use Lemma B7.4 to strengthen q_n to q' such that

- q_n and q' are identical below L^*,
- the nm norms of q' remain at least $n+2$ starting from $4k^*$, and
- there is an $m^* > i_n$ such that $q' \Vdash m^* \geq \dot m$.

Hence Lemma B12.2 is forced to hold for this m^* as well, and there is a name $\dot F \subseteq \dot T_n \cap 2^{m^*}$ for a "large" set of fat nodes. This m^* will be our i_{n+1}.

Step 2: We apply Lemma B7.4 a second time to strengthen q' to q'' such that

- (q_n and) q' and q'' are identical below L^*,
- the nm norms of q'' remain at least $n+1$ starting from $4k^*$, and
- q'' essentially decides $\dot F$, i. e. q'' decides $\dot F$ below some height $L^{**} = (4k^{**}, 0)$.

Since we already know $\dot T$ is read continuously by p (and thus also by any stronger condition), we pick L^{**} large enough such that q'' decides $\dot T_n \cap 2^{i_{n+1}}$ below L^{**}, and also such that the nm norms of q'' are at least $n+4$ starting from $4k^{**}$ and $4k^{**} > 2^{i_{n+1}+2}$. This $L^{**} = (4k^{**}, 0)$ will be our $L_{n+1} = (4k_{n+1}, 0)$. So far, we have defined L_{n+1} and i_{n+1} and fulfilled properties (i), (iv) and (v).

[38] The reason for the term "+2" will become apparent in Step 3 of the construction below.

Step 3: The set $\dot{F}$ is forced to be a subset of $\dot{T}_n \cap 2^{i_{n+1}}$ of relative size at least $1-\varepsilon$, and both $\dot{F}$ and $\dot{T}_n \cap 2^{i_{n+1}}$ are decided by q'' below L_{n+1}. We also already know that $\dot{T}_n \cap 2^{i_{n+1}}$ does not depend on any indices in $A_{\mathrm{cn}} \cup A_{\mathrm{slalom}}$ below L_n. Hence we can construct a name $\dot{F}' \subseteq \dot{F}$, also not depending on such indices, such that $\dot{F}'$ has relative size at least $1 - \varepsilon \cdot n^P_{<L_n} = 1 - 1/n^P_{<L^*} \geq 1/2$, as follows:

Each $\eta \in \mathrm{poss}(q'', {<}L_{n+1})$ determines objects $F_\eta \subseteq S_\eta$ in the sense that

$$q'' \wedge \eta \Vdash \dot{F} = F_\eta \text{ and } \dot{T}_n \cap 2^{i_{n+1}} = S_\eta.$$

We call two possibilities $\eta, \eta' \in \mathrm{poss}(q'', {<}L_{n+1})$ equivalent if they differ only on indices in $A_{\mathrm{cn}} \cup A_{\mathrm{slalom}}$ below L_n. (Note that this implies $S_\eta = S_{\eta'}$.) Obviously, each equivalence class $[\eta]$ has size at most $n^P_{<L_n}$; for each such equivalence class, let $F'_{[\eta]} := \bigcap_{\vartheta \in [\eta]} F_\vartheta$; the relative size of any such $F'_{[\eta]}$ then is at least $1 - \varepsilon \cdot n^P_{<L_n}$. Hence the function mapping each η to $F'_{[\eta]}$ defines a name $\dot{F}'$ (not depending on any indices in $A_{\mathrm{cn}} \cup A_{\mathrm{slalom}}$ below L_n) for a subset of $\dot{T}_n \cap 2^{i_{n+1}}$ of relative size at least $1/2$.

Since $\dot{T}_n$ is forced to have $\dot{r}_n \in 2^{i_n}$ as its stem and measure greater than $1/2 \cdot 2^{i_n}$, the size of $\dot{T}_n \cap 2^{i_{n+1}}$ is forced to be greater than $2^{i_{n+1}-(i_n+1)}$, and the size of $\dot{F}'$ is then forced to be greater than $2^{i_{n+1}-(i_n+1)} \cdot 1/2 = 2^{i_{n+1}-i_n-2}$, which is greater than $2^{i_{n+1}}/n^B_{L_n}$ by property (iv).

So far, we have achieved the following: $\dot{T}_n \cap 2^{i_{n+1}}$ and its subset $\dot{F}'$ are decided by q'' below L_{n+1} not using any indices in $A_{\mathrm{cn}} \cup A_{\mathrm{slalom}}$ below L_n; q'' forces each $s \in \dot{F}'$ to fulfil $\lambda([\dot{T}_n] \cap [s]) \geq (1-\varepsilon) \cdot 2^{-i_{n+1}}$; and as a subset of $2^{i_{n+1}}$, $\dot{F}'$ is forced to have measure greater than $1/n^B_{L_n}$.

Step 4: We define the condition $q^* \leq q''$ by replacing all lim sup creatures in q'' starting from L^* and below L_{n+1} by arbitrary singletons. So q^* is identical to q_n below L_n, and identical to q'' starting from L_{n+1}. Note that so far, the nm norms of q^* remain at least $n+1$ starting from $4k_n$. In the next few (lengthy) steps, we will define q_{n+1} from q^* by modifying the creatures in q^* starting from L_n and below L_{n+1} such that afterwards, the nm norms of q_{n+1} will remain at least n starting from $4k_n$, and there will be witnesses for lim sup norms at least n between L_n and L_{n+1}, as required to fulfil property (ii).

Since q^* decides both $\dot{T}_n \cap 2^{i_{n+1}}$ and $\dot{F}'$ below L_{n+1} not using any indices in $A_{\mathrm{cn}} \cup A_{\mathrm{slalom}}$ below L_n, we decompose the set of possibilities $\mathrm{poss}(q^*, {<}L_{n+1})$ into $U \times V \times W$ as follows:

- $U := \mathrm{poss}(q^*, {<}L_n) = \mathrm{poss}(q, {<}L_n)$,
- V are the possibilities of q^* starting from L_n and below L^*, and
- W are the possibilities of q^* starting from L^* and below L_{n+1}, for which we only have to consider the nm part, as the lim sup part has just been defined to be arbitrary singletons.

We will now proceed as follows: For each $\nu \in W$, we will perform an induction on the heights starting from L_n up to $(L^*)^-$ to arrive at a candidate $D(\nu)$ for the creatures of q_{n+1} between L_n and L^*; we will then use bigness to see that for many $\nu \in W$, the candidates $D(\nu)$ will be equal, and we will use that fact to finally

define q_{n+1}.

Step 5: Fix some $\nu \in W$. Recall that relevant heights (in the context of this proof) are those $L \in \mathrm{heights}_{\mathrm{tg}}$ for some $\mathrm{tg} \neq \mathrm{ct}$ such that there is some $\alpha_L \in \mathrm{supp}(q^*) \cap A_{\mathrm{tg}}$ with a non-trivial $q^*(\alpha_L, L)$. We will inductively go through all heights L with $L_n \leq L < L^*$ (although we will only have to do something for relevant heights) and successively define conditions $q^L \leq q^*$ such that for any $L_n \leq K < L < L^*$

- $q^L \leq q^K$ and q^K and q^L are identical up to (including) K,
- the norm of $q^K(\alpha_K, K)$ decreased by at most 1 when compared with the norm of $q^*(\alpha_K, K)$, and
- the norm of $q^K(\alpha_L, L)$ decreased by at most i/n_L^B when compared with the norm of $q^*(\alpha_L, L)$, where i is the number of steps already performed in the induction (i. e. the number of heights between L_n and K).

This means that the induction successively strengthens the non-trivial creature at height L until the induction height is L itself; after that step, the non-trivial creature at height L is final and will no longer be modified.

We will also define functions F^L mapping each $\eta \in U \times V$ to a subset $F^L(\eta)$ of $2^{i_{n+1}}$ such that

- $q^{L^-} \wedge (\eta, \nu) \Vdash F^L(\eta) \subseteq \dot{F}'$,
- $F^L(\eta)$ is of relative size at least $1/n_L^B$, and
- $F^L(\eta)$ does not depend on any indices in $A_{\mathrm{cn}} \cup A_{\mathrm{slalom}}$ below L.

The preparation for the induction (so that we can start with $L = L_n$) is simply to set $q^{L_n^-} := q^*$ and $F^{L_n} := \dot{F}'$.[39]

Now assume we are at some step $L_n \leq L < L^*$ of the iteration and have already defined q^{K^-} and F^K for all $L_n \leq K < L$. If L is not a relevant height or if the associated index is in $A_{\mathrm{nm}} \cup A_{\mathrm{nn}}$, we do not have to do anything and can set $q^L := q^{L^-}$ and $F^{L^+} := F^L$. So assume the creature $C := q^{L^-}(\alpha_L, L)$ associated with the relevant height L is of type cn or slalom.

We now further decompose V (restricted to just those possibilities which are compatible with C) into

- V^-, the part below L,
- C, the part at height L, and
- V^+, the part strictly above L (and below L^*).

Hence we can write every $\eta \in U \times V$ (which is compatible with C) as (η^-, η^L, η^+), where $\eta^- \in U \times V^-$, $\eta^L \in C$ and $\eta^+ \in V^+$.

If we now fix η^- and η^+, the function F^L is reduced to an $F^{(\eta^-,\eta^+)}$ mapping each $X \in C$ to a subset of $2^{i_{n+1}}$ of relative size at least $1/n_L^B$. Hence we can use (depending on the type of C) either Lemma B11.4 or Lemma B12.1 to strengthen the creature

[39] We ask the reader to excuse the abuse of notation here; a name and a function are, of course, not the same thing, but for all practical purposes, they might as well be in the context of this step of the proof.

C to $D(\eta^-, \eta^+)$, decreasing the norm by at most $1/n^B_L$, such that

$$F^*(\eta^-, \eta^+) := \bigcap_{X \in D(\eta^-, \eta^+)} F^{(\eta^-, \eta^+)}(X)$$

is a set of relative size at least $1/n^B_{L^+}$.

If we now fix only η^+ and successively iterate this strengthening for all $\eta^- \in U \times V^-$, we ultimately arrive at some $D(\eta^+) \subseteq C$ with the norm decreasing by at most $n^P_{<L}/n^B_L < 1$ in total. Note that since $n^S_L < n^B_{L^+}$, there are less than $n^B_{L^+}$ many possible values for $D(\eta^+)$ and we can apply strong bigness in the form of Lemma B5.8 on the V^+ part to strengthen all $q^{L^-}(\alpha_K, K)$ for $L^+ \leq K < L^*$ to $q^L(K)$, decreasing the norm by at most $1/n^B_K$ at each height K, such that for each η^+ in the resulting smaller $\overline{V}^+$, we get the same $D := D(\eta^+)$.[40] This D then will be the (final) value of $q^L(\alpha_L, L)$. If we now define

$$F^{L^+}(\eta) := \bigcap_{X \in D} F^L(\eta^-, X, \eta^+),$$

by the considerations above, this is a set of relative size at least $1/n^B_{L^+}$, does not depend on any indices in $A_{\mathrm{cn}} \cup A_{\mathrm{slalom}}$ below L^+, and is forced to be a subset of $\dot{F}'$ by $q^L \wedge (\eta, \nu)$.

Having now defined q^L and F^{L^+}, we can proceed with the next step of the inductive construction.

Step 6: We perform the construction in Step 5 independently for each $\nu \in W$ (i. e. starting with the original q^* each time). We thus get a (potentially) different $q_\nu^{(L^*)^-}$ for each ν. Since the number of possible values for $q_\nu^{(L^*)^-}$ is less than $n^B_{L^*}$, we can now apply Lemma B5.8 again to thin out the creatures $q^*(\alpha_K, K)$ for $L^* \leq K < L_{n+1}$ to $q_*(\alpha_K, K)$, decreasing the norm by at most $1/n^B_K$ at each height K, such that for each ν in the resulting smaller $\overline{W}$, we get the same $q_{**} := q_\nu^{(L^*)^-}$. We can then finally define $q_{n+1} := q_{**}^{<L^*} \frown q_*^{\geq L^*}$, which fulfils property (ii) by construction.

Step 7: Now, this q_{n+1} forces the family of "terminal" $F_\nu^{(L^*)^-}$ (for $\nu \in \overline{W}$) to constitute a name $\dot{F}''$ for a subset of $\dot{F}' \subseteq 2^{i_{n+1}}$ of relative size greater than 0, and q_{n+1} decides $\dot{F}''$ below L_{n+1} not using any indices in $A_{\mathrm{cn}} \cup A_{\mathrm{slalom}}$ – due to the fact that below L_n, even the name $\dot{F}'$ did not depend on such indices; from L_n up to L^*, we removed the dependence on such creatures height by height in Step 5; and from L^* up to L_{n+1}, by Step 4 only singletons remain for such lim sup creatures, anyway.

Hence we can pick some name $\dot{r}_{n+1}$ for an arbitrary fixed element of $\dot{F}''$ (e. g. the first element in the natural lexicographic order), and this name fulfils properties (iii) (by construction) and (vi) (since $\dot{r}_{n+1}$ is a node in $\dot{T}_n$, whose stem is forced to be $\dot{r}_n$ by q_n).

Step 8: Since q_{n+1} forces $\dot{r}_{n+1} \in \dot{F}''$, $\dot{r}_{n+1}$ is a fat node, which means $\dot{T}' := \dot{T}_n \cap [\dot{r}_{n+1}]$ is forced to have measure greater than $\frac{1-\varepsilon}{2^{i_{n+1}}}$. The tree $\dot{T}'$ is read continuously by

[40] Keep in mind that since we are working in $\mathbb{Q}_{\text{non-ct}}$, there will be no $K \in \mathrm{heights}_{\mathrm{ct}}$, and hence we can apply Lemma B5.8.

q_n and hence also by q_{n+1}; in particular, for each $j > i_{n+1}$, the finite initial tree $\dot{T}' \cap 2^j$ is decided below some L_j. For each $\eta \in \mathrm{poss}(q_{n+1}, {<}L_j)$, let T^j_η be the corresponding value of $\dot{T}' \cap 2^j$ (which is a subset of 2^j with at least $2^j \cdot \frac{1-\varepsilon}{2^{i_{n+1}}}$ many elements). It is clear that for $j < j'$ and $\eta \in \mathrm{poss}(q_{n+1}, {<}L_j)$, $\eta' \in \mathrm{poss}(q_{n+1}, {<}L_{j'})$ such that $\eta \subseteq \eta'$, it is forced that the corresponding finite trees are also nested, i. e. $T^{j'}_{\eta'} \subseteq T^j_\eta$.

We now implement a reduction similar to Step 3 to eliminate the dependency on indices in $A_{\mathrm{cn}} \cup A_{\mathrm{slalom}}$: We call two possibilities $\eta, \eta' \in \mathrm{poss}(q_{n+1}, {<}L_j)$ equivalent if they differ only on indices in $A_{\mathrm{cn}} \cup A_{\mathrm{slalom}}$ below L_{n+1}. Since from L^* up to L_{n+1}, there are only singletons for such lim sup creatures, each equivalence class $[\eta]$ has size at most $n^P_{<L^*}$. For each such equivalence class, let $T^j_{[\eta]} := \bigcap_{\vartheta \in [\eta]} T^j_\vartheta$. Note that by the nesting of the T^j_ϑ, the $T^j_{[\eta]}$ are also nested (for j, j', η, η' as above), and the size of $T^j_{[\eta]}$ is at least

$$\frac{2^j \cdot (1 - \varepsilon \cdot n^P_{<L^*})}{2^{i_{n+1}}} = \frac{2^j \cdot (1 - 1/n^P_{<L_n})}{2^{i_{n+1}}}.$$

So q_{n+1} forces the family of such $T^j_{[\eta]}$ (for $j > i_{n+1}$ and $\eta \in \mathrm{poss}(q_{n+1}, {<}L_j)$) to constitute a name $\dot{T}_{n+1}$ as required to fulfil property (vii). □

Corollary B12.4. $\mathbb{Q}$ *forces* $\mathrm{non}(\mathcal{N}) \leq \kappa_{\mathrm{nn}}$.

Proof. Fix a condition p and a sequence of names of null sets $\langle \dot{N}_i \mid i \in I \rangle$ which p forces to be a basis of null sets. As described above, for each $i \in I$, we can assume that $\dot{N}_i = N_{\dot{T}_i}$ for some name for a sturdy tree $\dot{T}_i$. Let $\dot{R}$ consist of all reals read rapidly only using indices in $A_{\mathrm{nm}} \cup A_{\mathrm{nn}}$.

By the preceding lemma, for each $\dot{T}_i$, there is a $q \leq p$ in $\mathbb{Q}_{\text{non-ct}}$ and an $\dot{r} \in \dot{R}$ such that $q \Vdash \dot{r} \in [\dot{T}_i]$ and hence $\mathbb{Q}_{\text{non-ct}} \Vdash \dot{r} \notin N_{\dot{T}_i}$; it follows that $\mathbb{Q}_{\text{non-ct}} \Vdash$ "$\dot{R}$ is not null" and hence also $\mathbb{Q} \Vdash$ "$\dot{R}$ is not null". □

This proves (M3) of Theorem B1.1, and hence completes the proof of that theorem entirely.

B13 Failed Attempts and Open Questions

To counteract the common habit of only talking about successes and withholding the failed attempts that went before, we want to give a very brief account of two results we attempted, but failed to achieve in the course of writing this chapter.

For one, we wanted to add κ_{rp} many factors which would carefully increase the cardinals $\mathfrak{r}$ and $\mathfrak{u}$ to κ_{rp}, a cardinal between κ_{cn} and κ_{ct}. The plan was to use a forcing poset $\mathbb{Q}_{\mathrm{rp}}$ (a variant of the forcing poset from [GS90]) in each factor. While it seemed quite simple to align the structure of $\mathbb{Q}_{\mathrm{rp}}^{\kappa_{\mathrm{rp}}}$ with the structure of $\mathbb{Q}_{\mathrm{ct},\,\kappa_{\mathrm{ct}}}$ to allow the proof of section B8 to function for both $\mathbb{Q}_{\mathrm{rp}}^{\kappa_{\mathrm{rp}}}$ and $\mathbb{Q}_{\mathrm{ct},\,\kappa_{\mathrm{ct}}}$, it was not clear why the κ_{ct} many Sacks-like reals would preserve $\mathfrak{u} \leq \kappa_{\mathrm{rp}}$, or indeed why the

old reals would be unreapable even after multiplying the forcing poset with the product of merely two copies of $\mathbb{Q}_{\mathrm{rp}}$.

The second idea we had was to add Cohen forcing to the construction to control the value of $\mathrm{cov}(\mathcal{M})$ in the resulting model. This would have complicated a lot of the proofs, since many things would then have turned into names dependent on the Cohen-generic filter; however, a more fundamental problem is that this approach destroys the Sacks property of the "upper" part of the construction:

Lemma B13.1. *Let $\mathbb{C}$ be the Cohen forcing poset and let $\mathbb{S}$ be the Sacks forcing poset. Then $V^{\mathbb{C}\times\mathbb{S}}$ does not have the Sacks property over $V^{\mathbb{C}}$.*

More generally, consider two forcing posets $\mathbb{X}$ and $\mathbb{Y}$, where $\mathbb{X}$ adds an unbounded real $\dot{x}$ and $\mathbb{Y}$ adds another new real $\dot{y}$. Then $V^{\mathbb{X}\times\mathbb{Y}}$ does not have the Sacks property over $V^{\mathbb{X}}$.

Proof. We prove the stronger claim. Let $\dot{\tau}$ denote the name of some code for $\dot{x}$-dependent initial segments of the other real $\dot{y}$, i. e. $\dot{\tau}(n) := \langle \dot{y}|_{\dot{x}(n)}\rangle$.

Assume that we have some sequence of $\mathbb{X}$-names $\dot{B}_k$ for a $(k+1)$-slalom catching $\dot{\tau}(k)$:

$$\Vdash_{\mathbb{X}} |\dot{B}_k| = k+1$$
$$(p,q) \Vdash \forall\, k<\omega\colon \dot{B}_k \subseteq 2^{\dot{x}(k)} \wedge \dot{\tau}(k) \in \dot{B}_k$$

Let n be the index of the first value of $\dot{x}$ not bounded by p. Let T be the tree of initial segments of $\dot{y}$; since $\dot{y}$ is a new real, T must have unbounded width. Hence there is some m such that the $\dot{\tau}(m)$ has at least $n+2$ many possible values. Fix such an m. Then find $p' \leq p$ forcing $\dot{x}(n) = m^* \geq m$, and let $p^* \leq p'$ be such that p^* decides $\dot{B}_n$, i. e. $p^* \Vdash \dot{B}_n = B$ for some B.

Since $\dot{\tau}(m^*)$ has at least $n+2$ many possible values, there is some possible value v that is not in B. But then there is a $q^* \leq q$ forcing $\dot{\tau}(m^*) = v$, and hence

$$(p^*, q^*) \Vdash \dot{\tau}(m^*) \notin B,$$

which is a contradiction. □

The point of this lemma is that if we do not have the Sacks property, $\mathrm{cof}(\mathcal{N})$ will increase.

We turn our attention towards related work and open questions. Several recent results [KTT17, GKS17, KST17] have constructed models in which eight or even all ten conceivably different cardinal characteristics in Cichoń's diagram take different values. The constructions involved are all finite support iterations, however, which necessarily means the left side of Cichoń's diagram must be less than or equal to the right side, in particular $\mathrm{non}(\mathcal{M}) \leq \mathrm{cov}(\mathcal{M})$ (since the cofinality of the iteration length lies between these two cardinal characteristics). In contrast, [FGKS17] and our improvement thereof have $\mathrm{non}(\mathcal{M}) > \mathrm{cov}(\mathcal{M})$.

However, as far as Cichoń's diagram is concerned, our creature forcing construction still has rather strict limitations as explained in the preceding section. Necessarily, $\mathfrak{d} = \aleph_1$ by the ω^ω-boundedness of the forcing posets involved; the only open question regarding Cichoń's diagram and our construction is whether it is possible to separate $\mathrm{cov}(\mathcal{N})$ from $\aleph_1$.

Question F. *Is it possible to modify the construction to achieve* $\aleph_1 < \mathrm{cov}(\mathcal{N})$*?*

Finally, our failed attempt to introduce $\mathfrak{r}$ and $\mathfrak{u}$ into the construction motivates the following general question:

Question G. *Are there any well-known cardinal characteristics which can be set via a* $\limsup$*-type creature forcing poset compatible with the structure of* $\mathbb{Q}$*?*

CHAPTER C

HALFWAY NEW CARDINAL CHARACTERISTICS

This chapter is based on [BHK+18], which is joint work with Jörg Brendle, Lorenz J. Halbeisen, Marc Lischka and Saharon Shelah.

C1 Introduction

Like the first two chapters, this research forms part of the study of cardinal characteristics of the continuum.

Based on the well-known cardinal characteristics $\mathfrak{s}$, $\mathfrak{r}$ and $\mathfrak{i}$,[41] we were inspired to define specialised variants of these (all of them related in some way to asymptotic density, in particular asymptotic density $1/2$) and successfully proved a number of bounds and consistency results for them.

We will use the following concept in a few of the proofs:

Definition C1.1. A *chopped real* is a pair (x, Π) where $x \in 2^\omega$ and Π is an interval partition of ω. We say a real $y \in 2^\omega$ *matches* (x, Π) if $y{\restriction_I} = x{\restriction_I}$ for infinitely many $I \in \Pi$.

We note that the set $\mathrm{Match}(x, \Pi)$ of all reals matching (x, Π) is a comeagre set (see [Bla10, Theorem 5.2]).

We remark that we will not rigidly distinguish between a real r in 2^ω and the set $R := r^{-1}(1)$, or conversely, between a subset of ω and its characteristic function.

This chapter is structured as follows.

[41] See the preface to this thesis for definitions and references.

- In section C2, we introduce and work on several cardinal characteristics related to $\mathfrak{s}$.
- In section C3, we conduct a particularly sophisticated proof for a consistency claim from the preceding section.
- In section C4, we introduce and work on cardinal characteristics mostly related to $\mathfrak{r}$ and $\mathfrak{i}$.
- The final section C5 summarises the open questions.

C2 Characteristics Related to $\mathfrak{s}$

Recall the following concepts from number theory.

Definition C2.1. For $X \in [\omega]^\omega$ and $0 < n < \omega$, define the *initial density* (of X up to n) as

$$d_n(X) := \frac{|X \cap n|}{n}$$

and the *lower* and *upper density* of X as

$$\underline{d}(X) := \liminf_{n\to\infty}(d_n(X)) \quad \text{and} \quad \bar{d}(X) := \limsup_{n\to\infty}(d_n(X)),$$

respectively. In case of convergence of $d_n(X)$, call

$$d(X) := \lim_{n\to\infty}(d_n(X))$$

the *asymptotic density* or just the *density* of X.

We define four relations on $[\omega]^\omega \times [\omega]^\omega$ and their associated cardinal characteristics.

Definition C2.2. Let $S, X \in [\omega]^\omega$. We define the following relations:

- *S bisects X in the limit* (or just *S bisects X*), written as $S \mid_{1/2} X$, if

$$\lim_{n\to\infty} \frac{|S \cap X \cap n|}{|X \cap n|} = \lim_{n\to\infty} \frac{d_n(S \cap X)}{d_n(X)} = \frac{1}{2}.$$

- For $0 < \varepsilon < 1/2$, *S ε-almost bisects X*, written as $S \mid_{1/2\pm\varepsilon} X$, if for all but finitely many $n < \omega$ we have

$$\frac{|S \cap X \cap n|}{|X \cap n|} = \frac{d_n(S \cap X)}{d_n(X)} \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right).$$

- *S weakly bisects X*, written as $S \mid^w_{1/2} X$, if for any $\varepsilon > 0$, for infinitely many $n < \omega$ we have

$$\frac{|S \cap X \cap n|}{|X \cap n|} = \frac{d_n(S \cap X)}{d_n(X)} \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right).$$

- *S bisects X infinitely often*, written as $S\ |^{\infty}_{1/2}\ X$, if for infinitely many $n < \omega$ we have

$$\frac{|S \cap X \cap n|}{|X \cap n|} = \frac{d_n(S \cap X)}{d_n(X)} = \frac{1}{2}.$$

Definition C2.3. We say a family $\mathcal{S}$ of infinite sets is

$$\begin{cases} \text{bisecting (in the limit)} \\ \varepsilon\text{-almost bisecting} \\ \text{weakly bisecting} \\ \text{infinitely often bisecting} \end{cases}$$

if for each $X \in [\omega]^\omega$ there is some $S \in \mathcal{S}$ such that

$$\begin{cases} S \text{ bisects } X \text{ (in the limit)} \\ S\ \varepsilon\text{-almost bisects } X \\ S \text{ weakly bisects } X \\ S \text{ bisects } X \text{ infinitely often} \end{cases}$$

and denote the least cardinality of such a family by $\mathfrak{s}_{1/2}$, $\mathfrak{s}_{1/2\pm\varepsilon}$, $\mathfrak{s}^w_{1/2}$, $\mathfrak{s}^\infty_{1/2}$, respectively.

Theorem C2.4. *The relations shown in Figure 10 hold.*

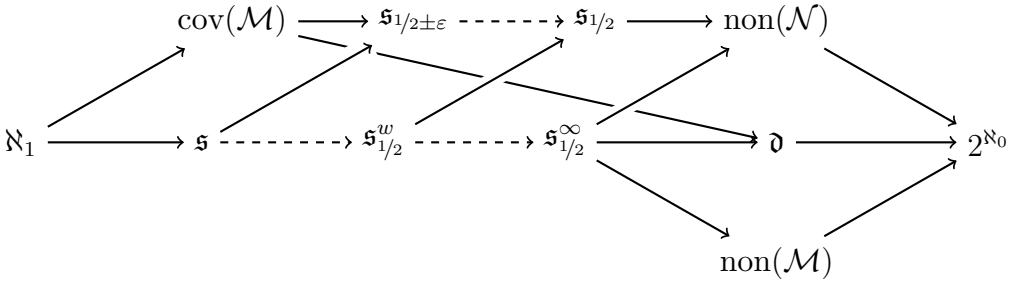

Figure 10: The ZFC-provable and/or consistent inequalities between $\mathfrak{s}_{1/2}$, $\mathfrak{s}_{1/2\pm\varepsilon}$, $\mathfrak{s}^w_{1/2}$, $\mathfrak{s}^\infty_{1/2}$ and other well-known cardinal characteristics, where $\longrightarrow$ means "$\leq$, consistently $<$" and $\dashrightarrow$ means "$\leq$, possibly $=$".

Proof. Recall that it is known that $\mathfrak{s} \leq \mathrm{non}(\mathcal{M})$ and $\mathfrak{s} \leq \mathrm{non}(\mathcal{N})$ (see e. g. [Bla10, Theorem 5.19]) as well as $\mathfrak{s} \leq \mathfrak{d}$ (see e. g. [Hal17, Theorem 9.4] or [Bla10, Theorem 8.13]).

$\mathfrak{s} \leq \mathfrak{s}^w_{1/2} \leq \mathfrak{s}^\infty_{1/2}$: An infinitely often bisecting real is a weakly bisecting real (being equal to $1/2$ infinitely often implies entering an arbitrary ε-neighbourhood of $1/2$

infinitely often), and a weakly bisecting real is a splitting real (if a real X does not split another real Y, the relative initial density of X in Y, that is

$$\frac{d_n(X \cap Y)}{d_n(Y)},$$

cannot be close to $1/2$ infinitely often). Hence a family witnessing the value of $\mathfrak{s}^{\infty}_{1/2}$ gives an upper bound for the value of $\mathfrak{s}^{w}_{1/2}$ (and analogously for $\mathfrak{s} \leq \mathfrak{s}^{w}_{1/2}$).

$\mathfrak{s} \leq \mathfrak{s}_{1/2\pm\varepsilon} \leq \mathfrak{s}_{1/2}$: The first claim follows since an ε-almost bisecting real is a splitting real by the fact that finite sets have density 0 and cofinite sets have density 1, and hence if X does not split Y, the relative initial densities of X and $\omega \setminus X$ in Y tend to 0 and 1, respectively (or vice versa). The second claim follows since a bisecting real is an ε-almost bisecting real by definition.

$\mathbf{cov}(\mathcal{M}) \leq \mathfrak{s}_{1/2\pm\varepsilon}$: Given a family $\mathcal{S}$ witnessing the value of $\mathfrak{s}_{1/2\pm\varepsilon}$, take $S \in \mathcal{S}$. Define a chopped real based on S with the interval partition having the partition boundaries at the $n!$-th elements of S; the sets matching this chopped real form a comeagre set which consists of reals not halved by S (as the matching intervals grow longer and longer, "pulling" the relative initial density above $1 - 1/n$). Hence the family $E(S)$ of those reals that *are* ε-almost bisected by S is a meagre set (as its complement is a superset of a comeagre set), and $\{E(S) \mid S \in \mathcal{S}\}$ is a 2^{ω}-covering consisting of meagre sets.

$\mathfrak{s}^{w}_{1/2} \leq \mathfrak{s}_{1/2}$: A bisecting real is a weakly splitting real – for the relative density to converge to $1/2$, it has to eventually be arbitrarily close to $1/2$, and hence also within an arbitrary ε-neighbourhood of $1/2$ infinitely often. The same argument using the families witnessing the cardinal characteristics holds.

$\mathfrak{s}^{\infty}_{1/2} \leq \mathbf{non}(\mathcal{M})$: For a given $X \in [\omega]^{\omega}$, we show that the set $B(X)$ of reals bisecting X infinitely often (contains and hence) is a comeagre set. For any $F \notin \mathcal{M}$, $F \cap B(X)$ is non-empty, hence it contains a real bisecting X infinitely often.

Given X as above, let $f(n) := \sum_{k=0}^{n} k!$ and define an interval partition Π with partition boundaries precisely after the $f(2n)$-th elements of X. Define a chopped real (S, Π) as follows: Let $S \cap (\omega \setminus X) = \varnothing$ (i. e. S contains no elements not in X). For each $0 < n < \omega$, the n-th interval $I_n \in \Pi$ contains at least $(2n-1)! + (2n)!$ elements of X. Let S skip the first $(2n-1)!$ of these elements and contain the rest. Any real that matches (S, Π) indeed has a lower relative density of 0 in X and an upper relative density of 1 in X and hence bisects X infinitely often. The set of all reals matching (S, Π) is comeagre, as required to finish the proof above.

$\mathfrak{s}^{\infty}_{1/2} \leq \mathfrak{d}$: Let $\mathcal{D}$ be a dominating family. Without loss of generality assume that every member g of $\mathcal{D}$ is strictly increasing and satisfies $g(0) > 0$. Let $X \in [\omega]^{\omega}$ and let f_X be its enumeration. Pick a $g_X =: g$ from $\mathcal{D}$ that dominates f_X and define $G \colon \omega \to \omega$ by $G(n) := g^{(n+1)}(0)$ for every $n < \omega$. Then, for sufficiently large n,

$$G(n) \leq f_X(G(n)) < g(G(n)) = G(n+1).$$

Hence (for sufficiently large n) every interval $[G(n), G(n+1))$ contains at least one element of X and at most $G(n+1) - G(n)$ many. Now iteratively define a function

$\Gamma\colon \omega \to \omega$ by $\Gamma(0) := 0$, $\Gamma(1) := G(0) = g(0)$ and $\Gamma(n+1) := G\big(\sum_{k=0}^{n}\Gamma(k)\big) = G(\Sigma_n)$ and consider the interval partition with partition boundaries $\langle \Gamma(n) \mid n < \omega\rangle$; for sufficiently large n, every interval

$$\begin{aligned} I_n :=& \Big[\Gamma(n), \Gamma(n+1)\Big) = \Big[G\big(\sum_{k=0}^{n-1}(\Gamma(k))\big), G\big(\sum_{k=0}^{n}(\Gamma(k))\big)\Big) \\ =& \Big[G(\Sigma_{n-1}), G(\Sigma_{n-1}+1)\Big) \cup \ldots \cup \Big[G(\Sigma_{n-1}+\Gamma(n)-1), G(\Sigma_{n-1}+\Gamma(n))\Big) \end{aligned}$$

contains at least $\Gamma(n)$ many elements of X and at most $\Gamma(n+1)-\Gamma(n)$ many of them.

The real defined as the union of every other interval, i. e. the intervals $I_{2k} = [\Gamma(2k), \Gamma(2k+1))$, will yield a real Y_X bisecting X infinitely often: Since the number of elements of X which are in any interval I_n is at least as large as the lower boundary of I_n, and since Y_X is defined to alternate between consecutive intervals, this means the relative initial density infinitely often reaches $1/2$, as each I_{2k} "pushes" the relative initial density above $1/2$ (and each I_{2k+1}, which is disjoint from Y_X, "pulls" it below $1/2$).

$\mathfrak{s}^{\infty}_{1/2} \leq \mathbf{non}(\mathcal{N})$: Given some $X \in [\omega]^\omega$ with enumerating function f_X and a Lebesgue-random set S (i. e. such that $\forall\, n < \omega\colon \Pr[n \in S] = 1/2$), the function $g(n) := |X \cap S \cap f_X(n)| - n/2$ defines a balanced random walk with step size $1/2$, since

$$g(n+1) - g(n) = \begin{cases} +1/2 & f_X(n) \in S, \\ -1/2 & f_X(n) \notin S. \end{cases}$$

From probability theory we know that for almost all S, $g(n)$ will be 0 infinitely often. Equivalently, almost surely,

$$\frac{g(n)}{n} + \frac{1}{2} = \frac{|X \cap S \cap f_X(n)|}{n}$$

will be $1/2$ infinitely often.

In other words, for any $X \in [\omega]^\omega$, the set of all S not bisecting X infinitely often is a null set. By contraposition, for any $X \in [\omega]^\omega$, any non-null set contains a set S that bisects X infinitely often.

$\mathfrak{s}_{1/2} \leq \mathbf{non}(\mathcal{N})$: Let $X \in [\omega]^\omega$ and $F \notin \mathcal{N}$. Enumerating $X =: \{x_0, x_1, x_2, \ldots\}$, we define functions $f_{X,n}$ and f_X as follows:

$$f_{X,n}\colon [\omega]^\omega \to \{0,1\}\colon Y \mapsto \begin{cases} 0 & x_n \notin Y \\ 1 & x_n \in Y \end{cases}$$

$$f_X\colon [\omega]^\omega \to [0,1]\colon Y \mapsto \begin{cases} \lim_{k\to\infty} \dfrac{\sum_{n=1}^{k} f_{X,n}(Y)}{k} & \text{if the limit exists} \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $\lambda(f_{X,n}^{-1}(\{1\})) = 1/2$. Hence, the $f_{X,n}$ are identically distributed random variables on the probability space $[\omega]^\omega$ with probability measure the Lebesgue

measure λ. Moreover, they are independent and have finite variance. By the law of large numbers it follows that f_X is almost surely equal to $1/2$, in other words $\lambda(f_X^{-1}(\{1/2\})) = 1$. This means that with

$$S_X := \{Y \in [\omega]^\omega \mid f_X(Y) = 1/2\} = \{Y \in [\omega]^\omega \mid Y \mid_{1/2} X\},$$

we have that $\lambda(S_X) = 1$ and hence $S_X \notin \mathcal{N}$. Hence $F \cap S_X \neq \varnothing$ and there is some $S \in F$ such that $S \mid_{1/2} X$. Since all this holds for any $X \in [\omega]^\omega$, we have $\mathfrak{s}_{1/2} \leq \mathrm{non}(\mathcal{N})$.

Con($\mathrm{non}(\mathcal{M}) < \mathfrak{s}_{1/2\pm\varepsilon}$): This is implied by Con($\mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$) as witnessed by the Cohen model.

Con($\mathfrak{s}^\infty_{1/2} < \mathfrak{s}_{1/2\pm\varepsilon}$): This also follows from Con($\mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$).

Con($\mathfrak{s}^\infty_{1/2} < \mathrm{non}(\mathcal{M})$), **Con($\mathfrak{s}^\infty_{1/2} < \mathfrak{d}$)** and **Con($\mathfrak{s}^\infty_{1/2} < \mathrm{non}(\mathcal{N})$)**: In the Cohen model, we have $\aleph_1 = \mathfrak{s} = \mathfrak{s}^\infty_{1/2} = \mathrm{non}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) = \mathfrak{d}$; and in the random model, we have $\aleph_1 = \mathfrak{s}^\infty_{1/2} = \mathfrak{d} < \mathrm{non}(\mathcal{M})$.

Con($\mathrm{cov}(\mathcal{M}) < \mathfrak{s} \leq \mathfrak{s}_{1/2}$): In the Mathias model, we have $\mathrm{cov}(\mathcal{M}) < \mathfrak{s} = 2^{\aleph_0}$, see [Hal17, Theorem 26.14].

Con($\mathfrak{s}_{1/2} < \mathrm{non}(\mathcal{N})$): See Theorem C3.5 in the subsequent section. □

Finally, we remark that $\mathfrak{b}$ is incomparable with all of our newly defined cardinal characteristics. This is because in the Cohen model, $\mathfrak{s}$ is strictly above $\mathfrak{b}$ and so are all of our characteristics; and in the Laver model, $\mathrm{non}(\mathcal{N})$ is strictly below $\mathfrak{b}$ and so are all of our characteristics.

C3 Separating $\mathfrak{s}_{1/2}$ and $\mathrm{non}(\mathcal{N})$

To prove Con($\mathfrak{s}_{1/2} < \mathrm{non}(\mathcal{N})$), we will use a typical creature forcing construction to increase $\mathrm{non}(\mathcal{N})$ and show that the forcing poset does not increase $\mathfrak{s}_{1/2}$.

We will not go into too much detail regarding creature forcing in this chapter; see [RS99] for the most general and most detailed explanation, or refer to the previous two chapters for more concise expositions. The specific forcing poset we use here also appears in [FGKS17] and chapter B.

Definition C3.1. We define a forcing poset $\mathbb{P}$ as follows: A condition $p \in \mathbb{P}$ is a sequence of *creatures* $p(k)$ such that each $p(k)$ is a non-empty subset of

$$\mathrm{POSS}_k := \left\{ F \subseteq 2^{I_k} \;\middle|\; \frac{|F|}{|2^{I_k}|} \geq 1 - \frac{1}{2^{a_k}} \right\}$$

for some sufficiently large consecutive intervals $I_k \subseteq \omega$ and strictly increasing $a_k < \omega$ (for our construction, let I_k be an interval of length 2^{2^k} and let $a_k := k$) and such that, letting the *norm* $\|\cdot\|$ of a creature C be defined by $\|C\| := \log_2 |C|$, p fulfils $\limsup_{k\to\infty} \|p(k)\| = \infty$. The order is $q \leq p$ iff $q(k) \subseteq p(k)$ for all $k < \omega$

(i. e. stronger conditions consist of smaller subsets of POSS_k). Note that $\mathbb{P} \neq \varnothing$ since $\limsup_{k\to\infty} \| \mathrm{POSS}_k \| = \infty$.

Given a condition p such as above, the finite initial segments in $p{\restriction}_{k+1}$ (for $k < \omega$) are sometimes referred to as *possibilities* and denoted by $\mathrm{poss}(p, {\leq}k) := \prod_{\ell\leq k}[p(\ell)]^1 = \{\langle\{z(\ell)\} \mid \ell \leq k\rangle \mid \forall\, \ell \leq k\colon z(\ell) \in p(\ell)\}$. We may also use the notation $\mathrm{poss}(p, {<}k) := \mathrm{poss}(p, {\leq}k-1)$. When $\eta \in \mathrm{poss}(p, {\leq}k)$, we write $p \wedge \eta$ to denote $\eta^\frown p{\restriction}_{[k+1,\omega)}$.[42]

Define the forcing poset $\mathbb{Q}$ as the countable support product $\mathbb{Q} := \prod_{\alpha<\omega_2} \mathbb{Q}_\alpha$, where each $\mathbb{Q}_\alpha = \mathbb{P}$. We will work with the dense subset of *modest* conditions of $\mathbb{Q}$, i. e. conditions $p \in \mathbb{Q}$ such that for each $k < \omega$, there is at most one index α_k such that $|p(\alpha_k, k)| > 1$. We call such creatures $p(\alpha_k, k)$ *non-trivial*. (An easy bookkeeping argument shows that the modest conditions do indeed form a dense subset of $\mathbb{Q}$.) Modest conditions p have the advantage that for each $k < \omega$, $\mathrm{poss}(p, {<}k)$ is finite and even bounded by $\mathrm{maxposs}({<}k) := \prod_{j<k} |\,\mathrm{POSS}_k\,|$, which makes iterating over all possibilities below a certain level possible.

By the usual Δ-system argument, CH implies that $\mathbb{Q}$ is $\aleph_2$-cc. (For details, see [FGKS17, Lemma 3.3.1] or Lemma B4.18.) By the usual creature forcing arguments, it is clear that $\mathbb{Q}$ satisfies the finite version of Baumgartner's axiom A and hence is proper and ω^ω-bounding, that $\mathbb{Q}$ continuously reads all reals and that $\mathbb{Q}$ preserves all cardinals and cofinalities. (For details, see [FGKS17, section 5] or section B6 and section B7 in chapter B.) In particular, given any condition $p \in \mathbb{Q}$ and any name $\dot r$ for a real, we can find $q \leq p$ such that each $\eta \in \mathrm{poss}(q, {<}k)$ already decides $\dot r{\restriction}_{\min(I_k)}$ (which we refer to as "*q reads $\dot r$ rapidly*"). We will reproduce an abbreviated version of the proof of $V^{\mathbb{Q}} \vDash \mathrm{non}(\mathcal{N}) \geq \aleph_2$ here:

Lemma C3.2. *Assuming* CH *in the ground model,* $\mathbb{Q}$ *forces that* $\mathrm{non}(\mathcal{N}) \geq \aleph_2$.

Proof. First, note that for $\alpha < \omega_2$, the generic object $\dot R_\alpha$ is a sequence of $\dot R_\alpha(k) \subseteq 2^{I_k}$ of relative size at least $1 - 1/2^{a_k}$. Since $\langle a_k \mid k < \omega\rangle$ is strictly increasing, it is clear that

$$\prod_{k<\omega} \left(1 - \frac{1}{2^{a_k}}\right) > 0$$

and hence the set

$$\{r \in 2^\omega \mid \forall\, k < \omega\colon r{\restriction}_{I_k} \in \dot R_\alpha(k)\}$$

is positive and

$$\dot N_\alpha := \{r \in 2^\omega \mid \exists^\infty k < \omega\colon r{\restriction}_{I_k} \notin \dot R_\alpha(k)\}$$

[42] The usual creature forcing notation, which is used quite a lot in chapter B, defines the set of possibilities more abstractly as $\mathrm{poss}(p, {\leq}k) := \prod_{\ell\leq k} p(\ell)$ and defines $p \wedge \eta$ as a condition with an extended *trunk* (a concept which we did not deem necessary to introduce in this third chapter). Since working with possibilities η as sequences of singletons suffices for our proofs and is conceptually easier, we instead opted for the simpler definition in this third chapter.

is a name for a null set.

Now, given a name $\dot{r} \in 2^\omega$ for a real and a $p \in \mathbb{Q}$ which reads $\dot{r}$ rapidly, we can pick an $\alpha < \omega_2$ not in the support of p and add it to the support to get a (without loss of generality) modest condition p'; then p' still reads $\dot{r}$ rapidly not using the index α. Since we only require the lim sup of the norms to go to infinity, one can then show that $p' \Vdash \dot{r} \in \dot{N}_\alpha$. From this fact and $\aleph_2$-cc, it follows that for any $\kappa < \omega_2$, any sequence of names of reals $\langle \dot{r}_i \mid i < \kappa \rangle$ is contained in a null set of $V^{\mathbb{Q}}$.[43] □

We will now prove that the ground model reals are a bisecting family in $V^{\mathbb{Q}}$. To show this, we will use the following combinatorial lemma.

Lemma C3.3. *If $R, S \subseteq \omega$ are disjoint finite sets of sizes r and s, respectively, $s = c \cdot r$ for some $c > 1$, and $A \subseteq R$, $B \subseteq S$ such that*

$$\frac{|B|}{|S|} \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)$$

for some $\varepsilon > 0$, then

$$\frac{|A \cup B|}{|R \cup S|} \in \left(\frac{1}{2} - \varepsilon - \frac{1}{c}, \frac{1}{2} + \varepsilon + \frac{1}{c}\right).$$

Proof. Since

$$\frac{1}{1 + 1/c} \geq 1 - \frac{1}{c},$$

we have the lower bound

$$\begin{aligned}\frac{|A \cup B|}{|R \cup S|} &> \frac{s \cdot (1/2 - \varepsilon)}{r + s} = \frac{s \cdot (1/2 - \varepsilon)}{s \cdot 1/c + s} = \frac{1/2 - \varepsilon}{1 + 1/c} \\ &\geq \left(\frac{1}{2} - \varepsilon\right)\left(1 - \frac{1}{c}\right) \geq \frac{1}{2} - \varepsilon - \frac{1}{c}.\end{aligned}$$

For the upper bound, we get

$$\begin{aligned}\frac{|A \cup B|}{|R \cup S|} &< \frac{r + s \cdot (1/2 + \varepsilon)}{r + s} = \frac{s \cdot 1/c + s \cdot (1/2 + \varepsilon)}{s \cdot 1/c + s} \\ &= \frac{1/2 + \varepsilon + 1/c}{1 + 1/c} \leq \frac{1}{2} + \varepsilon + \frac{1}{c}.\end{aligned}$$

□

[43] The actual argument for $p \Vdash \dot{r} \in \dot{N}_\alpha$ involves a slightly more complicated norm than we defined above; however, since the parameters of the creature forcing poset $\mathbb{P}$ are immaterial for the more complicated proof in Lemma C3.4 below, we opted to omit the details for this chapter. Details can be found in section B11.

Lemma C3.4. *$2^\omega \cap V$ is a bisecting family in $V^{\mathbb{Q}}$.*

Proof. We will show the following: Given a modest condition $p \in \mathbb{Q}$ and a name $\dot{Y}$ for a real, we can find $q \leq p$ and a ground model real X such that $q \Vdash X \mid_{1/2} \dot{Y}$.

In order to do this, we will construct $p^* \leq p$ as well as $m_0 := 0 < m_1 < m_2 < \dots$ and choose $\langle P_i \mid i < \omega\rangle$ with $P_0 := 1/2$, $P_i > 0$ for all $i < \omega$ and $\lim_{i\to\infty} P_i = 0$ such that the following statements hold:

(i) The condition p^* is not only modest, but even fulfils that for each interval $J_i := [m_i, m_{i+1})$, there is exactly one $k_i \in J_i$ such that $|p^*(\alpha_{k_i}, k_i)| > 1$, i. e. such that the creature $C_i := p^*(\alpha_{k_i}, k_i)$ is non-trivial.

(ii) Due to continuous reading, we can find for each $\eta \in \operatorname{poss}(p^*, {<}k_i)$ and each $S \in C_i$ finite sets $Y_{\eta,S} \subseteq m_{i+1}$ and $Z_{\eta,S} \subseteq J_i$ such that

$$p^* \wedge (\eta^\frown\{S\}) \Vdash \dot{Y}\restriction_{m_{i+1}} = Y_{\eta,S} \text{ and } \dot{Y}\restriction_{J_i} = Z_{\eta,S}.$$

(iii) Note that due to property (i), $N_{i+1} := |\operatorname{poss}(p^*, {<}m_{i+1})| = |\operatorname{poss}(p^*, {\leq}k_i)|$ only depends on the i-th creature $C_i = p^*(\alpha_{k_i}, k_i)$, since from $k_i + 1$ to m_{i+1}, there are only singletons in p^*. Hence we can choose m_{i+1} such that $m_{i+1} \gg N_{i+1}$.

(iv) For all $0 < i < \omega$, we have $N_i \geq i^6$. Additionally, let $N_1 = |C_0| \geq 100$. (This is possible without loss of generality since we can just "skip" creatures which do not have sufficiently many elements to fulfil these bounds.)

(v) Letting the name $\dot{M}_i$ denote the number of elements in $\dot{Y}\restriction_{[m_i,m_{i+1})}$, we can ensure that p^* forces for all $i < \omega$ that $\dot{M}_i \geq \max\{2i \cdot m_i, N_{i+1}\}$.

(vi) Letting $E_i := \lceil N_i \cdot P_i\rceil$, letting $e_i(\eta, S)$ be the E_i-th element of $Z_{\eta,S}$ and letting $e_i := \max_{\eta,S} e_i(\eta, S)$, we can finally choose m_{i+1} large enough such that $m_i + e_i < m_{i+1}$.

We now make a probabilistic argument using the following formulation of Chernoff's bound (see [AS16, Theorem A.1.1]): Given mutually independent random variables $\langle x_i \mid 1 \leq i \leq k\rangle$ with $\Pr[x_i = 0] = \Pr[x_i = 1] = 1/2$ for all $1 \leq i \leq k$ and letting $S_k := \sum_{1\leq i\leq k} x_i$, it follows that for any $a > 0$,

$$\Pr\left[S_k - \frac{k}{2} > a\right] < \exp\left(-\frac{a^2}{2k}\right).$$

We use this bound as follows: Fix $n < \omega$. Let X be some randomly chosen subset of J_n and denote the probability space by Ω. Fix $\eta \in \operatorname{poss}(p^*, {<}k_n)$, $S \in C_n$ and $m \in J_n$ with $m \geq m_n + e_n(\eta, S)$. We consider the probability that this randomly chosen X does *not* bisect $Z_{\eta,S} \cap m$ with error at most $\frac{1}{2n}$; denote this event by $\mathrm{FAIL}(X, \eta, S, m)$.

Let $k \geq E_n$ denote the number of elements in $Z_{\eta,S} \cap m$. Then the choice of X (or, more precisely, the choice of the initial part of X relevant for this argument) amounts to tossing k fair coins x_j with values in $\{0, 1\}$, summing up the results and dividing by k, and comparing the gap between the result and $1/2$. By Chernoff's

bound above we have

$$\Pr[\mathrm{FAIL}(X,\eta,S,m)] = \Pr\left[\sum_{1\le i\le k}\frac{x_i}{k}-\frac{1}{2}>\frac{1}{2n}\right] = \Pr\left[\sum_{1\le i\le k}x_i-\frac{k}{2}>\frac{k}{2n}\right]$$
$$< \exp\left(-\frac{(k/2n)^2}{2k}\right) = \exp\left(-\frac{k}{8n^2}\right).$$

Hence the probability of failing for at least one $m \in J_n$ (with $Z_{\eta,S}\cap m \geq E_n$) is bounded as follows (note that we only have to sum over the elements of $Z_{\eta,S}\cap m$):

$$\Pr[\mathrm{FAIL}(X,\eta,S)] := \Pr[\exists\, m \geq m_n + e_n(\eta,S)\colon \mathrm{FAIL}(X,\eta,S,m)]$$
$$< \sum_{k\geq E_n}\exp\left(-\frac{k}{8n^2}\right) = \frac{\exp(-E_n/8n^2)}{1-\exp(-1/8n^2)}$$

Using the fact that $\frac{1}{1-\exp(-x)} \leq \frac{2}{x}$ for $x \in (0,1)$, we get

$$\Pr[\mathrm{FAIL}(X,\eta,S)] < 16n^2\cdot\exp\left(-\frac{E_n}{2n^2}\right) = 16n^2\cdot\exp\left(-\frac{\lceil N_n\cdot P_n\rceil}{2n^2}\right).$$

For the final step of our probabilistic estimate, we want to bound the probability of failing for at least one η, and we get

$$\Pr[\mathrm{FAIL}(X,S)] := \Pr[\exists\,\eta\colon \mathrm{FAIL}(X,\eta,S)] \leq N_n\cdot 16n^2\cdot\exp(-\lceil N_n\cdot P_n\rceil/2n^2) =: \delta_n.$$

It is easy to see that $\delta_n < 1/2$ holds for e. g. $P_n := \max\{1/2, 1/n\}$ and $N_n \geq \min\{n^6, 100\}$, which holds by property (iv).

Now we make the following observation: If we count the number of pairs $\{\langle X,S\rangle \mid X\in\Omega, S\in C_n\}$ with $\mathrm{FAIL}(X,S)$, this total number of failures is bounded from above by $\delta_n\cdot|C_n|\cdot|\Omega|$. If we now assume that for each $X\in\Omega$, the number of $S\in C_n$ with $\mathrm{FAIL}(X,S)$ is at least F, then the total number of failures is bounded from below by $F\cdot|\Omega|$ – but this shows that $F \leq \delta_n\cdot|C_n| < |C_n|/2$.

Summing up the entire probabilistic argument, this means that we can find some $X =: X_n \subseteq J_n$ and some $D_n \subseteq C_n$ with $|D_n| > |C_n|/2$ (and hence $\|D_n\| > \|C_n\| - 1$) such that for each $\eta \in \mathrm{poss}(p^*, {<}k_n)$, each $S\in D_n$ and each $m \geq m_n + e_n(\eta,S)$, we have that

$$\frac{|X_n\cap Z_{\eta,S}\cap m|}{|Z_{\eta,S}\cap m|} \in \left(\frac{1}{2}-\frac{1}{2n},\frac{1}{2}+\frac{1}{2n}\right).$$

Now we perform the usual fusion construction, starting with $q_0 := p^*$, shrinking the creature C_n to D_n in the n-th step (and keeping everything below that from q_{n-1}), and constructing a fusion condition $q := \bigcap_{n<\omega} q_n$ as well as sets $X_n \subseteq J_n$. It is clear that the q constructed this way is a valid condition. We now claim that the set $X := \bigcup_{n<\omega} X_n$ is as required; in particular, we claim that for each $\varepsilon > 0$, there is an m_ε such that for all $m \geq m_\varepsilon$, we have

$$q \Vdash \frac{|X\cap\dot{Y}\cap m|}{|\dot{Y}\cap m|} \in \left(\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon\right).$$

We prove this inductively and will show that the error at any point $m < \omega$ is bounded by an expression that goes to 0 as n goes to infinity. Let $X_{<n} := \bigcup_{i<n} X_i$ for each $n < \omega$. For our induction hypothesis, assume that we already know that at m_n, the bisection error of $X_{<n}$ with each possible $Y_{\eta,S}{\restriction}_{m_n}$ is at most $1/n-1$. For each $m \in [m_n + 1, m_{n+1}]$, we now have to consider the bisection error of $X_{<n+1}$ at m with each such $Y_{\eta,S}$.

- For $m \in [m_n + 1, m_n + e_n(\eta, S))$, note that $Y_{\eta,S}{\restriction}_{m_n}$ has at least N_n elements by property (v), while $Y_{\eta,S}{\restriction}_{[m_n,m]}$ has at most $E_n = N_n \cdot P_n$ elements by property (vi). Thus we can apply Lemma C3.3 with $R := Y_{\eta,S}{\restriction}_{[m_n,m]}$, $S := Y_{\eta,S}{\restriction}_{m_n}$, $\varepsilon := 1/n-1$ and some $c > 1/P_n$ to get

$$\begin{aligned}\frac{|X_{<n+1} \cap Y_{\eta,S} \cap m|}{|Y_{\eta,S} \cap m|} &\in \left(\frac{1}{2} - \frac{1}{n-1} - \frac{1}{c}, \frac{1}{2} + \frac{1}{n-1} + \frac{1}{c}\right) \\ &\subseteq \left(\frac{1}{2} - \frac{1}{n-1} - P_n, \frac{1}{2} + \frac{1}{n-1} + P_n\right) \\ &\subseteq \left(\frac{1}{2} - \frac{2}{n-1}, \frac{1}{2} + \frac{2}{n-1}\right).\end{aligned}$$

- For $m \in [m_n + e_n(\eta, S), m_{n+1}]$, it is clear that

$$\frac{|X_{<n+1} \cap Y_{\eta,S} \cap m|}{|Y_{\eta,S} \cap m|} \in \left(\frac{1}{2} - \frac{1}{n-1}, \frac{1}{2} + \frac{1}{n-1}\right),$$

  since the error on $Y_{\eta,S}{\restriction}_{m_n}$ is at most $1/n-1$ and the error on $Y_{\eta,S}{\restriction}_{[m_n,m]}$ is at most $1/n$.
- For $m = m_{n+1}$, however, we have to show even more to ensure that our induction hypothesis remains true for the next step. So note that $Y_{\eta,S}{\restriction}_{m_n}$ has at most m_n elements, while $Y_{\eta,S}{\restriction}_{[m_n,m_{n+1}]}$ has at least $2n \cdot m_n$ elements by property (v). Thus we can apply Lemma C3.3 once more with $R := Y_{\eta,S}{\restriction}_{m_n}$, $S := Y_{\eta,S}{\restriction}_{[m_n,m_{n+1}]}$, $\varepsilon := 1/2n$ and some $c \geq 2n$ to get

$$\begin{aligned}\frac{|X_{<n+1} \cap Y_{\eta,S} \cap m_{n+1}|}{|Y_{\eta,S} \cap m_{n+1}|} &\in \left(\frac{1}{2} - \frac{1}{2n} - \frac{1}{c}, \frac{1}{2} + \frac{1}{2n} + \frac{1}{c}\right) \\ &\subseteq \left(\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}\right),\end{aligned}$$

  which is precisely the induction hypothesis for $n + 1$.

Given any $\varepsilon > 0$, pick some n_ε such that $\frac{2}{n_\varepsilon - 1} < \varepsilon$ and let $m_\varepsilon := m_{n_\varepsilon}$. Then for all $m \geq m_\varepsilon$, by the bounds above

$$q \Vdash \frac{|X \cap \dot{Y} \cap m|}{|\dot{Y} \cap m|} \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right),$$

finishing the proof. □

Theorem C3.5. Con($\mathfrak{s}_{1/2} < \text{non}(\mathcal{N})$).

Proof. Assume CH in the ground model; then the statement follows by combining Lemma C3.2 and Lemma C3.4. □

C4 Characteristics Related to $\mathfrak{r}$ and $\mathfrak{i}$

We define a second set of properties more closely related to $\mathfrak{i}$, although $\mathfrak{s}$ does reappear in this section.

Definition C4.1. A set $X \in [\omega]^\omega$ is *moderate* if $\underline{d}(X) > 0$ as well as $\bar{d}(X) < 1$.[44]

Definition C4.2. A family $\mathcal{I}_* \subseteq [\omega]^\omega$ is *statistically independent* or $*$*-independent* if for any set $X \in \mathcal{I}_*$ we have that X is moderate and for any finite subfamily $\mathcal{E} \subseteq \mathcal{I}_*$, the following holds:

$$\lim_{n\to\infty} \left(\frac{d_n\left(\bigcap_{E\in\mathcal{E}} E\right)}{\prod_{E\in\mathcal{E}} d_n(E)} \right) = 1$$

In the case of convergence of $d_n(X)$, this simplifies to asking for $0 < d(X) < 1$ to hold for all $X \in \mathcal{I}_*$ and

$$\prod_{E\in\mathcal{E}} d(E) = d\Big(\bigcap_{E\in\mathcal{E}} E\Big)$$

to hold for any finite subfamily $\mathcal{E} \subseteq \mathcal{I}_*$.

We denote the least cardinality of a maximal $*$-independent family by $\mathfrak{i}_*$.

Recall that a family $\mathcal{I}$ of subsets of ω is called *independent* if for any disjoint finite subfamilies $\mathcal{A}, \mathcal{B} \subseteq \mathcal{I}$, the set

$$\bigcap_{A\in\mathcal{A}} A \cap \bigcap_{B\in\mathcal{B}} (\omega \setminus B)$$

is infinite. Generalising this notion leads to the following definitions (which are more obviously related to the classical $\mathfrak{i}$):

Definition C4.3. Let $\rho \in (0,1)$. A family $\mathcal{I}_\rho \subseteq [\omega]^\omega$ is ρ*-independent* if for any disjoint finite subfamilies $\mathcal{A}, \mathcal{B} \subseteq \mathcal{I}_\rho$, the following holds:

$$d\left(\bigcap_{A\in\mathcal{A}} A \cap \bigcap_{B\in\mathcal{B}} (\omega \setminus B)\right) = \rho^{|\mathcal{A}|} \cdot (1-\rho)^{|\mathcal{B}|},$$

which simplifies to $= 1/2^{|\mathcal{A}|+|\mathcal{B}|}$ in the case of $\rho = 1/2$. This definition is equivalent to demanding that for any finite $\mathcal{A} \subseteq \mathcal{I}_\rho$, the following holds:

$$d\left(\bigcap_{A\in\mathcal{A}} A\right) = \rho^{|\mathcal{A}|}$$

We denote the least cardinality of a maximal ρ-independent family by $\mathfrak{i}_\rho$.

[44] Actually, it would suffice to demand $\bar{d}(X) > 0$ as well as $\underline{d}(X) < 1$, though one would have to modify a few of the subsequent proofs.

Recalling the definition of $\mathfrak{r}$ as the least cardinality of a family $\mathcal{R} \subseteq [\omega]^\omega$ such that no $S \in [\omega]^\omega$ splits every $R \in \mathcal{R}$, we naturally arrive at the following definition:

Definition C4.4. A family $\mathcal{R}_{1/2} \subseteq [\omega]^\omega$ is *1/2-reaping* if there is no $S \in [\omega]^\omega$ bisecting all $R \in \mathcal{R}_{1/2}$. We denote the least cardinality of a 1/2-reaping family by $\mathfrak{r}_{1/2}$.

Given the above, the natural question is: Can we define $\mathfrak{r}_*$ analogously? Consider the following definition:

Definition C4.5. A family $\mathcal{R}_* \subseteq [\omega]^\omega$ is *statistically reaping* or *$*$-reaping* if

$$\nexists\, S \in [\omega]^\omega \text{ moderate such that } \forall\, X \in \mathcal{R}_*\colon \lim_{n\to\infty} \left(\frac{d_n\left(S \cap X\right)}{d_n(S) \cdot d_n(X)} \right) = 1.$$

We denote the least cardinality of a $*$-reaping family by $\mathfrak{r}_*$.

The motivation for this is as follows: Considering the analogous definitions for $\mathfrak{r}$, we might call $\mathcal{I}$ *maximal quasi-independent* if there is no X such that for all $Y \in \mathcal{I}$ we have that X splits Y and X splits $\omega \setminus Y$ (i. e. X and Y are independent for all $Y \in \mathcal{I}$). It is obvious that a reaping family is also maximal quasi-independent; the converse can easily be derived by taking a maximal quasi-independent family and saturating it (without increasing its size) by adding the complements of all its sets, resulting in a reaping family. By this train of thought, it makes sense to take Definition C4.5 as the defining property of a $*$-reaping family.

Dualising the definition of $*$-reaping leads to the following, final definition:

Definition C4.6. A family $\mathcal{S}_* \subseteq [\omega]^\omega$ is *statistically splitting* or *$*$-splitting* if

$$\forall\, X \in [\omega]^\omega\ \exists\, S \in \mathcal{S}_* \text{ moderate}\colon \lim_{n\to\infty} \left(\frac{d_n\left(S \cap X\right)}{d_n(S) \cdot d_n(X)} \right) = 1.$$

We denote the least cardinality of a $*$-splitting family by $\mathfrak{s}_*$.

Theorem C4.7. *The relations shown in Figure 11 hold.*

Proof. $\mathbf{cov(\mathcal{N})} \leq \mathfrak{r}_{1/2}$ and $\mathfrak{s}_* \leq \mathbf{non(\mathcal{N})}$: Both proofs are analogous to the proof of $\mathfrak{s}_{1/2} \leq \mathrm{non}(\mathcal{N})$.

For the first claim, let $\mathcal{R}_{1/2}$ be a family witnessing the value of $\mathfrak{r}_{1/2}$. By the argument for $\mathfrak{s}_{1/2} \leq \mathrm{non}(\mathcal{N})$ in the proof of Theorem C2.4, the family

$$\{[\omega]^\omega \setminus \mathcal{S}_R \mid R \in \mathcal{R}_{1/2}\}$$

is a covering of $\mathcal{N}$. (Recall that $[\omega]^\omega \setminus \mathcal{S}_R \in \mathcal{N}$ for $R \in \mathcal{R}_{1/2}$.)

For the second claim, let $X \in [\omega]^\omega$ and $F \notin \mathcal{N}$. As seen above, letting

$$S_X = \{Y \in [\omega]^\omega \mid Y \mid_{1/2} X\},$$

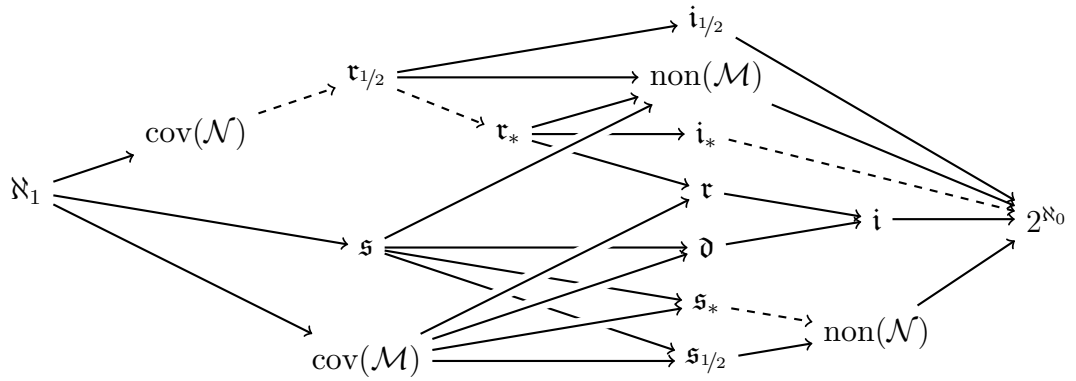

Figure 11: The ZFC-provable and/or consistent inequalities between $\mathfrak{i}_{1/2}$, $\mathfrak{i}_*$, $\mathfrak{r}_{1/2}$, $\mathfrak{r}_*$, $\mathfrak{s}_{1/2}$, $\mathfrak{s}_*$ and other well-known cardinal characteristics, where $\longrightarrow$ means "$\leq$, consistently $<$" and $\dashrightarrow$ means "$\leq$, possibly $=$".

we have that $\lambda(S_X) = 1$ and hence $S_X \notin \mathcal{N}$. Moreover, this is true in particular for $X = \omega$ and

$$S_\omega = \{Y \in [\omega]^\omega \mid Y \mid_{1/2} \omega\} = \{Y \in [\omega]^\omega \mid d(Y) = 1/2\}.$$

Since then $F \cap S_X \cap S_\omega \neq \varnothing$, there is some $S \in F$ such that $S \mid_{1/2} X$ and $d(S) = 1/2$, which implies $S \mid_* X$.

Since all this is true for any $X \in [\omega]^\omega$, we have $\mathfrak{s}_* \leq \mathrm{non}(\mathcal{N})$.

$\mathbf{\mathfrak{r}_{1/2} \leq \mathfrak{r}_*}$: Let $\mathcal{R}_*$ be a $*$-reaping family and let $\mathcal{R}_{1/2} := \mathcal{R}_* \cup \{\omega\}$; clearly, $|\mathcal{R}_{1/2}| = |\mathcal{R}_*|$. Now, any S which bisects all $R \in \mathcal{R}_{1/2}$ also $*$-splits all $R \in \mathcal{R}_*$ – this follows from the fact that $S \mid_{1/2} \omega$ implies $d(S) = 1/2$, and hence for any $R \in \mathcal{R}_*$, we now have

$$\frac{d_n(S \cap R)}{d_n(S) \cdot d_n(R)} = \frac{d_n(S \cap R)}{d_n(R)} \cdot \frac{1}{d_n(S)} \to 1,$$

since $S \mid_{1/2} R$ implies that the first factor converges to $1/2$, while $d(S) = 1/2$ implies that the second factor converges to 2.

$\mathbf{\mathfrak{r}_{1/2} \leq \mathrm{non}(\mathcal{M})}$: Since the set of all reals bisected by a fixed real S is a meagre set (by the argument for $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{s}_{1/2\pm\varepsilon}$), a non-meagre set contains some real not bisected by S and hence is $1/2$-reaping.

$\mathbf{\mathfrak{r}_* \leq \mathrm{non}(\mathcal{M})}$: This is analogous to the proof of $\mathfrak{r}_{1/2} \leq \mathrm{non}(\mathcal{M})$, since the set of all reals $*$-split by a fixed moderate real S is a meagre set, as well. To see this, define a chopped real based on S with the interval partition having the partition boundaries at the $n!$-th elements of S; the sets matching this chopped real form a comeagre set which consists of reals X not $*$-split by S: As the matching intervals grow longer and longer, they "pull" $\frac{d_n(S \cap X)}{d_n(X)}$ above $1 - 1/n$, which implies that $\frac{d_n(S \cap X)}{d_n(S) \cdot d_n(X)}$ cannot converge to 1 as $d_n(S)$ does not converge to 1 by the moderacy of S.

$\mathbf{\mathrm{cov}(\mathcal{M}) \leq \mathfrak{s}_*}$: This is analogous to the proof of $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{s}_{1/2}$ by the same argument as in the proof of $\mathfrak{r}_* \leq \mathrm{non}(\mathcal{M})$.

$\mathfrak{s} \leq \mathfrak{s}_*$: Let $\mathcal{S}_*$ be a family witnessing the value of $\mathfrak{s}_*$ and let $X \in [\omega]^\omega$ be arbitrary. We will prove by contradiction that there must be some $S \in \mathcal{S}_*$ splitting X. Suppose not, that is, suppose that for any $S \in \mathcal{S}_*$, either (a) $S \cap X$ is finite or (b) $S \cap X$ is cofinite. In case (a), we use the fact that S is moderate to see that $d_n(S)$ must eventually be bounded from below by some ε, and the fact that $S \cap X$ is finite to see that $|S \cap X \cap n|$ is bounded by some k^*. Letting $k_n := |X \cap n|$, this eventually yields

$$\frac{d_n(S \cap X)}{d_n(S) \cdot d_n(X)} \leq \frac{{}^{k^*}\!/_n}{\varepsilon \cdot {}^{k_n}\!/_n} = \frac{k^*}{\varepsilon \cdot k_n} \to 0.$$

Similarly, in case (b) we use the moderacy of S to see that $d_n(S)$ is eventually bounded from above by some $1 - \delta$, and the fact that $S \cap X$ is cofinite to see that $|S \cap X \cap n|$ is bounded from below by $k_n - k^*$ for some k^*. (This bound simply states that after some finite aberrations, S contains all elements of X.) Taken together, we eventually have

$$\begin{aligned}\frac{d_n(S \cap X)}{d_n(S) \cdot d_n(X)} &\geq \frac{{}^{(k_n - k^*)}\!/_n}{(1-\delta) \cdot {}^{k_n}\!/_n} \\ &= \frac{1}{1-\delta} - \frac{k^*}{(1-\delta) \cdot k_n} \to \frac{1}{1-\delta} = 1 + \varepsilon\end{aligned}$$

for some $\varepsilon > 0$. In summary, for all $S \in \mathcal{S}_*$ we have that S does not $*$-split X, and hence $\mathcal{S}_*$ could not have been a witness for the value of $\mathfrak{s}_*$.

$\mathfrak{r}_{1/2} \leq \mathfrak{i}_{1/2}$ and $\mathfrak{r}_* \leq \mathfrak{i}_*$: For the first claim, let $\mathcal{I}_{1/2}$ be a maximal $1/2$-independent family. Define

$$\mathcal{R}_{1/2} := \left\{ \bigcap_{A \in \mathcal{A}} A \cap \bigcap_{B \in \mathcal{B}} (\omega \setminus B) \;\middle|\; \mathcal{A}, \mathcal{B} \subseteq \mathcal{I}_{1/2}, \mathcal{A} \cap \mathcal{B} = \varnothing \right\}.$$

Then $\mathcal{R}_{1/2}$ is a $1/2$-reaping family, since the existence of an $S \in [\omega]^\omega$ bisecting each $R \in \mathcal{R}_{1/2}$ (in the limit) would contradict the maximality of $\mathcal{I}_{1/2}$.

The proof of the second claim is analogous: Take all finite tuples of sets in the witness $\mathcal{I}_*$ of the value of $\mathfrak{i}_*$ and collect their Boolean combinations in a family $\mathcal{R}_*$; this family must then be $*$-reaping, because a set S $*$-splitting each $R \in \mathcal{R}_*$ would violate the maximality of $\mathcal{I}_*$, and thus $\mathcal{R}_*$ witnesses $\mathfrak{r}_* \leq \mathfrak{i}_*$.

$\mathfrak{i}_\rho \leq 2^{\aleph_0}$ and $\mathfrak{i}_* \leq 2^{\aleph_0}$: For $\mathfrak{i}_\rho$, consider the collection $\mathcal{I}_\rho$ of all ρ-independent families. Now, $\mathcal{I}_\rho$ has finite character, i. e. for each $I \subseteq 2^{\aleph_0}$, I belongs to $\mathcal{I}_\rho$ iff every finite subset of I belongs to $\mathcal{I}_\rho$. Hence we can apply Tukey's lemma and see that $\mathcal{I}_\rho$ has a maximal element with respect to inclusion. Therefore, $\mathfrak{i}_\rho$ is well defined and hence $\mathfrak{i}_\rho \leq 2^{\aleph_0}$. The proof for $\mathfrak{i}_*$ is analogous.

Con($\mathfrak{r}_* < \mathfrak{r}$): This follows from Con(non($\mathcal{M}$) < cov($\mathcal{M}$)), but we also have an explicit proof of this.

We will show that Cohen forcing does not increase $\mathfrak{r}_*$ due to the ground model reals remaining $*$-reaping; we already know that Cohen forcing increases $\mathfrak{r}$, proving our consistency statement.

Let $\dot{X}$ be a $\mathbb{C}$-name for a real. We will construct a ground model real Y such that for any $q \in \mathbb{C}$, we can find $r \leq q$ such that $r \Vdash \dot{X} \not\ast Y$.

Let $\varphi(\dot{X})$ be the statement $\forall\, k\ \exists\, \ell_0, \ell_1 > k \colon \dot{X}(\ell_0) = 0 \wedge \dot{X}(\ell_1) = 1$. Let $D_{\text{good}} := \{p \in \mathbb{C} \mid p \Vdash \varphi(\dot{X})\}$ and $D_{\text{bad}} := \{p \in \mathbb{C} \mid p \Vdash \neg\varphi(\dot{X})\}$ and note that $D := D_{\text{good}} \cup D_{\text{bad}}$ is open dense in $\mathbb{C}$. Since it is clear that any $q \in D_{\text{bad}}$ already forces that $\dot{X}$ is not moderate, we only need to consider $q \in D_{\text{good}}$.

Now pick an enumeration $\langle p_k \mid k < \omega\rangle$ of D_{good} which enumerates each element infinitely often. In the following argument, for each $k < \omega$, let $L_k := \sum_{\ell \leq k} \ell_k$.

- For $k = 0$, we find $q_0 \leq p_0$, $\ell_0 \geq 2$ and $A_0 \subseteq [0, \ell_0)$ such that q_0 decides $\dot{X}{\restriction_{\ell_0}}$, $q_0 \Vdash \dot{X}{\restriction_{\ell_0}} = A_0$ and such that $|A_0| \geq 1$, $|[0, \ell_0) \setminus A_0| \geq 1$, and at least one of these two inequalities is an equality.
- For $0 < k < \omega$, we find $q_k \leq p_k$, $\ell_k < \omega$ and $A_k \subseteq [L_{k-1}, L_k)$ such that q_k decides $\dot{X}{\restriction_{L_k}}$, $q_k \Vdash \dot{X}{\restriction_{[L_{k-1}, L_k)}} = A_k$ and such that $|A_k| \geq 3L_{k-1}$, $|[L_{k-1}, L_k) \setminus A_k| \geq 3L_{k-1}$, and at least one of these inequalities is an equality.

Define Y piecewise by $Y{\restriction_{[L_{k-1}, L_k)}} := A_k$. Assume $\dot{X}$ $*$-splits Y; then there must be some $q \in \mathbb{C}$ forcing this. It is clear that $q \perp D_{\text{bad}}$. In particular, this means that q forces that for any $\varepsilon > 0$, there is some $m_\varepsilon < \omega$ such that for any $j > m_\varepsilon$,

$$\frac{d_j(\dot{X} \cap Y)}{d_j(\dot{X}) \cdot d_j(\dot{Y})} > 1 - \varepsilon.$$

Pick some sufficiently small ε, say $\varepsilon := 2/9$, and find $n < \omega$ such that $p_n = q$ and $L_n > m_{1/4}$. Letting O_n and I_n be the number of 0s and 1s in A_n, respectively, $q_n \leq q$ forces

$$\begin{aligned} d_{L_n}(\dot{X} \cap Y) &\leq \frac{L_{n-1}}{L_n}, \\ d_{L_n}(\dot{X}) &\geq \frac{I_n}{L_n}, \\ d_{L_n}(Y) &\geq \frac{O_n}{L_n}. \end{aligned}$$

Without loss of generality, $O_n = 3L_{n-1}$ and $I_n = 3L_{n-1} + \Delta$ for some $\Delta < \omega$. Then q_n forces

$$\begin{aligned} \frac{d_{L_n}(\dot{X} \cap Y)}{d_{L_n}(\dot{X}) \cdot d_{L_n}(\dot{Y})} &\leq \frac{\frac{L_{n-1}}{L_n}}{\frac{O_n I_n}{L_n^2}} = \frac{L_{n-1}L_n}{O_n I_n} = \frac{L_{n-1}(L_{n-1} + O_n + I_n)}{O_n I_n} \\ &= \frac{L_{n-1}(7L_{n-1} + \Delta)}{3L_{n-1}(3L_{n-1} + \Delta)} = \frac{7L_{n-1} + \Delta}{3 \cdot (3L_{n-1} + \Delta)}, \end{aligned}$$

which is strictly decreasing in Δ and is $7/9$ for $\Delta = 0$. This contradicts the assumption on q, proving that $\dot{X}$ does not $*$-split Y in $V^{\mathbb{C}}$.

Hence assuming CH in the ground model and forcing with $\mathbb{C}_\lambda$ for some $\lambda \geq \aleph_2$ with $\lambda = \lambda^{\aleph_0}$ gives us $V^{\mathbb{C}_\lambda} \vDash \mathfrak{r}_* = \aleph_1 < \lambda = \mathfrak{r} = \mathfrak{c}$.

Con($\mathfrak{r}_{1/2} < \mathbf{non}(\mathcal{M})$) and **Con($\mathfrak{r}_* < \mathbf{non}(\mathcal{M})$)**: This follows from Con($\mathfrak{r} < \mathrm{non}(\mathcal{M})$), see [BJ95, Model 7.5.9].

Con($\mathfrak{s} < \mathfrak{s}_*$): Just like Con($\mathfrak{r}_* < \mathfrak{r}$), this follows from Con(non($\mathcal{M}$) < cov($\mathcal{M}$)), but once more, we also have an explicit proof of this.

We will show that Cohen forcing increases $\mathfrak{s}_*$ due to the Cohen real not being $*$-split by any real from the ground model; we already know that Cohen forcing keeps $\mathfrak{s}$ small, proving our consistency statement.

The proof uses the same technique as the one for $\mathfrak{s} \leq \mathfrak{s}_*$: Given some moderate $X \in [\omega]^\omega \cap V$, with moderacy in the sense of $\bar{d}(X) = 1 - 2\varepsilon$ and $d_n(X) < 1 - \varepsilon$ for all $n \geq n_0$ for some n_0, we will show that the assumption that there is a condition forcing $X \mid_* \dot{C}$, i. e. that X $*$-splits the Cohen real, leads to a contradiction.

So suppose that there were some $p \in \mathbb{C}$ such that $p \Vdash X \mid_* \dot{C}$; more specifically, suppose that for some n_1, even $p \Vdash \frac{d_n(X \cap \dot{C})}{d_n(X) \cdot d_n(\dot{C})} < 1 - \delta$ for all $n \geq n_1$, where $\delta := \frac{\varepsilon/2}{1-\varepsilon}$.

We now define $q \leq p$ as follows: Let n_2 be large enough such that

$$\frac{|p|}{|X \cap n_2|} < \frac{\varepsilon}{2} \quad \iff \quad \frac{2 \cdot |p|}{\varepsilon} < |X \cap n_2|;$$

this is possible due to the moderacy of X (which implies X is infinite). Let $k := \max\{n_0, n_1, n_2\}$ and $q := p^\frown \chi_X{\restriction}_{[|p|+1,k]}$, that is, extend p by the next $k - |p|$ values of the characteristic function of X. Then we have

$$\frac{d_k(X \cap \dot{C})}{d_k(X) \cdot d_k(\dot{C})} > \frac{1}{1-\varepsilon} \cdot \frac{d_k(X \cap \dot{C})}{d_k(\dot{C})}$$

by the moderacy of X. By our choice of q, we have

$$q \Vdash \frac{d_k(X \cap \dot{C})}{d_k(\dot{C})} = \frac{|X \cap \dot{C} \cap k|}{|\dot{C} \cap k|} \geq \frac{|X \cap k| - |p|}{|X \cap k|} = 1 - \frac{|p|}{|X \cap k|} > 1 - \frac{\varepsilon}{2},$$

with the first inequality being an equality in the "worst case" of $X{\restriction}_{|p|+1} \equiv 1$ and $(p = q{\restriction}_{|p|+1} =) \dot{C}{\restriction}_{|p|+1} \equiv 0$. This implies that

$$q \Vdash \frac{d_k(X \cap \dot{C})}{d_k(X) \cdot d_k(\dot{C})} > \frac{1 - \varepsilon/2}{1 - \varepsilon} = 1 + \delta,$$

contradictory to the original assumption on p.

Con(cov($\mathcal{M}$) < $\mathfrak{s} \leq \mathfrak{s}_*$): Follows as in the proof of Con(cov($\mathcal{M}$) < $\mathfrak{s} \leq \mathfrak{s}_{1/2}$).

Con($\mathfrak{r}_{1/2} < \mathfrak{i}_{1/2}$) and **Con($\mathfrak{r}_* < \mathfrak{i}_*$)**: For these proofs, which are due to Jörg Brendle, see [BHK+18, Lemma 4.8 and Corollary 4.9].

Con($\mathfrak{i}_{1/2} < 2^{\aleph_0}$): This follows from Lemma C4.9 below. □

For the final proof of this section, we will require another combinatorial lemma.

Lemma C4.8. *If $R, S \subseteq \omega$, $0 < r < 1$, $\varepsilon > 0$ and $m < n$ are such that*

$$\frac{|R \cap m|}{m} \in (r - \varepsilon, r + \varepsilon)$$

and for all ℓ with $m \leq \ell \leq n$, we have

$$\frac{|S \cap \ell|}{\ell} \in (r - \varepsilon, r + \varepsilon)\,,$$

then for all ℓ with $m \leq \ell \leq n$, we have

$$\frac{|(R \cap m) \cup (S \cap [m, \ell))|}{\ell} \in (r - 3\varepsilon, r + 3\varepsilon)\,.$$

Proof. Suppose this were false for some $\ell^* \geq m$; then without loss of generality,

$$\frac{|(R \cap m) \cup (S \cap [m, \ell^*))|}{\ell^*} \geq r + 3\varepsilon.$$

Since

$$\frac{|R \cap m|}{m} < r + \varepsilon,$$

we get

$$\frac{|S \cap [m, \ell^*)|}{\ell^*} \geq r + 3\varepsilon - \frac{m}{\ell^*}(r + \varepsilon).$$

But then

$$\frac{|S \cap m|}{m} > r - \varepsilon$$

implies

$$\begin{aligned}\frac{|S \cap \ell^*|}{\ell^*} = \frac{|(S \cap m) \cup (S \cap [m, \ell^*))|}{\ell^*} &> \frac{m}{\ell^*}(r - \varepsilon) + r + 3\varepsilon - \frac{m}{\ell^*}(r + \varepsilon) \\ &= r + 3\varepsilon - \frac{2m}{\ell^*} \cdot \varepsilon \geq r + \varepsilon,\end{aligned}$$

which is a contradiction. □

Lemma C4.9. $\mathrm{Con}(\mathfrak{i}_{1/2} < \mathfrak{i})$.

Proof. The proof is analogous to the classical proof of $\mathrm{Con}(\aleph_1 = \mathfrak{a} < 2^{\aleph_0})$ (see e. g. [Hal17, Proposition 18.5]).

Assume CH in the ground model and let $\lambda \geq \aleph_2$. We force with the λ-Cohen forcing poset $\mathbb{C}_\lambda$; letting G be a $\mathbb{C}_\lambda$-generic filter, it is clear that $V[G] \vDash \mathfrak{i} = 2^{\aleph_0} = \lambda$. We will now show $V[G] \vDash \mathfrak{i}_{1/2} = \aleph_1$ by constructing a maximal $1/2$-independent family $\mathcal{A}$ in the ground model such that $\mathcal{A}$ remains maximal $1/2$-independent in $V[G]$. By the usual arguments, it suffices to consider what happens to a countably infinite $1/2$-independent family when forcing with just $\mathbb{C} := \langle 2^{<\omega}, \subseteq \rangle$.

Let $\mathcal{A}_0 := \{A_n \subseteq [\omega]^{\aleph_0} \mid n < \omega\}$ be such a family. Fix (in the ground model) an enumeration $\{(p_\alpha, \dot{X}_\alpha) \mid \omega \leq \alpha < \omega_1\}$ of all pairs $(p, \dot{X})$ such that $p \in \mathbb{C}$

and $\dot{X}$ is a nice name for a subset of ω.[45] In particular, this means that for any $\langle \check{n}, p_1\rangle, \langle \check{n}, p_2\rangle \in \dot{X}$, either $p_1 = p_2$ or $p_1 \perp p_2$. Note that since $V \vDash \text{CH}$, there are just $\aleph_1$ many nice names for subsets of ω in V.

We now construct $\mathcal{A}$ from $\mathcal{A}_0$ iteratively as follows: Let $\omega \leq \alpha < \omega_1$ and assume we have already defined sets $A_\beta \subseteq \omega$ for all $\beta < \alpha$. Below, we will construct $A_\alpha \subseteq \omega$ such that the following two properties hold:

(i) The family $\{A_\beta \mid \beta \leq \alpha\}$ is $1/2$-independent.
(ii) If $p_\alpha \Vdash |\dot{X}_\alpha| = \aleph_0 \wedge$ "$\{A_\beta \mid \beta < \alpha\} \cup \{\dot{X}_\alpha\}$ is $1/2$-independent", then for all $m < \omega$, the set $D^\alpha_m := \{q \in \mathbb{C} \mid \exists\, n \geq m \colon q \Vdash A_\alpha \cap [2^n, 2^{n+1}) = \dot{X}_\alpha \cap [2^n, 2^{n+1})\}$ is dense below p_α.

We first show that the $\mathcal{A} := \{A_\beta \mid \beta \leq \omega_1\}$ constructed this way is a maximal $1/2$-independent family in $V^{\mathbb{C}}$. Clearly, $\mathcal{A}$ is $1/2$-independent, so only maximality could fail. Suppose it were not maximal; then there is a condition p and a nice name $\dot{X}$ for a subset of ω such that $p \Vdash$ "$\mathcal{A} \cup \{\dot{X}\}$ is $1/2$-independent". Let α be such that $(p, \dot{X}) = (p_\alpha, \dot{X}_\alpha)$ and let $\varepsilon > 0$ be sufficiently small (e. g. $\varepsilon < 1/16$). We can then find $q \leq p_\alpha$ and $m < \omega$ such that

$$q \Vdash \frac{|A_\alpha \cap \dot{X}_\alpha \cap \ell|}{\ell} \in \left(\frac{1}{4} - \varepsilon, \frac{1}{4} + \varepsilon\right) \text{ for all } \ell \geq 2^m \tag{$*_{11}$}$$

(because p_α forces that $\{A_\alpha, \dot{X}_\alpha\}$ is $1/2$-independent) and

$$\frac{|A_\alpha \cap [2^n, 2^{n+1})|}{2^n} > \frac{1}{2} - \varepsilon \text{ for all } n \geq m.$$

Now by the density of D^α_m below p_α, we can find $r \leq q$ and some $n \geq m$ such that $r \Vdash A_\alpha \cap [2^n, 2^{n+1}) = \dot{X}_\alpha \cap [2^n, 2^{n+1})$. But this implies that

$$\begin{aligned} r \Vdash \frac{|A_\alpha \cap \dot{X}_\alpha \cap 2^{n+1}|}{2^{n+1}} &= \frac{1}{2} \cdot \frac{|A_\alpha \cap \dot{X}_\alpha \cap 2^n|}{2^n} + \frac{1}{2} \cdot \frac{|A_\alpha \cap \dot{X}_\alpha \cap [2^n, 2^{n+1})|}{2^n} \\ &> \frac{1/4 - \varepsilon}{2} + \frac{1/2 - \varepsilon}{2} = \frac{3}{8} - \varepsilon > \frac{1}{4} + \varepsilon, \end{aligned}$$

which contradicts Eq. ($*_{11}$).

We finally have to show that we can find such an A_α satisfying (i) and (ii) for any $\omega \leq \alpha < \omega_1$. We only have to consider those α such that $\dot{X}_\alpha$ satisfies the assumption in property (ii), since finding an A_α with property (i) is straightforward. Enumerate $\{A_\beta \mid \beta < \alpha\}$ as $\{B_n \mid n < \omega\}$. For $n < \omega$ and any partial function $f \colon n \to \{-1, 1\}$, we let

$$B^f := \bigcap_{i \in \mathrm{dom}(f)} B_i^{f(i)},$$

where $B_i^1 := B$ and $B_i^{-1} := \omega \setminus B$. We further pick some strictly decreasing sequence of real numbers $\langle \delta_n \mid n < \omega\rangle$ with $\delta_0 := 3$ and $\lim_{n\to\infty} \delta_n = 0$ and let

[45] The reason the index set of the enumeration is $[\omega, \omega_1)$ instead of $[0, \omega_1)$ is just to make the notation more convenient.

$\langle q_n \mid n < \omega\rangle$ be some sequence enumerating all conditions below p_α infinitely often. We will now construct, by induction on $n < \omega$, conditions $r_n \leq q'_n \leq q_n$, a strictly increasing sequence of natural numbers $\langle k_n \mid n < \omega\rangle$ and initial segments $Z_n = A_\alpha \cap 2^{k_n}$ of A_α such that for all $n < \omega$ and all partial functions $f\colon n \to \{-1, 1\}$, the following four statements will hold (with $F := |\operatorname{dom}(f)| + 1$)

(R1) $\dfrac{|B^f \cap Z_n \cap 2^{k_n}|}{2^{k_n}}, \dfrac{|(B^f \setminus Z_n) \cap 2^{k_n}|}{2^{k_n}} \in \left(\dfrac{1}{2^F} - \dfrac{\delta_n}{3}, \dfrac{1}{2^F} + \dfrac{\delta_n}{3}\right)$,

(R2) $q'_n \Vdash \dfrac{|B^f \cap \dot X_\alpha \cap \ell|}{\ell}, \dfrac{|(B^f \setminus \dot X_\alpha) \cap \ell|}{\ell} \in \left(\dfrac{1}{2^F} - \dfrac{\delta_n}{3}, \dfrac{1}{2^F} + \dfrac{\delta_n}{3}\right)$
for all ℓ with $2^{k_n} \leq \ell \leq 2^{k_{n+1}}$,

(R3) $\dfrac{|B^f \cap Z_{n+1} \cap \ell|}{\ell}, \dfrac{|(B^f \setminus Z_{n+1}) \cap \ell|}{\ell} \in \left(\dfrac{1}{2^F} - \delta_n, \dfrac{1}{2^F} + \delta_n\right)$
for all ℓ with $2^{k_n} \leq \ell \leq 2^{k_{n+1}}$, and

(R4) $r_n \Vdash Z_{n+1} \cap [2^{k_n}, 2^{k_{n+1}}) = \dot X_\alpha \cap [2^{k_n}, 2^{k_{n+1}})$.

It is clear that (R1)–(R4) taken together for all $n < \omega$ imply that $A_\alpha := \bigcup_{n<\omega} Z_n$ is as required by (i) and (ii).

For $n = 0$, let $k_0 := 0$, $q'_0 := q_0$ and $Z_0 := \varnothing$; then (R1) and (R2) hold vacuously by our choice of δ_0, and there is nothing to show yet for (R3) and (R4).

Now assume that we have obtained k_n, $q'_n \leq q_n$ and Z_n such that (R1) and (R2) hold for n; we will construct $r_n \leq q'_n$, k_{n+1}, $q'_{n+1} \leq q_{n+1}$ and Z_{n+1} such that (R3) and (R4) hold for n and such that (R1) and (R2) hold for $n+1$. We first find $q'_{n+1} \leq q_{n+1}$ and $k'_n \geq k_n$ such that for all partial functions $f\colon n+1 \to \{-1, 1\}$, we have that (with $F := |\operatorname{dom}(f)| + 1$)

$$q'_{n+1} \Vdash \frac{|B^f \cap \dot X_\alpha \cap \ell|}{\ell}, \frac{|(B^f \setminus \dot X_\alpha) \cap \ell|}{\ell} \in \left(\frac{1}{2^F} - \frac{\delta_{n+1}}{3}, \frac{1}{2^F} + \frac{\delta_{n+1}}{3}\right)$$

for all $\ell \geq 2^{k_n}$ (hence satisfying (R2) for $n+1$); this is possible since the assumption in property (ii) is true. Next we find $r_n \leq q'_n$ and a sufficiently large $k_{n+1} \geq k'_n$ such that for all partial functions $f\colon n+1 \to \{-1, 1\}$, we have that (still with $F := |\operatorname{dom}(f)| + 1$)

$$r_n \Vdash \frac{|B^f \cap \dot X_\alpha \cap 2^{k_{n+1}}|}{2^{k_{n+1}}}, \frac{|(B^f \setminus \dot X_\alpha) \cap 2^{k_{n+1}}|}{2^{k_{n+1}}} \in \left(\frac{1}{2^F} - \frac{\delta_{n+1}}{6}, \frac{1}{2^F} + \frac{\delta_{n+1}}{6}\right) \qquad (*_{12})$$

and that r_n decides $\dot X_\alpha \cap 2^{k_{n+1}}$; in particular, let $X_n \subseteq [2^{k_n}, 2^{k_{n+1}})$ be such that $r_n \Vdash \dot X_\alpha \cap [2^{k_n}, 2^{k_{n+1}}) = X_n$. All this is also possible since the assumption in property (ii) is true. Let $Z_{n+1} := Z_n \cup X_n$.

Now, (R4) holds for n by definition of Z_{n+1}. Apply Lemma C4.8 to $R := Z_n$, $S := \dot X_\alpha[r_n]$, $r := 1/2^F$, $\varepsilon := \delta_n$, $m := 2^{k_n}$ and $n := 2^{k_{n+1}}$ to see that (R3) for n follows from (R1) and (R2) for n and our choice of Z_{n+1}. Finally, (R1) for $n+1$ follows from Eq. ($*_{12}$), (R4) for n and the choice of a sufficiently large k_{n+1} (e. g. using the argument from Lemma C3.3).

By the usual arguments, our construction implies that $\mathcal{A}$ remains maximal $1/2$-independent in $V^{\mathbb{C}_\lambda}$. □

C5 Open Questions

While we have shown that several of our newly defined cardinal characteristics are, in fact, new, there are still a number of open questions.

Question H. *We summarise the open questions related to Figure 10:*

(Q1) *Does* $\mathrm{Con}(\mathfrak{d} < \mathfrak{s}_{1/2\pm\varepsilon} \leq \mathfrak{s}_{1/2})$ *hold or is* $\mathfrak{s}_{1/2} \leq \mathfrak{d}$*? (If it is the latter, we already know* $\mathrm{Con}(\mathfrak{s}_{1/2} < \mathfrak{d})$ *by* $\mathrm{Con}(\mathrm{non}(\mathcal{N}) < \mathfrak{d})$*.)*

(Q2) *Which of the following statements are true?*

$$\begin{aligned} \mathrm{Con}(\mathfrak{s} < \mathfrak{s}^w_{1/2}) \quad &\text{or} \quad \mathfrak{s} = \mathfrak{s}^w_{1/2} \\ \mathrm{Con}(\mathfrak{s}^w_{1/2} < \mathfrak{s}^\infty_{1/2}) \quad &\text{or} \quad \mathfrak{s}^w_{1/2} = \mathfrak{s}^\infty_{1/2} \\ \mathrm{Con}(\mathfrak{s}_{1/2\pm\varepsilon} < \mathfrak{s}_{1/2}) \quad &\text{or} \quad \mathfrak{s}_{1/2\pm\varepsilon} = \mathfrak{s}_{1/2} \end{aligned}$$

(Q3) *Given* $\varepsilon > \varepsilon'$ *and an* ε*-almost bisecting family, can one (finitarily) modify it to get an* ε'*-almost bisecting family of equal size? (If yes, then* $\mathfrak{s}_{1/2\pm\varepsilon}$ *is independent of* ε*. If not, then* $\inf_{\varepsilon\in(0,1/2)} \mathfrak{s}_{1/2\pm\varepsilon}$ *and* $\sup_{\varepsilon\in(0,1/2)} \mathfrak{s}_{1/2\pm\varepsilon}$ *might be interesting characteristics, as well.)*

(Q4) *Can characteristics in the upper row of the diagram consistently be smaller than ones in the lower row? Specifically, which of the following statements are true?*

$$\begin{aligned} \mathrm{Con}(\mathfrak{s}_{1/2\pm\varepsilon} < \mathfrak{s}^w_{1/2}) \quad &\text{or} \quad \mathfrak{s}_{1/2\pm\varepsilon} \geq \mathfrak{s}^w_{1/2} \\ \mathrm{Con}(\mathfrak{s}_{1/2\pm\varepsilon} < \mathfrak{s}^\infty_{1/2}) \quad &\text{or} \quad \mathfrak{s}_{1/2\pm\varepsilon} \geq \mathfrak{s}^\infty_{1/2} \\ \mathrm{Con}(\mathfrak{s}_{1/2} < \mathfrak{s}^\infty_{1/2}) \quad &\text{or} \quad \mathfrak{s}_{1/2} \geq \mathfrak{s}^\infty_{1/2} \end{aligned}$$

Question I. *We summarise the open questions related to Figure 11:*

(Q5) *Is it consistent that* $\mathfrak{i}_* < 2^{\aleph_0}$*?*

(Q6) *Which relations between* $\mathfrak{i}_{1/2}$*,* $\mathfrak{i}_*$ *and* $\mathfrak{i}$ *are true or consistent?*

(Q7) *Are there any smaller upper bounds for* $\mathfrak{i}_{1/2}$ *and* $\mathfrak{i}_*$*?*

(Q8) *Which relations between* $\mathfrak{s}_{1/2}$ *and* $\mathfrak{s}_*$ *are true or consistent?*

(Q9) *Which of the following statements are true?*

$$\begin{aligned} \mathrm{Con}(\mathrm{cov}(\mathcal{N}) < \mathfrak{r}_{1/2}) \quad &\text{or} \quad \mathrm{cov}(\mathcal{N}) = \mathfrak{r}_{1/2} \\ \mathrm{Con}(\mathfrak{r}_{1/2} < \mathfrak{r}_*) \quad &\text{or} \quad \mathfrak{r}_{1/2} = \mathfrak{r}_* \\ \mathrm{Con}(\mathfrak{s}_* < \mathrm{non}(\mathcal{N})) \quad &\text{or} \quad \mathfrak{s}_* = \mathrm{non}(\mathcal{N}) \end{aligned}$$

We suspect that (Q5) might be provable (via $\mathrm{Con}(\mathfrak{i}_* < \mathfrak{i})$) using the same idea as in Lemma C4.9. If the probabilistic argument from Lemma C3.4 can be reproduced for $\mathfrak{s}_*$, a similar approach as in section C3 might also work to answer the third part of (Q9) and prove $\mathrm{Con}(\mathfrak{s}_* < \mathrm{non}(\mathcal{N}))$. Finally, since it is not too difficult to ensure that a creature forcing poset keeps $\mathrm{cov}(\mathcal{N})$ small (compare [FGKS17, Lemma 5.4.2] or Lemma B7.7), a clever creature forcing construction might be able to answer the first part of (Q9) and prove $\mathrm{Con}(\mathrm{cov}(\mathcal{N}) < \mathfrak{r}_{1/2})$.

LIST OF FIGURES

1 The standard inequalities for the cardinal characteristics associated with a σ-ideal $\mathcal{I}$. 2

2 The ZFC-provable inequalities between localisation and anti-localisation cardinals and other well-known cardinal characteristics in Cichoń's diagram. 4

3 An example for the initial segment of a condition $p \in \mathbb{Q}_{c,h}$. 11

4 The proof of Lemma A5.6 up to the construction of p_3. 24

5 Cichoń's diagram with some exemplary $\mathfrak{c}_{f,g}$ added to it. 33

6 A diagram of the basic structure of the forcing poset $\mathbb{Q}$. 37

7 A visualisation of the segment $\mathrm{segm}(L)$ of the height L in a frame. 42

8 An example of a compound creature $C := p(4k)$ of a condition $p \in \mathbb{Q}_{\mathrm{nm},\,\kappa_{\mathrm{nm}}}$. 44

9 The construction in Lemma B7.3. 63

10 The ZFC-provable and/or consistent inequalities between $\mathfrak{s}_{1/2}$, $\mathfrak{s}_{1/2\pm\varepsilon}$, $\mathfrak{s}^{w}_{1/2}$, $\mathfrak{s}^{\infty}_{1/2}$ and other well-known cardinal characteristics. 99

11 The ZFC-provable and/or consistent inequalities between $\mathfrak{i}_{1/2}$, $\mathfrak{i}_{*}$, $\mathfrak{r}_{1/2}$, $\mathfrak{r}_{*}$, $\mathfrak{s}_{1/2}$, $\mathfrak{s}_{*}$ and other well-known cardinal characteristics. 110

BIBLIOGRAPHY

[AS16] Noga Alon and Joel Harold Spencer, *The Probabilistic Method*, Wiley Series in Discrete Mathematics and Optimization, John Wiley & Sons, Hoboken, NJ, [4]2016, DOI: 10.1002/9780470277331.

[Bar84] Tomek Bartoszyński, *Additivity of Measure Implies Additivity of Category*, Trans. Amer. Math. Soc. **281** (1984), no. 1, 209–213, DOI: 10.2307/1999530.

[BHK+18] Jörg Brendle, Lorenz J. Halbeisen, Lukas Daniel Klausner, Marc Lischka, and Saharon Shelah, *Halfway New Cardinal Characteristics*, arXiv: 1808.02442 [math.LO].

[BJ95] Tomek Bartoszyński and Haim Judah, *Set Theory: On the Structure of the Real Line*, A K Peters, Wellesley, MA, 1995, DOI: 10.1112/S0024609396222374.

[Bla93] Andreas Blass, *Simple Cardinal Characteristics of the Continuum*, Set Theory of the Reals (Haim Judah, ed.), Israel Mathematics Conference Proceedings, vol. 6, Gelbart Research Institute for Mathematical Sciences, Bar-Ilan University, Ramat-Gan, 1993, pp. 63–90, arXiv: 9405202 [math.LO].

[Bla10] ———, *Combinatorial Cardinal Characteristics of the Continuum*, Handbook of Set Theory (Matthew Foreman and Akihiro Kanamori, eds.), Springer, Dordrecht, 2010, pp. 395–489, DOI: 10.1007/978-1-4020-5764-9_7.

[BM14] Jörg Brendle and Diego Alejandro Mejía, *Rothberger Gaps in Fragmented Ideals*, Fund. Math. **227** (2014), no. 1, 35–68, DOI: 10.4064/FM227-1-4, arXiv: 1409.0222 [math.LO].

[CM17] Miguel Antonio Cardona and Diego Alejandro Mejía, *On Cardinal Characteristics of Yorioka Ideals*, arXiv: 1703.08634 [math.LO].

[FGKS17] Arthur Fischer, Martin Goldstern, Jakob Kellner, and Saharon Shelah, *Creature Forcing and Five Cardinal Characteristics in Cichoń's Diagram*, Arch. Math. Logic **56** (2017), no. 7–8, 1045–1103, DOI: 10.1007/S00153-017-0553-8, arXiv: 1402.0367 [math.LO].

[GK18] Martin Goldstern and Lukas Daniel Klausner, *Cichoń's Diagram and Localisation Cardinals*, arXiv: 1808.01921 [math.LO].

[GKS17] Martin Goldstern, Jakob Kellner, and Saharon Shelah, *Cichoń's Maximum*, arXiv: 1708.03691 [math.LO].

[GS90] Martin Goldstern and Saharon Shelah, *Ramsey Ultrafilters and the Reaping Number—Con($\mathfrak{r} < \mathfrak{u}$)*, Ann. Pure Appl. Logic **49** (1990), no. 2, 121–142, DOI: 10.1016/0168-0072(90)90063-8.

[GS93] ———, *Many Simple Cardinal Invariants*, Arch. Math. Logic **32** (1993), no. 3, 203–221, DOI: 10.1007/BF01375552, arXiv: MATH/9205208 [math.LO].

[Hal17] Lorenz J. Halbeisen, *Combinatorial Set Theory: With a Gentle Introduction to Forcing*, Springer Monographs in Mathematics, Springer, Cham, ^{2}2017, DOI: 10.1007/978-3-319-60231-8.

[Kec95] Alexander Sotirios Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, vol. 156, Springer, New York, 1995, DOI: 10.1007/978-1-4612-4190-4.

[Kel08] Jakob Kellner, *Even More Simple Cardinal Invariants*, Arch. Math. Logic **47** (2008), no. 5, 503–515, DOI: 10.1007/S00153-008-0094-2, arXiv: 0706.0319 [math.LO].

[KM18] Lukas Daniel Klausner and Diego Alejandro Mejía, *Many Different Uniformity Numbers of Yorioka Ideals*, arXiv: 1805.11005 [math.LO].

[KO08] Shizuo Kamo and Noboru Osuga, *The Cardinal Coefficients of the Ideal $\mathcal{I}_f$*, Arch. Math. Logic **47** (2008), no. 7–8, 653–671, DOI: 10.1007/S00153-008-0091-5.

[KO14] ———, *Many Different Covering Numbers of Yorioka's Ideals*, Arch. Math. Logic **53** (2014), no. 1, 43–56, DOI: 10.1007/S00153-013-0354-7.

[KS09] Jakob Kellner and Saharon Shelah, *Decisive Creatures and Large Continuum*, J. Symb. Log. **74** (2009), no. 1, 73–104, DOI: 10.2178/JSL/1231082303, arXiv: MATH/0601083 [math.LO].

[KS12] Jakob Kellner and Saharon Shelah, *Creature Forcing and Large Continuum: The Joy of Halving*, Arch. Math. Logic **51** (2012), no. 1–2, 49–70, DOI: 10.1007/S00153-011-0253-8, arXiv: 1003.3425 [math.LO].

[KST17] Jakob Kellner, Saharon Shelah, and Anda Ramona Tănasie, *Another Ordering of the Ten Cardinal Characteristics in Cichoń's Diagram*, arXiv: 1712.00778 [math.LO].

[KTT17] Jakob Kellner, Anda Ramona Tănasie, and Fabio Elio Tonti, *Compact Cardinals and Eight Values in Cichoń's Diagram*, arXiv: 1706.09638 [math.LO].

[Mil81] Arnold W. Miller, *Some Properties of Measure and Category*, Trans. Amer. Math. Soc. **266** (1981), no. 1, 93–114, DOI: 10.2307/1998389.

[RS99] Andrzej Rosłanowski and Saharon Shelah, *Norms on Possibilities I: Forcing with Trees and Creatures*, Mem. Amer. Math. Soc. **141** (1999), no. 671, xii+167, DOI: 10.1090/MEMO/0671, arXiv: MATH/9807172 [math.LO].

[Vau90] Jerry E. Vaughan, *Small Uncountable Cardinals and Topology*, Open Problems in Topology (Jan van Mill and George M. Reed, eds.), North-Holland, Amsterdam, 1990, pp. 195–218.

[Yor02] Teruyuki Yorioka, *The Cofinality of the Strong Measure Zero Ideal*, J. Symb. Log. **67** (2002), no. 4, 1373–1384, DOI: 10.2178/JSL/1190150290.