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FOUNDATIONS

The lattice of subspaces of a vector space over a finite field

Ivan Chajda[1] · Helmut Länger[1,2]

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Abstract

For finite m and q we study the lattice $\mathbf{L}(\mathbf{V}) = (L(\mathbf{V}), +, \cap, \{\vec{0}\}, V)$ of subspaces of an m-dimensional vector space $\mathbf{V}$ over a field $\mathbf{K}$ of cardinality q. We present formulas for the number of d-dimensional subspaces of $\mathbf{V}$, for the number of complements of a subspace and for the number of e-dimensional subspaces including a given d-dimensional subspace. It was shown in Eckmann and Zabey (Helv Phys Acta 42:420–424, 1969) that $\mathbf{L}(\mathbf{V})$ possesses an orthocomplementation only in case $m = 2$ and char $\mathbf{K} \neq 2$. Hence, only in this case $\mathbf{L}(\mathbf{V})$ can be considered as an orthomodular lattice. On the contrary, we show that a complementation $'$ on $\mathbf{L}(\mathbf{V})$ can be chosen in such a way that $(L(\mathbf{V}), +, \cap, ')$ is both weakly orthomodular and dually weakly orthomodular. Moreover, we show that $(L(\mathbf{V}), +, \cap, ^\perp, \{\vec{0}\}, V)$ is paraorthomodular in the sense of Giuntini et al. (Stud Log 104:1145–1177, 2016).

Keywords Vector space · Finite field · Lattice of subspaces · Antitone · Involution · Complementation · Orthocomplementation · Ortholattice · Orthomodular lattice · Weakly orthomodular lattice · Dually weakly orthomodular lattice · Paraorthomodular lattice

It is well known that in a Hilbert space $\mathbf{H}$ there exists a one-to-one correspondence between the set of projection operators and the set of closed subspaces. These subspaces form an orthomodular lattice $(L(\mathbf{H}), \vee, \cap, ^\perp, \{\vec{0}\}, H)$ where for $M, N \in L(\mathbf{H})$ we have $M \vee N = \overline{M + N}$.

Some doubts concerning the relevance of such an approach for an algebraic treatment of quantum mechanics arose when it was discovered that the class of orthomodular lattices arising from projections on Hilbert spaces does not generate the variety of orthomodular lattices showing that there are equational properties of event-state systems that are not correctly reflected by the proposed mathematical abstraction. Hence, alternative approaches appeared in the literature, see, e.g., the paper by Eckmann and Zabey (1969) on subspaces of a vector space over a finite field or the approach by Giuntini, Ledda and Paoli (Giuntini et al. 2016) concerning so-called paraorthomodular lattices and Kleene lattices.

The aim of the present paper is to describe the lattice $\mathbf{L}(\mathbf{V})$ of subspaces of a finite-dimensional vector space over a finite field with respect to the question of defining a suitable complementation. Similarly as in Giuntini et al. (2016), we do not restrict ourselves to orthomodular lattices, but we also consider so-called weakly orthomodular and dually weakly orthomodular lattices which were recently introduced and studied by the authors in Chajda and Länger (2018). It turns out that despite the fact that $\mathbf{L}(\mathbf{V})$ is orthomodular only in very exceptional cases, it is paraorthomodular with respect to orthogonality.

Throughout the paper let $m > 1$ be an integer and $\mathbf{V} = (V, +, \cdot)$ an m-dimensional vector space over some finite field $\mathbf{K} = (K, +, \cdot)$ of cardinality q. In the following, without loss of generality we identify $\mathbf{V}$ with $\mathbf{K}^m$. We denote the zero element of $\mathbf{V}$ by $\vec{0}$ and the zero element of $\mathbf{K}$ by 0. Moreover, we denote by $\mathbf{L}(\mathbf{V}) = (L(\mathbf{V}), +, \cap, \{\vec{0}\}, V)$ the lattice of subspaces of $\mathbf{V}$. For every $d \in \{0, \ldots, m\}$ let $L_d(\mathbf{V})$ denote the set of d-dimensional subspaces of $\mathbf{V}$. Finally, we define

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Helmut Länger
helmut.laenger@tuwien.ac.at

Ivan Chajda
ivan.chajda@upol.cz

1 Department of Algebra and Geometry, Faculty of Science, Palacký University Olomouc, 17. listopadu 12, 771 46 Olomouc, Czech Republic

2 Institute of Discrete Mathematics and Geometry, Faculty of Mathematics and Geoinformation, TU Wien, Wiedner Hauptstraße 8-10, 1040 Vienna, Austria

$a_0 := 1$ and

$$a_n := \prod_{i=1}^{n}(q^i - 1)$$

for every natural number n.

Theorem 1 *We have*

$$|L_d(\mathbf{V})| = \frac{a_m}{a_d a_{m-d}}$$

for all $d \in \{0, \ldots, m\}$.

Proof Let $d \in \{0, \ldots, m\}$. Put

$$A := \{(\vec{x}_1, \ldots, \vec{x}_d) \in V^d \mid \vec{x}_1, \ldots, \vec{x}_d \text{ are linearly independent}\}.$$

We want to determine $|A|$. For choosing $\vec{x}_1$ we have $q^m - 1$ possibilities. For every single one of these $q^m - 1$ possibilities for choosing $\vec{x}_1$ we have $q^m - q$ possibilities for choosing $\vec{x}_2$. Hence we have $(q^m - 1)(q^m - q)$ possibilities for choosing $(\vec{x}_1, \vec{x}_2)$. For every single one of these $(q^m - 1)(q^m - q)$ possibilities for choosing $(\vec{x}_1, \vec{x}_2)$ we have $q^m - q^2$ possibilities for choosing $\vec{x}_3$. Hence we have $(q^m - 1)(q^m - q)(q^m - q^2)$ possibilities for choosing $(\vec{x}_1, \vec{x}_2, \vec{x}_3)$. Going on in this way we finally obtain

$$|A| = \prod_{i=0}^{d-1}(q^m - q^i) = \frac{q^{d(d-1)/2} a_m}{a_{m-d}}.$$

Now let $(\vec{a}_1, \ldots, \vec{a}_d)$ be a fixed element of A. We want to determine the number of ordered bases of the subspace U of $\mathbf{V}$ generated by $\{\vec{a}_1, \ldots, \vec{a}_d\}$. It is easy to see that a subset $\{\vec{b}_1, \ldots, \vec{b}_d\}$ of V is a basis of U if and only if there exists a regular matrix $B \in K^{d \times d}$ with $(\vec{a}_1, \ldots, \vec{a}_d)B = (\vec{b}_1, \ldots, \vec{b}_d)$ and that the number of such ordered n-tuples $(\vec{b}_1, \ldots, \vec{b}_n)$ coincides with the number of regular $d \times d$-matrices B over $\mathbf{K}$. But this number can be easily computed. For choosing the first row of B we have $q^d - 1$ possibilities. For every single one of these $q^d - 1$ possibilities for choosing the first row of B we have $q^d - q$ possibilities for choosing the second row of B. Hence we have $(q^d - 1)(q^d - q)$ possibilities for choosing the first two rows of B. For every single one of these $(q^d - 1)(q^d - q)$ possibilities for choosing the first two rows of B we have $q^d - q^2$ possibilities for choosing the third row of B. Hence we have $(q^d - 1)(q^d - q)(q^d - q^2)$ possibilities for choosing the first three rows of B. Going on in this way we finally obtain

$$\prod_{i=0}^{d-1}(q^d - q^i) = q^{d(d-1)/2} a_d$$

possibilities for B. Hence

$$|L_d(\mathbf{V})| = \frac{q^{d(d-1)/2} a_m / a_{m-d}}{q^{d(d-1)/2} a_d} = \frac{a_m}{a_d a_{m-d}}.$$

□

Remark 2 Theorem 1 also holds in case $m \in \{0, 1\}$.

Lemma 3 $|L_d(\mathbf{V})| = |L_{m-d}(\mathbf{V})|$ *for all* $d = 0, \ldots, m$.

Proof We have

$$|L_d(\mathbf{V})| = \frac{a_m}{a_d a_{m-d}} = \frac{a_m}{a_{m-d} a_d} = |L_{m-d}(\mathbf{V})|$$

for all $d = 0, \ldots, m$. □

Lemma 4 *If m is even then* $|L_{m/2}(\mathbf{V})| = (q^{m/2} + 1)|L_{m/2}(\mathbf{K}^{m-1})|$.

Proof If m is even then

$$|L_{m/2}(\mathbf{V})| = \frac{a_m}{a_{m/2}^2} = \frac{(q^m - 1)a_{m-1}}{(q^{m/2} - 1)a_{m/2-1} a_{m/2}} = (q^{m/2} + 1)|L_{m/2}(\mathbf{K}^{m-1})|.$$

□

Theorem 5 $|L(\mathbf{V})|$ *is odd if and only if m is even and* char $\mathbf{K} = 2$.

Proof We have

$$|L(\mathbf{V})| = \sum_{d=0}^{m} |L_d(\mathbf{V})|.$$

If m is odd then

$$|L(\mathbf{V})| = 2 \sum_{d=0}^{(m-1)/2} |L_d(\mathbf{V})|$$

according to Lemma 3 showing evenness of $|L(\mathbf{V})|$. If m is even and char $\mathbf{K} = 2$ then a_m is odd, and hence, $|L_d(\mathbf{V})|$ is odd for every $d = 0, \ldots, m$ showing oddness of $|L(\mathbf{V})|$. If, finally, m is even and char $\mathbf{K} \neq 2$ then q is odd and

$$|L(\mathbf{V})| = 2 \sum_{d=0}^{m/2-1} |L_d(\mathbf{V})| + |L_{m/2}(\mathbf{V})| = 2 \sum_{d=0}^{m/2-1} |L_d(\mathbf{V})| + (q^{m/2} + 1)|L_{m/2}(\mathbf{K}^{m-1})|$$

according to Lemma 4 showing evenness of $|L(\mathbf{V})|$. □

Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice. A unary operation $'$ on L is called

- *antitone* if $x \le y$ implies $y' \le x'$ $(x, y \in L)$,
- an *involution* if it satisfies the identity $(x')' \approx x$,
- a *complementation* if it satisfies the identities $x \vee x' \approx 1$ and $x \wedge x' \approx 0$,
- an *orthocomplementation* if it is both a complementation and an antitone involution.

A bounded lattice with an orthocomplementation is called an *ortholattice*.

Lemma 6 *If $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ is a non-trivial finite bounded lattice with a complementation which is an involution then $|L|$ is even.*

Proof It is easy to see that the binary relation R defined by $x \mathrel{R} y$ if and only if $y = x$ or $y = x'$ $(x, y \in L)$ is an equivalence relation on L consisting of two-element classes only. □

Corollary 7 *If m is even and* $\operatorname{char} \mathbf{K} = 2$ *then $\mathbf{L}(\mathbf{V})$ has no complementation which is an involution and hence no orthocomplementation.*

Proof This follows from Theorem 5 and Lemma 6. □

A lattice $\mathbf{L} = (L, \vee, \wedge)$ is called *modular* if $(x \vee y) \wedge z = x \vee (y \wedge z)$ or all $x, y, z \in L$ with $x \le z$.

The following result is well known.

Proposition 8 *The lattice $\mathbf{L}(\mathbf{V})$ is modular.*

Definition 9 (cf. Chajda and Länger 2018) Let $\mathbf{L} = (L, \vee, \wedge, ')$ be a lattice with a unary operation $'$. $\mathbf{L}$ is called *weakly orthomodular* if $y = x \vee (y \wedge x')$ for all $x, y \in L$ with $x \le y$, and it is called *dually weakly orthomodular* if $x = y \wedge (x \vee y')$ for all $x, y \in L$ with $x \le y$. Now assume $\mathbf{L}$ to be bounded. The element b of L is called a *complement* of the element a of L if both $a \vee b = 1$ and $a \wedge b = 0$. An ortholattice is called an *orthomodular lattice* if it is weakly orthomodular or, equivalently, if it is dually weakly orthomodular. The corresponding condition is then called the *orthomodular law*.

Lemma 10 (cf. Chajda and Länger 2018) *Every bounded modular lattice $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ equipped with a complementation $'$ is both weakly orthomodular and dually weakly orthomodular. Hence every modular ortholattice is orthomodular.*

Proof Let $a, b \in L$ and assume $a \le b$. Then, using modularity,

$$a \vee (b \wedge a') = a \vee (a' \wedge b) = (a \vee a') \wedge b = 1 \wedge b = b,$$
$$b \wedge (a \vee b') = (a \vee b') \wedge b = a \vee (b' \wedge b) = a \vee 0 = a.$$

□

Theorem 11 *Every d-dimensional subspace of $\mathbf{V}$ has $q^{d(m-d)}$ complements. Hence, $\mathbf{L}(\mathbf{V})$ has*

$$q^{\sum_{d=0}^{m} (d(m-d)a_m/(a_d a_{m-d}))}$$

complementations.

Proof Let $d \in \{0, \ldots, m\}$, $U \in L_d(\mathbf{V})$ and $\{\vec{b}_1, \ldots, \vec{b}_d\}$ be a basis of U and put

$$A := \{(\vec{x}_{d+1}, \ldots, \vec{x}_m) \in V^{m-d} \mid \vec{b}_1, \ldots, \vec{b}_d, \vec{x}_{d+1}, \ldots, \vec{x}_m \text{ are linearly independent}\}.$$

We want to determine $|A|$. For choosing $\vec{x}_{d+1}$ we have $q^m - q^d$ possibilities. For every single one of these $q^m - q^d$ possibilities for choosing $\vec{x}_{d+1}$ we have $q^m - q^{d+1}$ possibilities for choosing $\vec{x}_{d+2}$. Hence we have $(q^m - q^d)(q^m - q^{d+1})$ possibilities for choosing $(\vec{x}_{d+1}, \vec{x}_{d+2})$. For every single one of these $(q^m - q^d)(q^m - q^{d+1})$ possibilities for choosing $(\vec{x}_{d+1}, \vec{x}_{d+2})$ we have $q^m - q^{d+2}$ possibilities for choosing $\vec{x}_{d+3}$. Hence we have $(q^m - q^d)(q^m - q^{d+1})(q^m - q^{d+2})$ possibilities for choosing $(\vec{x}_{d+1}, \vec{x}_{d+2}, \vec{x}_{d+3})$. Going on in this way we finally obtain

$$|A| = \prod_{i=d}^{m-1} (q^m - q^i) = q^{(m+d-1)(m-d)/2} a_{m-d}.$$

As in the proof of Theorem 1 we see that there are $q^{(m-d)(m-d-1)/2} a_{m-d}$ ordered bases of an $(m-d)$-dimensional subspace of $\mathbf{V}$. Hence U has

$$\frac{q^{(m+d-1)(m-d)/2} a_{m-d}}{q^{(m-d-1)(m-d)/2} a_{m-d}} = q^{d(m-d)}$$

complements. Together with Theorem 1 we conclude that $\mathbf{L}(\mathbf{V})$ has

$$\prod_{d=0}^{m} (q^{d(m-d)})^{(a_m/(a_d a_{m-d}))} = q^{\sum_{d=0}^{m} (d(m-d)a_m/(a_d a_{m-d}))}$$

complementations. □

Corollary 12 *For any complementation $'$ on $\mathbf{L}(\mathbf{V})$, $(L(\mathbf{V}), +, \cap, ')$ is both weakly orthomodular and dually weakly orthomodular.*

Proof This follows from Proposition 8 and Lemma 10. □

As pointed out in Eckmann and Zabey (1969), the fact that a complementation on $\mathbf{L}(\mathbf{V})$ is an orthocomplementation is very exceptional:

Theorem 13 *The lattice $\mathbf{L}(\mathbf{V})$ has an orthocomplementation if and only if $m = 2$ and* $\operatorname{char} \mathbf{K} \neq 2$.

Hence by defining a unary operation on $L(\mathbf{V})$ in a suitable way, $\mathbf{L}(\mathbf{V})$ can be transformed into an orthomodular lattice if and only if $m = 2$ and char $\mathbf{K} \neq 2$. The cases $m = 2$ and char $\mathbf{K} = 2$ as well as $m = 2$ and char $\mathbf{K} \neq 2$ will be shown in the next examples. At first, we recall some concepts from lattice theory.

In the following, for $n \geq 3$ let M_n denote the modular lattice whose Hasse diagram is visualized in Fig. 1

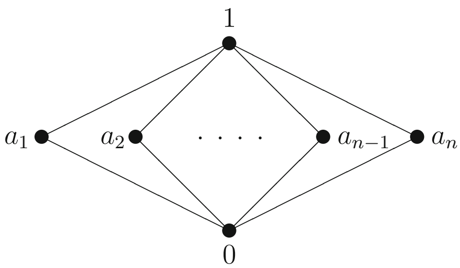

Fig. 1

and for $n \geq 2$ let MO_n denote the modular ortholattice whose Hasse diagram is visualized in Fig. 2.

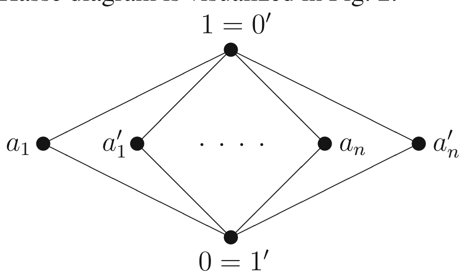

Fig. 2

The situation described by Theorems 1 and 5, Corollary 7, Proposition 8 and Theorem 11 is illustrated by the following examples.

Example 14 Let $(m, q) = (2, 2)$, i.e., char $\mathbf{K} = 2$. Then the Hasse diagram of $\mathbf{L}(\mathbf{V})$ looks as follows (see Fig. 3):

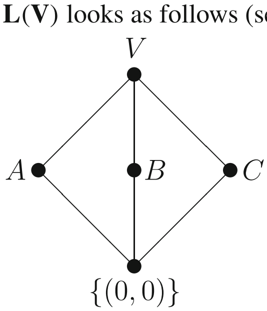

Fig. 3

where

$$A := \{(0,0), (0,1)\},$$
$$B := \{(0,0), (1,0)\},$$
$$C := \{(0,0), (1,1)\}.$$

Hence $\mathbf{L}(\mathbf{V}) \cong \mathrm{M}_3$. It is easy to see that there are the following eight possibilities for defining a complementation $'$ on $\mathbf{L}(\mathbf{V})$:

A'	B'	C'
B	A	A
B	A	B
B	C	A
B	C	B
C	A	A
C	A	B
C	C	A
C	C	B

This is in accordance with Theorem 11. Every single of these complementations is antitone, but none of them is an orthocomplementation.

More generally, we have

Theorem 15 *If $m = 2$ and char $\mathbf{K} = 2$ then $\mathbf{L}(\mathbf{V}) \cong \mathrm{M}_{q+1}$ and any complementation on $\mathbf{L}(\mathbf{V})$ is antitone, but none of them is an orthocomplementation.*

Proof Assume $m = 2$ and char $\mathbf{K} = 2$. Since $|L_1(\mathbf{V})| = q + 1$ according to Theorem 1 we have $\mathbf{L}(\mathbf{V}) \cong \mathrm{M}_{q+1}$. Clearly, any complementation on $\mathbf{L}(\mathbf{V})$ is antitone. That $\mathbf{L}(\mathbf{V})$ has no orthocomplementation follows from Corollary 7 and it follows from Theorem 13. □

Now let us introduce the concept of orthogonality in $\mathbf{V}$.

Let $\vec{a} = (a_1, \ldots, a_m), \vec{b} = (b_1, \ldots, b_m) \in V$. By $\vec{a}\vec{b}$ we denote the *inner* or *scalar product* $a_1 b_1 + \cdots + a_m b_m$ of $\vec{a}$ and $\vec{b}$. Define $\vec{a} \perp \vec{b}$ if $\vec{a}\vec{b} = 0$, and for any subset A of V put $A^\perp := \{\vec{x} \in V \mid \vec{x}\vec{y} = 0 \text{ for all } \vec{y} \in A\}$.

Lemma 16 *The mapping $^\perp : U \mapsto U^\perp$ is an antitone involution on $\mathbf{L}(\mathbf{V})$.*

Proof Let $U \in L(\mathbf{V})$. The definition of $U^\perp$ implies that $^\perp$ is antitone and $U \subseteq U^{\perp\perp}$. From the theory concerning the solving of systems of linear equations (Gaussian elimination method) one easily obtains that $\dim U + \dim U^\perp = m$. Hence also $\dim U^\perp + \dim U^{\perp\perp} = m$, and we obtain

$$\dim U^{\perp\perp} = m - \dim U^\perp = m - (m - \dim U) = \dim U$$

showing that $U = U^{\perp\perp}$, i.e., $^\perp$ is an involution on $\mathbf{L}(\mathbf{V})$. □

In general, $^\perp$ is not an orthocomplementation on $\mathbf{L}(\mathbf{V})$. For example, in Example 14 we have

U	A	B	C
$U^\perp$	B	A	C

Moreover, $\dim U^\perp = m - \dim U$ for every $U \in L(\mathbf{V})$.

Example 17 Let $(m, q) = (2, 3)$, i.e., char $\mathbf{K} \neq 2$. Then the Hasse diagram of $\mathbf{L}(\mathbf{V})$ looks as follows (see Fig. 4):

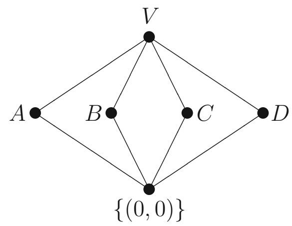

Fig. 4

where

$$\begin{aligned} A &:= \{(0,0),(0,1),(0,2)\}, \\ B &:= \{(0,0),(1,0),(2,0)\}, \\ C &:= \{(0,0),(1,1),(2,2)\}, \\ D &:= \{(0,0),(1,2),(2,1)\}. \end{aligned}$$

Hence $\mathbf{L}(\mathbf{V}) \cong \mathrm{M}_4$. It is easy to see that the possible orthocomplementations on $\mathbf{L}(\mathbf{V})$ are in one-to-one correspondence with the partitions of $\{A, B, C, D\}$ into two-element classes. For any of these orthocomplementations $'$, $(L(\mathbf{V}), +, \cap, ', \{\vec{0}\}, V) \cong \mathrm{MO}_2$ is a modular ortholattice. Moreover,

U	A	B	C	D
$U^\perp$	B	A	D	C

and hence $^\perp$ is an orthocomplementation. It should be remarked that in case $(m, q) = (2, 5)$, $^\perp$ is not an orthocomplementation on $\mathbf{L}(\mathbf{V})$ since $U^\perp = U$ for

$$U = \{(0, 0), (1, 3), (2, 1), (3, 4), (4, 2)\}.$$

More generally, we have

Theorem 18 *If* $m = 2$ *and* char $\mathbf{K} \neq 2$ *then there exist*

$$\frac{(q+1)!}{2^{(q+1)/2}((q+1)/2)!}$$

orthocomplementations $'$ *on* $\mathbf{L}(\mathbf{V})$. *With any of these* $(L(\mathbf{V}), +, \cap, ', \{\vec{0}\}, V) \cong \mathrm{MO}_{(q+1)/2}$ *is a modular ortholattice.*

Proof Assume $m = 2$ and char $\mathbf{K} \neq 2$. It is clear that there is a one-to-one correspondence between the set of all orthocomplementations on $\mathbf{L}(\mathbf{V})$ and the set of all partitions of $L_1(\mathbf{V})$ into two-element classes. It is easy to see by induction on n that for an arbitrary positive integer n there are exactly $(2n-1)(2n-3) \cdot \ldots \cdot 1$ different partitions of a $2n$-element set into two-element classes. Now we have

$$\begin{aligned} (2n-1)(2n-3) \cdot \ldots \cdot 1 &= \frac{(2n)(2n-1) \cdot \ldots \cdot 1}{(2n)(2n-2) \cdot \ldots \cdot 2} \\ &= \frac{(2n)!}{2^n n(n-1) \cdot \ldots \cdot 1} = \frac{(2n)!}{2^n n!}. \end{aligned}$$

This shows that there are

$$\frac{(q+1)!}{2^{(q+1)/2}((q+1)/2)!}$$

different partitions of the $(q+1)$-element set $L_1(\mathbf{V})$ into two-element classes. That with any of these $(L(\mathbf{V}), +, \cap, ', \{\vec{0}\}, V) \cong \mathrm{MO}_{(q+1)/2}$ is a modular ortholattice is clear. □

As mentioned in the introduction, another approach to the lattice $\mathbf{L}(\mathbf{V})$ was developed in Giuntini et al. (2016). We recall the following definition:

Definition 19 A bounded lattice $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ with an antitone involution $'$ is called *paraorthomodular* if

$$x = y \text{ for all } x, y \in L \text{ satisfying both } x \leq y \text{ and } x' \wedge y = 0. \tag{1}$$

It was shown in Giuntini et al. (2016) that for ortholattices (1) is equivalent to the orthomodular law. Note that in Definition 19 we do not ask $'$ to be a complementation, and we only ask $'$ to be an antitone involution.

The following result is taken from Giuntini et al. (2016). For the reader's convenience we provide a proof.

Proposition 20 *Every bounded modular lattice with an antitone involution is paraorthomodular.*

Proof If $(L, \vee, \wedge, ', 0, 1)$ is a bounded modular lattice with an antitone involution, $a, b \in L$, $a \leq b$ and $a' \wedge b = 0$ then $a \vee a' \geq a \vee b' = (a' \wedge b)' = 0' = 1$ and hence $a \vee a' = 1$ whence $a = a \vee 0 = a \vee (a' \wedge b) = (a \vee a') \wedge b = 1 \wedge b = b$. □

Corollary 21 *The lattice* $(L(\mathbf{V}), +, \cap, ^\perp, \{\vec{0}\}, V)$ *is paraorthomodular.*

Proof This follows from Proposition 8, Lemma 16 and Proposition 20. There exists also another proof of Corollary 21 not explicitly using modularity. If $U, W \in L(\mathbf{V})$, $U \subseteq W$ and $U^\perp \cap W = \{\vec{0}\}$ then

$$\begin{aligned} \dim U \leq \dim W &= \dim(U^\perp + W) - \dim U^\perp \\ &= \dim(U^\perp + W) - m + \dim U \leq \dim U \end{aligned}$$

and hence $\dim U = \dim W$, i.e., $U = W$. □

Example 22 Let $(m, q) = (3, 2)$, i.e., char $\mathbf{K} = 2$. Then the Hasse diagram of $\mathbf{L}(\mathbf{V})$ looks as follows (see Fig. 5):

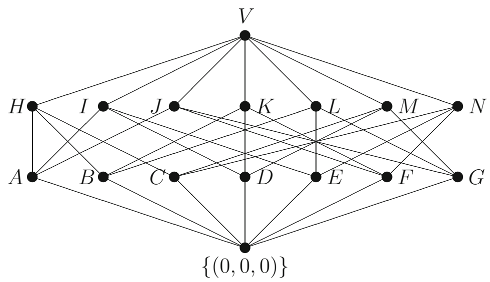

Fig. 5

where

$$\begin{aligned}A &:= \{(0,0,0),(0,0,1)\},\\ B &:= \{(0,0,0),(0,1,0)\},\\ C &:= \{(0,0,0),(0,1,1)\},\\ D &:= \{(0,0,0),(1,0,0)\},\\ E &:= \{(0,0,0),(1,0,1)\},\\ F &:= \{(0,0,0),(1,1,0)\},\\ G &:= \{(0,0,0),(1,1,1)\},\\ H &:= \{(0,0,0),(0,0,1),(0,1,0),(0,1,1)\},\\ I &:= \{(0,0,0),(0,0,1),(1,0,0),(1,0,1)\},\\ J &:= \{(0,0,0),(0,0,1),(1,1,0),(1,1,1)\},\\ K &:= \{(0,0,0),(0,1,0),(1,0,0),(1,1,0)\},\\ L &:= \{(0,0,0),(0,1,0),(1,0,1),(1,1,1)\},\\ M &:= \{(0,0,0),(0,1,1),(1,0,0),(1,1,1)\},\\ N &:= \{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\}.\end{aligned}$$

It is easy to see that the following table defines a complementation on $\mathbf{L}(\mathbf{V})$ which is an involution:

x	A	B	C	D	E	F	G
x'	K	I	J	H	M	L	N

but $'$ is not an orthocomplementation on $\mathbf{L}(\mathbf{V})$ since $A \subseteq J$, but $J' = C \nsubseteq K = A'$. Moreover, $^\perp$ is given by

U	A	B	C	D	E	F	G
$U^\perp$	K	I	M	H	L	J	N

Since $C + C^\perp = M \neq V$, $^\perp$ is not an orthocomplementation on $\mathbf{L}(\mathbf{V})$. If $'$ were an orthocomplementation on $\mathbf{L}(\mathbf{V})$ then

$$\bigcup_{(U,W)\in P} \{(U, W), (W', U')\}^2$$

would be an equivalence relation on

$$P := \{(U, W) \in L_1(\mathbf{V}) \times L_2(\mathbf{V}) \mid U \subseteq W\}$$

consisting of two-element classes only contradicting $|M| = 21$.

Finally, we want to present a new proof of the fact that $\mathbf{L}(\mathbf{V})$ has no orthocomplementation in case char $\mathbf{K} = 2$.

Theorem 23 *If $0 \le d \le e \le m$ then every d-dimensional subspace of $\mathbf{V}$ is included in*

$$\frac{a_{m-d}}{a_{m-e}a_{e-d}}$$

e-dimensional subspaces of $\mathbf{V}$.

Proof Assume $0 \le d \le e \le m$, let $U \in L_d(\mathbf{V})$ and $\{\vec{b}_1, \ldots, \vec{b}_d\}$ be a basis of U and put

$$A := \{(\vec{x}_{d+1}, \ldots, \vec{x}_e) \in V^{e-d} \mid \vec{b}_1, \ldots, \vec{b}_d, \vec{x}_{d+1}, \ldots, \vec{x}_e \text{ are linearly independent}\}.$$

We want to determine $|A|$. For choosing $\vec{x}_{d+1}$ we have $q^m - q^d$ possibilities. For every single one of these $q^m - q^d$ possibilities for choosing $\vec{x}_{d+1}$ we have $q^m - q^{d+1}$ possibilities for choosing $\vec{x}_{d+2}$. Hence we have $(q^m-q^d)(q^m-q^{d+1})$ possibilities for choosing $(\vec{x}_{d+1}, \vec{x}_{d+2})$. For every single one of these $(q^m - q^d)(q^m - q^{d+1})$ possibilities for choosing $(\vec{x}_{d+1}, \vec{x}_{d+2})$ we have $q^m - q^{d+2}$ possibilities for choosing $\vec{x}_{d+3}$. Hence we have $(q^m - q^d)(q^m - q^{d+1})(q^m - q^{d+2})$ possibilities for choosing $(\vec{x}_{d+1}, \vec{x}_{d+2}, \vec{x}_{d+3})$. Going on in this way we finally obtain

$$|A| = \prod_{i=d}^{e-1}(q^m - q^i) = \frac{q^{(e+d-1)(e-d)/2}a_{m-d}}{a_{m-e}}.$$

Now let $(\vec{a}_{d+1}, \ldots, \vec{a}_e)$ be a fixed element of A. We want to determine the number of ordered bases of the subspace of $\mathbf{V}$ generated by $\{\vec{b}_1, \ldots, \vec{b}_d, \vec{a}_{d+1}, \ldots, \vec{a}_e\}$ which are of the form $(\vec{b}_1, \ldots, \vec{b}_d, \vec{x}_{d+1}, \ldots, \vec{x}_e)$. Similarly as in the proof of Theorem 1 it is easy to see that the number of such ordered bases coincides with the number of regular $e \times e$-matrices B over $\mathbf{K}$ the first d columns of which coincide with the first d canonical unit vectors of K^e. But this number can be easily computed. For choosing the $(d+1)$-th column of B we have $q^e - q^d$ possibilities. For every single one of these $q^e - q^d$ possibilities for choosing the $(d+1)$-th column of B we have $q^e - q^{d+1}$ possibilities for choosing the $(d+2)$-th column of B. Hence we have $(q^e - q^d)(q^e - q^{d+1})$ possibilities for choosing the $(d+1)$-th and $(d+2)$-th column of B. For every single one of these $(q^e - q^d)(q^e - q^{d+1})$ possibilities for choosing the $(d+1)$-th and $(d+2)$-th column of B we have $q^e - q^{d+2}$ possibilities for choosing the $(d+3)$-th column of B. Hence we have $(q^e-q^d)(q^e-q^{d+1})(q^e-q^{d+2})$ possibilities for choosing the $(d+1)$-th, $(d+2)$-th and $(d+3)$-th column of B. Going on in this way we finally obtain

$$\prod_{i=d}^{e-1}(q^e - q^i) = q^{(e+d-1)(e-d)/2}a_{e-d}$$

possibilities for B. Hence U is included in

$$\frac{q^{(e+d-1)(e-d)/2}a_{m-d}/a_{m-e}}{q^{(e+d-1)(e-d)/2}a_{e-d}}=\frac{a_{m-d}}{a_{m-e}a_{e-d}}$$

e-dimensional subspaces of $\mathbf{V}$. □

Theorem 24 *If* char $\mathbf{K}=2$ *then* $\mathbf{L}(\mathbf{V})$ *has no orthocomplementation.*

Proof If char $\mathbf{K}=2$ and $'$ were an orthocomplementation on $\mathbf{L}(\mathbf{V})$ then

$$\bigcup_{(U,W)\in M}\{(U,W),(W',U')\}^2$$

would be an equivalence relation on

$$M:=\{(U,W)\in L_1(\mathbf{V})\times L_{m-1}(\mathbf{V})\mid U\subseteq W\}$$

consisting of two-element classes only contradicting oddness of $|M|$ which follows from

$$|M|=\frac{(q^m-1)(q^{m-1}-1)}{(q-1)^2}$$

according to Theorems 1 and 23. □

We can summarize our results as follows: Despite the fact that $(L(\mathbf{V}),+,\cap,{}',\{\vec{0}\},V)$ with an appropriate $'$ is an orthomodular lattice in exceptional cases only, we have shown that this lattice is weakly orthomodular, dually weakly orthomodular and paraorthomodular when $'$ is chosen in a appropriate way. This motivates further study of these structures.

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Compliance with ethical standards

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