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Numerical evidence of Kneser solutions and convergence of collocation applied to singular ODEs

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Abstract

In the first chapter of the work, we give the numerical evidence of Kneser solutions for the following problem class:

$$\big(p(t)u'(t)\big)' + q(t)f\left(u(t)\right) = 0, \quad u'(0) = 0, \quad \lim_{t\to\infty} u(t) = 0,$$

where $t \in [a, \infty)$. The main goal is to illustrate the existence theory of Kneser solutions provided in [6, 9] and focusing on the case when $p \equiv q$ and the underlying differential equation is singular, i.e. $a = 0$. We also discuss asymptotic behavior for $t \to \infty$. We try to computationally recover Kneser solutions for the case $p \neq q$, where the existence theory is still an open question. To this aim, we carefully choose the functions p, q, and the nonlinear term $f(u)$.

In the second part of the work, we test the convergence properties of the collocation schemes applied to solve the systems of linear ordinary differential equations with a singularity of the first kind,

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad t \in (0, 1],$$

where $M \in \mathbb{R}^{n\times n}$ and $f \in C[0, 1]$. We extend the collection of examples provided in [1] by more involved initial, terminal and boundary value problems.

Contents

1. Numerical evidence of Kneser solutions

1.1. Introduction

Throughout the work, we heavily rely on the theory developed in [6, 9], where the nonlinear ordinary differential equation (ODE),

$$\big(p(t)u'(t)\big)' + q(t)f\,(u(t)) = 0, \quad t \in [a,\infty), \quad u(a) \in (L_0, L), \quad a \geq 0, \tag{1.1}$$

have been studied. Bevor designing the model problems used in the numerical simulation, we recapitulate the available analytical results and specify the necessary properties the data functions p, q, and f have to satisfy.

The model problems have the following general form. Let the points $L_0 < B_0 < 0 < A_0 < L$ be chosen in such a way that $F(L_0) > F(L)$, where $F(x) = \int_0^x f(x)\,\mathrm{d}x$. For $|B_0| = A_0$ this means that $|L_0| > L$. Moreover, let $r > 1$ and

$$p(t) \;=\; t^\alpha,\;\; t \in [a,\infty),\;\; \alpha \in [2,\infty], \tag{1.2a}$$

$$q(t) \;=\; t^\alpha,\;\; t \in [a,\infty),\;\; \beta \in [\alpha-1, 2\alpha-3], \tag{1.2b}$$

$$f(u) \;=\; \begin{cases} \dfrac{|B_0|^r(L_0-u)}{B_0-L_0} & \text{for} \quad u \in [L_0, B_0), \\ |u|^r \operatorname{sgn} u & \text{for} \quad u \in [B_0, A_0], \\ \dfrac{A_0^r(u-L)}{A_0-L} & \text{for} \quad u \in (A_0, L]. \end{cases} \tag{1.2c}$$

We shall also test the following more involved data functions:

$$p(t) = t^\alpha \ln(1+t),\;\; q(t) = t^\beta \ln(1+t), \tag{1.3a}$$
$$p(t) = t^\alpha,\;\; q(t) = t^\beta(1+\exp(-t)), \tag{1.3b}$$
$$p(t) = t^\alpha(1+\exp(-t)),\;\; q(t) = t^\beta(1+\exp(-t)). \tag{1.3c}$$

As already mentioned, in [6, 9] the existence theory of Kneser solutions and their asymptotic behavior were described. In the context of these articles, we ask the following questions:

i) Can we give the numerical evidence of Kneser solutions for (1.1), with $p \equiv q$ given in (1.2a) and $a = 0$, supporting the statements from [6, 9] ?

ii) Can we numerically recover the asymptotic behavior of Kneser solutions specified in [6, 9] ?

iii) Can we numerically simulate the case of (1.1) with $p \neq q$, see $(1.3a)$–$(1.3c)$, and $a = 0$ which have not been covered analytically yet ?

We begin with a simple example, to show how we proceed.

Example 1.1. We first choose p, q and f, $p(t) = q(t) = t^3$, and

$$f(u) = \begin{cases} -12 - 2u & \text{for} \quad u < -2, \\ u^3 & \text{for} \quad -2 \leq u \leq 1, \\ 2 - u & \text{for} \quad u > 1, \end{cases}$$

see model problem 1.5 in [9]. Using these functions in (1.1), we obtain

$$u''(t) + \frac{3}{t}u'(t) + f(u) = 0, \quad t \in (0, \infty), \quad u'(0) = 0, \quad u(\infty) = 0. \tag{1.4}$$

In this case, we know the exact solution,

$$u(t) = \frac{8u_0}{8 + (u_0 t)^2},$$

where $u_0 = u(0)$. To approximate u, we use the open domain MATLAB code `bvpsuite` based on collocation [3]. Depending on the choice of u_0, we obtain a positive or a negative solution whose values are tending to zero for $t \to \infty$, see Figure 1.1.

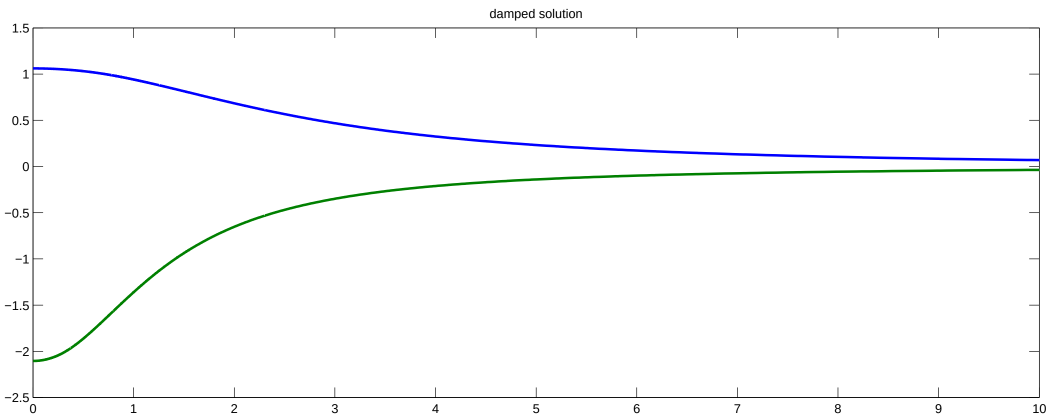

Figure 1.1.: Example 1.5 [9]: A positive, $u_0 \in (0, 1]$, and a negative, $u_0 \in (-2, 0]$, solution of the problem, damped to zero for $t \to \infty$.

1.2. Kneser solutions and their numerical realisation

In this section, we focus on the computation of Kneser solutions. We first recapitulate the theoretical results, see [6, 9] for details.

1.2.1. Existence theory

We study (1.1) on $[0, \infty)$ under the following assumptions:

$$L_0 < 0 < L, \quad f(L_0) = f(0) = f(L) = 0, \tag{1.5a}$$

$$f \in Lip_{loc}(\mathbb{R}), \tag{1.5b}$$

$$uf(u) > 0 \text{ for } u \in (L_0, L), \tag{1.5c}$$

$$F(L_0) > F(L), \text{ where } F(u) = \int_0^u f(x)dx, \tag{1.5d}$$

$$p \in C[0, \infty] \cap C^1(0, \infty), \quad p(0) = 0, \tag{1.5e}$$

$$p' > 0 \text{ on } (0, \infty), \quad \lim_{t \to \infty} \frac{p'(t)}{p(t)} = 0, \tag{1.5f}$$

where $Lip_{loc}(\mathbb{R})$ is defined below.

Definition 1.2. A function $f : \mathbb{R} \to \mathbb{R}$ is called locally Lipschitz continuous, i.e. $f \in Lip_{loc}(\mathbb{R})$, if there exists a constant L such that

$$|f(x_1) - f(x_2)| \leq L\,|x_1 - x_2|, \quad \forall x_1, x_2 \in U,$$

where U is an open subdomain of $\mathbb{R}$.

Definition 1.3. A function u is called a solution of problem (1.1) on $[a, \infty)$ if $u \in C^1[a, \infty)$, $pu' \in C^1[a, \infty)$ and (1.1) holds for all $t \in [a, \infty)$.

Definition 1.4. A solution u is called *oscillatory* solution of (1.1) if it has an unbounded set of isolated zeros. Otherwise it is called *non-oscillatory*.

Definition 1.5. A solution u of (1.1) with (1.5a) is called *damped* solution if

$$\sup\{u(t) : t \in [0, \infty)\} < L.$$

Definition 1.6. A solution u of the problem (1.1) is called *Kneser solution* if there exists $t_0 > a$ such that

$$u(t)u'(t) < 0 \text{ for } t \in [t_0, \infty). \tag{1.6}$$

In particular, there is a connection between damped solutions and Kneser solution which is stated in the following theorem.

Theorem 1.7. *Let u be a damped, non–oscillatory solution of* (1.1) *where $u_0 \in (L_0, 0) \cup (0, L)$. Then u is a Kneser solution of* (1.1).

Since we are interested in Kneser solutions, we do not deal with oscillatory solutions here.
The following important theorem discusses the existence of Kneser solutions to problem (1.1) under the assumption that $p \equiv q$.

Theorem 1.8. *Let* (1.5*f*) *hold and let* $p \equiv q$. *Moreover, let us assume that there exists a constant* $c > \frac{1}{2}$ *such that*

$$\frac{p'(t)P(t)}{p^2(t)} \geq c, \quad t \in (0, \infty),$$

and a point $B_0 \in (L_0, 0)$ *such that*

$$\frac{uf(u)}{F(u)} \geq \frac{2}{2c-1}, \quad u \in [B_0, 0).$$

Then, for every $u_0 \in [B_0, 0)$, *there exists a unique Kneser solution* u *of the singular*[1] *problem* (1.1). *This solution satisfies the following properties:*

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0, \quad u'(0) = 0, \quad u'(t) > 0, \;\; t \in [0, \infty).$$

Note, that the existence of Kneser solutions for (1.1) with $p \neq q$ is still an open question. However, as we will see later, we are able to calculate Kneser solutions numerically, which gives us valuable information towards the possible theory.

1.2.2. Asymptotic properties

In this section, we characterize asymptotic properties of Kneser solutions. Especially, we derive asymptotic formulas for Kneser solutions and their first derivatives. To this aim, we specify p and q as follows:

$$p(t) = t^{\alpha} g(t), \quad g \in C[a, \infty), \tag{1.7a}$$
$$q(t) = t^{\beta} h(t), \quad h \in C[a, \infty). \tag{1.7b}$$

Theorem 1.9. *Let us assume* (1.5*a*)–(1.5*f*), (1.7*a*), *and* (1.7*b*) *to hold and* $a \geq 0$. *Additionally, let* $\alpha > 0$, $\beta > 0$, $\beta - \alpha > -1$ *and*

$$\exists r > 1 : \liminf_{u\to 0} \frac{|f(u)|}{|u|^r} > 0, \quad \limsup_{u\to 0} \frac{|f(u)|}{|u|^r} < \infty.$$

Let u *be a Kneser solution of* (1.1). *Then,*

$$\lim_{t\to\infty} t^{\frac{\beta-\alpha+2}{r-1}-\varepsilon} |u(t)| = 0 \tag{1.8}$$

for all $\varepsilon > 0$.

[1] $a = 0$

Theorem 1.10. *Let all the assumptions of Theorem 1.9 be satisfied. Then, for all $\varepsilon > 0$, one of the following two formulas holds, in case that u is a Kneser solution of* (1.1).

1. *If $\beta > r\alpha - r - 1$, then*

$$\lim_{t\to\infty} t^{\alpha-\varepsilon} \left|u'(t)\right| = 0. \tag{1.9}$$

2. *If $\beta \le r\alpha - r - 1$, then*

$$\lim_{t\to\infty} t^{\frac{\beta-\alpha+r+1}{r-1}-\varepsilon} \left|u'(t)\right| = 0. \tag{1.10}$$

1.2.3. The code `bvpsuite` and its properties

For the computation of the Kneser solutions of (1.1), we use the open domain MATLAB code `bvpsuite`. This code is an efficient solver for boundary value problems (BVPs) in ODEs. Is scope of the code are problems written in implicit form with multi-point boundary conditions,

$$F(t, p_1, ..p_s, z_1(t), z_1'(t), ..., z_1^{(l_1)}(t), ..., z_n(t), z_n'(t), ..., z_n^{(l_n)}(t)) = 0, \quad (1.11a)$$

$$B(p_1, ..p_s, z_1(c_1), z_1'(c_1), ..., z_1^{(l_1-1)}(c_1), ..., z_n(c_1), z_n'(c_1), ..., z_n^{(l_n-1)}(c_1), ...,$$

$$z_1(c_q), z_1'(c_q), ..., z_1^{(l_1-1)}(c_q), ..., z_n(c_q), z_n'(c_q), ..., z_n^{(l_n-1)}(c_q)) = 0. \quad (1.11b)$$

Under the assumption that the problem is well posed and has a smooth solution, `bvpsuite` can be used to calculate the solution $z(t) = (z_1(t), ..., z_n(t))^T$ and the unknown parameters $p_i, i = 1...s$.

In particular, singular BVPs, cf. (1.4), can be dependably solved using `bvpsuite`. Its basic solver is based on a class of collocation methods whose order varies from two to eight. For detailed analysis of the collocation for singular problems see [2,10]. An asymptotically correct global error estimate of the numerical approximation is made available to the user and, for better efficiency, mesh adaptation procedure is provided. Also, a strategy to solve an eigenvalue problems is implemented. It is also possible to calculate a solution to a problem posed on a semi-infinite interval, by reducing the problem to a related one posed on a finite interval. Finally, a path-following module can be used to solve parameter dependent problems. The code and the manual can be downloaded from http://www.asc.tuwien.ac.at/∼ewa/ [4].

As we already said, our main goal is to give the numerical evidence for Kneser solutions of (1.1). In order to use `bvpsuite`, we need to specify the boundary conditions in such a way that the resulting BVP is well-posed. Therefore, we now collect all necessary properties of the Kneser solutions, see Theorem 1.8,

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0, \quad u'(0) = 0, \quad u'(t) > 0, \ t \in (0,\infty).$$

It is clear that a Kneser solution satisfies $u'(0) = 0$, so we only need to fix one more boundary condition since we deal with a scalar second order BVP. From the asymptotic behavior of a Kneser solution, the following boundary condition seems to be well motivated:

$$\lim_{t\to\infty} u(t) = u(\infty) = 0.$$

Hence, we are solving the singular BVP of the form

$$\big(p(t)u'(t)\big)' + q(t)f\,(u(t)) = 0, \quad t \in (0,\infty), \tag{1.12a}$$
$$u(\infty) = 0, \tag{1.12b}$$
$$u'(0) = 0. \tag{1.12c}$$

In the context of `bvpsuite` the condition $u(\infty) = 0$ can be easily realized. The main idea behind solving BVPs posed on semi-infinite intervals is to split the interval $(0,\infty)$ into two subintervals $(0,1]$ and $[1,\infty)$ first. The second subinterval is then transformed to $(0,1]$ by $\xi = \frac{1}{t}$. This way we obtain two differential equations for u_1 and u_2 to be solved on $(0,1]$, where

$$u(t) = u_1(t), \ t \in (0,1], \qquad u(t) = u_2\left(\frac{1}{\xi}\right), \ \xi \in (0,1].$$

These two ODEs are subject to the following boundary conditions, for details see [5]:

$$u_1'(0) = 0, \quad u_1(1) = u_2(1), \quad u_1'(1) = -u_2'(1), \quad u_2(0) = 0.$$

Finding a good starting profile:

Since the problems we aim at solving are posed on semi-infinite intervals, are nonlinear and singular, finding good starting values is crucial. Although `bvpsuite` is explicitly designed for a solution of such problems, experience and intensive testing showed that in many cases finding solutions is not an easy task. The main problem is finding a 'good' initial profile which enables the fast frozen Newton solver to converge. In this context two problems have to be addressed.

- What is a 'good' function for the initial profile?
- If we find a 'good' starting profile, can we then expect the Newton solver to converge?

Clearly, a 'good' function $f_s(t)$ serving as a starting profile could be characterized by the same properties and behavior as the Kneser solution itself. Therefore, it would be advantageous if $f_s(t)$ satisfied the boundary conditions (1.12b) and (1.12c). Additionally, $f_s(u)$ shall be monotonically decaying for all $t \in (0,\infty)$.

After some testing, we found two functions which used as staring profiles resulted in a very good convergence behavior of the Newton solver for nearly all model problems,

$$f_s(t) = \frac{c}{-t + \exp(t)} \quad \text{and} \quad g_s(t) = \frac{c}{1 + x^2},$$

where c is an arbitrary constant. We were successful however, first after changing the standard settings of the Newton procedure. In standard settings, the tolerances, absolute and relative, are chosen as

$$a_{tol} = 10^{-12} \quad \text{and} \quad r_{tol} = 10^{-12}.$$

These values turned out to be too strict for this very delicate class of differential equations. Hence, we used larger values and could observe the desired convergence behavior with

$$a_{tol} = 10^{-5} \quad \text{and} \quad r_{tol} = 10^{-5}.$$

In Example (1.1), the exact solution was known and therefore, we used it as the initial profile. The following example illustrates the approach in case that the exact solution is not given.

Example 1.11. We consider Example 3.8 in [9]. The model problem reads:

$$u''(t) + \frac{3t+3}{t^2+t}u'(t) + f(u) = 0, \ \ t \in [0, \infty), \quad u'(0) = 0, \quad u(\infty) = 0. \quad (1.13)$$

We are interested in monotone solutions satisfying $u'(0) = 0$ and $u(\infty) = 0$, so we have used

$$f(x) = \frac{c}{-x + \exp(x)}$$

on the grid with 100 equidistant subintervals. According to [9], there exists an Kneser solution to the problem, see Figure 1.2 for the result of the numerical simulation.

We stress that the choice of the constant c determines, whether we are able to find a solution or not. The main idea of finding a suitable value of c is to test possible many values of $c \in [1, c_{max}]$, where c_{max} is given by the user. It is clear that this is a brute force method, but intensive testing showed, that useful values were found already for moderate values of c_{max}, and so the time of validation were never long.
In [9], the existence of positive and negative Kneser solutions, depending on the initial value u_0, was shown. To obtain a negative solution, we need do choose $c \in [-c_{max}, -1]$.

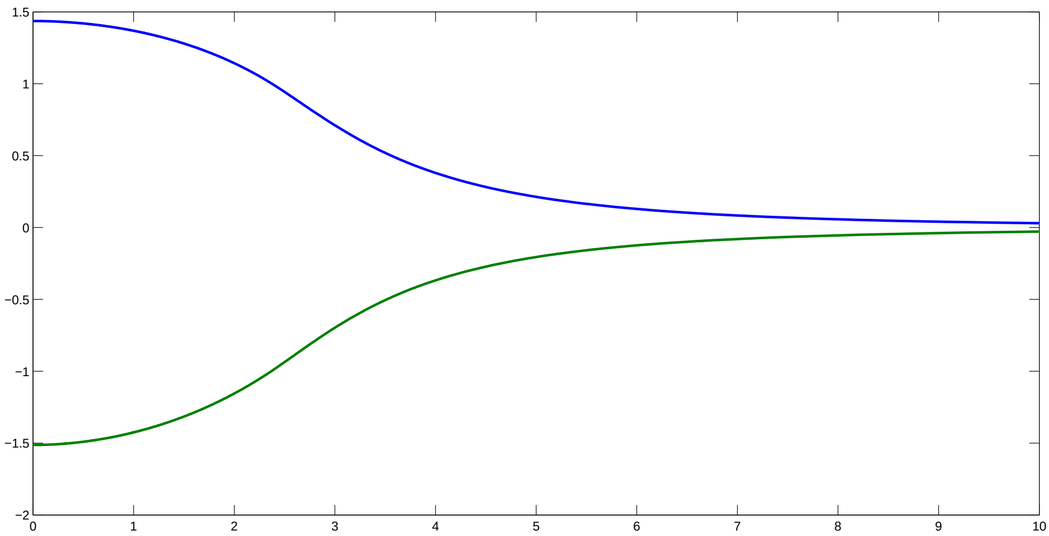

Figure 1.2.: Example 3.8 [9]: A positive, $u_0 \in (0, 1.5]$, and a negative, $u_0 \in (-2, 0]$, solution of the problem, damped to zero for $t \to \infty$.

1.3. Numerical results

1.3.1. Approach

After discussing the theoretical background of the problem, we now describe the results of the numerical simulation of the problem (1.12a) subject to the boundary condition (1.2a)–(1.3c). In particular, we use the following data functions $p(t)$ and $q(t)$:

$$p(t) = t^\alpha, \quad q(t) = t^\beta, \tag{1.14a}$$
$$p(t) = t^\alpha \ln(1+t), \quad q(t) = t^\beta \ln(1+t), \tag{1.14b}$$
$$p(t) = t^\alpha, \quad q(t) = t^\beta(1+\exp(-t)), \tag{1.14c}$$
$$p(t) = t^\alpha(1+\exp(-t)), \quad q(t) = t^\beta(1+\exp(-t)), \tag{1.14d}$$

where $\alpha \in [2, \infty]$ and $\beta \in [\alpha - 1, 2\alpha - 3]$. Moreover, as f we choose

$$f(u) = \begin{cases} -\frac{u+3}{2} & \text{for} \quad u < -1, \\ u^r & \text{for} \quad -1 \le u \le 1, \\ 2-u & \text{for} \quad u > 1, \end{cases} \tag{1.15a}$$
$$f(u) = \begin{cases} -12-2u & \text{for} \quad u < -2, \\ u^r & \text{for} \quad -2 \le u \le 1, \\ 2-u & \text{for} \quad u > 1, \end{cases} \tag{1.15b}$$
$$f(u) = \operatorname{sgn}(u) u^r (2+u)(1-u), \quad r \in [3, \infty). \tag{1.15c}$$

Clearly, we use only a small subset of all possible data combinations to give the required evidence.

Settings for Newton - solver

All solutions were computed using `bvpsuite`. The settings for the Newton solver are given in Figure 1.3.

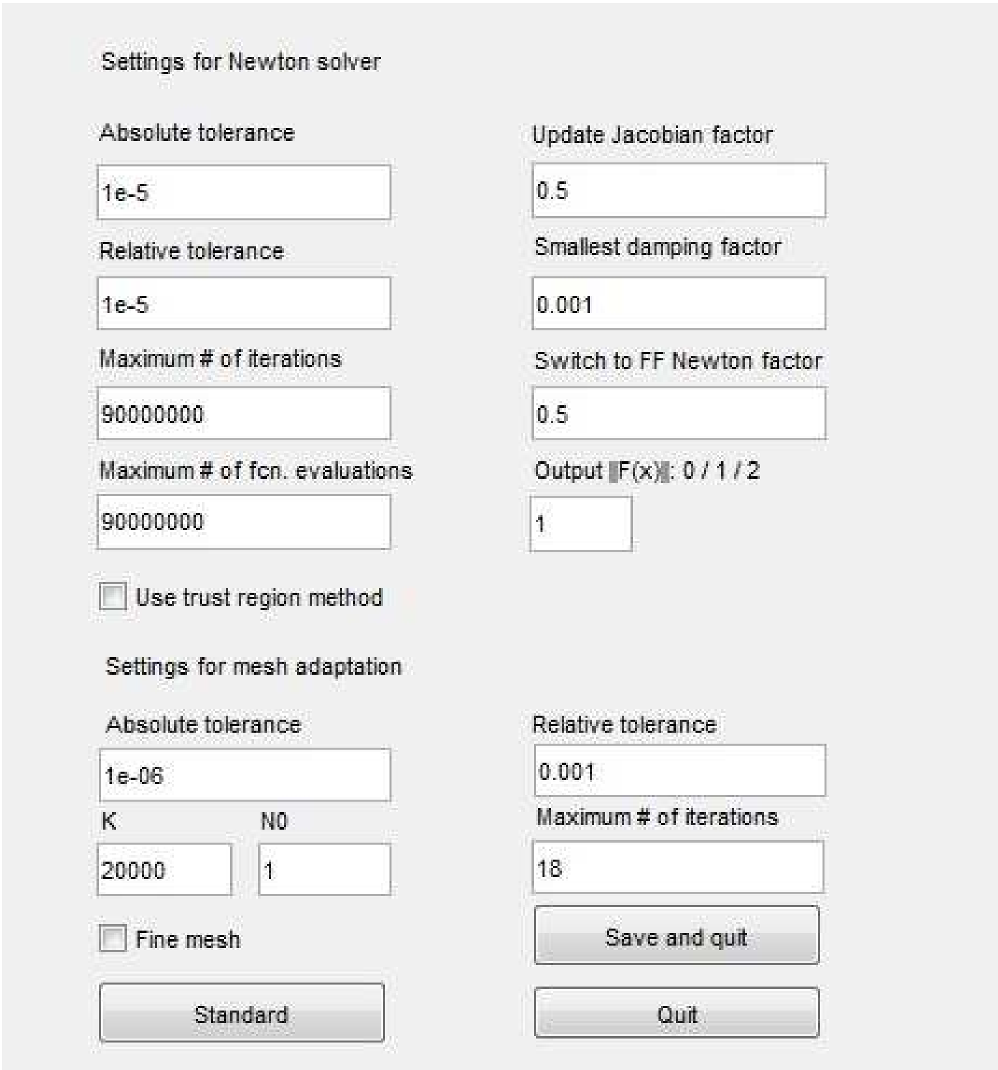

Figure 1.3.: Newton solver settings for the test runs.

Note, that in all experiments no mesh adaptation was used. This was motivated by the smoothness of the solutions we intended to recover. Due to their structure, most probably, the mesh adaptation would not hit in. All solutions where generated using equidistant grids with 100 subintervals and collocation with 8 Gaussian collocation points in each subinterval [4].

We now discuss some specific model problems to illustrate the approach. The rest of the data can be found in Appendix.

1.3.2. Introductory examples

Example 1.12. Let us consider (1.12a), where $p(t) = t^\alpha$, $q(t) = t^\beta$, and f is given by (1.15b) with $r = 3$,

$$u''(t) + \frac{\alpha}{t} u'(t) + t^{\beta-\alpha} f(u) = 0.$$

In particular, we choose $\alpha = 4$, $\beta = 5$ and obtain

$$u''(t) + \frac{4}{t} u'(t) + t f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0.$$

After computing the solution and its first derivative, we verify their analytically shown asymptotic behavior. Note, that there exist positive and negative Kneser solutions. However, we deal only with the asymptotics of the positive solution. In Figure 1.4 top left, we can see the Kneser solution (green) and its asymptotic behavior (blue), cf. Theorem 1.9 while in top right the same graphs are shown in double logarithmic scaling to better visualize the asymptotic behavior for large values of t. The related graphs of the first derivative are shown below, cf. Theorem 1.10. All further plots have the same structure.
We observe that the solution and its derivative decay faster than it was shown in Theorems 1.9 and 1.10. This important information may be used to refine the asymptotic analysis.
During the testing, we discovered that for certain data combinations we are not able to produce any Kneser solutions. This is the case for

$$\alpha \in [2, \infty], \quad \beta \in [\alpha - 1, 2\alpha - 3].$$

Note that here $\beta < \alpha$ and therefore, (1.12a) has the form

$$u''(t) + \frac{\alpha}{t} u'(t) + \frac{1}{t^\gamma} f(u) = 0, \tag{1.16}$$

where $\gamma = \beta - \alpha > 0$. This results in an additional singularity of the nonlinear term $f(u)$. We illustrate similar difficulty via the following example.

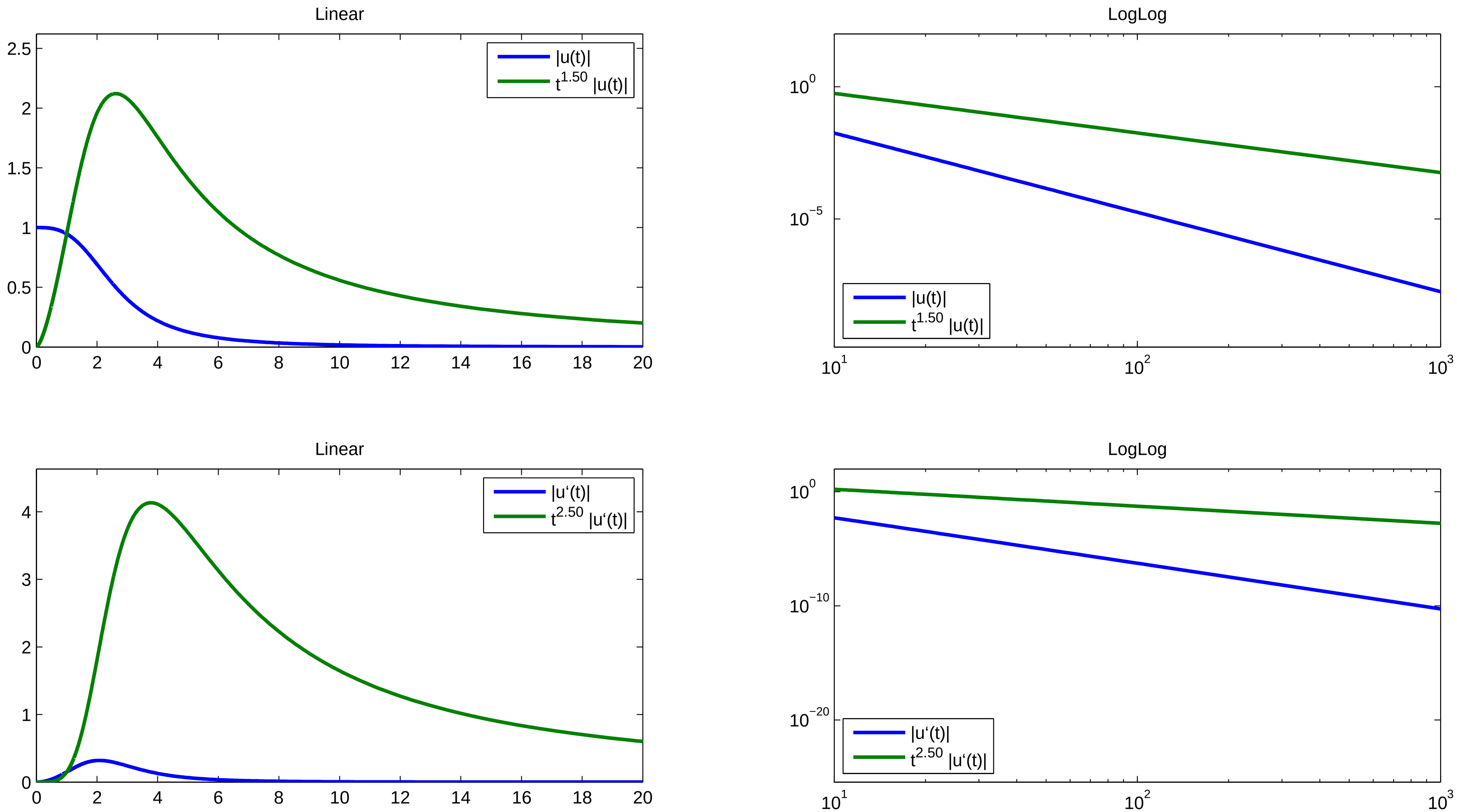

Figure 1.4.: Example 1.12: $\alpha = 4$, $\beta = 5$, $\quad u''(t) + \frac{4}{t}u'(t) + tf(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0$. The Kneser solution (green) and its asymptotic behavior (blue) top left, the same graphs in double logarithmic scaling top right. The related graphs of the solution and its first derivative are shown bottom left and right, respectively.

Example 1.13. Let us again consider (1.12*a*), where $p(t) = t^\alpha$, $q(t) = t^\beta$, and f is specified in (1.15*b*) with $r = 3$. Using $\alpha = 4$ and $\beta = 3$ yields

$$u''(t) + \frac{4}{t}u'(t) + \frac{1}{t}f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0.$$

Note, that now the singularity also occurs in the nonlinearity. This has the following effect. Although, we are able to calculate a solution, see Figure 1.5, with a monotone behavior similar to the one of a Kneser solution, a closer look shows that there is a difficulty with the initial condition $u'(0) = 0$. Their numerical values are $u'(0) \approx 4.55 \cdot 10^{-25}$ and in the first grid point $t_1 = 10^{-3}$, we have $u'(t_1) \approx -3.84 \cdot 10^{-2}$. This means that we are facing a very unsmooth behavior of the first derivative in the vicinity of $t = 0$. Therefore, we decided to make a global restriction concerning the numerical existence of Kneser solutions.

Restrictions in case of singular inhomogeneity

Figure 1.5 suggests that for the case $\beta < \alpha$ we, in general, cannot hope for finding the Kneser solutions that satisfy $u'(0) = 0$. Thus, the regions

$$\alpha \in [2, \infty], \quad \beta \in [\alpha - 1, 2\alpha - 3]$$

have to be restricted to

$$\alpha \in [2, \infty], \quad \beta \in [\alpha, 2\alpha - 3]. \tag{1.17}$$

We now specify in more detail the data used in the further testing. For $p(t)$ and $q(t)$ we use (1.14*a*)–(1.14*d*) and four different functions for $f(u)$ are used as specified below,

$$f_1(u) = \begin{cases} -\frac{u+3}{2} & \text{for} \quad u < -1, \\ u^3 & \text{for} \quad -1 \le u \le 1, \\ 2 - u & \text{for} \quad u > 1, \end{cases} \tag{1.18a}$$

$$f_2(u) = \begin{cases} -\frac{u+3}{2} & \text{for} \quad u < -1, \\ u^5 & \text{for} \quad -1 \le u \le 1, \\ 2 - u & \text{for} \quad u > 1, \end{cases} \tag{1.18b}$$

$$f_3(u) = \begin{cases} -12 - 2u & \text{for} \quad u < -2, \\ u^3 & \text{for} \quad -2 \le u \le 1, \\ 2 - u & \text{for} \quad u > 1, \end{cases} \tag{1.18c}$$

$$f_4(u) = \operatorname{sgn}(u)u^4(2 + u)(1 - u). \tag{1.18d}$$

The parameters α and β are combined as follows:

$$(\alpha, \beta) \in \{(4, 4), (4, 5), (5, 6), (5, 7), \} \tag{1.19}$$

cf. (1.17). All respective results can be found in Appendix. We now analyze the behavior of Kneser solutions for some different forms of (1.12*a*).

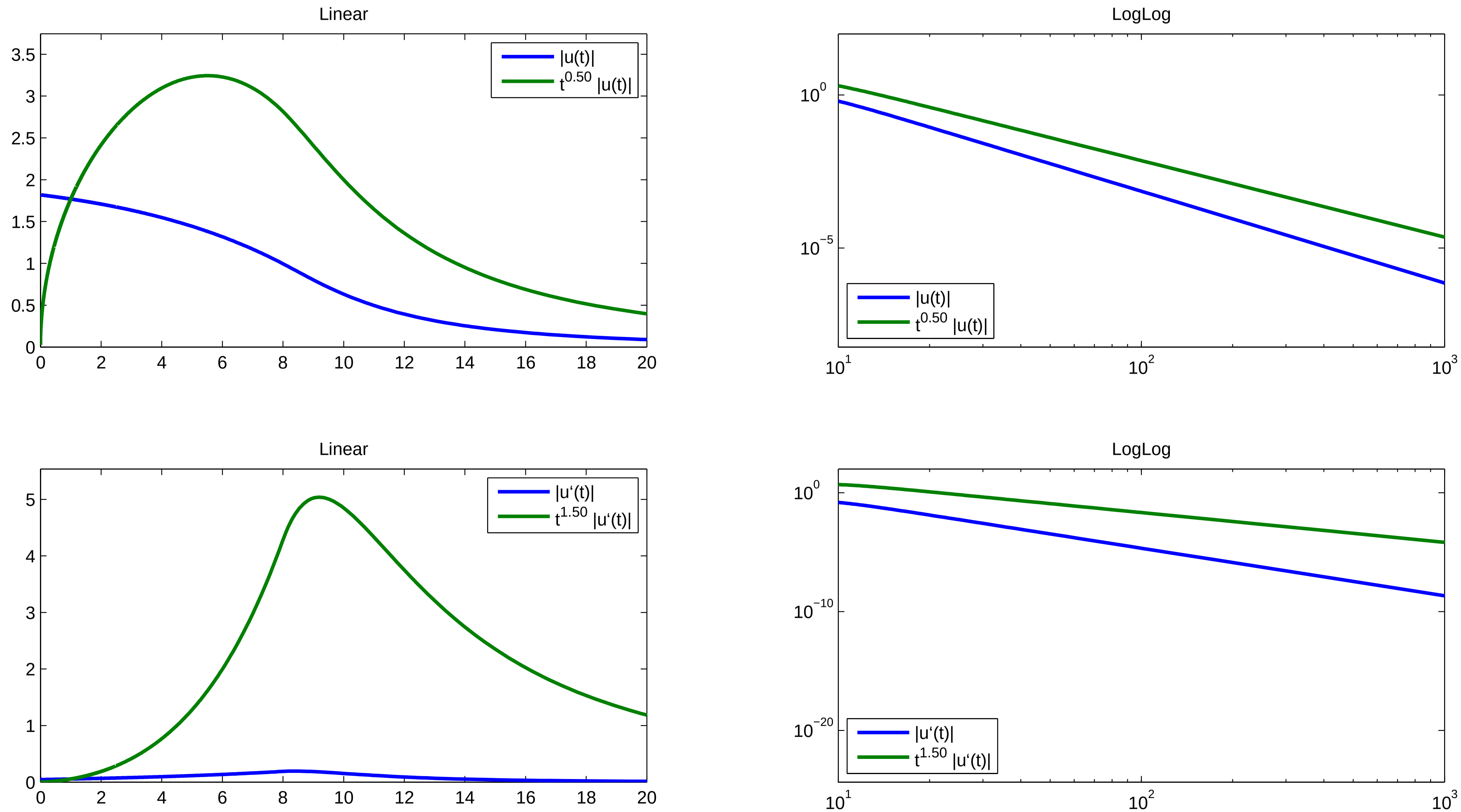

Figure 1.5.: Example 1.13: $\alpha = 4,\ \beta = 3,\quad u''(t) + \frac{4}{t}u'(t) + \frac{1}{t}f(u) = 0,\quad u'(0) = 0, u(\infty) = 0$. The Kneser solution (green) and its asymptotic behavior (blue) top left, the same graphs in double logarithmic scaling top right. The related graphs of the solution and its first derivative are shown bottom left and right, respectively. Not that the boundary condition $u'(0) = 0$ is not satisfied.

1.3.3. Dependence on parameters

In this section, we demonstrate how the qualitative behavior of Kneser solutions depends on the characteristic parameters α, β and $f(u)$. By varying these parameters,in particular the parameter r in $f(u)$, we shall derive some additional information about the solutions.

Example 1.14. Using this example, we want to analyze how the Kneser solutions depend on α. Therefore, we consider (1.12a) with $f(u)$ given in (1.18d). We choose (1.14b) for $p(t)$ and $q(t)$ and obtain (1.12a) in the following form:

$$u''(t) + \left(\frac{\alpha}{t} + \frac{1}{(1+t)\log(1+t)}\right) u'(t) + t^{\beta-\alpha} f(u) = 0.$$

By fixing the parameter $\beta = 10$, we study the problem

$$u''(t)+\left(\frac{\alpha}{t} + \frac{1}{(1+t)\log(1+t)}\right) u'(t)+t^{10-\alpha} f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0.$$

Moreover, we use the pairs of (α, β),

$$(\alpha, \beta) \in \{(5, 10), (6, 10), (7, 10), (8, 10)\}.$$

Note that this constellation means that the exponent in t in the non linear term decreases, while the multiplicative factor in front of the first derivative is growing with growing α.

In Figure 1.6 we can see the Kneser solutions for all pairs (α, β), top left, as well as their first derivatives, bottom left. The asymptotic behavior of the solutions and their first derivatives is demonstrated in double logarithmic scaling, top right and bottom right, respectively. We see that with the decreasing difference $\beta - \alpha$ the decay of the solution and its first derivative becomes stronger.

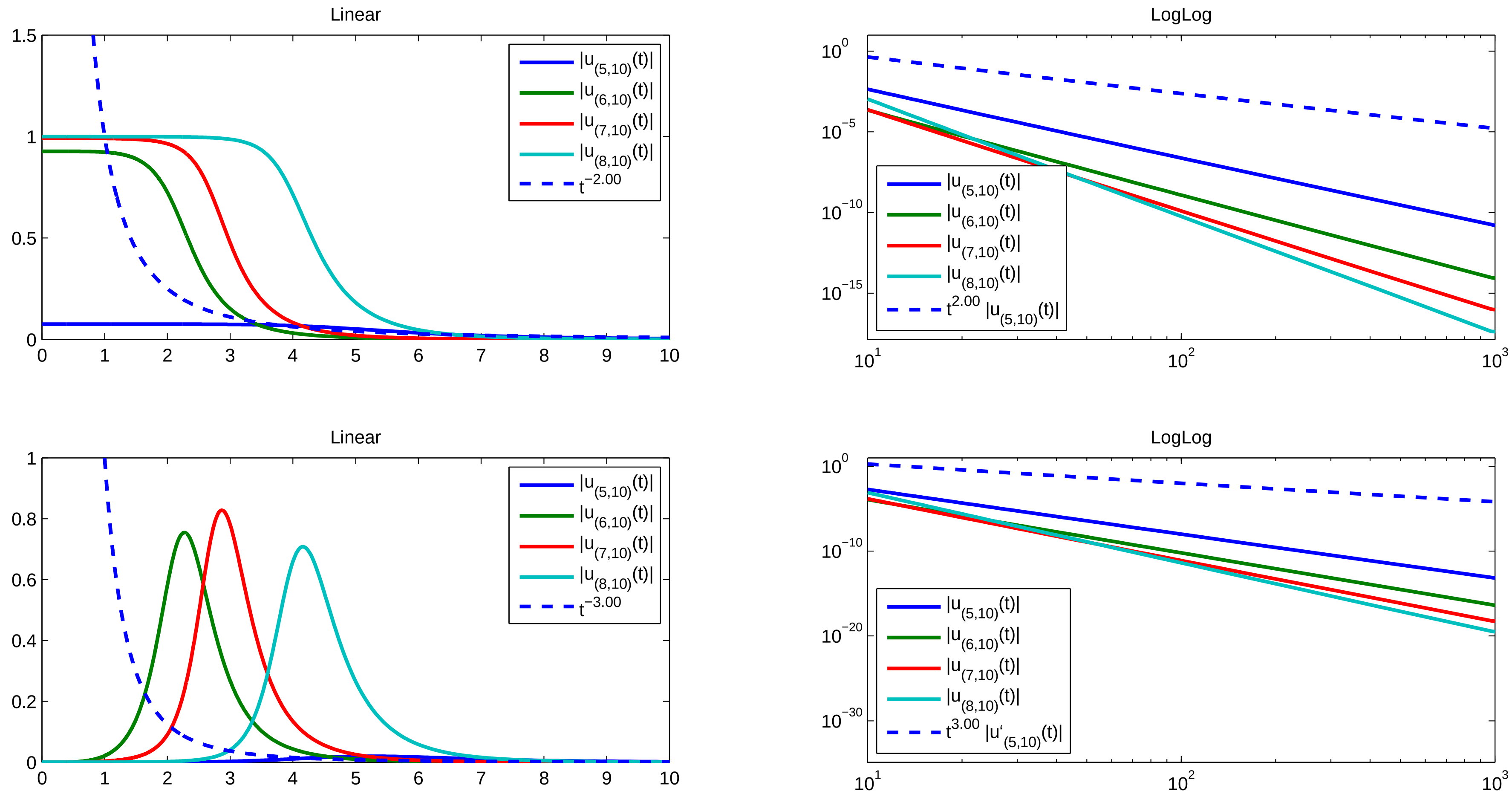

Figure 1.6.: Example 1.14: $\alpha \in \{5,6,7,8\}$, $\beta = 10$, $u''(t) + \left(\frac{\alpha}{t} + \frac{1}{(1+t)\log(1+t)}\right) u'(t) + t^{10-\alpha} f(u) = 0$, $u'(0) = 0$, $u(\infty) = 0$. Kneser solutions for all pairs $(\alpha, \beta = 10)$, top left, their first derivatives, bottom left. The asymptotic behavior of the solutions and their first derivatives is illustrated in double logarithmic scaling, top right and bottom right, respectively.

Example 1.15. The next example is similar to the previous one, but now, we fix the parameter $\alpha = 5$ and study problem (1.12a) for varying values of β. Using (1.14a) and (1.15b) with $r = 5$ yields

$$u''(t) + \frac{\alpha}{t} u'(t) + t^{\beta-\alpha} f(u) = 0.$$

The pairs (α, β) are now given by

$$(\alpha, \beta) \in \{(5,5), (5,6), (5,7), (5,8)\}.$$

Hence, we discuss the BVP

$$u''(t) + \frac{5}{t} u'(t) + t^{\beta-5} f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0.$$

Note, that here $\beta - \alpha = \beta - 5 \geq 0$ and so the nonlinearity stays regular. As a result, the exponent of the nonlinear term is decreasing for growing β and the multiplicative factor of the first derivative stays constant.

The results are shown in plots of Figure 1.7 which are arranged in the same way as related plots in Figure 1.6. The interesting conclusion which we can draw from this experiment is that although the exponent of the nonlinear term grows, this does not influence the solution decay which is the same for all values of α and β. This may be explained by the fact that the factor in front of the first derivative does not change either. Apparently, this factor is stronger influencing the asymptotic behavior of the Kneser solutions than the growth of the nonlinear term.

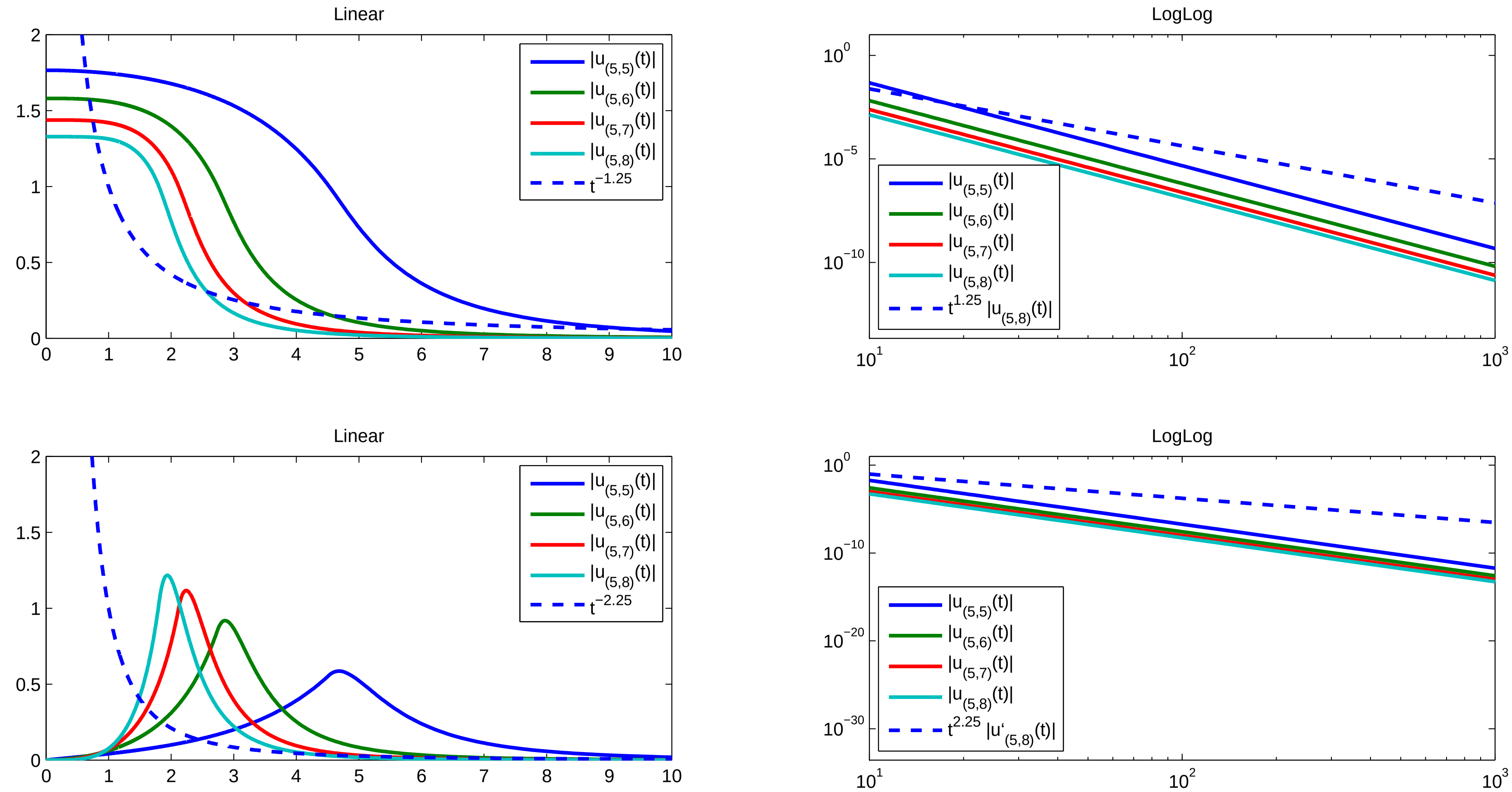

Figure 1.7.: Example 1.15: $\alpha = 5$, $\beta \in \{5, 6, 7, 8\}$, $u''(t) + \frac{5}{t}u'(t) + t^{\beta-5} f(u) = 0$, $u'(0) = 0$, $u(\infty) = 0$. Kneser solutions for all pairs $(\alpha = 5, \beta)$, top left, their first derivatives, bottom left. The asymptotic behavior of the solutions and their first derivatives is illustrated in double logarithmic scaling, top right and bottom right, respectively.

Example 1.16. The final example shall demonstrate how nonlinearity affects the behavior of the Kneser solutions. Therefore, we consider $(1.12a)$ in the following setting: we choose p and q according to $(1.14a)$ with fixed parameters $\alpha = 4$ and $\beta = 5$. The function $f(u)$ is given by $(1.15b)$ where the parameter r varies as shown below,

$$r \in \{4, 6, 8, 10, 20\}.$$

The resulting BVP reads:

$$u''(t) + \frac{4}{t}u'(t) + tf(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0.$$

The results of the simulation can be found in Figure 1.8. Usually, the nonlinearity has a crucial impact on the problem solutions, but in this case it does not seem to have much effect.

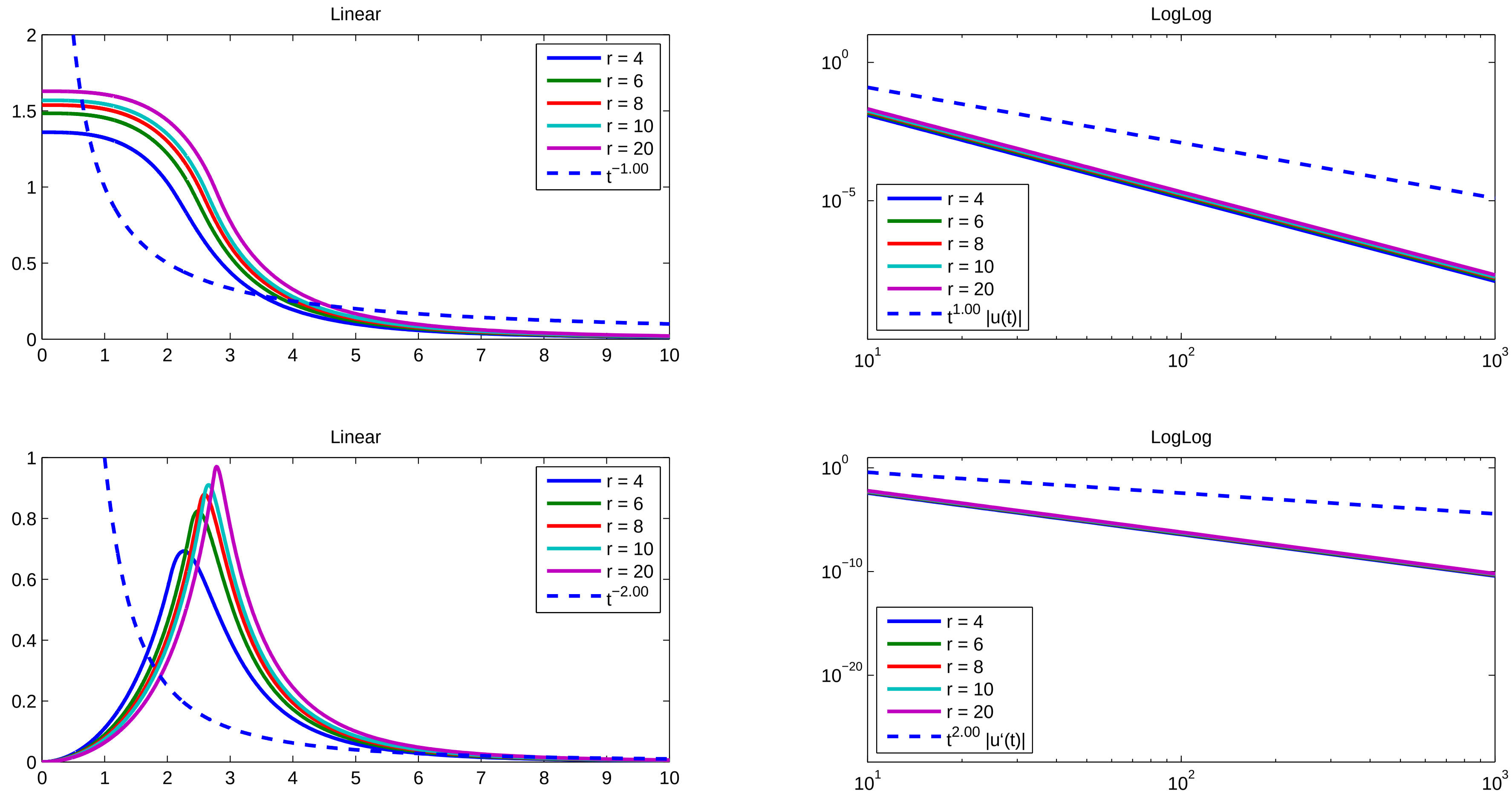

Figure 1.8.: Example 1.16: $\alpha = 4$, $\beta = 5$, $u''(t) + \frac{4}{t}u'(t) + tf(u) = 0$, $u'(0) = 0$, $u(\infty) = 0$. Kneser solutions for all values of $r \in \{4, 6, 8, 10, 20\}$, top left, their first derivatives, bottom left. The asymptotic behavior of the solutions and their first derivatives is illustrated in double logarithmic scaling, top right and bottom right, respectively.

1.3.4. Conclusions

In the previous chapter, we discussed the numerical solutions of problem (1.12a) subject to boundary condition (1.12b) and (1.12c),

$$\big(p(t)u'(t)\big)' + q(t)f\,(u(t)) = 0, \quad t \in (0,\infty), \quad u'(0) = 0, \quad u(\infty) = 0.$$

The underlying ODE is a nonlinear second-order differential equation with a singularity of the first kind. We designed variants of the above model BVP by choosing different functions p and q. Additionally, the nonlinearity $f(u)$ was subject to special attention and also here, we considered different forms of $f(u)$. The main goal of the work was to answer the questions formulated in i), ii) and iii).

We were able to calculate Kneser solutions for all prescribed specifications given via (1.14a)–(1.14d) and the nonlinearities specified in (1.18a)–(1.18d). Here, we remind that the restriction (1.17) was important in order to exclude the singularity in the nonlinear term.
The model BVPs were numerically solved using the MATLAB code `bvpsuite` and the results of the numerical simulations provided the necessary information to answer the questions i), ii), and iii). It is worth noting that although the theory for the case $p \neq q$ is not available yet, it was possible to find Kneser solutions also in this case. This is an important support towards the possible analysis. Note that some of the parameter combinations still constitute very difficult problems to solve numerically and remain open questions, as well as the cases with the singularity in the nonlinear term.

Finally, we shall mention that we also considered the following data:

$$p(t) = t^{\alpha}\ln(1+t), \; q(t) = t^{\beta}.$$

Here, we faced a logarithmic singularity at $t = 0$ in the nonlinearity and failed to find a Kneser solution. Clearly, this does not mean that such a solution does not exist, but at present neither the analysis nor the numerical approximation of such solutions are available.

2. Convergence of collocation schemes applied to solve singular ODEs

2.1. Introduction

In the second part of the thesis, we deal with systems of linear ODEs equations with a singularity of the first kind, given by

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \quad t \in (0, 1], \tag{2.1}$$

where $M \in \mathbb{R}^{n \times n}$ and $f \in C[0, 1]$. In order to solve this system numerically, we need to close it with a set of $2n$ properly chosen boundary conditions which guarantee that the resulting BVP is well-posed. The analysis of such boundary conditions have been provided in [2] for the smooth inhomogeneity and in [7] for the above setting. In this paper, we consider initial value, terminal value and BVPs, where the boundary conditions are posed at $t = 0$, $t = 1$, and at both interval endpoints, respectively.

The model examples used in this section were designed in [8]. The first collection of model problems has been provided for the constant matrix M in [1]. Here, the goal is to generalize these examples to cover the case of a variable coefficient matrices $M(t)$ depending on $t \in [0, 1]$ and provide the numerical illustration for the theory given in [7].

There are six new examples, whose solution was calculated using `bvpsuite`. The aim is to experimentally estimate the order of convergence of polynomial collocation applied to solve the model BVPs [3]. We execute the method with equidistant and Gaussian collocation points and coherently refined equidistant meshes. All results will be presented in tables described below.

2.2. Collocation method

In this section, we introduce a class of collocation methods applied to approximate the solution y of the BVP[1]

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \quad B_0 y(0) + B_1 y(1) = \beta. \tag{2.2}$$

[1]Note that with special choices of B_0 and B_1, we can cover IVPs and TVPs.

We assume that the BVP (2.2) has a unique solution which is at least in $C[0,1]$ and discretize problem (2.2). To this aim, let us consider a mesh

$$\Delta_h = \{t_0 = 0 < t_1 < t_2 < \ldots < t_i < t_{i+1} < \ldots < t_{N-1} < t_N = 1\}.$$

In each subinterval $[t_i, t_{i+1}]$, we introduce m equidistantly spaced collocation nodes $t_{il} := t_i + u_l h,\ i = 0, \ldots, N-1,\ l = 1, \ldots, m$, where

$$0 < u_1 < \ldots < u_m < 1.$$

The computational grid including the mesh points and the collocation points is shown in Figure 2.1.

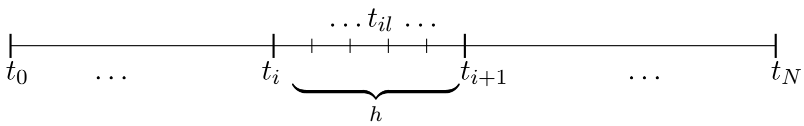

Figure 2.1.: The computational grid

By $\mathcal{P}_{m,h}$, we denote the class of piecewise polynomial functions which are globally continuous on $[0,1]$ and reduce in each subinterval $[t_i, t_{i+1}]$ to a polynomial of degree less or equal to m. We now approximate the analytical solution y by a polynomial function $p \in \mathcal{P}_{m,h}$, such that p satisfies system (2.1) at the collocation points, the boundary conditions, and the continuity relations,

$$p'(t_{il}) - \frac{M(t_{il})}{t_{il}} p(t_{il}) = \frac{f(t_{il})}{t_{il}}, \quad l = 1, \ldots, m, \quad i = 0, \ldots, N-1, \tag{2.3a}$$

$$B_0 p(0) + B_1 p(1) = \beta, \tag{2.3b}$$

$$p_{i-1}(t_i) = p_i(t_i), \quad i = 1, \ldots, N-1, \tag{2.3c}$$

where $p(t) := p_i(t),\ t \in [t_i, t_{i+1}]$.

2.3. Tables and parameters

Here, we describe how we present the numerical results. The characteristic properties of each test run are collected in tables shown below.

$h = 1$	Gauss			equidistant			uniform		
	$\Vert Yh - Y\Vert_\infty$	c	p	$\Vert Yh - Y\Vert_\infty$	c	p	$\Vert Yh - Y\Vert_\infty$	c	p
1/1	.	.	.	.	.	.	.	.	.
1/2	.	.	.	.	.	.	.	.	.
1/4	.	.	.	.	.	.	.	.	.

Gauss - equidistant - uniform

The three columns in each table are labelled with Gauss, equidistant and N = 1000. This means that the results in the first column are related to Gaussian collocation points and in the second column to equidistant collocation points.

For $N = 2$ and $m = 5$, we illustrate the distribution of both types of collocation points in Figure 2.2.

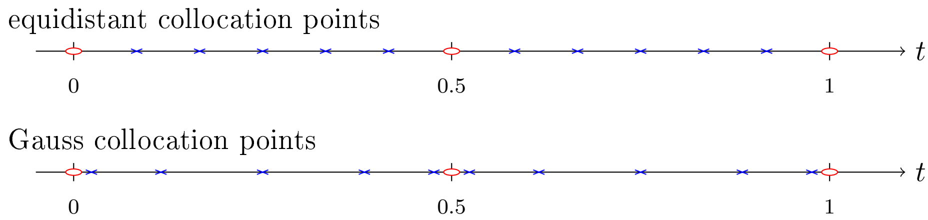

Figure 2.2.: Distribution of the collocation points for equidistant (top) and Gaussian (bottom) collocation.

Let us denote the collocation solution computed on Δ_h and evaluated at t_i by y_i^h. This means that the exact global error at t_i is given by $y_i^h - y(t_i) \in \mathbb{R}^n$. Moreover, let us denote the maximum vector norm by

$$|x| = \max_{1 \leq j \leq n} |x_j|,$$

where $x = (x_1, \dots, x_j \dots, x_n)^T \in \mathbb{R}^n$. Finally, with

$$Y_h := (y_0^h, y_1^h, \dots, y_i^h, y_{i+1}^h, \dots, y_{N-1}^h, y_N^h)$$

and

$$Y := (y(t_0), y(t_1), \dots, y(t_i), y(t_{i+1}), \dots, y(t_{N-1}), y(t_N)),$$

we can explain the quantity denoted by $\|Y_h - Y\|_\infty$, namely

$$\|Y_h - Y\|_\infty := \max_{0 \leq i \leq N} |y_i^h - y(t_i)|.$$

In the third column, we report on the collocation with equidistant collocation points but the maximal error $\|Y_h - Y\|_\infty$ is now calculated (uniformly) using 1001 equidistantly spaced points,

$$\|Y_h - Y\|_\infty := \max_{0 \leq j \leq 1000} |y_j^h - y(t_j)|,$$

where $t_j = j \cdot 10^{-3}$, $j = 0, 1, \dots, 10^3$. The reason for these different ways of measuring the global error is that we aim at observing different convergence rates: the possible superconvergence order in Δ_h for $h \to 0$ in case of the Gaussian collocation points (first column) and the uniform convergence order for any collocation points (third column). According to [2], for singular ODEs, we cannot expect the superconvergence to hold in general, not even in the case of smooth inhomogeneity, while the uniform convergence order is at least m (provided that the solution is appropriately smooth). Similar analysis for problem (2.1) can be found in [8].

Convergence rate p and error constant c

The estimates for p and c are obtained in the standard way. We first calculate the collocation solutions Y_h and $Y_{\frac{h}{2}}$ on two grids with the step sizes h and $h/2$, respectively. Then, we compute the global errors as explained above. Finally, from

$$\|Y_h - Y\|_\infty = c\, h^p \quad \text{and} \quad \|Y_{\frac{h}{2}} - Y\|_\infty = c \left(\frac{h}{2}\right)^p$$

we conclude

$$p = \log\left(\frac{\|Y_h - Y\|_\infty}{\|Y_{\frac{h}{2}} - Y\|_\infty}\right)\frac{1}{\log(2)}, \qquad c = \frac{\|Y_{\frac{h}{2}} - Y\|_\infty}{\left(\frac{h}{2}\right)^p}.$$

The goal is to numerically simulate model problems specified below and illustrate the theoretically deduced convergence rates from [8].

2.4. Results

2.4.1. Initial value problems (IVPs)

According to [7] for IVPs we need to assume, that the matrix $M(0)$ has only eigenvalues with non negative real parts and if $\sigma = 0$, then $\lambda = 0$. The following two examples are designed in such a way that these assumptions are satisfied.

Example 2.1. We consider the IVP,

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \quad t \in (0,1], \quad \begin{pmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix} y(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

where

$$M(t) = \begin{pmatrix} 3t - 2\sin t - 4 & -2t + \sin t + 2 & t - 1 \\ 3t^2 + 6t - 4\sin t - 8 & -2t^2 - 4t + 2\sin t + 4 & t^2 + 2t - 2 \\ 6t^2 + 6t - 4\sin t - 12 & -4t^2 - 4t + 2\sin t + 8 & 2t^2 + 2t - 4 \end{pmatrix},$$

$$M(0) = \begin{pmatrix} -4 & 2 & -1 \\ -8 & 4 & -2 \\ -12 & 8 & -4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 1 & \\ & -2 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix}}_{EJE^{-1}}$$

and

$$f(t) = \begin{pmatrix} 2t\cos^2(t) - t^2\sin(t) + \sin(t)\cos(t) + 2\exp(t) - t \\ (-t^2 + 4)\exp(t) + 4t\cos^2(t) - 2t^2\sin(t) + 2\sin(t)\cos(t) + 2t^2 - 2t \\ -2t^2\exp(t) + 2t\cos^2(t) - 2t^2\sin(t) + 4t^2 + 4\exp(t) - t. \end{pmatrix}.$$

The exact solution of the BVP reads:

$$y(t) = \begin{pmatrix} \exp(t) + \sin(t)\cos(t) \\ 2\exp(t) + 2\sin(t)\cos(t) + t^2 \\ 2\exp(t) + \sin(t)\cos(t) + 2t^2. \end{pmatrix} \in C^\infty[0,1].$$

Analysis provided in [7] specifies the convergence rate in context of IVPs. Let us assume that $y \in C^{m+1}[0,1]$ is the unique solution of (2.1) subject to appropriately stated initial conditions and $M \in C^1[0,1]$, $f \in C[0,1]$. Let the function $p \in P_{m,h}$ be the unique solutions of the collocation scheme ($2.3a$ - $2.3c$) executed on a grid with the step size h. Then, provided that h is sufficiently small,

$$\|p - y\|_\infty \le const.\, h^m,$$

where $P_{m,h}$ denotes the class of piecewise continuous polynomial functions reducing on each subinterval to a polynomial of degree at most m.

m=1

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	1.9e-01	0.0e+00	0.00	1.9e-01	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	4.5e-02	8.4e-01	2.11	4.5e-02	8.4e-01	2.11	1.6e-01	3.0e+00	2.10
1/4	1.6e-02	3.6e-01	1.50	1.6e-02	3.6e-01	1.50	4.8e-02	1.9e+00	1.78
1/8	4.6e-03	6.7e-01	1.80	4.6e-03	6.7e-01	1.80	1.3e-02	2.4e+00	1.88
1/16	1.2e-03	9.1e-01	1.91	1.2e-03	9.1e-01	1.91	3.4e-03	2.8e+00	1.94
1/32	3.2e-04	1.1e+00	1.96	3.2e-04	1.1e+00	1.96	8.6e-04	3.2e+00	1.97
1/64	8.0e-05	1.2e+00	1.98	8.0e-05	1.2e+00	1.98	2.2e-04	3.3e+00	1.99
1/128	2.0e-05	1.2e+00	1.99	2.0e-05	1.2e+00	1.99	5.5e-05	3.4e+00	1.99
1/256	5.1e-06	1.3e+00	1.99	5.1e-06	1.3e+00	1.99	1.4e-05	3.5e+00	2.00
1/512	1.3e-06	1.3e+00	2.00	1.3e-06	1.3e+00	2.00	3.4e-06	3.6e+00	2.00
1/1024	3.2e-07	1.3e+00	2.00	3.2e-07	1.3e+00	2.00	8.4e-07	4.5e+00	2.03
1/2048	7.9e-08	1.3e+00	2.00	7.9e-08	1.3e+00	2.00	2.1e-07	3.1e+00	1.98

m=2

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	1.4e-02	0.0e+00	0.00	4.3e-02	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	3.1e-03	6.8e-02	2.23	1.1e-02	1.6e-01	1.94	1.1e-02	1.6e-01	1.94
1/4	4.0e-04	1.9e-01	2.96	2.6e-03	2.0e-01	2.08	2.6e-03	2.0e-01	2.08
1/8	4.9e-05	2.1e-01	3.02	6.5e-04	1.7e-01	2.01	6.5e-04	1.7e-01	2.01
1/16	6.1e-06	2.1e-01	3.01	1.6e-04	1.6e-01	2.00	1.6e-04	1.6e-01	2.00
1/32	7.6e-07	2.0e-01	3.01	4.1e-05	1.6e-01	2.00	4.1e-05	1.6e-01	2.00
1/64	9.4e-08	2.0e-01	3.00	1.0e-05	1.7e-01	2.00	1.0e-05	1.7e-01	2.00
1/128	1.2e-08	2.0e-01	3.00	2.5e-06	1.7e-01	2.00	2.5e-06	1.7e-01	2.00
1/256	1.5e-09	2.0e-01	3.00	6.4e-07	1.7e-01	2.00	6.4e-07	1.7e-01	2.00
1/512	1.8e-10	2.0e-01	3.00	1.6e-07	1.7e-01	2.00	1.6e-07	1.7e-01	2.00
1/1024	2.3e-11	2.0e-01	3.00	4.0e-08	1.7e-01	2.00	4.0e-08	1.7e-01	2.00
1/2048	2.9e-12	2.1e-01	3.01	9.9e-09	1.7e-01	2.00	9.9e-09	1.7e-01	2.00

m=3

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	2.7e-04	0.0e+00	0.00	2.8e-02	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	3.3e-05	2.1e-03	2.99	1.6e-03	5.2e-01	4.19	1.6e-03	5.2e-01	4.19
1/4	1.3e-06	2.3e-02	4.70	9.4e-05	4.3e-01	4.05	9.4e-05	4.3e-01	4.05
1/8	4.2e-08	3.6e-02	4.93	5.8e-06	4.0e-01	4.01	5.8e-06	4.0e-01	4.01
1/16	1.3e-09	4.1e-02	4.98	3.6e-07	3.8e-01	4.00	3.6e-07	3.8e-01	4.00
1/32	4.2e-11	4.4e-02	4.99	2.3e-08	3.8e-01	4.00	2.3e-08	3.8e-01	4.00
1/64	1.3e-12	4.4e-02	4.99	1.4e-09	3.8e-01	4.00	1.4e-09	3.8e-01	4.00
1/128	3.7e-14	1.0e-01	5.17	8.8e-11	3.8e-01	4.00	8.8e-11	3.8e-01	4.00
1/256	3.9e-14	2.3e-14	-0.08	5.5e-12	3.8e-01	4.00	5.5e-12	3.8e-01	4.00
1/512	2.0e-14	1.6e-11	0.97	3.5e-13	3.5e-01	3.99	3.5e-13	3.5e-01	3.99
1/1024	2.4e-14	3.2e-15	-0.26	5.9e-14	1.9e-05	2.57	5.6e-14	3.0e-05	2.64
1/2048	2.4e-14	2.0e-14	-0.03	5.2e-14	2.0e-13	0.16	5.3e-14	9.6e-14	0.07

m=4

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	2.2e-05	0.0e+00	0.00	2.1e-03	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	9.7e-07	5.1e-04	4.51	1.1e-04	3.9e-02	4.21	1.3e-04	3.5e-02	4.05
1/4	2.6e-08	1.3e-03	5.22	6.1e-06	4.0e-02	4.22	6.8e-06	4.5e-02	4.23
1/8	7.3e-10	1.2e-03	5.16	3.6e-07	3.0e-02	4.08	3.8e-07	3.7e-02	4.14
1/16	2.1e-11	9.8e-04	5.09	2.2e-08	2.5e-02	4.02	2.3e-08	3.0e-02	4.06
1/32	6.5e-13	8.5e-04	5.05	1.4e-09	2.4e-02	4.01	1.4e-09	2.6e-02	4.02
1/64	2.3e-14	3.2e-04	4.81	8.7e-11	2.3e-02	4.00	8.8e-11	2.5e-02	4.01
1/128	9.3e-15	1.3e-11	1.31	5.4e-12	2.3e-02	4.00	5.5e-12	2.4e-02	4.00
1/256	2.9e-14	9.8e-19	-1.65	3.4e-13	2.3e-02	4.00	3.4e-13	2.4e-02	4.00
1/512	8.9e-15	1.4e-09	1.72	7.0e-14	5.0e-07	2.28	7.0e-14	5.0e-07	2.28
1/1024	2.7e-14	1.5e-19	-1.58	5.5e-14	7.9e-13	0.35	5.5e-14	7.9e-13	0.35
1/2048	3.3e-14	2.7e-15	-0.30	4.7e-14	3.1e-13	0.23	4.7e-14	3.1e-13	0.23

m=5

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	1.6e-06	0.0e+00	0.00	4.0e-04	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	5.9e-09	4.5e-04	8.12	5.2e-06	3.1e-02	6.26	5.2e-06	3.1e-02	6.26
1/4	3.0e-11	2.2e-04	7.60	7.8e-08	2.3e-02	6.06	7.8e-08	2.3e-02	6.06
1/8	2.0e-13	1.0e-04	7.23	1.2e-09	2.1e-02	6.01	1.2e-09	2.1e-02	6.01
1/16	1.5e-14	6.2e-09	3.73	1.9e-11	2.0e-02	6.00	1.9e-11	2.0e-02	6.00
1/32	1.6e-14	1.1e-14	-0.08	2.9e-13	2.1e-02	6.01	2.9e-13	2.2e-02	6.02
1/64	2.4e-14	1.4e-15	-0.58	3.3e-14	1.5e-07	3.16	3.8e-14	5.6e-08	2.93
1/128	9.8e-15	1.3e-11	1.30	1.3e-14	2.3e-11	1.35	1.4e-14	3.9e-11	1.43
1/256	1.2e-14	2.6e-15	-0.24	1.7e-14	1.5e-15	-0.39	1.9e-14	1.6e-15	-0.39
1/512	2.7e-14	5.4e-18	-1.23	2.0e-14	3.0e-15	-0.28	2.0e-14	8.2e-15	-0.13
1/1024	2.1e-14	3.0e-13	0.35	3.6e-14	6.3e-17	-0.83	3.6e-14	6.3e-17	-0.83
1/2048	7.3e-14	2.9e-20	-1.77	4.6e-14	2.7e-15	-0.34	4.6e-14	2.7e-15	-0.34

m=6

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	6.4e-08	0.0e+00	0.00	2.2e-05	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	2.6e-10	1.6e-05	7.97	3.4e-07	1.4e-03	6.00	3.5e-07	1.3e-03	5.95
1/4	1.4e-12	9.1e-06	7.55	4.4e-09	2.1e-03	6.28	5.0e-09	1.7e-03	6.13
1/8	2.2e-14	3.2e-07	5.95	6.7e-11	1.2e-03	6.04	7.1e-11	1.8e-03	6.14
1/16	3.8e-14	2.5e-15	-0.78	1.4e-12	3.7e-04	5.60	1.4e-12	5.3e-04	5.71
1/32	4.3e-14	2.2e-14	-0.16	8.8e-13	1.3e-11	0.65	8.4e-13	1.6e-11	0.70
1/64	3.8e-14	8.2e-14	0.16	8.7e-13	9.5e-13	0.02	8.5e-13	7.9e-13	-0.01
1/128	5.5e-14	2.9e-15	-0.53	8.4e-13	1.1e-12	0.04	8.3e-13	9.6e-13	0.03
1/256	2.6e-14	2.4e-11	1.10	8.7e-13	6.6e-13	-0.04	8.6e-13	6.2e-13	-0.05
1/512	2.0e-14	3.1e-13	0.40	8.6e-13	9.3e-13	0.01	8.6e-13	9.1e-13	0.01
1/1024	1.0e-13	9.9e-22	-2.42	8.5e-13	9.3e-13	0.01	8.5e-13	9.0e-13	0.01
1/2048	3.4e-14	2.7e-08	1.63	8.6e-13	7.5e-13	-0.02	8.6e-13	7.3e-13	-0.02

m=7

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	2.4e-09	0.0e+00	0.00	4.4e-06	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	1.6e-12	3.4e-06	10.51	1.4e-08	1.3e-03	8.24	1.4e-08	1.3e-03	8.24
1/4	3.4e-13	3.7e-11	2.25	5.1e-11	1.2e-03	8.15	5.0e-11	1.2e-03	8.18
1/8	3.4e-13	3.5e-13	0.02	4.7e-12	6.2e-08	3.42	5.7e-12	3.4e-08	3.14
1/16	3.6e-13	2.7e-13	-0.08	4.8e-12	4.3e-12	-0.03	5.3e-12	7.6e-12	0.11
1/32	3.7e-13	3.0e-13	-0.05	4.8e-12	4.8e-12	-0.00	5.1e-12	6.4e-12	0.06
1/64	3.9e-13	2.8e-13	-0.07	4.8e-12	5.0e-12	0.01	4.9e-12	5.9e-12	0.04
1/128	3.8e-13	4.3e-13	0.02	4.8e-12	5.2e-12	0.01	4.8e-12	5.8e-12	0.03
1/256	3.7e-13	5.1e-13	0.05	4.8e-12	4.6e-12	-0.01	4.8e-12	4.8e-12	-0.00
1/512	4.1e-13	1.4e-13	-0.15	4.8e-12	4.5e-12	-0.01	4.8e-12	4.6e-12	-0.01
1/1024	4.0e-13	5.6e-13	0.04	4.8e-12	5.2e-12	0.01	4.8e-12	5.3e-12	0.01
1/2048	3.4e-13	2.4e-12	0.23	4.9e-12	4.0e-12	-0.02	4.9e-12	4.1e-12	-0.02

m=8

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	7.0e-11	0.0e+00	0.00	1.9e-07	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	1.5e-12	3.2e-09	5.51	7.1e-10	5.2e-05	8.07	7.6e-10	4.9e-05	7.98
1/4	1.3e-12	2.3e-12	0.28	5.2e-11	1.4e-07	3.79	3.4e-11	3.9e-07	4.49
1/8	1.2e-12	1.4e-12	0.05	5.0e-11	5.8e-11	0.05	4.0e-11	2.0e-11	-0.25
1/16	1.2e-12	1.3e-12	0.03	5.0e-11	5.1e-11	0.00	4.5e-11	2.6e-11	-0.15
1/32	1.2e-12	1.4e-12	0.03	5.0e-11	5.1e-11	0.01	4.7e-11	3.6e-11	-0.06
1/64	1.2e-12	1.4e-12	0.05	5.0e-11	5.0e-11	0.00	4.8e-11	4.0e-11	-0.04
1/128	1.2e-12	1.1e-12	-0.01	5.0e-11	4.9e-11	-0.00	4.9e-11	4.3e-11	-0.02
1/256	1.1e-12	1.3e-12	0.03	4.9e-11	5.0e-11	0.00	4.9e-11	4.7e-11	-0.01
1/512	1.2e-12	6.4e-13	-0.09	4.9e-11	5.0e-11	0.00	4.9e-11	4.8e-11	-0.00
1/1024	1.1e-12	2.5e-12	0.11	4.9e-11	4.9e-11	-0.00	4.9e-11	4.8e-11	-0.00
1/2048	1.2e-12	7.2e-13	-0.06	4.9e-11	5.1e-11	0.00	4.9e-11	5.0e-11	0.00

Example 2.2. Since the next example is also an IVP, the same assumptions in terms of the eigenvalues of $M(0)$ have to be made. Here, we discuss the following IVP:

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \quad t \in (0,1], \quad \begin{pmatrix} 2 & 2 & 1 \\ 2 & 3 & 1 \\ -1 & -1 & 0 \end{pmatrix} y(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

where

$$M(t) = \begin{pmatrix} 4t\sin t + 2t - 2\cos t + 1 & 4t\sin t + 3t - 3\cos t + 3 & 3t - \cos t + 2 \\ -2t - 2 & -3t - 3 & -t - 1 \\ -8t\sin t + 4\cos t + 4 & -8t\sin t + 6\cos t + 3 & -4t\sin t + 2\cos t - 1 \end{pmatrix},$$

$$M(0) = \begin{pmatrix} -1 & 0 & 1 \\ -2 & -3 & -1 \\ 8 & 9 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & -3 \end{pmatrix} \begin{pmatrix} 2 & 2 & 1 \\ 2 & 3 & 1 \\ -1 & -1 & 0 \end{pmatrix}}_{EJE^{-1}}$$

and

$$f(t) = \begin{pmatrix} -4t\cos^2(t)\sin(t) + 4t\cos^2(t) - 2z\sin(t)\cos(t) + \cos(t)\exp(t)... \\ -2t\exp(t) + 3\cos^2(t) - 2t\cos(t) + 3t\sin(t) - 2\exp(t) - 4t - 3 \\ t\sin(t) + \exp(t) + 2t\exp(t) \\ 8t\sin(t)\cos(t) - 2\cos(t)\exp(t) - 6\cos^2(t) - t\sin(t) + \exp(t) + 6 \end{pmatrix}.$$

The exact solution is given by

$$y(t) = \begin{pmatrix} \cos(t) - \exp(t) - \sin^2(t) \\ -\cos(t) + \exp(t) \\ \cos(t) + 2\sin^2(t). \end{pmatrix} \in C^\infty[0,1].$$

As in the previous IVP, we expect the convergence order to be at least m, where m is the degree of the collocation polynomial or the number of collocation points.

m=1

$h=1$	Gauss			equidistant			uniform		
	$\lVert Y_h - Y\rVert_\infty$	c	p	$\lVert Y_h - Y\rVert_\infty$	c	p	$\lVert Y_h - Y\rVert_\infty$	c	p
1/1	1.4e+01	0.0e+00	0.00	1.4e+01	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	1.7e+00	1.2e+02	3.06	1.7e+00	1.2e+02	3.06	1.7e+00	1.2e+02	3.06
1/4	2.2e-01	1.0e+02	2.96	2.2e-01	1.0e+02	2.96	2.2e-01	1.0e+02	2.96
1/8	4.5e-02	2.7e+01	2.31	4.5e-02	2.7e+01	2.31	4.5e-02	2.7e+01	2.31
1/16	1.3e-02	6.4e+00	1.79	1.3e-02	6.4e+00	1.79	1.3e-02	6.4e+00	1.79
1/32	4.1e-03	4.0e+00	1.65	4.1e-03	4.0e+00	1.65	4.1e-03	4.0e+00	1.65
1/64	1.3e-03	3.9e+00	1.65	1.3e-03	3.9e+00	1.65	1.3e-03	3.9e+00	1.65
1/128	4.1e-04	4.7e+00	1.68	4.1e-04	4.7e+00	1.68	4.1e-04	4.7e+00	1.68
1/256	1.2e-04	5.9e+00	1.73	1.2e-04	5.9e+00	1.73	1.2e-04	5.9e+00	1.73
1/512	3.6e-05	7.4e+00	1.76	3.6e-05	7.4e+00	1.76	3.6e-05	7.4e+00	1.76
1/1024	1.0e-05	9.1e+00	1.79	1.0e-05	9.1e+00	1.79	1.0e-05	9.1e+00	1.79
1/2048	3.0e-06	1.1e+01	1.82	3.0e-06	1.1e+01	1.82	3.0e-06	1.1e+01	1.82

m=2

$h=1$	Gauss			equidistant			uniform		
	$\lVert Y_h - Y\rVert_\infty$	c	p	$\lVert Y_h - Y\rVert_\infty$	c	p	$\lVert Y_h - Y\rVert_\infty$	c	p
1/1	5.6e-02	0.0e+00	0.00	1.3e-01	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	1.7e-02	1.9e-01	1.73	2.8e-02	6.1e-01	2.22	2.8e-02	6.1e-01	2.22
1/4	3.0e-03	5.3e-01	2.49	5.5e-03	7.5e-01	2.36	5.5e-03	7.5e-01	2.36
1/8	4.4e-04	9.5e-01	2.77	1.1e-03	6.5e-01	2.30	1.1e-03	6.5e-01	2.30
1/16	5.9e-05	1.3e+00	2.88	2.4e-04	4.9e-01	2.20	2.4e-04	4.9e-01	2.20
1/32	7.8e-06	1.5e+00	2.93	5.7e-05	3.6e-01	2.11	5.7e-05	3.6e-01	2.11
1/64	1.0e-06	1.7e+00	2.96	1.4e-05	2.9e-01	2.05	1.4e-05	2.9e-01	2.05
1/128	1.3e-07	1.9e+00	2.97	3.4e-06	2.5e-01	2.03	3.4e-06	2.5e-01	2.03
1/256	1.6e-08	2.0e+00	2.98	8.3e-07	2.3e-01	2.01	8.3e-07	2.3e-01	2.01
1/512	2.0e-09	2.1e+00	2.99	2.1e-07	2.3e-01	2.01	2.1e-07	2.3e-01	2.01
1/1024	2.6e-10	2.1e+00	2.99	5.2e-08	2.2e-01	2.00	5.2e-08	2.2e-01	2.00
1/2048	3.2e-11	2.3e+00	3.01	1.3e-08	2.2e-01	2.00	1.3e-08	2.2e-01	2.00

m=3

$h=1$	Gauss			equidistant			uniform		
	$\lVert Y_h - Y\rVert_\infty$	c	p	$\lVert Y_h - Y\rVert_\infty$	c	p	$\lVert Y_h - Y\rVert_\infty$	c	p
1/1	1.8e-03	0.0e+00	0.00	3.7e-02	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	1.6e-04	2.0e-02	3.51	1.8e-03	7.8e-01	4.38	1.9e-03	7.5e-01	4.33
1/4	3.5e-05	3.1e-03	2.16	1.1e-04	4.8e-01	4.03	1.1e-04	5.3e-01	4.08
1/8	3.0e-06	5.6e-02	3.55	1.2e-05	9.5e-02	3.25	1.2e-05	9.5e-02	3.25
1/16	1.9e-07	1.6e-01	3.93	1.3e-06	8.0e-02	3.19	1.3e-06	8.0e-02	3.19
1/32	1.2e-08	2.1e-01	4.01	1.1e-07	2.9e-01	3.56	1.1e-07	2.9e-01	3.56
1/64	7.4e-10	2.2e-01	4.02	8.2e-09	5.5e-01	3.71	8.2e-09	5.5e-01	3.71
1/128	4.6e-11	2.0e-01	4.00	5.9e-10	7.9e-01	3.79	5.9e-10	7.9e-01	3.79
1/256	3.3e-12	6.7e-02	3.80	4.2e-11	9.5e-01	3.82	4.2e-11	9.5e-01	3.82
1/512	1.3e-13	1.4e+01	4.66	2.7e-12	2.0e+00	3.94	2.7e-12	2.0e+00	3.94
1/1024	1.3e-13	1.7e-13	0.04	7.1e-14	1.9e+04	5.26	7.1e-14	1.9e+04	5.26
1/2048	2.5e-13	7.2e-17	-0.98	3.3e-13	3.5e-21	-2.21	3.3e-13	3.5e-21	-2.21

m=4

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p
1/1	2.0e-03	0.0e+00	0.00	8.6e-03	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	3.1e-05	1.3e-01	6.00	2.8e-04	2.6e-01	4.93	3.1e-04	2.8e-01	4.89
1/4	6.1e-07	7.9e-02	5.66	9.0e-06	2.8e-01	4.97	1.1e-05	2.6e-01	4.86
1/8	1.6e-08	3.1e-02	5.21	5.1e-07	5.0e-02	4.15	5.7e-07	7.4e-02	4.25
1/16	4.9e-10	2.0e-02	5.06	3.1e-08	3.8e-02	4.05	3.3e-08	5.4e-02	4.13
1/32	1.5e-11	1.7e-02	5.01	1.9e-09	3.5e-02	4.02	2.0e-09	4.3e-02	4.06
1/64	2.0e-13	3.1e+00	6.26	1.2e-10	3.3e-02	4.01	1.2e-10	3.8e-02	4.03
1/128	2.1e-13	1.3e-13	-0.10	7.3e-12	3.2e-02	4.00	7.4e-12	3.4e-02	4.01
1/256	8.8e-14	2.6e-10	1.28	8.1e-13	3.4e-04	3.18	8.1e-13	3.6e-04	3.19
1/512	4.4e-13	4.1e-20	-2.33	4.6e-14	1.4e-01	4.15	4.7e-14	1.2e-01	4.12
1/1024	6.4e-13	1.2e-14	-0.53	2.9e-13	4.2e-22	-2.67	2.9e-13	5.1e-22	-2.64
1/2048	1.8e-13	6.3e-07	1.81	8.4e-14	2.6e-07	1.80	8.3e-14	2.7e-07	1.80

m=5

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p
1/1	9.6e-05	0.0e+00	0.00	1.9e-03	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	1.2e-06	7.6e-03	6.31	3.0e-05	1.2e-01	5.97	3.0e-05	1.2e-01	5.97
1/4	1.2e-08	1.2e-02	6.62	3.4e-07	2.3e-01	6.45	3.4e-07	2.3e-01	6.45
1/8	1.3e-10	9.5e-03	6.52	3.6e-09	2.9e-01	6.58	3.7e-09	2.6e-01	6.52
1/16	1.8e-12	4.3e-03	6.23	3.7e-11	3.0e-01	6.58	4.0e-11	2.6e-01	6.51
1/32	1.1e-13	1.7e-06	3.97	1.2e-12	9.3e-04	4.92	1.3e-12	1.2e-03	4.98
1/64	1.4e-14	3.1e-08	3.01	8.4e-13	1.2e-11	0.55	8.4e-13	1.7e-11	0.62
1/128	4.6e-13	4.1e-25	-5.00	6.4e-13	5.6e-12	0.39	6.4e-13	5.6e-12	0.39
1/256	4.9e-14	2.6e-05	3.22	3.8e-13	4.6e-11	0.77	3.8e-13	4.5e-11	0.77
1/512	3.9e-13	3.5e-22	-3.01	1.8e-13	3.1e-10	1.08	1.8e-13	3.1e-10	1.08
1/1024	2.6e-13	2.6e-11	0.61	2.9e-13	1.5e-15	-0.69	2.9e-13	1.6e-15	-0.68
1/2048	5.8e-13	3.3e-17	-1.18	1.1e-13	8.0e-09	1.34	1.1e-13	7.6e-09	1.34

m=6

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p
1/1	4.9e-06	0.0e+00	0.00	1.8e-04	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	9.7e-09	2.5e-03	8.99	6.2e-07	5.4e-02	8.20	7.3e-07	4.9e-02	8.02
1/4	6.3e-12	2.3e-02	10.58	6.9e-09	5.1e-03	6.50	8.0e-09	6.0e-03	6.51
1/8	6.7e-14	5.3e-06	6.56	9.9e-11	2.3e-03	6.12	1.1e-10	3.0e-03	6.17
1/16	2.0e-13	8.0e-16	-1.60	1.7e-12	1.2e-03	5.89	1.8e-12	1.6e-03	5.95
1/32	4.0e-13	7.0e-15	-0.97	2.0e-13	7.4e-08	3.08	2.0e-13	1.1e-07	3.17
1/64	3.9e-14	4.4e-07	3.35	1.2e-13	4.9e-12	0.77	1.1e-13	5.8e-12	0.81
1/128	4.9e-13	8.1e-22	-3.65	6.2e-13	9.8e-19	-2.41	6.2e-13	8.1e-19	-2.44
1/256	3.5e-13	7.8e-12	0.50	1.3e-13	1.5e-07	2.23	1.3e-13	1.6e-07	2.25
1/512	3.9e-13	1.1e-13	-0.18	3.7e-13	1.1e-17	-1.50	3.7e-13	1.1e-17	-1.50
1/1024	2.8e-13	1.0e-11	0.47	5.8e-13	4.3e-15	-0.64	5.8e-13	4.0e-15	-0.65
1/2048	2.2e-13	4.4e-12	0.36	2.7e-13	3.0e-09	1.12	2.7e-13	3.1e-09	1.13

m=7

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p
1/1	2.0e-07	0.0e+00	0.00	1.6e-05	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	7.3e-10	5.3e-05	8.08	6.1e-08	4.4e-03	8.07	6.1e-08	4.4e-03	8.07
1/4	1.8e-12	1.2e-04	8.69	1.5e-10	1.0e-02	8.67	1.6e-10	9.3e-03	8.61
1/8	3.4e-13	2.4e-10	2.36	7.9e-13	1.0e-03	7.57	9.0e-13	8.1e-04	7.44
1/16	1.3e-13	1.6e-11	1.39	9.1e-13	4.5e-13	-0.20	9.7e-13	6.5e-13	-0.11
1/32	8.6e-14	1.1e-12	0.61	2.2e-12	1.1e-14	-1.28	2.3e-12	1.5e-14	-1.21
1/64	1.1e-13	1.5e-14	-0.42	1.3e-12	5.5e-11	0.77	1.3e-12	6.1e-11	0.79
1/128	3.2e-13	8.3e-17	-1.49	1.7e-12	1.7e-13	-0.42	1.7e-12	1.8e-13	-0.41
1/256	2.4e-13	2.9e-12	0.40	1.1e-12	5.5e-11	0.62	1.1e-12	5.5e-11	0.63
1/512	6.7e-14	2.8e-08	1.87	9.1e-13	6.6e-12	0.28	9.1e-13	6.7e-12	0.29
1/1024	4.8e-13	1.7e-22	-2.85	2.4e-12	7.0e-17	-1.37	2.4e-12	7.2e-17	-1.36
1/2048	2.4e-13	1.2e-09	1.03	9.1e-13	7.9e-08	1.37	9.1e-13	7.8e-08	1.37

m=8

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p
1/1	1.6e-08	0.0e+00	0.00	1.5e-06	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	1.1e-11	2.1e-05	10.42	1.4e-09	1.5e-03	10.00	1.7e-09	1.4e-03	9.80
1/4	2.0e-13	3.7e-08	5.82	9.9e-12	2.9e-05	7.16	1.2e-11	3.4e-05	7.14
1/8	2.1e-13	1.8e-13	-0.06	7.4e-12	2.3e-11	0.41	7.4e-12	5.4e-11	0.72
1/16	2.5e-13	1.1e-13	-0.24	8.1e-12	5.2e-12	-0.13	8.1e-12	5.2e-12	-0.13
1/32	1.7e-13	1.6e-12	0.54	1.1e-11	2.2e-12	-0.38	1.2e-11	1.1e-12	-0.58
1/64	8.9e-13	9.0e-18	-2.37	1.0e-11	1.2e-11	0.03	1.1e-11	2.4e-11	0.16
1/128	1.7e-13	9.5e-08	2.39	1.2e-11	3.4e-12	-0.23	1.3e-11	3.8e-12	-0.22
1/256	2.1e-13	2.9e-14	-0.32	1.4e-11	3.5e-12	-0.22	1.4e-11	4.4e-12	-0.19
1/512	3.4e-13	3.5e-15	-0.66	1.3e-11	3.0e-11	0.12	1.3e-11	3.3e-11	0.13
1/1024	3.1e-13	7.4e-13	0.11	1.4e-11	6.3e-12	-0.10	1.4e-11	6.7e-12	-0.10
1/2048	5.5e-13	6.0e-16	-0.82	1.7e-11	1.5e-12	-0.29	1.7e-11	1.6e-12	-0.28

2.4.2. Terminal value problems (TVPs)

Example 2.3. According to [7], for TVPs we again have to make additional assumptions. For TVP we assume that the matrix $M(0)$ has only eingenvalues with nonnegative real parts and if the real part σ of an eigenvalue λ is zero then $\lambda = 0$. Additionally, if zero is an eigenvalue of $M(0)$ the algebraic and geometric multiplicity have to be equal. The following two examples are constructed according to those assumptions.

We first consider the TVP,

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad t \in (0,1], \quad \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} y(1) = \begin{pmatrix} e \\ e \\ 1/15 \end{pmatrix},$$

where

$$M(t) = \begin{pmatrix} 24+2t & 12+t & -12-t \\ -26t & 20-12t & 13t \\ 24-24t & 32-11t & -12+12t \end{pmatrix},$$

$$M(0) = \begin{pmatrix} 24 & 12 & -12 \\ 0 & 20 & 0 \\ 24 & 32 & -12 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 12 & & \\ & 20 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}}_{EJE^{-1}}$$

and

$$f(t) = \begin{pmatrix} -12\exp(t) + t^{15} \\ -6t\exp(t) \\ -6t\exp(t) - 12\exp(t) + 2t^{15} \end{pmatrix}.$$

The exact solution is

$$y(t) = \begin{pmatrix} \exp(t) + \frac{1}{15}t^{15} \\ t\exp(t) \\ \exp(t) + t\exp(t) + \frac{2}{15}t^{15} \end{pmatrix} \in C^{\infty}[0,1].$$

In [8], we find the respective information on the convergence rates in context of TVPs. Let us assume that $y \in C^{m+1}[0,1]$ is the unique solution of (2.1) subject to to correctly posed terminal conditions and $M \in C^1[0,1]$, $f \in C[0,1]$. Let the function $p \in P_{m,h}$ be the unique solutions of the associated collocation scheme ($2.3a$ - $2.3c$). Then, for h sufficiently small,

$$\|p - y\|_{\infty} \leq const.\, h^m,$$

where $P_{p,h}$ denotes the class of piecewise continuous polynomial functions reducing in each subinterval of length h to a polynomial of a maximal degree m.

m=1

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	1.2e+00	0.0e+00	0.00	1.2e+00	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	3.0e-01	5.1e+00	2.04	3.0e-01	5.1e+00	2.04	3.0e-01	5.1e+00	2.04
1/4	7.4e-02	5.2e+00	2.04	7.4e-02	5.2e+00	2.04	1.1e-01	2.5e+00	1.51
1/8	1.5e-02	9.0e+00	2.31	1.5e-02	9.0e+00	2.31	4.3e-02	1.7e+00	1.32
1/16	3.1e-03	7.5e+00	2.24	3.1e-03	7.5e+00	2.24	1.4e-02	3.7e+00	1.61
1/32	7.8e-04	3.4e+00	2.01	7.8e-04	3.4e+00	2.01	4.0e-03	6.8e+00	1.79
1/64	1.9e-04	3.3e+00	2.01	1.9e-04	3.3e+00	2.01	1.1e-03	1.0e+01	1.89
1/128	4.9e-05	3.2e+00	2.00	4.9e-05	3.2e+00	2.00	2.8e-04	1.3e+01	1.94
1/256	1.2e-05	3.2e+00	2.00	1.2e-05	3.2e+00	2.00	7.3e-05	1.6e+01	1.97
1/512	3.0e-06	3.2e+00	2.00	3.0e-06	3.2e+00	2.00	1.8e-05	1.7e+01	1.98
1/1024	7.6e-07	3.2e+00	2.00	7.6e-07	3.2e+00	2.00	3.8e-06	1.4e+02	2.29
1/2048	1.9e-07	3.2e+00	2.00	1.9e-07	3.2e+00	2.00	1.0e-06	5.2e+00	1.86

m=2

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	2.8e-01	0.0e+00	0.00	3.7e-01	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	6.1e-02	1.3e+00	2.22	1.4e-01	1.0e+00	1.42	1.4e-01	1.0e+00	1.42
1/4	6.8e-03	4.8e+00	3.15	4.5e-02	1.4e+00	1.64	4.5e-02	1.4e+00	1.64
1/8	6.2e-04	9.1e+00	3.46	1.2e-02	2.3e+00	1.89	1.2e-02	2.3e+00	1.89
1/16	6.0e-05	7.1e+00	3.37	3.1e-03	2.9e+00	1.98	3.1e-03	2.9e+00	1.98
1/32	6.3e-06	4.5e+00	3.24	7.7e-04	3.1e+00	2.00	7.7e-04	3.1e+00	2.00
1/64	7.2e-07	3.0e+00	3.14	1.9e-04	3.1e+00	2.00	1.9e-04	3.1e+00	2.00
1/128	8.5e-08	2.2e+00	3.07	4.8e-05	3.1e+00	2.00	4.8e-05	3.2e+00	2.00
1/256	1.0e-08	1.8e+00	3.04	1.2e-05	3.1e+00	2.00	1.2e-05	3.2e+00	2.00
1/512	1.3e-09	1.6e+00	3.02	3.0e-06	3.1e+00	2.00	3.0e-06	3.2e+00	2.00
1/1024	1.6e-10	1.5e+00	3.01	7.5e-07	3.1e+00	2.00	7.5e-07	3.1e+00	2.00
1/2048	2.0e-11	1.4e+00	3.00	1.9e-07	3.1e+00	2.00	1.9e-07	3.1e+00	2.00

m=3

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	1.3e-02	0.0e+00	0.00	8.0e-02	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	1.6e-03	1.1e-01	3.03	3.5e-02	1.8e-01	1.18	4.7e-02	2.8e-01	1.29
1/4	2.7e-05	5.8e+00	5.90	4.1e-03	2.6e+00	3.11	8.5e-03	1.5e+00	2.48
1/8	1.3e-06	2.7e-01	4.41	3.0e-04	1.0e+01	3.76	9.1e-04	6.9e+00	3.22
1/16	8.8e-08	6.1e-02	3.88	2.0e-05	1.7e+01	3.94	7.5e-05	2.0e+01	3.61
1/32	6.1e-09	5.4e-02	3.85	1.3e-06	2.0e+01	3.98	5.3e-06	4.0e+01	3.80
1/64	3.9e-10	8.9e-02	3.97	7.8e-08	2.1e+01	4.00	3.6e-07	6.0e+01	3.90
1/128	2.4e-11	1.0e-01	3.99	4.9e-09	2.1e+01	4.00	2.3e-08	7.6e+01	3.95
1/256	1.6e-12	9.3e-02	3.98	3.1e-10	2.1e+01	4.00	1.5e-09	8.7e+01	3.98
1/512	8.8e-14	2.6e-01	4.14	1.9e-11	2.1e+01	4.00	8.2e-11	2.7e+02	4.16
1/1024	4.0e-14	2.2e-10	1.13	1.3e-12	1.3e+01	3.93	5.3e-12	6.3e+01	3.95
1/2048	4.6e-14	9.9e-15	-0.18	8.1e-14	1.6e+01	3.96	3.5e-13	5.6e+01	3.93

m=4

$h = 1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	5.0e-03	0.0e+00	0.00	9.6e-02	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	8.1e-05	3.0e-01	5.93	2.6e-02	3.5e-01	1.87	2.6e-02	3.5e-01	1.87
1/4	1.0e-06	4.9e-01	6.28	2.9e-03	2.1e+00	3.16	2.9e-03	2.1e+00	3.16
1/8	2.3e-08	9.5e-02	5.49	2.1e-04	7.4e+00	3.77	2.1e-04	7.4e+00	3.77
1/16	6.7e-10	3.5e-02	5.13	1.4e-05	1.2e+01	3.94	1.4e-05	1.2e+01	3.94
1/32	2.0e-11	2.6e-02	5.04	8.8e-07	1.4e+01	3.99	8.8e-07	1.4e+01	3.99
1/64	6.4e-13	2.2e-02	5.00	5.5e-08	1.4e+01	4.00	5.5e-08	1.4e+01	4.00
1/128	3.3e-14	6.7e-04	4.28	3.4e-09	1.5e+01	4.00	3.4e-09	1.5e+01	4.00
1/256	2.5e-14	2.9e-13	0.39	2.1e-10	1.5e+01	4.00	2.1e-10	1.5e+01	4.00
1/512	3.0e-14	4.1e-15	-0.29	1.3e-11	1.5e+01	4.00	1.3e-11	1.5e+01	4.00
1/1024	3.2e-14	1.8e-14	-0.07	8.3e-13	1.6e+01	4.01	8.3e-13	1.6e+01	4.01
1/2048	6.2e-14	2.4e-17	-0.94	4.2e-14	1.6e+02	4.31	4.2e-14	1.6e+02	4.31

m=5

$h = 1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	1.7e-04	0.0e+00	0.00	5.2e-02	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	4.9e-07	6.0e-02	8.44	4.1e-03	6.6e-01	3.66	7.2e-03	5.1e-01	3.06
1/4	7.4e-09	2.2e-03	6.06	1.0e-04	6.4e+00	5.30	3.1e-04	3.9e+00	4.54
1/8	1.2e-10	1.9e-03	5.99	1.8e-06	1.9e+01	5.83	8.0e-06	1.9e+01	5.29
1/16	1.8e-12	2.2e-03	6.04	3.0e-08	2.7e+01	5.96	1.6e-07	5.1e+01	5.65
1/32	2.9e-14	1.4e-03	5.91	4.7e-10	3.1e+01	5.99	2.8e-09	9.4e+01	5.83
1/64	4.4e-15	2.4e-09	2.72	7.3e-12	3.2e+01	6.00	4.7e-11	1.3e+02	5.91
1/128	1.8e-14	3.0e-19	-1.98	1.4e-13	7.2e+00	5.69	7.3e-13	1.9e+02	5.99
1/256	4.4e-15	1.0e-09	1.98	3.3e-14	1.7e-08	2.11	3.3e-14	4.6e-02	4.48
1/512	4.2e-14	7.0e-24	-3.25	2.3e-14	8.6e-13	0.52	3.5e-14	2.0e-14	-0.08
1/1024	6.9e-14	3.0e-16	-0.72	2.8e-14	2.8e-15	-0.30	2.8e-14	2.7e-13	0.30
1/2048	4.8e-14	4.0e-12	0.53	1.1e-14	6.4e-10	1.32	1.1e-14	6.4e-10	1.32

m=6

$h = 1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	6.5e-06	0.0e+00	0.00	4.1e-02	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	1.5e-08	2.8e-03	8.76	3.0e-03	5.6e-01	3.78	3.0e-03	5.6e-01	3.78
1/4	9.3e-11	3.9e-04	7.33	7.4e-05	4.7e+00	5.32	7.4e-05	4.7e+00	5.32
1/8	7.2e-13	2.0e-04	7.02	1.3e-06	1.4e+01	5.83	1.3e-06	1.4e+01	5.83
1/16	5.3e-15	2.4e-04	7.07	2.1e-08	1.9e+01	5.96	2.1e-08	1.9e+01	5.96
1/32	2.6e-14	2.0e-18	-2.27	3.3e-10	2.2e+01	5.99	3.3e-10	2.2e+01	5.99
1/64	2.3e-14	5.6e-14	0.19	5.2e-12	2.3e+01	6.00	5.2e-12	2.2e+01	6.00
1/128	8.9e-15	1.6e-11	1.35	6.1e-14	1.5e+02	6.39	7.1e-14	5.6e+01	6.19
1/256	1.9e-14	2.3e-17	-1.07	8.3e-14	5.6e-15	-0.43	8.9e-14	1.2e-14	-0.33
1/512	4.3e-14	1.1e-17	-1.19	9.9e-14	1.7e-14	-0.26	1.0e-13	2.7e-14	-0.19
1/1024	1.8e-14	2.7e-10	1.26	9.2e-14	1.9e-13	0.09	9.5e-14	1.9e-13	0.09
1/2048	4.6e-14	4.6e-19	-1.39	1.0e-13	3.9e-14	-0.11	1.0e-13	5.2e-14	-0.08

m=7

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	7.4e-08	0.0e+00	0.00	1.3e-02	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	2.4e-10	2.3e-05	8.27	1.9e-04	9.1e-01	6.12	4.6e-04	6.8e-01	5.27
1/4	8.2e-13	2.1e-05	8.20	1.0e-06	6.6e+00	7.55	4.3e-06	5.3e+00	6.75
1/8	1.9e-14	6.9e-08	5.45	4.2e-09	1.3e+01	7.89	2.5e-08	2.1e+01	7.41
1/16	2.8e-14	3.5e-15	-0.61	1.7e-11	1.7e+01	7.99	1.2e-10	4.9e+01	7.71
1/32	3.2e-14	1.6e-14	-0.17	4.1e-13	1.8e-03	5.34	6.8e-13	2.0e+01	7.46
1/64	4.0e-14	8.4e-15	-0.32	4.8e-13	1.4e-13	-0.25	5.5e-13	2.5e-12	0.31
1/128	3.4e-14	1.3e-13	0.24	4.8e-13	4.9e-13	0.00	5.2e-13	8.8e-13	0.10
1/256	3.8e-14	1.3e-14	-0.18	5.0e-13	3.9e-13	-0.04	5.1e-13	5.6e-13	0.01
1/512	3.7e-14	4.7e-14	0.03	4.7e-13	7.5e-13	0.07	4.8e-13	9.0e-13	0.09
1/1024	5.5e-14	7.6e-16	-0.56	5.0e-13	2.7e-13	-0.08	5.1e-13	2.8e-13	-0.08
1/2048	7.3e-14	2.6e-15	-0.40	4.7e-13	1.1e-12	0.10	4.7e-13	1.1e-12	0.11

m=8

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	2.6e-09	0.0e+00	0.00	9.7e-03	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	3.9e-12	1.7e-06	9.38	1.3e-04	7.0e-01	6.17	1.3e-04	7.0e-01	6.17
1/4	5.2e-14	2.1e-08	6.21	7.1e-07	4.8e+00	7.56	7.1e-07	4.8e+00	7.56
1/8	8.7e-14	1.1e-14	-0.73	3.0e-09	9.6e+00	7.90	3.0e-09	9.5e+00	7.89
1/16	1.0e-13	4.3e-14	-0.26	1.4e-11	6.3e+00	7.75	1.9e-11	1.9e+00	7.31
1/32	1.2e-13	5.5e-14	-0.18	4.3e-12	4.6e-09	1.68	7.5e-12	1.9e-09	1.33
1/64	1.2e-13	1.2e-13	0.00	4.7e-12	2.6e-12	-0.12	6.3e-12	2.1e-11	0.25
1/128	1.2e-13	1.1e-13	-0.02	4.9e-12	3.5e-12	-0.06	5.7e-12	1.3e-11	0.14
1/256	1.3e-13	6.4e-14	-0.11	5.1e-12	4.1e-12	-0.03	5.4e-12	8.8e-12	0.08
1/512	1.3e-13	1.4e-13	0.01	5.1e-12	4.5e-12	-0.02	5.3e-12	6.6e-12	0.03
1/1024	1.4e-13	5.1e-14	-0.13	5.1e-12	4.9e-12	-0.01	5.2e-12	5.9e-12	0.02
1/2048	1.9e-13	4.9e-15	-0.44	5.1e-12	5.3e-12	0.00	5.2e-12	5.9e-12	0.02

Example 2.4. We now deal with a less smooth[2] TVP, where the same assumptions for the eigenvalues have to be made:

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad t \in (0,1], \quad \begin{pmatrix} 4 & -1 & 1 \\ 0 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} y(1) = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix},$$

where

$$M(t) = \begin{pmatrix} 28+14t^{12} & -8-4t^{12} & 8+4t^{12} \\ -112-7t^{12}\ln(t^8) & 40+2t^{12}\ln(t^8) & -32-4t^{12}\ln(t^8) \\ -196\ln(t^8)-56t^{12} & 64+\ln(t^8)+16t^{12} & -56\ln(t^8)-16t^{12} \end{pmatrix},$$

$$M(0) = \begin{pmatrix} 28 & -8 & 8 \\ -112 & 40 & -32 \\ -196 & 64 & -56 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & -2 \\ 4 & 1 & 0 \\ 1 & 1 & 7 \end{pmatrix} \begin{pmatrix} 4 & & \\ & 8 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 7 & -2 & 2 \\ -28 & 9 & -8 \\ 3 & -1 & 1 \end{pmatrix}}_{EJE^{-1}}$$

and

$$f(t) = \begin{pmatrix} 8-38t^{16}+4t^{12} \\ 32+17t^{16}\ln(t^8)+12t^{28}\ln(t^8)+8t^{20}\ln(t^8)+2t^{28}\ln^2(t^8)+8t^{12}\ln(t^8) \\ 8+11t^{16}\ln(t^8)+133t^{16}-14t^{12}-2t^{12}\ln(t^8) \end{pmatrix}.$$

The exact solution is

$$y(t) = \begin{pmatrix} -2t^{16}+3t^4-4 \\ -t^{16}+4t^8+12t^4+t^{16}\ln t^8-8 \\ 6t^{16}+4t^8+3t^4+t^{16}\ln t^8+5 \end{pmatrix} \in C^{15}[0,1].$$

In this case the exact solution is sufficiently smooth to guarantee

$$\|p-y\|_\infty \le const.\, h^m,$$

where m varies from 1 to 8.

[2]still sufficiently smooth for a high order method to work efficiently

m=1

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	3.9e+00	0.0e+00	0.00	3.9e+00	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	9.7e+00	1.5e+00	-1.32	9.7e+00	1.5e+00	-1.32	1.6e+01	5.7e+00	-0.74
1/4	1.0e+01	8.5e+00	-0.09	1.0e+01	8.5e+00	-0.09	1.0e+01	3.7e+01	0.62
1/8	3.0e+00	4.2e+02	1.78	3.0e+00	4.2e+02	1.78	3.0e+00	4.2e+02	1.78
1/16	7.4e-01	7.9e+02	2.01	7.4e-01	7.9e+02	2.01	7.5e-01	7.9e+02	2.01
1/32	1.8e-01	8.2e+02	2.02	1.8e-01	8.2e+02	2.02	2.0e-01	4.9e+02	1.87
1/64	4.6e-02	7.7e+02	2.01	4.6e-02	7.7e+02	2.01	5.5e-02	5.3e+02	1.89
1/128	1.1e-02	7.5e+02	2.00	1.1e-02	7.5e+02	2.00	1.4e-02	6.9e+02	1.94
1/256	2.8e-03	7.5e+02	2.00	2.8e-03	7.5e+02	2.00	3.6e-03	8.0e+02	1.97
1/512	7.1e-04	7.5e+02	2.00	7.1e-04	7.5e+02	2.00	9.2e-04	8.7e+02	1.99
1/1024	1.8e-04	7.5e+02	2.00	1.8e-04	7.5e+02	2.00	1.9e-04	5.9e+03	2.26
1/2048	4.4e-05	7.5e+02	2.00	4.4e-05	7.5e+02	2.00	5.2e-05	3.0e+02	1.87

m=2

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	4.8e+00	0.0e+00	0.00	2.3e+00	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	7.6e+00	3.0e+00	-0.67	1.6e+01	3.4e-01	-2.79	1.6e+01	9.7e+00	-0.37
1/4	9.1e-01	5.3e+02	3.06	4.7e+00	1.9e+02	1.78	4.7e+00	1.9e+02	1.78
1/8	4.4e-02	8.1e+03	4.37	1.2e+00	2.5e+02	1.92	1.2e+00	2.5e+02	1.92
1/16	1.1e-03	1.2e+05	5.36	3.2e-01	2.7e+02	1.94	3.2e-01	2.7e+02	1.94
1/32	4.4e-05	8.6e+03	4.59	8.2e-02	3.1e+02	1.98	8.2e-02	3.1e+02	1.98
1/64	2.4e-06	1.8e+03	4.21	2.1e-02	3.3e+02	1.99	2.1e-02	3.3e+02	1.99
1/128	1.4e-07	8.6e+02	4.06	5.2e-03	3.4e+02	2.00	5.2e-03	3.4e+02	2.00
1/256	8.9e-09	6.7e+02	4.02	1.3e-03	3.4e+02	2.00	1.3e-03	3.4e+02	2.00
1/512	5.5e-10	6.2e+02	4.00	3.2e-04	3.4e+02	2.00	3.2e-04	3.4e+02	2.00
1/1024	3.5e-11	6.2e+02	4.00	8.1e-05	3.4e+02	2.00	8.1e-05	3.4e+02	2.00
1/2048	2.2e-12	6.1e+02	4.00	2.0e-05	3.4e+02	2.00	2.0e-05	3.4e+02	2.00

m=3

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	1.8e+00	0.0e+00	0.00	4.1e+00	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	7.6e-01	4.4e+00	1.26	6.4e+00	2.6e+00	-0.64	7.1e+00	4.3e+01	1.30
1/4	6.3e-03	1.1e+04	6.90	8.8e-01	3.4e+02	2.86	8.8e-01	4.7e+02	3.02
1/8	1.9e-03	2.2e-01	1.71	7.1e-02	1.6e+03	3.62	7.2e-02	1.6e+03	3.62
1/16	5.5e-05	3.0e+03	5.14	4.6e-03	4.0e+03	3.95	5.9e-03	1.6e+03	3.61
1/32	9.9e-07	2.8e+04	5.79	2.9e-04	4.8e+03	4.00	4.3e-04	2.7e+03	3.76
1/64	1.7e-08	4.4e+04	5.89	1.8e-05	4.9e+03	4.00	2.9e-05	4.4e+03	3.88
1/128	3.0e-10	2.5e+04	5.78	1.1e-06	4.9e+03	4.00	1.9e-06	5.9e+03	3.94
1/256	1.8e-11	1.7e+00	4.05	7.1e-08	4.9e+03	4.00	1.2e-07	6.9e+03	3.97
1/512	1.0e-12	4.1e+00	4.19	4.4e-09	4.9e+03	4.00	6.9e-09	2.2e+04	4.15
1/1024	3.5e-13	4.3e-08	1.54	2.8e-10	4.9e+03	4.00	4.3e-10	6.6e+03	3.98
1/2048	5.0e-13	7.4e-15	-0.51	1.8e-11	3.5e+03	3.96	2.9e-11	4.2e+03	3.92

m=4

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p
1/1	5.2e-01	0.0e+00	0.00	7.4e+00	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	1.6e-02	1.7e+01	4.99	3.2e+00	1.7e+01	1.19	3.3e+00	2.5e+02	3.12
1/4	4.1e-03	2.6e-01	1.99	3.4e-01	2.9e+02	3.23	3.5e-01	3.0e+02	3.26
1/8	6.2e-05	1.2e+03	6.06	2.6e-02	8.3e+02	3.74	2.6e-02	8.4e+02	3.75
1/16	3.5e-07	5.7e+04	7.45	1.7e-03	1.3e+03	3.90	1.7e-03	1.3e+03	3.90
1/32	1.5e-09	2.5e+05	7.88	1.1e-04	1.6e+03	3.97	1.1e-04	1.6e+03	3.97
1/64	6.0e-12	3.7e+05	7.97	6.9e-06	1.8e+03	3.99	6.9e-06	1.8e+03	3.99
1/128	5.6e-13	9.4e-05	3.42	4.3e-07	1.8e+03	4.00	4.3e-07	1.8e+03	4.00
1/256	7.6e-13	4.8e-14	-0.44	2.7e-08	1.9e+03	4.00	2.7e-08	1.9e+03	4.00
1/512	4.1e-13	1.8e-10	0.88	1.7e-09	1.9e+03	4.00	1.7e-09	1.9e+03	4.00
1/1024	3.9e-13	6.9e-13	0.07	1.1e-10	1.7e+03	3.99	1.1e-10	1.7e+03	3.99
1/2048	9.2e-13	3.7e-17	-1.22	7.3e-12	6.3e+02	3.86	7.0e-12	1.1e+03	3.93

m=5

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p
1/1	5.6e-02	0.0e+00	0.00	3.4e+00	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	1.5e-02	2.1e-01	1.87	8.7e-01	1.4e+01	1.98	1.0e+00	1.5e+02	3.58
1/4	2.5e-04	5.9e+01	5.96	3.4e-02	5.7e+02	4.68	3.7e-02	8.3e+02	4.82
1/8	7.1e-07	1.0e+04	8.44	6.8e-04	4.2e+03	5.64	8.1e-04	3.5e+03	5.52
1/16	8.8e-10	3.0e+05	9.66	1.1e-05	1.0e+04	5.97	1.7e-05	4.0e+03	5.56
1/32	1.1e-12	2.4e+05	9.59	1.7e-07	1.2e+04	6.00	3.1e-07	8.5e+03	5.78
1/64	1.5e-13	2.1e-07	2.91	2.7e-09	1.2e+04	6.00	5.2e-09	1.4e+04	5.90
1/128	1.7e-13	8.1e-14	-0.13	4.1e-11	1.3e+04	6.02	8.4e-11	2.0e+04	5.97
1/256	2.2e-13	1.9e-14	-0.39	8.6e-13	1.1e+03	5.58	1.3e-12	2.3e+04	5.99
1/512	8.4e-13	1.1e-18	-1.96	4.2e-13	5.3e-10	1.03	4.3e-13	3.2e-08	1.62
1/1024	4.7e-13	3.0e-10	0.85	6.6e-13	4.8e-15	-0.64	6.6e-13	6.1e-15	-0.61
1/2048	5.4e-13	1.1e-13	-0.19	4.6e-13	3.1e-11	0.50	4.4e-13	4.8e-11	0.56

m=6

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p
1/1	2.5e-03	0.0e+00	0.00	1.5e+00	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	2.1e-03	3.0e-03	0.28	4.1e-01	5.6e+00	1.88	4.1e-01	6.8e+01	3.68
1/4	7.1e-06	1.8e+02	8.19	1.0e-02	6.7e+02	5.33	1.0e-02	6.7e+02	5.33
1/8	3.8e-09	4.4e+04	10.85	1.9e-04	1.7e+03	5.77	1.9e-04	1.7e+03	5.77
1/16	9.0e-13	1.3e+06	12.06	3.1e-06	2.5e+03	5.91	3.1e-06	2.5e+03	5.91
1/32	7.6e-13	2.0e-12	0.24	5.0e-08	3.1e+03	5.97	5.0e-08	3.1e+03	5.97
1/64	4.9e-13	1.1e-11	0.65	7.8e-10	3.4e+03	6.00	7.8e-10	3.4e+03	6.00
1/128	4.8e-13	5.5e-13	0.03	9.9e-12	1.5e+04	6.31	9.9e-12	1.5e+04	6.30
1/256	3.2e-13	1.2e-11	0.59	2.6e-12	4.7e-07	1.94	2.6e-12	4.9e-07	1.95
1/512	5.0e-13	5.3e-15	-0.66	2.6e-12	2.6e-12	0.00	3.7e-12	8.4e-14	-0.55
1/1024	5.7e-13	1.4e-13	-0.18	2.4e-12	4.1e-12	0.07	2.4e-12	2.9e-10	0.63
1/2048	5.6e-13	7.3e-13	0.03	1.9e-12	3.1e-11	0.33	1.9e-12	2.9e-11	0.33

m=7

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	6.6e-03	0.0e+00	0.00	6.6e-01	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	2.0e-04	2.2e-01	5.06	5.7e-02	7.7e+00	3.53	8.4e-02	7.5e+01	4.89
1/4	1.1e-07	6.2e+02	10.79	5.6e-04	6.0e+02	6.68	6.7e-04	1.3e+03	6.97
1/8	4.6e-12	1.6e+06	14.57	2.7e-06	5.3e+03	7.72	3.7e-06	4.1e+03	7.51
1/16	8.3e-13	4.3e-09	2.47	1.0e-08	1.1e+04	7.99	1.9e-08	5.4e+03	7.61
1/32	9.7e-13	3.8e-13	-0.22	2.7e-11	9.0e+04	8.59	8.3e-11	1.1e+04	7.82
1/64	1.1e-12	5.8e-13	-0.12	1.4e-11	1.6e-09	0.98	1.9e-11	5.9e-07	2.13
1/128	9.6e-13	2.0e-12	0.13	1.4e-11	1.1e-11	-0.04	1.5e-11	1.2e-10	0.38
1/256	1.7e-12	1.0e-14	-0.82	1.5e-11	7.3e-12	-0.12	1.7e-11	4.5e-12	-0.21
1/512	1.4e-12	1.1e-11	0.30	1.4e-11	3.3e-11	0.12	1.4e-11	8.4e-11	0.26
1/1024	8.8e-13	1.2e-10	0.64	1.5e-11	7.5e-12	-0.09	1.5e-11	7.3e-12	-0.10
1/2048	1.5e-12	3.0e-15	-0.75	1.4e-11	2.6e-11	0.07	1.4e-11	2.7e-11	0.07

m=8

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	1.1e-03	0.0e+00	0.00	3.8e-01	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	8.2e-06	1.5e-01	7.08	2.5e-02	5.8e+00	3.94	2.5e-02	6.9e+01	5.73
1/4	7.9e-10	8.8e+02	13.34	1.2e-04	9.7e+02	7.63	1.2e-04	9.7e+02	7.63
1/8	4.6e-12	3.9e-03	7.42	5.4e-07	1.5e+03	7.84	5.4e-07	1.5e+03	7.84
1/16	4.4e-12	5.7e-12	0.08	2.4e-09	1.5e+03	7.84	2.4e-09	1.5e+03	7.84
1/32	4.0e-12	6.4e-12	0.11	1.6e-10	1.9e-03	3.93	1.6e-10	1.7e-03	3.90
1/64	3.7e-12	7.0e-12	0.13	1.5e-10	2.1e-10	0.08	1.5e-10	2.5e-10	0.11
1/128	3.7e-12	4.0e-12	0.02	1.5e-10	1.4e-10	-0.01	1.5e-10	1.2e-10	-0.04
1/256	3.9e-12	2.3e-12	-0.08	1.5e-10	1.5e-10	0.00	1.5e-10	1.9e-10	0.04
1/512	3.5e-12	1.0e-11	0.16	1.5e-10	1.4e-10	-0.00	1.5e-10	1.5e-10	-0.00
1/1024	3.6e-12	2.6e-12	-0.04	1.5e-10	1.5e-10	-0.00	1.5e-10	1.5e-10	-0.00
1/2048	4.5e-12	3.0e-13	-0.33	1.5e-10	1.5e-10	0.00	1.5e-10	1.5e-10	0.00

2.4.3. Boundary value problems

Example 2.5. Finally, we turn to BVPs. In this case, according to [7], we do not need any additional assumptions for the spectrum of the matrix $M(0)$. The first problem reads:

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \quad t \in (0,1],$$

$$\begin{pmatrix} -2 & 3 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} y(0) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} y(1) = \begin{pmatrix} 1/2 \\ 0 \\ 4/5 \end{pmatrix},$$

where

$$M(t) = \begin{pmatrix} 1-2t & 3t - 3\exp(t) & 3\exp(t) - 3 \\ 2-2t & 3t - 2\exp(t) - 2 & 2\exp(t) - 2 \\ 2-2t & 3t - 2\exp(t) - 12 & 2\exp(t) + 8 \end{pmatrix},$$

$$M(0) = \begin{pmatrix} 1 & -3 & 0 \\ 2 & -4 & 0 \\ 2 & -14 & 10 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -2 & & \\ & -1 & \\ & & 10 \end{pmatrix} \begin{pmatrix} -2 & 3 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}}_{EJE^{-1}}$$

and

$$f(t) = \begin{pmatrix} 3t^{21/2} - t^{-1} + t^{-1}\exp(t) - 3t^{10}\exp(t) + 3t^{10} + \frac{3}{5}\exp(t) - \frac{3}{5} \\ 2t^{21/2} - t^{-1} + t^{-1}\exp(t) - 2t^{10}\exp(t) + 2t^{10} + \frac{2}{5}\exp(t) - \frac{2}{5} \\ 2t^{21/2} - t^{-1} + t^{-1}\exp(t) - 2t^{10}\exp(t) + 2t^{10} + \frac{2}{5}\exp(t) + \frac{8}{5} \end{pmatrix}.$$

The exact solution is

$$y(t) = \begin{pmatrix} t^{-2} + t^{-1}\exp(t) - t^{-2}\exp(t) + \frac{6}{23}t^{21/2} \\ t^{-2} + t^{-1}\exp(t) - t^{-2}\exp(t) + \frac{4}{23}t^{21/2} \\ t^{-2} + t^{-1}\exp(t) - t^{-2}\exp(t) + \frac{4}{23}t^{21/2} + t^{10} - \frac{1}{5} \end{pmatrix} \in C^{10}[0,1].$$

Let us assume that $y \in C^{m+1}[0,1]$ is the unique solution of (2.1) with appropriately posed boundary conditions and $M \in C^1[0,1]$, $f \in C[0,1]$. Let the function $p \in P_{m,h}$ be unique solutions of the associated collocation scheme ($2.3a$ - $2.3c$). Then

$$\|p - y\|_\infty \leq const.\, h^m,$$

provided that h is sufficiently small. Here, again $P_{m,h}$ denotes the class of piecewise continuous polynomial functions of maximal degree m.

m=1

$h = 1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	1.5e-01	0.0e+00	0.00	1.5e-01	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	5.8e-01	3.7e-02	-1.99	5.8e-01	3.7e-02	-1.99	5.8e-01	1.1e+00	0.49
1/4	2.8e-01	2.6e+00	1.08	2.8e-01	2.6e+00	1.08	2.8e-01	2.6e+00	1.08
1/8	7.6e-02	1.3e+01	1.86	7.6e-02	1.3e+01	1.86	1.0e-01	5.3e+00	1.43
1/16	1.6e-02	3.6e+01	2.22	1.6e-02	3.6e+01	2.22	3.6e-02	6.8e+00	1.52
1/32	3.9e-03	1.9e+01	2.04	3.9e-03	1.9e+01	2.04	1.1e-02	1.4e+01	1.71
1/64	9.9e-04	1.6e+01	2.00	9.9e-04	1.6e+01	2.00	3.0e-03	2.4e+01	1.85
1/128	2.5e-04	1.6e+01	2.00	2.5e-04	1.6e+01	2.00	8.1e-04	3.3e+01	1.91
1/256	6.2e-05	1.6e+01	2.00	6.2e-05	1.6e+01	2.00	2.1e-04	4.1e+01	1.96
1/512	1.5e-05	1.6e+01	2.00	1.5e-05	1.6e+01	2.00	5.3e-05	4.7e+01	1.98
1/1024	3.8e-06	1.6e+01	2.00	3.8e-06	1.6e+01	2.00	9.8e-06	1.1e+03	2.43
1/2048	9.6e-07	1.6e+01	2.00	9.6e-07	1.6e+01	2.00	2.9e-06	7.8e+00	1.78

m=2

$h = 1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	8.1e-02	0.0e+00	0.00	5.4e-02	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	1.8e-01	3.6e-02	-1.18	2.6e-01	1.1e-02	-2.30	2.6e-01	1.4e+00	1.22
1/4	2.7e-02	8.5e+00	2.77	7.9e-02	2.9e+00	1.73	7.9e-02	2.9e+00	1.73
1/8	2.0e-03	6.2e+01	3.72	1.9e-02	5.5e+00	2.04	2.0e-02	5.0e+00	1.99
1/16	1.2e-04	1.7e+02	4.09	4.1e-03	9.3e+00	2.23	4.6e-03	6.8e+00	2.10
1/32	7.4e-06	1.4e+02	4.02	1.0e-03	4.6e+00	2.02	1.1e-03	6.4e+00	2.09
1/64	4.6e-07	1.2e+02	4.00	2.5e-04	4.4e+00	2.01	2.6e-04	5.7e+00	2.06
1/128	2.9e-08	1.2e+02	4.00	6.2e-05	4.1e+00	2.00	6.4e-05	5.1e+00	2.04
1/256	1.8e-09	1.2e+02	4.00	1.6e-05	4.1e+00	2.00	1.6e-05	4.7e+00	2.02
1/512	1.1e-10	1.2e+02	4.00	3.9e-06	4.1e+00	2.00	3.9e-06	4.3e+00	2.00
1/1024	1.3e-10	3.8e-11	-0.16	9.8e-07	4.1e+00	2.00	9.8e-07	4.3e+00	2.01
1/2048	3.1e-10	6.5e-15	-1.29	2.4e-07	4.1e+00	2.00	2.4e-07	4.3e+00	2.00

m=3

$h = 1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	2.6e-02	0.0e+00	0.00	6.7e-03	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	3.4e-02	2.1e-02	-0.35	1.0e-01	4.5e-04	-3.91	1.0e-01	1.7e+00	2.04
1/4	1.2e-03	2.4e+01	4.75	1.4e-02	5.6e+00	2.90	1.4e-02	5.6e+00	2.90
1/8	2.2e-05	2.2e+02	5.81	9.9e-04	3.5e+01	3.78	1.2e-03	1.8e+01	3.45
1/16	3.4e-07	4.1e+02	6.04	5.8e-05	8.3e+01	4.09	1.0e-04	2.5e+01	3.57
1/32	5.2e-09	3.8e+02	6.01	3.6e-06	6.6e+01	4.02	7.6e-06	5.0e+01	3.78
1/64	8.2e-11	3.6e+02	6.00	2.2e-07	6.1e+01	4.01	5.2e-07	8.0e+01	3.89
1/128	1.3e-12	3.6e+02	6.00	1.4e-08	6.0e+01	4.00	3.4e-08	1.0e+02	3.94
1/256	6.2e-13	4.2e-10	1.04	8.7e-10	5.9e+01	4.00	2.1e-09	1.2e+02	3.97
1/512	3.8e-12	5.0e-20	-2.62	5.4e-11	6.0e+01	4.00	1.2e-10	3.9e+02	4.15
1/1024	1.2e-10	2.9e-27	-5.02	1.2e-10	1.3e-14	-1.20	6.5e-11	5.7e-08	0.89
1/2048	3.1e-10	6.0e-15	-1.30	3.1e-10	6.0e-15	-1.30	6.5e-11	6.4e-11	-0.00

m=4

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	2.8e-03	0.0e+00	0.00	4.0e-03	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	3.8e-03	2.0e-03	-0.45	3.0e-02	5.5e-04	-2.88	3.0e-02	2.1e+00	3.06
1/4	3.2e-05	5.4e+01	6.89	2.0e-03	6.3e+00	3.87	2.0e-03	6.3e+00	3.87
1/8	1.3e-07	4.7e+02	7.94	1.1e-04	1.3e+01	4.23	1.2e-04	8.6e+00	4.02
1/16	5.3e-10	4.7e+02	7.93	5.7e-06	1.3e+01	4.23	6.6e-06	1.6e+01	4.23
1/32	2.1e-12	5.8e+02	8.00	3.4e-07	7.5e+00	4.06	3.8e-07	1.0e+01	4.12
1/64	1.9e-13	4.1e-06	3.49	2.1e-08	6.1e+00	4.01	2.3e-08	9.1e+00	4.08
1/128	5.4e-13	1.1e-16	-1.53	1.3e-09	5.8e+00	4.00	1.4e-09	7.6e+00	4.05
1/256	6.3e-13	1.5e-13	-0.23	8.3e-11	5.7e+00	4.00	8.4e-11	6.6e+00	4.02
1/512	3.8e-12	6.1e-20	-2.59	5.2e-12	5.7e+00	4.00	6.5e-11	8.6e-10	0.37
1/1024	1.2e-10	2.6e-27	-5.04	1.2e-10	8.3e-26	-4.58	6.5e-11	6.5e-11	0.00
1/2048	3.1e-10	6.0e-15	-1.30	3.1e-10	5.9e-15	-1.30	6.5e-11	6.6e-11	0.00

m=5

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	9.8e-05	0.0e+00	0.00	6.3e-05	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	2.7e-04	3.5e-05	-1.48	6.3e-03	6.2e-07	-6.66	6.3e-03	2.4e+00	4.28
1/4	4.8e-07	8.9e+01	9.16	1.7e-04	8.8e+00	5.22	1.7e-04	8.8e+00	5.22
1/8	5.3e-10	3.5e+02	9.81	2.7e-06	4.1e+01	5.96	3.5e-06	2.0e+01	5.62
1/16	5.5e-13	4.7e+02	9.92	4.0e-08	5.5e+01	6.07	6.7e-08	2.4e+01	5.69
1/32	1.8e-13	1.4e-10	1.59	6.3e-10	4.6e+01	6.02	1.2e-09	4.2e+01	5.84
1/64	1.8e-13	1.9e-13	0.01	9.8e-12	4.2e+01	5.99	6.5e-11	3.9e-02	4.17
1/128	5.3e-13	1.0e-16	-1.54	5.6e-13	4.9e-03	4.13	6.5e-11	6.5e-11	0.00
1/256	6.3e-13	1.3e-13	-0.25	6.1e-13	3.0e-13	-0.11	6.5e-11	6.6e-11	0.00
1/512	3.8e-12	5.8e-20	-2.59	3.7e-12	4.9e-20	-2.62	6.5e-11	6.5e-11	-0.00
1/1024	1.2e-10	2.7e-27	-5.03	1.2e-10	2.2e-27	-5.06	6.6e-11	6.0e-11	-0.01
1/2048	3.1e-10	5.9e-15	-1.30	3.1e-10	5.7e-15	-1.31	6.5e-11	7.1e-11	0.01

m=6

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	9.0e-07	0.0e+00	0.00	1.5e-03	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	1.2e-05	6.6e-08	-3.77	9.3e-04	2.3e-03	0.67	9.3e-04	2.4e+00	5.67
1/4	1.1e-08	1.4e+01	10.08	1.1e-05	6.2e+00	6.35	1.2e-05	5.6e+00	6.27
1/8	1.1e-11	1.3e+01	10.04	1.2e-07	9.1e+00	6.53	1.7e-07	4.5e+00	6.17
1/16	3.0e-14	1.8e-01	8.50	1.6e-09	4.1e+00	6.25	2.1e-09	7.0e+00	6.33
1/32	1.8e-13	3.8e-18	-2.59	2.4e-11	2.2e+00	6.06	6.5e-11	6.9e-02	5.00
1/64	1.8e-13	1.7e-13	-0.02	3.5e-13	2.9e+00	6.13	6.5e-11	6.5e-11	-0.00
1/128	5.4e-13	9.5e-17	-1.56	5.0e-13	2.6e-14	-0.53	6.5e-11	6.5e-11	0.00
1/256	6.3e-13	1.5e-13	-0.23	6.4e-13	7.7e-14	-0.34	6.5e-11	6.5e-11	0.00
1/512	3.8e-12	6.3e-20	-2.58	3.9e-12	5.7e-20	-2.60	6.5e-11	6.5e-11	-0.00
1/1024	1.2e-10	2.5e-27	-5.04	1.2e-10	3.1e-27	-5.01	6.4e-11	7.3e-11	0.02
1/2048	3.1e-10	5.9e-15	-1.30	3.1e-10	5.9e-15	-1.31	6.4e-11	6.4e-11	-0.00

m=7

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p
1/1	2.9e-09	0.0e+00	0.00	6.4e-04	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	3.4e-07	2.4e-11	-6.87	9.2e-05	4.5e-03	2.80	9.2e-05	1.9e+00	7.16
1/4	3.1e-10	3.9e-01	10.08	4.5e-07	3.8e+00	7.68	4.5e-07	3.8e+00	7.68
1/8	3.0e-13	3.3e-01	10.00	1.6e-09	1.1e+01	8.16	2.0e-09	4.7e+00	7.77
1/16	2.0e-14	1.6e-08	3.91	6.3e-12	5.7e+00	7.94	6.5e-11	2.0e-03	4.97
1/32	1.8e-13	3.5e-19	-3.16	2.2e-13	1.2e-04	4.83	6.5e-11	6.5e-11	0.00
1/64	1.8e-13	1.7e-13	-0.02	2.5e-13	1.2e-13	-0.16	6.5e-11	6.5e-11	-0.00
1/128	5.4e-13	9.1e-17	-1.57	4.9e-13	2.2e-15	-0.98	6.5e-11	6.5e-11	0.00
1/256	6.3e-13	1.6e-13	-0.22	6.5e-13	5.0e-14	-0.41	6.5e-11	6.5e-11	-0.00
1/512	3.8e-12	6.2e-20	-2.59	3.7e-12	1.1e-19	-2.50	6.5e-11	6.5e-11	0.00
1/1024	1.2e-10	2.6e-27	-5.04	1.2e-10	1.9e-27	-5.08	6.5e-11	6.5e-11	-0.00
1/2048	3.1e-10	6.0e-15	-1.30	3.0e-10	6.4e-15	-1.30	6.5e-11	6.3e-11	-0.00

m=8

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p
1/1	5.0e-12	0.0e+00	0.00	2.6e-04	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	5.1e-09	4.8e-15	-10.00	5.0e-06	1.4e-02	5.72	5.0e-06	1.0e+00	8.83
1/4	4.7e-12	6.0e-03	10.08	9.1e-09	1.5e+00	9.09	1.3e-08	7.1e-01	8.56
1/8	4.3e-14	6.1e-06	6.77	3.1e-11	2.4e-01	8.22	6.7e-11	9.9e-02	7.62
1/16	4.9e-14	2.5e-14	-0.19	2.0e-12	1.7e-06	3.94	6.5e-11	7.4e-11	0.04
1/32	1.8e-13	7.7e-17	-1.86	1.9e-12	2.6e-12	0.08	6.5e-11	6.5e-11	-0.00
1/64	1.8e-13	1.5e-13	-0.04	2.1e-12	9.9e-13	-0.16	6.5e-11	6.5e-11	-0.00
1/128	5.3e-13	9.9e-17	-1.55	2.3e-12	1.2e-12	-0.12	6.5e-11	6.6e-11	0.00
1/256	6.3e-13	1.5e-13	-0.23	2.4e-12	1.6e-12	-0.06	6.5e-11	6.5e-11	-0.00
1/512	3.8e-12	5.5e-20	-2.60	3.2e-12	1.7e-13	-0.42	6.5e-11	6.5e-11	0.00
1/1024	1.2e-10	2.7e-27	-5.03	1.3e-10	3.8e-28	-5.29	6.4e-11	7.4e-11	0.02
1/2048	3.1e-10	6.0e-15	-1.30	3.1e-10	6.5e-15	-1.29	6.5e-11	5.2e-11	-0.03

Example 2.6. The final example has been already studied as Example 4.2 in [1]. Here, we recalculate this BVP,

$$y'(t) = \frac{M}{t} y(t) + \frac{f(t)}{t}, \quad t \in (0,1],$$

$$\begin{pmatrix} -2 & 3 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} y(0) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} y(1) = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{4}{5} \end{pmatrix},$$

where

$$M = \begin{pmatrix} 1 & -3 & 0 \\ 2 & -4 & 0 \\ 2 & -14 & 10 \end{pmatrix}$$

and

$$f(t) = \begin{pmatrix} \exp(t) + 3t^{\frac{21}{2}} \\ \exp(t) + 2t^{\frac{21}{2}} \\ \exp(t) + 2t^{\frac{21}{2}} + 2 \end{pmatrix}.$$

The solution of the system is given by

$$y(t) = \begin{pmatrix} t^{-2} + t^{-1}\exp(t) - t^{-2}\exp(t) + \frac{6}{23} t^{\frac{21}{2}} \\ t^{-2} + t^{-1}\exp(t) - t^{-2}\exp(t) + \frac{4}{23} t^{\frac{21}{2}} \\ t^{-2} + t^{-1}\exp(t) - t^{-2}\exp(t) + \frac{4}{23} t^{\frac{21}{2}} + t^{10} - \frac{1}{5} \end{pmatrix} \in C^{10}[0,1],$$

and we again expect the global error to be $O(h^m)$ for $m = 1, \dots, 8$.

m=1

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p
1/1	3.3e-01	0.0e+00	0.00	3.3e-01	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	5.5e-01	1.9e-01	-0.76	5.5e-01	1.9e-01	-0.76	5.5e-01	1.2e+00	0.56
1/4	2.4e-01	2.9e+00	1.21	2.4e-01	2.9e+00	1.21	2.4e-01	2.9e+00	1.21
1/8	6.8e-02	1.0e+01	1.81	6.8e-02	1.0e+01	1.81	9.2e-02	4.3e+00	1.38
1/16	1.5e-02	3.0e+01	2.20	1.5e-02	3.0e+01	2.20	3.2e-02	5.9e+00	1.50
1/32	3.7e-03	1.5e+01	1.99	3.7e-03	1.5e+01	1.99	9.9e-03	1.3e+01	1.72
1/64	9.3e-04	1.6e+01	2.01	9.3e-04	1.6e+01	2.01	2.7e-03	2.2e+01	1.85
1/128	2.3e-04	1.5e+01	2.00	2.3e-04	1.5e+01	2.00	7.2e-04	3.0e+01	1.92
1/256	5.8e-05	1.5e+01	2.00	5.8e-05	1.5e+01	2.00	1.9e-04	3.8e+01	1.96
1/512	1.4e-05	1.5e+01	2.00	1.4e-05	1.5e+01	2.00	4.7e-05	4.3e+01	1.98
1/1024	3.6e-06	1.5e+01	2.00	3.6e-06	1.5e+01	2.00	8.9e-06	8.1e+02	2.40
1/2048	9.0e-07	1.5e+01	2.00	9.0e-07	1.5e+01	2.00	2.6e-06	7.7e+00	1.79

m=2

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p
1/1	1.4e-01	0.0e+00	0.00	2.4e-01	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	1.7e-01	1.2e-01	-0.26	2.5e-01	2.2e-01	-0.08	2.5e-01	1.5e+00	1.30
1/4	2.2e-02	9.0e+00	2.88	6.7e-02	3.5e+00	1.90	6.7e-02	3.5e+00	1.90
1/8	1.6e-03	6.3e+01	3.82	1.4e-02	7.8e+00	2.29	1.5e-02	5.9e+00	2.15
1/16	9.3e-05	1.4e+02	4.09	2.8e-03	7.7e+00	2.28	3.2e-03	7.5e+00	2.24
1/32	5.7e-06	1.1e+02	4.03	6.6e-04	3.8e+00	2.08	7.2e-04	5.0e+00	2.13
1/64	3.6e-07	9.9e+01	4.01	1.6e-04	3.0e+00	2.02	1.7e-04	4.0e+00	2.07
1/128	2.2e-08	9.6e+01	4.00	4.1e-05	2.7e+00	2.01	4.2e-05	3.4e+00	2.04
1/256	1.4e-09	9.6e+01	4.00	1.0e-05	2.7e+00	2.00	1.0e-05	3.1e+00	2.02
1/512	8.7e-11	9.5e+01	4.00	2.5e-06	2.7e+00	2.00	2.5e-06	3.0e+00	2.02
1/1024	1.3e-10	2.3e-12	-0.53	6.3e-07	2.7e+00	2.00	6.3e-07	2.7e+00	2.00
1/2048	3.1e-10	6.5e-15	-1.29	1.6e-07	2.7e+00	2.00	1.6e-07	2.7e+00	2.00

m=3

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p
1/1	2.3e-02	0.0e+00	0.00	1.6e-01	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	3.0e-02	1.7e-02	-0.43	9.4e-02	2.9e-01	0.81	9.4e-02	1.8e+00	2.15
1/4	1.1e-03	2.5e+01	4.84	1.2e-02	5.9e+00	2.99	1.2e-02	5.9e+00	2.99
1/8	1.8e-05	2.1e+02	5.86	8.8e-04	2.9e+01	3.75	1.1e-03	1.3e+01	3.38
1/16	2.8e-07	3.4e+02	6.04	5.2e-05	7.2e+01	4.08	9.5e-05	2.3e+01	3.57
1/32	4.3e-09	3.1e+02	6.01	3.2e-06	5.8e+01	4.02	6.9e-06	4.7e+01	3.78
1/64	6.8e-11	2.9e+02	5.99	2.0e-07	5.3e+01	4.00	4.7e-07	7.4e+01	3.89
1/128	1.1e-12	3.0e+02	6.00	1.3e-08	5.4e+01	4.00	3.0e-08	9.5e+01	3.94
1/256	6.3e-13	7.3e-11	0.76	7.8e-10	5.4e+01	4.00	1.9e-09	1.1e+02	3.97
1/512	3.8e-12	6.0e-20	-2.59	4.9e-11	5.4e+01	4.00	1.1e-10	3.9e+02	4.17
1/1024	1.2e-10	2.6e-27	-5.04	1.2e-10	4.4e-15	-1.34	6.5e-11	1.6e-08	0.72
1/2048	3.1e-10	6.0e-15	-1.30	3.1e-10	6.0e-15	-1.30	6.5e-11	6.5e-11	-0.00

m=4

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	1.2e-03	0.0e+00	0.00	9.7e-02	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	3.4e-03	3.9e-04	-1.56	2.7e-02	3.5e-01	1.84	2.7e-02	2.2e+00	3.18
1/4	2.7e-05	5.6e+01	6.99	1.6e-03	7.6e+00	4.06	1.6e-03	7.6e+00	4.06
1/8	1.1e-07	4.4e+02	7.98	7.3e-05	1.8e+01	4.49	9.1e-05	9.4e+00	4.16
1/16	4.4e-10	3.6e+02	7.91	4.6e-06	4.5e+00	3.98	4.6e-06	1.4e+01	4.31
1/32	1.7e-12	4.9e+02	8.00	2.9e-07	4.7e+00	3.99	2.9e-07	4.7e+00	3.99
1/64	1.8e-13	1.3e-06	3.25	1.8e-08	4.8e+00	4.00	1.8e-08	4.8e+00	4.00
1/128	5.3e-13	1.0e-16	-1.54	1.1e-09	4.8e+00	4.00	1.1e-09	4.8e+00	4.00
1/256	6.3e-13	1.4e-13	-0.24	7.1e-11	4.9e+00	4.00	7.1e-11	4.9e+00	4.00
1/512	3.8e-12	6.0e-20	-2.59	4.4e-12	4.8e+00	4.00	6.5e-11	1.5e-10	0.12
1/1024	1.2e-10	2.6e-27	-5.04	1.2e-10	1.4e-26	-4.81	6.5e-11	6.5e-11	-0.00
1/2048	3.1e-10	6.0e-15	-1.30	3.1e-10	6.0e-15	-1.30	6.5e-11	6.5e-11	0.00

m=5

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	1.1e-05	0.0e+00	0.00	4.5e-02	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	2.5e-04	5.1e-07	-4.46	5.8e-03	3.4e-01	2.95	5.8e-03	2.6e+00	4.41
1/4	4.1e-07	9.1e+01	9.25	1.5e-04	8.7e+00	5.27	1.5e-04	8.7e+00	5.27
1/8	4.6e-10	2.7e+02	9.77	2.5e-06	3.4e+01	5.93	3.1e-06	1.6e+01	5.58
1/16	4.8e-13	4.1e+02	9.92	3.7e-08	4.9e+01	6.06	6.1e-08	2.2e+01	5.69
1/32	1.8e-13	6.2e-11	1.40	5.7e-10	4.2e+01	6.02	1.1e-09	3.8e+01	5.84
1/64	1.8e-13	1.8e-13	-0.01	8.9e-12	3.9e+01	6.00	6.5e-11	1.9e-02	4.02
1/128	5.3e-13	1.0e-16	-1.54	5.3e-13	3.3e-03	4.06	6.5e-11	6.5e-11	0.00
1/256	6.3e-13	1.4e-13	-0.24	6.3e-13	1.4e-13	-0.24	6.5e-11	6.5e-11	0.00
1/512	3.8e-12	6.0e-20	-2.59	3.8e-12	6.0e-20	-2.59	6.5e-11	6.5e-11	0.00
1/1024	1.2e-10	2.6e-27	-5.04	1.2e-10	2.6e-27	-5.04	6.5e-11	6.5e-11	-0.00
1/2048	3.1e-10	6.0e-15	-1.30	3.1e-10	6.0e-15	-1.30	6.5e-11	6.5e-11	0.00

m=6

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	5.8e-09	0.0e+00	0.00	1.7e-02	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	1.1e-05	3.0e-12	-10.91	8.4e-04	3.2e-01	4.29	8.4e-04	2.7e+00	5.82
1/4	1.1e-08	1.2e+01	10.00	8.7e-06	8.0e+00	6.61	9.7e-06	6.4e+00	6.44
1/8	1.1e-11	1.2e+01	10.00	1.4e-07	2.3e+00	6.00	1.4e-07	3.6e+00	6.16
1/16	3.0e-14	1.5e-01	8.44	2.1e-09	2.2e+00	5.99	2.1e-09	2.2e+00	5.99
1/32	1.8e-13	4.0e-18	-2.58	3.3e-11	2.3e+00	6.00	6.5e-11	8.0e-02	5.03
1/64	1.8e-13	1.8e-13	-0.01	4.5e-13	5.6e+00	6.22	6.5e-11	6.5e-11	0.00
1/128	5.3e-13	1.0e-16	-1.54	5.3e-13	1.3e-13	-0.25	6.5e-11	6.5e-11	0.00
1/256	6.3e-13	1.4e-13	-0.24	6.3e-13	1.4e-13	-0.24	6.5e-11	6.5e-11	0.00
1/512	3.8e-12	6.0e-20	-2.59	3.8e-12	6.0e-20	-2.59	6.5e-11	6.5e-11	-0.00
1/1024	1.2e-10	2.6e-27	-5.04	1.2e-10	2.6e-27	-5.04	6.5e-11	6.5e-11	0.00
1/2048	3.1e-10	6.0e-15	-1.30	3.1e-10	6.0e-15	-1.30	6.5e-11	6.5e-11	-0.00

m=7

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	3.7e-11	0.0e+00	0.00	4.2e-03	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	3.0e-07	4.5e-15	-13.00	8.4e-05	2.1e-01	5.66	8.4e-05	2.1e+00	7.29
1/4	3.0e-10	3.2e-01	10.00	4.1e-07	3.6e+00	7.69	4.1e-07	3.6e+00	7.69
1/8	3.0e-13	3.0e-01	9.96	1.5e-09	8.6e+00	8.11	1.8e-09	4.7e+00	7.82
1/16	2.9e-14	3.1e-09	3.34	5.9e-12	5.8e+00	7.97	6.5e-11	1.1e-03	4.79
1/32	1.8e-13	3.3e-18	-2.62	4.1e-13	3.6e-06	3.85	6.5e-11	6.5e-11	-0.00
1/64	1.8e-13	1.7e-13	-0.02	4.1e-13	3.8e-13	-0.01	6.5e-11	6.5e-11	0.00
1/128	5.3e-13	1.0e-16	-1.54	5.3e-13	7.0e-14	-0.37	6.5e-11	6.5e-11	-0.00
1/256	6.3e-13	1.4e-13	-0.24	6.3e-13	1.3e-13	-0.25	6.5e-11	6.5e-11	-0.00
1/512	3.8e-12	6.0e-20	-2.59	3.8e-12	6.2e-20	-2.59	6.5e-11	6.5e-11	0.00
1/1024	1.2e-10	2.6e-27	-5.04	1.2e-10	2.6e-27	-5.04	6.5e-11	6.5e-11	-0.00
1/2048	3.1e-10	6.0e-15	-1.30	3.1e-10	6.0e-15	-1.30	6.5e-11	6.5e-11	-0.00

m=8

$h=1$	Gauss			equidistant			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/1	7.5e-13	0.0e+00	0.00	7.8e-04	0.0e+00	0.00	0.0e+00	0.0e+00	0.00
1/2	4.6e-09	1.2e-16	-12.59	4.3e-06	1.4e-01	7.50	4.3e-06	1.2e+00	9.03
1/4	4.5e-12	4.9e-03	10.00	1.5e-08	3.7e-01	8.19	1.5e-08	3.7e-01	8.19
1/8	1.2e-13	2.3e-07	5.22	6.3e-11	2.0e-01	7.88	6.7e-11	1.6e-01	7.80
1/16	1.1e-13	1.6e-13	0.10	4.5e-12	2.4e-06	3.80	6.5e-11	7.4e-11	0.04
1/32	1.8e-13	1.1e-14	-0.68	4.3e-12	6.0e-12	0.08	6.5e-11	6.5e-11	0.00
1/64	1.8e-13	1.6e-13	-0.03	4.2e-12	4.3e-12	0.00	6.5e-11	6.5e-11	-0.00
1/128	5.3e-13	1.1e-16	-1.54	4.2e-12	4.3e-12	0.00	6.5e-11	6.5e-11	0.00
1/256	6.3e-13	1.4e-13	-0.24	4.2e-12	4.3e-12	0.00	6.5e-11	6.5e-11	-0.00
1/512	3.8e-12	6.0e-20	-2.59	4.2e-12	4.3e-12	0.00	6.5e-11	6.5e-11	0.00
1/1024	1.2e-10	2.6e-27	-5.04	1.2e-10	8.9e-27	-4.88	6.5e-11	6.5e-11	-0.00
1/2048	3.1e-10	6.0e-15	-1.30	3.1e-10	6.0e-15	-1.30	6.5e-11	6.5e-11	0.00

2.4.4. Conclusions

In the previous chapter, we used the collocation method to calculate solutions of singular ODEs of the form

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \quad t \in (0, 1],$$

where $M(t) \in \mathbb{R}^{n \times n}$ and $f \in C[0,1]$. We were especially interested in studying the convergence behaviour of the collocation schemes, with $m = 1, \dots, 8$ collocation points. In this work, we distinguished between IVPs, TVPs, and BVPs. We stress, that in case of singular ODEs certain assumptions on the spectrum of the matrix $M(0)$ [7] are necessary to obtain well-posed problems which can be successfully solved numerically. The convergence rate p and the error constant c were calculated for different types of the global error measures. We have used equidistant collocation points and Gaussian collocation points. Additionally, we computed p and c for the global error taken uniformly in $t \in [0, 1]$.

For all problems, analysis in [8] provides estimations for the convergence rate which is expected to be at least m in case of appropriately smooth problem data and smooth solutions. All numerical tests support this theoretically predicted convergence order. Collocation at Gaussian points shows the so-called small superconvergence, where the rate is $m+1$, even when the singularity is present. However, the Gaussian superconvergence order $2m$ cannot be expected to hold in general.

As a final conclusion, we see that we can expect at least the convergence order m uniformly in $t \in [0, 1]$. This means that we can consider collocation to be a robust, dependable, and high order method also in context of singular ODEs.

A. Further numerical results for Kneser solutions

In this section we collect further results obtained for different model problems of type (1.12a) with data specified via (1.14a)–(1.14d), (1.18a)–(1.18d), and (1.19). We first rewrite (1.12a) and obtain

$$u''(t) + \frac{p'(t)}{p(t)}u'(t) + \frac{q(t)}{p(t)} = 0, \tag{A.1}$$

$$u'(0) = 0, \tag{A.2}$$

$$u(\infty) = 0. \tag{A.3}$$

The plots related to various data settings are always arranged in the same way:

- **top left**: The solution of the problem and its asymptotic behavior, cf. Theorem 1.9.
- **top right**: The solution of the problem and its asymptotic behavior in a double logarithmic scaling, cf. Theorem 1.9.
- **bottom left**: The first derivative of the solution and its asymptotic behavior, cf. Theorems 1.9 and 1.10.
- **bottom right**: The first derivative of the solution and its asymptotic behavior in a double logarithmic scaling cf. Theorems 1.9 and 1.10.
- **captions:** All data functions and parameters are specified in the captions.

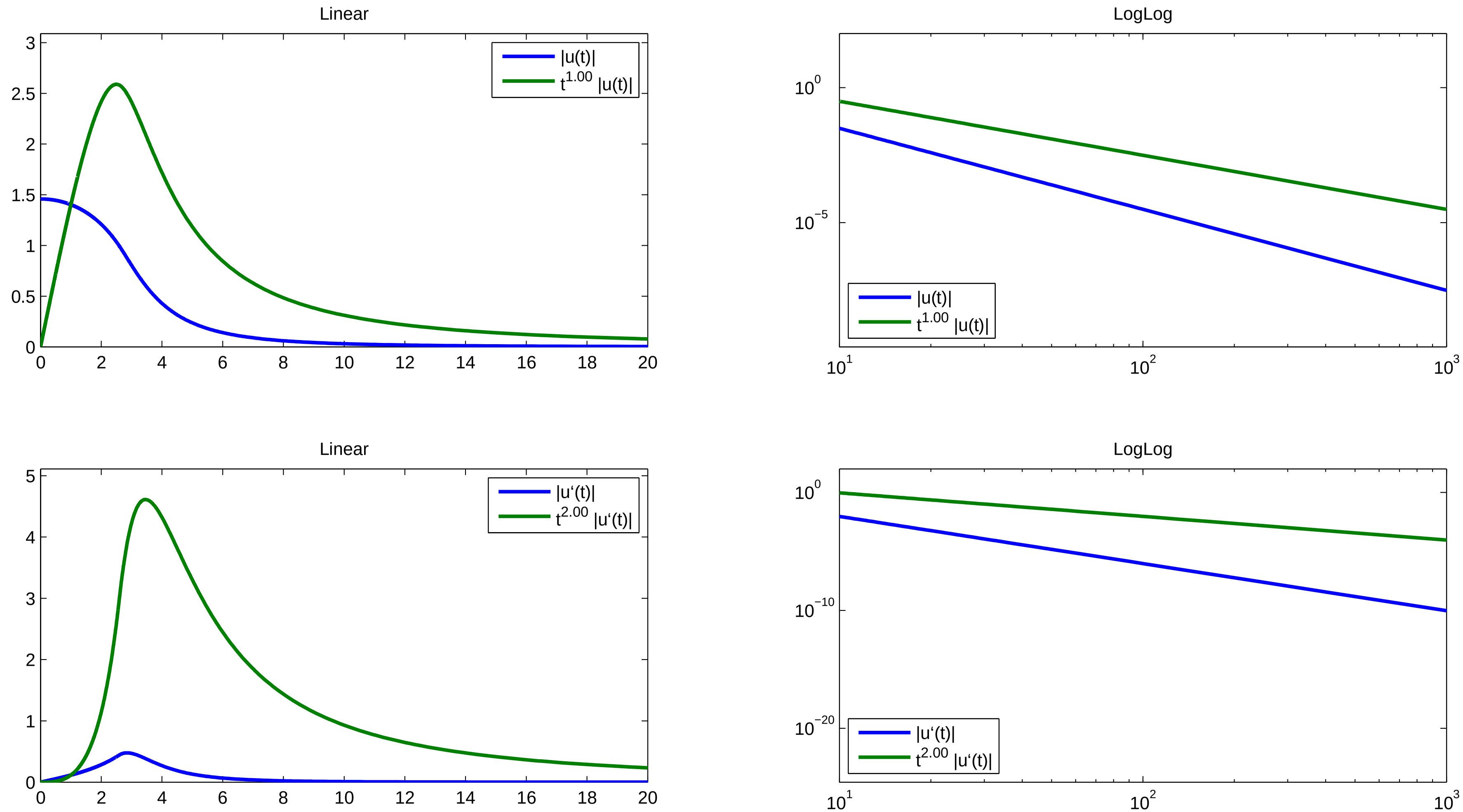

Figure A.1.: $\alpha = 4,\ \ \beta = 4, \quad u''(t) + \frac{4}{t}u'(t) + f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18c)$.

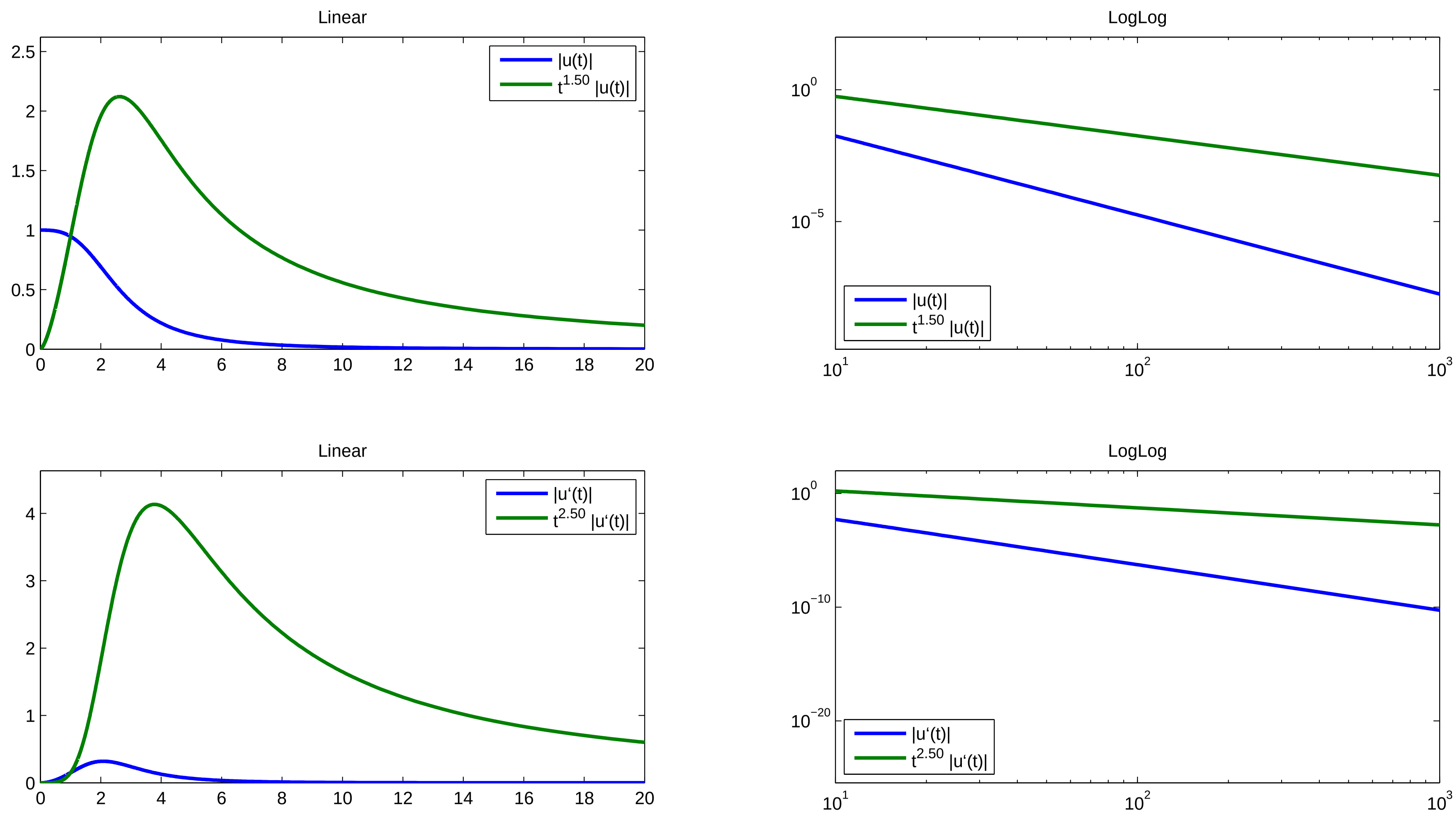

Figure A.2.: $\alpha = 4, \quad \beta = 5, \quad u''(t) + \frac{4}{t}u'(t) + t f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by (1.18c).

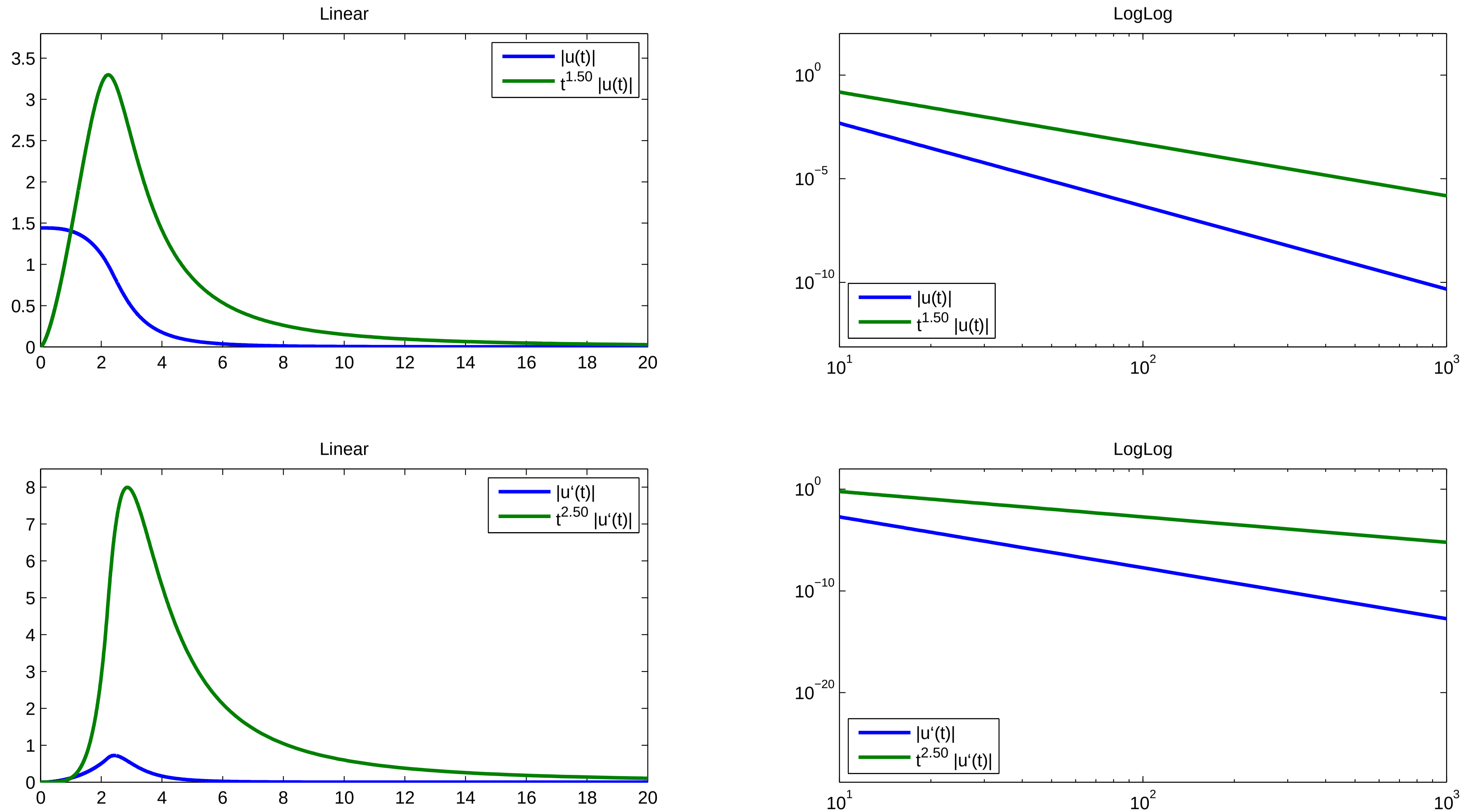

Figure A.3.: $\alpha = 5, \quad \beta = 6, \quad u''(t) + \frac{5}{t}u'(t) + tf(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by (1.18c).

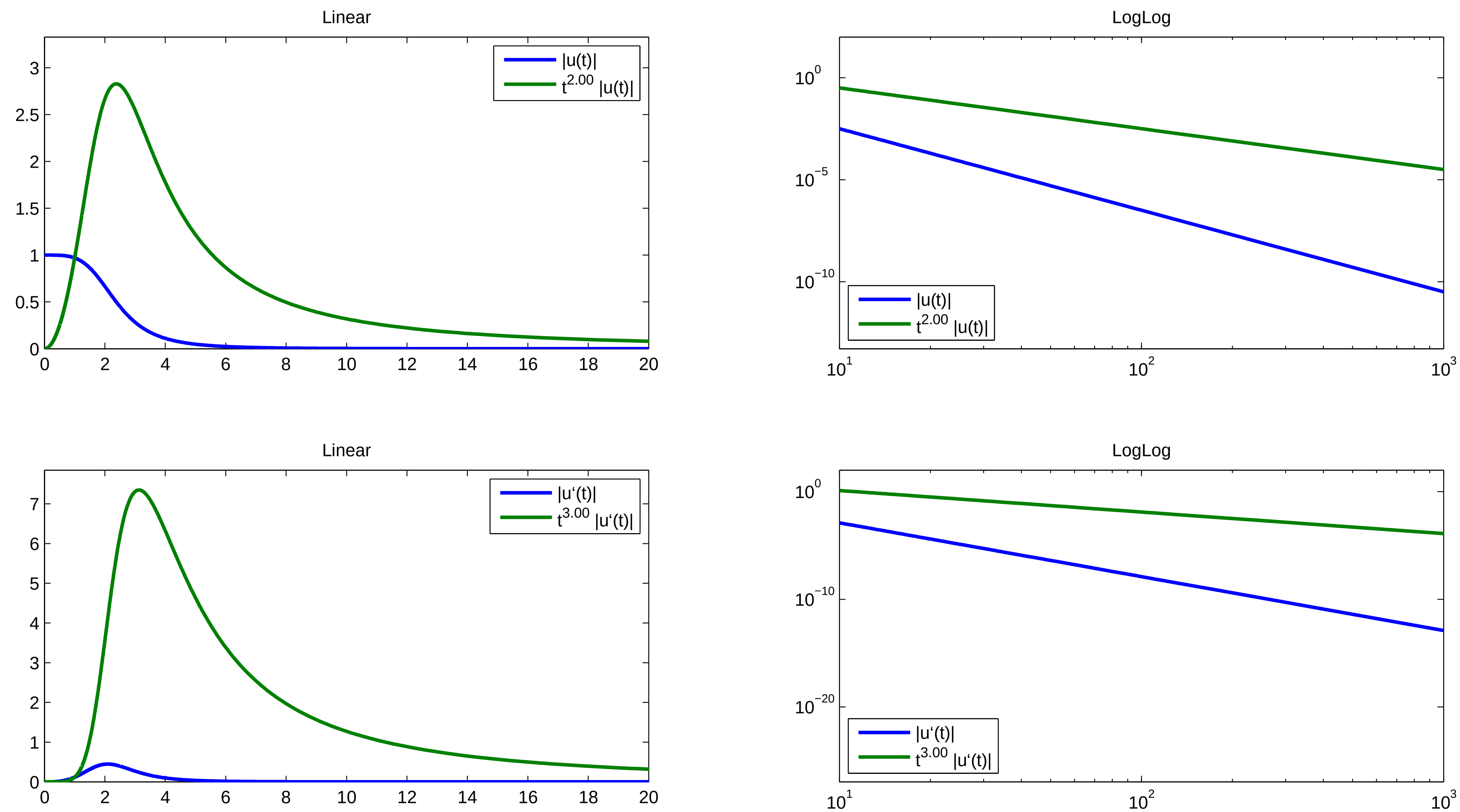

Figure A.4.: $\alpha = 5, \quad \beta = 7, \quad u''(t) + \frac{4}{t}u'(t) + t^2 f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18c)$.

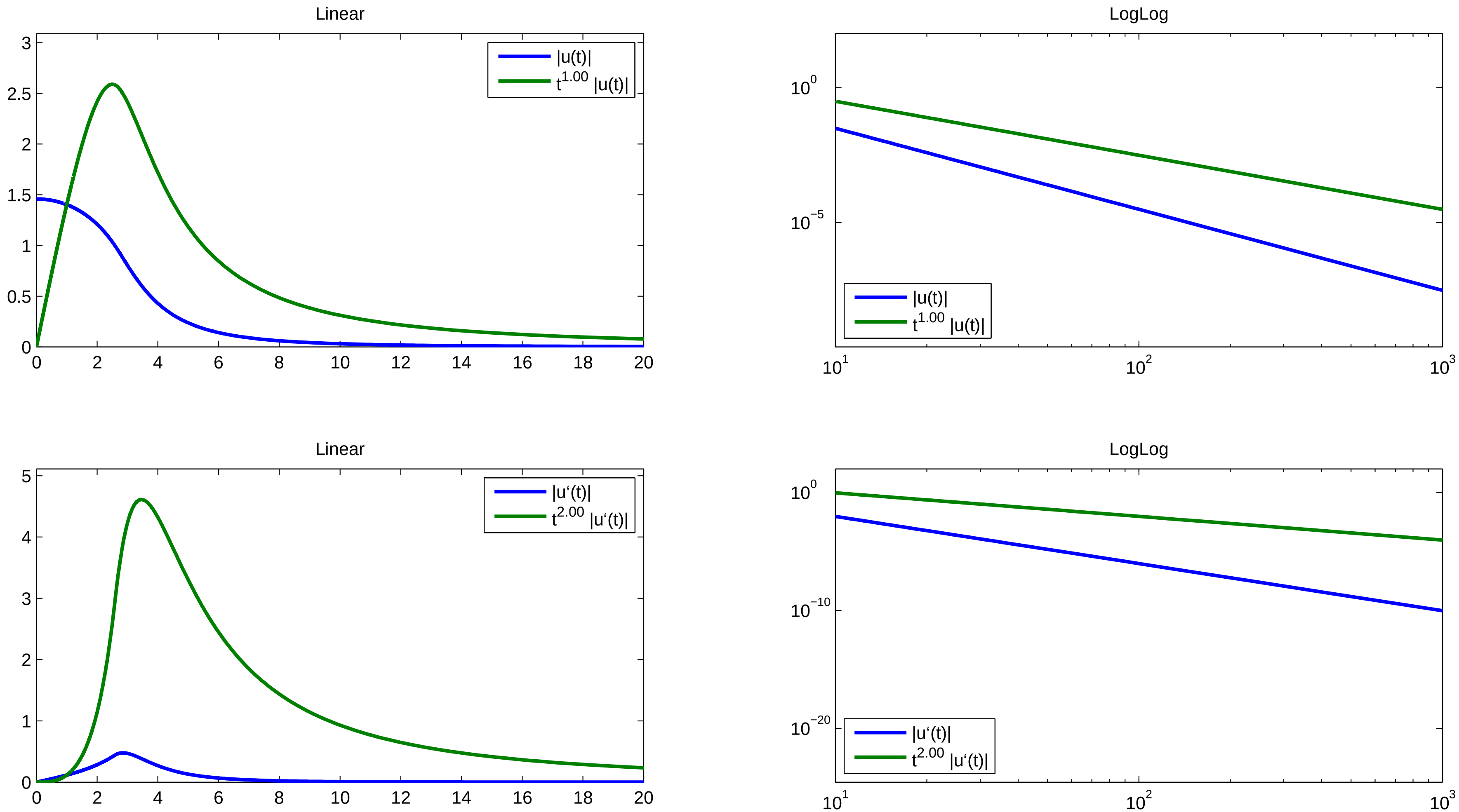

Figure A.5.: $\alpha = 4, \quad \beta = 4, \quad u''(t) + \frac{4}{t}u'(t) + f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18a)$.

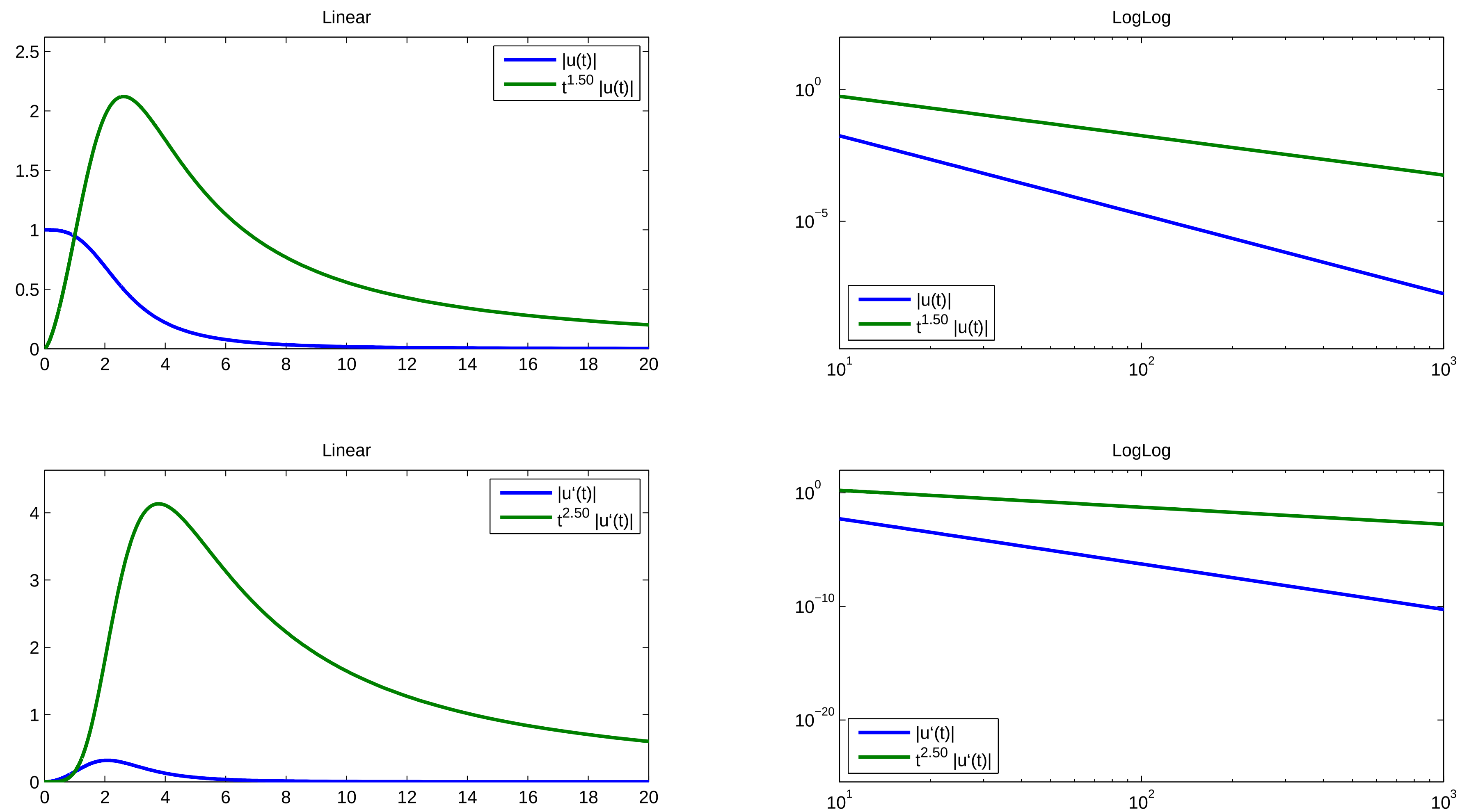

Figure A.6.: $\alpha = 4, \quad \beta = 5, \quad u''(t) + \frac{4}{t}u'(t) + t f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18a)$.

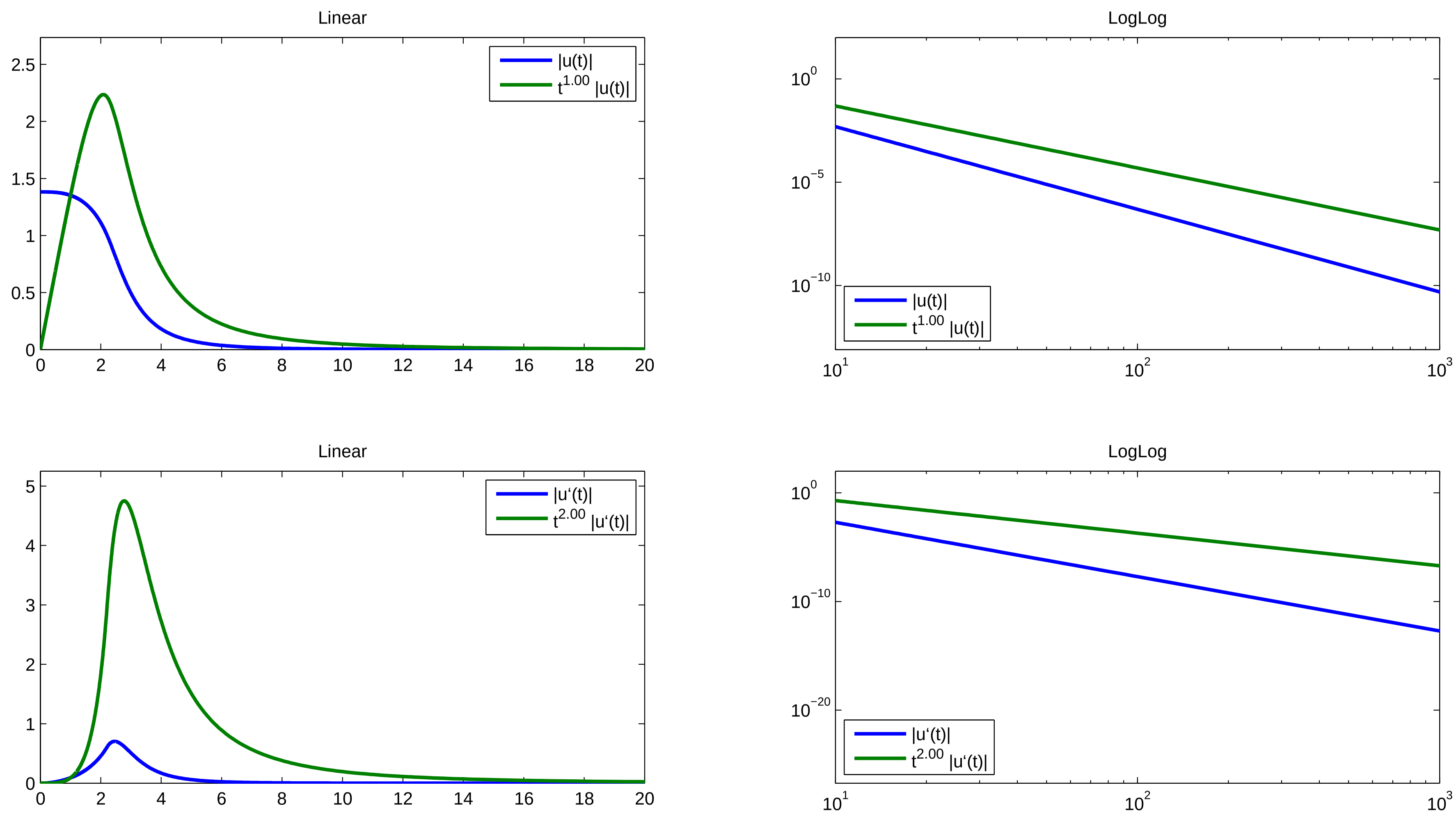

Figure A.7.: $\alpha = 5, \quad \beta = 6, \quad u''(t) + \frac{5}{t}u'(t) + tf(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18a)$.

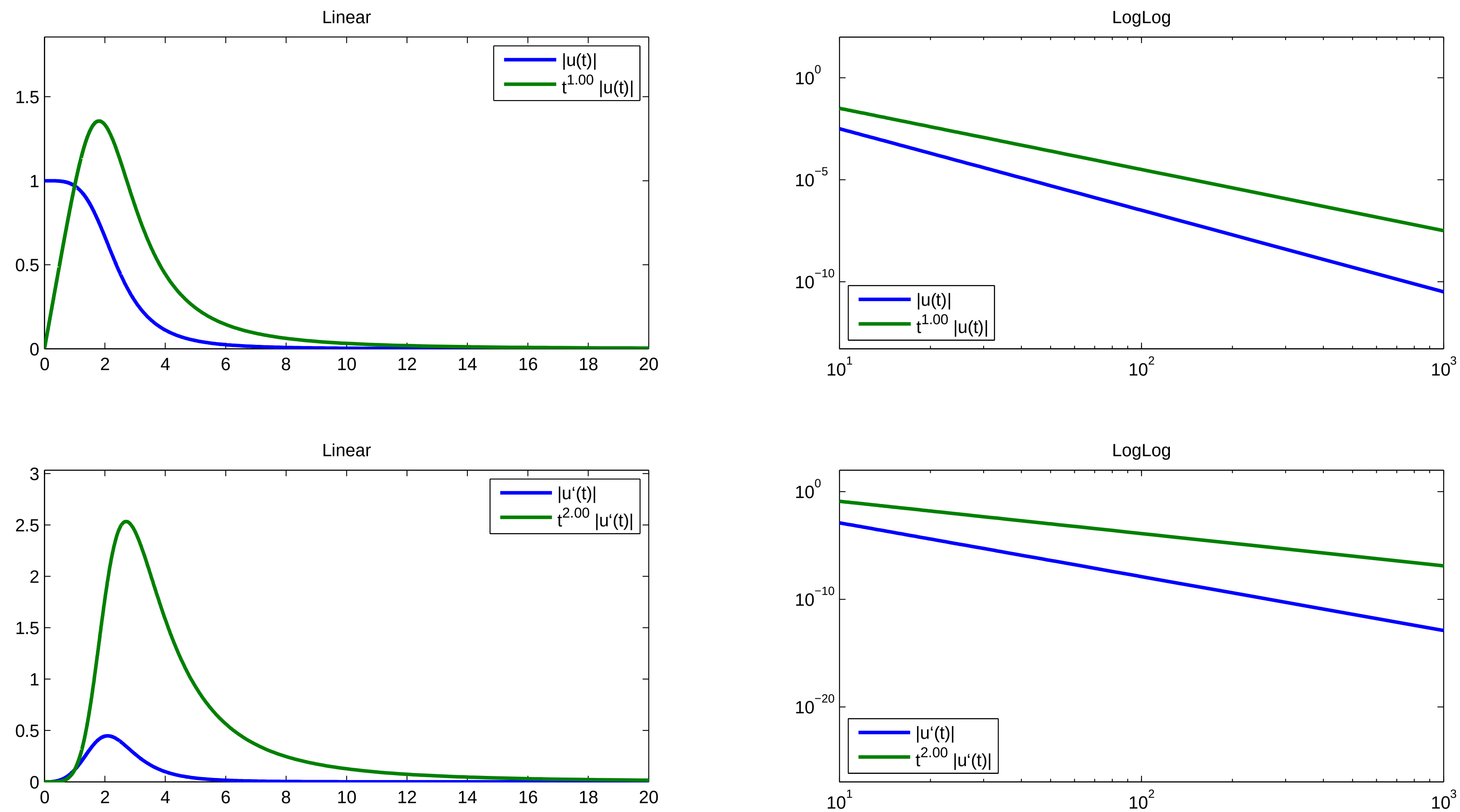

Figure A.8.: $\alpha = 5, \quad \beta = 7, \quad u''(t) + \frac{4}{t}u'(t) + t^2 f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18a)$.

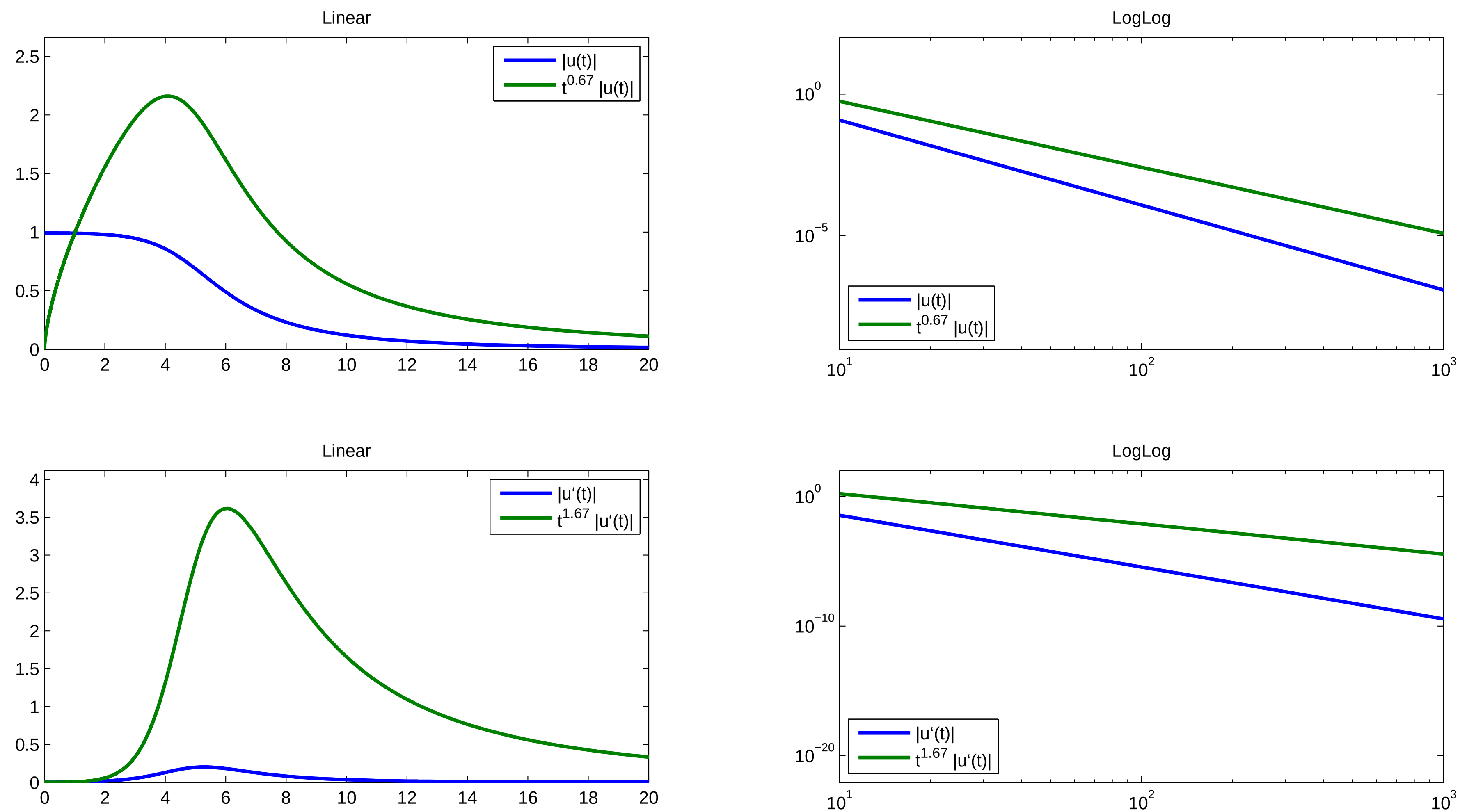

Figure A.9.: $\alpha = 4, \quad \beta = 4, \quad u''(t) + \frac{4}{t}u'(t) + f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18d)$.

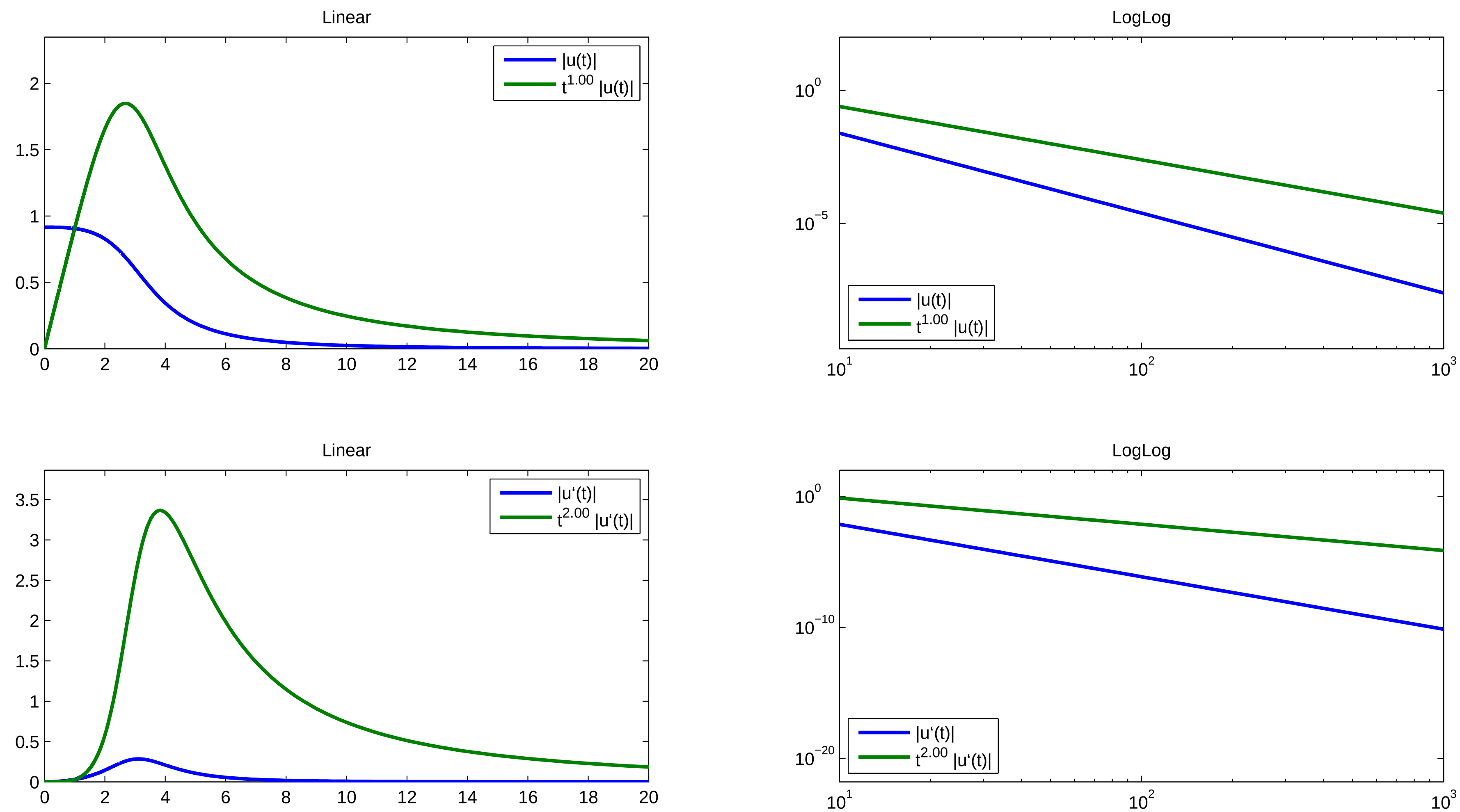

Figure A.10.: $\alpha = 4, \quad \beta = 5, \quad u''(t) + \frac{4}{t}u'(t) + tf(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by (1.18d).

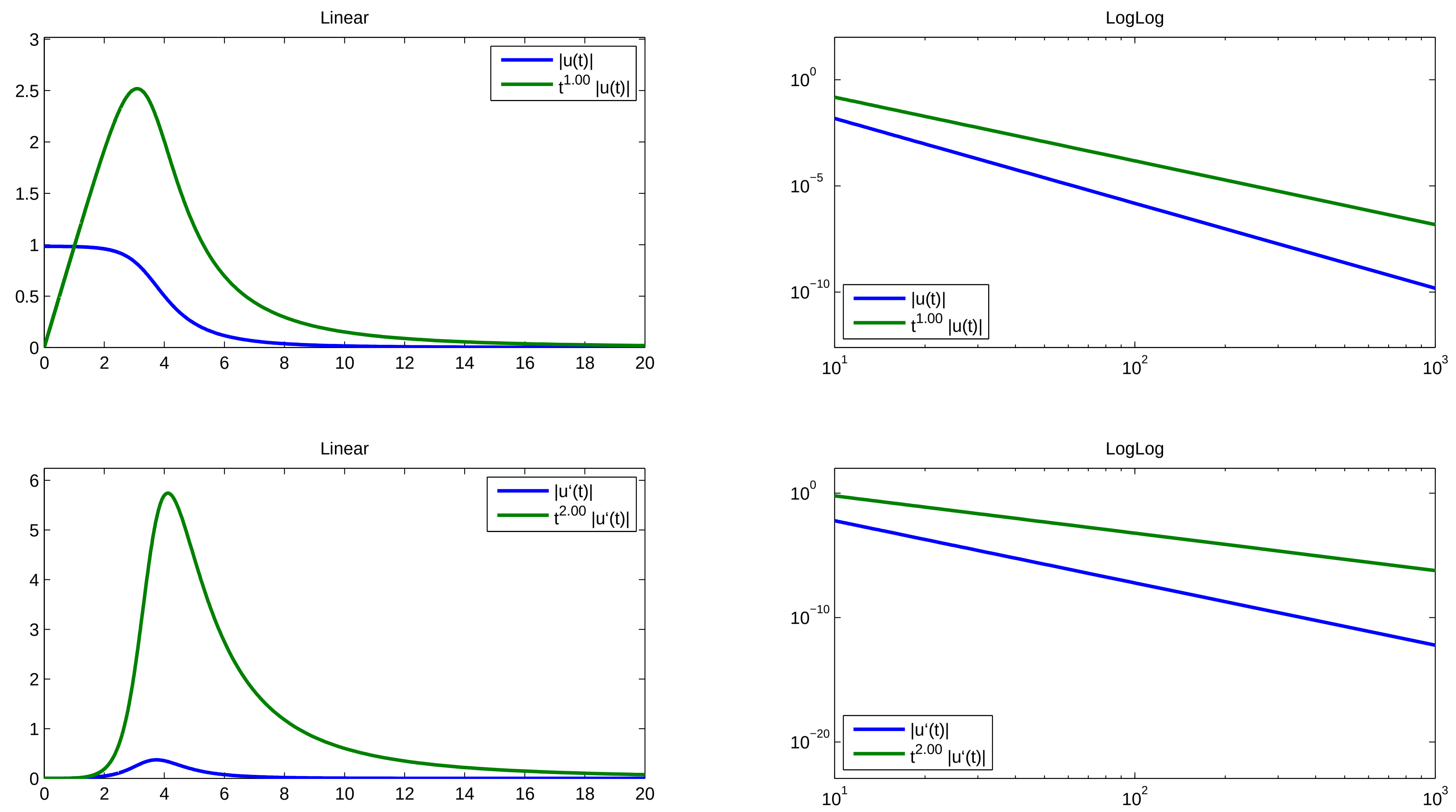

Figure A.11.: $\alpha = 5, \quad \beta = 6, \quad u''(t) + \frac{5}{t}u'(t) + t f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18d)$.

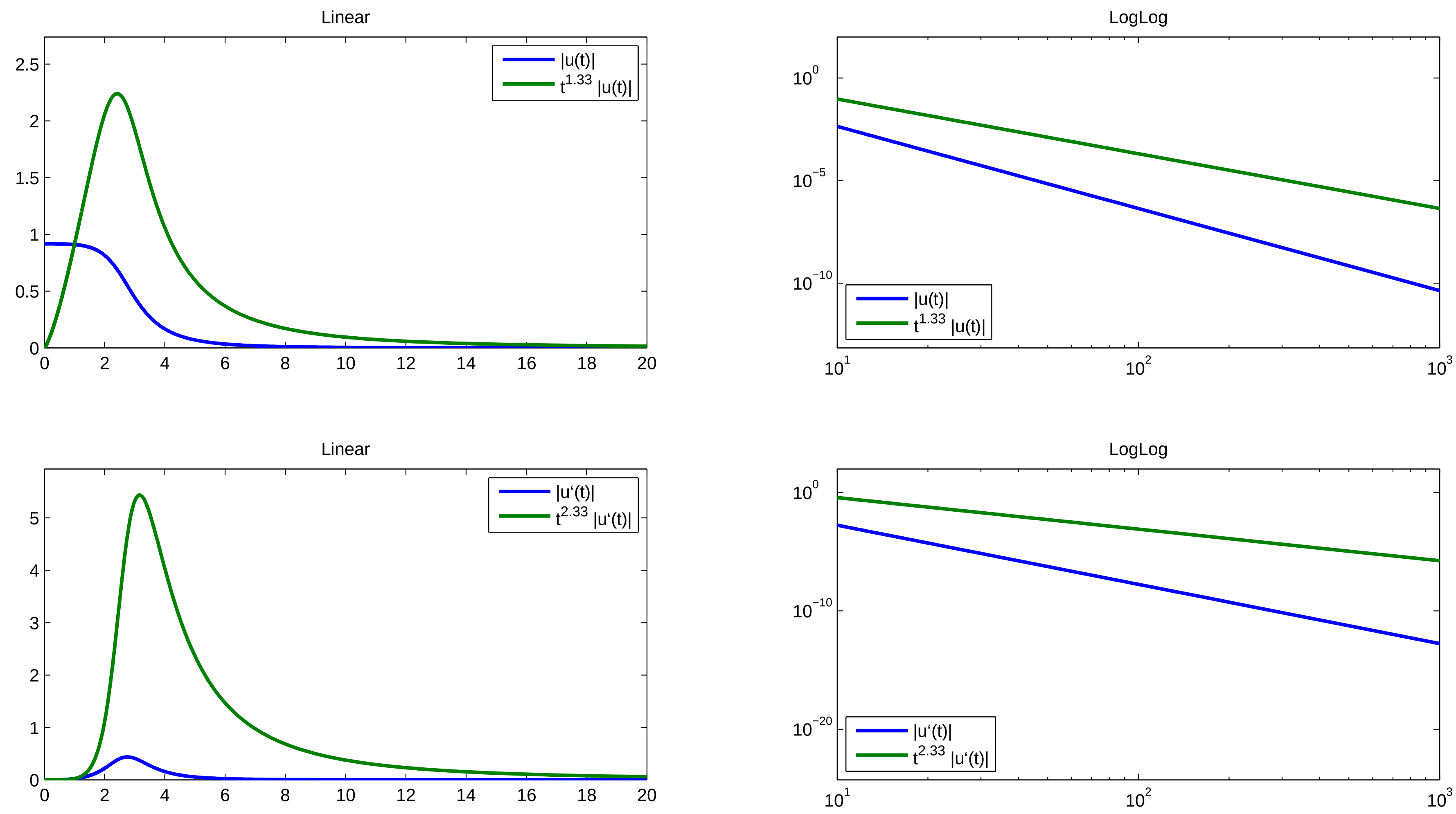

Figure A.12.: $\alpha = 5, \quad \beta = 7, \quad u''(t) + \frac{4}{t}u'(t) + t^2 f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18d)$.

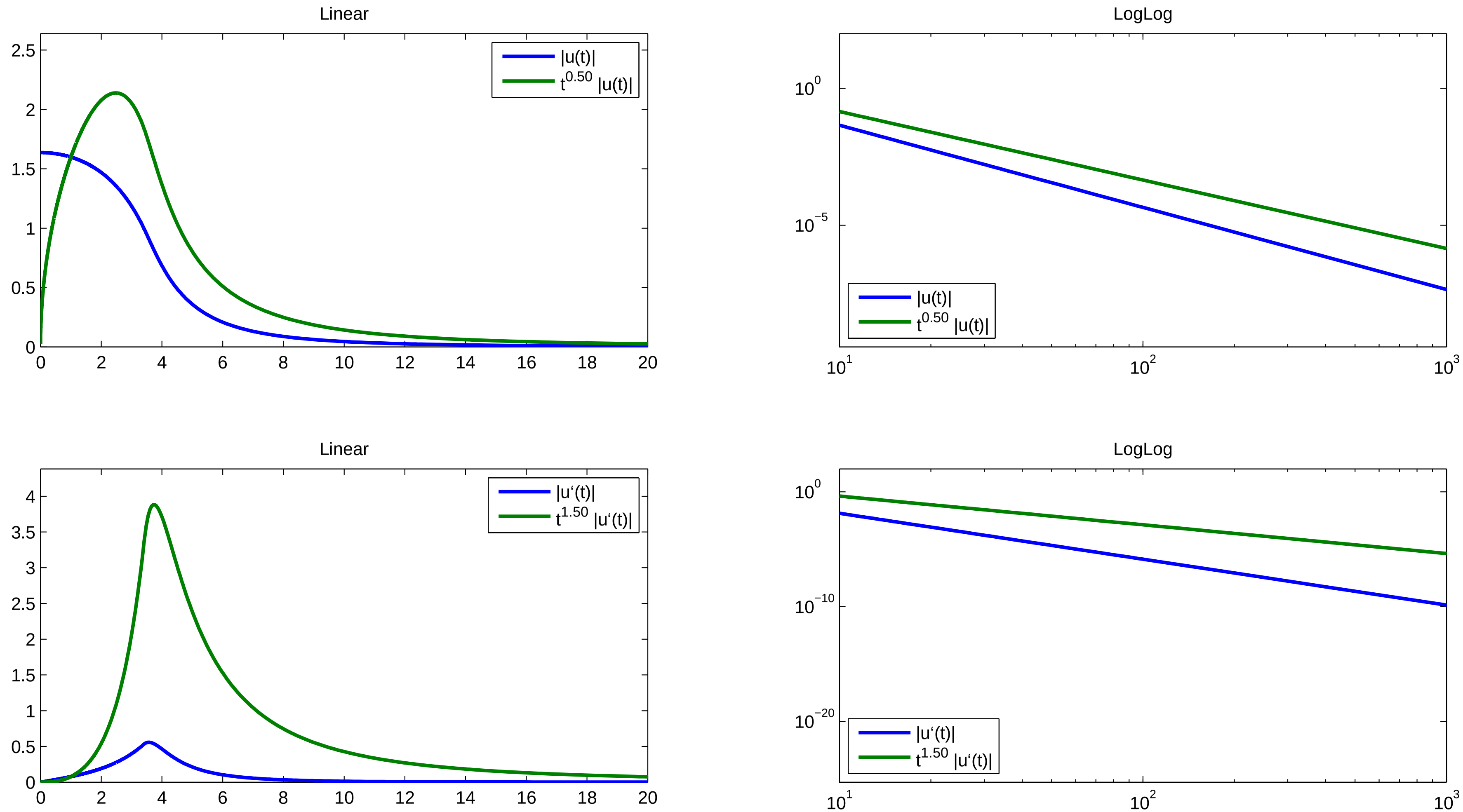

Figure A.13.: $\alpha = 4, \quad \beta = 4, \quad u''(t) + \frac{4}{t}u'(t) + f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18b)$.

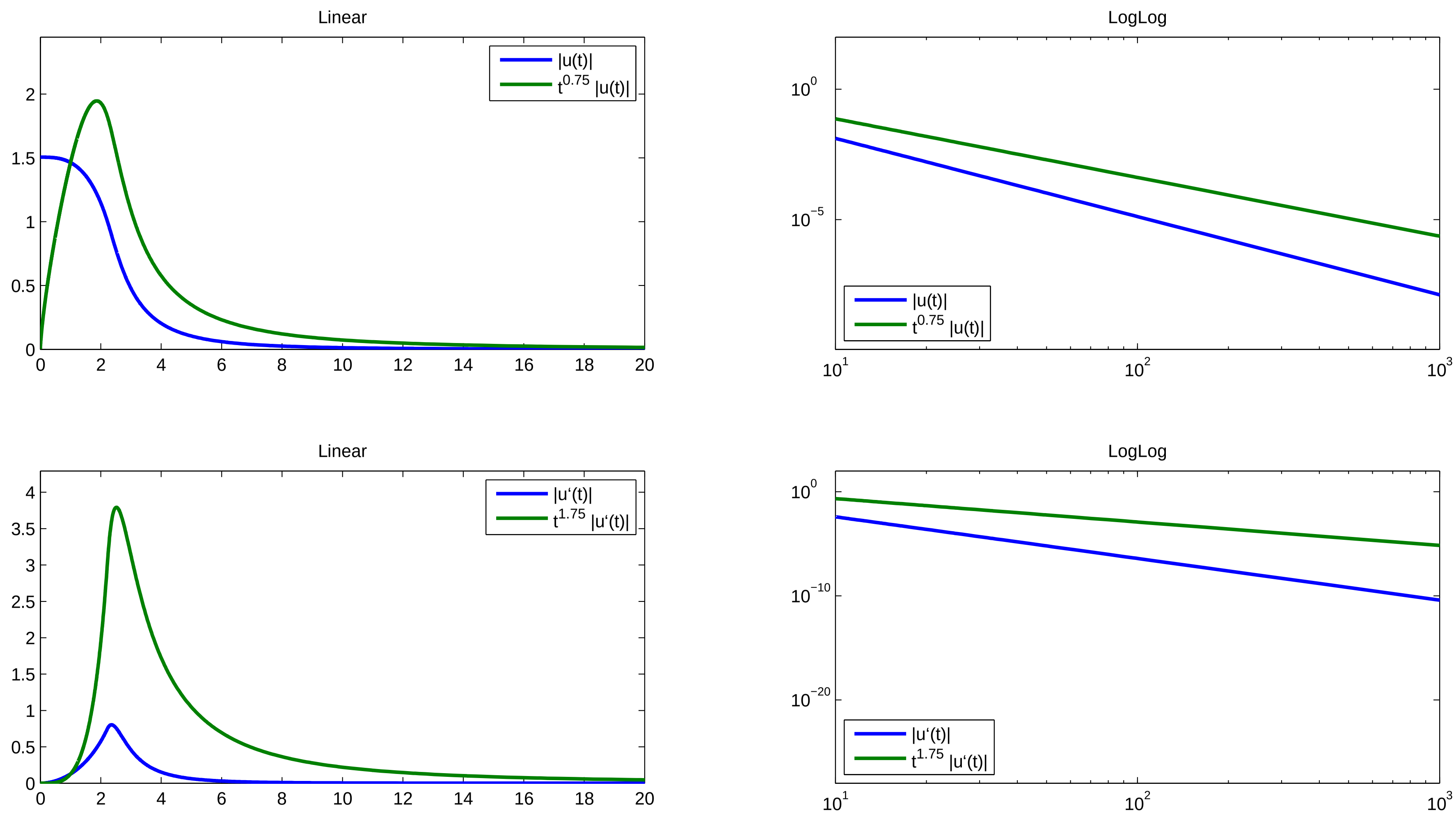

Figure A.14.: $\alpha = 4, \quad \beta = 5, \quad u''(t) + \frac{4}{t}u'(t) + t f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18b)$.

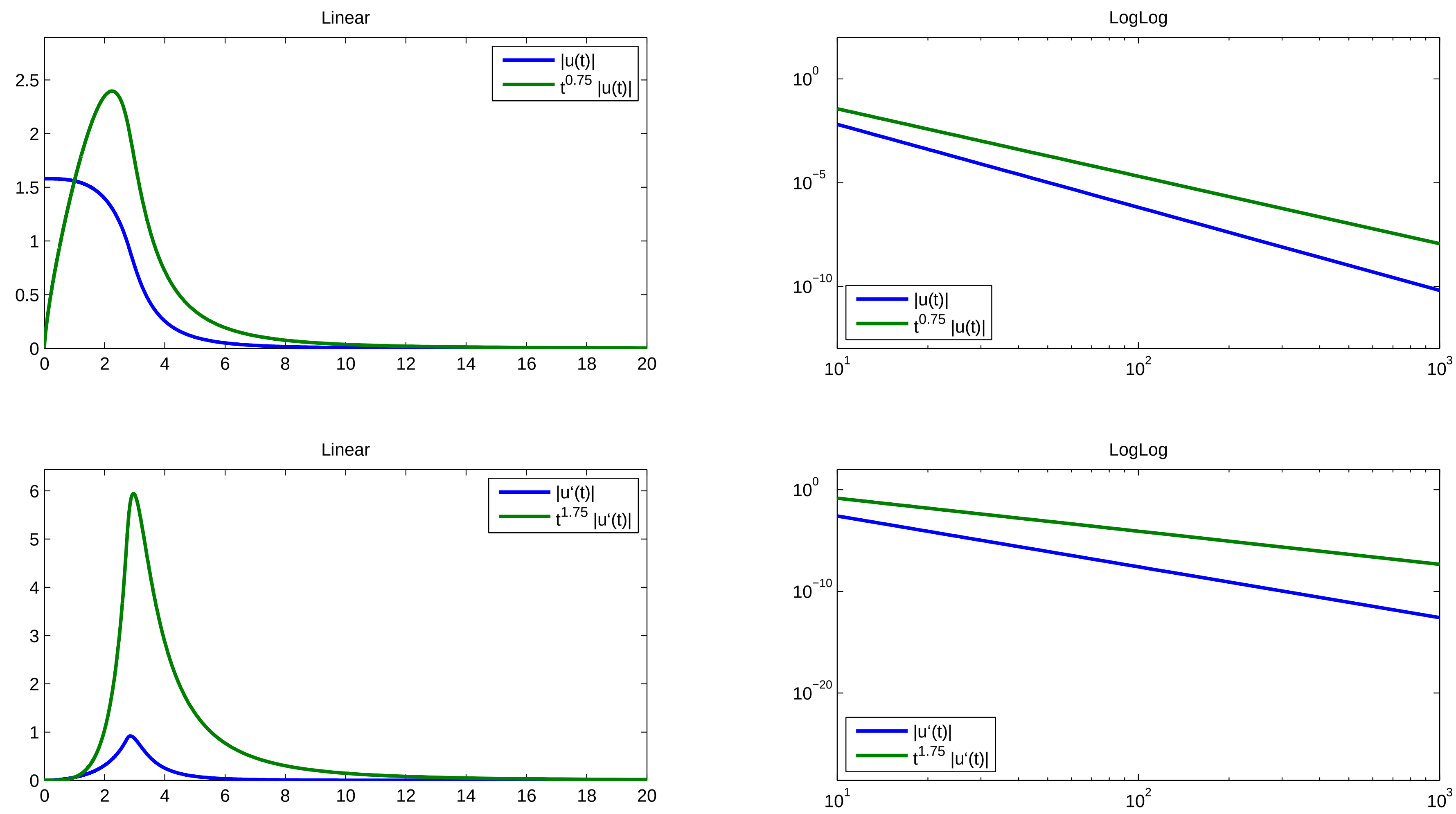

Figure A.15.: $\alpha = 5, \quad \beta = 6, \quad u''(t) + \frac{5}{t}u'(t) + t\,f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18b)$.

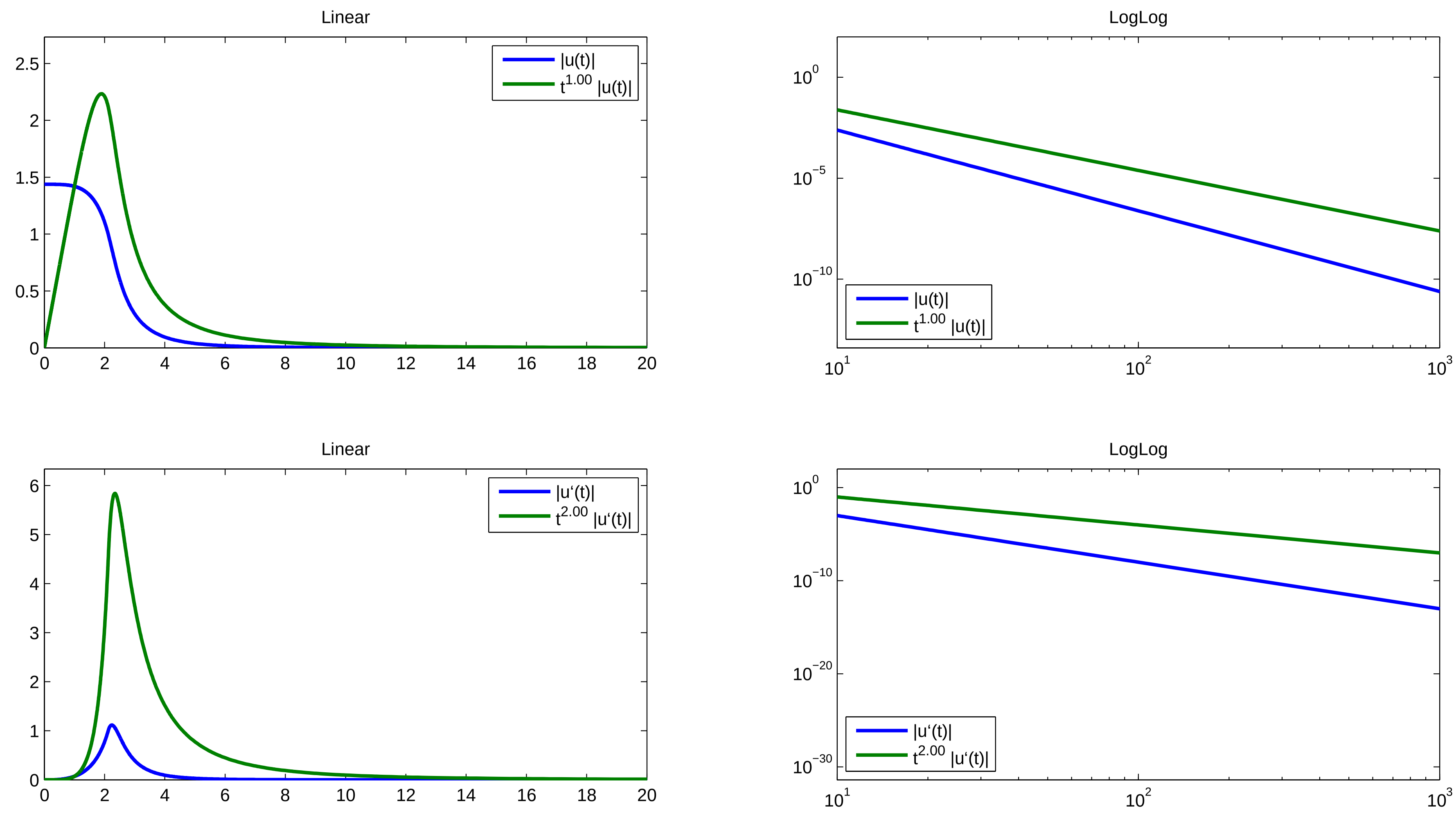

Figure A.16.: $\alpha = 5, \quad \beta = 7, \quad u''(t) + \frac{4}{t}u'(t) + t^2 f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18b)$.

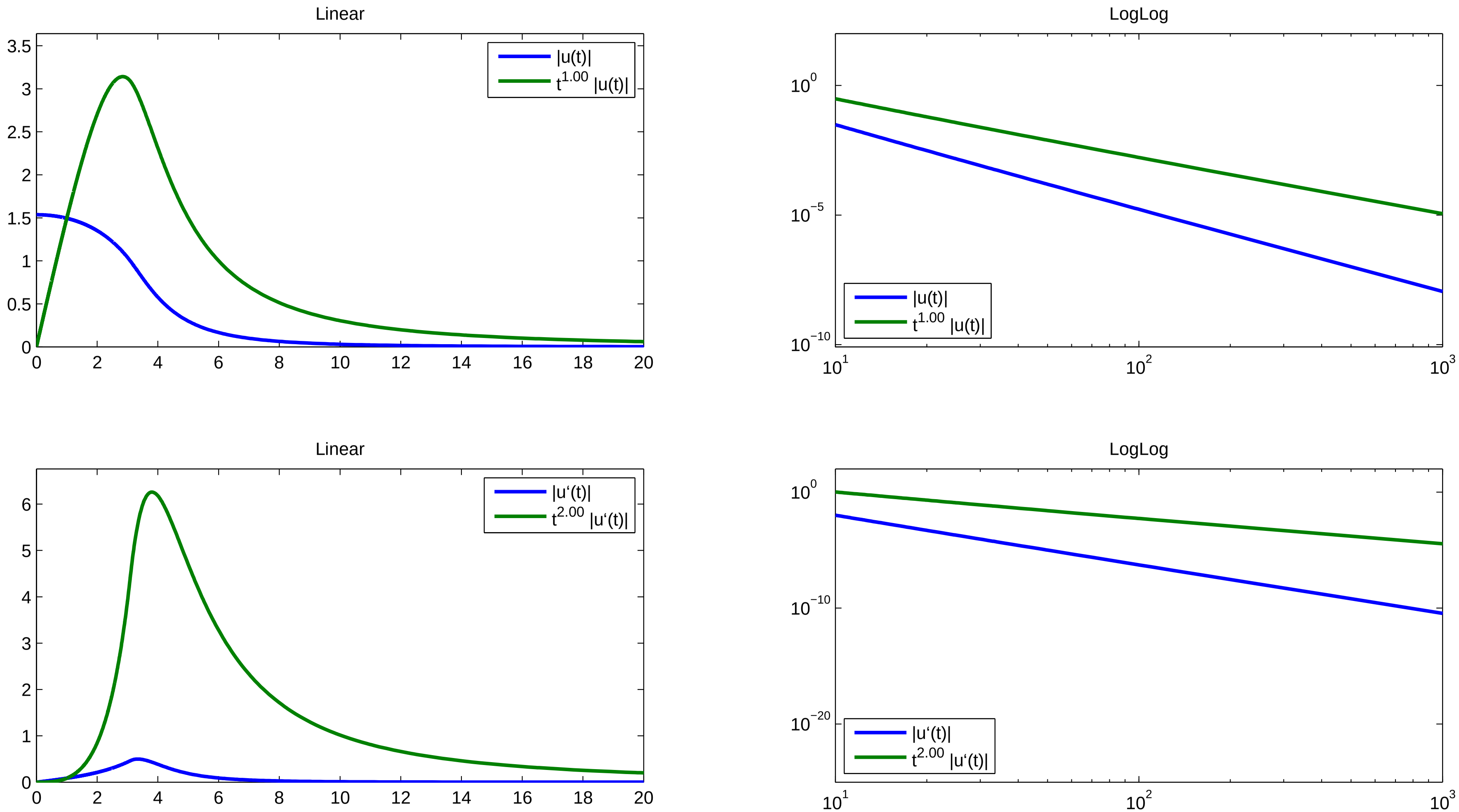

Figure A.17.: $\alpha = 4, \quad \beta = 4, \quad u''(t) + \left[\frac{4}{t} + \frac{1}{(1+t)(\log(1+t))}\right] u'(t) + f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18c)$.

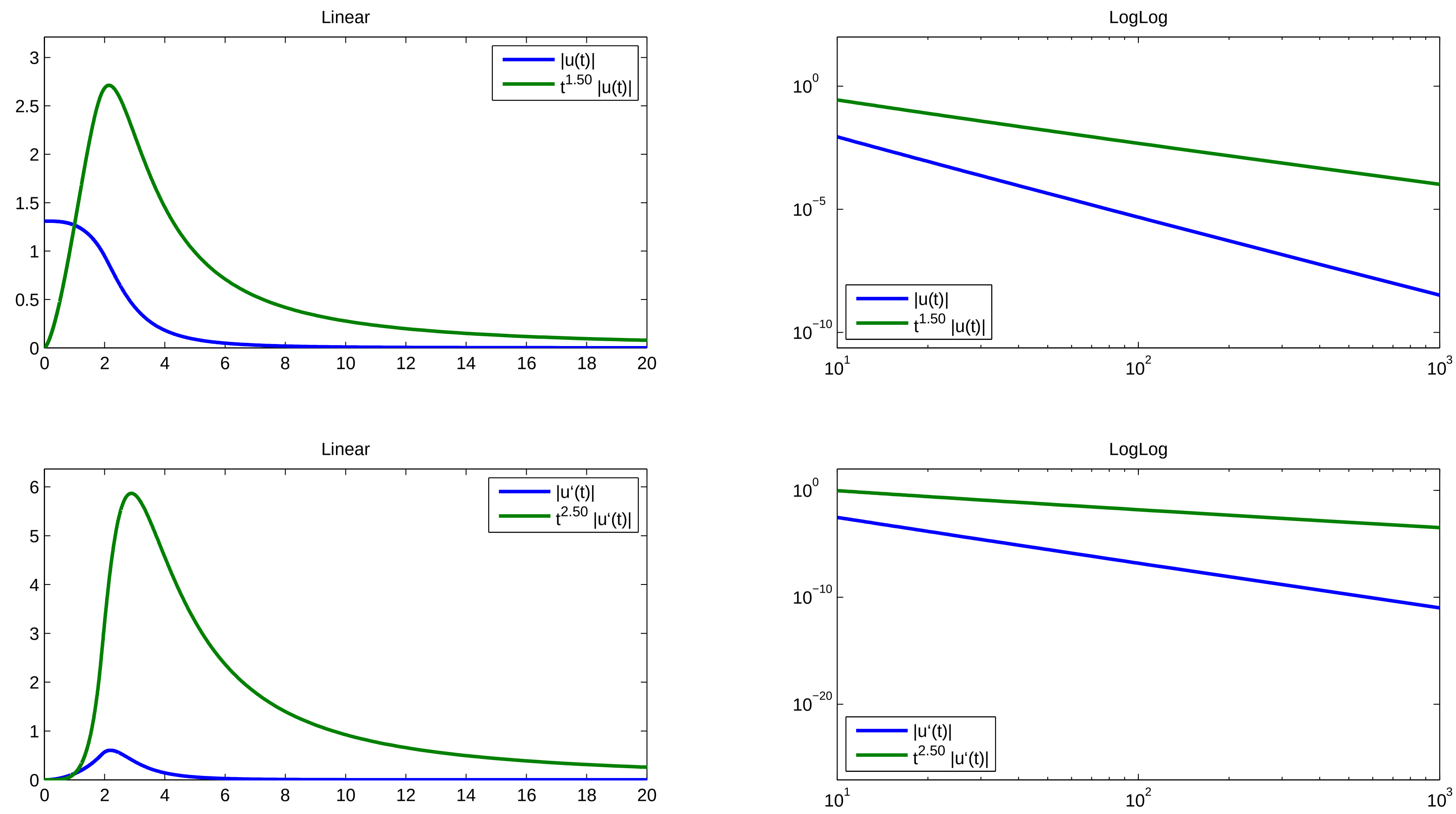

Figure A.18.: $\alpha = 4, \quad \beta = 5, \quad u''(t) + \left[\frac{4}{t} + \frac{1}{(1+t)(\log(1+t)}\right] u'(t) + tf(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by (1.18c).

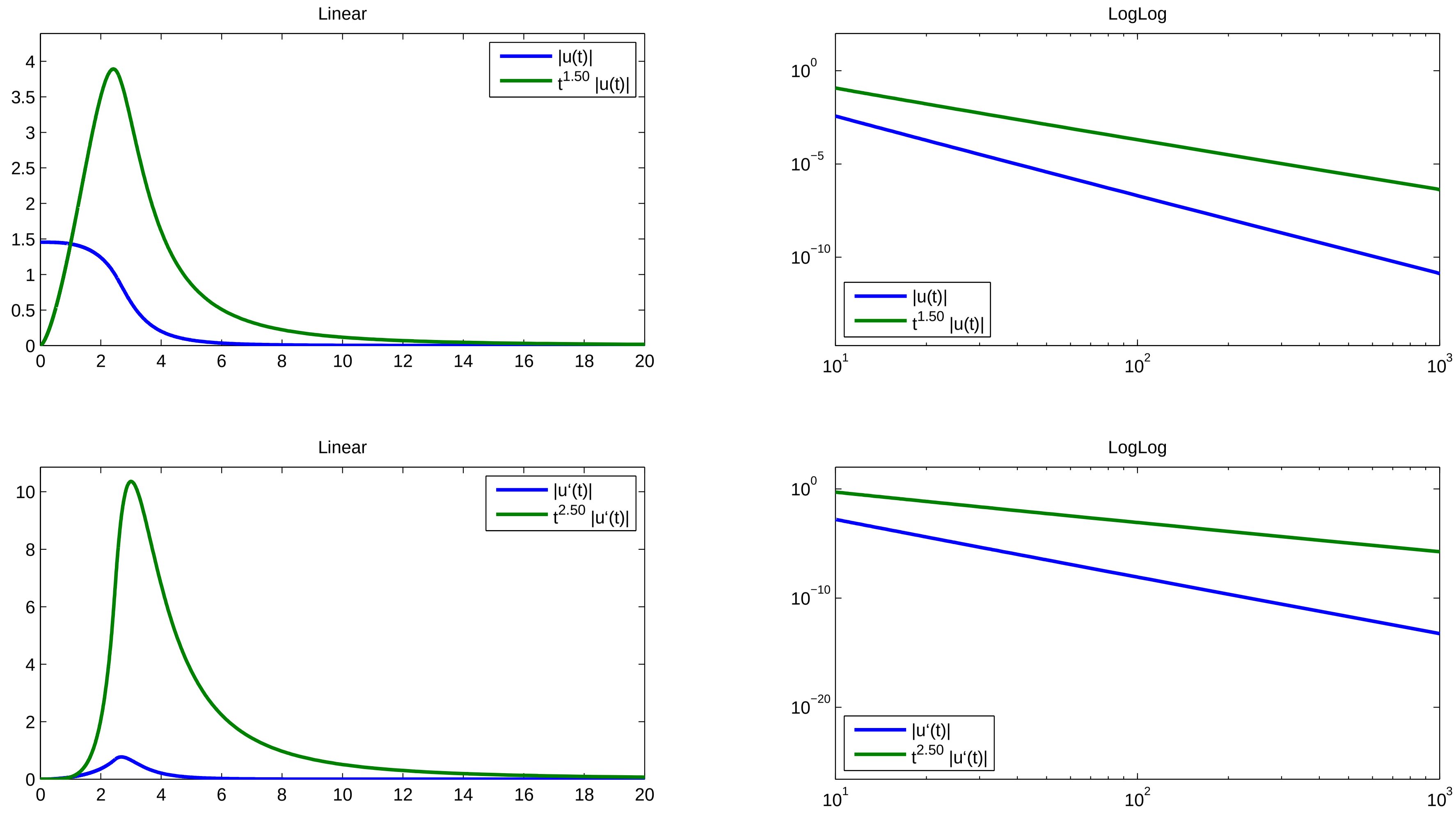

Figure A.19.: $\alpha = 5, \quad \beta = 6, \quad u''(t) + \left[\frac{5}{t} + \frac{1}{(1+t)(\log(1+t))}\right] u'(t) + t f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18c)$.

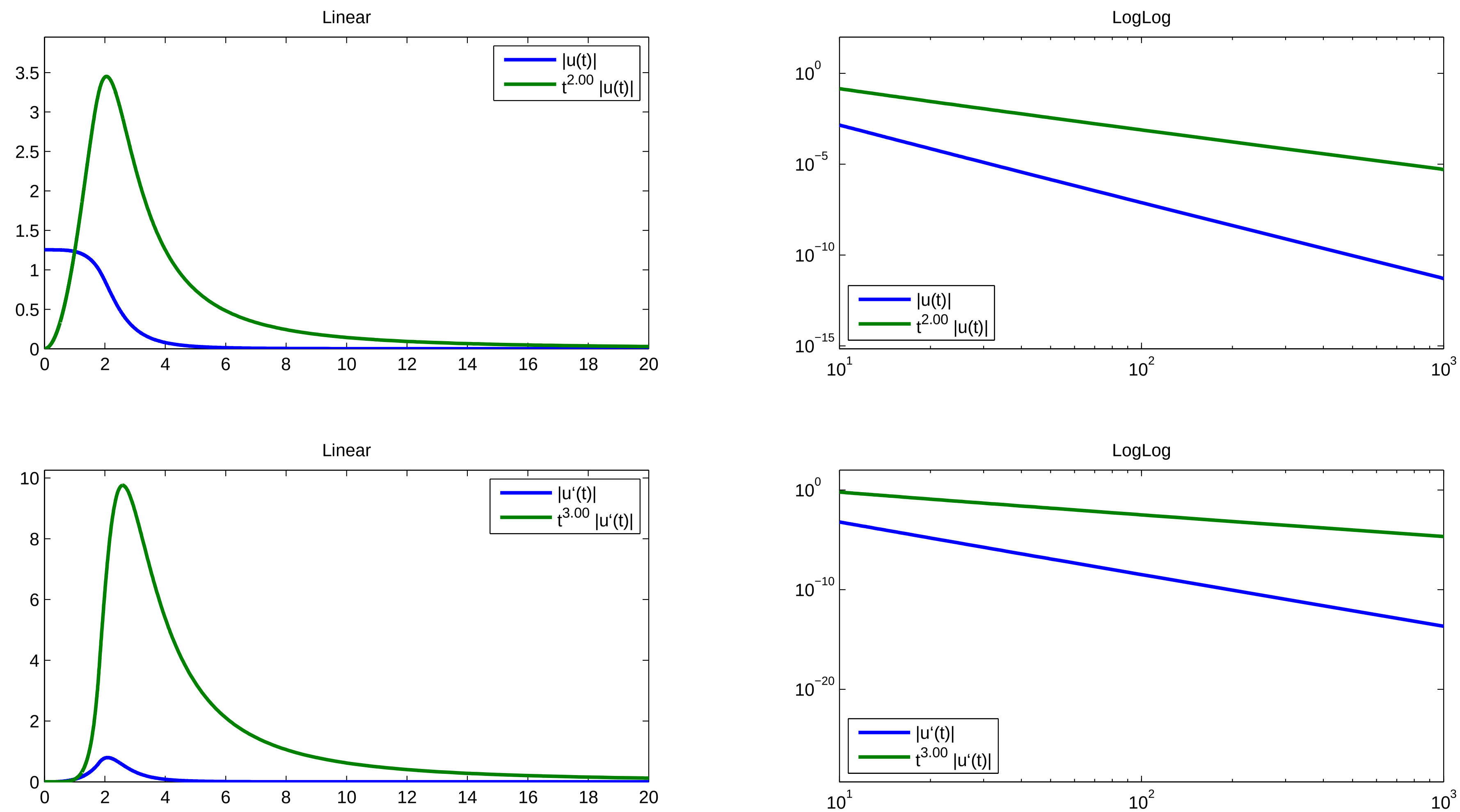

Figure A.20.: $\alpha = 5, \quad \beta = 7, \quad u''(t) + \left[\frac{5}{t} + \frac{1}{(1+t)(\log(1+t))}\right] u'(t) + t^2 f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18c)$.

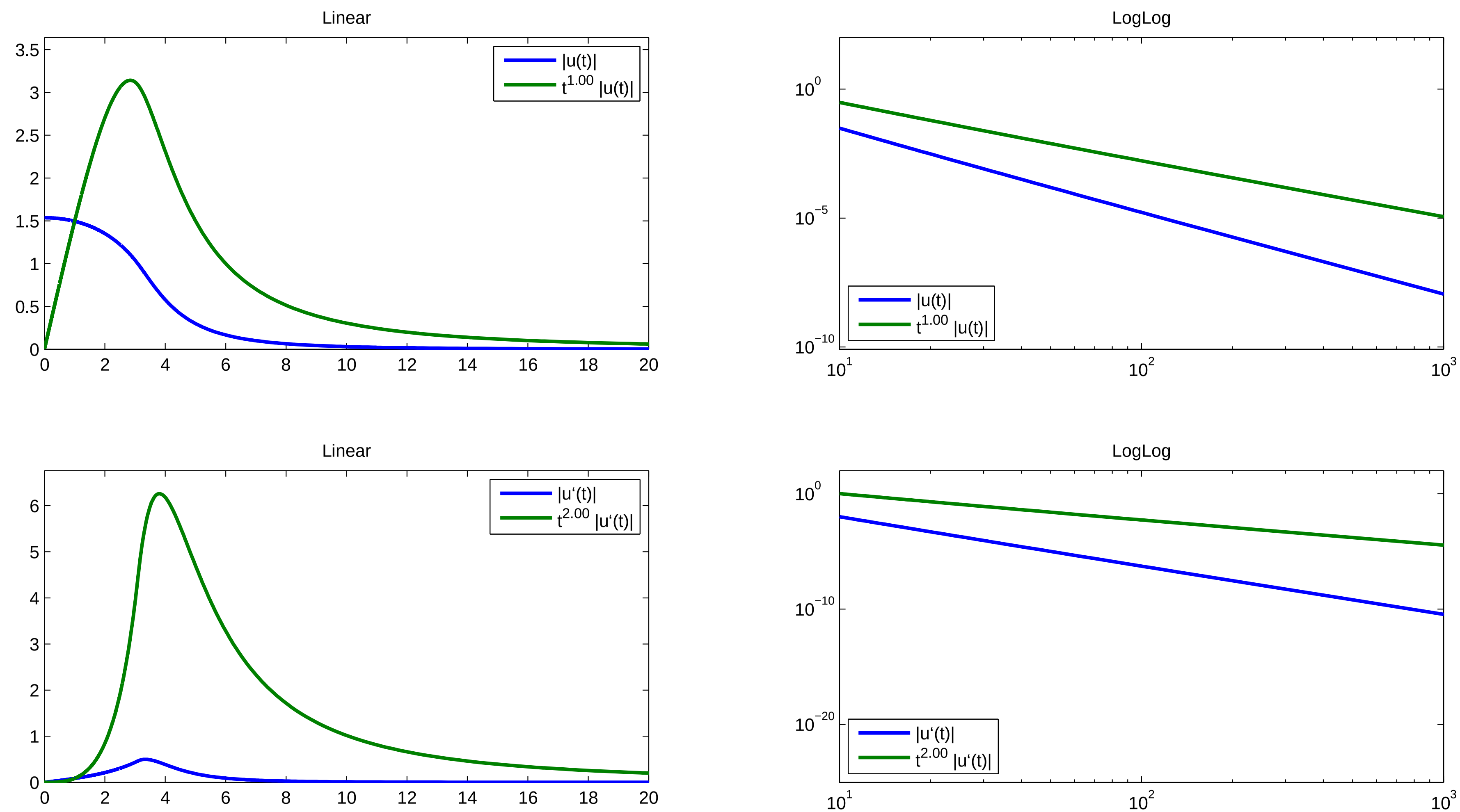

Figure A.21.: $\alpha = 4, \quad \beta = 4, \quad u''(t) + \left[\frac{4}{t} + \frac{1}{(1+t)(\log(1+t))}\right] u'(t) + f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18a)$.

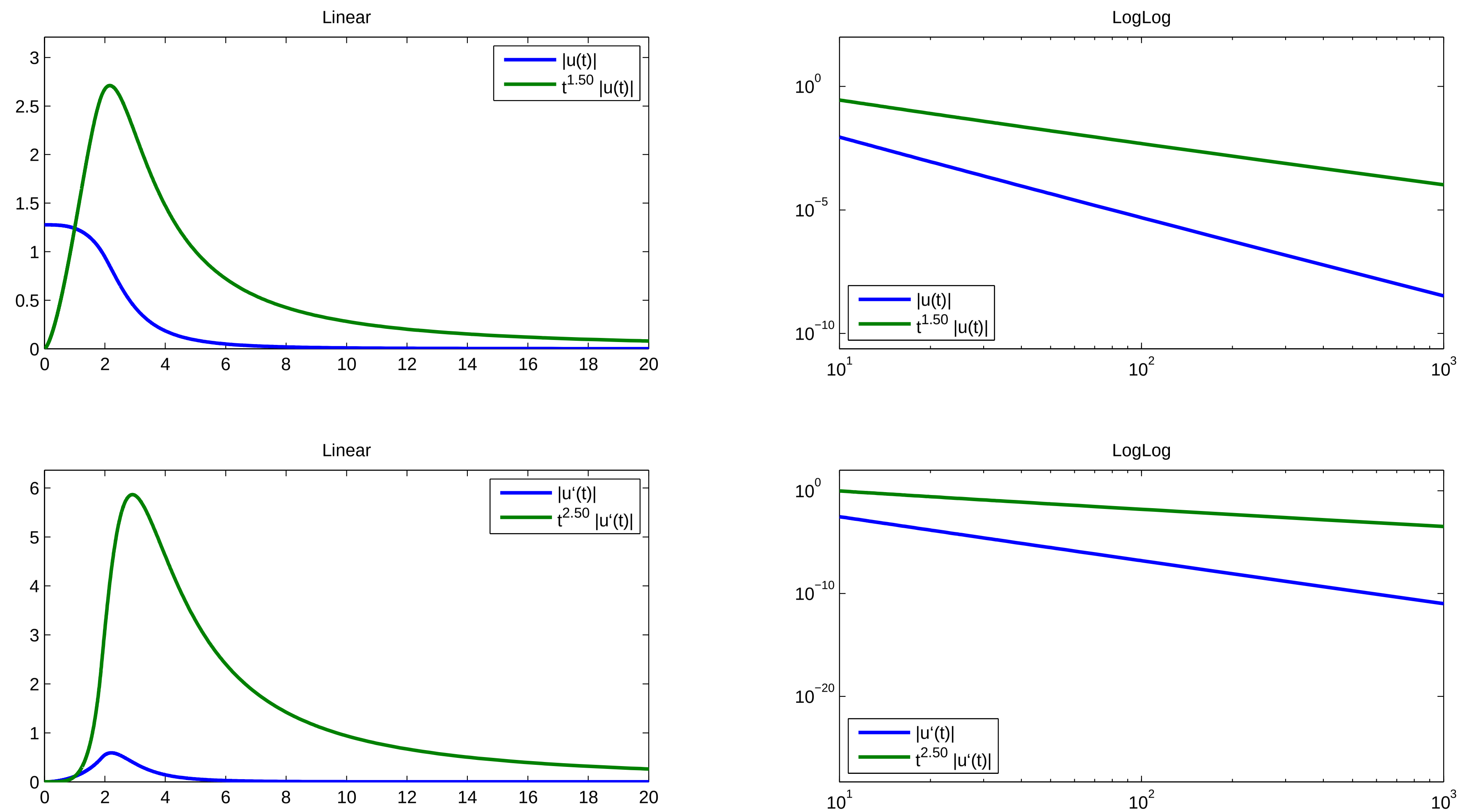

Figure A.22.: $\alpha = 4, \quad \beta = 5, \quad u''(t) + \left[\frac{4}{t} + \frac{1}{(1+t)(\log(1+t))}\right] u'(t) + t f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18a)$.

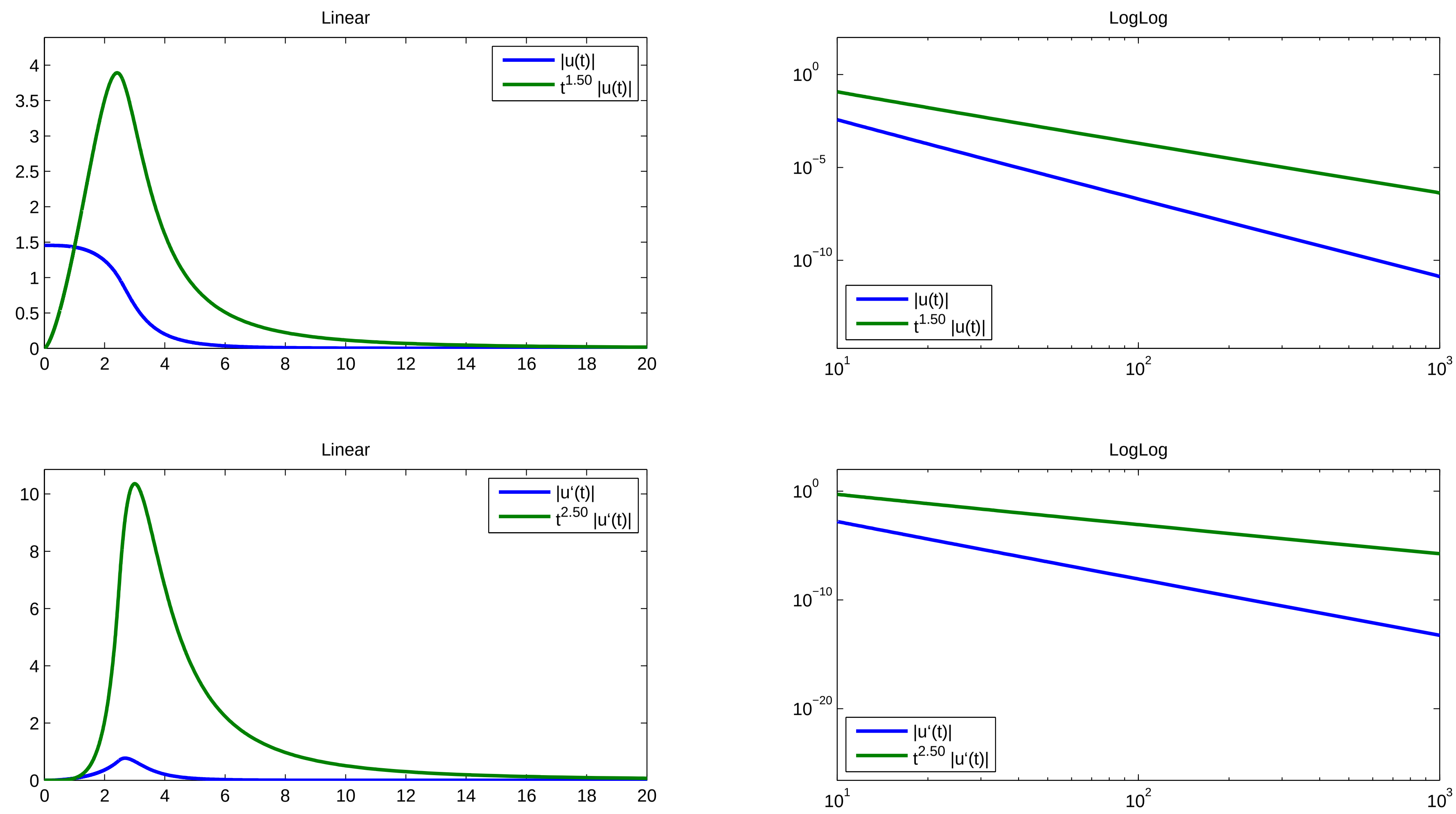

Figure A.23.: $\alpha = 5, \ \ \beta = 6, \quad u''(t) + \left[\frac{5}{t} + \frac{1}{(1+t)(\log(1+t)}\right] u'(t) + t f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18a)$.

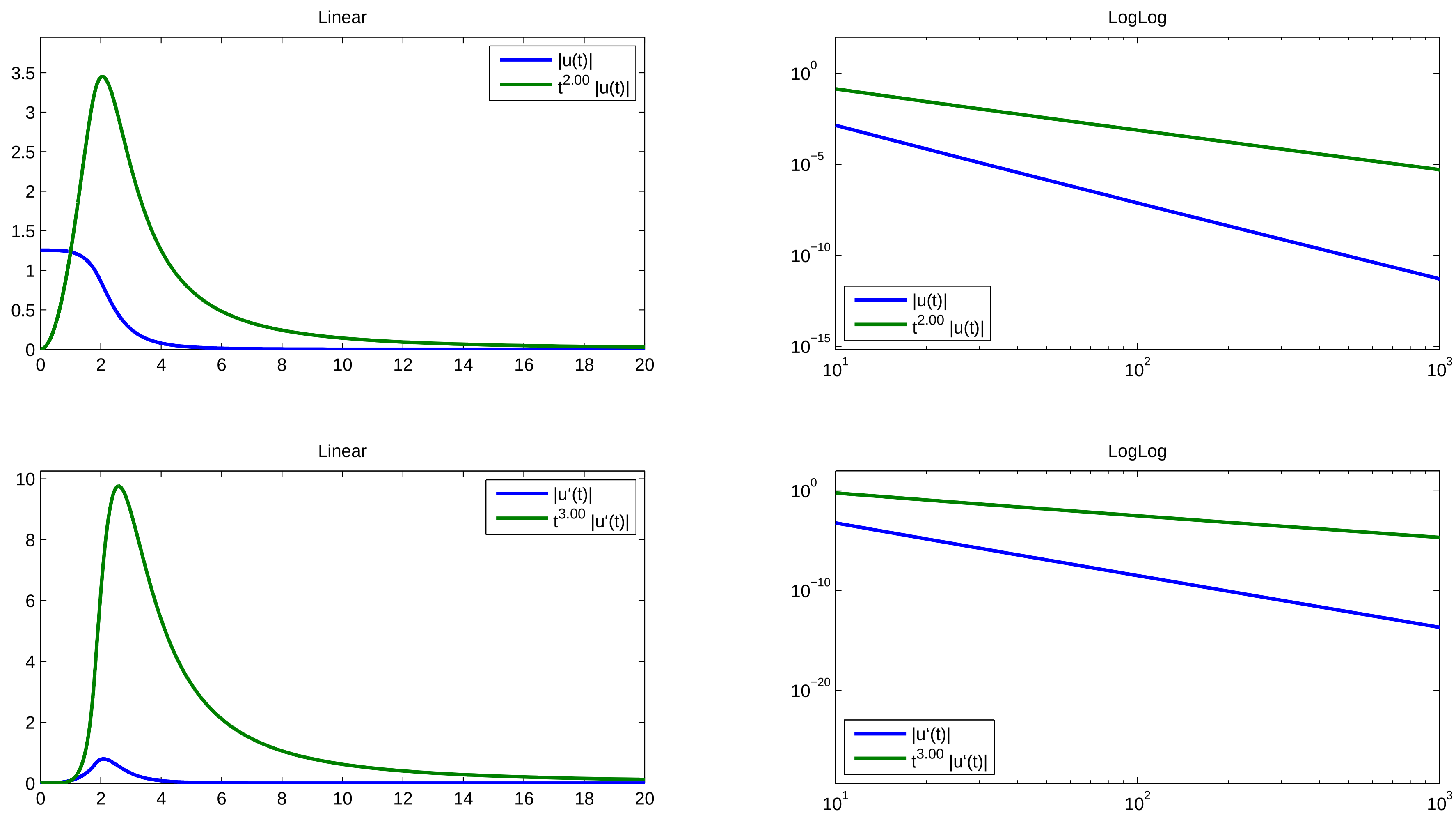

Figure A.24.: $\alpha = 5,\ \ \beta = 7, \quad u''(t) + \left[\frac{5}{t} + \frac{1}{(1+t)(\log(1+t))}\right] u'(t) + t^2 f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18a)$.

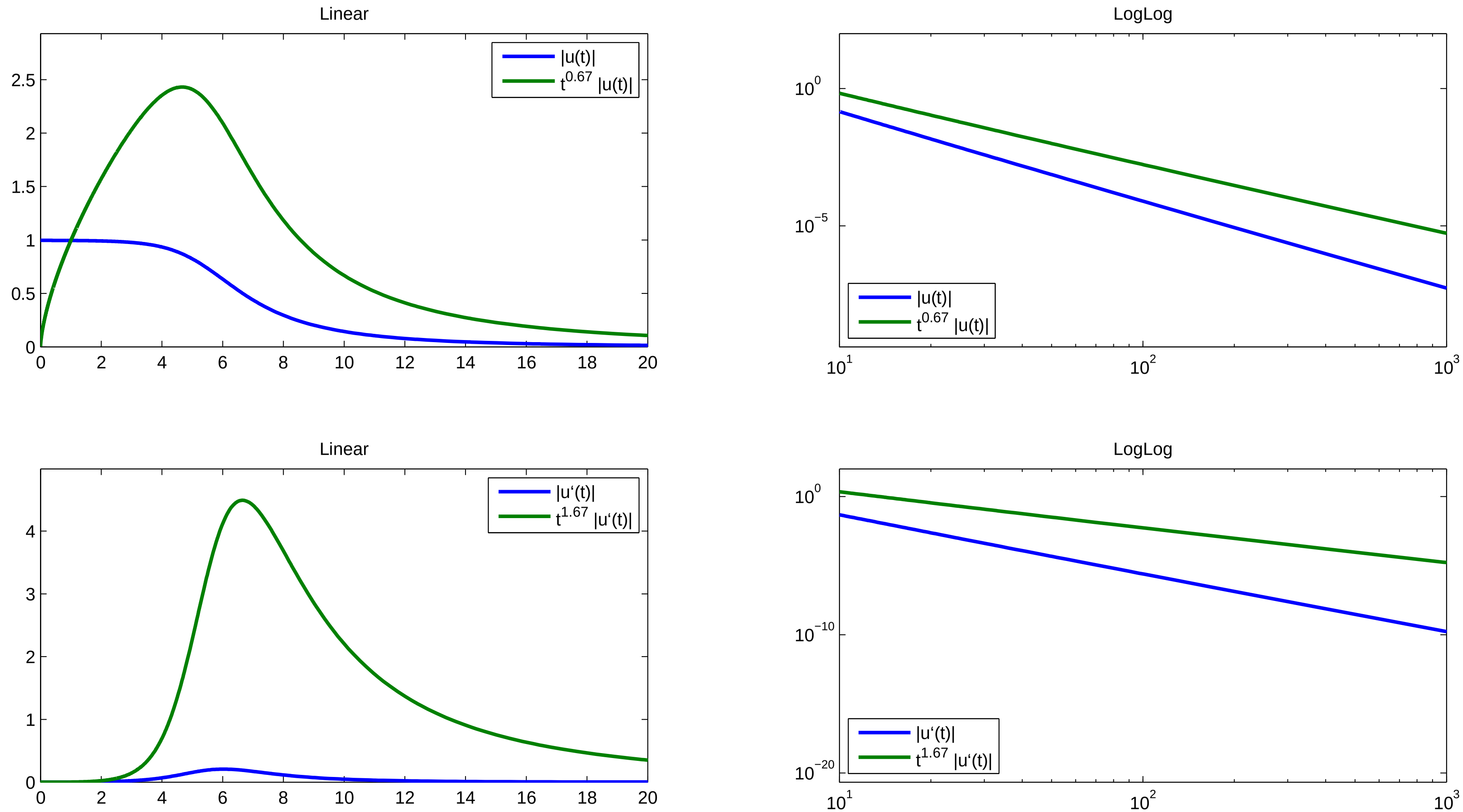

Figure A.25.: $\alpha = 4, \;\; \beta = 4, \quad u''(t) + \left[\frac{4}{t} + \frac{1}{(1+t)(\log(1+t))}\right] u'(t) + f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18d)$.

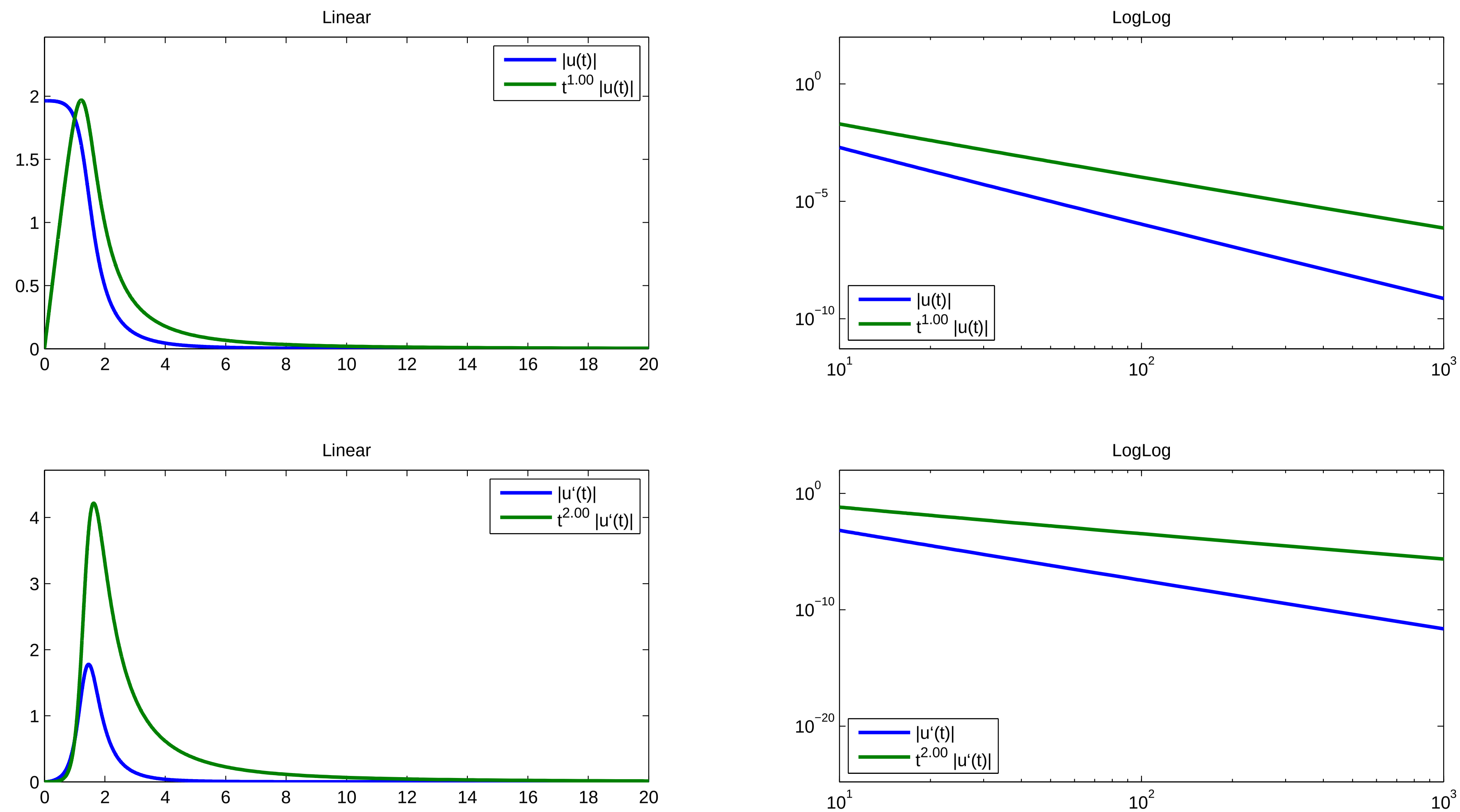

Figure A.26.: $\alpha = 4, \;\; \beta = 5, \quad u''(t) + \left[\frac{4}{t} + \frac{1}{(1+t)(\log(1+t))}\right] u'(t) + tf(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18d)$.

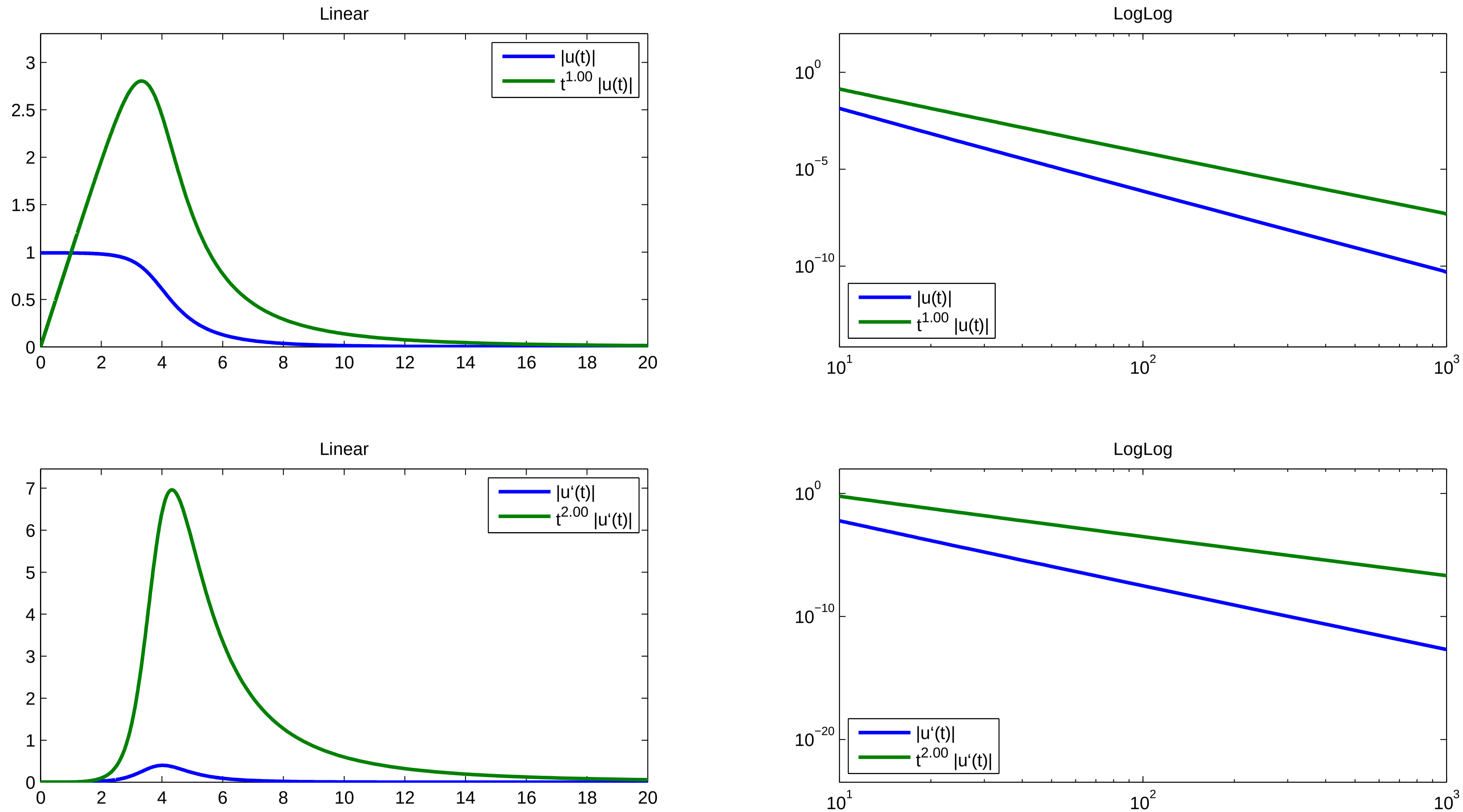

Figure A.27.: $\alpha = 5$, $\beta = 6$, $u''(t) + \left[\frac{5}{t} + \frac{1}{(1+t)(\log(1+t))}\right] u'(t) + tf(u) = 0$, $u'(0) = 0$, $u(\infty) = 0$, where $f(u)$ is given by $(1.18d)$.

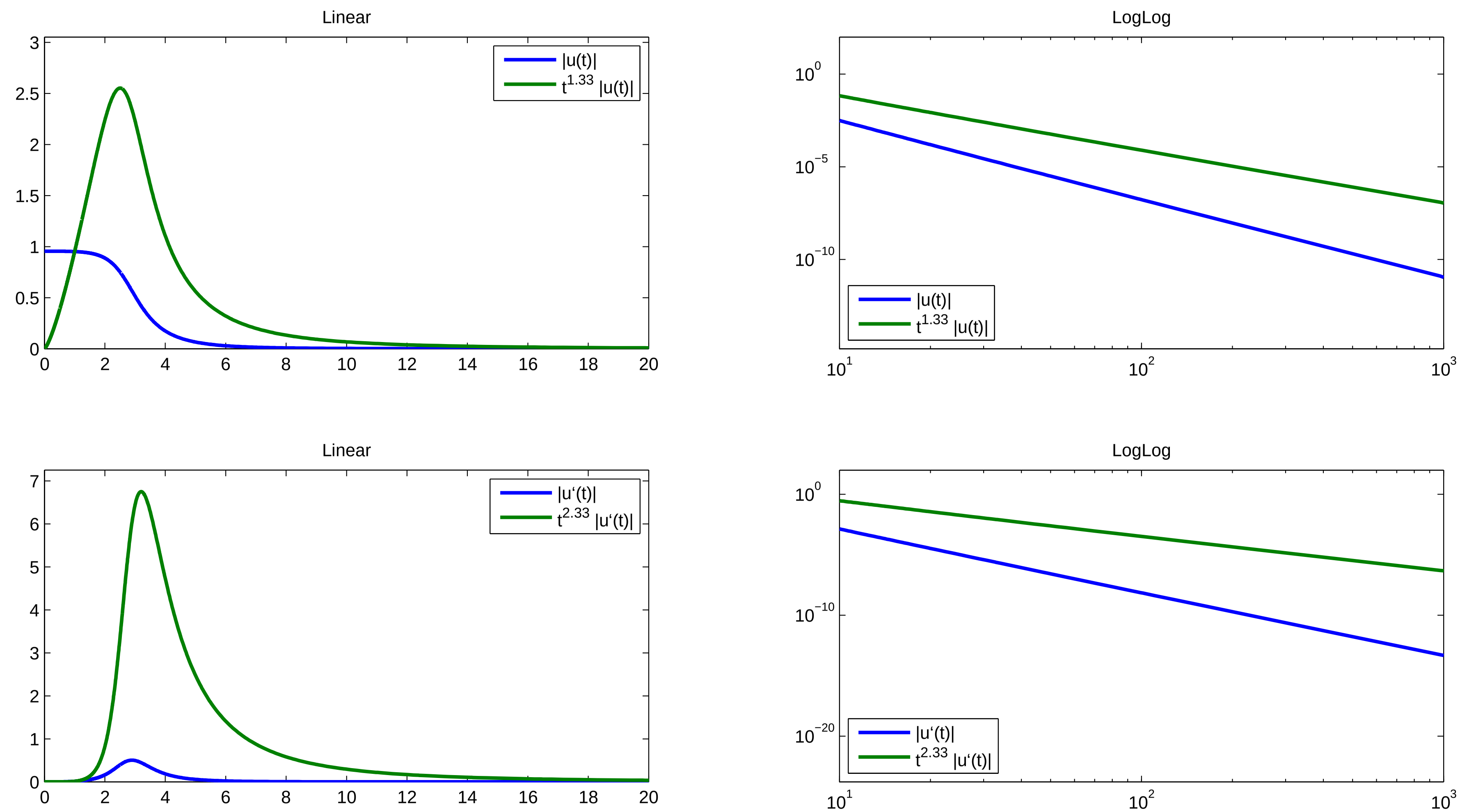

Figure A.28.: $\alpha = 5, \ \beta = 7, \quad u''(t) + \left[\frac{5}{t} + \frac{1}{(1+t)(\log(1+t))}\right] u'(t) + t^2 f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18d)$.

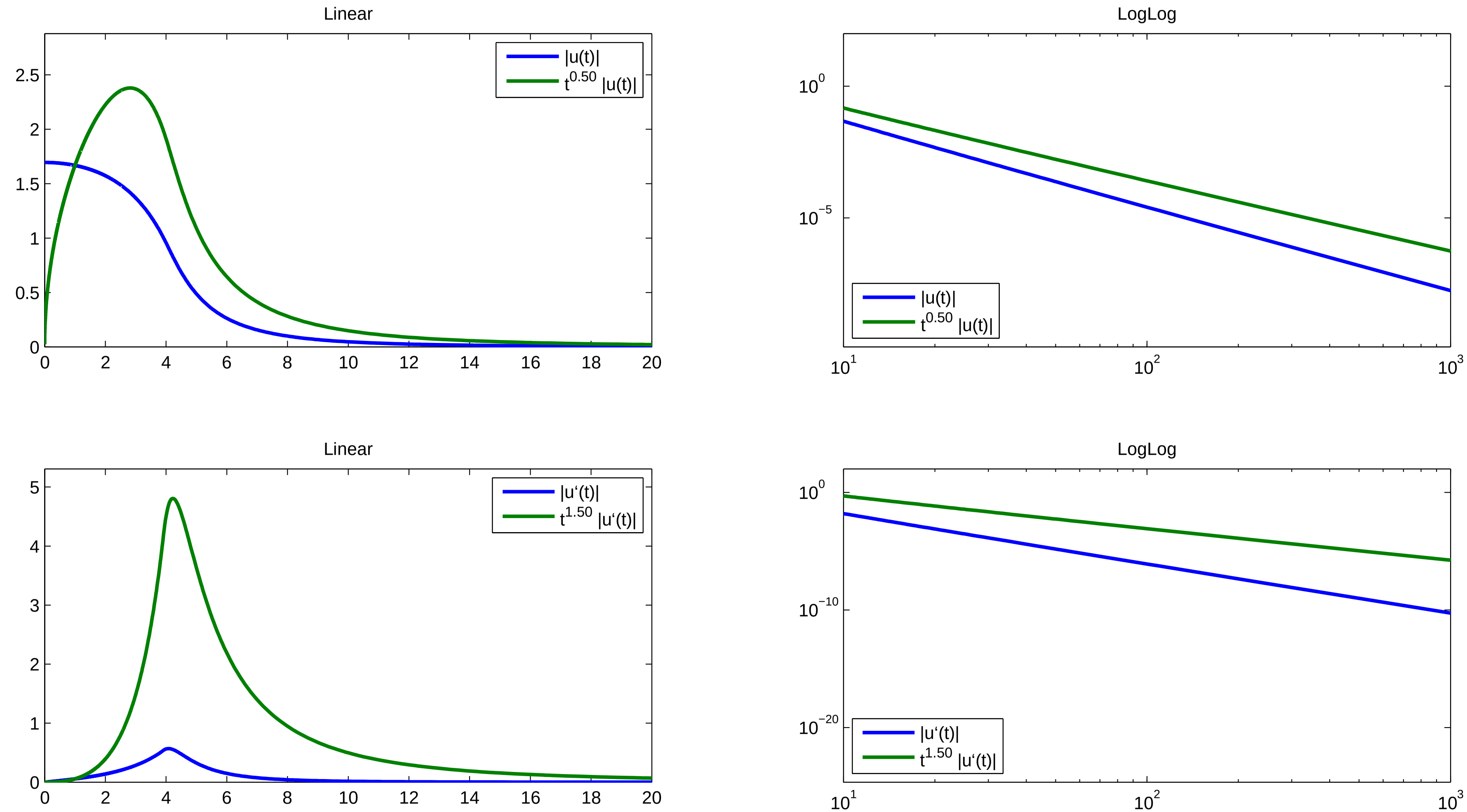

Figure A.29.: $\alpha = 4, \;\; \beta = 4, \quad u''(t) + \left[\frac{4}{t} + \frac{1}{(1+t)(\log(1+t))}\right] u'(t) + f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18b)$.

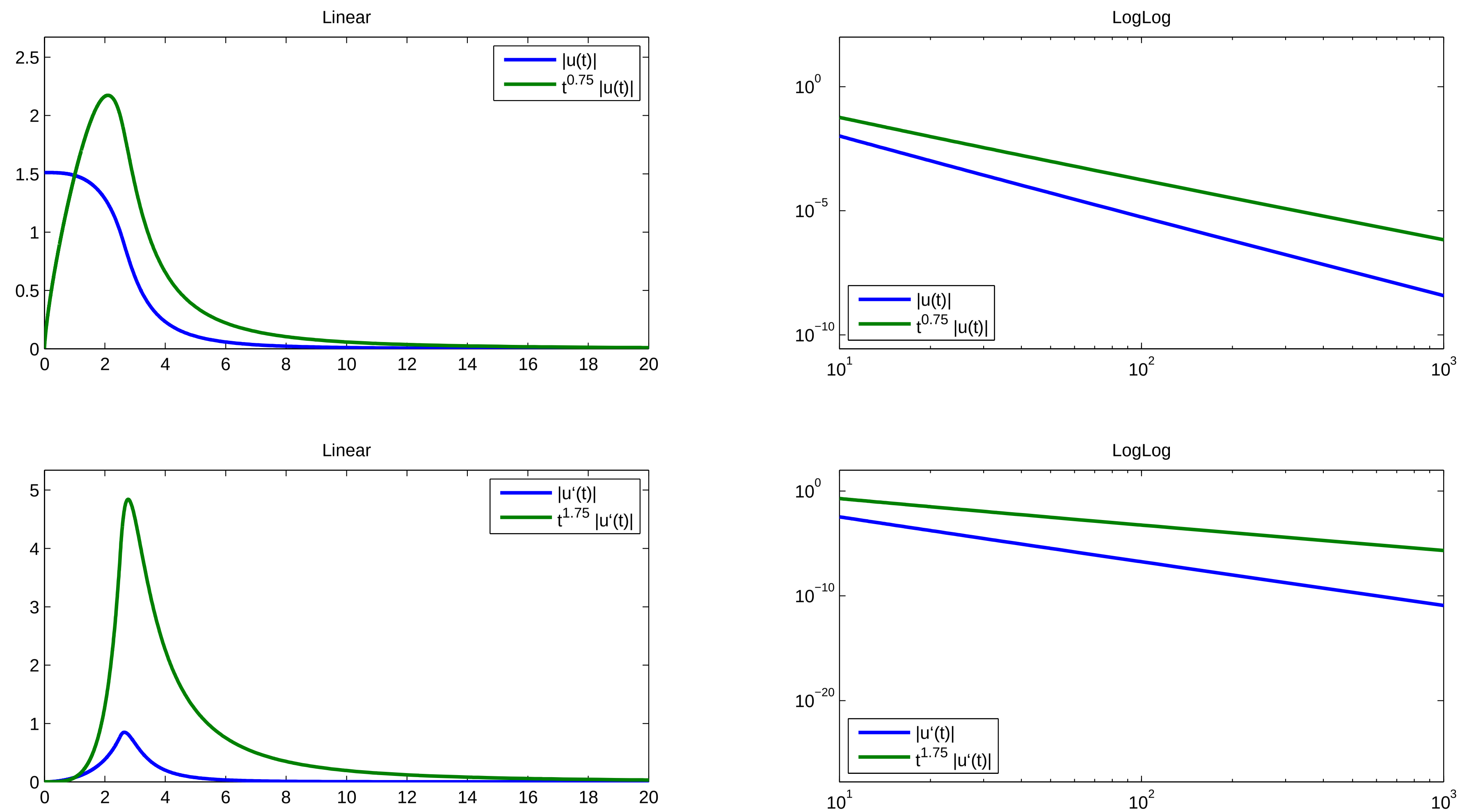

Figure A.30.: $\alpha = 4, \;\; \beta = 5, \quad u''(t) + \left[\frac{4}{t} + \frac{1}{(1+t)(\log(1+t)}\right] u'(t) + t f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18b)$.

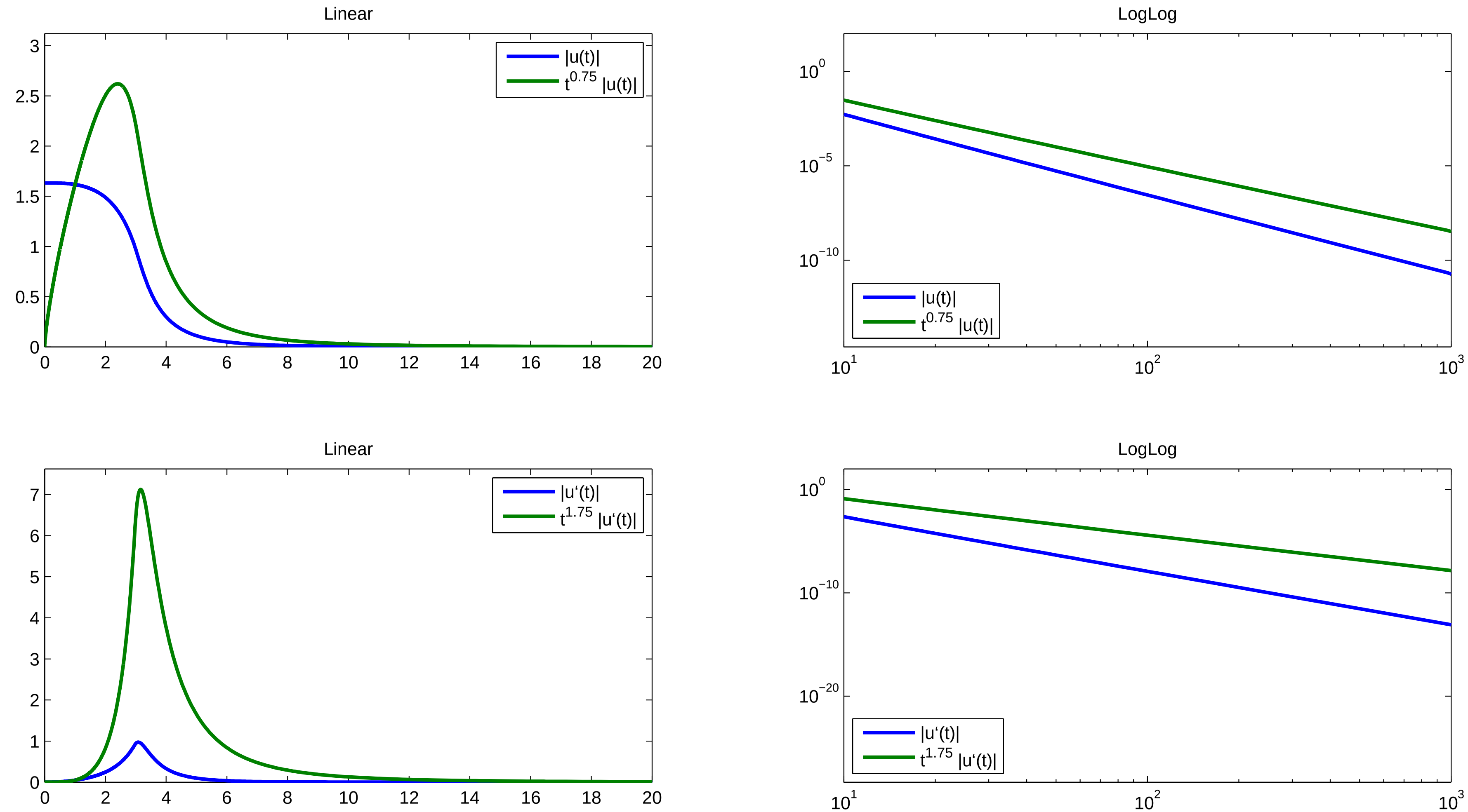

Figure A.31.: $\alpha = 5,\ \beta = 6,\quad u''(t) + \left[\frac{5}{t} + \frac{1}{(1+t)(\log(1+t))}\right] u'(t) + t f(u) = 0,\quad u'(0) = 0,\quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18b)$.

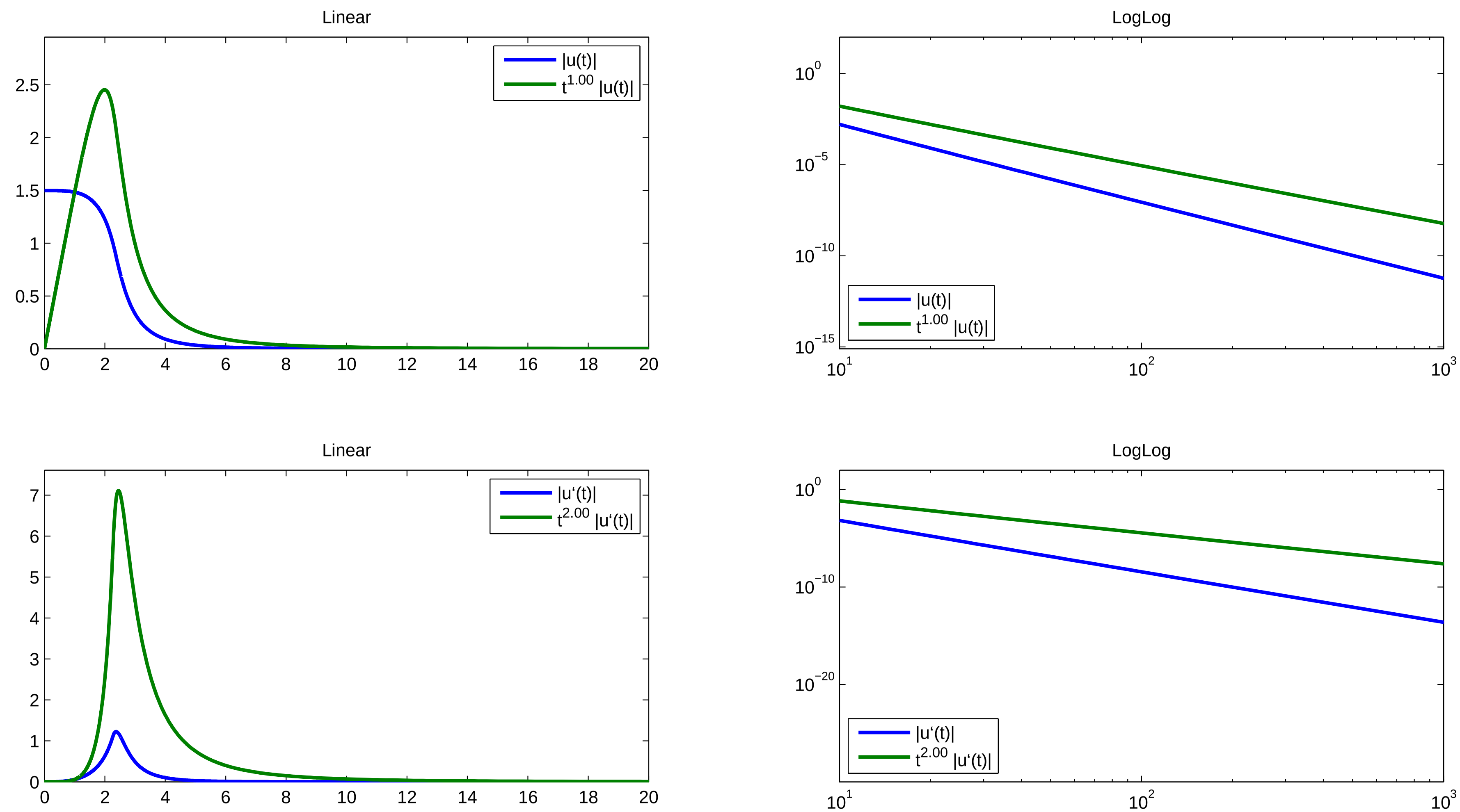

Figure A.32.: $\alpha = 5,\ \ \beta = 7, \quad u''(t) + \left[\frac{5}{t} + \frac{1}{(1+t)(\log(1+t))}\right] u'(t) + t^2 f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18b)$.

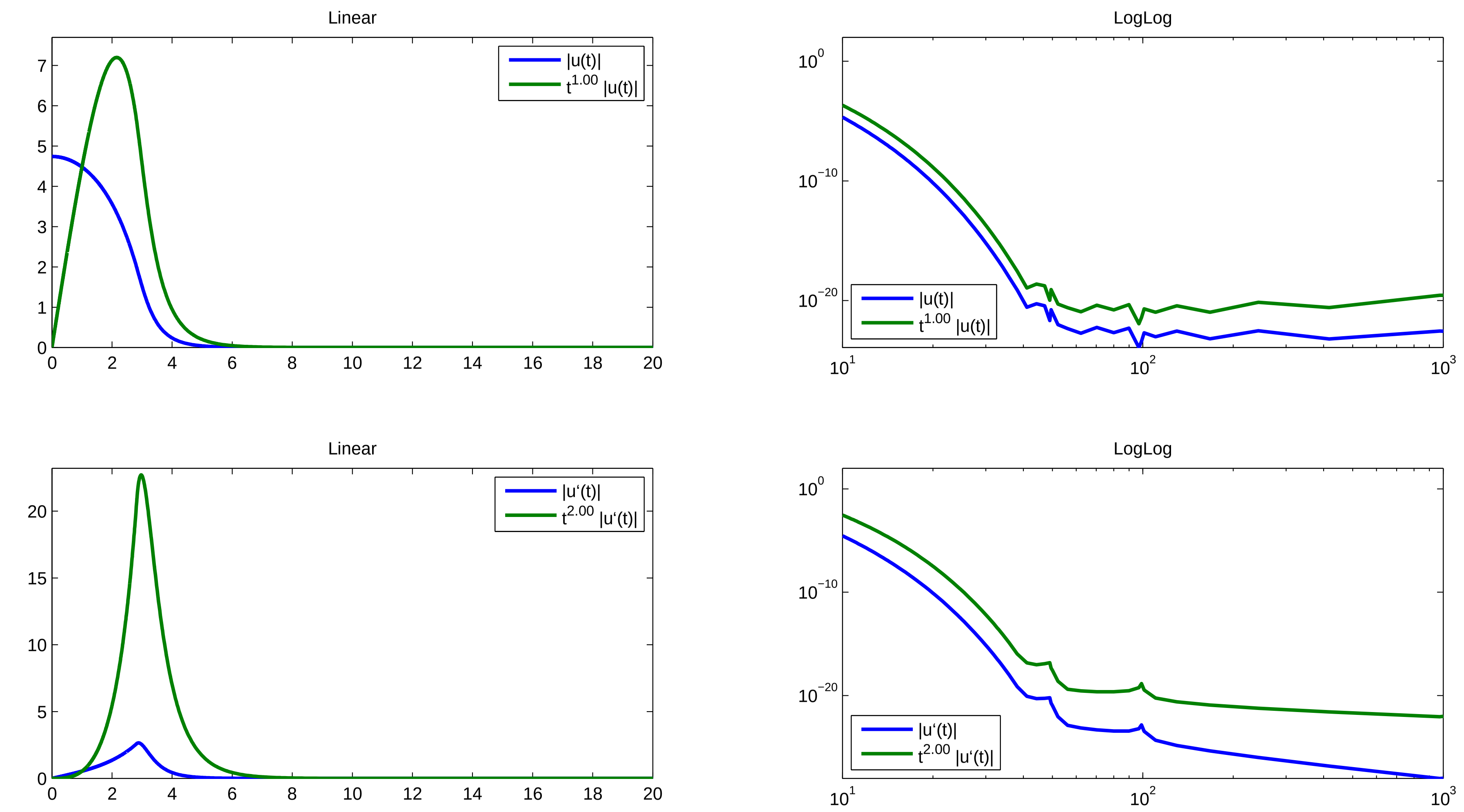

Figure A.33.: $\alpha = 4, \ \beta = 4, \quad u''(t) + \frac{4}{t}u'(t) + (1 + \exp(-t))f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by (1.18c).

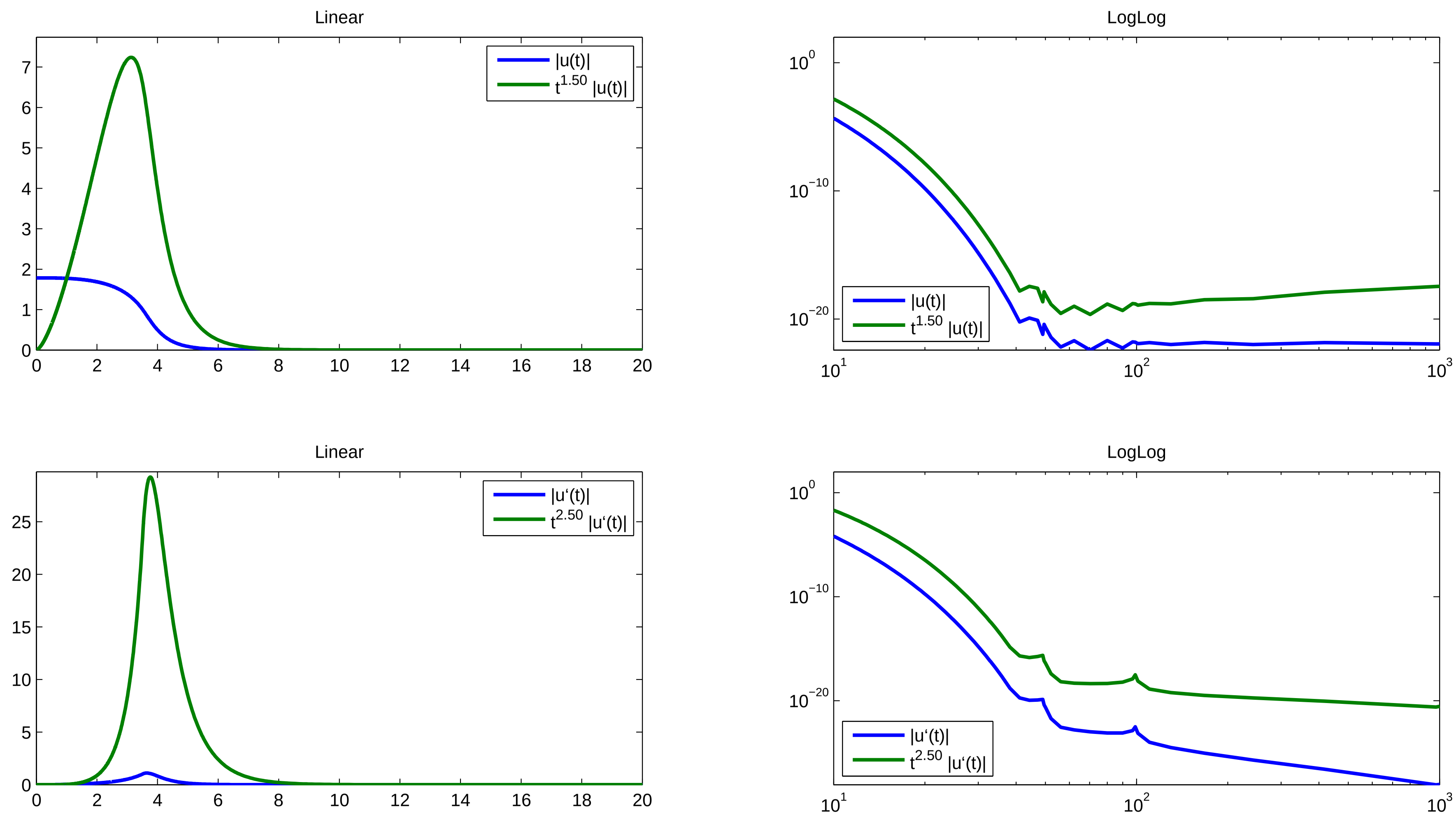

Figure A.34.: $\alpha = 4,\ \ \beta = 5, \quad u''(t) + \frac{4}{t}u'(t) + t(1+\exp(-t))f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by (1.18c).

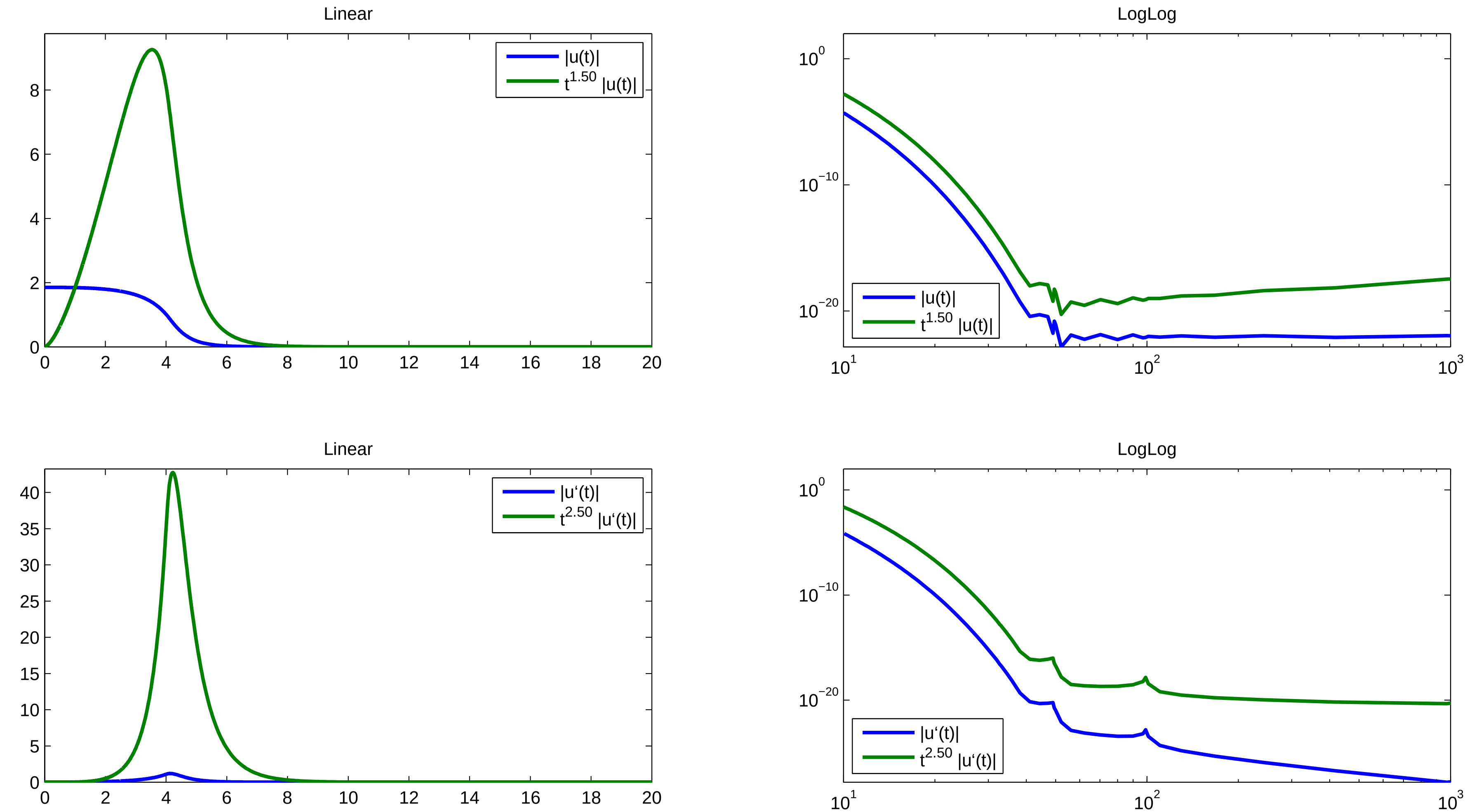

Figure A.35.: $\alpha = 5,\ \ \beta = 6,\quad u''(t) + \frac{5}{t}u'(t) + t(1 + \exp(-t))f(u) = 0,\quad u'(0) = 0,\quad u(\infty) = 0,$ where $f(u)$ is given by (1.18c).

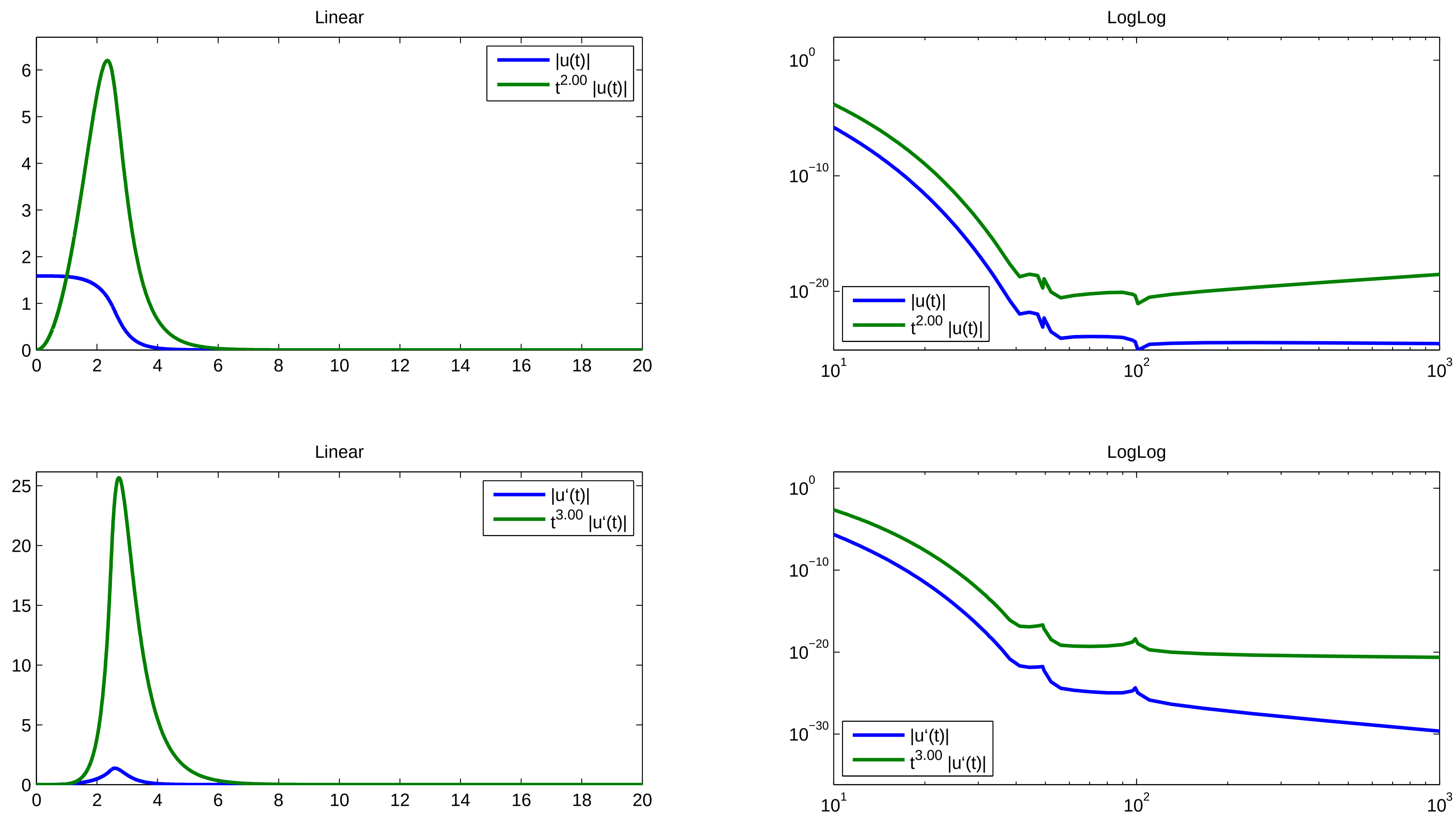

Figure A.36.: $\alpha = 5, \;\; \beta = 7, \quad u''(t) + \frac{5}{t}u'(t) + t^2(1+\exp(-t))f(u) = 0,, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18c)$.

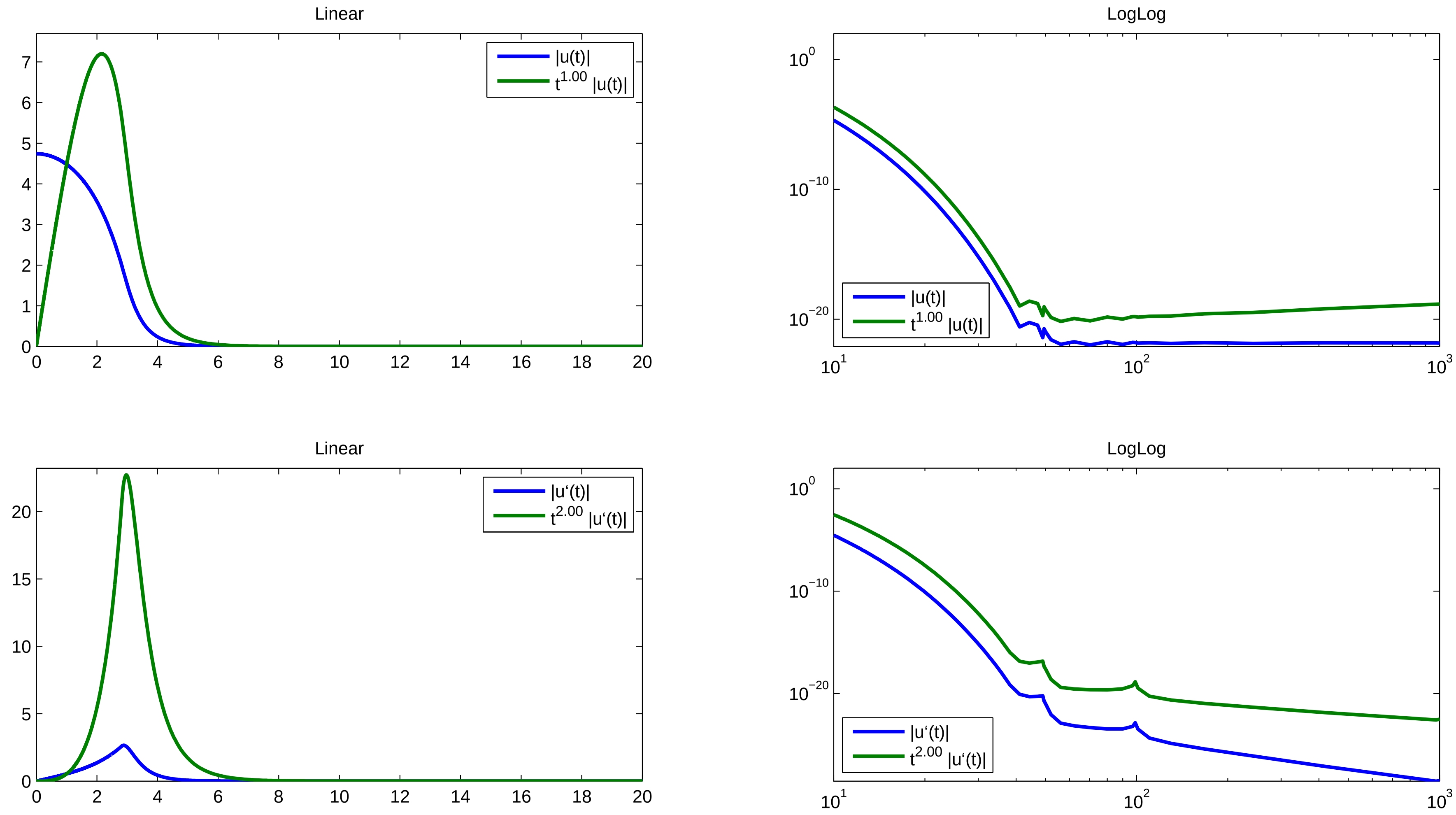

Figure A.37.: $\alpha = 4,\ \ \beta = 4, \quad u''(t) + \frac{4}{t}u'(t) + (1 + \exp(-t))f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18a)$.

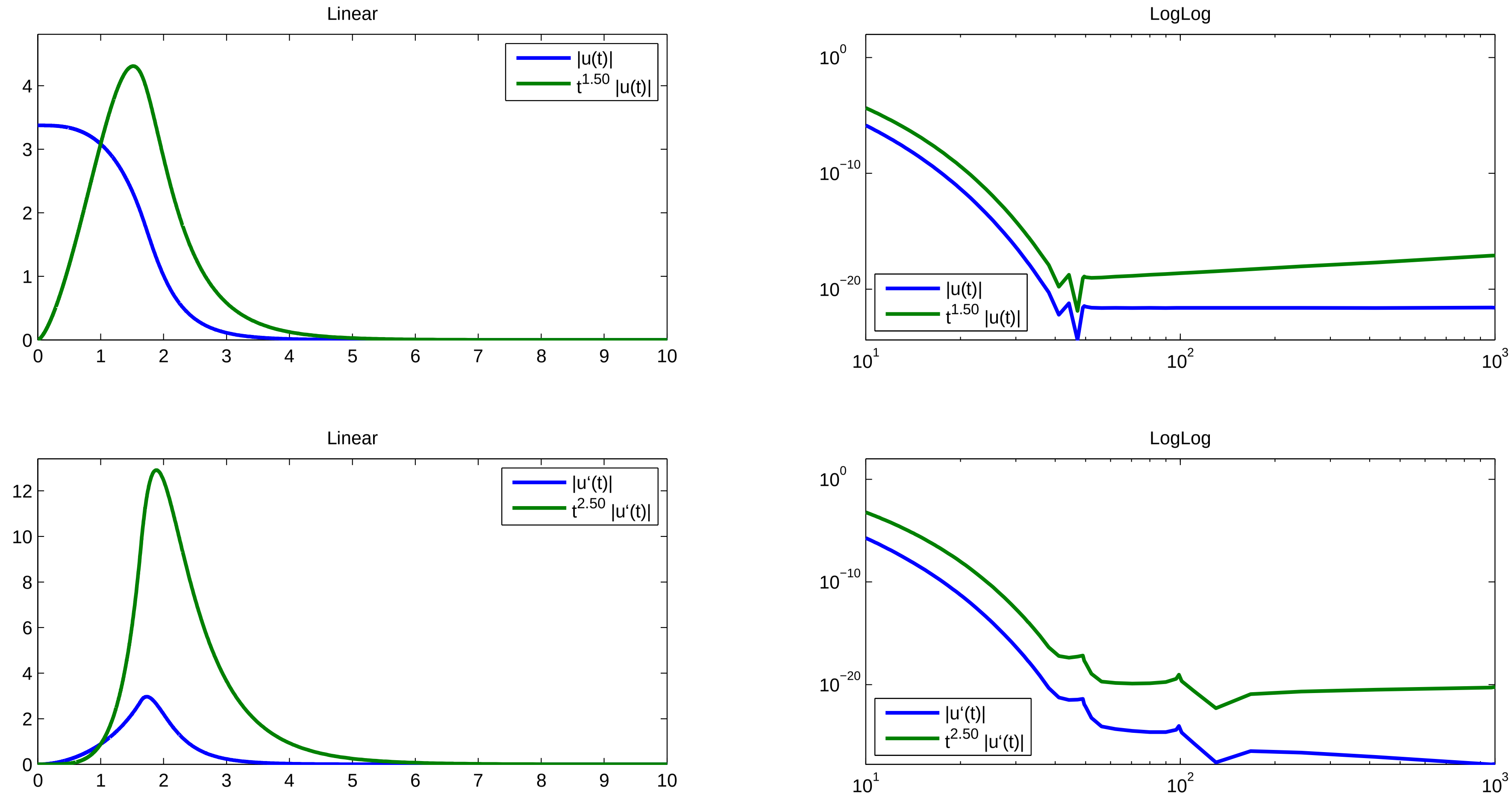

Figure A.38.: $\alpha = 4$, $\beta = 5$, $u''(t) + \frac{4}{t}u'(t) + t(1+\exp(-t))f(u) = 0$, $u'(0) = 0$, $u(\infty) = 0$, where $f(u)$ is given by (1.18a).

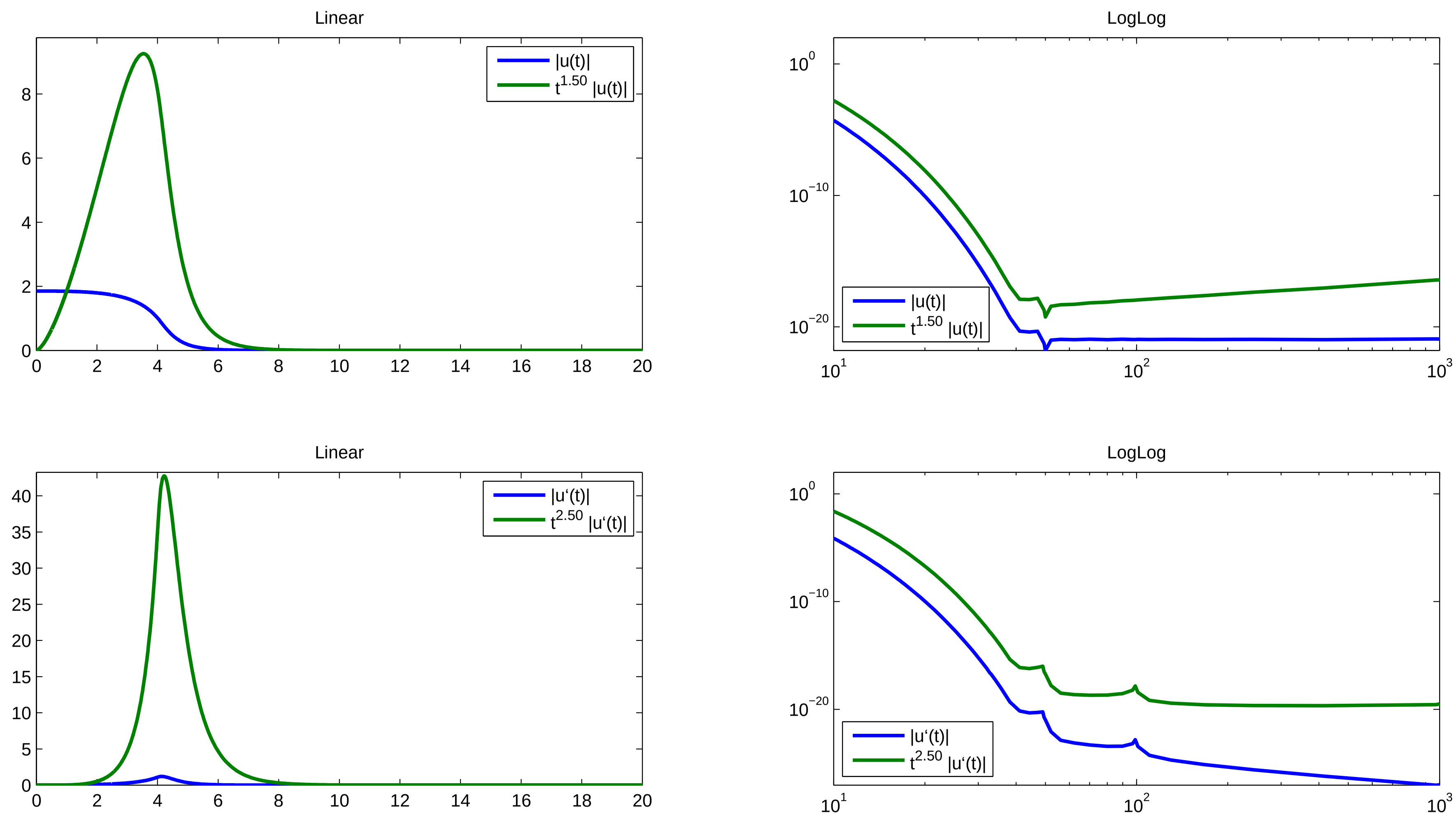

Figure A.39.: $\alpha = 5, \ \beta = 6, \quad u''(t) + \frac{5}{t}u'(t) + t(1 + \exp(-t))f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18a)$.

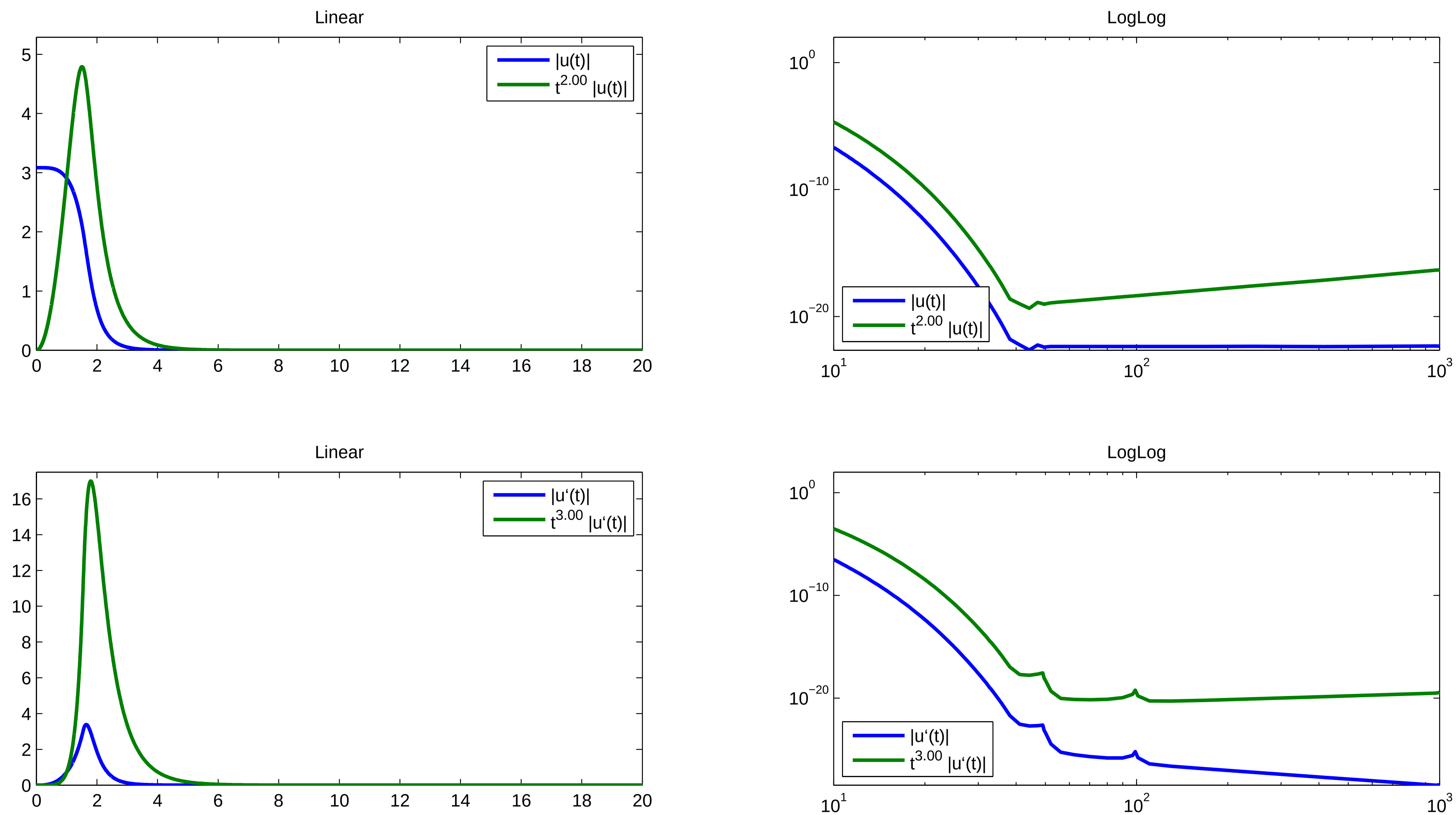

Figure A.40.: $\alpha = 5,\ \ \beta = 7, \quad u''(t) + \frac{5}{t}u'(t) + t^2(1+\exp(-t))f(u) = 0,, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18a)$.

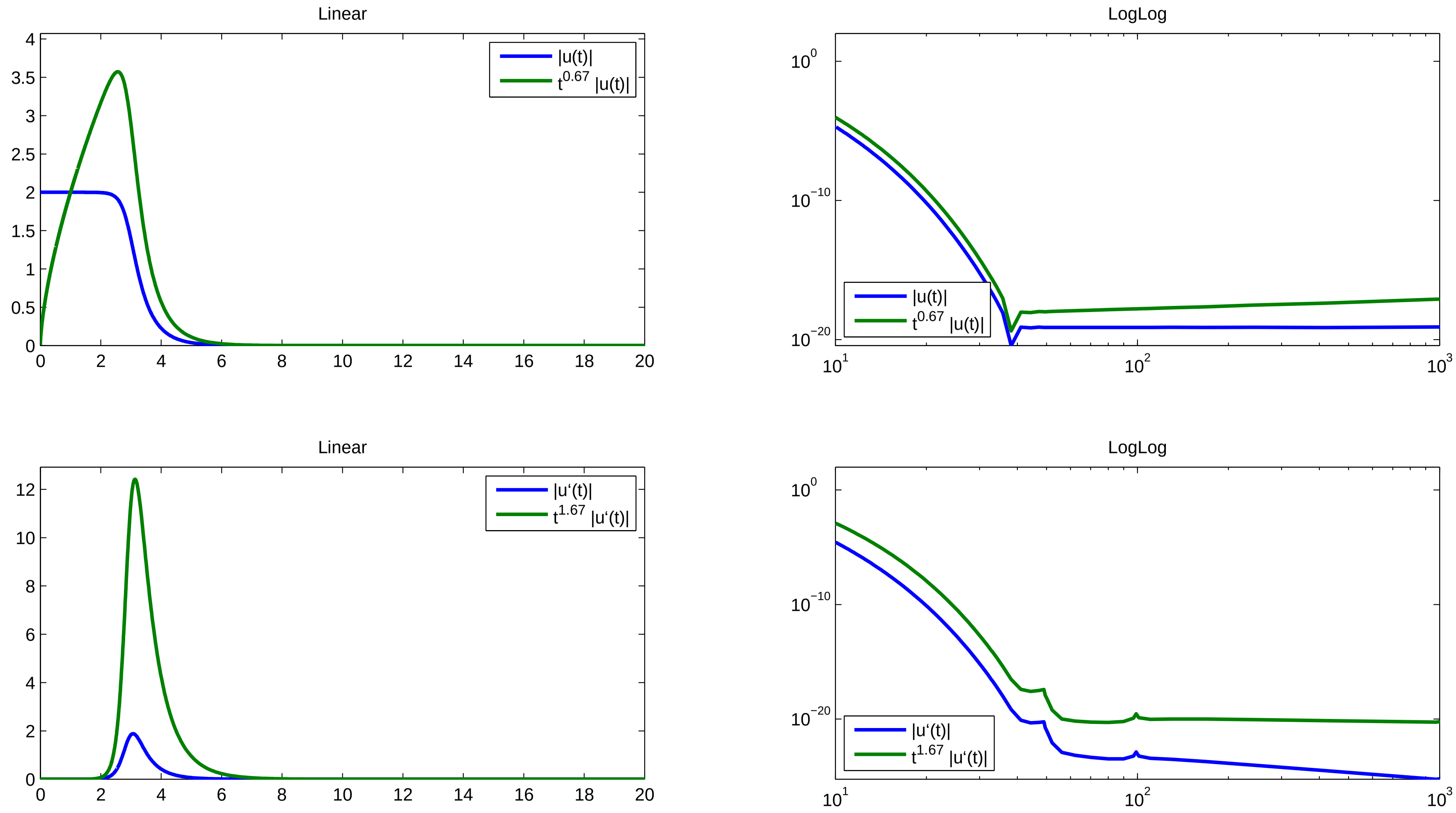

Figure A.41.: $\alpha = 4,\ \ \beta = 4, \quad u''(t) + \frac{4}{t}u'(t) + (1 + \exp(-t))f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18d)$.

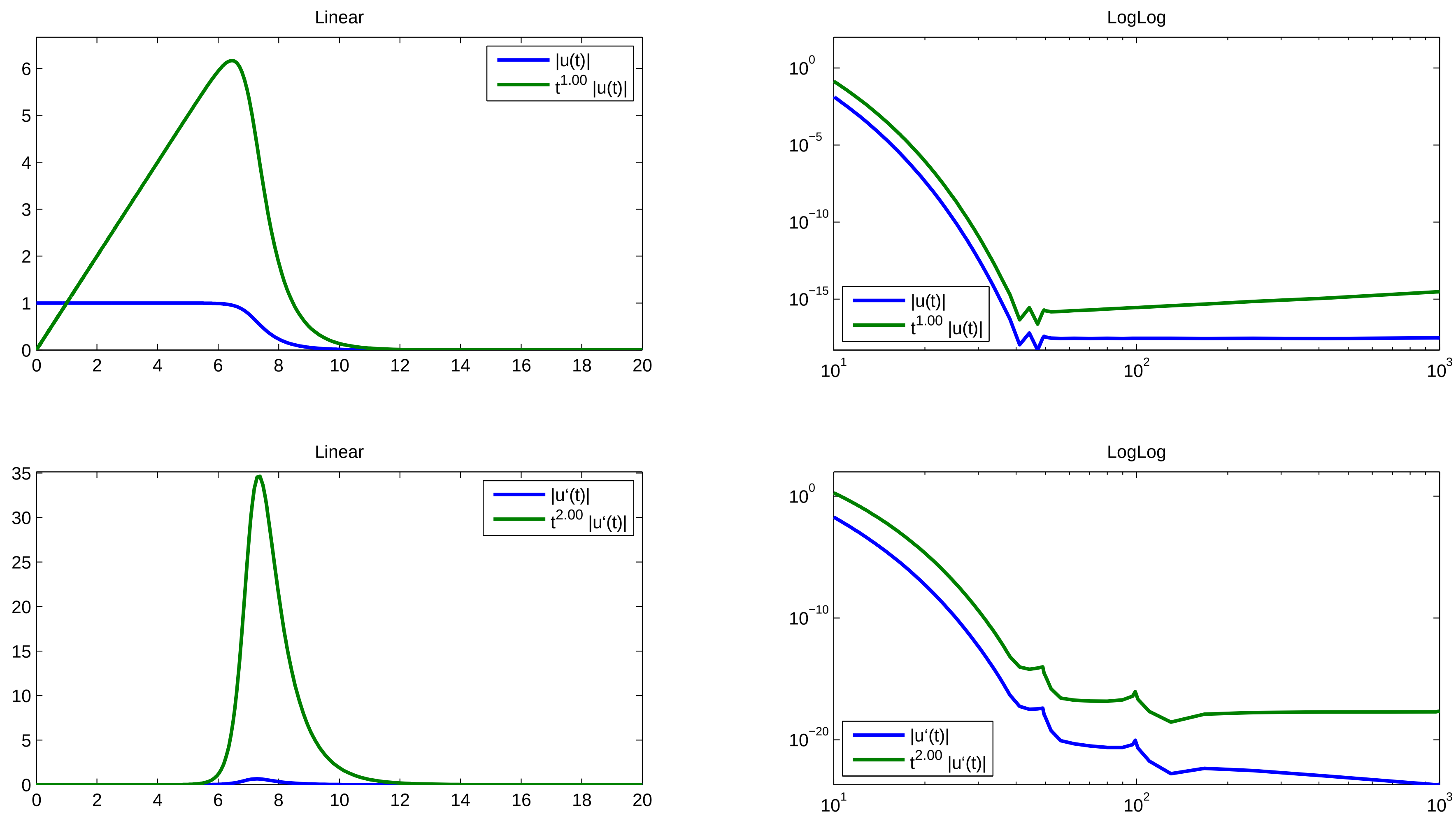

Figure A.42.: $\alpha = 4$, $\beta = 5$, $u''(t) + \frac{4}{t}u'(t) + t(1 + \exp(-t))f(u) = 0$, $u'(0) = 0$, $u(\infty) = 0$, where $f(u)$ is given by (1.18d).

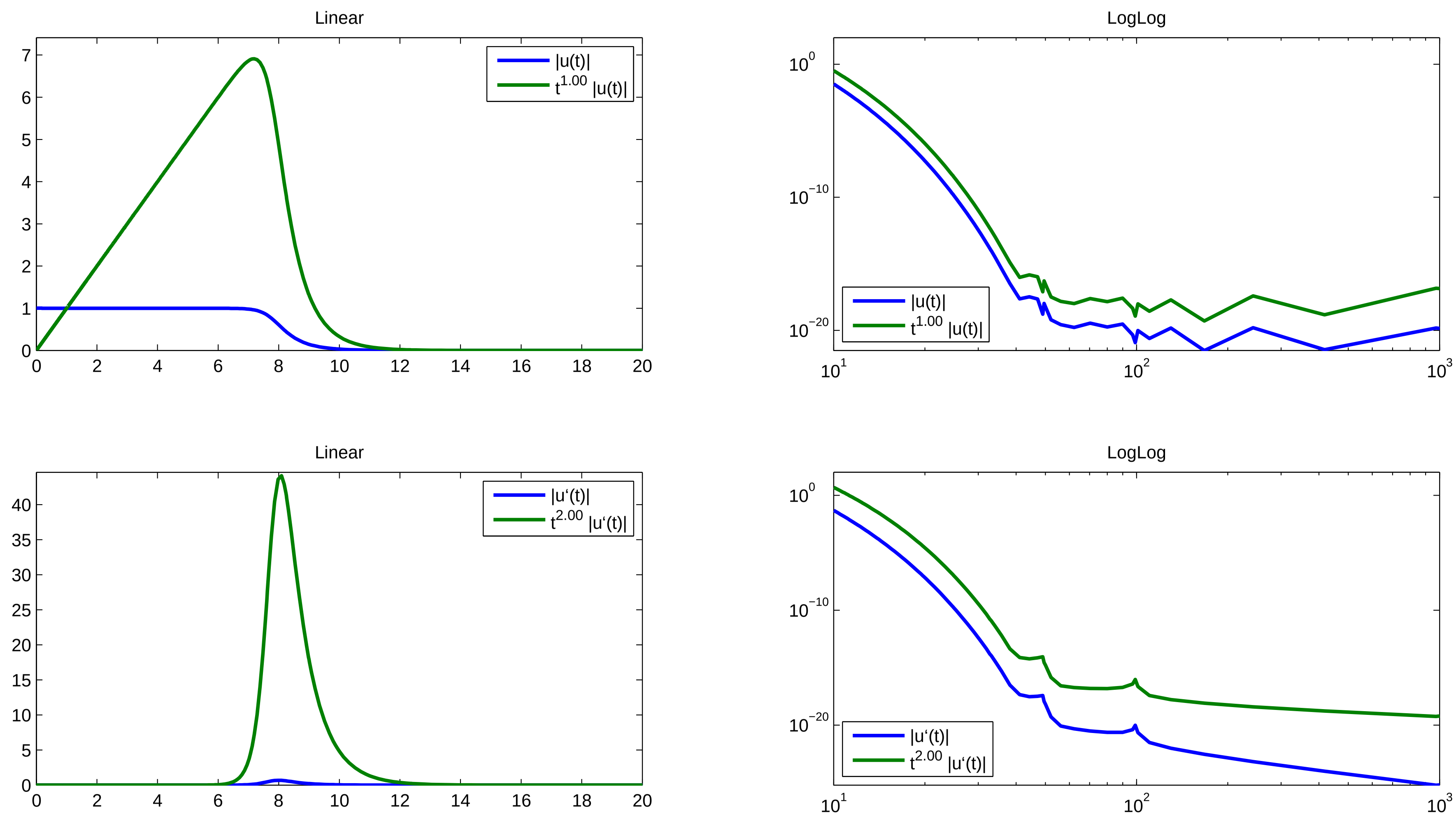

Figure A.43.: $\alpha = 5,\ \ \beta = 6, \quad u''(t) + \frac{5}{t}u'(t) + t(1+\exp(-t))f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18d)$.

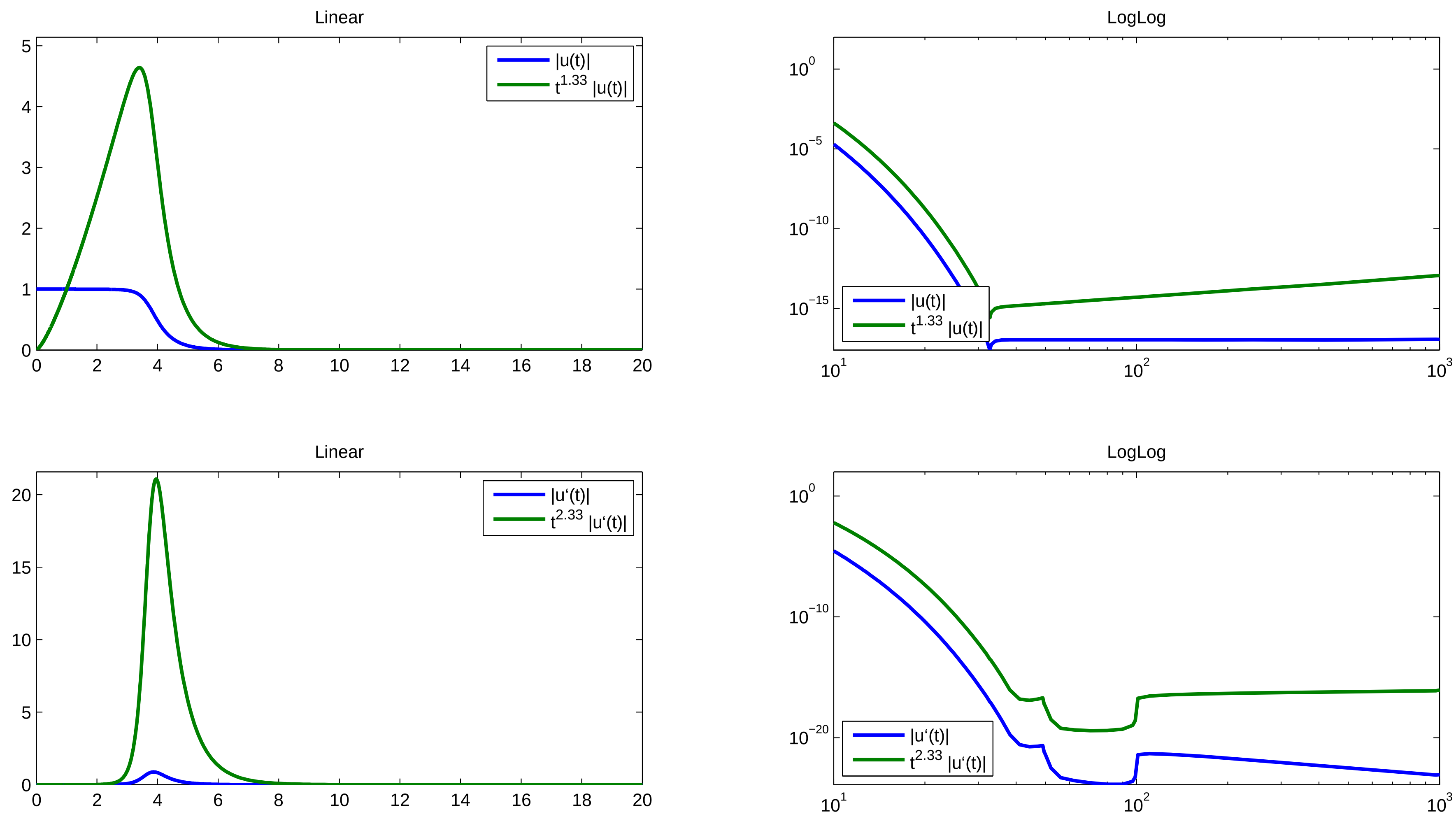

Figure A.44.: $\alpha = 5, \ \beta = 7, \quad u''(t) + \frac{5}{t}u'(t) + t^2(1+\exp(-t))f(u) = 0,, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18d)$.

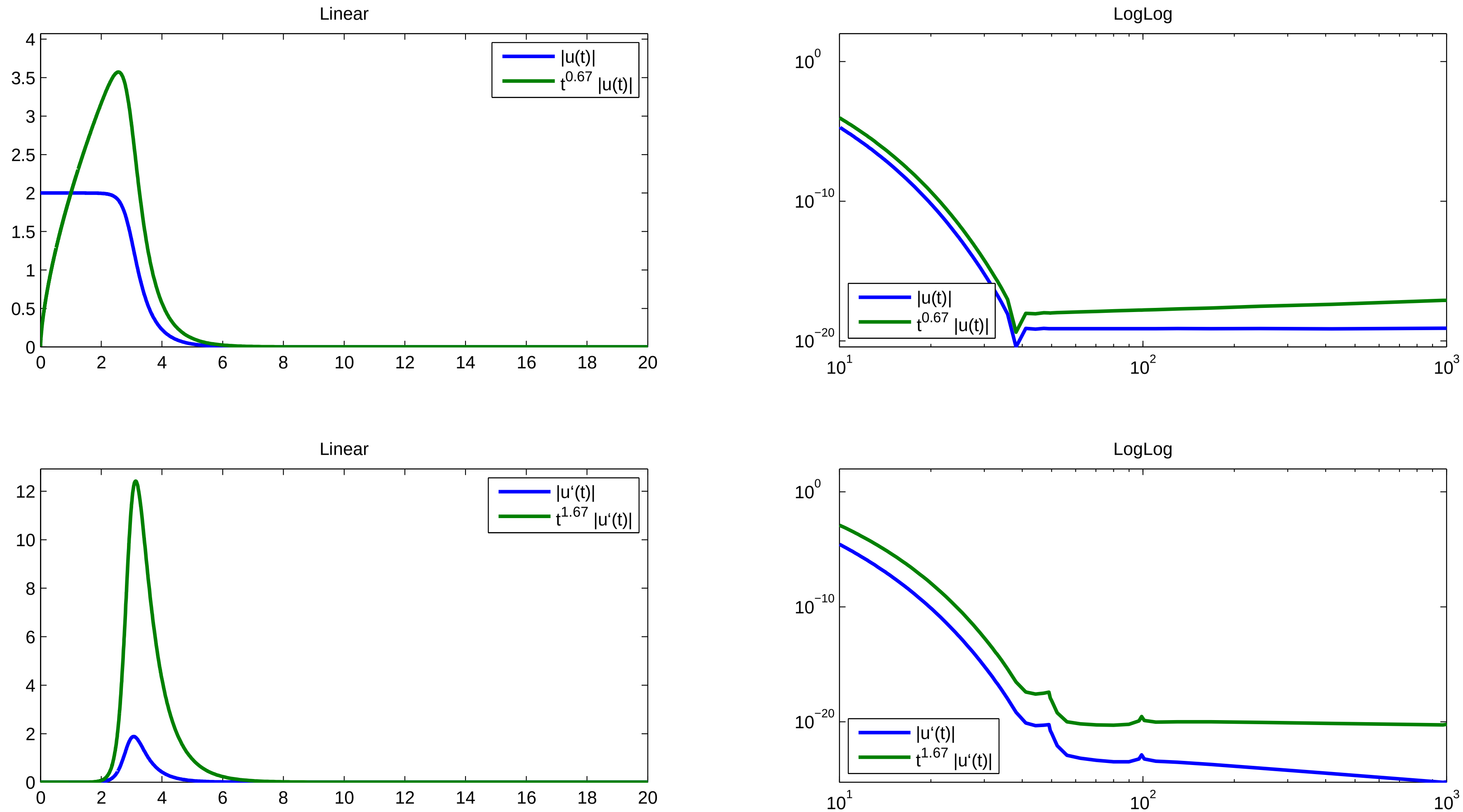

Figure A.45.: $\alpha = 4,\ \ \beta = 4, \quad u''(t) + \frac{4}{t}u'(t) + (1+\exp(-t))f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18b)$.

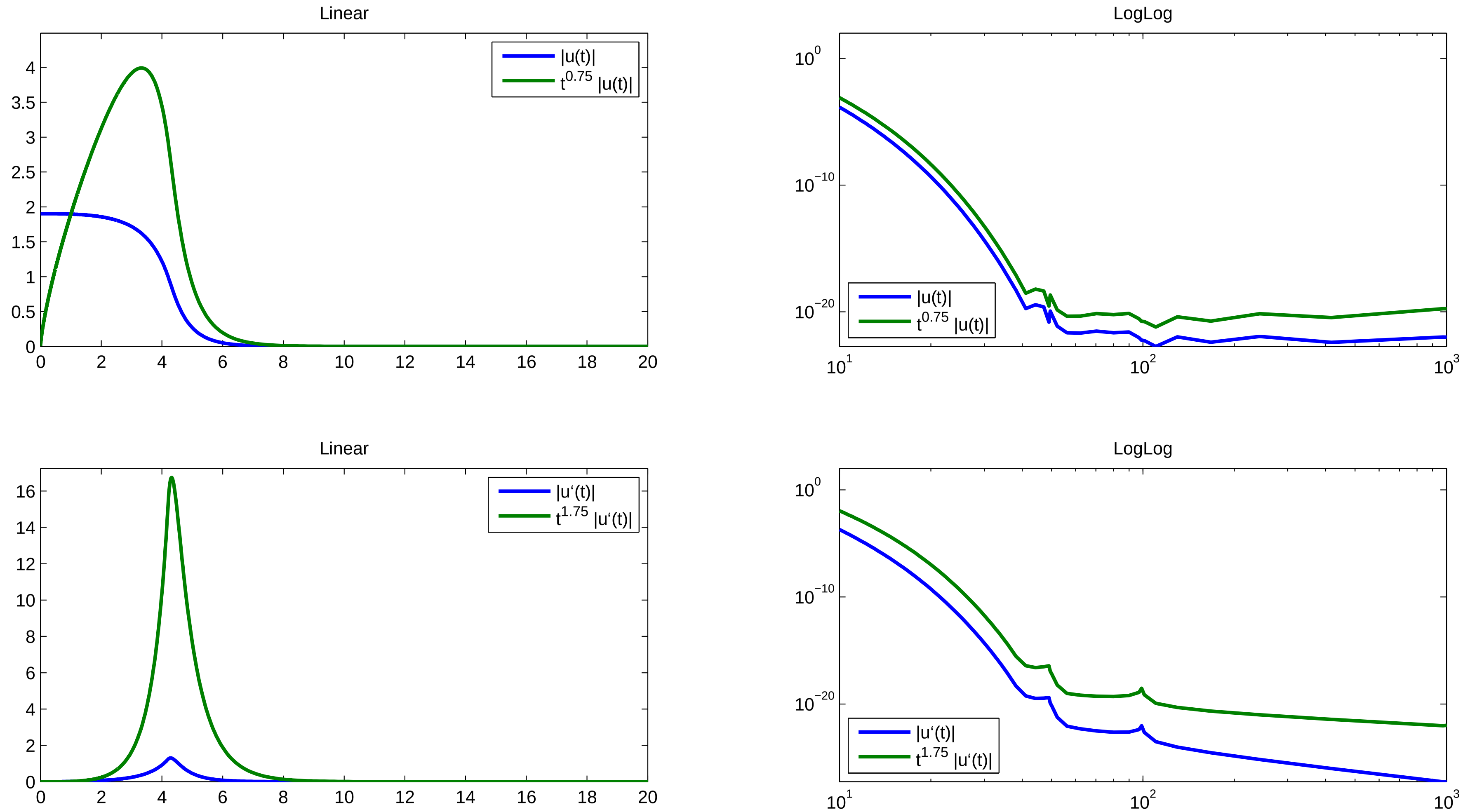

Figure A.46.: $\alpha = 4,\ \ \beta = 5, \quad u''(t) + \frac{4}{t}u'(t) + t(1 + \exp(-t))f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18b)$.

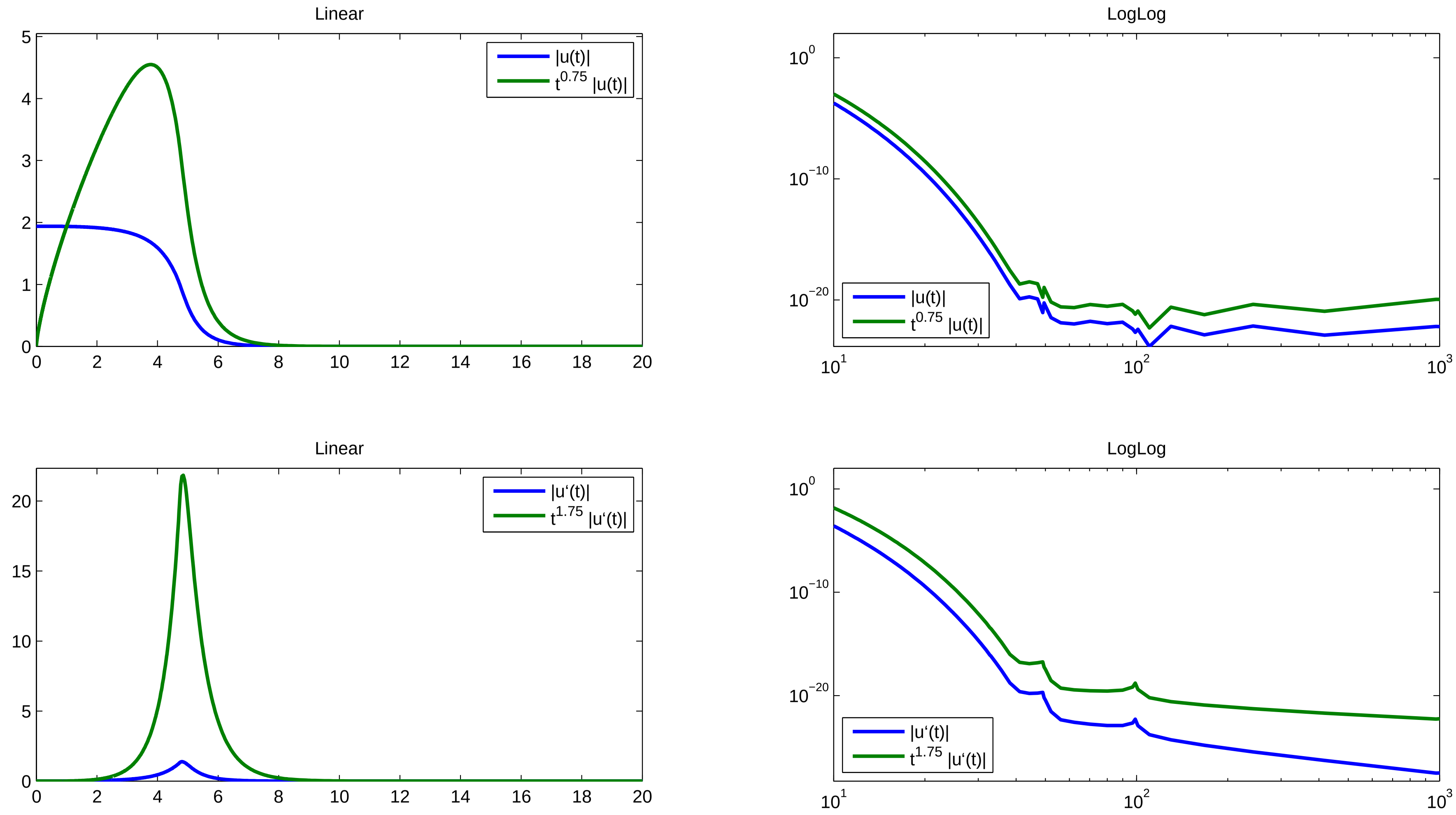

Figure A.47.: $\alpha = 5$, $\beta = 6$, $u''(t) + \frac{5}{t}u'(t) + t(1 + \exp(-t))f(u) = 0$, $u'(0) = 0$, $u(\infty) = 0$, where $f(u)$ is given by (1.18b).

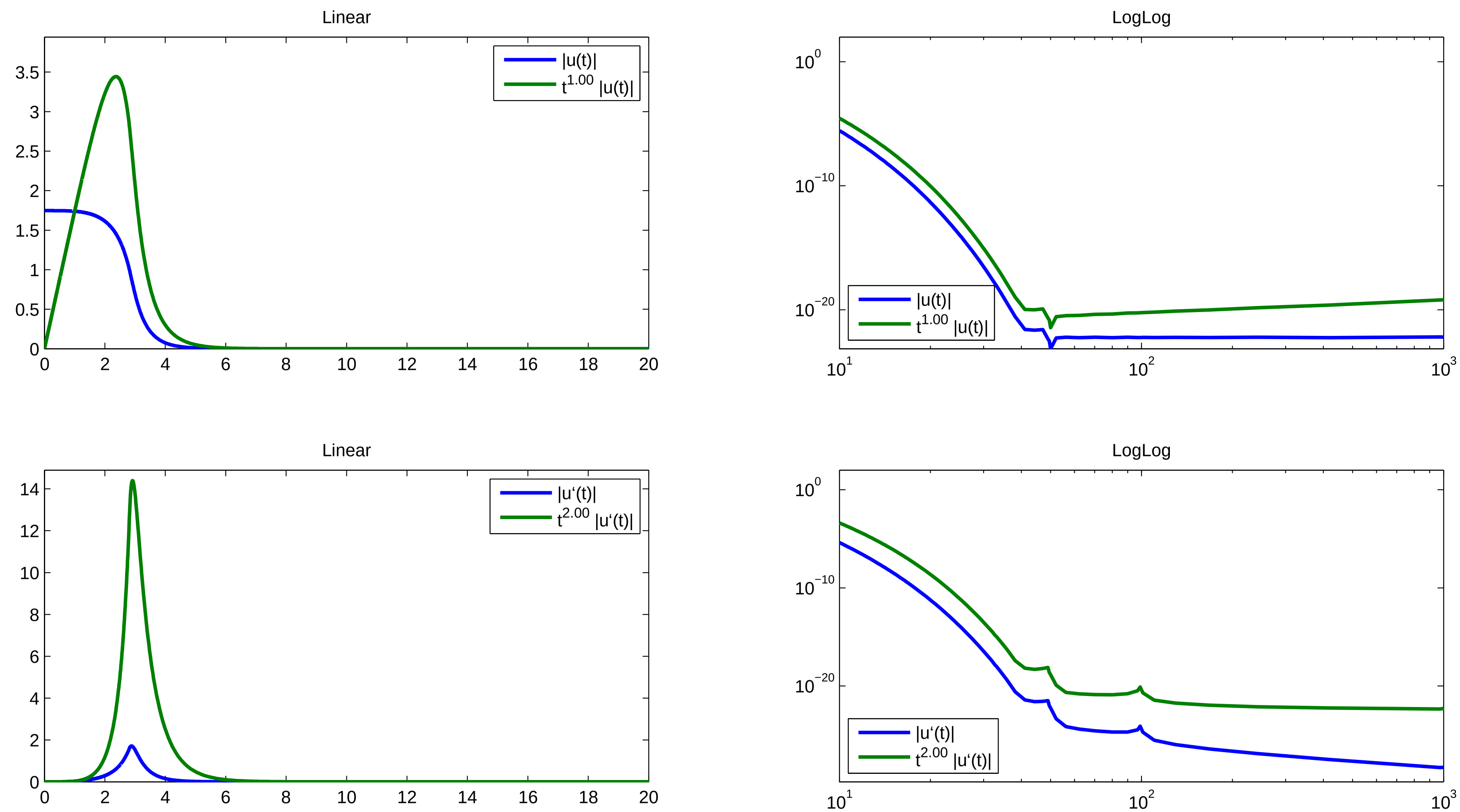

Figure A.48.: $\alpha = 5,\ \ \beta = 7, \quad u''(t) + \frac{5}{t}u'(t) + t^2(1+\exp(-t))f(u) = 0,, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18b)$.

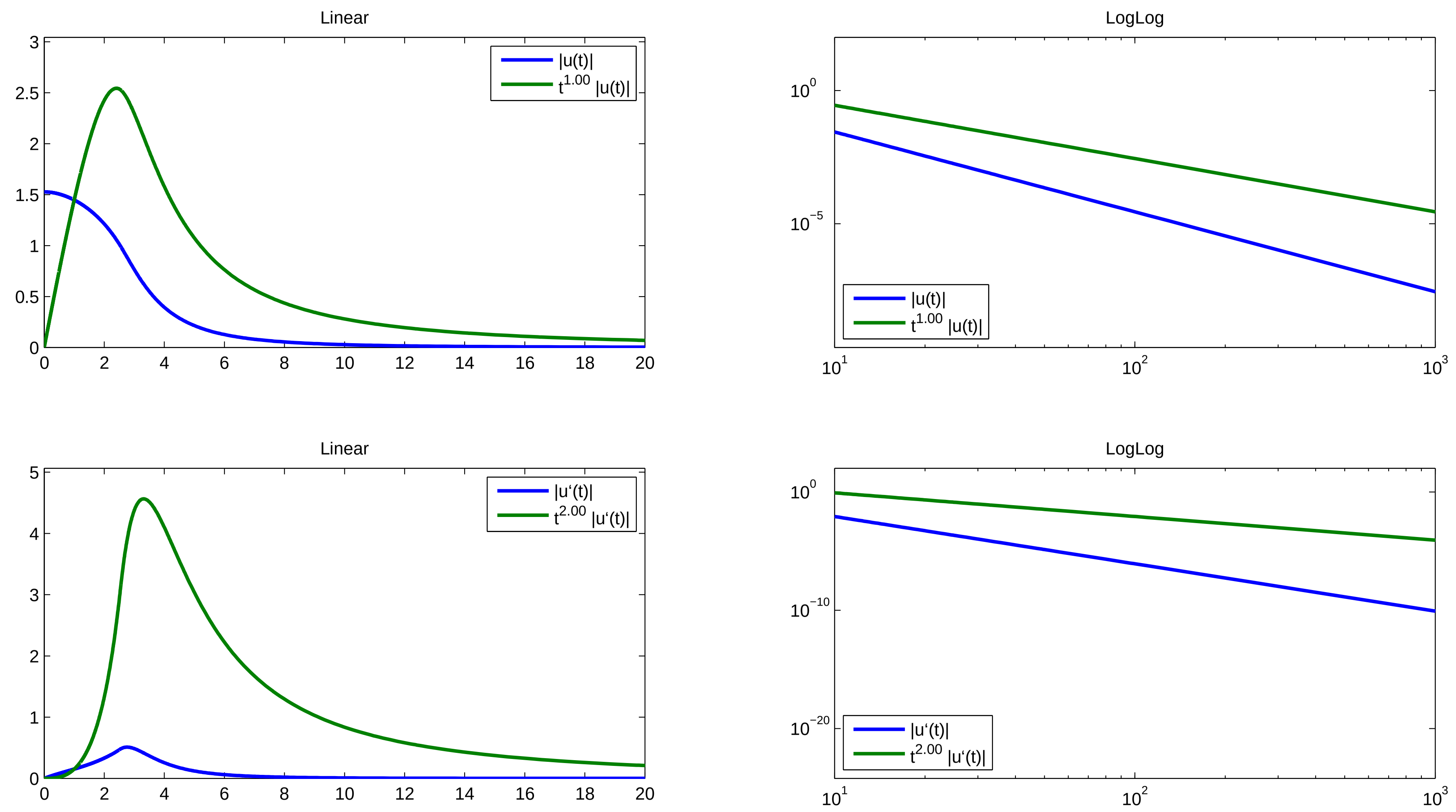

Figure A.49.: $\alpha = 4,\ \beta = 4, \quad u''(t) + \left[\frac{4}{t} - \frac{\exp(-t)}{1+\exp(-t)}\right] u'(t) + f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18c)$.

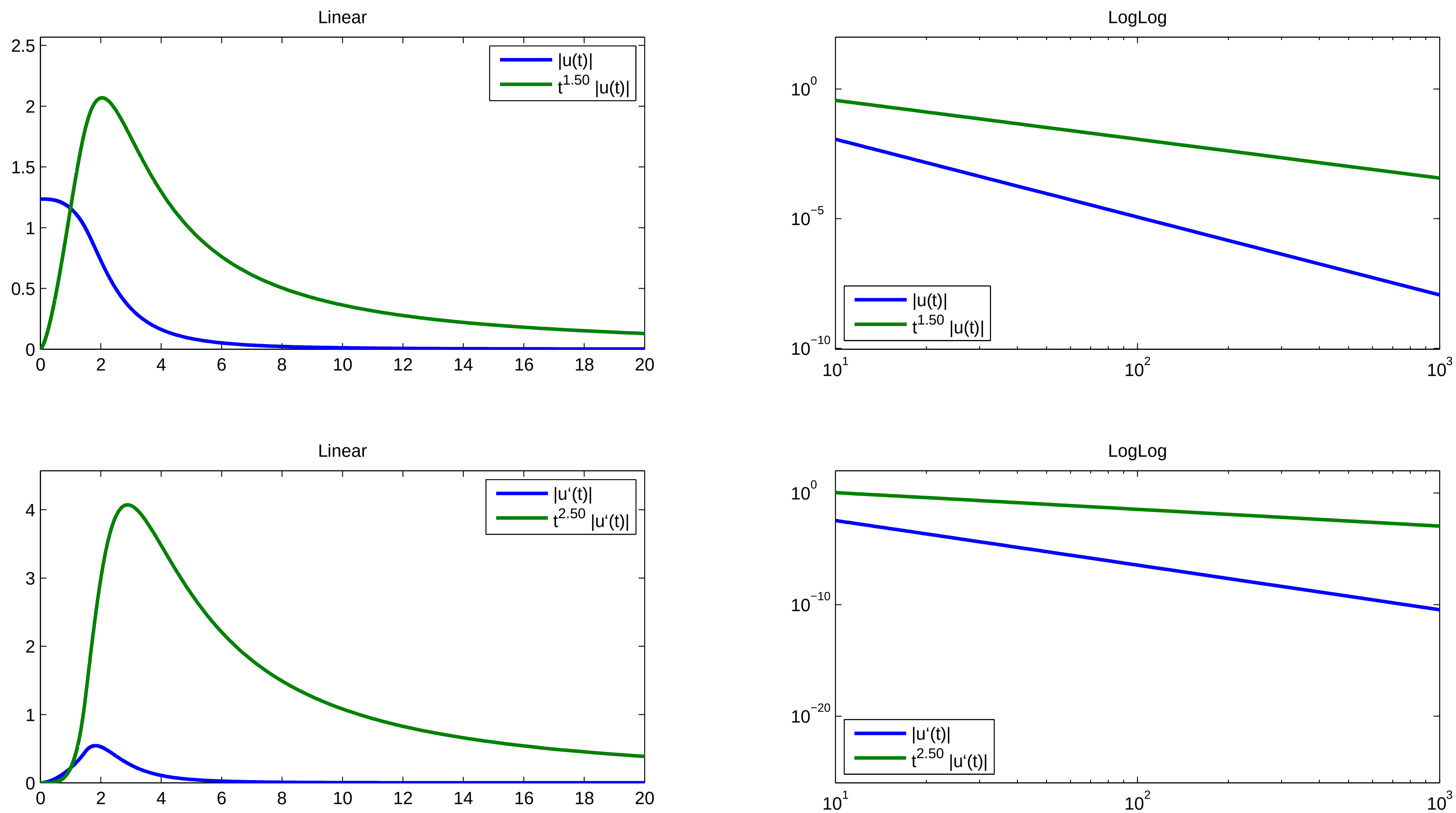

Figure A.50.: $\alpha = 4, \ \beta = 5, \quad u''(t) + \left[\frac{4}{t} - \frac{\exp(-t)}{1+\exp(-t)}\right] u'(t) + t f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by (1.18c).

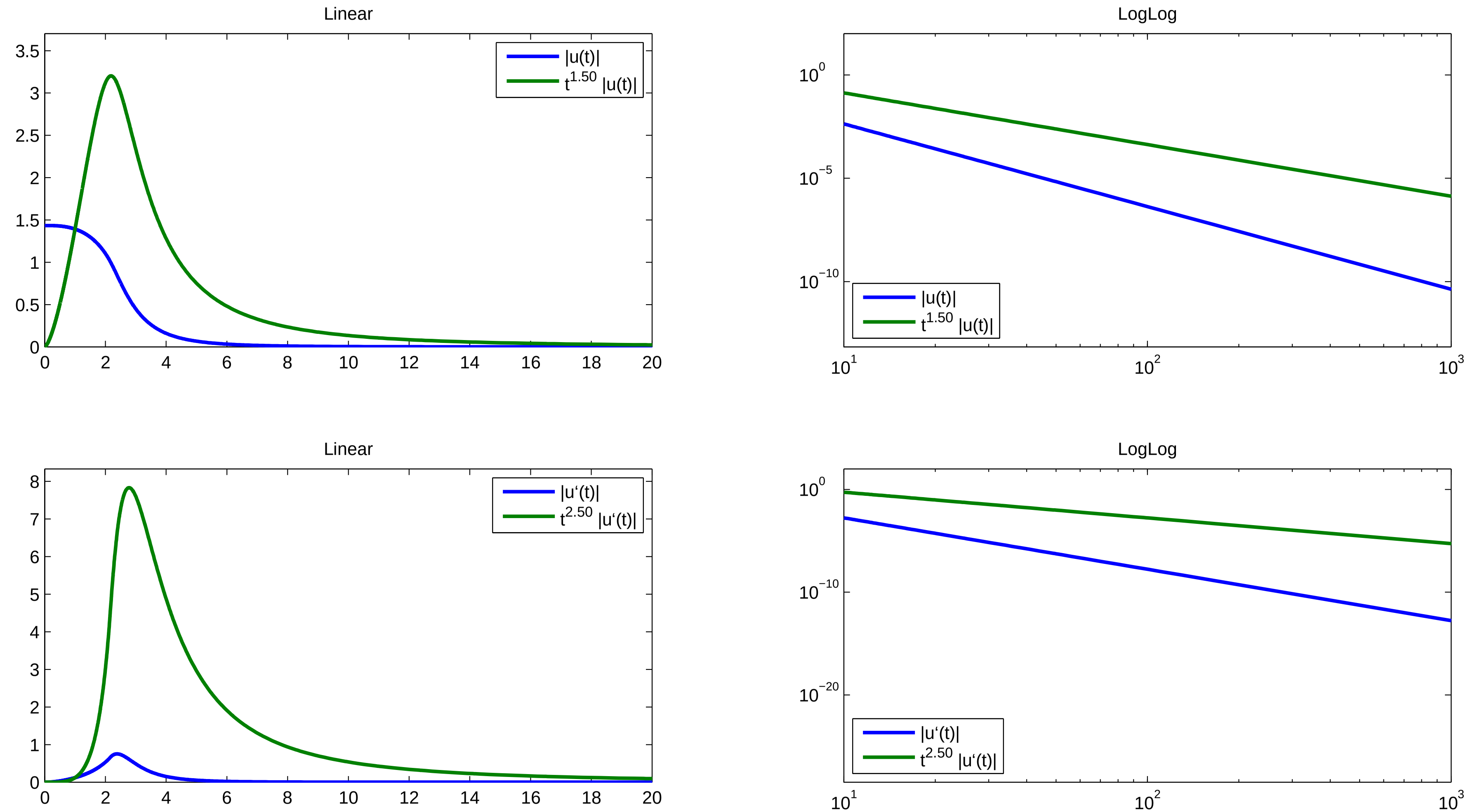

Figure A.51.: $\alpha = 5, \ \beta = 6, \quad u''(t) + \left[\frac{5}{t} - \frac{\exp(-t)}{1+\exp(-t)}\right] u'(t) + t f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18c)$.

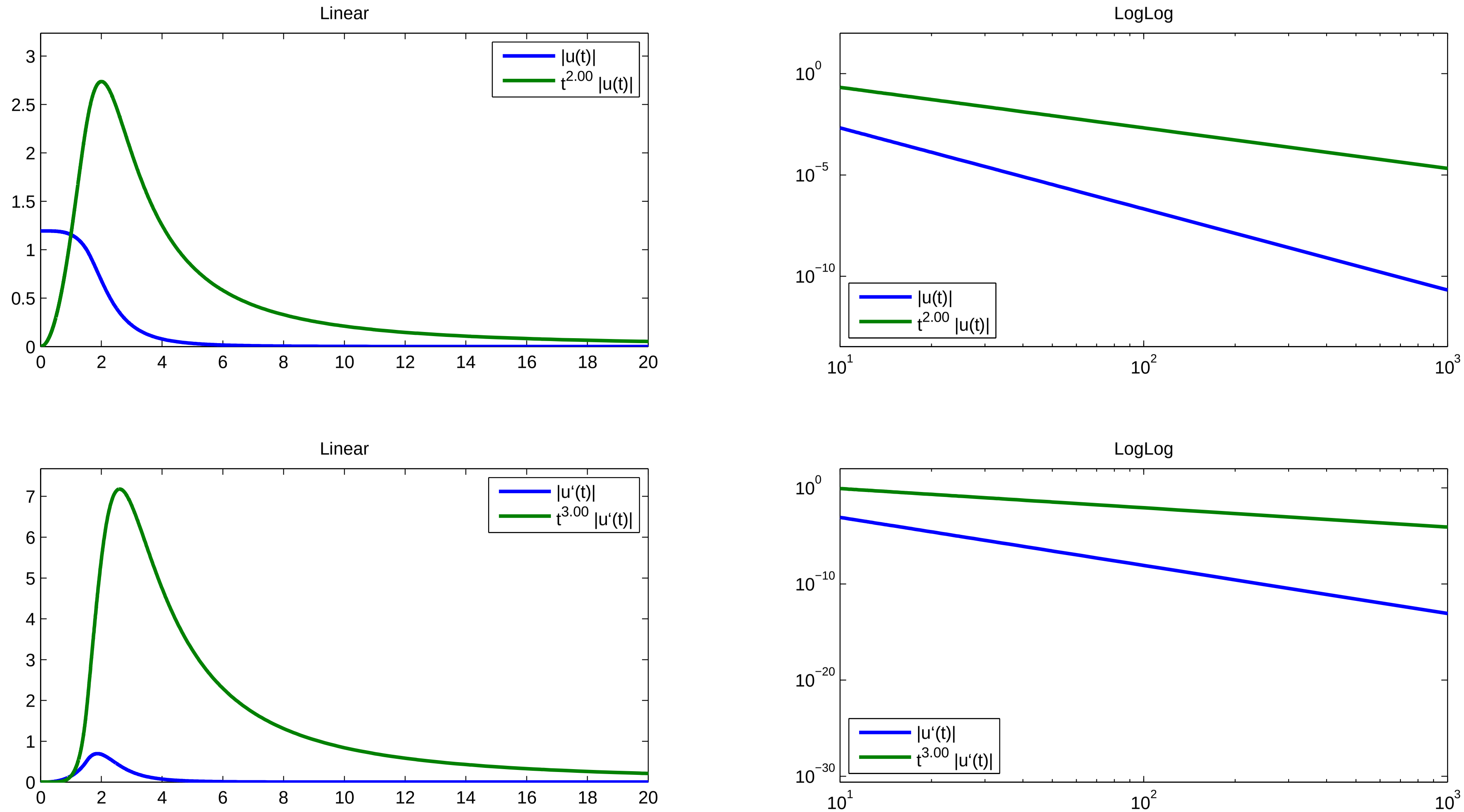

Figure A.52.: $\alpha = 5,\ \beta = 7, \quad u''(t) + \left[\frac{4}{t} - \frac{\exp(-t)}{1+\exp(-t)}\right] u'(t) + t^2 f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18c)$.

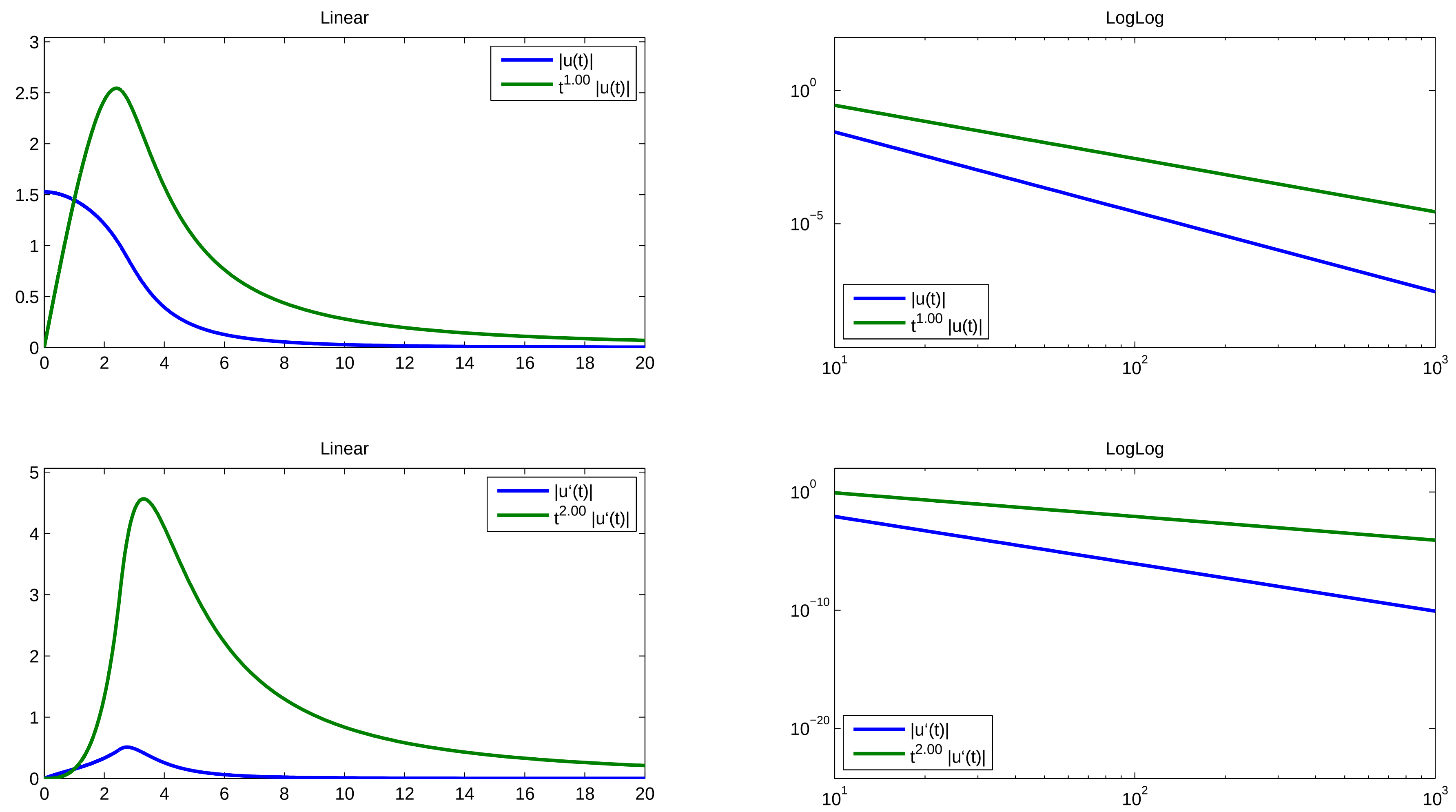

Figure A.53.: $\alpha = 4$, $\beta = 4$, $u''(t) + \left[\frac{4}{t} - \frac{\exp(-t)}{1+\exp(-t)}\right] u'(t) + f(u) = 0$, $u'(0) = 0$, $u(\infty) = 0$, where $f(u)$ is given by $(1.18a)$.

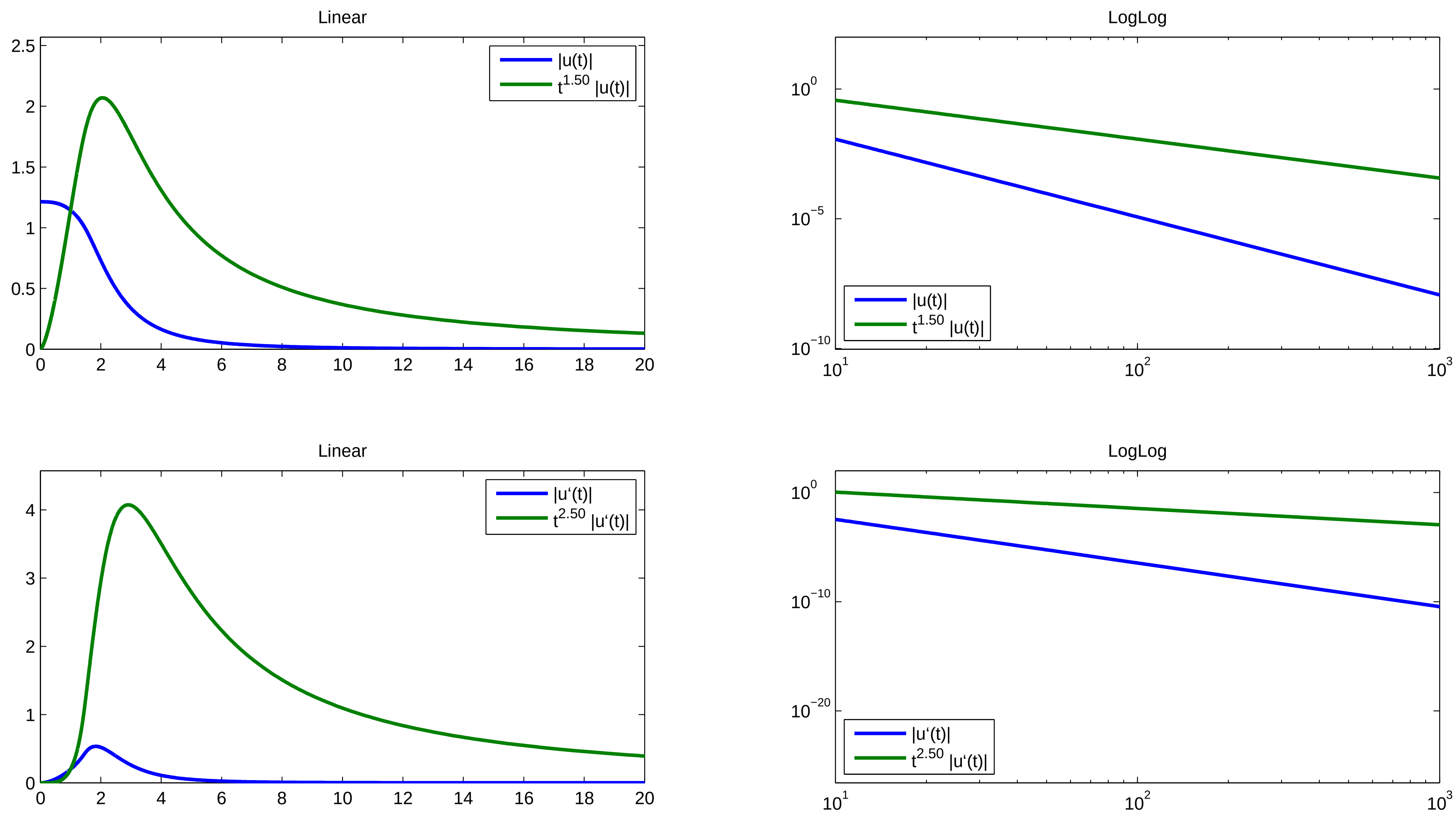

Figure A.54.: $\alpha = 4,\ \ \beta = 5, \quad u''(t) + \left[\frac{4}{t} - \frac{\exp(-t)}{1+\exp(-t)}\right] u'(t) + t f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18a)$.

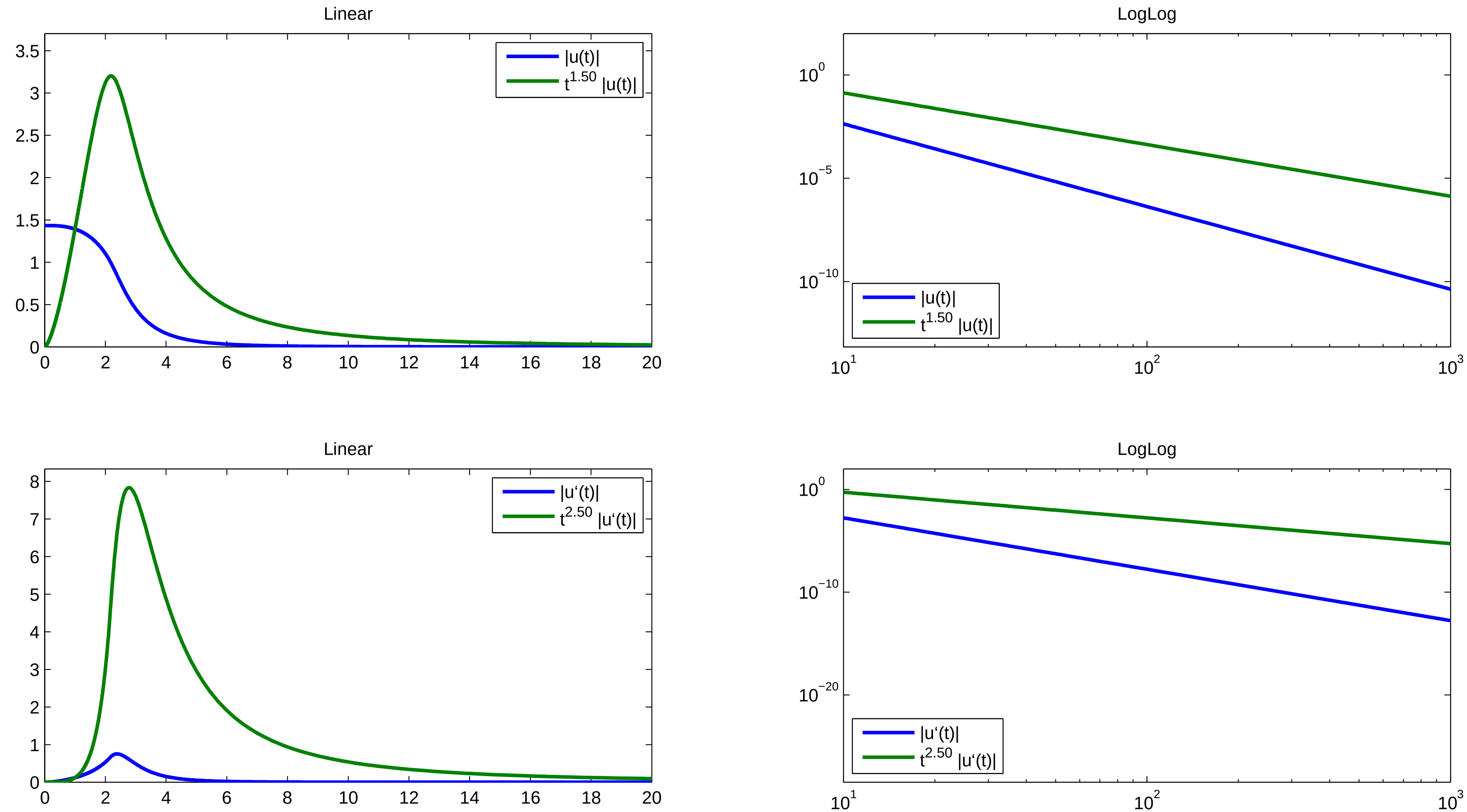

Figure A.55.: $\alpha = 5, \ \beta = 6, \quad u''(t) + \left[\frac{5}{t} - \frac{\exp(-t)}{1+\exp(-t)}\right] u'(t) + t f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18a)$.

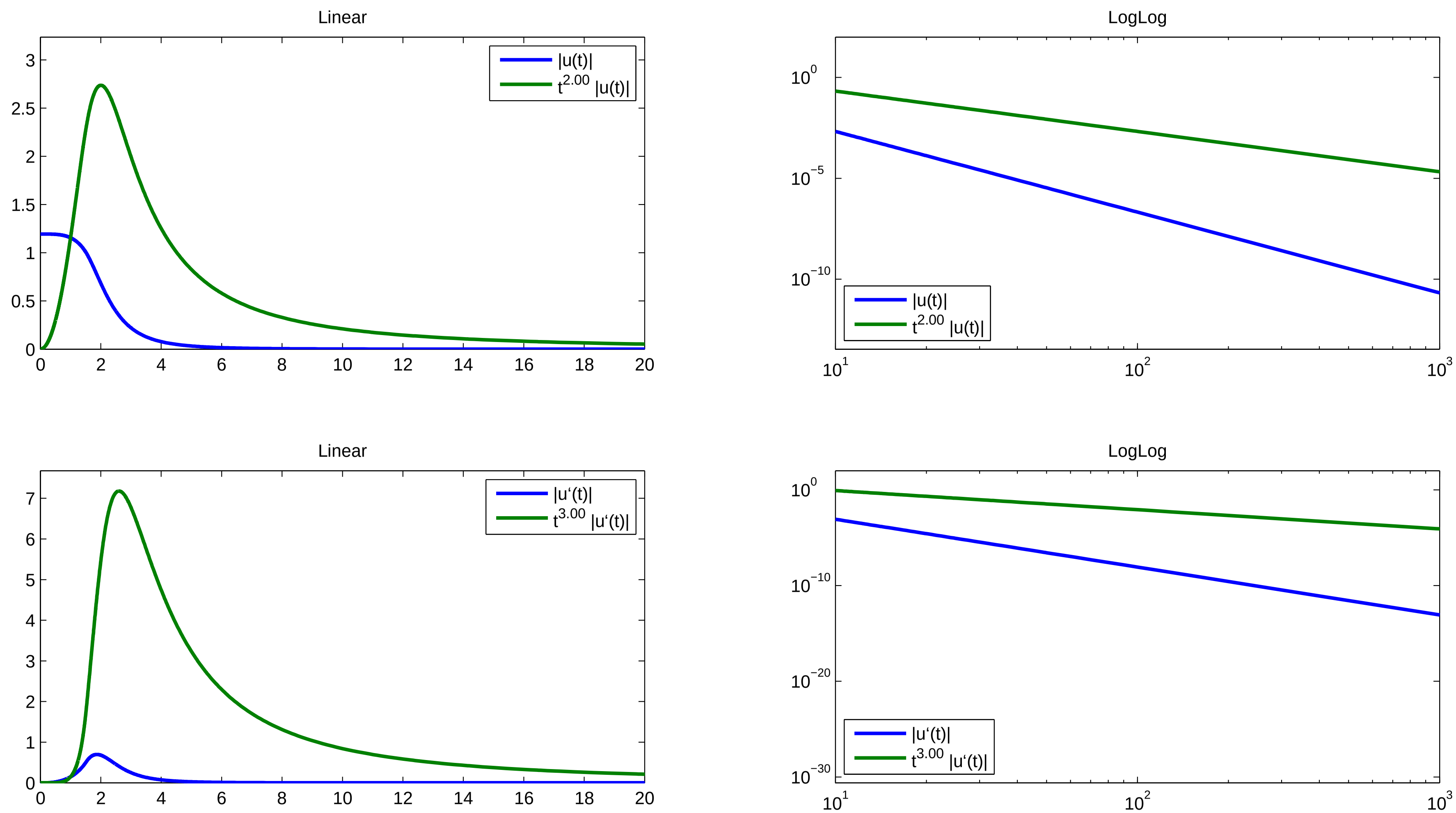

Figure A.56.: $\alpha = 5,\ \beta = 7, \quad u''(t) + \left[\frac{4}{t} - \frac{\exp(-t)}{1+\exp(-t)}\right] u'(t) + t^2 f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18a)$.

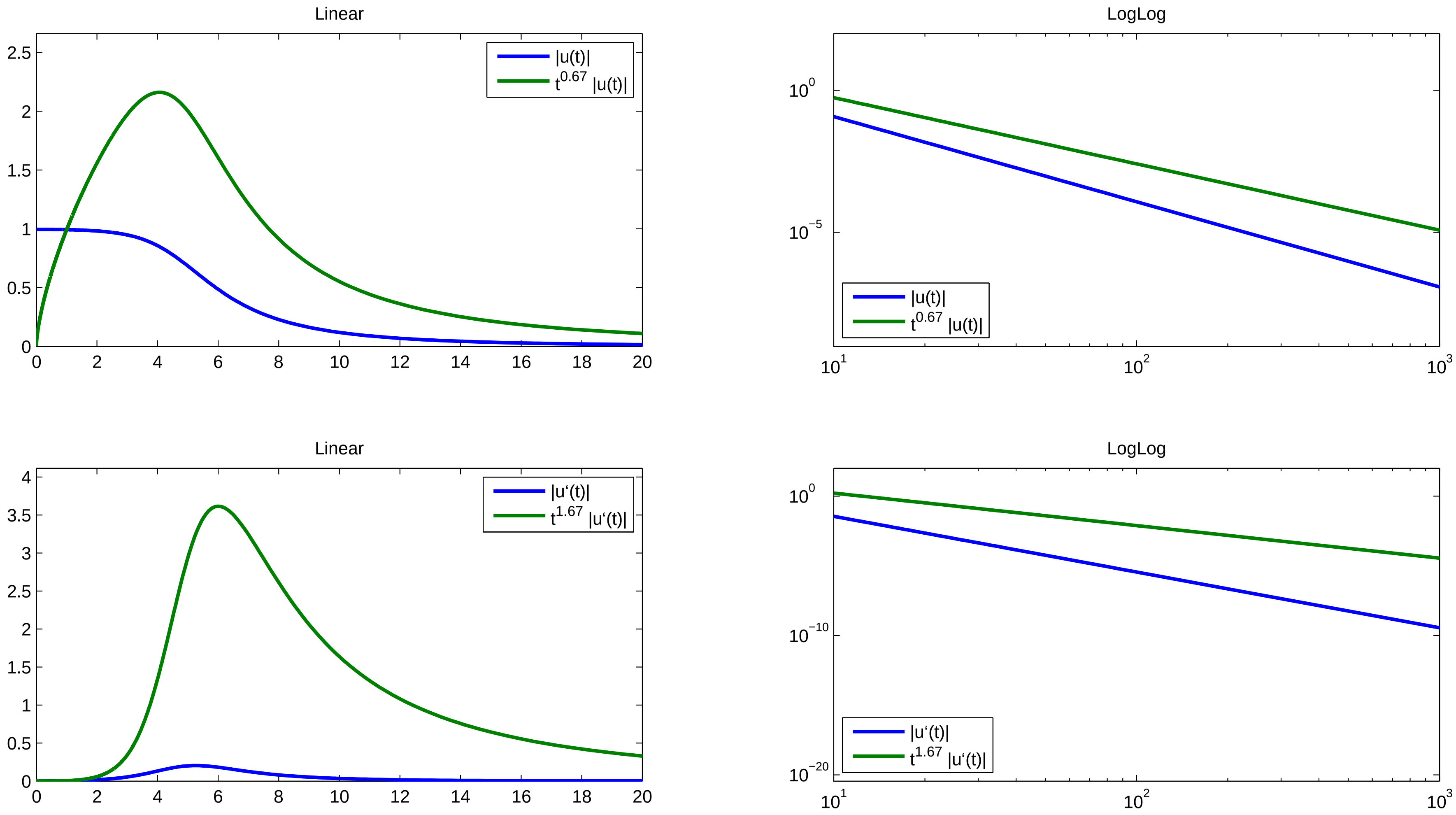

Figure A.57.: $\alpha = 4$, $\beta = 4$, $u''(t) + \left[\frac{4}{t} - \frac{\exp(-t)}{1+\exp(-t)}\right] u'(t) + f(u) = 0$, $u'(0) = 0$, $u(\infty) = 0$, where $f(u)$ is given by $(1.18d)$.

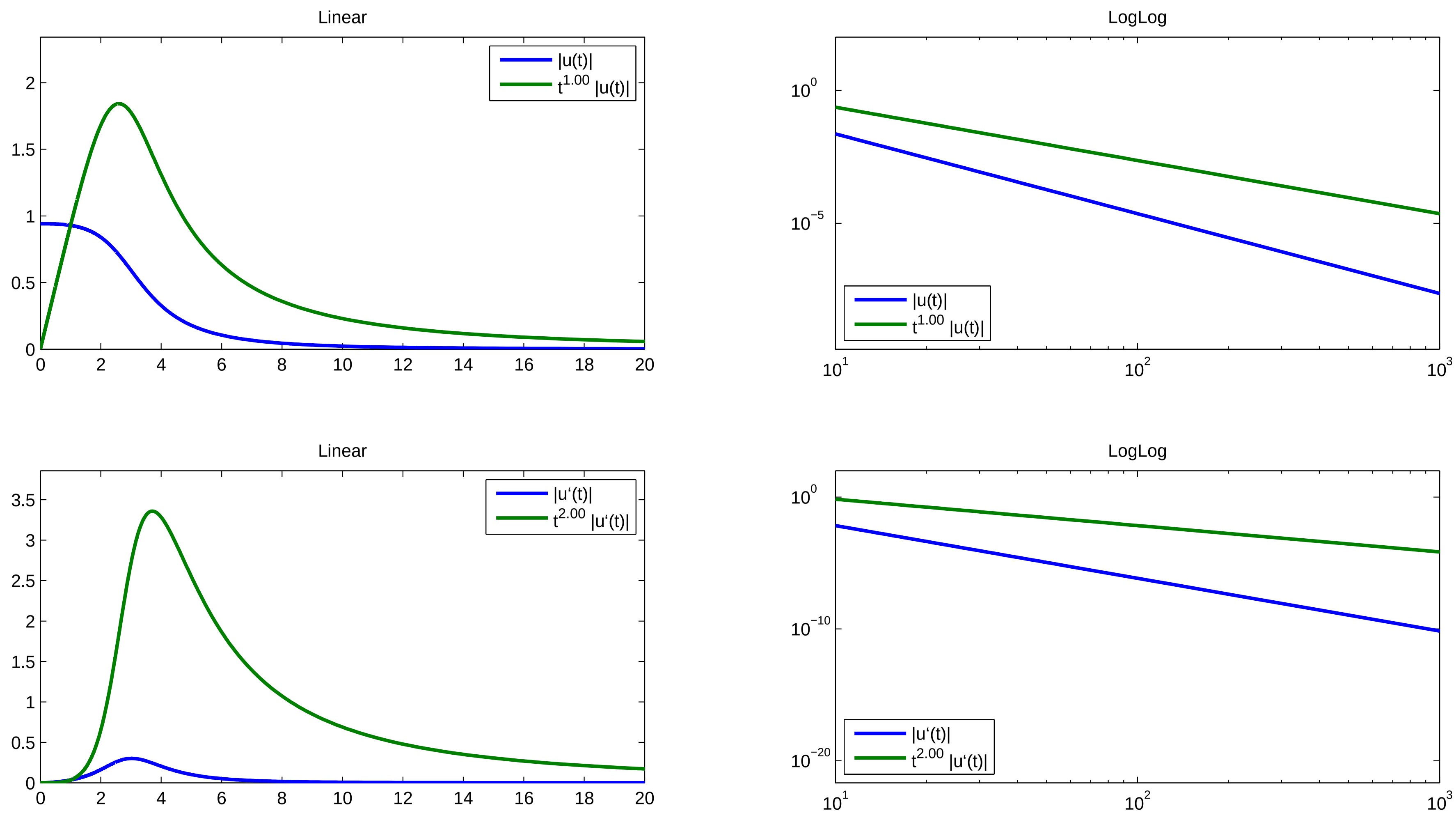

Figure A.58.: $\alpha = 4,\ \beta = 5, \quad u''(t) + \left[\frac{4}{t} - \frac{\exp(-t)}{1+\exp(-t)}\right] u'(t) + t f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18d)$.

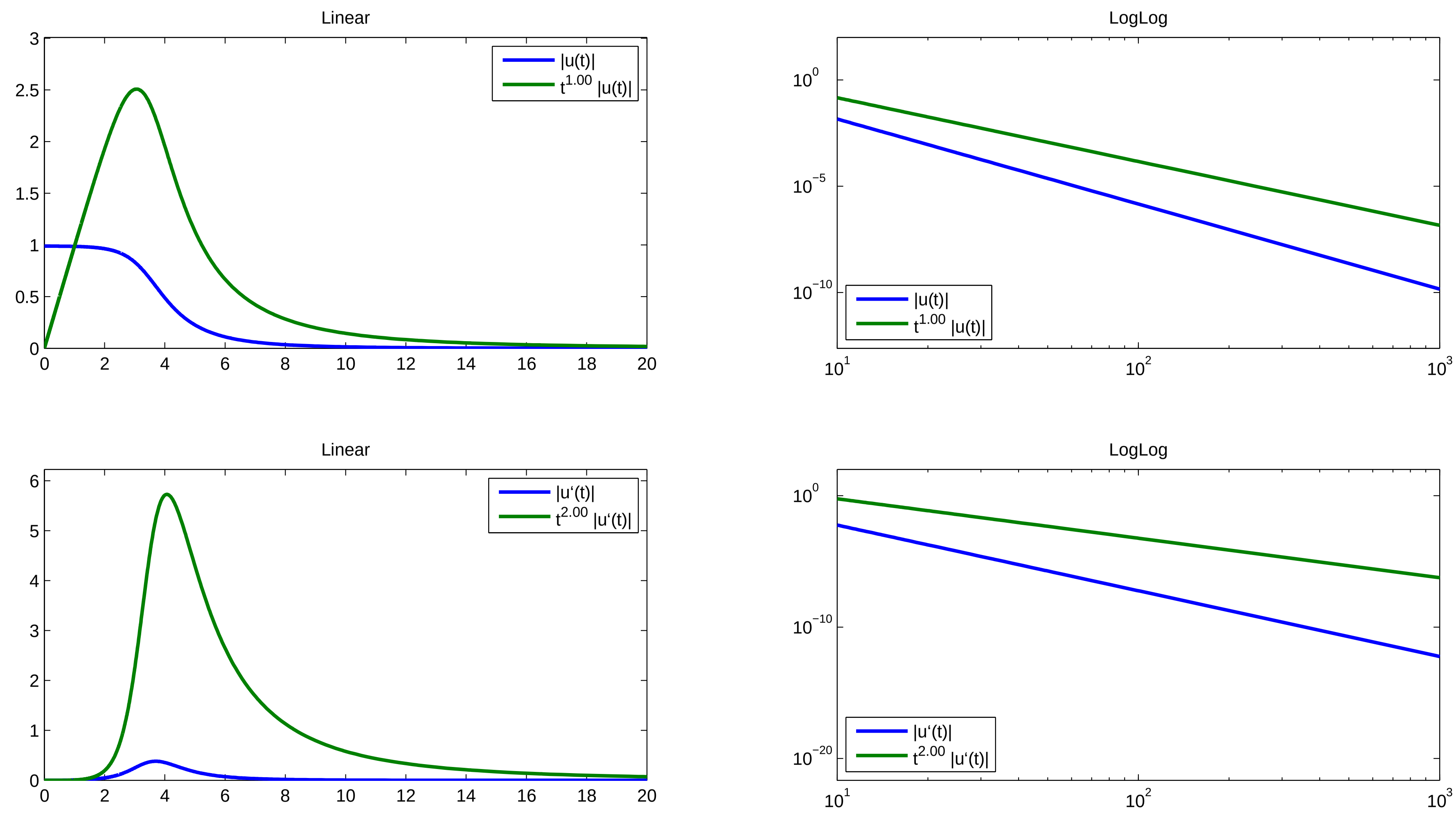

Figure A.59.: $\alpha = 5,\ \beta = 6, \quad u''(t) + \left[\frac{5}{t} - \frac{\exp(-t)}{1+\exp(-t)}\right] u'(t) + t f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18d)$.

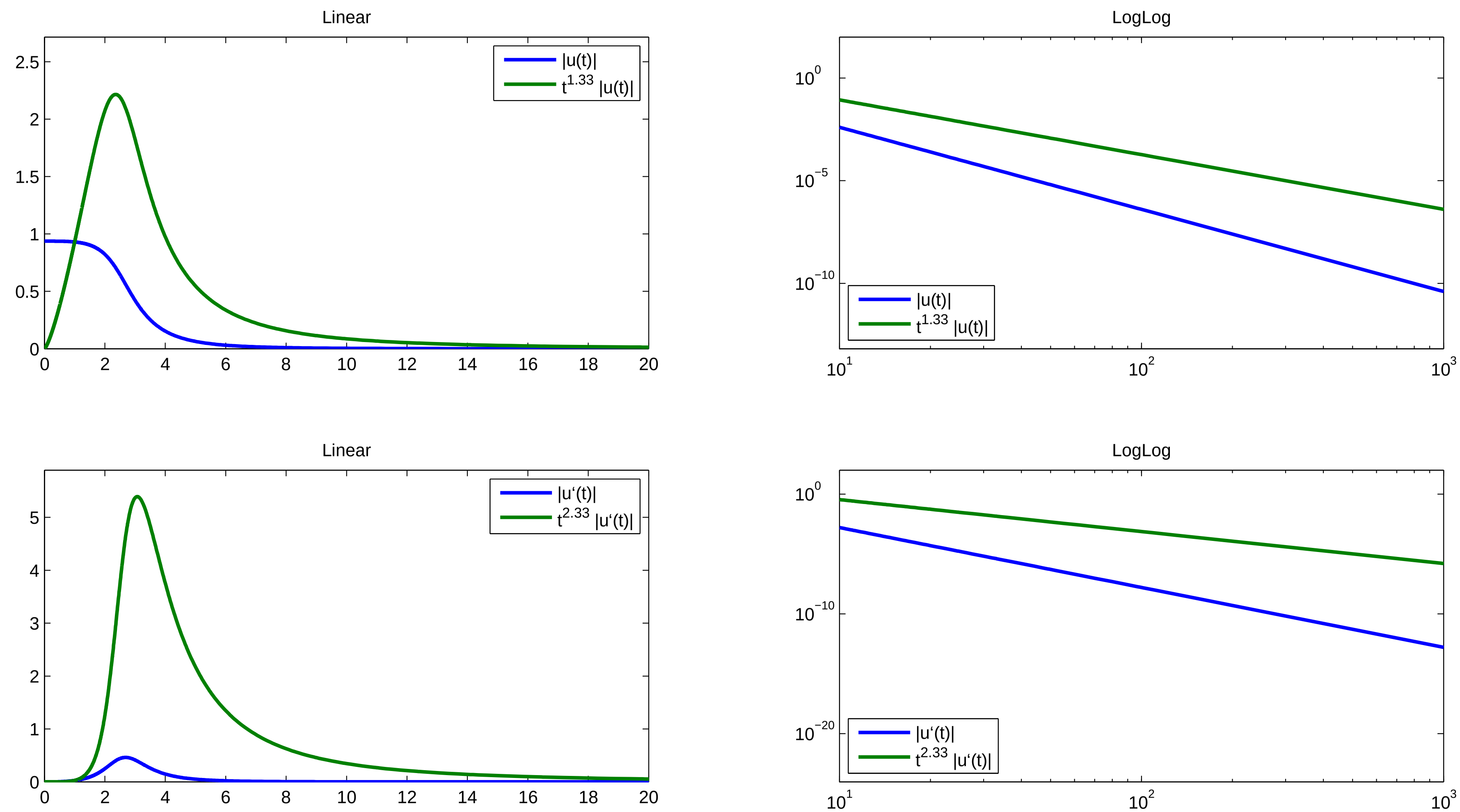

Figure A.60.: $\alpha = 5$, $\beta = 7$, $u''(t) + \left[\frac{4}{t} - \frac{\exp(-t)}{1+\exp(-t)}\right] u'(t) + t^2 f(u) = 0$, $u'(0) = 0$, $u(\infty) = 0$, where $f(u)$ is given by $(1.18d)$.

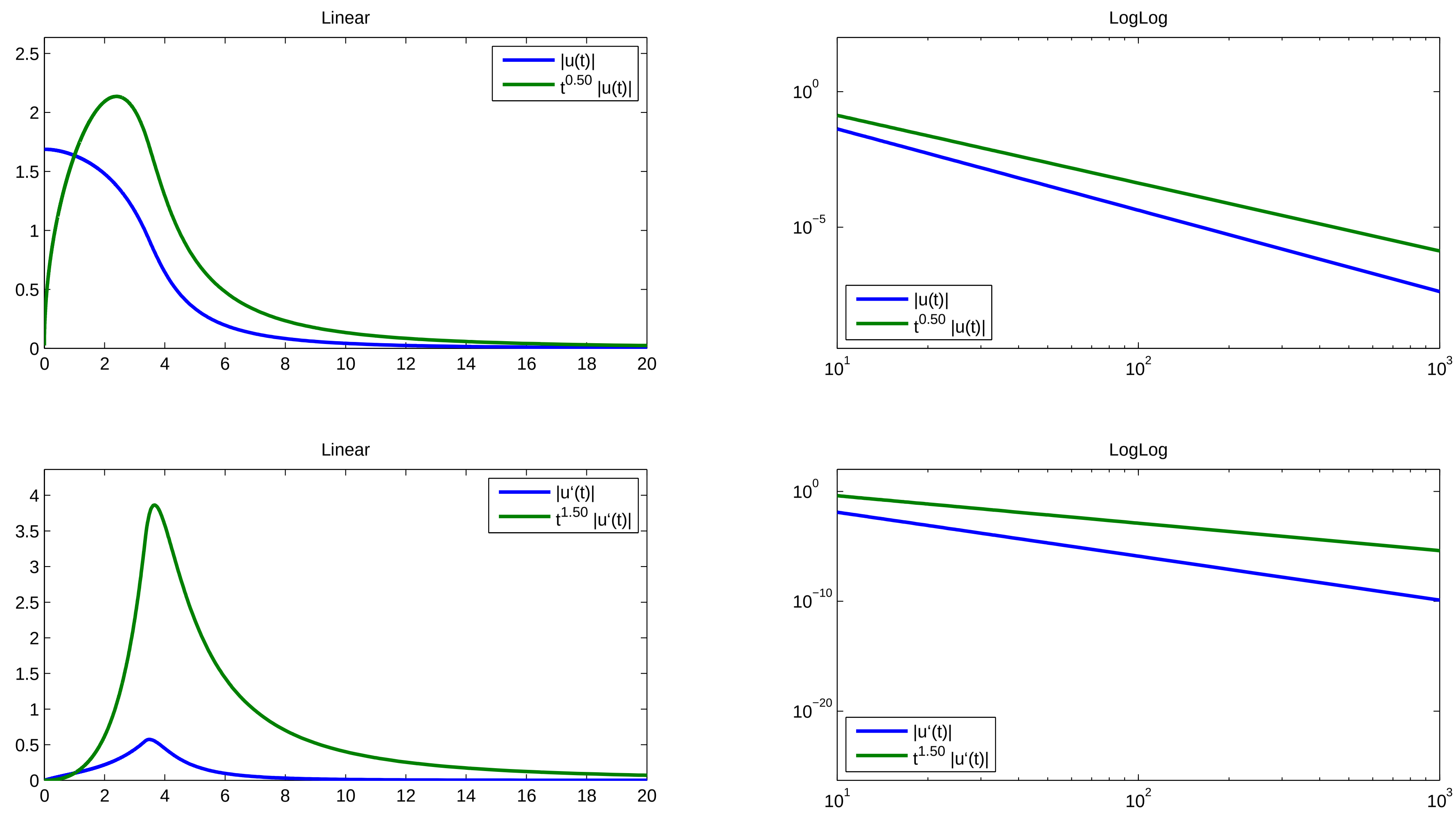

Figure A.61.: $\alpha = 4, \ \beta = 4, \quad u''(t) + \left[\frac{4}{t} - \frac{\exp(-t)}{1+\exp(-t)}\right] u'(t) + f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18b)$.

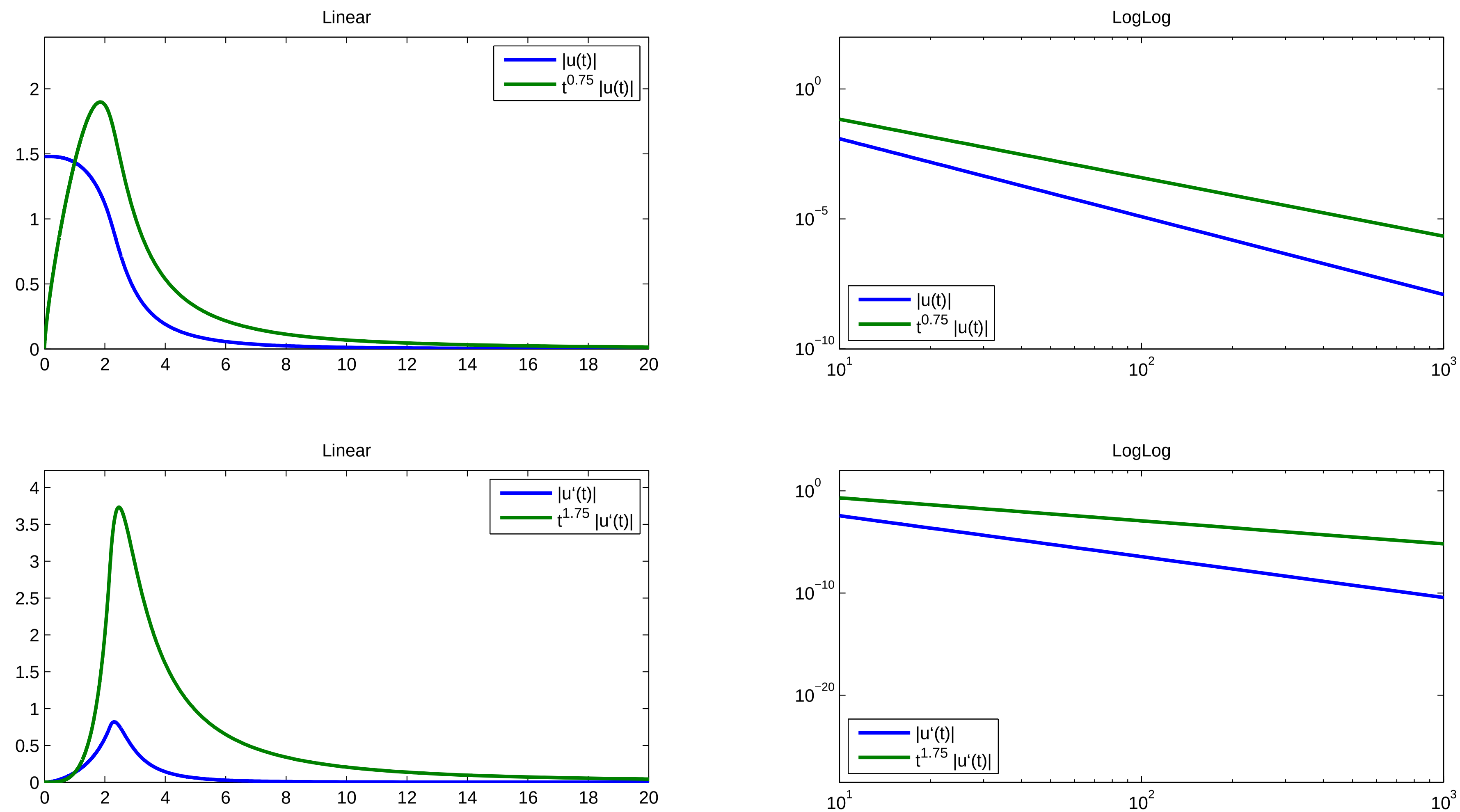

Figure A.62.: $\alpha = 4,\ \beta = 5,\quad u''(t) + \left[\frac{4}{t} - \frac{\exp(-t)}{1+\exp(-t)}\right] u'(t) + t f(u) = 0,\quad u'(0) = 0,\quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18b)$.

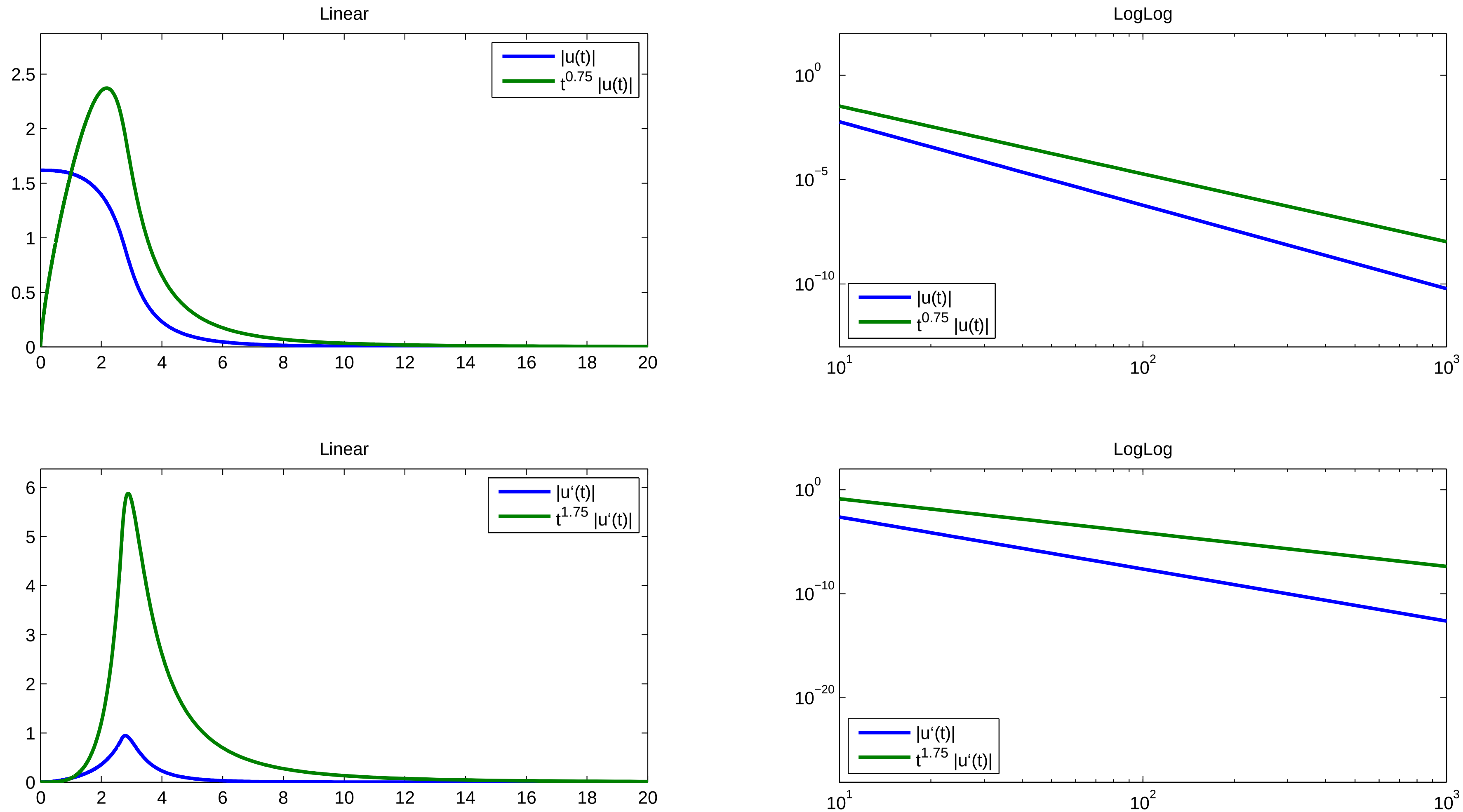

Figure A.63.: $\alpha = 5,\ \beta = 6, \quad u''(t) + \left[\frac{5}{t} - \frac{\exp(-t)}{1+\exp(-t)}\right] u'(t) + t f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18b)$.

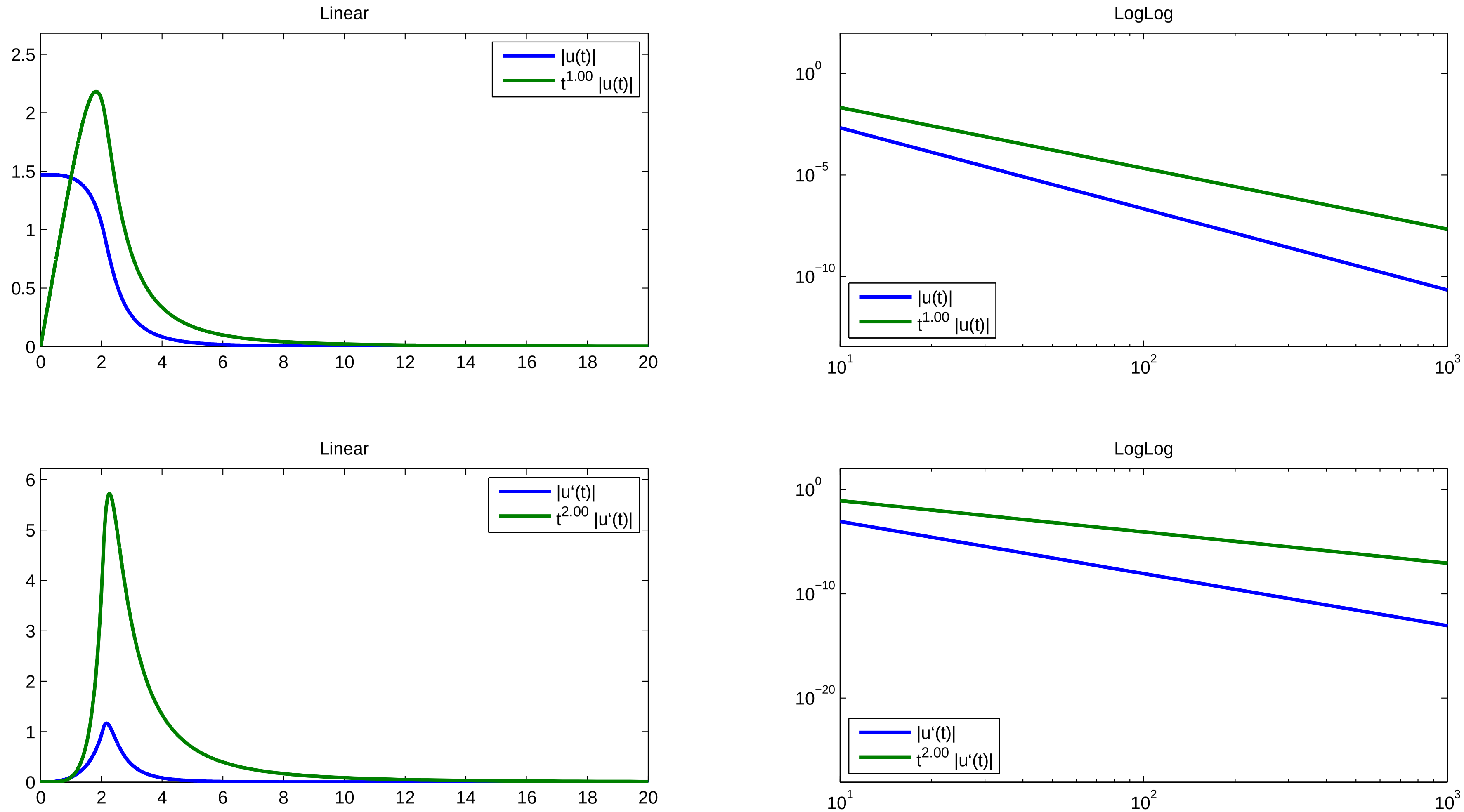

Figure A.64.: $\alpha = 5,\ \ \beta = 7, \quad u''(t) + \left[\frac{4}{t} - \frac{\exp(-t)}{1+\exp(-t)}\right] u'(t) + t^2 f(u) = 0, \quad u'(0) = 0, \quad u(\infty) = 0,$ where $f(u)$ is given by $(1.18b)$.

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