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Mathematical conditions for and physical meaning of a maximum of the determinant of $\widetilde{\boldsymbol{K}}_T$ in the prebuckling regime

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Dedicated to Professor Zdeněk Bittnar on the occasion of his 70th birthday.

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ABSTRACT

It is shown that the determinant of the tangent stiffness matrix has a maximum in the prebuckling regime if and only if the determinant of a specific linear combination of the first and the third derivative of this matrix with respect to a dimensionless load factor vanishes. The mathematical tool for this proof is the so-called consistently linearized eigenproblem in the frame of the Finite Element Method. The physical meaning of the mentioned maximum is the one of a minimum of the percentage bending energy of the total strain energy. The paper provides mathematical and physical background knowledge on numerical results that were obtained 35 years ago.

1. Introduction

The vanishing of the determinant of the tangent stiffness matrix $\widetilde{\boldsymbol{K}}_T$, i.e.

$$Det\widetilde{\boldsymbol{K}}_T = 0, \tag{1}$$

is an indispensable ingredient of academic teaching of computational structural stability with emphasis on the Finite Element Method (FEM). Eq. (1) represents a necessary and sufficient condition for loss of static stability of structures subjected to conservative loading $\lambda\boldsymbol{P}$, where $\boldsymbol{P}$ denotes the vector of node forces, in the context of the FEM, that are work-equivalent to the actual loading, and λ is a dimensionless load factor. In general, $\widetilde{\boldsymbol{K}}_T$ depends on the vector of node displacements $\boldsymbol{q}(\lambda)$, i.e.

$$\widetilde{\boldsymbol{K}}_T := \widetilde{\boldsymbol{K}}_T(\boldsymbol{q}(\lambda)). \tag{2}$$

The tilde in $\widetilde{\boldsymbol{K}}_T$ serves the purpose of distinguishing the tangent stiffness matrix from tensor functions $\boldsymbol{K}_T$ which refer to internal forces that are not in equilibrium with the external forces [1].

In contrast to Eq. (1), the exact physical meaning of

$$\frac{d}{d\lambda}(Det(\widetilde{\boldsymbol{K}}_T)) = 0, \quad \frac{d^2}{d\lambda^2}(Det(\widetilde{\boldsymbol{K}}_T)) < 0, \tag{3}$$

in the prebuckling regime seems to be unknown. This would explain why mathematical conditions, alternative to (3), analogous to the alternative condition

$$\widetilde{\boldsymbol{K}}_T \cdot \boldsymbol{v} = \boldsymbol{0} \tag{4}$$

to (1), where $\boldsymbol{v}$ is the eigenvector of $\widetilde{\boldsymbol{K}}_T$, could not be found in the literature.

The mathematical definition of

$$\frac{d}{d\lambda}(Det\boldsymbol{A}), \tag{5}$$

where each element of $Det\boldsymbol{A}$ which is of order n is a differentiable function of λ, reads as follows [2]: The derivative of $Det\boldsymbol{A}$ with respect to λ is equal to the sum of n derivatives, the ith one of which is identical with $Det\boldsymbol{A}$ except for the ith row, which consists of the derivatives of the elements of the ith row of $Det\boldsymbol{A}$. This definition is not very helpful, because normally the elements of $Det\boldsymbol{A}$ are numerical quantities that are obtained in the frame of incremental-iterative nonlinear Finite Element Analysis (FEA).

The objective of this work is to derive mathematical conditions for (3) and to explain the physical meaning of a maximum of $Det\widetilde{\boldsymbol{K}}_T$ in the prebuckling regime. The tool for this derivation is the so-called consistently linearized eigenproblem (CLE) [3].

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In Section 2, a brief summary of the CLE will be given. In Section 3, the aforementioned conditions will be derived, and in Section 4 the physical meaning of these conditions will be explained. Section 5 is devoted to a numerical example. Conclusions drawn from this work will be given in Section 6.

2. Consistently linearized eigenproblem

The formulation of the CLE for the first eigenpair $(\lambda_1^*(\lambda)-\lambda, \boldsymbol{v}_1^*(\lambda))$ reads as follows [3]:

$$\boldsymbol{A}_1(\lambda)\cdot\boldsymbol{v}_1^*(\lambda)=\boldsymbol{0},\quad \boldsymbol{A}_1 := \widetilde{\boldsymbol{K}}_T+(\lambda_1^*-\lambda)\dot{\widetilde{\boldsymbol{K}}}_T,\quad |\boldsymbol{v}_1^*|=1, \tag{6}$$

where $\dot{\widetilde{\boldsymbol{K}}}_T$ is the first derivative of $\widetilde{\boldsymbol{K}}_T$ with respect to λ along a direction parallel to the primary path, i.e.

$$\dot{\widetilde{\boldsymbol{K}}}_T=\frac{d}{d\lambda}\widetilde{\boldsymbol{K}}_T(\boldsymbol{q}(\lambda)):=\frac{d}{d\alpha}\bigg|_{\alpha=0}\widetilde{\boldsymbol{K}}_T(\boldsymbol{q}(\lambda)+\alpha\dot{\boldsymbol{q}}(\lambda)). \tag{7}$$

In contrast to the matrix $\widetilde{\boldsymbol{K}}_T$, which is positive definite in the prebuckling regime, $\dot{\widetilde{\boldsymbol{K}}}_T$ is an indefinite matrix. Apart from one special case [4], $\boldsymbol{v}_1^*(\lambda)$ does not become an eigenvector of $\dot{\widetilde{\boldsymbol{K}}}_T$ in the prebuckling regime. Hence, the eigenvalue $\lambda_1^*-\lambda$ remains finite. Fig. 1 illustrates a typical function $\lambda_1^*(\lambda)$ for bifurcation buckling from a nonlinear load–displacement path. At the starting point A, $\lambda = 0$. At the stability limit S:

$$\lambda_1^*-\lambda_S=0\Rightarrow\lambda_1^*=\lambda=\lambda_S,\quad \dot{\lambda}_1^*=0, \tag{8}$$

[3] and

$$\boldsymbol{v}_1^*(\lambda) := \boldsymbol{v}_1. \tag{9}$$

(For snap-through, λ is a nonmonotonic function of the displacements. This suggests replacing λ by a suitable parameter ξ.)

In order to derive the mathematical conditions for a maximum of $Det\widetilde{\boldsymbol{K}}_T$, the first three derivatives of (6.1) with respect to λ are needed. They are obtained as follows [4]:

$$\dot{\boldsymbol{A}}_1\cdot\boldsymbol{v}_1^*+\boldsymbol{A}_1\cdot\dot{\boldsymbol{v}}_1^*=\boldsymbol{0}, \tag{10}$$

$$\ddot{\boldsymbol{A}}_1\cdot\boldsymbol{v}_1^*+2\dot{\boldsymbol{A}}_1\cdot\dot{\boldsymbol{v}}_1^*+\boldsymbol{A}_1\cdot\ddot{\boldsymbol{v}}_1^*=\boldsymbol{0}, \tag{11}$$

$$\dddot{\boldsymbol{A}}_1\cdot\boldsymbol{v}_1^*+3\ddot{\boldsymbol{A}}_1\cdot\dot{\boldsymbol{v}}_1^*+3\dot{\boldsymbol{A}}_1\cdot\ddot{\boldsymbol{v}}_1^*+\boldsymbol{A}_1\cdot\dddot{\boldsymbol{v}}_1^*=\boldsymbol{0}, \tag{12}$$

where

$$\dot{\boldsymbol{A}}_1=\dot{\lambda}_1^*\dot{\widetilde{\boldsymbol{K}}}_T+(\lambda_1^*-\lambda)\ddot{\widetilde{\boldsymbol{K}}}_T, \tag{13}$$

$$\ddot{\boldsymbol{A}}_1=\ddot{\lambda}_1^*\dot{\widetilde{\boldsymbol{K}}}_T+\left(2\dot{\lambda}_1^*-1\right)\ddot{\widetilde{\boldsymbol{K}}}_T+(\lambda_1^*-\lambda),\dddot{\widetilde{\boldsymbol{K}}}_T, \tag{14}$$

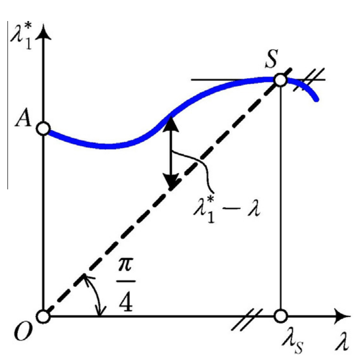

Fig. 1. λ_1^* versus λ, with $\lambda_1^*(\lambda)-\lambda$ as a typical eigenvalue function for bifurcation buckling [4].

$$\dddot{\boldsymbol{A}}_1=\dddot{\lambda}_1^*\dot{\widetilde{\boldsymbol{K}}}_T+3\ddot{\lambda}_1^*\ddot{\widetilde{\boldsymbol{K}}}_T+\left(3\dot{\lambda}_1^*-2\right)\dddot{\widetilde{\boldsymbol{K}}}_T+(\lambda_1^*-\lambda)\ddddot{\widetilde{\boldsymbol{K}}}_T, \tag{15}$$

and

$$\dot{\boldsymbol{v}}_1^*=\sum_{j=2}^{N}c_{1j}\boldsymbol{v}_j^*\quad\text{with}\quad c_{1j}=-\frac{\lambda_1^*-\lambda}{\lambda_1^*-\lambda_j^*}\frac{\boldsymbol{v}_j^*\cdot\ddot{\widetilde{\boldsymbol{K}}}_T\cdot\boldsymbol{v}_1^*}{\boldsymbol{v}_j^*\cdot\dot{\widetilde{\boldsymbol{K}}}_T\cdot\boldsymbol{v}_j^*}, \tag{16}$$

$$\ddot{\boldsymbol{v}}_1^*=\sum_{j=2}^{N}\left(\dot{c}_{1j}\boldsymbol{v}_j^*+c_{1j}\dot{\boldsymbol{v}}_j^*\right), \tag{17}$$

$$\dddot{\boldsymbol{v}}_1^*=\sum_{j=2}^{N}\left(\ddot{c}_{1j}\boldsymbol{v}_j^*+2\dot{c}_{1j}\dot{\boldsymbol{v}}_j^*+c_{1j}\ddot{\boldsymbol{v}}_j^*\right), \tag{18}$$

where $\lambda_1^*-\lambda_j^*$ represents the difference of the first and the jth eigenvalue and $\boldsymbol{v}_j^*$ denotes the jth eigenvector. Following from normalization of $\boldsymbol{v}_1^*$ according to (6.3):

$$\boldsymbol{v}_1^*\cdot\dot{\boldsymbol{v}}_1^*=0, \tag{19}$$

$$\dot{\boldsymbol{v}}_1^*\cdot\dot{\boldsymbol{v}}_1^*+\boldsymbol{v}_1^*\cdot\ddot{\boldsymbol{v}}_1^*=0, \tag{20}$$

$$3\dot{\boldsymbol{v}}_1^*\cdot\ddot{\boldsymbol{v}}_1^*+\boldsymbol{v}_1^*\cdot\dddot{\boldsymbol{v}}_1^*=0. \tag{21}$$

Substitution of (8.1) into (16.2) and insertion of the result into (16.1) gives

$$\dot{\boldsymbol{v}}_1^*(\lambda_S)=\boldsymbol{0}. \tag{22}$$

3. Derivation of mathematical conditions for a maximum of $Det\widetilde{\boldsymbol{K}}_T$

Premultiplication of (11) by $\boldsymbol{v}_1^*$ and use of (6.1) gives

$$\boldsymbol{v}_1^*\cdot\ddot{\boldsymbol{A}}_1\cdot\boldsymbol{v}_1^*+2\boldsymbol{v}_1^*\cdot\dot{\boldsymbol{A}}_1\cdot\dot{\boldsymbol{v}}_1^*=0. \tag{23}$$

Substitution of (13) and (14) into (23) yields

$$\boldsymbol{v}_1^*\cdot\left(\ddot{\lambda}_1^*\dot{\widetilde{\boldsymbol{K}}}_T+(2\dot{\lambda}_1^*-1)\ddot{\widetilde{\boldsymbol{K}}}_T+(\lambda_1^*-\lambda)\dddot{\widetilde{\boldsymbol{K}}}_T\right)\cdot\boldsymbol{v}_1^*+2\boldsymbol{v}_1^*\cdot\left(\dot{\lambda}_1^*\dot{\widetilde{\boldsymbol{K}}}_T+(\lambda_1^*-\lambda)\ddot{\widetilde{\boldsymbol{K}}}_T\right)\cdot\dot{\boldsymbol{v}}_1^*=0. \tag{24}$$

Substitution of (16) into the bilinear form $\boldsymbol{v}_1^*\cdot\dot{\widetilde{\boldsymbol{K}}}_T\cdot\dot{\boldsymbol{v}}_1^*$ in (24), consideration of the orthogonality relations

$$\boldsymbol{v}_j^*\cdot\dot{\widetilde{\boldsymbol{K}}}_T\cdot\boldsymbol{v}_1^*=0,\quad j=2,3,\ldots,N, \tag{25}$$

following from (6.1), and use of $\boldsymbol{v}_1^*\cdot\dot{\boldsymbol{A}}_1\cdot\boldsymbol{v}_1^*=0$, following from premultiplication of (10) by $\boldsymbol{v}_1^*$ and consideration of (6.1), gives

$$\frac{\ddot{\lambda}_1^*(\lambda_1^*-\lambda)-\dot{\lambda}_1^*\left(\dot{\lambda}_1^*-1\right)}{(\lambda_1^*-\lambda)^2}=-\frac{1}{\boldsymbol{v}_1^*\cdot\dot{\widetilde{\boldsymbol{K}}}_T\cdot\boldsymbol{v}_1^*}\cdot\left(2\boldsymbol{v}_1^*\cdot\ddot{\widetilde{\boldsymbol{K}}}_T\cdot\dot{\boldsymbol{v}}_1^*+\frac{\left(\boldsymbol{v}_1^*\cdot\dddot{\widetilde{\boldsymbol{K}}}_T\cdot\boldsymbol{v}_1^*\right)\left(\boldsymbol{v}_1^*\cdot\dot{\widetilde{\boldsymbol{K}}}_T\cdot\boldsymbol{v}_1^*\right)-\left(\boldsymbol{v}_1^*\cdot\ddot{\widetilde{\boldsymbol{K}}}_T\cdot\boldsymbol{v}_1^*\right)^2}{\boldsymbol{v}_1^*\cdot\dot{\widetilde{\boldsymbol{K}}}_T\cdot\boldsymbol{v}_1^*}\right). \tag{26}$$

Substitution of (8) into

$$\frac{d}{d\lambda}\left(\frac{\dot{\lambda}_1^*}{\lambda_1^*-\lambda}\right)=\frac{\ddot{\lambda}_1^*(\lambda_1^*-\lambda)-\dot{\lambda}_1^*(\dot{\lambda}_1^*-1)}{(\lambda_1^*-\lambda)^2}, \tag{27}$$

and

$$\frac{d^2}{d\lambda^2}\left(\frac{\dot{\lambda}_1^*}{\lambda_1^*-\lambda}\right)=\frac{\left(\dddot{\lambda}_1^*(\lambda_1^*-\lambda)-\dot{\lambda}_1^*\ddot{\lambda}_1^*\right)(\lambda_1^*-\lambda)^2-2\left(\ddot{\lambda}_1^*(\lambda_1^*-\lambda)-\dot{\lambda}_1^*(\dot{\lambda}_1^*-1)\right)(\lambda_1^*-\lambda)(\dot{\lambda}_1^*-1)}{(\lambda_1^*-\lambda)^4}, \tag{28}$$

and use of *de l'Hopital's* rule yields

$$\frac{d}{d\lambda}\left(\frac{\dot{\lambda}_1^*}{\lambda_1^*-\lambda}\right)\Bigg|_{\lambda=\lambda_S}=-\frac{1}{2}\left(\ddot{\lambda}_1^{*2}+\dddot{\lambda}_1^*\right), \tag{29}$$

and

$$\frac{d^2}{d\lambda^2}\left(\frac{\dot{\lambda}_1^*}{\lambda_1^*-\lambda}\right)\Bigg|_{\lambda=\lambda_S}=-\frac{1}{6}\left(15\ddot{\lambda}_1^*\dddot{\lambda}_1^*+2\ddddot{\lambda}_1^*\right), \tag{30}$$

respectively. For a general stress state, resulting from a combination of membrane and non-membrane action, in addition to (8):

$$\frac{d^2}{d\lambda^2}\left(\frac{\dot{\lambda}_1^*}{\lambda_1^*-\lambda}\right)\Bigg|_{\lambda=\lambda_S}=0, \tag{31}$$

indicating a point of inflection of the function $\lambda_1^*/(\lambda_1^*-\lambda)$ at $\lambda=\lambda_S$ (see the situation at point S in Fig. 2b). Substitution of (31) into (30) gives

$$15\ddot{\lambda}_1^*\dddot{\lambda}_1^*+2\ddddot{\lambda}_1^*=0. \tag{32}$$

For the limiting case of buckling from a state of pure bending (lateral torsional buckling):

$$\dddot{\lambda}_1^*=\ddddot{\lambda}_1^*=0. \tag{33}$$

For the limiting case of buckling from a membrane stress state, in addition to (31):

$$\frac{d}{d\lambda}\left(\frac{\dot{\lambda}_1^*}{\lambda_1^*-\lambda}\right)\Bigg|_{\lambda=\lambda_S}=0, \tag{34}$$

indicating a saddle point of the function $\lambda_1^*/(\lambda_1^*-\lambda)$ at $\lambda=\lambda_S$. Substitution of (34) into (29) gives

$$\ddot{\lambda}_1^{*2}+\dddot{\lambda}_1^*=0. \tag{35}$$

Substitution of (35) into (32) yields

$$-15\ddot{\lambda}_1^{*3}+2\ddddot{\lambda}_1^*=0, \tag{36}$$

For a maximum of $Det\widetilde{\boldsymbol{K}}_T$ in the prebuckling regime:

$$\frac{d}{d\lambda}\left(\frac{\dot{\lambda}_1^*}{\lambda_1^*-\lambda}\right)\Bigg|_{\lambda=\lambda_R}=0,\quad \frac{d^2}{d\lambda^2}\left(\frac{\dot{\lambda}_1^*}{\lambda_1^*-\lambda}\right)\Bigg|_{\lambda=\lambda_R}<0, \tag{37}$$

indicating a maximum of the function $\dot{\lambda}_1^*/(\lambda_1^*-\lambda)$ at $\lambda=\lambda_R$ (see the situation at point R in Fig. 2b). The rationale of this assertion becomes evident from substitution of (37.1) into (27), resulting in

$$\ddot{\lambda}_1^*(\lambda_1^*-\lambda)-\dot{\lambda}_1^*\left(\dot{\lambda}_1^*-1\right)=0. \tag{38}$$

This relation indicates a disintegration of (26), analogous to the disintegration of (26) at the stability limit, albeit one of a different kind. As will be shown later:

$$\dot{\lambda}_1^*\big|_{\lambda=\lambda_R}=\frac{1}{2}, \tag{39}$$

(see the situation at point R in Fig. 2a).

Representing part of the solution of the CLE, the eigenvalue $\lambda_1^*-\lambda$ is known for all values of λ in the prebuckling regime. In order to determine the two unknown quantities $\dot{\lambda}_1^*$ and $\ddot{\lambda}_1^*$, at $\lambda=\lambda_R$, in (38), an additional condition is needed. As part of a disintegration of (12), this condition is obtained as

$$\ddot{\boldsymbol{A}}_1\cdot\dot{\boldsymbol{v}}_1^*=\boldsymbol{0}. \tag{40}$$

In this context, it is worthy of note that (12) also disintegrates at the stability limit, albeit differently from the disintegration at point R.

Substitution of (14) into (40) gives

$$\left[\ddot{\lambda}_1^*\dot{\widetilde{\boldsymbol{K}}}_T+\left(2\dot{\lambda}_1^*-1\right)\ddot{\widetilde{\boldsymbol{K}}}_T+(\lambda_1^*-\lambda)\dddot{\widetilde{\boldsymbol{K}}}_T\right]\cdot\dot{\boldsymbol{v}}_1^*=\boldsymbol{0}. \tag{41}$$

Elimination of $\ddot{\lambda}_1^*$ in (41) by means of (38) yields

$$\left[\frac{\dot{\lambda}_1^*\left(\dot{\lambda}_1^*-1\right)}{\lambda_1^*-\lambda}\dot{\widetilde{\boldsymbol{K}}}_T+\left(2\dot{\lambda}_1^*-1\right)\ddot{\widetilde{\boldsymbol{K}}}_T+(\lambda_1^*-\lambda)\dddot{\widetilde{\boldsymbol{K}}}_T\right]\cdot\dot{\boldsymbol{v}}_1^*=\boldsymbol{0}. \tag{42}$$

Substitution of (16) into (42) and premultiplication by $\boldsymbol{v}_1^*$ results in

$$\boldsymbol{v}_1^*\cdot\left[\frac{\dot{\lambda}_1^*\left(\dot{\lambda}_1^*-1\right)}{\lambda_1^*-\lambda}\dot{\widetilde{\boldsymbol{K}}}_T+\left(2\dot{\lambda}_1^*-1\right)\ddot{\widetilde{\boldsymbol{K}}}_T+(\lambda_1^*-\lambda)\dddot{\widetilde{\boldsymbol{K}}}_T\right]\cdot\left(\sum_{j=2}^{N}c_{1j}\boldsymbol{v}_j^*\right)=0. \tag{43}$$

Consideration of the orthogonality condition (25) in (43) gives

$$\boldsymbol{v}_1^*\cdot\left[\left(2\dot{\lambda}_1^*-1\right)\ddot{\widetilde{\boldsymbol{K}}}_T+(\lambda_1^*-\lambda)\dddot{\widetilde{\boldsymbol{K}}}_T\right]\cdot\left(\sum_{j=2}^{N}c_{1j}\boldsymbol{v}_j^*\right)=0. \tag{44}$$

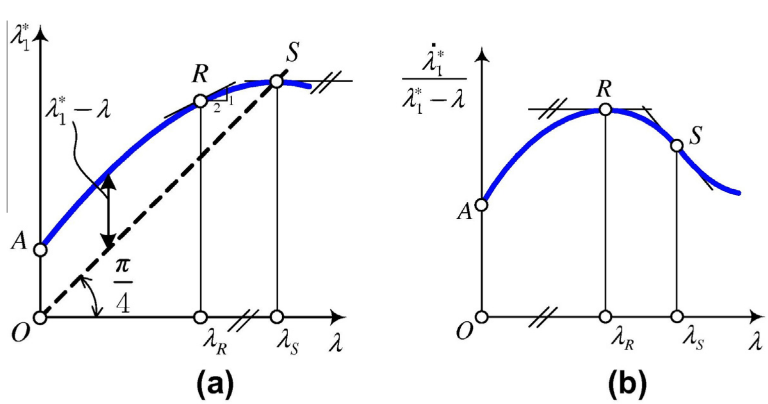

Fig. 2. (a) λ_1^* versus λ, with $\lambda_1^*(\lambda)-\lambda$ as an eigenvalue function for bifurcation buckling that contains point R, $\lambda_R<\lambda_S$, referring to a maximum of $Det\widetilde{\boldsymbol{K}}_T$, (b) $\dot{\lambda}_1^*/(\lambda_1^*-\lambda)$ versus λ, corresponding to (a).

Substitution of (38) into (26) yields

$$2\boldsymbol{v}_1^* \cdot \dddot{\tilde{\boldsymbol{K}}}_T \cdot \dot{\boldsymbol{v}}_1^* + \frac{(\boldsymbol{v}_1^* \cdot \dddot{\tilde{\boldsymbol{K}}}_T \cdot \boldsymbol{v}_1^*)(\boldsymbol{v}_1^* \cdot \dot{\tilde{\boldsymbol{K}}}_T \cdot \boldsymbol{v}_1^*) - (\boldsymbol{v}_1^* \cdot \ddot{\tilde{\boldsymbol{K}}}_T \cdot \boldsymbol{v}_1^*)^2}{\boldsymbol{v}_1^* \cdot \dot{\tilde{\boldsymbol{K}}}_T \cdot \boldsymbol{v}_1^*} = 0. \quad (45)$$

Hence:

$$\boldsymbol{v}_1^* \cdot \ddot{\tilde{\boldsymbol{K}}}_T \cdot \boldsymbol{v}_j^* \neq 0, \quad j = 2, 3, \ldots, N. \quad (46)$$

Consequently, satisfaction of (44) requires that

$$2\dot{\lambda}_1^* - 1 = 0, \quad (47)$$

which confirms (39), and that

$$\boldsymbol{v}_1^* \cdot \dddot{\tilde{\boldsymbol{K}}}_T \cdot \boldsymbol{v}_j^* = 0, \quad j = 2, 3, \ldots, N. \quad (48)$$

Substitution of (47) into (42) gives

$$\left[\dot{\tilde{\boldsymbol{K}}}_T - 4(\lambda_1^* - \lambda)^2 \dddot{\tilde{\boldsymbol{K}}}_T\right] \cdot \dot{\boldsymbol{v}}_1^* = \boldsymbol{0}, \quad (49)$$

which shows that the sign of the eigenvalue $\lambda_1^* - \lambda$ has no influence on this relation. Thus:

$$Det\left[\dot{\tilde{\boldsymbol{K}}}_T - 4(\lambda_1^* - \lambda)^2 \dddot{\tilde{\boldsymbol{K}}}_T\right] = \boldsymbol{0}, \quad (50)$$

which is a necessary and sufficient condition for a maximum of $Det\tilde{\boldsymbol{K}}_T$ in the prebuckling regime.

4. Physical interpretation of a maximum of $Det\tilde{\boldsymbol{K}}_T$

For the special case of a linear stability problem:

$$\ddot{\tilde{\boldsymbol{K}}}_T = \boldsymbol{0} \quad \forall\, \lambda, \quad (51)$$

which implies that

$$\tilde{\boldsymbol{K}}_T(\lambda) = \boldsymbol{K}_0 + \lambda\bar{\boldsymbol{K}}_\sigma \quad (52)$$

where $\boldsymbol{K}_0$ is the constant small-displacement stiffness matrix and $\bar{\boldsymbol{K}}_\sigma$ is the constant initial stress matrix, evaluated after the first step of incremental FEA [5]. Substitution of (52) into (6) gives

$$\left(\boldsymbol{K}_0 + \lambda_1^*\bar{\boldsymbol{K}}_\sigma\right) \cdot \boldsymbol{v}_1^* = \boldsymbol{0}, \quad (53)$$

resulting in

$$\lambda_1^* = \lambda_S, \quad \boldsymbol{v}_1^* = \boldsymbol{v}_1 \ \forall\, \lambda. \quad (54)$$

Hence:

$$\dot{\lambda}_1^* = 0, \quad \dot{\boldsymbol{v}}_1^* = \boldsymbol{0} \ \forall\, \lambda. \quad (55)$$

This permits interpretation of $\dot{\lambda}_1^*/(\lambda_1^* - \lambda)$ (see Fig. 2b) as the increase of the aggregate nonlinearity of the underlying stability problem, normalized with respect to the eigenvalue. Consequently, point R in Fig. 2b indicates a maximum of such an increase.

Linear stability problems are a special case of buckling from a membrane stress state [4]. To obtain a mathematical condition for buckling from such a stress state, (11) is premultiplied by $\ddot{\boldsymbol{v}}_1^*$:

$$\ddot{\boldsymbol{v}}_1^* \cdot \left(\ddot{\boldsymbol{A}}_1 \cdot \boldsymbol{v}_1^* + 2\dot{\boldsymbol{A}}_1 \cdot \dot{\boldsymbol{v}}_1^* + \boldsymbol{A}_1 \cdot \ddot{\boldsymbol{v}}_1^*\right) = 0. \quad (56)$$

For buckling from a membrane stress state, (56) disintegrates into

$$\ddot{\boldsymbol{v}}_1^* \cdot \ddot{\boldsymbol{A}}_1 \cdot \boldsymbol{v}_1^* = 0 \quad \text{and} \quad \ddot{\boldsymbol{v}}_1^* \cdot \left(2\dot{\boldsymbol{A}}_1 \cdot \dot{\boldsymbol{v}}_1^* + \boldsymbol{A}_1 \cdot \ddot{\boldsymbol{v}}_1^*\right) = 0. \quad (57)$$

Eq. (57.1) indicates that $\boldsymbol{v}_1^*$ is orthogonal to $\ddot{\boldsymbol{A}}_1 \cdot \ddot{\boldsymbol{v}}_1^*$ for all values of λ in the prebuckling regime. Disintegration of (56) into (57) does not imply disintegration of (11), as was originally assumed [6]. Comparing (57) with (40) reveals that buckling from a membrane stress state precludes the possibility of a maximum of $Det\tilde{\boldsymbol{K}}_T$ in the prebuckling regime.

Buckling from a state of pure bending (lateral torsional buckling) is characterized by [4]

$$\boldsymbol{v}_1^* = \boldsymbol{v}_1 \quad \forall\, \lambda. \quad (58)$$

Hence:

$$\dot{\boldsymbol{v}}_1^* = \boldsymbol{0} \quad \forall\, \lambda. \quad (59)$$

Substitution of (59) into (10) and consideration of (58) yields

$$\dot{\boldsymbol{A}}_1 \cdot \boldsymbol{v}_1 = \boldsymbol{0}. \quad (60)$$

Comparing (60) with (40) shows that also buckling from a state of pure bending precludes the possibility of a maximum of $Det\tilde{\boldsymbol{K}}_T$ in the prebuckling regime.

Consequently, such a maximum is only possible for buckling from a general stress state, characterized by

$$0 < \frac{U_B}{U_M + U_B} < 1 \quad \forall\, \lambda, \quad (61)$$

where U_M denotes the membrane energy and U_B stands for the complement of U_M to the total strain energy. For convenience's sake, U_B is termed as the bending energy. Recalling that a maximum of $Det\tilde{\boldsymbol{K}}_T$ in the prebuckling regime correlates with a maximum of the function $\dot{\lambda}_1^*/(\lambda_1^* - \lambda)$, i.e. with a maximum increase of the aggregate normalized nonlinearity of the underlying stability problem, such a maximum must correspond to a minimum percentage bending energy of the total strain energy, characterized by

$$\frac{d}{d\lambda}\left(\frac{U_B}{U_M + U_B}\right) = 0 \ \Rightarrow \ \dot{U}_B U_M - U_B \dot{U}_M = 0, \quad (62)$$

and

$$\frac{d^2}{d\lambda^2}\left(\frac{U_B}{U_M + U_B}\right) > 0 \ \Rightarrow \ \frac{\ddot{U}_B U_M - U_B \ddot{U}_M}{(U_M + U_B)^2} > 0, \quad (63)$$

(It is worthy of note that a maximum of $U_B/(U_M + U_B)$ occurs at the stability limit [4].) As follows from (62.2), for $\lambda = \lambda_R$:

$$\frac{\dot{U}_B}{\dot{U}_M} = \frac{dU_B}{dU_M} = \frac{U_B}{U_M}. \quad (64)$$

Substitution of (64) into (63) gives

$$\frac{d^2}{d\lambda^2}\left(\frac{U_B}{U_M + U_B}\right) = \frac{(\ddot{U}_B \dot{U}_M - \dot{U}_B \ddot{U}_M)}{(U_M + U_B)^2} \frac{U_M}{\dot{U}_M}. \quad (65)$$

Insertion of (65) into

$$\frac{d^2 U_B}{dU_M^2} = \frac{\ddot{U}_B \dot{U}_M - \dot{U}_B \ddot{U}_M}{\dot{U}_M^3} \quad (66)$$

yields

$$\frac{d^2 U_B}{dU_M^2} = \frac{d^2}{d\lambda^2}\left(\frac{U_B}{U_M + U_B}\right) \frac{(U_M + U_B)^2}{U_M \dot{U}_M^2}. \quad (67)$$

Fig. 3 elucidates the physical meaning of a maximum of $Det\tilde{\boldsymbol{K}}_T$ in the prebuckling regime. According to (64), the tangent to the curve $U_B(U_M)$ at point R is equal to the chord. As follows from (63) and (67), the curvature of this curve at point R is positive. Analogous to the situation at R, the tangent to the curve $U_B(U_M)$ at the stability limit S is also equal to the chord. However, at S, in contrast to the situation at R:

$$\frac{d^2 U_B}{dU_M^2} = \frac{d^3 U_B}{dU_M^3} = 0, \quad \frac{d^4 U_B}{dU_M^4} < 0, \quad (68)$$

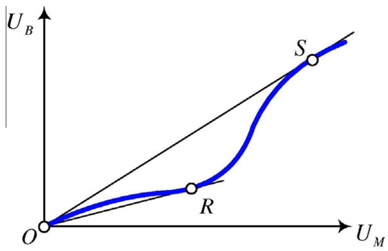

Fig. 3. U_B versus U_M for a situation characterized by a maximum of $Det\widetilde{\boldsymbol{K}}_T$ at point R in the prebuckling regime.

Fig. 4. Buckled test specimen of a pressure vessel head subjected to internal pressure [7].

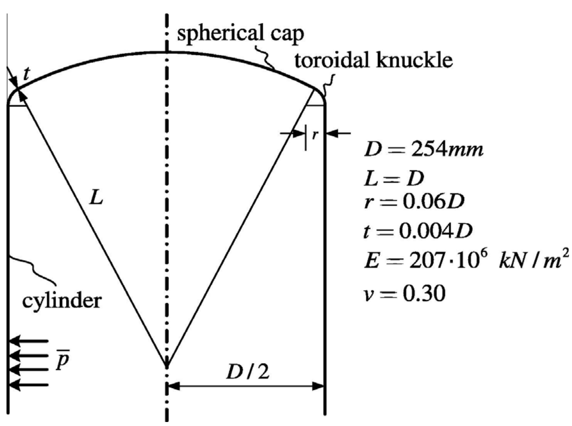

Fig. 5. Geometrical details of a cylindrical pressure vessel with a torispherical head [8].

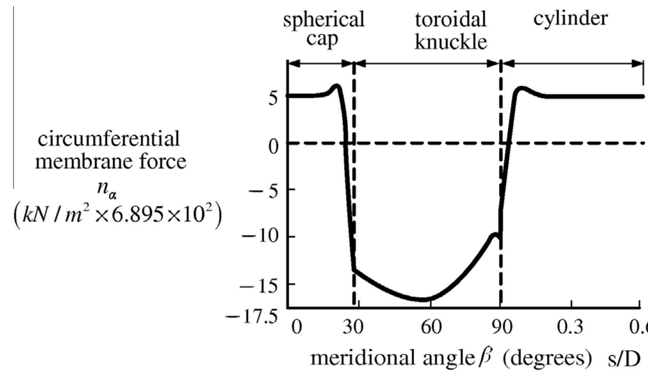

Fig. 6. Circumferential membrane force n_α due to internal pressure $\bar{p} = 703.8$ kN/m^2, in the meridional (axial) direction [8].

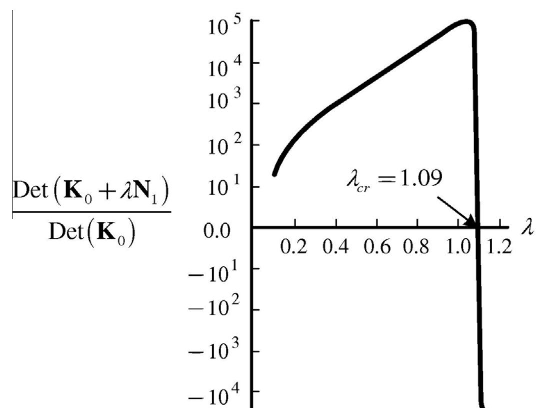

Fig. 7. Normalized determinant of a reduced form of the tangent stiffness matrix $\widetilde{\boldsymbol{K}}_T$ versus the load factor λ [8].

indicating a planar point of the curve $U_B(U_M)$ at the stability limit.

5. Numerical example

Following the failure of a large cylindrical fluid coker with a shallow spherical bottom head while undergoing its hydrostatic proof test at Avon, California, in 1956, interest in the problem of buckling of such pressure vessels under internal pressure was growing. Fig. 4 shows a buckled test specimen of a pressure vessel head that was subjected to internal pressure [7].

Fig. 5 illustrates geometrical details of a pressure vessel head that was investigated by means of the FEM 35 years ago by Kanodia, Gallagher, and the senior writer [8].

Fig. 6 shows the distribution of the circumferential membrane force n_α due to internal pressure in the meridional (axial) direction β and s/D, respectively, where s denotes the axial coordinate of the cylinder and D stands for its diameter. Fig. 7 shows a plot of the normalized determinant of a reduced form of the tangent stiffness matrix $\widetilde{\boldsymbol{K}}_T$, containing the small-displacement stiffness matrix $\boldsymbol{K}_0$ and the so-called first-order (linear) geometric stiffness matrix $\lambda \boldsymbol{N}_1$, versus the load factor λ. Normalization is performed with respect to $Det\widetilde{\boldsymbol{K}}_0$. It is seen that the determinant increases up to a value of the internal pressure close to the buckling pressure, indicating the initial load-hardening nature of the behavior of the structure. The conjecture that the vanishing of the determinant of $\ddot{\widetilde{\boldsymbol{K}}}_T$ is the condition for a maximum of the determinant of $\widetilde{\boldsymbol{K}}_T$ [9] is falsified by (50), according to which such a maximum occurs if the determinant of a specific linear combination of $\dot{\widetilde{\boldsymbol{K}}}_T$ and $\dddot{\widetilde{\boldsymbol{K}}}_T$ vanishes.

6. Conclusions

1. The determinant of the tangent stiffness matrix has a maximum in the prebuckling regime if and only if the determinant of a specific linear combination of the first and the third derivative of the tangent stiffness matrix vanishes.
2. A necessary condition for a maximum of the determinant in the prebuckling regime is a general stress state, resulting from a combination of membrane and bending action.
3. The physical meaning of a maximum of the determinant of the tangent stiffness matrix in the prebuckling regime is the one of a minimum of the percentage bending energy of the total strain energy.
4. The present work is a good example of the lasting usefulness of mathematical and physical background knowledge about numerical results that were obtained long ago.

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