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Proof Theory for Modal Logics:

Embedding between Hypersequent Calculi and Systems of Rules

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Wien, 10. Oktober 2018

Sanja Pavlović

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Abstract

The study of modal logics goes back to Aristotle and stems from the desire to finely qualify the truth of a proposition. Modal logics extend classical (or sometimes intuitionistic) logic with modal operators which enable us to express notions that cannot be well stated in classical logic. These commonly include statements like “it is possible...”, “an agent knows...”, “it might happen in the future...”. Due to their expressiveness and flexibility, modal logics are extensively used in many areas of computer science.

Formal proof systems are useful tools for investigating computational and meta-logical properties of logics; moreover, when analytic, they serve as the base for developing automated reasoning tools. In recent years, there has been a plethora of newly-developed formalisms yielding new proof systems for modal logics. These formalisms can have different expressive powers and thus be able to capture different classes of logics, or they could be more tailored towards certain applications. For example, resolution, sequent calculi, and the related tableau calculi are indispensable tools in the area of automated deduction, while Hilbert systems are a popular choice for syntactic characterization of logics due to their simplicity and modularity. Given a large number of formalisms, it is important to investigate the relationships between them and, in particular, relate their expressive powers. An effective method of doing so is through defining embeddings – procedures that given a calculus in one formalism produce a calculus for the same logic in another formalism. One-way embeddings can be used to show that one formalism subsumes another, while establishing that two formalisms are equally expressive can be done through bidirectional embeddings. In addition, embeddings allow us to transfer certain proof-theoretic results and thus avoid duplicate work. This thesis focuses on modal logic characterized by frames with simple properties, i.e., properties that can be described by first-order formulae of a restricted form. The starting point of our investigation will be analytic hypersequent calculi for these logics, a natural generalization of sequent calculi operating on finite multisets of sequents. Extending the methods of Ciabattoni and Genco, we provide an embedding between the hypersequent calculi for the aforementioned class of modal logics and sequent calculi extended by systems of rules – sets of sequent rules sharing schematic variables that can only be applied in a predetermined order. For our purposes, we restrict the vertical non-locality to at most two (unlabelled) sequent rules (2-systems). The embedding yields new analytic calculi for the considered modal logics which in turn can be used to formulate new natural deduction systems. Furthermore, by switching to hypersequent notation 2-system derivations are made local.

Kurzfassung

Das Studium der Modallogik entspringt dem Wunsch, die Wahrheit einer Aussage zu qualifizieren. Diese Erweiterung klassischer (oder manchmal intuitionistischer) Logik ermöglicht die Nutzung von modalen Operatoren, um Begriffe auszudrücken, die in klassischer Logik nicht klar beschrieben werden können. Auf Grund ihrer Ausdrucksfähigkeit und Flexibilitat wird Modallogik in vielen Bereichen der Informatik ausgiebig genutzt.

Formale Beweissysteme sind nützliche Werkzeuge zur Untersuchung von rechnerischen und metalogischen Eigenschaften von Logik. In den letzten Jahren gab es eine Fülle von neu entwickelten Formalismen, die darauf abzielen, Beweissysteme für modale Logiken zu liefern. Diese können unterschiedliche Ausdrucksstärken haben, oder sie könnten auf bestimmte Anwendungen zugeschnitten werden. Zum Beispiel sind Resolution und Sequenzenkalküle unentbehrliche Werkzeuge auf dem Gebiet der automatischen Deduktion, während Hilbert-Systeme aufgrund ihrer Einfachheit und Modularität eine beliebte Wahl für die syntaktische Charakterisierung von Logiken sind. Bei einer Vielzahl von Formalismen ist es wichtig, die Beziehungen zwischen ihnen zu untersuchen und insbesondere ihre Ausdruckskräfte zu verknüpfen. Eine effektive Methode dafür ist das Definieren von Einbettungen - Prozeduren die bei Existenz eines Kalküls in einem Formalismus einen Kalkül für die gleiche Logik in einem anderen Formalismus erzeugen. Einseitige Einbettungen können verwendet werden, um zu zeigen, dass ein Formalismus einen anderen subsumiert, während die Feststellung, dass zwei Formalismen gleich expressiv sind, durch bidirektionale Einbettungen erfolgt. Darüber hinaus ermöglichen uns Einbettungen, bestimmte beweistheoretische Ergebnisse zu übertragen und somit Doppelarbeit zu vermeiden. Diese Diplomarbeit konzentriert sich auf eine Klasse von Modallogiken, gekennzeichnet durch Frames mit bestimmten Eigenschaften, die durch Formeln erster Stufe in eingeschränkter Form beschrieben werden. Der Ausgangspunkt unserer Untersuchung werden analytische Hypersequenzkalküle für diese Logiken sein, eine natürliche Verallgemeinerung von Sequenzenkalkülen, die auf endlichen Multimengen von Sequenzen operieren. Indem wir die Methoden von Ciabattoni und Genco erweitern, bieten wir eine Einbettung zwischen diesen Kalkülen und Sequenzenkalkülen, die durch Regelsysteme erweitert werden. Diese bestehen aus einer Menge von Sequenzregeln, die sich schematische Variablen teilen, die nur in einer vorgegebenen Reihenfolge angewendet werden können. Wir beschränken diese vertikale Nichtlokalität auf höchstens zwei (unmarkierte) Regeln. Diese Einbettung liefert neue analytische Kalküle für diese Klasse der modalen Logik, die weiter verwendet werden können, um neue natürliche Deduktionssysteme für diese Logiken zu formulieren.

Contents

CHAPTER 1

Introduction

Why Modal Logic?

In everyday life we reason not only about the truth or the falsehood of statements, but also about *modes* in which statements can be true. Consider, for example, the following statement[1]:

John is happy.

In classical logic this statement is either true or false, i.e., John is either happy or he is not. However, we can think of many different ways in which we can qualify the truth of this statement. Is John *necessarily/possibly* happy? Do we *believe* John is happy? Is John *permitted* to be happy? Is John happy *now*? Is he *always* happy? Will he be happy *in the future*? All these additional words, called *modals*, alter our original statement and indicate the mode in which this statement is true.

Handling modalities in formal reasoning has a very long and rich history, having already been present in ancient logic in the form of Aristotle's modal syllogisms. As a branch of formal logic concerned with reasoning about modes of truth, contemporary modal logic was conceived in the early twentieth century through the pioneering work of C.I. Lewis and gained prominence with the introduction of relational semantics in the 1960s.

Strictly speaking, modal logic is an extension of classical logic with alethic modal operators $\Box$ and $\Diamond$, that allow us to reason about necessity and possibility. For example, if we denote by p the statement *"John is happy"*, $\Box p$ is understood as *"it is necessary that John is happy"* and $\Diamond p$ as *"it is possible that John is happy"*. In a broader sense, when we say "modal logic" we refer to a number of related logical systems that in addition to the usual sentential operators possess one or more modal operators expressing some

[1] This example was presented in [12].

kind of modality. Some examples of such systems include temporal logics used to reason about the time, deontic logics concerned with the notions of obligation and permission, epistemic and doxastic logics that are respectively used to reason about knowledge and belief, and many others.

Due to their flexibility and expressiveness, modal logics constitute a lively research area and have applications in various fields such as linguistics, epistemology, and computer science. One notable example for the use of modal logic in computer science is temporal logic, an indispensable tool in the field of formal verification used to model and analyze the behavior of programs and ensure their correctness. Another example includes a variety of (multi-modal) logics with epistemic/doxastic modalities, used in artificial intelligence for reasoning over knowledge and beliefs of intelligent agents, formalizing reasoning under uncertainty and ensuring security in distributed systems.

The Need for Analytic Proof Systems

Formal logic is primarily concerned with formalizing reasoning. As such, it is of uttermost importance to analyze which statements and arguments are considered valid in a particular logical system. This is commonly done through devising sound and complete formal proof systems, also called calculi, for the logic under investigation, which can prove exactly those statements (theorems) that are considered valid in this logic.

Hilbert-style proof systems are a popular choice for characterizing various systems of modal logics due to their simplicity and modularity. However, Hilbert-style systems do not possess the *subformula property*, i.e. a Hilbert-style proof of some theorem is not guaranteed to contain only those formulae that are subformulae of the theorem to be proven. The violation of this property makes Hilbert-style systems less than optimal for establishing essential meta-logical properties of the considered logic, such as consistency, decidability and interpolation. Further, as we might have to consider arbitrary formulae and not only those that occur as subformule of the theorem to be proven, proof search in these systems can be a rather tedious task, which makes them unsuitable for the use in automated deduction.

In fact, the subformula property (also called *analyticity*) is arguably one of the most desirable properties a proof system can possess [5, 23]. Moreover, the existence of analytic calculi is a prerequisite for developing automated reasoning methods and proving useful theorems about the logic itself. Gentzen's sequent calculs framework [13] has since its introduction been the preferred formalism for obtaining analytic calculi for a variety of logics. However, there are still logics, including some systems of modal logic, that do not possess an analytic sequent calculus. To overcome this issue, many generalizations have been introduced, like hypersequent calcului [3, 24], labelled sequent calculi [18], display calculi [7], and system of rules [20], to name some.

The Importance of Embeddings

In recent years, we have witnessed the introduction of many new formalisms aimed at yielding analytical proof systems for modal and other (non-classical) logics (see, e.g., a survey by Wansing [26] or a more recent survey by Negri [19]). These formalisms may differ in their expressive powers and thus might be able to capture a larger/different class of logics, or could be more tailored towards specific applications. For example, the aforementioned sequent calculi and Hilbert-style systems are both useful in their own way. Hilbert-style systems offer elegant syntactic characterization of logics that is often modular, while sequent calculi are optimized for proof search and can serve as the base for automated deduction. Furthermore, certain systems may be more suitable for revealing various properties of the logic at hand than others; for example consistency, decidability and complexity results can be obtained from termination of proof search in analytic calculi, which cannot be done in Hilbert proof system.

However, with a large number of diverse formalisms comes the increasing need to investigate the relationships between them and compare their expressive powers. This can be effectively achieved through defining *embeddings*, a well-known method that is extensively used in the literature, see e.g., [11, 22, 14, 9]. In our context, embeddings are procedures that given a calculus in one formalism produce a calculus for the same logic in another formalism. One-way embeddings can be used to show subsumption of one formalism by another, while bidirectional embeddings help us establish that two formalisms possess equal expressive powers. An additional advantage of providing embeddings is the fact that they allow us to transfer certain proof-theoretic results between calculi in different formalisms and thus avoid duplicate work. For example, when developing a new calculus for a logic with model-theoretical characterization, soundness and semantic completeness must be proven. While it is often easy to establish the soundness of the system, completeness proofs tend to be more involved and highly technical. However, having an embedding from derivations in a calculus complete for a given logic into another calculus whose semantic completeness has not yet been established, allows us to transfer this result without the need for a separate proof. Similarly, we can use embeddings to carry over analyticity results (e.g. the cut-elimination theorems), identify relevant information for countermodel generation, and possibly transfer computational properties.

1.1 Aim of the Thesis

The main aim of this thesis is to provide an embedding between two formalisims: hypersequent caluli and 2-systems. The former represent an extension of Gentzen's sequent calculi where the basic objects of inference are no longer sequents, but finite multisets of sequents. The latter are a restriction of Negri's systems of rules [20]. Systems of rules are sets of sequent rules that can only be applied in a predetermined order and might share schematic variables, and 2-systems restrict the vertical dependency between sequent rules in systems of rules to at most two rules.

We focus on modal logics that are characterized by classes of frames of possibly restricted shape or size. With the exception of transitivity and symmetry, all considered restrictions can be described by first order formulae of certain form and are referred to as simple frame properties. Many well-studied logics can be characterized in this way, including **KT**, **KD**, **S4**, **S5**, **S4.3**, **K4D**, **KBD**. The starting point of our investigation will be analytic hypersequent calculi for these modal logics obtained systematically from first order formulae expressing simple frame properties [16]. Extending the approach from [9, 10] developed for a different family of logics (intermediate logics), we first show how to obtain systems of rules for simple frame properties from the corresponding hypersequent rules. We then proceed to show how to transform hypersequent derivations into derivations of the same end-sequent in the corresponding sequent calculus extended by the newly-obtained 2-systems and vice versa. This embedding will yield new (unlabelled) 2-systems for the considered class of modal logic and transfer analyticity results from hypersequent calculi, thus alleviating the need for separate proofs. The 2-systems could be further used to formulate new natural deduction systems for these logics, however the normalization of such systems is still up for investigation. Further, by using hypersequent notation, derivations using 2-systems are made local.

1.2 Thesis Overview

The rest of this thesis is organized as follows: Chapter 2 introduces the important theoretical background of the thesis. We begin by formally introducing the class of modal logics to be considered in this thesis (Section 2.1). Definitions of basic notions related to sequent and hypersequent frameworks as well as the analytic hypersequent calculi for the considered class of modal logics are given in Section 2.2. The last section of Chapter 2 gives a short introduction to systems of rules. Chapter 3 presents the main results of the thesis. We first show how to transform hypersequent rules corresponding to simple frame properties into systems of rules (Section 3.1). Each of the remaining two sections is devoted to one direction of the embedding, namely translating hypersequent into 2-system derivations (Section 3.2) and vice versa (3.3). We summarize our results in Chapter 4 and give a few ideas for future research directions.

CHAPTER 2

Background

This chapter reviews briefly the background material relevant for this thesis. The first section represents a short introduction to propositional modal logic and identifies those logics that are of interest to us. In Section 2.2, we present hypersequent calculi – a well-established extension of sequent calculi that can capture many non-classical logics. We then proceed to show how to obtain sound and complete hypersequent calculi for the considered class of modal logics. Finally, we introduce in Section 2.3 the notion of system of rules, another more expressive extension of the standard sequent calculus.

2.1 Modal Logics with Simple Frame Properties

In Section 1, we gave a general and a rather informal introduction to modal logic and its relevance for computer science. As previously mentioned, *"modal logic"* usually does not refer to a single logical system. Instead, it is often used as an umbrella term for a number of related but also diverse logical systems. This thesis focuses on a particular family of modal logical systems with classical logic as their base, and the purpose of this section is to characterize the systems that are of interest to us. We begin by formally defining their syntax.

We assume a propositional modal language $\mathcal{L}$ that consists of the following elements: (i) a (countably) infinite set of propositional constants $P = \{p, q, r, \dots\}$, (ii) the logical constant $\bot$, (iii) primitive logical connectives $\neg, \wedge, \vee, \supset$, (iv) the modal operator $\Box$, and (v) the auxiliary symbols "(", ")".

Definition 1. A *well-formed formula*, also called simply a *formula* or a *sentence*, over the language $\mathcal{L}$ is defined according to the following conditions:

(i) $\bot$ is a sentence;

(ii) if $p \in P$, then p is a sentence;

(iii) if φ and ψ are sentences, then so are $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \supset \psi)$, and $\Box\varphi$; and

(iv) the only sentences are those given by (i), (ii) and (iii).

As usual, parentheses can be omitted if doing so does not lead to ambiguity.

We denote the set of all sentences over $\mathcal{L}$ by Φ. For the sake of brevity, the dual modal operator $\Diamond$ is introduced as the standard abbreviation for $\neg\Box\neg$. Hereinafter, unless stated otherwise, upper-case Greek letters (Γ, Δ, Σ, ...) denote multisets of formulae and lower-case Greek letters (φ, ψ, ...) denote single formulae. We write $\Box\Gamma$ as an abbreviation for the multiset $\{\Box\varphi \mid \varphi \in \Gamma\}$, and $\Box^{-1}\Gamma$ for $\{\varphi \mid \Box\varphi \in \Gamma\}$.

Intuitively, a term *"logical system"*, or *"logic"* for short, is understood as a system of axioms and reasoning principles that are used to distinguish valid statements and arguments from the invalid ones. As such, we can think of a logic as a set of sentences that are deemed valid. There are many ways to characterize this set of sentences. For modal logics, one popular approach is to give a proof-theoretic characterization, usually done by specifying Hilbert-style axioms and inference rules that are used to derive valid sentences. In this thesis, we focus on modal logics with model-theoretic characterization. Informally, this means that, given a class of models, the logic consists of exactly those sentences that are true in all models of the class. To this end, we define formally the semantics of the considered logics, usually given in terms of Kripke models.

Definition 2. A *Kripke frame* is a tuple $\langle W, \mathcal{R} \rangle$ where W is a non-empty set and $\mathcal{R}$ is a binary relation on W. W is usually referred to as the set of *possible worlds* and $\mathcal{R}$ is called the *accessibility* relation.

Definition 3. A *Kripke model* is a tuple $\mathcal{M} = \langle \mathcal{F}, \Vdash \rangle$, where $\mathcal{F}$ is a Kripke frame and a *forcing relation* $\Vdash$ is a binary relation between the set of possible worlds W in $\mathcal{F}$ and the set of $\mathcal{L}$-formulae Φ that satisfies the following conditions for each $w \in W$:

(K0) $\mathcal{M}, w \nVdash \bot$;

(K1) $\mathcal{M}, w \Vdash \neg\varphi$ iff $\mathcal{M}, w \nVdash \varphi$;

(K2) $\mathcal{M}, w \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, w \Vdash \varphi$ and $M, w \Vdash \psi$;

(K3) $\mathcal{M}, w \Vdash \varphi \vee \psi$ iff $\mathcal{M}, w \Vdash \varphi$ or $M, w \Vdash \psi$;

(K4) $\mathcal{M}, w \Vdash \varphi \supset \psi$ iff $\mathcal{M}, w \nVdash \varphi$ or $M, w \Vdash \psi$;

(K5) $\mathcal{M}, w \Vdash \Box\varphi$ iff $\mathcal{M}, u \Vdash \varphi$ for every $u \in W$ s.t. $w\mathcal{R}u$.

Given a Kripke frame $\mathcal{F}$, we say that a Kripke model $\mathcal{M}$ is based on $\mathcal{F}$ if $\mathcal{M} = \langle \mathcal{F}, \Vdash \rangle$, for some forcing relation $\Vdash$.

A formula φ is said to be *true in a world* w of a Kripke model $\mathcal{M}$ if $\mathcal{M}, w \Vdash \varphi$. Further, φ is *true in a Kripke model* $\mathcal{M}$ if it is true in every $w \in W$, and it is *valid in a Kripke frame* $\mathcal{F}$ if it is true in every Kripke model based on $\mathcal{F}$. Finally, we say that φ is valid in a class of Kripke frames, if it is valid in every frame in the class.

We consider modal logics that are characterized by a set of formulae valid in a certain class of Kripke frames and the differences between the logics arise precisely from the stipulations on the shape and size of the Kripke frames constituting this class. The weakest modal logic we consider is the logic **K**, obtained by adding to some Hilbert system for classical propositional logic the distribution axiom $\Box(A \supset B) \supset (\Box A \supset \Box B)$ and the inference rule of necessitation: *if* A *is a theorem of* ***K*** *then so is* $\Box A$. This logic corresponds to the logic of all Kripke frames. Most of the restrictions (also called properties) that we consider can be described using a certain form of first-order formulae. These are discussed in detail in [16] and we give a short summary of this work below.

Let $\mathcal{L}^1$ be a first-order language, consisting of (i) quantifiers $\forall$ and $\exists$, (ii) a (countably) infinite set of variables $V = \{u, w_1, w_2, \dots\}$, (iii) a binary predicate $\mathcal{R}$, (iv) logical connectives $\wedge, \vee, \neg, \supset$, (v) the equality symbol $=$, and (vi) the auxiliary symbols "(", and ")". Formulae over this language are built in the usual way, i.e., we have a set of terms that in this case consists only of elements in V and formulae over this set are built according to the following rules: (i) if $w, v \in V$ then $(w\mathcal{R}v)$ and $(w = v)$ are formulae, also called *atomic formulae* (ii) if φ and ψ are formulae then so are $\neg\varphi, (\varphi \wedge \psi), (\varphi \vee \psi)$, and $(\varphi \supset \psi)$, and (iii) if φ is a formula and $w \in V$, then $\forall w\varphi$ and $\exists w\varphi$ are formulae. From now on, we abbreviate $\forall w_1 \forall w_2 \dots \forall w_n$ by $\forall w_1, \dots, w_n$.

Definition 4. A formula over $\mathcal{L}^1$ is called *n-simple* if it is of form $\forall w_1, \dots, w_n \exists u\theta$, where θ consists of $\wedge$, $\vee$ and atomic $\mathcal{L}^1$-formulae of the form $w_i\mathcal{R}u$ or $w_i = u$, where $1 \leq i \leq n$.

A convenient way of representing n-simple formulae is by using *normal descriptors*. A normal descriptor is a set of pairs that specifies an n-simple formula θ through an equivalent formula θ' that is in disjunctive normal form, i.e., it is a disjunction of conjunctions. Intuitively, each pair describes one disjunct of θ'. The first (resp. second) element of the pair defines which conjuncts of form $w_i\mathcal{R}u$ (resp. $w_i = u$) occur in this disjunct. We next give a formal definition of normal descriptors.

Definition 5. A *normal descriptor* of an n-simple $\mathcal{L}^1$-formula $\forall w_1, \dots, w_n \exists u\theta$ is a non-empty finite set S of pairs of form $\langle S_R, S_= \rangle$, where S_R and $S_=$ are subsets of $\{1, \dots, n\}$ such that the following conditions hold:

- $S_R \cup S_= \neq \emptyset$;
- θ is equivalent to $\bigvee_{\langle S_R, S_= \rangle \in S} (\bigwedge_{i \in S_R} w_i\mathcal{R}u \wedge \bigwedge_{i \in S_=} w_i = u)$.

Proposition 1. Every n-simple formula can be expressed using a normal descriptor.

Seriality	$\forall w_1 \exists u (w_1 \mathcal{R} u)$	$\{\langle\{1\}, \emptyset\rangle\}$
Reflexivity	$\forall w_1 \exists u (w_1 \mathcal{R} u \wedge w_1 = u)$	$\{\langle\{1\}, \{1\}\rangle\}$
Directedness	$\forall w_1, w_2 \exists u (w_1 \mathcal{R} u \wedge w_2 \mathcal{R} u)$	$\{\langle\{1, 2\}, \emptyset\rangle\}$
Degenerateness	$\forall w_1, w_2 \exists u (w_1 = u \wedge w_2 = u)$	$\{\langle\emptyset, \{1, 2\}\rangle\}$
Universality	$\forall w_1, w_2 \exists u (w_1 \mathcal{R} u \wedge w_2 = u)$	$\{\langle\{1\}, \{2\}\rangle\}$
Linearity	$\forall w_1, w_2 \exists u (w_1 \mathcal{R} u \wedge w_2 = u) \vee (w_2 \mathcal{R} u \wedge w_1 = u)$	$\{\langle\{1\}, \{2\}\rangle, \langle\{2\}, \{1\}\rangle\}$
Bounded Cardinality	$\forall w_1, \ldots, w_n \exists u \bigvee_{1 \leq i < j \leq n} (w_i = u \wedge w_j = u)$	$\{\langle\emptyset, \{i, j\}\rangle : 1 \leq i < j \leq n\}$
Bounded Top Width	$\forall w_1, \ldots, w_n \exists u \bigvee_{1 \leq i < j \leq n} (w_i \mathcal{R} u \wedge w_j \mathcal{R} u)$	$\{\langle\{i, j\}, \emptyset\rangle : 1 \leq i < j \leq n\}$
Bounded Acyclic Subg.	$\forall w_1, \ldots, w_n \exists u \bigvee_{1 \leq i < j \leq n} (w_i \mathcal{R} u \wedge w_j = u)$	$\{\langle\{i\}, \{j\}\rangle : 1 \leq i < j \leq n\}$
Bounded Width	$\forall w_1, \ldots, w_n \exists u \bigvee_{1 \leq i, j \leq n; i \neq j} (w_i \mathcal{R} u \wedge w_j = u)$	$\{\langle\{i\}, \{j\}\rangle : 1 \leq i, j \leq n, i \neq j\}$

Table 2.1: Some simple frame properties [16].

We adopt the notation given in [16] and denote some normal descriptor of an n-simple $\mathcal{L}^1$-formula θ by $S(\theta)$.

Simple frame properties are properties of Kripke frames that can be described by n-simple $\mathcal{L}^1$-formulae. For example, reflexivity is a simple frame property, as it can be described by the formula $\forall w_1 \exists u (w_1 \mathcal{R} u \wedge w_1 = u)$, or by the normal descriptor $\{\langle\{1\}, \{1\}\rangle\}$. Other examples are listed in Table 2.1.

Unfortunately, there are other widely-studied frame properties that are not expressible by n-simple $\mathcal{L}^1$-formulae. Some of the most prominent examples include transitivity and symmetry, expressed by the following $\mathcal{L}^1$-formulae:

$$\theta_{tr} = \forall w_1, w_2, w_3 ((w_1 \mathcal{R} w_2 \wedge w_2 \mathcal{R} w_3) \supset w_1 \mathcal{R} w_2)$$

$$\theta_{sym} = \forall w_1, w_2 (w_1 \mathcal{R} w_2 \supset w_2 \mathcal{R} w_1)$$

As there are many useful transitive and/or symmetric modal logics, we take into consideration the two properties given above, even if they are not necessarily simple. We show in the next section how this affects the construction of the hypersequent calculi.

Definition 6. Let $\mathcal{F} = \langle W, \mathcal{R} \rangle$ be a Kripke frame and let Θ be a set of first-order formulae over $\mathcal{L}^1$. We say that $\mathcal{F}$ is a Θ-*frame* if the first-order $\mathcal{L}^1$-structure naturally induced by $\mathcal{F}$ is a model of every formula in Θ. A Kripke model is called a Θ-*model* if it is based on a Θ-frame.

In modal logic, we distinguish between two types of logical consequence, *local* and *global*. These relations are defined as follows:

Definition 7. Let Θ be a set of $\mathcal{L}^1$-formulae. For a set of formulae $\mathcal{T}$ and a formula φ we write:

1. $\mathcal{T} \vdash^l_{\mathbf{K}+\Theta} \varphi$ iff for every Θ-model $\mathcal{M} = \langle W, \mathcal{R}, \Vdash \rangle$ and every $w \in W$, either $\mathcal{M}, w \nVdash \psi$, for some $\psi \in \mathcal{T}$, or $\mathcal{M}, w \Vdash \varphi$. We say that φ is a *local consequence* of $\mathcal{T}$ in the logic $\mathbf{K}+\Theta$.

2. $\mathcal{T} \vdash^g_{\mathbf{K}+\Theta} \varphi$ iff for every Θ-model $\mathcal{M} = \langle W, \mathcal{R}, \Vdash \rangle$, if every formula in $\mathcal{T}$ is true in $\mathcal{M}$, then so is φ. We say that φ is a *global consequence* of $\mathcal{T}$ in the logic $\mathbf{K}$+Θ.

Adhering to the standard modal logic nomenclature, we use the following abbreviations:

- $\vdash^l_{\mathbf{K}+\{\Theta_{tr}\}} = \vdash^l_{\mathbf{K4}}$, $\vdash^g_{\mathbf{K}+\{\Theta_{tr}\}} = \vdash^g{}_{\mathbf{K4}}$
- $\vdash^l_{\mathbf{K}+\{\Theta_{sym}\}} = \vdash^l_{\mathbf{KB}}$, $\vdash^g_{\mathbf{K}+\{\Theta_{sym}\}} = \vdash^g_{\mathbf{KB}}$
- $\vdash^l_{\mathbf{K}+\{\Theta_{tr},\Theta_{sym}\}} = \vdash^l_{\mathbf{K4B}}$, $\vdash^g_{\mathbf{K}+\{\Theta_{tr},\Theta_{sym}\}} = \vdash^g_{\mathbf{K4B}}$.

Several well-studied logics are particular instances of $\vdash^l_{\mathbf{K}+\Theta}$ and $\vdash^g_{\mathbf{K}+\Theta}$. For example, adding the formula for universality to **K** yields **S5** and taking the formulae for degenerateness and reflexivity gives us classical logic. If we allow the use of non-simple properties of transitivity and symmetry we can capture many more logics like **S4**, **S4.3**, **KD4**, **K4.3**, **KDB**, **KTB** etc.

2.2 Sequent and Hypersequent Calculi

The sequent framework was originally introduced by Gentzen in 1935. In the first part of his influential paper *"Untersuchungen über das logische Schließen"* [13], Gentzen presented sequent calculi for classical and intuitionistic logic and proceeded to prove the *cut-elimination theorem*, also known as *Gentzen's Hauptsatz*, that states that every sequent derivable in these systems possesses a derivation in which no application of the cut-rule occurs. This result has many wide-reaching consequences that established the significance of the sequent calculus and made it a preferred framework for formulating well-behaved proof-theoretic characterizations for many different logics. For example, one of the most important corollaries of the cut-elimination theorem is the analyticity of these calculi in the sense that all formulae occurring in a proof are subformulae of what is to be proven.

We begin with formally defining the notions of sequents and derivations in a sequent calculus.

Definition 8. A *sequent* is a pair $\langle \Gamma, \Delta \rangle$, where Γ and Δ are finite multisets of formulae. Following the standard approach in the literature, we denote this pair by $\Gamma \Rightarrow \Delta$.

We abbreviate $\Gamma \cup \{\varphi\}$ by $\Gamma, \{\varphi\}$, for a multiset of formulae Γ and a single formula φ.

A sequent calculus consists of initial sequents (axioms) and inference rules of form:

$$\frac{S_1 \qquad \dots \qquad S_n}{S}$$

where $S, S_1, \dots, S_n$ are sequents.

Definition 9. A *derivation*, or a *proof*, in a sequent calculus of a sequent S from sequents $S_1, \ldots, S_n$ is a finite tree of sequents rooted in S, where each leaf is either an axiom or a sequent S_i, $1 \leq i \leq n$, and each non-leaf node is obtained from its immediate predecessors by an application of some inference rule in the system.

Unfortunately, not all logics have a cut-free and/or analytic sequent calculus. In modal logics, one of the most notable examples for which no cut-free sequent calculus is known is the logic **S5**. Various generalizations of the sequent calculus have been developed in order to overcome this issue. One such generalization is the hypersequent framework [24, 3] that instead of single sequents operates on a multiset of sequents. The additional expressive power comes from rules that manipulate simultaneously different components of one or more hypersequents, thus allowing us to capture logics that cannot be captured by the (ordinary) sequent framework. The hypersequent framework has yielded analytic calculi for many non-classical logics, including modal logics, see, e.g., [5, 16, 17, 15]. This section reviews the work of Lahav [16], in which he shows how to systematically construct hypersequent calculi for the modal logics with simple frame properties.

Definition 10. A *hypersequent* is a finite multiset of sequents. We adopt the usual notation and write $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ instead of $\{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n\}$.

We refer to the sequents in a hypersequent as the *components*. Further, we abbreviate $H \cup \{S\}$ by $H \mid \{S\}$, where H is a hypersequent and S is a sequent. Like sequent calculi, hypersequent calculi consist of initial hypersequents (axioms) and inference rules, and the notion of derivation remains the same, except that we work with trees of hypersequents instead of sequents.

We next present the hypersequent calculi for the logics **K**, **K4**, **KB**, and **K4B**, which will serve as bases for obtaining analytic hypersequent calculi for modal logics with simple frame properties.

The hypersequent calculus **HK** for the logic **K** is displayed in Table 2.2. This calculus is obtained by extending the sequent calculus for classical propositional logic with the rule $(\Rightarrow \Box)$ and decorating it with hypersequents. Note that for logic **K**, the hyperlevel is redundant, as the sequent version of this calclus is already sound, complete and cut-free for **K** (see, e.g., survery by Ono [21]). Nonetheless, we consider the hypersequent version as it will later be extended with hypersequent rules that manipulate multiple components at once – this is precisely what allows us to capture modal logics with simple frame properties that might not possess a cut-free sequent calculus.

We mentioned in the previous section that we will allow two non-simple frame properties: transitivity and symmetry. Unfortunately, these cannot be accommodated by simply adding hypersequent rules to **HK**, but rather require us to change the base calculus. Thus, we present the hypersequent calculi **HK4**, **HKB**, **HK4B** for logics **K4**, **KB**, and **K4B**, respectively.

Axioms: $\varphi \Rightarrow \varphi \quad \bot \Rightarrow$

Rules:

$$\frac{H|\Gamma \Rightarrow \Delta}{H|\Gamma, \varphi \Rightarrow \Delta}\,(IW \Rightarrow) \qquad \frac{H|\Gamma \Rightarrow \Delta}{H|\Gamma \Rightarrow \Delta, \varphi}\,(\Rightarrow\ IW) \qquad \frac{H}{H|\Gamma \Rightarrow \Delta}\,(EW)$$

$$\frac{H|\Gamma, \varphi, \varphi \Rightarrow \Delta}{H|\Gamma, \varphi \Rightarrow \Delta}\,(IC\ \Rightarrow) \qquad \frac{H|\Gamma \Rightarrow \Delta, \varphi, \varphi}{H|\Gamma \Rightarrow \Delta, \varphi}\,(\Rightarrow\ IC) \qquad \frac{H|\Gamma \Rightarrow \Delta|\Gamma \Rightarrow \Delta}{H|\Gamma \Rightarrow \Delta}\,(EC)$$

$$\frac{H|\Gamma \Rightarrow \Delta, \varphi}{H|\Gamma, \neg\varphi \Rightarrow \Delta}\,(\neg \Rightarrow) \qquad \frac{H|\Gamma, \varphi \Rightarrow \Delta}{H|\Gamma \Rightarrow \Delta, \neg\varphi}\,(\Rightarrow \neg) \qquad \frac{H|\Gamma \Rightarrow \varphi}{H|\Box\Gamma \Rightarrow \Box\varphi}\,(\Rightarrow\ \Box)$$

$$\frac{H|\Gamma, \varphi_1 \Rightarrow \varphi_2, \Delta}{H|\Gamma \Rightarrow \varphi_1 \supset \varphi_2, \Delta}\,(\Rightarrow \supset) \qquad \frac{H|\Gamma, \varphi_1, \varphi_2 \Rightarrow \Delta}{H|\Gamma\varphi_1 \wedge \varphi_2 \Rightarrow \Delta}\,(\wedge \Rightarrow) \qquad \frac{H|\Gamma \Rightarrow \Delta, \varphi_i}{H|\Gamma \Rightarrow \Delta, \varphi_1 \vee \varphi_2,}\,(\Rightarrow \vee)$$

$$\frac{H|\Gamma \Rightarrow \Delta, \varphi_1 \qquad H|\Gamma, \varphi_2 \Rightarrow \Delta}{H|\Gamma, \varphi_1 \supset \varphi_2 \Rightarrow \Delta}\,(\supset\Rightarrow) \qquad \frac{H|\Gamma \Rightarrow \Delta, \varphi_1 \qquad H|\Gamma \Rightarrow \Delta, \varphi_2}{H|\Gamma, \Rightarrow \Delta, \varphi_1 \wedge \varphi_2}\,(\Rightarrow \wedge)$$

$$\frac{H|\Gamma, \varphi_1 \Rightarrow \Delta \qquad H|\Gamma, \varphi_2 \Rightarrow \Delta}{H|\Gamma, \varphi_1 \vee \varphi_2 \Rightarrow \Delta}\,(\vee \Rightarrow) \qquad \frac{H|\Gamma \Rightarrow \Delta, \varphi \qquad H|\Gamma, \varphi \Rightarrow \Delta}{H|\Gamma \Rightarrow \Delta}\,(cut)$$

Table 2.2: The hypersequent calculus **HK** for **K**, where $i \in \{1, 2\}$

We obtain the calculus **HK4** for logic **K4** from **HK** by replacing the rule ($\Rightarrow\ \Box$) with the rule:

$$(\Rightarrow \Box_4)\ \frac{H|\Gamma, \Box\Gamma \Rightarrow \varphi}{H|\Box\Gamma \Rightarrow \Box\varphi}$$

Similarly, we get the calculus **HKB** by replacing the rule ($\Rightarrow\ \Box$) with:

$$(\Rightarrow \Box_B)\ \frac{H|\Gamma \Rightarrow \varphi, \Box\Delta}{H|\Box\Gamma \Rightarrow \Box\varphi, \Delta}$$

Finally, **HK4B** is obtained by replacing ($\Rightarrow\ \Box$) with:

$$(\Rightarrow \Box_{4B})\ \frac{H|\Gamma, \Box\Gamma \Rightarrow \varphi, \Box\Delta, \Box\Box^{-1}\Delta}{H|\Box\Gamma \Rightarrow \Box\varphi, \Delta}$$

Before we proceed with our agenda, we give a few remarks. Given a hypersequent rule

$$\frac{H|G_1 \qquad \ldots \qquad H|G_n}{H|G}$$

where $G, G_1, \ldots, G_n$, and H are hypersequents, we refer to the components of H as the *context components*, and the components in $G, G_1, \ldots, G_n, G$ are called *active components*. Notice that the calculi presented above have the following two properties:

(i) premisses within the same hypersequent rule all have the same context components,

(ii) each premiss in a hypersequent rule, with the exception of (EC), has one active component.

Each of the hypersequent systems induces two consequence relations, corresponding to the previously-described local and global consequence relations.

Definition 11. Consider a sequent calculus **HL**, where **L** is one of the following logics: **K**, **K4**, **KB**, **K4B**. Further, let $\mathcal{T}$ be a set of formulae, and φ be a single formula.

1. A *"local"* relation, denoted by $\vdash^{l}_{\mathbf{HL}}$, is defined as follows: $\mathcal{T} \vdash^{l}_{\mathbf{HL}} \varphi$ if there exists a derivation of $\Gamma \Rightarrow \varphi$ in **HL**, for some finite multiset Γ, where every formula in Γ is in $\mathcal{T}$.

2. A *"global"* relation, denoted by $\vdash^{g}_{\mathbf{HL}}$, is defined as follows: $\mathcal{T} \vdash^{g}_{\mathbf{HL}} \varphi$ if there exists a derivation of $\Rightarrow \varphi$ in **HL** from the assumptions $\{\Rightarrow \psi : \psi \in \mathcal{T}\}$.

The introduced calculi have the following properties:

Theorem 1. Let **L** be one of the following logics: **K**, **K4**, **KB**, **K4B**. The hypersequent calculus **HL** is sound and complete for **L**, i.e., $\mathcal{T} \vdash^{l}_{\mathbf{HL}} \varphi$ iff $\mathcal{T} \vdash^{l}_{\mathbf{L}} \varphi$, and $\mathcal{T} \vdash^{g}_{\mathbf{HL}} \varphi$ iff $\mathcal{T} \vdash^{g}_{\mathbf{L}} \varphi$.

We next formally define what it means for a (hyper)sequent calculus to enjoy cut-admissibility or analyticity.

Definition 12. A hypersequent calculus **HL** for the logic **L** enjoys *strong cut-admissibility* if, for every hypersequent H it holds that whenever there exists a derivation of H in **HL** from a set $\mathbb{H}$ of hypersequents, then there must exist a derivation of H in **HL** from $\mathbb{H}$ where only formulae from $\mathbb{H}$ serve as cut-formulae.

In the special case when $\mathbb{H} = \emptyset$, we have that if there exists a **HL** derivation of H, then there exists a *cut-free* **HL** derivation of H, i.e., a derivation that contains no cut rule applications.

Definition 13. A hypersequent calculus **HL** for the logic **L** is *analytic* if, for every hypersequent H it holds that whenever there exists a derivation of H in **HL** from a set $\mathbb{H}$ of hypersequents, then there must exist a derivation of H in **HL** from $\mathbb{H}$ containing only the subformulae of H and $\mathbb{H}$.

Theorem 2. **HK** and **HK4** enjoy strong cut-admissibility. Further, **HKB** and **HK4B** are analytic.

Note that the analyticity of **HK** and **HK4** follows from the fact that they enjoy strong cut-admissibility.

We now show, using the approach given by Lahav in [16], how to construct sound and complete hypersequent calculi for a wide class of modal logics, in particular, for those modal logics obtained from **K**, **K4**, **KB** or **K4B** by assuming additional simple frame properties.

Theorem 3. Let Θ be a set of n-simple $\mathcal{L}^1$-formulae and let $\mathbb{R} = \{r^{\mathbf{HK}}_{S(\theta)} : \theta \in \Theta\}$ be a set of hypersequent rules, where $r^{\mathbf{HK}}_{S(\theta)}$ is defined as

$$\frac{\{H \mid \bigcup_{i \in S_=} \Gamma_i, \bigcup_{i \in S_R} \Gamma'_i \Rightarrow \bigcup_{i \in S_=} \Delta_i : \langle S_R, S_= \rangle \in S(\theta)\}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \ \ldots \ \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n} \ (r^{\mathbf{HK}}_{S(\theta)}),$$

for any n-simple $\mathcal{L}^1$-formula $\theta \in \Theta$. Further, let **HK**+$\mathbb{R}$ denote the system obtained by augmenting **HK** with the rules from $\mathbb{R}$ and let the relations $\vdash^{l}_{\mathbf{HK}+\mathbb{R}}$ and $\vdash^{G}_{\mathbf{HK}+\mathbb{R}}$ be defined analogously to those in Definition 11. Then (i) $\vdash^{l}_{\mathbf{HK}+\mathbb{R}} = \vdash^{l}_{\mathbf{K}+\Theta}$, (ii) $\vdash^{g}_{\mathbf{HK}+\mathbb{R}} = \vdash^{g}_{\mathbf{K}+\Theta}$ and (iii) **HK**+$\mathbb{R}$ enjoys strong cut-admissibility.

Similar results hold for the other three hypersequent calculi:

Theorem 4. Let Θ be a set of n-simple $\mathcal{L}^1$-formulae and let $\mathbb{R} = \{r^{\mathbf{HK4}}_{S(\theta)} : \theta \in \Theta\}$ be a set of hypersequent rules, where $r^{\mathbf{HK4}}_{S(\theta)}$ is defined as

$$\frac{\{H \mid \bigcup_{i \in S_=} \Gamma_i, \bigcup_{i \in S_R} \Gamma'_i \cup \Box\Gamma'_i \Rightarrow \bigcup_{i \in S_=} \Delta_i : \langle S_R, S_= \rangle \in S(\theta)\}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \ \ldots \ \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n} \ (r^{\mathbf{HK4}}_{S(\theta)}),$$

for any n-simple $\mathcal{L}^1$-formula $\theta \in \Theta$. Further, let **HK4**+$\mathbb{R}$ denote the system obtained by augmenting **HK4** with the rules from $\mathbb{R}$ and let the relations $\vdash^{l}_{\mathbf{HK4}+\mathbb{R}}$ and $\vdash^{G}_{\mathbf{HK4}+\mathbb{R}}$ be defined analogously to those in Definition 11. Then (i) $\vdash^{l}_{\mathbf{HK4}+\mathbb{R}} = \vdash^{l}_{\mathbf{K4}+\Theta}$, (ii) $\vdash^{g}_{\mathbf{HK4}+\mathbb{R}} = \vdash^{g}_{\mathbf{K4}+\Theta}$ and (iii) **HK4**+$\mathbb{R}$ enjoys strong cut-admissibility.

Theorem 5. Let Θ be a set of n-simple $\mathcal{L}^1$-formulae and let $\mathbb{R} = \{r^{\mathbf{HKB}}_{S(\theta)} : \theta \in \Theta\}$ be a set of hypersequent rules, where $r^{\mathbf{HKB}}_{S(\theta)}$ is defined as

$$\frac{\{H \mid \bigcup_{i\in S_=}\Gamma_i, \bigcup_{i\in S_R}\Gamma'_i \Rightarrow \bigcup_{i\in S_=}\Delta_i, \bigcup_{i\in S_R}\Box\Delta'_i : \langle S_R, S_=\rangle \in S(\theta)\}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1, \Delta'_1 \mid \ \dots \ \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n, \Delta'_n}\ (r^{\mathbf{HKB}}_{S(\theta)}),$$

for any n-simple $\mathcal{L}^1$-formula $\theta \in \Theta$. Further, let **HKB**+$\mathbb{R}$ denote the system obtained by augmenting **HKB** with the rules from $\mathbb{R}$ and let the relations $\vdash^l_{\mathbf{HKB}+\mathbb{R}}$ and $\vdash^G_{\mathbf{HKB}+\mathbb{R}}$ be defined analogously to those in Definition 11. Then (i) $\vdash^l_{\mathbf{HKB}+\mathbb{R}}=\vdash^l_{\mathbf{KB}+\Theta}$,(ii) $\vdash^g_{\mathbf{HKB}+\mathbb{R}}=\vdash^g_{\mathbf{KB}+\Theta}$ and (iii) **HKB**+$\mathbb{R}$ is analytic, i.e., it has the subformula property.

Theorem 6. Let Θ be a set of n-simple $\mathcal{L}^1$-formulae and let $\mathbb{R} = \{r^{\mathbf{HK4B}}_{S(\theta)} : \theta \in \Theta\}$ be a set of hypersequent rules, where $r^{\mathbf{HK4B}}_{S(\theta)}$ is defined as

$$\frac{\{H \mid \bigcup_{i\in S_=}\Gamma_i, \bigcup_{i\in S_R}\Gamma'_i \cup \Box\Gamma'_i \Rightarrow \bigcup_{i\in S_=}\Delta_i, \bigcup_{i\in S_R}\Box\Delta'_i \cup \Box\Box^{-1}\Delta'_i : \langle S_R, S_=\rangle \in S(\theta)\}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1, \Delta'_1 \mid \ \dots \ \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n, \Delta'_n}\ (r^{\mathbf{HK4B}}_{S(\theta)}),$$

for any n-simple $\mathcal{L}^1$-formula $\theta \in \Theta$. Further, let **HK4B**+$\mathbb{R}$ denote the system obtained by augmenting **HK4B** with the rules from $\mathbb{R}$ and let the relations $\vdash^l_{\mathbf{HK4B}+\mathbb{R}}$ and $\vdash^G_{\mathbf{HK4B}+\mathbb{R}}$ be defined analogously to those in Definition 11. Then (i) $\vdash^l_{\mathbf{HK4B}+\mathbb{R}}=\vdash^l_{\mathbf{K4B}+\Theta}$, (ii) $\vdash^g_{\mathbf{HK4B}+\mathbb{R}}=\vdash^g_{\mathbf{K4B}+\Theta}$ and (iii) **HK4B**+$\mathbb{R}$ is analytic, i.e., it has the subformula property.

Definition 14. Let **L** be **K**, **K4**, **KB** or **K4B**, and let θ be some n-simple $\mathcal{L}^1$-formula. The rule $(r^{\mathbf{HL}}_{S(\theta)})$ is said to be *induced by a normal descriptor of θ for **HL***.

The proofs of all theorems presented in this section can be found in [16]. Table 2.3 shows examples of hypersequent rules induced by normal descriptors for **HK**, as given in [16]. In particular, these rules correspond to the frame properties listed in Table 2.1.

We next give a few examples in order to illustrate the approach given above.

Example 1. Consider the modal logic **S5** commonly characterized by a class of frames in which the accessibility relation $\mathcal{R}$ is an equivalence relation. However, **S5** can also be characterized by a class of universal frames, i.e., in which every two worlds are accessible from each other. This is convenient for us, as universality is a simple frame property described by the 2-simple $\mathcal{L}^1$-formula $\forall w_1, w_2 \exists u(w_1\mathcal{R}u \land w_2 = u)$ with a normal descriptor $\{\langle\{1\},\{2\}\rangle\}$.

We then obtain a hypersequent calculus for **S5** by adding to **HK** the hypersequent rule corresponding to the property of universality:

$$\frac{H|\Gamma_2, \Gamma'_1 \Rightarrow \Delta_2}{H|\Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1|\Gamma_2, \Box\Gamma'_2 \Rightarrow \Delta_2}\ (r^{\mathbf{HL}}_{S(\theta)})$$

The following is then the derivation of axiom $(5) : \neg\Box\neg\varphi \supset \Box\neg\Box\neg\varphi$ in **HK**+$(r^{\mathbf{HL}}_{S(\theta)})$, for any formula φ:

$$\frac{H|\Gamma_1' \Rightarrow}{H|\Gamma_1, \Box\Gamma_1' \Rightarrow \Delta_1}\ \text{Seriality} \qquad \frac{H|\Gamma_1', \Gamma_1' \Rightarrow \Delta_1}{H|\Gamma_1, \Box\Gamma_1' \Rightarrow \Delta_1}\ \text{Reflexivity} \qquad \frac{H|\Gamma_1', \Gamma_2' \Rightarrow}{H|\Gamma_1, \Box\Gamma_1' \Rightarrow \Delta_1|\Gamma_2, \Box\Gamma_2' \Rightarrow \Delta_2}\ \text{Directedness}$$

$$\frac{H|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{H|\Gamma_1, \Box\Gamma_1' \Rightarrow \Delta_1|\Gamma_2, \Box\Gamma_2' \Rightarrow \Delta_2}\ \text{Degenerateness} \qquad \frac{H|\Gamma_2, \Gamma_1' \Rightarrow \Delta_2}{H|\Gamma_1, \Box\Gamma_1' \Rightarrow \Delta_1|\Gamma_2, \Box\Gamma_2' \Rightarrow \Delta_2}\ \text{Universality}$$

$$\frac{H|\Gamma_2, \Gamma_1' \Rightarrow \Delta_2 \qquad H|\Gamma_1, \Gamma_2' \Rightarrow \Delta_1}{H|\Gamma_1, \Box\Gamma_1' \Rightarrow \Delta_1|\Gamma_2, \Box\Gamma_2' \Rightarrow \Delta_2}\ \text{Linearity} \qquad \frac{\{H|\Gamma_i, \Gamma_j \Rightarrow \Delta_i, \Delta_j : 1 \leq i < j \leq n\}}{H|\Gamma_1, \Box\Gamma_1' \Rightarrow \Delta_1|\ldots|\Gamma_n, \Box\Gamma_n' \Rightarrow \Delta_n}\ \text{Bounded Cardinality}$$

$$\frac{\{H|\Gamma_i', \Gamma_j' \Rightarrow : 1 \leq i < j \leq n\}}{H|\Gamma_1, \Box\Gamma_1' \Rightarrow \Delta_1|\ldots|\Gamma_n, \Box\Gamma_n' \Rightarrow \Delta_n}\ \text{Bounded Top Width} \qquad \frac{\{H|\Gamma_j, \Gamma_i' \Rightarrow \Delta_j : 1 \leq i < j \leq n\}}{H|\Gamma_1, \Box\Gamma_1' \Rightarrow \Delta_1|\ldots|\Gamma_n, \Box\Gamma_n' \Rightarrow \Delta_n}\ \text{Bounded Acyclic Subgraph}$$

$$\frac{\{H|\Gamma_j, \Gamma_i' \Rightarrow \Delta_j : 1 \leq i \leq j \leq n, i \neq j\}}{H|\Gamma_1, \Box\Gamma_1' \Rightarrow \Delta_1|\ldots|\Gamma_n, \Box\Gamma_n' \Rightarrow \Delta_n}\ \text{Bounded Width}$$

Table 2.3: Hypersequent rules induced by simple frame properties for **HK** [16].

$$(5) \quad \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\neg\varphi \Rightarrow \neg\varphi}{\Box\neg\varphi \Rightarrow \ \mid \ \Rightarrow \neg\varphi}\ (r^{\mathbf{HL}}_{S(\theta)})}{\Rightarrow \neg\Box\neg\varphi \mid \Rightarrow \neg\varphi}\ (\Rightarrow \neg)}{\Rightarrow \Box\neg\Box\neg\varphi \mid \Rightarrow \neg\varphi}\ (\Rightarrow \Box)}{\neg\Box\neg\varphi \Rightarrow \Box\neg\Box\neg\varphi \mid \Rightarrow \neg\varphi}\ (IW \Rightarrow)}{\neg\Box\neg\varphi \Rightarrow \Box\neg\Box\neg\varphi \mid \Rightarrow \Box\neg\varphi}\ (\Rightarrow \Box)}{\neg\Box\neg\varphi \Rightarrow \Box\neg\Box\neg\varphi \mid \neg\Box\neg\varphi \Rightarrow}\ (\neg \Rightarrow)}{\neg\Box\neg\varphi \Rightarrow \Box\neg\Box\neg\varphi \mid \neg\Box\neg\varphi \Rightarrow \Box\neg\Box\neg\varphi}\ (\Rightarrow IW)}{\neg\Box\neg\varphi \Rightarrow \Box\neg\Box\neg\varphi}\ (EC)}{\Rightarrow \neg\Box\neg\varphi \supset \Box\neg\Box\neg\varphi}\ (\Rightarrow\supset)$$

Example 2. Consider next the modal logic **S4.3**. This logic can be characterized by frames whose accessibility relation $\mathcal{R}$is a total preorder, i.e., for all worlds w, u, v we have:

- $w\mathcal{R}w$, i.e., $\mathcal{R}$ is reflexive,
- if $w\mathcal{R}u$ and $u\mathcal{R}v$, then $w\mathcal{R}v$, i.e., $\mathcal{R}$ is transitive and
- either $w\mathcal{R}u$ or $u\mathcal{R}w$, i.e., $\mathcal{R}$ is linear.

Since $\mathcal{R}$ is transitive, we choose **HK4** as our base calculus. We then augment this calculus with the following two rules:

$$\frac{H|\Gamma_1, \Gamma_1', \Box\Gamma_1' \Rightarrow \Delta_1}{H|\Gamma_1, \Box\Gamma_1' \Rightarrow \Delta_1} \, (r^{\mathbf{HK4}}_{S(\theta_{ref})}) \qquad \frac{\Gamma_2, \Gamma_1', \Box\Gamma_1' \Rightarrow \Delta_2 \qquad \Gamma_1, \Gamma_2', \Box\Gamma_2' \Rightarrow \Delta_1}{H|\Gamma_1, \Box\Gamma_1' \Rightarrow \Delta_1|\Gamma_2, \Box\Gamma_2' \Rightarrow \Delta_2} \, (r^{\mathbf{HK4}}_{S(\theta_{lin})})$$

where $\theta_{ref} = \forall w_1 \exists u (w_1 \mathcal{R} u \wedge w_1 = u)$ describes reflexivity and $\theta_{lin} = \forall w_1, w_2 \exists u (w_1 \mathcal{R} u \wedge u = w_2) \vee (w_2 \mathcal{R} u \wedge u = w_1)$ describes linearity.

The logic **S4.3** can also be characterized with the following Hilbert-style axiom schemata:

(4) $\Box A \supset \Box\Box A$

(T) $\Box A \supset A$

(H) $\Box(\Box A \supset B) \vee \Box(\Box B \supset A)$.

The following are the derivations of these axioms in **HK4**+$\{(r^{\mathbf{HK4}}_{S(\theta_{ref})}), (r^{\mathbf{HK4}}_{S(\theta_{lin})})\}$:

$$(4) \quad \cfrac{\cfrac{\cfrac{\Box\varphi \Rightarrow \Box\varphi}{\varphi, \Box\varphi \Rightarrow \Box\varphi} (IW \Rightarrow)}{\Box\varphi \Rightarrow \Box\Box\varphi} (\Rightarrow \Box_4)}{\Rightarrow \Box\varphi \supset \Box\Box\varphi} (\Rightarrow\supset) \qquad (T) \quad \cfrac{\cfrac{\cfrac{\varphi \Rightarrow \varphi}{\varphi, \Box\varphi \Rightarrow \varphi} (IW \Rightarrow)}{\Box\varphi \Rightarrow \varphi} (r^{\mathbf{HK4}}_{S(\theta_{ref})})}{\Rightarrow \Box\varphi \supset \varphi} (\Rightarrow\supset)$$

$$(H) \quad \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\varphi \Rightarrow \varphi}{\varphi, \Box\varphi \Rightarrow \varphi} (IW \Rightarrow) \qquad \cfrac{\psi \Rightarrow \psi}{\psi, \Box\psi \Rightarrow \psi} (IW \Rightarrow)}{\Box\varphi \Rightarrow \psi \mid \Box\psi \Rightarrow \varphi} (r^{\mathbf{HK4}}_{S(\theta_{lin})})}{\Box\varphi \Rightarrow \psi \mid \Rightarrow \Box\psi \supset \varphi} (\Rightarrow\supset)}{\Rightarrow \Box\varphi \supset \psi \mid \Rightarrow \Box\psi \supset \varphi} (\Rightarrow\supset)}{\Rightarrow \Box\varphi \supset \psi \mid \Rightarrow \Box(\Box\psi \supset \varphi)} (\Rightarrow \Box_4)}{\Rightarrow \Box(\Box\varphi \supset \psi) \mid \Rightarrow \Box(\Box\psi \supset \varphi)} (\Rightarrow \Box_4)}{\Rightarrow \Box(\Box\varphi \supset \psi) \mid \Rightarrow \Box(\Box\varphi \supset \psi) \vee \Box(\Box\psi \supset \varphi)} (\Rightarrow \vee)}{\Rightarrow \Box(\Box\varphi \supset \psi) \vee \Box(\Box\psi \supset \varphi) \mid \Rightarrow \Box(\Box\varphi \supset \psi) \vee \Box(\Box\psi \supset \varphi)} (\Rightarrow \vee)}{\Rightarrow \Box(\Box\varphi \supset \psi) \vee \Box(\Box\psi \supset \varphi)} (EC)$$

2.3 System of rules

The main task of this thesis is to define an embedding between the hypersequent calculi presented in the previous section and the corresponding sequent calculi possibly containing *systems of rules*. Systems of rules were introduced by Negri in [20] and they essentially consist of a set of sequent or labeled sequent rules that share schematic variables or labels and that have to be applied in a certain order.

We begin with an introductory example to systems of rules presented in [20]. In her work, Negri focuses on labeled sequent calculi for modal logics. The labeled sequent calculus for the logic **K** was introduced in [18] and is given in Table 2.4. Note that, previously, sequents were defined as objects $\Gamma \Rightarrow \Delta$, where Γ and Δ are multisets of formulae. In this labeled system, sequents still have the same form, however Γ and Δ are now multisets of atoms of form $x\mathcal{R}y$ and $x : \varphi$. Intuitively, x and y represent worlds in a Kripke model, the atom $x\mathcal{R}y$ states that the world y is accessible from the world x, and $x : \varphi$ states that formula φ is true in world x.

Example 3. Consider a modal logic obtained from some Hilbert-style system for the logic **K** by adding the axiom $A \supset \Diamond\Box\Diamond A$. This axiom corresponds to the (non-simple) frame property $\forall x \exists y(x\mathcal{R}y \wedge \forall z(y\mathcal{R}z \supset z\mathcal{R}x))$. We obtain a calculus for this logic by extending the labeled sequent calculus **G3K** with the following system of rules:

$$\left\{\begin{array}{l} \dfrac{z\mathcal{R}x, y\mathcal{R}z, \Gamma \Rightarrow \Delta}{y\mathcal{R}z, \Gamma \Rightarrow \Delta}\,(r_1) \\[2em] \dfrac{z\mathcal{R}x, y\mathcal{R}z, \Gamma \Rightarrow \Delta}{y\mathcal{R}z, \Gamma \Rightarrow \Delta}\,(r_2) \end{array}\right.$$

where y is not in Γ, Δ and the upper rule can only be applied on some branch if it is followed by an application of the lower rule, i.e., every branch using (r_1) must be of the following form:

$$\begin{array}{c} \mathcal{D} \\ \vdots \\ \dfrac{z\mathcal{R}x, y\mathcal{R}z, \Gamma' \Rightarrow \Delta'}{y\mathcal{R}z, \Gamma' \Rightarrow \Delta'}\,(r_1) \\ \vdots \\ \dfrac{z\mathcal{R}x, y\mathcal{R}z, \Gamma \Rightarrow \Delta}{y\mathcal{R}z, \Gamma \Rightarrow \Delta}\,(r_2) \end{array}$$

In this thesis, we focus on defining non-labeled sequent calculi with systems of rules for modal logics with simple frame properties. For our purposes, we further restrict ourselves to the usage of (non-labeled) *two-level systems of rules*, also called *2-systems* for short. These were introduced in [9, 10] for intermediate logics, and we extend the definition to capture propositional modal logics based on classical logics.

Axioms: $x : P, \Gamma \Rightarrow \Delta, x : P \quad x\mathcal{R}y, \Gamma \Rightarrow \Delta, x\mathcal{R}y$

Rules:

$$\frac{x : \varphi, x : \psi, \Gamma \Rightarrow \Delta}{x : \varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow) \qquad \frac{\Gamma \Rightarrow \Delta, x : \varphi \qquad \Gamma \Rightarrow \Delta, x : \psi}{\Gamma \Rightarrow \Delta, x : \varphi \wedge \psi}\ (\Rightarrow \wedge)$$

$$\frac{x : \varphi, \Gamma \Rightarrow \Delta \qquad x : \psi, \Gamma \Rightarrow \Delta}{x : \varphi \vee \psi, \Gamma \Rightarrow \Delta}\ (\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow \Delta, x : \varphi, x : \psi}{\Gamma \Rightarrow \Delta, x : \varphi \vee \psi}\ (\Rightarrow \vee)$$

$$\frac{\Gamma \Rightarrow \Delta, x : \varphi \qquad x : \psi, \Gamma \Rightarrow \Delta}{x : \varphi \supset \psi, \Gamma \Rightarrow \Delta}\ (\supset\Rightarrow) \qquad \frac{x : \varphi, \Gamma \Rightarrow \Delta, x : \psi}{\Gamma \Rightarrow \Delta, x : \varphi \supset \psi}\ (\Rightarrow\supset)$$

$$\frac{}{x : \bot, \Gamma \Rightarrow \Delta}\ (\bot \Rightarrow)$$

$$\frac{y : \varphi, x : \Box\varphi, x\mathcal{R}y, \Gamma \Rightarrow \Delta}{x : \Diamond\varphi, x\mathcal{R}y, \Gamma \Rightarrow \Delta}\ (\Box \Rightarrow) \qquad \frac{x\mathcal{R}y, \Gamma \Rightarrow \Delta, y : \varphi}{\Gamma \Rightarrow \Delta, x : \Box\varphi}\ (\Rightarrow \Box)$$

$$\frac{x\mathcal{R}y, y : \varphi\Gamma \Rightarrow \Delta}{x : \Diamond\varphi\Gamma \Rightarrow \Delta}\ (\Diamond \Rightarrow) \qquad \frac{, x\mathcal{R}y, \Gamma \Rightarrow \Delta, x : \Diamond\varphi, y : \varphi}{x\mathcal{R}y, \Gamma \Rightarrow \Delta, x\Diamond\varphi,}\ (\Rightarrow \Diamond)$$

where P is an arbitrary atomic formula and rules $(\Rightarrow \Box)$ and $(\Diamond \Rightarrow)$ have the condition that y is not in the conclusion.

Table 2.4: Labeled sequent calculus **G3K** for modal logic **K** [18].

Definition 15. A *two-level system of rules*, or a *2-system* for short, is a set of sequent rules $\{(t_1), \ldots, (t_n), (r_B)\}$ that can only be applied according to the following schema:

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \vdots \\ \Gamma \Rightarrow \Delta\end{array} \quad \ldots \quad \begin{array}{c}\mathcal{D}_n\\ \vdots \\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\ (r_B)$$

where each derivation $\mathcal{D}_i$, $1 \leq i \leq n$, may contain several applications of the rule

$$\frac{\Gamma_1, \Sigma_i, \Rightarrow \Delta_1, \Pi_i \qquad \ldots \qquad \Gamma_k, \Sigma_i, \Rightarrow \Delta_k, \Pi_i}{\Gamma_0, \Sigma_i, \Rightarrow \Delta_0, \Pi_i}\ (t_i)$$

where the multisets of formulae $\Gamma_0, \Delta_0, \Gamma_1, \Delta_1 \ldots, \Gamma_k, \Delta_k$ are shared among different (t_i) rules and Σ_i,Π_i are arbitrary multisets of formulae, for $1 \leq i \leq n$. The rule (r_B) is called *bottom rule*, and the rules $(t_1), \ldots, (t_n)$ are called *top rules*.

For our purposes, we introduce an additional constraint that each derivation $\mathcal{D}_i$, $1 \leq i \leq n$ must contain one or more applications of the top rule (t_i).

Let **S** be a sequent calculus and $\mathbb{S}$ be a set of 2-systems, $\mathbf{S}+\mathbb{S}$ denotes the system obtained by augmenting **S** with the 2-systems from $\mathbb{S}$ and the relation $\vdash^l_{\mathbf{S}+\mathbb{S}}$ and $\vdash^g_{\mathbf{S}+\mathbb{S}}$ are defined analogously to those in Definition 11. Further, let $\vdash_{\mathbf{S}+\mathbb{S}}$ denote the derivability relation of the system. In the next chapter, we show how to obtain sequent calculi with system of rules for the logics considered in the previous section and prove that their derivations can be embedded into derivations in the corresponding hypersequent calculi and vice versa.

Before we proceed with the main part of the thesis, we give a few notational remarks. Hereinafter, when we write **L** we refer to any of the basic logics introduced in Section 2.1, i.e., the logics **K**, **K4**, **KB**, and **K4B**. Further, we write **SL** to refer to the sequent version of the hypersequent calculs **HL** for the logic **L**. This calculus consists of the axioms and the sequent versions of **HL** rules, with the exception of (EW) and (EC). These rules are obtained by keeping the active components in premisses and the conclusion and leaving out the context components, i.e., given a hypersequent rule (r) of **HL**

$$\frac{H|\Gamma_1 \Rightarrow \Delta_1 \qquad \dots \qquad H|\Gamma_n \Rightarrow \Delta_n}{H|\Gamma \Rightarrow \Delta}\ (r)$$

the corresponding sequent rule is obtained as:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \qquad \dots \qquad \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}\ (r_s).$$

Finally, we give an example to guide the intuition and illustrate the use of 2-systems.

Example 4. Consider once again the modal logic **S4.3** from Example 2. The 2-system $sys^{\mathbf{HK4}}_{S(\theta_{lin})}$ that corresponds to the property of linearity in transitive frames is:

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\ (r_B)$$

where $\mathcal{D}_i$, for $1 \leq i \leq 2$, contains at least one application of (t_i):

$$\frac{\Gamma_1, \Gamma_2', \Box\Gamma_2' \Rightarrow \Delta_1}{\Gamma_1, \Box\Gamma_1' \Rightarrow \Delta_1}\ (t_1) \qquad \frac{\Gamma_2, \Gamma_1', \Box\Gamma_1' \Rightarrow \Delta_2}{\Gamma_2, \Box\Gamma_2' \Rightarrow \Delta_2}\ (t_2)$$

Further, note that the multisets Γ_1' and Γ_2' are the same in both rules.

The 2-system given above is obtained from the hypersequent rule induced for **HK4** by a normal descriptor of θ_{lin}, as explained in Section 3.1. The calculus for **S4.3** is then obtained by extending the sequent calculus **SK4** with the aforementioned 2-system. The soundness and completeness of this calculus (for the logic **S4.3**) follow from the embedding presented in the rest of this thesis.

We now have the following derivation of the axiom (H) in **SK4** extended with $sys^{\mathbf{HK4}}_{S(\theta_{lin})}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\psi \Rightarrow \psi}{\psi, \Box\psi \Rightarrow \psi}\,(IW\Rightarrow)}{\Box\varphi \Rightarrow \psi}\,(t_1)}{\Rightarrow \Box\varphi \supset \psi}\,(\Rightarrow\supset)}{\Rightarrow \Box(\Box\varphi \supset \psi)}\,(\Rightarrow \Box_4)}{\Rightarrow \Box(\Box\varphi \supset \psi) \vee \Box(\Box\psi \supset \varphi)}\,(\Rightarrow \vee) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi, \Box\varphi \Rightarrow \varphi}\,(IW\Rightarrow)}{\Box\psi \Rightarrow \varphi}\,(t_2)}{\Rightarrow \Box\psi \supset \varphi}\,(\Rightarrow\supset)}{\Rightarrow \Box(\Box\psi \supset \varphi)}\,(\Rightarrow \Box_4)}{\Rightarrow \Box(\Box\varphi \supset \psi) \vee \Box(\Box\psi \supset \varphi)}\,(\Rightarrow \vee)}{\Rightarrow \Box(\Box\varphi \supset \psi) \vee \Box(\Box\psi \supset \varphi)}\,(r_B)$$

CHAPTER 3

The Embedding

Here we define an embedding between the hypersequent calculi for the considered class of modal logics and the corresponding sequent calculi extended with 2-systems. In Section 3.1, we show how to obtain 2-systems from hypersequent rules induced by normal descriptors of n-simple $\mathcal{L}^1$-formulae describing simple frame properties. The first direction of the embedding is presented in Section 3.2, where we prove that each derivation in the hypersequent calculus for some modal logic with simple frame properties can be translated into a derivation of the same end-sequent in the corresponding sequent calculus extended with 2-systems. Section 3.3 is devoted to the other direction of the embedding and shows how 2-system derivations are transformed into derivations in the corresponding hypersequent calculus.

3.1 2-systems for Simple Frame Properties

In this section we show how to obtain 2-systems from hypersequent rules induced by normal descriptors of n-simple $\mathcal{L}^1$-formulae.

Consider a hypersequent rule $(r^{\mathbf{HL}}_{S(\theta)})$ induced by a normal descriptor of some n-simple $\mathcal{L}^1$-formula θ for **HL**. In Section 2.2, we presented four different definitions of hypersequent rules corresponding to simple frame properties, depending on the logic **L** that is being considered. Notice, however, that these definitions share a common thread – each premiss of the defined rule corresponds to one pair $\langle S_R, S_= \rangle$ in $S(\theta)$. We next give a definition that "links" these premisses to one or more active components in the conclusion of $(r^{\mathbf{HL}}_{S(\theta)})$.

Definition 16. Let θ be a n-simple $\mathcal{L}^1$-formula describing a simple frame property, $S(\theta)$ some normal descriptor of θ and $(r^{\mathbf{HL}}_{S(\theta)})$ the hypersequent rule induced by $S(\theta)$ for **HL**. We denote by C_i the i-th active component in the conclusion of $(r^{\mathbf{HL}}_{S(\theta)})$. Further, let $\langle S_R, S_= \rangle$ be a pair in $S(\theta)$ and let $H|C_{\langle S_R, S_= \rangle}$ denote a premiss of $(r^{\mathbf{HL}}_{S(\theta)})$ that corresponds to this pair in the definition of $(r^{\mathbf{HL}}_{S(\theta)})$. We say that $H|C_{\langle S_R,S_=\rangle}$ is *linked* to C_i, if one of the following conditions is satisfied:

(i) $i \in S_=$,

(ii) $S_= = \emptyset$ and $i \in S_R$, or

(iii) $i \in S_R$, $S_= \neq \emptyset$ and there is no other pair $\langle S'_R, S'_= \rangle \in S(\theta)$ such that the corresponding premiss $H|C_{\langle S'_R,S'_=\rangle}$ satisfies (i) or (ii).

Notice that every premiss of $(r^{\mathbf{HL}}_{S(\theta)})$ will be linked to at least one active component in its conclusion, since for each pair $\langle S_R, S_= \rangle \in S(\theta)$, $S_R \cup S_= \neq \emptyset$. In particular, if $S_= \neq \emptyset$ then $M_{\langle S_R,S_=\rangle}$ is linked to the conclusion component C_i, s.t. $i \in S_=$. Otherwise, $S_R \neq \emptyset$, and so $M_{\langle S_R,S_=\rangle}$ will satisfy condition (ii) in the definition above and is linked to C_i, where $i \in S_R$.

Definition 17. Given a rule $(r^{\mathbf{HL}}_{S(\theta)})$ induced by a normal descriptor of some n-simple $\mathcal{L}^1$-formula θ for **HL**, the corresponding 2-system $sys^{\mathbf{SL}}_{S(\theta)}$ is defined as:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array} \quad \dots \quad \begin{array}{c}\mathcal{D}_n\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\ (r_B)$$

where each derivation $\mathcal{D}_i$, $1 \leq i \leq n$, contains one or more applications of the rule

$$\dfrac{\{C_{\langle S_R,S_=\rangle} : H|C_{\langle S_R,S_=\rangle} \text{ is a premiss of } (r^{\mathbf{HL}}_{S(\theta)}) \text{ that is linked to } C_i\}}{C_i}\ (t_i)$$

We next give a small example that illustrate the approach given above.

Example 5. Recall the 2-system presented in Example 4. The simple frame property of linearity is expressed by a 2-simple $\mathcal{L}^1$-formula

$$\theta_{lin} := \forall w_1, w_2 \exists u (w_1 \mathcal{R} u \wedge w_2 = u) \vee (w_2 \mathcal{R} u \wedge w_1 = u)$$

and has a normal descriptor $S(\theta_{lin}) = \{\langle\{1\},\{2\}\rangle, \langle\{2\},\{1\}\rangle\}$.

Then, we have the following hypersequent rule $(r^{\mathrm{HK4}}_{S(\theta_{lin})})$ induced by $S(\theta_{lin})$ for **HK4**:

$$\dfrac{H|\Gamma_2, \Gamma'_1, \Box\Gamma'_1 \Rightarrow \Delta_2 \qquad H|\Gamma_1, \Gamma'_2, \Box\Gamma'_2 \Rightarrow \Delta_1}{H|\Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 | \Gamma_2, \Box\Gamma'_2 \Rightarrow \Delta_2}\ (r^{\mathbf{HK4}}_{S(\theta_{lin})})$$

Following the Definition 16, we have that the premiss $H|\Gamma_2, \Gamma'_1, \Box\Gamma'_1 \Rightarrow \Delta_2$, that corresponds to the pair $\langle\{1\},\{2\}\rangle$ is linked to $\Gamma_2, \Box\Gamma'_2 \Rightarrow \Delta_2$. Similarly, $H|\Gamma_1, \Gamma'_2, \Box\Gamma'_2 \Rightarrow \Delta_1$ is linked to $\Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1$.

Hence, following the Definition 17, we get the following 2-system ($sys^{\mathbf{SK4}}_{S(\theta_{lin})}$):

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,(r_B) \qquad \dfrac{\Gamma_1, \Gamma'_2, \Box\Gamma'_2 \Rightarrow \Delta_1}{\Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1}\,(t_1) \qquad \dfrac{\Gamma_2, \Gamma'_1, \Box\Gamma'_1 \Rightarrow \Delta_2}{\Gamma_2, \Box\Gamma'_2 \Rightarrow \Delta_2}\,(t_2)$$

where each D_i, for $1 \leq i \leq 2$ contains one or more applications of (t_i).

We finish this section with a few notational remarks. Given a set Θ of formulae describing simple frame properties, we denote by $\mathbb{R}(\Theta)$ the set of hypersequent rules induced by normal descriptors of formulae in Θ for **HL**, i.e., $\mathbb{R}(\Theta) = \{(r^{\mathbf{HL}}_{S(\theta)}) : \theta \in \Theta\}$. Further, we denote by $\mathbb{S}(\Theta)$ the set of 2-systems that were obtained from the rules in $\mathbb{R}(\theta)$, as explained in the Definition 17.

3.2 From Hypersequent to 2-system Derivations

The purpose of this section is to show that every hypersequent derivation in **HL**+$\mathbb{R}(\Theta)$ can be translated into a derivation in **SL**+$\mathbb{S}(\Theta)$ of the same end-sequent. Intuitively, we translate the applications of **HL** rules as applications of their corresponding **SL** rules, and applications of rules from $\mathbb{R}(\Theta)$ as applications of corresponding 2-systems in $\mathbb{S}(\Theta)$. As there are no rules in **SL**+$\mathbb{S}(\Theta)$ that are direct translations of (EC) and (EW), we only consider **HL**+$\mathbb{R}(\Theta)$ derivations that are of certain form, called *structured normal form*. In what follows, we show that this is not a restriction.

3.2.1 Structured Normal Form

We now prove that every hypersequent derivation in **HL**+$\mathbb{R}(\Theta)$ can be transformed into a derivation of the same end-hypersequent that is in the structured normal form, which will be crucial for embedding hypersequent derivations into 2-system derivations. We first introduce a some relevant notions and then proceed to show how to translate **HL**+$\mathbb{R}(\Theta)$ derivations into derivations that respect the structured normal form. The approach below follows closely the one from [10], which was introduced for logics intermediate between classical and intuitionistic logic.

Definition 18. Let (r) be a one-premiss hypersequent rule. We refer to any sequence of consecutive applications of (r) that is neither immediately preceded nor immediately followed by an application of (r) as a *queue of* (r).

Definition 19. Given a $\mathbf{HL}+\mathbb{R}(\Theta)$ derivation, a sequent C' is a *parent* of a sequent C, denoted by $p(C, C')$, if one of the following conditions holds:

- C' is active in a premiss, and C in the conclusion of some application of rule $(r) \in \mathbf{HL}$.
- C is active in the conclusion of some application of rule $(r) \in \mathbb{R}(\Theta)$, and C' is active in a premiss that is linked to C.
- C is a context component of the conclusion of any rule application, and C' is the corresponding context component in a premiss of such application.

The *ancestor relation*, denoted by $a\langle\cdot,\cdot\rangle$, is obtained as a transitive closure of the *parent relation*.

Definition 20. An $\mathbf{HL}+\mathbb{R}(\Theta)$ derivation $\mathcal{D}$ is in *structured normal form* if it satisfies the following two conditions:

(i) All applications of (EC) appear in a queue immediately above the root of $\mathcal{D}$.

(ii) All applications of (EW) occur in subderivations of the form

$$\dfrac{\dfrac{\dfrac{H_1|C_1}{\vdots}\,(EW)}{H|C_1}\,(EW) \quad \dots \quad \dfrac{\dfrac{H_n|C_n}{\vdots}\,(EW)}{H|C_n}\,(EW)}{H|C_0}\,(r)$$

where (r) is any multi-premiss hypersequent rule, $C_1, \dots, C_n$ are active components in the premisses of (r) and each component of H occurs in one of the H_i, for $1 \leq i \leq n$.

Intuitively, the second property states that (EW) applications are only used for introducing missing context components to the premisses of a multi-premiss rule (r) in order to ensure that all the premisses share the same context. Further, such (EW) applications must occur in a queue immediately above (r).

We begin with an example that shows how a hypersequent derivation violating the structured normal form can be transformed into one in which no violations occur.

Example 6. Consider the following derivation in the hypersequent calculus $\mathbf{HK}+(r^{\mathbf{HK}}_{s(\theta)})$ for **S5** (see Tables 2.2 and 2.3 for its rules), where $\theta = \forall w_1, w_2 \exists u (w_1 \mathcal{R} u \wedge w_2 = u)$ describes the simple property of universality:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi\Rightarrow\varphi}{\Box\varphi\Rightarrow\ |\ \Rightarrow\varphi}}{\Rightarrow\neg\Box\varphi\ |\ \Rightarrow\varphi}}{\Rightarrow\neg\Box\varphi\ |\ \Rightarrow\Box\varphi}}{\Rightarrow\neg\Box\varphi\ |\ \neg\Box\varphi\Rightarrow}}{\Rightarrow\Box\neg\Box\varphi\ |\ \neg\Box\varphi\Rightarrow}}{\Rightarrow\Box\neg\Box\varphi\ |\ \neg\Box\varphi\Rightarrow\Box\neg\Box\varphi}}{\neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\ |\ \neg\Box\varphi\Rightarrow\Box\neg\Box\varphi}
\quad
\dfrac{\dfrac{\dfrac{\varphi\Rightarrow\varphi}{\neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\ |\ \varphi\Rightarrow\varphi}\,(EW)}{\neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\ |\ \neg\Box\varphi,\varphi\Rightarrow\varphi}}{\neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\ |\ \neg\Box\varphi\Rightarrow\varphi\supset\varphi}}{\neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\ |\ \neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)}
\quad
\dfrac{\dfrac{\dfrac{\varphi\Rightarrow\varphi}{\varphi\Rightarrow\varphi\ |\ \neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)}\,(EW)}{\neg\Box\varphi,\varphi\Rightarrow\varphi\ |\ \neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)}}{\neg\Box\varphi\Rightarrow\varphi\supset\varphi\ |\ \neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)}}{\dfrac{\dfrac{\neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)\ |\ \neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)}{\neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)}\,(EC)}{\Rightarrow\neg\Box\varphi\supset\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)}\,(\Rightarrow\supset)}
$$

There are three violations in this derivation: the (EC) application does not occur in the queue of (EC) immediately above the root and both (EW) applications do not occur where immediately needed. We first shift the (EC) application below the last rule as follows:

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)\ |\ \neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)\end{array}}{\Rightarrow\neg\Box\varphi\supset\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)\ |\ \neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)}\,(\Rightarrow\supset)}{\Rightarrow\neg\Box\varphi\supset\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)\ |\ \Rightarrow\neg\Box\varphi\supset\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)}\,(\Rightarrow\supset)}{\Rightarrow\neg\Box\varphi\supset\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)}\,(EC)
$$

Next, we shift the two (EW) applications immediately above the two-premiss rules whose contexts they introduce. We then obtain the following derivation which is in structured normal form:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi\Rightarrow\varphi}{\Box\varphi\Rightarrow\ |\ \Rightarrow\varphi}}{\Rightarrow\neg\Box\varphi\ |\ \Rightarrow\varphi}}{\Rightarrow\neg\Box\varphi\ |\ \Rightarrow\Box\varphi}}{\Rightarrow\neg\Box\varphi\ |\ \neg\Box\varphi\Rightarrow}}{\Rightarrow\Box\neg\Box\varphi\ |\ \neg\Box\varphi\Rightarrow}}{\Rightarrow\Box\neg\Box\varphi\ |\ \neg\Box\varphi\Rightarrow\Box\neg\Box\varphi}}{\neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\ |\ \neg\Box\varphi\Rightarrow\Box\neg\Box\varphi}
\quad
\dfrac{\dfrac{\dfrac{\varphi\Rightarrow\varphi}{\neg\Box\varphi,\varphi\Rightarrow\varphi}}{\neg\Box\varphi\Rightarrow\varphi\supset\varphi}}{\neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\ |\ \neg\Box\varphi\Rightarrow\varphi\supset\varphi}\,(EW)}{\neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\ |\ \neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)}
\quad
\dfrac{\dfrac{\dfrac{\varphi\Rightarrow\varphi}{\neg\Box\varphi,\varphi\Rightarrow\varphi}}{\neg\Box\varphi\Rightarrow\varphi\supset\varphi}}{\neg\Box\varphi\Rightarrow\varphi\supset\varphi\ |\ \neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)}\,(EW)}{\neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)\ |\ \neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)}}{\Rightarrow\neg\Box\varphi\supset\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)\ |\ \neg\Box\varphi\Rightarrow\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)}\,(\Rightarrow\supset)}{\Rightarrow\neg\Box\varphi\supset\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)\ |\ \Rightarrow\neg\Box\varphi\supset\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)}\,(\Rightarrow\supset)}{\Rightarrow\neg\Box\varphi\supset\Box\neg\Box\varphi\wedge(\varphi\supset\varphi)}\,(EC)
$$

We now formalize the strategy shown in the example above. To this end, we introduce the notion of the *EC-rank* that tells us how far an application of (EC) is from the queue of (EC) occurring immediately above the root of the derivation.

Definition 21. Given a **HL**+$\mathbb{R}(\Theta)$ derivation $\mathcal{D}$, the *EC-rank* of an application E of (EC) in $\mathcal{D}$ is the number of applications of rules other than (EC) that occur between E and the root of $\mathcal{D}$.

For notational convenience, we denote by $(H)^n$ the hypersequent $H|\dots|H$ consisting of n copies of the hypersequent H. We begin with proving the following lemma:

Lemma 1. For any application of a hypersequent rule (r) with premisses $H|C_1, \dots, H|C_n$ and conclusion $H|G$ consider the following set of hypersequents:

$$\mathbb{L}_d = \{H|(G)^c|(C_1)^{x_1}|\dots|(C_n)^{x_n} : \sum_{i=1}^{n} x_i = d\},$$

where G, H are hypersequents, $C_1, \dots, C_n$ are active components in the premisses of (r), i.e., they are sequents, and c, d are natural numbers.

For any natural number e, $0 \leq e \leq d$, each element of the set

$$\mathbb{L}_{d-e} = \{H|(G)^{c+e}|(C_1)^{x'_1}|\dots|(C_n)^{x'_n} : \sum_{i=1}^{n} x'_i = d-e\},$$

is derivable from the hypersequents in $\mathbb{L}_d$ by repeatedly applying the rule (r).

Proof. Proof is given by induction on e.

BASE CASE: If $e = 0$, then $\mathbb{L}_d = \mathbb{L}_{d-e}$ and hence the claim holds trivially.

INDUCTIVE STEP: Assume $e > 0$ and that the claim holds for all $e' < e$. We show how we can obtain the derivation for each element of $\mathbb{L}_{d-e}$ from $\mathbb{L}_d$ that only uses (r).

Consider an arbitrary hypersequent in $\mathbb{L}_{d-e}$. This hypersequent must be of the following form:

$$H|(G)^{c+e}|G', \text{ where } G' = (C_1)^{x'_1}|\dots|(C_n)^{x'_n} \text{ and } \sum_{i=1}^{n} x'_i = d-e.$$

Consider the following set $S = \{H|(G)^{c+e-1}|G'|C_i : 1 \leq i \leq n\}$ of hypersequents. Note that, since we are dealing with multisets of sequents, the external exchange rule is implicit and so the hypersequent $H|(G)^{c+e-1}|(C_1)^{x'_1}|\dots|(C_i)^{x'_i+1}|\dots|(C_n)^{x'_n}$ is the same as the hypersequent $H|(G)^{c+e-1}|(C_1)^{x'_1}|\dots|(C_i)^{x'_i}|\dots|(C_n)^{x'_n}|C_i$. Therefore, each element of the considered set S is an element of $\mathbb{L}_{d-(e-1)}$ as

$$x'_1 + \dots + x'_i + 1 + \dots + x'_n = (\sum_{i=1}^{n} x'_i) + 1 = d - e + 1 = d - (e-1).$$

We then derive $H|(G)^{c+e}|G'$ from the elements of $\mathbb{L}_{d-(e-1)}$ as:

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ \vdots \\ H|(G)^{c+e-1}|G'|C_1\end{matrix} \quad \dots \quad \begin{matrix}\mathcal{D}_n \\ \vdots \\ H|(G)^{c+e-1}|G'|C_n\end{matrix}}{H|(G)^{c+e}|G'}\,(r)$$

where $\mathcal{D}_i$, $1 \leq i \leq n$, denotes the derivation of $H|(G)^{c+e-1}|G'|C_i$ from the elements of $\mathbb{L}_d$, that uses only the rule (r). By induction hypothesis, there are such derivations $\mathcal{D}_1, \dots, \mathcal{D}_n$, and so it follows that $H|(G)^{c+e}|G'$ can be derived from the elements of $\mathbb{L}_d$ by repeated applications of (r). □

We now prove that we can transform any **HL**+$\mathbb{R}(\Theta)$ derivation $\mathcal{D}$ into a derivation of the same end-hypersequent in which all applications of (EC) appear in the queue immediately above the root. We proceed in a stepwise manner, with each step bringing one or more applications of (EC) violating the first property in Definition 20 closer to the root. As $\mathcal{D}$ is finite, there can only be finitely many "violators" and their distance to the root must be of finite length as well. Hence, the algorithm will terminate and we will obtain the desired derivation. This approach is formalized in the lemma below.

Lemma 2. Each **HL**+$\mathbb{R}(\Theta)$ derivation $\mathcal{D}$ can be transformed into a derivation of the same end-hypersequent in which all applications of (EC) have EC-rank 0.

Proof. Let μ be the maximum EC-rank of any (EC) application in $\mathcal{D}$ and ν the number of (EC) applications in $\mathcal{D}$ with the EC-rank μ. The proof proceeds by a double induction on the lexicographically ordered pair $\langle \mu, \nu \rangle$.

Base Case: If $\mu = 0$, then all (EC) applications already appear in a queue above the root and so the claim trivially holds.

Inductive Step: If $\mu > 0$, we show that we can transform $\mathcal{D}$ into a derivation $\mathcal{D}'$ of the same end hypersequent such that:

- either the maximum EC-rank of $\mathcal{D}'$ is strictly smaller than μ, or
- the maximum EC-rank of $\mathcal{D}'$ is μ but the number of (EC) applications with such an EC-rank is strictly smaller than ν.

Consider an application of (EC) with EC-rank μ and the queue that contains it. As this application has the maximum rank, there can be no other applications of (EC) above this queue. Consider now the rule (r) that is applied immediately below this queue, i.e., it has as a premiss the conclusion of the last (EC) application in this queue. We make a case distinction depending on the type of (r).

1. $(r) = (EW)$. In this case, (EW) is simply applied immediately before the queue with the same active component followed by the queue of (EC) applications that are applied as before, but now with one additional context component.

2. (r) is a one-premiss rule other than (EW). In this case, we need to, once again, make a case distinction.

   2.1. If the active component of the premiss of (r) has an ancestor that was an active component in the conclusions of some EC application occurring in the queue, we apply (r) immediately before the queue as many times as there are ancestors of that component that occur in the premiss of the first (EC) application.

   To illustrate this, consider a small example where we transform the derivation on the left-hand side to the one on the right-hand side as follows:

$$
\begin{array}{cl}
\mathcal{D}_1 & \\
\vdots & \\
H|\Gamma \Rightarrow \Delta|\Gamma \Rightarrow \Delta & \\
\hline
\vdots & (EC) \\
\hline
H'|\Gamma \Rightarrow \Delta|\Gamma \Rightarrow \Delta & (EC) \\
\hline
H'|\Gamma \Rightarrow \Delta & (EC) \\
\hline
\vdots & (EC) \\
\hline
H''|\Gamma \Rightarrow \Delta & (EC) \\
\hline
H''|\Gamma' \Rightarrow \Delta' & (r)
\end{array}
\qquad
\begin{array}{cl}
\mathcal{D}_1 & \\
\vdots & \\
H|\Gamma \Rightarrow \Delta|\Gamma \Rightarrow \Delta & \\
\hline
H|\Gamma' \Rightarrow \Delta'|\Gamma \Rightarrow \Delta & (r) \\
\hline
H|\Gamma' \Rightarrow \Delta'|\Gamma' \Rightarrow \Delta' & (r) \\
\hline
\vdots & (EC) \\
\hline
H'|\Gamma' \Rightarrow \Delta'|\Gamma' \Rightarrow \Delta' & (EC) \\
\hline
H'|\Gamma' \Rightarrow \Delta' & (EC) \\
\hline
\vdots & (EC) \\
\hline
H''|\Gamma' \Rightarrow \Delta' & (EC)
\end{array}
$$

   where each component of H' and H'' occurs in H.

   2.2. Otherwise, the active component in the premiss of (r) is the context component of all (EC) applications in the queue. In that case, we simply apply (r) immediately before the queue with the same active component, as it does not disturb the applications of (EC).

Notice that all of the translations steps given above shift the application of (r) above the considered (EC) application and its associated queue. Thus, the EC-rank of the applications in the queue is reduced. If these are the only (EC) applications with the rank μ, then the maximum EC-rank of the derivation is reduced, i.e., $\mu' < \mu$. Otherwise, the number of (EC) applications with the maximum rank μ is reduced, i.e., $\mu' = \mu$ and $v' < \nu$.
Consider now the final case:

3. (r) is a multi-premiss rule. Then the considered (EC) application and its corresponding queue appear in a subderivation of $\mathcal{D}$ of the form:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots\\ H|H_1'|(C_1)^{m_1}\end{matrix}}{\vdots}\,(EC)}{H|C_1}\,(EC) \quad \dots \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_n\\ \vdots\\ H|H_n'|(C_n)^{m_n}\end{matrix}}{\vdots}\,(EC)}{H|C_n}\,(EC)}{}}{H|G}\,(r)$$

where H_i', for $1 \leq i \leq n$ only contains components in H. Note that $\mathcal{D}_1, \dots, \mathcal{D}_n$ contain no applications of (EC), otherwise the considered EC application would not have the maximum EC-rank. We begin by proving that the hypersequent

$$H|H'|(G)^q, \text{ where } H' = H_1'|\dots|H_n',\ q = 1 + \textstyle\sum_{i=1}^n (m_i - 1)$$

is derivable from $H|H_1'|(C_1)^{m_1}, \dots, H|H_n'|(C_n)^{m_n}$ using only (EW) and (r).

Let $\mathbb{Q}$ be the following set of hypersequents

$$\mathbb{Q} = \{H|H'|(G)^0|(C_1)^{x_1}|\dots|(C_n)^{x_n} : \textstyle\sum_{i=1}^n x_i = 1 + \sum_{i=1}^n (m_i - 1)\}.$$

Notice that for every hypersequent in $\mathbb{Q}$ there must be some i, $1 \leq i \leq n$, such that $x_i \geq m_i$, otherwise we would have $\sum_{i=1}^n x_i < 1 + \sum_{i=1}^n (m_i - 1)$ which is a contradiction. Hence, it is possible to derive this hypersequent from $H|H_i'|(C_i)^{m_i}$ using only (EW).

Now, by Lemma 1, we can derive every element of the set

$$\mathbb{Q}' = \{H|H'|(G)^q|(C_1)^{x_1}|\dots|(C_n)^{x_n} : \textstyle\sum_{i=1}^n x_i = 0\}, \text{ where } q = 1 + \sum_{i=1}^n (m_i - 1)$$

from the hypersequents in $\mathbb{Q}$ using only (r). Hence, the hypersequent $H|H'|(G)^q$ can be derived from $H|H'|(C_1)^{m_1}, \dots, H|H'|(C_n)^{m_n}$ using only (EW) and (r). Finally, we obtain $H|G$ from $H|H'|(G)^q$ by repeated application of EC, as all the components of H' are also in H'. This translation step eliminates the considered queues of (EC) with EC-rank μ and replaces them with a single queue of (EC) whose EC-rank is strictly smaller than μ. Hence, the number of (EC) applications with EC-rank μ was indeed reduced. If there are no other (EC) applications of rank μ, then the maximum rank is reduced. Therefore, we either have $\mu' < \mu$, or $\mu' = \mu$ and $v' < \nu$. □

Finally, we can proceed to prove the theorem below.

Theorem 7. Every **HL**+$\mathbb{R}(\Theta)$ derivation of a sequent can be transformed into a derivation of the same end-sequent that is in structured normal form.

Proof. Let $\mathcal{D}$ be an arbitrary derivation in **HL**+$\mathbb{R}(\Theta)$. We begin by translating $\mathcal{D}$ into a derivation $\mathcal{D}'$ of the same end-sequent that contains no (EC) applications of EC-rank greater than 0. By Lemma 2, we know that such a derivation must exist and how to obtain it.
Next, consider an application of (EW) in $\mathcal{D}'$ with the premiss H and conclusion $H|C$ that violates the structured normal form. First, observe that this application of (EW) cannot occur immediately above the root, since $H|C$ cannot be the end-sequent. We begin with identifying the ways in which an application of (EW) could violate the structured normal form:

(i) (EW) occurs in a queue of (EW) above a one-premiss rule.

(ii) (EW) occurs in a queue of (EW) above a multi-premiss rule (r), but instead of context components, it introduces the active component in some premiss of (r).

(iii) (EW) occurs in a queue of (EW) above a multi-premiss rule (r) and it introduces a context component C that occurs actively in the (EW) queues above all other premises of (r). This violates the structured normal form as it implies that the component C was not initially there, i.e., C was not present as a context component in the premiss of the topmost (EW) application in a queue occurring immediately above some premiss of (r).

Now, given an (EW) application that violates the structured normal form in one of the ways listed above, the main idea is to shift this (EW) application down the derivation tree until we find a position in which the normal form is not violated. Let (r) be the rule application immediately below the queue of (EW) in which the considered application occurs. We proceed as follows:

1. If $(r) \neq (EC)$ is an application of a one-premiss rule and (EW) introduces a context component of its premiss, then simply apply (r) first with one less context component, i.e., apply it to the premiss of the considered (EW) application, and then proceed to apply (EW). In the case that $(r) = (EC)$, the conclusion of the considered (EW) application must be the premiss of the topmost application of (EC) in the queue of (EC) occurring immediately above the root. Let us denote this premiss by H_{EC}. Notice now that in order to obtain a sequent from the hypersequent H_{EC} using only (EC) applications, all the components of H_{EC} must be identical. Hence, the considered (EW) application must introduce a component that already occurs in the premiss of the application. But then, there is no need to apply (EW) in the first place. We can simply remove this application together with one application of (EC) in the queue immediately below it and obtain the same end-sequent.

2. If (r) is an application of any one- or multi-premiss rule and the considered (EW) application introduces an active component of some premiss of (r), then this (EW) application, denoted by (EW_c), occurs in a subderivation of the following form:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{H}}{H|C_1}\,(EW_c)}{\vdots}\,(EW)}{H'|C_1}\,(EW) \quad \dots \quad \dfrac{\vdots}{H'|C_k}
\over H'|G
$$
(r)

where each component of H appears in H'. In this case we can simply obtain the conclusion of (r) by applying (EW) to the premiss of (EW_c), possibly multiple times, as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{H}{\vdots}\,(EW)}{H'}\,(EW)}{\vdots}\,(EW)}{H'|G}\,(EW)
$$

Notice that if $(r) = (EC)$, we can simply omit the application of both (EW_c) and (r) itself, as the (EW_c) introduces a component that will be eliminated again through the application of (EC).

3. Now, if the considered (EW) application occurs above a multi-premiss rule application (r) and the active component C in the conclusion of the considered (EW) application occurs actively in the queues of (EW) above each premiss of (r) we have the following situation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{H_1|C_1}}{\vdots}\,(EW)}{H_1'|C_1}\,(EW)}{H_1'|C|C_1}\,(EW_c)}{\vdots}}{H|C|C_1}\,(EW) \quad \dots \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{H_k|C_k}}{\vdots}\,(EW)}{H_k'|C_k}\,(EW)}{H_k'|C|C_k}\,(EW)}{\vdots}}{H|C|C_k}\,(EW)
\over H|C|G
$$
(r)

where H contains all components of $H_1, \dots, H_k, H_1', \dots, H_k'$. In this case, we can remove all applications of (EW) where C occurs actively, and apply (r) with one context component less, followed by an (EW) application to regain C. That is, we have the following:

$$
\dfrac{
\dfrac{\dfrac{\begin{array}{c}\vdots\\ H_1|C_1\end{array}}{\vdots}\,(EW)}{H|C_1}\,(EW)
\qquad \dots \qquad
\dfrac{\dfrac{\begin{array}{c}\vdots\\ H_k|C_k\end{array}}{\vdots}\,(EW)}{H|C_k}\,(EW)
}{
\dfrac{H|G}{H|C|G}\,(EW_c)
}\,(r)
$$

Hence, once again, (r) no longer occurs below the considered (EW) application.

Every translation step above reduces the amount of rules other than (EW) that occur below some violating (EW) application and brings it one step closer to where it needs to be. In case we shift the considered application all the way to the bottom, i.e., above the first (EC) application, it is deemed irrelevant and therefore is simply removed. Hence, each violating (EW) application will be resolved. As $\mathcal{D}'$ is finite, there is only a finite number of violating (EW) applications, and also a finite amount of shifts needed for each of the (EW) to reach either the correct position or be deemed irrelevant. Hence, this procedure terminates and gives us a derivation in **HL**+$\mathbb{R}(\Theta)$ that derives the same end-sequent as $\mathcal{D}$ and is in the structured normal form. □

3.2.2 Translation

We now turn to showing how to translate any hypersequent derivation in **HL**+$\mathbb{R}(\Theta)$ to a derivation of the same end-sequent in **SL**+$\mathbb{S}(\Theta)$, following once again the approach given in [10]. To this end, we introduce the notion of *partial derivation*.

Definition 22. A *partial derivation* in **SL**+$\mathbb{S}(\Theta)$ is a derivation in **SL** extended by the top rules of 2-systems from $\mathbb{S}(\Theta)$, without their applicability conditions relative to bottom rules.

Intuitively, we translate each application of **HL** as an application of a corresponding rule in **SL**. Since rules (EW) and (EC) have no corresponding rules in **SL**, we only consider derivations in structured normal form, which is not a limitation, as shown in Theorem 7. Notice that derivations in structured normal form can be divided into two parts: a part containing only (EC) applications and a part containing no applications of (EC). The two parts are separated by the premiss of the uppermost application of (EC), denoted by $\hat{H}_{\mathcal{D}}$. Now, each application of a rule in $\mathbb{R}(\Theta)$ is translated in two steps. First, we find a partial derivation of all the components of $\hat{H}_{\mathcal{D}}$ and then we obtain the required derivation by suitably applying the bottom rules of 2-systems.

Definition 23. The *ancestor tree* of a sequent C is the tree whose nodes are all sequents that are the ancestors of C and the edges between such nodes are defined by the $p\langle\cdot,\cdot\rangle$ relation (see Definition 19).

We first show how to construct a partial derivation of a component of $\hat{H}_{\mathcal{D}}$ that has the same structure as the ancestor tree of that component.

Lemma 3. Let $\mathbb{R}(\Theta)$ be a set of hypersequent rules induced by normal descriptors of simple frame properties for calculus **HL**. Given any **HL**+$\mathbb{R}(\Theta)$ derivation $\mathcal{D}$ in structured normal form, for each component $C \in \hat{H}_{\mathcal{D}}$ we can construct a partial derivation in **SL**+$\mathbb{S}(\Theta)$ that has the same structure as the ancestor tree of that component. This means that, with the exception of (EW), a rule application occurs in the ancestor tree of C in $\mathcal{D}$ if and only if its translation occurs in the partial derivation of C, and the translations of the rules occur in the order in which the original rules occur in the ancestor tree.

Proof. Let H be a hypersequent in $\mathcal{D}$ derived without using (EC). We show that we can construct a partial derivation in **SL**+$\mathbb{S}(\Theta)$ of each component of H that satisfies the required property. The proof is given by induction on the length l of the derivation of H, i.e., the number of rule applications occurring on any branch of the derivation of H plus one.

Base case: If $l = 1$, then H must be an axiom. Hence H has only one component and its required partial derivation consists simply of H itself.

Inductive step: Assume that the derivation of H is of length $l > 1$ and that for each hypersequent H' with a derivation of length $l' < l$ the claim given above holds. In particular, this means that for an arbitrary component C of H, the parents of C all have partial derivations in **SL**+$\mathbb{S}(\Theta)$ that have the same structure as their ancestor trees in $\mathcal{D}$. We show how to construct a required partial derivation of C. Consider the last rule $(r) \neq (EW)$ applied in the subderivation $\mathcal{D}'$ of H. We distinguish the following cases:

1. (r) is a rule in $\mathbb{R}(\Theta)$, i.e., $(r) = (r^{\mathbf{HL}}_{S(\theta)})$ is induced by a normal descriptor of some $\theta \in \Theta$ for **HL**. Then $H = G|\Gamma_1, \Box\Gamma'_1 \Rightarrow \Sigma_1| \dots |\Gamma_n, \Box\Gamma'_1 \Rightarrow \Sigma_n$, where $\Sigma_i = \Delta_i$ if **L**=**K**,**K4** and $\Sigma_i = \Delta_i, \Delta'_i$ if **L**=**KB**,**KB4**, for $1 \leq i \leq n$. Assume the derivation $\mathcal{D}'$ of H of length l is the following:

$$\dfrac{\begin{array}{ccc} \mathcal{D}_1 & & \mathcal{D}_k \\ \vdots & & \vdots \\ G|C_1 & \dots & G|C_k \end{array}}{G|\Gamma_1, \Box\Gamma'_1 \Rightarrow \Sigma_1| \dots |\Gamma_n, \Box\Gamma'_1 \Rightarrow \Sigma_n}\,(r^{\mathbf{HL}}_{S(\theta)})$$

where the premisses $G|C_1, \dots, G|C_k$ might be inferred by queues of (EW) from $G_1|C_1, \dots, G_k|C_k$, respectively. As each $G|C_j$, for $1 \leq j \leq k$, has a derivation strictly shorter than l, by induction hypothesis, we have that each C_j, for $1 \leq j \leq k$ and each component of $G_1, \dots, G_k$ has a partial derivation in **SL**+$\mathbb{S}(\Theta)$ that has the same structure as the ancestor tree of that component in $\mathcal{D}$. As $\mathcal{D}$ is in structured normal form, we know that each component of G appears in some G_i, $1 \leq i \leq k$, and thus each component of G has a partial derivation of the required form. Consider now the component $\Gamma_i, \Box\Gamma'_i \Rightarrow \Sigma_i$ of H. We can obtain $\Gamma_i, \Box\Gamma'_i \Rightarrow \Sigma_i$ from $G_1|C_1, \ldots G_k|C_k$ by applying the following rule:

$$\frac{\{C_j \ : \ G|C_j \text{ is linked to } \Gamma_i, \Box\Gamma'_i \Rightarrow \Sigma_i\}}{\Gamma_i, \Box\Gamma'_i \Rightarrow \Sigma_i}\ (t_i)$$

which is a top rule of the 2-system $sys^{\mathbf{SL}}_{S(\theta)} \in \mathbb{S}(\Theta)$. The partial derivation of $\Gamma_i, \Box\Gamma'_i \Rightarrow \Sigma_i$ then consists of the partial derivations of the premisses of the (t_i) application as illustrated above together with that application itself.

Note that if a component C of G occurs in more than one G_i, for $1 \leq i \leq k$, we have different partial derivations of C. We merge these with an application of a dummy bottom rule

$$\frac{C \qquad \ldots \qquad C}{C}$$

2. (r) is a multi-premiss rule in **HL**. In this case, in order to obtain a partial derivation of a component C of H we follow the approach from the previous case, but instead of using the top rules of 2-systems in $\mathbb{S}(\Theta)$ we now use the sequent version of (r).

3. (r) is a one-premiss rule in **HL**. Then we have the following derivation of $H = G|C$:

$$\begin{array}{c} \mathcal{D}_1 \\ \vdots \\ \dfrac{G|C'}{G|C}\ (r) \end{array}$$

As each component in G has a derivation of length strictly less than l, by induction hypothesis, it has a partial derivation of the required form. Consider now the component C of H. By induction hypothesis, the active component of the premiss of the (r) application must have a partial derivation in $\mathbf{SL}+\mathbb{S}(\Theta)$ that has the same shape as its ancestor tree. Hence, the partial derivation of C is obtained by taking the partial derivation of C' with an additional application of the sequent version of (r) to C'.

All partial derivations obtained in the ways described above clearly fulfill the requirement that with the exception of (EW) (and dummy bottom rules), a rule application occurs in the ancestor tree of the hypersequent component C if and only if its translation occurs in the partial derivation of C and the order of rule applications is respected. □

Notice that no two top rule applications translating the same application (r) of a hypersequent rule in $\mathbb{R}(\Theta)$ can occur in a partial derivation of the same component of $\hat{H}_{\mathcal{D}}$. In particular, let C_i and C_j be two active components in the conclusion of (r) translated, respectively, with top rules (t_i) and (t_j). Since there is no application of (EC) in the derivation of $\hat{H}_{\mathcal{D}}$ and all other hypersequent rules have premisses with a single active component, the descendants of C_i will not interact with the descendants of C_j in the derivation of $\hat{H}_{\mathcal{D}}$. This means that the top rules (t_i) and (t_j) translating C_i and C_j must occur in the partial derivations of different components of $\hat{H}_{\mathcal{D}}$. This property is important, as the next step involves applying a bottom rule for each group of top rule applications translating one application of some rule hypersequent rule in $\mathbb{R}(\Theta)$.

We give an example to guide the intuition:

Example 7. Recall the hypersequent derivation of $\Rightarrow \Box(\Box\varphi \supset \psi) \vee \Box(\Box\psi \supset \varphi)$ in Example 2:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi, \Box\varphi \Rightarrow \varphi}(IW\Rightarrow) \quad \dfrac{\psi \Rightarrow \psi}{\psi, \Box\psi \Rightarrow \psi}(IW\Rightarrow)}{\Box\varphi \Rightarrow \psi | \Box\psi \Rightarrow \varphi}(r^{\mathbf{HK4}}_{S(\theta_{lin})})}{\Box\varphi \Rightarrow \psi | \Rightarrow \Box\psi \supset \varphi}(\Rightarrow\supset)}{\Rightarrow \Box\varphi \supset \psi | \Rightarrow \Box\psi \supset \varphi}(\Rightarrow\supset)}{\Rightarrow \Box\varphi \supset \psi | \Rightarrow \Box(\Box\psi \supset \varphi)}(\Rightarrow \Box_4)}{\Rightarrow \Box(\Box\varphi \supset \psi) | \Rightarrow \Box(\Box\psi \supset \varphi)}(\Rightarrow \Box_4)}{\Rightarrow \Box(\Box\varphi \supset \psi) | \Rightarrow \Box(\Box\varphi \supset \psi) \vee \Box(\Box\psi \supset \varphi)}(\Rightarrow \vee)}{\Rightarrow \Box(\Box\varphi \supset \psi) \vee \Box(\Box\psi \supset \varphi) | \Rightarrow \Box(\Box\varphi \supset \psi) \vee \Box(\Box\psi \supset \varphi)}(\Rightarrow \vee)}{\Rightarrow \Box(\Box\varphi \supset \psi) \vee \Box(\Box\psi \supset \varphi)}(EC)
$$

We construct partial derivations of the two premisses of the (EC) application as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\psi \Rightarrow \psi}{\psi, \Box\psi \Rightarrow \psi}(IW\Rightarrow)}{\Box\varphi \Rightarrow \psi}(t_1)}{\Rightarrow \Box\varphi \supset \psi}(\Rightarrow\supset)}{\Rightarrow \Box(\Box\varphi \supset \psi)}(\Rightarrow \Box_4)}{\Rightarrow \Box(\Box\varphi \supset \psi) \vee \Box(\Box\psi \supset \varphi)}(\Rightarrow \vee)
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi, \Box\varphi \Rightarrow \varphi}(IW\Rightarrow)}{\Box\psi \Rightarrow \varphi}(t_2)}{\Rightarrow \Box\psi \supset \varphi}(\Rightarrow\supset)}{\Rightarrow \Box(\Box\psi \supset \varphi)}(\Rightarrow \Box_4)}{\Rightarrow \Box(\Box\varphi \supset \psi) \vee \Box(\Box\psi \supset \varphi)}(\Rightarrow \vee)
$$

where (t_1) and (t_2) are the top rules of $(sys^{\mathbf{SK4}}_{S(\theta_{lin})})$ (see Example 4).

Finally, we obtain the 2-system derivation of $\Rightarrow \Box(\Box\varphi \supset \psi) \vee \Box(\Box\psi \supset \varphi)$ by applying a bottom rule below the two partial derivations as follows:

$$
\begin{prooftree}
\AxiomC{$\psi \Rightarrow \psi$}
\RightLabel{$(IW \Rightarrow)$}
\UnaryInfC{$\psi, \Box\psi \Rightarrow \psi$}
\RightLabel{(t_1)}
\UnaryInfC{$\Box\varphi \Rightarrow \psi$}
\RightLabel{$(\Rightarrow\supset)$}
\UnaryInfC{$\Rightarrow \Box\varphi \supset \psi$}
\RightLabel{$(\Rightarrow \Box_4)$}
\UnaryInfC{$\Rightarrow \Box(\Box\varphi \supset \psi)$}
\RightLabel{$(\Rightarrow \vee)$}
\UnaryInfC{$\Rightarrow \Box(\Box\varphi \supset \psi) \vee \Box(\Box\psi \supset \varphi)$}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{$(IW \Rightarrow)$}
\UnaryInfC{$\varphi, \Box\varphi \Rightarrow \varphi$}
\RightLabel{(t_2)}
\UnaryInfC{$\Box\psi \Rightarrow \varphi$}
\RightLabel{$(\Rightarrow\supset)$}
\UnaryInfC{$\Rightarrow \Box\psi \supset \varphi$}
\RightLabel{$(\Rightarrow \Box_4)$}
\UnaryInfC{$\Rightarrow \Box(\Box\psi \supset \varphi)$}
\RightLabel{$(\Rightarrow \vee)$}
\UnaryInfC{$\Rightarrow \Box(\Box\varphi \supset \psi) \vee \Box(\Box\psi \supset \varphi)$}
\RightLabel{(r_B)}
\BinaryInfC{$\Rightarrow \Box(\Box\varphi \supset \psi) \vee \Box(\Box\psi \supset \varphi)$}
\end{prooftree}
$$

Unfortunately, this process is not always straightforward as we might be forced to apply a single bottom rule for more than one group of top rules translating different applications of hypersequent rules. Such a derivation is called a *mixed 2-system* and it occurs when there is a multi-premiss rule application (r) in the hypersequent derivation of $\hat{H}_{\mathcal{D}}$ such that a component in the conclusion of (r) has ancestors that occur above different premisses of (r) and serve as active components of two different hypersequent rule applications (h') and (h'') in $\mathbb{R}(\Theta)$.

Example 8. Consider the following derivation of $\Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi) \vee \Box(\Box\varphi \supset (\psi \wedge \theta))$ in the hypersequent calculus for **S4.3** presented in Example 2:

$$
\begin{prooftree}
\AxiomC{$\psi \Rightarrow \psi$}
\RightLabel{$(IW \Rightarrow)$}
\UnaryInfC{$\psi, \Box(\psi \wedge \theta) \Rightarrow \psi$}
\RightLabel{$(\wedge \Rightarrow)$}
\UnaryInfC{$\psi \wedge \theta, \Box(\psi \wedge \theta) \Rightarrow \psi$}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{$(IW \Rightarrow)$}
\UnaryInfC{$\varphi, \Box\varphi \Rightarrow \varphi$}
\RightLabel{$(r^{\mathbf{HK4}}_{S(\theta_{lin})})'$}
\BinaryInfC{$\Box(\psi \wedge \theta) \Rightarrow \varphi | \Box\varphi \Rightarrow \psi$}
\AxiomC{$\theta \Rightarrow \theta$}
\RightLabel{$(IW \Rightarrow)$}
\UnaryInfC{$\theta, \Box(\psi \wedge \theta) \Rightarrow \theta$}
\RightLabel{$(\wedge \Rightarrow)$}
\UnaryInfC{$\psi \wedge \theta, \Box(\psi \wedge \theta) \Rightarrow \theta$}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{$(IW \Rightarrow)$}
\UnaryInfC{$\varphi, \Box\varphi \Rightarrow \varphi$}
\RightLabel{$(r^{\mathbf{HK4}}_{S(\theta_{lin})})''$}
\BinaryInfC{$\Box(\psi \wedge \theta) \Rightarrow \varphi | \Box\varphi \Rightarrow \theta$}
\RightLabel{$(\Rightarrow \wedge)$}
\BinaryInfC{$\Box(\psi \wedge \theta) \Rightarrow \varphi | \Box\varphi \Rightarrow (\psi \wedge \theta)$}
\RightLabel{$(\Rightarrow\supset)$}
\UnaryInfC{$\Rightarrow \Box(\psi \wedge \theta) \supset \varphi | \Box\varphi \Rightarrow \psi \wedge \theta$}
\RightLabel{$(\Rightarrow\supset)$}
\UnaryInfC{$\Rightarrow \Box(\psi \wedge \theta) \supset \varphi | \Rightarrow \Box\varphi \supset (\psi \wedge \theta)$}
\RightLabel{$(\Rightarrow \Box_4)$}
\UnaryInfC{$\Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi) | \Rightarrow \Box\varphi \supset (\psi \wedge \theta)$}
\RightLabel{$(\Rightarrow \Box_4)$}
\UnaryInfC{$\Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi) | \Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta))$}
\RightLabel{$(\Rightarrow \vee)$}
\UnaryInfC{$\Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi) \vee \Box(\Box\varphi \supset (\psi \wedge \theta)) | \Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta))$}
\RightLabel{$(\Rightarrow \vee)$}
\UnaryInfC{$\Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi) \vee \Box(\Box\varphi \supset (\psi \wedge \theta)) | \Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi) \vee \Box(\Box\varphi \supset (\psi \wedge \theta))$}
\RightLabel{(EC)}
\UnaryInfC{$\Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi) \vee \Box(\Box\varphi \supset (\psi \wedge \theta))$}
\end{prooftree}
$$

We obtain the partial derivations of the two premisses of (EC) as follows:

$$
\begin{prooftree}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{$(IW \Rightarrow)$}
\UnaryInfC{$\varphi, \Box\varphi \Rightarrow \varphi$}
\RightLabel{$(t_1)'$}
\UnaryInfC{$\Box(\psi \wedge \theta) \Rightarrow \varphi$}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{$(IW \Rightarrow)$}
\UnaryInfC{$\varphi, \Box\varphi \Rightarrow \varphi$}
\RightLabel{$(t_1)''$}
\UnaryInfC{$\Box(\psi \wedge \theta) \Rightarrow \varphi$}
\RightLabel{$(dummy)$}
\BinaryInfC{$\Box(\psi \wedge \theta) \Rightarrow \varphi$}
\RightLabel{$(\Rightarrow\supset)$}
\UnaryInfC{$\Rightarrow \Box(\psi \wedge \theta) \supset \varphi$}
\RightLabel{$(\Rightarrow \Box_4)$}
\UnaryInfC{$\Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi)$}
\RightLabel{$(\Rightarrow \vee)$}
\UnaryInfC{$\Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi) \vee \Box(\Box\varphi \supset (\psi \wedge \theta))$}
\end{prooftree}
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\psi \Rightarrow \psi}{\psi, \Box(\psi \wedge \theta) \Rightarrow \psi}\,(IW \Rightarrow)}{\psi \wedge \theta, \Box(\psi \wedge \theta) \Rightarrow \psi}\,(\wedge \Rightarrow)}{\Box\varphi \Rightarrow \psi}\,(t_2)' \qquad \dfrac{\dfrac{\dfrac{\theta \Rightarrow \theta}{\theta, \Box(\psi \wedge \theta) \Rightarrow \theta}\,(IW \Rightarrow)}{\psi \wedge \theta, \Box(\psi \wedge \theta) \Rightarrow \theta}\,(\wedge \Rightarrow)}{\Box\varphi \Rightarrow \theta}\,(t_2)''}{\Box\varphi \Rightarrow (\psi \wedge \theta)}\,(\Rightarrow \wedge)}{\Rightarrow \Box\varphi \supset (\psi \wedge \theta)}\,(\Rightarrow\supset)}{\Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta))}\,(\Rightarrow \Box_4)}{\Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi) \vee \Box(\Box\varphi \supset (\psi \wedge \theta))}\,(\Rightarrow \vee)$$

where (t_1) and (t_2) are the top rules of $(sys^{\mathbf{SK4}}_{S(\theta_{lin})})$ (see Example 4).

If we now wanted to apply bottom rules for 2-system instances S' and S'' of $(sys^{\mathbf{SK4}}_{S(\theta_{lin})})$ containing top rules $(t_1)', (t_2)'$ and $(t_1)'', (t_2)''$, respectively, we get:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi, \Box\varphi \Rightarrow \varphi}\,(IW \Rightarrow)}{\Box(\psi \wedge \theta) \Rightarrow \varphi}\,(t_1)' \quad \dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi, \Box\varphi \Rightarrow \varphi}\,(IW \Rightarrow)}{\Box(\psi \wedge \theta) \Rightarrow \varphi}\,(t_1)''}{\Box(\psi \wedge \theta) \Rightarrow \varphi}\,(dummy)}{\Rightarrow \Box(\psi \wedge \theta) \supset \varphi}\,(\Rightarrow\supset)}{\Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi)}\,(\Rightarrow \Box_4)}{\Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi) \vee \Box(\Box\varphi \supset (\psi \wedge \theta))}\,(\Rightarrow \vee) \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\psi \Rightarrow \psi}{\psi, \Box(\psi \wedge \theta) \Rightarrow \psi}\,(IW \Rightarrow)}{\psi \wedge \theta, \Box(\psi \wedge \theta) \Rightarrow \psi}\,(\wedge \Rightarrow)}{\Box\varphi \Rightarrow \psi}\,(t_2)' \quad \dfrac{\dfrac{\dfrac{\theta \Rightarrow \theta}{\theta, \Box(\psi \wedge \theta) \Rightarrow \theta}\,(IW \Rightarrow)}{\psi \wedge \theta, \Box(\psi \wedge \theta) \Rightarrow \theta}\,(\wedge \Rightarrow)}{\Box\varphi \Rightarrow \theta}\,(t_2)''}{\Box\varphi \Rightarrow (\psi \wedge \theta)}\,(\Rightarrow \wedge)}{\Rightarrow \Box\varphi \supset (\psi \wedge \theta)}\,(\Rightarrow\supset)}{\Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta))}\,(\Rightarrow \Box_4)}{\Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi) \vee \Box(\Box\varphi \supset (\psi \wedge \theta))}\,(\Rightarrow \vee)}{\Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi) \vee \Box(\Box\varphi \supset (\psi \wedge \theta))}\,b_{(S',S'')}$$

Notice that the only way to apply the bottom rules for S' and S'' is to apply the same bottom rule for both 2-system instances and thus obtain a mixed 2-system. We show below that this and every other derivation containing mixed 2-systems can be transformed into one in which every 2-system instance has a separate bottom rule.

We begin by observing the following properties of mixed 2-systems:

(i) If two top rule applications belong to the same mixed 2-system, then they cannot occur on the same path of the derivation tree.

(ii) If we obtained a partial derivation of a component C using a dummy bottom rule, we can remove all but one premiss from it and still get a partial derivation of C.

(iii) if a pair of top rules that translate different applications of hypersequent rules occur in the same mixed 2-system above different premisses of a non-dummy rule application, all other pairs of top rules translating these two applications of hypersequent rules occur above different premisses of dummy bottom rules.

We say that we *split* a dummy bottom rule application if instead of the original partial derivation we consider a partial derivation where some (possibly all but one) of the premisses of this dummy bottom rule application have been removed.

Consider an application of a dummy bottom rule where two applications of top rules belonging to the same mixed 2-system occur above different premisses of the rule. Then, it follows from the properties above that we can remove one of the top rule applications from the partial derivation containing the other by splitting the dummy bottom rule.

We proceed to show that every mixed 2-system can be resolved, i.e., we can always transform a derivation containing mixed 2-systems into a derivation of the same end-sequent using only (proper) 2-systems.

Theorem 8. Given a set Θ of simple $\mathcal{L}^1$-formulae describing simple frame properties and a sequent $\Gamma \Rightarrow \Delta$ we have: if $\vdash_{\mathbf{HL}+\mathbb{R}(\Theta)} \Gamma \Rightarrow \Delta$ then $\vdash_{\mathbf{SL}+\mathbb{S}(\Theta)} \Gamma \Rightarrow \Delta$.

Proof. Let $\mathcal{D}$ be a derivation of $\Gamma \Rightarrow \Delta$ in $\vdash_{\mathbf{HL}+\mathbb{R}(\Theta)}$. By Theorem 7, we can assume that $\mathcal{D}$ is in structured normal form. We begin by constructing a partial derivation $\mathcal{D}_i$ for each component C_i of $\hat{H}_{\mathcal{D}}$ using the procedure described in Lemma 3.

We next apply bottom rules to the roots of these partial derivations, in order to complete 2-systems and obtain a $\vdash_{\mathbf{SL}+\mathbb{S}(\Theta)}$ derivation of $\Gamma \Rightarrow \Delta$. We group all top rule applications translating the same hypersequent, and apply a bottom rule below all partial derivations in which these rule applications occur. Assume that by doing this we are forced to apply the same bottom rule for n such groups, translating hypersequent rule applications $(h^1), \ldots, (h^n)$. We then get the following mixed 2-system:

$$\dfrac{\begin{array}{c}\mathcal{D}_1 \\ \vdots \\ \Gamma \Rightarrow \Delta\end{array} \quad \ldots \quad \begin{array}{c}\mathcal{D}_k \\ \vdots \\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\ (b^{\{1,\ldots,n\}})$$

where each $\mathcal{D}_i$, for $1 \leq i \leq k$, contains top rule applications $(t_i^1), \ldots, (t_i^n)$ and where top rule applications with the same superscript translate the same hypersequent rule application.

We begin by replacing this mixed 2-system with a new 2-system the following form:

$$\dfrac{\left\{ \begin{array}{c}\mathcal{D}_i' \\ \vdots \\ \Gamma \Rightarrow \Delta\end{array} \ : \ 1 \leq i \leq k \text{ and } \mathcal{D}_i \text{ contains } (t_i^1) \right\}}{\Gamma \Rightarrow \Delta}\ (b^1) \tag{3.1}$$

where each $\mathcal{D}_i'$ contains only applications of top rule (t_i^1) and those top rules that cannot be removed from $\mathcal{D}_i$ by splitting dummy bottom rules. In case we need to choose which top rules remain in the derivation we take the ones with the minimum superscript.

We next introduce the bottom rule for the group of top rules with superscript 2. For every subderivation $\mathcal{D}'_i$ in the 2-system given above that contains an application of a top rule with a superscript 2, we create a new subderivation of the form:

$$\dfrac{\begin{array}{c}\mathcal{D}''_1\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array} \quad \dots \quad \begin{array}{c}\mathcal{D}'_i\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array} \quad \dots \quad \begin{array}{c}\mathcal{D}''_k\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,(b^2)$$

where derivations $\mathcal{D}''_1, \dots, \mathcal{D}''_k$ are copies of $\mathcal{D}_1, \dots, \mathcal{D}_k$ that only contain the applications of $(t^2_1), \dots, (t^2_n)$ and those top rules that cannot be removed by splitting dummy bottom rules. In particular, if $\mathcal{D}_i$, for $1 \leq i \leq k$, does not contain an application of (t^2_i), it is not considered and the corresponding $\mathcal{D}''_i$ is left out from the subderivation above. Further, every subderivation $\mathcal{D}'_i$ in the 2-system (3.1) that does not contain an application of the top rule (t^2_i) remains unchanged.

We then obtain the following 2-system

$$\dfrac{\left\{\begin{array}{c}\mathcal{D}'_i\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array} \;:\; (t^2_i) \text{ not in } \mathcal{D}'_i\right\} \quad \cup \quad \left\{\dfrac{\begin{array}{c}\mathcal{D}''_1\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array} \quad \dots \quad \begin{array}{c}\mathcal{D}'_i\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array} \quad \dots \quad \begin{array}{c}\mathcal{D}''_k\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,(b^2) \;:\; (t^2_i) \text{ in } \mathcal{D}'_i\right\}}{\Gamma \Rightarrow \Delta}\,(b^1)$$

where each $\mathcal{D}''_i$ is a copy of $\mathcal{D}_i$ containing only (t^2_i) and those top rules that cannot be removed by splitting dummy bottom rules. We proceed to repeat the same steps until we either introduced bottom rules for all superscript indices $1, \dots, n$, or no more bottom rules are needed. Now, in order to be sure that the mixed system is resolved, we need to show that when creating the new derivation we never add a top rule application above the wrong premiss of the corresponding bottom rule. Assume, towards a contradiction, that we add an application of the top rule (t^i_p) above the wrong premiss of (b^i), say q. Then, this must have happened when we introduced the bottom rule (b^j), for $1 \leq i < j \leq n$ and we could not remove (t^i_p) from the derivation containing (t^j_p) by splitting a dummy bottom rule. But then, by property (iii), we can remove from a partial derivation containing (t^i_q), where $q \neq p$, any (t^j_q) by splitting dummy bottom rules. As (b^i) occurs bellow (b^j), that means that there is no application of the top rule (t^j_q) on this branch of the bottom rule (b^i). Since by property (i) we can know that, apart from i and j, no other 2-system instances are involved, we can conclude that (t^j_p) is not needed on this derivation at all and hence we do not need to add (t^i_p), contrarily to our assumption and so, the considered mixed 2-system is resolved.

Iterative application of this procedure will eventually resolve every mixed 2-system and we will be left with a $\mathbf{SL}{+}\mathbb{S}(\Theta)$ derivation $\mathcal{D}'$of $\Gamma \Rightarrow \Delta$. Notice that this derivation is still in line with our definition of 2-systems that requires each branch above a bottom rule of some 2-system application to contain at least one application of the corresponding top rule. Consider some application (h) of the hypersequent rule $(r^{\mathbf{HL}}_{S(\theta)}) \in \mathbb{R}(\Theta)$ in $\mathcal{D}$ with k active components in the conclusion. Then, the 2-system $sys^{\mathbf{SL}}_{S(\theta)}$ corresponding to $(r^{\mathbf{HL}}_{S(\theta)})$ has k different top rules. Further, each of these top rules is involved in a partial derivation of a different active component in the conclusion of (h). Now, assume it is possible to apply a bottom rule (b) for the group of top rules translating (h) without creating a mixed system. Obviously, the requirement that above the i^{th} premiss of (b), $1 \leq i \leq k$, there must be an application of the top rule (t_i) is fulfilled. Now if a mixed system is created, notice that every time we introduce a new bottom rule for the group of top rules translating the (h), we also take the (modified) copies of all (and only those) branches above the bottom rule of the mixed system that contain top rules belonging to this group. The fulfillment of the requirement above hence follows from the fact that each top rule (t_i), for $1 \leq i \leq k$, of the 2-system corresponding to $(r^{\mathbf{HL}}_{S(\theta)})$ must be applied at least once in one of the branches above the bottom rule of the mixed system. $\square$

Example 9. Recall the mixed 2-system from the Example 8. We show how to use the procedure described in Theorem 8 in order to obtain a proper 2-system derivation.

We begin with creating the following derivation that we obtain by removing $(t_1)''$ from the dervation above the first premiss of $(b_{S'})$ through splitting the *dummy* bottom rule.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi, \Box\varphi \Rightarrow \varphi}\,(IW\Rightarrow)}{\Box(\psi\wedge\theta) \Rightarrow \varphi}\,(t_1)'}{\Rightarrow \Box(\psi\wedge\theta) \supset \varphi}\,(\Rightarrow\supset)}{\Rightarrow \Box(\Box(\psi\wedge\theta)\supset\varphi)}\,(\Rightarrow\Box_4)}{\Rightarrow \Box(\Box(\psi\wedge\theta)\supset\varphi) \vee \Box(\Box\varphi \supset (\psi\wedge\theta))}\,(\Rightarrow\vee) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\psi\Rightarrow\psi}{\psi,\Box(\psi\wedge\theta)\Rightarrow\psi}\,(IW\Rightarrow)}{\psi\wedge\theta,\Box(\psi\wedge\theta)\Rightarrow\psi}\,(\wedge\Rightarrow)}{\Box\varphi\Rightarrow\psi}\,(t_2)' \quad \dfrac{\dfrac{\dfrac{\theta\Rightarrow\theta}{\theta,\Box(\psi\wedge\theta)\Rightarrow\theta}\,(IW\Rightarrow)}{\psi\wedge\theta,\Box(\psi\wedge\theta)\Rightarrow\theta}\,(\wedge\Rightarrow)}{\Box\varphi\Rightarrow\theta}\,(t_2)''}{\Box\varphi \Rightarrow (\psi\wedge\theta)}\,(\Rightarrow\wedge)}{\Rightarrow \Box\varphi \supset (\psi\wedge\theta)}\,(\Rightarrow\supset)}{\Rightarrow \Box(\Box\varphi\supset(\psi\wedge\theta))}\,(\Rightarrow\Box_4)}{\Rightarrow \Box(\Box(\psi\wedge\theta)\supset\varphi) \vee \Box(\Box\varphi \supset (\psi\wedge\theta))}\,(\Rightarrow\vee)}{\Rightarrow \Box(\Box(\psi\wedge\theta)\supset\varphi) \vee \Box(\Box\varphi \supset (\psi\wedge\theta))}\,b_{(S')}$$

Further, notice that we could not remove $(t_2)''$ from the derivation above the second premiss of $(b_{S'})$ in the same manner. Hence, we proceed as follows:

Let $\alpha = \Box(\Box(\psi \wedge \theta) \supset \varphi) \vee \Box(\Box\varphi \supset (\psi \wedge \theta))$, and denote by $\mathcal{D}'_1$ the following derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi, \Box\varphi \Rightarrow \varphi}\,(IW\Rightarrow)}{\Box(\psi\wedge\theta) \Rightarrow \varphi}\,(t_1)'}{\Rightarrow \Box(\psi\wedge\theta) \supset \varphi}\,(\Rightarrow\supset)}{\Rightarrow \Box(\Box(\psi\wedge\theta)\supset\varphi)}\,(\Rightarrow\Box_4)}{\Rightarrow \alpha}\,(\Rightarrow\vee)$$

Further, let $\mathcal{D}''_1$ denote

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi, \Box\varphi \Rightarrow \varphi}\,(IW\Rightarrow)}{\Box(\psi \wedge \theta) \Rightarrow \varphi}\,(t_1)''}{\Rightarrow \Box(\psi \wedge \theta) \supset \varphi}\,(\Rightarrow\supset)}{\Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi)}\,(\Rightarrow \Box_4)}{\Rightarrow \alpha}\,(\Rightarrow \vee)$$

We construct the following derivation that contains no mixed 2-systems:

$$\dfrac{\begin{matrix}\mathcal{D}_1' \\ \vdots \\ \Rightarrow \alpha\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_1'' \\ \vdots \\ \Rightarrow \alpha\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\psi \Rightarrow \psi}{\psi, \Box(\psi \wedge \theta) \Rightarrow \psi}\,(IW\Rightarrow)}{\psi \wedge \theta, \Box(\psi \wedge \theta) \Rightarrow \psi}\,(\wedge\Rightarrow)}{\Box\varphi \Rightarrow \psi}\,(t_2)' \quad \dfrac{\dfrac{\dfrac{\theta \Rightarrow \theta}{\theta, \Box(\psi \wedge \theta) \Rightarrow \theta}\,(IW\Rightarrow)}{\psi \wedge \theta, \Box(\psi \wedge \theta) \Rightarrow \theta}\,(\wedge\Rightarrow)}{\Box\varphi \Rightarrow \theta}\,(t_2)''}{\Box\varphi \Rightarrow (\psi \wedge \theta)}\,(\Rightarrow\wedge)}{\Rightarrow \Box\varphi \supset (\psi \wedge \theta)}\,(\Rightarrow\supset)}{\Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta))}\,(\Rightarrow \Box_4)}{\Rightarrow \alpha}\,(\Rightarrow \vee)}{\Rightarrow \alpha}\,b_{(S'')}}{\Rightarrow \alpha}\,b_{(S')}$$

3.3 From 2-systems to Hypersequent Derivations

In this section we focus on the other direction of the embedding. Following the approach from [10], we show that given a set Θ of formulae describing simple frame properties, we can translate every **SL**+$\mathbb{S}(\Theta)$ derivation into a derivation in **HL**+$\mathbb{R}(\Theta)$ of the same end-sequent. As in the previous section, the translation is defined only for 2-system derivations whose structure fulfills certain conditions, i.e., that are in *2-system normal form*.

3.3.1 2-system Normal Form

In what follows, we show that every **SL**+$Sys(\mathbb{R})$ derivation can be transformed into one that is in 2-system normal form. We begin with defining the conditions a 2-system derivation must fulfill in order to be in the 2-system normal form.

Definition 24. Given a derivation $\mathcal{D}$ in **SL**+$Sys(\mathbb{R})$, we say that $\mathcal{D}$ is in *2-system normal form* if

(i) there are no two applications of a top rule (t) that belong to the same 2-system instance and occur on the same path of the derivation tree, and

(ii) there are no two 2-system instances S_1 and S_2 that are *entangled* i.e., there are no two 2-system instances S_1 and S_2 such that there are applications of top rules belonging to S_1 above applications of top rules belonging to S_2 and vice versa.

Lemma 4. For every derivation $\mathcal{D}$ in $\mathbf{SL}+Sys(\mathbb{R})$, there is a derivation $\mathcal{D}'$ of the same end-sequent in which two applications of a top rule that belong to the same 2-system instance never occur on the same path of the derivation tree.

Proof. Consider the derivation in $\mathcal{D}$ of the 2-system instance where two applications $(t)^1$ and $(t)^2$ of a top rule (t) appear on the same path of the derivation tree. Assume w.l.o.g. that $(t)^1$ appears above the first premiss of $(t)^2$. Then this derivation must be of the following shape:

$$
\cfrac{\cfrac{\cfrac{\begin{matrix}\mathcal{D}_1\\ \vdots\\ \Gamma_1, \Sigma \Rightarrow \Delta_1, \Pi\end{matrix} \quad \dots \quad \begin{matrix}\mathcal{D}_k\\ \vdots\ \mathcal{G}\\ \Gamma_k, \Sigma \Rightarrow \Delta_k, \Pi\end{matrix}}{\Gamma_0, \Sigma \Rightarrow \Delta_0, \Pi}\,(t)^1}{\begin{matrix}\vdots\ \mathcal{G}\\ \Gamma_1, \Sigma' \Rightarrow \Delta_1, \Pi'\end{matrix}} \quad \dots \quad \begin{matrix}\mathcal{D}'_k\\ \vdots\\ \Gamma_k, \Sigma' \Rightarrow \Delta_k, \Pi'\end{matrix}}{\begin{matrix}\Gamma_0, \Sigma' \Rightarrow \Delta_0, \Pi'\\ \vdots\end{matrix}}\,(t)^2
$$

We transform this derivation into one in which $(t)^1$ no longer appears as follows:

$$
\cfrac{\begin{matrix}\mathcal{D}_1\\ \vdots\\ \cfrac{\cfrac{\Gamma_1, \Sigma \Rightarrow \Delta_1, \Pi}{\Gamma_0, \Gamma_1, \Sigma \Rightarrow \Delta_1, \Pi}\,|\Gamma_0| \times (IW \Rightarrow)}{\Gamma_0, \Gamma_1, \Sigma \Rightarrow \Delta_1, \Delta_0, \Pi}\,|\Delta_0| \times (\Rightarrow IW)\\ \vdots\ \mathcal{G}'\\ \cfrac{\cfrac{\Gamma_1, \Gamma_1, \Sigma' \Rightarrow \Delta_1, \Delta_1, \Pi'}{\Gamma_1, \Sigma' \Rightarrow \Delta_1, \Delta_1, \Pi'}\,|\Gamma_1| \times (IC \Rightarrow)}{\Gamma_1, \Sigma' \Rightarrow \Delta_1, \Pi'}\,|\Delta_1| \times (\Rightarrow IC)\end{matrix} \quad \dots \quad \begin{matrix}\mathcal{D}'_k\\ \vdots\\ \Gamma_k, \Sigma' \Rightarrow \Delta_k, \Pi'\end{matrix}}{\begin{matrix}\Gamma_0, \Sigma' \Rightarrow \Delta_0, \Pi'\\ \vdots\end{matrix}}\,(t)^2
$$

where the subderivation $\mathcal{G}'$ has the same structure as $\mathcal{G}$. This means that once we obtain the sequent $\Gamma_0, \Gamma_1, \Sigma \Rightarrow \Delta_1, \Delta_0, \Pi$ we proceed to apply the rules as they were applied in $\mathcal{G}$ such that for each sequent $\Sigma'' \Rightarrow \Pi''$ in $\mathcal{G}$ we have a sequent $\Gamma_1, \Sigma'' \Rightarrow \Pi'', \Delta_1$ in $\mathcal{G}'$.

By repeating this procedure as many times as necessary, we finally obtain a derivation $\mathcal{D}'$ in which there is no 2-system instance such that two applications of one of its top rules occur on the same path of the derivation tree. □

We now turn to the second condition.

First, it is easy to verify that in order for two arbitrary 2-system instances S_1 and S_2 to be entangled, their corresponding bottom rules (b_1) and (b_2) must occur on the same path of the derivation tree. Assume w.l.o.g. that (b_2) occurs below (b_1), i.e., (b_1) occurs in a derivation above some premiss of (b_2). Then, as all top rules belonging to S_1 must appear above (b_1), they also appear above this premiss of (b_2). As a single 2-system instance has only one bottom rule associated with it, this means that all of the top rules belonging to S_1 appear above exactly one premiss of (b_2).

We next show how we can disentangle derivations containing entangled 2-system instances. To this end we introduce the notion of *e-reduction*.

Definition 25. Let $\mathcal{D}$ be a 2-system derivation. By Lemma 4 we can assume that no two applications of a top rule belonging to the same 2-system instance appear on the same derivation path. Further, let S be a 2-system instance in $\mathcal{D}$ and let $\mathcal{S}_e = \{S_1, \ldots, S_n\}$ be a non-empty set of 2-system instances entangled with S in the following way:

- all bottom rules belonging to $S_1, \ldots, S_n$ appear above the i^{th} premiss of the bottom rule (b_S) belonging to S, and
- no top rule belonging to S appears in $\mathcal{D}$ between two top rules $(t^i), (t^j)$ belonging to some S_i, S_j, for $1 \leq i, j \leq n$, respectively.

Consider the following derivation of the conclusion of (b_S) in $\mathcal{D}$:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array} \quad \ldots \quad \begin{array}{c}\mathcal{D}_i\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array} \quad \ldots \quad \begin{array}{c}\mathcal{D}_n\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,(b_S)$$

where only $\mathcal{D}_i$ contains applications of top rules belonging to $S_1, \ldots, S_n$.

We perform an *e-reduction* by replacing the derivation of (b_S) in $\mathcal{D}$ by the following derivation:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array} \quad \ldots \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array} \quad \ldots \quad \begin{array}{c}\mathcal{D}_i'\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array} \quad \ldots \quad \begin{array}{c}\mathcal{D}_n\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,(b_{S'}) \quad \ldots \quad \begin{array}{c}\mathcal{D}_n\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,(b_{S''})$$

where S' and S'' are copies of S, $(b_{S'})$ and $(b_{S''})$ are their respective bottom rules and $\mathcal{D}_i'$ is a copy of $\mathcal{D}_i$ such that

- if a top rule application (t) in $\mathcal{D}_i$ that belongs to S occurs above a top rule application belonging to some S_j, for $1 \leq j \leq n$, then the top rule application in $\mathcal{D}'_i$ that corresponds to (t) belongs to S'
- if a top rule (t) in $\mathcal{D}_i$ that belongs to S occurs below a top rule application belonging to some S_j, for $1 \leq j \leq n$, then the top rule application in $\mathcal{D}'_i$ that corresponds to (t) belongs to S''.

By construction, in this newly-obtained derivation no top rule of S' can occur below a top rule of some 2-system in $\mathcal{S}_e$, and that no top rule of S'' can occur above a top rule of some 2-system in $\mathcal{S}_e$. Hence, we have the following:

Claim 1: S' and S'' are not entangled with any of the 2-system instances from $\mathcal{S}_e$. $(*)$

Furthermore, as 2-system derivations contain no applications of a top rule belonging to the same 2-system instance that occur on the same path of the derivation tree, we get:

Claim 2: The e-reduction causes neither S' and S'' nor two copies of the same 2-system instance to be entangled or have top rules on the same derivation path. $(\star)$

Originally, each derivation $\mathcal{D}_j$, $1 \leq j \leq n$, of the premisses of (b_S) contains an application of the top rule (t^S_j) belonging to S. Hence, we immediately know that all but the i^{th} branch of the $(b_{S'})$ (resp. $(b_{S''})$) contain an application of the appropriate top rule $(t^{S'})$ (resp. $(t^{S''}_i)$). Further, we know that $\mathcal{D}_i$ must contain at least two applications of (t^S_i), one below and the other above top rules belonging to another system in $\mathcal{S}_e$, otherwise S would not be entangled. This means that in the derivation of the i^{th} premiss of $(b_{S''})$, and hence also $(b_{S'})$, we have two applications of (t^S_i), one belonging to S' and the other to S''. Further, 2-system instances that appear in $\mathcal{D}_j$, for $1 \leq j \leq n$ and $i \neq j$, are copied together with their original derivations, so they still satisfy the condition that there is at least one top rule application above each premiss of their bottom rules. Those 2-systems occurring in $\mathcal{D}_i$ are also copied together with their derivations, where the only modification is the reassignment of top rules previously belonging to S to either S' or S''. As the shape of these derivations remains the same, we have the following claim:

Claim 3: The e-reduction of S yields a valid 2-system derivation. $(\circ)$

Now given an derivation containing entangled 2-system instances, we can obtain an entanglement-free derivation by repeated application of e-reductions. In order to guarantee termination, we do not apply these in an arbitrary manner, but rather follow a predetermined strategy given below. We next introduce some helpful notation.

Definition 26. Let $\mathcal{D}$ be a derivation, S be a 2-system instance in it and $\mathcal{D}'$ a derivation obtained from $\mathcal{D}$ by applying e-reductions. Further, let $\mathcal{S}$ and $\mathcal{S}'$ denote the set of all 2-system instances in $\mathcal{D}$ and $\mathcal{D}'$, respectively. The equivalence relation $\sim$ is then a transitive and symmetric relation that holds between $S \in \mathcal{S}$ and a 2-system in $S' \in \mathcal{S}'$ iff S' is a copy of S generated by e-reductions.

Now, given some 2-system derivation $\mathcal{D}$, we denote by $E^{\mathcal{D}}$ the set of 2-system instances that appear entangled in $\mathcal{D}$, and by $E^{\mathcal{D}}/_{\sim}$ the quotient set of $E^{\mathcal{D}}$ with respect to the equivalence relation $\sim$. Notice that the number of such equivalence classes in this set is bounded by the number of 2-system instances in the original derivation.

Next, we denote by S^{low} the entangled 2-system instance with the lowest and leftmost bottom rule in a derivation. Finally, we introduce the notion of the *entanglement number*.

Definition 27. Let $\mathcal{D}$ be a 2-system derivation and S a 2-system instance in it. The *entanglement number* of S, or *e-number* for short, is computed as follows: for each premiss of the bottom rule of S, we count the number of distinct equivalence classes in the derivation of this premiss that contain 2-systems entangled with S. We then take the sum over these numbers to obtain the e-number of S.

Theorem 9. Every $\mathbf{SL}{+}\mathbb{S}(\Theta)$ derivation can be transformed into a $\mathbf{SL}{+}\mathbb{S}(\Theta)$ derivation of the same-end sequent in which no two 2-system instances are entangled.

Proof. We give a proof by induction on the lexicographically ordered triple $\langle \kappa, \mu, \nu \rangle$ specifying the complexity of the given a derivation $\mathcal{D}$, where

- $\kappa = |E^{\mathcal{D}}/_{\sim}|$, i.e., κ is the number of equivalence classes that contain entangled 2-system instances,
- μ is the maximum e-number of the 2-system instances in $[S^{low}]_{\sim} \in E^{\mathcal{D}}/_{\sim}$, and
- ν is the number of 2-system instances in the equivalence class $[S^{low}]_{\sim} \in E^{\mathcal{D}}/_{\sim}$ that have the e-number μ.

Base case: In a derivation $\mathcal{D}$ that contains entanglement $E^{\mathcal{D}}/_{\sim}$ is not empty, i.e. $\kappa > 0$. Further, there must be at least one element of the $[S^{low}]_{\sim}$ with an e-number > 0, so we have $\mu, \nu > 0$. Hence, if any of the κ, μ, ν is equal to 0, the derivation does not contain any entanglement.

Inductive step: Let $\mathcal{D}$ be an arbitrary 2-system derivation with the complexity $\langle \kappa, \mu, \nu \rangle$ such that $\langle \kappa, \mu, \nu \rangle \geq \langle 1, 1, 1 \rangle$. We obtain a 2-system derivation $\mathcal{D}'$ by applying an (arbitrary) e-reduction to an uppermost element $S \in [S^{low}]_{\sim} \in E^{\mathcal{D}}/_{\sim}$ with the e-number μ. As $S \in [S^{low}]_{\sim}$, S is an entangled copy of the lowest bottom rule S^{low}, S must occur below the bottom rules of all 2-system instances it is entangled with. Hence, applying an e-reduction to S is possible. Notice that by performing an e-reduction we do not increase

κ. Indeed, it is easy to verify that if a 2-system instance S is not entangled in $\mathcal{D}$, S and its copies will not be entangled in $\mathcal{D}'$.

Further, by applying an e-reduction we either reduce ν without increasing μ or we reduce μ. By ($\star$) and ($*$) and the definition of $\sim$, two 2-systems belonging to the same class $[S']_\sim \in E^{\mathcal{D}}/_\sim$ cannot have top rules on the same derivation path or be entangled. As such, if S was entangled with elements from $[S']_\sim$ that all occur above the same premiss of (B_S), then after the e-reduction S will no longer be entangled with these elements. This means that the number of classes containing elements S is entangled with above this particular premiss of (b_S) is reduced, and as such the whole e-number of S is reduced.

If $\nu > 1$, then there are multiple 2-system instances in the $[S^{low}]_\sim$ equivalence class that have the e-number μ, including S. However, as explained above, the e-number of S decreases after the e-reduction, and so does the number of elements in $[S^{low}]_\sim$ that have the maximum e-number μ. However, if $\nu = 1$ (and $\mu > 1$), S was the only element of $[S^{low}]_\sim$ with the e-number μ, then after the e-reduction μ has decreased.

Notice that we never increase neither μ nor ν. This follows from the fact that if an application of an e-reduction generates the copy of some 2-system instance, we have the following possibilities:

- this 2-system instance did not belong to $[S^{low}]_\sim$ equivalence class to begin with. Then, obviously its copies do not belong to $[S^{low}]_\sim$.

Otherwise, as we always reduce the topmost 2-system we have that either:

- this 2-system belongs to the $[S^{low}]_\sim$ equivalence class but it did not have the maximum e-number, in which case its copies will also not have the maximum e-number, or
- this 2-system instance is S itself, and its copies have the e-number strictly lower than the e-number of S.

Finally, if $\mu = \nu = 1$, we are left with a single element of the class $[S^{low}]_\sim$ which is entangled only with one other class. The application of the e-reduction thus disentangles all elements of the $[S^{low}]_\sim$ equivalence class, and so $[S^{low}]_\sim \notin E^{\mathcal{D}}/_\sim$, which means that κ has reduced. Hence, the derivation $\mathcal{D}'$ obtained through an e-reduction from $\mathcal{D}$, which by the claim ($\circ$) is indeed a 2-system derivation, has a lower complexity than $\mathcal{D}$. □

3.3.2 Translation

We are now ready to introduce the procedure that translates any derivation in **SL** extended with 2-systems from $\mathbb{S}(\Theta)$ into a hypersequent derivation in **HL**+$\mathbb{R}(\Theta)$, where Θ is a set of simple $\mathcal{L}^1$-formulae describing simple frame properties.

Given a derivation **SL**+$\mathbb{S}(\Theta)$, we first transform it into a derivation $\mathcal{D}$ in 2-system normal form, as described in the previous section. We then apply the algorithm given below to $\mathcal{D}$ to obtain a derivation of the same end-sequent in **HL**+$\mathbb{R}(\Theta)$.

The Algorithm

Input: A **SL**+$\mathbb{S}(\Theta)$ derivation $\mathcal{D}$ in 2-system normal form.
Output: A **HL**+$\mathbb{R}(\Theta)$ derivation $\mathcal{D}'$ of the same end-sequent.

Stage 1: The leaves of $\mathcal{D}$ are copied as leaves of $\mathcal{D}'$ and marked.

While there are rules in $\mathcal{D}$ with marked premisses:

Stage i: Take a rule that has all its premisses marked and translate it. If we need to choose which rule to translate, we choose in the following order: first one-premiss rules, then two-premiss rules and bottom rules and lastly all top rules of one 2-system instance grouped together. Note that the 2-system normal form ensures that during the translation there will be a stage where all of the top rules belonging to the same 2-system instance will be marked. Once the rule is translated, unmark all its premisses and mark its conclusion. The actual translation is done as follows:

- if the rule application chosen for translation is a one-premiss **SL** rule, then the corresponding **HL** rule is simply applied.
- if the rule application chosen for translation is a multi-premiss **SL** rule, then the corresponding **HL** rule is applied, possibly preceeded by the queues of (EW) that introduces missing components to the contexts of the premisses.
- if we chose to translate all top rules belonging to S that is a 2-system instance of $sys^{\mathbf{SL}}_{S(\theta)}$, for an n-simple formula $\theta \in \Theta$, we proceed as follows:
  Consider the set

  $$\mathbb{H} = \{\langle (t_1)^{x_1}, \ldots, (t_n)^{x_n} \rangle : (t_i)^{x_i} \text{ is an application of top rule } (t_i) \text{ belonging to } S\}$$

  where, for $1 \leq i \leq n$, we have $1 \leq x_i \leq$ number of applications of (t_i) that belong to S. Then, for each n-tuple $\langle (t_1)^{x_1}, \ldots, (t_n)^{x_n} \rangle \in \mathbb{H}$ we introduce a different application of the hypersequent rule $(r^{\mathbf{HL}}_{S(\theta)})$ as follows:
  Let the following be the considered top rules of S

  $$\dfrac{\begin{matrix}\vdots & & \vdots \\ C_1^1 & \ldots & C_1^{m_1}\end{matrix}}{\Sigma_1}\,(t_1^{x_1}) \qquad \ldots \qquad \dfrac{\begin{matrix}\vdots & & \vdots \\ C_n^1 & \ldots & C_n^{m_n}\end{matrix}}{\Sigma_n}\,(t_n^{x_n})$$

where each sequent $C_1^1, \ldots, C_1^{m_1}, \ldots, C_n^1, \ldots, C_n^{m_n}$, is marked. As each of these sequents is marked, that means that we have already translated their derivations into hypersequent derivations of $H_1^1|C_1^1, \ldots, H_1^{m_1}|C_1^{m_1}, \ldots, H_n^1|C_n^1, \ldots, H_n^{m_n}|C_n^{m_n}$. We can then proceed to apply (EW) as many times as necessary until we get $H^|C_1^1, \ldots, H|C_1^{m_1}, \ldots, H|C_n^1, \ldots, H|C_n^{m_n}$ where each component of H occurs in some $H_1^1, \ldots, H_1^{m_1}, \ldots, H_n^1, \ldots, H_n^{m_n}$. Notice that since we apply a hypersequent rule for every possible combination of top rules belonging to one 2-system instance we guarantee that we always have the correct active and context components, and (EW) is only used for padding the contexts with existing components so all of the premisses share the same context.

We now apply the following instance of the hypersequent rule $(r^{\mathbf{HL}}_{S(\theta)})$:

$$\frac{\{H|C : \ C \text{ is a premiss of one some } (t_i)^{x_i} \text{ for } 1 \leq i \leq n\}}{H|\Sigma_1| \ldots |\Sigma_n}$$

Having applied an instance of $(r^{\mathbf{HL}}_{S(\theta)})$ for each n-tuple in $\mathbb{H}$, and thus potentially duplicating hypersequent derivations obtained so far, we unmark the premisses of the top rules and mark their conclusions. In this section we focus on the other direction of the embedding.

- if we chose to translate the bottom rule application

$$\frac{\begin{matrix}\vdots \\ \Gamma \Rightarrow \Delta\end{matrix} \quad \ldots \quad \begin{matrix}\vdots \\ \Gamma \Rightarrow \Delta\end{matrix}}{\Gamma \Rightarrow \Delta} \ (b_S)$$

  that belongs to the 2-system instance S of $sys^{\mathbf{SL}}_{S(\theta)}$ we proceed as follows: We assume that the top rules belonging to S have been applied above the premisses of (b_S), as otherwise we did not need to apply a 2-system in the first place. Hence, we have a hypersequent derivation of $H|\Gamma \Rightarrow \Delta| \ldots |\Gamma \Rightarrow \Delta$ and so we obtain $\Gamma \Rightarrow \Delta$ by repeated applications of (EC).

We next show that the algorithm described above terminates, but not before it translates the root of $\mathcal{D}$.

Theorem 10. Given a set Θ of simple $\mathcal{L}^1$-formulae describing simple frame properties and a sequent $\Gamma \Rightarrow \Delta$ we have: if $\vdash_{\mathbf{SL}+\mathbb{S}(\Theta)} \Gamma \Rightarrow \Delta$ then $\vdash_{\mathbf{HL}+\mathbb{R}(\Theta)} \Gamma \Rightarrow \Delta$.

Proof. Let $\mathcal{D}$ be a derivation of $\Gamma \Rightarrow \Delta$ in $\vdash_{\mathbf{SL}+\mathbb{S}(\Theta)}$. By Theorem 9, we can assume that $\mathcal{D}$ is in 2-system normal form. We obtain a $\mathbf{HL}+\mathbb{R}(\Theta)$ derivation $\mathcal{D}'$ of $\Gamma \Rightarrow \Delta$ by running the algorithm described above on $\mathcal{D}$.

Notice that at each stage the algorithm translates at least one rule application in $\mathcal{D}$. Thus, as there are only finitely many rule applications in $\mathcal{D}$, the algorithm must terminate. We show by induction on the number of u of 2-system instances whose top rules are not yet translated that the algorithm does not terminate before translating the root of $\mathcal{D}$.

Base case: $u = 0$. In this case we simply translate all remaining rules (all the way to the root) as soon as their premisses are marked.
Inductive step: $u \geq 1$. As $\mathcal{D}$ is in 2-system normal form and we do not allow two top rules belonging to the same 2-system instance on the same derivation path, we know that there must be a 2-system instance S in $\mathcal{D}$ whose top rules are still untranslated but do not occur below any other untranslated rules. Hence, in order to mark the premisses of S, we only need to translate rules that do not belong to any 2-system instance, and as such can be translated as soon as their premisses are marked. Having translated these, we translate S and decrease u. $\Box$

We next give a small example to illustrate the approach given above.

Example 10. Consider the derivation $\mathcal{D}$ of the formula $\Box(\Box\varphi \supset (\psi\wedge\theta))\vee\Box(\Box(\psi\wedge\theta) \supset \varphi)$ in the calculus $\mathbf{SK}{+}sys^{\mathbf{SL}}_{S(\theta)}$, where $\theta = \forall w_1, w_2 \exists u(w_1\mathcal{R}u \wedge w_2 = u) \vee (w_2\mathcal{R}u \wedge w_1 = u)$:

$$
\begin{prooftree}
\AxiomC{$\psi \Rightarrow \psi$}
\RightLabel{$(IW \Rightarrow)$}
\UnaryInfC{$\psi, \theta \Rightarrow \psi$}
\RightLabel{$(\wedge \Rightarrow)$}
\UnaryInfC{$\psi \wedge \theta \Rightarrow \psi$}
\RightLabel{$(t_1)'$}
\UnaryInfC{$\Box\varphi \Rightarrow \psi$}
\AxiomC{$\theta \Rightarrow \theta$}
\RightLabel{$(IW \Rightarrow)$}
\UnaryInfC{$\psi, \theta \Rightarrow \theta$}
\RightLabel{$(\wedge \Rightarrow)$}
\UnaryInfC{$\psi \wedge \theta \Rightarrow \theta$}
\RightLabel{$(t_1)''$}
\UnaryInfC{$\Box\varphi \Rightarrow \theta$}
\RightLabel{$(\Rightarrow \wedge)$}
\BinaryInfC{$\Box\varphi \Rightarrow \psi \wedge \theta$}
\RightLabel{$\Rightarrow\supset$}
\UnaryInfC{$\Rightarrow \Box\varphi \supset (\psi \wedge \theta)$}
\RightLabel{$(\Rightarrow \Box)$}
\UnaryInfC{$\Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta))$}
\RightLabel{$(\Rightarrow \vee)$}
\UnaryInfC{$\Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta)) \vee \Box(\Box(\psi \wedge \theta) \supset \varphi)$}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{(t_2)}
\UnaryInfC{$\Box(\psi \wedge \theta) \Rightarrow \varphi$}
\RightLabel{$(\Rightarrow\supset)$}
\UnaryInfC{$\Rightarrow \Box(\psi \wedge \theta) \supset \varphi$}
\RightLabel{$(\Rightarrow \Box)$}
\UnaryInfC{$\Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi)$}
\RightLabel{$(\Rightarrow \vee)$}
\UnaryInfC{$\Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta)) \vee \Box(\Box(\psi \wedge \theta) \supset \varphi)$}
\RightLabel{(b)}
\BinaryInfC{$\Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta)) \vee \Box(\Box(\psi \wedge \theta) \supset \varphi)$}
\end{prooftree}
$$

We begin the translation of $\mathcal{D}$ into a translation in $\mathbf{HK}{+}(r^{\mathbf{HL}}_{S(\theta)})$ by copying the three premisses as they are, and applying the translations of sequent rules as follows:

$$
\dfrac{\dfrac{\psi \Rightarrow \psi}{\psi, \theta \Rightarrow \psi}\,(IW \Rightarrow)}{\psi \wedge \theta \Rightarrow \psi}\,(\wedge \Rightarrow)
\qquad
\dfrac{\dfrac{\theta \Rightarrow \theta}{\psi, \theta \Rightarrow \theta}\,(IW \Rightarrow)}{\psi \wedge \theta \Rightarrow \theta}\,(\wedge \Rightarrow)
\qquad
\varphi \Rightarrow \varphi
$$

Notice that $\mathcal{D}$ contains only one 2-system instance, call it S, with two applications of the top rule (t_1) and a single application of (t_2). As all of these applications have their premisses marked, and there are no more sequent rules that can be translated first, we proceed to translate the top rules applications belonging to S.

We consider the following two pairs of top rule applications belonging to S: $\langle (t_1)', (t_2) \rangle$ and $\langle (t_1)'', (t_2) \rangle$. Each of these pairs is translated by a different application of the hypersequent rule $(r^{\mathbf{HL}}_{S(\theta)})$ in the following manner:

$$\dfrac{\dfrac{\dfrac{\psi \Rightarrow \psi}{\psi, \theta \Rightarrow \psi}\,(IW \Rightarrow)}{\psi \wedge \theta \Rightarrow \psi}\,(\wedge \Rightarrow) \qquad \varphi \Rightarrow \varphi}{\Box\varphi \Rightarrow \psi | \Box(\psi \wedge \theta) \Rightarrow \varphi}\,(r^{\mathbf{HL}}_{S(\theta)})' \qquad \dfrac{\dfrac{\dfrac{\theta \Rightarrow \theta}{\psi, \theta \Rightarrow \theta}\,(IW \Rightarrow)}{\psi \wedge \theta \Rightarrow \theta}\,(\wedge \Rightarrow) \qquad \varphi \Rightarrow \varphi}{\Box\varphi \Rightarrow \theta | \Box(\psi \wedge \theta) \Rightarrow \varphi}\,(r^{\mathbf{HL}}_{S(\theta)})''$$

We then translate the rest of the sequent rules. Since we duplicated $\varphi \Rightarrow \varphi$, the translation of each rule applied to the descendants of this sequent will have to be duplicated as well. Hence, we get the following translation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\psi \Rightarrow \psi}{\psi, \theta \Rightarrow \psi}\,(IW \Rightarrow)}{\psi \wedge \theta \Rightarrow \psi}\,(\wedge \Rightarrow) \quad \varphi \Rightarrow \varphi}{\Box\varphi \Rightarrow \psi | \Box(\psi \wedge \theta) \Rightarrow \varphi}\,(r^{\mathbf{HL}}_{S(\theta)})'}{\Box\varphi \Rightarrow \psi | \Rightarrow \Box(\psi \wedge \theta) \supset \varphi}\Rightarrow\supset}{\Box\varphi \Rightarrow \psi | \Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi)}\,(\Rightarrow \Box)}{\Box\varphi \Rightarrow \psi | \Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta)) \vee \Box(\Box(\psi \wedge \theta) \supset \varphi)}\,(\Rightarrow \vee) \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\theta \Rightarrow \theta}{\psi, \theta \Rightarrow \theta}\,(IW \Rightarrow)}{\psi \wedge \theta \Rightarrow \theta}\,(\wedge \Rightarrow) \quad \varphi \Rightarrow \varphi}{\Box\varphi \Rightarrow \theta | \Box(\psi \wedge \theta) \Rightarrow \varphi}\,(r^{\mathbf{HL}}_{S(\theta)})''}{\Box\varphi \Rightarrow \theta | \Rightarrow \Box(\psi \wedge \theta) \supset \varphi}\Rightarrow\supset}{\Box\varphi \Rightarrow \theta | \Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi)}\,(\Rightarrow \Box)}{\Box\varphi \Rightarrow \theta | \Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta)) \vee \Box(\Box(\psi \wedge \theta) \supset \varphi)}\,(\Rightarrow \vee)}{\Box\varphi \Rightarrow \psi \wedge \theta | \Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta)) \vee \Box(\Box(\psi \wedge \theta) \supset \varphi)}\,(\Rightarrow \wedge)}{\Rightarrow \Box\varphi \supset (\psi \wedge \theta) | \Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta)) \vee \Box(\Box(\psi \wedge \theta) \supset \varphi)}\,(\Rightarrow\supset)}{\Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta)) | \Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta)) \vee \Box(\Box(\psi \wedge \theta) \supset \varphi)}\,(\Rightarrow \Box)}{\Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta)) \vee \Box(\Box(\psi \wedge \theta) \supset \varphi) | \Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta)) \vee \Box(\Box(\psi \wedge \theta) \supset \varphi)}\,(\Rightarrow \vee)$$

Let us denote by $\mathcal{D}'$ the derivation of

$$\Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi) \vee \Box(\Box\varphi \supset (\psi \wedge \theta)) \mid \Rightarrow \Box(\Box(\psi \wedge \theta) \supset \varphi) \vee \Box(\Box\varphi \supset (\psi \wedge \theta)).$$

Lastly, we translate the application of the bottom rule (b) as a single (EC) application and obtain the following derivation of $\Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta)) \vee \Box(\Box(\psi \wedge \theta) \supset \varphi)$ in $\mathbf{HK}$+$(r^{\mathbf{HL}}_{S(\theta)})$:

$$\dfrac{\begin{array}{c}\mathcal{D}' \\ \vdots \\ \Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta)) \vee \Box(\Box(\psi \wedge \theta) \supset \varphi) | \Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta)) \vee \Box(\Box(\psi \wedge \theta) \supset \varphi)\end{array}}{\Rightarrow \Box(\Box\varphi \supset (\psi \wedge \theta)) \vee \Box(\Box(\psi \wedge \theta) \supset \varphi)}\,(EC)$$

CHAPTER 4

Conclusion

In this thesis, we showed how to obtain unlabeled sequent calculi extended with two-level systems of rules for modal logics that can be characterized by frames with simple properties, with examples including many well-studied modal logics like **KT**, **KD**, **S4**, **S5**, **S4.3**, **K4D**, **KBD** etc. We thus proved that the systems of rules framework is powerful enough to capture this class of modal logic. Extending the methods from [9, 10], we first defined the required 2-systems from the hypersequent rules encoding simple frame properties. We then proved constructively that every derivation in the hypersequent calculus can be transformed into a derivation of the same end-sequent using the corresponding 2-systems and vice versa.

As a corollary of the embedding we get that our newly-obtained calculi are sound and complete for the considered logics. Moreover, we get that the calculi obtained in this way are analytic, and in the case of non-symmetric modal logics even enjoy strong cut-admissibility. Indeed, the starting point of our investigation were cut-free and/or analytic hypersequent calculi, which means that any sequent $\Gamma \Rightarrow \Delta$ provable in these calculi also has a derivation that is cut-free and/or respects the subformula property. Notice now that the way in which we translate these derivations into 2-system derivations preserves the structure of the original derivations. In particular, this means that if the original derivation was cut-free, the translated one will also be cut-free. Furthermore, as we do not introduce any formulae that were not present in the original derivation, the property of analyticity is also preserved.

It is worth mentioning that 2-systems are in spirit very close to natural deduction systems. Indeed, it was shown in [10] for intermediate logics that once all the syntactic sugar is removed, 2-systems are in fact natural deduction systems with higher-level rules – rules that can discharge other rule applications instead of only discharging formulas (see e.g., [25]). In this sense, the embedding of hypersequent calculi into systems of rules may provide new natural deduction systems for many modal logics. These systems could in turn be used for extracting suitable parallel λ-calculi, as done in [1] for Gödel logic and in [2] for classical logic.

One disadvantage of 2-systems is that, unlike hypersequents, they are non-local objects – top rules must be applied above the bottom rule (vertical non-locality), and top rules across different branches belonging to the same 2-system are dependent on each other (horizontal non-locality). The translation of 2-system derivations into hypersequent derivations helps us recover locality.

It is important to note that there are still interesting research questions that remain open. We have shown how to capture modal logics with simple frame properties using systems of rules. Although many interesting and well-studied logics can be characterized using simple frame properties, there are also many that cannot. It would thus be important to generalize the results from this thesis to cover modal logics beyond this class. Another question concerns the addition of quantifiers. Besides the theoretical interest in knowing whether and how this can be done, embedding hypersequent derivations for first-order modal logics into 2-systems would be, much like in the propositional case, the first step towards extracting first-order parallel lambda calculi. However, as shown in Section 4.1, the methods used in this thesis do not suffice and some modifications are needed even in the case of first-order Gödel logic. This leads us to believe that the same holds for first-order modal logic, however, due to the lack of hypersequent formulations for modal logics, this intuition was not investigated further.

Finally, the focus of both our investigation and the one in [10] was on the systems of rules of level two which correspond to the hypersequent calculi. However, the expressive powers of systems of rules of levels higher than two are still not well-understood. The authors of [9] believe that they might be the key to climbing up the substructural hierarchy (see [8]) and capturing logics that still do not posses uniform analytic hypersequent calculi; whether this intuition is correct poses an open research question.

4.1 Adding Quantifiers: The Case of Gödel Logic

One of the most well-known logics intermediate between intuitionistic and classical logic is Gödel logic obtained from intuitionistic logic by adding the linearity axiom $(\varphi \supset \psi) \vee (\psi \supset \varphi)$. A hypersequent calculus **HG** for propositional Gödel logic consists of the hypersequent version of standard intuitionistic sequent calculus and the communication rule (*com*) introduced in [4]. Table 4.1 summarizes the rules and axioms of **HG**. As shown in [10], derivations in **HG** can be transformed into derivations in the corresponding sequent calculus extended with 2-systems using methods similar to the ones described in this thesis.

First-order Gödel logic is obtained by extending first-order intuitionistic logic with the aforementioned linearity axiom and $(\forall x)(\varphi(x) \vee \psi) \supset ((\forall x)\varphi(x) \vee \psi)$, where x does not occur in ψ. The hypersequent calculus **HIF** for first-order Gödel logic was defined in [6] and consists of **HG** extended with the following rules for quantifiers:

Axioms: $\varphi \Rightarrow \varphi \quad \bot \Rightarrow \Pi$

Rules:

$$\frac{H|\Gamma \Rightarrow \Pi}{H|\Gamma, \varphi \Rightarrow \Pi}\ (IW) \qquad \frac{H|\Gamma, \varphi, \varphi \Rightarrow \Pi}{H|\Gamma, \varphi \Rightarrow \Pi}\ (IC) \qquad \frac{H}{H|\Gamma \Rightarrow \Pi}\ (EW)$$

$$\frac{H|\Gamma \Rightarrow \Pi|\Gamma \Rightarrow \Pi}{H|\Gamma \Rightarrow \Pi}\ (EC) \qquad \frac{H|\Gamma, \varphi_1 \Rightarrow \varphi_2}{H|\Gamma \Rightarrow \varphi_1 \supset \varphi_2}\ (\Rightarrow \supset) \qquad \frac{H|\Gamma, \varphi_1, \varphi_2 \Rightarrow \Pi}{H|\Gamma \varphi_1 \wedge \varphi_2 \Rightarrow \Pi}\ (\wedge \Rightarrow)$$

$$\frac{H|\Gamma \Rightarrow \varphi_i}{H|\Gamma \Rightarrow \varphi_1 \vee \varphi_2,}\ (\Rightarrow \vee) \qquad \frac{H|\Gamma, \varphi_1 \Rightarrow \Pi \qquad H|\Gamma, \varphi_2 \Rightarrow \Pi}{H|\Gamma, \varphi_1 \vee \varphi_2 \Rightarrow \Pi}\ (\vee \Rightarrow)$$

$$\frac{H|\Gamma \Rightarrow \varphi_1 \qquad H|\Gamma, \varphi_2 \Rightarrow \Pi}{H|\Gamma, \varphi_1 \supset \varphi_2 \Rightarrow \Pi}\ (\supset\Rightarrow) \qquad \frac{H|\Gamma \Rightarrow \varphi_1 \qquad H|\Gamma \Rightarrow \varphi_2}{H|\Gamma, \Rightarrow \varphi_1 \wedge \varphi_2}\ (\Rightarrow \wedge)$$

$$\frac{H|\Phi, \Gamma_1 \Rightarrow \Pi_1 \qquad H|\Psi, \Gamma_2 \Rightarrow \Pi_2}{H|\Psi, \Gamma_1 \Rightarrow \Pi_1|\Phi, \Gamma_2 \Rightarrow \Pi_2}\ (com) \qquad \frac{H|\Gamma \Rightarrow \varphi \qquad H|\varphi, \Gamma' \Rightarrow \Pi}{H|\Gamma, \Gamma' \Rightarrow \Pi}\ (cut)$$

Table 4.1: The hypersequent calculus **HG** for propositional Gödel logic, where $i \in \{1, 2\}$

$$\frac{H|\varphi(t), \Gamma \Rightarrow \Pi}{H|(\forall x)\varphi(x), \Gamma \Rightarrow \Pi}\ (\forall \Rightarrow) \qquad \frac{H|\varphi(a), \Gamma \Rightarrow \Pi}{H|(\exists x)\varphi(x), \Gamma \Rightarrow \Pi}\ (\exists \Rightarrow)$$

$$\frac{H|\Gamma \Rightarrow \varphi(a)}{H|\Gamma \Rightarrow (\forall x)\varphi(x)}\ (\Rightarrow \forall) \qquad \frac{H|\Gamma \Rightarrow \varphi(t)}{H|\Gamma \Rightarrow (\exists x)\varphi(x)}\ (\Rightarrow \exists)$$

where free variable a in the rules $(\exists \Rightarrow)$ and $(\Rightarrow \forall)$ must not occur in the lower hypersequent.

Note that this condition makes prevents **HG** derivations to be transformed into the structured normal form. Indeed, it is impossible to shift all applications of (EC) to the queue occurring immediately above the root of the derivation. Consider the following **HIF** derivation of $(\forall x)((P(x) \supset Q(x)) \vee (Q(x) \supset P(x)))$, where $P(x)$ and $Q(x)$ are atomic formulas:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Q(a) \Rightarrow Q(a) \qquad P(a) \Rightarrow P(a)}{P(a) \Rightarrow Q(a)|Q(a) \Rightarrow P(a)}\ (com)}{\Rightarrow P(a) \supset Q(a)|Q(a) \Rightarrow P(a)}\ (\Rightarrow\supset)}{\Rightarrow P(a) \supset Q(a)| \Rightarrow Q(a) \supset P(a)}\ (\Rightarrow\supset)}{\Rightarrow (P(a) \supset Q(a)) \vee (Q(a) \supset P(a))| \Rightarrow (P(a) \supset Q(a)) \vee (Q(a) \supset P(a))}\ (\Rightarrow \vee)}{\Rightarrow (P(a) \supset Q(a)) \vee (Q(a) \supset P(a))}\ (EC)}{\Rightarrow (\forall x)(P(x) \supset Q(x)) \vee (Q(x) \supset P(x))}\ (\Rightarrow \forall)$$

Notice that if we try to push the application of (EC) closer to the root, we will be forced to apply the $(\Rightarrow \forall)$ rule twice – once to each active component of the premiss of (EC). Then, the derivation must be of the following form:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Q(a) \Rightarrow Q(a) \qquad P(a) \Rightarrow P(a)}{P(a) \Rightarrow Q(a) | Q(a) \Rightarrow P(a)}(com)}{\Rightarrow P(a) \supset Q(a) | Q(a) \Rightarrow P(a)}(\Rightarrow\supset)}{\Rightarrow P(a) \supset Q(a) | \Rightarrow Q(a) \supset P(a)}(\Rightarrow\supset)}{\Rightarrow (P(a) \supset Q(a)) \vee (Q(a) \supset P(a)) | \Rightarrow (P(a) \supset Q(a)) \vee (Q(a) \supset P(a))}(\Rightarrow \vee)}{\Rightarrow (\forall x)((P(x) \supset Q(x)) \vee (Q(x) \supset P(x))) | \Rightarrow (P(a) \supset Q(a)) \vee (Q(a) \supset P(a))}(\Rightarrow \forall)'}{\Rightarrow (\forall x)((P(x) \supset Q(x)) \vee (Q(x) \supset P(x))) | \Rightarrow (\forall x)((P(x) \supset Q(x)) \vee (Q(x) \supset P(x)))}(\Rightarrow \forall)''}{\Rightarrow (\forall x)(P(x) \supset Q(x)) \vee (Q(x) \supset P(x))}(EC)
$$

However, this is not a legal derivation since the application $(\Rightarrow \forall)'$ is blocked as a occurs freely in its conclusion. This example shows that we cannot transform every derivation in **HIF** into one that is in structured normal form and hence we cannot use the same technique as before. Indeed, embedding first-order Gödel logic into systems of rules is an open research question whose solution might pave the way for the introduction of a typed lambda calculus amenable for synthesis of parallel functional programs. This calculus would extend the one in [1] obtained via Curry-Howard correspondence starting from the natural deduction calculus for propositional Gödel logic arising from the embedding in [10].

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