TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

DIPLOMARBEIT

Spectral Theorem for Definitizable Linear Relations on Krein Spaces

ausgeführt am Institut für

Analysis und Scientific Computing
der Technischen Universität Wien

unter der Anleitung von

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Datum

Unterschrift

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Introduction

The purpose of this master thesis is to develop a spectral theorem for definitizable linear relations on Krein spaces, cf. Theorem 3.4.4. In 1981, Dr. Peter Jonas has already considered the case of a self-adjoint and densely defined linear operator $A : \operatorname{dom} A \leq \mathcal{K} \to \mathcal{K}$, cf. [Jon].

In the first chapter, we give a brief introduction to Krein spaces and discuss the notion of a linear relation, which is a generalization of linear operators.

In the second chapter, we show how $r(T)$ can be defined for a linear relation T on a Krein space $\mathcal{K}$ and a rational function r whose poles lie in $\rho(T)$. This elementary functional calculus can easily be obtained by using an extension of the Riesz-Dunford functional calculus for linear relations. We also outline an elementary approach in Section 2.2.

At the beginning of the third chapter, we recall the spectral theorem for self-adjoint operators on Hilbert spaces and show how it can be extended to self-adjoint linear relations, cf. Theorem 3.1.3.

In Section 3.2, we show how linear relations can be moved from one space to another if a linear mapping between theses spaces is on hand. The theory developed in this section is the main tool in the proof of Theorem 3.4.4.

In the next section, we introduce and study the notion of a definitizable linear relation A with definitizing polynomial p. Probably, there are definitizable linear relations which do not have a *real* definitizing polynomial, cf. Remark 3.3.4. In the following, we assumed to have a real definitizing polynomial. Furthermore, it seems that it is, in general, not possible to divide out the non-real zeros of a definitizing polynomial which do not belong to the spectrum of the operator, as done in [Jon]. This is the reason why we only get $\sigma(A) \subseteq \overline{\mathbb{R}} \cup p^{-1}(\{0\})$, whereas equality prevailed in the case of a densely defined operator for a certain polynomial p. Hence, there occur non-real zeros in general, which could belong to the spectrum.

In Section 3.4, we give the construction of our functional calculus E^I, and formulate the main-result Theorem 3.4.4. It states that for every definitizable linear relation on a Krein space with real definitizing polynomial, there exists a $*$-homomorphism from a certain space of function, namely $\mathcal{F}^I$, to the bounded operators on the Krein space, $\mathcal{B}(\mathcal{K})$, which is an extension of the functional calculus for rational functions. The proof, in a nutshell: We construct a Hilbert space and a linear mapping between the Krein space and this Hilbert space, and then use the theory developed in Section 3.2 to move our definitizable linear relation to a self-adjoint linear relation in the Hilbert space, where we have a spectral theorem at hand.

In the last section, we show that this functional calculus E is continuous, at least on a subspace $\mathcal{F}_U$ with respect to a certain norm $\|.\|_U$, and we show that the support of E coincides with the spectrum of A.

Danksagung

Ich bedanke mich bei Michael Kaltenbäck, dessen Betreuung von Beginn an gewissenhaft war und der bei jeder Version dieser Arbeit in kürzester Zeit mit neuen Vorschlägen und Ideen aufwarten konnte.

Ich bin eigentlich kein Freund von schmalzigen Worten, doch gibt es, auf die letzten Jahre zurückblickend, viele Menschen, die mich maßgeblich unterstützt haben, sei es in moralischer oder materieller Hinsicht, und diesen Personen gebührt mein Dank.

Viele Kommilitonen müssen sich ihr Leben, ihre Wohnung und ihr Auto selbst finanzieren. Vor allem meinem Vater Markus, meiner Mutter Gisela, meiner Omi und Familie Kleiner habe ich zu verdanken, dass mir all diese Dinge in meiner Studienzeit zur Verfügung gestanden haben und ich mich voll und ganz auf mein Studium konzentrieren konnte. Dafür möchte ich allen meinen aufrichtigen Dank aussprechen.

Meinen Schwestern Charlotte, Theres, Sara, Magdalena und Valentina sowie meinem Schwager Mitchell, meiner Nichte Emily und meinem Neffen Samuel habe ich zu verdanken, dass bei uns immer etwas los ist ☺. Edit, Kati und Arno, und Annemarie danke ich für viele kulinarische Genüsse.

Zu guter Letzt möchte ich mich bei Melanie bedanken, weil sie mir stets in stressigen Zeiten viel Kraft gibt und für mich da ist.

1 Preliminaries

1.1 Krein Spaces

Definition 1.1.1. Let $\mathcal{X}$ be a linear space together with a map $[.,.] : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$, such that for all $x, y, z \in \mathcal{X}$ and $\lambda, \mu \in \mathbb{C}$

$$\begin{aligned}&[\lambda x + \mu y, z] = \lambda[x, z] + \mu[y, z]\\&[x, y] = \overline{[y, x]}.\end{aligned}$$

We call $[.,.]$ an *inner product* and $(\mathcal{X}, [.,.])$ an *inner product space.*

An element $x \in \mathcal{X}$ is called *positive/negative/neutral* if the real number $[x, x]$ is positive/negative/zero. A linear subspace $\mathcal{Y} \leq \mathcal{X}$ is called *positive (semi)definite* if the inequality $[y, y] > (\geq) 0$ holds for all elements $0 \neq y \in \mathcal{Y}$. Accordingly, $\mathcal{Y}$ can be *negative (semi)definite* or *neutral.* By saying that $\mathcal{Y}$ is *(semi)definite*, we mean that it is either positive or negative (semi)definite. The inner product $[.,.]$ is called positive/negative (semi)definite if the subspace $\mathcal{X} \leq \mathcal{X}$ has the corresponding property.

An element $x \in \mathcal{X}$ is called *isotropic* if $[x, y] = 0$ for all $y \in \mathcal{X}$. By $(\mathcal{X}, [.,.])^\circ$, or shorter $\mathcal{X}^\circ$, we denote the set of all isotropic elements, called the *isotropic part* of $\mathcal{X}$. The inner product is called *degenerated* if $\mathcal{X}^\circ \neq \{0\}$, otherwise it is called *nondegenerated.*

Lemma 1.1.2. *Let $(\mathcal{X}, [.,.])$ be a semidefinite inner product space. Then, the* Schwarz inequality *holds,*

$$|[x, y]|^2 \leq [x, x] \cdot [y, y], \quad \textit{for all } x, y \in \mathcal{X}.$$

Proof. We assume that $\mathcal{X}$ is positive semidefinite.

Take $x, y \in \mathcal{X}$ and let $\lambda \in \mathbb{C}$, $|\lambda| = 1$ be such that $\lambda[y, x] = |[y, x]| = |[x, y]|$. We have

$$\begin{aligned}0 &\leq [x - \xi\lambda y, x - \xi\lambda y] = [x, x] - \xi\lambda[y, x] - \xi\overline{\lambda}[x, y] + \xi^2[y, y]\\&= [x, x] - 2\xi|[x, y]| + \xi^2[y, y]\,, \qquad \text{for all } \xi \in \mathbb{R}.\end{aligned}$$

If $[y, y] = 0$, this inequality applied to large ξ give $|[x, y]| = 0$. In the case $[y, y] \neq 0$, we can choose $\xi = \frac{|[x,y]|}{[y,y]} \in \mathbb{R}$. This reduces the above inequality to $0 \leq [x, x] - \frac{|[x,y]|^2}{[y,y]}$, i.e. $|[x, y]|^2 \leq [x, x] \cdot [y, y]$.

The case of a negative semidefinite inner product $\mathcal{X}$ can be handled by considering $(\mathcal{X}, -[.,.])$. □

Clearly, the Schwarz inequality does not hold in general inner product spaces, since the right-hand side of the inequality gets negative for a positive x and a negative y.

Corollary 1.1.3. *Let $(\mathcal{X}, [.,.])$ be a semidefinite inner product space. Then, the set of all neutral elements is a subspace and coincides with the isotropic part, i.e.*

$$\mathcal{N} := \{x \in \mathcal{X} : x \textit{ is neutral}\} = \mathcal{X}^\circ.$$

Proof. An isotropic element is always neutral. The other inclusion follows directly from the Schwarz inequality,

$$0 \leq |[x,y]| \leq [x,x] \cdot [y,y] = 0.$$

□

Definition 1.1.4. Let $(\mathcal{X}, [.,.])$ be an inner product space.

If $x, y \in \mathcal{X}$ satisfy $[x,y] = 0$, we call them *orthogonal*, and write $x[\perp]y$. Two subsets $A, B \subset \mathcal{X}$ are called *orthogonal* if $[x,y] = 0$ for all $x \in A$ and $y \in B$, denoted by $A[\perp]B$. For a subset $Y \subseteq \mathcal{X}$ we set $Y^{[\perp]} := \{x \in \mathcal{X} \,|\, [x,y] = 0 \text{ for all } y \in Y\}$, and call $Y^{[\perp]}$ the *orthogonal companion* of Y. If explicit notation of the inner product is not needed, we will write $x \perp y$, $A \perp B$ and $Y^\perp$.

Remark 1.1.5. Let $(\mathcal{H}, (.,.))$ be a Hilbert space, and let $M \leq \mathcal{H}$ be a closed linear subspace. Recall that there always exists a unique orthogonal projection $P : \mathcal{H} \to \mathcal{H}$ onto M, namely the one corresponding to the decomposition of $\mathcal{H}$ into the direct and orthogonal sum $\mathcal{H} = M(\dot{+})M^\perp$.

In inner product spaces $\mathcal{X}$, the existence and uniqueness of orthogonal projections, i.e. projections P with $\operatorname{ran} P \perp \ker P$, onto a given subspace M fail in general. We could have $M + M^\perp \neq \mathcal{X}$ or $M \cap M^\perp \neq \{0\}$, or both.

However, if a direct and orthogonal decomposition $\mathcal{X} = M[\dot{+}]N$ with $N \leq \mathcal{X}$ is available, we can, without difficulty, consider the corresponding projection $P : \mathcal{X} \to \mathcal{X}$ along N onto M, which is orthogonal.

Definition 1.1.6. Let $(\mathcal{X}, [.,.])$ be an inner product space.

A pair $(\mathcal{X}_+, \mathcal{X}_-)$ consisting of a positive definite subspace $\mathcal{X}_+$ and a negative definite subspace $\mathcal{X}_-$ such that $\mathcal{X}$ can be expressed as the direct and orthogonal sum

$$\mathcal{X} = \mathcal{X}_+ \,[\dot{+}]\, \mathcal{X}^\circ \,[\dot{+}]\, \mathcal{X}_-,$$

is called *fundamental decomposition* of $\mathcal{X}$. The space $\mathcal{X}$ is called *decomposable*, if there exists a fundamental decomposition. The orthogonal projections P_+ along $\mathcal{X}_- \,[\dot{+}]\, \mathcal{X}^\circ$ onto $\mathcal{X}_+$, and P_- along $\mathcal{X}_+ \,[\dot{+}]\, \mathcal{X}^\circ$ onto $\mathcal{X}_-$ are called *fundamental projections.* The linear map $J := P_+ - P_-$ is called *fundamental symmetry.* Furthermore, we set $(x,y)_J := [Jx, y]$ for $x, y \in \mathcal{X}$, and $\|x\|_J := \sqrt{(x,x)_J}$ for $x \in \mathcal{X}$.

The next lemma states some elementary properties, which reveal the usefulness of a fundamental decomposition.

Lemma 1.1.7. *Let $(\mathcal{X}, [.,.])$ be a decomposable inner product space, with fundamental symmetry J. Then the following assertions hold true.*

(*i*) $[Jx, y] = [x, Jy]$, $(Jx, y)_J = (x, Jy)_J$ *for all* $x, y \in \mathcal{X}$

(*ii*) $(.,.)_J$ *is a positive semidefinite inner product, with the property*

$$\|x\|_J = 0 \;\;\Leftrightarrow\;\; x \in (\mathcal{X}, [.,.])^\circ = (\mathcal{X}, (.,.)_J)^\circ.$$

Hence, $\|.\|_J$ is a norm, or equivalently $(.,.)_J$ is positive definite, if and only if the space $\mathcal{X}$ is nondegenerated.

(*iii*) $J|_{\mathcal{X}_+} = \mathrm{I}_{\mathcal{X}_+}$, $J|_{\mathcal{X}_-} = -\mathrm{I}_{\mathcal{X}_-}$ *and* $(J|_{\mathcal{X}_+ \dot{+} \mathcal{X}_-})^2 = \mathrm{I}_{\mathcal{X}_+ \dot{+} \mathcal{X}_-}$.

Especially, we have $J^2 = \mathrm{I}$ *if* $\mathcal{X}$ *is nondegenerated.*

For all $x, y \in \mathcal{X}$ *we have*

(*iv*) $[JJx, y] = [x, y]$, $(JJx, y)_J = (x, y)_J$

(*v*) $(Jx, y)_J = [x, y]$

(*vi*) $[Jx, Jy] = [x, y]$, $(Jx, Jy)_J = (x, y)_J$

Proof. Every $x \in \mathcal{X}$ can be written as $x = P_+x + P_-x + x_0$ for a certain isotropic element x_0. Since isotropic elements can be omitted in the inner product, we have for all $x, y \in \mathcal{X}$

$$\begin{aligned}[Jx, y] = [P_+x - P_-x, P_+y + P_-y] = [P_+x, P_+y] - [P_-x, P_-y] = \\ = [P_+x + P_-x, P_+y - P_-y] = [x, Jy].\end{aligned}$$

The same property holds true for $(.,.)_J$, due to

$$(Jx, y)_J = [JJx, y] = [Jx, Jy] = (x, Jy)_J. \tag{1.1}$$

Since J is linear, we clearly have that $(.,.)_J$ is linear in the first argument. In fact, $(.,.)_J$ is an inner product, due to

$$(x, y)_J = [Jx, y] = [x, Jy] = \overline{[Jy, x]} = \overline{(y, x)_J},$$

for all $x, y \in \mathcal{X}$. The inner product $(.,.)_J$ is positive semidefinite, since we have for all $x \in X$

$$(x, x)_J = [Jx, x] = [P_+x - P_-x, P_+x + P_-x] = \underbrace{[P_+x, P_+x]}_{\geq 0} - \underbrace{[P_-x, P_-x]}_{\leq 0} \geq 0.$$

Furthermore, an element x is neutral with respect to $(.,.)_J$, i.e. $(x, x)_J = 0$, if and only if $[P_+x, P_+x] = 0$ and $[P_-x, P_-x] = 0$. This is the case, by definition of a definite subspace, if and only if $P_+x = 0 = P_-x$, i.e. $x \in (\mathcal{X}, [.,.])^\circ$.

This proves that the neutral subspace with respect to $(.,.)_J$ coincides with $(\mathcal{X}, [.,.])^\circ$. By Corollary 1.1.3, this is further equal to $(\mathcal{X}, (.,.)_J)^\circ$. The calculation

$$J^2 = (P_+ - P_-)(P_+ - P_-) = P_+P_+ - P_+P_- - P_-P_+ + P_-P_- = P_+ + P_-,$$

shows (*iii*). This directly implies (*iv*), since isotropic elements can be omitted in both inner products. Item (*v*) follows from equation (1.1) together with (*iv*). Finally, (*vi*) follows from (*iv*) and (*i*)

$$\begin{aligned}[Jx, Jy] &= [JJx, y] = [x, y] \\ (Jx, Jy)_J &= (JJx, y)_J = (x, y)_J.\end{aligned}$$

$\square$

In order to study continuous linear operators, we need to endow an inner product space with a topology. A good topology on an inner product space fits not only to the vector space but also to the inner product.

Definition 1.1.8. A triple $(\mathcal{X}, [.,.], \mathcal{T})$ consisting of an inner product space $(\mathcal{X}, [.,.])$ and a topology $\mathcal{T}$ on $\mathcal{X}$ is called a *topological inner product space*, if

(i) $(\mathcal{X}, \mathcal{T})$ is a topological vector space, i.e. vector addition and scalar multiplication are continuous.

(ii) the map $[.,.] : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ is continuous with respect to the product topology on $\mathcal{X} \times \mathcal{X}$.

Remark 1.1.9. For a subset of an inner product space $Y \subseteq \mathcal{X}$, the orthogonal companion $Y^{[\perp]}$ can be written as

$$Y^{[\perp]} = \{x \in \mathcal{X} \mid [x, y] = 0 \text{ for all } y \in Y\} = \bigcap_{y \in Y} \ker(f_y),$$

using the linear maps

$$f_y : \left\{ \begin{array}{ccc} \mathcal{X} & \to & \mathbb{C} \\ x & \mapsto & [x, y] \end{array} \right. .$$

This shows that $Y^{[\perp]}$ is a linear subspace of $\mathcal{X}$. Moreover, if we have a topology on $\mathcal{X}$, such that f_y is continuous (for example, in a topological inner product space), we conclude that $Y^{[\perp]}$ is closed.

Under which conditions does a topology on a given inner product space exists such that $(\mathcal{X}, [.,.], \mathcal{T})$ is a topological inner product space and is it unique? We will not discuss this question in general, since we are mainly interested in the special case of a Krein space. We want to refer to [Bog, III, IV] for a general discussion of this topic.

One approach on decomposable inner product spaces $\mathcal{X}$ is to take the topology induced by the semi-norm $\|x\|_J := \sqrt{(x, x)_J}$, cf. Definition 1.1.6. It can be shown that $\mathcal{X}$ becomes a topological inner product space if endowed with that topology. But if one chooses a different fundamental decomposition, one may get a different semi-norm, which may not be equivalent to the previous one and could, therefore, induce a different topology. In order to overcome these difficulties, we need more assumptions on the inner product space.

Definition 1.1.10. An inner product space $(\mathcal{K}, [.,.])$ is called *Krein space*, if it is non-degenerated and decomposable, such that the positive inner product space $(\mathcal{K}, (.,.)_J)$ is complete for some fundamental symmetry J.

Remark 1.1.11. We know by Lemma 1.1.7 that $(.,.)_J$ is positive for every fundamental symmetry J if the space is nondegenerated. Thus, we have a family of pre-Hilbert spaces $(\mathcal{K}, (.,.)_{\tilde{J}})$ attached to a nondegenereated and decomposable inner product space $\mathcal{K}$ where $\tilde{J}$ runs through all fundamental symmetries. The inner product space is a Krein space, per definitionem, if one of these pre-Hilbert spaces is actually a Hilbert space.

In fact, the situation is as follows: Either none or all of those pre-Hilbert spaces are complete. In the last case, all Banach space norms $\|.\|_{\tilde{J}}$ are equivalent, and therefore induce the same topology on $\mathcal{K}$.

Theorem 1.1.12. *Let $\mathcal{K}$ be a Krein space, and let J denote the fundamental symmetry from Definition 1.1.10. Let J' be an arbitrary fundamental symmetry. Then the norms $\|.\|_J$ and $\|.\|_{J'}$ are equivalent.*

Consequently, also $\|.\|_{J'}$ is complete and Definition 1.1.10 does not depend on the specific fundamental decomposition.

Proof. At first, we want to show that both P'_+ and P'_- and, therefore, also $J' = P'_+ - P'_- : \mathcal{K} \to \mathcal{K}$ are continuous with respect to $\|.\|_J$. In fact, we will show that P'_+ as well as P'_- have closed graphs. Then, the continuity follows directly from the closed graph theorem, since $\|.\|_J$ is complete.

So take a sequence of pairs (x_n, P'_+x_n), such that $x_n \to x$ and $P'_+x_n \to y$ with respect to $\|.\|_J$ for $n \to \infty$. We have to show $P'_+x = y$. Note that both $\operatorname{ran}(P'_+) = \mathcal{K}'_+$ and $\ker(P'_+) = \mathcal{K}'_-$ are closed with respect to $\|.\|_J$. This follows from ${\mathcal{K}'_+}^{[\perp]} = \mathcal{K}'_-$ and ${\mathcal{K}'_-}^{[\perp]} = \mathcal{K}'_+$, using Remark 1.1.9. We deduce $y \in \mathcal{K}'_+$. Moreover, $x_n - P'_+x_n \in \mathcal{K}'_-$ for all $n \in \mathbb{N}$ implies $\lim_{n\to\infty} x_n - P'_+x_n = x - y \in \mathcal{K}'_-$, i.e. $P'_+x = P'_+y = y$. We conclude that P'_+ has a closed graph, and is, therefore, continuous with respect to $\|.\|_J$. An similar argument shows that P'_- is continuous as well.

The continuity of J' and the property $\|Jx\|_J = \|x\|_J$, cf. Lemma 1.1.7 (vi), give

$$\|x\|^2_{J'} = [J'x, x] = (JJ'x, x)_J \leq \|JJ'x\|_J\|x\|_J = \|J'x\|_J\|x\|_J \leq C\|x\|^2_J,$$

which yields to the first inequality

$$\|x\|_{J'} \leq C'\|x\|_J. \tag{1.2}$$

To achieve the other inequality, we consider the linear operator $T := J'J$. Clearly, T is continuous with respect to $\|.\|_J$, since we have just shown that J' is continuous and J is even an isometry with respect to $\|.\|_J$. We claim that T is also continuous with respect to $\|.\|_{J'}$. First, we note

$$(Tx, y)_{J'} = (J'Jx, y)_{J'} = [Jx, y] = [x, Jy] = [x, J'Ty] = [J'x, Ty] = (x, Ty)_{J'}.$$

From this, we derive for $m \in \mathbb{N}$

$$\|T^{2^m}x\|^2_{J'} = (T^{2^m}x, T^{2^m}x)_{J'} = (T^{2^{m+1}}x, x)_{J'} \leq \|T^{2^{m+1}}x\|_{J'}\|x\|_{J'}. \tag{1.3}$$

Now, we use induction to show

$$\|Tx\|^{2^n}_{J'} \leq \|T^{2^n}x\|_{J'}\|x\|^{2^n-1}_{J'}. \tag{1.4}$$

In the base case $n = 1$, (1.4) follows directly from (1.3), setting $m = 0$. Assuming (1.4) to be true for n, we get

$$\begin{aligned}\|Tx\|^{2^{n+1}}_{J'} &= \left(\|Tx\|^{2^n}_{J'}\right)^2 \overset{(1.4)}{\leq} \|T^{2^n}x\|^2_{J'}\|x\|^{2(2^n-1)}_{J'} \overset{(1.3)}{\leq} \\ &\leq \|T^{2^{n+1}}x\|_{J'}\|x\|^{2(2^n-1)+1}_{J'} = \|T^{2^{n+1}}x\|_{J'}\|x\|^{2^{n+1}-1}_{J'}.\end{aligned}$$

Thus, (1.4) is true for all $n \in \mathbb{N}$. We rewrite this identity in order to get the asserted continuity of T:

$$\|Tx\|_{J'} \leq \|T^{2^n}x\|_{J'}^{2^{-n}}\|x\|_{J'}^{1-2^{-n}} \overset{(1.2)}{\leq} C^{2^{-n}}\|T^{2^n}x\|_{J}^{2^{-n}}\|x\|_{J'}^{1-2^{-n}} \leq$$
$$\leq C^{2^{-n}}\left(\|T\|^{2^n}\|x\|\right)_J^{2^{-n}}\|x\|_{J'}^{1-2^{-n}} = C^{2^{-n}}\|T\|\|x\|_J^{2^{-n}}\|x\|_{J'}^{1-2^{-n}} \overset{n\to\infty}{\longrightarrow} \|T\|\|x\|_{J'}$$

Using the continuity of T

$$\|x\|_J^2 = [Jx,x] = [J'Tx,x] = (Tx,x)_{J'} \leq \|Tx\|_{J'}\|x\|_{J'} \leq \|T\|\|x\|_{J'}^2,$$

gives the desired inequality $\|x\|_J \leq D\|x\|_{J'}$, for some constant $D > 0$.

□

If not stated differently, we will always endow a Krein space $\mathcal{K}$ with the *Krein space topology* induced by $(.,.)_J$, for some fundamental symmetry J.

Lemma 1.1.13. *Let $(\mathcal{K}, [.,.])$ be a Krein space. Then, for all subsets $M \subseteq \mathcal{K}$, we have $M^{\perp\perp} = \overline{M}$.*

Proof. Let J be a fundamental symmetry. Due to $[x,y] = (x,Jy)_J = (Jx,y)_J$, we have $x \in M^\perp$ if and only if $x \in (JM)^{(\perp)}$, and if and only if $Jx \in M^{(\perp)}$. Here, $(\perp)$ denotes the orthogonal complement in the Hilbert space $(\mathcal{K}, (.,.)_J)$. The identity $M^\perp = (JM)^{(\perp)} = JM^{(\perp)}$ gives

$$M^{\perp\perp} = \left(JM^{(\perp)}\right)^{\perp} = \left(JJM^{(\perp)}\right)^{(\perp)} = M^{(\perp)(\perp)} = \overline{M}.$$

□

Example 1.1.14. We want to give an example of a Krein space.

Consider a Hilbert space $(\mathcal{H}, (.,.))$ and a self-adjoint bounded operator $G \in \mathcal{B}(\mathcal{H})$ with $0 \in \rho(G)$. The map $[x,y] := (Gx,y)$ defines an inner product on $\mathcal{H}$, since G is self-adjoint

$$[x,y] = (Gx,y) = (x,Gy) = \overline{(Gy,x)} = \overline{[y,x]}.$$

We claim that $(\mathcal{H}, [.,.])$ is a Krein space.

For a $x \in (\mathcal{H}, [.,.])^\circ$ we have $[x,y] = (Gx,y) = 0$ for all $y \in \mathcal{H}$, which implies $Gx = 0$. Since G is injective, we get $x = 0$. Therefore, $(\mathcal{H}, [.,.])$ is nondegenerated.

In order to show that our space is in fact a Krein space, we are going to construct a fundamental decomposition.

Due to the spectral theorem, there exists a spectral measure E on $\mathbb{R}$, such that $G = \int_{\sigma(G)} t\, dE(t)$. We define orthogonal projections via

$$P_+ := E(\mathbb{R}^+) \text{ and } P_- := E(\mathbb{R}^-).$$

Clearly, we have $P_+P_- = E(\mathbb{R}^+ \cap \mathbb{R}^-) = E(\emptyset) = 0$ and $P_+ + P_- = E(\mathbb{R}^+ \cup \mathbb{R}^-) = E(\mathbb{R} \setminus \{0\}) = E(\mathbb{R}) = \mathrm{I}$, as $\operatorname{ran} E(\{0\}) = \ker G = \{0\}$. This decomposes $\mathcal{H}$ into the direct and, with respect to $(.,.)$, orthogonal sum

$$\mathcal{H} = \operatorname{ran}(P_+) \dot{+} \operatorname{ran}(P_-).$$

This decomposition is also orthogonal w.r.t $[.,.]$, since we have

$$[P_+x, P_-y] = (GP_+x, P_-y) = (P_-GP_+x, y) = 0,$$

due to

$$P_-GP_+ = \int \mathbb{1}_{\mathbb{R}^-}(t)\, dE(t) \int t\, dE(t) \int \mathbb{1}_{\mathbb{R}^+}(t)\, dE(t) = \int t \cdot \mathbb{1}_{\emptyset}(t)\, dE(t) = 0.$$

Hence, we have found a fundamental decomposition with corresonding fundamental symmetry $J = P_+ - P_-$. It remains to show that the positive inner product

$$(x, y)_J = [Jx, y] = (GJx, y)$$

is complete. This is true, since $\|.\|$ and $\|.\|_J$ are equivalent:

$$\|x\|_J^2 = (x, x)_J = (GJx, x) \le \|G\|\|J\|\|x\|^2 = C\|x\|^2.$$

The other inequality follows from

$$\begin{aligned} \|x\|^2 =&(x, x) = [G^{-1}x, x] = (JG^{-1}x, x)_J \le \|JG^{-1}x\|_J\|x\|_J = \\ =&\|G^{-1}x\|_J\|x\|_J \le C\|G^{-1}x\|\|x\|_J \le C\|G^{-1}\|\|x\|\|x\|_J = D\|x\|\|x\|_J, \end{aligned}$$

after dividing by $\|x\|$.

This is not only an example but rather a different approach to Krein spaces. It is possible to start with a Hilbert space together with a so-called Gram operator G with certain properties, and then define an inner product like above, rather than starting with an inner product, like we did, and obtaining a Gram operator J and a Hilbert space in the aftermath.

1.2 Linear Relations

The main source for this section was [Kal2]. Without much effort, we generalized some notions and results to Krein spaces. Other brief introductions to linear relations are presented in [Sch] and [Neu].

Definition 1.2.1. Let X, Y be vector spaces. We call T a *linear relation* between X and Y if T is a linear subspace of the Cartesian product $X \times Y$, i.e. $T \leq X \times Y$.

Remark 1.2.2. Linear relations are a generalization of linear operators, simply by identifying an operator $T : X \to Y$ with its graph. Clearly, not all linear relations are graphs of operators. Just consider the linear relation $X \times Y$ or $\{0\} \times Y$.

Definition 1.2.3. For a linear relation $T \leq X \times Y$, we define

$$\begin{aligned} \operatorname{dom} T &:= \{x \in X : \exists y \in Y : (x; y) \in T\}, \text{ the } \textit{domain} \text{ of } T, \\ \operatorname{ran} T &:= \{y \in Y : \exists x \in Y : (x; y) \in T\}, \text{ the } \textit{range} \text{ of } T, \\ \ker T &:= \{x \in X : (x; 0) \in T\}, \text{ the } \textit{kernel} \text{ of } T, \\ \operatorname{mul} T &:= \{y \in Y : (0; y) \in T\}, \text{ the } \textit{multi-valued part} \text{ of } T. \end{aligned}$$

Remark 1.2.4. All these sets are linear subsets of X or Y. If T is in fact a linear operator, the above defined notions, excluding the multi-valued part, contain nothing new and match to the usual ones.

Clearly, we have $\operatorname{mul} T = \{0\}$ if T is an operator. On the other hand, the next lemma states that the condition $\operatorname{mul} T = \{0\}$ ensures that T can be interpreted to be the graph of a linear operator.

Lemma 1.2.5. *Let $T \leq X \times Y$ be a linear relation. For each $(x; y) \in T$ we have*

$$\{z \in Y : (x; z) \in T\} = y + \operatorname{mul} T.$$

Proof. For a $z \in Y$ with $(x; z) \in T$ we have $(x; z) - (x; y) = (0; z - y) \in T$. Therefore, $z - y \in \operatorname{mul} T$, i.e. $z \in y + \operatorname{mul} T$.

On the other hand, take $m \in \operatorname{mul} T$. The calculation $(x; y) + (0; m) = (x; y + m) \in T$ shows that $y + m$ is contained in the left-hand side. □

Definition 1.2.6. Let X, Y, Z be vector spaces, $S, T \leq X \times Y$, $R \leq Y \times Z$ and $\alpha \in \mathbb{C}$. Then we define

$$\begin{aligned} S + T &:= \{(f; g) \in X \times Y : \exists g_1, g_2 \in Y : g = g_1 + g_2, (f; g_1) \in S, (f; g_2) \in T\} \\ \alpha T &:= \{(f; \alpha g) \in X \times Y : (f; g) \in T\} \\ T^{-1} &:= \{(g; f) \in Y \times X : (f, g) \in T\} \text{ and} \\ RS &:= \{(f; h) \in X \times Z : \exists g \in Y : (f; g) \in S, (g; h) \in R\}. \end{aligned}$$

Remark 1.2.7. These notions are a direct generalization of the corresponding ones for linear operators. One easily verifies that each of these lines defines a linear relation. As usual, we are going to use the shortcut $T + \alpha$ for $T + \alpha I$.

Next, we want to present a way to transform linear relations, using 2-by-2 matrices.

Definition 1.2.8. For $M := \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathbb{C}^{2\times 2}$ we define

$$\tau_M : \begin{cases} X \times X & \to \quad X \times X \\ \quad (f;g) & \mapsto \quad (\delta f + \gamma g; \beta f + \alpha g) \end{cases}.$$

Lemma 1.2.9. *We have the following properties for $M, N \in \mathbb{C}^{2\times 2}$.*

(*i*) *τ_M is a linear map.*

(*ii*) $\tau_M \circ \tau_N = \tau_{MN}$

(*iii*) *If $\det M \neq 0$, we have $(\tau_M)^{-1} = \tau_{M^{-1}}$.*

(*iv*) *If X is a topological vector space, τ_M is continuous.*

Proof.

(*i*) This is obvious.

If one writes the elements of $X \times X$ as vectors, i.e. $(f;g) = \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$, we can write τ_M in block-operator form,

$$\tau_M \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \delta\, \mathrm{I}_X & \gamma\, \mathrm{I}_X \\ \beta\, \mathrm{I}_X & \alpha\, \mathrm{I}_X \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} \delta f & \gamma g \\ \beta f & \alpha g \end{pmatrix}.$$

(*ii*) Consider $M := \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ and $N := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. With the notation from (*i*), we have

$$\tau_N \circ \tau_M = \begin{pmatrix} \delta\, \mathrm{I}_X & \gamma\, \mathrm{I}_X \\ \beta\, \mathrm{I}_X & \alpha\, \mathrm{I}_X \end{pmatrix} \begin{pmatrix} d\, \mathrm{I}_X & c\, \mathrm{I}_X \\ b\, \mathrm{I}_X & a\, \mathrm{I}_X \end{pmatrix} =$$
$$= \begin{pmatrix} (\delta d + \gamma b)\, \mathrm{I}_X & (\delta c + \gamma a)\, \mathrm{I}_X \\ (\beta d + \alpha b)\, \mathrm{I}_X & (\beta c + \alpha a)\, \mathrm{I}_X \end{pmatrix} = \tau_{NM}.$$

(*iii*) This follows from (*ii*), due to

$$\tau_M \circ \tau_{M^{-1}} = \tau_{MM^{-1}} = \tau_I = \mathrm{I}_{X\times X} = \tau_{M^{-1}} \circ \tau_M.$$

(*iv*) The continuity of τ_M is clear, since its components consist of scalar multiplications and vector additions only.

□

Remark 1.2.10. For a linear relation $T \leq X \times X$ and a matrix $M \in \mathbb{C}^{2\times 2}$, we can look at the subset $\tau_M(T) \subseteq X \times X$. Since τ_M is a linear mapping, and since T is a linear subset, we get that $\tau_M(T)$ is also a linear subset of $X \times X$, i.e. a linear relation on X. Directly from the definition of τ_M we see

$$\operatorname{dom} \tau_M(T) = \{\delta f + \gamma g : (f;g) \in T\}, \ \operatorname{ran}(\tau_M(T)) = \{\beta f + \alpha g : (f;g) \in T\}.$$

Remark 1.2.11. One can use the properties stated in Lemma 1.2.9 to perform elementary calculations with linear relations. For example, we have

$$\begin{aligned}
T^{-1} &= \tau_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)}(T), \\
\lambda T &= \tau_{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)}(T), \\
(T+\mu) &= \tau_{\left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right)}(T), \\
(tT+s) &= \tau_{\left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)}(T) = \tau_{\left(\begin{smallmatrix} t & s \\ 0 & 1 \end{smallmatrix}\right)}(T), \\
(T-\mu)^{-1} &= \tau_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -\mu \\ 0 & 1 \end{smallmatrix}\right)}(T) = \tau_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & -\mu \end{smallmatrix}\right)} \\
I+\mu(T-\mu)^{-1} &= \tau_{\left(\begin{smallmatrix} \mu & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & -\mu \end{smallmatrix}\right)}(T) = \tau_{\left(\begin{smallmatrix} 1 & 0 \\ 1 & -\mu \end{smallmatrix}\right)}
\end{aligned}$$

Later on, we are going to expand this idea of "moving" linear relations, using more general linear mappings from $X \times X$ to $Y \times Y$, cf. Section 3.2.

Remark 1.2.12. We suppose that τ_M adapts its domain and codomain to its argument, in the sense that we write $\tau_M(R)$ and $\tau_M(S)$ for linear relations on different vector spaces, $R \le X \times X$ and $S \le Y \times Y$.

Furthermore, in the special case $M = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \delta \end{smallmatrix}\right)$, we can interpret τ_M to have domain and codomain $X \times Y$. Then, $\tau_M(T) \le X \times Y$ is well-defined for a $T \le X \times Y$.

Similarly, if $M = \left(\begin{smallmatrix} 0 & \beta \\ \gamma & 0 \end{smallmatrix}\right)$ and $T \le X \times Y$, we get $\tau_M(T) \le Y \times X$.

Definition 1.2.13. Let X be a Banach space, and let $T \le X \times X$ be a linear relation. Then we call

$\rho(T) := \{\lambda \in \mathbb{C} \cup \{\infty\} : (T-\lambda)^{-1} \in \mathcal{B}(X)\}$	the *resolvent set*,
$\sigma(T) := (\mathbb{C} \cup \{\infty\}) \setminus \rho(T)$	the *spectrum*,
$r(T) := \{\lambda \in \mathbb{C} \cup \{\infty\} : (T-\lambda)^{-1} \in \mathcal{B}(\operatorname{dom}(T))\}$	the set of points of *regular type*,

where we set $(T-\infty)^{-1} := T$, $\operatorname{dom}(T-\infty)^{-1} = \operatorname{dom} T$.

Remark 1.2.14. Notice that $(T-\lambda)^{-1}$ always exists as a linear relation. By definition, we have $\lambda \in \rho(T)$ if and only if this linear relation is in fact an everywhere defined bounded operator.

Furthermore, we want to remark that $\infty \in \rho(T)$ if and only if $T \in \mathcal{B}(X)$. Although it is not necessary to take also the point ∞ into consideration when studying the spectrum, this gives an harmonious overall picture.

For an invertible $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathbb{C}^{2\times 2}$, the mapping defined by

$$\phi_M(z) := \frac{\alpha z + \beta}{\gamma z + \delta}$$

is known as the *Möbius transformation.* By setting $\phi_M(\infty) := \frac{\alpha}{\gamma}$, $\phi_M(-\frac{\delta}{\gamma}) := \infty$, we get a bijection from $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$. For another matrix N, one can easily verify the well-known facts

$$\phi_M \circ \phi_N = \phi_{MN} \quad \text{and} \quad (\phi_M)^{-1} = \phi_{M^{-1}}.$$

Comparing these identities with Lemma 1.2.9, one could suspect that the objects τ_M and ϕ_M are related in some way. The proof of the next proposition can be found in [Kal2] (Korollar 4.2.10.).

Proposition 1.2.15. *Let T be a linear relation on X, and let $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathbb{C}^{2\times 2}$ be invertible. Then we have*

$$\begin{aligned} \sigma(\tau_M(T)) &= \phi_M(\sigma(T)) \\ \rho(\tau_M(T)) &= \phi_M(\rho(T)) \\ r(\tau_M(T)) &= \phi_M(r(T)) \end{aligned}$$

More relations between τ_M and ϕ_M will follow in Proposition 2.1.4 and Remark 2.1.7.

1.2.1 Linear Relations between Krein Spaces

Let $(\mathcal{K}_1, [.,.]_1)$ and $(\mathcal{K}_2, [.,.]_2)$ be two Krein spaces.

Definition 1.2.16. For a linear relation $T \leq \mathcal{K}_1 \times \mathcal{K}_2$ we denote by

$$T^* := \{(x; y) \in \mathcal{K}_2 \times \mathcal{K}_1 : [x, v]_2 = [y, u]_1 \text{ for all } (u; v) \in T\}$$

the *adjoint relation* of T. Where confusion may arise, we write $T^{[*]}$ or $T^{(*)}$ to indicate with respect to which inner product we take the adjoint.

Remark 1.2.17. If both T and T^* happen to be operators, which exactly is the case for operators T with dense domains, as we will learn in Lemma 1.2.18, we can write $v = Tu$ and $y = T^*x$ in Definition 1.2.16, and get the familiar-looking equation

$$[x, Tu]_2 = [T^*x, u]_1,$$

which holds for all $u \in \operatorname{dom} T$, $x \in \operatorname{dom} T^*$.

Lemma 1.2.18. *Let T be a linear relation between two Krein spaces, $T \leq \mathcal{K}_1 \times \mathcal{K}_2$.*

(i) $\operatorname{mul} T^* = (\operatorname{dom} T)^\perp$, $\ker T^* = (\operatorname{ran} T)^\perp$

(ii) $(T^{-1})^* = (T^*)^{-1}$

Proof.

(i)
$$\begin{aligned} \operatorname{mul} T^* &= \{y \in \mathcal{K}_1 : [0, v]_2 = [y, u]_1 \text{ for all } (u; v) \in T\} = (\operatorname{dom} T)^\perp \\ \ker T^* &= \{x \in \mathcal{K}_2 : [x, v]_2 = [0, u]_1 \text{ for all } (u; v) \in T\} = (\operatorname{ran} T)^\perp \end{aligned}$$

(ii) We have $(x; y) \in (T^{-1})^*$ if and only if $[x, u]_1 = [y, v]_2$ for all $(v; u) \in T^{-1}$, or equivalent $[x, u]_1 = [y, v]_2$ for all $(u; v) \in T$. This is just the definition of $(y; x) \in T^*$, or $(x; y) \in (T^*)^{-1}$.

□

We examine the situation for bounded operators.

Proposition 1.2.19. *Let $(\mathcal{K}, [.,.])$ be a Krein space and let T be a bounded operator $T \in \mathcal{B}(\mathcal{K})$. Let J be a fundamental symmetry, and let $T^{(*)}$ denote the adjoint of T in the Hilbert space $(\mathcal{K}, (.,.)_J)$.*

Then, we have $T^{[*]} = JT^{(*)}J$. In particular $T^{[*]}$ is again a bounded operator on $\mathcal{K}$. Furthermore, we have $\|T^{[*]}\| = \|T\|$, where $\|.\|$ denotes the operator norm with respect to $\|.\|_J$.

For $S, T \in \mathcal{B}(\mathcal{K})$ and $\lambda, \mu \in C$ we have

$$(\lambda S + \mu T)^{[*]} = \bar{\lambda} S^{[*]} + \bar{\mu} T^{[*]}, \quad (T^{[*]})^{[*]} = T, \quad (ST)^{[*]} = T^{[*]} S^{[*]}.$$

Proof. With Lemma 1.1.7, we have

$$[x, Tu] = (Jx, Tu)_J = (T^{(*)}Jx, u)_J = [JT^{(*)}Jx, u],$$

which gives $T^{[*]} \supseteq JT^{(*)}J$. Equality prevail due to $\operatorname{mul} T^{[*]} = (\operatorname{dom} T)^{\perp} = \{0\}$. By Lemma 1.1.7, J is an isometry with respect to $\|.\|_J$. This gives

$$\|T^{[*]}\| = \sup_{x \neq 0} \frac{\|T^{[*]}x\|_J}{\|x\|_J} = \sup_{x \neq 0} \frac{\|JT^{(*)}Jx\|_J}{\|x\|_J} = \sup_{x \neq 0} \frac{\|T^{(*)}Jx\|_J}{\|Jx\|_J} = \|T^{(*)}\|.$$

The proposed calculation rules follow directly from the appropriate ones in the Hilbert space. □

Remark 1.2.20. For bounded linear operators $T : \mathcal{K}_1 \to \mathcal{K}_2$ one can show, analogously to Proposition 1.2.19, $T^{[*]} = J_1 T^{(*)} J_2$. Here, J_1 and J_2 denote the fundamental symmetries of $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively, and $T^{(*)}$ denotes the Hilbert space adjoint of $T : (\mathcal{K}_1, (.,.)_{J_1}) \to (\mathcal{K}_2, (.,.)_{J_2})$.

The next lemma will be needed later.

Lemma 1.2.21. *For a bounded linear operator $R : \mathcal{K}_1 \to \mathcal{K}_2$ between two Krein spaces and a subset $L \subseteq \mathcal{K}_2$, we have*

$$(R^*(L))^{\perp} = R^{-1}\big(L^{\perp}\big).$$

Hereby, the symbol $\perp$ on the left-hand side refers to $\mathcal{K}_1$, and on the right-hand side refers to $\mathcal{K}_2$. With $R^{-1}(L^{\perp})$, we denote the inverse image of $L^{\perp}$.

Proof. Clearly, we can write both sets as

$$\begin{aligned} (R^*(L))^{\perp} &= \{x \in \mathcal{K}_1 : [x, R^*l]_1 = 0, \text{ for all } l \in L\} \\ R^{-1}\big(L^{\perp}\big) &= \{x \in \mathcal{K}_1 : [Rx, l]_2 = 0, \text{ for all } l \in L\}. \end{aligned}$$

Obviously, these sets coincide, due to $[Rx, l]_2 = [x, R^*l]_1$. □

We collect some more properties.

Lemma 1.2.22. *Let T be a linear relation between two Krein spaces, $T \leq \mathcal{K}_1 \times \mathcal{K}_2$.*

(*i*) $(RT)^* \supseteq T^*R^*$ *for all linear relations* $R \leq \mathcal{K}_2 \times \mathcal{K}_3$

(*ii*) $(RT)^* = T^*R^*$ *for all bounded linear operators* $R : \mathcal{K}_2 \to \mathcal{K}_3$

(*iii*) $(T + B)^* = T^* + B^*$ *for all bounded operator* $B \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$

(*iv*) T^* *is closed.*

(*v*) $(T^*)^* = \overline{T}$

Proof.

(*i*) Take an arbitrary pair $(x; z) \in T^*R^*$, i.e. $(x; y) \in R^*$ and $(y; z) \in T^*$ for a certain $y \in \mathcal{K}_2$. We want to show $(x; z) \in (RT)^*$. To this end, we need to verify

$$[x, e]_3 = [z, c] \quad \text{for all pairs } (c; e) \in RT.$$

For a fixed $(c; e) \in RT$ we have $(c; d) \in T$ and $(d; e) \in R$ with a certain $d \in \mathcal{K}_2$. Now, we have

$$[x, e]_3 = [y, d]_2 = [z, c]_1.$$

The first equality sign holds due to $(x; y) \in R^*$ and $(d; e) \in R$, and the second equality sign holds due to $(y, z) \in T^*$ and $(c; d) \in T$.

(*ii*) Fix an arbitrary pair $(x; z) \in (RT)^*$, i.e.

$$[x, c]_3 = [z, a]_1 \quad \text{for all pairs } (a; c) \in RT. \tag{1.5}$$

We have to find an $y \in \mathcal{K}_2$, such that $(x; y) \in R^*$ and $(y; z) \in T^*$, to conclude $(x; z) \in T^*R^*$. Since R^* is an everywhere-defined operator, we have to take $y := R^*x$. In order to show $(y; z) \in T^*$, we need to verify

$$[y, b]_2 = [z, a]_1 \quad \text{for all pairs } (a; b) \in T.$$

Due to $[x, Rb]_3 = [R^*x, b]_2 = [y, b]_2$, this follows from equation (1.5) applied to the pair $(a; Rb) \in RT$.

(*iii*) We have $(x; y) \in (T + B)^*$, per definition, if $[x, v]_2 = [y, u]_1$ for all $(u; v) \in B + T$, or $[x, v + Bu]_2 = [y, u]_1$ for all $(u; v) \in T$. Due to

$$[x, v + Bu]_2 = [y, u]_1 \;\Leftrightarrow\; [x, v]_2 = [y - B^*x, u]_1,$$

this is equivalent to $(x; y - B^*x) \in T^*$, and finally to $(x, y) \in T^* + B^*$.

(*iv*) By definition, we have $(x;y) \in T^*$ if and only if

$$[y,u]_1 - [x,v]_2 = [(x;y),(-v;u)]_{2,1} = 0, \text{ for all } (u;v) \in T,$$

where $[(a;b),(c;d)]_{2,1} := [a,c]_2 + [b,d]_1$ for $a,c \in \mathcal{K}_2$, $b,d \in \mathcal{K}_1$ denotes the sum inner product on the product space $\mathcal{K}_2 \times \mathcal{K}_1$. Since we can write $\{(-v;u) : (u;v) \in T\}$ as the image of T under the linear operator $\tau_M : \mathcal{K}_1 \times \mathcal{K}_2 \to \mathcal{K}_2 \times \mathcal{K}_1, (f;g) \mapsto (-g;f)$ with $M := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we get

$$T^* = (\tau_M(T))^{\perp},$$

where $\perp$ refers to the Krein space $\mathcal{K}_2 \times \mathcal{K}_1$. In particular, T^* is closed, cf. Remark 1.1.9.

(*v*) An easy calculation yields $(\tau_N)^* = \tau_{N^*}$ for any matrix $N = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, where $N^* := \left(\begin{smallmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{smallmatrix}\right)$.
We set again $M := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Elementary matrix calculus gives $\tau_{M^*} = \tau_{M^{-1}}$ and $(\tau_{M^*})^{-1} = \tau_M$. Lemma 1.2.21 applied to $R = \tau_{M^*}$ and $L = T$ yields

$$T^* = (\tau_M(T))^{\perp} = \big((\tau_{M^*})^*(T)\big)^{\perp} = (\tau_{M^*})^{-1}\big(T^{\perp}\big) = \tau_M\big(T^{\perp}\big). \tag{1.6}$$

Since for $M^2 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ we have $\tau_{M^2}(R) = R$ for all linear relations R, we get

$$(T^*)^* = (\tau_M(T^*))^{\perp} = \Big(\tau_M\big(\tau_M(T^{\perp})\big)\Big)^{\perp} = T^{\perp\perp} = \overline{T}.$$

The last identity follows from Lemma 1.1.13.

□

Lemma 1.2.23. *For $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathbb{C}^{2\times 2}$, $\det M \neq 0$ and a linear relation $T \leq \mathcal{K} \times \mathcal{K}$ we have*

$$\tau_M(T)^* = \tau_{\overline{M}}(T^*),$$

with $\overline{M} := \left(\begin{smallmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{smallmatrix}\right)$. In particular, this gives

$$\big((T-\lambda)^{-1}\big)^* = (T^* - \overline{\lambda})^{-1} \quad \textit{for all } \lambda \in \mathbb{C}, \tag{1.7}$$

which yields $\sigma(T^*) = \overline{\sigma(T)}$.

Proof. The elements of $\tau_{\overline{M}}(T^*)$ can be written as $(\bar{\delta}u + \bar{\gamma}v; \bar{\beta}u + \bar{\alpha}v)$ with $(u;v) \in T^*$. The elements of $\tau_M(T)$ can be written as $(\delta x + \gamma y; \beta x + \alpha y)$ with $(x;y) \in T$. The calculation

$$\begin{aligned}[\bar{\delta}u + \bar{\gamma}v\,,\, \beta x + \alpha y] &= \bar{\delta}\bar{\beta}[u,x] + \bar{\gamma}\bar{\beta}[v,x] + \bar{\delta}\bar{\alpha}[u,y] + \bar{\gamma}\bar{\alpha}[v,y] = \\ &= \bar{\delta}\bar{\beta}[u,x] + \bar{\gamma}\bar{\beta}[u,y] + \bar{\delta}\bar{\alpha}[v,x] + \bar{\gamma}\bar{\alpha}[v,y] = [\bar{\beta}u + \bar{\alpha}v\,,\, \delta x + \gamma y],\end{aligned}$$

shows that $\tau_{\overline{M}}(T^*) \subseteq \tau_M(T)^*$, for all matrices M and linear relations T.

In order to show the other inclusion, we use, what we have just proven, with the matrix M^{-1} and the linear relation $\tau_M(T)$,

$$\tau_{\overline{M^{-1}}}(\tau_M(T)^*) \subseteq \tau_{M^{-1}}(\tau_M(T))^* = T^*.$$

Applying $\tau_{\overline{M}}$ on both sides, and using $\overline{M^{-1}} = \overline{M}^{-1}$, reveals $\tau_M(T)^* \subseteq \tau_{\overline{M}}(T^*)$.

Equation (1.7) follows now by taking the matrix $M := \left(\begin{smallmatrix} 0 & 1 \\ 1 & -\lambda \end{smallmatrix}\right)$ for all $\lambda \in \mathbb{C}$, cf. Remark 1.2.11.

Furthermore, we have $\lambda \in \rho(T) \cap \mathbb{C}$ if and only if $(T-\lambda)^{-1} \in \mathcal{B}(\mathcal{K})$. By Proposition 1.2.19, this is equivalent to $\big((T-\lambda)^{-1}\big)^*$ being a bounded operator on $\mathcal{K}$, i.e. $\overline{\lambda} \in \rho(T^*)$. At last, $\infty \in \rho(T)$ means $T \in \mathcal{B}(\mathcal{K})$, which is equivalent to $T^* \in \mathcal{B}(\mathcal{K})$, i.e. $\infty \in \rho(T^*)$. ☐

Remark 1.2.24. We want to remark that equation (1.7) also directly follows from Lemma 1.2.18 and Lemma 1.2.22,

$$\big((T-\lambda)^{-1}\big)^* = ((T-\lambda)^*)^{-1} = (T^* - \overline{\lambda})^{-1}.$$

Definition 1.2.25. We call a linear relation $T \leq \mathcal{K}_1 \times \mathcal{K}_2$

·) *isometric* if $T^{-1} \subseteq T^*$.

·) *unitary* if $T^{-1} = T^*$.

In the case $\mathcal{K}_1 = \mathcal{K}_2$, we call T

·) *symmetric* if $T \subseteq T^*$.

·) *self-adjoint* if $T = T^*$.

By Lemma 1.2.23, we know that the spectrum of self-adjoint linear relations on Krein spaces is symmetric with respect to the real line. In Hilbert spaces, self-adjointness gives us even more.

Lemma 1.2.26. *For a self-adjoint linear relation $T \leq \mathcal{H} \times \mathcal{H}$ on a Hilbert space $\mathcal{H}$, we have $\sigma(T) \subseteq \mathbb{R} \cup \{\infty\}$.*

Sketch of proof. We will not give a detailed proof, since this result is well-known. Still, we want to give a sketch of this proof, because it is interesting to see why the proof fails for Krein spaces.

The Cayley-transform is nothing but τ_M for the matrix $M := \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right)$. More about the notion of the Cayley-transform and a detailed version of this proof can be found in [Kal2] (Korollar 4.3.16.).

The main idea is to look at the Cayley-transform of T, denoted by U. By the general theory, the Cayley-transform maps self-adjoint linear relations to unitary linear relations, i.e. U is unitary. One can show that all unitary linear relations on a Hilbert space are in fact everywhere defined unitary linear operators. Being on a Hilbert space, the spectrum of U is contained in the unit circle. One concludes the proof by calculating the spectrum of the inverse-Cayley-transform of U, which is T, with the help of Proposition 1.2.15. ☐

Remark 1.2.27. The proof of Lemma 1.2.26 fails for Krein space, because there actually are unitary linear relations U on Krein spaces whose spectrum is not contained in the unit circle.

Example 1.2.28. Let B be a bounded operator on a Hilbert space $\mathcal{H}$. We are going to construct a Krein space $\mathcal{K}$ and a bounded self-adjoint operator A on $\mathcal{K}$, such that $\sigma(A) = \sigma(B) \cup \overline{\sigma(B)}$.

This shows that the situation in Krein spaces really is more complex. Probably, one can understand now better, why we need further assumptions about the linear relation on a Krein space, besides being self-adjoint, in order to establish a spectral theorem.

As in Example 1.1.14, we start with a Hilbert space, namely the product space $\mathcal{H} \times \mathcal{H}$ endowed with the sum scalar product $(.,.)$, and define a self-adjoint Gram-operator G with $0 \in \rho(G)$ via $G((x;y)) := (y;x)$. The operator G can be written in block-operator form as

$$G = \begin{pmatrix} 0 & \mathrm{I}_{\mathcal{H}} \\ \mathrm{I}_{\mathcal{H}} & 0 \end{pmatrix}.$$

As in Example 1.1.14, we get a Krein space $(\mathcal{H} \times \mathcal{H}, [.,.])$ by setting $[.,.] := (G.,.)$. Consider the operator A defined by $A(x;y) := (Bx; B^*y)$, i.e.

$$A = \begin{pmatrix} B & 0 \\ 0 & B^* \end{pmatrix}.$$

Easy manipulations on $(A(x;y),(f;g))$ show that the $\mathcal{H} \times \mathcal{H}$-Hilbert space adjoint of A is given by

$$A^{(*)} = \begin{pmatrix} B^* & 0 \\ 0 & B \end{pmatrix}.$$

But with the Krein space inner product, one can verify $[A(x;y),(f;g)] = [(x;y), A(f;g)]$, i.e. A is self-adjoint in the Krein space $\mathcal{K}$.

Due to the block-diagonal-form we have $\sigma(A) = \sigma(B) \cup \sigma(B^*) = \sigma(B) \cup \overline{\sigma(B)}$.

2 The Functional Calculus for Rational Functions

Consider a linear relation T on a Banach space X with non-empty resolvent set, and let r be a rational function whose poles are in $\rho(T)$. The goal of this chapter is to define the expression $r(A) \in \mathcal{B}(X)$, and show some properties, cf. Corollary 2.1.11.

Definition 2.0.1. Let r be a rational function, and let $p, q \in \mathbb{C}[z]$ be two relatively prime polynomials such that $r = \frac{p}{q}$. We call $\lambda \in \mathbb{C}$ a *pole* of r, if $q(\lambda) = 0$. We define ∞ to be a pole of r if the condition $\deg p > \deg q$ holds true.

Remark 2.0.2. Let r be a rational function, and denote by $P \subseteq \mathbb{C} \cup \{\infty\}$ the set of all poles of r. Clearly, r is continuous on $\mathbb{C} \setminus P$, in fact even holomorphic.

We remark that r can be continuously extended on $(\mathbb{C} \cup \{\infty\}) \setminus P$, by setting $r(\infty) := \lim_{|z| \to \infty} r(z)$ if $\infty \notin P$. Technically, $\mathbb{C} \cup \{\infty\}$ is the one-point, or Alexandroff compactification of $\mathbb{C}$. The limit exists due to $\deg p \leq \deg q$, and the continuity of r at ∞ is clear by definition of $r(\infty)$. In the sequel, we will interpret r as a continuous function on $(\mathbb{C} \cup \{\infty\}) \setminus P$.

We present two approaches.

First, one can use an extension of the classical Riesz-Dunford functional calculus to linear relations with non-empty resolvent sets. We will give a brief introduction to the classical case and show how it can be extended to linear relations. This might feel a bit unsatisfying, since we are going to use this powerful tool with rational functions only. But if one is already familiar with this subject, it is a fast way to obtain the desired functional calculus with all needed properties.

Secondly, we give an elementary approach using the partial fraction decomposition.

2.1 The Riesz-Dunford Functional Calculus

2.1.1 Classical Version

Definition 2.1.1. Let A be a unital Banach algebra, and fix $a \in A$. If $f : G \to \mathbb{C}$ is holomorphic on an open set $G \supseteq \sigma(a)$, and Γ is a finite set of paths, such that

$$n(\Gamma, z) = \begin{cases} 0, & z \in \mathbb{C} \backslash G \\ 1, & z \in \sigma(a) \end{cases},$$

we define

$$f(a) := \frac{1}{2\pi i} \int_\Gamma f(\zeta)(\zeta e - a)^{-1} d\zeta \ \in A.$$

The mapping $f \mapsto f(a)$ is called Riesz-Dunford functional calculus. Formally, the domain of this calculus is defined as

$$\left(\dot{\bigcup_{\substack{\sigma(a)\subseteq G \\ G \text{ open}}}} \mathbb{H}(G) \right) \Big/ \sim \; =: \mathbb{H}(\sigma(a)),$$

where $\mathbb{H}(G)$, for an open set $G \subseteq \mathbb{C}$, denotes the set of all holomorphic functions $f : G \to \mathbb{C}$, and where $\sim$ is the equivalence relation

$$f \sim g \;:\Longleftrightarrow\; \exists O \subseteq \mathbb{C} \text{ open} : \sigma(a) \subseteq O \subseteq G_f \cap G_g \text{ and } f|_O = g|_O,$$

whereby G_f and G_g denote the domains of f and g respectively.

Lemma 2.1.2. *The Riesz-Dunford calculus is well-defined and has the following properties.*

(i) $(\lambda f + \mu g)(a) = \lambda f(a) + \mu g(a)$ *for* $f, g \in \mathbb{H}(\sigma(a))$ *and* $\lambda, \mu \in \mathbb{C}$

(ii) $(f \cdot g)(a) = f(a)g(a)$ *for* $f, g \in \mathbb{H}(\sigma(a))$

(iii) $f(a)^{-1} = f^{-1}(a)$ *for* $f \in \mathbb{H}(\sigma(a))$ *such that* $f(z) \neq 0$ *for all* $z \in \sigma(a)$

(iv) $\sigma(f(a)) = f(\sigma(a))$ *for* $f \in \mathbb{H}(\sigma(a))$

(v) $(f \circ g)(a) = f(g(a))$ *for* $g \in \mathbb{H}(\sigma(a)), f \in \mathbb{H}(\sigma(g(a)))$

(vi) $f(a) = a^k$ *if* $f(z) = z^k, k \in \mathbb{N}$

(vii) *If A is a Banach-*-algebra, then*

$$f(a)^* = f^{\#}(a^*),$$

with $f^{\#}(z) := \overline{f(\overline{z})}$.

Proof. These are classical results. We refer to [Rud, 240 ff.].

Properties (i) and (ii) are stated in [Rud, Theorem 10.27], for (iii) and (iv) see [Rud, Theorem 10.28], for (v) see [Rud, Theorem 10.29] and (vi) is located in [Rud, Theorem 10.25]. We are going to give an explicit proof for the last statement (vii).

Recall that the line integral is defined as the limit of Riemann sums with respect to tagged partitions such that the mesh converges to zero. Let $a = \xi_0 < \ldots < \xi_{n(\mathcal{R})} = b$ be a partition of the interval with tags $\alpha_i \in [\xi_i, \xi_{i+1}]$. We have

$$f(a)^* = \left(\frac{1}{2\pi i} \sum_{\gamma\in\Gamma} \int_\gamma f(\zeta)(\zeta e - a)^{-1} d\zeta \right)^* =$$

$$= -\frac{1}{2\pi i} \left(\sum_{\gamma\in\Gamma} \lim_{|\mathcal{R}|\to 0} \sum_{j=1}^{n(\mathcal{R})} f(\gamma(\alpha_j))(\gamma(\alpha_j)e - a)^{-1}(\gamma(\xi_j) - \gamma(\xi_{j-1})) \right)^* =$$

$$= -\frac{1}{2\pi i}\sum_{\gamma\in\Gamma}\lim_{|\mathcal{R}|\to 0}\sum_{j=1}^{n(\mathcal{R})}\overline{f(\gamma(\alpha_j))(\overline{\gamma(\alpha_j)}e-a^*)^{-1}(\overline{\gamma(\xi_j)}-\overline{\gamma(\xi_{j-1})})} =$$

$$= -\frac{1}{2\pi i}\sum_{\gamma\in\Gamma}\lim_{|\mathcal{R}|\to 0}\sum_{j=1}^{n(\mathcal{R})} f^{\#}(\overline{\gamma}(\alpha_j))(\overline{\gamma}(\alpha_j)e-a^*)^{-1}(\overline{\gamma}(\xi_j)-\overline{\gamma}(\xi_{j-1})) =$$

$$= -\frac{1}{2\pi i}\sum_{\gamma\in\Gamma}\int_{\overline{\gamma}} f^{\#}(\zeta)\,(\zeta e-a^*)^{-1}\,d\zeta =$$

$$= \frac{1}{2\pi i}\int_{-\overline{\Gamma}} f^{\#}(\zeta)\,(\zeta e-a^*)^{-1}\,d\zeta = f^{\#}(a^*),$$

with $-\overline{\Gamma} := \{-\overline{\gamma} \mid \gamma\in\Gamma\}$. In order to justify the last equality, we have to show that $-\overline{\Gamma}$ is a suitable set of paths for $a^*\in A$. This follows from $\sigma(a^*) = \overline{\sigma(a)} \subseteq \overline{G}$ together with $n(-\overline{\gamma},\overline{z}) = n(\gamma,z)$ for all $z\in\mathbb{C}\backslash\gamma([a,b])$, since we have

$$n(-\overline{\gamma},\overline{z}) = -\frac{1}{2\pi i}\int_a^b \frac{\overline{\gamma}'(t)}{\overline{\gamma}(t)-\overline{z}}\,dt = \overline{\frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t)-z}\,dt} = \overline{n(\gamma,z)} = n(\gamma,z).$$

□

Remark 2.1.3. Most commonly, Definition 2.1.1 is applied to the Banach algebra $\mathcal{B}(X)$ of bounded operators on a Banach space X.

In order to get a Banach-*-algebra and enrich the theory, cf. Lemma 2.1.2 (*vii*), consider $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$, which constitutes in fact a C^*-algebra.

Finally, one can look at the set $\mathcal{B}(\mathcal{K})$ for a Krein space $\mathcal{K}$. One possibility is to think of $\mathcal{K}$ as a Hilbert space $(\mathcal{K},(.,.)_J)$ and take the Banach-*-algebra $\mathcal{B}(\mathcal{K})$ as stated above. Alternatively, by replacing the Hilbert-space adjoint $T^{(*)}$ with the Krein-space adjoint $T^{[*]}$, one gets a different involution $T\mapsto T^{[*]}$, which also turns $\mathcal{B}(\mathcal{K})$ into a Banach-*-algebra, since we have $\|T^{[*]}\| = \|T\|$, cf. Proposition 1.2.19. Due to Lemma 2.1.2 (*vii*), we have $f(A)^{(*)} = f^{\#}(A^{(*)})$ and $f(A)^{[*]} = f^{\#}(A^{[*]})$.

For a bounded operator $B\in\mathcal{B}(X)$ and a regular matrix $M\in\mathbb{C}^{2\times 2}$, we have on the one hand the operator $\tau_M(B)$, cf. Definition 1.2.8, and on the other hand the bounded operator $\phi_M(B)$, using the Riesz-Dunford calculus with the Möbius transformation ϕ_M which fulfills $\phi_M\in\mathbb{H}(\sigma(B))$ if $\phi_{M^{-1}}(\infty)\in\rho(B)$. The same assumption gives $\infty\in\phi_M(\rho(B)) = \rho(\tau_M(B))$, cf. Lemma 1.2.15, which means that also $\tau_M(B)$ is a bounded operator. In fact, these operators coincide:

Proposition 2.1.4. *For a bounded operator $B\in\mathcal{B}(X)$ and a regular matrix $M\in\mathbb{C}^{2\times 2}$, such that $\phi_{M^{-1}}(\infty)\in\rho(B)$, holds*

$$\phi_M(B) = \tau_M(B).$$

Proof. It is a classical result, see e.g. [Con, III Proposition 3.6], that every Möbius transformation can be written as a composition of translations $z\mapsto z+\mu$, dilations $z\mapsto\lambda z$, and the inversion $z\mapsto\frac{1}{z}$. These mappings correlate to the matrices $\left(\begin{smallmatrix}1&\mu\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}\lambda&0\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$.

Therefore, we can write the matrix M as a product $M = LIR$, where $L, R \in \mathbb{C}^{2\times 2}$ are decompositions of dilations and translations and where I denotes the inversion, $I := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

At first, we point out that for $N := \left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right)$ we have $\phi_N(T) = \tau_N(T)$ for all bounded operators T and all $\mu \in \mathbb{C}$. Due to Remark 1.2.11, we have $\tau_M(T) = T + \mu$, and Lemma 2.1.2 applied to $\phi_M(z) = z + \mu$ gives exactly the same. For matrices N corresponding to a dilation, we can show $\phi_N(T) = \tau_N(T)$ analogously.

Recall that we have both $\tau_A \circ \tau_B = \tau_{AB}$ and $\phi_A \circ \phi_B = \phi_{AB}$ for all $A, B \in \mathbb{C}^{2\times 2}$. We start by deducing $\phi_R(B) = \tau_R(B)$.

For the inversion I, one has to be a bit more carefully. Since $z \mapsto \frac{1}{z}$ is not holomorphic at $z = 0$, we have to make sure that $0 \notin \sigma(\phi_R(B)) = \phi_R(\sigma(B))$. Assume $0 \in \phi_R(\sigma(B))$, i.e. $\phi_{R^{-1}}(0) \in \sigma(B)$. Due to $\phi_{I^{-1}}(\infty) = 0$, this would give $\phi_{R^{-1}}(\phi_{I^{-1}}(\infty)) \in \sigma(B)$. Since ∞ is a fixpoint of every dilation and translation, we have $\infty = \phi_{L^{-1}}(\infty)$ and arrive at the contradiction $\phi_{R^{-1}}(\phi_{I^{-1}}(\phi_{L^{-1}}(\infty))) = \phi_{M^{-1}}(\infty) \in \sigma(B)$.

Therefore, the expression $\phi_I(\phi_R(B))$ is well-defined and coincides with $(\phi_R(B))^{-1}$ due to Lemma 2.1.2 (*iii*). This is further equal to $\tau_I(\phi_R(B)) = \tau_I(\tau_R(B))$ due to Remark 1.2.11.

Since L is only a composition of dilations and translations, we conclude

$$\phi_M(B) = \phi_L(\phi_I(\phi_R(B))) = \phi_L(\tau_I(\tau_R(B))) = \tau_L(\tau_I(\tau_R(B))) = \tau_M(B).$$

□

2.1.2 An Extension to Linear Relations

There is a way to define the Riesz-Dunford calculus for linear relations $T \leq X \times X$ with non-empty resolvent set, although they do not form a Banach algebra anymore.

Definition 2.1.5. Let $T \leq X \times X$ be a linear relation with non-empty resolvent set, and let $f : G \to \mathbb{C}$ be a holomorphic function with $\sigma(T) \subseteq G \subseteq \mathbb{C} \cup \{\infty\}$, $f \in \mathbb{H}(G)$. For an invertible $M \in \mathbb{C}^{2\times 2}$, such that $\phi_{M^{-1}}(\infty) \in \rho(T)$, we define

$$f(T) := (f \circ \phi_{M^{-1}})\,(\tau_M(T)) \in \mathcal{B}(X).$$

Remark 2.1.6. If G contains the point ∞, which is part of the assumption in the above definition if T is not a bounded operator on X, we mean by "*f is holomorphic on G, $f \in \mathbb{H}(G)$*" that f is holomorphic on the open set $G \cap \mathbb{C}$ and that $z \mapsto f(\frac{1}{z})$ is holomorphic at $z = 0$.

We have to verify that $f(T)$ in Definition 2.1.5 is *well-defined*:

Since $\phi_{M^{-1}}(\infty) \in \rho(T)$, we have $\infty \in \phi_M(\rho(T)) = \rho(\tau_M(T))$, cf. Lemma 1.2.15. This means that $\tau_M(T)$ is a bounded operator on X. Now, we can apply the functional calculus from above since

$$f \circ \phi_{M^{-1}} : \phi_M(G)\backslash\{\infty\} \to \mathbb{C}$$

is holomorphic on the open set $\phi_M(G)\backslash\{\infty\}$, which contains $\sigma(\tau_M(T))$:

$$G \supseteq \sigma(T) \;\Rightarrow\; \phi_M(G) \supseteq \phi_M(\sigma(T)) = \sigma(\tau_M(T)).$$

Note that $\infty \notin \sigma(\tau_M(T))$. Thus, $f(T)$ is a well-defined element of $\mathcal{B}(X)$, for fixed M.

We have to show that the definition is independent of M. Take $N, M \in \mathbb{C}^{2\times 2}$ with $\phi_{N^{-1}}(\infty), \phi_{M^{-1}}(\infty) \in \rho(T)$. Due to Proposition 2.1.4, we have

$$\tau_N(T) = \tau_{NM^{-1}}(\tau_M(T)) = \phi_{NM^{-1}}(\tau_M(T)).$$

This gives

$$\begin{aligned}(f\circ\phi_{N^{-1}})(\tau_N(T)) = (f\circ\phi_{N^{-1}})(\phi_{NM^{-1}}(\tau_M(T))) &= \\ = (f\circ\phi_{N^{-1}}\circ\phi_{NM^{-1}})(\tau_M(T)) &= (f\circ\phi_{M^{-1}})(\tau_M(T)).\end{aligned}$$

Hence, the definition of $f(T)$ is independent of M.

Remark 2.1.7. The equality from Proposition 2.1.4 also holds true in this generalized setting. In fact, for $\phi_{M^{-1}}(\infty) \in \rho(T)$ we have

$$\phi_M(T) = (\phi_M \circ \phi_{M^{-1}})(\tau_M(T)) = (\mathrm{id}_{\mathbb{C}})(\tau_M(T)) = \tau_M(T).$$

Most properties, stated in Lemma 2.1.2, also hold true for linear relations.

Lemma 2.1.8. *Let $T \leq X \times X$ be a linear relation on X with non-empty resolvent set. Then, the following properties hold true.*

(i) $(\lambda f + \mu g)(T) = \lambda f(T) + \mu g(T)$ *for* $\lambda, \mu \in \mathbb{C}$ *and* $f, g \in \mathbb{H}(\sigma(T))$

(ii) $(f\cdot g)(T) = f(T)g(T)$ *for* $f, g \in \mathbb{H}(\sigma(T))$

(iii) $\sigma(f(T)) = f(\sigma(T))$ *for* $f \in \mathbb{H}(\sigma(T))$

(iv) $(f\circ g)(T) = f(g(T))$ *for* $g \in \mathbb{H}(\sigma(T)), f \in \mathbb{H}(\sigma(g(T)))$

(v) *If X is a Krein space and $f \in \mathbb{H}(\sigma(T))$, we have* $f(T)^* = f^{\#}(T^*)$

Proof. Most properties follow more or less directly using the corresponding property from Lemma 2.1.2.

(i)

$$\begin{aligned}(\lambda f + \mu g)(T) &= ((\lambda f + \mu g)\circ\phi_{M^{-1}})(\tau_M(T)) = \\ &= (\lambda(f\circ\phi_{M^{-1}}) + \mu(g\circ\phi_{M^{-1}}))(\tau_M(T)) = \\ &= \lambda(f\circ\phi_{M^{-1}})(\tau_M(T)) + \mu(g\circ\phi_{M^{-1}})(\tau_M(T)) = \\ &= \lambda f(T) + \mu g(T)\end{aligned}$$

(ii) Analogous to *(i)*.

(iii)

$$\begin{aligned}\sigma(f(T)) = \sigma\big((f\circ\phi_{M^{-1}})(\tau_M(T))\big) = (f\circ\phi_{M^{-1}})(\sigma(\tau_M(T))) &= \\ = (f\circ\phi_{M^{-1}})(\phi_M(\sigma(T))) &= f(\sigma(T))\end{aligned}$$

(iv)

$$(f\circ g)(T) = (f\circ g\circ\phi_{M^{-1}})(\tau_M(T)) = f\big((g\circ\phi_{M^{-1}})(\tau_M(T))\big) = f(g(T))$$

(v) $$f(T)^* = \big((f \circ \phi_{M^{-1}})(\tau_M(T))\big)^* = (f \circ \phi_{M^{-1}})^{\#}(\tau_M(T)^*)$$

Lemma 1.2.23 states $\tau_M(T)^* = \tau_{\overline{M}}(T^*)$. Furthermore, the following easy property

$$(g \circ h)^{\#}(z) = \overline{g(h(\overline{z}))} = g^{\#}(\overline{h(\overline{z})}) = g^{\#} \circ h^{\#}(z),$$

together with $\phi_N^{\#} = \phi_{\overline{N}}$, and $\overline{M^{-1}} = \overline{M}^{-1}$ gives

$$f(T)^* = (f^{\#} \circ \phi_{\overline{M}^{-1}})(\tau_{\overline{M}}(T^*)) = f^{\#}(T^*).$$

The last equality uses the fact, that $f^{\#}$ is holomorphic on $\sigma(T^*) = \overline{\sigma(T)}$ and that the matrix $\overline{M}$ is suitable for the operator T^*, in the sense that $\phi_{\overline{M}^{-1}}(\infty) \in \rho(T^*)$, which follows directly from $\phi_{M^{-1}}(\infty) \in \rho(T)$.

□

Remark 2.1.9. The property from Lemma 2.1.2, concerning the function $f(z) = z^k$ with $k \in \mathbb{N}$, can obviously not be transformed, since $f(z)$ is not holomorphic at ∞, and therefore $f(T)$ cannot be defined if $\infty \in \sigma(T)$.

Before we formulate the desired functional calculus, we give an almost trivial remark.

Remark 2.1.10. Let r be a rational function and let $K \subseteq \mathbb{C} \cup \{\infty\}$ be a closed and, therefore, compact subset. We claim that r is bounded on K, denoted by $r \in B(K)$, if and only if all poles of r are contained in $(\mathbb{C} \cup \{\infty\}) \setminus K$.

Assume $r \in B(K)$. Obviously, K does not contain any poles of r, since r is not even defined at poles.

On other hand, assume $P \subseteq (\mathbb{C} \cup \{\infty\}) \setminus K$, or equivalent $K \subseteq (\mathbb{C} \cup \{\infty\}) \setminus P$, where P denotes the set of all poles of r. Recall that $r : (\mathbb{C} \cup \{\infty\}) \setminus P \to \mathbb{C}$ is continuous, cf. Remark 2.0.2. Clearly, also the restriction $r : K \to \mathbb{C}$ is continuous. As an image of a compact set under a continuous map, $r(K)$ is a compact subset of $\mathbb{C}$, i.e. r is bounded on K.

Corollary 2.1.11. *Let $T \leq X \times X$ be a linear relation on a Banach space X, with nonempty resolvent set. Then, we have a functional calculus for rational functions,*

$$\Phi_{\mathrm{rat}} : \begin{cases} \mathbb{C}(z) \cap B(\sigma(T)) & \to \quad \mathcal{B}(X) \\ \qquad\qquad r & \mapsto \quad r(T) \end{cases}.$$

For all $r_1, r_2 \in \mathbb{C}(z) \cap B(\sigma(T))$ and $\lambda, \mu \in \mathbb{C}$, we have

(i) $(\lambda r_1 + \mu r_2)(T) = \lambda r_1(T) + \mu r_2(T)$.

(ii) $(r_1 r_2)(T) = r_1(T) r_2(T)$.

(iii) $\sigma(r(T)) = r(\sigma(T))$.

(iv) $r(T)^* = r^{\#}(T^*)$ *if X is a Krein space.*

In fact, (i) and (ii) state that Φ_{rat} constitutes an algebra homomorphism.

Proof. Take $r \in \mathbb{C}(z) \cap B(\sigma(T))$. In order to apply the extended version of the Riesz-Dunford calculus, we have to show that there exists a open subset G of $\sigma(T)$ such that r is holomorphic on G.

For G we naturally take $(\mathbb{C} \cup \{\infty\}) \setminus P$, where P denotes the set of all poles of r. Clearly, p is holomorphic on $\mathbb{C} \cap G$. In the case $\infty \in G$, i.e. ∞ is no pole of r, it is left to show that $z \mapsto r(\frac{1}{z})$ is holomorphic at $z = 0$. Setting $m := \deg q \geq \deg p$, we have

$$z \mapsto r\left(\tfrac{1}{z}\right) = \frac{p\left(\frac{1}{z}\right)}{q\left(\frac{1}{z}\right)} = \frac{p\left(\frac{1}{z}\right) z^m}{q\left(\frac{1}{z}\right) z^m}.$$

Note that $\tilde{p}(z) := p\left(\frac{1}{z}\right) z^m$ and $\tilde{q}(z) := q\left(\frac{1}{z}\right) z^m$ are again polynomials. Furthermore, the leading coefficient of q becomes the constant term of $\tilde{q}$, i.e. $\tilde{q}(0) \neq 0$. Due to the continuity of $\tilde{q}$ there is actually an open neighborhood of zero, such that $\tilde{q}$ is not zero on that neighborhood. Since $\tilde{p}$ is just a polynomial, this gives that r is holomorphic at $z = 0$.

This proves $\mathbb{C}(z) \cap B(\sigma(T)) \subseteq \mathbb{H}(\sigma(T))$. Also, $\mathbb{C}(z) \cap B(\sigma(T))$ is a linear subspace of the Banach-algebra $B(\sigma(T))$, endowed with the supremum norm. But note that this is not a closed subset. Also, $B(\sigma(T))$ is in general not a Banach-*-algebra with $r \mapsto r^{\#}$, since $r^{\#} \in B(\overline{\sigma(T)})$.

Still, $\Phi_{\text{rat}}(r) = r(T) \in \mathcal{B}(\mathcal{K})$ is well-defined, and all stated properties follow directly form Lemma 2.1.8. $\square$

Remark 2.1.12. For linear relations with empty resolvent set, we have that $\mathbb{C}(z) \cap B(\mathbb{C} \cup \{\infty\})$ consists only of constant functions. In this trivial case, Corollary 2.1.11 is still true if we manually set $(z \mapsto \lambda)(T) := \lambda \, \mathrm{I}$.

2.2 An Elementary Approach

Let $T \leq \mathcal{K} \times \mathcal{K}$ be a linear relation on a Krein space $\mathcal{K}$, and let $r \in \mathbb{C}(z) \cap B(\sigma(T))$ be a rational function. We assume $\infty \in \sigma(T)$. Denote by α_i for $i = 1, \ldots, n$ the poles of r with multiplicity γ_i. A partial fraction decomposition of r gives

$$r(z) = c + \sum_{i=1}^{n} \sum_{j=1}^{\gamma_i} \frac{c_{ij}}{(z - \alpha_i)^j} \,, \tag{2.1}$$

for some constants $c, c_{ij} \in \mathbb{C}$. Since we have $\alpha_i \in \rho(T)$ for all $i = 1, \ldots, n$, we can define $r(T)$ via

$$r(T) := c\,\mathrm{I} + \sum_{i=1}^{n} \sum_{j=1}^{\gamma_i} c_{ij}(T - \alpha_i)^{-j}.$$

Clearly, $r(T)$ is a bounded operator, and this definition coincides with the one from above. We have to check all properties, stated in Corollary 2.1.11, by hand.

Obviously, we have $\lambda r(T) = (\lambda r)(T)$ for all $\lambda \in \mathbb{C}$. To show the linearity, take $r_1, r_2 \in \mathbb{C}(z) \cap B(\sigma(T)$. Consider the partial fraction decomposition of both rational functions,

$$r_1(z) = c + \sum_{i=1}^{n} \sum_{j=1}^{\gamma_i} \frac{c_{ij}}{(z - \alpha_i)^j},$$
$$r_2(z) = d + \sum_{i=1}^{m} \sum_{j=1}^{\eta_i} \frac{d_{ij}}{(z - \beta_i)^j}.$$

Of course, we could have $\alpha_i = \beta_j$ for some i and j. If we add up these two equation, and sum up those fractions which have the same denominator, we get just the partial fraction decomposition of the sum $r_1 + r_2$. Trivially, rearrangement of that form can be done without difficulty, since we have for $\alpha_i = \beta_j =: \mu$

$$c_{il}(T - \mu)^{-l} + d_{jl}(T - \mu)^{-l} = (c_{il} + d_{jl})(T - \mu)^{-l}.$$

Therefore, we have $r_1(T) + r_2(T) = (r_1 + r_2)(T)$. Also, this calculus is compatible with taking adjoints, since we have

$$r(T)^* = \left(c\,\mathrm{I} + \sum_{i=1}^{n} \sum_{j=1}^{\gamma_i} c_{ij}(T - \alpha_i)^{-j} \right)^* =$$
$$= \bar{c}\,\mathrm{I} + \sum_{i=1}^{n} \sum_{j=1}^{\gamma_i} \bar{c_{ij}}(T^* - \bar{\alpha_i})^{-j} = r^{\#}(T^*).$$

The last equality sign follows directly from line (2.1) by complex conjugation. Finally, we show the multiplicity of the calculus. Again, let r_1 be

$$r_1(z) = c + \sum_{i=1}^{n} \sum_{j=1}^{\gamma_i} \frac{c_{ij}}{(z - \alpha_i)^j},$$

As a first step, we show

$$\left(r_1 \cdot \frac{1}{z-\alpha}\right)(T) = r_1(T)(T-\alpha)^{-1}, \tag{2.2}$$

for $\alpha \in \rho(T)$. By linearity, we have

$$\begin{aligned}
\left(r_1(z) \cdot \frac{1}{z-\alpha}\right)(T) &= \left(\left(c + \sum_{i=1}^{n}\sum_{j=1}^{\gamma_i} \frac{c_{ij}}{(z-\alpha_i)^j}\right) \cdot \frac{1}{z-\alpha}\right)(T) = \\
&= \left(c\,\frac{1}{z-\alpha} + \sum_{i=1}^{n}\sum_{j=1}^{\gamma_i} \frac{c_{ij}}{(z-\alpha_i)^j(z-\alpha)}\right)(T) = \\
&= c(T-\alpha)^{-1} + \sum_{i=1}^{n}\sum_{j=1}^{\gamma_i} \left(\frac{c_{ij}}{(z-\alpha_i)^j(z-\alpha)}\right)(T).
\end{aligned}$$

Therefore, in order to prove (2.2), it is sufficient to show

$$\left((z-\alpha_i)^{-j}(z-\alpha)^{-1}\right)(T) = (T-\alpha_i)^{-j}(T-\alpha)^{-1}. \tag{2.3}$$

Though this seems to be trivial, it requires a proof. Recall that the left-hand side is defined via the partial fraction decomposition of $(z-\alpha_i)^{-j}(z-\alpha)^{-1}$. We are not going to give a detailed proof of (2.3). One can compute the partial fraction decomposition of the left-hand side of (2.3). If you use j-times the resolvent identity on the right-hand side of (2.3), you end up with exactly the same expression.

By induction, we also have

$$\left(r_1(z) \cdot \frac{1}{(z-\alpha)^j}\right)(T) = r_1(T)(T-\alpha)^{-j}. \tag{2.4}$$

Now consider

$$r_2(z) = d + \sum_{i=1}^{m}\sum_{j=1}^{\eta_i} \frac{d_{ij}}{(z-\beta_i)^j}.$$

Since we already know that the calculus is linear, we get, by equation (2.4),

$$\begin{aligned}
(r_1 \cdot r_2)(T) &= \left(r_1(z) \cdot \left(d + \sum_{i=1}^{m}\sum_{j=1}^{\eta_i} \frac{d_{ij}}{(z-\beta_i)^j}\right)\right)(T) = \\
&= \left(d \cdot r_1(z) + \sum_{i=1}^{m}\sum_{j=1}^{\eta_i} r_1(z)\frac{d_{ij}}{(z-\beta_i)^j}\right)(T) = \\
&= dr_1(T) + \sum_{i=1}^{m}\sum_{j=1}^{\eta_i} \left(r_1(z)\frac{d_{ij}}{(z-\beta_i)^j}\right)(T) = \\
&= dr_1(T) + \sum_{i=1}^{m}\sum_{j=1}^{\eta_i} d_{ij} r_1(T)(T-\beta_i)^{-j} = \\
&= r_1(T)\left(d\,\mathrm{I} + \sum_{i=1}^{m}\sum_{j=1}^{\eta_i} d_{ij}(T-\beta_i)^{-j}\right) = r_1(T)r_2(T).
\end{aligned}$$

3 Spectral Theorem for Definitizable Linear Relations on Krein Spaces

We are going to make some final preparations and then approach the construction of the spectral theorem in Krein spaces.

3.1 Spectral Theorem for Self-Adjoint Linear Relations on Hilbert Spaces

We want to recall the well-known spectral theorem for unbounded self-adjoint operators $A : \operatorname{dom} A \subseteq \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$.

A proof can be found, for example, in [Rud, Theorem 13.30 on page 348], in [Wer, Theorem VII.3.2 on page 357], or in [Kal2, Satz 4.6.1 on page 96].

Theorem 3.1.1. *Let $\mathcal{H}$ be a Hilbert space, and let $A : \operatorname{dom} A \subseteq \mathcal{H} \to \mathcal{H}$ be a self-adjoint linear operator. Then, there exists a spectral measure E for $\langle \sigma(A) \cap \mathbb{R}, \mathfrak{B}(\sigma(A) \cap \mathbb{R}), \mathcal{H} \rangle$, such that*

$$A = \int_{\mathbb{R}} t \, dE(t). \tag{3.1}$$

Especially, for $\mu \in \rho(A) \backslash \{\infty\}$, we have

$$(A - \mu)^{-1} = \int_{\sigma(A) \cap \mathbb{R}} \frac{1}{t - \mu} \, dE(t). \tag{3.2}$$

Remark 3.1.2. Notice that the integrand on the right-hand side of (3.1) is not bounded. In the classical case for bounded operators on a Hilbert space, we did only declare how to integrate *bounded* and measurable functions with respect to a spectral measure. We want to point out that it is not trivial to expand this functional calculus to unbounded measurable functions.

Since we will not integrate unbounded measurable functions in the sequel, we are not going to introduce this notion here. Again, we want to refer to [Rud], [Wer] or [Kal2]. In fact, we are only going to use (3.2).

Theorem 3.1.3. *Let $\mathcal{H}$ be a Hilbert space, and let B be a self-adjoint linear relation on $\mathcal{H}$, with $\infty \in \sigma(B)$. Then, there exists a spectral measure E for $\langle \sigma(B), \mathfrak{B}(\sigma(B)), \mathcal{H} \rangle$ such that*

$$(B - \mu)^{-1} = \int_{\sigma(B)} \frac{1}{t - \mu} \, dE(t)$$

holds for all $\mu \in \rho(B)$. Moreover, we have

$$r(B) = \int_{\sigma(B)} r(t) \, dE(t),$$

for all rational functions r which are bounded on $\sigma(B)$, $r \in \mathbb{C}(z) \cap B(\sigma(B))$.

Proof. The idea of this proof, in a nutshell, is to look at the part of B which is actually a self-adjoint linear operator (later called C) by dropping the multi-valued part. The spectral measure for C, which exists by Theorem 3.1.1, can then be extended by taking the multi-valued part into account, in order to obtain a spectral measure for B.

By Lemma 1.2.18, we have $\operatorname{mul} B = \operatorname{mul} B^* = (\operatorname{dom} B)^\perp$. We can write the Hilbert space $\mathcal{H}$ as the direct and orthogonal sum $\mathcal{H} = \overline{\operatorname{dom} B}(\dot{+}) \operatorname{mul} B$. For $(x; y) \in B$, Lemma 1.2.5 gives

$$\{z : (x; z) \in B\} = y + \operatorname{mul} B = y_d + \operatorname{mul} B,$$

if one decomposes y in $y = y_d + y_m$ with $y_d \in \overline{\operatorname{dom} B}$ and $y_m \in \operatorname{mul} B$.

This motivates to look at the linear relation $C := B \cap \left(\overline{\operatorname{dom} B} \times \overline{\operatorname{dom} B}\right)$. In fact, C is a linear operator, since $(0, y) \in C$ gives $y \in \operatorname{mul} B \cap \overline{\operatorname{dom} B} = \{0\}$, i.e. $\operatorname{mul} C = \{0\}$. Furthermore, we have

$$B = C \mathbin{\dot{\boxplus}} (\{0\} \times \operatorname{mul} B)\,, \tag{3.3}$$

where $\boxplus$ denotes the sum of linear subspaces. (We cannot use the symbol $+$, because it already stands for the sum of two linear relations, cf. Definition 1.2.6.) The right-hand side of (3.3) is trivially contained in the left-hand side. To see the other inclusion, take $(x; y) \in B$ and mind the decomposition of y in $y = y_d + y_m$ with $y_d \in \overline{\operatorname{dom} B}$, $y_m \in \operatorname{mul} B$, which gives $(x; y) = (x; y_d) + (0; y_m)$.

Additionally we get from $C \subseteq \overline{\operatorname{dom} B} \times \overline{\operatorname{dom} B}$ and $\{0\} \times \operatorname{mul} B \subseteq \operatorname{mul} B \times \operatorname{mul} B$ that the sum in (3.3) is a direct sum. The linear mapping τ_M, cf. Definition 1.2.8, applied to equation (3.3) gives

$$\tau_M(B) = \tau_M(C) \mathbin{\dot{\boxplus}} \tau_M(\{0\} \times \operatorname{mul} B)\,, \tag{3.4}$$

for all $M \in \mathbb{C}^{2\times 2}$.

We want to apply Theorem 3.1.1 to the linear operator C. But C is not necessarily self-adjoint anymore, if you interpret C as a linear relation on $\mathcal{H}$. In general, C is only symmetric on $\mathcal{H}$, since we have $C \subsetneq B = B^* \subseteq C^*$. But we can interpret C as a linear relation on the Hilbert subspace $\overline{\operatorname{dom} B}$ with the associated adjoint,

$$C^{*,d} = \{(u; v) \in \overline{\operatorname{dom} B} \times \overline{\operatorname{dom} B} \mid (u, y) = (v, x) \text{ for all } (x; y) \in C\}. \tag{3.5}$$

Obviously, we have $C^{*,d} = C^* \cap \overline{\operatorname{dom} B} \times \overline{\operatorname{dom} B}$, which implies $C \subseteq C^{*,d}$.

Moreover, (3.4) applied to $M := \left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right)$ gives

$$(B + \mu) = (C + \mu) \mathbin{\dot{\boxplus}} (\{0\} \times \operatorname{mul} B)\,. \tag{3.6}$$

Since B is self-adjoint, we know that both defect indices of B are zero, i.e. $\dim \operatorname{ran}(B - i)^\perp = 0$ and $\dim \operatorname{ran}(B + i)^\perp = 0$. Equation (3.6) gives

$$\operatorname{ran}(B + \mu) = \operatorname{ran}(C + \mu) + \operatorname{mul} B,$$

for all $\mu \in \mathbb{C}$. Hence, $\operatorname{ran}(B + \mu)$ being dense in $\mathcal{H}$ is equivalent to $\operatorname{ran}(C + \mu)$ being dense in $\overline{\operatorname{dom} B}$. We conclude that the symmetric and closed operator C has also defect indices $(0, 0)$, and is, therefore, self-adjoint on the Hilbert space $\overline{\operatorname{dom} B}$.

In order to locate the spectrum of C, we make use of (3.4) one more time, with $M := \left(\begin{smallmatrix} 0 & 1 \\ 1 & -\mu \end{smallmatrix}\right)$, and get

$$(B-\mu)^{-1} = (C-\mu)^{-1} \dot{\boxplus} (\operatorname{mul} B \times \{0\}) . \tag{3.7}$$

Note that $\operatorname{mul} B \times \{0\}$, interpreted as a linear operator, is nothing but the zero-operator on $\operatorname{mul} B$ and is trivially continuous. Therefore, the linear relation $(C-\mu)^{-1} \leq \overline{\operatorname{dom} B} \times \overline{\operatorname{dom} B}$ is a bounded operator on $\overline{\operatorname{dom} B}$ if and only if the linear relation $(B-\mu)^{-1} \leq \mathcal{H} \times \mathcal{H}$ is a bounded operator on $\mathcal{H}$, i.e. $\sigma(C) \cup \{\infty\} = \sigma(B)$. Clearly, we have $\sigma(C) \cap \mathbb{R} = \sigma(B) \cap \mathbb{R}$ and $\rho(C) \setminus \{\infty\} = \rho(B)$.

Due to the spectral theorem for unbounded self-adjoint operators on Hilbert spaces, Theorem 3.1.1, applied to C, there exists a unique spectral measure $\tilde{E}$ for $\langle \sigma(B) \cap \mathbb{R}, \mathfrak{B}(\sigma(B) \cap \mathbb{R}), \overline{\operatorname{dom} B} \rangle$, such that

$$(C-\mu)^{-1} = \int_{\sigma(B)\cap\mathbb{R}} \frac{1}{t-\mu} \, d\tilde{E}(t) \tag{3.8}$$

holds for all $\mu \in \rho(B)$.

We extend $\tilde{E}$ to a spectral measure E for $\langle \sigma(B), \mathfrak{B}(\sigma(B)), \mathcal{H} \rangle$. To do so, denote by P the orthogonal projection $P : \mathcal{H} \to \operatorname{mul} B$ onto $\operatorname{mul} B$ along $\overline{\operatorname{dom} B}$ and set $E(\{\infty\}) := P$. More precisely, we define

$$E : \begin{cases} \mathfrak{B}(\sigma(B)) & \to \quad \mathcal{B}(\mathcal{H}) \\ \qquad \Delta & \mapsto \quad \begin{cases} \tilde{E}(\Delta)(I-P) & \text{if } \infty \notin \Delta \\ \tilde{E}(\Delta \cap \mathbb{R})(I-P) + P & \text{if } \infty \in \Delta \,. \end{cases} \end{cases}$$

It is elementary to check that E is indeed a spectral measure.

Finally, we are going to prove

$$r(B) = \int_{\sigma(B)} r(t) \, dE(t), \tag{3.9}$$

for all rational functions whose poles are in $\rho(B)$, i.e. $r \in \mathbb{C}(z) \cap B(\sigma(B))$. Considering a partial fraction decomposition of r, and having in mind that both the left- and right-hand side of (3.9) are linear and multiplicative in r, we see that it is enough to prove

$$(B-\mu)^{-1} = \int_{\sigma(B)} \frac{1}{t-\mu} \, dE(t), \quad \text{and} \quad I = \int_{\sigma(B)} 1 \, dE(t),$$

for all $\mu \in \rho(B)$. The second relation is clear due to

$$\int_{\sigma(B)} 1 \, dE(t) = E(\sigma(B)) = \tilde{E}(\sigma(B) \cap \mathbb{R})(I-P) + P = I_{\overline{\operatorname{dom} B}}(I-P) + P = I.$$

Now fix a $\mu \in \rho(B)$ and write (3.7) as $(B-\mu)^{-1} = (C-\mu)^{-1}(I-P)$. This gives

$$(B-\mu)^{-1} = \int_{\sigma(B)\cap\mathbb{R}} \frac{1}{t-\mu} \, d\tilde{E}(t) \, (I-P).$$

For the complex measures associated with the spectral measures, the relation

$$E_{x,y}(\Delta) = (E(\Delta)x, y) = (\tilde{E}(\Delta)(I-P)x, y) = \tilde{E}_{(I-P)x,y}(\Delta)$$

holds true, for $x, y \in \mathcal{H}$ and measurable $\Delta \subseteq \sigma(B) \cap \mathbb{R}$. Therefore, for a bounded measurable function $\phi \in B(\sigma(B))$ with $\phi(\infty) = 0$, we have

$$\left(\int \phi \, dE \, x, y\right) = \int \phi \, dE_{x,y} = \int \phi \, d\tilde{E}_{(I-P)x,y} = \left(\int \phi \, d\tilde{E} \, (I-P) \, x, y\right),$$

for all $x, y \in \mathcal{H}$, which implies $\int \phi \, dE = \int \phi \, d\tilde{E} \, (I-P)$. We conclude

$$\begin{aligned}(B-\mu)^{-1} = \int_{\sigma(B)\cap\mathbb{R}} \frac{1}{t-\mu} \, d\tilde{E}(t) \, (I-P) = \\ = \int_{\sigma(B)\cap\mathbb{R}} \frac{1}{t-\mu} \, dE(t) = \int_{\sigma(B)} \frac{1}{t-\mu} \, dE(t).\end{aligned}$$

□

3.2 Moving Linear Relations

Let $\mathcal{V}$ and $\mathcal{K}$ be two Krein spaces, and let $T : \mathcal{V} \to \mathcal{K}$ be a bounded linear operator. We can consider $T \times T : \mathcal{V} \times \mathcal{V} \to \mathcal{K} \times \mathcal{K}, (x; y) \mapsto (Tx; Ty)$. Clearly, also $T \times T$ is linear and bounded. Since both the inverse image and the image of a linear subspace under a linear mapping are again linear subspaces, the map $T \times T$ can be used to "move" linear subspaces, i.e. linear relations.

A linear relation A on $\mathcal{K}$ can be transformed to a linear relation $(T \times T)^{-1}(A)$ on $\mathcal{V}$. Analogously, a linear relation B on $\mathcal{V}$ can be moved to a linear relation $(T \times T)(B)$ on $\mathcal{K}$.

The study of this process yields useful tools, that we are going to use later in the construction of the functional calculus. The whole section originates from [Kal1].

Lemma 3.2.1. *Let $T : \mathcal{V} \to \mathcal{K}$ be a continuous linear mapping between two Krein spaces, and let A and B be linear relations on $\mathcal{K}$ and B respectively.*

(i) *$(T \times T)(B)$ coincides with the composition of linear relations TBT^{-1}.*

(ii) *$(T \times T)^{-1}(A)$ coincides with the composition of linear relations $T^{-1}AT$.*

(iii) *For all $M \in \mathbb{C}^{2\times 2}$ we have $(T \times T) \circ \tau_M = \tau_M \circ (T \times T)$.*

(iv) *For invertible $M \in \mathbb{C}^{2\times 2}$ we have $(T \times T)^{-1}\big(\tau_M(A)\big) = \tau_M\left((T \times T)^{-1}(A)\right)$.*

(v) *$(T \times T)^{-1}(A)$ is closed if A is closed.*

Proof.

(i) By the definition of compositions of relations, we have

$$\begin{aligned}(T \times T)(B) &= \{(Tx; Ty) : (x; y) \in B\} \\ &= \{(u; v) \in \mathcal{K}^2 \mid \exists x, y \in \mathcal{K} : (u; x) \in T^{-1}, (x; y) \in B, (y; v) \in T\} \\ &= TBT^{-1}.\end{aligned}$$

(ii) Similarly, we have

$$\begin{aligned}(T \times T)^{-1}(A) &= \{(x; y) \in \mathcal{V} \times \mathcal{V} : (Tx; Ty) \in A\} \\ &= \{(x; y) \in \mathcal{V}^2 \mid \exists u, v \in \mathcal{K} : (x; u) \in T, (u; v) \in A, (v; y) \in T^{-1}\} \\ &= T^{-1}AT.\end{aligned}$$

(iii) For $M = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$ we have

$$\begin{aligned}(T \times T) \circ \tau_M = (T \times T) \circ \begin{pmatrix}\delta I_{\mathcal{V}} & \gamma I_{\mathcal{V}} \\ \beta I_{\mathcal{V}} & \alpha I_{\mathcal{V}}\end{pmatrix} = \begin{pmatrix}\delta T & \gamma T \\ \beta T & \alpha T\end{pmatrix} = \\ = \begin{pmatrix}\delta I_{\mathcal{K}} & \gamma I_{\mathcal{K}} \\ \beta I_{\mathcal{K}} & \alpha I_{\mathcal{K}}\end{pmatrix} \circ (T \times T) = \tau_M \circ (T \times T).\end{aligned}$$

(*iv*) Due to $\tau_M^{-1} = \tau_{M^{-1}}$, we have

$$(T\times T)^{-1}\left(\tau_{M^{-1}}(A)\right) = (T\times T)^{-1}\left(\tau_M^{-1}(A)\right) = \left(\tau_M \circ (T\times T)\right)^{-1}(A) = \\ = \left((T\times T)\circ \tau_M\right)^{-1}(A) = \tau_M^{-1}\left((T\times T)^{-1}(A)\right) = \tau_{M^{-1}}\left((T\times T)^{-1}(A)\right).$$

Substitute M for M^{-1}.

(*v*) As a inverse image of a closed subset under a continuous function $(T\times T)^{-1}(A)$ is closed.

□

Corollary 3.2.2. *For a linear relation A on $\mathcal{K}$ we have*

$$\ker\left((T\times T)^{-1}(A) - \lambda\right) = T^{-1}\ker\left(A-\lambda\right),$$

for all $\lambda \in \mathbb{C}\cup\{\infty\}$. In particular, for $\lambda=\infty$, i.e. $\operatorname{mul}(T\times T)^{-1}(A) = T^{-1}(\operatorname{mul} A)$.
Furthermore, we have $\sigma_p\left((T\times T)^{-1}(A)\right) \subseteq \sigma_p(A)$ if T is injective.

Proof. For $\lambda=\infty$, the claim follows from

$$x\in \operatorname{mul}(T\times T)^{-1}(A) \Leftrightarrow (0,Tx)\in A \Leftrightarrow x\in T^{-1}(\operatorname{mul} A),$$

minding the convention $(R-\infty)^{-1} := R$. In order to treat the case $\lambda\in\mathbb{C}$, we set $M := \left(\begin{smallmatrix} 0 & 1 \\ 1 & -\lambda\end{smallmatrix}\right)$ such that $\tau_M(R) = (R-\lambda)^{-1}$ for all linear relation R, cf. Remark 1.2.11. Lemma 3.2.1 gives

$$\ker\left((T\times T)^{-1}(A)-\lambda\right) = \operatorname{mul}\tau_M\left((T\times T)^{-1}(A)\right) = \\ \operatorname{mul}(T\times T)^{-1}\left(\tau_M(A)\right) = T^{-1}\left(\operatorname{mul}\tau_M(A)\right) = T^{-1}\ker(A-\lambda).$$

The last statement holds, since $T^{-1}\ker(A-\lambda)\neq\{0\}$ implies $\ker(A-\lambda)\neq\{0\}$, for injective T. □

Lemma 3.2.3. *Let $T:\mathcal{V}\to\mathcal{K}$ be a bounded linear operator between two Krein space $\mathcal{V}$ and $\mathcal{K}$, and let A and B be linear relations on $\mathcal{K}$, $\mu\in\mathbb{C}\setminus\{0\}$. Then we have*

$$(T\times T)^{-1}(\mu A) = \mu(T\times T)^{-1}(A),$$
$$(T\times T)^{-1}(A+B) \supseteq (T\times T)^{-1}(A) + (T\times T)^{-1}(B),$$
$$(T\times T)^{-1}(AB) \supseteq (T\times T)^{-1}(A)(T\times T)^{-1}(B).$$

Proof. The first identity follows from Lemma 3.2.1, by considering τ_M for $M := \left(\begin{smallmatrix} 1 & 0 \\ 0 & \mu\end{smallmatrix}\right)$.
To verify the second one, take an element $(x;y)\in(T\times T)^{-1}(A) + (T\times T)^{-1}(B)$. We have $(Tx;Tu)\in A$ and $(Tx;Tv)\in B$ for some $u,v\in\mathcal{V}$ with $u+v=y$. Since $Tu+Tv=Ty$, we get $(Tx;Ty)\in A+B$, i.e. $(x;y)\in(T\times T)^{-1}(A+B)$.
The last claim follows from Lemma 3.2.1 and $TT^{-1}\subseteq I$:

$$(T\times T)^{-1}(A)(T\times T)^{-1}(B) = (T^{-1}AT)(T^{-1}BT) \subseteq T^{-1}ABT = (T\times T)^{-1}(AB)$$

□

Remark 3.2.4. Let $S : \mathcal{C} \to \mathcal{D}$ be a linear operator between two vector spaces $\mathcal{C}$ and $\mathcal{D}$, and let $A \leq \mathcal{C} \times \mathcal{C}$ and $B \leq \mathcal{D} \times \mathcal{D}$ be two linear relations.

The condition $(S \times S)(A) \subseteq B$, which will arise in the sequel, clearly is equivalent to $A \subseteq (S \times S)^{-1}B = S^{-1}BS$. Applying S from the left yields $SA \subseteq SS^{-1}BS \subseteq BS$, due to $SS^{-1} \subseteq I$, for operators S.

On the other hand, $SA \subseteq BS$ gives $S^{-1}SA \subseteq S^{-1}BS = (S \times S)^{-1}(B)$, which yields $A \subseteq (S \times S)^{-1}(B)$, since $I \subseteq S^{-1}S$ if $\operatorname{dom} S = \mathcal{C}$.

We have seen that $(S \times S)(A) \subseteq B$ is equivalent to the intertwining condition $SA \subseteq BS$. If A and B are everywhere defined operators, this condition is further equivalent to $SA = BS$.

Remark 3.2.5. With the notation from Remark 3.2.4, let $M \in \mathbb{C}^{2\times 2}$ be any regular matrix. We have

$$(S \times S)(A) \subseteq B \ \Leftrightarrow \ (S \times S)(\tau_M(A)) \subseteq \tau_M(B).$$

This equivalence holds true due to the fact that τ_M, from Definition 1.2.8, is bijective with ${\tau_M}^{-1} = \tau_{M^{-1}}$, and due to Lemma 3.2.1.

Together with Remark 3.2.4 we have shown $SA \subseteq BS$ if and only if $S\tau_M(A) \subseteq \tau_M(B)S$, for all regular matrix $M \in \mathbb{C}^{2\times 2}$.

We already know that the spectrum of self-adjoint linear relations on Krein spaces - in contrast to the situation in Hilbert spaces - is not contained in the real line. In fact, this is not even true for bounded self-adjoint operators on Krein spaces, cf. Example 1.2.28.

Recall that for a positive operator A on a Hilbert space, we have $\sigma(A) \subseteq [0, \infty)$. The next result states what is known about the spectrum of positive operators on Krein spaces.

Theorem 3.2.6. *Let $(\mathcal{K}, [.,.])$ be a Krein space, and let $A : \mathcal{K} \to \mathcal{K}$ be a bounded and self-adjoint linear operator, such that $[Ax, x] \geq 0$ for all $x \in \mathcal{K}$, i.e. A is positive . Then $\sigma(A) \subseteq \mathbb{R}$.*

Proof. We define by $\langle x, y\rangle := [Ax, y]$ a positive semidefinite inner product on $\mathcal{K}$. In order to get a pre-Hilbert space, we have to factor out the $\mathcal{N} := \{x \in \mathcal{K} \mid \langle x, x\rangle = 0\}$, which is a closed subspace since it coincides with the isotropic part, cf. Corollary 1.1.3.

Hence, the inner product, again denoted by $\langle .,.\rangle$, is well-defined on the factor space $\mathcal{K}/\mathcal{N}$ by

$$\langle x + \mathcal{N}, y + \mathcal{N}\rangle := \langle x, y\rangle \quad \text{ for } x, y \in \mathcal{K},$$

and is positive definite. The pre-Hilbert space $(\mathcal{K}/\mathcal{N}, \langle .,.\rangle)$ has a Hilbert space completion, denoted by $(\mathcal{V}, \langle .,.\rangle)$.

Now, consider the canonical embedding $\iota : (\mathcal{K}, [.,.]) \to (\mathcal{V}, \langle .,.\rangle)$ which denotes the composition of the canonical surjection $x \mapsto x + \mathcal{N}$ and the canonical embedding $\mathcal{K}/\mathcal{N} \to \mathcal{V}$ coming along with the completion. The linear operator ι is bounded and has dense range.

Thus, $T := \iota^* : (\mathcal{V}, \langle .,.\rangle) \to (\mathcal{K}, [.,.])$ is an injective operator due to $\ker T = \ker \iota^* = (\operatorname{ran} \iota)^{\perp} = \{0\}$, cf. Lemma 1.2.18. Furthermore, for x, y in $\mathcal{K}$ we have

$$[TT^*x, y] = \langle T^*x, T^*y\rangle = \langle \iota x, \iota y\rangle = \langle x, y\rangle = [Ax, y],$$

which gives $TT^* = A$. From

$$(T \times T)^{-1}(A) = T^{-1}AT = T^{-1}TT^*T = T^*T,$$

we learn that $(T \times T)^{-1}(A)$ is a bounded and self-adjoint operator on $\mathcal{V}$. Being on a Hilbert space, we have $\sigma((T \times T)^{-1}(A)) \subseteq \mathbb{R}$, cf. Lemma 1.2.26. For that reason, we can set $M := \left(\begin{smallmatrix} 1 & 0 \\ 1 & -\lambda \end{smallmatrix}\right)$, for arbitrary $\lambda \in \mathbb{C}\backslash\mathbb{R}$, and get a bounded operator on $\mathcal{V}$ via

$$\tau_M \left((T \times T)^{-1}(A)\right) = (T \times T)^{-1} \left(\tau_M(A)\right) =: B,$$

cf. Remark 1.2.11. Clearly, B can be written as $B = I + \lambda(T^*T - \lambda)^{-1} = T^*T(T^*T - \lambda)^{-1}$.

Until now, we constructed a Hilbert space $\mathcal{V}$, moved A to $\mathcal{V}$, and applied τ_M to get B. Now, we move B back to the Krein-space using $T^* : \mathcal{K} \to \mathcal{V}$. In fact, we set $C := (T^* \times T^*)^{-1}(B)$.

The idea is, basically, to calculate $\operatorname{dom} C$ in two different ways. By definition of C and by Lemma 3.2.1, we get

$$C = (T^*)^{-1}BT^* = (T^*)^{-1}T^*T(T^*T - \lambda)^{-1}T^* \supseteq T(T^*T - \lambda)^{-1}T^*.$$

Note that the right-hand side is a bounded operator on $\mathcal{V}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. This gives $\operatorname{dom} C = \mathcal{K}$, since the domain of a superset can only get larger.

Easy manipulations give

$$\begin{aligned} C &= (T^* \times T^*)^{-1}\tau_M \left((T \times T)^{-1}(A)\right) = \tau_M \left((T^* \times T^*)^{-1}(T \times T)^{-1}(A)\right) = \\ &= \tau_M \left((TT^* \times TT^*)^{-1}(A)\right) = \tau_M \left((A \times A)^{-1}(A)\right) = \tau_M \left(A^{-1}AA\right) = \\ &= \tau_M \left(\left(I \boxplus (\{0\} \times \ker A)\right)A\right) = \tau_M\left(A \boxplus (\{0\} \times \ker A)\right) \end{aligned}$$

Now, we can use Remark 1.2.10 to compute the domain of C, and get

$$\begin{aligned} \operatorname{dom} C &= \operatorname{dom} \tau_M\left(A \boxplus (\{0\} \times \ker A)\right) = \\ &= \{-\lambda f + g : (f; g) \in A \boxplus (\{0\} \times \ker A)\} = \operatorname{ran}(A - \lambda) + \ker A \end{aligned}$$

We have shown $\operatorname{ran}(A - \lambda) + \ker A = \mathcal{K}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Note that we have $\ker A \subseteq \operatorname{ran}(A - \lambda)$ for all $\lambda \neq 0$. For $x \in \ker A$ simply set $y := -\frac{1}{\lambda}x$ and note $(A - \lambda)y = x$, i.e. $\ker A \subseteq \operatorname{ran}(A - \lambda)$. We deduce $\operatorname{ran}(A - \lambda) = \mathcal{K}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Since A is self-adjoint, Lemma 1.2.18 gives $\ker(A - \overline{\lambda}) = \operatorname{ran}(A - \lambda)^{[\perp]} = \{0\}$ for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$. By the closed graph theorem, we get $(A - \lambda)^{-1} \in \mathcal{B}(\mathcal{K})$, i.e. $\mathbb{C}\backslash\mathbb{R} \subseteq \rho(A)$. $\square$

Lemma 3.2.7. *Let $T : \mathcal{V} \to \mathcal{K}$ be a bounded linear operator between two Krein spaces $\mathcal{V}$ and $\mathcal{K}$. For a linear relation $A \leq \mathcal{K} \times \mathcal{K}$, we have*

$$((T^* \times T^*)(A))^* = (T \times T)^{-1}(A^*).$$

In particular, $\left((T \times T)^{-1}(A^*)\right)^*$ is the closure of $(T^* \times T^*)(A)$.

Proof. For the proof, we need the following two facts.

First, we apply Lemma 1.2.21 to $R = T \times T : \mathcal{V} \times \mathcal{V} \to \mathcal{K} \times \mathcal{K}$, whereby $\mathcal{V} \times \mathcal{V}$ and $\mathcal{K} \times \mathcal{K}$ are endowed with the respective sum inner product, and to $L = A \subseteq \mathcal{K} \times \mathcal{K}$. This gives

$$\big((T^* \times T^*)(A)\big)^{\perp} = (T \times T)^{-1}(A^{\perp}).$$

Secondly, setting $M := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we have $S^* = \tau_M(S^{\perp}) = \tau_M(S)^{\perp}$, for any linear relation S, cf. (1.6). This gives

$$((T^* \times T^*)(A))^* = \tau_M\left(((T^* \times T^*)(A))^{\perp}\right) = \tau_M\left((T \times T)^{-1}(A^{\perp})\right) =$$
$$= (T \times T)^{-1}\left(\tau_M(A^{\perp})\right) = (T \times T)^{-1}(A^*).$$

□

Proposition 3.2.8. *Let $T : \mathcal{V} \to \mathcal{K}$ be a bounded linear operator between two Krein spaces $\mathcal{V}$ and $\mathcal{K}$, and let $A \leq \mathcal{K} \times \mathcal{K}$ be a closed linear relation on $\mathcal{K}$ satisfying*

$$(TT^* \times TT^*)(A^*) \subseteq A.$$

Then the linear relation $(T \times T)^{-1}(A)^*$ is symmetric and coincides with the closure of $(T^* \times T^*)(A^*)$.

If $\mathcal{V}$ is in fact a Hilbert space, T is injective, and if $\mathbb{C}\backslash\sigma_p(A)$ contains points from both $\mathbb{C}^+$ and $\mathbb{C}^-$, the relation $(T \times T)^{-1}(A)$ is self-adjoint.

Proof. $(TT^* \times TT^*)(A^*) = (T \times T)(T^* \times T^*)(A^*) \subseteq A$ implies $(T^* \times T^*)(A^*) \subseteq (T \times T)^{-1}(A)$. Since the right-hand side is closed, we obtain from Lemma 3.2.7, using $A^{**} = \overline{A} = A$,

$$\big((T \times T)^{-1}(A)\big)^* = \overline{(T^* \times T^*)(A^*)} \subseteq (T \times T)^{-1}(A).$$

Now let $\mathcal{V}$ be a Hilbert space, and assume that $(T \times T)^{-1}(A)^*$ is not self-adjoint, or equivalently, not both defect numbers are zero. We know that for an arbitrary linear relation R, we have $\operatorname{ran}(R - \lambda)^{\perp} = \ker(R^* - \overline{\lambda})$ for $\lambda \in \mathbb{C}$. For a symmetric linear relation R and non-real λ, the dimension of these subspaces of $\mathcal{V}$ only depends on the half-plane which contains λ. Thus, a defect number of $(T \times T)^{-1}(A)^*$ being unequal to zero, results in $\ker((T \times T)^{-1}(A) - \overline{\lambda})$ not being the nullspace for all λ in a half-plane, i.e. $\sigma_p((T \times T)^{-1}(A))$ contains a half-plane. Since T is injective, Corollary 3.2.2 gives $\sigma_p((T \times T)^{-1}(A)) \subseteq \sigma_p(A)$, which contradicts the assumption concerning $\mathbb{C}\backslash\sigma_p(A)$. □

Remark 3.2.9. We are going to to recall some well-known facts, which will be needed in the proof of the next lemma.

Let $P \in \mathcal{B}(\mathcal{H})$ be a positive operator on a Hilbert space $\mathcal{H}$, i.e. $(Px, x) \geq 0$ for all $x \in \mathcal{H}$. We write $P_1 \leq P_2$ if $P_2 - P_1$ is positive, i.e. $((P_2 - P_1)x, x) \geq 0$ or equivalently $(P_1 x, x) \leq (P_2 x, x)$ for all $x \in \mathcal{H}$. Every positive Operator has a unique square root, which can be declared via

$$\sqrt{P} := \int \sqrt{t}\, dE(t),$$

where E denotes the spectral measure of P. Loewner's Theorem states that $P_1 \leq P_2$ implies $\sqrt{P_1} \leq \sqrt{P_2}$ for all positive operators P_1, P_2. A proof of this theorem can be found here [Mur, Theorem 2.2.6].

We want to remark that the square of a self-adjoint operator is always positive. Its square root is denoted by $|A|$,

$$\sqrt{A^2} = \sqrt{\int t^2 \, dE(t)} = \int |t| \, dE(t) =: |A|.$$

Hereby, for all self-adjoint operators A and all $x \in \mathcal{H}$ the following inequality holds true,

$$|(Ax, x)| = \left| \int t \, dE_{x,x}(t) \right| \leq \int |t| \, dE_{x,x}(t) = (|A|x, x).$$

Lemma 3.2.10. *Let $T : \mathcal{V} \to \mathcal{K}$ be a bounded and injective linear operator between a Hilbert space $(\mathcal{V}, [.,.]_{\mathcal{V}})$ and a Krein space $(\mathcal{K}, [.,.]_{\mathcal{K}})$ and assume $A : \mathcal{K} \to \mathcal{K}$ to be a bounded linear operator, with $(TT^* \times TT^*)(A^*) \subseteq A$.*

Then $(T \times T)^{-1}(A)$ is a bounded and self-adjoint linear operator on $\mathcal{V}$ with

$$\|(T \times T)^{-1}(A)\| \leq \|A\|.$$

Here, $\|.\|$ on the left-hand side refers to the operator norm in $\mathcal{V}$, and $\|.\|$ on the right-hand side refers to the operator norm in $(\mathcal{K}, \|.\|_J)$ for any fundamental symmetry J on $\mathcal{K}$.

Proof. Since $\sigma_p(A) \subseteq \sigma(A) \subseteq K_{\|A\|}(0)$, we have $\mathbb{C} \backslash \sigma_p(A) \supseteq \mathbb{C} \backslash K_{\|A\|}(0)$. Therefore, $\mathbb{C} \backslash \sigma_p(A)$ contains points from both $\mathbb{C}^+$ and $\mathbb{C}^-$. By Proposition 3.2.8, the linear relation $(T \times T)^{-1}(A)$ is self-adjoint and is the closure of $(T^* \times T^*)(A^*)$. Due to Corollary 3.2.2, we have $\operatorname{mul}(T \times T)^{-1}(A) = T^{-1}(\operatorname{mul} A) = \ker T = \{0\}$. Hence, $(T \times T)^{-1}(A)$ is an operator.

Take an arbitrary pair $(x; y) \in (T^* \times T^*)(A^*)$. Since we have $x \in \operatorname{dom}(T^* \times T^*)(A^*) = \operatorname{dom} T^* A^* (T^*)^{-1} \subseteq \operatorname{dom} (T^*)^{-1} = \operatorname{ran} T^*$, we can write $x = T^* u$ for some $u \in \mathcal{K}$. Due to $(T^* \times T^*)(A^*) \subseteq (T \times T)^{-1}(A)$, we get $(Tx; Ty) = (TT^*u; Ty) \in A$, and hence

$$[y, x]_{\mathcal{V}} = [y, T^* u]_{\mathcal{V}} = [Ty, u]_{\mathcal{K}} = [ATT^* u, u]_{\mathcal{K}}.$$

Due to Remark 3.2.4, the assumption $(TT^* \times TT^*)(A^*) \subseteq A$ is equivalent to $TT^*A = ATT^*$, i.e. $B := ATT^* \in \mathcal{B}(\mathcal{K})$ is self-adjoint.

Take any fundamental symmetry J on $\mathcal{K}$. We have $B^{(*)} = JB^*J$, cf. Proposition 1.2.19, which gives $(BJ)^{(*)} = JB^{(*)} = JJB^*J = B^*J = BJ$, i.e. $BJ \in \mathcal{B}(\mathcal{K})$ is self-adjoint with respect to $(.,.)_J$. Moreover, by definition of B and by Remark 1.2.20, we have $BJ = ATT^*J = ATT^{(*)}$.

$$[ATT^* u, u]_{\mathcal{K}} = [Bu, u]_{\mathcal{K}} = (Bu, Ju)_J = (BJJu, Ju)_J = (ATT^{(*)}Ju, Ju)_J$$

Consider

$$(ATT^{(*)})^2 = ATT^{(*)}ATT^{(*)} = ATT^{(*)}JJATT^{(*)} = ATT^*JATT^{(*)} =$$

$$= TT^*A^*JATT^{(*)} = TT^*JJA^*JATT^{(*)} = TT^{(*)}A^{(*)}ATT^{(*)}. \quad (3.10)$$

As a square of a self-adjoint operator, the operator in (3.10) is positive. The calculation

$$(TT^{(*)}A^{(*)}ATT^{(*)}x, x)_J = (ATT^{(*)}x, ATT^{(*)}x)_J = \|ATT^{(*)}x\|_J^2 \leq$$
$$\leq \|A\|^2\|TT^{(*)}x\|_J^2 = \|A\|^2(TT^{(*)}x, TT^{(*)}x)_J = \|A\|^2(TT^{(*)}TT^{(*)}x, x)_J,$$

shows that we have $(ATT^{(*)})^2 \leq \|A\|^2 TT^{(*)}TT^{(*)}$, where $\|A\|$ is the operator norm of A with respect to $\|.\|_J$. By Loewner's Theorem, cf. Remark 3.2.9, we get $|ATT^{(*)}| \leq \|A\|TT^{(*)}$. Hence, for all $(x; y) \in (T^* \times T^*)(A^*)$ we have

$$|[y, x]_{\mathcal{V}}| = |(ATT^{(*)}Ju, Ju)_J| \leq (|ATT^{(*)}|Ju, Ju)_J \leq \|A\|(TT^{(*)}Ju, Ju)_J =$$
$$= \|A\|(TT^*u, Ju)_J = \|A\|[TT^*u, u]_{\mathcal{K}} = \|A\|[T^*u, T^*u]_{\mathcal{V}} = \|A\|[x, x]_{\mathcal{V}}.$$

Since $(T^* \times T^*)(A^*)$ is dense in $(T \times T)^{-1}(A)$, the inequality

$$|[y, x]_{\mathcal{V}}| \leq \|A\| \, [x, x]_{\mathcal{V}} \quad (3.11)$$

even holds true for all $(x; y) \in (T \times T)^{-1}(A)$.

The spectral theorem for unbounded self-adjoint operators in Hilbert spaces, Theorem 3.1.1 applied to $(T \times T)^{-1}(A)$ gives a spectral measure E for $\langle \mathbb{R}, \mathfrak{B}(\mathbb{R}), \mathcal{V}\rangle$. In the following, we need the well-known result that an element $g \in \mathcal{V}$ is in the domain of $\int \phi \, dE$ if and only if $\int |\phi|^2 \, dE_{g,g} < \infty$, see [Kal2, Lemma 4.5.4.] or [Rud, Lemma 13.23].

For any natural number $n > \|A\| + 1$, consider the interval $\Delta_n := [\|A\| + \frac{1}{n}, n\,] \subseteq \mathbb{R}$. For $x \in \operatorname{ran} E(\Delta_n)$, we have

$$\int |t|^2 \, dE_{x,x}(t) = \int_{\Delta_n} |t|^2 \, dE_{x,x}(t) < \infty,$$

which gives $x \in \operatorname{dom}(T \times T)^{-1}(A)$. Inequality (3.11) yields

$$\|A\|[x, x]_{\mathcal{V}} \geq |[(T \times T)^{-1}(A)x, x]_{\mathcal{V}}| = \int_{\Delta_n} t \, dE_{x,x}(t) \geq$$
$$\geq (\|A\| + \frac{1}{n})[E(\Delta_n)x, x]_{\mathcal{V}} = (\|A\| + \frac{1}{n})[x, x]_{\mathcal{V}}.$$

We conclude $x = 0$, i.e. $\operatorname{ran} E(\Delta_n) = 0$ and, therefore, $E(\Delta_n) = 0$ for all $n > \|A\|$. Since E is σ-additive we also get $E(\bigcup_{n>\|A\|} \Delta_n) = E((\|A\|, \infty)) = 0$.

In the same matter, we can show $E((-\infty, -\|A\|)) = 0$. So E and, therefore, all complex measures $E_{x,y}$ are concentrated on the interval $[-\|A\|, \|A\|]$. Also, for all measurable functions ϕ the operators $\int \phi \, dE$ depend only on the values of ϕ on $[-\|A\|, \|A\|]$. Due to

$$\int_{\mathbb{R}} |t|^2 \, dE_{x,x}(t) = \int_{[-\|A\|, \|A\|]} |t|^2 \, dE_{x,x}(t) < \infty,$$

we have $\operatorname{dom}(T \times T)^{-1}(A) = \mathcal{V}$. Since integrating bounded measurable functions with respect to a spectral measure always gives bounded everywhere-defined operators, we deduce $(T \times T)^{-1}(A) \in \mathcal{B}(\mathcal{V})$ from

$$(T \times T)^{-1}(A) = \int_{\mathbb{R}} t \, dE(t) = \int t \cdot \mathbb{1}_{[-\|A\|, \|A\|]}(t) \, dE(t)$$

The numerical radius of $(T \times T)^{-1}(A)$, i.e. the supremum of the absolute values of the numbers in the numerical range, coincides with the operator norm $\|(T \times T)^{-1}(A)\|$, because $(T \times T)^{-1}(A)$ is self-adjoint. Inequality 3.11 gives

$$\|(T \times T)^{-1}(A)\| = \sup \left\{ |[y, x]_{\mathcal{V}}| : (x; y) \in (T \times T)^{-1}(A), \|x\|_{\mathcal{V}} = 1 \right\} \leq \|A\|.$$

□

Theorem 3.2.11. *Let $T : \mathcal{V} \to \mathcal{K}$ be a bounded and injective linear operator between a Hilbert space $(\mathcal{V}, (.,.)_{\mathcal{V}})$ and a Krein space $(\mathcal{K}, [.,.]_{\mathcal{K}})$. Then*

$$\Theta : \begin{cases} (TT^*)' \subseteq \mathcal{B}(\mathcal{K}) & \to & (T^*T)' \subseteq \mathcal{B}(\mathcal{V}) \\ C & \mapsto & (T \times T)^{-1}(C) \end{cases}$$

constitutes a $*$-algebra homomorphism from the C^*-algebra $(TT^*)' \subseteq \mathcal{B}(\mathcal{K})$ of all bounded linear operators on $\mathcal{K}$ that commute with TT^*, into $(T^*T)' \subseteq \mathcal{B}(\mathcal{V})$ with $\|\Theta\| \leq 1$. Furthermore, we have $\Theta(I) = I$, $\Theta(TT^*) = T^*T$ and

$$\ker \Theta = \{C \in (TT^*)' : \operatorname{ran} C \subseteq \ker T^* = \ker TT^*\}.$$

Additionally, $(T^* \times T^*)(C)$ is densely contained in $\Theta(C)$ for all $C \in (TT^*)'$, and we have

$$T^*C = \Theta(C)T^*. \tag{3.12}$$

Proof. By Lemma 3.2.1, we have $\Theta(I) = (T \times T)^{-1}(I) = T^{-1}T = I$ and $\Theta(TT^*) = T^{-1}TT^*T = T^*T$. It is also elementary to check, that for $A, B \in (TT^*)'$ and $\mu \in \mathbb{C}$ we have $\mu A, A + B, AB, A^* \in (TT^*)'$. Likewise, also $(T^*T)'$ is a C*-subalgebra of $\mathcal{B}(\mathcal{V})$.

We start by taking a self-adjoint $A = A^* \in (TT^*)' \subseteq \mathcal{B}(\mathcal{K})$. In order to apply Lemma 3.2.10, we need to verify $(TT^* \times TT^*)(A^*) \subseteq A$. According to Remark 3.2.4 this is equivalent to $TT^*A^* = ATT^*$, which holds true for a self-adjoint operator $A \in (TT^*)'$. Hence, Lemma 3.2.10 gives that $(T \times T)^{-1}(A)$ is a bounded and self-adjoint linear operator on $\mathcal{V}$. Also, $(T^* \times T^*)(A)$ is densely contained in it. This yields

$$(T^*T \times T^*T)(T \times T)^{-1}(A) \subseteq (T^* \times T^*)(A) \subseteq (T \times T)^{-1}(A),$$

which is, again due to Remark 3.2.4, equivalent to $(T \times T)^{-1}(A) \in (T^*T)'$.

A general $C \in (TT^*)'$ can be written as $C = \operatorname{Re} C + i \operatorname{Im} C$, with

$$\operatorname{Re} C = \frac{C + C^*}{2}, \ \operatorname{Im} C = \frac{C - C^*}{2i}.$$

Since $\operatorname{Re} C$ and $\operatorname{Im} C$ are self-adjoint operators in $(TT^*)'$, we deduce that $(T \times T)^{-1}(\operatorname{Re} C)$ and $(T \times T)^{-1}(\operatorname{Im} C)$ are self-adjoint operators in $(T^*T)'$. Furthermore, Lemma 3.2.3 gives

$$(T \times T)^{-1}(C) = (T \times T)^{-1}(\operatorname{Re} C + i \operatorname{Im} C) \supseteq$$
$$(T \times T)^{-1}(\operatorname{Re} C) + i(T \times T)^{-1}(\operatorname{Im} C), \tag{3.13}$$

where the right-hand side is an everywhere defined operator, and the left-hand side is an operator, due to $\operatorname{mul}(T \times T)^{-1}(C) = T^{-1}(\operatorname{mul} C)$, cf. Corollary 3.2.2. Thus, we obtain equality in (3.13). This yields $\Theta(C) \in (T^*T)'$. Repeating the argument with $C^* = \operatorname{Re} C - i \operatorname{Im} C$ unveils

$$(T \times T)^{-1}(C^*) = (T \times T)^{-1}(\operatorname{Re} C - i \operatorname{Im} C) \supseteq \\ (T \times T)^{-1}(\operatorname{Re} C) - i(T \times T)^{-1}(\operatorname{Im} C) = (T \times T)^{-1}(C)^*.$$

Once more, we obtain equality and conclude $\Theta(C)^* = \Theta(C^*)$.

The linearity follows in the same manner. For a $\mu \in \mathbb{C}\setminus\{0\}$ and an $A \in (TT^*)'$ Lemma 3.2.3 gives $\Theta(\mu C) = \mu\Theta(C)$. For $\mu = 0$, $\Theta(0) = 0$ is checked by hand. Furthermore, for $A, B \in (TT^*)'$ we have $\Theta(A + B) \supseteq \Theta(A) + \Theta(B)$ and $\Theta(AB) \supseteq \Theta(A)\Theta(B)$. As before, we have everywhere defined operators on both sides, and, therefore, get equalities. Hence, Θ is well-defined, linear, multiplicative and compatible with taking adjoints.

In order to find an estimate for the operator norm of $\Theta(C)$, we can reduce the situation to self-adjoint C and use Lemma 3.2.10:

$$\|\Theta(C)x\|_{\mathcal{V}}^2 = \big(\Theta(C)x, \Theta(C)x\big)_{\mathcal{V}} = \big(\Theta(C^*C)x, x\big)_{\mathcal{V}} \leq \|\Theta(C^*C)x\|_{\mathcal{V}}\|x\|_{\mathcal{V}} \leq \\ \leq \|\Theta(C^*C)\|\|x\|_{\mathcal{V}}^2 \leq \|C^*C\|\|x\|_{\mathcal{V}}^2 \leq \|C\|^2\|x\|_{\mathcal{V}}^2.$$

This yields $\|\Theta\| \leq 1$.

Lemma 3.2.7 states

$$((T^* \times T^*)(C))^* = (T \times T)^{-1}(C^*) = \Theta(C^*) = \Theta(C)^*.$$

Taking adjoints gives $\overline{(T^* \times T^*)(C)} = \Theta(C)$. In particular, $\Theta(C) = 0$ is equivalent to $\operatorname{ran}((T^* \times T^*)(C)) = \{0\}$, i.e. $T^*y = 0$ for all $y \in \operatorname{ran}(C)$, or equivalently $\operatorname{ran}(C) \subseteq \ker T^* = \ker TT^*$.

For all $C \in (TT^*)'$ we have $(T^* \times T^*)(C) \subseteq \Theta(C)$. By Remark 3.2.4, this is equivalent to $T^*C = \Theta(C)T^*$. □

Corollary 3.2.12. *In the setting of the last theorem, we have $\sigma(\Theta(C)) \subseteq \sigma(C)$, or equivalently $\rho(\Theta(C)) \supseteq \rho(C)$, for all $C \in (TT^*)'$.*

Proof. We take a $\lambda \in \rho(C)$, so $(C - \lambda I_{\mathcal{K}})^{-1} \in \mathcal{B}(K)$. Further, this operator also belongs to $(TT^*)'$: Since $C, I_{\mathcal{K}} \in (TT^*)'$, we have that also $(C - \lambda I_{\mathcal{K}})$ is an element of the C*-algebra $(TT^*)'$, i.e.

$$TT^*(C - \lambda I_{\mathcal{K}}) = (C - \lambda I_{\mathcal{K}})TT^*.$$

We multiply this equation by $(C - \lambda I_{\mathcal{K}})^{-1}$ from the left and from the right-hand side, and obtain $(C - \lambda I_{\mathcal{K}})^{-1} \in (TT^*)'$.

Now consider $\Theta\big((C - \lambda I_{\mathcal{K}})^{-1}\big)$. The calculation,

$$\Theta\big((C - \lambda I_{\mathcal{K}})^{-1}\big)\Theta\big((C - \lambda I_{\mathcal{K}})\big) = \Theta\left((C - \lambda I_{\mathcal{K}})^{-1}(C - \lambda I_{\mathcal{K}})\right) = \Theta(I_{\mathcal{K}}) = I_{\mathcal{V}},$$

reveals that $\Theta\big((C - \lambda I_{\mathcal{K}})^{-1}\big) = \big(\Theta(C - \lambda I_{\mathcal{K}})\big)^{-1} = \big(\Theta(C) - \lambda I_{\mathcal{V}}\big)^{-1}$ hold, i.e. $\lambda \in \rho(\Theta(C))$. □

The next theorem is a version of Theorem 3.2.11 for linear relations.

Theorem 3.2.13. *Let $T : \mathcal{V} \to \mathcal{K}$ be a continuous and injective mapping between a Hilbert space $(\mathcal{V}, (.,.)_\mathcal{V})$ and a Krein space $(\mathcal{K}, [.,.]_\mathcal{K})$. Define the sets*

$$(TT^*)'_{\mathrm{rel}} := \{C \leq \mathcal{K} \times \mathcal{K} \,|\, \rho(C) \neq \emptyset, TT^*C \subseteq CTT^*\} \text{ and}$$
$$(T^*T)'_{\mathrm{rel}} := \{D \leq \mathcal{V} \times \mathcal{V} \,|\, \rho(D) \neq \emptyset, T^*TD \subseteq DT^*T\}.$$

These sets are closed under taking adjoints, and

$$\Theta_{\mathrm{rel}} : \begin{cases} (TT^*)'_{\mathrm{rel}} & \to & (T^*T)'_{\mathrm{rel}} \\ C & \mapsto & (T \times T)^{-1}(C) \end{cases}$$

fulfills $\Theta(C)^*_{\mathrm{rel}} = \Theta_{\mathrm{rel}}(C^*)$, and $\sigma(\Theta_{\mathrm{rel}}(C)) \subseteq \sigma(C)$.

Furthermore, for all $C \in (TT^*)'_{\mathrm{rel}}$ and all regular $M \in \mathbb{C}^{2\times 2}$ such that $\tau_M(C)$ is a bounded operator on $\mathcal{K}$, we have

$$T^*\tau_M(C) = \tau_M\left(\Theta_{\mathrm{rel}}(C)\right)T^*.$$

For all rational function h whose poles are in $\rho(C)$, i.e. $h \in \mathbb{C}(z) \cap B(\sigma(C))$, we have

$$T^*h(C) = h(\Theta_{\mathrm{rel}}(C))T^*.$$

Proof. Note that the sets $(TT^*)'_{\mathrm{rel}}$ and $(T^*T)'_{\mathrm{rel}}$, are not linear any more, since $\rho(C_1C_2)$ and $\rho(C_1 + C_2)$ could be empty for $C_1, C_2 \in (TT^*)'_{\mathrm{rel}}$. But still, we have $C^* \in (TT^*)'_{\mathrm{rel}}$ for $C \in (TT^*)'_{\mathrm{rel}}$. The resolvent set is not empty due to $\rho(C^*) = \overline{\rho(C)}$, cf. Lemma 1.2.23. Taking adjoints yields

$$C^*TT^* = (TT^*C)^* \supseteq (CTT^*)^* \supseteq TT^*C^*,$$

cf. Lemma 1.2.22. Analogous, $(T^*T)'_{\mathrm{rel}}$ is closed under taking adjoints.

For all $C \in (TT^*)'_{\mathrm{rel}}$ and all regular $M \in \mathbb{C}^{2\times 2}$, we have

$$\Theta_{\mathrm{rel}}(C) = \tau_{M^{-1}}\left(\Theta_{\mathrm{rel}}\left(\tau_M(C)\right)\right),$$

due to Lemma 3.2.1. This identity is especially useful if you choose M such that $\tau_M(C)$ is a bounded operator on $\mathcal{K}$. This can always be achieved by taking $M := \left(\begin{smallmatrix} 0 & 1 \\ 1 & -\mu \end{smallmatrix}\right)$ for $\mu \in \rho(C) \cap \mathbb{C}$. Note that the case $\rho(C) = \{\infty\}$ cannot occur, since the resolvent set is always an open subset of $\mathbb{C} \cup \{\infty\}$.

We claim that $\tau_M(C) \in (TT^*)'$, whenever $\tau_M(C)$ is a bounded operator on $\mathcal{K}$. Note that $C \in (TT^*)'_{\mathrm{rel}}$ gives $TT^*C \subseteq CTT^*$, which is, by Remark 3.2.5, equivalent to $TT^*\tau_M(C) \subseteq \tau_M(C)TT^*$. Under the assumption $\tau_M(C) \in \mathcal{B}(\mathcal{K})$, we have everywhere-defined operators on both sides and, therefore, equality holds, i.e. $\tau_M(C) \in (TT^*)'$. In this case, we can write

$$\Theta_{\mathrm{rel}}(C) = \tau_{M^{-1}}\left(\Theta\left(\tau_M(C)\right)\right). \tag{3.14}$$

Making use of Theorem 3.2.11, we get that $\Theta\left(\tau_M(C)\right) = \tau_M\left(\Theta_{\mathrm{rel}}(C)\right)$ is an element of $(T^*T)'$, i.e. $T^*T\,\tau_M\left(\Theta_{\mathrm{rel}}(C)\right) = \tau_M\left(\Theta_{\mathrm{rel}}(C)\right)T^*T$. Again, Remark 3.2.5 applied to $\tau_{M^{-1}}$ gives $T^*T\,\Theta_{\mathrm{rel}}(C) \subseteq \Theta_{\mathrm{rel}}(C)\,T^*T$, i.e. $\Theta_{\mathrm{rel}}(C) \in (T^*T)'_{\mathrm{rel}}$. This shows that $\Theta_{\mathrm{rel}} : (TT^*)'_{\mathrm{rel}} \to (T^*T)'_{\mathrm{rel}}$ is well-defined.

In order to show that Θ_{rel} is compatible with taking adjoints, fix $C \in (TT^*)'_{\mathrm{rel}}$ and choose M according to C, such that $\tau_M(C) \in (TT^*)'$. Equation (3.14), Lemma 1.2.23 and Theorem 3.2.11 give

$$\begin{aligned}\Theta_{\mathrm{rel}}(C)^* = \big(\tau_{M^{-1}} \circ \Theta \circ \tau_M(C)\big)^* = \tau_{\overline{M}^{-1}}\big(\Theta(\tau_M(C))\big)^* = \\ = \tau_{\overline{M}^{-1}}\big(\Theta\left(\tau_M(C)^*\right)\big) = \tau_{\overline{M}^{-1}}\left(\Theta\left(\tau_{\overline{M}}(C^*)\right)\right) = \Theta_{\mathrm{rel}}(C^*).\end{aligned}$$

In the same fashion, we get

$$\begin{aligned}\sigma(\Theta_{\mathrm{rel}}(C)) = \sigma\big(\tau_{M^{-1}} \circ \Theta \circ \tau_M(C)\big) = \phi_{M^{-1}}\Big(\sigma\big(\Theta(\tau_M(C))\big)\Big) \subseteq \\ \subseteq \phi_{M^{-1}}\big(\sigma\left(\tau_M(C)\right)\big) = \phi_{M^{-1}}\left(\phi_M\left(\sigma(C)\right)\right) = \sigma(C),\end{aligned}$$

cf. Proposition 1.2.15.

For the next statement, consider again $C \in (TT^*)'_{\mathrm{rel}}$ and $M \in \mathbb{C}^{2\times 2}$ such that $\tau_M(C) \in \mathcal{B}(\mathcal{K})$. Recall that we actually have $\tau_M(C) \in (TT^*)'$. Using equation (3.14) and the appropriate result from Theorem 3.2.11 gives

$$T^*\tau_M(C) = \Theta(\tau_M(C))T^* = \tau_M\big(\Theta_{\mathrm{rel}}(C)\big)T^*. \tag{3.15}$$

Finally, let h be a rational function whose poles are in $\rho(C) \subseteq \rho(\Theta_{\mathrm{rel}}(C))$. Both $h(C)$ and $h(\Theta_{\mathrm{rel}}(C))$ are bounded operators.

In the case $\infty \in \sigma(C)$, a partial fraction decomposition of h reveals that it is sufficient to prove

$$T^*(C-\mu)^{-1} = (\Theta_{\mathrm{rel}}(C) - \mu)^{-1}T^*, \tag{3.16}$$

for all $\mu \in \rho(C)$. For the constant term, which may appear in the partial fraction decomposition, we trivially have $T^*I = IT^*$. Equation (3.16) follows directly from (3.15), setting $M := \left(\begin{smallmatrix} 0 & 1 \\ 1 & -\mu \end{smallmatrix}\right)$

In the case $\infty \in \rho(C)$, i.e. $C \in \mathcal{B}(\mathcal{K})$, the situation is different, because h could have a pole at ∞. However, in this case equality holds in $TT^*C \subseteq CTT^*$, i.e. $C \in (TT^*)'$. The assertion follows now directly from Theorem 3.2.11.

□

3.3 Definitizable Linear Relations

Remark 3.3.1. For a linear relation A and a polynomial $p \in \mathbb{C}[z]$, we can define a linear relation $p(A)$ by using the definition of the sum and the composition of linear relations, as well as the multiplication by scalars, cf. Definition 1.2.6.

In fact, for $p(z) = \sum_{i=0}^{n} b_i z^i$ with $b_n \neq 0$, we define the $p(A) := \sum_{i=0}^{n} b_i A^i$. Clearly, we have $\operatorname{dom} p(A) = \bigcap_{i=0}^{n} \operatorname{dom} A^i = \operatorname{dom} A^n$.

Be warned that e.g. $p(A) + q(A) \neq (p+q)(A)$ if $\deg(p+q) < \max\{\deg p, \deg q\}$.

Definition 3.3.2. A self-adjoint linear relation $A \leq \mathcal{K} \times \mathcal{K}$ on a Krein space $\mathcal{K}$ is called *definitizable* if there exists a polynomial $p(z) \in \mathbb{C}\,[z]$ of degree n, such that

$$[y, x] \geq 0 \tag{3.17}$$

holds for all pairs $(x; y) \in p(A)$. In this case, p is called a *definitizing polynomial of* A.

The next lemma is a generalization of [Jon, Lemma 1].

Lemma 3.3.3. *Let $p \in \mathbb{C}[z]$ be a polynomial with $\deg p = n$, and let $A \leq \mathcal{K} \times \mathcal{K}$ be a self-adjoint linear relation on a Krein space $\mathcal{K}$ with non-empty resolvent set. For an arbitrary $\mu \in \rho(A)$, consider the rational function*

$$r(z) := \frac{p(z)}{(z-\mu)^n (z-\overline{\mu})^n}$$

and the corresponding bounded operator $r(A)$.

Then we state that $r(A)$ is positive, i.e. $[r(A)x, x] \geq 0$ for all $x \in \mathcal{K}$, if and only if A is definitizable, with definitizing polynomial p.

In this case, and under the additional assumption $p \in \mathbb{R}[z]$, we have $\sigma(A) \subseteq \overline{\mathbb{R}} \cup p^{-1}(\{0\})$. In particular, $\sigma(A) \backslash \overline{\mathbb{R}}$ consists of only finitely many points lying symmetrically with respect to the real axis.

Proof. For $x \in K$ we have $\big((A-\mu)^{-k-1}x; (A-\mu)^{-k}x + \mu(A-\mu)^{-k-1}x\big) \in A$, and in turn

$$\Big((A-\mu)^{-k}x; \sum_{j=0}^{k} \binom{k}{j} \mu^j (A-\mu)^{-j} x\Big) \in A^k \,.$$

Writing $p(z) = b_0 + \cdots + b_n z^n$ with $b_n \neq 0$, this gives

$$\Big((A-\mu)^{-n}x; \sum_{k=0}^{n} b_k \sum_{j=0}^{k} \binom{k}{j} \mu^j (A-\mu)^{-j-n+k} x\Big) \in p(A)\,. \tag{3.18}$$

To improve the readability, we denote the pair in (3.18) as $((A-\mu)^{-n}x; \Psi(A)x)$, with the bounded linear operator $\Psi(A) := \sum_{k=0}^{n} b_k \sum_{j=0}^{k} \binom{k}{j} \mu^j (A-\mu)^{-j-n+k}$.

In fact, $\Psi(z)$ for a complex variable z, is nothing but the partial fraction decomposition of the rational function $z \mapsto \frac{p(z)}{(z-\mu)^n}$, and $\Psi(A)$ is the corresponding bounded operator, cf. Corollary 2.1.11. What we have done so far, basically, was utilizing the fact $p(A)(A-$

$\mu)^{-n} \supseteq \Psi(A)$, which follows from $(A-\mu)(A-\mu)^{-1} \supseteq \mathrm{I}$. Because the functional calculus for rational functions is multiplicative, cf. Corollary 2.1.11, we have

$$[\Psi(A)x, (A-\mu)^{-n}x] = [(A-\bar{\mu})^{-n}\Psi(A)x, x] = [r(A)x, x] \quad \text{for all } x \in \mathcal{K}. \tag{3.19}$$

Now suppose that p is a definitizing polynomial of A. For all $x \in \mathcal{K}$, we have $[r(A)x, x] \geq 0$ due to (3.19) and by definition of a definitizing polynomial applied to the pairs in (3.18).

Before we prove the other implication, fix $z \in \sigma(A) \setminus \overline{\mathbb{R}}$ and assume $p(z) \neq 0$ for a *real* definitizing polynomial p. As we have just seen, $r(A)$ is positive for all $\mu \in \rho(A)$. In order to point out the dependence of μ, we are going to write $r_\mu := r$. Due to $r_\mu(A)^* = r_\mu^\#(A^*) = r_\mu(A)$, cf. Corollary 2.1.11, $r_\mu(A)$ is also self-adjoint. Theorem 3.2.6 gives $\sigma(r_\mu(A)) \subseteq \mathbb{R}$. Due to $\sigma(r_\mu(A)) = r_\mu(\sigma(A))$, cf. again Corollary 2.1.11, we deduce $r_\mu(z) \in \mathbb{R}$ for all $\mu \in \rho(A)$.

Express r_μ as

$$r_\mu(z) = \frac{p(z)(\bar{z}-\mu)^n(\bar{z}-\bar{\mu})^n}{|z-\mu|^{2n}|z-\bar{\mu}|^{2n}}. \tag{3.20}$$

The mapping $\rho(A) \setminus \{\mathbb{R}, z, \bar{z}\} \ni \mu \mapsto (\bar{z}-\mu)(\bar{z}-\bar{\mu})$, as a mapping from an open subset of $\mathbb{R}^2$ into $\mathbb{R}^2$, is a local diffeomorphism, due to the inverse function theorem. Hence, it is open, as well as $\rho(A) \setminus \{\mathbb{R}, z, \bar{z}\} \ni \mu \mapsto p(z)(\bar{z}-\mu)^n(\bar{z}-\bar{\mu})^n$. Since the denominator is real, this contradicts $r_\mu(z) \in \mathbb{R}$ for all $\mu \in \rho(A)$. We conclude $\sigma(A)\setminus\overline{\mathbb{R}} \subseteq p^{-1}(\{0\})$.

Now, for the proof of the other implication, assume that $r(A)$ is positive. We have to show that $[v, u] \geq 0$ for all $(u; v) \in p(A)$. For pairs of the form (3.18), this is immediately clear due to (3.19). We conclude the proof by showing that indeed *all* pairs of $(u; v) \in p(A)$ can be written as (3.18) for suitable $x \in \mathcal{K}$:

Fix $(u; v) \in p(A)$, and set $\Psi := \Psi(A)$. Since $u \in \operatorname{dom} p(A) = \operatorname{dom} A^n = \operatorname{dom}(A-\mu)^n = \operatorname{ran}(A-\mu)^{-n}$, we can find a $x' \in \mathcal{K}$ such that $u = (A-\mu)^{-n}x'$. So far, we have

$$((A-\mu)^{-n}x'; \Psi x') - (u; v) = (0; w) \in p(A),$$

for $w := \Psi x' - v \in \operatorname{mul} p(A) = \ker(A-\mu)^{-n}$. We want to achieve $w = 0$ by choosing $x' \in \mathcal{K}$ more carefully. Define $l := \min\{i \in \mathbb{N} : (A-\mu)^{-i}w = 0\}$, $l \leq n$, and let L be the linear span of the set of vectors

$$L := \operatorname{span}\{w, (A-\mu)^{-1}w, (A-\mu)^{-2}w, \dots, (A-\mu)^{-l+1}w\}.$$

Clearly, we have $L \leq \operatorname{mul} p(A)$. Note that these vectors are linearly independent: If we have

$$\sum_{i=0}^{l-1} \lambda_i (A-\mu)^{-i}w = 0, \; \lambda_i \in \mathbb{C}, \tag{3.21}$$

we can apply the linear operator $(A-\mu)^{-l+1}$ on both sides and get $\lambda_0(A-\mu)^{-l+1}w = 0$, which implies $\lambda_0 = 0$, due to the definition of l. Similarly, the operator $(A-\mu)^{-l+2}$ applied to equation (3.21) gives $\lambda_1 = 0$, and induction shows the asserted linear independence.

Since Ψ is just a linear combination of the operators $(A-\mu)^{-i}$, $i = 0, \dots n$, we have $\Psi(L) \leq L$. Furthermore, it is clear that Ψ commutes with the operator $(A-\mu)^{-1}$. Now we claim $\Psi|_L : L \to L$ to be bijective. Consider an arbitrary element of the kernel of $\Psi|_L$.

$$0 = \Psi\Big(\sum_{i=0}^{l-1} \lambda_i (A-\mu)^{-i} w\Big) = \sum_{i=0}^{l-1} \lambda_i \Psi\left((A-\mu)^{-i} w\right) = \sum_{i=0}^{l-1} \lambda_i (A-\mu)^{-i} \Psi w$$

This is similar to (3.21). In fact, we can repeat the same argumentation once we know $l = \min\{i \in \mathbb{N} : (A-\mu)^{-i}\Psi w = 0\}$. This is true, since we have $(A-\mu)^{-l}\Psi w = \Psi(A-\mu)^{-l} w = \Psi 0 = 0$ and $(A-\mu)^{-l+1}\Psi w = \Psi(A-\mu)^{-l+1} w = b_n (A-\mu)^{-l+1} w \neq 0$.

Hence, $\Psi|_L$ is bijective. So, we find an $m \in L \leq \operatorname{mul} p(A)$, such that $\Psi m = w$. Now $x := x' - m \in \mathcal{K}$ has the desired property,

$$((A-\mu)^{-n}x; \Psi x) - (u; v) = ((A-\mu)^{-n}x' - u; \Psi x' - w - v) = (0; 0).$$

This completes the proof, since all pairs $(u; v) \in p(A)$ can now be written as (3.18) and the positivity of $r(A)$ gives $[v, u] \geq 0$, due to (3.19). $\square$

Remark 3.3.4. In the following, we need the property $\sigma(A) \subseteq \overline{\mathbb{R}} \cup p^{-1}(\{0\})$, i.e. we would like to have a *real* definitizing polynomial. Given a definitizable linear relation $A \leq \mathcal{K} \times \mathcal{K}$ with a definitizing polynomial $p \in \mathbb{C}(z)$ - is it possible to pass over to a real definitizing polynomial?

With the notation from Lemma 3.3.3, let r denote the rational function corresponding to p. Due to

$$0 \leq [r(A)x, x] = [x, r^{\#}(A)x] = [r^{\#}(A)x, x],$$

we see that $r^{\#}(A)$ is also positive. Since $r^{\#}$ is the rational function corresponding to $p^{\#}$, Lemma 3.3.3 states that $p^{\#}$ is also definitizing for A.

We are tempted to proclaim that a real definitizing polynomial was found with $p + p^{\#}$. Obviously it is real, but it is not clear whether it is still definitizing for A or not.

In the special case $\deg(p + p^{\#}) = \max\{\deg p, \deg p^{\#}\}$, i.e. the leading coefficient of p is *not* an element of $i\mathbb{R}$, we can actually show that $p + p^{\#}$ is again definitizing for A.

In fact, let p and q be two arbitrary definitizing polynomials for A, and assume $\deg(p + q) = \max\{\deg p, \deg q\} =: n$. We claim that $p + q$ is definitizing for A. Without loss of generality assume $\deg q =: m \leq n = \deg p$. Consider

$$r(z) := \frac{p(z) + q(z)}{(z-\mu)^n (z-\bar{\mu})^n} = \frac{p(z)}{(z-\mu)^n (z-\bar{\mu})^n} + \frac{p(z)}{(z-\mu)^n (z-\bar{\mu})^n}.$$

By Lemma 3.3.3, it is sufficient to show that $r(A)$ is positive. Let r_p and r_q denote the rational functions corresponding to p and q, i.e.

$$r_p(z) := \frac{p(z)}{(z-\mu)^n (z-\bar{\mu})^n}, \quad r_q(z) := \frac{q(z)}{(z-\mu)^m (z-\bar{\mu})^m}.$$

Obviously, we have $r(z) = r_p(z) + r_q(z)(z-\mu)^{-(n-m)}(z-\bar{\mu})^{-(n-m)}$. Due to

$$[r(A)x, x] = [r_p(A)x, x] + [r_q(A)(A-\mu)^{-(n-m)}(A-\bar{\mu})^{-(n-m)}x, x] =$$

$$= [r_p(A)x, x] + [r_q(A)(A-\mu)^{-(n-m)}x, (A-\mu)^{-(n-m)}x] \geq 0,$$

$p+q$ is definitizable for A.

This proof fails if the leading coefficients cancel out. There may be definitizable linear relations which only have definitizing polynomials with leading coefficient in $i\mathbb{R}$.

Remark 3.3.5. In order to overcome the difficulties discussed in Remark 3.3.4, one could try to change the definition of a definitizable linear relation. In fact, we could define a self-adjoint linear relation $A \leq \mathcal{K} \times \mathcal{K}$ to be a *definitizable* if there exists a rational function $r \in \mathbb{C}(z) \cap B(\sigma(A))$ such that $r(A)$ is positive. The rational function r could be called *definitizing rational function* of A.

In this setting, $r^{\#}$ is also a definitizing rational function of A, since $r^{\#}(A)$ is again positive. We can consider the sum $r + r^{\#}$, which is definitizing for A as well, due to the linearity of the functional calculus for rational functions. Hence, without loss of generality, we can assume to $r = r^{\#}$.

It should also be possible to establish a spectral theorem for definitizable linear relations with this generalized notion.

3.4 The Functional Calculus

Let A be a definitizable linear relation on $\mathcal{K}$ with a real definitizing polynomial p of degree n. By Lemma 3.3.3 we know $\sigma(A) \subseteq \overline{\mathbb{R}} \cup p^{-1}(\{0\})$.

We fix the following *notation* for the rest of the chapter:

Denote by $N(p) := \{\alpha_1, \dots, \alpha_M\}$ the set of all distinct zeros of p which belong to $\sigma(A)$, and let γ_i be the multiplicity of the zero α_i, $i = 1, \dots, M$. The zeros of p which belong to $\rho(A)$ will only play a minor part. We denote them by $\beta_1, \dots, \beta_m$ and write η_i for the multiplicity of the zero β_i. We also set $q(z) := \prod_{i=1}^{m}(z - \beta_i)^{\eta_i}$. Then

$$p(z) = \prod_{i=1}^{m}(z - \beta_i)^{\eta_i} \prod_{i=1}^{M}(z - \alpha_i)^{\gamma_i} = q(z) \prod_{i=1}^{M}(z - \alpha_i)^{\gamma_i}.$$

Fix a non-real point $\lambda_0 \in \rho(A) \backslash N(p)$ and consider, as in Lemma 3.3.3,

$$r(z) := \frac{p(z)}{(z - \lambda_0)^n (z - \overline{\lambda_0})^n} \in \mathbb{C}(z).$$

We want to treat ∞ as a zero of r. Thus, we set $\alpha_{M+1} := \infty$, $\gamma_{M+1} := n$ and $N(r) = N(p) \cup \{\infty\}$.

Definition 3.4.1. For a Borel set $L \subseteq \mathbb{C} \cup \{\infty\}$, we denote by $B(L)$ the space of all bounded and measurable functions $g : L \to \mathbb{C}$.

We define the space of functions for which we will develop our functional calculus,

$$\mathcal{F}^I := \mathbb{C}(z) \cap B(\sigma(A) \cup I) + r \cdot B(\sigma(A) \cup I)_c.$$

Hereby, let I be a subset of $\rho(A)$, which is symmetric with respect to the real line, such that all points in $N(r)$ are accumulation points of $\sigma(A) \cup I$. We define $B(\sigma(A) \cup I)_c$ to be the set of all bounded and measurable functions on $\sigma(A) \cup I$ which are continuous at all points of $N(r)$. See Figure 3.1 and Figure 3.2 for a schematic illustration.

Remark 3.4.2. The aim of this remark is to give an idea why we introduced the set I.

The representation of a function $f \in \mathcal{F}^I$ as $f = h + rg$ with $h \in \mathbb{C}(z) \cap B(\sigma(A) \cup I)$ and $g \in B(\sigma(A) \cup I)_c$ is not unique, since we have

$$h + rg = (h + rk) + r(g - k),$$

for all $k \in \mathbb{C}(z) \cap B(\sigma(A) \cup I)$. Thus, given a $f \in \mathcal{F}^I$, we cannot expect to find a unique g. But, up to a rational function, the function g is uniquely determined,

$$h_1 + rg_1 = h_2 + rg_2 \;\Rightarrow\; g_1 - g_2 \in \mathbb{C}(z) \cap B(\sigma(A) \cup I). \tag{3.22}$$

This implication, which will later on guarantee that our functional calculus is well-defined, does not hold true if one does not introduce the set I. The main reason for the need of the set I is that the continuity of g at isolated points of $\sigma(A)$, in particular non-real points of $\sigma(A)$, is a trivial condition.

We also want to point out that the continuity of g at the points $N(r)$ is necessary, because we divide by r in the proof of the implication in line (3.22), which, by the way, is contained in the proof of Theorem 3.4.4, cf. (3.27).

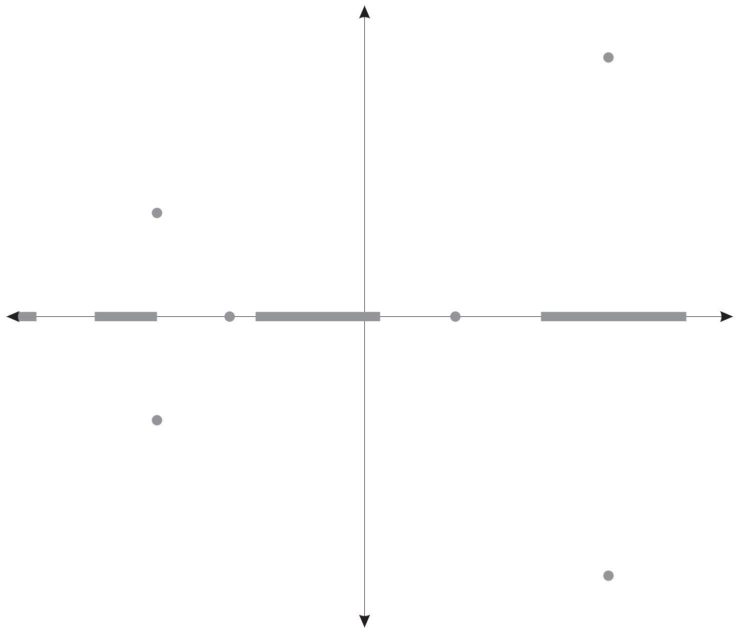

Figure 3.1: $\sigma(A)$

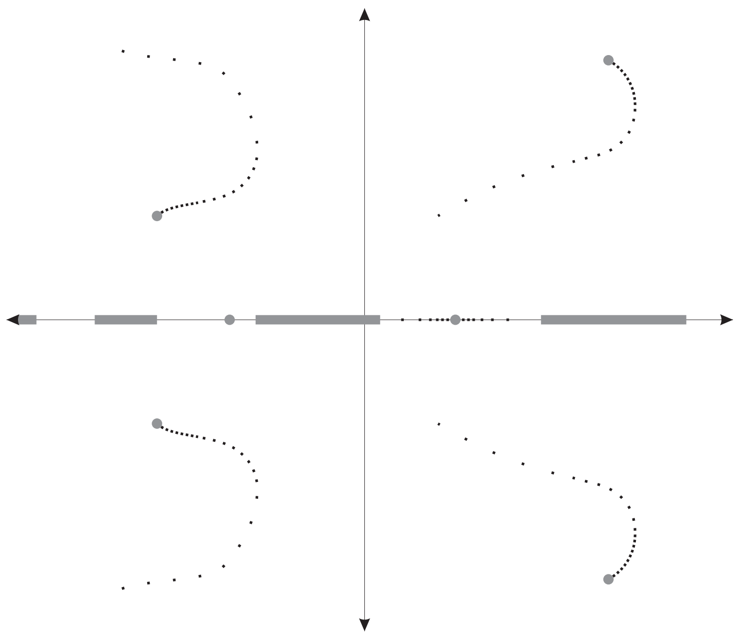

Figure 3.2: $\sigma(A) \cup I$

Remark 3.4.3. The space $\mathcal{F}^I$ is a vector space. Once endowed with the pointwise multiplication of functions and the mapping $\mathcal{F}^I \to \mathcal{F}^I, f \mapsto f^\#$, it actually becomes a *-algebra:

We check that $\mathcal{F}^I$ is closed under multiplication by taking $f_1 = h_1 + rg_1$, $f_2 = h_2 + rg_2 \in \mathcal{F}^I$ and calculating

$$f_1 f_2 = h_1 h_2 + r\,(g_1 h_2 + g_2 h_1 + r g_1 g_2)\,.$$

Clearly, $h_1 h_2$ is in $\mathbb{C}(z) \cap B(\sigma(A) \cup I)$. Since $g_1 h_2 + g_2 h_1 + r g_1 g_2$ is bounded and measurable on $\sigma(A) \cup I$ and is continuous at every point of $N(r)$, we conclude $f_1 f_2 \in \mathcal{F}$.

It is easy to check that $f \mapsto f^\#$ is a conjugate-linear involution. In order to show that $\mathcal{F}^I$ is closed under this mapping, consider

$$(h + rg)^\# = h^\# + r^\# g^\# = h^\# + r g^\#.$$

Note that we have $r = r^\#$, since both enumerator and denominator are real polynomials. Clearly, $h^\#$ has no poles in $\overline{\sigma(A) \cup I} = \sigma(A) \cup I$, and also $g^\#$ is still bounded and measurable.

Now, we will give the algebraic construction of the functional calculus.

Theorem 3.4.4. *Let A be a definitizable linear relation on a Krein space $\mathcal{K}$. Then, there exists a *-homomorphism $E^I : \mathcal{F}^I \to \mathcal{B}(\mathcal{K})$, which is an extension of Φ_{rat}, i.e. $E^I(h) = h(A)$ for all $h \in \mathbb{C}(z) \cap B(\sigma(A) \cup I)$.*

Moreover, we have $\sigma(\Theta_{\mathrm{rel}}(A)) \subseteq \operatorname{supp} E^I \subseteq \sigma(A)$. Hereby, Θ_{rel} was defined in Lemma 3.2.13 and the support *of E^I is defined as the smallest closed subset $C \subseteq \sigma(A) \cup I$ with the property*

$$f \in \mathcal{F}^I, \operatorname{supp} f \cap C = \emptyset \quad \Rightarrow \quad E^I(f) = 0. \tag{3.23}$$

Remark 3.4.5. Note that there exists the smallest closed set C with that property, namely the intersection of all closed C that fulfill (3.23). The fact that this intersection actually has property (3.23) is not trivial, but can be shown using a smooth partition of unity.

Proof. (of Theorem 3.4.4)
We define for $x, y \in \mathcal{K}$,

$$\langle x, y \rangle := [r(A)x, y].$$

The inner product $\langle ., . \rangle$ is positive semidefinite due to Lemma 3.3.3. We construct a Hilbert space by factoring out the neutral subspace $\mathcal{N} := \{x \in \mathcal{K} \mid \langle x, x \rangle = 0\}$, which is a closed linear subspace since it coincides with the isotropic part, cf. Corollary 1.1.3. Hence, the inner product, again denoted by $\langle ., . \rangle$, is well-defined on the factor space $\mathcal{K}/\mathcal{N}$ by

$$\langle x + \mathcal{N}, y + \mathcal{N} \rangle := \langle x, y \rangle \qquad \text{for } x, y \in \mathcal{K},$$

and is positive definite. The pre-Hilbert space $(\mathcal{K}/\mathcal{N}, \langle ., . \rangle)$ has a Hilbert space completion, denoted by $(\mathcal{V}, \langle ., . \rangle)$.

Now consider the canonical embedding $\iota : (\mathcal{K}, [., .]) \to (\mathcal{V}, \langle ., . \rangle)$ which denotes the composition of the canonical surjection $x \mapsto x + \mathcal{N}$ and the canonical embedding $\mathcal{K}/\mathcal{N} \to \mathcal{V}$ coming along with the completion. The linear operator ι is bounded and has dense range.

Thus, $T := \iota^* : (\mathcal{V}, \langle.,.\rangle) \to (\mathcal{K}, [.,.])$ is an injective operator $\ker T = \ker \iota^* = (\operatorname{ran} \iota)^\perp = \{0\}$, cf. Lemma 1.2.18. Furthermore, for x, y in $\mathcal{K}$ we have

$$[TT^*x, y] = \langle T^*x, T^*y\rangle = \langle \iota x, \iota y\rangle = \langle x, y\rangle = [r(A)x, y],$$

which gives $TT^* = r(A)$.

In Theorem 3.2.13, we have studied a way of moving linear relations on a Krein space to linear relations on a Hilbert space, with the help of a linear and injective map between these spaces. We apply Theorem 3.2.13 to the map $T : \mathcal{V} \to \mathcal{K}$ which we have just constructed and obtain the mapping $\Theta_{\text{rel}} : (TT^*)'_{\text{rel}} \to (T^*T)'_{\text{rel}}$.

We have to verify that A is in the domain of this mapping. The condition $\rho(A) \neq \emptyset$ is obviously true. It is left to show $TT^*A \subseteq ATT^*$, i.e. $r(A)A \subseteq Ar(A)$. Due to Remark 3.2.5 applied to $M := \left(\begin{smallmatrix} 0 & 1 \\ 1 & -\mu \end{smallmatrix}\right)$ for $\mu \in \rho(A)$, this is equivalent to

$$r(A)(A-\mu)^{-1} = r(A)\tau_M(A) \subseteq \tau_M(A)r(A) = (A-\mu)^{-1}r(A).$$

This holds true, since the functional calculus for rational functions is multiplicative, cf. Corollary 2.1.11.

We set $B := \Theta_{\text{rel}}(A)$. By Theorem 3.2.13, we have $B^* = \Theta_{\text{rel}}(A)^* = \Theta_{\text{rel}}(A^*) = B$, i.e. B is self-adjoint in the Hilbert space $\mathcal{V}$. Furthermore, we have $\sigma(B) \subseteq \sigma(A)$.

By the spectral theorem for self-adjoint linear relations on Hilbert spaces, Theorem 3.1.3, applied to B, we get a spectral measure F for $\langle \sigma(B), \mathfrak{B}(\sigma(B)), \mathcal{V}\rangle$, such that

$$r(B) = \int_{\sigma(B)} r(t)\, dF(t) \tag{3.24}$$

holds for all rational functions $r \in C(z) \cap B(\sigma(B))$. Recall that $B(\sigma(B))$ denotes the set of all bounded and measurable functions $g : \sigma(B) \to \mathbb{C}$. Now consider

$$G : \left\{ \begin{array}{rcl} B(\sigma(B)) & \to & \mathcal{B}(\mathcal{K}) \\ g & \mapsto & T \int_{\sigma(B)} g(t)\, dF(t)\, T^* . \end{array} \right. \tag{3.25}$$

Clearly, G is a well-defined linear operator, which is even bounded with norm less or equal to $\|T\|^2$. Moreover, G is compatible with taking the complex conjugate of a function, with taking adjoints respectively, meaning

$$G(\overline{g}) = T \int_{\sigma(\Theta(A))} \overline{g}(t)\, dF(t)\, T^* = T \left(\int_{\sigma(\Theta(A))} g(t)\, dF(t) \right)^* T^* = G(g)^*.$$

Furthermore, take a rational function $h \in \mathbb{C}(z) \cap B(\sigma(A))$ and look at $G(h)$. Equation (3.24) and the last statement from Theorem 3.2.13 reveal

$$G(h) = Th(B)T^* = TT^*h(A) = r(A)h(A), \tag{3.26}$$

for all $h \in \mathbb{C}(z) \cap B(\sigma(A))$.

Finally, we are able to define our functional calculus. We write $f \in \mathcal{F}^I$ as $f = h + rg$, with $h \in \mathbb{C}(z) \cap B(\sigma(A) \cup I)$ and $g \in B(\sigma(A) \cup I)_c$, and define

$$E^I(f) := h(A) + G(g|_{\sigma(B)}).$$

We have to show that E is well-defined. For $f = h_1 + rg_1 = h_2 + rg_2$, we need to verify $h_1(A) + G(g_1) = h_2(A) + G(g_2)$. Since both the rational functional calculus and G are linear, it is sufficient to show $h(A) + G(g|_{\sigma(B)}) = 0$, whenever $h + rg = 0$.

So let us assume $h + rg = 0$. The equality

$$g(z) = -\frac{h(z)}{r(z)} \tag{3.27}$$

clearly holds for all z, with $r(z) \neq 0$, i.e. $z \in (\sigma(A) \cup I)\backslash N(r)$.

We want to show that the rational function $-\frac{h}{r}$ does not have any poles in $\sigma(A) \cup I$. All potential poles which belong to $\sigma(A) \cup I$ have to originate from zeros of r. Assume that $\alpha \in N(r)$ is a pole of $-\frac{h}{r}$. Since α is a accumulation point of $\sigma(A) \cup I$, we can find a sequence $z_n \in \sigma(A) \cup I$ with $z_n \to \alpha$ for $n \to \infty$. We get

$$|g(z_n)| = \left|\frac{h(z_n)}{r(z_n)}\right| \overset{n\to\infty}{\longrightarrow} \infty,$$

which contradicts the boundedness of g.

The right-hand side of (3.27) is a rational function, whose poles are not in $\sigma(A) \cup I$, and is therefore continuous on $\sigma(A) \cup I$. By definition, the left-hand side of (3.27) is continuous at all points of $N(r)$. Thus, the identity (3.27) holds actually for all $z \in \sigma(A) \cup I$.

Since g turns out to be equal to a rational function, in particular on $\sigma(B) \subseteq \sigma(A) \cup I$, we can use (3.26) to get

$$h(A) + G(g|_{\sigma(B)}) = h(A) - G\big(\tfrac{h}{r}|_{\sigma(B)} \big) = h(A) - r(A) \left(\tfrac{h}{r}\right)(A) = 0. \tag{3.28}$$

Thus $E^I : \mathcal{F}^I \to \mathcal{B}(\mathcal{K})$ is a well-defined linear operator, which obviously is an extension of Φ_{rat}.

Next we prove that E^I is a *-algebra homomorphism. It is compatible with taking the complex conjugate of a function, with taking adjoints respectively, since for $h + rg = f \in \mathcal{F}^I$ we have

$$E(f^{\#}) = E(h^{\#} + rg^{\#}) = h^{\#}(A) + G(\overline{g}|_{\sigma(B)}) = h(A)^* + G(g|_{\sigma(B)})^* = E(f)^*.$$

In particular, $E^I(f)$ is self-adjoint for every real-valued $f \in \mathcal{F}^I$.

In order to show that E^I is multiplicative, take two functions $f_1 = h_1 + rg_1$ and $f_2 = h_2 + rg_2$, with $f_1, f_2 \in \mathcal{F}^I$. The decomposition of the product f_1f_2 into

$$f_1f_2 = h_1h_2 + r\,(g_1h_2 + g_2h_1 + rg_1g_2)$$

gives

$$\begin{aligned} E(f_1f_2) &= (h_1h_2)(A) + G(g_1h_2|_{\sigma(B)} + g_2h_1|_{\sigma(B)} + rg_1g_2|_{\sigma(B)}) = \\ &= h_1(A)h_2(A) + G(g_1h_2|_{\sigma(B)}) + G(g_2h_1|_{\sigma(B)}) + G(rg_1g_2|_{\sigma(B)}). \end{aligned} \tag{3.29}$$

On the other side, we have

$$E(f_1)E(f_2) = \big[h_1(A) + G(g_1|_{\sigma(B)})\big]\,\big[h_2(A) + G(g_2|_{\sigma(B)})\big] =$$

$$= h_1(A)h_2(A) + G(g_1|_{\sigma(B)})h_2(A) + \tag{3.30}$$
$$+ h_1(A)G(g_2|_{\sigma(B)}) + G(g_1|_{\sigma(B)})G(g_2|_{\sigma(B)}). \tag{3.31}$$

In order to see that the lines (3.29) and (3.31) actually coincide, we need two more properties.

First, we argue that $G(gh|_{\sigma(B)}) = G(g|_{\sigma(B)})h(A) = h(A)G(g|_{\sigma(B)})$ for all $g \in B(\sigma(A))$ and $h \in \mathbb{C}(z) \cap B(\sigma(A))$. Using $h(B)T^* = T^*h(A)$, cf. Theorem 3.2.13, we obtain

$$\begin{aligned} G(gh|_{\sigma(B)}) &= T \int_{\sigma(B)} g(t)\, dF(t) \int_{\sigma(B)} h(t)\, dF(t)\, T^* = \\ &= T \int_{\sigma(B)} g(t)\, dF(t)\, h(B)\, T^* = T \int_{\sigma(B)} g(t)\, dF(t)\, T^*\, h(A) = \\ &= G(g|_{\sigma(B)})h(A). \end{aligned} \tag{3.32}$$

In order to show also $G(gh|_{\sigma(B)}) = h(A)G(g|_{\sigma(B)})$, we take adjoints in $h(B)T^* = T^*h(A)$ and get

$$Th^{\#}(B) = h^{\#}(A)T, \tag{3.33}$$

for all $h \in \mathbb{C}(z) \cap B(\sigma(A))$. Now we can more or less repeat (3.32), but pull out $h(A)$ on the left-hand side,

$$\begin{aligned} G(gh|_{\sigma(B)}) &= T \int_{\sigma(B)} h(t)\, dF(t) \int_{\sigma(B)} g(t)\, dF(t)\, T^* = \\ &= Th(B) \int_{\sigma(B)} g(t)\, dF(t)\, T^* = h(A)\, T \int_{\sigma(B)} g(t)\, dF(t)\, T^* = \\ &= h(A)G(g|_{\sigma(B)}). \end{aligned} \tag{3.34}$$

Secondly, we claim $G(rg_1g_2|_{\sigma(B)}) = G(g_1|_{\sigma(B)})G(g_2|_{\sigma(B)})$ for all $g_1, g_2 \in B(\sigma(A))$. We apply (3.33) to $h^{\#} = r$ and get

$$Tr(B) = r(A)T = TT^*T.$$

Since T is injective, this gives $r(B) = T^*T$. The claim follows from

$$\begin{aligned} G(rg_1g_2|_{\sigma(B)}) &= T \int_{\sigma(B)} g_1(t)r(t)g_2(t)\, dF(t)\, T^* = \\ &= T \int_{\sigma(B)} g_1(t)\, dF(t)\, r(B) \int_{\sigma(B)} g_2(t)\, dF(t)\, T^* = \\ &= T \int_{\sigma(B)} g_1(t)\, dF(t)\, T^*T \int_{\sigma(B)} g_2(t)\, dF(t)\, T^* = \\ &= G(g_1|_{\sigma(B)})G(g_2|_{\sigma(B)}). \end{aligned} \tag{3.35}$$

Finally, (3.32), (3.34) and (3.35) imply that (3.29) and (3.31) coincide, i.e. E is multiplicative.

Lastly, we want to show $\sigma(\Theta_{\mathrm{rel}}(A)) \subseteq \operatorname{supp} E \subseteq \sigma(A)$.

To this end, take a function $f \in \mathcal{F}^I$ with $\operatorname{supp} f \cap \sigma(A) = \emptyset$. Since $\mathbb{C} \cup \{\infty\}$ is normal, we can find an open superset $U \supseteq \sigma(A)$ such that even $\operatorname{supp} f \cap U = \emptyset$, i.e. $f(z) = 0$ for all $z \in U$. Now, the set $I' := I \cap U$ has again the property that all points of $N(r)$ are accumulation points of $\sigma(A) \cup I'$. Hence, we can consider $E^{I'} : \mathcal{F}^{I'} \to \mathcal{B}(\mathcal{K})$. Note that the function $f|_{\sigma(A)\cup I'} \in \mathcal{F}^{I'}$ is zero on $\sigma(A) \cup I'$. Hence, we are exactly in the situation of (3.27). Just as in (3.28), we deduce $E^{I'}(f|_{\sigma(A)\cup I'}) = 0$, and also get $E^I(f) = 0$, cf. Remark 3.4.6.

Now, we take an arbitrary closed $C \subseteq \sigma(A) \cup I$, such that property (3.23) holds.

Fix a $\lambda \in (\sigma(A) \cup I)$ which is *not* in the closed set C, and choose an open set $\Delta \subseteq \sigma(A) \cup I$, open in the subspace topology of $\sigma(A) \cup I$, with $\lambda \in \Delta$ and $\Delta \cap C = \emptyset$. Since $N(r)$ contains only finitely many points, we can also find such a Δ with $\partial\Delta \cap N(r) = \emptyset$. The function $z \mapsto \mathbb{1}_\Delta(z)$ is obviously bounded and measurable, and due to $\partial\Delta \cap N(r) = \emptyset$ also continuous at all points in $N(r)$. This gives that $f := r\mathbb{1}_\Delta \in \mathcal{F}^I$ for all possible sets I, with $\operatorname{supp}(f) \cap C = \Delta \cap C = \emptyset$. So

$$0 = E(f) = G(\mathbb{1}_\Delta) = TF\left(\Delta \cap \sigma(\Theta_{\mathrm{rel}}(A))\right) T^*.$$

Since T is injective, we have $0 = F\left(\Delta \cap \sigma(\Theta_{\mathrm{rel}}(A))\right) T^*$. Recalling that $T^* = \iota$ has dense range implies $F\left(\Delta \cap \sigma(\Theta_{\mathrm{rel}}(A))\right) = 0$. We conclude $\Delta \cap \sigma(\Theta_{\mathrm{rel}}(A)) = \emptyset$, or equivalently $\Delta \subseteq \rho(\Theta_{\mathrm{rel}}(A))$, since the spectral measure of a non-empty set, which is open in the subspace topology of $\sigma(\Theta_{\mathrm{rel}}(A))$, always gives an operator unequal to the null operator, which can be seen by Urysohn's Lemma. In particular, we have $\lambda \in \rho(\Theta_{\mathrm{rel}}(A))$, which implies $C \supseteq \sigma(\Theta_{\mathrm{rel}}(A))$.

□

Remark 3.4.6. Does the functional calculus $E^I : \mathcal{F}^I \to \mathcal{B}(\mathcal{K})$ depend on the set I? Obviously, at least the domain, $\mathcal{F}^I$, depends on I.

We recall that for $h + rg = f \in \mathcal{F}^I$ we defined $E^I(f) = h(A) + G(g|_{\sigma(B)})$. The expression $h(A)$ is well-defined for all $h \in \mathbb{C}(z) \cap B(\sigma(A))$ and independent of the choice of I. The term $G(g|_{\sigma(B)})$ does obviously depend on the values of g on $\sigma(B)$ only. So $G(g)$ is also independent of the choice of I.

Note that it is not possible, at least not for general sets I, to pass over to equivalence classes of functions by defining $f \sim g$ if f coincides with g on $\sigma(A) \cup (I_f \cap I_g)$, since the intersection of I_f and I_g may be empty.

Still, for $I_1 \subseteq I_2$ and $f \in \mathcal{F}^{I_2}$, we have $f|_{\sigma(A)\cup I_1} \in \mathcal{F}^{I_1}$ and

$$E^{I_1}(f|_{\sigma(A)\cup I_1}) = E^{I_2}(f).$$

3.5 The Continuity of the Functional Calculus

We are going to prove that our functional calculus is continuous, at least on a subclass $\mathcal{F}_{U,\epsilon}$ with respect to a certain norm $\|.\|_{U,\epsilon}$.

We want to recall some notation. Still, $A \leq \mathcal{K} \times \mathcal{K}$ denotes a definitizable linear relation on a Krein space $\mathcal{K}$ with a real definitizing polynomial p with degree n. We split the set of all zeros of p into those zeros which belong to $\sigma(A)$, denoted by α_i for $i = 1, \dots M$ with multiplicity γ_i, and the remaining zeros of A, which belong to $\rho(A)$, denoted by β_i for $i = 1, \dots, m$ with multiplicity η_i. We set $N(p) := \{\alpha_1, \dots, \alpha_M\}$, and we defined the function $q(z) := \prod_{i=1}^{m}(z - \beta_i)^{\eta_i}$ and $r(z) := \frac{p(z)}{(z-\lambda_0)^n(z-\bar{\lambda}_0)^n}$, for $\lambda_0 \in \rho(A)$.

In the sequel, we will need the functions $f \in B(\sigma(A))$ to be not only continuous, but even differentiable, at all points of $N(r)$. In order to achieve that the continuity is a non-trivial condition, we introduced a set I, such that all points of $N(r)$ are accumulation points of $\sigma(A) \cup I$.

Now, the set I even needs to contain some form of neighborhoods of these points, in order to be able to talk about differentiability. We could ask for open neighborhoods in $\mathbb{C}$ and take holomorphic functions, but we only need a weaker notion of differentiability.

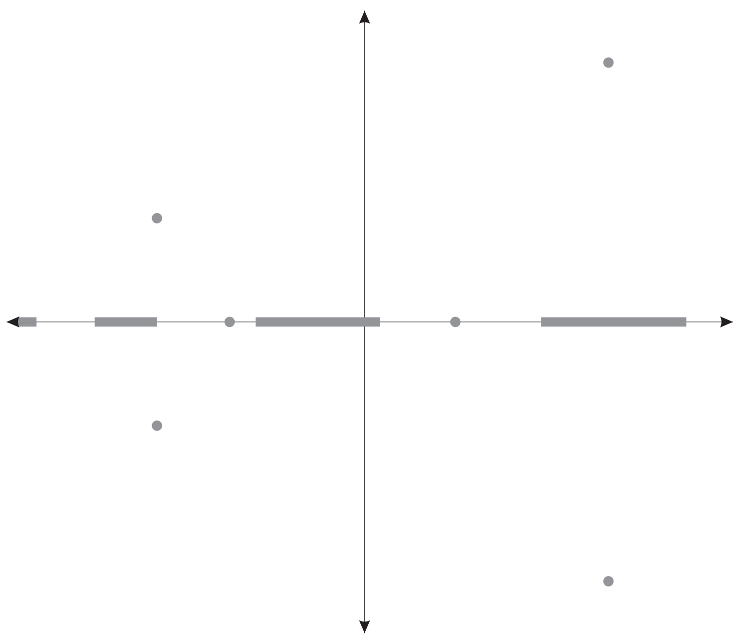

Figure 3.3: $\sigma(A)$

Definition 3.5.1. Set $I_{\alpha_i} := \{\alpha_i + t : t \in \mathbb{R}, |t| < \epsilon\}$, for $i = 1, \dots, M$, $\epsilon > 0$. Additionally, set $I_\infty := \overline{\mathbb{R}} \setminus [-\frac{1}{\epsilon}, \frac{1}{\epsilon}]$. We assume that $\epsilon > 0$ is small enough such that the sets I_{α_i} do not overlap and such that $\frac{1}{\epsilon} > \max_{i=1,\dots,M} |\alpha_i| + 1$. By $I(\epsilon)$ we denote the union $I(\epsilon) := \bigcup_{i=1}^{M+1} I_{\alpha_i}$. We further assume that $I(\epsilon)$ does not contain any zeros

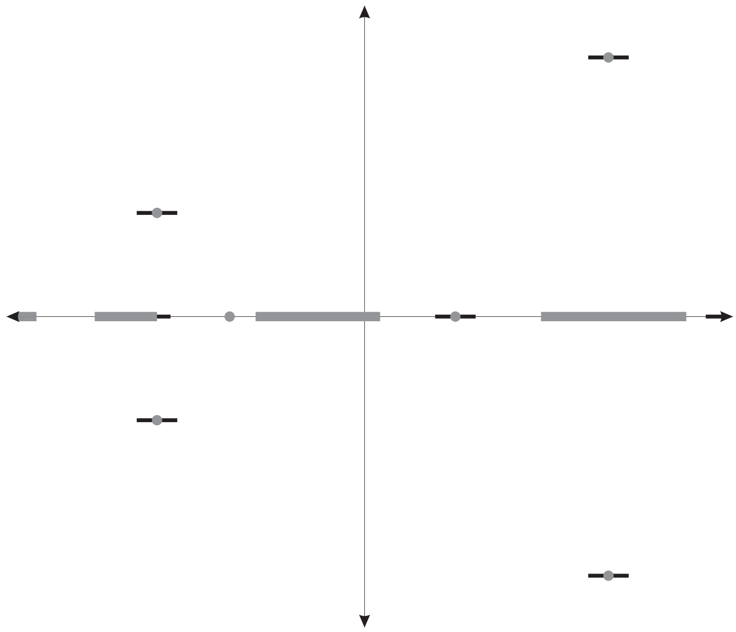

Figure 3.4: $\sigma(A) \cup I(\epsilon)$

of p which belong to $\rho(A)$, which is the case for sufficiently small ϵ. See Figure 3.3 and Figure 3.4 for a schematic illustration.

Define the charts $\phi_i : (-\epsilon, \epsilon) \to \mathbb{C} \cup \{\infty\}, t \mapsto \alpha_i + t$ for $i = 1, \ldots, M$ as well as $\phi_{M+1} : (-\epsilon, \epsilon) \to \mathbb{C} \cup \{\infty\}, t \mapsto \frac{1}{t}$. Note $\operatorname{ran} \phi_i = I_{\alpha_i}$.

We define $\mathcal{F}_{U,\epsilon}$ as the set of all functions $f \in B\left(\sigma(A) \cup I(\epsilon)\right)_c$, such that f is differentiable at all points in $N(r)$, in the following sense:

$$\mathcal{F}_{U,\epsilon} := \left\{ f \in B\left(\sigma(A) \cup I(\epsilon)\right)_c \;\middle|\; f \circ \phi_i \in C^{\gamma_i}(-\epsilon, \epsilon) \text{ for } i = 1, \ldots, M+1 \right\}.$$

We declare a norm on the linear space $\mathcal{F}_{U,\epsilon}$, by setting for $f \in \mathcal{F}_{U,\epsilon}$

$$\|f\|_{U,\epsilon} := \max \left(\left\{ \|f\|_\infty \right\} \cup \left\{ \left\| (f \circ \phi_i)^{(l)} \right\|_\infty : i = 1, \ldots, M+1; l = 1, \ldots, \gamma_i \right\} \right).$$

Here we mean by $\|.\|_\infty$ on the left-hand side the supremum norm on $\sigma(A) \cup I(\epsilon)$ and on the right-hand side the supremum norm on the interval $(-\epsilon, \epsilon)$.

Finally, we set

$$\mathcal{E} := \left\{ \frac{q(z)a(z)}{(z-\lambda_0)^n (z-\overline{\lambda_0})^n} \;\middle|\; a(z) \in \mathbb{C}[z], \deg a < n - \deg q \right\} \dot{+}$$
$$\left\{ r(z)b(z) \;\middle|\; b(z) \in \mathbb{C}[z], \deg b \le n, b(0) = 0 \right\}.$$

Remark 3.5.2. We want to show that the sum, in the definition of $\mathcal{E}$, is actually a direct sum. Suppose

$$\frac{q(z)a(z)}{(z-\lambda_0)^n(z-\overline{\lambda_0})^n} = r(z)b(z),$$

or equivalently $q(z)a(z) = p(z)b(z)$. Note that the left-hand side is a polynomial of degree at most $n-1$. This means that b has to be the zero-polynomial, because otherwise the right-hand side would be a polynomial of degree at least n, which implies that a and, therefore, b are zero.

Remark 3.5.3. We do not want to introduce the general notion of a differentiable manifold. However, there is no way around this topic, since the domain of the functions $f \in \mathcal{F}^{I(\epsilon)}$ is a subset of the manifold $\mathbb{C} \cup \{\infty\}$. The subspace $\mathcal{F}_{U,\epsilon}$ consists, basically, of those functions in $\mathcal{F}^{I(\epsilon)}$ which are γ_i-times continuously differentiable in a neighborhood of α_i for $i = 1, \ldots, M+1$. A satisfying answer to the question of how to differentiate f at ∞, is given by the theory of differentiable manifolds.

Of course, it is possible to choose a compatible atlas and adopt the definition of $\mathcal{F}_{U,\epsilon}$ and $\|.\|_{U,\epsilon}$ accordingly.

Definition 3.5.4. We define an equivalence relation on $\bigcup_{\epsilon>0} \mathcal{F}^{I(\epsilon)}$ via

$$f \sim g :\Longleftrightarrow \; f(z) = g(z) \text{ for all } z \in \sigma(A) \cup (I(\epsilon_f) \cap I(\epsilon_g)).$$

An equivalence class is also called *germ of a function*. We denote the set of all germs by

$$\mathcal{F} := \left(\bigcup_{\epsilon>0} \mathcal{F}^{I(\epsilon)} \right) \Big/ \sim .$$

Now, we can define $E : \mathcal{F} \to \mathcal{B}(\mathcal{K})$ by setting $E([f]_\sim) := E^{I(\epsilon)}(f)$, where f denotes any representative of the equivalence class $[f]_\sim$. Note that E is well-defined due to Remark 3.4.6.

Remark 3.5.5. All rational functions in $\mathcal{E}$ have at most one pole, namely λ_0. An important property of the set $\mathcal{E}$ is that it is a *finite dimensional* subspace of $\mathbb{C}(z) \cup B(\sigma(A) \cup I(\epsilon))$. Thus all norms are equivalent on $\mathcal{E}$ and every linear operator on $\mathcal{E}$ is continuous.

We are going to replace the space $\mathcal{F}^{I(\epsilon)} = \mathbb{C}(z) \cap B(\sigma(A) \cup I(\epsilon)) + r \cdot B(\sigma(A) \cup I(\epsilon))_c$ by $\mathcal{E} + r \cdot B(\sigma(A) \cup I(\epsilon))_c$, which helps to obtain continuity. Proposition 3.5.6 tells us, among other things, that this replacement does not make the space of functions smaller.

Proposition 3.5.6. *The following holds true.*

(i) $\mathcal{F}_{U,\epsilon} \subseteq \mathcal{E} + r \cdot B(\sigma(A) \cup I(\epsilon))_c$

(ii) $\mathcal{F}^{I(\epsilon)} = \mathcal{E} \dot{+} r \cdot B(\sigma(A) \cup I(\epsilon))_c$

(iii) *Let λ denote the bijection $\lambda : \mathcal{F}^{I(\epsilon)} \to \mathcal{E} \times B(\sigma(A) \cup I(\epsilon))_c$, which maps $f = h + rg$ to $\lambda(f) = (h, g)$. Then, the restriction $\lambda|_{\mathcal{F}_{U,\epsilon}}$ is continuous.*

The space $\mathcal{F}_{U,\epsilon}$ is equipped with the norm $\|.\|_{U,\epsilon}$, cf. Definition 3.5.1. The product space $\mathcal{E} \times B(\sigma(A) \cup I(\epsilon))_c$ is endowed with the sum norm, where both $\mathcal{E}$ and $B(\sigma(A) \cup I(\epsilon))_c$ carry the supremum norm on $\sigma(A) \cup I(\epsilon)$.

(*iv*) *For a Borel set* $\Delta \subseteq \mathbb{C} \cup \{\infty\}$, *we have* $\mathbb{1}_\Delta \in \mathcal{F}$ *if and only if*

$$\partial_\epsilon \left(\Delta \cap (\sigma(A) \cup I(\epsilon))\right) = \emptyset$$

for some $\epsilon > 0$. *The symbol* ∂_ϵ *refers to the topological boundary in the topological subspace* $\sigma(A) \cup I(\epsilon)$ *of* $\mathbb{C} \cup \{\infty\}$.

For the proof, we need the following Lemma.

Lemma 3.5.7. *Let* $I \subseteq \mathbb{R}$ *be an interval,* $\alpha \in I$ *and* $\gamma \in \mathbb{N}$. *Every bounded* $f : \mathrm{dom}(f) \to \mathbb{C}$, *such that* $I \subseteq \mathrm{dom}(f) \subseteq \mathbb{C} \cup \{\infty\}$, $0 < \delta := \inf_{z \in \mathrm{dom}(f) \setminus I} |\alpha - z|$ *and* $f \in C^\gamma(I)$, *can be written as*

$$f(z) = h(z) + (z - \alpha)^\gamma g(z), \; z \in \mathrm{dom}(f) \setminus \{\infty\},$$

with the $(\gamma - 1)$*-th Taylor polynomial* h *of* f *at the point* α, *i.e.*

$$h(z) := \sum_{l=0}^{\gamma-1} \tfrac{f^{(l)}(\alpha)}{l!} (z - \alpha)^l,$$

and a bounded function $g : \mathrm{dom}(f) \to \mathbb{C}$, *which is continuous on* I *with* $\|g|_I\|_\infty \leq \frac{2}{\gamma!} \|f|_I^{(\gamma)}\|_\infty$.

Proof. First consider a real-valued function f. We have to show that $g(z) := \frac{f(z)-h(z)}{(z-\alpha)^\gamma}$, $z \in \mathrm{dom}(f) \setminus \{\alpha\}$ can be extended to $\mathrm{dom}(f)$, such that $g|_I$ is continuous. Notice, that if $\infty \in \mathrm{dom}(f)$, the above definition conveys $g(\infty) := 0$, since f is bounded and $\deg h < \gamma$.

Taylor's Theorem, in fact the Lagrange form of the remainder, states that for every $t \in I$ there exists a $\xi \in I$ between t and α, such that $g(t) = \frac{1}{\gamma!} f^{(\gamma)}(\xi)$. From $\lim_{t \to \alpha} g(t) = \frac{1}{\gamma!} f^{(\gamma)}(\alpha)$, we conclude that g can be continuously extended to I.

The estimation $\|g|_I\|_\infty \leq \frac{1}{\gamma!} \|f|_I^{(\gamma)}\|_\infty$ is also a consequence of the Lagrange form of the remainder. The boundedness of g on $\mathrm{dom}(f) \setminus I$ follows from

$$\begin{aligned}
\|g|_{\mathrm{dom}(f)\setminus I}\|_\infty &= \sup_{z \in \mathrm{dom}(f)\setminus I} \left| \frac{f(z)}{(z-\alpha)^\gamma} - \frac{h(z)}{(z-\alpha)^\gamma} \right| \\
&\leq \frac{\|f\|_\infty}{\delta^\gamma} + \sup_{z \in \mathrm{dom}(f)\setminus I} \left| \frac{h(z)}{(z-\alpha)^\gamma} \right| \\
&\leq \frac{\|f\|_\infty}{\delta^\gamma} + \sum_{k=0}^{\gamma-1} \frac{1}{k!} \frac{|f^{(k)}(\alpha)|}{\delta^{\gamma-k}}. \qquad (3.36)
\end{aligned}$$

In the general case of a complex-valued f, we apply the first part of this proof to $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ and get $\mathrm{Re}\, f(z) = h_1(z) + (z-\alpha)^\gamma g_1(z)$ and $\mathrm{Im}\, f(z) = h_2(z) + (z-\alpha)^\gamma g_2(z)$ respectively. Taking the sum of the first and i-times the second equation, gives the desired decomposition $h := h_1 + i h_2$, $g := g_1 + i g_2$. Note that $f^{(l)} = \mathrm{Re}\, f^{(l)} + i\, \mathrm{Im}\, f^{(l)}$. One can easily give a bound of g on I

$$\|g|_I\|_\infty = \|g_1|_I + i g_2|_I\|_\infty \leq \frac{2}{\gamma!} \|f^{(\gamma)}\|_\infty.$$

To verify that g is bounded on $\mathrm{dom}(f) \setminus I$, use the boundedness of g_1 and g_2, or simply repeat the calculation done in (3.36). □

Proof. (of Proposition 3.5.6) Take an arbitrary $f \in \mathcal{F}_{U,\epsilon}$. We are going to construct a decomposition $f = h + rg$ with $h \in \mathcal{E}$ and $g \in B(\sigma(A) \cup I(\epsilon))_c$.

We are going to use the functions $\varphi_i : \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}, z \mapsto \alpha_i + z$ for $i = 1, \dots, M$ as well as $\varphi_{M+1} : \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}, z \mapsto \frac{1}{z}$.

We apply Lemma 3.5.7 to the function $\tilde{f} := f \circ \varphi_i$ with $\mathrm{dom}(\tilde{f}) := \varphi_i^{-1}(\sigma(A) \cup I(\epsilon))$, $0 \in (-\epsilon, \epsilon) \subseteq \mathrm{dom}(\tilde{f})$. As required, we have $\tilde{f} \in C^\gamma(-\epsilon, \epsilon)$ for $i = 1, \dots M$ and $\gamma = 1, \dots, \gamma_i$. Moreover, due to our assumption on I_{α_i}, the condition $0 < \delta := d\left(\alpha_i, (\sigma(A) \cup I(\epsilon)) \setminus I_{\alpha_i}\right)$ is satisfied and also holds true when everything is translated by α_i.

Lemma 3.5.7 gives a polynomial $\tilde{h}_{i,\gamma}$ of degree at most $\gamma - 1$ and a bounded function $\tilde{g}_{i,\gamma} : \mathrm{dom}(\tilde{f}) \to \mathbb{C}$, which is continuous on $(-\epsilon, \epsilon)$, such that

$$\tilde{f}(w) = \tilde{h}_{i,\gamma}(w) + w^\gamma \tilde{g}_{i,\gamma}(w), \quad w \in \mathrm{dom}(\tilde{f})\setminus\{\infty\},$$

or, after the substitution $w = \varphi_i^{-1}(z) = z - \alpha_i$,

$$f(z) = \tilde{h}_{i,\gamma}(z - \alpha_i) + (z - \alpha_i)^\gamma \tilde{g}_{i,\gamma}(z - \alpha_i), \quad z \in (\sigma(A) \cup I(\epsilon))\setminus\{\infty\},$$

holds. Setting $h_{i,\gamma}(z) := \tilde{h}_{i,\gamma}(z - \alpha_i)$ and $g_{i,\gamma}(z) := \tilde{g}_{i,\gamma}(z - \alpha_i)$ gives

$$f(z) = h_{i,\gamma}(z) + (z - \alpha_i)^\gamma g_{i,\gamma}(z), \quad z \in (\sigma(A) \cup I(\epsilon)) \setminus\{\infty\}, \tag{3.37}$$

with $\deg h_{i,\gamma} < \gamma$ and a bounded function $g_{i,\gamma} : \sigma(A) \cup I(\epsilon) \to \mathbb{C}$, which is continuous on $\varphi_i((-\epsilon, \epsilon)) = I_{\alpha_i}$.

To show that $g_{i,\gamma}$ is also continuous at the points α_j for $j \in \{1, \dots, M\}$, $j \neq i$, simply rearrange (3.37) to

$$g_{i,\gamma}(z) = \frac{f(z) - h_{i,\gamma}(z)}{(z - \alpha_i)^\gamma}, \quad z \in \sigma(A) \cup I(\epsilon). \tag{3.38}$$

By definition of $\mathcal{F}_U$, f is continuous on I_{α_j}. The denominator is not zero on I_{α_j}, and $h_{i,\gamma}$ is just a polynomial. So $g_{i,\gamma}$ is continuous at α_j, and equation (3.38) also reveals that $g_{i,\gamma}$ is measurable.

In order to get $g_{i,\gamma} \in B(\sigma(A) \cup I(\epsilon))_c$, we have to show the continuity of $g_{i,\gamma}$ at ∞. Take a sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ with $|z_n| \to \infty$. The right-hand side of (3.38) converges to zero, since f is bounded and the degree of $h_{i,\gamma}$ is strictly smaller than γ. Notice that $(z_n)_{n\geq M} \subset \sigma(A) \cup I(\epsilon)$ if M is large enough. Together with $g_{i,\gamma}(\infty) = 0$, cf. the beginning of the proof of Lemma 3.5.7, we get the continuity of $g_{i,\gamma}$ at ∞, and conclude $g_{i,\gamma} \in B(\sigma(A) \cup I(\epsilon))_c$.

We treat the point $\alpha_{M+1} = \infty$ analogously. Again, we make use of Lemma 3.5.7 applied to the function $\tilde{f}(z) := f \circ \varphi_{M+1}(z) = f(\frac{1}{z})$ with $\mathrm{dom}(\tilde{f}) := \varphi_{M+1}(\sigma(A) \cup I(\epsilon)) \supseteq I_\infty^{-1} = (-\epsilon, \epsilon) \ni 0$. We have $\tilde{f} \in C^\gamma(-\epsilon, \epsilon)$ for $\gamma = 1, \dots, n$. The condition $0 < \delta := d\big(0, (\sigma(A) \cup I(\epsilon))^{-1}\setminus(-\epsilon, \epsilon)\big)$ is fulfilled too, because $|z| > \frac{1}{\epsilon}$ for $z \in \sigma(A) \cup I(\epsilon)$ implies $z \in \overline{\mathbb{R}}\setminus[-\frac{1}{\epsilon}, \frac{1}{\epsilon}] = I_\infty$.

Lemma 3.5.7 gives a polynomial $h_{M+1,\gamma}$ of degree at most $\gamma - 1$ and a bounded function $\tilde{g}_{M+1,\gamma} : \mathrm{dom}(\tilde{f}) \to \mathbb{C}$, which is continuous on $(-\epsilon, \epsilon)$, such that

$$\tilde{f}(w) = h_{M+1,\gamma}(w) + w^\gamma \tilde{g}_{M+1,\gamma}(w), \quad w \in \mathrm{dom}(\tilde{f})\setminus\{\infty\},$$

or, after the substitution $z := \varphi_{M+1}(w) = \frac{1}{w}$,

$$f(z) = h_{M+1,\gamma}\Big(\frac{1}{z}\Big) + z^{-\gamma}\tilde{g}_{M+1,\gamma}\Big(\frac{1}{z}\Big), \quad z \in (\sigma(A) \cup I(\epsilon)) \setminus \{0\},$$

holds. Setting $g_{M+1,\gamma}(z) := \tilde{g}_{M+1,\gamma}(\frac{1}{z})$, with $\operatorname{dom}(g_{M+1,\gamma}) := \sigma(A) \cup I(\epsilon)$, gives

$$f(z) = h_{M+1,\gamma}\Big(\frac{1}{z}\Big) + z^{-\gamma}g_{M+1,\gamma}(z), \quad z \in (\sigma(A) \cup I(\epsilon)) \setminus \{0\}. \tag{3.39}$$

Clearly, $g_{M+1,\gamma}$ is bounded on $\sigma(A) \cup I(\epsilon)$, and continuous on $\varphi_{M+1}((-\epsilon, \epsilon)) = I_\infty$. Rearranging (3.39) gives

$$z^\gamma f(z) = z^\gamma h_{M+1,\gamma}\Big(\frac{1}{z}\Big) + g_{M+1,\gamma}(z), \quad z \in (\sigma(A) \cup I(\epsilon)), \tag{3.40}$$

and

$$g_{M+1,\gamma}(z) = f(z)z^\gamma - h_{M+1,\gamma}\Big(\frac{1}{z}\Big)z^\gamma, \quad z \in (\sigma(A) \cup I(\epsilon)). \tag{3.41}$$

Notice, that $h_{M+1,\gamma}\left(\frac{1}{z}\right) z^\gamma$ is indeed a polynomial of degree at most γ, lacking a constant term, i.e. $z = 0$ is a root. Equation (3.41) directly reveals that $g_{M+1,\gamma}$ is measurable, and continuous at the points α_i for $i = 1, \dots, M$. We note $g_{M+1,\gamma} \in B(\sigma(A))_c$.

A partial fraction decomposition of $\frac{1}{r(t)}$ yields

$$\frac{1}{r(z)} = \sum_{i=1}^{M}\sum_{\gamma=1}^{\gamma_i} \frac{c_{i\gamma}}{(z-\alpha_i)^\gamma} + \sum_{i=1}^{m}\sum_{\gamma=1}^{\eta_i} \frac{d_{i\gamma}}{(z-\beta_i)^\gamma} + a(z), \tag{3.42}$$

with constants $c_{j\gamma}, d_{j\gamma} \in \mathbb{C}$ and a polynomial a of degree at most n. We split the partial fraction decomposition into two parts. The first one is related to the zeros of p which belong to $\sigma(A)$, i.e. $N(p)$. The second part is related to the remaining zeros, denoted by β_i with corresponding multiplicity η_i, $i = 1, \dots, m$. We write $a(z) = \sum_{\gamma=0}^{n} a_\gamma z^\gamma$. Bringing together (3.37), (3.40) and (3.42) gives

$$\begin{aligned} f(z) = r(z)\frac{f(z)}{r(z)} &= \\ &= \underbrace{r(z)\Bigg(\sum_{i=1}^{M}\sum_{\gamma=1}^{\gamma_i} \frac{c_{i\gamma}h_{i,\gamma}(z)}{(z-\alpha_i)^\gamma} + \sum_{\gamma=1}^{n} a_\gamma z^\gamma h_{M+1,\gamma}(\tfrac{1}{z})}_{=:h_f(z)} + \qquad (3.43) \\ &\quad + \underbrace{\sum_{i=1}^{M}\sum_{\gamma=1}^{\gamma_i} c_{i\gamma}g_{i,\gamma}(z) + \sum_{i=1}^{m}\sum_{\gamma=1}^{\eta_i} \frac{d_{i\gamma}}{(z-\beta_i)^\gamma}f(z) + a_0 f(z) + \sum_{\gamma=1}^{n} a_\gamma g_{M+1,\gamma}(z)}_{=:g_f(z)}\Bigg), \qquad (3.44) \end{aligned}$$

for all $z \in (\sigma(A) \cup I(\epsilon)) \setminus \{\infty\}$. For these z we have, by definition,

$$f(z) = h_f(z) + r(z)g_f(z). \tag{3.45}$$

As a linear combination of elements of $B(\sigma(A) \cup I(\epsilon))_c$ we have $g_f \in B(\sigma(A) \cup I(\epsilon))_c$. Since both sides of (3.45) are continuous at ∞, we see that this equation actually holds true for all $z \in \sigma(A) \cup I(\epsilon)$.

We have to show that $h_f \in \mathcal{E}$. We observe that h_f is a linear combination of the rational functions

$$r(z)\frac{h_{i,\gamma}(z)}{(z-\alpha_i)^\gamma} = \frac{q(z)h_{i,\gamma}(z)(z-\alpha_i)^{\gamma_i-\gamma}\prod_{i\neq j=1}^{M}(z-\alpha_j)^{\gamma_j}}{(z-\lambda_0)^n(z-\overline{\lambda_0})^n}, \tag{3.46}$$

for $i = 1, \dots, M$ and $\gamma = 1, \dots, \gamma_i$, as well as

$$r(z)z^\gamma h_{M+1,\gamma}(\tfrac{1}{z}), \qquad \text{for } \gamma = 1, \dots, n. \tag{3.47}$$

The numerators in all fractions of (3.46) are polynomials of degree at most $n-1$, and always contain the factor $q(z)$. Also the rational functions in (3.47) are in $\mathcal{E}$. Recall that $z \mapsto z^\gamma h_{M+1,\gamma}(\frac{1}{z})$ is actually a polynomial of degree at most γ, which has a zero at $z = 0$. Since $\mathcal{E}$ is linear, we conclude $h_f \in \mathcal{E}$, which completes the proof of $\mathcal{F}_{U,\epsilon} \subseteq \mathcal{E} + r \cdot B(\sigma(A) \cup I(\epsilon))_c$.

The inclusion $\mathcal{E} + r \cdot B(\sigma(A) \cup I(\epsilon))_c \subseteq \mathcal{F}^{I(\epsilon)}$ is trivial. In order to prove the other inclusion, it is sufficient to show $\mathbb{C}(z) \cap B(\sigma(A) \cup I(\epsilon)) \subseteq \mathcal{E} + r \cdot B(\sigma(A) \cup I(\epsilon))_c$. This follows from the first step, since we have $\mathbb{C}(z) \cap B(\sigma(A) \cup I(\epsilon)) \subseteq \mathcal{F}_{U,\epsilon}$.

Now take an $s \in \mathcal{E} \cap r \cdot B(\sigma(A) \cup I(\epsilon))_c$. For all $z \in \sigma(A) \cup I(\epsilon)$ we have

$$s(z) = \frac{q(z)a(z)}{(z-\lambda_0)^n(z-\overline{\lambda_0})^n} + r(z)b(z) = r(z)g(z),$$

for polynomials a and b, with $\deg a < n - \deg q$ and $\deg b \leq n$, $b(0) = 0$, and a function $g \in B(\sigma(A) \cup I(\epsilon))_c$. Dividing by $r(z)$ gives

$$\frac{q(z)a(z)}{p(z)} + b(z) = g(z) \qquad \text{for all } z \in (\sigma(A) \cup I(\epsilon)) \setminus N(r). \tag{3.48}$$

Take a sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \in \mathbb{R}$, $|z_n| \geq \frac{1}{\epsilon}$ and $|z_n| \to \infty$ for $n \to \infty$. Since $(z_n)_{n\in\mathbb{N}} \subset (\sigma(A) \cup I(\epsilon)) \setminus N(r)$, we can look at equation (3.48) for z_n. Since g is bounded, we have $|g(z_n)| \leq C$ for all $n \in \mathbb{N}$, for a constant $C > 0$. The term $\frac{q(z_n)a(z_n)}{p(z_n)}$ converges to zero, since $\deg q + \deg a < n = \deg p$. We get to know that $|b(z_n)|$ is also bounded for all $n \in \mathbb{N}$. This can only be the case if $\deg b = 0$. Due to $b(0) = 0$, we have $b \equiv 0$. The continuity of g at ∞ gives $0 = g(\infty)$.

The boundedness of the rational function $\frac{q(z)a(z)}{p(z)}$ on $I_{\alpha_i} \setminus \{\alpha_i\}$ for $i = 1, \dots M$, which follows from (3.48), implies that α_i is not a pole of $\frac{q(z)a(z)}{p(z)}$. This means that both sides of the equation (3.48) are continuous at α_i, and hence (3.48) even holds true for all $z \in \sigma(A) \cup I(\epsilon)$.

Furthermore, the fact that $\frac{q(z)a(z)}{p(z)}$ has no pole at α_i implies that α_i is a zero of the polynomial a with multiplicity at least γ_i for $i = 1, \dots M$. We learn that the polynomial $q(z)a(z)$ has the same zeros as $p(z)$, with the same or even higher multiplicities. In view of $\deg q + \deg a < n = \deg p$ this means $a \equiv 0$, and thus $s \equiv 0$.

Now consider the linear bijection $\lambda : \mathcal{F}^{I(\epsilon)} \to \mathcal{E} \times B(\sigma(A) \cup I(\epsilon))_c$ which corresponds to the direct sum $\mathcal{F}^{I(\epsilon)} = \mathcal{E} \dotplus r \cdot B(\sigma(A) \cup I(\epsilon))_c$. We claim that there exists a constant $C > 0$ such that $\|\lambda(f)\| \leq C\|f\|_{U,\epsilon}$ holds for all $f \in \mathcal{F}_{U,\epsilon}$, i.e. $\lambda|_{\mathcal{F}_{U,\epsilon}}$ is continuous.

Since we endow $\mathcal{E} \times B(\sigma(A) \cup I(\epsilon))_c$ with the sum norm, where $\mathcal{E}, B(\sigma(A) \cup I(\epsilon))_c \leq B(\sigma(A) \cup I(\epsilon))$ are equipped with the supremum norm on $\sigma(A) \cup I(\epsilon)$, we have

$$\|\lambda(f)\| = \|(h; g)\| = \|h\|_\infty + \|g\|_\infty = \sup_{z \in \sigma(A) \cup I(\epsilon)} |h(z)| + \sup_{z \in \sigma(A) \cup I(\epsilon)} |g(z)|.$$

Take an arbitrary $f \in \mathcal{F}_{U,\epsilon}$, and consider the decomposition of f in $f = h_f + rg_f$, which we constructed at the beginning of this proof. Clearly, we have $(h_f; g_f) = \lambda(f)$. We are going to show $\|h_f\|_\infty \leq C\|f\|_{U,\epsilon}$ and $\|g_f\|_\infty \leq C\|f\|_{U,\epsilon}$ separately.

First, consider the bounded and measurable function g_f, defined in (3.44). We see that g_f is a linear combination of the functions $g_{i,\gamma}$ for $i \in \{1, \ldots, M+1\}$ and $\gamma \in \{1, \ldots, \gamma_i\}$, as well as $(z-\beta_i)^{-\gamma} f(z)$ for $i = \{1, \ldots, m\}$ and $\gamma \in \{1, \ldots, \eta_i\}$. Note that the coefficients of this linear combination, namely $c_{i\gamma}$, $d_{i\gamma}$ and a_γ, originate from the partial fraction decomposition of $\frac{1}{r}$, cf. (3.42), and are, therefore, independent of f.

For the term $(z - \beta_i)^{-\gamma} f(z)$, we have the estimate $|(z - \beta_i)^{-\gamma} f(z)| \leq \delta^{-\gamma}\|f\|_\infty \leq \delta^{-\gamma}\|f\|_{U,\epsilon}$ for all $z \in \sigma(A) \cup I(\epsilon)$, where $\delta > 0$ denotes the distance from β_i to $\sigma(A) \cup I(\epsilon)$. Recall that the numbers β_i denote the zeros of p which belong to $\rho(A)$. The distance δ does not depend on f but only on ϵ.

Fix an $i \in \{1, \ldots M\}$ and $\gamma \in \{1, \ldots, \gamma_i\}$ and consider $g_{i,\gamma}$, declared in (3.37). On the set I_{α_i}, an estimation is on hand coming from Lemma 3.5.7,

$$\left\|g_{i,\gamma}|_{I_{\alpha_i}}\right\|_\infty \leq \frac{2}{\gamma!} \left\|f|_{I_{\alpha_i}}^{(\gamma)}\right\|_\infty \leq C\|f\|_{U,\epsilon}.$$

In fact, we have seen in the proof of Lemma 3.5.7, cf. equation (3.36), that $g_{i,\gamma}$ is even bounded on $\sigma(A) \cup I(\epsilon)$,

$$\left\|g_{i,\gamma}|_{(\sigma(A) \cup I(\epsilon)) \setminus I_{\alpha_i}}\right\|_\infty \leq \frac{\|f\|_\infty}{\delta^\gamma} + \sum_{k=0}^{\gamma-1} \frac{|f^{(k)}(\alpha_i)|}{k!\, \delta^{\gamma-k}} \leq C\|f\|_{U,\epsilon}.$$

The case $i = M+1$ and $\gamma \in \{1, \ldots, n\}$ follows in a similar matter. With the notation introduced in the paragraph above (3.39), Lemma 3.5.7 gives

$$\left\|g_{M+1,\gamma}|_{I_\infty}\right\|_\infty = \left\|\tilde{g}_{M+1,\gamma}|_{I_\infty^{-1}}\right\|_\infty \leq \frac{2}{\gamma!} \left\|\tilde{f}|_{I_\infty^{-1}}^{(\gamma)}\right\|_\infty \leq C\|f\|_{U,\epsilon}.$$

The estimate from (3.36) gives

$$\begin{aligned} \left\|g_{M+1,\gamma}|_{(\sigma(A) \cup I(\epsilon)) \setminus I_\infty}\right\|_\infty = \left\|\tilde{g}_{M+1,\gamma}|_{(\sigma(A) \cup I(\epsilon))^{-1} \setminus I_\infty^{-1}}\right\|_\infty \leq \\ \leq \frac{\|\tilde{f}\|_\infty}{\delta^\gamma} + \sum_{k=0}^{\gamma-1} \frac{|\tilde{f}^{(k)}(0)|}{k!\, \delta^{\gamma-k}} \leq C\|f\|_{U,\epsilon} \end{aligned}$$

All in all, we have shown $\|g_f\|_\infty \leq C\|f\|_{U,\epsilon}$.

Now, we consider the rational function h_f, defined in (3.43). We can use the equality $f(z) = h_f(z) + r(z)g_f(z)$, cf. (3.45), to get

$$\|h_f\|_\infty \leq \|f\|_\infty + \|r\|_\infty \|g_f\|_\infty \leq \|f\|_{U,\epsilon} + \|r\|_\infty C \|f\|_{U,\epsilon} \leq C' \|f\|_{U,\epsilon}.$$

This completes the proof of the continuity of $\lambda|_{\mathcal{F}_{U,\epsilon}}$.

Let $\Delta \subseteq \mathbb{C} \cup \{\infty\}$ be a Borel set, and assume $\mathbb{1}_\Delta \in \mathcal{F}$, i.e. $\mathbb{1}_\Delta \in \mathcal{F}^{I(\epsilon)}$ for some $\epsilon > 0$. Hence, we can write $\mathbb{1}_\Delta = h + rg$ for a $h \in \mathbb{C}(z) \cap B(\sigma(A) \cup I(\epsilon))$ and $g \in B(\sigma(A) \cup I(\epsilon))_c$. Then, we have

$$g(z) = \begin{cases} \frac{1-h(z)}{r(z)} & , \quad \text{for } z \in \Delta \cap (\sigma(A) \cup I(\epsilon)), \quad z \notin N(r) \\ -\frac{h(z)}{r(z)} & , \quad \text{for } z \in \Delta^c \cap (\sigma(A) \cup I(\epsilon)), \quad z \notin N(r)\,. \end{cases}$$

Now suppose that there exists an $\alpha_i \in N(r)$ which is an accumulation point of both $\Delta \cap (\sigma(A) \cup I(\epsilon))$ and $\Delta^c \cap (\sigma(A) \cup I(\epsilon))$, i.e. $N(r) \cap \partial_\epsilon (\Delta \cap (\sigma(A) \cup I(\epsilon))) \neq \emptyset$. The symbol ∂_ϵ refers to the subspace topology on $\sigma(A) \cup I(\epsilon)$.

The boundedness of g on I_{α_i} and the continuity at α_i would imply that both $\frac{1-h(z)}{r(z)}$ and $-\frac{h(z)}{r(z)}$ are bounded near α_i. The denominator of these fractions is zero at $z = \alpha_i$, which implies $h(\alpha_i) = 1$ and $h(\alpha_i) = 0$. This contradiction proves that we cannot find such an $\alpha_i \in N(r)$, i.e. $\partial_\epsilon \Delta \cap N(r) = \emptyset$.

On the other hand, $\partial_\epsilon((\Delta \cap (\sigma(A) \cup I(\epsilon))) = \emptyset$ for some $\epsilon > 0$ means that we have either $\alpha_i \in \overline{\Delta}^c$ or $\alpha_i \in \Delta^\circ$ for all $i = 1, \ldots, M$. Here, the closure and the interior refer to the subspace topology on $\sigma(A) \cup I(\epsilon)$. We can make $\epsilon > 0$ sufficiently small, such that we have even $I_{\alpha_i} \subseteq \overline{\Delta}^c$ or $I_{\alpha_i} \subseteq \Delta^\circ$ respectively. Obviously, this gives $\mathbb{1}_\Delta \in \mathcal{F}_{U,\epsilon} \subseteq \mathcal{F}^{I(\epsilon)} \subseteq \mathcal{F}$, since $\mathbb{1}_\Delta$ is now constant on every set I_{α_i}.

□

Definition 3.5.8. With the equivalence relation $\sim$ from Definition 3.5.4, we set

$$\mathcal{F}_U := \left(\bigcup_{\epsilon > 0} \mathcal{F}_{U,\epsilon} \right) \Big/ \sim .$$

Since $\mathcal{F}_{U,\epsilon}$ is a linear subspace of $\mathcal{F}^{I(\epsilon)}$, $\mathcal{F}_U$ is a linear subspace of $\mathcal{F}$.

Hereby, we equip the disjoint union $\bigcup_{\epsilon>0} \mathcal{F}_{U,\epsilon}$ with the final topology of all canonical injections $\iota_\epsilon : \mathcal{F}_{U,\epsilon} \to \bigcup_{\epsilon>0} \mathcal{F}_{U,\epsilon}$, and equip $\mathcal{F}_U$ with the final topology of the canonical projection $\pi : \bigcup_{\epsilon>0} \mathcal{F}_{U,\epsilon} \to \mathcal{F}_U$.

Corollary 3.5.9. *Let A be a definitizable linear relation on a Krein space $\mathcal{K}$ with a real definitizing polynomial.*

The restriction of the functional calculus $E : \mathcal{F} \to \mathcal{B}(\mathcal{K})$, constructed in Theorem 3.4.4 and Definition 3.5.4, to the subspace $\mathcal{F}_U$ is continuous with respect to the topology on $\mathcal{F}_U$ described in Definition 3.5.8, and the operator norm on $\mathcal{B}(\mathcal{K})$.

Proof. Proposition 3.5.6 states that the map $\lambda : \mathcal{F}_{U,\epsilon} \to \mathcal{E} \times B(\sigma(A) \cup I(\epsilon))_c$, with $\lambda(f) = (h, g)$ for $f = h + rg$, is continuous. Hereby, $\mathcal{F}_{U,\epsilon}$ is equipped with $\|.\|_{U,\epsilon}$, and

$\mathcal{E} \times B(\sigma(A) \cup I(\epsilon)_c$ is endowed with the product topology of the supremum norm on $\sigma(A) \cup I(\epsilon)$.

We can write $E^{I(\epsilon)}|_{\mathcal{F}_{U,\epsilon}}$ as

$$E^{I(\epsilon)}|_{\mathcal{F}_{U,\epsilon}}(f) = \Phi_{\mathrm{rat}}(\lambda_1(f)) + G(\lambda_2(f)), \quad \text{for } f \in \mathcal{F}_{U,\epsilon}.$$

Clearly, $\Phi_{\mathrm{rat}} : \mathcal{E} \to \mathcal{B}(\mathcal{K})$ is continuous, since $\mathcal{E}$ is finite dimensional. The continuity of G was already remarked in the proof of Theorem 3.4.4, cf. (3.25). Therefore, $E^{I(\epsilon)}|_{\mathcal{F}_{U,\epsilon}}$ is continuous for all $\epsilon > 0$ since it can be written as a composition of continuous operators.

By a well-known property for the final topology, the continuity of $E|_{\mathcal{F}_U} : \mathcal{F}_U \to \mathcal{B}(\mathcal{K})$ is equivalent to the continuity of all functions $E|_{\mathcal{F}_U} \circ \pi \circ \iota_\epsilon = E^{I(\epsilon)}|_{\mathcal{F}_{U,\epsilon}}$. □

Using the continuity of our functional calculus, we can now precisely determine the support of E.

Corollary 3.5.10. *We have* $\operatorname{supp} E = \sigma(A)$.

Here, we define the support of E as the smallest closed subset $C \subseteq \mathbb{C} \cup \{\infty\}$ with the property

$$f \in \bigcup_{\epsilon>0} \mathcal{F}^{I(\epsilon)}, \operatorname{supp} f \cap C = \emptyset \;\Rightarrow\; E([f]_\sim) = 0. \tag{3.49}$$

Proof. First, we claim $\operatorname{supp} E \subseteq \sigma(A)$, i.e. $C = \sigma(A)$ fulfills (3.49). Take $f \in \mathcal{F}^{I(\epsilon)}$ for some $\epsilon > 0$ and assume $\operatorname{supp} f \cap \sigma(A) = \emptyset$. By Theorem 3.4.4 applied to $I = I(\epsilon)$, this gives $0 = E^{I(\epsilon)}(f) = E([f]_\sim)$.

In order to show the other inclusion, take $\lambda \notin \operatorname{supp} E$ and assume $\lambda \in \sigma(A)$. Since the support of E is closed, there is an $s > 0$ such that the ball with radius s and center λ, denoted by $U_s(\lambda)$, is still disjoint from $\operatorname{supp} E$.

In the case $\lambda \in N(r)$, take an $\epsilon > 0$ sufficiently small such that $I_\lambda \subseteq U_{\frac{s}{2}}(\lambda)$. If $\lambda \notin N(r)$, we may assume $I(\epsilon) \cap U_{\frac{s}{2}}(\lambda) = \emptyset$ for sufficiently small $s > 0$ and sufficiently small $\epsilon > 0$.

Let $\psi : \mathbb{C} \cup \{\infty\} \to [0, 1]$ be a bump function, i.e. a smooth function such that $\psi(U_{\frac{s}{2}}(\lambda)) = \{0\}$ and $\psi((\mathbb{C} \cup \{\infty\}) \backslash U_s(\lambda)) = \{1\}$. Take a sequence $\lambda_n \in \rho(A) \setminus I(\epsilon)$ which converges to λ, such that $(\lambda_n)_{n\in\mathbb{N}} \subset U_{\frac{s}{2}}(\lambda)$. Consider the sequence of functions $f_n(z) := \psi(z)\frac{1}{z-\lambda_n}$. Since the functions f_n are smooth everywhere, we have $f_n \in \mathcal{F}_{U,\epsilon}$. The pointwise limit of f_n clearly is $f(z) := \psi(z)\frac{1}{z-\lambda} \in \mathcal{F}_{U,\epsilon}$. Note that the bump function ψ is necessary, since $z \mapsto \frac{1}{z-\lambda} \notin \mathcal{F}_{U,\epsilon}$.

One can elementarily verify that f_n converges not only pointwise to f, but even with respect to $\|.\|_{U,\epsilon}$. This gives

$$\begin{aligned} E([f]_\sim) &= E(\lim_{n\to\infty} [f_n]_\sim) = \lim_{n\to\infty} E([f_n]_\sim) = \lim_{n\to\infty} E\left(\left[z \mapsto \psi(z)\frac{1}{z-\lambda_n}\right]_\sim\right) = \\ &= \lim_{n\to\infty} E\left(\left[z \mapsto \frac{1}{z-\lambda_n}\right]_\sim\right) = \lim_{n\to\infty} E^{I(\epsilon)}\left(z \mapsto \frac{1}{z-\lambda_n}\right) = \lim_{n\to\infty} (A-\lambda_n)^{-1}. \end{aligned}$$

Hereby, we used the fact that $\frac{1}{z-\lambda_n}$ and $\psi(z)\frac{1}{z-\lambda_n}$ only distinguish themselves on $U_s(\lambda) \subseteq (\operatorname{supp} E)^c$. We realize that $(A - \lambda_n)^{-1}$ converges to the bounded operator $E([f]_\sim)$ in the operator norm and arrive at the contradiction

$$\frac{1}{\operatorname{dist}(\lambda_n, \sigma(A))} - \|E([f]_\sim)\| \leq \left\|(A-\lambda_n)^{-1}\right\| - \|E([f]_\sim)\| \leq$$
$$\leq \left\|(A-\lambda_n)^{-1} - E([f]_\sim)\right\| \overset{n\to\infty}{\longrightarrow} 0,$$

since $(\operatorname{dist}(\lambda_n, \sigma(A)))^{-1} \to \infty$ for $n \to \infty$.
Therefore, we have $\lambda \in \rho(A)$, and in turn $\operatorname{supp} E = \sigma(A)$. □

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