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Soft hairy Schrödinger black holes

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Kurzfassung

Wir finden Randbedingungen am Ereignishorizont von dreidimensionalen Schrödinger schwarzen Löchern in topologisch massiver Gravitation. Wir formulieren diese als Felder erster Ordnung einer Chern-Simons-ähnlichen Formulierung. Weiters berechnen wir die zugehörigen Ladungen, finden deren Symmetriealgebra und drücken die Entropie mittels dieser Ladungen aus. Schlussendlich versuchen wir die Symmetriealgebra am Ereignishorizont mit der Algebra im asymptotisch Unendlichen zu verknüpfen - der Virasoroalgebra.

Abstract

We find near horizon boundary conditions for three dimensional Schrödinger black holes in topologically massive gravity. These are expressed in terms of first order fields of a Chern-Simons-like formalism. Then we calculate the associated charges, find their symmetry algebra and express the entropy in terms of these charges. Finally, we attempt to link the near horizon symmetry algebra to the algebra at asymptotic infinity - the Virasoro algebra.

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Contents

Chapter 1

Introduction

Since Albert Einstein published his general theory of relativity in 1915, it remains the best experimentally verified theory of gravity to this day. The essence of the purely geometric theory has been famously captured in a quote of John Archibald Wheeler: “Spacetime tells matter how to move; matter tells spacetime how to curve.” [1]

However, the existence of singularities and inconsistencies with quantum theory suggest that a full theory of quantum gravity is needed. A very fertile playing ground for testing toy models of quantum gravity has been three dimensional spacetime. It is not only simpler due to the reduction of dimensions, but it actually has some unexpected advantages over theories in four dimensions. In 1986, Achucarro and Townsend discovered that general relativity in three dimensions with negative cosmological constant $\Lambda < 0$ can be reformulated as a so-called Chern-Simons theory [2]. In this formulation, one can apply techniques familiar from other gauge theories. In the same year, Brown and Henneaux constructed boundary conditions for three dimensional anti-de Sitter space that led to two towers of canonical boundary charges generating two copies of the Virasoro algebra, the symmetry algebra of two dimensional conformal field theory [3]. This result was a precursor of the famous AdS/CFT correspondence, which states that a theory of gravity in anti-de Sitter space can be equivalently described by a conformal field theory living on its boundary [4–6].

It was not always clear however, that three dimensional gravity would be interesting to study. In fact, one could be tempted to discard it as trivial since there are no propagating degrees of freedom in three dimensional Einstein gravity. However, in 1992 Bañados, Teitelboim and Zanelli found a black hole solution [7] with very similar properties as its four dimensional analogue, the Kerr black hole.

Very recently, it has been shown [8] that one can impose boundary conditions for BTZ black holes at the horizon which lead to a near horizon symmetry algebra of two $\mathfrak{u}(1)$ current algebras, which is equivalent to infinitely many copies of the

Heisenberg algebra. The entropy was then shown to be only given in terms of the zero mode charges of the $\mathfrak{u}(1)$ current algebra:

$$S = 2\pi(J_0^+ + J_0^-)\,. \tag{1.1}$$

This formula has been shown to hold for flat space, anti-de Sitter space, for Einstein gravity, theories with higher derivatives [9], for theories with higher spin [10,11] and more recently for the first non-maximally symmetric case, warped black holes in topologically massive gravity [12,13]. Topologically massive gravity is an extension of three dimensional Einstein gravity with one propagating degree of freedom. It is constructed by adding the so-called gravitational Chern-Simons term to the Einstein-Hilbert action. In [14], all stationary axi-symmetric of TMG have been classified in four categories:

- *Einstein* - This class of solutions solves the three dimensional Einstein equations of motion. All of them are locally AdS$_3$ and they obey Brown-Henneaux boundary conditions.
- *Warped* - This class of solutions is asymptotically or locally warped AdS$_3$.
- *Schrödinger* - This class of solutions is either asymptotically AdS$_3$ or asymptotically Schrödinger.
- *Generic* - All solutions that are neither of the above.

While, as already mentioned, for the first two sectors, near horizon boundary conditions that lead to a near horizon symmetry algebra of two $\mathfrak{u}(1)$ currents and the entropy formula (1.1) have been found, it is the goal of this thesis to provide an example that lives in the Schrödinger sector.

This thesis is structured as follows: in chapter 2, the basic concepts underlying the calculations of this thesis are reviewed. These include a review of three dimensional gravity, the Chern-Simons formulation of Einstein gravity, the global AdS$_3$ solution and BTZ black holes, a near horizon analysis of BTZ black holes, the introduction of Chern-Simons-like theories of gravity, topologically massive gravity and a review of the Schrödinger spacetime. In chapter 3, the near horizon behaviour of three dimensional Schrödinger black holes is studied. We present a first order form of the black hole solution, present boundary conditions and calculate the associated charges as well as their symmetry algebra and the entropy. Finally we study how the near horizon charges and algebra are related to the asymptotic ones. In chapter 4, we briefly summarize the results of this thesis and give a short outlook on possible future research directions. In appendix A, the conventions used in this thesis are stated.

Chapter 2

Basic concepts

The purpose of this chapter is to give an overview of the topics needed for the calculations in chapter 3. We start in section 2.1 by discussing the advantages of working in three dimensions while in 2.2 the Chern-Simons formulation of three dimensional Einstein gravity is discussed. In section 2.3, we discuss Global AdS$_3$ and BTZ black holes as solutions to three dimensional gravity. In section 2.4, near horizon boundary conditions that lead to $\mathfrak{u}(1)$ current algebras, or, equivalently, infinitely many Heisenberg algebras as near horizon symmetry algebras of BTZ black holes are presented. Then, in section 2.5, we discuss Chern-Simons-like theories, a generalization of Chern-Simons theory to a whole family of gravity theories. In section 2.6 we discuss topologically massive gravity and its solution space. Finally, in section 2.7, Schrödinger spacetime, a solution to topologically massive gravity, is presented.

2.1 Why gravity in 3 dimensions?

Three dimensional gravity is interesting to study for a variety of reasons. First of all, it is useful for studying models of quantum theory simply because it is less involved from a technical point of view. More importantly however, it has some very interesting features unique to three spacetime dimensions.

At first sight one may fear that three dimensional general relativity might be trivial. The Riemann curvature tensor $R_{\rho\sigma\mu\nu}$ in three dimensions can be expressed in terms of the Ricci tensor $R_{\mu\nu}$ as

$$R_{\rho\sigma\mu\nu} = g_{\rho\mu}R_{\sigma\nu} + g_{\sigma\nu}R_{\rho\mu} - g_{\rho\nu}R_{\sigma\nu} - g_{\sigma\mu}R_{\rho\nu} - \frac{1}{2}R(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\nu})\,. \tag{2.1}$$

Einstein's field equations are given by

$$R_{\mu\nu} + \left(\Lambda - \frac{R}{2}\right) g_{\mu\nu} = 8\pi G\, T_{\mu\nu}\,, \tag{2.2}$$

where Λ is the cosmological constant and G is Newton's constant, allowing to express the Ricci tensor in terms of a constant times the metric and the stress-energy tensor $T_{\mu\nu}$. Therefore, in the absence of matter $T_{\mu\nu}$, the curvature of three dimensional spacetime in Einstein gravity is determined only by the value of the cosmological constant. Either spacetime is locally flat when $\Lambda = 0$ or of constant curvature when $\Lambda \neq 0$.

We could have arrived at the same conclusion by counting independent components of the aforementioned tensors. The Riemann tensor in d dimensions, taking into account all its symmetries under index permutation as well as the first Bianchi identity, has $d^2(d^2-1)/12$ independent components, while the symmetric Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{2.3}$$

has $d(d+1)/2$ independent components. It is only for $d = 3$, that both of them have the same number of independent components, namely 6.

The above results imply that there are no local propagating degrees of freedom, i.e. no gravitons in three dimensional Einstein gravity. Even though it might therefore be tempting to dismiss Einstein gravity in three dimensions, the theory turns out to have non-trivial solutions thanks to global effects. This was most famously realized in a black hole solution found by Bañados, Teitelboim and Zanelli in 1992 [7, 15]. The BTZ solution has very similar properties as the four dimensional Kerr solution as it possesses a singularity, inner and outer horizon and it admits a no-hair theorem.

Furthermore, in 1986 Brown and Henneaux found consistent boundary conditions for AdS$_3$ and showed that their canonical charges are the generators of two copies of the Virasoro algebra with central charge $c = 3\ell/2G$ where ℓ is called AdS radius [3]. This led to the conclusion that AdS$_3$ can equivalently be described by a two-dimensional conformal field theory at the boundary.

2.2 The Chern-Simons formulation of gravity

A particularly nice feature of three dimensional Einstein gravity with $\Lambda < 0$ is that it admits a reformulation as Chern-Simons theory. This was first discovered by Achúcarro and Townsend in 1986 [16] and Witten attempted a quantization in 1988 [17].

In the metric formulation of gravity, the symmetric tensor $g_{\mu\nu}$ is the fundamental field of a given theory and usually the directional derivatives ∂_μ and the differentials $\mathrm{d}x^\mu$ with respect to some coordinates x^μ are chosen as the basis vectors of the tangent space or its dual respectively. Due to the equivalence principle however, we can also find local frames at each point of a manifold in which the metric is locally flat. Then, as a natural basis we can use the orthonormal vector-valued 1-forms e^a called the dreibein (or vielbein in arbitrary dimensions). They obey

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = e^a e^b \eta_{ab}\,, \tag{2.4}$$

with the Minkowski metric $\eta_{ab} = \mathrm{diag}(-1, +1, +1)$. Here, latin indices represent local Lorentz indices that are raised and lowered with the Minkowski metric, while greek spacetime indices are raised or lowered with the metric $g_{\mu\nu}$. The relationship (2.4) is preserved under local Lorentz transformations $\Lambda^a{}_{a'}$, which means that as long as eq. (2.4) is satisfied, the choice of the dreibein is independent of chosen coordinates.

As a next step we can define a covariant derivative on tensors with Lorentz indices. Consider some arbitrary tensor $T^a{}_\mu$. We can then write the covariant derivative as

$$\nabla_\mu T^a{}_\nu = \partial_\mu T^a{}_\nu + \omega^a{}_{b\mu} T^b{}_\nu - \Gamma^\sigma{}_{\nu\mu} T^a{}_\sigma\,, \tag{2.5}$$

with the Christoffel connection $\Gamma^\sigma{}_{\nu\mu}$ and we introduced the spin-connection $\omega^a{}_b$, that can be thought of as a tensor-valued 1-form. Metric compatibility of the connection implies $\omega_{ab} = -\omega_{ba}$. Now, what is special about three dimensions is the fact that antisymmetric matrices are dual to vectors. Hence, we can define the dualised spin connection

$$\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc}. \tag{2.6}$$

This is important because now the dualised spin connection and the dreibein are both vector-valued 1-forms and we can construct linear combinations of them:

$$A = \omega^a J_a + e^a P_a \tag{2.7}$$

J_a and P_a generate the Lie algebra

$$[J_a, J_b] = \eta^{cd}\epsilon_{abc}J_d, \quad [J_a, P_b] = \eta^{cd}\epsilon_{abc}P_d, \quad [P_a, P_b] = -\Lambda\eta^{cd}\epsilon_{abc}J_d. \tag{2.8}$$

For $\Lambda > 0$, this is $so(3,1)$, the symmetry algebra of de Sitter spacetime in three dimensions. For $\Lambda = 0$, it is $isl(2,\mathbb{R})$, the Poincaré algebra in three dimensions. For the case $\Lambda < 0$, it is $so(2,2)$, the symmetry algebra of three dimensional Anti-de Sitter spacetime.

We are now going to focus on the last case, $\Lambda < 0$.
With the connection A, the Chern-Simons action can be written down:

$$I_{CS}[A] = \frac{k}{4\pi}\int \mathrm{tr}\left[A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A\right] \tag{2.9}$$

Here, k is the Chern-Simons level and the trace is taken on the space of the $so(2,2)$ matrices. This action, up to boundary terms, is equivalent to the Einstein-Hilbert action

$$I_{\mathrm{EH}} = \frac{1}{16\pi G}\int \mathrm{d}^3x\sqrt{-g}(R - 2\Lambda)\,, \tag{2.10}$$

which yields Einstein's field equations (2.3) upon variation with regard to the metric tensor $g_{\mu\nu}$. In order to see this, one can write the Einstein-Hilbert action in terms of our two new first-order fields

$$I_{\mathrm{EH}} = \frac{1}{8\pi G}\int\left[e_a \wedge R^a - \frac{\Lambda}{6}\epsilon_{abc}e^a \wedge e^b \wedge e^c\right]\,, \tag{2.11}$$

where R^a is the curvature two-form

$$R^a = \mathrm{d}\omega^a + \frac{1}{2}\epsilon^a{}_{bc}\,\omega^b \wedge \omega^c\,. \tag{2.12}$$

With $\Lambda < 0$, the underlying gauge symmetry can be written as the direct sum $so(2,2) \sim sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$. This convenient feature of AdS$_3$ allows to introduce new generators

$$Y_a^{\pm} = \frac{1}{2}(J_a \pm P_a). \tag{2.13}$$

An explicit realization of these 4×4 matrices is given by

$$Y_a^{+} = \begin{pmatrix} X_a^{+} & 0 \\ 0 & 0 \end{pmatrix}, \quad Y_a^{-} = \begin{pmatrix} 0 & 0 \\ 0 & X_a^{-} \end{pmatrix}, \tag{2.14}$$

where $X_a^{\pm}$ are $sl(2,\mathbb{R})$ matrices. Therefore, we can decompose the connection A as the direct sum of two $sl(2,\mathbb{R})$ valued connections

$$A = A^{+} \oplus A^{-}, \tag{2.15}$$

with

$$A^{\pm} = \left(\omega^a \pm \frac{e^a}{\ell}\right)X_a^{\pm}. \tag{2.16}$$

It follows that we can now write the Chern-Simons action (2.9) as the difference of two copies of the Chern-Simons action constructed from each copy of the $sl(2,\mathbb{R})$ valued connections:

$$I = I_{\mathrm{CS}}[A^+] - I_{\mathrm{CS}}[A^-]. \tag{2.17}$$

A common choice for the matricial basis $\{X_a^\pm\}$ makes use of the Pauli matrices:

$$T_0 = \frac{i}{2}\sigma_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad T_1 = -\frac{1}{2}\sigma_1 = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad T_2 = \frac{1}{2}\sigma_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}. \tag{2.18}$$

Their commutator reads

$$[T_a, T_b] = \eta^{cd}\epsilon_{abc}T_d\,. \tag{2.19}$$

Another choice for a basis of $sl(2,\mathbb{R})$ is given by

$$L_{-1}- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad L_{+1} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \tag{2.20}$$

where the generators satisfy

$$[L_i, L_j] = (i-j)L_{i+j}\,, \tag{2.21}$$

with $i, j = -1, 0, 1$.
These two bases can be conveniently transformed into one another as

$$L_i = T_a\Sigma^a{}_i, \tag{2.22}$$

using the transformation matrix

$$(\Sigma^a{}_i) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{2.23}$$

The only bilinear or quadratic combinations that do not vanish under the trace are

$$\mathrm{tr}\,(L_+L_-) = -1\,, \tag{2.24}$$

$$\mathrm{tr}\,(L_0L_0) = \frac{1}{2}\,. \tag{2.25}$$

The Chern-Simons formulation is advantageous in many aspects, one of them being that we can apply ordinary techniques commonly used in other gauge theories. A finite gauge transformation can be written as

$$A' = g^{-1}(A + \mathrm{d})g\,, \tag{2.26}$$

where g is a group element of the gauge group. Its infinitesimal version $g = \mathbb{1} + \xi$ is given by

$$\delta_\xi A = \mathrm{d}\xi + [A, \xi]\,. \tag{2.27}$$

If we take the special choice $\xi = \zeta^\nu A_\nu$ as our gauge parameter, we obtain

$$\delta_{(\zeta^\nu A_\nu)} A_\mu = \mathcal{L}_\zeta A_\mu + \zeta^\nu F_{\mu\nu}\,, \tag{2.28}$$

where the Lie derivative $\mathcal{L}_\zeta A_\mu$ of the gauge field A_μ is given by

$$\mathcal{L}_\zeta A_\mu = \zeta^\nu \partial_\nu A_\mu + A_\nu \partial_\mu \zeta^\nu \tag{2.29}$$

and

$$F = \mathrm{d}A + A \wedge A = 0 \tag{2.30}$$

are the equations of motion of Chern-Simons theory. Thus we see that on-shell, i.e. for $F = 0$, gauge transformations with this gauge parameter are equivalent to diffeomorphisms.

2.3 Global AdS$_3$ and BTZ black holes

Under Brown-Henneaux boundary conditions any asymptotically AdS$_3$ metric can be written in a form with two arbitrary functions $\mathcal{L}_+(x^+)$ and $\mathcal{L}_-(x^-)$ [18]:

$$\mathrm{d}s^2 = \ell^2 \left[\mathrm{d}\rho^2 + (\mathcal{L}_+(\mathrm{d}x^+)^2 + \mathcal{L}_-(\mathrm{d}x^+)^2) - \left(e^{2\rho} + \mathcal{L}_+\mathcal{L}_- e^{-2\rho}\right)\mathrm{d}x^+\mathrm{d}x^-\right] \tag{2.31}$$

One choice for the corresponding connections is given by

$$A^+ = +\left(\frac{1}{2}e^\rho L_{+1} - \frac{2}{k}\mathcal{L}_+(x^+)e^{-\rho}L_{-1}\right)\mathrm{d}x^+ + L_0\mathrm{d}\rho \tag{2.32a}$$

$$A^- = -\left(\frac{1}{2}e^\rho L_{-1} - \frac{2}{k}\mathcal{L}_-(x^-)e^{-\rho}L_{+1}\right)\mathrm{d}x^- - L_0\mathrm{d}\rho\,. \tag{2.32b}$$

The metric (2.31) and the connections (2.32) both are exact solutions of three dimensional Einstein gravity, i.e. of Einstein's field equations or of the Chern-Simons field equations (2.30) respectively.

Special cases of the above spacetime include global AdS$_3$ and BTZ black holes. The first case corresponds to the constant values $\mathcal{L}_+ = \mathcal{L}_- = 1/4$ whereas the latter is given by

$$\mathcal{L}_+ = \frac{2G}{\ell}(J - \ell M)\,, \qquad \mathcal{L}_- = -\frac{2G}{\ell}(J + \ell M)\,, \tag{2.33}$$

where M and J are the mass and the angular momentum of the black hole respectively. Under the coordinate transformations

$$x^\pm = \frac{1}{\ell}t \pm \phi\,, \tag{2.34a}$$

$$r^2 = r_+^2 \cosh^2(\rho - \rho_0) - r_-^2 \sinh^2(\rho - \rho_0)\,, \tag{2.34b}$$

the line element of BTZ black holes can then be brought into ADM form

$$\mathrm{d}s^2 = -N(r)^2 \mathrm{d}t^2 + N(r)^{-2}\mathrm{d}r^2 + r^2(N^\phi(r)\mathrm{d}t + \mathrm{d}\phi)^2, \tag{2.35}$$

where

$$N(r)^2 = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2} \tag{2.36a}$$

$$N^\phi(r) = -\frac{J}{2r^2}\,. \tag{2.36b}$$

These black holes have very similar properties to their four dimensional counterpart, namely Kerr black holes. In particular, they possess an inner and an outer horizon $r = r_-$ and $r = r_+$, in terms of which the black hole's mass and angular momentum can be expressed as

$$M = \frac{r_+^2 + r_-^2}{\ell^2}\,, \quad J = \frac{2r_+ r_-}{\ell}\,. \tag{2.37}$$

The question if the BTZ black hole possesses an ergosphere is a somewhat subtle question and depends on the definition of an ergoregion. If one defines the ergosphere by a sign change of the asymptotically timelike Killing vector ∂_t, then the answer is yes, there exists an ergoregion inside [19]

$$r_{\text{erg}} = \sqrt{r_+^2 + r_-^2}\,. \tag{2.38}$$

This argument holds for BTZ and also for Kerr black holes. However, for BTZ (and not so for Kerr black holes), there exists a Killing vector, given by $\partial_t + (r_-/r_+)\partial_\varphi$, that stays timelike everywhere between asymptotic infinity and the event horizon r_+. Thus, the observer moving along this Killing vector would not be able to see any of the effects normally associated with an ergosphere, i.e. energy extraction or frame dragging.

2.4 Soft hair on BTZ black holes

The line element (2.35) has been shown to approach Rindler spacetime at the horizon [8, 20]

$$\mathrm{d}s^2 = -a^2 r^2 \mathrm{d}t^2 + \mathrm{d}r^2 + \gamma^2 \mathrm{d}\varphi^2 + \dots \tag{2.39}$$

in coordinates where $r = 0$ corresponds to the horizon. The constant a is the Rindler acceleration and the horizon area is given by $A = \oint \mathrm{d}\varphi\gamma$. For the BTZ solution, these constants are given by $\gamma = r_+$ and $|\omega| = r_-/\ell$.
More generally, a whole family of spacetimes given by

$$\begin{aligned}\mathrm{d}s^2 = {} & \mathrm{d}r^2 - \left((a^2 - \Omega^2)\cosh^2(r) - a^2\right) \mathrm{d}t^2 \\ & + 2(\gamma\Omega\cosh^2(r) + a\omega\sinh^2(r))\,\mathrm{d}t\,\mathrm{d}\phi \\ & + (\gamma^2\cosh^2(r) - \omega^2\sinh^2(r))\,\mathrm{d}\phi^2\,,\end{aligned} \tag{2.40}$$

approaches (2.39) near the horizon. This line element solves the equations of motion of Einstein gravity provided the conditions

$$\dot{\gamma} = \Omega'\,, \qquad \dot{\omega} = -a'\,, \tag{2.41}$$

are met. Then, in [8], boundary conditions are proposed. There, the gauge fields that are compatible with these proposed boundary conditions are written in the form

$$A^\pm = b_\pm^{-1}(\mathrm{d} + \mathfrak{a}^\pm)b_\pm\,, \tag{2.42}$$

with gauge group elements $b_\pm$ that depend only on the radial coordinate. The choice

$$b_\pm = \exp\left(\pm\frac{r}{2\ell}(L_1 - L_{-1})\right)\,, \tag{2.43}$$

allows to express the auxiliary connections $\mathfrak{a}^\pm$ in the following way:

$$\mathfrak{a}^\pm = L_0\left(\pm\mathcal{J}^\pm \mathrm{d}\varphi + \zeta^\pm \mathrm{d}t\right) \tag{2.44}$$

The two novel functions are given by

$$\mathcal{J}^\pm = \gamma\ell^{-1} \pm \omega\,, \qquad \zeta^\pm = -a \pm \Omega\ell^{-1}\,. \tag{2.45}$$

From the proposed boundary conditions one can calculate the associated canonical charges. They are given by

$$\mathcal{Q}^\pm[\eta^\pm] = \mp\frac{k}{4\pi}\oint \mathrm{d}\varphi\,\eta^\pm\mathcal{J}^\pm\,, \tag{2.46}$$

and are therefore finite and conserved in time. When the charges are expanded in terms of Fourier modes

$$J_n^{\pm} = \frac{k}{4\pi} \int \mathrm{d}\varphi e^{+in\varphi} \mathcal{J}^{\pm} \,, \tag{2.47}$$

it can be checked that these satisfy the algebra

$$[J_n^{\pm},\, J_m^{\pm}] = \tfrac{1}{2} k\, n \delta_{n+m,0} \tag{2.48a}$$
$$[J_n^{+},\, J_m^{-}] = 0\,. \tag{2.48b}$$

These are two $\mathfrak{u}(1)$ current algebras. Equivalently, this can be rewritten as infinitely many copies of the Heisenberg algebra with two Casimir operators X_0 and P_0:

$$\begin{aligned} &[X_n,\, X_m] = [P_n,\, P_m] = [X_0, P_n] = [P_0,\, X_n] = 0 \\ &[X_n,\, P_m] = i\,\delta_{n,m} \quad \text{for} \quad n \neq 0 \end{aligned} \tag{2.49}$$

where

$$\begin{aligned} P_0 &= J_0^+ + J_0^- \\ P_n &= \tfrac{i}{kn}(J_{-n}^+ + J_n^-) \quad \text{for} \quad n \neq 0 \\ X_n &= J_n^+ - J_{-n}^-\,. \end{aligned} \tag{2.50}$$

The Hamiltonian of the system is defined as the charge associated with unit time translations

$$H = Q[\eta^{\pm}|_{\partial_t}] \equiv \mathcal{Q}^+[\eta^+|_{\partial_t}] - \mathcal{Q}^-[\eta^-|_{\partial_t}]\,, \tag{2.51}$$

where $\eta^{\pm}|_{\partial_t} = \mathfrak{a}_t^{\pm} = L_0\zeta^{\pm}$ according to eq. (2.44). For the choice $\zeta^{\pm} = -a$, or equivalently $\Omega = 0$, this yields

$$H = aP_0 = a(J_0^+ + J_0^-)\,, \tag{2.52}$$

which, according to eqs. (2.48) or (2.49), is a Casimir operator. A remarkable consequence of this fact is that there exist “soft hair” excitations

$$|\psi\rangle \sim \prod_{i,j} (J_{n_i^+}^+)^{m_i^+} (J_{n_j^-}^-)^{m_j^-}\; |0\rangle \tag{2.53}$$

of the vacuum state $|0\rangle$. All states (2.53) have the same energy as the vacuum. The name “soft hair” was introduced in 2016 by Hawking, Perry and Strominger [21], while the zero energy “soft hair” excitations of horizons were first introduced in [8]. Furthermore, the entropy is then found to be

$$S = 2\pi(J_0^+ + J_0^-)\,, \tag{2.54}$$

i.e. it is only given in terms of the zero mode charges.

2.5 Chern-Simons-like theories of gravity

In [22,23], the Chern-Simons action (2.17) is generalized to include a whole family of gravity theories, including Einstein gravity. This is done by attaching an additional field space index to the $sl(2,\mathbb{R})$ valued one-form fields. Then, the action of a CS-like theory is given by

$$I = \frac{k}{2\pi}\int \mathrm{tr}\left(g_{\mathtt{pq}} a^{\mathtt{p}} \wedge \mathrm{d}a^{\mathtt{q}} + \frac{1}{3} f_{\mathtt{pqr}} a^{\mathtt{p}} \wedge a^{\mathtt{q}} \wedge a^{\mathtt{r}}\right), \tag{2.55}$$

where $g_{\mathtt{pq}}$ and $f_{\mathtt{pqr}}$ are a completely symmetric metric on the field space and structure constants respectively. The theory one is interested in can then be specified with the choice of the components of the field space metric and the structure constants. For example, the choice

$$g_{e\omega} = -1\,, \qquad f_{e\omega\omega} = -1\,, \qquad f_{eee} = -1\,, \tag{2.56}$$

yields the first-order action of Einstein gravity. Under the assumption that $g_{\mathtt{pq}}$ is invertible, its inverse can be used to raise indices on the field space. Gauge-like transformations of the fields $a^{\mathtt{p}} \to a^{\mathtt{p}} + \delta_\xi a^{\mathtt{p}}$ can be written in a form similar to the case of pure Chern-Simons theory:

$$\delta_\xi a^{\mathtt{p}} = \mathrm{d}\xi^{\mathtt{p}} + f^{\mathtt{p}}{}_{\mathtt{qr}}[a^{\mathtt{q}}, \xi^{\mathtt{r}}]. \tag{2.57}$$

Furthermore, diffeomorphisms are generated by the above formula when the specific choice

$$\xi^{\mathtt{p}} = a_\nu{}^{\mathtt{p}} \zeta^\nu \tag{2.58}$$

for the gauge parameters is employed:

$$\delta_\zeta a_\mu{}^{\mathtt{p}} = \zeta^\nu \partial_\mu a_\nu{}^{\mathtt{p}} + a_\nu{}^{\mathtt{p}} \partial_\mu \zeta^\nu + \cdots \overset{\mathrm{on-shell}}{=} \mathcal{L}_\zeta a_\mu{}^{\mathtt{p}}\,. \tag{2.59}$$

For this thesis, it will be useful to establish a Hamiltonian formulation of CS-like theories. This can be done starting with a split of time and space components of the fields

$$a^{\mathtt{p}} = a_t{}^{\mathtt{p}} \mathrm{d}t + a_i{}^{\mathtt{p}} \mathrm{d}x^i\,, \tag{2.60}$$

with spatial indices $i, j, \ldots$. This leads to a Lagrangian density of the form

$$\mathcal{L} = \mathrm{tr}(-\epsilon^{ij} g_{\mathtt{pq}} a_i{}^{\mathtt{p}} \partial_t a_j{}^{\mathtt{q}} + 2 a_t{}^{\mathtt{p}} \phi_{\mathtt{p}})\,. \tag{2.61}$$

The time components thus serve as Lagrange multipliers for the primary constraints $\phi_{\mathtt{p}}$, given explicitly by

$$\phi_{\mathtt{p}} = \epsilon^{ij}(g_{\mathtt{pq}}\partial_i a_j{}^{\mathtt{q}} + \tfrac{1}{2} f_{\mathtt{pqr}}[a_i{}^{\mathtt{q}},\, a_j^{\mathtt{r}}])\,. \tag{2.62}$$

Here, the definition $\epsilon^{ij} \equiv \epsilon^{tij}$ was used.
The Hamiltonian density is given by the sum of the primary constraints and Lagrange multipliers

$$\mathcal{H} = -\int \mathrm{d}^2 x\, \mathrm{tr}(a_t{}^{\mathtt{p}}\phi_{\mathtt{p}})\,. \tag{2.63}$$

The Poisson brackets of the canonical fields are given by

$$\{a_i^{n\mathtt{p}}(x),\, a_j^{m\mathtt{q}}(y)\} = \frac{\pi}{k}\epsilon_{ij}\, g^{\mathtt{pq}}\gamma^{nm}\delta^{(2)}(x-y) \tag{2.64}$$

where γ^{nm} denotes the inverse of $\gamma_{nm} = \mathrm{antidiag}(-1, \frac{1}{2}, -1)_{nm}$. It is useful to associate "smeared" functions $\phi[\xi^{\mathtt{p}}]$ with the constraints by integrating them with a test function $\xi^{\mathtt{p}}$ as

$$\phi[\xi^{\mathtt{p}}] = \frac{k}{\pi}\int \mathrm{d}^2 x\, \mathrm{tr}(\xi^{\mathtt{p}}(x)\phi_{\mathtt{p}}(x))\,. \tag{2.65}$$

Now, it turns out that the Poisson brackets of this newly defined function with the spatial components of the fields are given by

$$\{\phi[\xi^{\mathtt{q}}],\, a_i{}^{\mathtt{p}}(y)\} = \delta_\xi a_i{}^{\mathtt{p}}(y) \tag{2.66}$$

and thus generate the spatial part of the gauge-like transformations (2.57).
Choosing the specific form of the gauge fields (2.58) yields that the Poisson brackets of the smeared constraint function and the spatial fields generate diffeomorphisms:

$$\{\phi[a_\mu{}^{\mathtt{q}}\zeta^\mu],\, a_i{}^{\mathtt{p}}(y)\} = \mathcal{L}_\zeta a_i{}^{\mathtt{p}}(y) \tag{2.67}$$

It is noteworthy that this equation does not rely on the equations of motions to hold, i.e. it is also true off-shell.

2.5.1 Boundary charges

On manifolds with a boundary there might appear boundary terms that would lead to ill-defined constraint functions. We therefore need to add a boundary term to them

$$\Phi[\xi^{\mathtt{p}}] = \frac{k}{\pi}\int \mathrm{d}^2 x\, \mathrm{tr}\,(\xi^{\mathtt{p}}(x)\phi_{\mathtt{p}}(x)) + Q[\xi^{\mathtt{p}}]\,. \tag{2.68}$$

The boundary term $Q[\xi^{\mathrm{p}}]$ should be of a form such that it cancels all boundary terms that appear under variation of the constraints with respect to the fields. This requirement is fulfilled by

$$\delta Q[\xi^{\mathrm{p}}] = -\frac{k}{\pi}\oint \mathrm{d}\varphi \, \mathrm{tr}(g_{\mathrm{pq}}\xi^{\mathrm{p}}\delta a^{\mathrm{q}}_{\varphi}) \,. \tag{2.69}$$

These functions are the boundary charges of a theory. With a suitable choice as boundary conditions they should be finite, integrable and conserved.

2.6 Topologically massive gravity

The specific theory of interest for this thesis is topologically massive gravity [24, 25]. As discussed, three dimensional Einstein gravity does not have propagating degrees of freedom. TMG, however, extends general relativity so it propagates a single degree of freedom. It does so by adding an extra term, the so-called gravitational Chern-Simons term, to the Einstein Hilbert action:

$$I_{\mathrm{TMG}} = I_{\mathrm{EH}} + \frac{1}{16\pi G} I_{\mathrm{gCS}} \tag{2.70}$$

The Einstein-Hilbert action is given by

$$I_{\mathrm{EH}} = \frac{1}{16\pi G}\int \mathrm{d}^3x\sqrt{-g}(R - 2\Lambda) \tag{2.71}$$

and the gravitational Chern-Simons term reads

$$I_{\mathrm{gCS}} = \frac{1}{2\mu}\int \mathrm{d}^3x \, \epsilon^{\mu\nu\rho}\left(\Gamma^{\alpha}{}_{\mu\beta}\partial_\nu\Gamma^{\beta}{}_{\rho\alpha} + \frac{2}{3}\Gamma^{\alpha}{}_{\mu\gamma}\Gamma^{\gamma}{}_{\nu\beta}\Gamma^{\beta}{}_{\rho\alpha}\right) , \tag{2.72}$$

with the Christoffel connection Γ and the mass parameter μ as coupling constant. The equations of motion of TMG can then be obtained by varying the action (2.70) with respect to the metric $g_{\mu\nu}$:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0 \,, \tag{2.73}$$

where the Cotton tensor $C_{\mu\nu}$ is given by

$$C_{\mu\nu} = \epsilon_{\mu}{}^{\alpha\beta}\nabla_\alpha\left(R_{\beta\nu} - \frac{1}{4}R\, g_{\beta\nu}\right) = \epsilon_{\mu}{}^{\alpha\beta}\nabla_\alpha S_{\beta\nu} \,, \tag{2.74}$$

with the Schouten tensor $S_{\beta\nu}$. The Cotton tensor can be used to write the Bianchi identity in the elegant form

$$C_{[\mu\nu]} = 0\,. \tag{2.75}$$

Therefore, we are left with only its symmetric part

$$C_{\mu\nu} = \frac{1}{2}\left(\epsilon_\mu{}^{\rho\sigma}\nabla_\rho R_{\sigma\nu} + \epsilon_\nu{}^{\rho\sigma}\nabla_\rho R_{\sigma\mu}\right) \tag{2.76}$$

and see that its trace vanishes, $C^\mu_\mu = 0$. Now, taking the trace of (2.73) yields the coordinate invariant quantity $R = -6/\ell^2$. Here, ℓ is called AdS length and is related to the cosmological constant as $\Lambda = -1/\ell^2$.

One can formulate this theory in a first order form using as basic variables the triad e, dualized spin-connection ω and auxiliary field f by using (2.55) with field space metric and structure constants

$$g_{e\omega} = -1 \quad g_{\omega\omega} = \frac{1}{\mu} \quad g_{ef} = \frac{1}{\mu} \tag{2.77a}$$

$$f_{e\omega\omega} = -1 \quad f_{eee} = -1 \quad f_{e\omega f} = \frac{1}{\mu} \quad f_{\omega\omega\omega} = \frac{1}{\mu} \tag{2.77b}$$

leading to the first order action

$$\begin{aligned} I = -\frac{1}{4\pi G}\int \mathrm{tr}\Big[e \wedge \Big(\mathrm{d}\omega + \frac{1}{2}\,[\omega \overset{\wedge}{,}\ \omega] + \frac{1}{6}\,[e \overset{\wedge}{,}\ e]\Big) \\ -\frac{1}{\mu}\Big(f \wedge (\mathrm{d}e + [\omega \overset{\wedge}{,}\ e]) + \frac{1}{2}\omega \wedge (\mathrm{d}\omega + \frac{1}{3}\,[\omega \overset{\wedge}{,}\ \omega])\Big)\Big]\,. \end{aligned} \tag{2.78}$$

The auxiliary field f serves as a Lagrange multiplier for the torsion constraint. Varying this action with respect to the fields yields the equations of motion

$$\mathrm{d}e + [\omega \overset{\wedge}{,}\ e] = 0 \tag{2.79a}$$

$$\mathrm{d}\omega + \tfrac{1}{2}\,[\omega \overset{\wedge}{,}\ \omega] + [e \overset{\wedge}{,}\ f] = 0 \tag{2.79b}$$

$$\mathrm{d}f + [\omega \overset{\wedge}{,}\ f] + \mu\,[e \overset{\wedge}{,}\ f] - \tfrac{\mu}{2}\,[e \overset{\wedge}{,}\ e] = 0\,. \tag{2.79c}$$

The first equation is nothing but the torsion constraint and can be solved for the spin connection. The second equation of motion can then be solved for f and shows that f is on-shell essentially the Schouten tensor. The third equation is then the first order version of eq. (2.73).

A classification of solutions of TMG is given in e.g. [26]. In [14], a classification of all stationary axi-symmetric solutions to TMG is given. There are no non-trivial static solutions, and all solutions with timelike hypersurface-orthogonal Killing vector are Einstein, i.e. also solve pure Einstein gravity. The authors distinguish between four sectors:

- *Einstein* - This class of solutions solves the three dimensional Einstein equations of motion. All of them are locally AdS$_3$ and they obey Brown-Henneaux boundary conditions.
- *Warped* - This class of solutions is asymptotically or locally warped AdS$_3$.
- *Schrödinger* - This class of solutions is either asymptotically AdS$_3$ or asymptotically Schrödinger.
- *Generic* - All solutions that are neither of the above.

The entropy formula (2.54) has been shown to hold for solutions of the Einstein sector, namely BTZ black holes, and more recently also for warped black holes [12].

2.7 Schrödinger spacetime

Spacetimes with the Schrödinger group as their symmetry group are studied in the context of holography with non-relativistic field theories. For instance, Schrödinger spacetime has been proposed as holographic duals to non-relativistic conformal field theories describing cold atoms at unitarity [27]. In [28], a bulk dual of non-relativistic CFTs is found. This is done by considering the algebra of generators of the non-relativistic conformal group and invariance under non-relativistic scaling

$$t \to \lambda^z t \tag{2.80}$$

$$x^i \to \lambda x^i \tag{2.81}$$

is demanded. Here, x^i are the spatial coordinates and z is called the dynamical critical exponent. Note that for $z = 2$, this is the symmetry of the free particle Schrödinger equation.

The symmetry algebra that is used by the authors of [28] is given by

$$\begin{aligned}
&[M^{ij},\, M^{kl}] = i(\delta^{ik}M^{jl} + \delta^{jl}M^{ik} - \delta^{il}M^{jk} - \delta^{jk}M^{il})\,,\\
&[M^{ij}, N] = [M^{ij}, D] = 0\,, \quad [M^{ij},\, P^k] = i(\delta^{ik}P^j - \delta^{jk}P^i)\,,\\
&[M^{ij},\, K^k] = i(\delta^{ik}K^j - \delta^{jk}K^i)\,,\\
&[P^i, P^j] = [K^i, K^j] = 0\,, \quad [D,\, P^i] = +i\, P^i\,,\\
&[D,\, K^i] = (1-z)iK^i\,, \quad [P^i,\, K^j] = -i\,\delta^{ij}M\,,\\
&[H, N] = [H, P^i] = [H, M^{ij}] = 0\,, \quad [H, K^i] = iP^i\,,\\
&[D,\, H] = izH\,, \quad [D,\, N] = i(2-z)N\,.
\end{aligned} \tag{2.82}$$

The generators of this group consist of

- temporal translations H
- spatial translations P^i
- rotations M^{ij}
- Galilean boosts K^i
- dilatations D
- conserved rest mass or particle number N
- a mass operator M .

In the special case $z = 2$, an additional conformal generator C with commutators

$$[M^{ij}, C] = 0\,, \quad [K^i, C] = 0\,, \quad [D, C] = -2iC\,, \quad [H, C] = -iD\,, \tag{2.83}$$

appears. The spacetime that is invariant under the isometries generated by the above algebra is given by

$$\mathrm{d}s^2 = -\frac{\mathrm{d}t^2}{r^{2z}} + \frac{1}{r^2}\Big(- 2\mathrm{d}t\mathrm{d}\xi + \mathrm{d}r^2 + (\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^d)^2\Big)\,. \tag{2.84}$$

It is invariant under scale transformations

$$x^i \to \lambda x^i\,, \quad t \to \lambda^z t\,, \quad r \to \lambda r\,, \quad \xi \to \lambda^{2-z}\xi\,. \tag{2.85}$$

It should also be noted that the metric (2.84) is not supported by vacuum Einstein gravity by itself, one has to consider a theory with a non-zero stress tensor describing dust.

Gravity duals of non-relativistic field theories have been studied extensively in the literature, see e.g. [28–33].

In this thesis, we are interested in three dimensional gravity and the metric of our interest is given by [34–37]

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{4r^2} + 2r\,\mathrm{d}t\mathrm{d}\varphi + \frac{1}{2}\left(b^2 + ar + sr^z\right)\mathrm{d}\varphi^2\,. \tag{2.86}$$

These black holes have been extensively studied in [34] where they are called z-warped black holes by the authors. Their line element asymptotes to null z-warped AdS$_3$, the three dimensional analogue of Schrödinger spacetime. One can think of these black holes as the double analytical continuation, i.e. exchanging time with a space direction, of the Schrödinger geometry truncated to three dimensions.

Schrödinger spacetime is invariant under an anisotropic scaling $(t, x^i) \to (\lambda^z t, \lambda x^i)$, which is not the case for these black holes, even asymptotically. The parameter z is therefore not to be thought of as a critical exponent here.
When the parameters a and b are zero and φ is non-compact, the line element (2.86) is invariant under dilatations

$$D \equiv -\frac{2}{z} r \frac{\partial}{\partial r} + \varphi \frac{\partial}{\partial \varphi} + \frac{2-z}{z} t \frac{\partial}{\partial t} \,. \tag{2.87}$$

Thus, it is more sensible to interpret the scaling of (null) time with respect to the spacial coordinate φ as critical exponent:

$$Z = \frac{2-z}{z} \tag{2.88}$$

Chapter 3

Soft hairy Schrödinger black holes

In this chapter, our objective is to study the near horizon behaviour of null warped black holes. In section 3.2, we present the metric that will be subject to the calculations that follow. In section 3.3, we find a set of first order fields corresponding to the metric introduced in the first section. In section 3.4, we focus on the case $\mu\ell = 1$ while we find near horizon boundary conditions and gauge transformations that preserve these boundary conditions. In section 3.5, we discuss how the results for $\mu\ell > 1$ relate to the previous ones. Finally, in section 3.6, we calculate the charges, the symmetry algebra and the entropy and then relate the charges to the symmetry algebra at asymptotic infinity.

3.1 General procedure

In this chapter, we will follow a rather straightforward procedure of finding the charges and their algebra near the horizon of the considered black holes. A flow chart depicting the process is given in fig. 3.1 . The first step is to find a set of boundary conditions. These should also include information about which functions are allowed to vary and which are kept fixed under variation. These boundary conditions can be formulated in a first order form, using the triad e_μ, or expressed in terms of the metric $g_{\mu\nu}$. Then, one has to find the transformations that preserve the specified boundary conditions. Once these transformations are found, it is straightforward to calculated the charges associated to the boundary conditions. Now, if the charges turn out to be non-trivial, it has to be checked whether they are integrable, finite and if they are conserved. Should this not be the case, the boundary conditions have to be altered. They could be either too weak or too strong. Once suitable boundary conditions and their charges have been found, one can then expand the charges in Fourier modes and then find the algebra generated by those modes, the near horizon symmetry algebra.

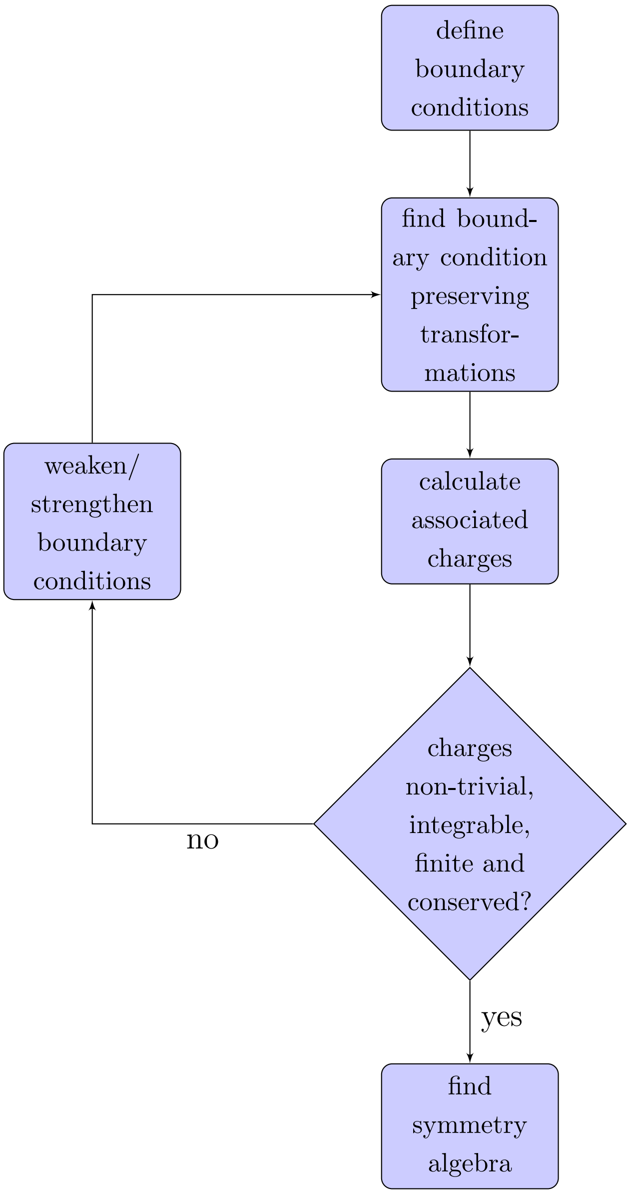

Figure 3.1: Flow chart of the procedure of finding the near horizon symmetry algebra

3.2 Metric

The black holes subject to this thesis can be described by the line element [34–37]

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{4r^2} + 2r\,\mathrm{d}t\mathrm{d}\varphi + \frac{1}{2}f(r)\mathrm{d}\varphi^2\,, \tag{3.1}$$

with

$$f(r) = b^2 + ar + sr^z \tag{3.2}$$

where b and a are constants, s is a sign and z can be given in terms of the coupling constant of TMG:

$$z = \frac{\mu\ell + 1}{2}\,. \tag{3.3}$$

The angular coordinate is periodic $\varphi \sim \varphi + 2\pi$ and the time coordinate is unrestricted. From now on, we will choose $s = +1$. Furthermore, we require $a \geq 0$, which ensures that there are no positive roots of $f(r)$. Then, the range of the radial coordinate is $r_s < r < \infty$ with $r_s < 0$ being the largest real root of $f(r)$. This line element solves the equations of motion of TMG (2.73) for every z. The black holes described by (3.1) possess two isometries, generated by either a null Killing vector ∂_t or a spacelike compact $U(1)$ Killing vector ∂_φ. They are called null warped black holes for $z = 2$ and null z-warped black holes for $z > 2$. When $f(r)$ is negative, the black holes have closed timelike curves, but they are then hidden inside the horizon $r < 0$. The existence of these causal singularities is what justifies the name black hole for these spacetimes. Depending on the value of z, the solution has different asymptotic behaviour, as discussed in e.g. [34]. It can be characterized as

- asymptotically null warped AdS$_3$ for $z = 2$ ($\mu\ell = 3$)
- asymptotically null z-warped AdS$_3$ for $z > 2$
- asymptotically AdS$_3$ under Brown-Henneaux boundary conditions or anti-de Sitter boundary conditions in TMG for $z \leq 1$ ($\mu\ell \leq 1$) .

In the limit $r \to 0$, the situation is different. The term of order $\mathcal{O}(r^z)$ gains more weight the lower z. For the near horizon considerations in the following calculations I am going to distinguish between two cases: $z = 1$ and $z > 1$ Furthermore, for all the following calculations I set $\ell = 1$.

3.2.1 ADM form and extremality

Before we go on to find a first order description of the metric (3.1) we are first going to find where the horizons are located by putting the metric into ADM form. This can be done by introducing the new coordinates

$$\tau = \frac{1}{\sqrt{2}}\left(\varphi - \frac{1}{2}t\right) \qquad \phi = -\frac{1}{\sqrt{2}}\left(\varphi + \frac{1}{2}t\right) \tag{3.4}$$

which makes it possible to write (3.1) in ADM form

$$\mathrm{d}s^2 = -N^2(r)\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{N^2(r)R^2(r)} + R^2(r)(\mathrm{d}\phi - N^\phi(r)\mathrm{d}\tau)^2 \tag{3.5}$$

where

$$N^2(r) = \frac{16r^2}{b^2 + 8r + ar + r^z} \tag{3.6a}$$

$$R^2(r) = \tfrac{1}{4}(b^2 + 8r + ar + r^z) \tag{3.6b}$$

$$N^\phi(r) = \frac{b^2 + ar + r^z}{b^2 + 8r + ar + r^z}\,. \tag{3.6c}$$

The horizons are given by the zeros of $N^2(r)$, hence, they are both located at $r = 0$.

Killing vectors for the metric (3.5) are given by ∂_τ and ∂_ϕ. The event horizon $r = 0$ is a Killing horizon for the Killing vector

$$K = \partial_\tau + \Omega\,\partial_\phi \tag{3.7}$$

when $\Omega = 1$.

Using the definition of surface gravity [38]

$$\kappa^2 = -\frac{1}{2}(\nabla^\mu K^\nu)(\nabla_\mu K_\nu)\Big|_{\text{Horizon}} \tag{3.8}$$

it turns out that the surface gravity vanishes, $\kappa = 0$, and therefore the black holes described by (3.1) describe an extremal black hole.

3.3 Chern-Simons like formulation

In section 2.6, we discussed how TMG, using the metric (2.77a) and structure constants (2.77b), can be reformulated as a CS-like theory which yields the first order action

$$I = -\frac{1}{4\pi G}\int \mathrm{tr}\Big[e \wedge \Big(\mathrm{d}\omega + \frac{1}{2}\,[\omega \overset{\wedge}{,}\ \omega] + \frac{1}{6}\,[e \overset{\wedge}{,}\ e]\Big) \\ -\frac{1}{\mu}\Big(f \wedge (\mathrm{d}e + [\omega \overset{\wedge}{,}\ e]) + \frac{1}{2}\omega \wedge (\mathrm{d}\omega + \frac{1}{3}\,[\omega \overset{\wedge}{,}\ \omega])\Big)\Big]\,. \tag{3.9}$$

We now express the spacetime described by the metric (3.1) in terms of first order variables, i.e. dreibein e, spin-connection ω and auxiliary field f.
A possible dreibein is given by

$$e_t = -\frac{\sqrt{2}\,r}{\sqrt{b^2 + a\,r + r^z}}(T^0 + T^2) \tag{3.10a}$$

$$e_r = \frac{1}{2r}T^1 \tag{3.10b}$$

$$e_\varphi = -\sqrt{\tfrac{1}{2}(b^2 + a\,r + r^z)}\,T_2\,, \tag{3.10c}$$

where the $sl(2,\mathbb{R})$ generators (2.18) have been used. Of course, any dreibein related to eqs. (3.10) by a local Lorentz transformation is allowed as long as

$$g_{\mu\nu} = 2\,\mathrm{tr}(e_\mu e_\nu) \tag{3.11}$$

is satisfied. The choice (3.10) happens to be a nice choice when we want to find boundary conditions later on. The only combinations of the matrices (2.18) that survive under the trace are

$$\mathrm{tr}(T_0^2) = -\frac{1}{2}\,, \qquad \mathrm{tr}(T_1^2) = \frac{1}{2}\,, \qquad \mathrm{tr}(T_2^2) = \frac{1}{2}\,. \tag{3.12}$$

Thus, it is not hard to see that the triad (3.10) reproduces the metric (3.1). The combination of generators in the e_t component may seem odd at first sight, it is necessary to ensure that the g_{tt} component of the metric vanishes however. Using the equations of motion of TMG (2.79) yields spin connection

$$\omega_t = -\frac{\sqrt{2}\,r}{\sqrt{ar + b^2 + r^z}}(T^0 + T^2) \tag{3.13a}$$

$$\omega_r = \frac{b^2 - (z-1)r^z}{2r\,(ar + b^2 + r^z)}T^1 \tag{3.13b}$$

$$\omega_\varphi = -\frac{ar + zr^z}{\sqrt{2}\sqrt{ar + b^2 + r^z}}T^0 + \frac{(z-1)r^z - b^2}{\sqrt{2}\sqrt{ar + b^2 + r^z}}T^2 \tag{3.13c}$$

and auxiliary field

$$f_t = -\frac{r}{\sqrt{2}\sqrt{ar + b^2 + r^z}}(T^0 + T^2) \tag{3.14a}$$

$$f_r = \frac{1}{4r}T^1 \tag{3.14b}$$

$$f_\varphi = -\frac{\sqrt{2}\,(z-1)zr^z}{\sqrt{ar + b^2 + r^z}}T^0 - \frac{ar + b^2 + (1-2z)^2 r^z}{2\sqrt{2}\sqrt{ar + b^2 + r^z}}T^2\,. \tag{3.14c}$$

3.4 The case $z = 1$

We now focus on the case $z = 1$. We will see later on that most of the results that follow generalize to the case $z > 1$. In order to find suitable boundary conditions, the radial coordinate will be rescaled in the following way for the rest of this section:

$$r \to \frac{b^2}{a+1}r \tag{3.15}$$

3.4.1 Boundary conditions

Based on the findings of the last section, we propose the following boundary conditions:

$$e_t = -\sqrt{2}\,\mathcal{C}(t,\varphi)\,r\;(T^0 + T^2) + \mathcal{O}(r^2) \tag{3.16a}$$

$$e_r = \frac{1}{2r}T^1 + \mathcal{O}(1) \tag{3.16b}$$

$$e_\varphi = -\tfrac{1}{\sqrt{2}}\,\mathcal{B}(t,\varphi)\left(1 + \frac{1}{2}r\right)T^2 + \mathcal{O}(r^2), \tag{3.16c}$$

As part of the boundary conditions, the restriction $\delta\mathcal{C}(t,\varphi) = 0$ is imposed while the variation of $\mathcal{B}(t,\varphi)$ is unrestricted.

In order for a spacetime described by (3.16) at the boundary to fulfill the equations of motion of TMG (2.79) however, the following on-shell restrictions also have to hold:

$$\partial_t\,\mathcal{B}(t,\varphi) = 0\,, \qquad \partial_\varphi\,\mathcal{C}(t,\varphi) = 0\,, \tag{3.17}$$

or stated differently

$$\mathcal{B}(t,\varphi) \equiv \mathcal{B}(\varphi)\,, \qquad \mathcal{C}(t,\varphi) \equiv \mathcal{C}(t)\,. \tag{3.18}$$

An exact solution that satisfies the above boundary conditions is given by

$$e_t = -\sqrt{2}\,\mathcal{C}(t,\varphi)\,\frac{r}{\sqrt{1+r}}\,(T^0+T^2) \tag{3.19a}$$

$$e_r = \frac{1}{2r}T^1 \tag{3.19b}$$

$$e_\varphi = -\tfrac{1}{\sqrt{2}}\,\sqrt{1+r}\,\mathcal{B}(t,\varphi)\,T^2, \tag{3.19c}$$

which corresponds to the metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{4r^2} + 2r\mathcal{B}(t,\varphi)\,\mathcal{C}(t,\varphi)\mathrm{d}t\mathrm{d}\varphi + \frac{1}{2}\,(1+r)\,\mathcal{B}(t,\varphi)^2\,\mathrm{d}\varphi^2\,. \tag{3.20}$$

The solution (3.1) is clearly contained in the general metric (3.20) for $z=1$ after the coordinate rescaling (3.15). If the functions $\mathcal{B}$ and $\mathcal{C}$ take on the constant values

$$\mathcal{B} = b\,, \qquad \mathcal{C} = \frac{b}{a+1}\,, \tag{3.21}$$

we recover the black hole solution.

3.4.2 Boundary condition preserving transformations

Now, let's find the gauge transformations $a^{\mathtt{p}} \to a^{\mathtt{p}} + \delta_\xi a^{\mathtt{p}}$ that preserve the boundary conditions above. In section 2.5 it was stated that a gauge transformation in a Chern-Simons-like theory is given by

$$\delta_\xi a^{\mathtt{p}} = \mathrm{d}\xi^{\mathtt{p}} + f^{\mathtt{p}}{}_{\mathtt{qr}}[a^{\mathtt{q}},\,\xi^{\mathtt{r}}]. \tag{3.22}$$

For TMG specifically, plugging in the metric (2.77a) structure constants (2.77b) yields

$$\delta_\xi e = \mathrm{d}\xi^e + [\omega,\,\xi^e] + [e,\,\xi^\omega] \tag{3.23a}$$

$$\delta_\xi \omega = \mathrm{d}\xi^\omega + [\omega,\,\xi^\omega] + [e,\,\xi^f] + [f,\,\xi^e] \tag{3.23b}$$

$$\delta_\xi f = \mathrm{d}\xi^f + [\omega,\,\xi^f] + [f,\,\xi^\omega] + \mu([e,\,\xi^f] + [f,\,\xi^e] - [e,\,\xi^e])\,. \tag{3.23c}$$

We are then searching for the parameters $\xi^{\mathtt{p}}$ that preserve the proposed near horizon boundary conditions (3.16). Hence, they need to satisfy

$$\delta_\xi e_\varphi = -\tfrac{1}{\sqrt{2}}\,\delta\mathcal{B}(\varphi)\,\left(1+\frac{1}{2}\,r\right)T^2 + \mathcal{O}(r^2) \tag{3.24a}$$

$$\delta_\xi \omega_\varphi = -\tfrac{1}{\sqrt{2}}\,\delta\mathcal{B}(\varphi)\,r\,T^0 + \tfrac{1}{\sqrt{2}}\,\delta\mathcal{B}(\varphi)\,\left(1-\frac{1}{2}\,r\right)T^2 + \mathcal{O}(r^2) \tag{3.24b}$$

$$\delta_\xi f_\varphi = -\tfrac{1}{\sqrt{2}}\,\delta\mathcal{B}(\varphi)\,\left(\frac{1}{2}+\frac{1}{4}\,r\right)T^2 + \mathcal{O}(r^2)\,. \tag{3.24c}$$

Variations on all other field components must vanish up to orders

$$\delta_\xi a_t^{\mathtt{p}} = \mathcal{O}(r^2)\,, \qquad \delta_\xi a_r^{\mathtt{p}} = \mathcal{O}(1)\,, \quad \text{for all } \mathtt{p}\,. \tag{3.25}$$

Solving for the components of ξ^p leads to a solution that can be written in terms of two arbitrary functions $\eta(\varphi)$ and $\epsilon(\varphi)$:

$$\xi^e = \frac{1}{\sqrt{2}}\left(\eta(\varphi) + \frac{1}{2}\epsilon(\varphi)r\right)T^2 - \frac{\eta(\varphi) - \epsilon(\varphi)}{2\sqrt{2}} r\, T^0 + \mathcal{O}(r^2) \tag{3.26a}$$

$$\xi^\omega = \frac{1}{\sqrt{2}}\left(-\eta(\varphi) + \frac{1}{2}\epsilon(\varphi)r\right)T^2 + \frac{\eta(\varphi) + \epsilon(\varphi)}{2\sqrt{2}} r\, T^0 + \mathcal{O}(r^2) \tag{3.26b}$$

$$\xi^f = \frac{1}{2}\xi^e \tag{3.26c}$$

However, the parameters above only preserve the first order fields under the condition that the state dependent function $\mathcal{B}(\varphi)$ transforms as

$$\delta\mathcal{B}(\varphi) = \partial_\varphi \eta(\varphi) = \partial_\varphi \epsilon(\varphi)\,. \tag{3.27}$$

Thus, the two functions η and ϵ are not independent. They differ by a constant λ:

$$\epsilon(\varphi) = \eta(\varphi) + \lambda \tag{3.28}$$

Hence, we can write

$$\xi^e = \frac{1}{\sqrt{2}}\left(\eta(\varphi)\left(1 + \frac{1}{2}r\right) + \frac{1}{2}\lambda\, r\right)T^2 + \frac{\lambda}{2\sqrt{2}} r\, T^0 + \mathcal{O}(r^2) \tag{3.29a}$$

$$\xi^\omega = \frac{1}{\sqrt{2}}\left(-\eta(\varphi)\left(1 - \frac{1}{2}r\right) + \frac{1}{2}\lambda\, r\right)T^2 + \frac{2\eta(\varphi) + \lambda}{2\sqrt{2}} r\, T^0 + \mathcal{O}(r^2) \tag{3.29b}$$

$$\xi^f = \frac{1}{2}\xi^e \tag{3.29c}$$

We can also search for parameters $\xi^{\mathtt{p}}$ that leave the full dreibein unaltered under gauge transformations. The parameters that leave the exact dreibein (3.19) unchanged are given by

$$\xi^e = \frac{1}{2\sqrt{2}}\left(\frac{r}{\sqrt{1+r}}\lambda + 2\sqrt{1+r}\,\eta(\varphi)\right)T^2 + \frac{1}{2\sqrt{2}}\frac{r}{\sqrt{1+r}}\,\lambda\, T^0\,, \tag{3.30a}$$

$$\xi^\omega = \frac{1}{2\sqrt{2}}\left(\frac{r}{\sqrt{1+r}}\lambda - \frac{2}{\sqrt{1+r}}\,\eta(\varphi)\right)T^2 + \frac{1}{2\sqrt{2}}\frac{r}{\sqrt{1+r}}(2\eta(\varphi) + \lambda)T^0\,, \tag{3.30b}$$

$$\xi^f = \frac{1}{2}\xi^e\,. \tag{3.30c}$$

They include the near horizon parameters (3.29) in the limit $r \to 0$.
We stated in section 2.5 that diffeomorphisms are generated by gauge parameters of the form $\xi^{\mathtt{p}} = a_\nu{}^{\mathtt{p}}\zeta^\nu$. Thus, we can find a corresponding near horizon Killing vector to the above transformations. It is given by

$$\zeta = \zeta^\mu \partial_\mu = \frac{\lambda}{4\mathcal{C}}\,\partial_t - \frac{\eta(\varphi)}{\mathcal{B}(\varphi)}\,\partial_\varphi\,. \tag{3.31}$$

It should be noted that eqs. (3.29) include the time components of the fields $a_t^{\mathtt{p}}$ as the special case

$$\lambda = 4\mathcal{C}\,, \qquad \eta(\varphi) = 0\,. \tag{3.32}$$

Therefore, we see that we could add a second chemical potential to our time fields. Replacing the time components of the fields with the gauge fields (3.29) as

$$a_t^{\mathtt{p}} \to \xi^{\mathtt{p}}\,, \tag{3.33}$$

yields as on-shell conditions the more familiar form of the Ward identities

$$\partial_t \mathcal{B} = -\partial_\varphi \eta\,. \tag{3.34}$$

3.5 The case $z > 1$

For this case, instead of using (3.15), the radial coordinate now is rescaled as

$$r \to \frac{b^2}{a} r\,. \tag{3.35}$$

The boundary conditions take the form

$$e_t = -\sqrt{2}\,\mathcal{C}(t,\varphi)\,r\;(T^0 + T^2) + \mathcal{O}(r^2) \tag{3.36a}$$

$$e_r = \frac{1}{2r}T^1 + \mathcal{O}(1) \tag{3.36b}$$

$$e_\varphi = -\tfrac{1}{\sqrt{2}}\,\mathcal{B}(t,\varphi)\left(1 + \frac{1}{2}r\right)\,T^2 + \begin{cases} \mathcal{O}(r^z) & \text{for } 1 < z < 2 \\ \mathcal{O}(r^2) & \text{for } z \geq 2 \end{cases} \tag{3.36c}$$

for this case. The black hole solution now corresponds to

$$\mathcal{B} = b\,, \qquad \mathcal{C} = \frac{b}{a}\,, \tag{3.37}$$

where again the restriction $\delta\mathcal{C} = 0$ is imposed while the variation of $\mathcal{B}$ is unrestricted.
The rest of the results is completely analogous to the case $z = 1$. The boundary condition preserving transformations are again given by

$$a^{\mathtt{p}} \to a^{\mathtt{p}} + \delta_\xi a^{\mathtt{p}} \tag{3.38}$$

with the gauge parameter (3.29) and a corresponding Killing vector is given by (3.31).
However, a very significant difference to the case $z = 1$ is the fact that only the perturbative results of the last section hold.

3.6 Charges, symmetry algebra and entropy

The variation of the charges of a general CS-like theory is given by

$$\delta Q[\xi^{\mathtt{p}}] = -\frac{k}{\pi} \oint \mathrm{tr}(g_{\mathtt{pq}} \xi^{\mathtt{p}} \delta a^{\mathtt{q}}_\varphi) \, \mathrm{d}\phi \, . \tag{3.39}$$

Evaluated for this case, i.e. Schrödinger black holes in TMG with the results (3.26) and the fields in section 3.3, the above expression yields

$$\delta Q = \frac{2z}{2z-1} \frac{k}{2\pi} \oint \delta\mathcal{B}(\varphi)\eta(\varphi) \, \mathrm{d}\varphi \, . \tag{3.40}$$

Now, from eq. (3.31) we can see that $\zeta = \partial_\varphi$ amounts to the condition $\eta = -\mathcal{B}$. The conserved charge associated to the asymptotic symmetry ∂_φ is therefore given by

$$Q_{\partial_\varphi} = -\frac{2z}{2z-1} \frac{k}{4\pi} \oint \mathcal{B}^2(\varphi) \, \mathrm{d}\varphi \, . \tag{3.41}$$

which means that the result (up to a sign and a factor of 1/2) agrees with [34].
Assuming $\delta\eta = 0$ instead, the expression (3.40) integrates to

$$Q = \frac{2z}{2z-1} \frac{k}{2\pi} \oint \mathcal{B}(\varphi)\eta(\varphi) \, \mathrm{d}\varphi \, . \tag{3.42}$$

If we expand this expression in Fourier modes

$$J_n = Q[\eta = e^{in\varphi}] = \frac{2z}{2z-1} \frac{k}{2\pi} \oint \mathcal{B}(\varphi) e^{in\varphi} \, \mathrm{d}\varphi \, , \tag{3.43}$$

we find that they satisfy the $\mathfrak{u}(1)$ current algebra

$$[J_n, J_m] = \frac{2z}{2z-1}\, k\, n\, \delta_{n+m,0}\,. \tag{3.44}$$

It should be noted here that this case is different to previous cases since it is not possible to rewrite this algebra as infinitely many copies of the Heisenberg algebra because such a change of basis would require two copies of $\mathfrak{u}(1)$ current algebras.
The entropy was found in [34] to be

$$S = 2\pi\, \frac{2z}{2z-1} k\, \mathcal{B}\,, \tag{3.45}$$

which yields that once again the entropy can be given in terms of a zero mode of a $\mathfrak{u}(1)$ current:

$$S = 2\pi\, J_0 \tag{3.46}$$

The boundary Hamiltonian is defined as the charge associated with unit time-translations. Hence, it is in this case trivially realized by

$$H = Q[\eta = 0, \lambda = 4\mathcal{C}] = 0\,, \tag{3.47}$$

and it obviously commutes with all other charges. All excitations

$$|\psi\rangle \sim \prod_i (J_{n_i})^{m_i} \left|0\right\rangle\,, \tag{3.48}$$

therefore have the same energy as the original one, analogous to the previous black holes studied, albeit realized in a rather trivial manner. These zero energy excitations are again interpreted as "soft hair".

3.6.1 Relationship to asymptotic symmetry algebra

As discovered in [34], the asymptotic symmetry algebra is given by the Virasoro algebra

$$[L_n,\, L_m] = (n-m)L_{n+m} + \frac{c_L}{12} n^3\, \delta_{m+n,0} \tag{3.49}$$

with central extension

$$c_L = \frac{3\ell}{2G}\frac{2z}{2z-1}\,. \tag{3.50}$$

It turns out that the Sugawara construction

$$\mathcal{L} = \frac{1}{4}\frac{2z-1}{2z}\mathcal{J}^2 + \sqrt{\frac{c_L}{12k}\frac{2z-1}{2z}}\mathcal{J}' \tag{3.51}$$

$$= \frac{1}{4}\frac{2z-1}{2z}\mathcal{J}^2 + \frac{1}{\sqrt{2}}\mathcal{J}' \tag{3.52}$$

where

$$\mathcal{J} \equiv 2\,\frac{2z}{2z-1}\,\mathcal{B} \tag{3.53}$$

relates the near horizon algebra (3.44) to the asymptotic algebra (3.49) when expanded in Fourier modes

$$kL_n = \frac{1}{2}\frac{2z-1}{2z}\sum_{p\in\mathbb{Z}} J_{n-p}J_p + ikn\frac{1}{\sqrt{2}}J_n\,. \tag{3.54}$$

Here,

$$\mathcal{J} = \frac{2}{k}\sum_n e^{in\varphi}J_n \tag{3.55}$$

was used. Now, we know that, since we expect a Virasoro algebra at asymptotic infinity, the function $\mathcal{L}$ transforms as

$$\delta\mathcal{L} = 2\mathcal{L}\gamma' + \mathcal{L}'\gamma - \frac{2z}{2z-1}\gamma''' \tag{3.56}$$

with an arbitrary function γ and the transformation properties of $\mathcal{B}$ were found in section 3.4.2 to be $\delta\mathcal{B} = -\partial_\varphi\eta$. The Ansatz

$$\eta = c_0\gamma' + c_1\gamma\mathcal{B} \tag{3.57}$$

$$\mathcal{L} = c_2\mathcal{B}^2 + c_3\mathcal{B}' \tag{3.58}$$

then leads to

$$\eta = \frac{1}{\sqrt{2}}\gamma' - \gamma\mathcal{B} \tag{3.59}$$

$$\mathcal{L} = \frac{2z}{2z-1}\mathcal{B}^2 + \sqrt{2}\frac{2z}{2z-1}\mathcal{B}'\,. \tag{3.60}$$

In section 3.4.2 it was mentioned that we could introduce an additional chemical potential by replacing the time components of the fields with the gauge parameter.

For $z = 1$, this yields

$$(g_{\mu\nu}) = \begin{array}{cc} \begin{array}{ccc} t & r & \varphi \end{array} & \\ \begin{pmatrix} 2r\,\eta\,\mathcal{C} + \frac{1}{2}(1+r)\eta^2 & 0 & \frac{1}{2}r\,\mathcal{B}\,\mathcal{C} + \frac{1}{4}(1+r)\eta\mathcal{B} \\ 0 & \frac{1}{4r^2} & 0 \\ \frac{1}{2}r\,\mathcal{B}\,\mathcal{C} + \frac{1}{4}(1+r)\eta\mathcal{B} & 0 & \frac{1}{2}(1+r)\mathcal{B}^2 \end{pmatrix} & \begin{array}{c} t \\ r \\ \varphi \end{array} \end{array} \tag{3.61}$$

Since we know how the gauge parameter γ transforms, we know also how the corresponding chemical potential μ needs to transform:

$$\eta = \frac{1}{\sqrt{2}}\mu' - \mu\mathcal{B} \tag{3.62}$$

For constant μ and $\mathcal{B}$, it is easy to see that (3.61) expressed in terms of the asymptotic functions $\mathcal{L}$ and μ can be written as

$$(g_{\mu\nu}) = \begin{array}{cc} \begin{array}{ccc} t & r & \varphi \end{array} & \\ \begin{pmatrix} -\sqrt{2}r\mu\sqrt{\mathcal{L}}\,\mathcal{C} + \frac{1}{4}(1+r)\mu^2\mathcal{L} & 0 & \frac{\sqrt{2}}{4}r\,\mathcal{C}\sqrt{\mathcal{L}} - \frac{1}{8}(1+r)\mu\mathcal{L} \\ 0 & \frac{1}{4r^2} & 0 \\ \frac{\sqrt{2}}{4}r\,\mathcal{C}\sqrt{\mathcal{L}} - \frac{1}{8}(1+r)\mu\mathcal{L} & 0 & \frac{1}{2}(1+r)\mathcal{L} \end{pmatrix} & \begin{array}{c} t \\ r \\ \varphi \end{array} \end{array} \tag{3.63}$$

In the more general case, i.e. when $\mathcal{B}' \neq 0$ and $\mu' \neq 0$, the situation becomes much more complex and an expression of the metric in terms of the asymptotic charges and chemical potentials is yet to be found. Things that can be tried are gauge transformations

$$g_{\mu\nu} = 2\,\mathrm{tr}\,((e_\mu + \delta e_\mu)(e_\nu + \delta e_\nu)) \tag{3.64}$$

with

$$\delta e_\mu = \delta_{\xi=\zeta^\nu a_\nu} e_\mu\,, \tag{3.65}$$

as well as ordinary coordinate transformations.

Chapter 4

Conclusion

In this thesis, we studied the near horizon symmetries of Schrödinger black holes in three dimensional topologically massive gravity. We translated the black hole metric into a first order formalism, namely that of Chern-Simons like theories. In terms of the first order fields we were able to find boundary conditions that were then used to calculate the associated boundary charges. It turns out that their Fourier modes satisfy a $\mathfrak{u}(1)$ current algebra

$$[J_n, J_m] = \frac{2z}{2z-1} k\, n\, \delta_{n+m,0}\,. \tag{4.1}$$

Furthermore, the entropy is given only in terms of the zero mode of the $\mathfrak{u}(1)$ charge:

$$S = 2\pi\, J_0\,. \tag{4.2}$$

This is one of the main results of this thesis, as the main goal was to provide further evidence for the universality of the above result. Finally, we tried to connect the obtained $\mathfrak{u}(1)$ current algebra to the asymptotic symmetry algebra given by the Virasoro algebra. We find a map between the near horizon charges and the asymptotic charges heuristically and then attempt a derivation from first principles.

It would be very rewarding to completely understand the map of the near horizon charges given in the last subsection from first principles.

Furthermore, it would be very interesting to study the near horizon behaviour of higher dimensional black holes on backgrounds dual to non-relativistic field theories, see e.g. [39–48]. Then, one could study the implications of these results on the dual field theories and therefore on physical systems such as cold atoms at unitarity [27].

Appendix A

Conventions

Natural units $\hbar = c = 1$ are used throughout this thesis.

For the metric, the mostly plus sign convention $(-, +, +)$ is used.

The Levi-Civita-symbol, denoted by ϵ, is understood with the sign convention

$$\epsilon_{tr\varphi} = +1\,. \tag{A.1}$$

The sign convention for the Ricci-tensor

$$R_{\mu\nu} = +\partial_\lambda \Gamma^\lambda{}_{\mu\nu} - \dots \tag{A.2}$$

is employed.

The wedge product of some Lorentz vectors X_μ and Y_μ is defined as

$$(X \wedge Y)_{\mu\nu} = \tfrac{1}{2}(X_\mu Y_\nu - X_\nu Y_\mu)\,. \tag{A.3}$$

The wedged commutators can be written in index notation as

$$[B \overset{\wedge}{,}\, A] = [A \overset{\wedge}{,}\, B] \equiv T_a\, \epsilon^{abc}\, A_b \wedge B_c \tag{A.4}$$

and relate to the crossproduct notation of [23] as

$$[B \overset{\wedge}{,}\, A] = [A \overset{\wedge}{,}\, B] \equiv A \times B = B \times A\,. \tag{A.5}$$

The following identity is useful for e.g. varying the action:

$$A \wedge [B \overset{\wedge}{,}\, C] = B \wedge [C \overset{\wedge}{,}\, A] = C \wedge [A \overset{\wedge}{,}\, B]\,. \tag{A.6}$$

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