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The Optimal Extraction of Non-Renewable Resources Under Hyperbolic Discounting

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Kurzfassung

In der vorliegenden Arbeit untersuche ich, wie sich hyperbolische Diskontierung auf den optimalen Abbau von nicht erneuerbaren Ressourcen auswirkt. Dazu löse ich ein einfaches Modell des Ressourcenabbaus und das Dasgupta-Heal-Solow-Stiglitz (DHSS) Modell auf zwei verschiedene Arten. Einmal wird klassischer exponentielle Diskontierung verwendet und einmal eine zeitkonsistente Art der hyperbolischen Diskontierung. Anschließend vergleiche ich die Resultate unter der Normierung, dass der Gegenwartswert eines konstanten Nutzenniveaus unter beiden Diskontierungsarten gleich groß ist.
Die Analyse der Ergebnisse zeigt, dass Ressourcen unter hyperbolischer Diskontierung langsamer abgebaut werden, obwohl die kurzfristige Diskontrate der Haushalte viel höher ist als bei exponentieller Diskontierung. Das liegt daran, dass die niedrige langfristige Diskontrate der Haushalte die Effekte ihrer Gegenwartspräferenz ausgleicht. Dadurch sind sie bereit, mittelfristig auf Konsum zu verzichten um langfristig mehr konsumieren zu können. Sowohl der Kapitalstock als auch der Ressourcenbestand sind in der Nähe des Steady-States mit hyperbolischer Diskontierung höher. Dies führt dazu, dass auch die langfriste Wachstumsrate des Konsums höher ist als bei exponentieller Diskontierung. Insbesondere wird gezeigt, dass unter exponentieller Diskontierung ein deutlich schnellerer technologischer Fortschritt benötigt wird, um langfristiges Konsumwachstum zu garantieren.

Abstract

In this thesis, I investigate the effects of hyperbolic discounting on the optimal extraction of non-renewable resources. For this purpose, I apply a time-consistent method of hyperbolic discounting to a simple model of resource extraction and to the Dasgupta-Heal-Solow-Stiglitz (DHSS) model of resource extraction and capital accumulation. I then compare the results to those obtained using conventional exponential discounting applying the normalization that the present value of a constant utility stream is the same in both frameworks. The analysis shows that resource use is more conservative under hyperbolic discounting, even though households' short-term discount rate is much larger compared to exponential discounting. This is due to the fact that their low long-run discount rate offsets the effects of their present-bias, making them willing to forgo some utility in the medium run in order to consume more in the long run. Both the capital stock and the resource stock tend to be larger in the scenario with hyperbolic discounting than with exponential discounting near the steady state. This in turn leads to higher long-run consumption growth rate. In particular, I show that with exponential discounting much higher technological growth is needed to achieve positive consumption growth at all.

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Eidesstattliche Erklärung

Ich erkläre eidesstattlich, dass ich die vorliegende Diplomarbeit selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Wien, am 18.07.2018

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1 Introduction

The appropriate choice of the social discount rate is of great importance for the evaluation of policies with very long-run impacts. This is specifically true for environmental policies, whose costs are typically borne in the near future while their benefits occur in the distant future. In economic analysis, the standard discounting method applied is exponential discounting, which corresponds to a constant discount rate. However, empirical evidence from the fields of psychology and behavioural economics suggests that the individual rate of time preference declines over time (see DellaVigna, 2009; Frederick, Loewenstein, and O'Donoghue, 2002; Loewenstein and Prelec, 1992). This suggests that the social discount rate, which is closely related to the individual rate of time preference, declines over time as well.

The same result is achieved by assuming that shocks to the consumption rate are uncertain but positively correlated. Gollier (2013) shows that in this case the efficient result is a declining discount rate as well.

Some countries, namely the United Kingdom and France, already apply declining discount rates in the evaluation of public projects (Lebègue, 2005; Treasury, 2003). In the United States, the Office of Management and Budget (OMB) still recommends the use of a constant discount rate. However, a group of leading environmental economists recently suggested the use of a declining discount rate for the costs and benefits of long-run projects in the USA (Arrow et al., 2014).

A concern that often arises in the context of hyperbolic discounting is that it may lead to time-inconsistent behaviour. Indeed, hyperbolic discounting is oftentimes used in behavioural economics to explain time-inconsistent decision making, which leads to many people believing that hyperbolic discounting cannot be time-consistent. However, it can be shown that decision making is time-consistent if and only if the discount function is multiplicatively separable in decision time t_0 and pay-off time t (Burness, 1976; Drouhin, 2009).

Strulik (2017) introduces a time-consistent hyperbolic discount function which is multiplicatively separable in decision time t_0 and pay-off time t and therefore leads to time-consistent decision making. He then uses this discount function to solve three environmental problems for hyperbolic discounting: the optimal use of a renewable resource, the non-cooperative use of an open access resource, and the problem of optimal growth and pollution.

In this thesis, I apply the method of hyperbolic discounting proposed by Strulik (2017) to the well-known Dasgupta-Heal-Solow-Stiglitz (DHSS) model (Dasgupta and Heal, 1974; Solow, 1974; Stiglitz, 1974) in order to understand how hyperbolic discounting affects the extraction of non-renewable resources.

In order to gain a basic understanding of how hyperbolic discounting changes the mechanisms of the model, I first introduce a simple model of resource extraction that does not take capital accumulation into account. Utility is gained from resources extracted, which are immediately consumed. The model can be solved analytically for both exponential and hyperbolic discounting. I then compare the results using the equivalent-present-value argument (Myerson, Green, and Warusawitharana, 2001), which gives me a parameter restriction for which both discounting methods provide the same present value of a constant flow.

Contrary to what one might expect, the comparison shows that resource depletion is not faster with hyperbolic discounting compared to exponential discounting. While the level of the intertemporal rate of substitution determines whether resource extraction is higher for exponential or hyperbolic discounting in the short run, resource extraction is always lower with hyperbolic discounting in the medium run. This leads to a higher resource stock with hyperbolic discounting in the long run. Therefore, the economy can afford to extract more resources in the long run compared to exponential discounting.

This seemingly counter-intuitive result can be explained by the fact that even though the discount rate is higher for hyperbolic discounting in the short run, it drops below the level of the discount rate for exponential at some point and remains lower thereafter. Since complete depletion of the resource in finite time can never be optimal, households inevitably are more patient with hyperbolic discounting after some point. This leads to a more conservative resource use.

I then solve the DHSS model of resource extraction and capital accumulation for both discounting methods analytically and numerically using the relaxation algorithm of Trimborn, Koch, and Steger (2008). For the comparison between the discounting methods, I again use the equivalent-present-value approach.

The analysis of the results shows a pattern of resource extraction similar to the simple model. Again, resource extraction is lower in the medium run and higher in the long run for hyperbolic discounting. However, short-term resource extraction is always higher for hyperbolic discounting in the short run, reflecting households' present-bias.

The DHSS model further provides results concerning the evolution of consumption and capital stock. The long-run growth rate of consumption is always higher for hyperbolic discounting. In particular, much higher technological growth is needed for consumption

growth to be positive with exponential discounting.
After the qualitative comparison, I calibrate the model parameters to fit data from the United States of America in order to make a quantitative comparison of the discounting methods. The model predicts resource extraction with hyperbolic discounting to drop below the level of the scenario with exponential discounting after 15 years. However, after only 70 years, the resource extraction curves intersect again and resource extraction remains higher with hyperbolic discounting thereafter. A sensitivity analysis shows that both the qualitative and the quantitative results are robust with respect to parameter variations.
The remainder of this thesis is organized as follows. In chapter 2 I solve the simplified model and compare both discounting methods. Chapter 3 and chapter 4 solve the DHSS model for conventional exponential discounting and hyperbolic discounting. In chapter 5 the model parameters are calibrated and the results are compared in chapter 6, again using the equivalent-present-value argument. Chapter 7 performs the sensitivity analysis and chapter 8 concludes.

2 The Simplified Model

Before turning to the more complex DHSS model, we will first study the differences between exponential and hyperbolic discounting in a simple resource model.
Consider a social planner who maximizes households' welfare U understood as utility from resource extraction $u(R(t))$ experienced over an infinite time horizon and discounted to the present using the discount function $D(t)$

$$U = \int_0^\infty D(t)u(R(t))dt. \tag{2.1}$$

The resources are extracted from a resource stock S, the initial resources stock S_0 is given. We will first solve the model using exponential discounting; afterwards, we will turn to hyperbolic discounting and compare the solutions.

2.1 Exponential Discounting

Using exponential discounting and a utility function of the CRRA type, the model can be written as

$$\underset{R}{max} \int_0^\infty \frac{R^{1-\sigma}-1}{1-\sigma}\mathrm{e}^{-\rho t}dt \tag{2.2}$$

$$s.t.\ \dot{S} = -R,\ \ S(0) = S_0$$

The associated Hamiltonian and the first order conditions are given by

$$\begin{aligned} H &= \frac{R^{1-\sigma}-1}{1-\sigma}\mathrm{e}^{-\rho t} + \lambda_S(-R) \\ H_R &= R^{-\sigma}\mathrm{e}^{-\rho t} - \lambda_S = 0 \\ H_S &= 0 = -\dot{\lambda_S} \\ \dot{S} &= -R \end{aligned}$$

To ensure that the value of resource stock S approaches 0 in the long run, the optimal trajectory must further fulfil the transversality condition

$$\lim_{t\to\infty} \lambda_S(t)S(t) = 0$$

By combining the first order conditions we get the growth rates of S and R:

$$
\begin{aligned}
&\frac{\dot{S}}{S} = -\frac{R}{S} \\
&R^{-\sigma}\mathrm{e}^{-\rho t} = \lambda_S && |\ln() \\
&-\sigma\ln(R) - \rho t = \ln(\lambda_S) && |\frac{\partial}{\partial t} \\
&-\sigma\frac{\dot{R}}{R} - \rho = \frac{\dot{\lambda_S}}{\lambda_S} = 0 \\
&\frac{\dot{R}}{R} = -\frac{\rho}{\sigma}
\end{aligned}
$$

The optimal path of resource extraction can easily be calculated:

$$
\begin{aligned}
&\frac{\dot{R}}{R} = -\frac{\rho}{\sigma} && |\int \\
&\ln(R) = -\frac{\rho}{\sigma}t + C && |\exp() \\
&R = \mathrm{e}^{-\frac{\rho}{\sigma}t}C_1
\end{aligned}
$$

Because of

$$S_0 = \int_0^\infty R(t)dt = \int_0^\infty \mathrm{e}^{-\frac{\rho}{\sigma}t}C_1 dt = \mathrm{e}^{-\frac{\rho}{\sigma}t}(-\frac{\rho}{\sigma})C_1 \Big|_0^\infty = \frac{\sigma}{\rho}C_1$$

we set C_1 equal to $\frac{\rho}{\sigma}$ in order to normalize S_0 to 1. The optimal path of resource extraction is therefore given by

$$R = \mathrm{e}^{-\frac{\rho}{\sigma}t}\frac{\rho}{\sigma} \tag{2.3}$$

2.2 Hyperbolic Discounting

We will now solve the same problem with hyperbolic discounting instead of exponential discounting. The problem reads

$$\max_R \int_0^\infty \frac{R^{1-\sigma}-1}{1-\sigma}\left(\frac{1}{1+at}\right)^b dt \tag{2.4}$$
$$s.t.\ \dot{S} = -R,\ \ S(0) = S_0$$

Hamiltonian and first order conditions are given by

$$\begin{aligned}
&H = \frac{R^{1-\sigma}-1}{1-\sigma}\left(\frac{1}{1+at}\right)^b + \lambda_S(-R)\\
&H_R = R^{-\sigma}(\frac{1}{1+at})^b - \lambda_S = 0\\
&H_S = 0 = -\dot{\lambda_S}\\
&\dot{S} = -R
\end{aligned}$$

Similarly to the model with exponential discounting, we can now solve for the growth rates of S and R.

$$\begin{aligned}
&\frac{\dot{S}}{S} = -\frac{R}{S}\\
&R^{-\sigma}(\frac{1}{1+at})^b = \lambda_S && |\ln()\\
&-\sigma\ln(R) - b(\ln(1+at)) = \ln(\lambda_S) && |\frac{\partial}{\partial t}\\
&-\sigma\frac{\dot{R}}{R} - \frac{ba}{1+at} = \frac{\dot{\lambda_S}}{\lambda_S} = 0\\
&\frac{\dot{R}}{R} = -\frac{1}{\sigma}(\frac{ba}{1+at})
\end{aligned}$$

For the optimal path of resource extraction we get

$$\begin{aligned}
&\frac{\dot{R}}{R} = -\frac{1}{\sigma}(\frac{ba}{1+at}) && |\int\\
&\ln(R) = -\frac{b}{\sigma}\ln(1+at) + C && |\exp()\\
&R = (1+at)^{-\frac{b}{\sigma}}C_2
\end{aligned}$$

In order to normalize S_0 to 1 too, we calculate

$$S_0 = \int_0^\infty R(t)dt = \int_0^\infty (1+at)^{-\frac{b}{\sigma}} C_2 dt = \frac{\sigma}{a(\sigma - b)} (1+at)^{1-\frac{b}{\sigma}} C_2 \Big|_0^\infty$$

There is no solution for $b < \sigma$, but for $b > \sigma$ we get

$$S_0 = \int_0^\infty R(t)dt = \frac{\sigma}{a(b-\sigma)} C_2$$

Therefore, C_2 has to be set equal to $\frac{a(b-\sigma)}{\sigma}$. The optimal path of resource extraction is given by

$$R = (1+at)^{-\frac{b}{\sigma}} \frac{a(b-\sigma)}{\sigma} \tag{2.5}$$

2.3 Comparing Discounting Methods

In order to make a fair comparison between hyperbolic and exponential discounting, we will apply the equivalent present-value approach (Myerson, Green, and Warusawitharana, 2001), which requires the parameters to be chosen such that the present value of a constant utility stream is the same for both discounting methods, $\int_0^\infty e^{-\rho t} dt = \int_0^\infty (\frac{1}{1+at})^{-b} dt$. Solving the integrals leads to the parameter requirement

$$\rho = a(b-1) \tag{2.6}$$

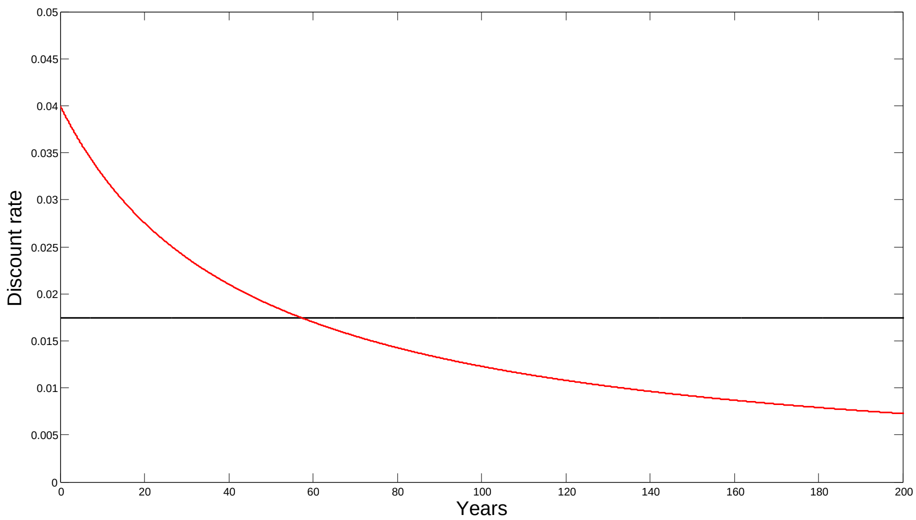


Figure 2.1: Discount rates for exponential (black line) and hyperbolic (red line) discounting

We set $a = 0.0225$ and $b = 1.777$ (see Strulik, 2017) - this means that $\rho = 0.0175$ - to illustrate the differences between the two discounting methods.
Figure 2.1 shows how the discount rates evolve over time. While the discount rate stays constant at ρ for exponential discounting, it declines and approaches zero in the long run for hyperbolic discounting.
We will now compare the paths of optimal resource extraction for both discounting methods. Setting the inverse of the elasticity of intertemporal substitution, σ, equal to 1, we can plot the paths of optimal resource extraction, which are shown in figure 2.2.

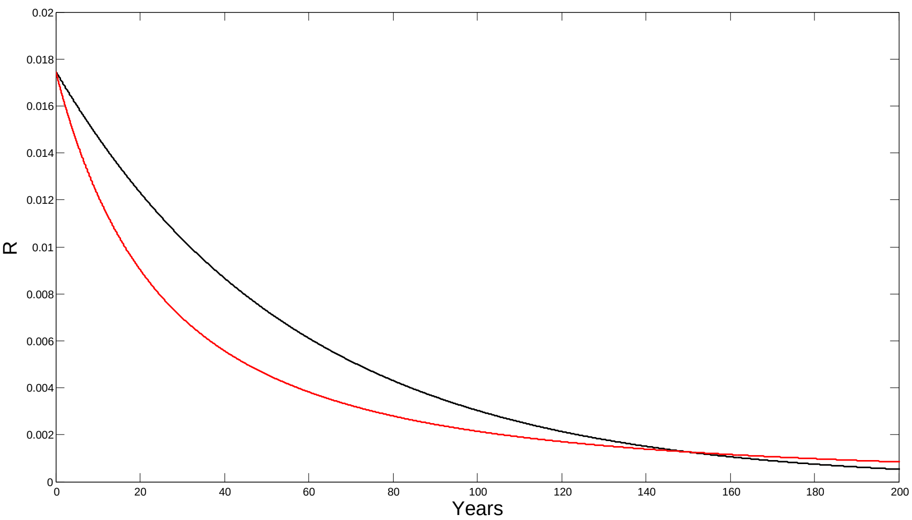


Figure 2.2: Optimal paths of resource extraction for exponential (black line) and hyperbolic (red line) discounting

The black line shows the optimal path of resource extraction for exponential discounting, the red line the one for hyperbolic discounting. Resource extraction is higher for exponential discounting at first, but after around 150 years the curves intersect. After that point, resource extraction is higher with hyperbolic discounting.
The initial amount of resource extracted is the same for both discounting methods in our example. This is due to the fact that the inverse of the elasticity of intertemporal substitution was set to 1, as $R_0^{exp} = \frac{\rho}{\sigma}$ and $R_0^{hyp} = \frac{a(b-\sigma)}{\sigma}$ are the same for $\sigma = 1$.
Concluding, it can be said that households are more patient in the model with hyperbolic discounting. Even though their discount rate is larger in the short run, they attach greater importance to the distant future than the households in the model with exponential discounting. Therefore, they are willing to forgo some utility in the short run in order to be able to extract more resources in the long run.

3 The DHSS Model with Exponential Discounting

In 1972, the Club of Rome published its much debated report on the ”Limits to Growth” (Meadows et al., 1972), predicting a bleak future for the world and its natural resources. The main argument was that increasing levels of pollution and rising resource scarcity would eventually lead to a significant decline in standard of living.

The report was heavily criticised by other economists, who countered the arguments made by Meadows et al. (1972) in a series of papers, which resulted in the well-known Dasgupta-Heal-Solow-Stiglitz (DHSS) model (Dasgupta and Heal, 1974; Solow, 1974; Stiglitz, 1974). In the DHSS model output is produced using man-made capital and a non-renewable resource. The main argument against the limits to growth proposed by Meadows et al. (1972) was that resources can be substituted by man-made capital, leading to sustainable growth. Over the last decades, numerous variations of the DHSS model have been analysed. However, it is yet to be investigated how deviations from standard exponential discounting affect the outcome of the model.

In the following two chapters I will analyse the DHSS model under both exponential discounting and a time-consistent method of hyperbolic discounting introduced by Strulik (2017). The comparison in chapter 6 shows resource use is more conservative in the model with hyperbolic discounting compared to the model with exponential discounting. In particular, much higher technological growth is needed for positive consumption growth with exponential discounting.

3.1 The Model

We will analyse the DHSS model from the perspective of a social planner who maximises the discounted utility stream of a representative household over an infinite time horizon:

$$\max_{C,R} \int_{t_0}^{\infty} u(C) \mathrm{e}^{-\rho(t-t_0)} dt$$

The model includes two decision variables, consumption C and natural resources extracted R.

The social planner has two constraints to take into account. The first one is the capital accumulation equation which describes how capital K evolves over time:

$$\dot{K} = F(C, L, K) - C - \delta K$$

where $F(C, L, K)$ denotes the production function. The size of the labour force L is assumed to grow exogenously at rate n: $\frac{\dot{L}}{L} = n$. The initial values of capital stock $K(0) = K_0$ and labour $L(0) = L_0$ are assumed to be given.

The second constraint describes the development of the resource stock S:

$$\dot{S} = -R, \qquad S(0) = S_0$$

As the resource regarded is assumed to be non-renewable, the resource stock S shrinks by exactly R at every point in time. An initial condition $S(0) = S_0$, $R \geq 0$ and $S \geq 0$ have to be taken into consideration as well.

In the following, we will assume a Cobb-Douglas production function

$$Y = F(C, L, K) = AK^{\alpha} L^{\beta} R^{\gamma}, \qquad \alpha, \beta, \gamma > 0, \qquad \alpha + \beta + \gamma = 1$$

which means that the capital accumulation equation can be rewritten as

$$\dot{K} = AK^{\alpha} L^{\beta} R^{\gamma} - C - \delta K$$

Technological progress is exogenous, the level of technology A grows at rate g, $\frac{\dot{A}}{A} = g$. We further assume a CRRA type utility function

$$u(C) = \frac{C^{1-\sigma} - 1}{1 - \sigma}$$

with $\sigma \geq 1$.
The complete model reads

$$\max_{C,R} \int_{t_0}^{\infty} \frac{C^{1-\sigma}-1}{1-\sigma} \mathrm{e}^{-\rho(t-t_0)} dt \tag{3.1}$$

$$\begin{aligned} s.t.\ \dot{K} &= AK^{\alpha}L^{\beta}R^{\gamma} - C - \delta K \\ \dot{S} &= -R \\ \frac{\dot{A}}{A} &= g \\ \frac{\dot{L}}{L} &= n \end{aligned}$$

3.2 The Solution

Setting $t_0 = 0$ the current value Hamiltonian of the problem is given by:

$$H = \frac{C^{1-\sigma}-1}{1-\sigma} + \lambda_K(AK^{\alpha}L^{\beta}R^{\gamma} - C - \delta K) - \lambda_S R + \lambda_A gA + \lambda_L nL$$

in which λ_K, λ_S, λ_A and λ_L are the shadow prices of capital, the resource, technology and labour, respectively. The first order conditions (FOCs) are given by

$$\frac{\partial H}{\partial C} = 0 \Leftrightarrow C^{-\sigma} - \lambda_K = 0 \tag{3.2}$$

$$\frac{\partial H}{\partial R} = 0 \Leftrightarrow \gamma \lambda_K AK^{\alpha}L^{\beta}R^{\gamma-1} - \lambda_S = 0 \tag{3.3}$$

$$\dot{K} = AK^{\alpha}L^{\beta}R^{\gamma} - C - \delta K \tag{3.4}$$

$$\dot{S} = -R \tag{3.5}$$

$$\dot{A} = gA \tag{3.6}$$

$$\dot{L} = nL \tag{3.7}$$

$$\frac{\partial H}{\partial K} = \alpha \lambda_K AK^{\alpha-1}L^{\beta}R^{\gamma} - \lambda_K \delta = \rho \lambda_K - \dot{\lambda_K} \tag{3.8}$$

$$\frac{\partial H}{\partial S} = 0 = \rho \lambda_S - \dot{\lambda_S} \tag{3.9}$$

$$\frac{\partial H}{\partial A} = \lambda_K K^{\alpha}L^{\beta}R^{\gamma} + \lambda_A g = \rho \lambda_A - \dot{\lambda_A} \tag{3.10}$$

$$\frac{\partial H}{\partial L} = \beta \lambda_K AK^{\alpha}L^(\beta - 1)R^{\gamma} + \lambda_L n = \rho \lambda_L - \dot{\lambda_L} \tag{3.11}$$

The FOCs are supplemented by four transversality conditions for the capital stock, the resource stock, technology and labour which ensure that their values approach zero in the long run.

$$\lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda_K K = 0$$
$$\lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda_S S = 0$$
$$\lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda_A A = 0$$
$$\lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda_L L = 0$$

From equation (3.9) we get

$$\frac{\dot{\lambda_S}}{\lambda_S} = \rho \tag{3.12}$$

This means that the shadow price of a marginal unit of the resource grows at a rate of ρ. From equation (3.8) we can derive a condition for λ_K:

$$\frac{\dot{\lambda_K}}{\lambda_K} = \rho + \delta - \alpha A K^{\alpha-1} L^{\beta} R^{\gamma} \tag{3.13}$$

Equation (3.3) establishes a relation between the two costate variables λ_K and λ_S. The equation can be converted to growth rates by taking the logarithm of each side and their derivatives with respect to time:

$$\lambda_S = \gamma \lambda_K A K^{\alpha} L^{\beta} R^{\gamma-1} \qquad |ln(.), \frac{\partial}{\partial t}$$
$$\frac{\dot{\lambda_S}}{\lambda_S} = \frac{\dot{\lambda_K}}{\lambda_K} + g + \alpha\frac{\dot{K}}{K} + \beta n + (\gamma - 1)\frac{\dot{R}}{R} \tag{3.14}$$

The Keynes-Ramsey rule

By taking the logarithm of equation (3.2) and taking the derivative with respect to time of the result, we get:

$$-\sigma\frac{\dot{C}}{C} = \frac{\dot{\lambda_K}}{\lambda_K}$$

Combing this equation with equation (3.13) yields the familiar Keynes-Ramsey rule:

$$\frac{\dot{C}}{C} = \frac{\alpha A K^{\alpha-1} L^{\beta} R^{\gamma} - \rho - \delta}{\sigma} \tag{3.15}$$

The $\dot{R}$-equation

By substituting the growth rates of λ_K and λ_S into equation (3.14) we get

$$\begin{aligned}\rho &= \rho + \delta - \alpha A K^{\alpha-1} L^{\beta} R^{\gamma} + g + \alpha \frac{\dot{K}}{K} + \beta n + (\gamma - 1)\frac{\dot{R}}{R} \\ &= \rho + \delta - \alpha A K^{\alpha-1} L^{\beta} R^{\gamma} + g + \alpha A K^{\alpha-1} L^{\beta} R^{\gamma} - \alpha \frac{C}{K} - \delta\alpha + \beta n + (\gamma - 1)\frac{\dot{R}}{R}\end{aligned}$$

Solving the equation for $\frac{\dot{R}}{R}$ yields

$$\frac{\dot{R}}{R} = -\frac{\alpha}{1-\gamma}\frac{C}{K} + \frac{(1-\alpha)}{1-\gamma}\delta + \frac{1}{1-\gamma}(g + \beta n) \tag{3.16}$$

The dynamic system

We can now write down the dynamic system that describes the economy:

$$\frac{\dot{K}}{K} = A K^{\alpha-1} L^{\beta} R^{\gamma} - \frac{C}{K} - \delta = \frac{Y}{K} - \frac{C}{K} - \delta \tag{3.17}$$

$$\frac{\dot{S}}{S} = -\frac{R}{S} \tag{3.18}$$

$$\frac{\dot{A}}{A} = g \tag{3.19}$$

$$\frac{\dot{L}}{L} = n \tag{3.20}$$

$$\frac{\dot{C}}{C} = \frac{\alpha A K^{\alpha-1} L^{\beta} R^{\gamma} - \rho - \delta}{\sigma} = \frac{\alpha\frac{Y}{K} - \frac{C}{K} - \delta}{\sigma} \tag{3.21}$$

$$\frac{\dot{R}}{R} = -\frac{\alpha}{1-\gamma}\frac{C}{K} + \frac{(1-\alpha)}{1-\gamma}\delta + \frac{1}{1-\gamma}(g + \beta n) \tag{3.22}$$

To solve the system, we will use the relaxation algorithm by Trimborn, Koch, and Steger (2008). The algorithm can only be used if the long-run growth rate of all variables is 0. As this is not the case in the dynamic system (3.17)-(3.22), the variables need to be scaled appropriately. In order to do so, first, the steady state growth rates have to be calculated.

The steady state

At the steady state, all variables grow at constant rates. From the dynamic system (3.17)-(3.22) we can see that this means that in the steady state, the ratios $\frac{C}{K}$, $\frac{Y}{K}$ and $\frac{R}{S}$ have to be constant as well. Consequently, the steady state growth rates of the variables have to satisfy the following equations:

$$g_R^* = g_S^*, \quad g_C^* = g_K^* = g_Y^*$$

To calculate the steady state growth rates first consider the production function $Y = AK^\alpha L^\beta R^\gamma$. Dividing both sides by K and converting to growth rates yields:

$$\begin{aligned}
\frac{Y}{K} &= AK^{\alpha-1}L^\beta R^\gamma \qquad\qquad |ln(.), \frac{\partial}{\partial t} \\
\underbrace{g_Y^* - g_K^*}_{=0} &= g + (\alpha - 1)g_K^* + \beta n + \gamma g_R^* \\
(1-\alpha)g_K^* &= \gamma g_R^* + g + \beta n
\end{aligned} \tag{3.23}$$

Substituting $\frac{\dot{K}}{K}$ and $\frac{\dot{R}}{R}$ into this equation we have

$$\begin{aligned}
& (1-\alpha)\left(\frac{Y}{K} - \frac{C}{K} - \delta\right) = \gamma\left(-\frac{\alpha}{1-\gamma}\frac{C}{K} + \frac{(1-\alpha)}{1-\gamma}\delta + \frac{1}{1-\gamma}(g+\beta n)\right) + g + \beta n \\
\Leftrightarrow \quad & (1-\alpha)\frac{Y}{K} = \left((1-\alpha) - \frac{\gamma\alpha}{1-\gamma}\right)\frac{C}{K} + \left((1-\alpha) + \frac{(1-\alpha)\gamma}{1-\gamma}\right)\delta + \left(\frac{\gamma}{1-\gamma} + 1\right)(g+\beta n) \\
& \qquad = \frac{1-\alpha-\gamma}{1-\gamma}\frac{C}{K} + \frac{1-\alpha}{1-\gamma}\delta + \frac{1}{1-\gamma}(g+\beta n)
\end{aligned} \tag{3.24}$$

This first relation between $\frac{Y}{K}$ and $\frac{C}{K}$ can be supplemented by a second one by equating g_K and g_C:

$$\frac{Y}{K} - \frac{C}{K} - \delta = \frac{\alpha\frac{Y}{K} - \rho - \delta}{\sigma} \tag{3.25}$$

Equating (3.24) and (3.25) and solving for $\frac{Y}{K}$ and $\frac{C}{K}$ yields the following steady state values:

$$\begin{aligned}
\frac{C}{K} &= \frac{(1-\gamma)(1-\alpha)}{\alpha(1-\alpha-\gamma+\sigma\gamma)}\rho + \frac{1-\alpha}{\alpha}\delta + \frac{\sigma-\alpha}{\alpha(1-\alpha-\gamma+\sigma\gamma)}(g+\beta n) \\
\frac{Y}{K} &= \frac{1-\alpha-\gamma}{(1-\alpha-\gamma+\sigma\gamma)\alpha}\rho + \frac{1}{\alpha}\delta + \frac{\sigma}{\alpha(1-\alpha-\gamma+\sigma\gamma)}(g+\beta n)
\end{aligned}$$

Equating g_R and g_S results in:

$$\begin{aligned}\frac{R}{S} &= \frac{\alpha}{1-\gamma}\frac{C}{K} - \frac{(1-\alpha)}{1-\gamma}\delta - \frac{1}{1-\gamma}(g+\beta n)\\ &= \frac{1-\alpha}{1-\alpha-\gamma+\sigma\gamma}\rho + \left(\frac{\sigma-\alpha}{(1-\gamma)(1-\alpha-\gamma+\sigma\gamma)} - \frac{1}{1-\gamma}\right)(g+\beta n) + \frac{1-\alpha}{1-\gamma}\delta - \frac{1-\alpha}{1-\gamma}\delta\\ &= \frac{1-\alpha}{1-\alpha-\gamma+\sigma\gamma}\rho - \frac{1-\sigma}{1-\alpha-\gamma+\sigma\gamma}(g+\beta n) \qquad (3.26)\end{aligned}$$

We get the following steady state growth rates:

$$g_R^* = g_S^* = -\frac{R}{S} = -\frac{1-\alpha}{1-\alpha-\gamma+\sigma\gamma}\rho + \frac{1-\sigma}{1-\alpha-\gamma+\sigma\gamma}(g+\beta n) \qquad (3.27)$$

$$\begin{aligned}g_Y^* = g_K^* = g_C^* &= \frac{\alpha\frac{Y}{K} - \rho - \delta}{\sigma}\\ &= \frac{1-\alpha-\gamma}{(1-\alpha-\gamma+\sigma\gamma)\sigma}\rho + \frac{\delta}{\sigma} + \frac{1}{1-\alpha-\gamma+\gamma\sigma}(g+\beta n) - \frac{\rho}{\sigma} - \frac{\delta}{\sigma}\\ &= -\frac{\gamma}{1-\alpha-\gamma+\gamma\sigma}\rho + \frac{1}{1-\alpha-\gamma+\gamma\sigma}(g+\beta n) \qquad (3.28)\end{aligned}$$

As the resource in our model is non-renewable, the long-run growth rate of S, g_S^*, which is equal to the growth rate of R, g_R^*, must be negative. This condition is fulfilled if $(1-\alpha)\rho > (1-\sigma)(g+\beta n)$. As the left-hand side of the inequality condition is positive, the condition is always fulfilled for $\sigma \geq 1$.

When we look at $g_Y^* = g_K^* = g_C^*$, we see that the long-run growth rate of consumption can only be positive for

$$\gamma\rho < (g+\beta n) \qquad (3.29)$$

The scaled system

In order to apply the relaxation algorithm we construct a stationary system by scaling the variables according to their long-run growth rate. We obtain the scale-adjusted variables

$$x(t) = X(t)\mathrm{e}^{-g_x t}$$

where x denotes the scaled variable, X the unscaled variable and g_x the steady state growth rate of the unscaled variable.

The dynamic system for the scale-adjusted variables is given by:

$$\frac{\dot{k}}{k} = \frac{y}{k} - \frac{c}{k} - \delta - g_k^* \tag{3.30}$$

$$= ak^{\alpha-1}l^{\beta}r^{\gamma} - \frac{c}{k} - \delta + \frac{\gamma}{1-\alpha-\gamma+\gamma\sigma}\rho - \frac{1}{1-\alpha-\gamma+\gamma\sigma}(g+\beta n) \tag{3.31}$$

$$\frac{\dot{s}}{s} = -\frac{r}{s} - g_s^* \tag{3.32}$$

$$= -\frac{r}{s} + \frac{1-\alpha}{1-\alpha-\gamma+\sigma\gamma}\rho - \frac{1-\sigma}{1-\alpha-\gamma+\sigma\gamma}(g+\beta n) \tag{3.33}$$

$$\frac{\dot{a}}{a} = 0 \tag{3.34}$$

$$\frac{\dot{l}}{l} = 0 \tag{3.35}$$

$$\frac{\dot{c}}{c} = \frac{\alpha\frac{y}{k} - \frac{c}{k} - \delta}{\sigma} - g_c^* = \frac{\alpha ak^{\alpha-1}l^{\beta}r^{\gamma} - \rho - \delta}{\sigma} - g_c^* \tag{3.36}$$

$$= \frac{\alpha ak^{\alpha-1}l^{\beta}r^{\gamma} - \rho - \delta}{\sigma} + \frac{\gamma}{1-\alpha-\gamma+\gamma\sigma}\rho - \frac{1}{1-\alpha-\gamma+\gamma\sigma}(g+\beta n) \tag{3.37}$$

$$\frac{\dot{r}}{r} = -\frac{\alpha}{1-\gamma}\frac{c}{k} + \frac{(1-\alpha)}{1-\gamma}\delta + \frac{1}{1-\gamma}(g+\beta n) - g_r^* \tag{3.38}$$

$$= -\frac{\alpha}{1-\gamma}\frac{c}{k} + \frac{(1-\alpha)}{1-\gamma}\delta + \frac{1-\alpha}{1-\alpha-\gamma+\sigma\gamma}\rho + \frac{\sigma-\alpha}{(1-\gamma)(1-\alpha-\gamma+\sigma\gamma)}(g+\beta n) \tag{3.39}$$

As a and l do not change over time, we can normalize them to unity and omit the respective equations. We will thus continue only using the dynamic system (3.31)-(3.33) and (3.37)-(3.39).

The relaxation algorithm

We can now apply the relaxation algorithm (Trimborn, Koch, and Steger, 2008), which allows us to analyse how the economy converges towards its steady state. The parameter values used for the following graphs are listed in table 3.1. For information on their calibration see chapter 5.

Parameter	Description	Value
α	Output elasticity of capital	0.3
δ	Depreciation rate	0.08
γ	Output elasticity of resources	0.1
β	Output elasticity of labour	0.6
ρ	Discount rate	0.024
σ	Inverse of the intertemporal elasticity of substitution	2
g	Growth rate of the level of technology	0.015
n	Growth rate of population	0.01

Table 3.1: Parameter values used for numerical simulation

In order to understand the economy's development, we consider the phase diagram of the scaled system in the (k, s) plane, see figure 3.1. The balanced growth path of the original system becomes a (saddle-point stable) center manifold in the scaled system. For each combination of initial values (k_0, s_o) the economy converges towards a point on the center manifold.

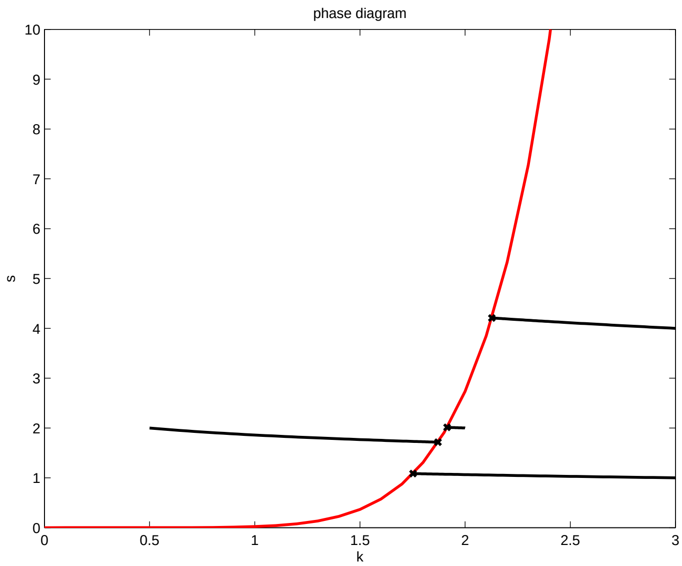


Figure 3.1: Phase diagram of the scaled system

The center manifold can be computed by dividing the production function by k:

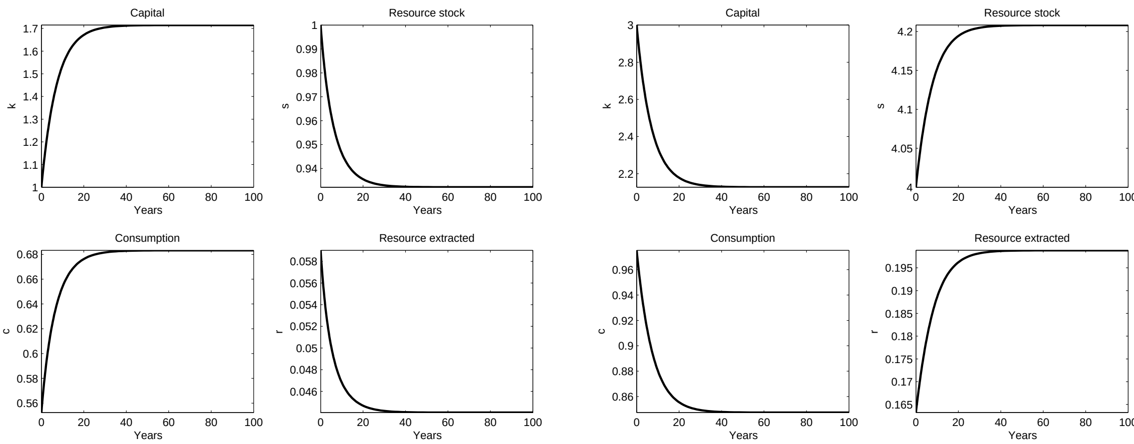
$$\frac{y}{k} = k^{\alpha-1} r^{\gamma}$$

Inserting the steady state values of $\frac{y}{k}$ and r

$$\frac{y}{k} = \frac{1-\alpha-\gamma}{(1-\alpha-\gamma+\sigma\gamma)\alpha}\rho + \frac{1}{\alpha}\delta + \frac{\sigma}{\alpha(1-\alpha-\gamma+\sigma\gamma)}(g+\beta n)$$
$$\frac{r}{s} = \frac{1-\alpha}{1-\alpha-\gamma+\sigma\gamma}\rho - \frac{1-\sigma}{1-\alpha-\gamma+\sigma\gamma}(g+\beta n)$$

and solving for s yields:

$$s = \frac{\left(\frac{1-\alpha-\gamma}{(1-\alpha-\gamma+\sigma\gamma)\alpha}\rho + \frac{\delta}{\alpha} + \frac{\sigma}{(1-\alpha-\gamma+\sigma\gamma)\alpha}(g+\beta n)\right)^{\frac{1}{\gamma}}}{\frac{1-\alpha}{(1-\alpha-\gamma+\sigma\gamma)}\rho - \frac{1-\sigma}{1-\alpha-\gamma+\sigma\gamma}(g+\beta n)} \tag{3.40}$$



(a) (k_0, s_0) above center manifold

(b) (k_0, s_0) below center manifold

Figure 3.2: Development of the variables of the scaled system

The development of the variables depend on the initial values k_0 and s_0. If (k_0, s_0) lies above the center manifold, k grows over time while s shrinks, see 3.2a. If (k_0, s_0) lies below the center manifold, the opposite happens: k shrinks over time and k grows, see 3.2b[1]. For $(k_0, s_0) = (3, 1)$ (see chapter 5), the adjustment process of the variables is depicted in figure 3.3. As the initial value lies below the center manifold, the capital stock k decreases while the resource stock s increases.

[1]Even though the constraint $R \geq 0$ was not explicitly taken into account in the calculations, it is always fulfilled for the parameter combinations we used. Therefore, in the unscaled system $\dot{S} \leq 0$.

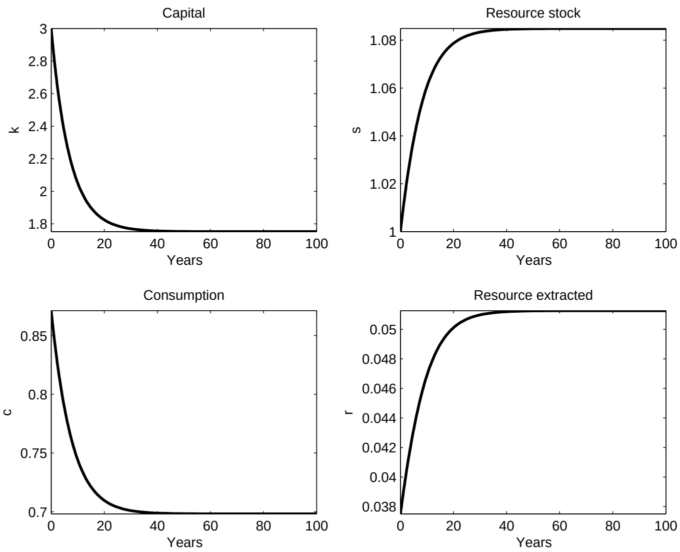


Figure 3.3: Development of the variables for $(k_0, s_0) = (3, 1)$

4 The DHSS Model with Hyperbolic Discounting

We will now turn to the DHSS model under hyperbolic discounting. In contrast to the simplified model presented in chapter 2, where the discount function was given by $D(t_0, t) = \left(\frac{1+at_0}{1+at}\right)^b$ with $t_0 = 0$, we will now use the discount function

$$D(t_0, t) = \mathrm{e}^{-\bar{\rho}(t-t_0)} \left(\frac{1+at_0}{1+at}\right)^b \tag{4.1}$$

The discount rate, defined as $-\frac{\partial D(t_0,t)}{\partial t}/D(t_0,t)$, changes from $\frac{ab}{(1+at)}$ to

$$\bar{\rho} + \frac{ab}{(1+at)} \tag{4.2}$$

While the discount rate still declines over time, this combination of exponential and hyperbolic discounting ensures that the discount rate does not vanish, but approaches $\bar{\rho}$ in the long run. A small but positive long-run growth rate accounts for the possibility of humanity being extinguished (Stern, 2007).

4.1 The Model

Similarly to chapter 3, we will analyse the DHSS from the perspective of a social planner who maximises the discounted stream of a representative consumer's utility over an infinite time horizon by choosing optimal paths of consumption C and extraction of resources R. While we used a standard exponential discount function $\mathrm{e}^{-\rho t}$ in chapter 3, utility will now be discounted by the discount function (4.1), which yields

$$\max_{C,R} \int_{t_0}^{\infty} u(C)\mathrm{e}^{-\bar{\rho}(t-t_0)} \left(\frac{1+at_0}{1+at}\right)^b dt$$

Apart from the discounting method, the model does not differ from the model presented in chapter 3. The social planner faces the same two constraints as before:

$$\dot{K} = F(C, L, K) - C - \delta K \tag{4.3}$$

$$\dot{S} = -R, \qquad S(0) = S_0 \tag{4.4}$$

Equation (4.3) describes the process of capital accumulation in the economy. The production function $F(C, L, K)$ depends on consumption C, capital K and labour L. The labour force is assumed to grow exogenously at rate n. The initial values of capital stock and labour are given by $K(0) = K_0$ and $L(0) = L_0$, respectively.

Equation (4.4) describes the evolution of the resource stock, which shrinks by R at every point in time as the resource is assumed to be non-renewable.

Using a Cobb-Douglas production function $Y = F(C, L, K) = AK^{\alpha}L^{\beta}R^{\gamma}, \quad \alpha, \beta, \gamma > 0, \quad \alpha + \beta + \gamma = 1$ with the level of technology A growing at rate g, a CRRA type utility function $u(C) = \frac{C^{1-\sigma}-1}{1-\sigma}$ and $t_0 = 0$, the problem reads:

$$\max_{C,R} \int_0^{\infty} \frac{C^{1-\sigma}-1}{1-\sigma} \mathrm{e}^{-\bar{\rho}t} \left(\frac{1}{1+at}\right)^b dt \tag{4.5}$$

$$\begin{aligned} s.t.\ \dot{K} &= AK^{\alpha}L^{\beta}R^{\gamma} - C - \delta K \\ \dot{S} &= -R \\ \frac{\dot{A}}{A} &= g \\ \frac{\dot{L}}{L} &= n \end{aligned}$$

4.2 The Solution

In the case of hyperbolic discounting, we have to consider the present value Hamiltonian:

$$H = \mathrm{e}^{-\bar{\rho}t} \left(\frac{1}{1+at}\right)^b \frac{C^{1-\sigma}-1}{1-\sigma} + \lambda_K(AK^{\alpha}L^{\beta}R^{\gamma} - C - \delta K) - \lambda_S R + \lambda_A gA + \lambda_L nL$$

The first order conditions are given by:

$$\frac{\partial H}{\partial C} = 0 \Leftrightarrow \mathrm{e}^{-\bar{\rho}t}\left(\frac{1}{1+at}\right)^{b} C^{-\sigma} - \lambda_K = 0 \tag{4.6}$$

$$\frac{\partial H}{\partial R} = 0 \Leftrightarrow \gamma\lambda_K A K^{\alpha} L^{\beta} R^{\gamma-1} - \lambda_S = 0 \tag{4.7}$$

$$\dot{K} = AK^{\alpha}L^{\beta}R^{\gamma} - C - \delta K \tag{4.8}$$

$$\dot{S} = -R \tag{4.9}$$

$$\dot{A} = gA \tag{4.10}$$

$$\dot{L} = nL \tag{4.11}$$

$$\frac{\partial H}{\partial K} = \alpha\lambda_K A K^{\alpha-1} L^{\beta} R^{\gamma} - \lambda_K \delta = -\dot{\lambda_K} \tag{4.12}$$

$$\frac{\partial H}{\partial S} = 0 = -\dot{\lambda_S} \tag{4.13}$$

$$\frac{\partial H}{\partial A} = \lambda_K K^{\alpha} L^{\beta} R^{\gamma} + \lambda_A g = -\dot{\lambda_A} \tag{4.14}$$

$$\frac{\partial H}{\partial L} = \beta\lambda_K A K^{\alpha} L^{(}\beta - 1) R^{\gamma} + \lambda_L n = -\dot{\lambda_L} \tag{4.15}$$

In order to get a unique solution, again, the FOCs have to be complemented by four transversality conditions:

$$\lim_{t\to\infty} \lambda_K K = 0$$
$$\lim_{t\to\infty} \lambda_S S = 0$$
$$\lim_{t\to\infty} \lambda_A A = 0$$
$$\lim_{t\to\infty} \lambda_L L = 0$$

The Keynes-Ramsey rule

In order to get the Keynes-Ramsey rule, we take the logarithm of equation (4.6) and differentiate the result with respect to time:

$$ln(\lambda_K) = -b(ln(1+at)) - \bar{\rho}t - \sigma ln(C) \qquad |\frac{\partial}{\partial t}$$

$$\frac{\dot{\lambda_K}}{\lambda_K} = -\frac{ba}{1+at} - \bar{\rho} - \sigma\frac{\dot{C}}{C} \tag{4.16}$$

Dividing equation (4.12) by λ_K yields

$$-\frac{\dot{\lambda_K}}{\lambda_K} = \alpha A K^{\alpha-1} R^{\gamma} - \delta \tag{4.17}$$

Combing equation (4.16) and equation (4.17) we get the Keynes-Ramsey rule:

$$\frac{\dot{C}}{C} = \frac{\alpha A K^{\alpha-1} R^{\gamma} - \delta - \bar{\rho} - \frac{ba}{1+at}}{\sigma} \tag{4.18}$$

The dynamic system

The remainder of the dynamic system can be derived analogously to chapter 3. The complete dynamic system reads:

$$\frac{\dot{K}}{K} = A K^{\alpha-1} L^{\beta} R^{\gamma} - \frac{C}{K} - \delta = \frac{Y}{K} - \frac{C}{K} - \delta \tag{4.19}$$

$$\frac{\dot{S}}{S} = -\frac{R}{S} \tag{4.20}$$

$$\frac{\dot{A}}{A} = g \tag{4.21}$$

$$\frac{\dot{L}}{L} = n \tag{4.22}$$

$$\frac{\dot{C}}{C} = \frac{\alpha A K^{\alpha-1} R^{\gamma} - \delta - \bar{\rho} - \frac{ba}{1+at}}{\sigma} = \frac{\frac{Y}{K} - \delta - \bar{\rho} - \frac{ba}{1+at}}{\sigma} \tag{4.23}$$

$$\frac{\dot{R}}{R} = -\frac{\alpha}{1-\gamma}\frac{C}{K} + \frac{(1-\alpha)}{1-\gamma}\delta + \frac{1}{1-\gamma}(g + \beta n) \tag{4.24}$$

In contrast to the dynamic system (3.17)-(3.22) in chapter 3, the system (4.19)-(4.24) is non-autonomous. Therefore, calculating the steady state in order to apply the relaxation algorithm is not as easy as in the model with exponential discounting. However, as the term containing t approaches 0 in the long run, we can calculate an asymptotic steady state, which is enough to be able to apply the relaxation algorithm.

The steady state

At the steady state, all variables grow at a constant rate. As the term containing t converges to 0, this means that at the steady state $\frac{Y}{K}$, $\frac{C}{K}$ and $\frac{R}{S}$ have to be constant, which is equivalent to:

$$g_R^* = g_S^* \text{ and } g_C^* = g_K^* = g_Y^*$$

Similar to chapter 3 we can now calculate the steady state values of $\frac{Y}{K}$, $\frac{C}{K}$ and $\frac{R}{S}$. We get

$$\frac{Y}{K} = \lim_{t\to\infty}\left[\frac{1-\alpha-\gamma}{(1-\alpha-\gamma+\sigma\gamma)\alpha}\left(\bar{\rho}+\frac{ba}{1+at}\right)+\frac{1}{\alpha}\delta+\frac{\sigma}{\alpha(1-\alpha-\gamma+\sigma\gamma)}(g+\beta n)\right]$$
$$\frac{C}{K} = \lim_{t\to\infty}\left[\frac{(1-\gamma)(1-\alpha)}{\alpha(1-\alpha-\gamma+\sigma\gamma)}\left(\bar{\rho}+\frac{ba}{1+at}\right)+\frac{1-\alpha}{\alpha}\delta+\frac{\sigma-\alpha}{\alpha(1-\alpha-\gamma+\sigma\gamma)}(g+\beta n)\right]$$
$$\frac{R}{S} = \lim_{t\to\infty}\left[\frac{1-\alpha}{1-\alpha-\gamma+\sigma\gamma}\left(\bar{\rho}+\frac{ba}{1+at}\right)-\frac{1-\sigma}{1-\alpha-\gamma+\sigma\gamma}(g+\beta n)\right]$$

The corresponding steady state growth rates are:

$$g_R^* = g_S^* = -\frac{R}{S} = \lim_{t\to\infty}\left[-\frac{1-\alpha}{1-\alpha-\gamma+\sigma\gamma}\left(\bar{\rho}+\frac{ba}{1+at}\right)+\frac{1-\sigma}{1-\alpha-\gamma+\sigma\gamma}(g+\beta n)\right] \tag{4.25}$$

$$g_Y^* = g_K^* = g_C^* = \lim_{t\to\infty}\left[-\frac{\gamma}{1-\alpha-\gamma+\gamma\sigma}\left(\bar{\rho}+\frac{ba}{1+at}\right)+\frac{1}{1-\alpha-\gamma+\gamma\sigma}(g+\beta n)\right] \tag{4.26}$$

Just like in the model with exponential discounting, we need to make sure that the growth rate of the resource stock S converges towards a non-positive value. From equation (4.25) we see that this is the case for $(1-\alpha)\bar{\rho} > (1-\sigma)(g+\beta n)$. This condition is always fulfilled for $\bar{\rho} \geq 0$ and $\sigma \geq 1$.

For the growth rate of consumption to approach a positive value, the parameter restriction $\gamma\bar{\rho} < (g+\beta n)$ must hold. Note that this condition is not as restrictive as the corresponding condition in the model with exponential discounting (see equation (3.29)), as $\bar{\rho}$ will typically be much smaller than ρ. For $\bar{\rho} = 0$ the growth rate of consumption is always positive.

The scaled system

Just like in the model with exponential discounting, we will now scale the variables according to their long-run growth rates such that the relaxation algorithm (Trimborn, Koch, and Steger, 2008) can be applied. However, this time we do not take the steady state growth rates, but time-dependent scaling variables. We define the scaled variables as:

$$x(t) = X(t)\mathrm{e}^{-g_x(t)}$$

where X denotes the original variable and $g_x(t)$ the growth rate given in equations (4.25) and (4.26) without taking the limit $t \to \infty$.

As the scaled versions of the level of technology A and labour L grow at rate zero, we can normalize them to unity and omit their respective equations.

Thus, the dynamic system of the scaled variables reads:

$$\frac{\dot{k}}{k} = \frac{y}{k} - \frac{c}{k} - \delta - g_k^* \tag{4.27}$$

$$= k^{\alpha-1} r^{\gamma} - \frac{c}{k} - \delta + \frac{1}{1-\alpha-\gamma+\gamma\sigma}\left(\gamma\left(\bar{\rho} + \frac{ba}{1+at}\right) - (g+\beta n)\right) \tag{4.28}$$

$$\frac{\dot{s}}{s} = -\frac{r}{s} - g_s^* \tag{4.29}$$

$$= -\frac{r}{s} + \frac{1}{1-\alpha-\gamma+\sigma\gamma}\left((1-\alpha)\left(\bar{\rho} + \frac{ba}{1+at}\right) - (1-\sigma)(g+\beta n)\right) \tag{4.30}$$

$$\frac{\dot{c}}{c} = \frac{\alpha\frac{y}{k} - \frac{c}{k} - \delta}{\sigma} - g_c^* = \frac{\alpha k^{\alpha-1} r^{\gamma} - \left(\bar{\rho} + \frac{ba}{1+at}\right) - \delta}{\sigma} - g_c^* \tag{4.31}$$

$$= \frac{\alpha k^{\alpha-1} r^{\gamma} - \delta}{\sigma} - \frac{1-\alpha-\gamma}{(1-\alpha-\gamma+\gamma\sigma)\sigma}\left(\bar{\rho} + \frac{ba}{1+at}\right) - \frac{1}{1-\alpha-\gamma+\gamma\sigma}(g+\beta n) \tag{4.32}$$

$$\frac{\dot{r}}{r} = -\frac{\alpha}{1-\gamma}\frac{c}{k} + \frac{(1-\alpha)}{1-\gamma}\delta + \frac{1}{1-\gamma}(g+\beta n) - g_r^* \tag{4.33}$$

$$= -\frac{\alpha}{1-\gamma}\frac{c}{k} + \frac{(1-\alpha)}{1-\gamma}\delta + \frac{1}{1-\alpha-\gamma+\sigma\gamma}\left((1-\alpha)\left(\bar{\rho} + \frac{ba}{1+at}\right) + \frac{\sigma-\alpha}{1-\gamma}(g+\beta n)\right) \tag{4.34}$$

We can now use the relaxation algorithm to solve the system (4.30)-(4.34). The results are presented in chapter 6.

5 Calibration

The model is calibrated to fit data for the United States of America as there is good data on the natural resource base available.

The capital share in the Cobb-Douglas production function is set to $\alpha = 0.3$, which is in line with (common/current/standard) literature (suggesting a plausible range for the capital share of 0.3 to 0.4) (see Collins, Bosworth, and Rodrik, 1996; Maddison, 1987; Englander and Gurney, 1994). The output elasticity of resources, γ, is set to 0.1. We assume constant returns to scale which means that $\alpha + \beta + \gamma = 1$. Thus, the labour share in the production function, β, is set to 0.6.

The depreciation rate, δ, is set to 0.08 which is a standard value in literature.

For the calibration of the inverse of the elasticity of intertemporal substitution, σ, we use a meta-analysis of 2735 published estimates of the elasticity of intertemporal substitution by Havranek et al. (2015). They find the average value of the elasticity of intertemporal substitution to be 0.5. Accordingly, we set $\sigma = 2$.

The growth rates of the population and technology are set to 1% and 1.5%, respectively.

In order to calibrate the discount rate for the model with exponential discounting, ρ, we use the assumption that the average growth rate of consumption is 1.8%. From the Keynes-Ramsey rule we get

$$\frac{\dot{C}}{C} = \frac{r - \rho}{\sigma} = g_c = g_y = 1.8\%$$

Using $r = 6\%$ and $\sigma = 2$ yields $\rho = 2.4\%$.

Finally, we need to calibrate the parameters for the model with hyperbolic discounting, a, b and $\bar{\rho}$. Following Stern (2007), we set $\bar{\rho}$ to 0.1% to account for the possibility of humanity being extinguished. The parameters a and b have to be chosen such that the present value of a constant utility stream is the same for both discounting methods, see equation (6.1). However, this condition is not sufficient to get unique values for a and b. Therefore, we use values which provide a good fit to the responses Weitzman (2001) obtained from asking over two thousand PhD-level economists "what real interest rate should be used to discount over time the (expected) benefits and (expected) costs of projects being proposed to mitigate the possible effects of global climate change". The forward rates he suggested are presented in table 5.1.

Time Period	Name	Marginal Discount Rate (Percent)
Within years 1 to 5 hence	*Immediate* Future	4
Within years 6 to 25 hence	*Near* Future	3
Within years 26 to 75 hence	*Medium* Future	2
Within years 76 to 300 hence	*Distant* Future	1
Within years more than 300 hence	*Far-Distant* Future	0

Table 5.1: Forward Discount Rate Schedule, Source: Weitzman (2001)

Using equation (6.1) and table 5.1, we obtain $b = 2$ and $a \approx 0.02$.
For the calibration of K_0 we first take the United States' 2016 GDP which amounts to 18.624 trillion dollars (The World Bank, 2018). In order to calibrate K_0 we use the estimate of the capital-output ratio given by Strulik and Trimborn (2010), $\frac{K}{Y} = 2.16$. We get $K_0 = 40.2$ trillion dollars.
The initial resource stock is calibrated by adding up the worth of the United States' proved crude oil reserves, their recoverable coal reserves and their natural gas reserves which yields $S_0 = 13.5$ trillion dollars.
Finally, we normalise S_0 to unity and scale K_0 accordingly, yielding $S_0 = 1$ and $K_0 = 3$.

6 Comparing Discounting Methods

Just like in the simplified model, we will apply the equivalent present-value approach (Myerson, Green, and Warusawitharana, 2001) to compare exponential and hyperbolic discounting. This means that the parameters have to be chosen such that the present value of a constant utility stream is the same for both discounting methods:

$$\int_{t_0}^{\infty} \mathrm{e}^{-\rho(t-t_0)} dt = \int_{t_0}^{\infty} \mathrm{e}^{-\bar{\rho}(t-t_0)} \left(\frac{1+at_0}{1+at}\right)^b dt \tag{6.1}$$

Setting $t_0 = 0$ and solving this equation for ρ yields:

$$\rho = \frac{\bar{\rho}}{\Gamma(1-b, \frac{\bar{\rho}}{a})\mathrm{e}^{\frac{\bar{\rho}}{a}}(\frac{\bar{\rho}}{a})^b} \tag{6.2}$$

where $\Gamma(.,.)$ is the incomplete gamma function. This equation, while more complex than its counterpart in the simplified model (2.6), is numerically solvable.

Parameter	Description	Value
α	Output elasticity of capital	0.3
β	Output elasticity of labour	0.6
γ	Output elasticity of resources	0.1
δ	Depreciation rate	0.08
σ	Inverse of the intertemporal elasticity of substitution	2
g	Growth rate of the level of technology	0.015
n	Growth rate of population	0.01
ρ	Discount rate (exponential discounting)	0.024
a	Parameter for the discount rate (hyperbolic discounting)	0.021
b	Parameter for the discount rate (hyperbolic discounting)	2
$\bar{\rho}$	Parameter for the discount rate (hyperbolic discounting)	0.001

Table 6.1: Parameter values used for numerical simulation

In order to compare the exponential and hyperbolic discounting, we set the parameters according to table 6.1 (for further information on the calibration of the parameters see

chapter 5).
First, we will have a look at the differences between the discount rates for the two methods of discounting. Figure 6.1 illustrates the discount rates for exponential and hyperbolic discounting.
Like in the simplified model, the discount rate does not change over time for exponential discounting; it is constant at $\rho = 2.4\%$. For hyperbolic discounting, the discount rate declines over time. However, it does not vanish in the long run like it did in the simplified model but converges towards $\bar{\rho} = 0.1\%$, indicated by the dotted line.

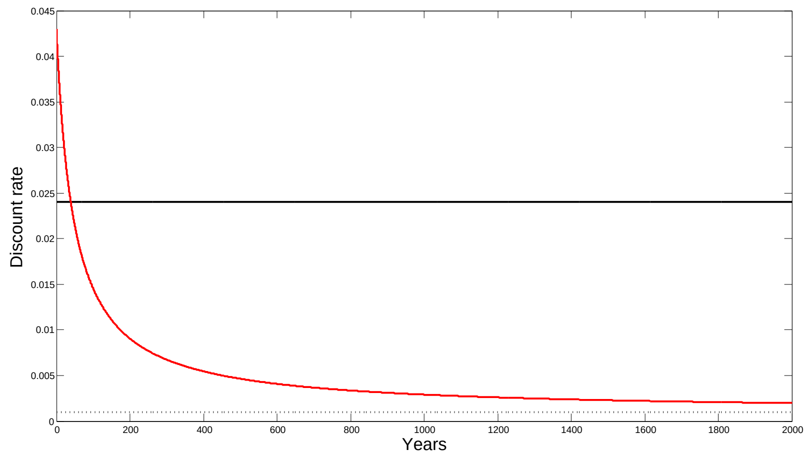
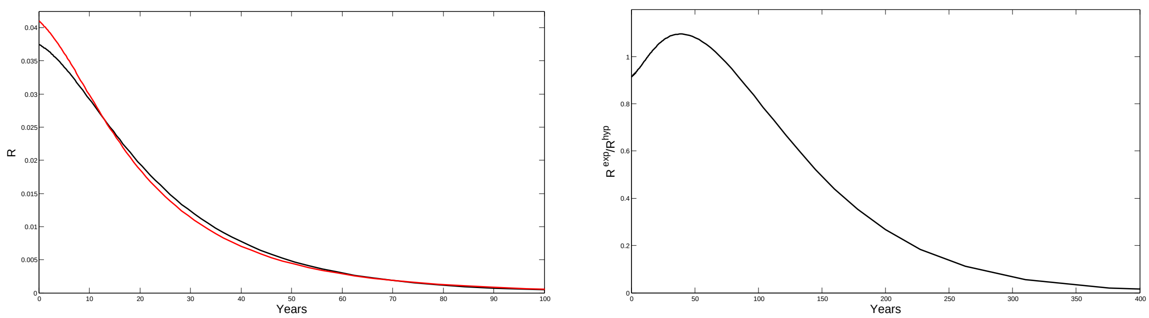


Figure 6.1: Discount rates for exponential (black solid line) and hyperbolic (red solid line) discounting

We will now compare the optimal paths of resource extraction for both discounting methods. Using the parameter values specified in table 6.1 we get the paths shown in figure 6.2a.

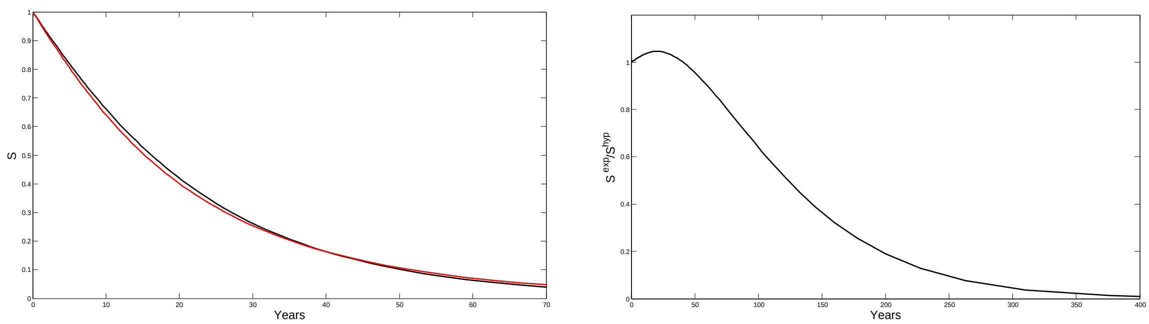


(a) Resource extraction for exponential (black line) and hyperbolic (red line) discounting

(b) Ratio of resource extraction with exponential and hyperbolic discounting R^{exp}/R^{hyp}

Figure 6.2: Comparison of resource extraction levels

Resource extraction is higher in the model with hyperbolic discounting at first due to the high short-run discount rate. However, after around 15 years, the curves intersect and the level of resource extraction in the model with hyperbolic discounting drops below the level of resource extraction in the model with exponential discounting. After around 70 years, the curves intersect again. From this point onwards, resource extraction is higher with hyperbolic discounting. This behaviour is similar to the behaviour described in the simplified model, see chapter 2. Households attach greater importance to the distant future in the model with hyperbolic discounting. Therefore, they are willing to sacrifice some short-term utility in order to be able to extract more resources in the long run.
Figure 6.2b shows the ratio of resource extraction with exponential discounting and resource extraction with hyperbolic discounting R^{exp}/R^{hyp}. It rises to 1.1 after 40 years and declines thereafter. The impatience of the households in the model with exponential discounting does not only lead to lower resource extraction in the long run in absolute terms; additionally, the ratio of the levels of extraction approaches zero in the long run.



(a) Resource stock for exponential (black line) and hyperbolic (red line) discounting

(b) Ratio of resource stock with exponential and hyperbolic discounting S^{exp}/S^{hyp}

Figure 6.3: Comparison of resource stocks

Figure 6.3 shows the evolution of resource stocks with both discounting methods. Due to higher short-run resource extraction the resource stock is smaller in the model with hyperbolic discounting at first. However, the curves intersect after around 40 years. After that, the resource stock is higher in the model with hyperbolic discounting.
Figure 6.3b shows the ratio of resource stocks S^{exp}/S^{hyp} which looks similar to the ratio of resources extracted in figure 6.2b. The ratio of resource stocks rises at first, but converges towards zero in the long run.
The phase diagram represented in figure 6.4 shows the evolution of both R and S. Resource extraction is higher with hyperbolic discounting when the resource stock is high, but falls

below the level of resource extraction with exponential discounting for lower levels of the resource stock.

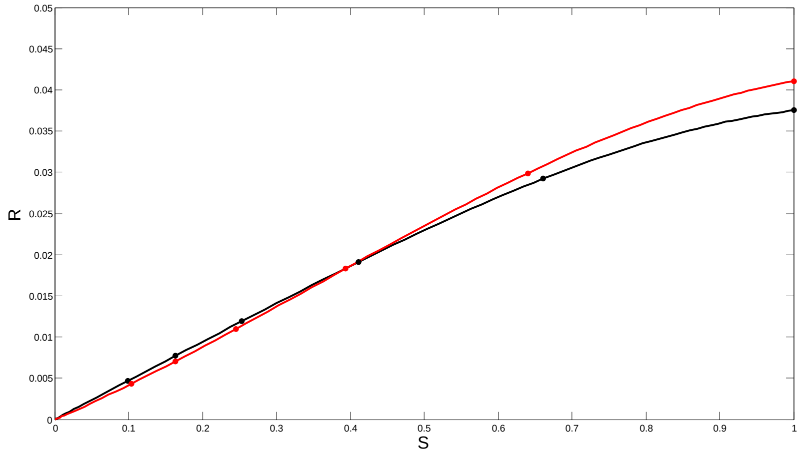


Figure 6.4: Phase diagram in the S/R-plane for exponential (black line) and hyperbolic (red line) discounting

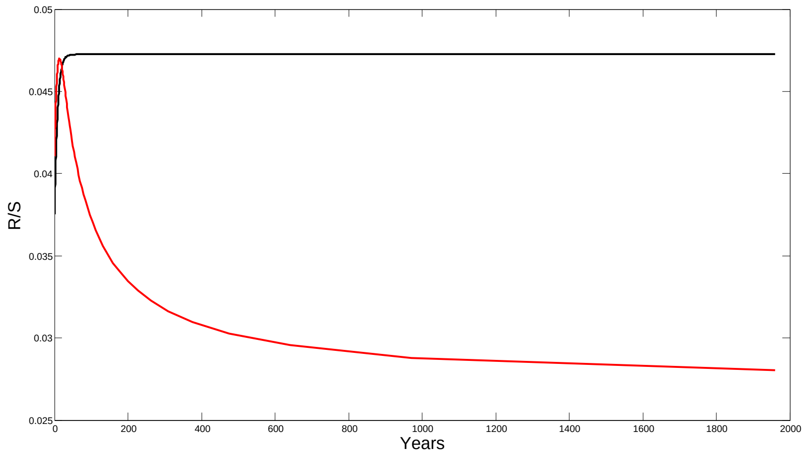


Figure 6.5: Resource extraction rate for exponential (black line) and hyperbolic (red line) discounting

Finally, we will compare the resource extraction rate for both discounting methods. Figure 6.5 shows the evolution of the resource extraction rate for exponential (black line) and hyperbolic (red line) discounting. While the paths of resource extraction R and resource stock S shown in figures 6.2-6.4 are similar for both discounting methods, the resource extraction rate looks very different. The extraction rate rises in both cases in the short run. We already know that the initial extraction rate is higher for hyperbolic discounting and it remains higher for the first years. However, after around 15 years it starts declining while the extraction rate for exponential discounting further rises towards its steady state value $\frac{1-\alpha}{1-\alpha-\gamma+\sigma\gamma}\rho - \frac{1-\sigma}{1-\alpha-\gamma+\sigma\gamma}(g+\beta n) \approx 0.0473$, see equation (3.26). For hyperbolic discounting, the extraction rate converges towards the much smaller steady state value $\frac{1-\alpha}{1-\alpha-\gamma+\sigma\gamma}\bar{\rho} - \frac{1-\sigma}{1-\alpha-\gamma+\sigma\gamma}(g+\beta n) \approx 0.0271$. This means that in the medium and long run the share of the remaining resource stock extracted at each point in time is much smaller with hyperbolic discounting.

7 Sensitivity Analysis

We will now investigate the sensitivity of our results to deviations from the benchmark parameter values introduced in chapter 5. In order to do so, we vary the values of certain parameters ceteris paribus and check how these deviations affect the depletion of the resource stock S, the extraction rate $\frac{R}{S}$ and the initial resource extraction R_0. More precisely, we analyse how the changes affect the key figures $S_{intersect}$ and $\frac{R}{S}_{intersect}$, which indicate the time it takes until the resource stock and extraction rate lines intersect for exponential and hyperbolic discounting. Additionally, we will check how changes in the parameter values affect the ratio of initial resource extraction, R_0^{exp}/R_0^{hyp}. Table 7.1 shows $S_{intersect}$, $\frac{R}{S}_{intersect}$ and R_0^{exp}/R_0^{hyp} for each scenario.

Scenario	$S_{intersect}$	$\frac{R}{S}_{intersect}$	R_0^{exp}/R_0^{hyp}
-	40.73	20.24	0.91
$\gamma+$	39.98	19.95	0.91
$\gamma-$	41.45	20.54	0.91
$\sigma+$	48.48	23.93	0.92
$\sigma-$	27.27	13.87	0.92
$g+$	45.98	22.50	0.91
$g-$	33.83	17.15	0.92
$n+$	46.94	22.88	0.91
$n-$	36.83	18.52	0.92
$a+$	23.02	12.03	0.93
$a-$	105.42	48.41	0.92
$b+$	29.82	15.38	0.93
$b-$	56.36	26.80	0.90
$\frac{K}{S}+$	42.40	20.07	0.91
$\frac{K}{S}-$	39.64	20.38	0.92

Table 7.1: Sensitivity analysis

In our first scenario we increase the value of the output elasticity of resources γ from 1 to

1.1 ($\gamma+$) and decrease it to 0.9 ($\gamma-$). The effect on our key figures is minimal, $S_{intersect}$ and $\frac{R}{S}_{intersect}$ both change by less than one year while the ratio of initial resource extraction does not change at all.
Next, we vary the value of the inverse of the elasticity of intertemporal substitution, σ. An increase of σ to 2.5 increases $S_{intersect}$ by almost 8 years, a decrease of σ to 1.5 decreases $S_{intersect}$ by more than 12 years. $\frac{R}{S}_{intersect}$ changes by 3 and 7 years, respectively. The ratio of initial resource extraction is similar to the benchmark case for both scenarios.
Changes in the growth rate of technology g and the growth rate of population n have similar effects on the key figures as they only appear together in the expression $(g + \beta n)$ in the dynamic system (4.28)-(4.34). Varying g between 0.01 and 0.02 and n between 0.005 and 0.2 leads to changes in $S_{intersect}$ and $\frac{R}{S}_{intersect}$ of up to 7 years. In these scenarios, too, the changes in R_0^{exp}/R_0^{hyp} are minimal.
We next analyse changes in the initial capital-resource ratio $\frac{K_0}{S_0}$. Varying the initial capital-resource ratio between 1.5 and 4.5 does lead to changes in $S_{intersect}$ of less than two years, while $\frac{R}{S}_{intersect}$ and R_0^{exp}/R_0^{hyp} hardly change at all. Considering that $\frac{K_0}{S_0}$ is changed by 50%, the sensitivity of the results with respect to the initial capital-resource ratio seems to be rather small.
Finally, we investigate the effects of changes in the parameters a and b, which determine the discount function in the hyperbolic model. Increasing a drastically to 0.03 - which corresponds to a increase in ρ from 2.4% to 3.3% - significantly reduces both $S_{intersect}$ and $\frac{R}{S}_{intersect}$ to 23.02 and 12.03 years, respectively. A decrease in a to 0.01 ($\equiv \rho = 1.25$)% increases $S_{intersect}$ and $\frac{R}{S}_{intersect}$ to 105.42 and 48.41 years, respectively. The impact of changes in b is smaller. Varying b between 1.75 and 2.25 leads to changes in $S_{intersect}$ and $\frac{R}{S}_{intersect}$ of up to 16 years. The ratio of initial resource extraction is similar to the benchmark case for all scenarios.
Concluding, it can be said that the results are robust with respect to changes in some parameters such as the output elasticity of resources, γ and the initial capital-resource ratio $\frac{K_0}{S_0}$. Other parameters, such as a and b, do impact the quantitative results quite significantly when changed enough. However, none of the parameter alterations changed the qualitative results of the model. Initial resource extraction is always higher with hyperbolic discounting, which means that the resource stock declines faster in the short run. In the medium-term, the level of resource extraction in the model with hyperbolic discounting drops below the level of resource extraction in the model with exponential discounting. Therefore, the resource stock declines slower in the model with hyperbolic discounting and remains higher after a unique point of intersection, $S_{intersect}$. The extraction rate is also higher in the model with hyperbolic discounting initially, but falls below the extraction rate of the model with exponential discounting after $\frac{R}{S}_{intersect}$.

8 Conclusion

In my thesis, I applied the time-consistent method of hyperbolic discounting proposed by Strulik (2017) and applied it to the Dasgupta-Heal-Solow-Stiglitz (DHSS) model of capital accumulation and optimal depletion of a non-renewable resource. I then compared the results to those obtained under conventional exponential discounting. In order to be able to make a fair comparison between both discounting methods, I applied the equivalent-present-value argument which ensures that both discounting methods provide the same present value of a constant utility flow.
The comparison showed that resource use is more conservative under hyperbolic discounting, even though households' discount rate is much larger in the short run than with exponential discounting. However, their low long-run discount rate offsets the effects of their present-bias, making them more patient in the long-run. Both the capital stock and the resource stock tend to be larger in the scenario with hyperbolic discounting than with exponential discounting in the long-run. This in turn leads to higher long-run consumption growth rate. In particular, I showed that with exponential discounting much higher technological growth is needed to achieve positive consumption growth at all.
Finally, I calibrated the model parameters to fit data from the United States of America. The simulation shows that the resource extraction rate is only higher for the first twenty years with hyperbolic discounting. After that, it is always lower compared to the model with exponential discounting. The resource stock is lower with hyperbolic discounting at first. Due to the lower resource extraction rate this changes after around 40 years. The resource stock remains higher compared to exponential discounting thereafter.
A sensitivity analysis shows that the results are robust with respect to parameter variations.

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