Aspects of conformal gravity

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ABSTRACT

Conformal gravity is a higher derivative gravitational theory that is conformally invariant, in addition to its diffeomorphism invariance. In four dimensions the conformal gravity Lagrangian contains up to four derivatives of the metric. Like most higher derivative theories, a naive analysis yields that conformal gravity is power-counting renormalizable at the prize of introducing ghost degrees of freedom, in contrast to general relativity, which is power-counting non-renormalizable but has no ghosts. The theory has been considered in several contexts in the literature, such as a quantum gravity, cosmology and holography.

Throughout this thesis, conformal gravity is examined in holographic, classical and semi-classical contents. At first, in order to establish the structure of a possible holographic dual the theory is considered in the holographic approach. In particular, conformal gravity is formulated with new, generalized asymptotic boundary conditions which allow for a term compatible with the most general spherically symmetric solution of the theory, namely an asymptotically subleading Rindler term. The conformal gravity action with the proposed asymptotic boundary conditions is proven to constitute a well-defined variational principle and the corresponding response functions are shown to be finite. Therefore, no additional boundary terms or holographic counterterms are required to be added at the level of the action. The obtained results for the response functions are applied to phenomenologically interesting examples. Furthermore, the asymptotic symmetry algebras of the dual field theory are constructed and they are classified according to their number of generators. It turns out that the highest-dimensional subalgebra consists of 5 generators. Then, classical aspects of conformal gravity are examined via the Hamiltonian formulation of the theory. Namely, exploiting the constraint analysis, the generator of gauge symmetries is derived and then, using slightly more generalized boundary conditions compared to the ones of the holographic analysis, the canonical charges associated with asymptotic symmetries are constructed. No charges associated with local Weyl rescalings are found. Thus, the obtained charges are associated with asymptotic spacetime diffeomorphisms and their asymptotic symmetry algebra is the algebra of boundary conditions preserving diffeomorphisms. Finally, conformal gravity is considered in the semi-classical approximation. This is done by analytically evaluating the 1-loop partition function of the theory, using heat kernel techniques.

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1

INTRODUCTION, SUMMARY AND OUTLINE

General Relativity, describing the dynamics of the gravitational field or equivalently of spacetime, has been established as a successful theory of nature since it makes predictions that were tested experimentally. Namely, it makes predictions for phenomena e.g. at solar system scales, at galactic scales, for the Cosmic Microwave Background, etc. In particular, concerning solar system scales, the Schwarzschild solution, which describes the exact exterior field of a static and spherical body, predicts precession of the perihelia of the orbit of Mercury, gravitational red-shift of spectral lines, deflection of light by the sun, time delay of radar echoes passing the sun. Furthermore, General Relativity together with assumptions of homogeneity, isotropy and about the matter content of the universe, predicts the cosmic abundance of helium and the existence of the cosmic microwave radiation. Additionally, an analysis of the linearized Einstein's equations predicts gravitational waves. All the previously mentioned predictions have been accurately confirmed by precise measurements which were detected experimentally [1], [2]. Furthermore, there has also been experimental verifications of the underlying principles which the theory relies on, e.g. equivalence principle, local Lorentz invariance, and others [3], [4], [5], [6]. Lastly, General Relativity has the correct behavior in the slow moving (non-relativistic) and weak field limit. That is, its predictions reduce to those of Newtonian gravity in this limit.

Although present experimental technology is very far from detecting quantum gravitational effects, it is nevertheless a crucial theoretical task to investigate a quantum theory of gravity. Then, if the principles of quantum theory are to be applied to the gravitational field, General Relativity must be the classical approximation to a fundamental theory of quantum gravity. Enormous attempts have been made towards the direction of quantization of the gravitational field. According to the standard quantization methods, all such attempts fail in quantizing the gravitational field. At first, following perturbation theory, General Relativity is rendered non-renormalizable. Firstly, this can be obtained by power counting arguments: the inverse Newton's constant, i.e. the coupling constant of the Einstein-Hilbert action, has negative mass dimensions and thus renders the theory power counting non-renormalizable. Indeed, an explicit calculation reveals that it is 2-loop non-renormalizable [7]. Additionally, each perturbative order requires a new counterterm, with the nth-order one being a function of the nth-power of the Riemann tensor and its contractions. Thus, with the series not terminating, Einstein gravity is rendered non-renormalizable. Furthermore, adopting the path integral approach to quantization, the Euclidean form of the General Relativity action is not positive definite [8], not even for real and positive metrics. Hence the Euclidean path integral does not converge and thus, it is ill-defined. Only after restricting to particular classes of solutions, e.g. asymptotically Euclidean metrics and others, a positive action is obtained [9] and the Euclidean path integral is expected to converge. Nevertheless, restricting the space of solutions is not a satisfactory resolution to the problem of the convergence of the Euclidean path integral of General Relativity. Lastly, following the canonical (Dirac) quantization, one ends up with the Wheeler-Dewitt equation [10] which has technical difficulties on actually finding its solutions, as well as conceptual issues concerning the (no) time evolution of the gravitational wave function, or wave function of the universe.

Adopting perturbation theory and as an attempt to resolve the problems of renormalization of General Relativity, higher derivatives in the metric were added in the action. The justification was that, since higher derivative terms in the Einstein-Hilbert Lagrangian appear as counterterms at the 1-loop level, an obvious resolution was to add such terms in the original action, in order to make it renormalizable. In particular, adding two higher derivative terms in the Einstein-Hilbert action, consisting of the square of the Ricci scalar and the Ricci tensor, it turns out that the action is renormalizable [11], [12] for appropriate values of the coupling constants of these terms. Now some classical aspects of generic higher derivative theories are the following: they have a well-defined initial value formulation [13] and Schwarzschild solution is among the wide class of solutions of these theories. Also, the Newtonian limit is recovered in the weak field approximation, but with fine tuning of the parameters [11].

The problem of higher derivative theories, which renders them unsatisfactory as fundamental theories of (quantum) gravity, already appears at similar theories in classical mechanics. That is, addition of higher time derivative terms in the Lagrangian causes an instability [14]. This is a linear instability in the Hamiltonians associated with Lagrangians depending on more that one time derivative, in a way that the higher derivatives cannot be removed by partial integration. Such Hamiltonians depend linearly on the canonical momenta and thus, the system is unbounded from below. This problem is maintained when passing to the higher derivative gravitational theories. Indeed, considering a theory consisting of the Einstein-Hilbert Lagrangian and additionally the square of the Ricci scalar and the Ricci tensor, and choosing the value of the coupling constants for which this theory is renormalizable, a linearized analysis reveals 8 physical degrees of freedom [15]. Two of them correspond to the standard massless spin-2 graviton and from the remaining 6, 5 correspond to a massive spin-2 field and the 6th is the Boulware-Deser ghost [16]. This massive spin-2 field appears with a minus sign relative to the other fields and this can never be changed. Classically, this means that the corresponding excitation has negative energy which leads to a breakdown of causality, since propagation of negative energy waves occur outside the light cone. At the quantum level, the theory can be reconstructed by having positive energy but negative norm on the state vector space. These negative norms cannot be discarded without destroying the unitarity property of the S-matrix. Therefore, facing such obscure conceptual problems, such theories cannot be considered as fundamental theories of (quantum) gravity. A survival of such theories is, maybe, in some sense viable when considering an effective theory approach, but again with the cost of fine tuning of the coupling constants of the higher derivative terms. If those coupling constants were small enough to make the ghost fields only important on distance scales near the Planck length and if there was a breakdown of causality at this scale, a higher derivative model could represent an effective theory of gravitation at more familiar lengths.

Throughout this thesis, a particular higher derivative gravitational theory in 4 spacetime dimensions is researched. The name that is attributed to this theory is conformal gravity, due to its conformal invariance as its characteristic feature. In other words, the theory depends on Lorentz angles, but not on distances. The conformal gravity action consists of the square of the Weyl tensor. Since the Weyl tensor has the property of being invariant under local rescalings of the metric, the resulting action in 4 spacetime dimensions is conformally invariant. As it was already mentioned for generic higher derivative theories, conformal gravity seems to have a better behavior than General Relativity in the quantum regime. Adopting perturbation theory as a scheme of quantization, conformal gravity is power-counting renormalizable due to its dimensionless coupling constant. An actual calculation reveals that it is 1-loop renormalizable [17]. However, the open question is whether presence of conformally invariant counterterms are required at all orders of perturbation. Along the line of the path integral approach to quantization, the conformal gravity action is positive definite assuming real and positive metrics and a positive coupling constant. Therefore, the Euclidean path integral converges [18]. Lastly, when adopting the canonical quantization scheme, the presence of the Hamiltonian constraint does not seem to resolve the problematics of the corresponding Wheeler-Dewitt equation [19]. As far as classical aspects of conformal gravity are concerned, the initial value formulation of the theory is well-defined [20]. The most general static, spherically symmetric solution of the theory [21], [22] contains one additional parameter as compared to $Schwarzschild - AdS_4$ solution of General Relativity. This parameter is known as Rindler acceleration [23]. It turns out that $Schwarzschild - AdS_4$ spacetime is obtained as a solution of conformal gravity, as a special case with vanishing Rindler acceleration. Additionally, the theory with some fine-tuning of the parameters fits well to galactic rotation curves without need for dark matter [24]. Also, the gravitational potential of a source characterized by the above mentioned conformal gravity solution, is linear in the Rindler term and at small distances this term is negligible and the Newtonian limit is recovered.

Furthermore in the literature, conformal gravity has been studied in a quantum gravity context [25], as a possible UV completion of gravity [26], [27], [28], it emerges theoretically from twistor string theory [29] and as a counter term in the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [30], [31].

Of course, like all higher derivative theories, conformal gravity contains ghosts. It has been conjectured that the theory may admit an alternative quantization that preserves unitarity [27], [32], nevertheless it has received some criticism [33], [34]. The problem of ghosts is beyond the scope of this thesis and in any case, conformal gravity should not be considered as a fundamental theory of (quantum) gravity. Instead, the focus here is on the classical formulation of the theory and on establishing the framework for a possible holographic dual.

Interesting results were obtained in the literature, when attempting to find connections between conformal and Einstein gravity. In particular, it was observed that the renormalized on-shell Einstein-Hilbert action of 4-dimensional asymptotically hyperbolic Einstein spaces is given by the action of conformal gravity in these spaces [35], [36], [37]. Following this approach to connect both theories, it was demonstrated [38] that a class of solutions of conformal gravity with an appropriate boundary condition leads to solutions of Einstein gravity with a cosmological constant term. Particularly, considering the conformal gravity action and if a Neumann boundary condition is imposed on the metric at the boundary, then for spherically symmetric configurations $Schwarzschild - (A)dS_4$ spacetime arises as a black hole solution. Moreover, the Euclidean conformal gravity action of $Schwarzschild - (A)dS_4$ black hole, with a particular choice for the coupling constant of the theory, matches with the Euclidean Einstein-Hilbert action with a cosmological constant of the same black hole solution. Subsequently, the corresponding black hole entropy of both theories coincides as well. Therefore, according to these findings, it can be concluded that this Neumann boundary condition eliminates, in a way, the ghosts that conformal gravity possesses by picking only those solutions that are also solutions of Einstein gravity plus a cosmological constant.

The scope of the present thesis is to investigate holographic and classical aspects of conformal gravity. In particular, the theory is analyzed in a holographic context by implementing appropriate asymptotic boundary conditions, in order to evaluate the finite response functions of the dual field theory and to specify the asymptotic symmetry algebras. Furthermore, conformal gravity is formulated classically by exploiting the Hamiltonian formalism, in order for the dynamics as well as the gauge symmetries of the theory to be revealed. Furthermore, adopting a semi-classical approximation the 1-loop corrections of the theory are evaluated.

The organization of the thesis is as follows: in Part I, all basic concepts which are essential in the formulation of the research on aspects of conformal gravity are introduced. At first, in chapter 2, the Hamiltonian formulation of gauge theories is presented and in 2.4, the Hamiltonian analysis of General Relativity is presented. Then, in chapter 3, the path integral approach to gauge theories and 1-loop corrections is displayed and the relevant analysis on General Relativity is performed in 3.2. Finally, in chapter 4, comments on the AdS-CFT conjecture are mentioned. Then, Part II is the main part of the research on aspects of conformal gravity. At first, in chapter 5, a holographic calculation with the presence of asymptotically $(A)dS_4$ boundary conditions is formulated. The variational principle of the conformal gravity action and the corresponding holographic response functions are examined and the results are applied to particular solutions of conformal gravity. Finally, the asymptotic symmetry algebras are analyzed. Continuing further with chapter 6, the Hamiltonian formalism of conformal gravity is constructed. The constraint analysis is performed, the Poisson bracket algebra of the constraints is discussed, the gauge generator is constructed and after the implementation of boundary conditions, the associated canonical charges are evaluated. In chapter 7, the 1-loop partition function of the theory is computed. A linearized analysis is adopted and the corresponding linearized equations of motion are extracted. Then, after evaluating the Faddeev-Popov determinant, the 1-loop partition function is presented and evaluated analytically using heat kernel techniques. Part II ends with chapter 8, where summary of the research and its main conclusions are explicitly presented. Part III consists of appendices and Part IV contains the bibliography.

Part I

BASIC CONCEPTS

2

HAMILTONIAN FORMULATION OF GAUGE THEORIES

2.1 GENERAL SETUP

The most explicit treatment of a gauge system is the Hamiltonian formulation, originally formulated by Dirac [39]. The formalism is constructed in a way to accommodate the characteristic feature of gauge theories: the presence of arbitrary functions of time in the solutions of the equations of motion. This implies that the canonical variables are not all independent but there are constraint relations between them. The Hamiltonian formulation basically consists of handling these constraints.

Throughout this chapter, the Hamiltonian formalism is presented for a gauge system consisting of finite degrees of freedom (mechanics) in 2.1. Then, in 2.2, the formalism is generalized for a gauge system consisting of infinite degrees of freedom (field theory). In this case, the presence of boundaries and the construction of canonical charges is discussed. The Hamiltonian formalism is then applied to two particular examples in 2.3, the first one is a mechanical system and the second one is Electrodynamics. Finally, in 2.4, the Hamiltonian analysis of General Relativity is presented.

2.1.1 *Primary constraints*

The mechanical system in question is characterized by the Lagrangian function $L = L(q, \dot{q})$, which is a function of positions $q = \{q_i(t)\}$ and velocities $\dot{q} = \{\dot{q}_i(t)\}$, with $i = 1, \ldots, N$ denoting the physical degrees of freedom. The action functional is of the form

$$S = \int_{t_1}^{t_2} dt\; L(q, \dot{q})\,. \tag{2.1}$$

In order to find the classical equations of motion one requires Hamilton's principle to hold: that is, action (2.1) must be stationary under arbitrary variations of the positions δq_i, i.e.

$$0 \stackrel{!}{=} \delta S = \int_{t_1}^{t_2} dt\; \delta q_i \Big[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big)\Big] + \frac{\partial L}{\partial \dot{q}_i}\delta q_i\Big|_{t_1}^{t_2} \tag{2.2}$$

where Einstein summation has been adopted and is considered from now on. The above is satisfied when

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) = 0 \quad , \forall\; i = 1, \ldots, N \tag{2.3}$$

assuming fixed end points, i.e. $\delta q_i(t_1) = \delta q_i(t_2) = 0$. These are the Euler-Lagrange equations of motion and can be rewritten as

$$\ddot{q}_j \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = \frac{\partial L}{\partial q_i} - \dot{q}_j \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} \tag{2.4}$$

with $j = 1, \ldots, N$. It is clear from the above expression that, at a given time, the accelerations $\ddot{q}_j$ are uniquely determined by positions and velocities (at that time) if and only if the matrix $\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$ is invertible i.e. if and only if $\det\left[\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right] \neq 0$. If, on the contrary, this determinant vanishes then the accelerations $\ddot{q}_j$ cannot be uniquely determined by positions and velocities at that given time. This implies that the equations of motion (2.4) will contain arbitrary functions of time. And this is exactly the characteristic feature of the theories that exhibit gauge invariance. In other words, for a gauge theory one has

$$\det\left[\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right] = 0\,. \tag{2.5}$$

The next step is to construct the canonical Hamiltonian from the Lagrangian. The canonical momenta are defined as

$$p^i \equiv \frac{\partial L}{\partial \dot{q}_i}\,. \tag{2.6}$$

With this definition, condition (2.5) takes the form $\det\left[\frac{\partial p^j}{\partial \dot{q}_i}\right] = 0$ and implies that the velocities are non-invertible functions of coordinates and momenta. Therefore, the canonical momenta (2.6) are not all independent but there are relations between them of the form

$$\phi_m(q,p) = 0 \quad , m = 1, \ldots, M \tag{2.7}$$

with $M < N$. These relations are the primary constraints. The name primary is justified in the sense that these constraints follow directly from the definition of the canonical momenta and no use of the equations of motion has been made to obtain them.

It is usually assumed for simplicity that the rank of the matrix $\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$ is constant in $(q, \dot{q})$ space and that the constraint equations (2.7) define a submanifold smoothly embedded in the phase space (q,p). This submanifold is the primary constraint surface. To continue with the Hamiltonian formulation it is necessary to impose some conditions on this primary constraint surface, also known as regularity conditions. These are not explicitly mentioned here but are assumed to hold throughout the following sections. A detailed description of these regularity conditions can be found in section 1.1.2. in [40].

2.1.2 *Canonical Hamiltonian*

The canonical Hamiltonian is defined as a Legendre transformation of the Lagrangian

$$H_c = \dot{q}_i p^i - L\,. \tag{2.8}$$

Variation of (2.8), after using (2.6), yields

$$\delta H_c = \dot{q}_i \delta p^i + \delta \dot{q}_i p^i - \delta \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - \delta q_i \frac{\partial L}{\partial q_i}\,. \tag{2.9}$$

This implies that H_c can be expressed as a function of coordinates and momenta and not as function of velocities. Rewriting the l.h.s. of (2.9) as $\delta H_c = \frac{\partial H_c}{\partial q_i}\delta q_i + \frac{\partial H_c}{\partial p^i}\delta p^i$ one gets

$$\Big(\frac{\partial H_c}{\partial q_i} + \frac{\partial L}{\partial q_i}\Big)\delta q_i + \Big(\frac{\partial H_c}{\partial p^i} - \dot{q}_i\Big)\delta p^i = 0\,. \tag{2.10}$$

At this point it is necessary to make use of a theorem that is stated here without proof (this can be found again in section 1.1.2. in [40]). First one assumes that the primary constraint surface fulfills the regularity conditions. The theorem then is

If $\lambda^i \delta q_i + \mu_i \delta p^i = 0$ *for arbitrary variations* $\delta q_i, \delta p^i$ *tangent to the constraint surface, then*

$$\lambda^i = u^m \frac{\partial \phi_m}{\partial q_i} \tag{2.11a}$$

$$\mu_i = u^m \frac{\partial \phi_m}{\partial p^i} \tag{2.11b}$$

for some u^m. The equalities are equalities on the primary constraint surface.

Combining (2.10) and (2.11a) one gets

$$\dot{q}_i = \frac{\partial H_c}{\partial p^i} + u^m \frac{\partial \phi_m}{\partial p^i} \tag{2.12}$$

$$-\frac{\partial L}{\partial q_i} = \frac{\partial H_c}{\partial q_i} + u^m \frac{\partial \phi_m}{\partial q_i} \tag{2.13}$$

with u^m being arbitrary. Now, using the definition of the canonical momenta (2.6) and the Euler-Lagrange equations of motion (2.3) one finds

$$\dot{q}_i = \frac{\partial H_c}{\partial p^i} + u^m \frac{\partial \phi_m}{\partial p^i} \tag{2.14}$$

$$\dot{p}^i = -\frac{\partial H_c}{\partial q_i} - u^m \frac{\partial \phi_m}{\partial q_i} \,. \tag{2.15}$$

These are the Hamilton equations of motion. As it has been demonstrated, they follow directly from the Euler-Lagrange equations of motion. Conversely, it is straightforward to show that the Hamilton equations (2.14), (2.15) together with the constraints (2.7) give the Euler-Lagrange equations of motion (2.3). This complete equivalence between the canonical Hamiltonian and the Lagrangian is lost, at the later construction of the extended Hamiltonian function.

The Hamilton equations of motion (2.14), (2.15) can be derived from an action principle of the form

$$S_c = \int_{t_1}^{t_2} dt \left(p^i \dot{q}_i - H_c - u^m \phi_m \right) \tag{2.16}$$

where ϕ_m are the primary constraints and u^m are arbitrary Lagrange multipliers. Requiring this action to be stationary under arbitrary variations δq_i, δp^i and δu^m, one finds

$$\begin{aligned} 0 \stackrel{!}{=} \delta S_c = \int_{t_1}^{t_2} dt \Big(\delta p^i \Big[\dot{q}_i - u^m \frac{\partial \phi_m}{\partial p^i} - \frac{\partial H_c}{\partial p^i} \Big] + \delta q_i \Big[-\dot{p}^i - u^m \frac{\partial \phi_m}{\partial q_i} - \frac{\partial H_c}{\partial q_i} \Big] \\ - \delta u^m \phi_m \Big) + \delta q_i p^i \Big|_{t_1}^{t_2} \end{aligned} \tag{2.17}$$

which are the Hamilton equation of motion (2.14), (2.15) and the primary constraints $\phi_m(q, p) = 0$, subject to the boundary conditions $\delta q_i(t_1) = \delta q_i(t_2) = 0$ (the same boundary conditions that were used in the Lagrangian formulation). Another observation is made at this point: performing a displacement in the canonical Hamiltonian as $H_c \to H_c + v^m \phi_m$, it is obvious from action principle (2.16) and variation (2.17) that one gets again equations of motion (2.14), (2.15) up to a redefinition of the Lagrange multipliers as $u^m \to u^m + v^m$. Thus, there is not a unique Hamiltonian function describing a gauge theory.

2.1.3 *Poisson bracket formalism*

There is a more systematic way to perform the Hamiltonian analysis. In particular, for arbitrary functions f, g of the canonical variables (q, p) there exists a bracket operation defined as

$$\{f, g\} \equiv \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q_i} \,. \tag{2.18}$$

This is the Poisson bracket. It has the following properties, arising directly from its definition:

$$\{f, g\} = -\{g, f\} \quad \text{antisymmetry} \tag{2.19}$$

$$\{f_1 + f_2, g_1 + g_2\} = \{f_1, g_1\} + \{f_1, g_2\} + \{f_2, g_1\} + \{f_2, g_2\} \quad \text{linearity} \tag{2.20}$$

$$\{f_1 f_2, g\} = f_1\{f_2, g\} + \{f_1, g\} f_2 \quad \text{product law} \tag{2.21}$$

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0 \quad \text{Jacobi identity}\,. \tag{2.22}$$

With this tool in hand, it is possible to express the Hamilton equations (2.14), (2.15) in a compact way. Indeed, considering the total time derivative

$$\dot{f} = \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p^i}\dot{p}^i \tag{2.23}$$

and then substituting $\dot{q}_i$ and $\dot{p}_i$ the r.h.s. of the above expression with the Hamilton equations (2.14), (2.15) and finally using the Poisson brackets' properties (2.20), (2.21) one finds that

$$\dot{f} = \{f, H_c\} + u^m\{f, \phi_m\} \,. \tag{2.24}$$

It is straightforward to verify that for $f = q_i$ and $f = p^i$, one gets the Hamilton equations of motion (2.14), (2.15).

2.1.4 *Secondary constraints*

A consistency requirement one has to impose is that the primary constraints ϕ_m have to be preserved in time. This is because they have to hold during the whole time evolution of the system. Using $f = \phi_m$ in (2.24) one gets

$$\dot{\phi}_m = \{\phi_m, H_c\} + u^r\{\phi_m, \phi_r\} \overset{!}{=} 0 \quad \forall \; m \tag{2.25}$$

where $r = 1, \dots, M$. The above set of equations gives consistency conditions for the constraints which can be categorized as follows: the set (2.25) gives

1. relations between the canonical variables (q, p), independent of the Lagrange multipliers u^m If
   - these relations reduce to primary constraints, then equations (2.25) are trivially satisfied.
   - these relations are independent of primary constraints, then they are called secondary constraints and one has to proceed further to examine their consistency in time. The name secondary is justified, in the sense that primary constraints are just consequences of the definition of canonical momenta, whereas for the secondary constraints one has to make use of the equations of motion as well. For example, considering $X(q, p) = 0$ as a secondary constraint, this also need to be preserved in time and according to (2.24) one gets
   $$\dot{X} = \{X, H_c\} + u^m\{X, \phi_m\} \overset{!}{=} 0 \,. \tag{2.26}$$
   Of course now, one has to examine again whether this new consistency condition implies new secondary constraints or if one falls into either category $1i.$, i.e. one gets an expression dependent on the constraints found so far, or category $2)$, i.e. one gets restrictions on the Lagrange multipliers u^m. This case is analyzed in 2.1.4.2, that follows, after introducing the weak equality notation. When this process is terminated, one ends up with the secondary constraints
   $$X_k = 0 \,, \quad k = M+1, \dots, M+K \tag{2.27}$$
   where K is their total number.
2. relations between the Lagrange multipliers u^m. Then, the set (2.25) has to be solved with respect to these u's. Again, this case will be analyzed in 2.1.4.2.
3. a mixture of categories $1)$ and $2)$ (this of course is also the case $1ii.$, when exhausting the constraints' consistency conditions). In this case, one first has to terminate the consistency conditions for the constraints that are not u-dependent and then determine the u's.

For later convenience, it is better from now on to denote all constraints, primary and secondary, as

$$\phi_j \equiv \begin{pmatrix} \phi_m \\ X_k \end{pmatrix} = 0 \,, \qquad j = \underbrace{1, \ldots, M}_{\text{m}}, \underbrace{M+1, \ldots, M+K}_{\text{k}} \equiv J \,. \tag{2.28}$$

1. Weak equality

At this moment, it is useful to distinguish between equations that hold on the constraint surface and equations that hold on the entire phase space. For this, one uses the weak equality symbol $\approx$ for equations on the constraint surface. Therefore, one can write (2.28) as

$$\phi_j \approx 0 \,. \tag{2.29}$$

Whenever a quantity is weakly zero, this implies that it is a combination of constraints and need not necessarily vanish in the entire phase space. In this perspective, after the consistency algorithm for the constraints (2.25) has been terminated one should get

$$\dot{\phi}_j \approx 0 \quad \forall \;\, j \tag{2.30}$$

either because the l.h.s. is a combination of constraints either because the u's have been restricted either because there is a combination of both these cases.

2. Restrictions on Lagrange multipliers

What is left of the above discussion is to analyze the case where one gets restrictions on the Lagrange multipliers u^m. For this case one assumes firstly that the consistency process arising from (2.25) has been exhausted. In this way, both categories 2 and 3 which have been stated before are covered. The consistency conditions for the constraints (2.25) are written as

$$\dot{\phi}_j = \{\phi_j, H_c\} + u^m \{\phi_j, \phi_m\} \approx 0 \,. \tag{2.31}$$

These are J inhomogeneous equations of the $M \leq J$ unknowns u^m, with coefficients that are functions of the canonical variables (q, p). Notice that this set of equations must be solvable, otherwise the mechanical system described by the Lagrangian (2.1) is inconsistent in the first place. The general solution of (2.31) is of the form

$$u^m = U^m + V^m \tag{2.32}$$

where U^m is a particular solution of the system of inhomogeneous equations

$$\{\phi_j, H_c\} + U^m \{\phi_j, \phi_m\} \approx 0 \tag{2.33}$$

and V_m is the general solution of the system of homogeneous equations

$$V^m \{\phi_j, \phi_m\} \approx 0 \tag{2.34}$$

where V^m consists of a linear combination of the linearly dependent solutions V_{am} of (2.34), denoted as $v^a V_{am}$, with v^a being arbitrary and $a = 1, \ldots, A$. Thus, the general solution (2.32) can further be written as

$$u^m \approx U^m + v^a V_{am} \,. \tag{2.35}$$

2.1.5 *Total Hamiltonian*

Once the consistency algorithm of the constraints is terminated, one can rewrite the equations of motion $\dot{f} = \{f, H_c + u^m \phi_m\}$ as

$$\begin{aligned} \dot{f} &= \{f, H_c + U^m \phi_m + v^a \phi_a\} \\ &\approx \{f, H_c\} + U^m \{f, \phi_m\} + v^a \{f, \phi_a\} \end{aligned} \tag{2.36}$$

with $\phi_a \equiv V_{am}\phi_m$. Defining

$$H_T \equiv H_c + U^m \phi_m + v^a \phi_a \tag{2.37}$$

as the total Hamiltonian, the equations of motion can be written as

$$\dot{f} \approx \{f, H_T\}\,. \tag{2.38}$$

These are again equivalent with the Euler-Lagrange equations (2.3) because the total Hamiltonian is by construction equivalent to the Lagrangian and both describe the same gauge symmetries. Furthermore, (2.38) can be derived from an action principle. This is

$$S_T = \int_{t_1}^{t_2} dt \big(p^i \dot{q}_i - H_c - U^m \phi_m - u^m \phi_m\big) \tag{2.39}$$

where ϕ_m are primary constraints. After redefining $U^m + u^m \rightarrow u^m$ and requiring this action to be stationary for arbitrary variations δq_i, δp^i and δu^m subject to the boundary conditions $\delta q_i(t_1) = \delta q_i(t_2) = 0$, one gets exactly variation (2.17) i.e. the Hamilton equations (2.14), (2.15) and the primary constraints $\phi_m \approx 0$.

2.1.6 *Classification of constraints into 1st and 2nd class*

As soon as the consistency algorithm of the constraints has been exhausted, the distinction between primary and secondary constraints is of minor importance. This is understood in the sense that no new constraints are generated. Another type of classification is now of crucial importance. This is the classification of constraints into 1st and 2nd class. This is of great significance because it reveals the gauge symmetries of the system. In particular, as it will be explicitly demonstrated, 1st class constraints are the generators of gauge symmetries while 2nd class constraints are not and they are removed by treating them as strong equations, after performing the Dirac bracket formulation.

An arbitrary function f of the canonical variables (q, p) is 1st class if its Poisson bracket with every constraint vanishes weakly, i.e. if

$$\{f, \phi_j\} = c_j^l \phi_l \approx 0\,, \quad \forall j\,. \tag{2.40}$$

A function of the canonical variables that is not 1st class is then called 2nd class and its Poisson bracket with at least one constraint is of the form

$$\{f, \phi_j\} = d_j \tag{2.41}$$

where d_j is a constant, which does not depend on the canonical variables. Some important features arise from the above classification:

- The Poisson bracket of two 1st class functions is 1st class.
  Proof:
  Since f, g are 1st class

$$\{f, \phi_j\} = c_j^l \phi_l \quad , \quad \{g, \phi_j\} = d_j^l \phi_l\,. \tag{2.42}$$

Then

$$
\begin{aligned}
\Big\{\{f,g\},\phi_j\Big\} &= \Big\{f,\{g,\phi_j\}\Big\} - \Big\{g,\{f,\phi_j\}\Big\} \\
&= \Big(\{f,c^l_j\} - \{g,d^l_j\} + c_{jm}c^{ml} - d_{jm}d^{ml}\Big)\phi_l \\
&\approx 0 \\
&\rightarrow \{f,g\} \text{ is } 1^{\text{st}} \text{ class (q. e. d.)}\,. \qquad (2.43)
\end{aligned}
$$

- The constraints $\phi_a \equiv V^m_a \phi_m$ are 1^{st} class.
  Proof:

  $$\{\phi_a, \phi_j\} = \{V^m_a \phi_m, \phi_j\} \approx V^m_a \{\phi_m, \phi_j\} \approx 0\,, \quad \forall j \qquad (2.44)$$

  since V^m_a is a solution of (2.34) (q. e. d.). Moreover, ϕ_a form a complete set of 1^{st} class constraints because $v^a V^m_a$ is the most general solution of (2.34) on the constraint surface.

- The total Hamiltonian (2.37) is 1^{st} class.
  Proof:

  $$
  \begin{aligned}
  \{H_T, \phi_j\} &\approx \{H_c, \phi_j\} + U^m\{\phi_m, \phi_j\} + v^a\{\phi_a, \phi_j\} \\
  &\overset{(2.33),\,(2.34)}{\approx} 0\,, \quad \forall j \quad \text{(q. e. d.)}\,. \qquad (2.45)
  \end{aligned}
  $$

After the above classification, the analysis continues with the interpretation of 1^{st} class constraints as gauge generators and the treatment of 2^{nd} class constraints with the introduction of the Dirac bracket.

2.1.7 *1^{st} class constraints as gauge generators*

In a gauge system, if one examines the final state of the time evolution of canonical variables from a given initial state, then a very interesting feature is revealed. This feature is the fundamental difference between gauge theories and those that do not exhibit gauge invariance.

A general dynamical variable f with initial value f_1 at t_1 is considered to represent the initial state. Then, the final state is obtained after a short time interval $\delta t = t_2 - t_1$ and is characterized by the value f_2 at later time t_2. One can write f_2 as

$$
\begin{aligned}
f_2 &= f_1 + \dot{f}\,\delta t \\
&= f_1 + \{f, H_T\}\delta t \\
&= f_1 + \{f, H_c + U^m\phi_m\}\delta t + v^a\{f, \phi_a\}\delta t \qquad (2.46)
\end{aligned}
$$

where ϕ_a are primary 1^{st} class constraints and v^a are completely arbitrary by construction as was stated in (2.35). At this point, one makes the following observation: even though the initial state f_1 is characterized uniquely by the set (q_1, p^1), one has to choose some values for the v^a's at time t_1 in order to get a set of (q_2, p^2) that represents the final state f_2. This implies that choosing different values for these coefficients at time t_1 one gets different values for (q_2, p^2) in the final state. In particular, it is assumed that one chooses v'^a as different values from v^a. Then, the difference in f_2 is

$$
\begin{aligned}
\Delta f_2 &= (v_a - v'^a)\,\{f, \phi_a\}\,\delta t \\
&= \epsilon^a\{f, \phi_a\} \qquad (2.47)
\end{aligned}
$$

with $\epsilon^a \equiv (v^a - v'^a)\delta t$. Now, since one requires a well-defined initial value formulation, the initial state must determine uniquely the final state. Thus, it is clear that these different values of f_2 (2.47) at final time t_2 must represent the same final state. This implies that the final state must be characterized not only by one set (q_2, p^2) but rather by the set $(\Delta q_2, \Delta p^2)$. This ambiguity is a physically irrelevant ambiguity. This is the characteristic feature of gauge theories and it is exactly the dissimilarity with theories which are not gauge invariant. The transformations $(\Delta q_2, \Delta p^2)$ between the different values of the canonical variables, which correspond to the same physical state, are gauge transformations and as showed in (2.47) they are generated by 1st class primary constraints.

In general, gauge transformations are not uniquely generated by expressions of the form (2.47). In fact, the following statement holds:

- The Poisson bracket of two 1st class primary constraints generates a gauge transformation.
  Proof:
  Suppose that one applies in succession two transformations of the form (2.47) with parameters ϵ^a and η^a respectively. The difference in the final state is

$$\Delta_\epsilon \Delta_\eta f_2 = f_1 + \epsilon^a \{f, \phi_a\} + \eta^b \Big\{ f + \epsilon^a \{f, \phi_a\}, \phi_b \Big\} \tag{2.48}$$

  where $b = 1, \ldots, A$ and terms of the orders $\mathcal{O}(\epsilon^2)$, $\mathcal{O}(\eta^2)$ have been neglected. Applying the two transformations in reverse order one gets

$$\Delta_\eta \Delta_\epsilon f_2 = f_1 + \eta^b \{f, \phi_b\} + \epsilon^a \Big\{ f + \eta^b \{f, \phi_b\}, \phi_a \Big\} \tag{2.49}$$

  Subtracting (2.48) and (2.49) and applying the Jacobi identity (2.22), the result is

$$(\Delta_\eta \Delta_\epsilon - \Delta_\epsilon \Delta_\eta) f_2 = \epsilon^a \eta^b \Big\{ f, \{\phi_a, \phi_b\} \Big\} . \tag{2.50}$$

  Since $(\Delta_2 \Delta_1 - \Delta_1 \Delta_2) f_2$ corresponds to a change of final values f_2 that do not alter the final physical state, this must be the also case for the r.h.s. of the above expression. Thus, $\{\phi_a, \phi_b\}$ generates a gauge transformation (q. e. d.).

This statement allows for possible generalization of the fact that 1st class primary constraints generate gauge transformations. According to (2.43), the Poisson bracket of two primary 1st class constraints is also 1st class. But this by no means implies that this is necessarily a primary constraint. Thus, in principle, a secondary 1st class constraint could also generate gauge transformations. In practice it really does so, e. g. in the case of General Relativity and Electrodynamics. Under this consideration, Dirac conjectured that all secondary 1st class constraints generate gauge symmetries. Although this has not been proved to be true and despite the fact that there are counter-examples to this conjecture, it is generally postulated for all practical and realistic examples that all 1st constraints generate gauge symmetries. Then, the gauge generator takes the form

$$G = \epsilon^a \phi_a \tag{2.51}$$

where ϕ_a are *all* the 1st class constraints and coefficients ϵ^a are arbitrary.

Castellani algorithm

In addition to Dirac's conjecture, there is an algorithm which formally constructs gauge generators by fixing the arbitrary coefficients ϵ^a in (2.51). In some cases this is useful and instructive. This algorithm is known as Castellani algorithm [52] and it can be applied when one is interested in gauge symmetries of the Lagrangian.

The starting point is to consider a physical state (q_i, p^i) of the system that satisfies the Hamilton equations of motion. Then, it is assumed that there exists a second state that again obeys the

Hamilton equations and differs by small variations (η_i, ξ^i) from the first one. For the varied state $(q_i + \eta_i, p^i + \xi^i)$, the Hamilton equations read

$$\begin{aligned} \dot{q}_i + \dot{\eta}_i &= \frac{\partial H_T}{\partial p^i}(q_j + \eta_j, p^l + \xi^l) - (\dot{q}_m + \dot{\eta}_m)\frac{\partial \phi_m}{\partial p^i}(q_j + \eta_j, p^l + \xi^l) \\ \dot{p}^i + \dot{\xi}^i &= -\frac{\partial H_T}{\partial q_i}(q_j + \eta_j, p^l + \xi^l) + (\dot{q}_m + \dot{\eta}_m)\frac{\partial \phi_m}{\partial q_i}(q_j + \eta_j, p^l + \xi^l) \end{aligned} \tag{2.52}$$

and the constraints are

$$p^m + \xi^m = \phi^m(q_j + \eta_j, p^l + \xi^l)\,. \tag{2.53}$$

Expanding the r.h.s. of (2.52) and (2.53) to first order in the small variations $\eta_i(t)$ and $\xi^i(t)$ and then using the Hamilton equations of motion for the first state (q_i, p^i) and its constraint equation one finds

$$\begin{aligned} \frac{\partial^2 H_T}{\partial q_i \partial p^j}\eta_i + \frac{\partial^2 H_T}{\partial p^i \partial p_j}\xi^i - \frac{\partial \phi_m}{\partial p^j}\dot{\eta}_m - \dot{\eta}_j &= 0 \\ \frac{\partial^2 H_T}{\partial q_i \partial q_j}\eta_i + \frac{\partial^2 H_T}{\partial p^i \partial q_j}\xi^i - \frac{\partial \phi_m}{\partial q_j}\dot{\eta}_m + \dot{\xi}^j &= 0 \\ \frac{\partial \phi_m}{\partial q_i}\eta_i + \frac{\partial \phi_m}{\partial p^i}\xi^i - \xi^m &= 0\,. \end{aligned} \tag{2.54}$$

This set of equations give the necessary and sufficient conditions for the varied state $(q_i + \eta_i, p^i + \xi^i)$ to be physical: the small variations $\eta_i(t)$ and $\xi^i(t)$ which satisfy the above set, correspond to gauge degrees of freedom. Now, the next step is clear: one is interested in finding the corresponding conditions for a function to be a generator of these gauge degrees of freedom. This construction involves two assumptions: (i) the small variations (η_i, ξ^i) are generated by arbitrary but finite phase space functions $G_n(q,p)$ with $n = 0, 1, \ldots, k$, and (ii) each generating function $G_n(q,p)$ is parametrized by infinitesimal quantities ϵ^n, with ϵ^n being the $n - th$ time derivative such that

$$\eta_i = \sum_{n=0}^{k} \epsilon^n \{q_i, G_n\} = \sum_{n=0}^{k} \epsilon^n \frac{\partial G_n}{\partial p^i} \tag{2.55}$$

$$\xi^i = \sum_{n=0}^{k} \epsilon^n \{p^i, G_n\} = -\sum_{n=0}^{k} \epsilon^n \frac{\partial G_n}{\partial q_i} \tag{2.56}$$

where $\frac{\partial G_n}{\partial p^i}$ and $\frac{\partial G_n}{\partial q_i}$ are calculated along the first state (q_i, p^i). Substitution of (2.55), (2.56) to the set (2.54) yields

$$\begin{aligned} \Big(\frac{\partial}{\partial q_i} + \frac{\partial \phi_\rho}{\partial q_i}\frac{\partial}{\partial p^\rho}\Big)\Big(\{G_n, H_T\} + G_{n-1}\Big) &= 0 \\ \Big(\frac{\partial}{\partial p^i} + \frac{\partial \phi_m}{\partial p^i}\frac{\partial}{\partial p^m}\Big)\Big(\{G_n, H_T\} + G_{n-1}\Big) &= 0 \\ \{G_n, \text{any constraint}\} &= \text{constraints} \end{aligned} \tag{2.57}$$

or more explicitly

$$\begin{aligned} \{G_0, H_T\} &= \text{PFC} \\ G_0 + \{G_1, H_T\} &= \text{PFC} \\ G_1 + \{G_2, H_T\} &= \text{PFC} \\ &\vdots \\ G_{k-1} + \{G_k, H_T\} &= \text{PFC} \\ G_k &= \text{PFC} \end{aligned} \tag{2.58}$$

where *PFC* stands for "primary first class constraints". An underlying assumption in this expression is that the chain of G_n is finite. This is equivalent with the assumption that there is a finite number of secondary constraints. Actually, k is the number of generations of secondary constraints of the theory. Finally, the gauge generator has the general form

$$G = \sum_{n=0}^{k} \epsilon^n G_n \, . \tag{2.59}$$

The Castellani gauge generator contains by construction all symmetries of the Lagrangian. As it will be explained in what follows, in the quantization procedure one should in principle allow for more gauge degrees of freedom and extend the total Hamiltonian. In this case, the Castellani algorithm can not be applied any more and then, the gauge generator, adopting Dirac's conjecture, can be expressed as (2.51).

2.1.8 *Extended Hamiltonian*

Given the significance of 1st class constraints on account of them being gauge generators, it is important to involve them all in the Hamiltonian formulation. The total Hamiltonian (2.37) does not contain, in principle, all 1st class constraints. Therefore, it is useful to construct a Hamiltonian function in order to exhibit all the existing gauge symmetries of the theory. This is the extended Hamiltonian, constructed as

$$H_E = H_c + U^m \phi_m + u^a \gamma_a \tag{2.60}$$

where γ_a are *all* 1st class constraints. It can be straightforwardly verified, using (2.33), that the extended Hamiltonian (2.60) is 1st class, i.e.

$$\{H_E, \phi_j\} \approx 0 \, , \quad \forall j \tag{2.61}$$

where ϕ_j are all the constraints.

It is evident from expression (2.60) that the extended Hamiltonian is not equivalent with the original Lagrangian (2.1). The extended action principle takes the form

$$S_E = \int_{t_1}^{t_2} dt(\, p^i \dot{q}_i - H_c - U^m \phi_m - u^j \phi_j) \tag{2.62}$$

with ϕ_j being all the constraints. The corresponding equations of motion are

$$\dot{f} \approx \{f, H_E\} \tag{2.63}$$

$$\phi_j \approx 0 \tag{2.64}$$

with $u_j = u^a A_{aj}$ and A_{aj} is such that 1st class constraints can be written as $\gamma_a = A_{aj}\phi_j$. It should be emphasized again that these equations of motion are not equivalent with the ones of the total Hamiltonian (2.38). On the contrary, there is an extension of the Lagrangian theory to a construction such that all gauge freedom becomes manifest. And this is the reason that it is preferable to work with the extended Hamiltonian in the canonical quantization.

2.1.9 *Treatment of 2nd class constraints and Dirac bracket*

According to the previous definition (2.41), a constraint ϕ_j is 2nd class when its Poisson bracket with at least one other constraint does not vanish weakly, i.e. when

$$\{\phi_i, \phi_j\} = d_{ij} \tag{2.65}$$

where d_{ij} are constants, not depending on the canonical variables. From now on, the 2nd class constraints are denoted as $\mathcal{X}_\alpha$ and indices $\alpha, \beta, \ldots$ denote the number of them. It is obvious from the above expression that the contact transformation generated by a 2nd class constraint does not preserve all the constraints $\phi_j \approx 0$ and thus maps an allowed state to a non-allowed state. Therefore, 2nd class constrains cannot be interpreted as gauge generators, they are unphysical and they have to be treated differently. This requires the introduction of the Dirac bracket. The Poisson bracket is then abandoned and all equations of the theory are formulated in terms of this Dirac bracket.

The construction of the Dirac bracket starts as follows: it is stated here without proof that there is an equivalent description of the constraint surface $\phi_j \approx 0$ in terms of 1st class (γ_a) and 2nd class ($\mathcal{X}_\alpha$) constraints (the proof can be found in 1.3.1. of [40]). In this description, one obtains a Poisson bracket matrix as

$$\begin{pmatrix} \{\gamma_a, \gamma_b\} & \{\gamma_a, \mathcal{X}_\beta\} \\ \{\mathcal{X}_\alpha, \gamma_b\} & \{\mathcal{X}_\alpha, \mathcal{X}_\beta\} \end{pmatrix} \approx \begin{pmatrix} 0 & 0 \\ 0 & C_{\alpha\beta} \end{pmatrix} \tag{2.66}$$

where $C_{\alpha\beta}$ is antisymmetric and invertible in the constraint surface, such that

$$C^{\alpha\beta} C_{\beta\gamma} = \delta^\alpha{}_\gamma \,. \tag{2.67}$$

An interesting observation here is that the number of 2nd class constraints must be even, otherwise $C_{\alpha\beta}$ has a zero determinant. The Dirac bracket is now defined as

$$\{f, g\}^\star \equiv \{f, g\} - \{f, \mathcal{X}_\alpha\} \, C^{\alpha\beta} \, \{\mathcal{X}_\beta, g\} \tag{2.68}$$

for any phase space functions f, g. Its properties are

$$\{f, g\}^\star = -\{g, f\}^\star \quad \text{antisymmetry} \tag{2.69}$$

$$\{f_1 f_2, g\}^\star = f_1 \{f_2, g\}^\star + \{f_1, g\}^\star f_2 \quad \text{product law} \tag{2.70}$$

$$\{f, \{g, h\}^\star\}^\star + \{h, \{f, g\}^\star\}^\star + \{g, \{h, f\}^\star\}^\star = 0 \quad \text{Jacobi identity} \,. \tag{2.71}$$

Additionally, one obtains

$$\{\mathcal{X}_\alpha, f\}^\star = 0 \quad \text{for any } f \tag{2.72}$$

$$\{f, g\}^\star \approx \{f, g\} \quad \text{for } g \text{ 1st class and } f \text{ arbitrary} \tag{2.73}$$

$$\Big\{h, \{f, g\}^\star\Big\}^\star \approx \Big\{h, \{f, g\}\Big\} \quad \text{for } g, f \text{ 1st class and } h \text{ arbitrary} \,. \tag{2.74}$$

Relation (2.72) holds in the entire phase space (strong equality). Thus, the 2nd class constraints can be set strongly equal to zero, i.e.

$$\mathcal{X}_\alpha = 0 \tag{2.75}$$

either before of after the evaluation of a Dirac bracket. Now, the equations of motion must be evaluated in terms of the Dirac bracket. Since the extended Hamiltonian (2.63) is 1st class, using property (2.73) one finds that the equations of motion of the extended Hamiltonian take the form

$$\dot{f} \approx \{f, H_E\} \approx \{f, H_E\}^\star \,. \tag{2.76}$$

Therefore, the extended Hamiltonian (2.63) still generates the correct equations of motion. Furthermore, the effect of a gauge transformation can also be evaluated in terms of the Dirac bracket as

$$\{f, \gamma_a\} \approx \{f, \gamma_a\}^\star \,. \tag{2.77}$$

So now the original Poisson bracket is discarded and all equations of the theory are formulated in terms of the Dirac bracket. As it was stated in (2.75), the 2nd class constraints are strongly set to zero and they just become identities, expressing some canonical variables in terms of others. In some cases, including the case of conformal gravity as it is described in Appendix C.3, setting 2nd class constraints strongly equal to zero can actually be used to eliminate some of the canonical variables of the formalism.

2.1.10 *Counting of degrees of freedom*

Since gauge freedom indicates that there is more than one set of canonical variables that corresponds to the same state, it is always allowed to choose one of these sets among the rest. This can be done by some gauge fixing condition. In this way, one can completely fix the gauge by eliminating all 1^{st} class constraints and end up with 2^{nd} class constraints. Then, one arrives at the following counting of physical degrees of freedom:

$$\begin{aligned} 2 \times N =& \big(\text{Number of independent canonical variables}\big) \\ =& \big(\text{Total number of canonical variables}\big) - \big(\text{Number of second class constraints}\big) \\ -& \big(\text{Number of first class constraints}\big) - \big(\text{Number of gauge fixing conditions}\big) \\ =& \big(\text{Total number of canonical variables}\big) - 2 \times \big(\text{Number of first class constraints}\big) \\ -& \big(\text{Number of second class constraints}\big) \; . \end{aligned} \tag{2.78}$$

2.2 GENERALIZATION TO FIELD THEORIES

The described Hamiltonian formalism for gauge systems consisting of finite degrees of freedom can be formally generalized for gauge systems that possess infinite degrees of freedom i.e. for field theories. This formal generalization is performed after generalizing appropriately the notions of the dynamical variables, of summation, of partial differentiation and of the Poisson bracket operation. In particular, a field theory can be described as a mechanical system in which the dynamical variables $\big(q_i(t), p^i(t)\big)$ are defined at each point $x \equiv (\vec{x}, t)$ in spacetime with each index $i, j, \ldots$ taking on continuous values as well, i.e.

$$i \to (i, \vec{x}) \tag{2.79}$$

$$q_i(t) \to q_i(t, \vec{x}) \equiv q_i(x) \tag{2.80}$$

$$p^i(t) \to p^i(t, \vec{x}) \equiv p^i(x) \, . \tag{2.81}$$

Summation of any phase space functions $f_i = f_i(x), g^i = g^i(x)$ is promoted to integration and summation as

$$f_i \, g^i \to \int_{\Sigma_t} d^3x f_i(x) g^i(x) \, . \tag{2.82}$$

Additionally, partial differentiation of a phase space function $f_i(x)$ with respect to the canonical variables $q_i(x), p^j(x)$ is defined as

$$\frac{\partial f_i(x)}{\partial \varrho_k(y)} \to \frac{\partial f_i}{\partial \varrho_k}(x,y) \delta^{(3)}(\vec{x} - \vec{y}) \quad , t = t' \tag{2.83}$$

with $y \equiv (\vec{y}, t')$ and $\varrho_k(x)$ is any of the canonical variables $q_i(x), p^j(x)$. Using (2.82) and (2.83), the (equal time) Poisson bracket (2.18) between two phase space functions $f_i(x), g^j(y)$ becomes

$$\{f_i(x), g^j(y)\} \equiv \int_{\Sigma_t} d^3z \left[\frac{\partial f_i}{\partial q_k}(x,z) \frac{\partial g^j}{\partial p^k}(y,z) - \frac{\partial f_i}{\partial p^k}(x,z) \frac{\partial g^j}{\partial q_k}(y,z) \right] \delta^{(3)}(\vec{x} - \vec{z}) \, \delta^{(3)}(\vec{y} - \vec{z}) \tag{2.84}$$

$$= \left[\frac{\partial f_i}{\partial q_k} \frac{\partial g^j}{\partial p^k} - \frac{\partial f_i}{\partial p^k} \frac{\partial g^j}{\partial q_k} \right] (x,y) \, \delta^{(3)}(\vec{x} - \vec{y}) \quad , t = t' \, . \tag{2.85}$$

Using the above definition, the Poisson bracket between the canonical variables takes the form

$$\{q_i(x), p^j(y)\} = \delta_i^j \, \delta^{(3)}(\vec{x} - \vec{y}) \quad , t = t' \, . \tag{2.86}$$

Phase space functions like the Lagrangian (2.1), the total and the extended Hamiltonian (2.37), (2.60) and the gauge generator (2.51) are now promoted to phase space functionals of the form

$$f = \int_{\Sigma_t} d^3x \; f\left[q_i(x), p^j(x)\right] . \tag{2.87}$$

Partial differentiation of phase functionals with respect to the canonical variables, appearing for example in the definition of the canonical momenta and in the Poisson bracket formalism, is generalized to functional differentiation as

$$\frac{\delta f}{\delta \varrho_k(x)} = \int_{\Sigma_t} d^3y \frac{\partial f\left[q_i, p^j\right]}{\partial \varrho_k}(x,y)\delta^{(3)}(\vec{x}-\vec{y}) \quad , t = t' . \tag{2.88}$$

Exploiting the above relation, the (equal time) Poisson bracket of any phase space functionals f, g is now defined with the above functional derivative (2.88) replacing the partial derivative in (2.18) and using the summation (2.82) as

$$\{f, g\} \equiv \int_{\Sigma_t} d^3x \left[\frac{\delta f}{\delta q_i(x)}\frac{\delta g}{\delta p^i(x)} - \frac{\delta f}{\delta p^i(x)}\frac{\delta g}{\delta q_i(x)}\right] \quad , t = t' . \tag{2.89}$$

The Poisson bracket between a phase space function $f_i(x)$ and a functional g can be derived from the above Poisson bracket (2.89) by setting $f = f_i(x)$. This is found to be

$$\{f_i(x), g\} = \int_{\Sigma_t} d^3y \left[\frac{\partial f_i}{\partial q_k}(x,y)\frac{\delta g}{\delta p^k}(y) - \frac{\partial f_i}{\partial p^k}(x,y)\frac{\delta g}{\delta q_k}(y)\right] \delta^{(3)}(\vec{x}-\vec{y}) \tag{2.90}$$

$$= \left[\frac{\partial f_i}{\partial q_k}\frac{\partial g\left[q_i, p^j\right]}{\partial p^k} - \frac{\partial f_i}{\partial p^k}\frac{\partial g\left[q_i, p^j\right]}{\partial q_k}\right](x) \quad , t = t' \tag{2.91}$$

after using (2.83) and (2.88). It is obvious that this Poisson bracket yields Hamilton equations of motion for $f_i(x)$ being a canonical variable and g being the Hamiltonian.

Under the above considerations, all Poisson bracket relations, equations of motions, interpretations, results and consequently the whole Hamiltonian setup which was presented previously for a gauge system of finite degrees of freedom can be directly applied in the case of gauge field theories. In this way, the Hamiltonian formalism provides a tool of major importance because it can be used to analyze field theories of actual physical interest, since almost all of them possess gauge symmetries.

2.2.1 *Presence of boundaries*

A closer look on the aforementioned rules of generalization of the Hamiltonian formalism to field theories reveals a potential ambiguity. This is obtained as follows: functional differentiation as defined in (2.88) consists of boundary integrals on Σ_t. Now, functionals appearing in the Hamiltonian formalism, from the initial canonical Lagrangian up to the Hamiltonians and the gauge generator, depend on the canonical variables but also on their derivatives. Thus, their functional differentiation consists of boundary integrals on Σ_t and inevitably, after performing an integration by parts, consists as well of surface integrals on $\partial\Sigma_t$. Therefore, consistency with definition (2.88) requires those surface integrals on $\partial\Sigma_t$ to vanish. This is not problematic per se: when the boundary Σ_t is closed (compact), the surface terms on $\partial\Sigma_t$ vanish identically. In such cases, functional differentiation of the Hamiltonians and the gauge generator is well-defined, in the sense that it is of the form (2.88). On the contrary, when the boundary Σ_t is open (non-compact), surface integrals on $\partial\Sigma_t$ resulting from integration by parts do not vanish. Then, the Hamiltonians and the gauge generator have not well-defined functional derivatives, i.e. they are not of the form (2.88). This fact has several consequences: Hamilton equations of motion are defined up to surface integrals on $\partial\Sigma_t$ and thus, they are ill-defined. Likewise, the gauge generator, when acting on the canonical variables via the Poisson brackets, generates gauge transformations of the canonical variables up to surface integrals on $\partial\Sigma_t$. These are not merely technical ambiguities: it turns out that they are crucial for the correct notion of energy, momentum and other conserved quantities of the physical system.

Presence of open boundaries is a completely realistic situation in field theories. This is because one is mostly interested in examining solutions of equations of motion of the theory or, in the language of the Hamiltonian setup, the phase space of solutions: in the case of gravitational theories, where the dynamical field is the spacetime itself, the boundary Σ_t of phase space of a solution is the spatial geometry of this solution. In many cases, this is an open boundary. For instance, asymptotically flat and asymptotically AdS solutions of General Relativity, where asymptotically refers to spatial infinity, have open spatial surfaces. In those cases, functional derivatives of the Hamiltonians and the gauge generator fail to be of the form (2.88) and thus, they are not well-defined.

Under the above considerations, it is obvious that in presence of open boundaries surface integrals on $\partial\Sigma_t$, arising from integration by parts, have to be treated in such a way that functional derivatives end up being of the form (2.88), i.e. they must consist only of boundary integrals on Σ_t. In other words, the Hamiltonians and the gauge generator have to be improved.

2.2.2 *Improved Hamiltonian and gauge generator, boundary conditions and charges*

Throughout this thesis, in order for the Hamiltonians and the gauge generator to be improved in such a way to have well-defined functional derivatives of the form (2.88), the Regge-Teitelboim approach [41] is adopted. Its characteristic feature is the fact that it maintains the (natural) foliation of spacetime into constant time slices (Σ_t), as Hamiltonian formalism does. There exist as well other methods which follow a covariant approach [42], [43].

It was discussed in the previous subsection that, in the case of gravitational theories, the boundary Σ_t of phase space of a solution is the spatial geometry of this solution. This implies that (asymptotic) boundary conditions that coincide with this spatial geometry have to be imposed on the canonical variables. Exploiting such boundary conditions, the Regge-Teitelboim approach improves the Hamiltonians and the gauge generator in order to have well-defined functional derivatives of the form (2.88) as follows: appropriate surface integrals are added, the variation of which cancels the already existing surface integrals, under the boundary conditions. In this way, functional derivatives of the Hamiltonians and the gauge generator consist only on boundary integrals on Σ_t. Thus, they are indeed of the form (2.88) and they are rendered well-defined. Additionally, variation of the improved Hamiltonians and gauge generator vanishes on-shell, on the constraint surface and under the boundary conditions. That is, denoting the improved quantities as H' and Γ, their variation takes the form

$$\delta H'|_{\text{on-shell}} = \delta H - \delta Q' \approx 0 \tag{2.92}$$

$$\delta\Gamma|_{\text{on-shell}} = \delta G - \delta Q \approx 0 \tag{2.93}$$

after the boundary conditions have been imposed and Q, Q' are the appropriate surface integrals on $\partial\Sigma_t$. They are usually referred to as canonical charges. Since they appear under imposition of boundary conditions, they are associated with the corresponding asymptotic symmetries of the theory. Indeed, charges turn out to be the energy, the momentum, or any other conserved quantity of the physical system.

In this chapter, an explicit derivation of the improved Hamiltonians, gauge generator and canonical charges is presented in 2.3.2, where the case of Electrodynamics is discussed and in 2.4, where the case of General Relativity is considered.

With the Hamiltonian formulation in hand, one is now equipped with all the necessary tools to pass into the canonical quantization of theory. The canonical quantization was first performed by

Dirac [44], [45] and its major advantage is the fact that canonical coordinates, conjugate momenta and their corresponding Poisson (or Dirac) brackets have a simple quantum analogue. Therefore, transition from the classical Hamiltonian setup to the quantum theory is almost straightforward. Now the importance of the Hamiltonian formulation becomes extremely significant: it is the starting point of the canonical quantization of classical field theories of physical interest.

2.3 EXAMPLES

The Hamiltonian formulation is now applied to two systems that exhibit gauge symmetries. The first one is a mechanical system which describes conformal mechanics and the second one is a field theory example, in particular Electrodynamics.

2.3.1 *Conformal mechanics*

The action is

$$S = -\int dt \, \frac{1}{2\mu} \, e^{-2}\epsilon_{ijk} \, x^i \, y^j \, z^k \tag{2.94}$$

where e, x^i, y^j, z^k are the dynamical fields, $i, j, k = 1, 2, 3$ and μ is a coupling constant. The above expression is the symmetrical reduction of the conformal gravity action in 3 dimensions, i.e. of $S = \frac{-1}{2\mu}\int_M d^3x \, \epsilon^{abc}\Gamma^d_{ae}(\partial_b\Gamma^e_{cd} + \frac{2}{3}\Gamma^e_{bf}\Gamma^f_{cd})$ [46]. The first order form of the action (2.94) is

$$S = \int dt \Big[p^x_i \dot{x}^i + p^y_i \dot{y}^i - H_c(e, x^i, y^i, z^i, p^x_i, p^y_i) \Big] \tag{2.95}$$

where the canonical coordinates are (e, x^i, y^i, z^i) and the corresponding canonical momenta are $(p^e, p^x_i, p^y_i, p^z_i)$. In order to avoid a larger phase space with more constraints, the appropriate momenta have already been identified [47] and p^x_i, p^y_i are Lagrange multipliers enforcing the relations $\dot{x}^i = ey^i$, $\dot{y}^i = ez^i$. Assuming further that z^i is a linear combination of x^i and y^i of the form $z^i = z_x x^i + z_y y^i$, the phase space becomes 18-dimensional, consisting of the canonical variables (e, x^i, y^i, z_x, z_y) and the canonical momenta $(p^e, p^x_i, p^y_i, p^{zx}, p^{zy})$. Then, the primary constraints are

$$p^e = \frac{\partial L}{\partial \dot{e}} = 0 \tag{2.96}$$

$$p^{zx} = \frac{\partial L}{\partial \dot{z}_x} = 0 \tag{2.97}$$

$$p^{zy} = \frac{\partial L}{\partial \dot{z}_y} = 0 \, . \tag{2.98}$$

The canonical Hamiltonian in (2.95) takes the form

$$H_c = eG \tag{2.99}$$

where

$$G = p^x_i y^i + p^y_i (z_x x^i + z_y y^i) \, . \tag{2.100}$$

The total Hamiltonian is

$$H_T = eG + u_1 p^e + u_2 p^{zx} + u_3 p^{zy} \tag{2.101}$$

where $u_1 = u_1(t), u_2 = u_2(t), u_3 = u_3(t)$ are arbitrary functions of time. The consistency conditions for the primary constraints (2.96), (2.97), (2.98) reveal the secondary constraints

$$\{p^e, H_T\} = -G \tag{2.102}$$

$$\{p^{zx}, H_T\} = -ep^y_i x^i \equiv -e\Pi_x \tag{2.103}$$

$$\{p^{zy}, H_T\} = -ep^y_i y^i \equiv -e\Pi_y \, . \tag{2.104}$$

Proceeding further, the time evolution of the above secondary constraints (2.102), (2.103), (2.104) is

$$\{G, H_T\} = u_2\Pi_x + u_3\Pi_y \approx 0 \tag{2.105}$$

$$\begin{aligned} \{\Pi_x, H_T\} &= e\Big(- z_y\Pi_x + \Pi_y - p^x_i x^i \Big) \\ &\approx -e \, p^x_i x^i \equiv -e \, \Phi_x \end{aligned} \tag{2.106}$$

$$\begin{aligned} \{\Pi_y, H_T\} &= e\Big(- 2p^x_i y^i - z_y\Pi_y + G \Big) \\ &\approx -2e \, p^x_i y^i \equiv -2e \, \Phi_y \, . \end{aligned} \tag{2.107}$$

Thus, Φ_x in (2.106) and Φ_y in (2.107) are ternary constraints. Their time evolution is

$$\{\Phi_x, H_T\} = e\Big(-z_x\Pi_x + \Phi_y\Big) \approx 0 \tag{2.108}$$

$$\{\Phi_y, H_T\} = ez_x\Big(\Pi_y + \Phi_x\Big) \approx 0 \tag{2.109}$$

and therefore, there are no further constraints.

The next step in the analysis is to classify all constraints (2.96)-(2.98), (2.102)-(2.104), (2.106) and (2.107) into 1^{st} and 2^{nd} class. An explicit calculation yields

$$\begin{aligned}
&\{p^e, \phi_m\} = 0 \ , \ \{p^{zx}, \phi_m\} = 0 \ , \ \{p^{zy}, \phi_m\} = 0 \ , \ \forall m = 1, \dots 8 && (2.110)\\
&\{G, p^{zx}\} = \Pi_x \approx 0 \ , \ \{G, p^{zy}\} = \Pi_y \approx 0 \ , \ \{G, \Pi_x\} = \Phi_x + z_y\Pi_x - \pi_y \approx 0 \\
&\{G, \Pi_y\} = \Phi_y - z_x\Pi_x \approx 0 \ , \ \{G, \Phi_x\} = z_x\Pi_x - \phi_y \approx 0 \\
&\{G, \Phi_y\} = z_x\Pi_y - z_x\Phi_x - z_y\Phi_y \approx 0 && (2.111)\\
&\{\Pi_x, \Pi_y\} = -\Pi_x \approx 0 \ , \ \{\Pi_x, \Phi_x\} = \Pi_x \approx 0 \ , \ \{\Pi_x, \Phi_y\} = -\Phi_x + \Pi_y \approx 0 && (2.112)\\
&\{\Pi_y, \Phi_x\} = 0 \ , \ \{\Pi_y, \Phi_y\} = -\Phi_y \approx 0 && (2.113)\\
&\{\Phi_x, \Phi_y\} = \Phi_y \approx 0 && (2.114)
\end{aligned}$$

where ϕ_m in (2.110) are all constraints with $m = 1, \dots 8$. It is obvious from the above Poisson bracket algebra that all constraints are 1^{st} class. The physical degrees of freedom are $\frac{1}{2}(2 \times 9 - 0 - 2 \times 8) = 1$.

Using the total Hamiltonian (2.101), the Hamilton equations of motion are

$$\begin{aligned}
&\dot{e} = \{e, H_T\} = u_1 \ , \ \dot{p}^e = \{p^e, H_T\} = -G && (2.115)\\
&\dot{x}^i = \{x^i, H_T\} = ey^i \ , \ \dot{p}^x_i = \{p^x_i, H_T\} = -e\, p^y_i z_x && (2.116)\\
&\dot{y}^i = \{y^i, H_T\} = e\big(z_x x^i + z_y y^i\big) \ , \ \dot{p}^y_i = \{p^y_i, H_T\} = -e\big(p^x_i + p^y_i z_y\big) && (2.117)\\
&\dot{z}_x = \{z_x, H_T\} = u_2 \ , \ \dot{p}^{zx} = \{p^{zx}, H_T\} = -e\Pi_x && (2.118)\\
&\dot{z}_y = \{z_y, H_T\} = u_3 \ , \ \dot{p}^{zy} = \{p^{zy}, H_T\} = -e\Pi_y\,. && (2.119)
\end{aligned}$$

One notices the characteristic feature of gauge theories, namely the fact that the equations of motion contain arbitrary functions of time. That is, presence of the arbitrary functions u_1, u_2, u_3 in the first equations of (2.115), (2.118), (2.119) respectively states that the canonical variables e, z_x, z_y are not fixed by the equations of motion. Subsequently, the rest equations of motion contain as well arbitrary functions of time and the rest of the canonical variables are partially fixed.

The extended Hamiltonian is obtained when adding the secondary (2.103), (2.104) and ternary constraints (2.106), (2.107) to the total Hamiltonian (2.101). Thus, the extended Hamiltonian takes the form

$$\begin{aligned}
H_E &= H_T + v_4\Pi_x + v_5\Pi_y + v_6\Phi_x + v_7\Phi_x \\
&= (v_1 + e)G + v_2p^e + v_3p^{zx} + v_4p^{zy} + v_5\Pi_x + v_6\Pi_y + v_7\Phi_x + v_8\Phi_y && (2.120)
\end{aligned}$$

where $v_a = v_a(t)$ with $a = 1, \dots 8$, are arbitrary functions of time. The corresponding equations of motion are

$$\begin{aligned}
&\dot{e} = \{e, H_E\} = v_2 \ , \ \dot{p}^e = \{p^e, H_T\} = -G && (2.121)\\
&\dot{x}^i = \{x^i, H_E\} = (v_1 + v_8 + e)y^i + v_7x^i \ , \ \dot{p}^x_i = \{p^x_i, H_E\} = -(v_1 + e)p^y_i z_x - v_5p^y_i - v_7p^x_i && (2.122)\\
&\dot{y}^i = \{y^i, H_E\} = (v_1 + e)\big(z_x x^i + z_y y^i\big) + v_5x^i + v_6y^i \ , \\
&\dot{p}^y_i = \{p^y_i, H_E\} = -(e + v_1)\big(p^x_i + p^y_i z_y\big) - v_6p^y_i - v_8p^x_i && (2.123)\\
&\dot{z}_x = \{z_x, H_E\} = v_3 \ , \ \dot{p}^{zx} = \{p^{zx}, H_T\} = -(e + v_1)\Pi_x && (2.124)\\
&\dot{z}_y = \{z_y, H_E\} = v_4 \ , \ \dot{p}^{zy} = \{p^{zy}, H_T\} = -(e + v_1)\Pi_y\,. && (2.125)
\end{aligned}$$

Once again, one notices the presence of the arbitrary functions of time $v_a(t)$ in the above equations of motion. In particular, presence of v_2, v_3, v_4 in the first equations of (2.121), (2.124), (2.125) implies that the canonical variables e, z_x, z_y are not fixed by the time evolution. As a result, the rest of the canonical variables, the equations of motion of which contain these arbitrary functions of time, are partially fixed.

The gauge generator is constructed by adding all 1st class constraint. In this case, they are contained already in the extended Hamiltonian (2.120). Thus, the gauge generator is simply the extended Hamiltonian (2.120) and generates the gauge symmetries which are described by equations (2.121)-(2.125).

2.3.2 *Electrodynamics*

The Maxwell Lagrangian is

$$L = -\frac{1}{4}\int_{\Sigma_t} d^3x F_{ab}F^{ab} \tag{2.126}$$

with $F_{ab} = \partial_a A_b - \partial_b A_a$ and $a, b = 0, 1, 2, 3$. The vector potential A_a is considered to be the canonical variable. The canonical momenta are

$$\pi^a = \frac{\delta L}{\delta \dot{A}_a} = F^{a0} \tag{2.127}$$

with the primary constraint being

$$\pi^0 = F^{00} = 0\,. \tag{2.128}$$

The canonical Hamiltonian takes the form

$$\begin{aligned} H_c &= \int_{\Sigma_t} d^3x\ \pi^a \dot{A}_a - L \\ &= \int_{\Sigma_t} d^3x \Big(\frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}\pi^i\pi_i - A_0\partial_i\pi^i\Big) \end{aligned} \tag{2.129}$$

with $i, j = 1, 2, 3$. The total Hamiltonian is

$$\begin{aligned} H_T &= H_c + \int_{\Sigma_t} d^3x\ u\,\pi_0 \\ &= \int_{\Sigma_t} d^3x \Big(\frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}\pi^i\pi_i - A_0\partial_i\pi^i + u\pi_0\Big)\,. \end{aligned} \tag{2.130}$$

where $u = u(x)$ is arbitrary. The consistency condition for the primary constraint (2.128) generates the secondary constraint

$$\{\pi^0(x), H_T\} = -\int_{\Sigma_t} d^3y\ \delta^{(3)}(\vec{x} - \vec{z})\frac{\delta}{\delta A_0(z)}\int_{\Sigma_t} d^3y(-A_0\partial_i\pi^i) = \partial_i\pi^i\,. \tag{2.131}$$

Additionally, there are not further secondary constraints since

$$\{\partial_i\pi^i, H_T\} = \partial_i\partial_j F^{ij} = 0 \tag{2.132}$$

and both constraints (2.128), (2.131) are 1st class because

$$\{\pi^0, \partial_i A^i\} = 0\,. \tag{2.133}$$

The physical degrees of freedom are $\frac{1}{2}(2\times 4 - 2\times 2) = 2$. The equations of motion are

$$\dot{A}_0 = \{A_0, H_T\} = u \tag{2.134}$$

$$\dot{A}_i = \{A_i, H_T\} = \pi_i + \partial_i A^0 \tag{2.135}$$

$$\dot{\pi}^0 = \{\pi^0, H_T\} = \partial_i\pi^i \approx 0 \tag{2.136}$$

$$\dot{\pi}^i = \{\pi^i, H_T\} = \partial_j F^j{}_i \tag{2.137}$$

One notices the characteristic feature of gauge theories, namely the presence of arbitrary functions of time in the equations of motion. Indeed, (2.134) contains the arbitrary function $u = u(x)$ and thus, the time evolution of A_0 is not determined. The 0-th component of the vector potential A_μ is left arbitrary.

The 1st class constraints (2.128), (2.131) generate the gauge transformations $\delta A_\mu = \epsilon_a\{A_\mu, \phi_a\}$ and $\delta\pi^\mu = \epsilon_a\{\pi^\mu, \phi_a\}$ with $\phi_1 \equiv \pi^0, \phi_2 \equiv \partial_i\pi^i$, $a = 1,2$ and $\epsilon_a = \epsilon_a(x)$ being arbitrary. These transformations explicitly read

$$\delta A_0 = \epsilon_1 \tag{2.138}$$

$$\delta A_i = -\partial_i\epsilon_2 \tag{2.139}$$

$$\delta\pi^\mu = 0\,. \tag{2.140}$$

with $\epsilon_1 = \epsilon_1(x)$ and $\epsilon_2 = \epsilon_2(x)$ being arbitrary. Finally, the gauge generator of (2.138), (2.139), (2.140) is

$$G = \int_{\Sigma_t} d^3x\Big(\epsilon_1\,\pi^0 + \epsilon_2\,\partial_i\pi^i\Big)\,. \tag{2.141}$$

Now it is interesting to examine the symmetries of the extended Hamiltonian. Since, the 1st class secondary constraint (2.131) is already present in the total Hamiltonian (2.130), it is sufficient to view A_0 as a Lagrange multiplier and not as a canonical variable. Notice that is also implied by the first equation of motion (2.134). The extended Hamiltonian is simply

$$H_E = \int_{\Sigma_t} d^3x\Big(\frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}\pi^i\pi_i + \lambda\partial_i\pi^i\Big) \tag{2.142}$$

where $\lambda = \lambda(x)$ is an arbitrary function. The equations of motions with respect to the extended Hamiltonian are

$$\dot{A}_i = \{A_i, H_E\} = \pi_i - \partial_i\lambda \tag{2.143}$$

$$\dot{\pi}^i = \{\pi^i, H_E\} = \partial_j F^{ji}\,. \tag{2.144}$$

One notices again the presence of the arbitrary function $\lambda = \lambda(x)$ in equation of motion (2.143).

Now, the 1st class secondary constraint (2.131) generates the gauge transformations $\delta A_i = \epsilon\{A_i, \phi\}$ and $\delta\pi^i = \epsilon\{\pi^i, \phi\}$ with $\phi \equiv \partial_i\pi^i$ and $\epsilon = \epsilon(x)$ being arbitrary. These transformations explicitly read

$$\delta A_i = -\partial_i\epsilon \tag{2.145}$$

$$\delta\pi^i = 0\,. \tag{2.146}$$

The generator of the above gauge symmetries is

$$G = \int_{\Sigma_t} d^3x\;\epsilon\,\partial_i\pi^i\,. \tag{2.147}$$

It is now investigated whether the above gauge generator has well-defined functional derivatives of the form (2.88). Its variation takes the form

$$\delta G = \int_{\Sigma_t} d^3x\;\Big[\partial_i\pi^i\delta\epsilon - \partial_i\epsilon\delta\pi^i\Big] + \int_{\partial\Sigma_t} d^2x_i\;\epsilon\,\delta\pi^i \tag{2.148}$$

where an integration by parts has been performed. It is obvious that, since the variation consists of surface integrals on $\partial\Sigma_t$, the gauge generator (2.147) has not well-defined functional derivatives. Thus, it needs to be improved. This is done by adding to (2.148) the surface integral

$$\delta Q = -\int_{\partial\Sigma_t} d^2x_i\;\epsilon\,\delta\pi^i\,. \tag{2.149}$$

Then, the variation (2.93) of the improved generator Γ becomes

$$\begin{aligned} \delta\Gamma|_{\text{on-shell}} &= \delta G + \delta Q \\ &= \int_{\Sigma_t} d^3x \left[\partial_i \pi^i \delta\epsilon - \partial_i \epsilon \delta\pi^i\right] \\ &\approx 0 \end{aligned} \tag{2.150}$$

after use of the constraint (2.131), subject to the boundary condition

$$\delta\pi^i|_{\Sigma_t} = 0\,. \tag{2.151}$$

The charges are evaluated by integrating (2.149) in phase space. The result is

$$Q = -\int_{\partial\Sigma_t} d^2x_i\ \epsilon\,\pi^i\,. \tag{2.152}$$

Identifying the canonical momenta π^i as the electric field E^i, the 0-th component of the above charge takes the form

$$Q = -\int_{\partial\Sigma_t} d^2x_i\ E^i \tag{2.153}$$

which is simply the electric charge, since the above expression is the integrated form of Gauss' law.

2.4 THE CASE OF GENERAL RELATIVITY

2.4.1 *General setup*

The Einstein-Hilbert action is

$$S = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}R\,. \tag{2.154}$$

The Gibbons-Hawking-York boundary term [48], [49] that is required for a well-defined variational principle does not contribute to the dynamical evolution of the system. Therefore, from now on, this and other boundary terms will be neglected, being dynamically irrelevant. Using the ADM decomposition [50], which is presented in Appendix A.3, the Lagrangian of (2.154) can be rewritten as

$$L = \int_{\Sigma_t} d^3x\sqrt{h}N\Big(\mathcal{R} + K_{ab}K^{ab} - K^2 + 2\nabla_a(n^a\nabla_c n^c) - 2\nabla_a(n^c\nabla_c n^a)\Big)\,. \tag{2.155}$$

The derivation of the above is given in Appendix A.2. The last two terms in the parentheses yield boundary terms in (2.154) and are dropped. The components of the canonical momenta $\pi^{ab} = \frac{\delta L}{\delta \dot{g}_{ab}}$ are

$$\pi^{ab} = \frac{\delta L}{\delta \dot{h}_{ab}} = \sqrt{h}(K^{ab} - Kh^{ab}) \tag{2.156}$$

$$\pi = \frac{\delta L}{\delta \dot{N}} = 0 \tag{2.157}$$

$$\pi_a = \frac{\delta L}{\delta \dot{N}^a} = 0\,. \tag{2.158}$$

The last two expressions (2.157), (2.158) are primary constraints. The Lagrangian (2.155) in terms of canonical variables takes the form

$$L = \int_{\Sigma_t} d^3x\sqrt{h}N\Big(\mathcal{R} + \frac{1}{h}\pi^{ab}\pi_{ab} - \frac{1}{2h}\pi_a^{a2}\Big) \tag{2.159}$$

and therefore the canonical Hamiltonian is

$$\begin{aligned} H_c &= \int_{\Sigma_t} d^3x\Big[\pi^{ab}\dot{h}_{ab} + \pi^a\dot{N}_a + \pi\dot{N}\Big] - L \\ &= \int_{\Sigma_t} d^3x\sqrt{h}N\Big(-\mathcal{R} + \frac{1}{h}\pi^{ab}\pi_{ab} - \frac{1}{2h}\pi_a^{a2}\Big) - \int_{\Sigma_t} d^3x\,2\,N_b\,D_a\pi^{ab} + \int_{\Sigma_t} d^3x\,2\,D_a\Big(N_b\pi^{ab}\Big) \end{aligned} \tag{2.160}$$

using (2.156), (2.157) and (2.158). The last expression is a boundary term and is dropped. The total Hamiltonian is

$$\begin{aligned} H_T &= H_c + \int_{\Sigma_t} d^3x\Big(u\pi + u^a\pi_a\Big) \\ &= \int_{\Sigma_t} d^3x\sqrt{h}N\Big(-\mathcal{R} + \frac{1}{h}\pi^{ab}\pi_{ab} - \frac{1}{2h}\pi_a^{a2}\Big) + \int_{\Sigma_t} d^3x\Big(-2N_b\,D_a\pi^{ab} + u\pi + u^a\pi_a\Big) \end{aligned} \tag{2.161}$$

with $u = u(x)$, $u^a = u^a(x)$ being arbitrary. The consistency conditions for the primary constraints (2.157) and (2.158) reveal the secondary constraints

$$\{\pi, H_T\} = \sqrt{h}\Big(\mathcal{R} - \frac{1}{h}\pi^{ab}\pi_{ab} + \frac{1}{2h}\pi_a^{a2}\Big) \equiv -\mathcal{H}_\perp \tag{2.162}$$

$$\{\pi_a, H_T\} = 2D_b\pi_a^b \equiv -\mathcal{H}_a\,. \tag{2.163}$$

At this stage, it is convenient to calculate their Poisson brackets. They are

$$\{\mathcal{H}_\perp(x), \mathcal{H}_\perp(y)\} = h^{ab}\Big(\mathcal{H}_b(x) + \mathcal{H}_b(y)\Big)\,\partial_a\delta^{(3)}(\vec{x}-\vec{y}) \tag{2.164}$$

$$\{\mathcal{H}_a(x), \mathcal{H}_\perp(y)\} = \mathcal{H}_\perp(x)\,\partial_a\delta^{(3)}(\vec{x}-\vec{y}) \tag{2.165}$$

$$\{\mathcal{H}_a(x), \mathcal{H}_b(y)\} = \mathcal{H}_a(y)\,\partial_b\delta^{(3)}(\vec{x}-\vec{y}) + \mathcal{H}_b(x)\,\partial_a\delta^{(3)}(\vec{x}-\vec{y}) \tag{2.166}$$

with the partial derivative always acting on the x argument. It is easily recognized that the above relations form an algebra. The discussion and analysis of this Poisson bracket algebra is postponed until subsection 2.4.2 that follows.

The total Hamiltonian (2.161) can now compactly written as

$$H_T = \int_{\Sigma_t} d^3x \Big(N\mathcal{H}_\perp + N^a \mathcal{H}_a + u\pi + u^a \pi_a \Big) \,. \tag{2.167}$$

Using the algebra (2.164), (2.165), (2.166), the time evolution of the secondary constraints $\mathcal{H}_\perp$, $\mathcal{H}_a$ terminates since

$$\{\mathcal{H}_a(x), H_T\} = \int_{\Sigma_t} d^3y \{\mathcal{H}_a(x), N\mathcal{H}_\perp + N^c \mathcal{H}_c\,(y)\} = \mathcal{H}_\perp \partial_a N + \partial_c(N^c \mathcal{H}_a) + \mathcal{H}_c \partial_a N^c \approx 0 \tag{2.168}$$

$$\begin{aligned}\{\mathcal{H}_\perp(x), H_T\} = \int_{\Sigma_t} d^3y \{\mathcal{H}_\perp(x), N\mathcal{H}_\perp + N^a \mathcal{H}_a\,(y)\} &= \mathcal{H}^a \partial_a N + \partial_a(N\mathcal{H}^a) \\ &+ \partial_a(\mathcal{H}_\perp N^a) \approx 0 \,.\end{aligned} \tag{2.169}$$

Thus, there exist no further secondary constraints.

The next step is to classify all constraints into 1st and 2nd class. One finds

$$\begin{aligned}\{\pi, \pi_a\} = \{\pi, \mathcal{H}_\perp\} = \{\pi, \mathcal{H}_a\} &= 0 \\ \{\pi_a, \mathcal{H}_\perp\} = \{\pi_a, \mathcal{H}_b\} &= 0 \\ \{\mathcal{H}_\perp(x), \mathcal{H}_a(y)\} = \mathcal{H}_\perp(x)\ \partial_a \delta^{(3)}(\vec{x} - \vec{y}) &\approx 0\end{aligned} \tag{2.170}$$

and therefore all constraints are 1st class. The physical degrees of freedom are $\frac{1}{2}(2\times 10 - 0 - 2\times 8) = 2$, corresponding to the two different polarizations of the graviton.

Using the total Hamiltonian (2.167), the Hamilton's equations of motion are

$$\dot{h}_{ab} = \{h_{ab}, H_T\} = \frac{2N}{\sqrt{h}}(\pi_{ab} - \frac{1}{2} h_{ab} \pi^c_c) + 2D_{(a}N_{b)} \tag{2.171}$$

$$\dot{N} = \{N, H_T\} = u \tag{2.172}$$

$$\dot{N}^a = \{N^a, H_T\} = u^a \tag{2.173}$$

$$\begin{aligned}\dot{\pi}^{ab} = \{\pi^{ab}, H_T\} &= -N\sqrt{h}(\mathcal{R}^{ab} - \frac{1}{2}h^{ab}\mathcal{R}) + \frac{N}{2\sqrt{h}} h^{ab}(\pi_{cd}\pi^{cd} - \frac{1}{2}\pi^{c2}_c) \\ &- \frac{2N}{\sqrt{h}}(\pi^a_c \pi^{bc} - \frac{1}{2}\pi^{ab}\pi^c_c) + \sqrt{h}(D^a D^b N - h^{ab} D^c D_c N) - 2\pi^{c(a} D_c N^{b)} + D_c(N^c \pi^{ab})\end{aligned} \tag{2.174}$$

$$\dot{\pi} = \{\pi, H_T\} = -\mathcal{H}_\perp \approx 0 \tag{2.175}$$

$$\dot{\pi}_a = \{\pi_a, H_T\} = -\mathcal{H}_a \approx 0 \,. \tag{2.176}$$

It is obvious from (2.172) and (2.173) that N, N^a remain arbitrary functions. Subsequently, equations of motion (2.171), (2.174) contain as well arbitrary functions, a characteristic feature of General Relativity as a gauge theory.

As discussed in 2.1.8, the extended Hamiltonian is obtained by adding all secondary 1st class constraints (2.162), (2.163) to the total Hamiltonian (2.167), i.e.

$$\begin{aligned}H_E &= H_T + \int_{\Sigma_t} d^3x \Big(\alpha\, \mathcal{H}_\perp + \alpha^a\, \mathcal{H}_a \Big) \\ &= \int_{\Sigma_t} d^3x \Big((\alpha + N)\, \mathcal{H}_\perp + (\alpha^a + N^a)\, \mathcal{H}_a + u\pi + u^a \pi_a \Big)\end{aligned} \tag{2.177}$$

where $\alpha = \alpha(x), \alpha^a = \alpha^a(x)$ are arbitrary functions. Since the equations of motion of N, N^a (2.172), (2.173) state that N, N^a are, as well, arbitrary, the terms in the above parentheses can be redefined as

$$H_E = \int_{\Sigma_t} d^3x \Big(\epsilon^\perp \mathcal{H}_\perp + \varepsilon^a \mathcal{H}_a + u\pi + u^a \pi_a \Big) \tag{2.178}$$

with $\epsilon^\perp = \epsilon^\perp(x), \varepsilon^a = \varepsilon^a(x)$ being arbitrary functions. As it was argued in 2.1.8, one is generally interested in the extended Hamiltonian when quantizing the theory, where all gauge freedom becomes manifest. That is because secondary 1st are added in the total Hamiltonian, which were not present before. But in the present case, it is obvious that the extended Hamiltonian (2.179) is the same with the total one (2.167) and generates the gauge symmetries of the Lagrangian (2.155), as the total Hamiltonian does. Nevertheless, for distinguishing the true dynamics of the theory from parts characterizing merely how the coordinate system evolves in time (sections 3.3 and 3.4 of [50]), it is customary to reduce the phase space as follows: from the analytic expressions of the secondary constraints $\mathcal{H}_\perp$ (2.162) and $\mathcal{H}_a$ (2.163), one observes that they do not depend neither on N, N^a nor their conjugate momenta π, π_a and thus the extended Hamiltonian (2.178) can be viewed as describing two distinct systems: the first system, consisting of $\int_{\Sigma_t} d^3x \left(\epsilon^\perp \mathcal{H}_\perp + \varepsilon^a \mathcal{H}_a\right)$ in which N, N^a are not dynamical variables anymore, reveals the true dynamics. The second one, consisting of $\int_{\Sigma_t} d^3x \left(u\pi + u^a \pi_a\right)$ characterizes the evolution of the coordinate system in time and is discarded. The rest of the analysis continues with the extended Hamiltonian to be

$$H_E = \int_{\Sigma_t} d^3x \left(\epsilon^\perp \mathcal{H}_\perp + \varepsilon^a \mathcal{H}_a\right). \tag{2.179}$$

Now the phase space has been reduced since N, N^a and their conjugate momenta are not canonical variables anymore. The remaining dynamical variables are h_{ab} and π^{ab}. The corresponding equations of motion are

$$\dot{h}_{ab} = \{h_{ab}, H_E\} = \frac{2\epsilon^\perp}{\sqrt{h}}(\pi_{ab} - \frac{1}{2}h_{ab}\pi^c_c) + 2D_{(a}\varepsilon_{b)} \tag{2.180}$$

$$\begin{aligned}\dot{\pi}^{ab} = \{\pi^{ab}, H_E\} &= -\epsilon^\perp\sqrt{h}(\mathcal{R}^{ab} - \frac{1}{2}h^{ab}\mathcal{R}) + \frac{\epsilon^\perp}{2\sqrt{h}}h^{ab}(\pi_{cd}\pi^{cd} - \frac{1}{2}\pi^{c2}_c) \\ &- \frac{2\epsilon^\perp}{\sqrt{h}}(\pi^a_c\pi^{bc} - \frac{1}{2}\pi^{ab}\pi) + \sqrt{h}(D^aD^b\epsilon^\perp - h^{ab}D^cD_c\epsilon^\perp) - 2\pi^{c(a}D_c\varepsilon^{b)} + D_c(\varepsilon^c\pi^{ab}).\end{aligned} \tag{2.181}$$

One obtains, once again, the characteristic form of equations of motion of a gauge theory, namely the fact that they contain arbitrary functions, in this case these functions being $\epsilon^\perp$ and ε^a. Additionally, varying the extended Hamiltonian (2.179) with respect to $\epsilon^\perp$ and ε^a one gets the constraint equations $\mathcal{H}_\perp = 0$, $\mathcal{H}_a = 0$.

2.4.2 *Poisson bracket algebra of the constraints*

It was mentioned before that the algebra of the constraints is

$$\{\mathcal{H}_\perp(x), \mathcal{H}_\perp(y)\} = h^{ab}\left(\mathcal{H}_b(x) + \mathcal{H}_b(y)\right)\partial_a\delta^{(3)}(\vec{x} - \vec{y}) \tag{2.182}$$

$$\{\mathcal{H}_a(x), \mathcal{H}_\perp(y)\} = \mathcal{H}_\perp(x)\,\partial_a\delta^{(3)}(\vec{x} - \vec{y}) \tag{2.183}$$

$$\{\mathcal{H}_a(x), \mathcal{H}_b(y)\} = \mathcal{H}_a(y)\,\partial_b\delta^{(3)}(\vec{x} - \vec{y}) + \mathcal{H}_b(x)\,\partial_a\delta^{(3)}(\vec{x} - \vec{y}). \tag{2.184}$$

Their geometrical interpretation was particularly important for understanding the dynamics of General Relativity: the third relation (2.184) states that $\mathcal{H}_a$ are generators of spatial diffeomorphisms on the surface Σ_t. The second relation (2.183) states that $\mathcal{H}_\perp$ is a scalar density. This does not at all contain any new information since it was already defined as such in (2.162). Finally, the first relation (2.182) states that $\mathcal{H}_\perp$ is generator of deformations of the surface Σ_t normal to itself, as it is embedded in $\Sigma_t \times \mathbf{R} \simeq M$. This means that the evolution of the dynamical variables of the theory can be represented as motion of the 3-dimensional surface Σ_t in the 4-dimensional manifold of hyperbolic signature. A very interesting discussion concerning this geometric interpretation of the above Poisson bracket algebra can be found in [51]. A last observation is that the presence of h^{ab}

in the r.h.s. of (2.182) shows that the Poisson bracket algebra of constraints is not a true Lie algebra at all. Interestingly, as it will be explained later, it becomes a Lie algebra asymptotically.

Another useful version of this Poisson bracket algebra can be derived using smeared variables as

$$H_\perp[\eta] \equiv \int_{\Sigma_t} d^3x \; \mathcal{H}_\perp \eta \tag{2.185}$$

$$H[\xi^a] \equiv \int_{\Sigma_t} d^3x \; \mathcal{H}_a \xi^a \tag{2.186}$$

where $\eta = \eta(x)$ and $\xi^a = \xi^a(x) = h^a_b \xi^b$ are a scalar and a tangent vector field on Σ_t respectively. Then, the algebra (2.182), (2.183), (2.184) becomes

$$\{H_\perp[\eta_1], H_\perp[\eta_2]\} = H[h^{ab}(\eta_1 D_b \eta_2 - \eta_2 D_b \eta_1)] \tag{2.187}$$

$$\{H[\xi^a], H_\perp[\eta]\} = H_\perp[{}^3\pounds_{\xi^a}\eta] \tag{2.188}$$

$$\{H[\xi^a_1], H[\xi^b_2]\} = H\big[[\xi^a_1, \xi^b_2]^c\big] \tag{2.189}$$

where

$$[\xi^a_1, \xi^b_2]^c = \xi^a_1 \partial_a \xi^c_2 - \xi^a_2 \partial_a \xi^c_1 \tag{2.190}$$

is the usual expression for the Lie bracket of two vector fields.

Furthermore, the extended Hamiltonian (2.179) can be written in the smeared version as

$$H_E[\epsilon] = \int_{\Sigma_t} d^3x \; \epsilon^a \mathcal{H}_a \tag{2.191}$$

with $\epsilon^a \equiv \{\epsilon^\perp, \varepsilon^a\}$. Then, the algebra (2.187), (2.188), (2.189) can be compactly written as

$$\{H_E[\epsilon_1], H_E[\epsilon_2]\} = H_E\big[[\epsilon_1, \epsilon_2]\big] \tag{2.192}$$

with

$$[\epsilon_1, \epsilon_2]^\perp = \varepsilon^a_1 \partial_a \epsilon^\perp_2 - \varepsilon^a_2 \partial_a \epsilon^\perp_1 \tag{2.193}$$

$$[\epsilon_1, \epsilon_2]^a = h^{ab}(\epsilon^\perp_1 \partial_b \epsilon^\perp_2 - \epsilon^\perp_2 \partial_b \epsilon^\perp_1) + \varepsilon^b_1 \partial_b \varepsilon^a_2 - \varepsilon^b_2 \partial_b \varepsilon^a_1 \tag{2.194}$$

also known as surface deformation algebra [51].

2.4.3 *Gauge generator*

Firstly, the case of the gauge symmetries of the Lagrangian (2.154) is explored. For this, one uses the Castellani algorithm [52]: starting with the primary first class constraint π, one has the following chain

$$\begin{aligned} G_1 &= \pi \\ G_0 + \{\pi, H_T\} &= PFC \\ \{G_0, H_T\} &= PFC \end{aligned} \tag{2.195}$$

with

$$PFC = \int_{\Sigma_t} d^3y \Big(\alpha(x,y)\pi(y) + \alpha^a(x,y)\pi_a(y)\Big)\,. \tag{2.196}$$

The coefficients $\alpha(x,y)$ and $\alpha^a(x,y)$ are determined via chain (2.195) to be

$$\begin{aligned} \alpha(x,y) &= N^a \partial_a(y) \delta^{(3)}(\vec{x} - \vec{y}) \\ \alpha^a(x,y) &= \partial^a N(y)\, \delta^{(3)}(\vec{x} - \vec{y}) + N(y) \partial^a \delta^{(3)}(\vec{x} - \vec{y})\,. \end{aligned} \tag{2.197}$$

The corresponding gauge generator is associated with diffeomorphisms orthogonal to the spatial hypersurface ∂M and it takes the form

$$G_\perp(\zeta^\perp, \dot\zeta^\perp) = \int_{\Sigma_t} d^3x \Big[\zeta^\perp\Big(\mathcal{H}_\perp + {}^3\pounds_{\vec N}\pi + \pi_a D^a N + D_a(\pi^a N)\Big) + \dot\zeta^\perp\pi\Big]\,. \tag{2.198}$$

Likewise, starting with the primary first class constraint π_a the algorithm gives the following chain

$$\begin{aligned} G_{1a} &= \pi_a \\ G_{0a} + \{\pi_a, H_T\} &= PFC_a \\ \{G_{0a}, H_T\} &= PFC_a \end{aligned} \tag{2.199}$$

with

$$PFC = \int d^3y\Big(\beta_a(x,y)\pi(y) + \beta_a^b(x,y)\pi_b(y)\Big)\,. \tag{2.200}$$

The coefficients $\beta_a(x,y)$ and $\beta_a^b(x,y)$ are found from (2.199) to be

$$\begin{aligned} \beta_a(x,y) &= \partial_a N(y)\delta^{(3)}(\vec x - \vec y) \\ \beta_a^b(x,y) &= \partial_a N^b \delta^{(3)}(\vec x - \vec y) + N^c \delta_a^b \partial_c \delta^{(3)}(\vec x - \vec y)\,. \end{aligned} \tag{2.201}$$

The corresponding gauge generator is associated with spatial diffeomorphisms on ∂M and it takes the form

$$G_D(\zeta^a, \dot\zeta^a) = \int_{\Sigma_t} d^3x\Big[\zeta^a\Big(\mathcal{H}_a + \pi D_a N + {}^3\pounds_{\vec N}\pi_a\Big) + \dot\zeta^a\pi_a\Big]\,. \tag{2.202}$$

Hence, the gauge generator of symmetries of the Lagrangian (2.154) is

$$\begin{aligned} G &= G_\perp + G_D \\ &= \int_{\Sigma_t} d^3x\Big[\zeta^\perp\Big(\mathcal{H}_\perp + {}^3\pounds_{\vec N}\pi + \pi_a D^a N + D_a(\pi^a N)\Big) + \dot\zeta^\perp\pi \\ &\quad + \zeta^a\Big(\mathcal{H}_a + \pi D_a N + {}^3\pounds_{\vec N}\pi_a\Big) + \dot\zeta^a\pi_a\Big]\,. \end{aligned} \tag{2.203}$$

Lastly, as it is shown in Appendix B.1.1, it can be straightforward verified that the Castellani generator (2.203) produces the correct gauge symmetries of the Lagrangian (2.155). That is, it generates the transformations

$$\delta g_{ab} = \pounds_{\zeta^c} g_{ab} \tag{2.204}$$

under diffeomorphisms of the coordinates, $x'^a = x^a + \zeta^a(x^b)$.

Now, the case of the gauge symmetries of the extended Hamiltonian in the form of (2.179) is analyzed. Since 1st class constraints generate gauge symmetries, the gauge generator is simply the extended Hamiltonian

$$H_E = \int_{\Sigma_t} d^3x\Big(\epsilon^\perp\mathcal{H}_\perp + \varepsilon^a\mathcal{H}_a\Big)\,. \tag{2.205}$$

At this point, it is reminded that the phase space has been reduced by discarding N, N^a and their canonical momenta. Thus, the gauge symmetries in this case are the transformations of the spatial metric

$$\delta h_{ab} = \pounds_{\epsilon^c} h_{ab} \tag{2.206}$$

under diffeomorphisms of the coordinates x^a on M of the form $x'^a = x^a + \epsilon^a(x^b)$. It is demonstrated that the extended Hamiltonian (2.205) indeed generates the correct gauge transformations (2.206). The task is to find δh_{ab} under the action of the gauge generator (2.205), i.e. to find $\delta_{H_E} h_{ab}$, and then verify that it generates the r.h.s. of (2.206). Indeed, $\delta_{H_E} h_{ab}$ is the Hamilton's equation of motion (2.180), which is rewritten as

$$\begin{aligned} \delta_{H_E} h_{ab} &= \{h_{ab}, H_E\} \\ &= \frac{2\epsilon^\perp}{\sqrt h}(\pi_{ab} - \frac{1}{2}h_{ab}\pi_c^c) + 2D_{(a}\varepsilon_{b)} \\ &= 2\epsilon^\perp K_{ab} + 2D_{(a}\varepsilon_{b)} \end{aligned} \tag{2.207}$$

after use of (2.156) and its trace. Now, focusing on the r.h.s. of (2.206), the vector field ϵ^a on M can be decomposed in the ADM basis as

$$\epsilon^a = \epsilon^\perp n^a + \varepsilon^a \tag{2.208}$$

with

$$\epsilon^\perp = -n_a \epsilon^a \quad \text{and} \quad \varepsilon^a = h^a_b \varepsilon^b \tag{2.209}$$

Then, using (2.208), the r.h.s. of (2.206) is found to be

$$\begin{aligned} \pounds_{\epsilon^c} h_{ab} &= \pounds_{\epsilon^\perp n^c + \varepsilon^c} h_{ab} = \pounds_{\epsilon^\perp n^c} h_{ab} + {}^3\pounds_{\varepsilon^c} h_{ab} \\ &= \epsilon^\perp n^c \nabla_c h_{ab} + 2h_{c(a}\nabla_{b)}(\epsilon^\perp n^c) + 2D_{(a}\varepsilon_{b)} \\ &= \epsilon^\perp \left[n^c \nabla_c h_{ab} + 2h_{c(a}\nabla_{b)} n^c \right] + 2D_{(a}\varepsilon_{b)} \\ &= \epsilon^\perp \pounds_{n^c} h_{ab} + 2D_{(a}\varepsilon_{b)} \\ &= 2\epsilon^\perp K_{ab} + 2D_{(a}\varepsilon_{b)} \end{aligned} \tag{2.210}$$

after using $n^a h_{ab} = 0$. This is exactly the r.h.s of (2.207) and thus $\delta_{H_E} = \pounds_{\epsilon^c} h_{ab}$. Therefore, it is concluded that the extended Hamiltonian (2.205) indeed generates the gauge transformations (2.206) of the spatial metric.

2.4.4 *Improved Hamiltonian and generator*

It was already discussed in 2.2.1 that, in presence of boundaries, the Hamiltonian functions and the gauge generator have not, in principle, well-defined functional derivatives of the form (2.88). Indeed, this holds true in the case of General Relativity as well. Namely, the extended Hamiltonian or gauge generator (2.205) has not well-defined functional derivatives. Thus, it has to be improved by adding appropriate surface integrals, such that they cancel those arising from its variation. This inevitably leads to imposition of asymptotic boundary conditions which, here, are chosen to be asymptotically AdS_4.

At first, it is demonstrated that the extended Hamiltonian or gauge generator (2.205) has not well-defined functional derivatives of the form (2.88). Writing Hamilton equations of motion of (2.205) compactly as

$$\dot{h}_{ab} = \{h_{ab}, H_E\} = \frac{\delta H_E}{\delta \pi^{ab}} \tag{2.211}$$

$$\dot{\pi}^{ab} = \{h_{ab}, H_E\} = -\frac{\delta H_E}{\delta h_{ab}} \tag{2.212}$$

one observes that the functional derivatives appearing in the r.h.s. are the coefficients of the variation of the extended Hamiltonian (2.205), i.e.

$$\delta H_E = \int_{\Sigma_t} d^3x \left[A^{ab} \delta h_{ab} + B_{ab} \delta \pi^{ab} \right] \tag{2.213a}$$

with

$$\frac{\delta H_E}{\delta h_{ab}} \equiv A^{ab} \tag{2.213b}$$

$$\frac{\delta H_E}{\delta \pi^{ab}} \equiv B_{ab}\,. \tag{2.213c}$$

Thus, in order for the Hamilton equations of motion (2.211), (2.212) to be defined at all it is compulsory that variation δH_E is of the form (2.213) for arbitrary changes of the phase space functions (h_{ab}, π^{ab}). But an explicit variation of (2.205) yields

$$\begin{aligned} \delta H_E = &\int_{\Sigma_t} d^3x \left[A^{ab} \delta h_{ab} + B_{ab} \delta \pi^{ab} \right] - \int_{\partial\Sigma_t} d^2 s_d \left[G^{abcd} (\epsilon^\perp D_c \delta h_{ab} - D_c \epsilon^\perp \delta h_{ab}) \right] \\ &- \int_{\partial\Sigma_t} d^2 s_d \left[2\varepsilon_c \delta \pi^{cd} + (2\varepsilon^c \pi^{bd} - \varepsilon^d \pi^{bc}) \delta h_{bc} \right] \end{aligned} \tag{2.214}$$

with $G^{abcd} = \frac{1}{2}\sqrt{h}(h^{ac}h^{bd} + h^{ad}h^{bc} - 2h^{ab}h^{cd})$ and the exact expressions of A^{ab}, B_{ab} are the r.h.s. of the Hamilton equations (2.180) and (2.181) respectively. Thus, the surface integral on $\partial\Sigma_t$ which appears in the above variation renders the extended Hamiltonian (2.205) not being of the form (2.88) and thus not-well defined. For the Hamilton equations to make sense, the surface integrals in the above variation must vanish. Therefore, the extended Hamiltonian (2.205) has to be improved. This is done by adding an appropriate term, the variation of which cancels these unwanted surface terms. The explicit form of this term depends on the asymptotic boundary conditions one imposes.

2.4.5 *Asymptotically AdS_4 boundary conditions and canonical charges*

The extended Hamiltonian (2.205) is improved to have well-defined functional derivatives, after imposing asymptotically AdS_4 boundary conditions. Then, the corresponding canonical, asymptotic charge is constructed. As it is shown, these charges describe asymptotic symmetries of the theory, namely the symmetries of $O(3,2)$ which is the isometry group of AdS_4.

Firstly, the AdS_4 spacetime is considered: the solution of Einstein's equations with a cosmological constant term Λ that possesses the maximum number of isometries is AdS_4 spacetime and can be written as

$$ds_0^2 = -(1+\rho^2)dt^2 + (1+\rho^2)^{-1}dr^2 + r^2 d\Omega_2^2 \tag{2.215}$$

where $\rho \equiv \frac{r}{\ell}$, $d\Omega_2^2 = d\theta^2 + \sin^2\theta\phi^2$ and $\Lambda = -\frac{3}{\ell^2}$, with ℓ being the radius of curvature of AdS_4. The group of isometries is $O(3,2)$. Its ten generators are the Killing vectors U_{AB} with $A, B = 1, \ldots, 5$ and they have the explicit form

$$\begin{aligned}
U_{51} &= \frac{\partial}{\partial\tau} \\
U_{21} &= -\rho\sin\tau\sin\theta\cos\phi(1+\rho^2)^{-1/2}\frac{\partial}{\partial\tau} + (1+\rho^2)^{1/2}\cos\tau\sin\theta\cos\phi\frac{\partial}{\partial\rho} \\
&\quad + \rho^{-1}(1+\rho^2)^{1/2}\cos\tau\Big(\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\Big) \\
U_{31} &= -\rho\sin\tau\sin\theta\sin\phi(1+\rho^2)^{-1/2}\frac{\partial}{\partial\tau} + (1+\rho^2)^{1/2}\cos\tau\sin\theta\sin\phi\frac{\partial}{\partial\rho} \\
&\quad + \rho^{-1}(1+\rho^2)^{1/2}\cos\tau\Big(\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\Big) \\
U_{41} &= -\rho\sin\tau\cos\theta(1+\rho^2)^{-1/2}\frac{\partial}{\partial\tau} + (1+\rho^2)^{1/2}\cos\tau\cos\theta\frac{\partial}{\partial\rho} - \rho^{-1}(1+\rho^2)^{1/2}\cos\tau\sin\theta\frac{\partial}{\partial\theta} \\
U_{25} &= \rho\cos\tau\sin\theta\cos\phi(1+\rho^2)^{-1/2}\frac{\partial}{\partial\tau} + (1+\rho^2)^{1/2}\sin\tau\sin\theta\cos\phi\frac{\partial}{\partial\rho} \\
&\quad + \rho^{-1}(1+\rho^2)^{1/2}\sin\tau\Big(\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\Big) \\
U_{35} &= \rho\cos\tau\sin\theta\sin\phi(1+\rho^2)^{-1/2}\frac{\partial}{\partial\tau} + (1+\rho^2)^{1/2}\sin\tau\sin\theta\sin\phi\frac{\partial}{\partial\rho} \\
&\quad + \rho^{-1}(1+\rho^2)^{1/2}\sin\tau\Big(\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\Big) \\
U_{45} &= \rho\cos\tau\cos\theta(1+\rho^2)^{-1/2}\frac{\partial}{\partial\tau} + (1+\rho^2)^{1/2}\sin\tau\cos\theta\frac{\partial}{\partial\rho} - \rho^{-1}(1+\rho^2)^{1/2}\sin\tau\sin\theta\frac{\partial}{\partial\theta} \\
U_{23} &= \frac{\partial}{\partial\phi} \\
U_{34} &= -\sin\phi\frac{\partial}{\partial\theta} - \cot g\theta\cos\phi\frac{\partial}{\partial\phi} \\
U_{42} &= \cos\phi\frac{\partial}{\partial\theta} - \cot g\theta\sin\phi\frac{\partial}{\partial\phi}
\end{aligned} \tag{2.216}$$

where $\tau = t\ell^{-1}$. These Killing vectors obey the $o(3,2)$ Lie algebra

$$[U_{AB}, U_{CD}] = C^{EF}{}_{ABCD} U_{EF} \tag{2.217}$$

where

$$C^{EF}{}_{ABCD} = \frac{1}{2}\eta_{BC}(\delta^E_A\delta^F_D - \delta^F_A\delta^E_D) - \frac{1}{2}\eta_{BD}(\delta^E_A\delta^F_C - \delta^F_A\delta^E_C) - (A \leftrightarrow B) \tag{2.218}$$

and the five-dimensional metric η_{AB} has signature $(-,+,+,+,-)$.

Since the interest here is for an asymptotically AdS_4 spacetime, one can write such a line element as

$$ds^2 = ds_0^2 + h_{ab}dx^a dx^b\,. \tag{2.219}$$

For example, such a spacetime describes the $Kerr - AdS_4$ and the $Schwarzschild - AdS_4$ spacetimes (with appropriate h_{ab} in each case). Now, boundary conditions at infinity (h_{ab} as $r \to \infty$) are usually imposed by hand. However, they should be such that they meet some reasonable and desirable criteria. In this example, adopting the argument of [53], boundary conditions should be such that they fulfill the following requirements: 1) they should contain asymptotically AdS spacetimes of physical interest, for example the $Kerr - AdS_4$ solution, otherwise they would be too restrictive, 2) they should be invariant under the $O(3,2)$ group, otherwise an allowed configuration under a symmetry transformation would be mapped into an non-allowed one, and 3) they should make the surface integrals associated with generators of the $O(3,2)$ group finite asymptotically. To fulfill condition 1), one starts with the $Kerr - AdS_4$ metric in the form (2.219) and acts on it with the generators of $O(3,2)$ (2.216). The generated metric perturbations h_{ab} are

$$\begin{pmatrix} h_{tt} = r^{-1}f_{tt} + \mathcal{O}(r^{-2}) & h_{tr} = r^{-4}f_{tr} + \mathcal{O}(r^{-5}) & h_{t\theta} = r^{-1}f_{t\phi} + \mathcal{O}(r^{-2}) & h_{t\phi} = r^{-1}f_{t\phi} + \mathcal{O}(r^{-2}) \\ h_{rt} = h_{tr} & h_{rr} = r^{-5}f_{rr} + \mathcal{O}(r^{-6}) & h_{r\theta} = r^{-4}f_{r\theta} + \mathcal{O}(r^{-5}) & h_{r\phi} = r^{-4}f_{r\phi} + \mathcal{O}(r^{-5}) \\ h_{\theta r} = h_{t\theta} & h_{\theta r} = h_{r\theta} & h_{\theta\theta} = r^{-1}f_{\theta\theta} + \mathcal{O}(r^{-2}) & h_{\theta\phi} = r^{-1}f_{\phi\theta} + \mathcal{O}(r^{-2}) \\ h_{\phi t} = h_{t\phi} & h_{\phi r} = h_{r\phi} & h_{\phi\theta} = h_{\theta\phi} & h_{\phi\phi} = r^{-1}f_{\phi\phi} + \mathcal{O}(r^{-2}) \end{pmatrix} \tag{2.220}$$

where the functions $f_{tt}, f_{tr}, f_{rr}, f_{r\theta}, f_{r\phi}, f_{\theta\phi}, f_{\phi\phi}$ depend on r, θ, ϕ. It turns out that the above asymptotic boundary conditions are $O(3,2)$ invariant, i.e. also fulfill condition 2). Additionally, the above set of boundary conditions preserves asymptotic isometries of the metric h_{ab} generated by vector fields ϵ^c (${}^3\pounds_{\epsilon^c} h_{ab} = r^{-n} f_{ab}$, with $n = 1, 4, 5$ according to (2.220)) obeying

$$\lim_{r\to\infty}\left[\epsilon^c - \epsilon^{AB}U^c_{AB}\right] \longrightarrow 0 \tag{2.221}$$

where ϵ^{AB} are the constant components of ϵ^c along the Killing vector fields components U^c_{AB}. The precise fall off behavior of (2.221) cam be found in [53].

The next step is to find, under the asymptotic boundary conditions (2.220), (2.221), the appropriate surface term the variation of which cancels the unwanted surface terms in (2.214). Additionally, to meet condition 3), this surface term must be finite asymptotically. Using $K_{ab} = \frac{1}{2N}(-\dot{h}_{ab} + D_{(a}N_{b)})$, the asymptotic behavior of canonical momenta is found to be

$$\begin{pmatrix} \pi^{rr} = r^{-1}p^{rr} + \mathcal{O}(r^{-2}) & \pi^{r\theta} = r^{-2}p^{r\theta} + \mathcal{O}(r^{-3}) & \pi^{r\phi} = r^{-2}p^{r\phi} + \mathcal{O}(r^{-3}) \\ \pi^{\theta r} = \pi^{r\theta} & \pi^{\theta\theta}(x) = r^{-5}p^{\theta\theta} + \mathcal{O}(r^{-6}) & \pi^{\theta\phi} = r^{-5}p^{\theta\phi} + \mathcal{O}(r^{-6}) \\ \pi^{\phi r} = \pi^{r\phi} & \pi^{\phi\theta} = \pi^{\theta\phi} & \pi^{\phi\phi} = r^{-5}p^{\phi\phi} + \mathcal{O}(r^{-6}) \end{pmatrix} \tag{2.222}$$

where the functions $p^{rr}, p^{r\theta}, p^{r\phi}, p^{\theta\theta}, p^{\theta\phi}, p^{\phi\phi}$ depend on θ, ϕ and all $\pi^{ta} = 0\ \forall a$. Using the asymptotic boundary conditions (2.220), (2.221), (2.222), the appropriate term to cancel the unwanted surface integrals in (2.214) is found, up to a constant, to be

$$\frac{1}{2}\lim_{r\to\infty}\epsilon^{AB}Q_{AB} \tag{2.223}$$

where

$$Q_{AB} = \int_{\partial\Sigma_t} d^2 S_d \Big(\tilde{W}^{abcd} (U^{\perp}_{AB} \tilde{D}_b h_{ac} - \tilde{D}_b U^{\perp}_{AB} h_{ac}) + 2 U^c_{AB} \pi^d_c \Big) \tag{2.224}$$

and the quantities with $\tilde{}$ refer to the spatial AdS metric (2.215) and $\tilde{W}^{abcd} = \frac{1}{2}\sqrt{\tilde{h}}(\tilde{h}^{ac}\tilde{h}^{bd} + \tilde{h}^{ad}\tilde{h}^{bc} - 2\tilde{h}^{ab}\tilde{h}^{cd})$. Moreover, it turns out that this term is finite asymptotically, therefore condition 3) is satisfied. Thus, one has now achieved the initial aim: the improved Hamiltonian or equivalently the improved gauge generator can be written as

$$\begin{aligned} H' &\equiv H_E + \frac{1}{2} \lim_{r\to\infty} \epsilon^{AB} Q_{AB} \\ &= \int_{\Sigma_t} d^3x \Big(\epsilon^{\perp}\mathcal{H} + \varepsilon^a \mathcal{H}_a \Big) + \frac{1}{2} \lim_{r\to\infty} \int_{\partial\Sigma_t} d^2 S_d \Big(\tilde{W}^{abcd} (U^{\perp}_{AB} \tilde{D}_b h_{ac} - \tilde{D}_b U^{\perp}_{AB} h_{ac}) + 2 U^c_{AB} \pi^d_c \Big) . \end{aligned} \tag{2.225}$$

One notices immediately that $H' \approx Q_{AB}$. Therefore, the improved Hamiltonian (2.225) can be interpreted now as the (correct) energy of the system, being charge (2.224).

Lastly, it is interesting to calculate charge (2.224) of a particular solution e.g. for the $Schwarzschild - AdS_4$ spacetime

$$ds^2 = -\Big(1 + \rho^2 - \frac{2M}{r}\Big) dt^2 + \Big(1 + \rho^2 - \frac{2M}{r}\Big)^{-1} dr^2 + r^2 d\Omega^2_2 \tag{2.226}$$

with $\rho \equiv \frac{r}{\ell}$. In this case, the asymptotic boundary conditions (2.220), (2.222) are

$$h_{rr} = \frac{2M\ell^4}{r^5} + \mathcal{O}(r^{-8}) \, , \text{ other } h_{ab} = 0 \tag{2.227}$$

$$\pi^{ab} = 0 \tag{2.228}$$

and charge (2.224) associated with the Killing vector $U_{51} = \partial\tau$ reads

$$Q_{51} = \int_{\partial\Sigma_t} d^2 s_a \, \tilde{W}^{abcd} U^{\perp}_{51} \tilde{D}_b h_{cd} = 8\pi M\ell \int_0^{\pi} d\theta \sin\theta = 16\pi M\ell \tag{2.229}$$

and all other $Q_{AB} = 0$. This is indeed the mass (energy) of the system at spatial infinity.

2.4.6 *Asymptotic symmetry algebra*

Now it is demonstrated that charge (2.224) generates asymptotic symmetries of the theory, namely that is isomorphic to $o(3,2)$ algebra (2.217): since the extended Hamiltonian has been improved to $H' = H_E + \frac{1}{2}\lim_{r\to\infty} \epsilon^{AB} Q_{AB}$, the smeared version of the Poisson bracket algebra of the constraints (2.192) takes the form

$$\{H'[\epsilon_1], H'[\epsilon_2]\} = H'\big[[\epsilon_1, \epsilon_2]\big] \, . \tag{2.230}$$

Since $H'[\epsilon]$ now has well-defined functional derivatives, so has $\{H'[\epsilon_1], H'[\epsilon_2]\}$, according to [54]: the Poisson bracket of two well-defined differentiable generators is also a well-defined differentiable generator. The algebra (2.230) on the constraint surface becomes

$$\{Q[\epsilon_1], Q[\epsilon_2]\} \approx Q\big[[\epsilon_1, \epsilon_2]\big] \, . \tag{2.231}$$

The asymptotic part of the Lie bracket $[\epsilon_1, \epsilon_2]$ is expressed in terms of the asymptotic parts of ϵ_1, ϵ_2 according to the surface deformation algebra (2.193), (2.194). Then, using the asymptotic boundary conditions (2.220), (2.221), one finds that the Lie bracket $[\epsilon_1, \epsilon_2]$ takes also the form (2.221) asymptotically, i.e.

$$\lim_{r\to\infty} \Big[[\epsilon_1, \epsilon_2]^c - [\epsilon_1, \epsilon_2]^{AB} U^c_{AB} \Big] \longrightarrow 0 \tag{2.232}$$

and additionally that

$$\lim_{r\to\infty} [\epsilon_1, \epsilon_2]^{AB} = C^{AB}{}_{CDEF} \, \epsilon_1^{CD} \, \epsilon_2^{EF} \tag{2.233}$$

with $C^{AB}{}_{CDEF}$ being the structure constants of $o(3,2)$ defined in (2.218). Thus, substituting (2.232) and (2.233) into the r.h.s. of (2.231), one finds that

$$\{Q[\epsilon_1], Q[\epsilon_2]\} \approx C^{AB}{}_{CDEF}\, \epsilon_1^{CD}\, \epsilon_2^{EF}\, Q[U_{AB}] \tag{2.234}$$

i.e. the algebra of the charges is isomorphic to the Lie algebra $o(3,2)$ (2.217).

3

PATH INTEGRAL APPROACH TO GAUGE THEORIES AND 1-LOOP CORRECTIONS

3.1 GENERAL SETUP

It was already discussed in chapter 2 that the Hamiltonian formulation is the starting point of the canonical quantization of a classical (gauge or not) theory. One major advantage of this approach is the simple analogue of the whole Hamiltonian setup to the quantum theory. Nevertheless, the choice of a particular time slice, mandatory in the Hamiltonian setup, makes the canonical quantization approach not manifestly relativistic.

There are alternative approaches to quantization. One of them is the path integral quantization which is formally provided by a Hamiltonian formulation of the theory. This still maintains the non-relativistic feature. However, it can equivalently be provided by a Lagrangian formulation. This is desired because it has the obvious advantage that the Lagrangian can be expressed manifestly relativistically, on account on the action being relativistically invariant. The Lagrangian approach to quantization is equivalent with the Hamiltonian approach for the cases where the Hamiltonian function is schematically of the form $H = \frac{p^2}{2m} + V(q)$, where p are the conjugate momenta and $V(q)$ a potential term. As it is obvious, this form of Hamiltonian already covers a wide variety of realistic systems. But there are cases, such as non-Abelian gauge theories, in which the Hamiltonian function is not in the above mentioned form. Thus, the Lagrangian path integral approach is not subsequently obtained from the initial Hamiltonian formalism. Nevertheless, due to the relativistic feature the Lagrangian possesses, it is customary to ignore the presence or not of an equivalent Hamiltonian path integral approach. Instead, the starting point of path integral quantization is considered to be provided by the Lagrangian.

Throughout this chapter, the path integral approach to field theories which possess gauge symmetries starts in 3.1, by analyzing initially a non-relativistic quantum mechanical system. The generalization to field theories is then performed gradually and the linearization and 1-loop corrections are presented. Finally, in 3.2, the whole setup is applied in the case of General Relativity.

3.1.1 *Quantum probability amplitude and classical Lagrangian*

All equalities between quantum amplitudes and the path integral are to be understood up to (infinite) overall factors. This point is emphasized when it appears for the first time.

Non-relativistic quantum mechanics

Throughout this subsection, $\hbar$ is restored, for clarifying the classical limit of the path integral.

The original idea of connecting the quantum mechanical amplitude between two states with the classical Lagrangian was first stated by Dirac [55]. This connection was inspired by the following fundamental concepts: classical canonical transformations and their validity in the quantum level as well, the Hamilton-Jacobi equation and the classical limit of wave mechanics. The main statement, or postulate, was that the probability amplitude of an initial state $|q_1, t_1\rangle$ to be found in a final state $|q_2, t_2 >$ must satisfy the following relation:

$$\langle q_2, t_2 | q_1, t_1 \rangle \;\; \textit{corresponds to} \;\; e^{i\int_{t_1}^{t_2} dtL/\hbar} \tag{3.1}$$

where $L = L(q, \dot{q})$ is the classical Lagrangian with $q = \{q_i\}$, $\dot{q} = \{\dot{q}_i\}$, $i = 1, \ldots N$ being the degrees of freedom of the system and $q_1 \equiv q_i(t_1)$, $q_2 \equiv q_i(t_2)$. The expression *corresponds to* is used to point out the relation between the quantum mechanical amplitude and the classical theory, and throughout Dirac's work expression (3.1) is treated as an equality. The assumption that in the r.h.s. appears the exponential of time integral of the Lagrangian, i.e. the classical action, is reasonable and is justified: expressing the wavefunction $\psi(q, t)$ as $\psi(q, t) = \rho(q, t)e^{iS[q]/\hbar}$ where $\rho(q, t)$ is the probability density and $S[q]$ any real functional, it turns out that the classical limit of Schroedinger equation is simply the Hamilton-Jacobi equation of classical mechanics $H(q, \frac{\partial S[q]}{\partial q}, t) + \frac{\partial S[q]}{\partial t} = 0$ where $S[q] = \int_{t_1}^{t_2} dtL + \text{constant}$.

Dirac's statement was then further developed by Feynman [56] to the formal formulation of the path integral. Namely, in an attempt to investigate what "corresponds to" accounts for in (3.1), he formulated a spacetime approach to quantum mechanics based on path integrals: the notion of a unique classical path (trajectory) describing motion of a particle between two fixed end points and satisfies Hamilton's principle is radically altered. In quantum theory, between the two fixed end points there exist infinite paths in spacetime plane along which motion takes place. Hence, all of them must be taken into account in the dynamics of the theory. In particular, the r.h.s. of Dirac's statement (3.1) must be integrated over all these spacetime paths as follows

$$\langle q_2, t_2 | q_1, t_1 \rangle = \int_{q_1}^{q_2} D[q(t)] \;\; e^{i\int_{t_1}^{t_2} dtL/\hbar} \quad . \tag{3.2}$$

This expression is known as Feynman's path integral. $D[q(t)]$ is the path integral measure and denotes functional integration along spacetime paths in configuration space. The boundary conditions (fixed end points in space) in the path integral are denoted as $q(t_2) = q_2$, $q(t_1) = q_1$. It should also be mentioned that the above equality is up to an (infinite) normalizing factor (further details can be found in the original work).

This path integral approach is mathematically equivalent with other formulations of quantum mechanics, like the Schroedinger equation and the Heisenberg operator algebra. Furthermore, it can be verified that the Feynman path integral has the correct limit as $\hbar \to 0$, indeed singling out the classical path contribution. A heuristic argument is the following: considering $\hbar \to 0$, paths away from the classical one will have large phase difference and will interfere destructively. On the contrary, paths that are near the classical one will have small phase difference and will interfere constructively. Thus, as $\hbar \to 0$, the most important contribution to the path integral comes from the region around the path which extremizes the action. According to Hamilton's principle, that is nothing else than the classical path. This conclusion can also be formally derived by use of the stationary-phase method.

Relativistic quantum field theory

In what follows, $\hbar$ is again set to unity.

The next step is to find an expression of the quantum amplitude in terms of the Feynman path integral (3.2) for systems with infinite degrees of freedom, i.e. for a field theory. The simplest case will be considered here, that of a scalar field. Later, more involved examples will be analyzed, such as the case of Electrodynamics and the case of General Relativity. Using the correspondence

$$q \to \phi \tag{3.3}$$

$$i \to \vec{x} \tag{3.4}$$

$$q_i(t) \to \phi(t, \vec{x}) = \phi(x), \quad \text{with } x \equiv (\vec{x}, t) \tag{3.5}$$

where $\phi(x)$ is the scalar field, the Feynman path integral (3.2) takes the form

$$\langle \phi_2(\vec{x}), t_2 | \phi_1(\vec{x}), t_1 \rangle = \int_{\phi_1(\vec{x})}^{\phi_2(\vec{x})} D[\phi(x)] \; e^{i \int d^4x \mathcal{L}(\phi)} \tag{3.6}$$

where $\mathcal{L} = \mathcal{L}(\phi(x), \eta_{ab})$ is the Lagrangian density and η_{ab} the Minkowski metric. The boundary conditions (fixed end points in spacetime) in the path integral are denoted as $\phi(\vec{x}, t_1) = \phi_1(\vec{x})$, $\phi(\vec{x}, t_2) = \phi_2(\vec{x})$.

In quantum field theory, it is usual to take the initial and final states $\phi_1(\vec{x})$, $\phi_2(\vec{x})$ to be the vacuum state. This can be denoted as $< 0, t_2 | 0, t_1 >$. This is a convenient but a natural choice as well, since this is precisely what is measured in experiments. Then, this quantum transition amplitude from vacuum to vacuum (for a free of interactions Lagrangian) represents the energy of the ground state. But this is not of particular interest, since the aim is to calculate and measure energies of excited states relative to the energy of vacuum state. Such cases are for example, when there is creation of particles which propagate for a while and then at a later point they annihilate. This is formally interpreted by a source at which particles can be created and annihilated. A typical Lagrangian density for a scalar field with a source function $J = J(x)$ is of the form

$$\mathcal{L}(\phi) = \frac{1}{2}\eta^{ab}\partial_a\phi\partial_b\phi - V(\phi) + J\phi \tag{3.7}$$

where $V(\phi)$ being a general potential term. Here and from now on, x-dependence is not explicitly written in expressions inside the path integral. Now, the quantum amplitude (3.6) takes the form

$$\langle 0, t_2 | J | 0, t_1 \rangle = \int D[\phi] \; e^{i \int d^4x \left[\frac{1}{2}\eta^{ab}\partial_a\phi\partial_b\phi - V(\phi) + J\phi \right]} . \tag{3.8}$$

Of course, the Lagrangian density (3.7) can also contain additional interaction terms between the fields like e.g. $-\frac{\lambda}{4!}\phi^4$ and so forth, with λ being the coupling constant of the interaction. For simplicity, these additional terms are not written explicitly here.

The ultimate target is to analytically evaluate this quantum amplitude in its most general form, i.e. for the most generic Lagrangian density. This is done as follows: it is expanded in a perturbation (Taylor) series in J and in λ and the resulting expression is a series of functional integrals. This series can also be schematically represented by the known Feynman diagrams. Each of the functional integrals of the series must be, in principle, analytically evaluated and, again in principle, they should not be infinite. It is important to state though, that even the functional integrals that appear in the lowest orders of perturbative expansion are in general highly non-trivial to calculate.

During the direct, straightforward evaluation of these functional integrals that arise from the Feynman path integral (3.8) an important identity is widely used. This identity is an infinite dimensional generalization of the Gaussian-type integral

$$\int_{-\infty}^{\infty} dx \; e^{-\frac{1}{2}ax^2 + bx} = \left(\frac{2\pi}{a}\right)^{1/2} e^{b^2/2a} . \tag{3.9}$$

The infinite dimensional generalization is obtained when integrating by parts the exponential in (3.8) and assuming as always boundary conditions, such as the fields to vanish fast enough at the boundaries. Then, the analogue of the identity (3.9) takes the form

$$\int D[\phi] \; e^{-\frac{1}{2}\phi \cdot K \cdot \phi - V + J \cdot \phi} = e^{-V(\delta/\delta J)} \, e^{\frac{1}{2} J \cdot K^{-1} \cdot J} \tag{3.10}$$

where $K = \partial^2$ here, but in general it contains all quadratic terms in the dynamical fields which appear in the Lagrangian density. Also, $\cdot$ denotes tensor multiplication in a generic case.

Relativistic quantum field theory with gauge symmetries

Since most of theories that actually describe nature do exhibit gauge invariance, it is of major importance to calculate the corresponding of the quantum amplitude (3.8) in these cases. In doing so, things seem to be not so straightforward. Namely, a small difficulty appears: whenever needed, implementation of the identity (3.10) is problematic. In particular, K has no inverse. Of course, this is not at all surprising since the Lagrangian (density) in gauge theories is singular: this is was already encountered in 2.1.1 where it was deduced that for a gauge theory, one has $\det\left[\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right] = 0$. This non invertibility of K appears at least to next-to leading order terms in the perturbative expansion of the path integral and consequently affects the unitarity property of the corresponding quantum amplitude [57].

A characteristic example of this complication is the case of Electrodynamics. The Maxwell action with presence of a current J_a, i.e.

$$S[A] = \int d^4x \Big[-\frac{1}{4} F_{ab}\, F^{ab} + A_a J^a \Big] \tag{3.11}$$

with $F_{ab} \equiv 2\partial_{[a} A_{b]}$, is invariant with respect to the gauge transformation $A_a \to A'_a = A_a - \partial_a \Lambda$. When evaluating the path integral, in the attempt to bring it to form (3.10), a partial integration in the action must be performed. This gives

$$S[A] = \int d^4x \Big[-\frac{1}{2} A_a \big(-\eta^{ab}\partial^2 + \partial^a \partial^b \big) A_b + A_a J^a \Big]\,. \tag{3.12}$$

Therefore here, K in (3.10) is proportional to $-\eta^{ab}\partial^2 + \partial^a\partial^b \equiv B^{ab}$. But one immediately observes that B^{ab} is singular, because acting on the vector $\partial_a \Lambda$ it has zero eigenvalues, i.e. $B^{ab}\partial_b \Lambda = 0$. Thus, K has no inverse.

Although there exist several tricks to overpass this difficulty, they are successful only at lowest order of the perturbative expansion of the path integral and only in the above case of Electrodynamics. The most general and successful method to attack this problem is that of Faddeev and Popov [58], [59]. Its success relies on the facts that it is successful at any order of perturbative expansion and at cases of non-Abelian gauge theories, such as Yang-Mills and also at the case of General Relativity.

The Faddeev-Popov method is now performed for the above case of Electrodynamics. It essentially consists of factoring out of the path integral the redundant integration over Λ (where K is singular), up to an overall infinite factor. Since quantum amplitudes at all orders will contain this infinite overall factor, it is usual to ignore it and throw it away. This can be schematically written as

$$\int D[A]\, e^{iS[A]} = \Big(\int D[\Lambda] \Big)\, J \tag{3.13}$$

where J is a path integral, independent of Λ and is to be specified. It is worthwhile mentioning that this is reminiscent of the choice of a particular gauge in the canonical quantization of Electrodynamics.

The method introduces a functional $\Delta(A)$, known as the Faddeev-Popov determinant. It is defined by writing unity as

$$1 \equiv \Delta[A] \int D[\Lambda] \delta[f(A')] \tag{3.14}$$

where $f[(A'_a)]$ is any gauge fixing function depending on $A'_a = A_a - \partial_a\Lambda$. It can be shown that the Faddeev-Popov determinant is gauge invariant, i.e. $\Delta(A) = \Delta(A')$ (a proof can be found in III.4 of [60]). Then, inserting (3.14), the l.h.s of (3.13) becomes

$$\begin{aligned}\int D[A]\, e^{iS[A]} &= \int\int D[A]D[\Lambda]\, e^{iS[A]}\Delta[A]\delta[f(A')] \\ &= \int\int D[A]D[\Lambda]\, e^{iS[A]}\Delta[A]\delta[f(A)] \\ &= \Big(\int D[\Lambda]\Big)\int D[A]\, e^{iS[A]}\Delta[A]\delta[f(A)]\end{aligned} \tag{3.15}$$

where in the second equality the inverse gauge transformation $A'_a \to A_a$ has been performed. Also, it has been used that the Faddeev-Popov determinant, the path integral measure $D[A]$ and the action are gauge invariant. Now this expression is indeed of form (3.13). The integration over Λ has been factored out in the overall infinite volume element $\int D[\Lambda]$. It is infinite because it is the volume of group of gauge transformations $A'_a = A_a - \partial_a\Lambda$. But as was stated before, this is not problematic since all quantum amplitudes are considered to be normalized to common infinite factors.

To see specifically that the singular behavior of K in identity (3.12) disappears with the Faddeev-Popov method, it is instructive to choose $f[A] = \partial_a A^a - \sigma$, with σ being a scalar function. Then, from (3.14) the inverse of the Faddeev-Popov determinant takes the form

$$[\Delta(A)]^{-1} = \int D\Lambda\delta(\partial_a A^a - \partial^2\Lambda - \sigma)\,. \tag{3.16}$$

But since in (3.15) $\Delta[A]$ is multiplied with $\delta[f(A)]$, one can heuristically set $f[A] = \partial_a A^a - \sigma$ to zero. Thus, $[\Delta(A)]^{-1} \sim [\int D\Lambda\delta(\partial^2\Lambda)]^{-1}$. This does not depend on A_a and is thrown away. In other words, the Faddeev-Popov determinant here can be set to 1. Therefore, up to overall factors that only depend on Λ, (3.15) takes the form

$$\begin{aligned}\int D[A]\, e^{iS[A]}\Delta[A]\delta[f(A)] &= \int D[A]\, e^{iS[A]}\delta(\partial_a A^a - \sigma) \\ &= \int D[\sigma]e^{-\frac{i}{2\xi}\int d^4x\,\sigma}\int D[A]\, e^{iS[A]}\delta(\partial_a A^a - \sigma) \\ &= \int D[A]e^{iS[A]-\frac{i}{2\xi}\int d^4x(\partial_a A^a)^2}\end{aligned} \tag{3.17}$$

where an additional integration over σ has been performed in order to compensate with the delta function $\delta(\partial_a A^a - \sigma)$ and ξ is a number. Then finally, the original Maxwell action (3.11) takes the form

$$\begin{aligned}\tilde{S}[A] &= S[A] - \frac{1}{2\xi}\int d^4x(\partial_a A^a)^2 \\ &= \int d^4x\Big[-\frac{1}{2}A_a\Big(-\eta^{ab}\partial^2 + (1-\frac{1}{\xi})\partial^a\partial^b\Big)A_b + A_a J^a\Big]\,.\end{aligned} \tag{3.18}$$

It is straightforward to check that K of (3.10) is now proportional to $\tilde{B}^{ab} = -\eta^{ab}\partial^2 + (1-\frac{1}{\xi})\partial^a\partial^b$ which does have an inverse and is no longer singular. Therefore, the Faddeev-Popov method in this case fixes the problem by "adding" a gauge fixing term in the action.

Consequently, for a relativistic theory that possesses a particular gauge symmetry the corresponding expression of the quantum amplitude (3.8) can be written as

$$\langle 0, t_2|J|0, t_1\rangle = \int D[\phi]\ \Delta(\phi)\ e^{iS[\phi]} \tag{3.19}$$

where $\Delta(\phi)$ is the Faddeev-Popov determinant, relevant with the gauge symmetry of the scalar field action $S[\phi]$.

Gravitational theories

Since the aim of this thesis is to analyze extensively aspects of conformal gravity, it is desirable to extend the above discussion in this direction. And since the most physical and illustrative example to start with is General Relativity, the action $S[g]$ will be left arbitrary and generic in order to cover both these cases.

The gauge symmetry that General Relativity possesses is that of diffeomorphism invariance, that is invariance under changes of coordinates as $x^a \to x^{a\prime} = x^a + \xi^a(x^b)$. Conformal gravity is diffeomorphic invariant as well. But it exhibits additionally scale invariance as a gauge symmetry. Therefore, for a generic gravity theory an analogue of expression (3.19) with the Faddeev-Popov determinant should be derived.

Before proceeding with the actual derivation and exactly because of diffeomorphism invariance, it is important to emphasize on boundary conditions or end points of the path integral. Namely, an initial and a final state can be characterized by a metric g_1 on a spacetime M_1 and a metric g_2 on a spacetime M_2 respectively. But, due to diffeomorphism invariance not all components of g_1 and g_2 are physically relevant. That is, the components $g^{ab}n_b$, where n_b is the normal vector on the induced surfaces S_1 and S_2 of M_1 and M_2, can have by gauge transformations arbitrary values which move points in the interior of spacetime but leave the boundary fixed. Therefore at the end surfaces S_1 and S_2, it is sufficient to specify only the induced 3-dimensional metric h_1 and h_2 respectively.

Taking this into account, the quantum amplitude from an initial state with metric h_1 on a surface S_1 to a final state with metric h_2 on a surface S_2 is [61]

$$\langle h_2, S_2 | h_1, S_1 \rangle = \int_{h_1}^{h_2} D[g]\ \Delta(g)\ e^{iS[g]} \tag{3.20}$$

with $\Delta(g)$ being the Faddeev-Popov determinant, associated with gauge symmetries of the gravitational action $S[g]$.

3.1.2 *Euclidean path integral and partition function*

There is a technical issue concerning the Feynman path integral that was described before. And this is the issue related to its convergence. In particular, the action that appears in the exponential of path integral (3.19) is real, for real ϕ. Likewise, the Einstein-Hilbert action as appearing in the exponential of (3.20) is real when g is a real Lorentzian metric, i.e. having signature $(-,+,+,+)$. Therefore in both these cases, the exponential terms oscillate and cause convergence problems in the path integral.

A resolution to this problem is to turn the oscillating exponentials into decaying ones. In order to achieve that, the time parameter t is analytically extended into the complex set. This is done by performing a Wick rotation $t \to -i\tau$, with $\tau \in \mathcal{R}$. Then, the section of the complex plane characterized by real coordinates (τ, x, y, z) is the Euclidean spacetime (τ is also known as the Euclidean time). The spacetime metrics η and g with initial Lorentzian signature are changed to the Euclidean signature $(+,+,+,+)$. Then, all path integrals can be performed in Euclidean spacetime and resultant expressions can be always analytically continued back to the initial Lorentzian section.

At first, the Euclidean continuation is applied in the case of scalar field theory. The path integral in (3.19) takes the form

$$\int D[\phi]\ \Delta(\phi)\ e^{-\hat{S}[\phi]} \tag{3.21}$$

with $\hat{S}[\phi] = -iS[\phi]$ being the Euclidean version of $S[\phi]$. For real ϕ on Euclidean spacetime, $\hat{S}[\phi]$ is positive semi-definite. Now the initial aim has been achieved: the path integral over all such field configurations is indeed exponentially damped and thus, it is expected to tend to converge. The above expression is also known as Euclidean path integral.

Another important application of the Euclidean path integral is related with aspects of statistical mechanics. That is, a canonical ensemble of the scalar field ϕ and its partition function. Such a relation can be obtained as follows: one can consider the scalar field ϕ without the presence of a source, i.e. to consider the free action $S = \int d^4x \left[\frac{1}{2}\eta^{ab}\partial_a\phi\partial_b\phi - V(\phi)\right]$. Then, the quantum amplitude between an initial state $|\phi_1(\vec{x}), t_1 >$ and a final state $|\phi_2(\vec{x}), t_2 >$ is given by the Feynman path integral (3.6), which is repeated here:

$$\langle\phi_2(\vec{x}), t_2|\phi_1(\vec{x}), t_1\rangle = \int_{\phi_1(\vec{x})}^{\phi_2(\vec{x})} D[\phi]\ e^{iS[\phi]}\,. \tag{3.22}$$

Using the Schroedinger picture, the l.h.s. can be written as

$$\langle\phi_2(\vec{x})|e^{-iH(t_2-t_1)}|\phi_1(\vec{x})\rangle\,. \tag{3.23}$$

Assuming that $t_2 - t_1 = -i\beta$, $\phi_1(\vec{x}) = \phi_2(\vec{x})$ and summing over a complete orthonormal basis of ϕ_n, one obtains from the above the partition function of the canonical ensemble of the field ϕ at temperature $T = \beta^{-1}$, which is

$$Z = \sum_n \langle\phi_n(\vec{x})|e^{-\beta E_n}|\phi_n(\vec{x})\rangle \tag{3.24}$$

where E_n is the energy eigenvalue of the state ϕ_n. But as it was stated before, the Feynman path integral can also be replaced by its Euclidean version. Thus, the partition function (3.24) can also be represented as a Euclidean path integral

$$Z = \int_{\phi_1(\vec{x})}^{\phi_2(\vec{x})} D[\phi]\ e^{-\hat{S}[\phi]} \tag{3.25}$$

where $\hat{S}[\phi]$ is the Euclidean action and the path integration is performed over all ϕ's that are real in the Euclidean section and periodic in τ with period β.

Now it is of interest to discuss the above considerations in the case of gravitational theories. That is, to derive the corresponding expression for the Euclidean path integral and investigate the canonical ensemble and partition function of the gravitational field. With analytic continuation to the Euclidean section, the spacetime metric g takes the Euclidean signature $(+,+,+,+)$ and the quantum amplitude (3.20) becomes

$$\langle h_2, S_2|h_1, S_1\rangle = \int_{h_1}^{h_2} D[g]\,\Delta(g)\ e^{-\hat{S}[g]} \tag{3.26}$$

where $\hat{S}[g] = -iS[g]$ is the Euclidean version of the gravitational action. The path integration is performed over all metric configurations that are real in Euclidean spacetime and have given values h_1, h_2 at the boundary surfaces S_1, S_2 respectively. Now the question is whether the issue of convergence of the path integral has been resolved, like in the case of the scalar field. One obtains that

- the Euclidean Einstein-Hilbert action is not positive definite. This conclusion comes form the fact that there is a particular example of a metric, that is $g' = \Omega^2 g$, which makes the action arbitrarily negative for Ω varying rapidly. Therefore, the issue of convergence of the path integral in this case is not resolved. It is unavoidable to adopt a "Positive Action Conjecture" [8], which nevertheless has been proven to hold for asymptotically Euclidean metrics [9] and other special cases.

- the conformal gravity action (5.1) consists of a Lagrangian density which is quadratic in curvature tensors of the metric. Assuming that i) the dimensionless coupling constant α_{CG} is positive and ii) the metric g_{ab} is real and positive, then the conformal gravity action is positive. Thus, and only under these assumptions, its path integral is expected to converge.

In any case, given the Euclidean path integral (3.26) it is natural to define the partition function of a gravitational canonical ensemble in an analogous manner like in the scalar field. But the situation is quite different: the universal attractive nature of gravity is source of many instabilities, which make the canonical ensemble ill-defined. This fact was already known from Newtonian gravity, where a static and homogeneous fluid is unstable under gravitational perturbations [62]. Similar problems appear when considering a gas of gravitons in a finite volume, which are in thermal equilibrium under their own Newtonian gravitational field: the system is thermodynamically unstable. All these problems arise due to the attractive nature of the gravitational force.

In the case of General Relativity the problem of instabilities not only is not resolved, but becomes even more severe. A gravitational system in equilibrium does not have a spatially constant temperature, but there is only a local notion of temperature depending on the local observer. This is concluded from the equivalence principle and implies that the temperature is blue- or red-shifted. Additionally, a gravitational system due to influence of its own gravitational field will experience gravitational collapse which will inevitably lead to the formation of a black hole [63]. The black hole has a negative specific heat and thus, it cannot be in stable equilibrium with thermal radiation at a fixed temperature [64]. Consequently, the black hole formation renders the canonical ensemble ill-defined. Similar instability results are deduced when restricting the ensemble into a particular metric configuration, such as flat spacetime: the canonical ensemble of flat spacetime at a finite temperature is not well-defined [65]. Other approaches that consider the gravitational microcanonical ensemble are not discussed here.

The problem of instabilities though is handled, when imposing certain assumptions in the whole setup. That is, when enclosing the system in a box and imposing particular type of asymptotic boundary conditions to the black hole to be formed. Such an example is the case of an asymptotically flat black hole inside a spherical cavity. The enclosing of the system inside the cavity is crucial for defining the notion of temperature of the black hole ensemble: this is defined to be the uniform temperature of the wall of the spherical cavity and depends on its boundary radius. In this case, the canonical ensemble and its partition function turn out to be well defined [66]. Another example is the case of an asymptotically AdS black hole. It is interesting to notice that this system does not require the confinement in any kind of cavity, because the gravitational potential of the AdS space acts like a box (of finite volume) itself. Then, an asymptotically AdS black hole has a positive specific heat and thus, makes the canonical ensemble and its partition function well-defined [67].

Under the above considerations, a definition of the partition function of a canonical ensemble for the case of gravitational theories should be done with great care on the (asymptotic) boundary conditions imposed on the Euclidean path integral. In the cases when those allow a definition of a gravitational canonical ensemble, following the analogue of (3.22), (3.23), (3.24) the gravitational partition function is

$$Z = \int_{h_1}^{h_2} D[g]\,\Delta(g)\;\; e^{-\hat{S}[g]} \tag{3.27}$$

where g is periodic in τ with period $\beta = T^{-1}$, i.e. $g(\tau, \vec{x}) = g(\tau + \beta, \vec{x})$, and has given boundary values h_1, h_2.

3.1.3 *Linearization and 1-loop correction to the classical action*

The present and following section are entirely focused on the case of gravitational theories. As it was already described before, their path integral is, in a sense, ill-defined. And this is due to the problem of convergence of the path integral, even in its Euclidean version (3.20), (3.27). But even if this issue is neglected and one attempts to actually calculate the whole perturbative series of the path integral, it turns out that it is extremely non-trivial. This complication arises due to non-linearity of the equations of motion that gravitational theories exhibit.

Therefore, in the path integral formulation of gravitational theories it is legitimate to use approximation methods. It is natural to expect that the dominant contribution to the path integral will come from metric configurations that are near to those who extremize the action. That is, near to solutions of the classical equations of motion. This method is called stationary-phase approximation and it basically exploits perturbation theory to approximate the non-linear equations of motion into a simpler, linearized version.

The starting point is to assume that there is an exact, known solution of the equations of motion and a deviation around this solution. The spacetime metric g_{ab} can then be written as

$$g_{ab} = \bar{g}_{ab} + g_{ab}(\epsilon) \tag{3.28}$$

where $\bar{g}_{ab}$ is the exact known solution and $g_{ab}(\epsilon)$ is the deviation around it. This can also be expressed in a perturbative (Taylor) series as

$$g_{ab} = \bar{g}_{ab} + \epsilon \frac{dg_{ab}}{d\epsilon}|_{\epsilon=0} + \frac{\epsilon^2}{2!}\frac{d^2 g_{ab}}{d\epsilon^2}|_{\epsilon=0} + \dots \tag{3.29}$$

$$= \bar{g}_{ab} + \epsilon\gamma_{ab} + \dots \tag{3.30}$$

with $\frac{dg_{ab}}{d\epsilon}|_{\epsilon=0} \equiv \gamma_{ab}$. The parameter ϵ denotes the degree of perturbation, in the sense that $g(\epsilon)$ depends differentiably on ϵ and $g_{ab}(0) = \bar{g}_{ab}$.

The aim is to find $S[\bar{g} + \epsilon\gamma + \dots]$. Using the functional analogue of the Taylor series expansion, i.e.

$$f[x+a] = \exp\left[a\frac{\delta}{\delta x}\right] f[x] = \sum_n \frac{1}{n!}\frac{\delta^{(n)} f[x]}{\delta x^n} a^n \tag{3.31}$$

one arrives at

$$S[\bar{g} + \epsilon\gamma + \dots] = S[\bar{g}] + \epsilon\frac{\delta S[g]}{\delta g}|_{\bar{g}}\gamma + \frac{1}{2!}\epsilon^2\frac{\delta^{(2)} S[g]}{\delta g^2}|_{\bar{g}}\gamma^2 + \dots \tag{3.32}$$

$$= S[\bar{g}] + \frac{1}{2!}\epsilon^2\frac{\delta^{(2)} S[g]}{\delta g^2}|_{\bar{g}}\gamma^2 + \dots \tag{3.33}$$

provided that the action has a well-defined variation principle. In the stationary-phase approximation that is considered here, one focuses up to the term of order ϵ^2. This term contains the second variation of the action and is quadratic in the perturbation γ_{ab}. After a partial integration, it can be brought into an equivalent form of an operator linear in γ_{ab}, which is more convenient for the analytic evaluation that follows in the next sections. This is always possible and to understand why, it is instructive to derive the analytic form of $\frac{\delta^{(2)} S[g]}{\delta g^2}|_{\bar{g}}$. This is done by studying the corresponding equations of motion, i. e. the equations of motion at order ϵ. That is, one performs the perturbative analysis (3.28), (3.29), (3.30) at the level of the equations of motion and picks the ones of order ϵ.

Denoting the equations of motion of g_{ab} as $\mathcal{E}[g] = 0$, one is interested for a one-parameter family of solutions $g_{ab}(\epsilon)$ of the form

$$\mathcal{E}\left[g(\epsilon)\right] = 0\,. \tag{3.34}$$

To extract the desired order (ϵ), it is necessary to differentiate them with respect to ϵ and set this to zero, i.e.

$$\frac{d\mathcal{E}\left[g(\epsilon)\right]}{d\epsilon}|_{\epsilon=0} = 0\,. \tag{3.35}$$

Then, the above is linear in

$$\frac{dg_{ab}}{d\epsilon}|_{\epsilon=0} \equiv \gamma_{ab} \tag{3.36}$$

which is the deviation term in metric expansion (3.30). On account of (3.35) being linear, it can be expressed in the form

$$\mathcal{L}(\gamma_{ab}) = 0 \tag{3.37}$$

where $\mathcal{L}$ is a linear operator. These are the linearized equations of motion about $\bar{g}_{ab}$. Now, substituting this back in the perturbative expansion of the action (3.33), one finds

$$S[\bar{g} + \epsilon\gamma + \ldots] = S[\bar{g}] - \frac{1}{2!}\int d^4x \; \epsilon^2 \; \gamma^{ab}\mathcal{L}(\gamma_{ab}) + \ldots \tag{3.38}$$

$$= S[\bar{g}] + \epsilon^2 \; S[\bar{g},\gamma]_{\text{1-loop}} + \ldots \tag{3.39}$$

where a partial integration has been performed and

$$S[\bar{g},\gamma]_{1-loop} \equiv -\frac{1}{2!}\int d^4x \; \gamma^{ab}\mathcal{L}(\gamma_{ab}) \tag{3.40}$$

also known as the 1-loop correction to the classical action. This is referred to as 1-loop because in the Feynman diagrams it corresponds to any number of external lines joined to a single closed loop. This 1-loop term is assumed to describe self-gravitational interactions of the theory.

3.1.4 *1-loop partition function*

The stationary-phase method described previously can be applied to extract 1-loop corrections of several quantities of physical interest. In this way, one improves their classical values. Throughout this thesis, such a quantity will be the partition function of the canonical gravitational ensemble.

The partition function (3.27) after substituting (the Euclidean form of) (3.39) can be written as

$$Z = Z_{\bar{g}} \times Z_{\text{1-loop}} \times \ldots = e^{-\hat{S}[\bar{g}]} \times \int D[g]\,\Delta(g) \;\; e^{-\hat{S}[\bar{g},\gamma]_{\text{1-loop}}} \times \ldots \tag{3.41}$$

and thus the 1-loop partition function takes the form

$$Z_{\text{1-loop}} = \int D[g]\,\Delta(g) \;\; e^{-\hat{S}[\bar{g},\gamma]_{\text{1-loop}}} \,. \tag{3.42}$$

The non-trivial task is to evaluate this expression analytically. One way of performing that is via heat-kernel techniques which are analyzed in detail in chapter 7 and analytic expressions are derived.

Since the 1-loop partition function has been defined, it is straightforward to obtain a relevant expression for the 1-loop correction to the free (Helmholtz) energy of the ensemble. That is

$$F_{1-\text{loop}} = -\frac{1}{\beta}\ln Z_{1-\text{loop}} \,. \tag{3.43}$$

On account of the gravitational canonical ensemble of asymptotically AdS black holes being well-defined [67], it is natural to perform the linearized analysis around the AdS_4 background and expect that its partition function will be well-defined. The purpose of this section is to derive the expression of the 1-loop correction to the partition function of such an ensemble.

3.2 THE CASE OF GENERAL RELATIVITY

3.2.1 *Linearized equations of motion and second variation of the action*

The Euclidean version of the Einstein-Hilbert action with a cosmological constant is

$$\hat{S}[g] = -\frac{1}{16\pi}\int_M d^4x\,\sqrt{g}\,\left(R+6\right) - \frac{1}{8\pi}\int_{\partial M} d^3x\,\sqrt{h}\,K \tag{3.44}$$

with g_{ab} having the Euclidean signature $(+,+,+,+)$. The first variation gives

$$\delta\hat{S}[g] = \frac{1}{16\pi}\int_M d^4x\,\sqrt{g}\,\Big(G_{ab} - 3g_{ab}\Big)\delta g^{ab} \tag{3.45}$$

yielding Einstein's equations of motion. At this point the perturbative approach starts: as it was described previously, the metric g_{ab} is decomposed into a fixed background solution $\bar{g}_{ab}$ of (3.45) and a perturbation γ_{ab} as a deviation from this background as follows

$$g_{ab} = \bar{g}_{ab} + \epsilon\,\gamma_{ab} \tag{3.46}$$

where ϵ is the perturbation parameter. The choice here for the fixed background solution $\bar{g}_{ab}$ of (3.45) will be AdS_4 spacetime. From now on quantities with a bar will denote their value on this background. Inserting the above perturbation expansion (3.46) into the Einstein's equations of motion arising from (3.45), one gets at first order on ϵ

$$2\bar{\nabla}_{(a}\bar{\nabla}_{|c|}\gamma^{c}_{b)} - \bar{\nabla}^2\gamma_{ab} - \bar{\nabla}_b\bar{\nabla}_a\gamma - 2\gamma_{ab} + \bar{g}_{ab}\big(\bar{\nabla}^2\gamma - \bar{\nabla}_d\bar{\nabla}_c\gamma^{cd} - \gamma\big) = 0 \tag{3.47}$$

with $\gamma \equiv \bar{g}^{ab}\gamma_{ab}$. These are the linearized equations of motion.

In order for diffeomorphism invariance and the true dynamics of the linearized theory to be revealed, it is customary to perform a York-decomposition to γ_{ab} into a transverse-traceless part (γ^{TT}_{ab}), a "trace" part ($\hat{\gamma}$) and a vector part (v_a) as

$$\gamma_{ab} = \gamma^{TT}_{ab} + \frac{1}{4}\bar{g}_{ab}\hat{\gamma} + 2\bar{\nabla}_{(a}v_{b)} \tag{3.48}$$

where $\bar{\nabla}^a\gamma^{TT}_{ab} = 0 = \bar{g}^{ab}\gamma^{TT}_{ab}$. Now the diffeomorphism invariance of the linearized theory is unveiled: it is represented by the transformation $\gamma_{ab} \to \gamma_{ab} + 2\bar{\nabla}_{(a}v_{b)}$. Thus, absence of the vector part v_a in expressions such as the equations of motion, the partition function, etc. denotes diffeomorphism invariance. This is the reason that v_a is sometimes referred to as gauge part. Therefore, this convenient York-decomposition of γ_{ab} is adopted throughout the rest of the calculation.

Inserting (3.48) into the linearized equations of motion (3.47), one arrives at the equivalent expression

$$4\bar{\nabla}_c\bar{\nabla}_{(a}\gamma^{cTT}_{b)} - 2\bar{\nabla}^2\gamma^{TT}_{ab} - \bar{\nabla}_b\bar{\nabla}_a\hat{\gamma} + 12\gamma^{TT}_{ab} + \bar{g}_{ab}\big(\bar{\nabla}^2\hat{\gamma} - 3\hat{\gamma}\big) = 0\,. \tag{3.49}$$

It is emphasized once again that the vector part v_a of decomposition (3.48) is absent, denoting diffeomorphism invariance.

The next step in the analysis is to find the second variation of the Euclideanized Einstein-Hilbert action (3.44). Varying (3.45) and inserting (3.46) one arrives at

$$\delta^{(2)}\hat{S}[\bar{g},\gamma] = \int_M d^4x\,\sqrt{\bar{g}}\,\Big[(\delta G_{ab} - 3\gamma_{ab})\gamma^{ab} + (\bar{G}_{ab} - 3\bar{g}_{ab})\delta\gamma^{ab}\Big]\,. \tag{3.50}$$

The second term in the parenthesis above vanishes on shell and the first term is just the linearized expression (3.47). Therefore, after using the York-decomposition (3.48) and some partial integration the second variation (3.50) or 1-loop correction takes the form

$$\hat{S}[\bar{g},\gamma]_{\text{1-loop}} = \int_M d^4x\,\sqrt{\bar{g}}\,\Big[\gamma_{TT}^{ab}\big(-\bar{\nabla}^2-2\big)\gamma_{ab}^{TT}+\hat{\gamma}\big(-\bar{\nabla}^2+4\big)\hat{\gamma}\Big] \tag{3.51}$$

where the subscripts $(0),(2)$ denote the transverse-traceless modes γ_{ab}^{TT} and the scalar modes $\hat{\gamma}$ respectively. From now on the bar is omitted from the Laplacian operator.

3.2.2 *Path integral measure, gauge fixing and Faddeev-Popov determinant*

The gauge group that yields an infinite volume factor in the path integral is the group of diffeomorphisms. In order to get rid of this infinite factor consistently, as it was already mentioned in 3.1.2, the Faddeev-Popov method should be applied. In the present case, the path integral measure $D[g]$ is divided by the infinite volume of the group of diffeomorphisms and it is expressed in terms of the Faddeev-Popov determinant $\Delta(g)$. Then, $\Delta(g)$ is given by the Jacobian of the York-decomposition transformation $\gamma_{ab}\to(\gamma_{ab}^{TT},v_a,\hat{\gamma})$ (3.48). The resulting expression can be written schematically as

$$\frac{D[\gamma]}{V_{\text{diff}}} = \Delta(g)\,D[\gamma^{TT}]\,D[v]\,D[\hat{\gamma}] \tag{3.52}$$

where V_{diff} is the volume of the group of diffeomorphisms.

The standard procedure to evaluate $\Delta(g)$ consists of picking a suitable gauge for the metric variables v_a and $\hat{\gamma}$ of the York-decomposition (3.48) and then expressing $\Delta(g)$ in terms of the Jacobian matrices of these transformations. This is done as follows: one assumes orthonormality for γ_{ab}, i.e.

$$\begin{aligned} 1 &= \int D[\gamma]\ \exp\Big[-\int d^4x\sqrt{\bar{g}}\ \gamma_{ab}\gamma^{ab}\Big] \\ &= \int \Delta(g)D[\gamma^{TT}]D[v]D[\hat{\gamma}]\ \exp\Big[-\int d^4x\sqrt{\bar{g}}\ \gamma_{ab}(\gamma^{TT},v,\hat{\gamma})\gamma^{ab}(\gamma^{TT},v,\hat{\gamma})\Big] \end{aligned} \tag{3.53}$$

and the same for each mode of the York-decomposition of γ_{ab} (3.48), i.e.

$$1 = \int D[\gamma^{TT}]\ \exp\Big[-\int d^4x\sqrt{\bar{g}}\ \gamma_{ab}^{TT}\gamma_{TT}^{ab}\Big] \tag{3.54}$$

$$1 = \int D[v]\ \exp\Big[-\int d^4x\sqrt{\bar{g}}\ v^a v_a\Big] \tag{3.55}$$

$$1 = \int D[\hat{\gamma}]\ \exp\Big[-\int d^4x\sqrt{\bar{g}}\ \hat{\gamma}^2\Big]\,. \tag{3.56}$$

Due to the mixing between modes of different types in the inner product in (3.53), it is convenient to choose as a gauge fixing condition one which cancels this mixing. Such a premise is met by decomposing v_a into a transverse (v_a^T) and a scalar part (σ) and $\hat{\gamma}$ into two scalar parts $(\tilde{\gamma},\sigma)$ as follows:

$$v_a = v_a^T + \nabla_a\sigma \tag{3.57}$$

$$\hat{\gamma} = \tilde{\gamma} - 2\nabla^2\sigma \tag{3.58}$$

with $\nabla^a v_a^T = 0$. With this gauge choice, the mixing is canceled and the decomposition of the inner product in (3.53) is indeed orthogonal

$$\gamma_{ab}\gamma^{ab} = \gamma_{ab}^{TT}\gamma_{TT}^{ab} - 2v_T^a(\nabla^2-3)v_a^T + 3\sigma(-\nabla^2)(-\nabla^2+4)\sigma + \frac{1}{4}\tilde{\gamma}^2\,. \tag{3.59}$$

Now these gauge-fixing transformations (3.57), (3.58) result a Jacobian, denoted as J_2, in the path integral measure $D[\gamma]$, i.e. $\frac{D[\gamma]}{V_{\text{diff}}} = J_2\, D[\gamma^{TT}]\, D[v^T]\, D[\sigma] D[\tilde{\gamma}]$. Using the orthonormality condition (3.53) one finds

$$1 = \int D[\gamma^{TT}] D[v^T] D[\sigma] D[\tilde{\gamma}]\, J_2 \exp\Big[-\int d^4x \sqrt{\bar{g}}\, \Big(\gamma^{TT}_{ab}\gamma^{ab}_{TT} - 2v^a_T(\nabla^2 - 3)v^T_a + 3\sigma(-\nabla^2) \\ (-\nabla^2 + 4)\Big)\Big] \quad \Rightarrow J_2 = \Big[\det(-\nabla^2+3)^T_{(1)}\, \det(-\nabla^2)_{(0)}\, \det(-\nabla^2+4)_{(0)} \Big]^{\frac{1}{2}}. \tag{3.60}$$

The notation $^T{}_{(1)}$ denotes the transverse vector mode v^T_a and the subscript (0) denotes the scalar part σ. What is left of the above analysis is to find the Jacobians of each of the gauge-fixing transformations (3.57), (3.58) in the corresponding path integral measures. That is, to find J_1 and J_0 for $D[v] = J_1\, D[v^T]\, D[\sigma]$ and $D[\hat{\gamma}] = J_0\, D[\tilde{\gamma}]\, D[\sigma]$ respectively, where J_1 and J_0 are the Jacobians. A straightforward calculation shows that $J_0 = 1$ where as for J_1 using (3.55) one obtains

$$1 = \int D[v^T] D[\sigma] J_1 \;\exp\Big[-\int d^4x \sqrt{\bar{g}}\, (v^a_T v^T_a - \sigma\nabla^2\sigma) \Big] \\ \Rightarrow J_1 = \Big[\det(-\nabla^2)_{(0)} \Big]^{\frac{1}{2}}. \tag{3.61}$$

Now the analysis is completed since the Faddeev-Popov determinant can be expressed in terms of the Jacobians J_1 and J_2. Considering again the expression for the path integral measure $D[\gamma]$ after the gauge-fixing transformations (3.57), (3.58), i.e.

$$\frac{D[\gamma]}{V_{\text{diff}}} = J_2\, D[\gamma^{TT}]\, D[v^T]\, D[\sigma] D[\tilde{\gamma}] = \frac{J_2}{J_1}\, D[\gamma^{TT}]\, D[v]\, D[\hat{\gamma}] \tag{3.62}$$

and comparing with (3.52), the Faddeev-Popov determinant is found to be

$$\Delta(g) = \frac{J_2}{J_1} = \Big[\det(-\nabla^2+3)^T_{(1)}\, \det(-\nabla^2+4)_{(0)} \Big]^{\frac{1}{2}}. \tag{3.63}$$

3.2.3 *1-loop partition function*

According to (3.42), all the necessary ingredients are now in hand to evaluate the 1-loop partition function of the theory. Using (3.51) and (3.63) the 1-loop partition function takes the form

$$Z_{1-\text{loop}} = \int \frac{D[\gamma]}{V_{\text{diff}}} e^{-\hat{S}[\bar{g},\gamma]_{\text{1-loop}}} \\ = \left[\frac{\det(-\nabla^2+3)^T_{(1)}}{\det(-\nabla^2-2)^{TT}_{(2)}} \right]^{\frac{1}{2}} \tag{3.64}$$

$$= Z_{(1)}\, Z^{-1}_{(2)} \tag{3.65}$$

where $Z_{(s)}$ being the partition functions of the modes of spins $s = 1, 2$. The interpretation of the above expression is the following: the partition function $Z_{(1)}$ that appears in the nominator, i.e. the determinant of the vector part, corresponds to diffeomorphism invariance of the theory as already discussed in 3.2.1. Thus, it is pure gauge. The true dynamics of the theory are expressed via the $s = 2$ modes, the partition function of which ($Z_{(2)}$) appears in the denominator. According to the canonical analysis of 2.4, the dynamical degrees of freedom for Einstein gravity are 2, corresponding to the massless spin-2 graviton, and are expressed here via the partition function of the spin-2 transverse traceless modes.

The story does not end here of course, since one would like to analytically calculate the 1-loop partition function (3.64). This is desirable for many purposes such as: comparing with the classical contribution and determine if indeed the 1-loop correction is small- according to the stationary-phase approximation that was employed, evaluating 1-loop corrections to thermodynamical quantities, etc. One method of attacking the analytic calculation are the heat kernel techniques which will be described in detail in 7.3. The corresponding expression for (3.64) will be presented then.

4

FEW WORDS ON THE ADS-CFT CONJECTURE

The $AdS - CFT$ conjecture is a particular realization of a generic holographic principle or holography [68]. Originated from considerations of black hole entropy, holography, roughly, suggests a correspondence between a theory in some space with a theory defined on the boundary of that space. In this perspective, both theories are considered to be dual because they generate the same physical content. Now, the $AdS - CFT$ conjecture proposes a correspondence between a gravity theory on AdS_{d+1} spacetime and a conformal field theory (CFT) without gravity in d spacetime dimensions. The conjecture was originally proposed [69] in the following form: the large N limit of a conformally invariant theory in d dimensions corresponds to a supergravity theory on AdS_{d+1} spacetime times a compact manifold. An example to which this correspondence applies is $\mathcal{N} = 4$ super Yang-Mills in $d = 4$ spacetime dimensions with gauge group $SU(N)$ and coupling constant g_{YM}. This theory is conjecturally equivalent to type IIB superstring theory on $AdS_5 \times \mathcal{S}^5$. In the large N limit with $g_{YM}^2 N$ fixed but large, super Yang-Mills is strongly coupled whereas the string theory is weakly coupled and is approximated well enough by the corresponding (classical) supergravity theory. Some piece of evidence that the conjecture might be true comes from entropy and symmetry considerations. In particular,

1. the Bekenstein-Hawking entropy of a 10-dimensional supergravity black hole has a T^3 dependence, where T is the temperature. Using arguments of statistical mechanics, one finds that a $U(N)\,\mathcal{N} = 4$ supermultiplet, which consists of a gauge field, $6N^2$ massless scalars and $4N^2$ Weyl fermions, has the same T^3 dependence.

2.
   - $SO(2,4)$ is the isometry group of AdS_5. This is the conformal group in 4 spacetime dimensions.
   - $SU(4) \sim SO(6)$ is the isometry group of S^5. This is the R-symmetry of the $\mathcal{N} = 4$ Super Yang-Mills theory.
   - $SU(2,2|4)$ is the full isometry group of $AdS_5 \times \mathcal{S}^5$. This is also the $\mathcal{N} = 4$ superconformal symmetry.

Elaborating the idea of the duality, a later proposal [70] suggests that there exists a precise correspondence between conformal field theory observables and the supergravity theory. Namely, correlation functions in the d-dimensional conformal field theory are given by the dependence of the supergravity partition function on the asymptotic behavior of the fields at spatial infinity (boundary) of AdS_{d+1}. In particular, for a massless scalar field $\phi(x)$, the proposed relation takes the form

$$Z_{\text{gravity}}[\phi_0(x)] = \langle \exp \int_{S^{d-1}} \phi_0(x)\, \mathcal{O}(x) \rangle_{\text{CFT}} \tag{4.1}$$

where $Z_{\text{gravity}}[\phi_0(x)]$ is the partition function of the supergravity action at $\phi(x) = \phi_0(x)$, $\phi_0(x)$ is the value of the field $\phi(x)$ at the boundary of AdS_{d+1}, S_{d-1} is identified as boundary of AdS_{d-1}, $\mathcal{O}(x)$ is the CFT operator and $\phi_0(x)$ is considered to be coupled to $\mathcal{O}(x)$ via a coupling $\int_{S^{d-1}} \phi_0(x)\mathcal{O}(x)$. Subsequently, one can evaluate correlation functions of CFT operators [71]. That is, considering the leading contribution on the gravitational partition function as $Z_{\text{gravity}}[\phi_0(x)] = e^{-S_{\text{gravity}}[\phi_0(x)]}$, it is deduced from (4.1) that the correlation functions of CFT operators are given by

$$\langle \mathcal{O}(x_1)\, \mathcal{O}(x_2) \ldots \mathcal{O}(x_n) \rangle = \frac{\delta}{\delta\phi_0(x_1)} \frac{\delta}{\delta\phi_0(x_2)} \cdots \frac{\delta}{\delta\phi_0(x_n)} Z_{\text{gravity}}[\phi_0(x)] \,. \tag{4.2}$$

Thus, the fields $\phi_0(x_1)$, ..., $\phi_0(x_n)$ can be interpreted as sources and the correlation functions $\mathcal{O}(x_1)$, ..., $\mathcal{O}(x_n)$ as holographic response functions to the corresponding sources. The above relation provides a calculational tool that can be applied in numerous examples.

Throughout this thesis, the $AdS - CFT$ conjecture is applied from the gravity theory side. Additionally, this gravity theory is regarded as classical. In particular, considering the case of conformal gravity in $(A)dS_4$ and imposing a particular set of boundary conditions, holographic response functions of the dual field theory are deduced, following the logic of (4.2).

Part II

THE CASE OF CONFORMAL GRAVITY

5

ASYMPTOTICALLY (A)DS BOUNDARY CONDITIONS AND HOLOGRAPHY

Throughout this chapter, conformal gravity is considered in a holographic content. Starting with the general setup in 5.1, the boundary value problem of the theory is investigated. Then, in 5.2, adopting a Fefferman-Graham type expansion for the boundary metric, the holographic response functions are evaluated. Also, under some proposed set of asymptotic boundary conditions, the variational principle is examined. Then, the analysis continues in 5.3 by applying the results found so far to three solutions of the theory. Finally, in 5.4, the asymptotic symmetry algebras of the dual field theory are constructed, those which are allowed by the proposed set of asymptotic boundary conditions.

5.1 GENERAL SETUP AND BOUNDARY VALUE PROBLEM

The conformal gravity action is

$$S = \alpha_{\mathrm{CG}} \int_M d^4x \, \sqrt{|g|} \, C_{abcd} \, C^{abcd} \tag{5.1}$$

where

$$C_{abcd} = R_{abcd} - g_{a[c} \, R_{b]d} + g_{b[c} \, R_{d]a} + \frac{1}{3} R \, g_{a[c} \, g_{d]b} \tag{5.2}$$

is the Weyl tensor and α_{CG} is a dimensionless coupling constant. This is the only free parameter of the theory and here it is set to unity. Using the analytic expression (5.2), the action (5.1) can be rewritten in an equivalent form which consists of a Ricci-squared and a total derivative term as

$$S = \int_M d^4x \, \sqrt{|g|} \left[2R_{ab}R^{ab} - \frac{2}{3}R^2 + 32\pi^2 \chi(M) \right] . \tag{5.3}$$

The total derivative term $\chi(M)$ is the Euler density

$$\chi(M) = \frac{1}{32\pi^2} \left(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2 \right) . \tag{5.4}$$

In order to examine the boundary value problem of the action (5.3) and later on its holographic response functions, it is convenient to decompose the spacetime (M, g_{ab}) into hypersurfaces ∂M located at constant coordinate ρ. This requires the choice of a normal vector field n^a to ∂M, which can be either timelike or spacelike. Then, the hypersurfaces are either spacelike or timelike respectively. The purpose here is to keep track of both these cases because the aim is to perform an $(A)dS - CFT$ analysis: the normal vector is timelike in the dS spacetime (like in the standard ADM foliation) and in the AdS spacetime it is spacelike. Thus, the normal vector is chosen to have the following norm

$$n^a n^b g_{ab} = -\sigma \tag{5.5}$$

where $\sigma = \pm 1$. When $\sigma = 1$, the normal vector is timelike and refers to the dS_4 case and when $\sigma = -1$ the normal vector is spacelike and refers to the AdS_4 case. With this notation, the induced metric γ_{ab} on each ∂M takes the form

$$\gamma_{ab} = g_{ab} + \sigma n_a n_b \, . \tag{5.6}$$

Since the task is to implement an $(A)dS - CFT$ analysis, it is natural to consider a 4-dimensional metric ansatz such that it coincides with the $(A)dS_4$ spacetime in some appropriate limit. This limit is considered to be the conformal boundary of M at $\rho = 0$, which coincides with the hypersurface ∂M. Such a metric ansatz can be described by the line element

$$ds^2 = \frac{\ell^2}{\rho^2}\Big(-\sigma d\rho^2 + \gamma_{ij}(\rho, x^k) dx^i dx^j \Big) \tag{5.7}$$

with $0 < \rho \ll \ell$ and ℓ is a length scale, which in the case of Einstein gravity is related to the cosmological constant as $\Lambda = \frac{3\sigma}{\ell^2}$. The above line element reduces to $(A)dS_4$ spacetime (with ℓ then being the radius of curvature of AdS_4) when the induced metric $\gamma_{ij}(\rho, x^k)$ is expanded in a generalized Fefferman-Graham form, as it will be shown in the next section of the holographic analysis. The above spacetime satisfies (5.5) and (5.6) with

$$n^a = -\frac{\rho}{\ell}\delta^a_\rho \tag{5.8}$$

$$\gamma_{ij} = g_{ij} \tag{5.9}$$

where n^a and $\gamma_{ij} = \gamma_{ij}(\rho, x^k)$ being the normal vector and the induced metric on ∂M respectively. Under these conventions, the extrinsic curvature tensor is defined as

$$K_{ij} = -\frac{\sigma}{2}\pounds_{n^a}\gamma_{ij}\,. \tag{5.10}$$

The boundary value problem is treated by varying the action (5.3) with respect to the induced metric γ_{ij} and the extrinsic curvature K_{ij} independently. Thus, the boundary conditions consist of keeping the metric and the extrinsic curvature on the boundary ∂M fixed, i.e. setting

$$\delta\gamma_{ij}|_{\partial M} = 0 \tag{5.11}$$

$$\delta K_{ij}|_{\partial M} = 0\,. \tag{5.12}$$

According to the holographic analysis of the next section, it turns out that the action (5.3) has a well-defined variational principle when a boundary term is added. Therefore, adding this boundary term at this stage the action (5.3) takes the form

$$\begin{aligned} S = &\int_M d^4x\sqrt{|g|}\Big[2R_{ab}R^{ab} - \frac{2}{3}R^2\Big] + 32\pi\chi(M) + \int_{\partial M} d^3x\sqrt{|\gamma|}\Big[-8\sigma\mathcal{G}^{ij}K_{ij} + \frac{4}{3}K^2 - 4KK^{ij}K_{ij} \\ &+ \frac{8}{3}K^{ij}K^l_jK_{il}\Big]\,. \end{aligned} \tag{5.13}$$

Then, the first variation of the above on-shell action reads

$$\delta S|_{\text{on-shell}} = \int_{\partial M} d^3x\sqrt{|\gamma|}\Big[p^{ij}\delta\gamma_{ij} + P^{ij}\delta K_{ij}\Big] \tag{5.14}$$

where

$$\begin{aligned} p^{ij} = &\frac{\sigma}{4}\big(\gamma^{ij}K^{kl} - \gamma^{kl}K^{ij}\big)f_{kl} + \frac{\sigma}{4}f^\rho_\rho\big(\gamma^{ij}K - K^{ij}\big) - \frac{1}{2}\gamma^{ij}D_k(n_\rho f^k_\rho) + \frac{1}{2}D^i(n_\rho f^{\rho j}) - \frac{1}{4}\big(\gamma^{ik}\gamma^{jl} \\ &- \gamma^{ij}\gamma^{kl}\big)\pounds_n f_{kl} + \sigma\Big(2K\mathcal{R}^{ij} - 4K^{ik}\mathcal{R}^j_k + 2\gamma^{ij}K_{kl}K^{kl} - \gamma^{ij}K\mathcal{R} + 2D^2K^{ij} - 4D^iD_kK^{kj} \\ &+ 2D^iD^jK + \gamma^{ij}\big(D_kD_lK^{kl} - D_kD^kK\big)\Big) + \frac{2}{3}\gamma^{ij}K^k_mK^{lm}K_{kl} - 4K^{ik}K^{jl}K_{kl} + K^{ij}K^{kl}K_{kl} \\ &+ \frac{1}{3}\gamma^{ij}K^3 - 2K^{ij}K^2 - \gamma^{ij}KK_{kl}K^{kl} + 4K^i_kK^{jk} + i \leftrightarrow j \end{aligned} \tag{5.15}$$

and

$$P^{ij} = -8\sigma\mathcal{G}^{ij} - \sigma\big(f^{ij} - \gamma^{ij}f^k_k\big) + 4\gamma^{ij}\big(K^2 - K_{kl}K^{kl}\big) - 8KK^{ij} + 8K^i_kK^{jk} \tag{5.16}$$

and finally f_{ab} is proportional to the 4-dimensional Schouten tensor

$$f_{ab} = -4\big(R_{ab} - \frac{1}{6}g_{ab}R\big)\,. \tag{5.17}$$

Therefore, the action (5.13) and the boundary conditions (5.11), (5.12) constitute a well-posed boundary value problem.

5.2 HOLOGRAPHIC ANALYSIS

After verifying the fact that the action (5.1) has a well-defined boundary value problem, the discussion continues with the holographic analysis. Namely, it is examined whether the first variation of the on-shell action (5.14) vanishes for some appropriate asymptotic boundary conditions. It turns out that it indeed vanishes. Thus the whole setup yields a well-defined variational principle. Additionally, the holographic response functions are calculated and it is shown that they are finite. Thus there is no need to add holographic counterterms.

5.2.1 *Generalized Fefferman-Graham expansion*

The hypersurface ∂M that is characterized by (5.8), (5.9) is identified with the conformal boundary of M, which is located at $\rho = 0$. Close to this conformal boundary, the 3-dimensional induced metric γ_{ij} is assumed to have a generalized Fefferman-Graham or Starobinsky expansion [72] of the form

$$\gamma_{ij} = \gamma_{ij}^{(0)} + \frac{\rho}{\ell}\gamma_{ij}^{(1)} + \frac{\rho^2}{\ell^2}\gamma_{ij}^{(2)} + \frac{\rho^3}{\ell^3}\gamma_{ij}^{(3)} + \dots \tag{5.18}$$

where $\gamma_{ij}^{(0)}$ is the boundary metric. Adopting this asymptotic expansion, the line element (5.7) indeed yields as $\rho \to 0$ the Poincare patch of $(A)dS_4$ for $\gamma_{ij}^{(0)} = \eta_{ij}$.

The next step in the holographic analysis consists of finding the asymptotic expansion of all the tensors that appear in the first variation of the on-shell action (5.14). Namely, to insert the line element (5.7) and the asymptotic expansion of the 3-dimensional induced metric (5.18) into the expression of (5.14). Consequently, it is necessary to express all the relevant tensors in terms of $\gamma_{ij}^{(0)}$, $\gamma_{ij}^{(1)}$, $\gamma_{ij}^{(2)}$, … and in terms of the curvature tensors of the boundary metric $\gamma_{ij}^{(0)}$, i.e. in terms of $R_{ij}^{(0)}$, $R^{(0)}$, $G_{ij}^{(0)}$, …. This calculation is performed with the *xAct* package of Mathematica [73] and all the results are given in the Appendix C.2. Here, due to their significance in the rest of the analysis, the asymptotic expansion of the electric and magnetic part of the Weyl tensor is presented explicitly: adopting the spacetime split (5.8), (5.9), the Weyl tensor can be decomposed into an electric (E_{ij}) and a magnetic part (B_{ijk}) as

$$E_{ij} \equiv n_a n^b C^a{}_{ibj} = C^{\rho}_{i\rho j} = \frac{1}{2}\left(\gamma_i^m \gamma_j^l - \frac{1}{3}\gamma_{ij}\gamma^{ml}\right)\left(\sigma\mathcal{R}_{ml} + K_{ml}K - \pounds_{n^\rho} K_{ml}\right) \tag{5.19}$$

$$B_{ijk} \equiv n_a C^a{}_{ijk} = C^{\rho}_{ijk} = 2D_{[i}K_{j]k} + D_l K^l_{[i}\gamma_{j]k} - D_{[i}K\gamma_{j]k}\,. \tag{5.20}$$

After inserting the line element (5.7) and the generalized Fefferman-Graham expansion of the boundary metric (5.18), their asymptotic expansion takes compactly the form

$$E_{ij} = E_{ij}^{(2)} + \frac{\rho}{\ell}E_{ij}^{(3)} + \dots \tag{5.21}$$

$$B_{ijk} = \frac{\ell}{\rho}B_{ijk}^{(1)} + B_{ijk}^{(2)} + \dots \tag{5.22}$$

The leading order terms of the above expansions, namely the orders $\mathcal{O}(\rho^{-2})$ and $\mathcal{O}(\rho^{-1})$ of the electric part and the order $\mathcal{O}(\rho^{-2})$ of the magnetic part, all vanish, i.e. $E_{ij}^{(0)} = E_{ij}^{(1)} = B_{ijk}^{(0)} = 0$. Also, the term $B_{ijk}^{(2)}$ is not of relevance for the present calculation. The non-vanishing terms $E_{ij}^{(2)}$ and $E_{ij}^{(3)}$ of the electric part (5.21) take the form

$$E_{ij}^{(2)} = -\frac{1}{2\ell^2}\psi_{ij}^{(2)} + \frac{\sigma}{2}(\mathcal{R}_{ij}^{(0)} - \frac{1}{3}\gamma_{ij}^{(0)}\mathcal{R}) + \frac{1}{8\ell^2}\gamma^{(1)}\psi_{ij}^{(1)} \tag{5.23}$$

$$\begin{aligned} E_{ij}^{(3)} = &-\frac{3}{4\ell^2}\psi_{ij}^{(3)} - \frac{1}{12\ell^2}\gamma_{ij}^{(0)}\psi_{kl}^{(1)}\psi^{kl}_{(2)} - \frac{1}{16\ell^2}\psi_{ij}^{(1)}\psi_{kl}^{(1)}\psi^{kl}_{(1)} - \frac{\sigma}{12}(\mathcal{R}^{(0)}\psi_{ij}^{(1)} - \gamma_{ij}^{(0)}\mathcal{R}_{kl}^{(0)}\psi^{kl}_{(1)} \\ &+ \gamma_{ij}^{(0)}D_l D_k\psi^{kl}_{(1)} + \frac{3}{2}D_k D^k\psi_{ij}^{(1)} - 3D_k D_i\psi_j^{(1)k}) + \frac{1}{24\ell^2}E_{ij}^{\gamma} \end{aligned} \tag{5.24}$$

with

$$E^{\gamma}_{ij} \equiv \gamma_{(1)}\big(3\psi^{(2)}_{ij} + \frac{1}{2}\gamma^{(0)}_{ij}\psi^{(1)}_{kl}\psi^{kl}_{(1)} - \gamma_{(1)}\psi^{(1)}_{ij}\big) + 5\gamma_{(2)}\psi^{(1)}_{ij} - \sigma\ell^2\big(D_jD_i\gamma_{(1)} - \frac{1}{3}\gamma^{(0)}_{ij}D^kD_k\gamma_{(1)}\big) \quad (5.25)$$

and

$$\gamma^{(n)}_{ij} = \frac{1}{3}\gamma^{(0)}_{ij}\gamma^{(n)} + \psi^{(n)}_{ij} \quad (5.26)$$

with $n = 1, 2, 3$. Finally, the non-vanishing term $B^{(1)}_{ijk}$ of the magnetic part (5.22) is

$$B^{(1)}_{ijk} = \frac{1}{2\ell}\big(D_j\psi^{(1)}_{ik} - \frac{1}{2}\gamma^{(0)}_{ij}D^l\psi^{(1)}_{kl}\big) - j \leftrightarrow k\,. \quad (5.27)$$

At this point, the following observation is made: the Weyl tensor is traceless in any pair of its indices, i.e. $\gamma^{ij}E_{ij} = 0$ and $\gamma^{ij}B_{ijk} = 0$. Consequently, this holds at all orders of the asymptotic expansion of these expressions, i.e.

$$\gamma^{ij}_{(0)}E^{(2)}_{ij} = 0 \quad (5.28)$$

$$\gamma^{ij}_{(0)}E^{(3)}_{ij} = 3\psi^{ij}_{(1)}E^{(2)}_{ij} \quad (5.29)$$

$$\gamma^{ij}_{(0)}B^{(1)}_{ijk} = 0\,. \quad (5.30)$$

These relations can also be verified by a straightforward calculation, using the analytic expressions (5.23), (5.24), (5.27). From now on they are referred to as trace conditions.

5.2.2 *Holographic response functions*

Using the generalized Fefferman-Graham expansion (5.18) and the corresponding asymptotic expansions of all the relevant tensors in (5.15), (5.16), the on-shell variation (5.14) for a compact region $\rho_c \leq \rho$ becomes

$$\delta S|_{\text{on-shell}} = \int_{\partial M} d^3x\sqrt{|\gamma^{(0)}|}\left[\tau^{ij}\delta\gamma^{(0)}_{ij} + \mathcal{P}^{ij}\delta\gamma^{(1)}_{ij}\right] \quad (5.31)$$

with

$$\tau_{ij} = \sigma\Big[\frac{2}{\ell}\big(E^{(3)}_{ij} - \frac{1}{3}E^{(2)}_{ij}\gamma^{(1)}\big) - \frac{4}{\ell}E^{(2)}_{ik}\psi^{k(1)}_{j} + \frac{1}{\ell}\gamma^{(0)}_{ij}E^{(2)}_{kl}\psi^{kl}_{(1)} + \frac{1}{2\ell^3}\psi^{(1)}_{ij}\psi^{(1)}_{kl}\psi^{kl}_{(1)}$$
$$- \frac{1}{\ell^3}\psi^{(1)}_{kl}\big(\psi^{(1)k}_{i}\psi^{(1)l}_{j} - \frac{1}{3}\gamma^{(0)}_{ij}\psi^{(1)k}_{m}\psi^{lm}_{(1)}\big)\Big] - D^kB^{(1)}_{ijk} + i \leftrightarrow j \quad (5.32)$$

$$\mathcal{P}_{ij} = -\frac{4\sigma}{\ell}E^{(2)}_{ij} \quad (5.33)$$

as $\rho_c \to 0$. The metric variations $\delta\gamma^{(0)}_{ij}$ and $\delta\gamma^{(1)}_{ij}$ are part of the specification of the proposed boundary conditions, which are analyzed in the following section. For the moment it is sufficient to obtain that they are finite as $\rho_c \to 0$. The tensors τ^{ij} and $\mathcal{P}^{ij}$ are interpreted as the holographic response functions conjugate to the sources $\gamma^{(0)}_{ij}$ and $\gamma^{(1)}_{ij}$ respectively. One major conclusion is made at this point: according to their analytic expression above, the holographic response functions are finite as $\rho_c \to 0$. Thus, they do not require addition of holographic counterterms. Furthermore, one obtains the following facts: firstly, the response function τ^{ij} (5.32) is proportional to the Brown-York stress tensor of the Einstein action [74], [75]. Secondly, the next-to leading order term $\gamma^{(1)}_{ij}$ exhibits partial masslessness in the sense of [76], [77] when plugged into the linearized equations of motion of the conformal gravity action (5.1) around an $(A)dS_4$ background. Therefore, the response function $\mathcal{P}^{ij}$ (5.33) which has $\gamma^{(1)}_{ij}$ as its source is called the “partially massless response" (PMR).

Additionally, the holographic response functions (5.32), (5.33) satisfy certain relations due to the trace conditions (5.28), (5.29), (5.30). In particular, taking their trace one finds that

$$\gamma^{ij}_{(0)}\tau_{ij} = \frac{\sigma}{\ell}\psi^{(1)}_{ij}E^{ij}_{(2)} \tag{5.34}$$

$$\gamma^{ij}_{(0)}P_{ij} = 0 \tag{5.35}$$

where use of (5.28), (5.29), (5.30) has been made. Substituting (5.33), the first condition (5.34) becomes

$$\gamma^{ij}_{(0)}\tau_{ij} + \frac{1}{2}\psi^{ij}_{(1)}\mathcal{P}_{ij} = 0\,. \tag{5.36}$$

The above expressions (5.35) and (5.36) are also referred to as trace conditions and play a crucial role in the forthcoming section.

5.2.3 *Asymptotic boundary conditions and variational principle*

The proposed asymptotic boundary conditions consist of fixing the leading ($\gamma^{(0)}_{ij}$) and the next-to leading ($\gamma^{(1)}_{ij}$) order terms in the generalized Fefferman-Graham expansion (5.18) as follows

$$\delta\gamma^{(0)}_{ij}|_{\partial M} = 2\lambda\gamma^{(0)}_{ij} \tag{5.37}$$

$$\delta\gamma^{(1)}_{ij}|_{\partial M} = \lambda\gamma^{(1)}_{ij} \tag{5.38}$$

where λ is a regular function on ∂M. The higher order terms in the asymptotic expansion (5.18) are allowed to vary freely, i.e. $\delta\gamma^{(n)}_{ij}|_{\partial M} \neq 0$ for $n \geq 2$.

It is now demonstrated that the first variation of the on-shell action (5.31) vanishes for the proposed asymptotic boundary conditions (5.37), (5.38). Indeed, inserting them in the on-shell variation (5.31) one finds

$$\delta S|_{\text{on-shell}} = \int_{\partial M} d^3x\sqrt{|\gamma^{(0)}|}\left[\tau^{ij}\delta\gamma^{(0)}_{ij} + \mathcal{P}^{ij}\delta\gamma^{(1)}_{ij}\right] \tag{5.39}$$

$$= \int_{\partial M} d^3x\sqrt{|\gamma^{(0)}|}\,\lambda\left[2\,\tau^{ij}\gamma^{(0)}_{ij} + \mathcal{P}^{ij}\gamma^{(1)}_{ij}\right] \tag{5.40}$$

$$= \int_{\partial M} d^3x\sqrt{|\gamma^{(0)}|}\,\lambda\left[2\,\tau^{ij}\gamma^{(0)}_{ij} + \mathcal{P}^{ij}\psi^{(1)}_{ij} + \frac{1}{3}\gamma^{(0)}_{ij}\mathcal{P}^{ij}\,\gamma_{(1)}\right] \tag{5.41}$$

$$= 0 \tag{5.42}$$

with use of the trace conditions (5.35), (5.36). Thus it is concluded that the action (5.1) and the asymptotic boundary conditions (5.37), (5.38) constitute a well-defined variational principle.

5.2.4 *An alternative formulation*

An alternative formulation is obtained when performing a Legendre transformation of the action (5.1) which exchanges the role of the PMR and its source. Namely, by adding a Weyl invariant boundary term as

$$\tilde{S} = S + \int_{\partial M} d^3x\sqrt{|\gamma|}K^{ij}E_{ij} \tag{5.43}$$

where E_{ij} is the electric part of the Weyl tensor (5.19). The first on-shell variation of the above action for a compact region $\rho_c \leq \rho$ takes the form

$$\delta\tilde{S}|_{\text{on-shell}} = \int_{\partial M} d^3x\sqrt{|\gamma^{(0)}|}\left[\tilde{\tau}^{ij}\delta\gamma^{(0)}_{ij} + \tilde{\mathcal{P}}^{ij}\delta E^{(2)}_{ij}\right] \tag{5.44}$$

with

$$\tilde{\tau}_{ij} = \tau_{ij} + \frac{2\sigma}{\ell} E^{(2)}_{lm} \psi^{lm}_{(1)} \gamma^{(0)}_{ij} + \frac{8\sigma}{3\ell} E^{(2)}_{ij} \gamma_{(1)} - \frac{8\sigma}{\ell} E^{(2)}_{m(i} \psi^{(1)m}_{j)} \tag{5.45}$$

$$\tilde{\mathcal{P}}_{ij} = \frac{4\sigma}{\ell} \gamma^{(1)}_{ij} \,. \tag{5.46}$$

The holographic response functions $\tilde{\tau}^{ij}$ and $\tilde{\mathcal{P}}^{ij}$ are now conjugate to the sources $\gamma^{(0)}_{ij}$ and $E^{(2)}_{ij}$ respectively. Indeed, the PMR has now exchanged its source, from $\gamma^{(1)}_{ij}$ to $E^{(2)}_{ij}$. Finally, the holographic response functions (5.45), (5.46) are finite as $\rho_c \to 0$ and thus, like before, no additional counterterms are required.

5.2.5 *Currents and charges*

A boundary diffeomorphism $x^{i\prime} = x^i + \xi^i_{(0)}$ on ∂M produces the following changes in the boundary metric $\gamma^{(0)}_{ij}$ and in the next to leading order term $\gamma^{(1)}_{ij}$:

$$\delta\gamma^{(0)}_{ij} = \pounds_{\xi^k_{(0)}} \gamma^{(0)}_{ij} = 2\, D_{(i}\xi^{(0)}_{j)} \tag{5.47}$$

$$\delta\gamma^{(1)}_{ij} = \pounds_{\xi^k_{(0)}} \gamma^{(1)}_{ij} = \xi^k_{(0)}\, D_k \gamma^{(1)}_{ij} + 2\, D_{(i}\xi^k_{(0)}\, \gamma^{(1)}_{j)k} \,. \tag{5.48}$$

Inserting the above transformations in the on-shell variation (5.31) and performing a partial integration, one finds

$$\begin{aligned}
\delta S|_{\text{on-shell}} &= \int_{\partial M} d^3x \sqrt{|\gamma^{(0)}|} \Big[D_i \Big(\big(2\tau^i_j + 2\mathcal{P}^{ij}\gamma^{(1)}_{jk}\big)\, \xi^k_{(0)} \Big) + \xi^k_{(0)} \Big(- D_i \big(2\tau^i_k + 2\mathcal{P}^{ij}\gamma^{(1)}_{jk}\big) \\
&\quad + \mathcal{P}^{ij} D_k \gamma^{(1)}_{ij} \Big) \Big] \\
&= \int_{\partial M} d^3x \sqrt{|\gamma^{(0)}|} D_i\, J^i
\end{aligned} \tag{5.49}$$

with

$$J^i = (2\tau^i_j + 2\mathcal{P}^{il}\gamma^{(1)}_{jl})\xi^j \tag{5.50}$$

being a conserved current if and only if

$$D_i \big(2\tau^i_k + 2\mathcal{P}^{ij}\gamma^{(1)}_{jk}\big) = \mathcal{P}^{ij} D_k \gamma^{(1)}_{ij} \,. \tag{5.51}$$

From the conserved current (5.50), a modified stress tensor T^i_j can be identified as $J^i \equiv T^i_j\, \xi^j_{(0)}$ [78] with

$$T^i_j = 2\tau^i_j + 2\mathcal{P}^{il}\gamma^{(1)}_{jl} \,. \tag{5.52}$$

Then, condition (5.51) implies that this modified stress tensor is not conserved but satisfies

$$D_i T^i_j = \mathcal{P}^{ik} D_j \gamma^{(1)}_{ik} \,. \tag{5.53}$$

Finally, one can construct the conserved charge that generates asymptotic symmetries. Considering the case of $\sigma = -1$ such that the boundary surface ∂M is timelike, the asymptotic charge becomes

$$Q[\xi^k_{(0)}] = \int_C d^2x \sqrt{h} u_i J^i \tag{5.54}$$

where C is a constant time surface, h is the metric on C and u^i is the future-pointing normal vector on C.

5.3 APPLICATIONS

The previous results are applied to three examples which are solutions of the conformal gravity action (5.1). In particular, the holographic response functions (5.32), (5.33) are calculated for each of these solutions.

5.3.1 *Solutions for which* $\gamma^{(1)}_{ij} = 0$

The first example that is considered are the solutions that have $\gamma^{(1)}_{ij} = 0$. Among these are the asymptotically $(A)dS_4$ solutions of Einstein gravity. It is found that

$$E^{(2)}_{ij} = 0 \tag{5.55}$$

$$B^{(1)}_{ijk} = 0\,. \tag{5.56}$$

The first expression is implied by the asymptotic equations of motion of the action (5.1) and the second one results from (5.22). Therefore, the holographic response functions (5.32), (5.33) become

$$\tau_{ij} = \frac{4\sigma}{\ell} E^{(3)}_{ij} \tag{5.57}$$

$$\mathcal{P}_{ij} = 0\,. \tag{5.58}$$

Subsequently, the trace condition (5.36) renders τ_{ij} traceless. Additionally, the modified stress tensor (5.53) is now conserved. Thus, one recovers the traceless and conserved Brown-York stress tensor of Einstein gravity [75], also in agreement with later analysis [38].

5.3.2 *The MKR solution*

The Mannheim-Kazanas-Riegert (MKR) spacetime [21], [22] is a static, spherically symmetric solution of the action (5.1) and it is the analogue of the $Schwarzschild - (A)dS_4$ solution of Einstein gravity. The additional parameter that appears except the mass and the cosmological constant is now the Rindler acceleration. The corresponding line element takes the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \tag{5.59}$$

where

$$f(r) = \sqrt{1 - 12\alpha M} - \frac{2M}{r} + 2\alpha r - \frac{\Lambda}{3} r^2 \tag{5.60}$$

and M, Λ, α are the mass, the cosmological constant and the Rindler acceleration respectively. Adopting the decomposition of a spacetime g_{ab} into hypersurfaces according to (5.5), (5.6), (5.7) for the above line element, setting $\sigma = -1$ for concreteness and then, expressing it in the generalized Fefferman-Graham form (5.18), the leading term $\gamma^{(0)}_{ij}$ is found to be

$$\gamma^{(0)}_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \ell^2 & 0 \\ 0 & 0 & \ell^2 \sin^2\theta \end{pmatrix} \tag{5.61}$$

with ℓ being the radius of curvature of AdS_4, whereas the next to leading order term $\gamma^{(1)}_{ij}$ takes the form

$$\gamma^{(1)}_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2\alpha\ell^3 & 0 \\ 0 & 0 & -2\alpha\ell^3 \sin^2\theta \end{pmatrix}\,. \tag{5.62}$$

The higher order terms of the induced metric expansion are

$$\gamma_{ij}^{(2)} = \begin{pmatrix} \alpha^2\ell^2 - \sqrt{1-12\alpha M} & 0 & 0 \\ 0 & 3\alpha^2\ell^4 - \ell^2\sqrt{1-12\alpha M} & 0 \\ 0 & 0 & 3\alpha^2\ell^4 - \ell^2\sqrt{1-12\alpha M}\sin^2\theta \end{pmatrix} \tag{5.63}$$

$$\gamma_{ij}^{(3)} = \begin{pmatrix} \frac{8M}{\ell} & 0 & 0 \\ 0 & -3\alpha^3\ell^5 + 4\ell M + 3\alpha\ell^3\sqrt{1-12\alpha M} & 0 \\ 0 & 0 & -3\alpha^3\ell^5 + 4\ell M + 3\alpha\ell^3\sqrt{1-12\alpha M}\sin^2\theta \end{pmatrix} . \tag{5.64}$$

Using the above values for the metric coefficients, the holographic response functions (5.32), (5.33) become

$$\tau_{ij} = -\frac{8m}{\ell^2} r_{ij} + 8\frac{\alpha\, a_M}{\ell^2} q_{ij} \tag{5.65}$$

$$\mathcal{P}_{ij} = \frac{8a_M}{\ell^2} r_{ij} \tag{5.66}$$

with

$$r_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} , \quad q_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{5.67}$$

and

$$a_M = \frac{1 - \sqrt{1-12\alpha M}}{6} \tag{5.68}$$

$$m = \frac{M}{\ell^2} . \tag{5.69}$$

It is noticed that when the Rindler acceleration vanishes, i.e. $\alpha = a_M = 0$, the PMR (5.66) vanishes and the response function (5.65) reduces to the previous case (5.57). Therefore, the *Schwarzschild* $-$ $(A)dS_4$ solution of Einstein gravity is recovered in this case. On the other hand, for non vanishing Rindler acceleration α, the PMR (5.66) is linear in α. Additionally, when $a_M \ll 1$ the trace of the response function (5.65) is quadratic in α. From these observations, it is deduced that the Rindler acceleration in the MKR solution can be interpreted as coming from a partially massless graviton condensate.

The Killing vectors of the MKR solution (5.59) are explicitly analyzed in section 5.4.4. Here, only the Killing vector of time translations $\xi^k_{(0)} = \partial_t$ will be used. Using the normalization $\alpha_{CG} = \frac{1}{64\pi}$ for the dimensionless coupling constant of the theory, the asymptotic charge (5.54) associated with the asymptotic killing vector ∂_t takes the form

$$Q[\partial_t] = m - \alpha a_M . \tag{5.70}$$

Then, the Wald entropy [79] is

$$S = \frac{A_h}{4\ell^2} \tag{5.71}$$

where

$$A_h = 4\pi r_h^2 \tag{5.72}$$

is the area of the event horizon $f(r_h) = 0$. It is noteworthy that the entropy obeys an area law despite the fact that the action (5.1) is a higher-derivative theory.

5.3.3 *The Rindler-Kerr-(A)dS solution*

The Rindler-Kerr-(A)dS spacetime [80] is a rotating black hole solution of the action (5.1) with vanishing mass parameter. Its line element takes the form

$$ds^2 = \rho^2\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(a dt - (r^2+a^2)\frac{d\phi}{\Xi}\right)^2 - \frac{\Delta_r}{\rho^2}\left(dt - a\sin^2\theta\frac{d\phi}{\Xi}\right)^2 \tag{5.73}$$

where

$$\Delta_r = (r^2 + a^2)(1 - \frac{1}{3}\Lambda r^2) - 2\mu r^3 \tag{5.74}$$

$$\Delta_\theta = 1 + \frac{1}{3}\Lambda a^2 \cos^2\theta \tag{5.75}$$

$$\Xi = 1 + \frac{1}{3}\Lambda a^2 \tag{5.76}$$

$$\rho^2 = r^2 + a^2 \cos^2\theta \tag{5.77}$$

with a being the rotation parameter, μ the Rindler acceleration and Λ the cosmological constant. Expressing the above line element in the generalized Fefferman-Graham form (5.18) and setting $\sigma = -1$ for concreteness, it is found that the boundary metric is

$$\gamma_{ij}^{(0)} = \begin{pmatrix} -1 & 0 & \frac{a\sin^2\psi}{1-\frac{a^2}{\ell^2}} \\ 0 & \frac{\ell^4}{\ell^2 - a^2\cos^2\psi} & 0 \\ \frac{a\sin^2\psi}{1-\frac{a^2}{\ell^2}} & 0 & \frac{\ell^4\sin^2\psi}{\ell^2-a^2} \end{pmatrix} \tag{5.78}$$

and the leading term $\gamma_{ij}^{(0)}$ is

$$\gamma_{ij}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2\mu\ell^2}{1-\frac{a^2}{\ell^2}\cos^2\psi} & 0 \\ 0 & 0 & \frac{2\mu(1-\frac{a^2}{\ell^2}\cos^2\psi)\sin^2\psi}{(1-\frac{a^2}{\ell^2})^2} \end{pmatrix}. \tag{5.79}$$

The higher order terms of the induced metric expansion are also calculated but they are not presented here explicitly. Consequently, the holographic response functions (5.32), (5.33) take the form

$$\tau_{ij} = \begin{pmatrix} \frac{24\frac{a^2\mu}{\ell}\cos^2\psi}{\ell^3} & 0 & -\frac{12a\mu(\cos^2\psi-1)\left(3\frac{a^2}{\ell^2}\cos^2\psi-1\right)}{a^2-\ell^2} \\ 0 & -\frac{12\frac{a^2\mu}{\ell}\cos^2\psi}{\frac{a^2}{\ell}\cos^2\psi-\ell} & 0 \\ -\frac{12a\mu(\cos^2\psi-1)\left(3\frac{a^2}{\ell^2}\cos^2\psi-1\right)}{a^2-\ell^2} & 0 & \frac{12\frac{a^2\mu}{\ell}(\cos^2\psi-1)\left(5\frac{a^2}{\ell^2}\cos^4\psi-4\frac{a^2}{\ell^2}\cos^2\psi-3\cos^2\psi+2\right)}{\ell(\frac{a}{\ell}-1)^2(\frac{a}{\ell}+1)^2} \end{pmatrix} \tag{5.80}$$

$$\mathcal{P}_{ij} = 0\,. \tag{5.81}$$

The vanishing of the PMR (5.81) implies that a non-zero Rindler term $\mu \neq 0$, or equivalently $\gamma_{ij}^{(1)} \neq 0$ from (5.79), is necessary but not sufficient for a non-zero PMR.

The Killing vectors of the Rindler-Kerr-(A)dS solution (5.73) are explicitly presented in section 5.4.5. Here, 2 Killing vectors will be used, namely the one of time translations $\xi^k_{(0)} = \partial_t$ and the one of rotations $\xi^k_{(0)} = \partial_\phi$. Using again the normalization $\alpha_{CG} = \frac{1}{64\pi}$ like in the case of the MKR solution, the asymptotic charges (5.54) are evaluated. The first charge associated with the asymptotic killing vector ∂_t is the energy and takes the form

$$E = -\frac{a^2\mu}{\ell^2\Xi^2} \tag{5.82}$$

whereas the second charge is the angular momentum associated with the asymptotic killing vector ∂_ϕ and becomes

$$J = -\frac{a\mu}{\Xi^2}\,. \tag{5.83}$$

5.4 ASYMPTOTIC SYMMETRY ALGEBRAS

The purpose of this section is to exploit under the asymptotic boundary conditions (5.37), (5.38) the symmetry algebra of the dual field theory and examine all possible subalgebras that are allowed by these boundary conditions. Namely, to analyze the asymptotic symmetries on the conformal boundary ∂M such that they admit the assumed boundary conditions and they preserve the gauge transformations of the conformal gravity action (5.1).

The procedure consists of analyzing the asymptotic expansion of the gauge transformations at the leading order $\mathcal{O}(1)$ and at the sub-leading order $\mathcal{O}(\rho)$. Higher order terms are not considered. As before, the generalized Fefferman-Graham expansion (5.18) is assumed for the induced metric γ_{ij}. The analysis at order $\mathcal{O}(1)$ is expected to yield the conformal algebra $o(3,2)$ for $\gamma_{ij}^{(0)} = \eta_{ij}$ and the analysis at order $\mathcal{O}(\rho)$ gives subalgebras of the conformal algebra by imposing restrictions on $\gamma_{ij}^{(1)}$.

Lastly, the asymptotic symmetry algebras of two particular solutions of conformal gravity are investigated. Namely, of the MKR solution which describes a static, spherically symmetric black hole and of the Rindler-Kerr-AdS solution which is describes a rotating black hole.

5.4.1 *Boundary conditions preserving gauge transformations analysis*

The additional gauge symmetry of the theory, in comparison to diffeomorphism invariance of General Relativity, is the local Weyl rescalings of the metric, i.e.

$$g_{ab} \to g'_{ab} = e^{2\Omega} g_{ab} \tag{5.84}$$

where $\Omega = \Omega(x^a)$. Thus, under the gauge symmetries of the theory, i.e. the transformation of the coordinates $x^a \to x'^a = x^a + \xi^a(x^b)$ and the local Weyl transformations (5.84), the consequent infinitesimal change in the spacetime metric is

$$\delta g_{ab} = \left(\mathcal{L}_{\xi^c} + 2\Omega\right) g_{ab} \,. \tag{5.85}$$

The aim is to exploit the above gauge transformations for the spacetime metric (5.7) asymptotically, under the boundary conditions (5.37), (5.38). The procedure consists of expanding (5.85) near the conformal boundary $\rho = 0$ under the following considerations: the 4-dimensional metric ansatz is taken to be

$$g_{ab} = \begin{pmatrix} \frac{-\sigma}{\rho^2} & 0 \\ 0 & g_{ij} \end{pmatrix} \tag{5.86}$$

and

$$g_{ij} = \frac{1}{\rho^2} \gamma_{ij} \tag{5.87}$$

with $\gamma_{ij} = \gamma_{ij}(\rho, x^k)$. Then, g_{ab} is of the form (5.7) with $\ell = 1$. The 3-dimensional induced metric γ_{ij} is assumed to have the generalized Fefferman-Graham expansion of the form (5.18). This is repeated here for clarity

$$\gamma_{ij} = \gamma_{ij}^{(0)} + \rho\gamma_{ij}^{(1)} + \rho^2\gamma_{ij}^{(2)} + \rho^3\gamma_{ij}^{(3)} + \dots \tag{5.88}$$

Additionally, one allows for Ω a similar expansion to the Fefferman-Graham type (5.88), i.e.

$$\Omega = \Omega^{(0)} + \rho\Omega^{(1)} + \rho^2\Omega^{(2)} + \dots \tag{5.89}$$

Finally, the boundary conditions (5.37), (5.38) are considered, which are repeated here

$$\delta\gamma_{ij}^{(0)}|_{\partial M} = 2\lambda\gamma_{ij}^{(0)} \tag{5.90}$$

$$\delta\gamma_{ij}^{(1)}|_{\partial M} = \lambda\gamma_{ij}^{(1)} \,. \tag{5.91}$$

Now, inserting (5.86), (5.87), (5.88), (5.89) and (5.90), (5.91) into the r.h.s and the l.h.s. of the gauge transformations of the spacetime metric (5.85) respectively, the resulting expression consists of terms of order $\mathcal{O}(1)$, terms of order $\mathcal{O}(\rho)$ and terms of higher order in ρ. These results can be classified into the components of (5.85) as follows:

•$\rho\rho$-component

$$\left(\mathcal{L}_{\xi^c} + 2\Omega\right) g_{\rho\rho} = 0 \Rightarrow \xi^\rho \partial_\rho g_{\rho\rho} + 2 g_{\rho\rho} \partial_\rho \xi^\rho + 2\Omega g_{\rho\rho} = 0\,. \tag{5.92}$$

Solving for $\xi^\rho = \xi^\rho_{\text{hom}} + \xi^\rho_{\text{inhom}}$ one gets

$$\xi^\rho_{hom} = \lambda\rho \tag{5.93}$$

$$\partial_\rho \xi^\rho_{\text{inhom}} - \frac{1}{\rho}\xi^\rho_{\text{inhom}} + \Omega = 0 \tag{5.94}$$

where $\lambda = \lambda(x)$. Inserting the asymptotic expansion (5.89) in (5.94) the solution for ξ^ρ_{inhom} is

$$\xi^\rho_{\text{inhom}} = \Omega^{(0)} \rho\, \log(\rho) + \rho\tilde{\lambda} + \rho^2 \lambda^{(2)} + \ldots \tag{5.95}$$

In order to avoid logarithmic terms one sets $\Omega^{(0)} = 0$. Therefore, the expansions for ξ^ρ and Ω finally take the form

$$\begin{aligned} \xi^\rho &= \xi^\rho_{\text{hom}} + \xi^\rho_{\text{inhom}} \\ &= \rho(\tilde{\lambda} + \lambda) + \rho^2 \lambda^{(2)} + \ldots \\ &= \rho\kappa + \rho^2\lambda^{(2)} + \ldots \end{aligned} \tag{5.96}$$

$$\Omega = \rho\Omega^{(1)} + \rho^2 \Omega^{(2)} + \ldots \tag{5.97}$$

with $\kappa \equiv \tilde{\lambda} + \lambda$.

•$i\rho$-component

$$\left(\mathcal{L}_{\xi^c} + 2\Omega\right) g_{i\rho} = 0 \Rightarrow g_{\rho\rho}\partial_i \xi^\rho + g_{ij}\partial_\rho \xi^j = 0 \tag{5.98}$$

Substituting (5.96) in the above equation, the solution for ξ^i is

$$\xi^i = \int d\rho\ \sigma\gamma^{ij}\left(\rho\partial_i\kappa + \rho^2 \partial_i \lambda^{(2)} + \ldots\right) \tag{5.99}$$

which, after inserting the inverse metric expansion one finds that the lowest order of the above integrals is $\mathcal{O}(\rho^2)$. Therefore, they appear at order $\mathcal{O}(\rho)^2$ of expression (5.85). This order will not be further analyzed in the forthcoming sections and thus, the asymptotic expansion of ξ^i (5.99) can be compactly written as

$$\xi^i = \xi^i_{(1)} + \rho^2 \xi^i_{(2)} + \ldots \tag{5.100}$$

where $\xi^i_{(2)} \equiv \sigma\gamma^{ij}_{(0)}\partial_j\kappa$ after solving the integrals in (5.99) in the lowest order.

•ij-component

$$\left(\mathcal{L}_{\xi^c} + 2\Omega\right) g_{ij} = \delta g_{ij} \Rightarrow \xi^\rho\, \partial_\rho g_{ij} + \mathcal{L}_{\xi^k} g_{ij} + 2\Omega\, g_{ij} = \delta g_{ij} \tag{5.101}$$

with

$$\delta g_{ij} = \frac{1}{\rho^2}\left(\delta\gamma^{(0)}_{ij} + \rho\delta\gamma^{(1)}_{ij} + \ldots\right) = \frac{1}{\rho^2}\left(2\lambda\gamma^{(0)}_{ij} + \rho\lambda\gamma^{(1)}_{ij} + \ldots\right) \tag{5.102}$$

under the boundary conditions (5.90), (5.91). Finally, inserting (5.87), (5.88), (5.96), (5.97), (5.100) and (5.102), expression (5.101) takes the form

$$\begin{aligned} &(\rho\kappa + \rho^2\,\lambda^{(2)} + \ldots)\,\frac{-2}{\rho^3}(\gamma^{(0)}_{ij} + \rho\,\gamma^{(1)}_{ij} + \ldots) + (\rho\,\kappa + \rho^2\,\lambda^{(2)} + \ldots)\,\frac{1}{\rho^2}(\gamma^{(1)}_{ij} + \ldots) \\ &+ \frac{1}{\rho^2}\,(\xi^k_{(0)} + \rho^2\,\xi^k_{(2)} + \ldots)(\partial_k\gamma^{(0)}_{ij} + \rho\,\partial_k\gamma^{(1)}_{ij} + \ldots) + \frac{1}{\rho^2}\,(\gamma^{(0)}_{kj} + \rho\,\gamma^{(1)}_{kj} + \ldots)(\partial_i\xi^k_{(0)} + \rho^2\,\partial_i\xi^k_{(2)} + \ldots) \\ &+ \frac{1}{\rho^2}\,(\gamma^{(0)}_{ki} + \rho\,\gamma^{(1)}_{ki} + \ldots)(\partial_j\xi^k_{(0)} + \rho^2\,\partial_j\xi^k_{(2)} + \ldots) + \frac{2}{\rho^2}\,(\rho\,\Omega^{(1)} + \ldots)(\gamma^{(0)}_{ij} + \rho\,\gamma^{(1)}_{ij} + \ldots) = 0\,. \end{aligned} \tag{5.103}$$

This expression is now explicitly analyzed at order $\mathcal{O}(1)$ and at order $\mathcal{O}(\rho)$. The leading order $\mathcal{O}(1)$ determines the asymptotic symmetry algebra at the conformal boundary of ∂M, which is expected to be the conformal algebra $o(3,2)$. The sub-leading order $\mathcal{O}(\rho)$ determines the asymptotic symmetry subalgebras of $o(3,2)$ by restricting $\gamma^{(1)}_{ij}$.

5.4.2 *Asymptotic symmetry algebra of the $\mathcal{O}(1)$ equation*

At $\mathcal{O}(1)$, (5.103) takes the form

$$\mathcal{L}_{\xi^k_{(0)}} \gamma^{(0)}_{ij} - 2\kappa\gamma^{(0)}_{ij} = 0 \,. \tag{5.104}$$

Taking the trace one gets

$$\kappa = \frac{1}{3} D_k \xi^k_{(0)} \tag{5.105}$$

and substituting back in (5.104), its final form becomes

$$\mathcal{L}_{\xi^k_{(0)}} \gamma^{(0)}_{ij} - \frac{2}{3} \gamma^{(0)}_{ij} D_k \xi^k_{(0)} = 0 \,. \tag{5.106}$$

Considering now the case of $\gamma^{(0)}_{ij} = \eta_{ij}$, the above expression is

$$\partial_i \xi^{(0)}_j + \partial_j \xi^{(0)}_i - \frac{2}{3}\eta_{ij}\, \partial_k \xi^k_{(0)} = 0 \tag{5.107}$$

which is the conformal Killing equation that characterizes the conformal algebra $o(3,2)$ as expected. It admits (up to constants) the following 10 conformal Killing vectors:

$$\xi^1_{(0)} = \partial_t \ ,\ \xi^2_{(0)} = \partial_x \ ,\ \xi^3_{(0)} = \partial_y \tag{5.108}$$

$$\xi^4_{(0)} = -y\partial_x + x\partial_y \ ,\ \xi^5_{(0)} = y\partial_t + t\partial_y \ ,\ \xi^6_{(0)} = x\partial_t + t\partial_x \tag{5.109}$$

$$\xi^7_{(0)} = t\partial_t + x\partial_x + y\partial_y \tag{5.110}$$

$$\begin{aligned} \xi^8_{(0)} &= 2xt\partial_t + (x^2 - y^2 + t^2)\partial_x + 2xy\partial_y \ ,\ \xi^9_{(0)} = 2yt\partial_t + 2yx\partial_x + (y^2 - x^2 + t^2)\partial_y \\ \xi^{10}_{(0)} &= (t^2 + x^2 + y^2)\partial_t + 2tx\partial_x + 2ty\partial_y \,. \end{aligned} \tag{5.111}$$

In particular, these conformal Killing vectors are the generators of 3 translations (5.108) ($P_i \equiv \{\xi^1_{(0)}, \xi^2_{(0)}, \xi^3_{(0)}\}$ with $i = t, x, y$), 2 boosts and 1 rotation (5.109) ($J_{ij} \equiv \{\xi^4_{(0)}, \xi^5_{(0)}, \xi^6_{(0)}\}$ with $i, j = t, x, y$), 1 dilatation (5.110) ($D \equiv \xi^7_{(0)}$) and 3 special conformal transformations (5.111) ($K_i \equiv \{\xi^8_{(0)}, \xi^9_{(0)}, \xi^{10}_{(0)}\}$ with $i = t, x, y$). These generators obey the usual commutation relations of the 3-dimensional conformal algebra:

$$\begin{aligned} &[P_i, P_j] = 0 \ ,\ [J_{ij}, J_{kl}] = \eta_{il} J_{jk} - \eta_{ik} J_{jl} - \eta_{jl} J_{ik} + \eta_{jk} J_{il} \ ,\ [P_i, J_{jk}] = \eta_{ij} P_k - \eta_{ik} P_j \\ &[D, P_i] = P_i \ ,\ [D, J_{ij}] = 0 \ ,\ [K_i, P_j] = 2(\eta_{ij} D - J_{ij}) \ ,\ [K_i, J_{jk}] = \eta_{ij} K_k - \eta_{ik} K_j \\ &[D, K_i] = -K_i \ ,\ [K_i, K_j] = 0 \,. \end{aligned} \tag{5.112}$$

5.4.3 *Subalgebras of $o(3,2)$, analysis of the $\mathcal{O}(\rho)$ equation and conditions on $\gamma^{(1)}_{ij}$*

At this stage, it is instructive to explore the subalgebras of $o(3,2)$ algebraically. That is, to try to construct possible subalgebras from the above commutation relations (5.112). This is done before analyzing the order $\mathcal{O}(\rho)$ of expression (5.85), which by imposing restrictions on $\gamma^{(1)}_{ij}$ admits particular subalgebras of $o(3,2)$. In other words, the following algebraic analysis enables one to know what subalgebras to look for. Then, by analyzing explicitly the $\mathcal{O}(\rho)$ of (5.85) one can specify if each particular subalgebra is actually realized by imposing conditions on $\gamma^{(1)}_{ij}$. A formal and exhaustive derivation of all the subalgebras of $o(3,2)$ in general can be found in Tables IV-XI of [81].

The commutation relations (5.112) lead to the following observations:

1. The largest subalgebras of $o(3,2)$ are 7-dimensional. Furthermore, they cannot contain both the subalgebra of translations and the subalgebra of special conformal transformations (SCTs). It consequently turns out that these 7-dimensional subalgebra are either the so called extended Poincare algebra (i.e. the Poincare transformations and the dilation which is also known as similitude algebra LSim(2,1)), or the analogue of this where the role of translations is replaced by the SCTs.

   Proof: Starting with the 3 SCTs and adding a translation one gets from $[K_i, P_t]$ the dilatation and two Lorentz rotations. Then from $[J_{ij}, J_{kl}]$, these two Lorentz rotations imply the third one. But from $[P_t, J_{ij}]$ one gets the remaining two translations and therefore one ends up with $o(3,2)$. This implies that the 3-dimensional subalgebra that contains the SCTs cannot be enlarged by adding any translation. Therefore, in order to enlarge this 3-dimensional subalgebra one has to add the rest of the generators except the translations. By adding the dilatation the resulting 4-dimensional subalgebra is trivially closing. By adding to this one Lorentz rotation, the resulting 5-dimensional subalgebra is also trivially closing: one possible subalgebra then is exactly this, namely containing 3 SCTs, 1 Lorentz generator and the dilation. But this is not the largest one. If one more Lorentz rotation is added, one gets automatically the third one i.e. one has the 7-dimensional analogue of the extended Poincare algebra. Likewise, the same logic can be followed by starting with the subalgebra of the 3 translations and attempt to include one SCT. Firstly, one obtains that this consequently leads to $o(3,2)$ and secondly, by adding the rest of the generators except the translations one gets the extended Poincare algebra.

2. The next lower dimensional subalgebras of $o(3,2)$ are 6-dimensional. They can be either the Poincare algebra or an algebra that contains 2 SCTs, 2 translations, 1 Lorentz rotation and the dilatation.

   Proof: The fact that one of the two 6-dimensional subalgebras is the Poincare algebra is evident. As far as the second 6-dimensional algebra is concerned, starting with 2 SCTs and adding the dilatation one has 3 generators. Then, one can add either (i) one Lorentz rotation or (ii) one translation. As it is explained, both cases (i) and (ii) give the same result:

   (i) Adding 1 Lorentz rotation, one gets a 4-dimensional subalgebra. It should be mentioned here that the appropriate Lorentz rotation should be added, i.e. the one which via $[K_i, J_{jk}]$ gives the 2 already assumed SCTs and not the third one. Otherwise, one falls into the 7-dimensional subalgebra of case 1. Then, one notices that this 4-dimensional subalgebra cannot be enlarged by adding one more Lorentz rotation. If one does so, one gets the third one as well and the commutation relation of each of them with the two SCTs now gives the third SCT, falling again into the case 1. So there is nothing left to add to this 4-dimensional subalgebra than 1 translation. But the appropriate one, such that via $[K_i, P_j]$ no new Lorentz rotation appears. Now the subalgebra is 5-dimensional, but still unclosed. For this then to close via $[P_i, J_{jk}]$, one more translation has to be added. Any attempt to include anything more ends up with $o(3,2)$. Thus, in this case the subalgebra is 6-dimensional and contains 2 SCTs, the dilatation, 1 Lorentz rotation and 2 translations.

   (ii) Adding 1 translation, one gets via $[K_i, P_j]$ 1 Lorentz rotation. This translation has to be the appropriate such that via $[K_i, P_j]$ one does not get two Lorentz rotations which imply the third one and consequently end up to $o(3,2)$. Now this subalgebra is 5-dimensional, but still not closed. For the closure it is necessary via $[P_i, J_{jk}]$ to get 1 more translation. If anything else is added in order to enlarge this subalgebra leads to $o(2,3)$. Therefore, in this case the subalgebra is 6-dimensional and has exactly the same content with case (i).

Similar arguments can be applied to obtain lower dimensional subalgebras of $o(3,2)$, that is the 5-, 4-, and lower-dimensional subalgebras. Those are not mentioned here explicitly and an exhaustive list of them can be found in Tables IV-XI of [81]. Later in the analysis, it is examined specifically which of the 5-, 4-, and lower-dimensional subalgebras are realized in the content of this thesis, that is for a particular choice of $\gamma_{ij}^{(1)}$.

Having obtained algebraically some of the subalgebras of $o(3,2)$, the purpose is now to investigate whether and which of them are actually realized by the order $\mathcal{O}(\rho)$ of (5.103). This procedure inevitably imposes restrictions on $\gamma_{ij}^{(1)}$. As a starting point two assumptions are made: i) the boundary metric $\gamma_{ij}^{(0)}$ is considered to be the 3-dimensional Minkowski metric η_{ij} and ii) the leading order term $\gamma_{ij}^{(1)}$ is taken to be traceless, i.e. $\gamma_{(1)} \equiv \eta^{ij}\gamma_{ij}^{(1)} = 0$. The first assumption is rather natural whereas the second one simply imposes relations between the coefficients of $\gamma_{ij}^{(1)}$ and thus it is a gauge choice. Then, at $\mathcal{O}(\rho)$, (5.103) becomes

$$-2\lambda^{(2)}\eta_{ij} - \kappa\gamma_{ij}^{(1)} + \mathcal{L}_{\xi^k_{(0)}}\gamma_{ij}^{(1)} + 2\Omega^{(1)}\eta_{ij} = 0 \tag{5.113}$$

which by setting $\lambda^{(2)} = -\Omega^{(1)}$ and using (5.105) becomes

$$\mathcal{L}_{\xi^k_{(0)}}\gamma_{ij}^{(1)} - \frac{1}{3}\gamma_{ij}^{(1)}\partial_k\xi^k_{(1)} + 4\Omega^{(1)}\eta_{ij} = 0\,. \tag{5.114}$$

Taking the trace and solving for $\Omega^{(1)}$ one finds

$$\Omega^{(1)} = -\frac{1}{12}\left(\xi^k_{(0)}\partial_k\gamma_{(1)} + 2\gamma^i_{k(1)}\partial_i\xi^k_{(0)} - \frac{1}{3}\gamma_{(1)}\partial_k\xi^k_{(0)}\right). \tag{5.115}$$

Taking into account the gauge choice $\gamma_{(1)} = 0$, the above equation takes the simpler form

$$\Omega = -\frac{1}{6}\gamma^i_{k(1)}\partial_i\xi^k_{(0)} \tag{5.116}$$

and (5.114) becomes

$$\mathcal{L}_{\xi^k_{(0)}}\gamma_{ij}^{(1)} - \frac{1}{3}\gamma_{ij}^{(1)}\partial_k\xi^k_{(1)} - \frac{2}{3}\eta_{ij}\,\gamma^i_{(1)k}\,\partial_i\xi^k_{(0)} = 0\,. \tag{5.117}$$

Then, the procedure consists of inserting particular ansatz for $\gamma_{ij}^{(1)}$ in the above equation and examine if any subalgebra is obtained. Of course it is a crucial point in the analysis that this ansatz is as general as possible.

7-dimensional subalgebras

For the trivial case of $\gamma_{ij}^{(1)} = 0$ one recovers $o(3,2)$, in the sense that (5.117) is trivially satisfied and one is left only with (5.107). For $\gamma_{ij}^{(1)}$ being conformally flat, i.e. $\gamma_{ij}^{(1)} = f(t,x,y)\,\eta_{ij}$ one also obtains $o(3,2)$.

The procedure then consists of finding the most general form of $\gamma_{ij}^{(1)}$ that through (5.117) accommodates the translations (5.108). Then, one gradually includes the rest of the generators (5.109), (5.110), (5.111). Demanding (5.117) to be satisfied by the translations (5.108) one finds that the following condition has to hold

$$\partial_m\gamma_{ij}^{(1)} = 0 \tag{5.118}$$

for every $m = t,x,y$. This expression is referred to as translations condition. Imposing this in (5.117) and inserting the boosts and the rotation in (5.109) one gets

$$\gamma_{xy}^{((1))} = 0\;,\;\gamma_{ty}^{(1)} = 0\;,\;\gamma_{tx}^{(1)} = 0\;,\;\gamma_{xx}^{(1)} = \gamma_{yy}^{(1)} = -\gamma_{tt}^{(1)} = c \tag{5.119}$$

or equivalently

$$\gamma_{ij}^{(1)} = c\,\eta_{ij} \tag{5.120}$$

where c is a constant. But due to this gauge fixing condition $\gamma_{(1)} = 0$ one gets $c = 0$ which implies $\gamma_{ij}^{(1)} = 0$ and leads trivially to $o(3,2)$. Therefore, the 7-dimensional extended Poincare subalgebra cannot be realized for any choice of $\gamma_{ij}^{(1)} \neq 0$.

The other 7-dimensional subalgebra, which is the analogue of the extended Poincare where the translations are replaced by the SCTs, was not obtained in the present analysis. In particular, the system of partial differential equations arising from (5.117) when one inserts the SCTs, could not be solved for $\gamma_{ij}^{(1)}$.

6-dimensional subalgebras

From the analysis above it is obvious that neither the 6-dimensional Poincare subalgebra can be realized: the conditions (5.118) and (5.120) for the translations and the Lorentz transformations respectively give $c = 0$, which implies $\gamma_{ij}^{(1)} = 0$ and thus inevitably leads to $o(3,2)$.

The 6-dimensional subalgebra that contains 2 SCTs, 2 translations, 1 Lorentz rotation and the dilatation cannot be realized. Any attempt to include a SCT in the 4-dimensional subalgebra containing the 2 translations, 1 Lorentz rotation and the dilatation leads to $o(3,2)$. A derivation of this is given in the forthcoming paragraph of the 4-dimensional subalgebras.

5-dimensional subalgebra

Among the wide list of the subalgebras of $o(3,2)$ with 5 generators there is exactly one which is realized, subject to equation (5.117). This is obtained by starting with the subalgebra of the translations, i.e. from $P_i = \{\xi^1_{(0)}, \xi^2_{(0)}, \xi^3_{(0)}\}$, with $i = t, x, y$. It was found previously that the translations conditions is $\partial_m \gamma_{ij}^{(1)} = 0$ for every $m = t, x, y$, or equivalently

$$\gamma_{ij}^{(1)} = \begin{pmatrix} c_1 + c_2 & c_3 & c_4 \\ c_3 & c_1 & c_5 \\ c_4 & c_5 & c_2 \end{pmatrix} . \tag{5.121}$$

This 3-dimensional subalgebra can then be enlarged by including generators that are suitable linear combinations of the Lorentz Killing vectors (5.109) and the dilatation (5.110). In particular, these linear combinations are between the dilatation $(D \equiv \xi^7_{(0)})$ and the ty-boost $(J_{ty} \equiv \xi^5_{(0)})$ as $\tilde{D} = \xi^7_{(0)} + \frac{1}{2}\xi^5_{(0)}$, and between the xy-rotation $(J_{xy} \equiv \xi^4_{(0)})$ and the tx-boost $(J_{tx} \equiv \xi^6_{(0)})$ as $\tilde{J} = \xi^4_{(0)} - \xi^6_{(0)}$. Indeed, inserting these and the above expression for $\gamma_{ij}^{(1)}$ in (5.117) one finds that

$$\gamma_{ij}^{(1)} = \begin{pmatrix} -c & 0 & c \\ 0 & 0 & 0 \\ c & 0 & -c \end{pmatrix} \tag{5.122}$$

and the commutation relations take the form

$$[\xi^1_{(0)}, \tilde{J}] = \xi^2_{(0)} \;\; , \;\; [\xi^1_{(0)}, \tilde{D}] = -\xi^1_{(0)} - \frac{1}{2}\xi^3_{(0)} \tag{5.123}$$

$$[\xi^2_{(0)}, \tilde{J}] = \xi^1_{(0)} + \xi^3_{(0)} \;\; , \;\; [\xi^2_{(0)}, \tilde{D}] = \xi^2_{(0)} \tag{5.124}$$

$$[\xi^3_{(0)}, \tilde{J}] = -\xi^2_{(0)} \;\; , \;\; [\xi^3_{(0)}, \tilde{D}] = -\xi^3_{(0)} - \frac{1}{2}\xi^1_{(0)} \tag{5.125}$$

$$[\tilde{J}, \tilde{D}] = -\frac{1}{2}\tilde{J} \tag{5.126}$$

$$[P_i, P_j] = 0 \;\; \forall\, i, j = t, x, y\,. \tag{5.127}$$

Similar results can be obtained when again starting with the translations and now including different but again suitable linear combinations between the Lorentz generators and the dilatation. That is, considering now the dilatation with the ty-boost ($J_{ty} \equiv \xi^6_{(0)}$) as $\tilde{D} = \xi^7 - \frac{1}{2}\xi^6_{(0)}$ and then the xy-rotation ($J_{xy} \equiv \xi^4_{(0)}$) with the tx-boost ($J_{tx} \equiv \xi^6_{(0)}$) as $\tilde{J} = \xi^4_{(0)} - \xi^6_{(0)}$. In this case, (5.117) restricts $\gamma^{(1)}_{ij}$ to be

$$\gamma^{(1)}_{ij} = \begin{pmatrix} c & c & 0 \\ c & c & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{5.128}$$

This algebra with 5 generators is the highest dimensional realizable subalgebra of $o(3,2)$. That means that for any other choice of $\gamma^{(1)}_{ij}$ no other 5-dimensional example can be constructed because it gives $\gamma^{(1)}_{ij} = 0$ via (5.117) and consequently leads to $o(3,2)$.

4-dimensional subalgebras

Some of the 4-dimensional subalgebras can be straightforwardly constructed by adding suitably one more generator to the subalgebra of the translations $P_i = \{\xi^1_{(0)}, \xi^2_{(0)}, \xi^3_{(0)}\}$, with $i = t, x, y$. Inserting the translations conditions (5.118) in (5.117) and demanding to contain 1 Lorentz rotation, e.g. $J_{xy} \equiv \xi^4_{(0)}$, one gets that $c_3 = c_4 = c_5 = 0$ and $c_1 = c_2 \equiv c$. Thus $\gamma^{(1)}_{ij}$ takes the form

$$\gamma^{(1)}_{ij} = \begin{pmatrix} 2c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \tag{5.129}$$

and the commutations relations of this subalgebra with 4 generators are

$$[P_i, P_j] = 0 \tag{5.130}$$

$$[P_i, J_{xy}] = \eta_{ix} P_y - \eta_{iy} P_x . \tag{5.131}$$

Similar results can be obtained when considering again the subalgebra of the translations and adding another Lorentz rotation, e.g. $J_{ty} \equiv \xi^5_{(0)}$ or $J_{tx} \equiv \xi^6_{(0)}$. Following the same technique as in the previous paragraph, the form of $\gamma^{(1)}_{ij}$ becomes

$$\gamma^{(1)}_{ij} = \begin{pmatrix} -c & 0 & 0 \\ 0 & -2c & 0 \\ 0 & 0 & c \end{pmatrix} \tag{5.132}$$

when the boost J_{ty} is contained and

$$\gamma^{(1)}_{ij} = \begin{pmatrix} -c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & -2c \end{pmatrix} \tag{5.133}$$

when the boost J_{tx} is contained. The commutation relations in these cases take the analogue form of (5.130), (5.131).

Another subalgebra with 4 generators is obtained when considering the 3 translations and a linear combination of the dilatation $D \equiv \xi^7_{(0)}$ and 1 boost, e.g. $J^5_{(0)} \equiv \xi^5_{(0)}$, of the form $\tilde{D} = \xi^7_{(0)} - \frac{1}{2}\xi^5_{(0)}$. The translations condition (5.118) imply constant $\gamma^{(1)}_{ij}$ and the addition of the generator $\tilde{D}$ restricts it further via (5.117) to be

$$\gamma^{(1)}_{ij} = \begin{pmatrix} c & 0 & c \\ 0 & 0 & 0 \\ c & 0 & c \end{pmatrix} . \tag{5.134}$$

The commutation relations in this case are

$$[\xi^1_{(0)}, \tilde{D}] = \xi^1_{(0)} + \frac{1}{2}\xi^3_{(0)} \tag{5.135}$$

$$[\xi^3_{(0)}, \tilde{D}] = \frac{1}{2}\xi^1_{(0)} + \xi^3_{(0)} \tag{5.136}$$

$$[P_i, P_j] = 0 \ \ \forall i, j = t, x, y \ \ , \ \ [\xi^2_{(0)}, \tilde{D}] = 0 \, . \tag{5.137}$$

Other 4-dimensional subalgebras with the 3 translations as a starting point can be also realized in a slightly different way. In particular, when assuming suitable linear combinations between these translations and between the Lorentz generators and the dilatation (5.109)-(5.110). That is, one considers the translation $\xi^2_{(0)}$ and two linear combinations of the rest of the translations as $P_+ = \xi^1_{(0)} + \xi^3_{(0)}$ and $P_- = \xi^1_{(0)} - \xi^3_{(0)}$. Inserting these in (5.117), one gets the translations condition (5.118). Then, another linear combination of the dilatation $D \equiv \xi^7_{(0)}$ and 1 boost, e.g. $J_{ty} \equiv \xi^5_{(0)}$ can be included, namely $\tilde{D} = \xi^7_{(0)} - \xi^5_{(0)}$. This implies via (5.117) that

$$\gamma^{(1)}_{ij} = \begin{pmatrix} 0 & c & 0 \\ c & 0 & c \\ 0 & c & 0 \end{pmatrix} \tag{5.138}$$

and the commutation relations are

$$[P_+, P_-] = 0 \ \ , \ \ [P_+, \tilde{D}] = 0 \ \ , \ \ [P_+, \xi^2_{(0)}] = 0 \ \ , \ \ [P_-, \xi^2_{(0)}] = 0 \tag{5.139}$$

$$[P_-, \tilde{D}] = 2P_- \tag{5.140}$$

$$[\xi^2_{(0)}, \tilde{D}] = \xi^2_{(0)} \, . \tag{5.141}$$

An analogous subalgebra is realized when starting again with the translation $\xi^2_{(0)}$ and a linear combination of the rest 2 of the form of the previous paragraph. Then, including a linear combination of the rotation and a boost as $\tilde{J} = \xi^4_{(0)} - \xi^5_{(0)}$ and inserting in (5.103), $\gamma^{(1)}_{ij}$ becomes

$$\gamma^{(1)}_{ij} = \begin{pmatrix} c & -c & 0 \\ -c & c & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.142}$$

whereas the commutation relations take the form

$$[P_+, \tilde{J}] = -2\xi^2_{(0)} \tag{5.143}$$

$$[\xi^2_{(0)}, \tilde{J}] = -P_- \tag{5.144}$$

$$[P_+, P_-] = 0 \ , \ [P_+, \xi^2_{(0)}] = 0 \ , \ [P_-, \xi^2_{(0)}] = 0 \ , \ [P_-, \tilde{J}] = 0 \, . \tag{5.145}$$

Another interesting subalgebra can be obtained by excluding one translation, e.g. $P_t \equiv \xi^1_{(0)}$, from (5.117). Then, the rest of the translations are maintained for $\partial_m \gamma^{(1)}_{ij} = 0$ for $m = x, y$ and the general form of $\gamma^{(1)}_{ij}$ is now

$$\gamma^{(1)}_{ij} = \begin{pmatrix} f_1(t) + f_2(t) & g(t) & w(t) \\ g(t) & f_1(t) & h(t) \\ w(t) & h(t) & f_2(t) \end{pmatrix} \, . \tag{5.146}$$

Inserting this in (5.117) and additionally including the xy-rotation ($J_{xy} \equiv \xi^4_{(0)}$), one finds that $g(t) = w(t) = h(t) = 0$ and $f_1(t) = f_2(t) \equiv f(t)$. Thus, expression (5.146) now becomes

$$\gamma^{(1)}_{ij} = \begin{pmatrix} 2f(t) & 0 & 0 \\ 0 & f(t) & 0 \\ 0 & 0 & f(t) \end{pmatrix} \, . \tag{5.147}$$

This 3-dimensional subalgebra that contains 2 translations and 1 Lorentz rotation is the Poincare algebra in 2 spatial dimensions. It can be further extended by adding the dilation $D \equiv \xi^7_{(0)}$. Indeed, inserting (5.147) and the dilatation generator in the non-vanishing components (xx, tt, yy) of (5.117) one gets the relation

$$t\partial_t f(t) + f(t) = 0 \tag{5.148}$$

which then implies for (5.147) that

$$\gamma^{(1)}_{ij} = \frac{1}{t}\begin{pmatrix} 2c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} . \tag{5.149}$$

Thus, for this choice of $\gamma^{(1)}_{ij}$ the extended 2-dimensional Poincare algebra is realized with the following commutation relations

$$[P_I, P_J] = 0 \ , \ \ [D, J_{xy}] = 0 \tag{5.150}$$

$$[P_I, J_{xy}] = \eta_{Ix} P_y - \eta_{Iy} P_x \tag{5.151}$$

$$[D, P_I] = P_I \tag{5.152}$$

where $I, J = x, y$. This is of particular interest because the corresponding theory has scale invariance, due to the dilatation generator. Then it is natural to ask if the special conformal transformations can be included to exhibit conformal invariance. Interestingly, the answer is no: inserting (5.149) and one of the special conformal transformations in (5.117) implies $c = 0$, which leads to $o(3, 2)$. Therefore, the extended 2-dimensional Poincare algebra corresponds to a theory that asymptotically has scale invariance but not conformal invariance. Additionally, this result is excluding the realization of the 6-dimensional subalgebra of $o(3, 2)$ that was discussed before. That is, the subalgebra that contains 2 translations, 2 SCTS, 1 Lorentz rotation and the dilatation: this cannot be realized because attempting to include 1 SCT inevitably leads to $o(3, 2)$.

Similar results can be derived when excluding from (5.117) another translation e.g. $P_x \equiv \xi^2_{(0)}$ or $P_y \equiv \xi^3_{(0)}$. Demanding then the rest of the translations, 1 (appropriate) Lorentz rotation and the dilatation to be contained in (5.117), the form of $\gamma^{(1)}_{ij}$ becomes

$$\gamma^{(1)}_{ij} = \frac{1}{x}\begin{pmatrix} -c & 0 & 0 \\ 0 & -2c & 0 \\ 0 & 0 & c \end{pmatrix} \tag{5.153}$$

for the exclusion of $P_x \equiv \xi^2_{(0)}$ and

$$\gamma^{(1)}_{ij} = \frac{1}{y}\begin{pmatrix} -c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & -2c \end{pmatrix} \tag{5.154}$$

for the exclusion of $P_y \equiv \xi^3_{(0)}$. The commutation relations in these cases take the analogue form of (5.150), (5.151) , (5.152).

It should be mentioned at this point that the subalgebras with 4 generators that were considered here do not yield an exhaustive list. They are the ones that were realized during this work but there may be more amongst the wide list of the subalgebras of $o(3, 2)$, by imposing a condition on $\gamma^{(1)}_{ij}$.

3-dimensional subalgebras

It is evident from the previous analysis that a realizable subalgebra with 3 generators consists of the translations with

$$\gamma^{(1)}_{ij} = c_{ij} \tag{5.155}$$

where c_{ij} are constant coefficients and the commutation relations are

$$[P_i, P_j] = 0 \;\; \forall\, i, j = t, x, y\,. \tag{5.156}$$

Following a similar logic with that in the investigation of the 4-dimensional subalgebras, i.e. excluding 1 translation from (5.117) one arrives at a realization of two more 3-dimensional subalgebras. The first one is obtained when excluding e.g. $P_t \equiv \xi^1_{(0)}$ and including the dilatation $D \equiv \xi^7_{(0)}$. Then, $\gamma^{(1)}_{ij}$ is restricted to be

$$\gamma^{(1)}_{ij} = \frac{1}{t} c_{ij} \tag{5.157}$$

with c_{ij} being a constant matrix and the commutation relations take the form

$$[P_x, P_y] = 0 \;\;,\;\; [D, P_I] = 0 \;\; \forall\, I = x, y\,. \tag{5.158}$$

Analogous results arise when excluding one of the rest of the translations, e.g. $P_x \equiv \xi^2_{(0)}$ and $P_y \equiv \xi^3_{(0)}$.

The second 3-dimensional subalgebra that is realized when removing one of the translations from (5.117) was already obtained in the construction of the 4-dimensional extended Poincare subalgebra in 2 spatial dimensions. Indeed, excluding the translation $P_t \equiv \xi^1_{(0)}$ and adding the rotation $J_{xy} \equiv \xi^4_{(0)}$ one arrives at the Poincare algebra in 2 spatial dimensions with the condition (5.147) for $\gamma^{(1)}_{ij}$. It is repeated here for clarity

$$\gamma^{(1)}_{ij} = \begin{pmatrix} 2f(t) & 0 & 0 \\ 0 & f(t) & 0 \\ 0 & 0 & f(t) \end{pmatrix}\,. \tag{5.159}$$

The commutation relations in this case are

$$[P_I, J_{xy}] = \eta_{Ix} P_y - \eta_{Iy} P_x \;\;, I, J = x, y \tag{5.160}$$

$$[P_I, P_J] = 0 \;\; \forall\, I, J\,. \tag{5.161}$$

Finally, it is mentioned that the Lorentz subalgebra is generated from (5.109). Inserting these generators into (5.117), one gets a system of partial differential equations which has to be solved for $\gamma^{(1)}_{ij}$. In the present analysis, this system was not solved analytically and thus, the Lorentz subalgebra is not realized for a particular form of $\gamma^{(1)}_{ij}$. Further subalgebras with 3 generators and lower, e. g. 2-dimensional, are not examined in the present work.

A last but important remark is that it is straightforward to verify that the generators of each of the above mentioned subalgebras satisfy the Jacobi identities. All results obtained so far, namely the subalgebras of $o(3,2)$ which arise via (5.117) for a particular form of $\gamma^{(1)}_{ij}$, are summarized in the following table. It should be also mentioned that this table does not provide an exhaustive list.

Generators	Form of $\gamma^{(1)}_{ij}$	Dimension of the subalgebra	Commutation relations
Linear combination of the Lorentz algebra, the dilatation and the translations	$\gamma^{(1)}_{ij} = \begin{pmatrix} -c & 0 & c \\ 0 & 0 & 0 \\ c & 0 & -c \end{pmatrix}$	5	$[\xi^1_{(0)}, \tilde{J}] = \xi^2_{(0)}$ $[\xi^1_{(0)}, \tilde{D}] = -\xi^1_{(0)} - \frac{1}{2}\xi^3_{(0)}$ $[\xi^2_{(0)}, \tilde{J}] = \xi^1_{(0)} + \xi^3_{(0)}$ $[\xi^2_{(0)}, \tilde{D}] = \xi^2_{(0)}$ $[\xi^3_{(0)}, \tilde{D}] = -\xi^3_{(0)} - \frac{1}{2}\xi^1_{(0)}$ $[\xi^3_{(0)}, \tilde{J}] = -\xi^2_{(0)}$ $[\tilde{J}, \tilde{D}] = -\frac{1}{2}\tilde{J}$ $[P_i, P_j] = 0 \;\; \forall$ i, j= t, x, y where $\tilde{D} = \xi^7_{(0)} + \frac{1}{2}\xi^6_{(0)}$ $\tilde{J} = -\xi^5_{(0)} + \xi^4_{(0)}$
Poincare in 2 spatial dimensions and dilatation	$\gamma^{(1)}_{ij} = \frac{1}{t}\begin{pmatrix} 2c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}$	4	$[D, P_I] = P_I$ $[P_I, J_{xy}] = \eta_{Ix}P_y - \eta_{Iy}P_x$ $[P_I, P_J] = [D, J_{xy}] = 0$
Translations and 1 rotation	$\gamma^{(1)}_{ij} = \begin{pmatrix} 2c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}$	4	$[P_i, J_{xy}] = \eta_{ix}P_y - \eta_{iy}P_x$ $[P_i, P_j] = 0$
Translations and linear combination of 1 boost and the dilatation	$\gamma^{(1)}_{ij} = \begin{pmatrix} c & 0 & c \\ 0 & 0 & 0 \\ c & 0 & c \end{pmatrix}$	4	$[\xi^1_{(0)}, \tilde{D}] = \xi^1_{(0)} + \frac{1}{2}\xi^3_{(0)}$ $[\xi^3_{(0)}, \tilde{D}] = \frac{1}{2}\xi^1_{(0)} + \xi^3_{(0)}$ $[P_i, P_j] = 0 \, \forall i, j = t, x, y$ $[\xi^2_{(0)}, \tilde{D}] = 0$ where $\tilde{D} = \xi^7_{(0)} - \frac{1}{2}\xi^5_{(0)}$
Linear combination of the translations, 1 boost and the dilatation	$\gamma^{(1)}_{ij} = \begin{pmatrix} 0 & c & 0 \\ c & 0 & c \\ 0 & c & 0 \end{pmatrix}$	4	$[P_-, \tilde{D}] = 2P_-$, $[\xi^2_{(0)}, \tilde{D}] = \xi^2_{(0)}$ $[P_+, P_-] = [P_+, \tilde{D}] = [P_+, \xi^2_{(0)}] = 0$ $[P_-, \xi^2_{(0)}] = 0$ where $P_+ = \xi^1_{(0)} + \xi^3_{(0)}$ $P_- = \xi^1_{(0)} - \xi^3_{(0)}$ $\tilde{D} = \xi^7_{(0)} - \xi^5_{(0)}$
Linear combination of the translations and 2 Lorentz generators	$\gamma^{(1)}_{ij} = \begin{pmatrix} c & -c & 0 \\ -c & c & 0 \\ 0 & 0 & 0 \end{pmatrix}$	4	$[P_+, \tilde{J}] = -2\xi^2_{(0)}$ $[\xi^2_{(0)}, \tilde{J}] = -P_-$ $[P_+, P_-] = [P_+, \xi^2_{(0)}] = [P_-, \xi^2_{(0)}] = [P_-, \tilde{J}] = 0$ where $P_+ = \xi^1_{(0)} + \xi^3_{(0)}$ $P_- = \xi^1_{(0)} - \xi^3_{(0)}$ $\tilde{J} = \xi^4_{(0)} - \xi^5_{(0)}$

Generators	Form of $\gamma_{ij}^{(1)}$	Dimension of the subalgebra	Commutation relations
Translations	$\gamma_{ij}^{(1)} = c_{ij}$	3	$[P_i, P_j] = 0$
2 translations + dilatation	$\gamma_{ij}^{(1)} = \frac{1}{t} c_{ij}$	3	$[P_I, P_J] = 0$, $[D, P_I] = P_I$
Poincare in 2 spatial dimensions	$\gamma_{ij}^{(1)} = \begin{pmatrix} 2f(t) & 0 & 0 \\ 0 & f(t) & 0 \\ 0 & 0 & f(t) \end{pmatrix}$	3	$[P_I, P_J] = 0$ $[P_I, J_{xy}] = \eta_{Ix} P_y - \eta_{Iy} P_x$

5.4.4 *The MKR solution*

As it was already mentioned in section 5.3.2, the MKR spacetime is a black hole solution of the action (5.1). Its line element takes the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \tag{5.162}$$

where

$$f(r) = \sqrt{1 - 12\alpha M} - \frac{2M}{r} + 2\alpha r - \frac{\Lambda}{3} r^2 \tag{5.163}$$

and M, Λ, α are the mass, the cosmological constant and the Rindler acceleration respectively.

Before proceeding with the analysis of the asymptotic symmetry algebras of (5.162), it is constructive to exploit its spacetime symmetries. It is immediately recognized that these symmetries consist of a static, spherically symmetric spacetime or equivalently the $\mathbb{R} + so(3)$ algebra. The Killing vectors which generate these symmetries and satisfy $\pounds_{\xi^c} g_{ab} = 0$ are

$$\xi^1 = \partial_t \tag{5.164}$$

$$\xi^2 = \partial_\phi \tag{5.165}$$

$$\xi^3 = \cos\phi \partial_\theta - \cot\theta \sin\phi \partial_\phi \tag{5.166}$$

$$\xi^4 = -\sin\phi \partial_\theta - \cot\theta \cos\phi \partial_\phi \tag{5.167}$$

and their commutation relations take the form

$$[\xi^2, \xi^3] = \xi^4 \ , \ [\xi^2, \xi^4] = -\xi^3 \tag{5.168}$$

$$[\xi^3, \xi^4] = \xi^2 \tag{5.169}$$

$$[\xi^1, \xi^k] = 0 \ \ \forall\, k = 2, 3, 4\,. \tag{5.170}$$

As it will be obtained in what follows, this 4-dimensional algebra of the spacetime symmetries can be enlarged asymptotically, at the leading order $\mathcal{O}(1)$ of (5.103).

Proceeding now with the asymptotic analysis, it is reminded that the following steps must be performed: the line element (5.162) can be rewritten suitably when following the procedure that was described in the beginning of this chapter. Namely the decomposition of a spacetime g_{ab} into hypersurfaces according to (5.5), (5.6), (5.7). This split allows one to expand (5.162) in terms of the generalized Fefferman-Graham form (5.18) and determine the leading order term $\gamma_{ij}^{(0)}$ and the next to leading order term $\gamma_{ij}^{(1)}$. Then, one ends up with the expressions (5.61) and (5.62) respectively, for $\ell = 1$. Additionally, the group that describes the asymptotic symmetries of the MKR spacetime (5.162) is considered to be the conformal symmetry group $O(3, 2)$. That is, the procedure consists of the boundary conditions preserving gauge transformations analysis of section 5.4.1. Consequently, the asymptotic symmetry algebras at order $\mathcal{O}(1)$ and at order $\mathcal{O}(\rho)$ for the MKR solution are determined by the orders $\mathcal{O}(1)$ and $\mathcal{O}(\rho)$ of (5.103) respectively.

Asymptotic symmetry algebra of the $\mathcal{O}(1)$ equation

The induced boundary metric (5.61) for $\ell = 1$ takes the form

$$\gamma^{(0)}_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin^2\theta \end{pmatrix} . \tag{5.171}$$

The leading order $\mathcal{O}(1)$ of (5.103) takes the form (5.106), which is repeated here for clarity

$$\mathcal{L}_{\xi^k_{(0)}} \gamma^{(0)}_{ij} - \frac{2}{3}\gamma^{(0)}_{ij} D_k \xi^k_{(0)} = 0 . \tag{5.172}$$

Inserting (5.171) into the above equation one finds that it admits the 10 conformal Killing vectors (5.108)-(5.111) expressed in spherical coordinates. In particular, they are

$$\xi^1_{(0)} = \partial_t \tag{5.173}$$
$$\xi^2_{(0)} = \partial_\phi \ , \ \xi^3_{(0)} = \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi \ , \ \xi^4_{(0)} = -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi \tag{5.174}$$
$$\xi^5_{(0)} = \cos t\cos\theta\,\partial_t - \sin t\sin\theta\,\partial_\theta \tag{5.175}$$
$$\xi^6_{(0)} = \cos t\sin\theta\cos\phi\,\partial_t + \sin t\cos\theta\cos\phi\,\partial_\theta - \sin t\frac{\sin\phi}{\sin\theta}\,\partial_\phi \tag{5.176}$$
$$\xi^7_{(0)} = \cos t\sin\theta\sin\phi\,\partial_t + \sin t\cos\theta\sin\phi\partial_\theta + \sin t\frac{\cos\phi}{\sin\theta}\,\partial_\phi \tag{5.177}$$
$$\xi^8_{(0)} = -\sin t\sin\theta\sin\phi\,\partial_t + \cos t\cos\theta\sin\phi\,\partial_\theta + \cos t\frac{\cos\phi}{\sin\theta}\,\partial_\phi \tag{5.178}$$
$$\xi^9_{(0)} = -\sin t\cos\theta\,\partial_t - \cos t\sin\theta\,\partial_\theta \tag{5.179}$$
$$\xi^{10}_{(0)} = -\sin t\sin\theta\cos\phi\,\partial_t + \cos t\cos\theta\cos\phi\,\partial_\theta - \cos t\frac{\sin\phi}{\sin\theta}\,\partial_\phi \tag{5.180}$$

with $\xi^1_{(0)}$ being the time translation, $\xi^2_{(0)}, \xi^3_{(0)}, \xi^4_{(0)}$ being the $o(3)$ generators and the rest conformal Killing vectors are linear combinations of the dilatation, the boosts and the SCTs in spherical coordinates. The non-vanishing commutation relations are

$$[\xi^1_{(0)}, \xi^5_{(0)}] = \xi^9_{(0)} \ , \ [\xi^1_{(0)}, \xi^6_{(0)}] = \xi^{10}_{(0)} \ , \ [\xi^1_{(0)}, \xi^7_{(0)}] = \xi^8_{(0)} \ , \ [\xi^1_{(0)}, \xi^8_{(0)}] = -\xi^7_{(0)} \ , \ [\xi^1_{(0)}, \xi^9_{(0)}] = -\xi^5_{(0)} \ , \ [\xi^1_{(0)}, \xi^{10}_{(0)}] = -\xi^6_{(0)} \tag{5.181}$$
$$[\xi^2_{(0)}, \xi^3_{(0)}] = \xi^4_{(0)} \ , \ [\xi^2_{(0)}, \xi^4_{(0)}] = -\xi^3_{(0)} \ , \ [\xi^2_{(0)}, \xi^6_{(0)}] = -\xi^7_{(0)} \ , \ [\xi^2_{(0)}, \xi^7_{(0)}] = -\xi^6_{(0)} \ , \ [\xi^2_{(0)}, \xi^8_{(0)}] = -\xi^{10}_{(0)} \ , \ [\xi^2_{(0)}, \xi^{10}_{(0)}] = -\xi^8_{(0)} \tag{5.182}$$
$$[\xi^3_{(0)}, \xi^4_{(0)}] = \xi^2_{(0)} \ , \ [\xi^3_{(0)}, \xi^5_{(0)}] = -\xi^6_{(0)} \ , \ [\xi^3_{(0)}, \xi^6_{(0)}] = \xi^5_{(0)} \ , \ [\xi^3_{(0)}, \xi^9_{(0)}] = -\xi^{10}_{(0)} \ , \ [\xi^3_{(0)}, \xi^{10}_{(0)}] = \xi^9_{(0)} \tag{5.183}$$
$$[\xi^4_{(0)}, \xi^5_{(0)}] = \xi^7_{(0)} \ , \ [\xi^4_{(0)}, \xi^7_{(0)}] = -\xi^5_{(0)} \ , \ [\xi^4_{(0)}, \xi^8_{(0)}] = -\xi^9_{(0)} \ , \ [\xi^4_{(0)}, \xi^9_{(0)}] = \xi^8_{(0)} \tag{5.184}$$
$$[\xi^5_{(0)}, \xi^6_{(0)}] = \xi^3_{(0)} \ , \ [\xi^5_{(0)}, \xi^7_{(0)}] = -\xi^4_{(0)} \ , \ [\xi^5_{(0)}, \xi^9_{(0)}] = -\xi^1_{(0)} \tag{5.185}$$
$$[\xi^6_{(0)}, \xi^7_{(0)}] = \xi^2_{(0)} \ , \ [\xi^6_{(0)}, \xi^{10}_{(0)}] = -\xi^1_{(0)} \tag{5.186}$$
$$[\xi^7_{(0)}, \xi^8_{(0)}] = -\xi^1_{(0)} \tag{5.187}$$
$$[\xi^8_{(0)}, \xi^9_{(0)}] = \xi^4_{(0)} \ , \ [\xi^8_{(0)}, \xi^{10}_{(0)}] = -\xi^2_{(0)} \tag{5.188}$$
$$[\xi^9_{(0)}, \xi^{10}_{(0)}] = \xi^3_{(0)} . \tag{5.189}$$

Therefore, the Killing vectors of $\mathbb{R} + so(3)$ which correspond to the spacetime symmetries of the MKR solution are indeed enlarged asymptotically, at the leading order, to the conformal Killing vectors of $o(3,2)$.

Asymptotic symmetry algebra of the $\mathcal{O}(\rho)$ equation

For $\ell = 1$, the next to leading order term (5.62) becomes

$$\gamma_{ij}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2\alpha & 0 \\ 0 & 0 & -2\alpha \sin^2\theta \end{pmatrix} . \tag{5.190}$$

The next to leading order $\mathcal{O}(\rho)$ of (5.103) becomes

$$-2\lambda^{(2)}\gamma_{ij}^{(0)} - \kappa\gamma_{ij}^{(1)} + \mathcal{L}_{\xi^k_{(0)}}\gamma_{ij}^{(1)} + 2\Omega^{(1)}\gamma_{ij}^{(0)} = 0\,. \tag{5.191}$$

Setting $\lambda^{(2)} = -\Omega^{(1)}$ like in the case of a flat boundary metric and using (5.105), the above expression takes the form

$$\mathcal{L}_{\xi^k_{(0)}}\gamma_{ij}^{(1)} - \frac{1}{3}\gamma_{ij}^{(1)}\partial_k\xi^k_{(1)} + 4\Omega^{(1)}\gamma_{ij}^{(0)} = 0\,. \tag{5.192}$$

Taking the trace and solving for $\Omega^{(1)}$ one finds

$$\Omega^{(1)} = -\frac{1}{6}\gamma^i_{k(1)}D_i\xi^k_{(0)}\,. \tag{5.193}$$

Thus, the final form of (5.192) is

$$\mathcal{L}_{\xi^k_{(0)}}\gamma_{ij}^{(1)} - \frac{1}{3}\gamma_{ij}^{(1)}\partial_k\xi^k_{(1)} - \frac{2}{3}\gamma_{ij}^{(0)}\gamma^i_{(1)k}\partial_i\xi^k_{(0)} = 0\,. \tag{5.194}$$

Îd'he time translation (5.173) and the $so(3)$ generators (5.174) satisfy (5.194) and constitute of the subalgebra $\mathbb{R} + so(3)$ with 4 generators. The commutation relations are

$$[\xi^2_{(0)}, \xi^3_{(0)}] = \xi^4_{(0)} \quad , \quad [\xi^2_{(0)}, \xi^4_{(0)}] = -\xi^3_{(0)} \tag{5.195}$$
$$[\xi^3_{(0)}, \xi^4_{(0)}] = \xi^2_{(0)} \tag{5.196}$$
$$[\xi^1_{(0)}, \xi^i_{(0)}] = 0 \quad \forall\, i = 2,3,4\,. \tag{5.197}$$

It is noteworthy that the attempt to enlarge the above 4-dimensional subalgebra fails, in the sense that it requires $\gamma_{ij}^{(1)} = 0$ and thus yields $o(3,2)$.

5.4.5 *The Rindler-Kerr-(A)dS solution*

The Rindler-Kerr-(A)dS spacetime is a rotating black hole solution of the conformal gravity action (5.1) with vanishing mass parameter. Its line element is

$$ds^2 = \rho^2\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(a dt - (r^2+a^2)\frac{d\phi}{\Xi}\right)^2 - \frac{\Delta_r}{\rho^2}\left(dt - a\sin^2\theta\frac{d\phi}{\Xi}\right)^2 \tag{5.198}$$

where

$$\Delta_r = (r^2+a^2)(1 - \frac{1}{3}\Lambda r^2) - 2\mu r^3 \tag{5.199}$$
$$\Delta_\theta = 1 + \frac{1}{3}\Lambda a^2\cos^2\theta \tag{5.200}$$
$$\Xi = 1 + \frac{1}{3}\Lambda a^2 \tag{5.201}$$
$$\rho^2 = r^2 + a^2\cos^2\theta \tag{5.202}$$

with a being the rotation parameter, μ the Rindler acceleration and Λ the cosmological constant.

The spacetime (5.198) is stationary and axisymmetric. Thus, the generators of these symmetries satisfying $\pounds_{\xi^c}\, g_{ab} = 0$ are the time translation and one rotation, i.e.

$$\xi^1 = \partial_t \tag{5.203}$$
$$\xi^2 = \partial_\phi \tag{5.204}$$

and they commute with each other. In what follows it will be shown that this 2-dimensional algebra of the spacetime symmetries can be enlarged asymptotically, at the leading order $\mathcal{O}(1)$ of (5.103).

Proceeding now with the asymptotic analysis, one follows the exact same steps like in the case of the MKR solution: applying the procedure in the beginning of the chapter, the spacetime metric of (5.198) is rewritten suitably and is expanded in terms of the generalized Fefferman-Graham form (5.18). Consequently, the leading order term $\gamma_{ij}^{(0)}$ and the next to leading order term $\gamma_{ij}^{(1)}$ are determined and one ends up with the expressions (5.78) and (5.79) respectively, for $\ell = 1$. It is then shown that $\gamma_{ij}^{(0)}$ is conformally equivalent with the MKR boundary metric (5.171) and thus they admit the same symmetry group, that is $O(3,2)$. Finally, according to the boundary conditions preserving gauge transformations analysis of section 5.4.1, the realized subalgebra of $o(3,2)$ is determined by the order $\mathcal{O}(\rho)$ of (5.103).

Asymptotic symmetry algebra of the $\mathcal{O}(1)$ equation

For $\ell = 1$, the Rindler-Kerr-(A)dS boundary metric (5.78) becomes

$$\gamma_{ij}^{(0)} = \begin{pmatrix} -1 & 0 & \frac{a\sin^2\psi}{1-a^2} \\ 0 & \frac{1}{1-a^2\cos^2\psi} & 0 \\ \frac{a\sin^2\psi}{1-a^2} & 0 & \frac{\sin^2\psi}{1-a^2} \end{pmatrix} \tag{5.205}$$

in the generalized Fefferman-Graham expansion (5.18). It will be now shown that under a coordinate transformation this is conformally equivalent to the MKR boundary metric (5.171). Therefore it admits the same asymptotic symmetry algebra, namely $o(3,2)$. Then, the conformal Killing vectors of (5.205) are constructed by applying this coordinate transformation to the conformal Killing vectors (5.173)-(5.180) of the MKR boundary metric.

Starting with the coordinate transformation

$$t = \frac{\tau}{\sqrt{Aa}} \tag{5.206}$$
$$\phi = \frac{\phi' - a\tau}{\sqrt{Aa}} \tag{5.207}$$

where $A = \frac{1}{1-a^2}$, the boundary metric (5.205) is brought into the diagonal form

$$\gamma_{ij}^{(0)} = \begin{pmatrix} -\frac{1}{aA} - a\sin^2\psi & 0 & 0 \\ 0 & \frac{1}{1-a^2\cos^2\psi} & 0 \\ 0 & 0 & \frac{\sin^2\psi}{a} \end{pmatrix}. \tag{5.208}$$

Performing now a Weyl rescaling to the above metric such that

$$\gamma_{ij}^{(0)} \left(\frac{1}{aA} + a\sin^2\psi\right)^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{a}{(1-a^2\cos^2\psi)^2} & 0 \\ 0 & 0 & \frac{\sin^2\psi}{1-a^2\cos^2\psi} \end{pmatrix} \equiv \gamma_{ij}'^{(0)} \tag{5.209}$$

and then transforming the coordinate ψ as

$$\psi = -\cos^{-1}\left[\sqrt{\frac{1+\cos^2 u - \sin^2 u}{-2 + a - a^2\cos^2 u + a^2\sin^2 u}}\right] \tag{5.210}$$

one finds that $\gamma'^{(0)}_{ij}$ becomes

$$\gamma'^{(0)}_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & \sin^2 u \end{pmatrix} \tag{5.211}$$

with $B = aA = \frac{a}{1-a^2}$. Finally, after a redefinition of the coordinates as $\tau \to \sqrt{B}\tau$, $\phi' \to \sqrt{B}\phi'$, the final form of (5.211) is

$$\gamma^{(1)}_{ij} = B \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin^2 u \end{pmatrix} . \tag{5.212}$$

This is indeed the MKR boundary metric (5.171) up to the constant B . Therefore, the Rindler-Kerr-(A)dS boundary metric (5.205) is conformally equivalent with the MKR boundary metric, subject to the Weyl rescaling (5.209) and the coordinate transformations (5.206), (5.207), (5.210).

Consequently, transforming the conformal Killing vectors (5.173)-(5.180) according to (5.206), (5.207), (5.210) one finds the 10 conformal Killing vectors of the Rindler-Kerr-(A)dS metric (5.208) to be

$$\xi^1_{(0)} = \sqrt{B}\partial_\tau \ , \ \xi^2_{(0)} = \sqrt{B}\partial_{\phi'} \tag{5.213}$$

$$\xi^3_{(0)} = \cos(\frac{\phi'}{\sqrt{B}})\, F(\psi)\, \partial_\psi - \cot u(\psi) \sin(\frac{\phi'}{\sqrt{B}}) \sqrt{B}\, \partial_{\phi'} \tag{5.214}$$

$$\xi^4_{(0)} = -\sin(\frac{\phi'}{\sqrt{B}})\, F(\psi)\, \partial_\psi - \cot u(\psi) \cos(\frac{\phi'}{\sqrt{B}}) \sqrt{B}\, \partial_{\phi'} \tag{5.215}$$

$$\xi^5_{(0)} = \sqrt{B} \cos(\frac{\tau}{\sqrt{B}}) \cos u(\psi)\, \partial_\tau - \sin(\frac{\tau}{\sqrt{B}}) \sin u(\psi) F(\psi)\, \partial_\psi \tag{5.216}$$

$$\begin{aligned} \xi^6_{(0)} = {} & \sqrt{B} \cos(\frac{\tau}{\sqrt{B}}) \sin u(\psi) \cos(\frac{\phi'}{\sqrt{B}})\, \partial_\tau + \sin(\frac{\tau}{\sqrt{B}}) \cos u(\psi) \cos(\frac{\phi'}{\sqrt{B}}) F(\psi)\, \partial_\psi \\ & - \sin(\frac{\tau}{\sqrt{B}}) \frac{\sin(\frac{\phi'}{\sqrt{B}})}{\sin u(\psi)} \sqrt{B}\, \partial_{\phi'} \end{aligned} \tag{5.217}$$

$$\begin{aligned} \xi^7_{(0)} = {} & \sqrt{B} \cos(\frac{\tau}{\sqrt{B}}) \sin u(\psi) \sin(\frac{\phi'}{\sqrt{B}})\, \partial_\tau + \sin(\frac{\tau}{\sqrt{B}}) \cos u(\psi) \sin(\frac{\phi'}{\sqrt{B}}) F(\psi) \partial_\psi \\ & + \sin(\frac{\tau}{\sqrt{B}}) \frac{\cos(\frac{\phi'}{\sqrt{B}})}{\sin u(\psi)} \sqrt{B}\, \partial_{\phi'} \end{aligned} \tag{5.218}$$

$$\begin{aligned} \xi^8_{(0)} = {} & -\sqrt{B} \sin(\frac{\tau}{\sqrt{B}}) \sin u(\psi) \sin(\frac{\phi'}{\sqrt{B}})\, \partial_\tau + \cos(\frac{\tau}{\sqrt{B}}) \cos u(\psi) \sin(\frac{\phi'}{\sqrt{B}}) F(\psi)\, \partial_\psi \\ & + \cos(\frac{\tau}{\sqrt{B}}) \frac{\cos(\frac{\phi'}{\sqrt{B}})}{\sin u(\psi)} \sqrt{B}\, \partial_{\phi'} \end{aligned} \tag{5.219}$$

$$\xi^9_{(0)} = -\sqrt{B} \sin(\frac{\tau}{\sqrt{B}}) \cos u(\psi)\, \partial_\tau - \cos(\frac{\tau}{\sqrt{B}}) \sin u(\psi) F(\psi)\, \partial_\psi \tag{5.220}$$

$$\begin{aligned} \xi^{10}_{(0)} = {} & -\sqrt{B} \sin(\frac{\tau}{\sqrt{B}}) \sin u(\psi) \cos(\frac{\phi'}{\sqrt{B}})\, \partial_\tau + \cos(\frac{\tau}{\sqrt{B}}) \cos u(\psi) \cos(\frac{\phi'}{\sqrt{B}}) F(\psi)\, \partial_\psi \\ & - \cos(\frac{\tau}{\sqrt{B}}) \frac{\sin(\frac{\phi'}{\sqrt{B}})}{\sin u(\psi)} \sqrt{B}\, \partial_{\phi'} \end{aligned} \tag{5.221}$$

where $u(\psi) = \sin^{-1}\left[\sqrt{\frac{\sin^2\psi}{1-a^2\cos^2\psi}}\right]$ and $F(\psi) = \frac{(1-a^2\cos\psi^2)\cos\psi\sin\psi}{\sqrt{(a^2-1)\cos^2\psi\sin^2\psi}}$. Their commutation relations take the form (5.181)-(5.189). Thus, the Killing vectors of the spacetime symmetries of the Rindler-Kerr-(A)dS line element (5.198) are indeed enlarged asymptotically, at the leading order, to the conformal Killing vectors of $o(3,2)$.

Asymptotic symmetry algebra of the $\mathcal{O}(\rho)$ equation

For $\ell = 1$, the next to leading order term (5.79) becomes

$$\gamma_{ij}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2\mu}{1-a^2\cos^2\psi} & 0 \\ 0 & 0 & \frac{2\mu(1-a^2\cos^2\psi)\sin^2\psi}{(1-a^2)^2} \end{pmatrix} \tag{5.222}$$

in the generalized Fefferman-Graham gauge (5.18). After performing the coordinate transformations (5.206), (5.207) this becomes

$$\gamma_{ij}^{(1)} = \begin{pmatrix} \frac{a\mu(-2+a^2+a^2\cos 2\psi)\sin^2\psi}{-1+a^2} & 0 & -\frac{\mu(-2+a^2+a^2\cos 2\psi)\sin^2\psi}{-1+a^2} \\ 0 & \frac{2\mu}{1-a^2\cos^2\psi} & 0 \\ -\frac{\mu(-2+a^2+a^2\cos 2\psi)\sin^2\psi}{-1+a^2} & 0 & \frac{\mu(-2+a^2+a^2\cos 2\psi)\sin^2\psi}{-1+a^2} \end{pmatrix}. \tag{5.223}$$

The order $\mathcal{O}(\rho)$ of (5.103) takes the form (5.191). Following the same steps as in the corresponding case of the MKR spacetime, i.e. setting $\lambda^{(2)} = -\Omega^{(1)}$ and solving for $\Omega^{(1)}$, the final expression is (5.194). This is repeated once again here

$$\mathcal{L}_{\xi^k_{(0)}}\gamma_{ij}^{(1)} - \frac{1}{3}\gamma_{ij}^{(1)}\partial_k\xi^k_{(1)} - \frac{2}{3}\gamma_{ij}^{(0)}\gamma^i_{(1)k}\partial_i\xi^k_{(0)} = 0\,. \tag{5.224}$$

Inserting (5.223) into the above, it is found that it admits the two conformal Killing vectors (5.213), i.e. the ones that generate time translation and rotation around the ϕ-axis. Thus, the subalgebra of $o(3,2)$ in this case is 2-dimensional and the Killing vectors commute with each other. Any attempt to enlarge this 2-dimensional subalgebra fails, in the sense that it gives $\gamma_{ij}^{(1)} = 0$ and thus $o(3,2)$ is recovered.

5.5 SUMMARY AND CONCLUSIONS

In this chapter the conformal gravity action was considered in the concept of holographic analysis and the investigation of the asymptotic symmetry algebras of the dual field theory. The original work concerning the holographic setup can be found in [82] and the analysis about the asymptotic symmetry algebras can be found in [83].

It was found that the first on-shell variation of action (5.1) vanishes, under the proposed asymptotic boundary conditions (5.37), (5.38). Thus, the setup renders a well-defined variational principle. Additionally, the holographic response functions conjugate to the sources $\gamma_{ij}^{(0)}$ and $\gamma_{ij}^{(1)}$ were evaluated and were found to be finite. Therefore, no addition of holographic counterterms is required. Then, three concrete solutions of the equations of motion of (5.1) were considered, namely solutions with $\gamma_{ij}^{(1)} = 0$, the MKR and the Rindler-Kerr-(A)dS spacetime. Their holographic response functions and physical quantities, like energy, angular momentum and entropy, were calculated. For the MKR solution it was found that the entropy obeys an area law, despite the fact that the conformal gravity action (5.1) constitutes a higher derivative theory. Later on, the asymptotic symmetry algebra of the dual field theory was explored under the proposed asymptotic boundary conditions (5.37), (5.38). This was done following a boundary conditions preserving gauge transformations analysis, which led to solving (5.85) at order $\mathcal{O}(1)$. The asymptotic symmetry algebra was found to be $o(3,2)$. It was also examined which of the subalgebras of $o(3,2)$ were realized from the order $\mathcal{O}(\rho)$ of (5.85), for a particular $\gamma_{ij}^{(1)}$. The highest dimensional subalgebra that was obtained was 5-dimensional, consisting of 3 translations, a linear combination of the dilatation and a boost and a linear combination of the other boost and the rotation. Lower dimensional subalgebras were also realized, e.g. 4- and 3-dimensional. Lastly, the MKR and Rindler-Kerr-(A)dS solutions were considered once again in order to obtain their asymptotic symmetry algebras. These were found to be $o(3,2)$, at order $\mathcal{O}(1)$

of (5.85). At order $\mathcal{O}(\rho)$ of (5.85), it was found that both solutions admit one subalgebra each, that of their original spacetime symmetries, i.e. $\mathbb{R} + so(3)$ for the MKR solution and the 2-dimensional algebra consisting of a time translation and one rotation for the Rindler-Kerr-(A)dS solution.

6

HAMILTONIAN ANALYSIS

Throughout this chapter, the Hamiltonian formulation of conformal gravity is performed. The analysis starts with formulating the general setup in 6.1, by introducing the canonical variables, the constraints of the theory and the construction of the Hamiltonian. Then, the Poisson bracket algebra of the constraints and their geometric interpretation is discussed in 6.2. The generator of gauge symmetries of the theory is presented in 6.3 and in 6.4, after establishing a set of boundary conditions the canonical charges associated with asymptotic symmetries are derived. Finally, in 6.5, the asymptotic symmetry algebra of the canonical charges is presented.

6.1 GENERAL SETUP

The conformal gravity action is

$$S = -\frac{1}{4}\int_M d^4x\sqrt{-g}\; C^a{}_{bcd}\; C_a{}^{bcd}\,. \tag{6.1}$$

The dimensionless coupling constant of the theory has been set to $\alpha_{\text{CG}} = -\frac{1}{4}$ for convenience, as it is described in Appendix A.3. Using the ADM foliation [50] the Lagrangian in (6.1) becomes

$$\begin{aligned} L = \int_{\Sigma_t} d^3x\,\sqrt{|h|}N\Big[&- \Big(h^c_a h^d_b - \frac{1}{3}h_{ab}h^{cd}\Big)\Big(\mathcal{R}_{cd} + K_{cd}K - \frac{1}{N}(\dot{K}_{cd} - {}^3\pounds_{N^a}K_{cd} - D_cD_dN)\Big) \\ &\times \frac{1}{2}\Big(h^{ca}h^{db} - \frac{1}{3}h^{ab}h^{cd}\Big)\Big(\mathcal{R}_{cd} + K_{cd}K - \frac{1}{N}(\dot{K}_{cd} - {}^3\pounds_{N^a}K_{cd} - D_cD_dN)\Big) \\ &+ \Big(2D_{[a}K_{b]c} + D_dK^d_{[a}h_{b]c} - D_{[a}Kh_{b]c}\Big) \times \Big(2D^{[a}K^{b]c} + D_dK^{d[a}h^{b]c} - D_{[a}Kh^{b]c}\Big)\Big]\,. \end{aligned} \tag{6.2}$$

This expression is derived explicitly in Appendix A.3. It is obvious that it contains second-order time derivatives of the metric h_{ab} and therefore the equations of motion will have up to fourth-order time derivatives. This is not problematic in the Lagrangian formulation but in the Hamiltonian setup the equations of motion contain only first-order time derivatives. This problem is handled by introducing additional fields in the Lagrangian and therefore enlarging the phase space in such a way as to construct an action that is first-order in time derivatives. Here, this is done by regarding h_{ab} and K_{ab} as independent canonical variables and the relation

$$\dot{h}_{ab} = 2(NK_{ab} + D_{(a}N_{b)}) \tag{6.3}$$

between them is treated as a constraint, multiplied with the additional auxiliary field λ^{ab}. The Lagrangian (6.2) now takes the form

$$\begin{aligned} L = \int_{\Sigma_t} d^3x\,\sqrt{h}\,N\Big[&- \Big(h^c_a h^d_b - \frac{1}{3}h_{ab}h^{cd}\Big)\Big(\mathcal{R}_{cd} + K_{cd}K - \frac{1}{N}(\dot{K}_{cd} - {}^3\pounds_{N^a}K_{cd} - D_cD_dN)\Big) \\ &\times \frac{1}{2}\Big(h^{ca}h^{db} - \frac{1}{3}h^{ab}h^{cd}\Big)\Big(\mathcal{R}_{cd} + K_{cd}K - \frac{1}{N}(\dot{K}_{cd} - {}^3\pounds_{N^a}K_{cd} - D_cD_dN)\Big) \\ &+ \Big(2D_{[a}K_{b]c} + D_dK^d_{[a}h_{b]c} - D_{[a}Kh_{b]c}\Big) \times \Big(2D^{[a}K^{b]c} + D_dK^{d[a}h^{b]c} - D_{[a}Kh^{b]c}\Big) \\ &+ \lambda^{ab}\Big(\frac{1}{N}(\dot{h}_{ab} - {}^3\pounds_{N^c}h_{ab}) - 2K_{ab}\Big)\Big]\,. \end{aligned} \tag{6.4}$$

The canonical momenta $\pi_g^{ab} = \frac{\delta L}{\delta \dot{g}_{ab}}$, $\Pi_K^{ab} = \frac{\delta L}{\delta \dot{K}_{ab}}$ and $p_{ab} = \frac{\delta L}{\delta \dot{\lambda}^{ab}}$ are

$$\pi_h^{ab} = \frac{\delta L}{\delta \dot{h}_{ab}} = \sqrt{|h|}\,\lambda^{ab} \tag{6.5}$$

$$\pi = \frac{\delta L}{\delta \dot{N}} = 0 \tag{6.6}$$

$$\pi_a = \frac{\delta L}{\delta \dot{N}^a} = 0 \tag{6.7}$$

$$\Pi_K^{ab} = \frac{\delta L}{\delta \dot{K}_{ab}} = -\sqrt{h}G^{abcd}\left({}^3\pounds_{N^a}K_{cd} - \mathcal{R}_{cd} - KK_{cd} - \frac{1}{N}D_cD_dN\right) \tag{6.8}$$

$$p_{ab} = 0 \tag{6.9}$$

where $G^{abcd} = \frac{1}{2}(h^{ac}h^{bd} + h^{ad}h^{bc}) - \frac{1}{3}h^{ab}h^{cd}$. At this point, it is observed that in (6.5), $\sqrt{|h|}\lambda^{ab}$ appears as the conjugate momentum π_h^{ab}. Thus, if the auxiliary field λ^{ab} and its conjugate momentum p_{ab} are to be treated as canonical variables in the analysis, then, the additional primary constraints $\phi_1 \equiv \pi_h^{ab} - \sqrt{|h|}\,\lambda^{ab} \approx 0$ and $\phi_2 \equiv p_{ab} \approx 0$ appear. It is shown in Appendix C.3 that the Hamiltonian analysis including ϕ_1 and ϕ_2 as primary constraints, i.e. considering λ^{ab} and p_{ab} as canonical variables, is equivalent with substituting $\pi_h^{ab} = \sqrt{|h|}\,\lambda^{ab}$ and $p_{ab} = 0$ in (6.4). With this substitution, the auxiliary field and its conjugate momentum are eliminated from the Lagrangian (6.4) and they are not canonical variables anymore. Expressions (6.6), (6.7) are primary constraints. One also observes by taking the trace of (6.8) that

$$\mathcal{P} \equiv h_{ab}\Pi_K^{ab} = 0 \tag{6.10}$$

which is a primary constraint as well. The origin of this constraint is due the fact that the Weyl tensor is traceless in any pair of its indices: the r.h.s of (6.8) is part of $P_h[n^cn^dC_{acbd}]$ (electric part of the Weyl tensor), as it is shown in (A.48) in Appendix A.3, and thus renders Π_K^{ab} traceless.

In terms of canonical variables, the Lagrangian (6.4) after a partial integration becomes

$$\begin{aligned} L = \int_{\Sigma_t} d^3x\, \Big[&\Pi_K^{ab}\dot{K}_{ab} + \pi_h^{ab}\dot{h}_{ab} - N\Big(-\frac{1}{\sqrt{|h|}}\frac{\Pi_K^{ab}\Pi_{ab}^K}{2} + D_aD_b\Pi_K^{ab} + \Pi_K^{ab}(\mathcal{R}_{ab} + K_{ab}K) + 2\pi_h^{ab}K_{ab} \\ &- \sqrt{|h|}\big(2D_{[a}K_{b]c} + D_dK^d_{[a}h_{b]c} - D_{[a}Kh_{b]c}\big) \times \big(2D^{[a}K^{b]c} + D_dK^{d[a}h^{b]c} - D^{[a}Kh^{b]c}\big)\Big) \\ &- N^a\Big(\Pi_K^{cd}D_aK_{cd} - 2D_d(\Pi_K^{cd}K_{ac}) - D_d\pi_h^{cd}h_{ac}\Big)\Big] \\ &- \int_{\partial\Sigma} d^2S_a\, \Big[\Pi_K^{ab}D_bN - D_b\Pi_K^{ab}N + 2N^c(\pi_h^{ab}h_{bc} + \Pi_K^{ab}K_{bc})\Big]. \end{aligned} \tag{6.11}$$

Although the surface term in the above expression is dynamically irrelevant, it will be kept explicitly throughout the analysis. The reason for doing this will become evident later, when the canonical charges subject to appropriate boundary conditions are constructed. It is worthwhile to be mentioned that the surface term in (6.11) renders the conformal gravity Lagrangian (6.11) to have well-defined functional derivatives, under some suitable boundary conditions. Not surprisingly, the resulting total Hamiltonian of (6.11) is already an improved gauge generator under these boundary conditions. Additionally, due to the specific form of the secondary constraint $\mathcal{W}$ which is derived later in the analysis, the extended Hamiltonian arising from (6.11) has also well-defined functional derivatives, under these boundary conditions. These aspects are not surprising, since the fact that the conformal gravity action (6.1) plus some appropriate boundary conditions has well-defined functional derivatives was already demonstrated in sections 5.1 and 5.2.3, following the Lagrangian formalism: it was stated that the action (6.1) and some suitable boundary conditions constitute a well-defined boundary value problem.

The canonical Hamiltonian is

$$H_c = \int_{\Sigma_t} d^3x\, \left(\Pi_K^{ab}\dot{K}_{ab} + \pi_h^{ab}\dot{h}_{ab}\right) - L = \int_{\Sigma_t} d^3x \Big(N^a\mathcal{H}_a + N\mathcal{H}_\perp\Big) + \int_{\partial\Sigma_t} d^2S_a\, \left(\mathcal{Q}_\perp^a + \mathcal{Q}^a\right) \tag{6.12}$$

where

$$\begin{aligned}
\mathcal{H}_\perp &\equiv -\frac{1}{\sqrt{|h|}}\frac{\Pi_K^{ab}\Pi^K_{ab}}{2} + D_aD_b\Pi_K^{ab} + \Pi_K^{ab}(\mathcal{R}_{ab} + K_{ab}K) + 2\pi_h^{ab}K_{ab} \\
&\quad - \sqrt{|h|}\big(2D_{[a}K_{b]c} + D_dK^d_{[a}h_{b]c} - D_{[a}Kh_{b]c}\big) \times \big(2D^{[a}K^{b]c} + D_dK^{d[a}h^{b]c} - D^{[a}Kh^{b]c}\big) && (6.13)\\
\mathcal{H}_a &\equiv \Pi_K^{bc}D_aK_{bc} - 2D_c(\Pi_K^{cd}K_{ad}) - 2D_c\pi_h^{cd}h_{ad} && (6.14)\\
\mathcal{Q}^a_\perp &\equiv \Pi_K^{ab}D_bN - D_b\Pi_K^{ab}N && (6.15)\\
\mathcal{Q}^a &\equiv 2N^c(\pi_h^{ab}h_{bc} + \Pi_K^{ab}K_{bc})\,. && (6.16)
\end{aligned}$$

Finally, the total Hamiltonian takes the form

$$H_T = \int_{\Sigma_t} d^3x\Big(N^a\mathcal{H}_a + N\mathcal{H}_\perp + \lambda\pi + \lambda^a\pi_a + \lambda_{\mathcal{P}}\mathcal{P}\Big) + \int_{\partial\Sigma_t} d^2S_a\,\Big(\mathcal{Q}^a_\perp + \mathcal{Q}^a\Big) \tag{6.17}$$

where $\lambda = \lambda(x)$, $\lambda^a = \lambda^a(x)$, $\lambda_{\mathcal{P}} = \lambda_{\mathcal{P}}(x)$ are arbitrary functions.

The consistency conditions for the primary constraints (6.6), (6.7), (6.10) reveal the secondary constraints

$$\begin{aligned}
\{\pi, H_T\} &= -\mathcal{H}_\perp && (6.18)\\
\{\pi_a, H_T\} &= -\mathcal{H}_a && (6.19)\\
\{\mathcal{P}, H_T\} &= -NK\mathcal{P} + D_a(N^a\mathcal{P}) - N(\Pi_K^{ab}K_{ab} + 2\pi_h^{ab}h_{ab}) \\
&\approx -N(\Pi_K^{ab}K_{ab} + 2\pi_h^{ab}h_{ab}) = -N\mathcal{W} && (6.20)
\end{aligned}$$

with

$$\mathcal{W} \equiv \Pi_K^{ab}K_{ab} + 2\pi_h^{ab}h_{ab}\,. \tag{6.21}$$

One notices that the first two secondary constraints (6.18), (6.19) already exist in the total Hamiltonian (6.17), exactly like in the case of General Relativity. The time evolution of all (6.18), (6.19), (6.20) gives

$$\begin{aligned}
\{\mathcal{H}_\perp(x), H_T\} &= \Big[-N^a\mathcal{H}_\perp(x) + N\Big(\mathcal{H}^a(x) + \mathcal{H}^a(y) + (\mathcal{P}(x)+\mathcal{P}(y))(D_bK^b_a - D_aK)\Big)\Big]\partial_a\delta^{(3)}(\vec{x}-\vec{y}) \\
&\quad + \lambda_{\mathcal{P}}\Big[\mathcal{W}(y) + \mathcal{P}(y)K(y)\Big]\delta^{(3)}(\vec{x}-\vec{y}) \approx 0 && (6.22)\\
\{\mathcal{H}_a(x), H_T\} &= N^b\Big[\mathcal{H}_a(y)\,\partial_b\delta^{(3)}(\vec{x}-\vec{y}) + \mathcal{H}_b(x)\,\partial_a\delta^{(3)}(\vec{x}-\vec{y})\Big] \\
&\quad + \Big[N\mathcal{H}_\perp(x) + \lambda_{\mathcal{P}}\mathcal{P}(x)\Big]\,\partial_a\delta^{(3)}(\vec{x}-\vec{y}) \approx 0 && (6.23)\\
\{\mathcal{W}(x), H_T\} &= 2N\mathcal{P}(y)\partial^2\delta^{(3)}(\vec{x}-\vec{y}) + \Big[-N^a\mathcal{W}(x) + 3N\partial^a\mathcal{P}(x)\Big]\partial_a\delta^{(3)}(\vec{x}-\vec{y}) \\
&\quad + \Big[N\Big(\mathcal{H}_\perp(x) + \partial^2\mathcal{P}(x)\Big) - \lambda_{\mathcal{P}}\mathcal{P}(y)\Big]\delta^{(3)}(\vec{x}-\vec{y}) \approx 0\,. && (6.24)
\end{aligned}$$

No new constraints are generated and thus, there are no further secondary constraints. All are 1st class since

$$\{\mathcal{H}_\perp(x),\mathcal{H}_\perp(y)\} = \Big[\mathcal{H}^a(x)+\mathcal{H}^a(y)+\Big(\mathcal{P}(x)+\mathcal{P}(y)\Big)\Big(D_bK^b_a - D_aK\Big)(x)\Big]\,\partial_a\delta^{(3)}(\vec{x}-\vec{y}) \tag{6.25}$$

$$\{\mathcal{H}_a(x),\mathcal{H}_\perp(y)\} = \mathcal{H}_\perp(x)\,\partial_a\delta^{(3)}(\vec{x}-\vec{y}) \tag{6.26}$$

$$\{\mathcal{H}_a(x),\mathcal{H}_b(y)\} = \mathcal{H}_a(y)\,\partial_b\delta^{(3)}(\vec{x}-\vec{y}) + \mathcal{H}_b(x)\,\partial_a\delta^{(3)}(\vec{x}-\vec{y}) \tag{6.27}$$

$$\{\mathcal{H}_a(x),\mathcal{P}(y)\} = \mathcal{P}(x)\,\partial_a\delta^{(3)}(\vec{x}-\vec{y}) \tag{6.28}$$

$$\{\mathcal{H}_a(x),\mathcal{W}(y)\} = \mathcal{W}(x)\,\partial_a\delta^{(3)}(\vec{x}-\vec{y}) \tag{6.29}$$

$$\{\mathcal{P}(x),\mathcal{W}(y)\} = \mathcal{P}(y)\delta^{(3)}(\vec{x}-\vec{y}) \tag{6.30}$$

$$\{\mathcal{P}(x),\mathcal{H}_\perp(y)\} = -\Big(\mathcal{W}(y)+\mathcal{P}(y)K(y)\Big)\delta^{(3)}(\vec{x}-\vec{y}) \tag{6.31}$$

$$\begin{aligned}\{\mathcal{W}(x),\mathcal{H}_\perp(y)\} &= \Big(\mathcal{H}_\perp(x)+\nabla^2\mathcal{P}(x)\Big)\delta^{(3)}(\vec{x}-\vec{y}) + 2\mathcal{P}(y)\nabla^2\delta^{(3)}(\vec{x}-\vec{y})\\ &\quad + 3\partial_a\mathcal{P}(x)\partial^a\delta^{(3)}(\vec{x}-\vec{y})\,.\end{aligned} \tag{6.32}$$

The above relations form an algebra. The discussion and analysis of this Poisson bracket algebra is postponed until section 6.2. Additionally, Hamilton equations of motion using the total Hamiltonian (6.17) are given in Appendix C.4.

The extended Hamiltonian is obtained by adding all secondary 1st class constraints (6.18), (6.19), (6.20) to the total Hamiltonian (6.17), i.e.

$$\begin{aligned}H_E &= H_T + \int_{\Sigma_t} d^3x\Big(\alpha\mathcal{H}_\perp + \alpha^a\mathcal{H}_a + w\mathcal{W}\Big)\\ &= \int_{\Sigma_t} d^3x\Big((\alpha+N)\mathcal{H}_\perp + (\alpha^a+N^a)\mathcal{H}_a + w\mathcal{W} + \lambda\pi + \lambda^a\pi_a + \lambda_{\mathcal{P}}\mathcal{P}\Big)\\ &\quad + \int_{\partial\Sigma_t} d^2S_a\,\Big(\mathcal{Q}^a_\perp + \mathcal{Q}^a\Big)\end{aligned} \tag{6.33}$$

where $\alpha=\alpha(x)$, $\alpha^a=\alpha^a(x)$, $\omega=\omega(x)$ are arbitrary functions. Additionally, according to Hamilton equations of motion of N, N^a using the total Hamiltonian (6.17), which are given in Appendix C.2, N, N^a remain arbitrary functions as well. This can also be deduced from the time evolution (6.22)-(6.24) of the secondary constraints. Indeed, their time evolution is a linear combination of constraints and thus it vanishes on the constraint surface, without restricting N, N^a. Thus, α, α^a, N, N^a can be redefined in (6.33) as

$$H_E = \int_{\Sigma_t} d^3x\Big(\epsilon^\perp\mathcal{H}_\perp + \varepsilon^a\mathcal{H}_a + \lambda\pi + \lambda^a\pi_a + \lambda_{\mathcal{P}}\mathcal{P} + w\mathcal{W}\Big) + \int_{\partial\Sigma_t} d^2S_a\,\Big(\mathcal{Q}^a_\perp + \mathcal{Q}^a\Big) \tag{6.34}$$

where now

$$\mathcal{Q}^a_\perp \equiv \Pi^{ab}_K D_b\epsilon^\perp - \epsilon D_b\Pi^{ab}_K \tag{6.35}$$

$$\mathcal{Q}^a \equiv 2\varepsilon^c(\pi^{ab}_h h_{bc} + \Pi^{ab}_K K_{bc}) \tag{6.36}$$

with $\epsilon=\epsilon(x)$ and $\varepsilon^a=\varepsilon^a(x)$ being arbitrary. It is observed that, in comparison with the total Hamiltonian (6.17), the extended one (6.34) contains additionally the 1st class constraint $\mathcal{W}$. Thus, since 1st class constraints generate gauge symmetries, the extended Hamiltonian (6.34) captures all the gauge freedom of the theory. Nevertheless, for distinguishing the true dynamics of the theory from parts characterizing merely how the coordinate system evolves in time (sections 3.3 and 3.4 of [50]), it is customary to reduce the phase space as follows: the secondary constraints (6.13), (6.14), (6.21) depend neither on N, N^a nor on their conjugate momenta π, π_a. Then, the extended Hamiltonian (6.34) can be considered as consisting of two distinct systems: the first one, being $\int_{\Sigma_t} d^3x\Big(\epsilon^\perp\mathcal{H}_\perp + \varepsilon^a\mathcal{H}_a + \lambda_{\mathcal{P}}\mathcal{P} + w\mathcal{W}\Big) + \int_{\partial\Sigma_t} d^2S_a\,\Big(\mathcal{Q}^a_\perp + \mathcal{Q}^a\Big)$ in which N, N^a are not dynamical variables anymore, reveals the true dynamics of the theory. The second one, consisting of

$\int_{\Sigma_t} d^3x\Big(\lambda\pi + \lambda^a\pi_a\Big)$ characterizes the evolution of the coordinate system in time and is discarded. The extended Hamiltonian (6.34) on the reduced phase space is considered to be

$$H_E = \int_{\Sigma_t} d^3x\Big(\epsilon^{\perp}\mathcal{H}_{\perp} + \varepsilon^a\mathcal{H}_a + \lambda_{\mathcal{P}}\mathcal{P} + w\mathcal{W}\Big) + \int_{\partial\Sigma_t} d^2S_a\,\Big(\mathcal{Q}^a_{\perp} + \mathcal{Q}^a\Big). \tag{6.37}$$

Subsequently, the Hamilton equations of motion are

$$\dot{h}_{ab} = \{h_{ab}, H_E\} = 2D_{(a}\varepsilon_{b)} + 2\epsilon^{\perp}K_{ab} + 2wh_{ab} \tag{6.38}$$

$$\begin{aligned}\dot{\pi}^{ab}_h = \{\pi^{ab}_h, H_E\} = {}& {}^3\pounds_{\varepsilon^c}\pi^{ab}_h - \lambda_{\mathcal{P}}\Pi^{ab}_K - \epsilon^{\perp}\Big[\frac{1}{\sqrt{h}}\Big(\frac{1}{4}\Pi^{cd}_K\Pi^K_{cd}h^{ab} - \Pi^a_{Kc}\Pi^{bc}_K\Big) - \Pi^{cd}_K K_{cd}K^{ab} \\ & + D_cD^{(b}\Pi^{a)c}_K - \frac{1}{2}h^{ab}D_cD_d\Pi^{cd}_K - \frac{1}{2}D_cD^c\Pi^{ab}_K + 2\sqrt{h}\Big(-\frac{1}{4}B^{cde}B_{cde}h^{ab} + B^{acd}B^b_{\ cd} \\ & + \frac{1}{2}B^{cda}B_{cd}{}^b + B^{c(ab)}D_cK - B^{c(ab)}D_dK^d_c - D_c\big(B^{d(ab)}K^c_d + B^{cd(a}K^{b)}_d + B^{(a|dc|}K^{b)}_d\big)\Big)\Big] \\ & - D_c\epsilon^{\perp}\Big[2D_d\Pi^{d(a}_K h^{b)c} + D^{(b}\Pi^{a)c}_K - \frac{3}{2}D^c\Pi^{ab}_K - D_d\Pi^{cd}_K h^{ab} - 2\sqrt{h}\Big(B^{d(ab)}K^c_d + B^{cd(a}K^{b)}_d \\ & + B^{(a|dc|}K^{b)}_d\Big)\Big] - D_cD_d\epsilon^{\perp}\Big[2\Pi^{(d|(a}_K h^{b)|d)} - \Pi^{ab}_K h^{cd} - \frac{1}{2}\Pi^{cd}_K h^{ab}\Big] - 2w\pi^{ab}_h\end{aligned} \tag{6.39}$$

$$\dot{K}_{ab} = \{K_{ab}, H_E\} = \lambda_{\mathcal{P}}h_{ab} + {}^3\pounds_{\varepsilon^c}K_{ab} + \epsilon^{\perp}\Big[\mathcal{R}_{ab} + KK_{ab} - \frac{1}{\sqrt{h}}\Pi^K_{ab}\Big] + D_aD_b\epsilon^{\perp} + wK_{ab} \tag{6.40}$$

$$\begin{aligned}\dot{\Pi}^{ab}_K = \{\Pi^{ab}_K, H_E\} = {}& {}^3\pounds_{\varepsilon^c}\Pi^{ab}_K - \epsilon^{\perp}\Big[\Pi^{ab}_K K + \Pi^{cd}_K K_{cd}h^{ab} + 2\pi^{ab}_h + 4\sqrt{h}D_cB^{cab}\Big] \\ & - 4B^{cab}D_c\epsilon^{\perp} - wK_{ab}\end{aligned} \tag{6.41}$$

where $B_{abc} \equiv 2D_{[a}K_{b]c} + D_dK^d_{[a}h_{b]c} - D_{[a}Kh_{b]c}$ is the magnetic part of the Weyl tensor, as explained in Appendix A.3. Also, varying (6.37) with respect to $\epsilon^{\perp}$, ε^a, $\lambda_{\mathcal{P}}$ and w one gets the constraint equations $\mathcal{H}_{\perp} \approx 0$, $\mathcal{H}_a \approx 0$, $\mathcal{P} \approx 0$, $\mathcal{W} \approx 0$. The physical degrees of freedom are $\frac{1}{2}(2 \times 12 - 2 \times 6) = 6$. The interpretation of these 6 physical degrees is as follows: 2 of these describe a massless graviton, like in the case of General Relativity and the remaining 4 a "partially massless" graviton [76], [77].

6.2 POISSON BRACKET ALGEBRA OF THE CONSTRAINTS

The Poisson bracket algebra of the constraints (6.25)-(6.32) can be reformulated compactly by introducing smeared variables as

$$H_{\perp}[\eta] \equiv \int_{\Sigma_t} d^3x\ \mathcal{H}_{\perp}\eta \tag{6.42}$$

$$H[\xi^a] \equiv \int_{\Sigma_t} d^3x\ \mathcal{H}_a\xi^a \tag{6.43}$$

$$P[\chi] \equiv \int_{\Sigma_t} d^3x\ \mathcal{P}\chi \tag{6.44}$$

$$W[\zeta] \equiv \int_{\Sigma_t} d^3x\ \mathcal{W}\zeta \tag{6.45}$$

where $\eta = \eta(x)$, $\chi = \chi(x)$, $\zeta = \zeta(x)$ are scalars and $\xi^a = \xi^a(x) = h^a_b \xi^b$ is a tangent vector field on Σ_t. Then, algebra (6.25)-(6.32) takes the form

$$\{H_\perp[\eta_1], H_\perp[\eta_2]\} = H[h^{ab}(\eta_1 D_b \eta_2 - \eta_2 D_b \eta_1)] + P[h^{ab}(\eta_1 D_b \eta_2 - \eta_2 D_b \eta_1)\mathcal{K}_a] \tag{6.46}$$

$$\{H[\xi^a], H_\perp[\eta]\} = H_\perp[{}^3\pounds_{\xi^a}\, \eta] \tag{6.47}$$

$$\{H[\xi_1^a], H[\xi_2^b]\} = H\big[[\xi_1^a, \xi_2^b]^c\big] \tag{6.48}$$

$$\{H[\xi^a], P[\chi]\} = P[{}^3\pounds_{\xi^a}\, \chi] \tag{6.49}$$

$$\{H[\xi^a], W[\zeta]\} = W[{}^3\pounds_{\xi^a} \zeta] \tag{6.50}$$

$$\{P[\chi], W[\zeta]\} = P[\chi\zeta] \tag{6.51}$$

$$\{H_\perp[\eta], P[\chi]\} = W[\eta\chi] + P[\eta\chi K] \tag{6.52}$$

$$\{W[\zeta], H_\perp[\eta]\} = H_\perp[\zeta\eta] + P[\zeta D^2 \eta + \eta D^2 \zeta - D^a \zeta D_a \eta] \tag{6.53}$$

where

$$\mathcal{K}_a \equiv h^{bc} D_b K_{ac} - D_a K \tag{6.54}$$

and

$$[\xi_1^a, \xi_2^b]^c = \xi_1^a \partial_a \xi_2^c - \xi_2^a \partial_a \xi_1^c \tag{6.55}$$

being the usual expression for the Lie bracket of two vector fields. At this point, it is instructive to compare the above Poisson bracket algebra (6.46)-(6.53) with the one of General Relativity (2.187)-(2.189). As expected, in the absence of constraints $P[\chi]$, $W[\zeta]$, there is complete equivalence with (2.187)-(2.189). The geometrical interpretation of the above algebra is analogous with that of General Relativity: relations (6.47), (6.49) and (6.50) state that $\mathcal{H}_\perp$, $\mathcal{P}$ and $\mathcal{W}$ are scalar densities. Of course, this is not new information since they were defined as such, according to (6.8), (6.10),(6.13) and (6.21). Relation (6.48) states that $H[\xi^a]$ are generators of spatial diffeomorphisms on the surface Σ_t. Additionally, the first relation (6.46) states that $H_\perp[\eta]$ is generator of deformations of the surface Σ_t normal to itself, as it is embedded in $\Sigma_t \times \mathbf{R} \simeq M$. Last but not least, there is the presence of constraint $P[\chi]$ as a result of tracelessness of the Weyl tensor, indicating the additional gauge symmetry of the theory, namely local Weyl rescalings.

The Poisson brackets between constraints $H_\perp[\eta]$, $H[\xi^a]$ (6.46), (6.47), (6.48) can be rewritten more compactly when considering the ADM decomposition of two vector fields k_1^a, k_2^a on M as

$$k_1^a = n^a k_1^\perp + \kappa_1^a \tag{6.56}$$

$$k_2^a = n^a k_2^\perp + \kappa_2^a \tag{6.57}$$

with $k_1^\perp = -n_a k_1^a$, $k_2^\perp = -n_a k_2^a$ and $\kappa_1^a = h^a_b \kappa_1^b$, $\kappa_2^a = h^a_b \kappa_2^b$ being their normal and parallel components respectively. Then, setting $H[k_1] \equiv H_\perp[k_1^\perp] + H[\kappa_1^a]$ and $H[k_2] \equiv H_\perp[k_2^\perp] + H[\kappa_2^a]$ the Poisson brackets (6.46), (6.47), (6.48) take the form

$$\{H[k_1], H[k_2]\} = H\big[[k_1, k_2]\big] + P\big[(k_1^\perp D^a k_2^\perp - k_2^\perp D^a k_1^\perp)\mathcal{K}_a\big] \tag{6.58}$$

with

$$[k_1, k_2]^\perp = n_a\, \pounds_{k_1^b} k_2^a \tag{6.59}$$

$$[k_1, k_2]^a = h^a_b\, \pounds_{\kappa_1^c} \kappa_2^b\,. \tag{6.60}$$

As expected, comparison of (6.58) with the corresponding Poisson bracket of General Relativity (2.192), shows clearly the modification of the surface deformation algebra (2.193)-(2.194) into (6.59)-(6.60) due to the presence of the primary P constraint.

6.3 GAUGE GENERATOR

Since 1st class constraints generate gauge symmetries, the gauge generator is simply the extended Hamiltonian (6.37), i.e.

$$H_E = \int_{\Sigma_t} d^3x \left(\epsilon^\perp \mathcal{H}_\perp + \varepsilon^a \mathcal{H}_a + \lambda_{\mathcal{P}} \mathcal{P} + w\mathcal{W}\right) + \int_{\partial\Sigma_t} d^2S_a \left(\mathcal{Q}^a_\perp + \mathcal{Q}^a\right). \tag{6.61}$$

Now, it is demonstrated that the gauge generator (6.61) generates the correct gauge transformations of the theory and that $\mathcal{H}_\perp$, $\mathcal{H}_a$ and $\mathcal{W}$ are generators of normal displacements, spatial diffeomorphisms and local conformal transformations of the spatial metric h_{ab}. It is reminded that the phase space has been reduced by discarding N, N^a and their canonical momenta. Thus, gauge symmetries are transformations of the spatial metric

$$\delta h_{ab} = (\pounds_{\epsilon^c} + 2w) h_{ab} \tag{6.62}$$

under diffeomorphisms of the coordinates x^a on M of the form $x'^a = x^a + \epsilon^a(x^b)$ and local Weyl rescalings of the form $h_{ab} \to h'_{ab} = w h_{ab}$ with $w = w(x)$. Therefore, the aim is to find δh_{ab} under the action of the gauge generator (6.61), i.e. to find $\delta_{H_E} h_{ab}$, and then verify that it generates the r.h.s. of (6.62). Indeed, $\delta_{H_E} h_{ab}$ is the Hamilton equation of motion (6.38), which is rewritten here explicitly as

$$\begin{aligned}
\delta_{H_E} h_{ab} &= \{h_{ab}, H_E\} \\
&= \int_{\Sigma_t} d^3x \epsilon^\perp \{h_{ab}, \mathcal{H}_\perp\} + \int_{\Sigma_t} d^3x + \varepsilon^c \{h_{ab}, \mathcal{H}_c\} + \int_{\Sigma_t} d^3x w \{h_{ab}, \mathcal{W}\} && (6.63) \\
&= 2\epsilon^\perp K_{ab} + 2D_{(a}\varepsilon_{b)} + 2w h_{ab} && (6.64)
\end{aligned}$$

where

$$\begin{aligned}
\{h_{ab}, \mathcal{H}_\perp\} &= 2\epsilon^\perp K_{ab} && (6.65) \\
\{h_{ab}, \mathcal{H}_c\} &= 2D_{(a}\varepsilon_{b)} && (6.66) \\
\{h_{ab}, \mathcal{W}\} &= 2w h_{ab}\,. && (6.67)
\end{aligned}$$

Then, focusing on the r.h.s. of (6.62), the vector field ϵ^a on M can be decomposed in the ADM basis as

$$\epsilon^a = \epsilon^\perp n^a + \varepsilon^a \tag{6.68}$$

with

$$\epsilon^\perp = -n_a \epsilon^a \quad \text{and} \quad \varepsilon^a = h^a_b \varepsilon^b\,. \tag{6.69}$$

Then, using (6.68), the r.h.s. of (6.62) is found to be

$$\begin{aligned}
\pounds_{\epsilon^c} h_{ab} &= (\pounds_{\epsilon^\perp n^c + \varepsilon^c} + 2w) h_{ab} = (\pounds_{\epsilon^\perp n^c} h_{ab} + {}^3\pounds_{\varepsilon^c} + 2w) h_{ab} && (6.70) \\
&= \epsilon^\perp n^c \nabla_c h_{ab} + 2h_{c(a} \nabla_{b)} (\epsilon^\perp n^c) + 2D_{(a}\varepsilon_{b)} + 2w h_{ab} \\
&= \epsilon^\perp \left[n^c \nabla_c h_{ab} + 2h_{c(a} \nabla_{b)} n^c \right] + 2D_{(a}\varepsilon_{b)} + 2w h_{ab} \\
&= \epsilon^\perp \pounds_{n^c} h_{ab} + 2D_{(a}\varepsilon_{b)} + 2w h_{ab} = 2\epsilon^\perp K_{ab} + 2D_{(a}\varepsilon_{b)} + 2w h_{ab} && (6.71)
\end{aligned}$$

after exploiting $n^a h_{ab} = 0$. This is exactly the r.h.s of (6.64) and therefore $\delta_{H_E} h_{ab} = (\pounds_{\epsilon^c} + 2w) h_{ab}$. Thus, it is concluded that the extended Hamiltonian (6.61) indeed generates the gauge transformations (6.62) of the spatial metric. Additionally, it is deduced from (6.65), (6.66), (6.67) and (6.70) that $\mathcal{H}_\perp$ generates normal displacements, $\mathcal{H}_a$ generates spatial diffeomorphisms whereas $\mathcal{W}$ generates local conformal transformations of the spatial metric h_{ab}.

6.4 BOUNDARY CONDITIONS, IMPROVED GAUGE GENERATOR AND CANONICAL CHARGES

The next step in the analysis is to construct the improved gauge generator and the canonical charges of the theory. For this task, it is necessary to impose boundary conditions. The approach that will be followed is that of Regge-Teitelboim [41], which was explicitly stated in 2.2. Briefly, this approach consists of improving the gauge generator in such a way that it has well-defined functional derivatives, given particular boundary conditions. Namely, the variation of the gauge generator must include only volume integrals on Σ_t yielding Hamilton equations of motion, after the imposition of boundary conditions. Equivalently, the surface integrals on $\partial\Sigma$ arising from partial integration of the volume terms must vanish, after the implementation of those boundary conditions. This requires 1) the addition of suitable surface terms on the original gauge generator, which takes the name improved after this addition, and 2) the imposition of those boundary conditions. Then, the improved gauge generator has well-defined functional derivatives i.e. its variation on $\partial\Sigma$ vanishes on-shell when the boundary conditions are imposed.

It is now demonstrated that the gauge generator (6.61) has well-defined functional derivatives, under the proposed set of boundary conditions and thus no improvement at all is required. In other words, only imposition of boundary conditions (condition 2) above) is necessary. The expression of the gauge generator (6.61) is repeated here

$$H_E = \int_{\Sigma_t} d^3x\Big(\epsilon^\perp \mathcal{H}_\perp + \varepsilon^a \mathcal{H}_a + \lambda_{\mathcal{P}}\mathcal{P} + w\mathcal{W}\Big) + \int_{\partial\Sigma_t} d^2S_a\, \Big(\mathcal{Q}^a_\perp + \mathcal{Q}^a\Big) \tag{6.72}$$

where

$$\mathcal{Q}^a_\perp = \Pi^{ab}_K D_b\epsilon^\perp - \epsilon D_b \Pi^{ab}_K \tag{6.73}$$

$$\mathcal{Q}^a = 2\varepsilon^c(\pi^{ab}_h h_{bc} + \Pi^{ab}_K K_{bc})\,. \tag{6.74}$$

It is emphasized again that the last two surface terms $\mathcal{Q}^\perp$, $\mathcal{Q}^a$ in (6.72) arise from partial integration of the original Lagrangian (6.2). An explicit variation of (6.72) evaluated on-shell, i.e. applying Hamilton equations (6.38)-(C.44), and on the constraint surface yields

$$\delta H_E|_{\text{on-shell}} \approx \int_{\Sigma_t} d^3x\Big(\epsilon^\perp\delta\mathcal{H}_\perp + \varepsilon^a\delta\mathcal{H}_a + \lambda_{\mathcal{P}}\delta\mathcal{P} + w\delta\mathcal{W}\Big) + \int_{\partial\Sigma} d^2S_a\Big(\delta\mathcal{Q}^a_\perp + \delta\mathcal{Q}^a\Big) \tag{6.75}$$

$$\begin{aligned}
&\approx \int_{\partial\Sigma} d^2S_a\Bigg\{\Big[\varepsilon^a\pi^{bc}_h - 2\pi^{a(c}_h\varepsilon^{b)} + 2\epsilon^\perp\sqrt{h}\Big(B^{ad(c}K^{b)}_d + B^{(c|da}K^{b)}_d + B^{d(cb)}K^a_d\Big)\\
&+\epsilon^\perp\Big(-D^c\Pi^{ab}_K + \frac{1}{2}D^a\Pi^{bc}_K + \frac{1}{2}D_d\Pi^{cd}_K h^{ab}\Big) - 2\Pi^{ab}_K D^c\epsilon^\perp + \Pi^{bc}_K D^a\epsilon^\perp + \frac{1}{2}\Pi^{ad}_K D_d\epsilon^\perp h^{bc}\Big]\delta h_{bc}\\
&-2\varepsilon^c h_{bc}\delta\pi^{ab}_h + \Big[\varepsilon^a\Pi^{bc}_K - 2\Pi^{ac}_K\varepsilon^b + 4\epsilon^\perp\sqrt{h}B^{acb}\Big]\delta K_{cb} + \Big[-2\varepsilon^c K_{cb} - D_b\epsilon^\perp\Big]\delta\Pi^{ab}_K\\
&+\epsilon^\perp D_b\delta\Pi^{bc}_K + \epsilon^\perp\Big(2\delta C^a_{bc}\Pi^{ab}_K - \delta C^b_{bc}\Pi^{ac}_K\Big)\Bigg\} + \int_{\partial\Sigma} d^2S_a\Big(\delta\mathcal{Q}^a_\perp + \delta\mathcal{Q}^a\Big)
\end{aligned} \tag{6.76}$$

where C^a_{bc} denotes the difference tensor of two neighboring Levi-Civita connections and

$$\delta\mathcal{Q}^a_\perp = \delta\Pi^{ab}_K D_b\epsilon^\perp + \Pi^{ab}_K D_b\delta\epsilon^\perp - \delta\epsilon^\perp D_b\Pi^{ab}_K - \epsilon^\perp D_b\delta\Pi^{ab}_K \tag{6.77}$$

$$\delta\mathcal{Q}^a = 2\delta\varepsilon^c(\pi^{ab}_h h_{bc} + \Pi^{ab}_K K_{bc}) + 2\varepsilon^c(\delta\pi^{ab}_h h_{bc} + \pi^{ab}_h\delta h_{bc} + \delta\Pi^{ab}_K K_{bc} + \Pi^{ab}_K\delta K_{bc})\,. \tag{6.78}$$

It is noteworthy that all surface integrals (except those of $\delta\mathcal{Q}^a_\perp$ and $\mathcal{Q}^a$) arise from partial integration of the volume integrals of $\delta\mathcal{H}_\perp$ and $\delta\mathcal{H}_a$. The variations $\delta\mathcal{P}$ and $\delta\mathcal{W}$ produce only volume integrals which contribute to Hamilton equations. Therefore, the gauge generator $\mathcal{W}$ (6.21) of conformal transformations of the spatial metric h_{ab} has well-defined functional derivatives, before the imposition of boundary conditions. This is expected, since from the analytic form of the constraints (6.10), (6.13), (6.14), (6.21), one observes that only $\mathcal{H}_\perp$ (6.13) and $\mathcal{H}_a$ (6.14) depend on derivatives of the canonical variables and thus, will produce surface integrals in the variation, while $\mathcal{P}$ (6.10) and $\mathcal{W}$ (6.21) depend linearly on the canonical variables.

Now the set of boundary conditions is implemented. The proposed set is constructed by considering local conformal rescalings of the spatial metric h_{ab} and the arbitrary functions $\epsilon^{\perp}$, ϵ^{a}, ω. That is, letting the boundary Σ_t having a topology $\mathbb{R} \times \partial\Sigma$, one assumes that h_{ab}, $\epsilon^{\perp}$, ε^{a}, ω can be written near the surface $\partial\Sigma$ as

$$h_{ab} = \Omega^2 \bar{h}_{ab} \tag{6.79}$$
$$\epsilon^{\perp} = \Omega \bar{\epsilon}^{\perp} \tag{6.80}$$
$$\varepsilon^{a} = \bar{\varepsilon}^{a} \tag{6.81}$$
$$\omega = \bar{\omega} \tag{6.82}$$

with Ω being arbitrary and $\bar{h}_{ab}, \bar{\epsilon}^{\perp}, \bar{\varepsilon}^{a}$, $\bar{\omega}$ being finite near $\partial\Sigma$. The purpose is now to find the corresponding conformal transformations of the rest of the canonical variables, i.e. of K_{ab}, π_h^{ab} and Π_K^{ab}. This is done as follows: inserting the above set of conformal transformations (6.79)-(6.81) into $K_{ab} = \frac{1}{2\epsilon^{\perp}}(\dot{h}_{ab} - 2D_{(a}\varepsilon_{b)})$ and using Hamilton equations of h_{ab} (6.38), one finds that K_{ab} transforms as

$$K_{ab} = \Omega \Big[\frac{\bar{h}_{ab}}{\bar{\epsilon}^{\perp}} \big[(\partial_t - {}^3\pounds_{\bar{\varepsilon}^c}) \ln\Omega + \bar{\omega} \big] + \bar{K}_{ab} \Big] . \tag{6.83}$$

Explicit formulas that are required for the above derivation and the ones that follow throughout this section are given in Appendix C.5. Afterwards, the conformal behavior of Π_K^{ab} is specified by expressing Π_K^{ab} via Hamilton equation of motion of K_{ab} (6.40) and then, by inserting the conformal transformations (6.79)-(6.81) and (6.83). The result is

$$\begin{aligned}\Pi_K^{ab} &= \Omega^{-1}\sqrt{\bar{h}}\bar{G}^{abcd}\Big[\bar{\mathcal{R}}_{cd} + \bar{K}\bar{K}_{cd} + \frac{1}{\bar{\epsilon}}\bar{D}_c\bar{D}_d\bar{\epsilon} - \frac{1}{\bar{\epsilon}}\big(\partial_t - ^3\pounds_{\bar{\epsilon}^e} - \bar{\omega}\big)\bar{K}_{cd}\Big] \\ &= \Omega^{-1}\bar{\Pi}_K^{ab}\end{aligned} \tag{6.84}$$

where $\bar{G}^{abcd} = \frac{1}{2}(\bar{h}^{ac}\bar{h}^{bd} + \bar{h}^{ad}\bar{h}^{bc}) - \frac{1}{3}\bar{h}^{ab}\bar{h}^{cd}$ and use of $\bar{G}^{abcd}\bar{h}_{ab} = 0$ has been made. Finally, expressing π_h^{ab} via Hamilton equation of motion of Π_K^{ab} (6.40) and inserting the rescalings (6.79)-(6.81) and (6.83), (6.84), the conformal behavior of π_h^{ab} is found to be

$$\pi_h^{ab} = \Omega^{-2}\Big[-\frac{1}{\bar{\epsilon}^{\perp}}\Big((\partial_t - {}^3\pounds_{\bar{\varepsilon}^c}) \ln\Omega + \frac{3}{2}\bar{\omega}\Big)\bar{\Pi}_K^{ab} + \bar{\pi}_h^{ab} \Big] \tag{6.85}$$

where use of $\bar{\mathcal{P}} = \bar{h}_{ab}\bar{\Pi}_K^{ab} = 0$ has been made. Using the above conformal rescalings (6.79), (6.83), (6.85), (6.84), the variations of the canonical variables on the surface $\partial\Sigma$ can be specified. These are found to be

$$\delta h_{ab} = \Omega^2(2\delta \ln \Omega \bar{h}_{ab} + \delta \bar{h}_{ab}) \tag{6.86}$$

$$\begin{aligned}\delta K_{ab} &= \Omega\delta\ln\Omega\Big[\frac{\bar{h}_{ab}}{\bar{\varepsilon}^{\perp}}\big((\partial_t - {}^3\pounds_{\bar{\varepsilon}^c})\ln\Omega + \bar{\omega}\big) + \bar{K}_{ab}\Big] + \Omega\frac{\bar{h}_{ab}}{\bar{\varepsilon}^{\perp}}(\partial_t - {}^3\pounds_{\bar{\varepsilon}^c})\delta\ln\Omega \\ &+ \Omega\left[\frac{\delta\bar{h}_{ab}}{\bar{\varepsilon}^{\perp}}\Big((\partial_t - {}^3\pounds_{\bar{\varepsilon}^c})\ln\Omega + \bar{\omega}\Big) + \delta\bar{K}_{ab}\right. \\ &\left. + \frac{\bar{h}_{ab}}{\bar{\varepsilon}^{\perp}}\Big(- (\partial_t - {}^3\pounds_{\bar{\varepsilon}^c})\ln\Omega\frac{\delta\bar{N}}{\bar{N}^2} - \delta\bar{\varepsilon}^a\bar{D}_a\ln\Omega + \delta\bar{\omega}\Big)\right]\end{aligned} \tag{6.87}$$

$$\begin{aligned}\delta\pi_h^{ab} &= -\frac{2\delta\ln\Omega}{\Omega^2}\Big[- \frac{1}{\bar{\varepsilon}^{\perp}}\Big((\partial_t - {}^3\pounds_{\bar{\varepsilon}^c})\ln\Omega + \frac{3}{2}\bar{\omega}\Big)\bar{\Pi}_K^{ab} + \bar{\pi}_h^{ab}\Big] - \frac{1}{\Omega^2\bar{\varepsilon}^{\perp}}(\partial_t - {}^3\pounds_{\bar{\varepsilon}^c})\delta\ln\Omega\bar{\Pi}_K^{ab} \\ &+ \Omega^{-2}\left[- \frac{1}{\bar{\varepsilon}^{\perp}}\Big((\partial_t - {}^3\pounds_{\bar{\varepsilon}^c})\ln\Omega + \frac{3}{2}\bar{\omega}\Big)\delta\bar{\Pi}_K^{ab} + \delta\bar{\pi}_h^{ab} + \frac{\bar{\pi}_h^{ab}}{\bar{\varepsilon}^{\perp}}\Big((\partial_t - {}^3\pounds_{\bar{\varepsilon}^c})\ln\Omega\frac{\delta\bar{\varepsilon}^{\perp}}{\bar{\varepsilon}^{\perp 2}}\right. \\ &\left. + \delta\bar{\varepsilon}^a\bar{D}^a\ln\Omega + \frac{3}{2}\delta\bar{\omega}\Big)\right]\end{aligned} \tag{6.88}$$

$$\delta\Pi_K^{ab} = \Omega^{-1}(-\delta\ln\Omega\bar{\Pi}_K^{ab} + \delta\bar{\Pi}_K^{ab}) \tag{6.89}$$

where both δ and ln only act on the first term on the right. Now, the proposed set of boundary conditions is specified: inspection of the variation of the gauge generator (6.76) indicates that the set of boundary conditions must consist of fixing the variation of the canonical variables $\bar{h}_{ab}$, $\bar{K}_{ab}$, $\bar{\pi}_h^{ab}$, $\bar{\Pi}_K^{ab}$ and the arbitrary functions $\bar{\varepsilon}^{\perp}$, $\bar{\varepsilon}^a$, $\bar{\omega}$ on the surface $\partial\Sigma$. At first, one assumes that the variation of $\bar{h}_{ab}$, $\bar{\varepsilon}^{\perp}$, $\bar{\varepsilon}^a$ and of their derivatives are fixed on $\partial\Sigma$, i.e. setting

$$\delta\bar{h}_{ab}|_{\partial\Sigma} = \bar{D}_c\delta\bar{h}_{ab}|_{\partial\Sigma} = 0 \tag{6.90}$$

$$\delta\bar{\varepsilon}^{\perp}|_{\partial\Sigma} = \bar{D}_c\delta\bar{\varepsilon}^{\perp}|_{\partial\Sigma} = 0 \tag{6.91}$$

$$\delta\bar{\varepsilon}^{a}|_{\partial\Sigma} = \bar{D}_c\delta\bar{\varepsilon}^{a}|_{\partial\Sigma} = 0\,. \tag{6.92}$$

Subsequently, demanding consistency with Hamilton equations (6.38), (C.44) and (6.40), it is deduced that

$$\delta\bar{K}_{ab}|_{\partial\Sigma} = 0 \tag{6.93}$$

$$\delta\bar{\omega}|_{\partial\Sigma} = 0 \tag{6.94}$$

$$\delta\bar{\Pi}_K^{ab}|_{\partial\Sigma} \;\; \text{can be arbitrary but finite} \tag{6.95}$$

$$\delta\bar{\pi}_h^{ab}|_{\partial\Sigma} \;\; \text{can be arbitrary but finite}\,. \tag{6.96}$$

Thus, the proposed of boundary conditions consists of the set (6.90)-(6.96).

Now the variation of the gauge generator (6.76) under the imposition of the above conformal transformations and boundary conditions is examined. Namely, after inserting the rescalings of all the canonical variables (6.79), (6.83), (6.85), (6.84) and of the arbitrary functions (6.80), (6.81), (6.82), as well as the variations (6.86)-(6.89), the variation of the gauge generator (6.76) becomes

$$\begin{aligned}\delta H_E|_{\text{on-shell}} &\approx \int_{\partial\Sigma} d^2S_a\Big[2\delta\bar{\varepsilon}^c(\bar{\pi}_h^{ab}\bar{h}_{bc} + \bar{\Pi}_K^{ab}\bar{K}_{bc}) + 2\bar{\varepsilon}^c(\bar{\pi}_h^{ab}\delta\bar{h}_{bc} + \bar{\Pi}_K^{ab}\delta\bar{K}_{bc}) + \bar{\Pi}_K^{ab}\bar{D}_b\delta\bar{\varepsilon}^{\perp} - \delta\bar{\varepsilon}^{\perp}\bar{D}_b\bar{\Pi}_K^{ab}\Big] \\ &\approx 0\end{aligned} \tag{6.97}$$

on the constraint surface, after imposition of the boundary conditions (6.90)-(6.92), (6.93)-(6.96). Thus, the variation of the gauge generator vanishes on-shell with no requirement of additional boundary terms. That is, the boundary terms (6.73), (6.74) arising from partial integration of the original Lagrangian (6.1) are necessary and sufficient, together with the implemented boundary conditions (6.90)-(6.92), (6.93)-(6.96), to render a well-defined variational principle for the extended Hamiltonian (6.76). As far as each generator in (6.76) is concerned, it is emphasized again that the gauge generator of conformal transformations $\mathcal{W}$ was already well-defined before the imposition of boundary conditions, as it was stated before. The gauge generators $\mathcal{H}_\perp$ and $\mathcal{H}_a$, which generate normal displacements and spatial diffeomorphisms of the spatial metric h_{ab}, as it was described in section 6.3, are well-defined generators with the presence of the surface terms $\mathcal{Q}^a_\perp$ and $\mathcal{Q}^a$ respectively.

Finally, the canonical charges are evaluated. Not surprisingly, they are, up to a constant, the surface terms of the extended Hamiltonian (6.72), i.e.

$$H_E \approx \int_{\partial\Sigma} d^2S_a \left(\mathcal{Q}^a_\perp + \mathcal{Q}^a\right) \equiv Q \tag{6.98}$$

where

$$\mathcal{Q}^a_\perp = \Pi_K^{ab} D_b \epsilon^\perp - D_b \Pi_K^{ab} \epsilon^\perp \tag{6.99}$$

is the charge associated with normal displacements and

$$\mathcal{Q}^a_\perp = \Pi_K^{ab} D_b \varepsilon - D_b \Pi_K^{ab} \varepsilon \tag{6.100}$$

is the charge associated with spatial diffeomorphisms of the spatial metric h_{ab}. It is now demonstrated that they are finite on $\partial\Sigma$. Indeed, imposing the conformal transformations (6.79)-(6.81) and (6.83), (6.84), (6.85) in the above charges (6.99), (6.100), a straightforward calculation reveals that

$$\begin{aligned}\mathcal{Q}^a_\perp &= \Pi_K^{ab} D_b \epsilon^\perp - D_b \Pi_K^{ab} \epsilon^\perp = \bar{\Pi}_K^{ab} \bar{D}_b \bar{\epsilon}^\perp - \bar{D}_b \bar{\Pi}_K^{ab} \bar{\epsilon}^\perp + \bar{\epsilon}^\perp \bar{\mathcal{P}} \bar{D}^a \ln \Omega \\ &\approx \bar{\Pi}_K^{ab} \bar{D}_b \bar{\epsilon}^\perp - \bar{D}_b \bar{\Pi}_K^{ab} \bar{\epsilon}^\perp\end{aligned} \tag{6.101}$$

and

$$\begin{aligned}\mathcal{Q}^a &= \Pi_K^{ab} D_b \varepsilon - D_b \Pi_K^{ab} \varepsilon \\ &= 2\bar{\varepsilon}^c (\bar{\pi}_h^{ab} \bar{h}_{bc} + \bar{\Pi}_K^{ab} \bar{K}_{bc}) \,.\end{aligned} \tag{6.102}$$

Thus, the charges (6.101), (6.102) are finite at $\partial\Sigma$. It is expected that they generate symmetries at $\partial\Sigma$ (asymptotic symmetries), namely normal displacements and spatial diffeomorphisms of the rescaled spatial metric $\bar{h}_{ab}$ with respect to a vector field k^a on M which is finite at $\partial\Sigma$. An important observation is made at this point: there exists no charge associated with local Weyl rescalings, independent of the imposition of boundary conditions. This statement is a direct result of the fact that the generator $\mathcal{W}$ of conformal transformations is already improved and does not require boundary terms, which yield a charge, to be a well-defined generator, independently of boundary conditions. Therefore, in view of the above charges (6.101), (6.102), it is concluded that the gauge symmetries of the spatial h_{ab}, i.e. normal displacements and spatial diffeomorphisms, become symmetries at $\partial\Sigma$ while conformal transformations remain (proper [84]) gauge symmetries even at the boundary Σ_t.

6.5 ASYMPTOTIC SYMMETRY ALGEBRA

Considering two vector fields k_1^a, k_2^a on M, their ADM decomposition is

$$k_1^a = n^a k_1^\perp + \kappa_1^a \tag{6.103}$$

$$k_2^a = n^a k_2^\perp + \kappa_2^a \tag{6.104}$$

with $k_1^\perp = -n_a k_1^a$, $k_2^\perp = -n_a k_2^a$ and $\kappa_1^a = h^a_b \kappa_1^b$, $\kappa_2^a = h^a_b \kappa_2^b$ being their normal and parallel components respectively. Performing a conformal rescaling according to (6.80), (6.81) for the normal and parallels components of k_1^a and k_2^a respectively and using $n^a = \Omega^{-1}\bar{n}^a$, it turns out that the conformal rescaling of (6.103), (6.104) takes the form

$$k_1^a = \bar{k}_1^a \quad , \quad k_2^a = \bar{k}_2^a \, . \tag{6.105}$$

Thus, the two vector fields $\bar{k}_1$, $\bar{k}_2$ are boundary conditions preserving symmetries at $\partial\Sigma$. Using the smeared version of the constraints $\mathcal{H}_\perp$, $\mathcal{H}_a$ according to (6.42), (6.43), (6.44), one sets $H[\bar{k}_1] \equiv H_\perp[\bar{k}_1^\perp] + H[\bar{\kappa}_1^a]$ and $H[\bar{k}_2] \equiv H_\perp[\bar{k}_2^\perp] + H[\bar{\kappa}_2^a]$. But then, according to [54], Poisson brackets of the form (6.58) remain true also in the case where k_1^a, k_2^a are boundary conditions preserving symmetries, like here. Thus, (6.58) can be written in the present case as

$$\{H[\bar{k}_1], H[\bar{k}_2]\} = H\big[[\bar{k}_1, \bar{k}_2]\big] + P\big[(\bar{k}_1^\perp \bar{D}^a \bar{k}_2^\perp - \bar{k}_2^\perp \bar{D}^a \bar{k}_1^\perp)\bar{\mathcal{K}}_a\big] \, . \tag{6.106}$$

Following [85], one might fix a gauge which turns the 1st class constraints into 2nd class. Then, these are required to vanish strongly and the Poisson brackets are converted to Dirac brackets. Additionally, on the constraint surface, $H[\bar{k}^a]$, is simply the charge (6.98), where k is any of k_1, k_2. Thus, $\{H[\bar{k}_1], H[\bar{k}_2]\} \approx \{Q[k_1], Q[k_2]\}$. Then, in terms of the Dirac brackets, the symmetry algebra of the charges on $\partial\Sigma$ can be written as

$$\{Q[\bar{k}_1], Q[\bar{k}_2]\}^* = Q\big[[\bar{k}_1, \bar{k}_2]\big] \tag{6.107}$$

where it has been assumed that there is not any central extension. Therefore, the algebra of the charges at $\partial\Sigma$ (asymptotic symmetry algebra) is isomorphic to the Lie algebra of the boundary conditions preserving diffeomorphisms.

6.6 SUMMARY AND CONCLUSIONS

In this chapter the conformal gravity action was formulated in the Hamiltonian formalism. The original work, in a slightly different approach than here, can be found in [86]. Earlier than this, a Hamiltonian formulation of generic higher derivative theories, including conformal gravity, was considered in [87].

By identifying the canonical variables and the primary constraints (6.6), (6.7), (6.10), the total Hamiltonian (6.17) was derived and then, the secondary constraints $\mathcal{H}_\perp$ (6.13), $\mathcal{H}_a$ (6.14) and (6.21) were deduced. Their Poisson bracket algebra (6.46)-(6.53) was discussed: constraints $\mathcal{H}_\perp$, $\mathcal{H}_a$ are generators of normal deformations and spatial diffeomorphisms of the spatial metric h_{ab} respectively and constraints $\mathcal{P}$, $\mathcal{W}$ are related with local conformal transformations of the spatial metric h_{ab}. Then, it was found that all constraints are 1st class and subsequently the physical degrees of freedom of the theory were detected: these are 6, 2 corresponding to the massless spin-2 graviton and the rest 4 corresponding to the PMR. Then, the extended Hamiltonian (6.37) in a reduced phase space was considered and the corresponding Hamilton equations (6.38)-(C.44) were presented. Furthermore, the gauge generator (6.61) of the theory was presented as being the reduced phase space extended Hamiltonian and it was shown that it generates the correct gauge transformations of the spatial metric h_{ab} (6.62). Additionally, assuming boundary conditions such as conformal transformations of the canonical variables and the rest of the parameters (6.90)-(6.92), (6.93)-(6.96), it was demonstrated that the variation of the gauge generator vanishes on-shell on the constraint surface (6.97). Therefore, the gauge generator (6.61) has well-defined functional derivatives and is already an improved generator. No additional boundary terms are required, other than (6.35), (6.36) which arise initially from partial integration of the conformal gravity Lagrangian (6.2), in order to render a well-defined variational principle. This is in complete agreement with the results of sections 5.1 and 5.2.3. Then, the canonical charges were derived (6.99), (6.100), associated with symmetries at the surface $\partial\Sigma$, i.e. normal displacements and spatial diffeomorphisms of the spatial rescaled metric $\bar{h}_{ab}$. There were proved to be finite (6.101), (6.102). There was not found a Weyl charge because the generator $\mathcal{W}$ of conformal transformations of h_{ab} has well-defined functional derivatives. Thus, conformal transformations of h_{ab} remain gauge symmetries even at the boundary Σ_t . Then, the asymptotic symmetry algebra of the charges (6.107) was deduced. Since there is no Weyl charge, the asymptotic symmetry algebra is isomorphic to the Lie algebra of the boundary conditions preserving symmetries at $\partial\Sigma$.

7

1-LOOP PARTITION FUNCTION

The issue of consistency of a gravitational canonical ensemble was discussed in 3.1.2. There it was mentioned that it is very important to specify asymptotic boundary conditions of the system in question, in order for the ensemble to make sense at all. In particular, in the case of General Relativity asymptotically AdS black holes have a positive specific heat and this renders the canonical ensemble and its partition function well-defined [67]. Thus, in the present case of conformal gravity it is reasonable to expect that considering the AdS background in the linearized approach, the canonical ensemble will be well-defined as well. Additionally, the Euclidean path integral is expected to converge for a positive coupling constant α_{CG} and for real and positive metrics, since the conformal gravity action is a quadratic functional of curvature tensors of the metric. Considering such a canonical ensemble, the forthcoming analysis starts by linearizing the theory in 7.1, continues with evaluating the path integral measure and fixing the gauge in 7.2 and finally, the 1-loop correction to the partition function of the ensemble is evaluated analytically using heat kernel techniques in 7.3.

7.1 LINEARIZED EQUATIONS OF MOTION AND SECOND VARIATION OF THE ACTION

The first step towards the evaluation of the 1-loop partition function of the theory is to find the second variation of the Euclidean version of the conformal gravity action, i.e. of

$$\hat{S}[g] = \frac{1}{4}\int_M d^4x\,\sqrt{g}\,C_{abcd}\,C^{abcd} \tag{7.1}$$

with g_{ab} having the Euclidean signature $(+,+,+,+)$. The coupling constant of the theory has been set to $\alpha_{CG} = \frac{1}{4}$. The first variation of (7.1) gives

$$\delta\hat{S}[g] = \frac{1}{4}\int_M d^4x\,\sqrt{g}\,B_{ab}\delta g^{ab} \tag{7.2}$$

and the resulting equations of motion are

$$B_{ab} \equiv 2\nabla^d\nabla^c C_{cabd} + C_{cabd}R^{cd} = 0 \tag{7.3}$$

where B_{ab} is the Bach tensor. On account of the perturbative approach that is followed throughout this analysis, it is convenient to consider the metric tensor g_{ab} as consisting of a fixed background field $\bar{g}_{ab}$ and a fluctuation γ_{ab}. The fixed background solution of the equations of motion is chosen to be AdS_4 spacetime. The decomposition of the metric g_{ab} is then written as

$$g_{ab} = \bar{g}_{ab} + \epsilon\gamma_{ab} \tag{7.4}$$

where ϵ is the perturbation parameter. From now on quantities with a bar will denote their value on the AdS_4 background.

In order for gauge symmetries and true dynamics of the linearized theory to be revealed, it is customary to perform a York-decomposition. That is, to decompose fluctuation γ_{ab} into a transverse-traceless part (γ^{TT}_{ab}), a "trace" part ($\hat{\gamma}$) and a vector part (v_a) as

$$\gamma_{ab} = \gamma^{TT}_{ab} + \frac{1}{4}\bar{g}_{ab}\hat{\gamma} + 2\bar{\nabla}_{(a}v_{b)} \tag{7.5}$$

with $\bar{\nabla}^a \gamma^{TT}_{ab} = 0 = \bar{g}^{ab}\gamma^{TT}_{ab}$. Under this decomposition, the gauge symmetries of the linearized theory are now uncovered. Namely, conformal invariance: diffeomorphism invariance is represented by the transformation $\gamma_{ab} \to \gamma_{ab} + 2\bar{\nabla}_{(a}v_{b)}$. Therefore, absence of the vector v_a in all forthcoming expressions indicates diffeomorphism invariance. Likewise, the local rescaling of the full, unperturbed metric $g_{ab} \to e^{2\omega} g_{ab}$ is represented at the linearized level by the transformation $\gamma_{ab} \to \gamma_{ab} + 2\omega\bar{g}_{ab}$. This is in accordance with the York-decomposition (7.5) for $\omega = \frac{1}{8}\hat{\gamma}$. Thus, absence of the "trace" part $\hat{\gamma}$ in all forthcoming expressions denotes local scale invariance.

Inserting (7.4) and the York-decomposition (7.5) into the equations of motion (7.3), one finds at first order in ϵ

$$-16\gamma^{TT}_{ab} + 8\bar{\nabla}_c\bar{\nabla}_{(a}\gamma^{TTc}_{b)} - 6\bar{\nabla}^2\gamma^{TT}_{ab} + 2\bar{\nabla}_c\bar{\nabla}_d\bar{\nabla}_{(a}\bar{\nabla}_{b)}\gamma^{cd}_{TT} - 6\bar{\nabla}^2\bar{\nabla}_c\bar{\nabla}_{(a}\gamma^{TTc}_{b)} + 3\bar{\nabla}^2\bar{\nabla}^2\gamma^{TT}_{ab} = 0\,. \quad (7.6)$$

These are the linearized equations of motion. It is stressed again that they depend neither on the vector v_a, stating the diffeomorphism invariance, nor the "trace" part $\hat{\gamma}$, stating local scale invariance. The dynamics of the linearized theory are fully encompassed in the transverse-traceless modes γ^{TT}_{ab}.

The next step in the analysis is to find the second variation of (7.2) which after inserting (7.4) becomes

$$\delta^{(2)}\hat{S}[\bar{g},\gamma] = \frac{1}{4}\int_M d^4x\,\sqrt{\bar{g}}\,\left(\delta B_{ab}\gamma^{ab} + \bar{B}_{ab}\delta\gamma^{ab}\right). \quad (7.7)$$

The second term above vanishes on shell for the AdS_4 background solution and in the first term δB^{ab} is the linearized expression (7.6). After using the York-decomposition (7.5) and some partial integration, the second variation (7.7) or 1-loop correction takes the form

$$\hat{S}[\bar{g},\gamma]_{\text{1-loop}} = \int_M d^4x\,\sqrt{\bar{g}}\,\gamma^{ab}_{TT}\Big(-8 - 6\bar{\nabla}_c\bar{\nabla}^c - \bar{\nabla}_c\bar{\nabla}^c\bar{\nabla}_d\bar{\nabla}^d\Big)\gamma^{TT}_{ab} \quad (7.8)$$

where the notation ${}^{TT}{}_{(2)}$ denotes the transverse-traceless modes γ^{TT}_{ab}. From now on the bar is omitted from the Laplacian operator.

7.2 PATH INTEGRAL MEASURE, GAUGE FIXING AND FADDEEV-POPOV DETERMINANT

The gauge group that provides an infinite volume factor in the path integral is the group of conformal gauge symmetries. In order to remove this infinite factor consistently, it is mandatory to pursue the Faddeev-Popov method. In the present case, this is done as follows: the path integral measure $D[\gamma]$ is divided by the infinite volume of the group of conformal gauge symmetries and it is expressed in terms of the Faddeev-Popov determinant $\Delta(g)$. Then, exploiting the York decomposition (7.5), $\Delta(g)$ is given by the Jacobian of the transformation $\gamma_{ab} \to (\gamma^{TT}_{ab}, v_a, \hat{\gamma})$. The resulting expression can be written schematically as

$$\frac{D[\gamma]}{V_{\text{conf}}} = \Delta(g)\,D[\gamma^{TT}]\,D[v]\,D[\hat{\gamma}]\,. \quad (7.9)$$

In order to evaluate $\Delta(g)$, the standard procedure consists of choosing a suitable gauge for the metric variables v_a and $\hat{\gamma}$ of the York-decomposition (7.5). Then, $\Delta(g)$ is expressed in terms of the Jacobian matrices of these gauge-fixing transformations. This is done as follows: one requires orthonormality for γ_{ab}, i.e.

$$\begin{aligned} 1 &= \int D[\gamma]\ \exp\Big[-\int d^4x\sqrt{\bar{g}}\ \gamma_{ab}\gamma^{ab}\Big] \\ &= \int \Delta(g)D[\gamma^{TT}]D[v]D[\hat{\gamma}]\ \exp\Big[-\int d^4x\sqrt{\bar{g}}\ \gamma_{ab}(\gamma^{TT},v,\hat{\gamma})\gamma^{ab}(\gamma^{TT},v,\hat{\gamma})\Big] \end{aligned} \quad (7.10)$$

and the same as well for each mode of the York-decomposition of γ_{ab} (7.5), i.e.

$$1 = \int D[\gamma^{TT}] \ \exp\left[-\int d^4x\sqrt{\bar{g}}\ \gamma^{TT}_{ab}\gamma^{ab}_{TT}\right] \tag{7.11}$$

$$1 = \int D[v] \ \exp\left[-\int d^4x\sqrt{\bar{g}}\ v^a v_a\right] \tag{7.12}$$

$$1 = \int D[\hat{\gamma}] \ \exp\left[-\int d^4x\sqrt{\bar{g}}\ \hat{\gamma}^2\right]. \tag{7.13}$$

Because of the mixing between modes of different types in the inner product in (7.10), it is convenient to select as a gauge-fixing condition one which cancels this mixing. Such a condition is met by decomposing v_a into a transverse (v_a^T) and a scalar part (σ) and $\hat{\gamma}$ into two scalar parts ($\tilde{\gamma}, \sigma$) as

$$v_a = v_a^T + \nabla_a\sigma \tag{7.14}$$

$$\hat{\gamma} = \tilde{\gamma} - 2\nabla^2\sigma \tag{7.15}$$

with $\nabla^a v_a^T = 0$. Such a gauge choice cancels the mixing and makes the decomposition of the inner product in (7.10) indeed orthogonal, i.e.

$$\gamma_{ab}\gamma^{ab} = \gamma^{TT}_{ab}\gamma^{ab}_{TT} - 2v^a_T(\nabla^2 - 3)v^T_a + 3\sigma(-\nabla^2)(-\nabla^2+4)\sigma + \frac{1}{4}\hat{\gamma}^2. \tag{7.16}$$

The gauge-fixing transformations (7.14), (7.15) produce a Jacobian, denoted as J_2, in the path integral measure $D[\gamma]$, i.e. $\frac{D[\gamma]}{V_{\text{conf}}} = J_2\, D[\gamma^{TT}]\, D[v^T]\, D[\sigma] D[\tilde{\gamma}]$. Using the orthonormality condition (7.10) one arrives at

$$\begin{aligned} 1 = \int D[\gamma^{TT}]D[v^T]D[\sigma]D[\tilde{\gamma}]\, J_2 \exp\Big[-\int d^4x\sqrt{\bar{g}}\ \Big(\gamma^{TT}_{ab}\gamma^{ab}_{TT} - 2v^a_T(\nabla^2-3)v^T_a \\ + 3\sigma(-\nabla^2)(-\nabla^2+4)\Big)\Big] \quad \Rightarrow J_2 = \Big[\det(-\nabla^2+3)^T_{(1)}\ \det(-\nabla^2)_{(0)}\ \det(-\nabla^2+4)_{(0)}\Big]^{\frac{1}{2}} \end{aligned} \tag{7.17}$$

where the notation ${}^T{}_{(1)}$ denotes the transverse vector mode v_a^T and the subscript (0) denotes the scalar part $\tilde{\gamma}$.

The next step in the analysis is to find the Jacobians of each of the gauge-fixing transformations (7.14), (7.15) in the corresponding path integral measures. That is, to find J_1 and J_0 for $D[v] = J_1\, D[v^T]\, D[\sigma]$ and $D[\hat{\gamma}] = J_0\, D[\tilde{\gamma}]\, D[\sigma]$ respectively, where J_1 and J_0 are the Jacobians of the corresponding transformations. A straightforward calculation yields $J_0 = 1$ where as for J_1 using (7.12) one finds

$$\begin{aligned} 1 &= \int D[v^T]D[\sigma]J_1 \ \exp\left[-\int d^4x\sqrt{\bar{g}}\ (v^a_T v^T_a - \sigma\nabla^2\sigma)\right] \\ &\Rightarrow J_1 = \left[\det(-\nabla^2)_{(0)}\right]^{\frac{1}{2}}. \end{aligned} \tag{7.18}$$

Now, the Faddeev-Popov determinant can be finally expressed in the terms of the Jacobians J_1 and J_2. Recalling the expression for the path integral measure $D[\gamma]$ after the gauge-fixing transformations (7.14), (7.15), i.e.

$$\frac{D[\gamma]}{V_{\text{conf}}} = J_2\, D[\gamma^{TT}]\, D[v^T]\, D[\sigma]D[\tilde{\gamma}] = \frac{J_2}{J_1}\, D[\gamma^{TT}]\, D[v]\, D[\hat{\gamma}] \tag{7.19}$$

and comparing with (7.9), the Faddeev-Popov determinant is

$$\Delta(g) = \frac{J_2}{J_1} = \left[\det(-\nabla^2+3)^T_{(1)}\,\det(-\nabla^2+4)_{(0)}\right]^{\frac{1}{2}}. \tag{7.20}$$

7.3 1-LOOP PARTITION FUNCTION AND HEAT KERNEL

Having in hand all necessary ingredients, one can now find the 1-loop canonical partition function of the theory according to (3.42). After the use of (7.8) and (7.20) the 1-loop partition function takes the form

$$\begin{aligned} Z_{1-\text{loop}} &= \int \frac{D[\gamma]}{V_{\text{conf}}} e^{-\hat{S}[\bar{g},\gamma]_{\text{1-loop}}} \\ &= \Big[\frac{\det(-\nabla^2+3)^T_{(1)}\det(-\nabla^2+4)_{(0)}}{\det(-\nabla^2-4)^{TT}_{(2)}\det(-\nabla^2-2)^{TT}_{(2)}}\Big]^{\frac{1}{2}} \qquad (7.21) \\ &= Z_{(1)}\, Z_{(0)}\, Z^{-1}_{(2),4}\, Z^{-1}_{(2),2} \qquad (7.22) \end{aligned}$$

where $Z_{(s)}$ are the partition functions of the modes of spins $s = 0, 1, 2$ and $Z_{(2),r}$ are the partition functions of the $s = 2$ modes with $r = 2$ and $r = 4$. The interpretation of the above expression is the following: the partition functions $Z_{(0)}$ and $Z_{(1)}$ that appear in the nominator, i.e. the determinants of the "trace" and of the vector part respectively, correspond to gauge symmetries of the theory as already emphasized in 7.1. Namely, they describe diffeomorphism and local Weyl rescalings. Therefore, $Z_{(0)}$ and $Z_{(1)}$ correspond to pure gauge degrees of freedom. The true dynamics of the theory are expressed via the $s = 2$ modes, the partition functions of which ($Z_{(2),4}$ and $Z_{(2),2}$) appear in the denominator. According to the canonical analysis in chapter 6, the dynamical degrees of freedom for conformal gravity are 6, 2 of which describe the massless spin-2 graviton and the rest 4 describe a "partially massless" spin-2 graviton (PMR). To determine which of $Z_{(2),r}$ describes what, it is helpful to compare with the 1-loop partition function of General Relativity (3.64). The common contribution of course is the massless graviton, described by the partition function $Z_{(2),2}$. Therefore one deduces that the remaining partition function $Z_{(2),4}$ in (7.22) corresponds to the PMR.

The next non-trivial thing is to evaluate analytically the determinants in (7.21). For this purpose, heat kernel techniques are adopted. An extensive overview of heat kernel methods in general can be found in [88]. Adopting this philosophy, the partition function and the determinant of an operator are related to the trace of the heat kernel $K^{(s)}(t)$ via

$$\ln Z_{(s)} = -\frac{1}{2}\ln\det(-\nabla^2+m^2)_{(s)} = -\frac{1}{2}\text{Tr}\ln(-\nabla^2+m^2)_{(s)} = -\frac{1}{2}\int_0^\infty \frac{dt}{t}K^{(s)}(t) \qquad (7.23)$$

where the traced heat kernel is defined as

$$K^{(s)}(t) \equiv Tr\, e^{-t(-\nabla^2+m^2)_{(s)}} \qquad (7.24)$$

and m is a constant. One way to evaluate this traced heat kernel coefficient is by using group theoretical techniques that are explicitly described in [89]. In this reference, the calculation is performed for the case of odd-dimensional hyperboloids. Here, the same approach is adopted but it is applied to even-dimensional hyperboloids and in particular to the case of a thermal quotient of AdS_4.

7.3.1 *The traced heat kernel on a thermal quotient of AdS_4*

To find the traced heat kernel on a thermal quotient of AdS_4 one considers the quotient space $\mathbb{H}^4 \simeq SO(4,1)/SO(4)$ obtained by analytic continuation of the 4-sphere $S^4 \simeq SO(5)/SO(4)$. The traced heat kernel coefficient is then given by

$$K^{(s)}(t) = \frac{\beta}{2\pi}\sum_{k\in\mathbb{Z}}\sum_{\vec{m}}\int_0^\infty d\lambda\, \chi_{\lambda,\vec{m}}(\gamma^k)\, e^{tE_R^{(s)}} \qquad (7.25)$$

where $E_R^{(s)}$ are the eigenvalues of spin s Laplacian operator on the quotient space $\mathbb{H}^4$, $\chi_{\lambda,\vec{m}}(\gamma^k)$ is the Harish-Chandra character in the principal series of $SO(4,1)$, γ is an element of the thermal quotient of S^4, β is the inverse temperature and $(\lambda, \vec{m})$ denotes the principal series representation. The next step is to find the eigenvalues $E_R^{(s)}$ and the character $\chi_{\lambda,\vec{m}}(\gamma^k)$ in the case of the symmetric transverse traceless tensors that are of interest here.

The unitary irreducible representations of $SO(4,1)$, denoted as R, are characterized via the array R and the unitary irreducible representations of $SO(4)$, denoted as S, are characterized via the array S as follows:

$$R = (i\lambda - \frac{3}{2}, m_2) \quad \text{with} \quad \lambda \in \mathbb{R}_+ \tag{7.26}$$

$$S = (s_1, s_2) \quad \text{with} \quad s_1 \geq s_2 \geq 0 \tag{7.27}$$

where m_2 is a non-negative (half-)integer and s_1, s_2 are (half-) integers. For the special case of the symmetric transverse traceless tensors, one finds that $s_2 = 0$ and $\lambda \geq s_1 = m_2 \geq 0$, which arise as branching rules. Further details and an explicit derivation of this can be found in [89]. Then, the above representations further simplify to

$$R = (i\lambda - \frac{3}{2}, s) \tag{7.28}$$

$$S = (s, 0) \tag{7.29}$$

where $s \equiv s_1$. Now, the eigenvalues of the spin-s Laplacian operator in the quotient space $SO(4,1)/SO(4)$ are given by

$$E_R^{(s)} = -C_2(R) + C_2(S) \tag{7.30}$$

where $C_2(R)$ and $C_2(S)$ are the quadratic Casimirs for R and S respectively:

$$C_2(R) = m \cdot m + 2r_{SO(4,1)} \cdot m\,, \qquad C_2(S) = s \cdot s + 2r_{SO(4)} \cdot s\,. \tag{7.31}$$

Here the dot product is the usual Euclidean one and $r_{i,SO(4,1)} = \frac{5}{2} - i$, $r_{i,SO(4)} = 2 - i$ and $i = 1, 2, 3$. For the special case of symmetric transverse traceless tensors, when substituting (7.28), (7.29) the quadratic Casimirs (7.31) become

$$C_2(R) = \lambda^2 + s^2 + 3s + \frac{9}{4}\,, \qquad C_2(S) = s^2 + 2s\,. \tag{7.32}$$

Therefore, the corresponding eigenvalues (7.30) take the form

$$E_R^{(s)} = -(\lambda^2 + \frac{9}{4} + s)\,. \tag{7.33}$$

The Harish-Chandra character in the principal series of SO(4,1) is [90]

$$\chi_{\lambda,\vec{m}}(\beta, \phi_1) = \frac{(e^{-i\beta\lambda} + e^{i\beta\lambda})\chi_{\vec{m}}^{SO(3)}(\phi_1)}{e^{-\frac{3\beta}{2}}|e^{\beta} - 1||e^{\beta} - e^{i\phi_1}|^2} \tag{7.34}$$

where $\chi_{\vec{m}}^{SO(3)}(\phi_1)$ is the character of the representation of $SO(3)$. For the thermal quotient that is considered here, one gets $\beta \neq 0$, $\phi_1 = 0$ and $\chi_{\vec{m}}^{SO(3)}(0) = 1 + 2s$ [91] and therefore the above expression simplifies to

$$\chi(\beta, \phi_1) = (1 + 2s)\frac{\cos(\beta\lambda)}{4\sinh^3\left(\frac{\beta}{2}\right)}\,. \tag{7.35}$$

Finally, using (7.33), (7.35), the traced heat kernel (7.25) takes the form

$$K^{(s)}(t) = \frac{\beta(1+2s)}{8\sqrt{\pi t}} \sum_{k \in \mathbb{Z}_+} \frac{1}{\sinh^3 \frac{k\beta}{2}} e^{-\frac{k^2\beta^2}{4t} - t(\lambda^2 + s)}\,. \tag{7.36}$$

At this point all necessary ingredients exist in order to evaluate a partition function that consists of spin-s operators. Substituting (7.36) in (7.23) and performing the integral the result is

$$\ln Z_{(s)} = -(1+2s) \sum_{k\in\mathbb{Z}_+} \frac{e^{-k\beta\left(\frac{3}{2}+\sqrt{\frac{9}{4}+m^2+s}\right)}}{(1-e^{-k\beta})^3 k} \,. \tag{7.37}$$

Finally, using the above in (7.22), the 1-loop partition function of the theory is found to be

$$\ln Z_{1-\text{loop}} = -\sum_{k\in\mathbb{Z}_+} \frac{e^{-2k\beta}(-5+4e^{-2k\beta}-5e^{-k\beta})}{(1-e^{-k\beta})^3 k} \tag{7.38}$$

$$= -\sum_{k\in\mathbb{Z}_+} \frac{q^{2k}(-5+4q^{2k}-5q^k)}{(1-q^k)^3 k} \tag{7.39}$$

where $q = e^{-\beta}$ and β is the inverse temperature.

Lastly, the 1-loop correction to the free energy of the ensemble takes the form

$$F_{1-\text{loop}} = \sum_{k\in\mathbb{Z}_+} \frac{e^{-2k\beta}(-5+4e^{-2k\beta}-5e^{-k\beta})}{(1-e^{-k\beta})^3 k\beta} \,. \tag{7.40}$$

For clarity, the corresponding expressions are now derived for the case of General Relativity . During the above calculation, the eigenvalues $E_R^{(s)}$ (7.33) and the character $\chi_{\lambda,\vec{m}}(\gamma^k)$ (7.35) have been derived considering the quotient space $\mathbb{H}^4 \simeq SO(4,1)/SO(4)$ and symmetric transverse traceless tensors. Therefore, the traced heat kernel (7.36) is characterized by the space one considers and the symmetries of the operators. This implies that it is independent on the type of the partition function (or gravitational theory) and thus, the resulting expression (7.37) can be used to evaluate the 1-loop partition function of General Relativity around AdS_4 spacetime. Applying (7.37) to the 1-loop partition function (3.64), the result is

$$\ln Z^{GR}_{1-\text{loop}} = -\sum_{k\in\mathbb{Z}_+} \frac{e^{-3k\beta}(5-3e^{-3k\beta})}{(1-e^{-k\beta})^3 k} \tag{7.41}$$

and the 1-loop correction of the free energy takes the form

$$F^{GR}_{1-\text{loop}} = \sum_{k\in\mathbb{Z}_+} \frac{e^{-3k\beta}(5-3e^{-3k\beta})}{(1-e^{-k\beta})^3 k\beta} \,. \tag{7.42}$$

7.4 SUMMARY AND CONCLUSIONS

In this chapter, the conformal gravity action was considered in the concept of the path integral approach. The original work can be found in [92].

Exploiting the Euclidean path integral, the purpose was to evaluate the 1-loop partition function of the theory. This required a number of steps. At first, a linearized analysis was performed around a background solution, which was chosen to be AdS_4 spacetime. Additionally, in order for the gauge symmetries and the true dynamics of the theory to be uncovered, a York decomposition was employed. The corresponding linearized equations of motion (7.6) and the 1-loop correction of the classical action (7.8) were derived. It is verified that the gauge symmetries of the theory are maintained at the linearized level, by the absence of the gauge parts $v_a, \hat{\gamma}$ of the York decomposition (7.5) in the linearized equations of motion (and the 1-loop correction of the classical action). Then, on account of the singular conformal gravity Lagrangian, it was necessary to perform the Faddeev-Popov method in order to remove consistently this infinite contribution from the path integral. Choosing a particular gauge like (7.14), (7.15), the corresponding Faddeev-Popov determinant (7.20) was evaluated. Subsequently, the path integral measure was defined properly with the Faddeev-Popov determinant as (7.9) and the 1-loop partition function was presented in (7.21) in terms of determinants of various modes. The dynamical modes of conformal gravity, i.e. the massless graviton and the PMR, as well as the pure gauge contributions are identified with the relevant determinants of (7.21). Lastly, heat kernel techniques were exploited, in order to actually evaluate the 1-loop partition function (7.21). This was possible via its relation (7.23) with the traced heat kernel (7.24). The traced heat kernel was evaluated on a thermal quotient of AdS_4 and finally, the 1-loop partition function of the theory took the final form (7.38).

8

SUMMARY, CONCLUSIONS AND ELABORATIONS

Throughout this thesis, the theory of conformal gravity has been explored in general concepts and formulations, such as the holographic analysis, the Hamiltonian formalism and the path integral approach.

In the holographic analysis, the theory was formulated by imposing asymptotically $(A)dS_4$ boundary conditions and evaluating the corresponding response functions of the dual field theory, which were found to be finite. Furthermore, under the proposed set of asymptotic boundary conditions, the variational principle of the conformal gravity action was found to be well-defined. Then, the holographic response functions of particular solutions of the theory were evaluated, such as a spherically symmetric and an axisymmetric black hole. Lastly, the asymptotic symmetry algebras of the dual field theory were explored, those which are allowed under the proposed asymptotic boundary conditions.

Thereafter, the theory was studied in the Hamiltonian formalism. Performing the constraint analysis, all constraints of the theory were classified as 1^{st} class and their Poisson bracket algebra was derived. Subsequently, the 6 physical degrees of freedom, ascribed to conformal gravity, were specified and the Hamilton equations of motion were derived. Then, the generator associated with gauge symmetries of the theory, namely diffeomorphisms and conformal transformations, was deduced and it was found to have well-defined functional derivatives under the imposition of particular boundary conditions. The charges associated with diffeomorphisms were constructed, and were found to be finite, while there was no charge associated with local Weyl rescalings. Consequently, the algebra of the charges was found to be isomorphic to the algebra of boundary conditions preserving symmetry diffeomorphisms.

Lastly, the theory was formulated in the Euclidean path integral approach to evaluate the 1-loop partition function. Following a linearized analysis around an AdS_4 background and performing a particular metric decomposition, the corresponding linearized equations of motion were derived. Their form confirmed the fact that gauge symmetries are maintained at the linearized level. Then, the corresponding Faddeev-Popov determinant was evaluated and a primary expression for the 1-loop partition function was derived, in which the contribution of the 6 physical degrees of freedom of the theory, as well as the gauge ones, were identified. Finally, the 1-loop partition function was analytically evaluated using heat kernel techniques.

The main conclusions that were obtained throughout the above analysis are:

- The conformal gravity action constitutes a well-defined variational principle, together with the proposed (asymptotic) boundary conditions. These are more general as compared to the Starobinsky ones [72], since they allow for an additional linear term, and when they are imposed, no additional boundary terms or counterterms are required in order for the action to have well-defined functional derivatives. This aspect is due to the fact that the boundary terms arising from partial integration of the original conformal gravity Lagrangian are necessary and sufficient to render the variational principle well-defined. In other words, the equations of motion constitute a well-defined boundary value problem in the presence of a boundary.

  This feature of the theory was demonstrated exploiting both the Lagrangian and Hamiltonian formulations. On the one hand, this was shown in the holographic analysis where the Lagrangian formulation was adopted. In this case, when the proposed generalized asymptotic boundary conditions are considered, the first on-shell variation of the action vanishes and thus, no additional counterterms are required and the corresponding holographic response functions are finite. And on the other hand, in the Hamiltonian setup, the extended Hamiltonian has well-defined functional derivatives when the proposed boundary conditions are imposed. These are more general than those of the holographic analysis, in the sense that they allow for arbitrary Weyl rescalings as well. Finally, the extended Hamiltonian is already an improved gauge generator and the resulting canonical charges are finite.

The result that no additional boundary terms or counterterms are required in the conformal gravity action confirms several indications that this might be the case, such as the finite on-shell action for a metric compatible with the proposed asymptotic boundary conditions and also the fact that the free energy derived from the on-shell action is consistent with the ADM mass [93] and Wald's definition of entropy [79]. Additionally, in the proposed asymptotic boundary conditions in the holographic analysis, the presence of the additional linear term allows for solutions of the conformal gravity action, as opposed to [38] where from the conformal gravity action only the Einstein gravity solutions are selected when this linear term vanishes.

- There exist no canonical charges associated with conformal transformations of the metric, independently of boundary conditions. Therefore, conformal transformations remain (proper [84]) gauge symmetries even at the boundary, on the contrary with diffeomorphisms which become symmetries at the boundary.

  This fact was demonstrated in the Hamiltonian formulation of the theory. It was immediately obtained that the generator of conformal transformations of the spatial metric depends linearly on the canonical variables and does not depend on their derivatives. Thus, it has well-defined functional derivatives, already before the implementation of boundary conditions, and there is not an associated charge. As a result of the non-existence of this Weyl charge, the asymptotic symmetry algebra of the charges associated with diffeomorphisms is isomorphic to the Lie algebra of the boundary conditions preserving symmetry diffeomorphisms.

  The result that there is no Weyl charge is in accordance with [94], where it was shown that the superpotential associated with Weyl rescalings is vanishing, implying that conformal symmetry has a trivial conserved charge. Additionally, in the Weyl invariant scalar-tensor model analyzed in [95], the Noether current associated with Weyl rescalings is also vanishing, consequently shows that Weyl symmetry does not play any dynamical role. Nevertheless, in view of analysis of the conformally invariant gravitational Cherns-Simons theory in three dimensions [96], the non-existence of Weyl charge is unexpected. It was shown there that the Weyl charge vanishes as well, but only if the Weyl factor is not allowed to vary freely. As soon as the Weyl factor is allowed to fluctuate, there exists Weyl charge.

- The sources of the holographic response functions are consistent with the physical degrees of freedom of the theory. Namely, the source conjugate to the analogue of the Brown-York tensor is identified as the spin-2 massless graviton and the source conjugate to the partially massless response (PMR) [76], [77] is identified as the partially massless spin-2 graviton.

  After performing the Hamiltonian analysis of the theory, the 6 physical degrees of freedom were detected. Then, in the holographic analysis content, 2 of those were identified as the source conjugate to the analogue of the Brown-York tensor and the rest 4 were identified with the source of the PMR. These identifications were also made in the 1-loop partition function of the theory, in terms of determinants.

- The proposed asymptotic boundary conditions for the well-defined variational principle of the conformal gravity action are consistent with the Mannheim-Kazanas-Riegert (MKR) [21], [22] solution. In particular, the Rindler acceleration in the MKR solution is interpreted as coming from the conjugate source of the PMR.

This was stated during the holographic analysis, where the holographic response functions of the MKR solution were derived. In particular, it was found that the PMR is linear and the analogue of the trace of the Brown-York tensor is quadratic in the Rindler acceleration for small mass and Rindler acceleration. Therefore, the Rindler acceleration in the MKR solution was identified as arising from a partially massless graviton condensate.

The consistency of the proposed asymptotic boundary conditions with the MKR solution and in particular the identification of the Rindler acceleration with a partially massless graviton renders conformal gravity an example of a theory that allows a non-trivial Rindler term in its solutions. Theories that admit non-trivial Rindler term were discussed in [97] as effective models for gravity at large distances. Thus, in this perspective, conformal gravity can be regarded as an effective model for gravity at large distances.

- The asymptotic symmetry algebra of the dual field theory is $o(3,2)$. Its highest dimensional subalgebra for non-trivial asymptotic boundary conditions was found to admit 5 generators.

  This was derived throughout the holographic setup and it was expected, since the imposed asymptotic boundary conditions were asymptotically $(A)dS_4$. Indeed, the 10 conformal Killing vectors of $o(3,2)$ were the solutions of the asymptotic gauge transformations of the boundary metric (conformal Killing equation). Furthermore, subalgebras of $o(3,2)$ were also obtained for particular choices of the next-to leading order boundary metric. The largest one was found to be 5-dimensional, consisting of 3 translations, a linear combination of the dilatation and a boost and a linear combination of the other boost and the rotation.

In view of the presented research and the above conclusions, further elaborations and future work can be outlined. In the holographic analysis content, an immediate application would be to use the presented results for the response functions and the resulting asymptotic charges to further solutions of conformal gravity. Among those solutions are for example, a rotating black hole with non-vanishing mass and a charged, rotating black hole [80], axisymmetric and spherically symmetric cosmological solutions [98], [99] and also solutions in the presence of additional fields in the Lagrangian [100]. Continuing in the direction of the holographic analysis, a later application that requires more labor would be to calculate higher n-point functions of the dual field theory, given the 1-point functions that were derived in the present work. Indeed, 2-point functions have been considered in [101] and therefore calculation of 3- and higher point functions can be further performed. Such computations can be useful for understanding the underlying degrees of freedom of the dual field theory, specifying in this way its operator content.

Continuing with the results of the Hamiltonian analysis of the present thesis, it is reminded at this point that, when deriving the Dirac algebra of the charges, it was assumed that this algebra admits no central extensions. Thus, it would be interesting to investigate whether non-trivial central extensions could be realizable in this algebra. Along the lines of [85], this can be done by choosing a particular gauge and thus formulating the entire Hamiltonian setup in terms of the Dirac brackets. Then, explicit evaluation of the Dirac bracket of the resulting canonical charges reveals the presence or not of central extensions. Additionally, according to the findings of the present work, local conformal rescalings are a trivial gauge symmetry even at the boundary. It would be useful to investigate further this classical aspect of the theory and possibly relate it with the presence or not of central extensions of the Dirac algebra of the charges.

Lastly, concerning the 1-loop partition function of the theory, the formulation in the present work was considered in an AdS_4 background. It would be of interest to perform the exact same analysis considering an arbitrary background solution. Then, this would be in the direction of including all solutions of the theory in the path integral and thus tracking the full partition function of the theory. Furthermore, either considering a particular background solution or not, future work can be done in verifying the fact that 1-loop corrections of conformal gravity are finite [17]. This would require to adopt a particular regularization scheme for the 1-loop partition function that was derived in the present work. And of course, placing the presented corrections in a more general aspect, it would be of obvious importance to evaluate explicitly higher order corrections of the partition function, as well as of scattering amplitudes. This would then reveal whether conformal gravity is actually renormalizable.

Part III

APPENDIX

A

CONVENTIONS AND PRELIMINARIES

A.1 CONVENTIONS

The conventions for the fundamental constants are taken to be $\hbar = c = G = 1$. Whenever each of these constants is restored, it is explicitly stated in the text.

Latin letters from the beginning of the alphabet, i.e. $a, b, c, \ldots$ denote 4-dimensional indices and Latin letters from the middle of the alphabet, i.e. $i, j, k, \ldots$ denote 3-dimensional indices, unless it is explicitly stated otherwise.

A 4-dimensional metric is denoted by g. The corresponding induced metric in a hypersurface ∂M is denoted by h. Perturbative metric coefficients are denoted by γ. The signature of every 4-dimensional metric is $(-,+,+,+)$. Additionally, the covariant derivative compatible with g is denoted with ∇ whereas the covariant derivative compatible with h is denoted by D.

All 4-dimensional tensors constructed from g_{ab} are denoted by normal letters, e.g. R, R_{ab}, G_{ab}, etc. and purely tangential 3-dimensional tensors constructed from h_{ab} are denoted by calligraphic letters, e.g. $\mathcal{R}, \mathcal{R}_{ab}, \mathcal{G}_{ab}$, etc. Similarly, the same notation is adopted for purely tangential tensors constructed from h_{ij}, e.g. $\mathcal{R}_{ij}, \mathcal{G}_{ij}$, etc. For clarity, whenever a Lie derivative is 3-dimensional, it will be denoted by a subscript, i.e. ${}^{3}\pounds$.

The notations $A_{[a}B_{b]}$, $A_{(a}B_{b)}$, $A_{[a|c}B_{d|b]}$ and $A_{(a|c}B_{d|b)}$ are defined as follows

$$A_{[a}B_{b]} = \frac{1}{2}(A_a B_b - A_b B_a) \tag{A.1}$$

$$A_{(a}B_{b)} = \frac{1}{2}(A_a B_b + A_b B_a) \tag{A.2}$$

$$A_{[a|c}B_{d|b]} = \frac{1}{2}(A_{ac} B_{db} - A_{bc} B_{da}) \tag{A.3}$$

$$A_{(a|c}B_{d|b)} = \frac{1}{2}(A_{ac} B_{db} + A_{bc} B_{da}) \,. \tag{A.4}$$

Lastly, throughout this thesis, Einstein summation convention is adopted.

A.2 DECOMPOSITION WITH RESPECT TO A SPACELIKE OR TIMELIKE HYPERSURFACE

A spacetime (M, g_{ab}), with g_{ab} having signature $(-,+,+,+,)$, can be decomposed into hypersurfaces ∂M of constant θ, where θ is a parameter which is chosen accordingly for spacelike and timelike hypersurfaces. That is, f is taken to be a global time function for spacelike hypersurfaces whereas for timelike ones it is taken to be a function of one spatial coordinate of g_{ab}. A normal vector field n^a to ∂M is assumed to have the norm

$$n^a n^b g_{ab} = -\sigma \tag{A.5}$$

where $\sigma = \pm 1$. Then, g_{ab} induces a metric h_{ab} on each ∂M via

$$h_{ab} = g_{ab} + \sigma n_a n_b \tag{A.6}$$

where $\sigma = 1$ for spacelike hypersurfaces and $\sigma = -1$ for timelike ones. It is deduced from (A.5) that the normal vector is timelike ($n^a n^b g_{ab} = -1$) for spacelike hypersurfaces and spacelike ($n^a n^b g_{ab} = 1$) for timelike ones.

The acceleration is defined as

$$\alpha_a = \frac{g^{bc} n_c \nabla_b n_a}{-n_a n_b g^{ab}} = \frac{g^{bc} n_c \nabla_b n_a}{\sigma} . \tag{A.7}$$

Any tensor on spacetime (M, g_{ab}) can be projected to the hypersurface ∂M by the projector h^a_b as

$$\mathcal{A}_{ab...} = P_h[A_{ab...}] \equiv h^c_a h^d_b A_{cd...} . \tag{A.8}$$

The covariant derivative D_a, compatible with h_{ab}, of any tensor $\mathcal{T}^{cd...}_{ef...}$ on ∂M is related with the covariant derivative ∇_a compatible with g_{ab} by

$$D_a \mathcal{T}^{cd...}_{ef...} = P_h[\nabla_a (P_h T^{cd...}_{ef...})] \tag{A.9}$$

where $T^{cd...}_{ef...}$ is any tensor on M. The extrinsic curvature is defined to be the projection of the covariant derivative of the normal vector, i.e.

$$K_{ab} \equiv P_h[\nabla_a n_b] = h^c_a \nabla_c n_b \tag{A.10}$$

where ∇ is the covariant derivative compatible with the spacetime metric g_{ab}. The decomposition of Riemann tensor into components tangent and normal to the hypersurface ∂M is given by the following relations:

- Gauss relation
$$P_h[R_{abcd}] = \mathcal{R}_{abcd} + \sigma\big(K_{ca}K_{db} - K_{cb}K_{da}\big) \tag{A.11}$$
- Codazzi relation
$$P_h[n^d\, R_{abcd}] = \sigma\big(D_a K_{bc} - D_b K_{ac}\big) \tag{A.12}$$
- Ricci relation
$$P_h[n^b n^d\, R_{abcd}] = K_{ab}K^b_c - \sigma \pounds_{n^b} K_{ac} + D_a \alpha_c + \alpha_a \alpha_c \tag{A.13}$$

Contractions of these relations give the projections of Ricci tensor. In particular one finds:

$$P_h[R_{ab}] = \mathcal{R}_{ab} + \sigma\big(K K_{ab} - 2K_{ac}K^c_b\big) + \pounds_{n^c} K_{ab} - \sigma D_a \alpha_b - \sigma \alpha_a \alpha_b \tag{A.14}$$
$$P_h[n^b R_{ab}] = \sigma\big(D_b K^b_a - D_a K\big) \tag{A.15}$$
$$n^a n^b\, R_{ab} = K_{ab}K^{ab} - \sigma h^{ab} \pounds_{n^c} K_{ab} + D_a \alpha^a + \alpha_a \alpha^a . \tag{A.16}$$

Then, the decomposition of Ricci scalar is given by

$$\begin{aligned} R &= h^{ab} P_h[R_{ab}] - \sigma n^a n^b R_{ab} \\ &= \mathcal{R} + \sigma\big(K^2 - 3K_{ab}K^{ab}\big) + 2h^{ab} \pounds_{n^c} K_{ab} - 2\sigma D_a \alpha^a - 2\sigma \alpha_a \alpha^a . \end{aligned} \tag{A.17}$$

It is usual to rewrite the above expression, when considering the Einstein-Hilbert action, up to a total derivative term. This is done as follows: focusing on a spacelike surface ∂M, i.e. setting $\sigma = 1$ in (A.17), one uses

$$\pounds_{n^c} K_{ab} = n^c \nabla_c K_{ab} + 2K^c_{(a} K_{b)c} - 2n_{(a} \alpha^c) K_{b)c} \tag{A.18}$$

to find its trace as

$$h^{ab} \pounds_{n^c} K_{ab} = n^c \nabla_c K + 2K_{ab}K^{ab} = \nabla_c (n^c K) - K^2 + 2K_{ab}K^{ab} . \tag{A.19}$$

Using the above trace and

$$D_a \alpha^a + \alpha_a \alpha^a = \nabla_a \alpha^a \tag{A.20}$$

in (A.17), the decomposition of the Ricci scalar becomes

$$R = \mathcal{R} + K_{ab}K^{ab} - K^2 + 2\nabla_a(n^a\nabla_c n^c) - 2\nabla_a(n^c\nabla_c n^a)\,. \tag{A.21}$$

This is the expression that is used in (2.155).

The Weyl tensor decomposition is

$$P_h[n^d C_{abcd}] = 2D_{[a}K_{b]c} + D_d K^d_{[a}h_{b]c} - D_{[a}K h_{b]c} \equiv B_{abc} \tag{A.22}$$

$$P_h[n^b n^d C_{abcd}] = \frac{\sigma}{2}\left(h^b_a h^d_c - \frac{1}{3}h_{ac}h^{bd}\right)\left(\mathcal{R}_{bd} - K_{bd}K - \pounds_{n^e}K_{bd} + D_{(b}\alpha_{d)} + \alpha_b\alpha_d\right) \equiv E_{ac} \tag{A.23}$$

The projection $P_h[C_{abcd}]$ into purely tangent components is derived later using the ADM foliation, i.e. in the case of a spacelike surface ∂M.

A.3 ADM DECOMPOSITION

The spacetime (M, g_{ab}) is now decomposed with the standard ADM foliation [50]. This consists of specifying a global time function t on M which by assumption allows to foliate M with a family of spatial surfaces $\partial M \equiv \Sigma_t$, defined by $t \equiv x^0 = $ constant. The tangent bundle $\mathcal{T}\Sigma$ is identified with the 1-form $\nabla_a t$, where ∇ is the covariant derivative on M. Then, the surfaces Σ_t are spacelike if

$$g^{ab}\nabla_a t\nabla_b t < 0\,. \tag{A.24}$$

The future pointing surface normal vector field is taken to be

$$n_a = \beta\,\nabla_a t \tag{A.25}$$

where β is a normalization constant. From (A.24) it follows that for the surfaces Σ_t to be spacelike, the normal vector is timelike, i.e.

$$g^{ab}n_a n_b = g^{ab}\,\beta^2\,\nabla_a t\nabla_b t < 0\,. \tag{A.26}$$

From (A.5), the above is satisfied for $\sigma = 1$. Additionally, the normalization constant β of the normal vector (A.25) is chosen as follows: one assumes that there exists a vector field t^a on M such that $t^a\nabla_a t = 1$. Then, this can be decomposed as

$$t^a = Nn^a + N^a \tag{A.27}$$

with N being the lapse function and N^a the shift vector. From the above, it follows that $t^a n_a = -N$ and from (A.25), it follows that $t^a n_a = \beta$. Thus, one sets $\beta = -N$ and the normal vector (A.25) becomes

$$n_a = -N\nabla_a t \tag{A.28}$$

Choosing $\nabla_a t = (1, 0, 0, 0)$, the normal vector (A.28) can be written as

$$n_a = (-N, 0, 0, 0) \quad , \quad n^a = \left(\frac{1}{N}, -\frac{N^a}{N}\right) \tag{A.29}$$

The acceleration (A.7) takes the form

$$\alpha_a = \frac{D_a N}{N}\,. \tag{A.30}$$

According to the previous conventions, any vector field on M can be decomposed as

$$\xi^a = n^a\xi^\perp + \Xi^a \tag{A.31}$$

with $\xi^\perp = -n_a\xi^a$ and $\Xi^a = h^a_b\Xi^b$ being its normal and parallel components respectively.

The induced metric h_{ab} on Σ_t is obtained from (A.6) for $\sigma = 1$ as

$$h_{ab} = g_{ab} + n_a n_b \,. \tag{A.32}$$

The invariant line element of spacetime M is written as

$$ds^2 = -N^2 dt^2 + h_{ab}(dx^a + N^a dt)(dx^b + N^b dt)\,. \tag{A.33}$$

Thus, using (A.32) and (A.33) the components of g_{ab} take the form

$$\begin{aligned}
&g_{00} = -N^2 + N^a N^b h_{ab} \;,\; g_{0a} = g_{a0} = N_a \;,\; g_{ab} = h_{ab} \\
&g^{00} = -\frac{1}{N^2} \;,\; g^{0a} = g^{a0} = \frac{N^a}{N^2} \;,\; g^{ab} = h^{ab} - \frac{N^a N^b}{N^2} \\
&\sqrt{g} = N\sqrt{h}\,.
\end{aligned} \tag{A.34}$$

The extrinsic curvature (A.10) and its trace take the form

$$K_{ab} = \frac{1}{2} P_h[\pounds_{n^c} h_{ab}] = \frac{1}{2N} h^a_c h^b_d \left(\pounds_{t^e} h_{cd} - \pounds_{N^e} h_{cd}\right) = \frac{1}{2N}\left(\dot{h}_{ab} - 2D_{(a}N_{b)}\right) \tag{A.35}$$

$$K = h^{ab} K_{ab} = -\frac{1}{2N}\left(h^{ab}\dot{h}_{ab} - 2D_a N^a\right) \tag{A.36}$$

where

$$\dot{h}_{ab} \equiv P_h[\pounds_{t^c} h_{ab}]\,. \tag{A.37}$$

Additionally, the time derivatives of the extrinsic curvature and its trace are

$$\dot{K}_{ab} = P_h[\pounds_{t^e} K_{ab}] = N P_h[\pounds_{n^e} K_{ab}] + {}^3\pounds_{N^c} K_{ab} \tag{A.38}$$

$$\dot{K} = P_h[\pounds_{t^e} h^{ab} K_{ab}] = N P_h[\pounds_{n^e} K] + {}^3\pounds_{N^e} K\,. \tag{A.39}$$

Now the Gauss, the Codazzi and the Ricci relations (A.11), (A.12), (A.13) become

$$P_h[R_{abcd}] = \mathcal{R}_{abcd} + K_{ca}K_{db} - K_{cb}K_{da} \tag{A.40}$$

$$P_h[n^d R_{abcd}] = D_a K_{bc} - D_b K_{ac} \tag{A.41}$$

$$P_h[n^b n^d R_{abcd}] = K^b_a K_{cd} + \frac{D_a D_c N}{N} + \frac{1}{N}\,{}^3\pounds_{N^b} K_{ac} - \frac{1}{N}\dot{K}_{ac}\,. \tag{A.42}$$

The decompositions of Ricci tensor (A.14), (A.15), (A.16) are

$$P_h[R_{ab}] = \mathcal{R}_{ab} + K K_{ab} - 2K^c_a K_{bc} - \frac{1}{N}\,{}^3\pounds_{N^c} K_{ab} + \frac{1}{N}\dot{K}_{ab} - \frac{D_b D_a N}{N} \tag{A.43}$$

$$P_h[n^b R_{ab}] = D_b K^b_a - D_a K \tag{A.44}$$

$$n^a n^b R_{ab} = K^{ab}K_{ab} - \frac{h^{ab}}{N}(\dot{K}_{ab} - {}^3\pounds_{N^c} K_{ab}) + \frac{1}{N} D_a D^a N \tag{A.45}$$

and the decomposition of Ricci scalar (A.17) takes the form

$$R = \mathcal{R} + K^2 - 3K_{ab}K^{ab} - 2\frac{D_a D^a N}{N} + 2\frac{h^{ab}}{N}(\dot{K}_{ab} - {}^3\pounds_{N^c} K_{ab})\,. \tag{A.46}$$

The magnetic and the electric part of the Weyl tensor (A.22), (A.23) are

$$B_{abc} \equiv P_h[n^d C_{abcd}] = 2D_{[a}K_{b]c} + D_a K^d_{[a} h_{b]c} - D_{[a} K h_{b]c} \tag{A.47}$$

$$\begin{aligned}
E_{ac} \equiv P_h[n^b n^d C_{abcd}] = \frac{1}{2}\left(h^b_a h^d_c - \frac{1}{3} h_{ac} h^{bd}\right)\Big(\mathcal{R}_{bd} + K_{bd}K - \frac{1}{N}(\dot{K}_{bd} - {}^3\pounds_N K_{bd} \\
- D_b D_d N)\Big)\,.
\end{aligned} \tag{A.48}$$

The decomposition into purely tangential components can be grouped into two parts after using the symmetries of the Weyl tensor, that is its traceless in any pair of indices. Then, the two parts consist of a traceless one, denoted as K_{abcd}, and a trace one as

$$\begin{aligned} P_h[C_{abcd}] &= K_{abcd} + h_{ac}P_h[n^e n^f C_{bedf}] - h_{ad}P_h[n^e n^f C_{becf}] - h_{bc}P_h[n^e n^f C_{aedf}] \\ &\quad + h_{bd}P_h[n^e n^f C_{aecf}] \\ &= K_{abcd} + h_{ac}E_{bd} - h_{ad}E_{bc} - h_{bc}E_{ad} + h_{bd}E_{ac} \end{aligned} \tag{A.49}$$

with

$$\begin{aligned} K_{abcd} &= \frac{1}{2}K_{ac}K_{bd} + h_{ac}(K^e_b K_{de} - K_{bd}K) - \frac{1}{4}h_{ac}h_{bd}(K_{ef}K^{ef} + K^2) \\ &\quad + (a \leftrightarrow b, c \leftrightarrow d) - (a \leftrightarrow b) - (c \leftrightarrow d)\,. \end{aligned} \tag{A.50}$$

Then, the square of the Weyl tensor which appears in the conformal gravity Lagrangian (6.1) is decomposed as

$$\begin{aligned} C_{abcd}C^{abcd} &= P_h[C_{abcd}]P_h[C^{abcd}] + 4E_{ab}E^{ab} - 4B_{abc}\,B^{abc} \\ &= K_{abcd}K^{abcd} + 8E_{ab}E^{ab} - 4B_{abc}\,B^{abc} \end{aligned} \tag{A.51}$$

after using (A.49). From the term $K_{abcd}K^{abcd}$ in the above expression, only the term $2K^a_{bcd}\,K^c_a\,K^{bd}$ survives. But due to Cayley- Hamilton theorem, it turns out that this also vanishes: in particular $-\frac{1}{3}K^a_{bcd}\,K^{bd}$ reads

$$K^a_b K^{bd} K_{dc} - K^a_b K^b_c K + \frac{1}{2}K^a_c\Big(K^2 - K^{bd}K_{bd}\Big) - \frac{1}{6}\delta^a_c\Big(K^b_d K^{de} K_{be} - 3K_{bd}K^{bd} + 2K^3\Big)\,. \tag{A.52}$$

Now, the characteristic polynomial of K^a_c with K^a_c as its argument is exactly expression (A.52) and thus, due to Cayley- Hamilton theorem it vanishes. Thus, the Weyl tensor decomposition (A.51) becomes

$$C_{abcd}C^{abcd} = 8E_{ab}E^{ab} - 4B_{abc}\,B^{abc}\,. \tag{A.53}$$

Consequently, choosing $\alpha_{\rm CG} = -\frac{1}{4}$ for convenient cancellations, the conformal gravity action (6.1) takes the form

$$S = \int_M d^4x\sqrt{-g}\Big[-2E_{ab}E^{ab} + B_{abc}\,B^{abc}\Big] \tag{A.54}$$

and then, by substituting (A.47) and (A.48), it explicitly reads

$$\begin{aligned} S = \int_M d^4x\sqrt{h}\,N\Big[&- \Big(h^c_a h^d_b - \frac{1}{3}h_{ab}h^{cd}\Big)\Big(\mathcal{R}_{cd} + K_{cd}K - \frac{1}{N}(\dot{K}_{cd} - {}^3\pounds_N K_{cd} - D_c D_d N)\Big) \\ &\times \frac{1}{2}\Big(h^{ca}h^{db} - \frac{1}{3}h^{ab}h^{cd}\Big)\Big(\mathcal{R}_{cd} + K_{cd}K - \frac{1}{N}(\dot{K}_{cd} - {}^3\pounds_N K_{cd} - D_c D_d N)\Big) \\ &+ \Big(2D_{[a}K_{b]c} + D_d K^d_{[a}h_{b]c} - D_{[a}K h_{b]c}\Big) \times \Big(2D^{[a}K^{b]c} + D_d K^{d[a}h^{b]c} - D_{[a}K h^{b]c}\Big)\Big]\,. \end{aligned} \tag{A.55}$$

B

BASIC CONCEPTS

B.1 THE CASE OF GENERAL RELATIVITY

B.1.1 *Gauge transformations of the metric with respect to Castellani generator*

Having constructed the Castellani generator (2.203), it is straightforward to verify that it indeed generates the correct gauge transformations of the metric: that is, diffeomorphisms generated by a vector field ζ^c on M as

$$\delta g^{ab} = \pounds_{\zeta^c} g^{ab} \,. \tag{B.1}$$

After decomposing the vector field ζ^a and the metric g^{ab} according to the ADM basis (A.31), (A.34), δg^{ab} takes the explicit form

$$\delta g^{00} = -(2\partial^0\zeta^0 - \zeta^\perp\partial_\perp g^{00} - \zeta^a\partial_a g^{00}) \tag{B.2}$$

$$\delta g^{0a} = -(\partial^0\zeta^a + \partial^a\zeta^0 + 2\zeta^\perp\frac{N^a}{N^3}\partial_\perp N - 2N^a\partial^0\zeta^0 - \zeta^c\partial_c g^{0a}) \tag{B.3}$$

$$\delta g^{ab} = -(\partial^a\zeta^b + \partial^b\zeta^a - \zeta^\perp\partial_\perp g^{ab} - \zeta^c\partial_c g^{ab}) \tag{B.4}$$

Now one can calculate the transformations which are produced by the Castellani generator (2.203), by evaluating $\delta_G g^{ab} = \{g^{ab}, G\}$ for each metric component. It is straightforward but tedious to show that $\delta_G(h^{ab} - \frac{N^a N^b}{N^2})$ indeed yields (B.4), using Hamilton's equations of motion (2.171), (2.172), (2.173). This calculation is not presented here. The rest of the components of $\delta_G g^{ab} = \{g^{ab}, G\}$, that is $\delta_G g^{00}$ and $\delta_G g^{0a}$ are now derived. An explicit calculation of $\delta_G g^{00} = \{g^{00}, G\}$ gives

$$\begin{aligned}\delta_G g^{00} &= \int d^3x \{\frac{1}{N^2}(y), \pi(x)\}(N^a\partial_a\epsilon^\perp - \dot{\epsilon^\perp} - \epsilon^a\partial_a N)(x) \\ &= -2\partial^0\epsilon^0 + \epsilon^\perp\partial_\perp g^{00} + \epsilon^a\partial_a g^{00}\end{aligned} \tag{B.5}$$

in complete agreement with (B.2). Likewise, for $\delta_G g^{0a} = \{g^{0a}, G\}$ one finds

$$\begin{aligned}\delta_G g^{0a} &= \int d^3x N^a(y)\{\frac{1}{N^2}(y), \pi(x)\}(-N^b\partial_b\epsilon^\perp + \dot{\epsilon^\perp} + \epsilon^b\partial_b N)(x) \\ &+ \int d^3x \frac{1}{N^2}(y)\{N^a(y), \pi^b(x)\}(\partial_b N\epsilon^\perp - N\pi^b\partial_b\epsilon^\perp + \epsilon^c\partial_c N_b - N^c\partial_c\epsilon^b + \dot{\epsilon}_b)(x) \\ &= -\partial^0\epsilon^a - \partial^a\epsilon^0 - 2\epsilon^\perp\frac{N^a}{N^3}\partial_\perp N + 2N^a\partial^0\epsilon^0 + \epsilon^c\partial_c g^{0a}\end{aligned} \tag{B.6}$$

in complete agreement with (B.3). Thus, it is concluded that the Castellani gauge generator (2.203) indeed gives the transformations (B.2)-(B.4) or equivalently (B.1).

C

THE CASE OF CONFORMAL GRAVITY

C.1 $(A)dS_4$ ASYMPTOTICS

The decomposition with respect to a spacelike or timelike hypersurface that was mentioned in section A is now performed for a particular type of metric g_{ab}. This is required for the holographic calculation of chapter 6. The ansatz for the line element is taken to be

$$ds^2 = g_{ab}\, dx^a\, dx^b = -\sigma \frac{\ell^2}{\rho^2} d\rho^2 + \frac{1}{\rho^2}\gamma_{ij}\, dx^i\, dx^j \tag{C.1}$$

where $\gamma_{ij} = \gamma_{ij}(x^k, \rho)$, $i, j, k, \ldots = 1, 2, 3$, ℓ is the $(A)dS$ radius and $\sigma = 1$ for dS_4 and $\sigma = -1$ for AdS_4. The conformal boundary is at $\rho \to 0$ and is treated as a constant ρ surface. The outward -or future- pointing unit normal vector to this surface is chosen to be

$$n^a = -\frac{\rho}{\ell}\,\delta^a_\rho \quad , \quad n_a = \frac{\sigma\ell}{\rho}\delta^\rho_a \tag{C.2}$$

such that it satisfies (A.5). Then, for the induced metric (A.6) one finds

$$\gamma_{\rho a} = g_{\rho a} + \sigma n_\rho n_a \tag{C.3}$$

$$\gamma_{ia} = \gamma_{ia} \tag{C.4}$$

and consequently from the line element one gets (C.1)

$$\gamma_{\rho\rho} = 0 \tag{C.5}$$

$$\gamma_{\rho i} = \gamma_{i\rho} = 0 \tag{C.6}$$

$$\gamma_{ij} = g_{ij}\,. \tag{C.7}$$

The non-vanishing components of the extrinsic curvature (A.10) are

$$K_{ij} = -\frac{\sigma}{2}\pounds_{n^\rho}\gamma_{ij} = \sigma\frac{\rho}{2\ell}\partial_\rho\gamma_{ij} \tag{C.8}$$

and its trace is

$$K = -\frac{\sigma}{2}\pounds_{n^\rho}\gamma_{ij} = \sigma\frac{\rho}{2\ell}\partial_\rho\gamma_{ij}\,. \tag{C.9}$$

The Gauss (A.11), the Codazzi (A.12) and the Ricci relation (A.13) become

$$P_\gamma[R_{abcd}] = h^a_i h^b_j h^c_k h^d_l R_{abcd} = \mathcal{R}_{ijkl} + \sigma\left(K_{ik}K_{jl} - K_{il}K_{jk}\right) \tag{C.10}$$

$$P_\gamma[n^d R_{abcd}] = h^a_i h^b_j h^c_k n^\rho R_{abc\rho} = \sigma\left(D_i K_{jk} - D_j K_{ik}\right) \tag{C.11}$$

$$P_\gamma[n^b n^d R_{abcd}] = h^a_i h^c_k n^\rho n^\rho R_{a\rho c\rho} = K_{ik}K^k_j + \sigma\pounds_{n^\rho}K_{ij}\,. \tag{C.12}$$

Contraction of the above give the components of the Ricci tensor (A.14), (A.15), (A.16). Here, they take the form

$$P_\gamma[R_{ab}] = h^a_i h^b_j R_{ab} = \mathcal{R}_{ij} + \sigma\left(KK_{ij} - 2K_{ik}K^k_j\right) - \pounds_{n^a}K_{ij} \tag{C.13}$$

$$P_\gamma[n^b R_{ab}] = h^a_i n^\rho R_{a\rho} = \sigma\left(D_i K - D_k K^k_i\right) \tag{C.14}$$

$$n^a\, n^b R_{ab} = n^\rho\, n^\rho R_{\rho\rho} = K_{ij}K^{ij} + \sigma\gamma^{ij}\pounds_{n^\rho}K_{ij}\,. \tag{C.15}$$

Then, the decomposition of the Ricci scalar (A.17) becomes

$$R = \mathcal{R} + \sigma\big(K^2 - 3K_{ij}K^{ij}\big) - 2\gamma^{ij}\pounds_{n^\rho}K_{ij}\,. \tag{C.16}$$

The magnetic and electric part of the Weyl tensor (A.22), (A.23) take the form

$$B_{ijk} \equiv P_\gamma[n^d C_{abcd}] = h_i^a h_j^b h_k^c n^\rho C_{abc\rho} = 2D_{[i}K_{j]k} + D_l K^l_{[i}\gamma_{j]k} - D_{[i}K\gamma_{j]k} \tag{C.17}$$

$$E_{ij} \equiv P_\gamma[n^b n^d C_{abcd}] = h_i^a h_k^c n^\rho n^\rho C_{a\rho c\rho} = \frac{\sigma}{2}\Big(\gamma_i^m\gamma_j^l - \frac{1}{3}\gamma_{ij}\gamma^{ml}\Big)\Big(\mathcal{R}_{ml} - K_{ml}K + \pounds_{n^a}K_{ml}\Big) \tag{C.18}$$

C.2 ASYMPTOTIC EXPANSION OF TENSORS

The asymptotic expansion of all the tensors, which are required for the calculation of the first variation of the on-shell action (5.14) and the holographic response functions that arise from (5.15), (5.16), is presented. This is done as follows: the induced, boundary metric (A.6) is assumed to have a generalized Fefferman-Graham expansion at the conformal boundary $\rho = 0$ as

$$\gamma_{ij} = \gamma_{ij}^{(0)} + \frac{\rho}{\ell}\gamma_{ij}^{(1)} + \frac{\rho^2}{\ell^2}\gamma_{ij}^{(2)} + \frac{\rho^3}{\ell^3}\gamma_{ij}^{(3)} + \dots \tag{C.19}$$

where the dots denote terms of order ρ^2 or/and higher. Then, one inserts the line element (C.1) and the above asymptotic expansion of the boundary metric into the tensors and express them in terms of $\gamma_{ij}^{(0)}$, $\gamma_{ij}^{(1)}$, $\gamma_{ij}^{(2)}$, ... and in terms of curvature tensors of the metric $\gamma_{ij}^{(0)}$, i.e. $\mathcal{R}_{ij}^{(0)}$, $\mathcal{R}^{(0)}$, ... This calculation is performed with the *xAct* package of Mathematica [73].

The inverse metric takes the form

$$\gamma^{ij} = \gamma^{ij}_{(0)} - \frac{\rho}{\ell}\gamma^{ij}_{(1)} + \frac{\rho^2}{\ell^2}\big(\gamma^{im}_{(1)}\gamma^{j}_{(1)m} - \gamma_{ij}^{(2)}\big) + \dots \tag{C.20}$$

such that $\gamma_{im}\,\gamma^{mj} = \delta_i^j + \mathcal{O}(\rho^5)$. From (C.8) and (C.19) the extrinsic curvature reads

$$K_{ij} = \sigma\Big[\frac{\rho}{2\ell^2}\gamma_{ij}^{(1)} + \frac{\rho^2}{\ell^3}\gamma_{ij}^{(2)} + \frac{3\rho^3}{2\ell^4}\gamma_{ij}^{(3)} + \dots\Big] \tag{C.21}$$

and its trace is

$$K = \gamma^{ij}K_{ij} = \sigma\Big[\frac{\rho}{2\ell}\gamma_{(1)} + \frac{\rho^2}{\ell^3}\Big(\gamma_{(2)} - \frac{1}{2}\gamma_{ij}^{(1)}\gamma^{ij}_{(1)}\Big) + \dots\Big]\,. \tag{C.22}$$

The Christoffel symbol of the metric (C.19) is

$$\Gamma^k_{ij} = \Gamma^{(0)k}_{ij} + \frac{\rho}{2\ell}\big(2D_{(i}\gamma^{(1)k}_{j)} - D^k\gamma^{(1)k}_{ij}\big) + \dots\,. \tag{C.23}$$

The components (C.13)-(C.15) of the Ricci tensor take the form

$$\begin{aligned} R_{ij} = {} & \frac{1}{\ell^2}\Big(\mathcal{R}_{ij}^{(0)} + \sigma\big(\frac{1}{2}\gamma_i^{(1)m}\gamma_{jm}^{(1)} - \frac{1}{4}\gamma_{ij}^{(1)}\gamma_{(1)} - \gamma_{ij}^{(2)}\big)\Big) + \frac{\rho}{\ell^3}\Big(\ell^2 D_m D_{(i}\gamma_{j)}^{(1)m} - \frac{1}{2}\ell^2 D_i D_j\gamma^{(1)} \\ & - \frac{1}{2}\ell^2 D^2\gamma_{ij}^{(1)} + \sigma\big(\frac{1}{4}\gamma_{ml}^{(1)}\gamma^{ml}_{(1)}\gamma_{ij}^{(1)} - \frac{1}{2}\gamma_{ml}^{(1)}\gamma_i^{(1)m}\gamma_j^{l(1)} + 2\gamma^{(1)}_{m(i}\gamma^{(2)m}_{j)} - \frac{1}{2}\gamma_{ij}^{(2)}\gamma^{(1)} - \frac{1}{2}\gamma_{ij}^{(1)}\gamma^{(2)} \\ & - 3\gamma_{ij}^{(3)}\big)\Big) + \dots \end{aligned} \tag{C.24}$$

$$\begin{aligned} R_{i\rho} = {} & \frac{1}{2\ell}\Big(D_m\gamma^m_{(1)i} - D_i\gamma^{(1)}\Big) + \frac{\rho}{4\ell^2}\Big(3\gamma^{ml}_{(1)}D_i\gamma^{(1)}_{ml} - 2\gamma^{ml}_{(1)}D_l\gamma^{(1)}_{mi} - 2\gamma^l_{(1)i}D_m\gamma_l^{(1)m} + 4\gamma^{(1)}_{im}D^m\gamma^{(1)} \\ & + 4D_m\gamma_i^{(2)m} - 4D_i\gamma^{(2)}\Big) + \dots \end{aligned} \tag{C.25}$$

$$R_{\rho\rho} = \frac{1}{2\ell^2}\Big(\gamma_{ij}^{(1)}\gamma^{ij}_{(1)} - 4\gamma_{(2)}\Big) + \frac{\rho}{2\ell^3}\Big(-\gamma_i^{(1)k}\gamma^{ij}_{(1)}\gamma^{(1)}_{jk} + 4\gamma^{ij}_{(1)} - 6\gamma_{(3)}\Big) + \dots \tag{C.26}$$

where the covariant derivative D is compatible with the metric $\gamma_{ij}^{(0)}$. The Ricci scalar (C.16) becomes

$$R = \frac{1}{\ell^2}\Big[\mathcal{R}^{(0)} + \sigma\big(\frac{3}{4}\gamma_{ij}^{(1)}\gamma_{(1)}^{ij} - \gamma_{(1)}^2 - 2\gamma_{(1)}\big)\Big] + \frac{\rho}{\ell^3}\Big[- \ell^2\gamma_{(1)}^{ij}R_{ij}^{(0)} + \ell^2 D_i D_j \gamma_{(1)}^{ij} - \ell^2 D^i D_j \gamma_{(1)} + \sigma\big(- \frac{3}{2}\gamma_m^{(1)i}\gamma_i^{(1)l}\gamma_l^{(1)m} + \frac{1}{2}\gamma_{(1)}^{ij}\gamma_{ij}^{(1)}\gamma_{(1)} + 5\gamma_{(1)}^{ij}\gamma_{ij}^{(2)}\gamma_{(1)} - \gamma_{(1)}\gamma_{(2)} - 6\gamma_{(3)}\big)\Big] + \dots \,. \tag{C.27}$$

One more tensor that is required for the calculation is the 3-dimensional Einstein tensor with tangential indices. It takes the form

$$\mathcal{G}_{ij} = \mathcal{G}_{ij}^{(0)} + \frac{\rho}{2\ell}\Big(2D_m D_{(i}\gamma_{j)}^{(1)m} - D_i D_j \gamma^{(1)} - D_m D^m \gamma_{ij}^{(0)} + \gamma_{ij}^{(0)}\big(D_m D^m \gamma^{(1)} - D_m D_l \gamma_{(1)}^{ml}\gamma_{(1)}^{ij}\, {}^3R_{ij}^{(0)}\big) - \gamma_{ij}^{(1)}\, {}^3R^{(0)}\Big) + \dots \,. \tag{C.28}$$

The asymptotic expansions of the magnetic and the electric part of the Weyl tensor (A.22) , (A.23) are given in the main text in (5.21) and in (5.22).

C.3 ELIMINATION OF THE AUXILIARY FIELD VARIABLES

Considering the auxiliary field λ^{ab} and its conjugate momentum p_{ab} as canonical variables in the Hamiltonian formulation of 6.1 implies that two additional primary constraints appear, namely

$$\phi_1 \equiv \pi_h^{ab} - \sqrt{|h|}\,\lambda^{ab} \approx 0 \tag{C.29}$$

$$\phi_2 \equiv p_{ab} \approx 0\,. \tag{C.30}$$

These primary constraints are 2$^{\text{nd}}$ class, since

$$\{\phi_1, \phi_2\} = -\sqrt{|h|}\,\delta_c^{(a}\delta_b^{d)}\,\delta^{(3)}(x-y)\,. \tag{C.31}$$

One now has to proceed by considering the Dirac bracket. The antisymmetric tensor $C_{\alpha\beta}$, which was introduced in (2.66) of 2.1.9, takes the matrix form

$$C = \sqrt{|h|}\delta^{(3)}(x-y)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{C.32}$$

and subsequently, for any phase space functions f, g, the Dirac bracket (2.68) is

$$\{f,g\}^\star = \{f,g\} - \int_{\Sigma_t} d^3x \frac{1}{\sqrt{|h|}}\{f,\phi_1(x)\}\,\{\phi_2(x),g\} + \int_{\Sigma_t} d^3x \frac{1}{\sqrt{|h|}}\{f,\phi_2(x)\}\,\{\phi_1(x),g\}\,. \tag{C.33}$$

According to the arguments of 2.1.9 that lead to (2.75), one can set the 2$^{\text{nd}}$ constraints (C.29), (C.30) strongly equal to zero

$$\lambda^{ab} = \frac{1}{\sqrt{|h|}}\pi_h^{ab} \tag{C.34}$$

$$p_{ab} = 0\,. \tag{C.35}$$

Using the above substitutions, the auxiliary field λ^{ab} and its conjugate momentum p_{ab} are eliminated from the conformal gravity Lagrangian (6.4). In other words, the 2$^{\text{nd}}$ class constraints (C.29), (C.30) have been solved everywhere as (C.34), (C.35). Thus, any phase space functions f, g do not depend on λ^{ab} or its conjugate momentum p_{ab} but they depend on the rest of the canonical variables, i.e. h_{ab}, N, N^a, K_{ab} and their conjugate momenta. Consequently, the last two terms in (C.33) vanish and the Dirac bracket (C.33) simplifies to

$$\{f,g\}^\star = \{f,g\} \tag{C.36}$$

and the Hamiltonian formulation in 6.1 can be performed using the Poisson bracket for the rest of the canonical variables, i.e. of h_{ab}, N, N^a, K_{ab} and their conjugate momenta.

C.4 HAMILTON'S EQUATIONS OF MOTION WITH THE TOTAL HAMILTONIAN

The Hamilton's equations of motion, using the total Hamiltonian (6.17), take the form

$$
\begin{aligned}
\dot{h}_{ab} &= \{h_{ab}, H_T\} = 2D_{(a}N_{b)} + 2NK_{ab} && \text{(C.37)}\\
\dot{\pi}_h^{ab} &= \{\pi_h^{ab}, H_T\} = {}^3\pounds_{N^a}\pi_h^{ab} - \lambda_{\mathcal{P}}\Pi_K^{ab} - N\Big[\frac{1}{\sqrt{h}}\Big(\frac{1}{4}\Pi_K^{cd}\Pi^K_{cd}h^{ab} - \Pi^a_{Kc}\Pi_K^{bc}\Big) - \Pi_K^{cd}K_{cd}K^{ab}\\
&\quad + D_cD^{(b}\Pi_K^{a)c} - \frac{1}{2}h^{ab}D_cD_d\Pi_K^{cd} - \frac{1}{2}D_cD^c\Pi_K^{ab} + 2\sqrt{h}\Big(-\frac{1}{4}B^{cde}B_{cde}h^{ab} + B^{acd}B^b_{\ cd}\\
&\quad + \frac{1}{2}B^{cda}B_{cd}{}^b + B^{c(ab)}D_cK - B^{c(ab)}D_dK^d_c - D_c\big(B^{d(ab)}K^c_d + B^{cd(a}K^{b)}_d + B^{(a|dc|}K^{b)}_d\big)\Big)\Big]\\
&\quad - D_cN\Big[2D_d\Pi_K^{d(a}h^{b)c} + D^{(b}\Pi_K^{a)c} - \frac{3}{2}D^c\Pi_K^{ab} - D_d\Pi_K^{cd}h^{ab} - 2\sqrt{h}\Big(B^{d(ab)}K^c_d + B^{cd(a}K^{b)}_d\\
&\quad + B^{(a|dc|}K^{b)}_d\Big)\Big] - D_cD_dN\Big[2\Pi_K^{(d|(a}h^{b)|d)} - \Pi_K^{ab}h^{cd} - \frac{1}{2}\Pi_K^{cd}h^{ab}\Big] && \text{(C.38)}\\
\dot{N} &= \{N, H_T\} = \lambda && \text{(C.39)}\\
\dot{\pi} &= \{\pi, H_T\} = -\mathcal{H}_\perp && \text{(C.40)}\\
\dot{N}^a &= \{N^a, H_T\} = \lambda^a && \text{(C.41)}\\
\dot{\pi}_a &= \{\pi_a, H_T\} = -\mathcal{H}_a && \text{(C.42)}\\
\dot{K}_{ab} &= \{K_{ab}, H_E\} = \lambda_{\mathcal{P}}h_{ab} + {}^3\pounds_{N^a}K_{ab} + N\Big[\mathcal{R}_{ab} + KK_{ab} - \frac{1}{\sqrt{h}}\Pi^K_{ab}\Big] + D_aD_bN && \text{(C.43)}\\
\dot{\Pi}_K^{ab} &= \{\Pi_K^{ab}, H_E\} = {}^3\pounds_{N^a}\Pi_K^{ab} - N\Big[\Pi_K^{ab}K + \Pi_K^{cd}K_{cd}h^{ab} + 2\pi_h^{ab} + 4\sqrt{h}D_cB^{cab}\Big] - 4D_cNB^{cab} && \text{(C.44)}
\end{aligned}
$$

Equations (C.39) and (C.41) state that N, N^a are arbitrary functions. Since N, N^a are contained in the remaining equations, the characteristic feature of conformal gravity as a gauge theory is revealed: that is, the dynamical evolution of the system is not completely determined by the equations of motion but depends on arbitrary functions.

C.5 CONFORMAL TRANSFORMATIONS

Expressions that are used in 6.4 are presented here analytically. Considering a local conformal transformation of the spatial metric h_{ab} on Σ_t as

$$h_{ab} = \Omega^2\bar{h}_{ab} \tag{C.45}$$

with Ω being arbitrary, the transformation for the inverse metric is

$$h^{ab} = \Omega^{-2}\bar{h}_{ab} \tag{C.46}$$

by demanding $h_{ab}h^{bc} = \bar{h}_{ab}\bar{h}^{bc} = \bar{h}^c_a = \delta^c_a$. Also, it is deduced that the square root of the determinant of the spatial metric transforms as

$$\sqrt{h} = \Omega^3\sqrt{\bar{h}}\,. \tag{C.47}$$

The normal vector (A.29) takes the form

$$n_a = \Omega\bar{n}_a \quad , \quad n^a = \Omega^{-1}\bar{n}^a\,. \tag{C.48}$$

The relation between the covariant derivative which is compatible with h_{ab} and the one which is compatible with $\bar{h}_{ab}$ is

$$D_a v^{de\ldots}_{bc\ldots} = \bar{D}_a v^{de\ldots}_{bc\ldots} - C^f_{ba}v^{de\ldots}_{fc\ldots} - C^f_{ac}v^{de\ldots}_{bf\ldots} + \ldots + C^d_{af}v^{fe\ldots}_{bc\ldots} + C^e_{af}v^{df\ldots}_{bc\ldots} + \ldots \tag{C.49}$$

where $v^{de\cdots}_{bc\cdots}$ is any tensor on Σ_t or $\partial\Sigma$ and

$$C^c_{ab} = \frac{1}{2}h^{cd}(2\bar{D}_{(a}h_{b)d} - \bar{D}_d h_{ab}) \\ = 2\delta^c_{(a}\bar{D}_{a)} \ln\Omega - \bar{h}_{ab}\bar{D}^c \ln\Omega \tag{C.50}$$

is the difference tensor. Using (C.49) and (C.50) the conformal behavior of the 3-dimensional Ricci tensor is found to be

$$\mathcal{R}_{ab} = \bar{\mathcal{R}}_{ab} - \bar{D}_a\bar{D}_b \ln\Omega + \bar{D}_a \ln\Omega\bar{D}_b \ln\Omega - \bar{h}_{ab}\big(\bar{D}^c\bar{D}_c \ln\Omega + \bar{D}_c \ln\Omega\bar{D}^c \ln\Omega\big)\,. \tag{C.51}$$

The Weyl tensor C^a_{bcd} is invariant under local conformal transformations of the form (C.45), so using (C.46) one finds

$$C_{abcd} = \Omega^2\bar{C}_{abcd}\,. \tag{C.52}$$

Using the above expression and (C.48), the magnetic and the electric part of the Weyl tensor (A.47), (A.48) become

$$B_{abc} = \Omega\bar{B}_{abc} \tag{C.53}$$

$$E_{ab} = \Omega^{-1}\bar{E}_{ab}\,. \tag{C.54}$$

The arbitrary functions $\epsilon^\perp$, ε^c are chosen to transform as

$$\epsilon^\perp = \Omega\bar{\epsilon}^\perp \tag{C.55}$$

$$\varepsilon^a = \bar{\varepsilon}^a\,. \tag{C.56}$$

Additionally, one further expression that is required is

$$\frac{1}{\epsilon^\perp}D_aD_a\epsilon^\perp = \bar{D}_a\bar{D}_b \ln\Omega - \bar{D}_a \ln\Omega\bar{D}_b \ln\Omega + \frac{1}{\bar{\epsilon}^\perp}\bar{D}_a\bar{D}_b\bar{\epsilon}^\perp + h_{ab}\big(\bar{D}_c \ln\Omega\bar{D}^c \ln\Omega \\ + \frac{1}{\bar{\epsilon}^\perp}\bar{D}^c \ln\Omega\bar{D}_c\bar{\epsilon}^\perp\big)\,. \tag{C.57}$$

Part IV

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BIBLIOGRAPHY

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Part V

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