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D I P L O M A R B E I T

Hadwiger Integration

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Introduction

A *valuation* on the set of all compact convex sets $\mathcal{K}^n \subset \mathbb{R}^n$ is a real-valued function μ that fulfills

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$
$$\mu(\emptyset) = 0,$$

whenever $A, B, A \cup B \in \mathcal{K}^n$. A classic result is Hadwiger's classification theorem which classifies the *intrinsic volumes* $\mu_0, \ldots, \mu_n$ as special valuations on $\mathcal{K}^n$. Here μ_0 is the Euler characteristic χ. Furthermore, μ_k acts as the k-dimensional Lebesgue measure on all sets of dimension less or equal to k. Besides their importance for various results in integral geometry, valuations can be seen as finitely additive measures. Therefore, one can try to establish an integration theory with respect to the intrinsic volumes, which is called *Hadwiger integration.* As it turns out, there is no unique way to extend the Hadwiger integrals of simple functions to Hadwiger integrals of continuous functions. Moreover, these integrals are valuations on functionals and a similar result to Hadwiger's classification theorem can be achieved. Subsequently, various integral transforms will be examined and applied to the setting of sensor networks. The integral with respect to the Euler characteristic will be of utmost importance for this purpose.

In Chapter 1 we will start with some definitions and basic results on valuations. On this basis, the Euler characteristic and further on the intrinsic volumes are introduced and examined. We proof Hadwiger's classification theorem to highlight the intrinsic volumes as a basis of the vector space of convex-continuous rigid motion invariant valuations. The main reference for this chapter is the book by Klain and Rota [16].

Chapter 2 introduces *definable sets* as a collection of subsets of $\mathbb{R}^n$. Examples include the semialgebraic and semilinear sets. This allows us to define the set of *definable functions* $\mathrm{Def}(\mathbb{R}^n)$ as real-valued functions with definable graph and compact support. Furthermore, the corresponding integer-valued simple functions are called *constructible functions* $\mathrm{CF}(\mathbb{R}^n)$. Using a straightforward integral of constructible functions with respect to the intrinsic volumes, we can interpret these integrals as a special class of valuations on functionals, whereby we follow [31]. Furthermore, we want to extend the integrals to continuous functions by approximating them with step functions. However, it turns out that it makes a difference if we approximate from below or from above. Hence, we obtain *lower* and *upper Hadwiger integrals* and study their properties.

In order to state a classification theorem for valuations on $\mathrm{Def}(\mathbb{R}^n)$ we need corresponding topologies to distinguish between the lower and upper Hadwiger integrals, which is done in Chapter 3. Therefore we introduce currents, which are continuous linear functionals on differential forms. By assigning each definable set a special current - the so-called *conormal cycle* - we obtain the lower and upper flat metrics on definable functions. Consequently, we can proof a result similar to Hadwiger's classification theorem (cf. Theorem 3.16) which was published in [5]. It states that any rigid motion invariant lower valuation ν on $\mathrm{Def}(\mathbb{R}^n)$ can be written as

$$\nu(f) = \sum_{i=0}^{n} \int_{\mathbb{R}^n} c_i(f) \lfloor \mathrm{d}\mu_i \rfloor,$$

for all $f \in \mathrm{Def}(\mathbb{R}^n)$ where the $c_i : \mathbb{R} \to \mathbb{R}$ are continuous increasing functions with $c_i(0) = 0$.

Chapter 4 focuses on integral transforms, where we start with Fubini's theorem and Euler convolution. A duality transform on $\mathrm{CF}(\mathbb{R}^n)$ then provides a deconvolution. Moreover, we consider Radon, Bessel and Fourier transforms. Most importantly, Schapira's inversion formula [26] allows to invert Radon transforms under certain conditions.

In Chapter 5 we follow a recent series of papers ([1], [2], [4] and [12]) to utilize the integral transforms of Chapter 4 in the setting of sensor networks. Proper Euler integrals then allow us to count and even localize targets. However, those results use a continuous field of sensors but we can show that they also apply to a sensor network based on a sufficiently dense and regular triangulation of the target space.

A perspective to other applications of Hadwiger integration is given in Chapter 6.

Acknowledgments

I want to use this opportunity to express my gratitude to Monika Ludwig for providing an interesting topic and for her conscientious supervision. She always helped me with problems and gave useful comments on every version of this thesis within the shortest time.
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Contents

1 Valuations

This chapter introduces valuations, which are set functions and the understanding of which is necessary for the rest of this thesis. We will start with the definition and consider some important valuations before we go on to proof a main classification theorem for valuations, Hadwiger's theorem (Theorem 1.41). The setup of this chapter mostly follows the book by Klain and Rota (see [16]). Some of the proofs are left out - they can be found in the book. A short version of the proof of Hadwiger's theorem is also given in [15].

1.1 Basic results on valuations

Before we can define the class of set functions, known as valuations, we need some basic results.

Definition 1.1 A relation $\leq$ on a set L is called a *partial ordering* if the following conditions are satisfied for all $x, y, z \in L$.

- $x \leq x$. (reflexivity)
- From $x \leq y$ and $y \leq x$ follows that $x = y$. (antisymmetry)
- From $x \leq y$ and $y \leq z$ follows that $x \leq z$. (transitivity)

The set L is then also called a *partially ordered set*.

Definition 1.2 A partially order set L is called a *lattice* if there exist a greatest lower bound $x \wedge y \in L$ and a least upper bound $x \vee y \in L$ for all $x, y \in L$. Furthermore L is said to be a *distributive lattice* if the following additional identities hold for all $x, y, z \in L$:

- $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.
- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Example 1.3 Let S be a set, and let L be a family of subsets of S closed under finite unions and finite intersections. It can easily be seen, that L is a distributive lattice. The partial ordering is then given by subset inclusion, the greatest lower bound is realized by the intersection of sets and the least upper bound is the union of sets.

Definition 1.4 (Valuation) We call a real-valued function μ on a lattice L of sets, $\mu : L \to \mathbb{R}$, a *valuation* if μ satisfies the following properties for all $A, B \in L$:

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) \tag{1.1}$$

$$\mu(\emptyset) = 0. \tag{1.2}$$

Example 1.5 Well known examples for valuations can be found in measure theory, where L would be a σ-algebra over S and μ would be a measure. Besides the restriction to non-negativity we have a difference in additivity: Measures have to fulfill σ-additivity whereas valuations only have to be finitely additive.

This property is also reflected in the conditions on the set L: a σ-algebra has to be closed under countable unions and intersections, while a lattice of sets only has to be closed under finite numbers of operations. So in a certain sense, valuations are generalizations of measures.

An instant basic result is the following.

Lemma 1.6 (Inclusion-exclusion principle) *For a valuation μ on a lattice L, $n \in \mathbb{N}$ and $A_1, A_2, \ldots, A_n \in L$ we get*

$$\begin{aligned} &\mu(A_1 \cup A_2 \cup \cdots \cup A_n) \\ &\quad= \sum_{1\le i\le n} \mu(A_i) - \sum_{1\le i<j\le n} \mu(A_i \cap A_j) + \sum_{1\le i<j<k\le n} \mu(A_i \cap A_j \cap A_k) \\ &\qquad - \cdots + (-1)^{n-1}\mu(A_1 \cap \cdots \cap A_n) \\ &\quad= \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1\le i_1<\cdots<i_k\le n} \mu(A_{i_1} \cap \cdots \cap A_{i_k}) \right) \\ &\quad= \sum_{\emptyset\ne I\subseteq\{1,2,\ldots,n\}} (-1)^{|I|-1} \mu(\cap_{i\in I} A_i), \end{aligned} \tag{1.3}$$

where $|I|$ denotes the cardinality of I.

Proof. Induction on n and use of the additive property (1.1). ■

Definition 1.7 Let $\mathbb{1}_A$ denote the indicator function of a set $A \subset \mathbb{R}^n$, that is

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

A function $f : L \to \mathbb{R}$ is called *L-simple* if there exist $A_i \in L$ and $\alpha_i \in \mathbb{R}$ for each $1 \le i \le k$ such that

$$f = \sum_{i=1}^{k} \alpha_i \mathbb{1}_{A_i}.$$

Using basic properties of indicator functions we obtain the inclusion-exclusion formula for indicators,

$$\begin{aligned} \mathbb{1}_{A_1\cup A_2\cup\cdots\cup A_n} &= \sum_{1\le i\le n} \mathbb{1}_{A_i} - \sum_{1\le i<j\le n} \mathbb{1}_{A_i\cap A_j} + \sum_{1\le i<j<k\le n} \mathbb{1}_{A_i\cap A_j\cap A_k} \\ &\quad - \cdots + (-1)^{n-1}\mathbb{1}_{A_1\cap\cdots\cap A_n} \\ &= \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1\le i_1<\cdots<i_k\le n} \mathbb{1}_{A_{i_1}\cap\cdots\cap A_{i_k}} \right) \\ &= \sum_{\emptyset\ne I\subseteq\{1,2,\ldots,n\}} (-1)^{|I|-1} \mathbb{1}_{\cap_{i\in I} A_i}. \end{aligned} \tag{1.4}$$

Definition 1.8 Let G be a subset of a lattice L that is closed under finite intersections. We call G a *generating set of* L when every element of L can be represented as a finite union of elements of G.

Example 1.9 Let $\mathcal{K}^n$ denote the set of all compact convex subsets of $\mathbb{R}^n$, which is closed under finite intersections. Furthermore, we call a finite union of compact convex sets a *polyconvex* set and denote the collection of all polyconvex sets in $\mathbb{R}^n$ by Polycon(n). Since the union and intersection of two polyconvex sets is again polyconvex, we obtain that Polycon(n) is in fact a lattice. Obviously $\mathcal{K}^n$ is a generating set of Polycon(n).

With the inclusion-exclusion formula for indicator functions (1.4) we get that every L-simple function f can be written as

$$f = \sum_{i=1}^{r} \beta_i \mathbb{1}_{B_i}$$

with $B_i \in G$ and $\beta_i \in \mathbb{R}$.

Definition 1.10 Let G be a family of sets that is closed under finite intersections. We call a real-valued function μ on G, $\mu : G \to \mathbb{R}$, a *valuation* on G if μ satisfies (1.1) and (1.2) for all $A, B \in G$ such that $A \cup B \in G$.

Since the definition of a generating set G of a lattice L demands that every set $B \in L$ can be expressed as a finite union of sets in G, we can attempt to extend a valuation defined on G to a valuation on L. Before we state a corresponding result we need another definition.

Definition 1.11 Let μ be a valuation on a generating set G of a lattice L and let f be an L-simple function with $f = \sum_{i=1}^{k} \alpha_i \mathbb{1}_{A_i}$ with $A_i \in G$ and $\alpha_i \in \mathbb{R}$. We define the *integral* of f with respect to μ as

$$\int f \, \mathrm{d}\mu := \sum_{i=1}^{k} \alpha_i \mu(A_i).$$

Since the representation of the form $f = \sum_{i=1}^{k} \alpha_i \mathbb{1}_{A_i}$ does not have to be unique, one has to check that the integral is well defined.

Theorem 1.12 (Groemer's integral theorem) *Let G be a generating set for a lattice L, and let μ be a valuation on G. The following statements are equivalent:*

1. *The valuation μ on G extends uniquely to a valuation on L.*
2. *The valuation μ satisfies the inclusion-exclusion identity on G, namely*

$$\mu(\cup_{i=1}^{n} B_i) = \sum_{\emptyset \neq I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|-1} \mu(\cap_{i \in I} B_i), \tag{1.5}$$

for all $n \geq 2$ and whenever $B_i \in G$ and $\cup_{i=1}^{n} B_i \in G$.

3. *The valuation μ defines an integral on the vector space of linear combinations of indicator functions of sets in L.*

Proof sketch.
1. $\Rightarrow$ 2.: Since μ extends uniquely to a valuation on L, the inclusion-exclusion principle holds for all sets in L and therefore for all sets $B_i \in G$ such that $\cup_{i=1}^n B_i \in G$.

3. $\Rightarrow$ 1.: Since μ defines an integral on the space of L-simple functions, we can define for $A \in L$

$$\mu(A) = \int \mathbb{1}_A \,\mathrm{d}\mu.$$

Using the fact that

$$\mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A\cap B},$$

as well as the linearity of the integral, we obtain a valuation on L.

2. $\Rightarrow$ 3.: Suppose that there exist non-empty sets $K_1, \dots, K_m \in G$ and non-zero real number $\alpha_1, \dots, \alpha_m$ such that

$$\sum_{i=1}^{m} \alpha_i \mathbb{1}_{K_i} \equiv 0.$$

Suppose, on the contrary, that the integral for this representation of the zero function would give

$$\sum_{i=1}^{m} \alpha_i \mu(K_i) \neq 0.$$

Furthermore we set

$$L_1 = K_1, \dots, L_m = K_m, L_{m+1} = K_1 \cap K_2, \dots, L_p = K_1 \cap K_2 \cap \dots \cap K_m,$$

so that we get all possible intersections of the sets K_i. Since generating sets are closed under intersections, we have that $L_i \in G$ for all i. Now we suppose that

$$\sum_{i=q}^{p} \alpha_i \mathbb{1}_{L_i} \equiv 0,$$

while

$$\sum_{i=q}^{p} \alpha_i \mu(L_i) \neq 0,$$

for $\alpha_q \neq 0$. We chose q to be maximal so that this is possible. Using the inclusion-exclusion principle, one can contradict the maximality of q. //

We want to find a convenient property of valuations on $\mathcal{K}^n$ to admit a unique extension to a valuation on $\mathrm{Polycon}(n)$. We will find this property to be a certain kind of continuity.

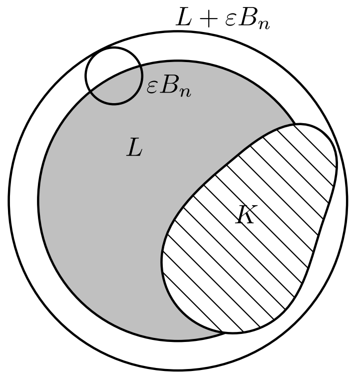


Figure 1: Example for $K \subseteq L + \varepsilon B_n$.

Definition 1.13 For two sets $A, B \subseteq \mathbb{R}^n$, the *Hausdorff distance* $\delta(A, B)$ is defined by

$$\delta(A, B) = \max \left(\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right),$$

where $d(x, A) = \inf_{a \in A} |x - a|$ denotes the distance from the point $x \in \mathbb{R}^n$ to the set A.

The following lemma gives us a more intuitively accessible description of the Hausdorff distance δ. A proof is given in [16], Chapter 4.

Lemma 1.14 *Let $K, L \subseteq \mathbb{R}^n$ be compact sets. Then $\delta(K, L) \leq \varepsilon$ iff $K \subseteq L + \varepsilon B_n$ and $L \subseteq K + \varepsilon B_n$, where B_n denotes the n-dimensional unit ball.*

Furthermore it is a well known fact that the Hausdorff distance defines a metric.

Theorem 1.15 *The distance δ defines a metric - the so-called* Hausdorff metric *- on the set of all compact subsets of $\mathbb{R}^n$.*

Using this metric we can define a notion of continuity for valuations on $\mathcal{K}^n$.

Definition 1.16 We call a valuation μ *convex-continuous* or simply *continuous* if

$$\mu(A_k) \xrightarrow{k \to \infty} \mu(A)$$

for all compact convex sets A_k, A such that $A_k \xrightarrow{k \to \infty} A$ with respect to the Hausdorff metric.

The following result shows that under certain conditions we can restrict ourselves to valuations on the generating set $\mathcal{K}^n$.

Theorem 1.17 (Groemer's extension theorem) *Every convex-continuous valuation μ on $\mathcal{K}^n$ admits a unique extension to a valuation on* Polycon(n).

Proof sketch. Using Groemer's integral theorem (Theorem 1.12) it is sufficient to show that every convex-continuous valuation μ defines an integral on the space of linear combinations of indicator functions of sets in Polycon(n). This can be done by induction on the dimension n.

Starting with $n = 0$ the statement is trivial. Now suppose that μ gives us a well defined integral in dimension $n-1$. For dimension n we assume the contradiction of our statement. Suppose that there exist sets $K_1, \dots, K_m \in \mathcal{K}^n$ such that

$$\sum_{i=1}^{m} \alpha_i \mathbb{1}_{K_i} \equiv 0.$$

On the other hand we suppose that the integral for this representation of the zero function would give us

$$\sum_{i=1}^{m} \alpha_i \mu(K_i) = 1.$$

Furthermore, we assume that m is the smallest number greater than zero such that this is possible. Using the induction hypothesis and the continuity of μ we can bring this to a contradiction by intersecting the K_i with closed half-spaces. //

1.2 Euler characteristic

The probably most important valuation is the so-called Euler characteristic (sometimes also referred to as Euler-Poincaré characteristic). In this section we will discuss some of the more convenient (and equivalent) definitions of the Euler characteristic as well as some of its properties.

Definition 1.18 Let E_n denote the Euclidean group on $\mathbb{R}^n$, that is the group generated by all translations and orthogonal transformations. Furthermore, for $A \subseteq \mathbb{R}^n$ and $g \in E_n$, we write

$$gA = g(A) = \{g(a) : a \in A\}.$$

We now call a valuation $\mu : \text{Polycon}(n) \to \mathbb{R}$ *rigid motion invariant* or simply *invariant* if

$$\mu(A) = \mu(gA)$$

for all $A \in \text{Polycon}(n)$ and all $g \in E_n$.

Definition 1.19 (Euler characteristic) The *Euler characteristic* is the unique convex-continuous invariant valuation $\chi : \text{Polycon}(n) \to \mathbb{R}$ such that

$$\chi(K) = 1 \tag{1.6}$$

for every non-empty compact convex set $K \subseteq \mathbb{R}^n$.

This definition seems to be quite simple but at a first glance it is not clear that there exists a (unique) valuation that fulfills (1.6).

Theorem 1.20 (The existence of the Euler characteristic)
There exists a unique convex-continuous invariant valuation χ defined on Polycon(n) *that is independent of the dimension n (normalized) and fulfills (1.6).*

Proof. We start by proofing the existence of valuations χ^n for each dimension by induction on n. Using Groemer's extension theorem (Theorem 1.17) it is sufficient to show the existence of a proper valuation on $\mathcal{K}^n$. Furthermore, by Groemer's integral theorem (Theorem 1.12) it suffices to show the existence of a linear functional L_n on the $\mathcal{K}^n$-simple functions, such that $L_n(\mathbb{1}_K) = 1$ whenever K is a non-empty compact convex set.

Starting with $n = 1$ we set

$$L_1(f) := \sum_{x\in\mathbb{R}} (f(x) - f(x+0)),$$

where $f(x+0) = \lim_{a\to 0^+} f(x+a)$. Since simple functions only have finitely many points of discontinuity, the sum above is finite. Now if K is a non-empty compact convex set in $\mathbb{R}$, K has to be an interval of the form $[a, b]$. For the corresponding indicator function $\mathbb{1}_K$ we get

$$L_1(\mathbb{1}_K) = \mathbb{1}_K(b) - \mathbb{1}_K(b+0) = 1.$$

This gives us the existence of the valuation $\chi^1(K) := L_1(\mathbb{1}_K)$, so that

$$L_1(f) = \int f \,\mathrm{d}\chi^1.$$

We now assume that the statement is true for $n-1$. For dimension n we choose an orthogonal coordinate system $x_1, x_2, \ldots, x_n$. Moreover, for any real number x, let H_x be the hyperplane parallel to the coordinate axes $x_2, \ldots, x_n$ passing through the point $(x, 0, \ldots, 0)$. If $f = f(x_1, x_2, \ldots, x_n)$ is a simple function we define

$$f_x(x_2, \ldots, x_n) := f(x, x_2, \ldots, x_n)$$

as a simple function in H_x. Using the induction hypothesis we assume that $L_{n-1}(f_x)$ has been defined in H_x, since H_x is isomorphic to $\mathbb{R}^{n-1}$. We set $F(x) = L_{n-1}(f_x)$ and get a simple function in $\mathbb{R}$. Using F we can define

$$L_n(f) := L_1(F).$$

Now for every compact convex set K, the function $f_x = (I_K)_x$ is the indicator function of a slice of K by the hyperplane $x_1 = x$. F is then the indicator function of the projection of K onto the x_1-coordinate axis (see Figure 2). It

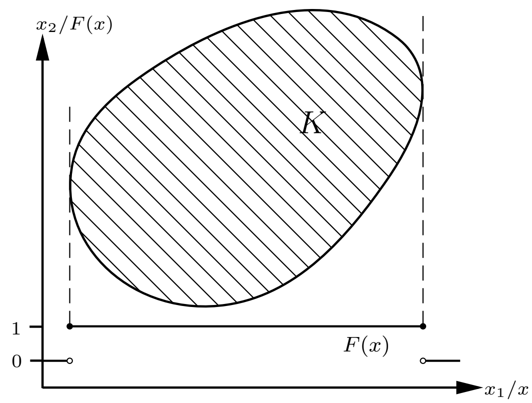


Figure 2: F is the indicator function of the projection of K onto the x_1-coordinate axis.

easily follows that $L_n(f) = L_1(F) = 1$. This gives us a valuation χ^n with the desired properties via

$$L_n(f) = \int f \,\mathrm{d}\chi^n.$$

To proof that the Euler characteristic is normalized, we assume that $k \leq j \leq n$ and $K \subseteq V \subseteq \mathbb{R}^n$, where K is a polyconvex set of dimension k and V is a plane of dimension j. We now have to show that $\chi^j(K)$ computed within V is equal to $\chi^n(K)$ computed in $\mathbb{R}^n$. Since we have a representation of K as

$$K = K_1 \cup K_2 \cup \cdots \cup K_m,$$

with the K_i being compact convex sets, we obtain by the inclusion-exclusion formula

$$\mu(K) = \sum_{\emptyset \neq I \subseteq \{1,2,\dots,m\}} (-1)^{|I|-1} \mu(\cap_{i \in I} K_i),$$

for any valuation μ. Since both χ^j and χ^n have the property to attain the value 1 on all non-empty compact convex sets - particularly on the sets $\cap_{i \in I} K_i$ - we get that $\chi^j(K) = \chi^n(K) = \chi(K)$. ■

Remark 1.21 The approach of using points of discontinuity and projections can also be used to give an inductive definition of the Euler characteristic. An example for that can be found in [13].

A first property of the Euler characteristic that can be seen using the inclusion-exclusion formula, is the fact that χ is only integer-valued, in particular χ can also become negative on certain sets.

Example 1.22 We consider an interval $[a, b] \subset \mathbb{R}$ which we can partition into

$$[a, b] = \{a\} \dot\cup (a, b) \dot\cup \{b\},$$

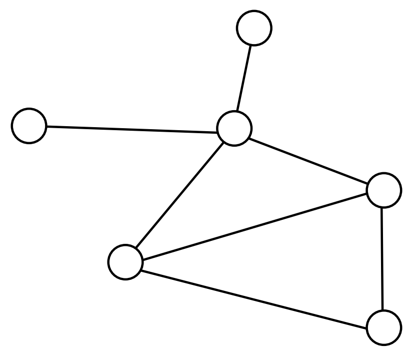


Figure 3: A graph with $V = 6$, $E = 7$, $F = 3$ and $V - E + F = 2$.

where $\dot\cup$ indicates the union of disjoint sets. Using the additive property we get

$$\chi([a, b]) = \chi(\{a\}) + \chi((a, b)) + \chi(\{b\}).$$

Since $[a, b], \{a\}, \{b\}$ are compact convex sets we obtain that $\chi((a, b)) = -1$.

Example 1.23 Let $A = \{a_1, \dots, a_k\}$ be a set of distinct points in $\mathbb{R}^n$. Since points are compact convex sets of dimension zero we get

$$\chi(A) = \sum_{i=1}^{k} \chi(a_i) = k = \#A.$$

Hence the Euler characteristic can be used to count points.

The first use of the Euler characteristic was for results on planar graphs and polyhedra.

Definition 1.24 The Euler characteristic χ for a planar graph or the surface of a polyhedra in three dimensions is defined as

$$\chi = V - E + F,$$

where V denotes the number of vertices, E the number of edges, and F the number of faces, including the exterior face in case of a planar graph.

For planar connected graphs and convex polyhedra we get the following result, see also Figure 3.

Theorem 1.25 (Euler's (polyhedron) formula)
The Euler characteristic of any planar connected graph or the surface of a convex polyhedra is 2.

For more general sets one can calculate the Euler characteristic of its surface by laying a map onto the object and thus finding a polygonization of the surface.

The torus for example has Euler characteristic 0. For more details see [10], Chapter 6.

In general one can consider the distributive sublattice of Polycon(n) that is generated by compact convex polytopes. For that we will recall some definitions and give some new ones respectively.

Definition 1.26 A *convex polytope* is the intersection of a finite collection of closed half-spaces. Moreover, we call a finite union of convex polytopes a *polytope*. For a compact convex polytope P of dimension k in $\mathbb{R}^n$, we consider the k-dimensional plane V containing P. The *relative interior of* P, denoted by relint(P), is the interior of P relative to the topology of V. Now we can define a *system of faces* $\mathcal{F}$ for a polytope P as a family with the following properties:

- The elements of $\mathcal{F}$ are convex polytopes.
- $\bigcup_{Q\in\mathcal{F}} \operatorname{relint}(Q) = P$.
- For $Q, R \in \mathcal{F}$ with $Q \neq R$ one has $\operatorname{relint}(Q) \cap \operatorname{relint}(R) = \emptyset$.

A result which can also be used as a definition of the Euler characteristic is the following.

Theorem 1.27 (The Euler-Schläfli-Poincaré formula)
Let $\mathcal{F}$ be a system of faces of a polytope P, and denote by f_i the number of elements of $\mathcal{F}$ of dimension i. Then

$$\chi(P) = f_0 - f_1 + f_2 - \cdots$$

One of the most important properties of the Euler characteristic is the fact that χ is a homotopy invariant. A significant consequence of this is that two sets have the same Euler characteristic if they are homeomorphic.

1.3 Intrinsic volumes

We will now introduce a special class of valuations, the Euler characteristic being one of them.

Let Mod(n) denote the set of all linear subspaces of $\mathbb{R}^n$. Under the relation of inclusion of linear subspaces Mod(n) is a partially ordered set. Furthermore Mod(n) is a lattice since for every two subspaces x, y there exist the subspace spanned by x and y, as well as the intersection of x and y.

Definition 1.28 We define the so-called *Grassmannian* Gr(n, k) as the set of all elements of Mod(n) of dimension k.

Up to a common factor there exists a unique Haar measure ν_k^n on Gr(n, k) that is invariant under rotations and reflections of the orthogonal group O(n). We choose this factor so that

$$\nu_k^n(\operatorname{Gr}(n,k)) = \binom{n}{k} \frac{\omega_n}{\omega_k \omega_{n-k}}, \tag{1.7}$$

where ω_i denotes the volume of the unit ball B_n in $\mathbb{R}^n$.
Next we consider the partially ordered set of all linear varieties in $\mathbb{R}^n$, $\mathrm{Aff}(n)$, that don't necessarily have to pass through the origin in $\mathbb{R}^n$. Note that the minimal element in $\mathrm{Mod}(n)$ is the zero subspace $\{0\}$ while the minimal element of $\mathrm{Aff}(n)$ is the empty set $\emptyset$.

Definition 1.29 The *affine Grassmannian* $\mathrm{Graff}(n,k)$ is the set of all elements of $\mathrm{Aff}(n)$ of dimension k.

Again we want to have an invariant measure on $\mathrm{Graff}(n,k)$. In this case the measure λ_k^n shall act invariant under the Euclidean group E_n. For that let $V^\perp$ be the maximal linear subspace of $\mathbb{R}^n$ orthogonal to $V \in \mathrm{Graff}(n,k)$ that contains the origin. Furthermore for a real-valued measurable function f on $\mathrm{Graff}(n,k)$, let $\bar{f} : \mathrm{Gr}(n,k) \times \mathbb{R}^n \to \mathbb{R}$ be given by

$$\bar{f}(V_0, p) := f(V_0 + p).$$

We can define λ_k^n via

$$\int_{\mathrm{Graff}(n,k)} f \,\mathrm{d}\lambda_k^n = \int_{\mathrm{Gr}(n,k)} \int_{V_0^\perp} \bar{f}(V_0, p) \,\mathrm{d}p \,\mathrm{d}\nu_k^n(V_0), \tag{1.8}$$

where $\mathrm{d}p$ denotes the ordinary Lebesgue measure on $V_0^\perp \cong \mathbb{R}^{n-k}$. The invariance of $\mathrm{d}\lambda_k^n$ then follows from the invariance of $\mathrm{d}p$.

Definition 1.30 (Intrinsic volumes) We define the *intrinsic volumes* μ_i^n, $0 \le i \le n$ on $\mathcal{K}^n$ as the valuations defined by

$$\mu_{n-k}^n(K) = \lambda_k^n(\mathrm{Graff}(K;k)), \tag{1.9}$$

for all $K \in \mathcal{K}^n$, where $\mathrm{Graff}(K;k)$ denotes the set of all $V \in \mathrm{Graff}(n,k)$ such that $K \cap V \neq \emptyset$.

It can be shown that this definition gives us indeed a family of continuous valuations. Hence, by Groemer's extension theorem (Theorem 1.17) μ_{n-k}^n has a unique extension to $\mathrm{Polycon}(n)$, which we will again denote by μ_{n-k}^n. Since (1.9) is only valid for compact convex sets we would like to have a representation, valid for all sets in $\mathrm{Polycon}(n)$. Known as *Hadwiger's formula* we get for all $A \in \mathrm{Polycon}(n)$

$$\mu_{n-k}^n(A) = \int_{\mathrm{Graff}(n,k)} \chi(A \cap V) \,\mathrm{d}\lambda_k^n(V). \tag{1.10}$$

For compact convex sets this gives us again (1.9). Hadwiger's formula also delivers us another justification why the intrinsic volumes are valuations, since (1.1) and (1.2) directly follow from the fact that the Euler characteristic is a valuation. Indeed, we could use (1.10) as a definition for μ_i^n. Moreover, since the integral of the affine Grassmannian is invariant under rigid motions on $\mathbb{R}^n$, we obtain that intrinsic volumes are a collection of invariant valuations.

It is also easy to see that

$$\mu_0^n(A) = \int_{\mathrm{Graff}(n,n)} \chi(A \cap V)\,\mathrm{d}\lambda_n^n(V) = \chi(A \cap \mathbb{R}^n)\lambda_n^n(\mathbb{R}^n) = \chi(A).$$

Furthermore, using (1.8) we obtain

$$\mu_{n-k}^n(A) = \int_{\mathrm{Gr}(n,k)} \int_{V_0^\perp} \chi(A \cap (V_0 + p))\,\mathrm{d}p\,\mathrm{d}\nu_k^n(V_0).$$

For $k = 0$ this gives us

$$\begin{aligned} \mu_n^n(A) &= \int_{\mathrm{Gr}(n,0)} \int_{V_0^\perp} \chi(A \cap (V_0 + p))\,\mathrm{d}p\,\mathrm{d}\nu_0^n(V_0) \\ &= \nu_0^n(\{0\}) \int_{\mathbb{R}^n} \chi(A \cap p)\,\mathrm{d}p \\ &= \int_A \mathrm{d}p. \end{aligned}$$

Hence, μ_n^n is the Lebesgue measure on $\mathbb{R}^n$. Besides that, we have

$$\mu_{n-1}^n(K) = \tfrac{1}{2}S(K),$$

where $S(K)$ denotes the surface area of $K \in \mathcal{K}^n$.

Definition 1.31 We call a valuation μ on Polycon(n) *homogeneous of degree* $k > 0$, if

$$\mu(\alpha K) = \alpha^k \mu(K)$$

for all $K \in$ Polycon(n) and all $\alpha \geq 0$.

We now refer to the orthogonal projection of a set $K \subset \mathbb{R}^n$ onto the subspace $V_0^\perp$ as $K|V_0^\perp$ and get for $\alpha \geq 0$

$$\begin{aligned} \lambda_k^n(\mathrm{Graff}(\alpha K; k)) &= \int_{\mathrm{Graff}(n,k)} \mathbb{1}_{\mathrm{Graff}(\alpha K;k)}\,\mathrm{d}\lambda_k^n \\ &= \int_{\mathrm{Gr}(n,k)} \int_{V_0^\perp} \underbrace{\mathbb{1}_{\mathrm{Graff}(\alpha K;k)}(V_0 + p)}_{=\mathbb{1}_{\alpha K|V_0^\perp}(p)}\,\mathrm{d}p\,\mathrm{d}\nu_k^n(V_0) \\ &= \int_{\mathrm{Gr}(n,k)} \mathrm{vol}_{n-k}(\alpha K|V_0^\perp)\,\mathrm{d}\nu_k^n(V_0) \\ &= \alpha^{n-k} \int_{\mathrm{Gr}(n,k)} \mathrm{vol}_{n-k}(K|V_0^\perp)\,\mathrm{d}\nu_k^n(V_0) \\ &= \alpha^{n-k}\lambda_k^n(\mathrm{Graff}(K;k)), \end{aligned}$$

whereas vol_{n-k} denotes the $(n-k)$-dimensional volume. Consequently, μ_i^n is homogeneous of degree i.

Definition 1.32 For a fixed Cartesian coordinate system in $\mathbb{R}^n$ we define an *orthogonal parallelotope* $P \subset \mathbb{R}^n$ to be a set of the form

$$P = [a_1, b_1] \times \cdots \times [a_n, b_n],$$

with $a_i \leq b_i$. In other words, a parallelotope is a rectilinear box. The distributive lattice that consists of finite unions and intersections of orthogonal parallelotopes shall be denoted by $\mathrm{Par}(n)$.

To get a better understanding of the intrinsic volumes, one can consider them on $\mathrm{Par}(n)$. We obtain that for each parallelotope $P \in \mathrm{Par}(n)$ with sides of length $x_1, x_2, \ldots, x_n$

$$\mu_k^n(P) = \sum_{1 \leq i_1 < \cdots < i_k \leq n}^{n} x_{i_1} x_{i_2} \cdots x_{i_n}, \tag{1.11}$$

for $1 \leq k \leq n$. For details, see [16], Chapter 4.

Remark 1.33 If a parallelotope P has dimension $k < n$, then

$$\mu_i^m(P) = \mu_i^n(P),$$

for all $k \leq m < n$. Hence the intrinsic volumes seem to be independent of the dimension of the ambient space. We will see that this is true in general in the next section. Note that the key to this normalization lies in the specific choice of the Haar measure ν_k^n in (1.7). Other choices of the Haar measure on $\mathrm{Mod}(n)$ lead to the so-called *quermaßintegrale* $W_{n,k}$, which are related to the intrinsic volumes via

$$W_{n,k}(K) = \omega_k \binom{n}{k}^{-1} \mu_{n-k}^n(K).$$

The most important difference is that the *quermaßintegrale* really depend on the dimension of the ambient space. See also [20].

The following theorem demonstrates that like the Euler characteristic all intrinsic volumes are sensible to whether a set is open or not. A proof is given in [16], Chapter 7.

Theorem 1.34 *Let P be a compact convex polytope. Then*

$$\mu_k^n(\mathrm{relint}(P)) = (-1)^{\dim P - k} \mu_k^n(P),$$

for $0 \leq k \leq n$ whereas $\dim P$ denotes the dimension of P.

1.4 Hadwiger's theorem

The purpose of this section is the formulation and proof of a characterization theorem for the intrinsic volumes.

Definition 1.35 We say that the set of points $\{x_0, \ldots, x_m\} \subset \mathbb{R}^n$ is *affinely independent*, if there doesn't exist any $(m-1)$-dimensional affine subspace $V \subseteq \mathbb{R}^n$ that contains those points. We define an m-dimensional *simplex* Δ as the convex hull of $m+1$ affinely independent points, that is

$$\Delta = \left\{ x \in \mathbb{R}^n : x = \sum_{i=0}^{m} a_i x_i, a_i \in [0,1], \sum_{i=0}^{m} a_i = 1 \right\}.$$

Furthermore, the convex hull of every subset of $\{x_0, \ldots, x_m\}$, consisting of m elements, is called a *facet of* Δ.

Definition 1.36 We call a valuation μ on $\mathcal{K}^n$ *simple*, if

$$\mu(K) = 0,$$

for all K of dimension less than n.

With these definitions we can give a characterization for the volume.

Theorem 1.37 (The volume characterization theorem)
Let μ be a continuous translation invariant simple valuation on $\mathcal{K}^n$. Then there exists $c \in \mathbb{R}$ such that

$$\mu(K) + \mu(-K) = c\mu_n^n(K),$$

for all $K \in \mathcal{K}^n$.

Proof sketch. For a continuous translation invariant simple valuation μ on $\mathcal{K}^n$ consider

$$\nu(K) := \mu(K) + \mu(-K) - 2\mu([0,1]^n)\mu_n^n(K),$$

for all $K \in \mathcal{K}^n$. If we can show, that $\nu(K) = 0$ for all $K \in \mathcal{K}^n$, we are finished. This can be done by induction on n, where the one dimensional case follows from the fact, that compact convex subsets of $\mathbb{R}$ are simply the closed line segments.

Next, the induction step starts with the case of $K = [0,1]^n$ for which the statement is obvious. This implies that the statement holds for all sets of the type $[0, 1/r]^n$ with an integer $r > 0$. Now let K be a box of rational dimensions with sides parallel to the coordinate axes. Since such boxes can be built up out of cubes of the form $[0, 1/r]^n$, the result holds for said boxes. The continuity of ν then implies that the statement holds for every box of positive real dimension, with sides parallel to the coordinate axes. Consequently, one can discuss the case of a box with sides parallel to a different set of orthogonal axes, by cutting the box into a finite number of pieces, translations of which form a box with sides parallel to the original coordinate axes.

Next, define a valuation τ on $\mathcal{K}^{n-1}$ via

$$\tau(K) := \nu(K \times [0,1]),$$

for all $K \in \mathcal{K}^{n-1}$. By the induction hypothesis one has that $\tau \equiv 0$. Using the continuity of ν again, one can deduce that $\nu(K \times [a, b]) = 0$ for any convex body $K \subseteq \mathbb{R}^{n-1}$ and for all $a, b \in \mathbb{R}$. By the preceding argument this holds for right cylinders with a convex base of any orientation.

Consequently the case of K being a prism can be reduced to the case of right cylinders by cutting and rearrangement. This can be utilized to show that $\nu(P + \overline{v}) = \nu(P)$ for all convex polytopes P and all line segments $\overline{v}$. By induction over finite Minkowski sums of line segments it follows that $\nu(Z) = 0$ and $\nu(P + Z) = \nu(P)$ for all zonotopes (finite Minkowski sums of straight line segments) Z and P as above. By continuity of ν, this implies that $\nu(Y) = 0$ and $\nu(K+Y) = \nu(K)$ for all $K \in \mathcal{K}^n$ and all zonoids (sets that can be approximated by zonotopes) Y.
Since for smooth symmetric convex sets K there exist zonoids Y_1, Y_2 such that $K + Y_2 = Y_1$, the result holds for all smooth symmetric convex sets and furthermore by approximation and continuity for all centered convex bodies.

Next, let Δ by an n-dimensional simplex with one vertex at the origin and let v be the vector sum of the vertices of Δ. One can find a proper centered parallelotope P and a centered set of points P_* such that

$$P = \Delta \cup P_* \cup (-\Delta + v),$$

which leads to $\nu(\Delta) = -\nu(-\Delta)$. Since $\nu(\Delta) = \nu(-\Delta)$ and ν is translation invariant, this gives $\nu(\Delta) = 0$ for any simplex Δ.
Now any convex polytope P can be expressed as a finite union of simplices that have pairwise intersections of dimension less than n. Additivity and simplicity of ν yield that $\nu(P) = 0$ for any convex polytope P. Since convex polytopes are dense in $\mathcal{K}^n$, the continuity of ν finally implies that $\nu(K) = 0$ for all $K \in \mathcal{K}^n$. //

Proposition 1.38 (Sah) *For each n-dimensional simplex Δ there exist polytopes $P_1, \dots, P_m$, where each of the P_i is symmetric under a reflection across a hyperplane and where each of the intersections $P_i \cap P_j$ is at most $(n-1)$-dimensional, such that*

$$\Delta = P_1 \cup \dots \cup P_m.$$

Proof. For the n-dimensional simplex Δ let $x_0, \dots, x_n$ be its vertices, and let Δ_i be the facet of Δ opposite to x_i. Since each simplex has a unique inscribed sphere, let z be the center of that inscribed in Δ. By z_i we denote the foot point in Δ_i of the perpendicular line from z to Δ_i. We now set $A_{i,j} := \text{conv}\{z, z_i, z_j, \Delta_i \cap \Delta_j\}$ for all $i < j$, where conv denotes the convex hull, see also Figure 4. Consequently we get that

$$\Delta = \bigcup_{0 \leq i < j \leq n} A_{i,j},$$

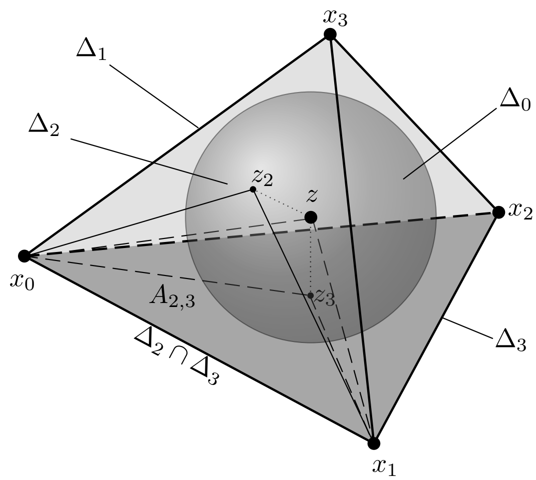


Figure 4: Example of a 3-dimensional simplex with the polytope $A_{2,3}$.

and as well that the intersections of the $A_{i,j}$ are at most $(n-1)$-dimensional. Besides that, each of the polytopes $A_{i,j}$ is also symmetric under the reflection across the $(n-1)$-dimensional hyperplane determined by z and $\Delta_i \cap \Delta_j$. By relabeling the $A_{i,j}$ we get the desired result

$$\Delta = P_1 \cup \cdots \cup P_m,$$

with $m = \frac{1}{2}n(n+1)$. ■

Theorem 1.39 (The volume theorem) *Let μ be a continuous rigid motion invariant simple valuation on $\mathcal{K}^n$ or on* Polycon(n). *Then there exists $c \in \mathbb{R}$ such that*

$$\mu(K) = c\mu_n^n(K)$$

for all $K \in \mathcal{K}^n$ or $K \in$ Polycon(n) *respectively.*

Proof. Since by Groemer's extension theorem (Theorem 1.17) a continuous valuation is well defined on Polycon(n) iff it is well defined on $\mathcal{K}^n$, we will restrict ourselves to the latter.
We start by using the volume characterization theorem (Theorem 1.37) and obtain the existence of a $\tilde{c} \in \mathbb{R}$ such that

$$\mu(K) + \mu(-K) = \tilde{c}\mu_n^n(K),$$

for all $K \in \mathcal{K}^n$. So especially for an arbitrary simplex $\Delta \subset \mathbb{R}^n$ we have

$$\mu(\Delta) + \mu(-\Delta) = \tilde{c}\mu_n^n(\Delta).$$

We distinguish between the following cases.

- <u>n is even:</u> In this case Δ and $-\Delta$ only differ by a rotation. Using the rotation invariance of μ we get that $\mu(\Delta) = \mu(-\Delta)$ and further

$$\mu(\Delta) = (\tilde{c}/2)\mu_n^n(\Delta).$$

- <u>n is odd:</u> By Proposition 1.38 there exist polytopes $P_1, \dots, P_m$, where each of the P_i is symmetric under a reflection across a hyperplane and where each of the intersections $P_i \cap P_j$ is at most $(n-1)$-dimensional, such that

$$\Delta = P_1 \cup \dots \cup P_m.$$

  Since $n-1$ is even, it follows that each P_i differs from $-P_i$ by a corresponding rigid motion, i.e. a rotation followed by a translation. Again by the invariance of μ we get that $\mu(-P_i) = \mu(P_i)$. Using that μ is simple and therefore vanishes on the $(n-1)$-dimensional intersections of the P_i we get as in the even case

$$\mu(-\Delta) = \sum_{i=1}^{m} \mu(-P_i) = \sum_{i=1}^{m} \mu(P_i) = \mu(\Delta) = (\tilde{c}/2)\mu_n^n(\Delta).$$

Now set $c = \tilde{c}/2$ and consider an arbitrary convex polytope P in $\mathbb{R}^n$. Since every convex polytope can be expressed as a finite union of simplices,

$$P = \Delta_1 \cup \dots \cup \Delta_k,$$

such that the intersections $\Delta_i \cap \Delta_j$ have dimensions less than n for all $i \neq j$, we get by using the simplicity of μ

$$\mu(P) = \sum_{i=1}^{k} \mu(\Delta_i) = \sum_{i=1}^{k} c\mu_n^n(\Delta_i) = c\mu_n^n(P).$$

Considering the fact that the set of all convex polytopes is dense in $\mathcal{K}^n$ and using the continuity of μ, it follows that

$$\mu(K) = c\mu_n^n(K),$$

for all $K \in \mathcal{K}^n$. ∎

Before we go on and proof Hadwiger's theorem (Theorem 1.41), we will justify the naming of the *intrinsic* volumes.

Theorem 1.40 (The universal normalization theorem)
The intrinsic volumes $\mu_0, \mu_1, \dots, \mu_n$ on Polycon(n) *are normalized independently of the dimension n.*

Proof. We have to show that for $n > k$, μ_k^n restricts to μ_k^l for all $k \leq l \leq n$. The statement is obvious for $k = 0$, since $\mu_0^n = \chi$. Now we suppose that μ_k^{n-1} restricts to μ_k^l for all $k \leq l \leq n-1$. We have to show that μ_k^n restricts to μ_k^l

as well. For that we will first prove that μ_k^n restricts to μ_k^k. Since this is only sufficient for $k = n-1$, we have to go further and show that if μ_k^n restricts to μ_k^l for some $k \leq l < n-1$, then μ_k^n restricts to μ_k^{l+1} as well. Induction on l and n then gives us the statement.

We start to show that μ_k^n restricts to μ_k^k. The valuation μ_k^n vanishes in dimension less than k, therefore the restriction of μ_k^n to a hyperplane in $\mathbb{R}^n$ is a continuous invariant simple valuation on Polycon(k). Thus, by the volume theorem (Theorem 1.39) we get that $\mu_k^n(K) = c\mu_k^k(K)$ for all $K \in$ Polycon(k), with a $c \in \mathbb{R}$. Since $\mu_k^n = \mu_k^k$ on the set of all parallelotopes in Par(k), see (1.11), we have that $c = 1$ and $\mu_k^n = \mu_k^k$ on Polycon(k).

Finally, we have to show that μ_k^n restricts to μ_k^{l+1} if μ_k^n already restricts to μ_k^l for some $k \leq l < n-1$. We denote by ν the restriction of μ_k^n to Polycon($l+1$). By our assumption, ν restricts to μ_k^l on Polycon(l) and furthermore μ_k^{l+1} restricts to μ_k^l on Polycon(l) by our induction hypothesis. Consequently, $\nu - \mu_k^{l+1}$ vanishes on Polycon(l), which gives us that $\nu - \mu_k^{l+1}$ is a continuous invariant simple valuation on Polycon($l+1$). Again, by the volume theorem (Theorem 1.39) we get that $\nu - \mu_k^{l+1} = c\mu_{l+1}^{l+1}$ on Polycon($l+1$) for a $c \in \mathbb{R}$. Since we know that $\nu - \mu_k^{l+1}$ vanishes on the set of all parallelotopes in Par($l+1$), we have $c = 0$ and furthermore $\nu = \mu_k^{l+1}$, which gives us the statement. ■

Considering the last statement, we will simplify the notation μ_k^n to μ_k, since the ambient space does not make any difference.

We will now conclude the characterization of invariant valuations on Polycon(n) with the following important theorem.

Theorem 1.41 (Hadwiger's characterization theorem)
The set of all convex-continuous rigid motion invariant valuations defined on Polycon(n) *is an* $(n+1)$*-dimensional vector space and the intrinsic volumes* $\mu_0, \mu_1, \ldots, \mu_n$ *form a basis for this space.*

Proof. We will proof the theorem by induction on the dimension n. For $n = 0$ we have to consider a zero dimensional space which only consists of a single point. Obviously, any valuation that does not vanish on that point, i.e. the Euler characteristic $\chi = \mu_0$, forms a basis for the 1-dimensional vector space of continuous invariant valuations.
We now assume that the theorem holds for dimension $n-1$ and go on to proof the statement for dimension n. For that we consider an arbitrary valuation μ that satisfies the hypothesis of the theorem. We restrict μ to polyconvex sets in $\mathbb{R}^{n-1}$ and obtain from the induction hypothesis the existence of $c_0, \ldots, c_{n-1} \in \mathbb{R}$ such that

$$\mu(A) = \sum_{i=0}^{n-1} c_i \mu_i(A),$$

for all polyconvex sets A of dimension $n-1$. Thereby the valuation

$$\mu - \sum_{i=0}^{n-1} c_i \mu_i$$

is a continuous rigid motion invariant simple valuation on Polycon(n). By the volume theorem (Theorem 1.39) we get the existence of $c_n \in \mathbb{R}$ such that

$$\mu - \sum_{i=0}^{n-1} c_i \mu_i = c_n \mu_n.$$

Consequently

$$\mu = \sum_{i=0}^{n} c_i \mu_i.$$

■

Remark 1.42 In the previous proof we have shown, that Hadwiger's theorem follows from the volume theorem (Theorem 1.39). It can be shown that the reverse implication holds as well. A proof can be found in [15].

Corollary 1.43 *If μ is a convex-continuous rigid motion invariant valuation defined on the polyconvex sets in $\mathbb{R}^n$ and homogeneous of degree k, for some $0 \leq k \leq n$, then there exists $c \in \mathbb{R}$ such that*

$$\mu(K) = c\mu_k(K)$$

for all $K \in$ Polycon(n).

Remark 1.44 We consider the characteristic description of the intrinsic volumes, given in Hadwiger's theorem, and try to find corresponding measures on $\mathbb{R}^n$ with the Borel σ-algebra. Clearly, the corresponding property to convex-continuity would be Borel regularity. If we furthermore ask for translation-invariance and admittance of finite values on compact sets - which the intrinsic volumes fulfill on $\mathcal{K}^n$ - we would get the Haar measure on $(\mathbb{R}^n, +)$. Up to a unique factor, this is the Lebesgue measure. In fact, the translation-invariance alone would be enough to characterize the multiples of the Lebesgue measure.

2 Hadwiger Integration

We now want to extend valuations as linear functionals on families of sets to linear functionals on functions on sets. Whilst the valuations introduced in the first chapter are classical geometric objects we will now consider a much younger field in mathematics. We will thereby extend the approach we have used to define the integral with respect to a valuation μ in Definition 1.11. Again, the integrals that come from the intrinsic volumes will play a vital role. Furthermore, in a recent work (see [5]) by Baryshnikov, Ghrist and Wright a classification theorem, similar to Hadwiger's theorem, was established.

2.1 Valuations on definable functions

Instead of the lattice Polycon(n) we will now consider valuations on a different family of sets. For that we need the concept of an o-minimal structure as defined in [29], Chapter 1.

Definition 2.1 We define a *structure* on $\mathbb{R}$ as a sequence $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ such that for each $n \geq 0$ we have the following:

- $\mathcal{S}_n$ is a Boolean algebra of subsets of $\mathbb{R}^n$, i.e. $\emptyset \in \mathcal{S}_n$ and $\mathcal{S}_n$ is closed under unions and complements.
- For each $A \in \mathcal{S}_n$ the sets $\mathbb{R} \times A$ and $A \times \mathbb{R}$ belong to $\mathcal{S}_{n+1}$.
- $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 = x_2 = \cdots = x_n\} \in \mathcal{S}_n$.
- For each $A \in \mathcal{S}_n$ the set $\pi(A)$ belongs to $\mathcal{S}_{n-1}$, whereas $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection map on the first $n-1$ coordinates.

Following this, we define an *o-minimal structure* on $(\mathbb{R}, <)$ as a structure $\mathcal{S}$ on $\mathbb{R}$ such that

- $\{(x, y) \in \mathbb{R}^2, x < y\} \in \mathcal{S}_2$,
- the sets in $\mathcal{S}_1$ are exactly the finite unions of open (possibly unbounded) intervals and points.

Example 2.2 An example for an o-minimal structure, given by van den Dries in [29], Chapter 2, are the *semilinear sets*. For that we consider *affine functions on* $\mathbb{R}^n$, which are functions $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x_1, \ldots, x_n) = \lambda_1 x_1 + \cdots + \lambda_n x_n + a,$$

with $\lambda_i, a \in \mathbb{R}$ fixed. Following this, we consider the *basic semilinear sets in* $\mathbb{R}^n$

$$\{x \in \mathbb{R}^n : \ f_1(x) = \cdots = f_k(x) = 0, \ g_1(x) > 0, \ldots, g_l(x) > 0\},$$

with the f_i and g_j being affine functions on $\mathbb{R}^n$. The semilinear sets on $\mathbb{R}^n$ are now simply the finite unions of the basic semilinear sets in $\mathbb{R}^n$.

Example 2.3 Denote by $\mathbb{R}[x_1, \ldots, x_n]$ the ring of real polynomial functions on $\mathbb{R}^n$. A set is called a *semialgebraic set* if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^n : \ f_1(x) = \cdots = f_k(x) = 0, \ g_1(x) > 0, \ldots, g_l(x) > 0\},$$

with $f_i, g_j \in \mathbb{R}[x_1, \ldots, x_n]$. For $n = 1$ these are just the finite unions of intervals and points. Note that each semilinear set is a semialgebraic set since the affine functions on $\mathbb{R}^n$ are the elements of $\mathbb{R}[x_1, \ldots, x_n]$ with degree less or equal to 1. The Tarski-Seidenberg theorem states that the projection of a semialgebraic set $A \subseteq \mathbb{R}^n$ onto $\mathbb{R}^k \subset \mathbb{R}^n$ with $k < n$ is again semialgebraic. Furthermore, the closure, interior, and convex hull of a semialgebraic set are semialgebraic. This shows that they are an example for an o-minimal structure. See also [29], Chapter 3.

Further well-known examples are the semianalytic (described by analytic functions) and subanalytic sets (locally projections of relatively compact semianalytic sets).

Definition 2.4 For the remainder of this work we fix an o-minimal structure $\mathcal{S}$. Elements of $\mathcal{S}_n$ are called *definable sets*. A *definable map* is then a map $f : \mathbb{R}^n \to \mathbb{R}^m$ the graph of which is a definable subset of $\mathbb{R}^{n+m}$. Furthermore, we define a *definable function* as a definable map $f : X \to \mathbb{R}$ with $X \subset \mathbb{R}^n$ compact. The set of all definable functions is denoted by $\mathrm{Def}(\mathbb{R}^n)$.

We now extend the Euler characteristic and the intrinsic volumes to definable sets. The *o-minimal Euler characteristic* χ is then defined as the valuation χ that fulfills

$$\chi(\sigma) = (-1)^k, \tag{2.1}$$

for any open k-dimensional simplex σ. One can show that every definable set is definably homeomorphic to a disjoint union of open simplices. Therefore, by (2.1) the Euler characteristic is defined on $\mathcal{S}$. Furthermore the intrinsic volumes μ_i for definable sets $K \in \mathcal{S}_n$ are now defined using Hadwiger's formula, that is

$$\mu_{n-k}(K) = \int_{\mathrm{Graff}(n,k)} \chi(K \cap V) \, \mathrm{d}\lambda_k^n(V). \tag{2.2}$$

Many of the results on valuations that we introduced in the last chapter are still valid for valuations on definable sets. Most important, Hadwiger's theorem still holds for valuations that are defined on definable sets. See also [31], Chapter 2.

Definition 2.5 A real-valued function $\nu : \mathrm{Def}(\mathbb{R}^n) \to \mathbb{R}$ is called *valuation* on $\mathrm{Def}(\mathbb{R}^n)$, if the following properties hold for all $f, g \in \mathrm{Def}(\mathbb{R}^n)$:

$$\nu(f \vee g) + \nu(f \wedge g) = \nu(f) + \nu(g), \tag{2.3}$$

$$\nu(\mathbf{0}) = 0 \tag{2.4}$$

where $\mathbf{0}$ denotes the zero function and $\vee$ and $\wedge$ denote the pointwise maximum and minimum of f and g, respectively.

These conditions are clearly similar to the properties a valuation on a lattice of sets has to satisfy. Note, that (2.4) together with the additivity of ν implies that a valuation on $\mathrm{Def}(\mathbb{R}^n)$ is independent of the support of a function.

Proposition 2.6 *A function* $\nu : \mathrm{Def}(\mathbb{R}^n) \to \mathbb{R}$ *is a valuation on* $\mathrm{Def}(\mathbb{R}^n)$ *iff*

$$\nu(f) = \nu(f \cdot \mathbb{1}_A) + \nu(f \cdot \mathbb{1}_{A^c}), \tag{2.5}$$

for all $f \in \mathrm{Def}(\mathbb{R}^n)$ *and all definable sets* $A \subset \mathbb{R}^n$*.*

For the following proof and the rest of this work we will use the notation of *excursion sets*, e.g. $\{f \geq s\} := \{x \in \mathbb{R}^n | f(x) \geq s\}$, for a function $f : \mathbb{R}^n \to \mathbb{R}$ and $s \in \mathbb{R}$.

Proof.
$\underline{\Longrightarrow:}$ If ν is a valuation on $\mathrm{Def}(\mathbb{R}^n)$, we have for any definable function $f \in \mathrm{Def}(\mathbb{R}^n)$ and any definable set $A \subset \mathbb{R}^n$

$$\begin{aligned} \nu(f \cdot \mathbb{1}_A) + \nu(f \cdot \mathbb{1}_{A^c}) &= \nu(f \cdot \mathbb{1}_A \vee f \cdot \mathbb{1}_{A^c}) + \nu(f \cdot \mathbb{1}_A \wedge f \cdot \mathbb{1}_{A^c}) \\ &= \nu(f \cdot \mathbb{1}_{\{f \geq 0\}}) + \nu(f \cdot \mathbb{1}_{\{f \leq 0\}}) = \nu(f \vee \mathbf{0}) + \nu(f \wedge \mathbf{0}) \\ &= \nu(f) + \nu(\mathbf{0}) = \nu(f). \end{aligned}$$

$\underline{\Longleftarrow:}$ Suppose that ν satisfies (2.5). To get equation (2.4) consider for any definable set $A \subset \mathbb{R}^n$

$$\nu(\mathbf{0}) = \nu(\mathbf{0} \cdot \mathbb{1}_A) + \nu(\mathbf{0} \cdot \mathbb{1}_{A^c}) = \nu(\mathbf{0}) + \nu(\mathbf{0}).$$

Therefore, $\nu(\mathbf{0}) = 0$.
Now take any two functions $f, g \in \mathrm{Def}(\mathbb{R}^n)$ and consider $A = \{f \geq g\}$, which is a definable subset of $\mathbb{R}^n$, since f and g are definable. Using (2.5) we get

$$\begin{aligned} \nu(f) + \nu(g) &= \nu(f \cdot \mathbb{1}_A) + \nu(f \cdot \mathbb{1}_{A^c}) + \nu(g \cdot \mathbb{1}_A) + \nu(g \cdot \mathbb{1}_{A^c}) \\ &= \nu((f \vee g) \cdot \mathbb{1}_A) + \nu((f \wedge g) \cdot \mathbb{1}_{A^c}) + \nu((f \wedge g) \cdot \mathbb{1}_A) + \nu((f \vee g) \cdot \mathbb{1}_{A^c}) \\ &= \nu(f \vee g) + \nu(f \wedge g). \end{aligned}$$

■

Induction on (2.5) gives us that

$$\nu(f) = \sum_{i=1}^{m} \nu(f \cdot \mathbb{1}_{A_i}),$$

for any valuation ν on $\mathrm{Def}(\mathbb{R}^n)$ and any finite partition of $\mathbb{R}^n$ into definable sets $\{A_i\}_{i=1}^m$.

2.2 Hadwiger integral of a constructible function

Similar to defining the integral of simple functions (Definition 1.11) we will introduce the Hadwiger integral of "$\mathcal{S}_n$-simple functions".

Definition 2.7 A *constructible function* $f : X \subseteq \mathbb{R}^n \to \mathbb{Z}$ is an integer-valued function with definable level sets, that is

$$f = \sum_{i\geq 0} c_i \mathbb{1}_{A_i},$$

with $c_i \in \mathbb{Z}$ and $A_i \in \mathcal{S}_n$. Moreover, the set of all constructible functions with compact support shall be denoted by $\mathrm{CF}(\mathbb{R}^n)$.

The domain of a constructible function f has a locally finite triangulation which means that f is constant on each simplex of the triangulation. This implies that constructible functions with compact support also have bounded range.

Definition 2.8 Let $f : X \to \mathbb{Z}$ be a constructible function where the domain X is a compact subset of $\mathbb{R}^n$. Consequently, there exist disjoint definable sets $A_i \subset \mathbb{R}^n$ and constants $c_i \in \mathbb{Z}$ such that $f = \sum_{i\geq 0} c_i \mathbb{1}_{A_i}$. The *Hadwiger integral* of f with respect to the intrinsic volume μ_k is then defined as

$$\int_X f \, \mathrm{d}\mu_k = \int_X \sum_{i\geq 0} c_i \mathbb{1}_{A_i} \, \mathrm{d}\mu_k := \sum_{i\geq 0} c_i \mu_k(A_i).$$

The case $k = 0$ is also known as *Euler integral.* For $k = n$ we obtain the Lebesgue integral.

Lemma 2.9 *For any constructible function $f : X \to \mathbb{Z}$ one has*

$$\int_X f \, \mathrm{d}\mu_k = \sum_{s=-\infty}^{\infty} s\mu_k\{f = s\} \tag{2.6}$$

$$= \sum_{s=0}^{\infty} \mu_k\{f > s\} - \mu_k\{f < -s\}. \tag{2.7}$$

Proof. For any constructible function f the following equations hold

$$f = \sum_{s=-\infty}^{\infty} s\mathbb{1}_{\{f=s\}} = \sum_{s=0}^{\infty} s(\mathbb{1}_{\{f\geq s\}} - \mathbb{1}_{\{f>s\}}) + \sum_{s=-\infty}^{0} s(\mathbb{1}_{\{f\leq s\}} - \mathbb{1}_{\{f<s\}}) =$$

$$\sum_{s=0}^{\infty} s(\mathbb{1}_{\{f>s-1\}} - \mathbb{1}_{\{f>s\}}) - s(\mathbb{1}_{\{f<-s+1\}} - \mathbb{1}_{\{f<-s\}}) = \sum_{s=0}^{\infty} \mathbb{1}_{\{f>s\}} - \mathbb{1}_{\{f<-s\}}.$$

■

In order to give a first result towards a classification theorem for valuations on definable functions we consider valuations on $\mathrm{CF}(\mathbb{R}^n) \subset \mathrm{Def}(\mathbb{R}^n)$, which are restrictions of valuations on $\mathrm{Def}(\mathbb{R}^n)$ to $\mathrm{CF}(\mathbb{R}^n)$.

Definition 2.10 A real-valued function $\nu : \mathrm{CF}(\mathbb{R}^n) \to \mathbb{R}$ is called a *valuation* on $\mathrm{CF}(\mathbb{R}^n)$if (2.3) and (2.4) hold for all $f, g \in \mathrm{CF}(\mathbb{R}^n)$.

Definition 2.11 A valuation ν on $\mathrm{Def}(\mathbb{R}^n)$ is *rigid motion invariant* if

$$\nu(f) = \nu(f \circ \phi),$$

for any $f \in \mathrm{Def}(\mathbb{R}^n)$ and any Euclidean motion ϕ on $\mathbb{R}^n$. The definition of a rigid motion invariant valuation on $\mathrm{CF}(\mathbb{R}^n)$ is analogous.

Lemma 2.12 *Let ν be a rigid motion invariant valuation on $\mathrm{CF}(\mathbb{R}^n)$ that satisfies*

$$\nu(\mathbb{1}_{K_k}) \xrightarrow{k\to\infty} \nu(\mathbb{1}_K),$$

for all compact convex sets K_k, K in $\mathcal{S}_n$ such that $K_k \xrightarrow{k\to\infty} K$ with respect to the Hausdorff metric. Then there exist coefficient functions $c_i : \mathbb{R} \to \mathbb{R}$ with $c_i(0) = 0$ such that

$$\nu(f) = \sum_{i=0}^{n} \int_{\mathbb{R}^n} c_i(f)\,\mathrm{d}\mu_i,$$

for all $f \in \mathrm{CF}(\mathbb{R}^n)$.

Proof. We start with multiples of characteristic functions and consider $\nu_r(A) := \nu(r \cdot \mathbb{1}_A)$ for any $r \in \mathbb{Z}$, which gives us a valuation on $\mathcal{S}_n$. Since convex definable sets are dense among convex sets in $\mathbb{R}^n$ we can apply Hadwiger's theorem to get the existence of constants $c_{i,r}$ that only depend on ν, such that

$$\nu(r \cdot \mathbb{1}_A) = \nu_r(A) = \sum_{i=0}^{n} c_{i,r}\mu_i(A). \tag{2.8}$$

Next, we consider a constructible function

$$f = \sum_{j=1}^{m} r_j \mathbb{1}_{A_j}$$

for some integer constants $r_1 < r_2 < \cdots < r_m$ and disjoint definable subsets $A_1, \ldots, A_m$ of $\mathbb{R}^n$. Using the additivity of ν, we obtain

$$\nu(f) = \sum_{j=1}^{m} \nu(r_j \mathbb{1}_{A_j}) = \sum_{j=1}^{m}\sum_{i=0}^{n} c_{i,r_j}\mu_i(A_j). \tag{2.9}$$

Now let $B_j = \bigcup_{i\geq j} A_i = \{f \geq r_j\} = \{f > r_{j-1}\}$. Equation (2.9) can then be rewritten as

$$\nu(f) = \sum_{j=1}^{m}\sum_{i=0}^{n}(c_{i,r_j} - c_{i,r_{j-1}})\mu_i(B_j),$$

with $c_{i,r_0} = 0$. This means, that a valuation of a constructible function can be expressed as a sum of finite differences of valuations of its excursion sets.

Changing the order of summation and setting $c_i(f) := \sum_{j=1}^{m}(c_{i,r_j} - c_{i,r_{j-1}})\mathbb{1}_{B_j}$, we obtain

$$\nu(f) = \sum_{i=0}^{n} \int_{\mathbb{R}^n} c_i(f)\,\mathrm{d}\mu_i.$$

■

Remark 2.13 It is easy to see that the last result also holds for functions of the form $f = \sum_{i=1}^{m} r_i \mathbb{1}_{A_i}$ with $r_i \in \mathbb{R}$ and definable sets $A_i \subset \mathbb{R}^n$.

2.3 Extending the integral

In one of the first works on Euler integration [30], Viro thought of the Euler characteristic as a finitely-additive measure (see also Example 1.5). Whilst one can see the similarities in defining an integral like in measure theory, the lack of σ-additivity and positivity are an issue when it comes to defining an integral for continuous functions. Without positivity for example one does not get monotonicity, e.g. $\chi(\{0\}) = 1$ but $\chi((-1, 1)) = -1$. Also, due to the lack of σ-additivity one cannot try to solve this by seeing χ as a signed measure. For example for a signed measure ν on $\mathbb{R}$ with the Borel σ-algebra, the Hahn decomposition theorem states, that there exist two measurable sets P and N, such that

1. $P \cup N = \mathbb{R}$ and $P \cap N = \emptyset$,
2. for each Borel set $A \subseteq P$ one has $\nu(A) \geq 0$,
3. for each Borel set $A \subseteq N$ one has $\nu(A) \leq 0$.

Since $\chi(\{x\}) = 1$ for every $x \in \mathbb{R}$, the positive set P of χ has to be $\mathbb{R}$. But then χ would not admit any negative values.

Another issue occurs when we consider Lebesgue's decomposition theorem. Applied to the Lebesgue measure and another σ-finite signed measure ν on $\mathbb{R}^1$ it states that there exist two σ-finite signed measures ν_0 and ν_1 such that:

1. ν_0 is absolutely continuous with respect to the Lebesgue measure,
2. ν_1 is singular,
3. $\nu = \nu_0 + \nu_1$.

Furthermore, if ν is locally finite, the function $x \mapsto \int_a^x \mathrm{d}\nu$ only has finitely many points of discontinuity on every bounded set. These points ask for special attention when one integrates with respect to ν. However, if we tried to consider the Euler characteristic instead of ν, every point would be a point of discontinuity. Especially, by the rigid motion invariance of the intrinsic volumes, one point of discontinuity implies discontinuity everywhere. Recall that μ_n is a simple valuation, which means that it vanishes on sets of dimension less than n. The intrinsic volumes μ_i, on the other hand, are not simple for $i < n$. This causes problems when one tries to approximate a continuous function with lower and upper step functions. See also Lemma 2.16 and Example 2.20.

Definition 2.14 (Hadwiger Integral) We define the *lower Hadwiger integrals* of a definable function $f \in \mathrm{Def}(\mathbb{R}^n)$ with support $X \subset \mathbb{R}^n$ as

$$\int_X f \lfloor \mathrm{d}\mu_k \rfloor := \lim_{m\to\infty} \frac{1}{m} \int_X \lfloor mf \rfloor \,\mathrm{d}\mu_k. \tag{2.10}$$

Similar the *upper Hadwiger integrals* are defined as

$$\int_X f \lceil \mathrm{d}\mu_k \rceil := \lim_{m\to\infty} \frac{1}{m} \int_X \lceil mf \rceil \,\mathrm{d}\mu_k. \tag{2.11}$$

Note that initially it is not clear if these limits even exist. The following theorem will give us equivalent expressions of the Hadwiger integrals and thereby existence of the limits.

Theorem 2.15 *For $f \in \mathrm{Def}(\mathbb{R}^n)$ with support $X \subset \mathbb{R}^n$ we get the following equivalent expressions of the lower Hadwiger integrals*

$$\int_X f \lfloor \mathrm{d}\mu_k \rfloor = \int_0^\infty (\mu_k\{f \geq s\} - \mu_k\{f < -s\})\,\mathrm{d}s \qquad \textit{excursion sets} \tag{2.12}$$

$$= \int_{\mathrm{Graff}(n,n-k)} \int_{X\cap V} f \lfloor \mathrm{d}\chi \rfloor \,\mathrm{d}\lambda^n_{n-k}(V). \qquad \textit{slices} \tag{2.13}$$

Similarly we get for the upper Hadwiger integrals

$$\int_X f \lceil \mathrm{d}\mu_k \rceil = \int_0^\infty (\mu_k\{f > s\} - \mu_k\{f \leq -s\})\,\mathrm{d}s \qquad \textit{excursion sets} \tag{2.14}$$

$$= \int_{\mathrm{Graff}(n,n-k)} \int_{X\cap V} f \lceil \mathrm{d}\chi \rceil \,\mathrm{d}\lambda^n_{n-k}(V). \qquad \textit{slices} \tag{2.15}$$

Proof. We will show the results only for the lower Hadwiger integrals since the proofs for the corresponding identities for the upper Hadwiger integrals are analogous.
For the first equation we consider that for $m > 0$

$$\begin{aligned}\lfloor mf \rfloor &= \sum_{i=-\infty}^{\infty} i\mathbb{1}_{\{\lfloor mf \rfloor = i\}} = \sum_{i=1}^{\infty} i\mathbb{1}_{\{\lfloor mf \rfloor = i\}} - \sum_{i=1}^{\infty} i\mathbb{1}_{\{\lfloor mf \rfloor = -i\}} \\ &= \sum_{i=1}^{\infty} \mathbb{1}_{\{mf \geq i\}} - \sum_{i=0}^{\infty} \mathbb{1}_{\{mf < -i\}} = -\mathbb{1}_{\{f<0\}} + \sum_{i=1}^{\infty} \mathbb{1}_{\{mf \geq i\}} - \mathbb{1}_{\{mf < -i\}}.\end{aligned} \tag{2.16}$$

Now we set $T = \sup(|f|)$ and furthermore $N = mT$. Considering the fact that $|\mu_k\{f < 0\}| < \infty$ and using the definition of the lower Hadwiger integrals together with (2.16) we have

$$\begin{aligned}\int_X f \lfloor \mathrm{d}\mu_k \rfloor &= \lim_{m\to\infty} \frac{1}{m} \int_X \lfloor mf \rfloor \, \mathrm{d}\mu_k \\ &= \lim_{m\to\infty} \frac{1}{m} \left(-\mu_k\{f < 0\} + \sum_{i=1}^{\infty} \mu_k\{mf \geq i\} - \mu_k\{mf < -i\} \right) \\ &= \lim_{N\to\infty} \frac{T}{N} \sum_{i=1}^{N} \mu_k \left\{ f \geq \frac{iT}{N} \right\} - \mu_k \left\{ f < -\frac{iT}{N} \right\} \\ &= \int_0^T \mu_k\{f \geq s\} - \mu_k\{f < -s\} \, \mathrm{d}s,\end{aligned}$$

which shows (2.12).

To prove (2.13), we use Hadwiger's formula (2.2) to obtain

$$\begin{aligned}&\int_0^\infty \mu_k\{f \geq s\} - \mu_k\{f < -s\} \, \mathrm{d}s \\ &\qquad = \int_0^\infty \int_{\mathrm{Graff}(n,n-k)} \chi(\{f \geq s\} \cap V) - \chi(\{f < -s\} \cap V) \, \mathrm{d}\lambda^n_{n-k}(V) \, \mathrm{d}s.\end{aligned}$$

We now want to apply Fubini's theorem. First, we note that the excursion sets $\{f \geq s\}$ and $\{f < -s\}$ are definable, which implies that they have finite Euler characteristic and thus the integrand is finite. Since f has compact support $X \subset \mathbb{R}^n$, the inner integral is over the bounded set $\mathrm{Graff}(X; n-k) \subset \mathrm{Graff}(n, n-k)$. Also the outer integral is over the bounded set $(0, T) \subset \mathbb{R}$. Thereby we can change the order of integration and the integral becomes

$$\begin{aligned}&\int_{\mathrm{Graff}(n,n-k)} \int_0^\infty \chi(\{f \geq s\} \cap V) - \chi(\{f < -s\} \cap V) \, \mathrm{d}s \, \mathrm{d}\lambda^n_{n-k}(V) \\ &\qquad\qquad = \int_{\mathrm{Graff}(n,n-k)} \int_{X\cap V} h \lfloor \mathrm{d}\chi \rfloor \, \mathrm{d}\lambda^n_{n-k}(V),\end{aligned}$$

which concludes the proof. ■

We now have multiple ways to think of the Hadwiger integrals. The definitions of the lower and upper integrals use limits of step functions, see also Figure 5. This makes clear, that both integrals with respect to μ_n are in fact n-dimensional Lebesgue integrals. Furthermore, a illustrations of excursion sets and slices are given in Figure 6.

A direct consequence of the definition of the Hadwiger integrals is the following.

Lemma 2.16 *The Hadwiger integrals*

$$\int_X \cdot \lfloor \mathrm{d}\mu_k \rfloor : \mathrm{Def}(\mathbb{R}^n) \to \mathbb{R}$$

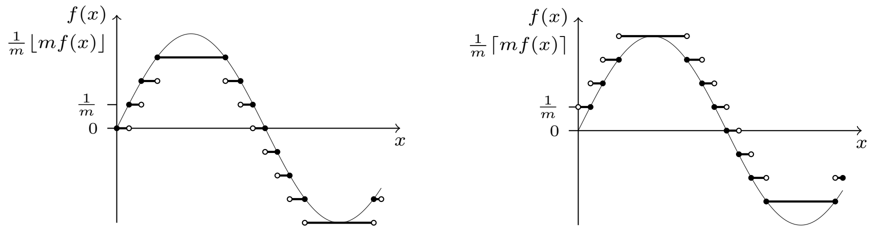


Figure 5: The lower and upper Hadwiger integrals are by definition limits of integrals of step functions.

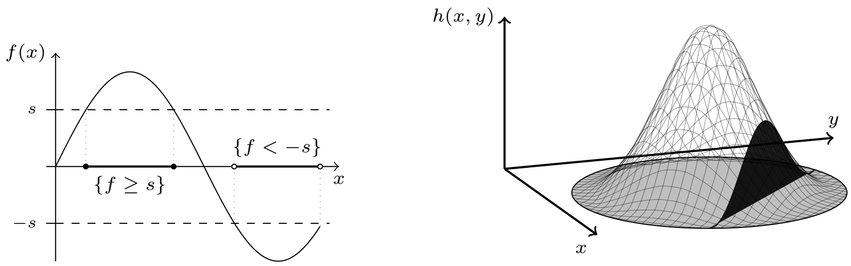


Figure 6: Illustration of excursion sets for a function $f : \mathbb{R} \to \mathbb{R}$, as well as slices of h by planes perpendicular to the domain, for $h : \mathbb{R}^2 \to \mathbb{R}$.

and

$$\int_X \cdot \lceil \mathrm{d}\mu_k \rceil : \mathrm{Def}(\mathbb{R}^n) \to \mathbb{R}$$

are not homomorphisms for $\dim X > 0$ *and* $k < n$.

Proof. We have to show that the equality

$$\int_X f + g \lfloor \mathrm{d}\mu_k \rfloor = \int_X f \lfloor \mathrm{d}\mu_k \rfloor + \int_X g \lfloor \mathrm{d}\mu_k \rfloor,$$

for $f, g \in \mathrm{Def}(\mathbb{R}^n)$ does not hold in general. This is simply due to the fact that $\lim_{m\to\infty} \lfloor m \cdot (f+g) \rfloor$ differs from $\lim_{m\to\infty}(\lfloor mf \rfloor + \lfloor mg \rfloor)$ only on a set of Lebesgue measure zero but not necessarily "μ_k-measure" zero. The same argumentation applies to $\lceil \mathrm{d}\mu_k \rceil$. ■

Remark 2.17 Note, that the integrals are homomorphisms for μ_n, since the Hadwiger integrals with respect to μ_n are Lebesgue integrals.

Example 2.18 Consider the functions $f(x) = x$ and $g(x) = -x$ on $[0,1] \subset \mathbb{R}$. Using the excursion set representation (2.12) we have

$$\int_{[0,1]} f \lfloor \mathrm{d}\chi \rfloor = \int_0^\infty \chi\{f \geq s\} - \chi\{f < -s\} \,\mathrm{d}s = \int_0^1 \chi([s,1]) \,\mathrm{d}s = 1$$

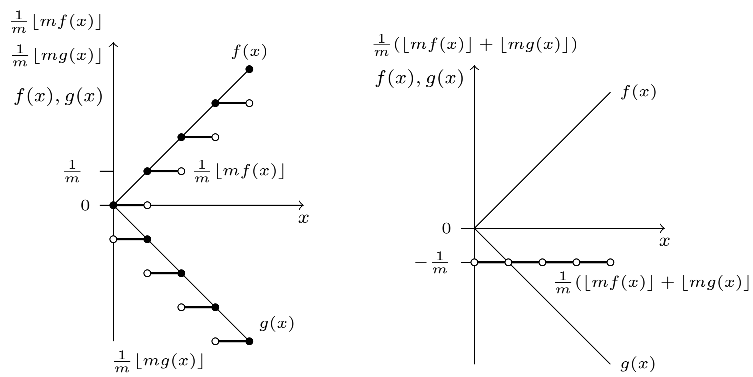


Figure 7: The sum of the lower Euler integrals of f and g does not vanish.

$$\int_{[0,1]} g \lfloor \mathrm{d}\chi \rfloor = \int_0^\infty \chi\{g \geq s\} - \chi\{g < -s\} \, \mathrm{d}s = -\int_0^1 \chi((s,1]) \, \mathrm{d}s = 0$$

However,

$$\int_{[0,1]} f + g \lfloor \mathrm{d}\chi \rfloor = \int_{[0,1]} \mathbf{0} \lfloor \mathrm{d}\chi \rfloor = 0.$$

This is also easy to see using the definition of the lower Euler integral:

$$\tfrac{1}{m} \lfloor mf(x) \rfloor = \sum_{i=0}^{m-1} \tfrac{i}{m} \mathbb{1}_{[i \cdot m, (i+1) \cdot m)} + \mathbb{1}_{\{1\}}$$

$$\tfrac{1}{m} \lfloor mg(x) \rfloor = 0 \cdot \mathbb{1}_{\{0\}} - \sum_{i=0}^{m-1} \tfrac{i+1}{m} \mathbb{1}_{(i \cdot m, (i+1) \cdot m]}$$

Addition gives

$$\tfrac{1}{m} \left(\lfloor mf(x) \rfloor + \lfloor mg(x) \rfloor \right) = -\tfrac{1}{m} \sum_{i=0}^{m-1} \mathbb{1}_{(i \cdot m, (i+1) \cdot m)}$$

and consequently

$$\begin{aligned} \int_{[0,1]} f \lfloor \mathrm{d}\chi \rfloor + \int_{[0,1]} g \lfloor \mathrm{d}\chi \rfloor &= \lim_{m \to \infty} -\tfrac{1}{m} \sum_{i=0}^{m-1} \chi((i \cdot m, (i+1) \cdot m)) \\ &= \lim_{m \to \infty} -\tfrac{1}{m} \cdot m \cdot (-1) = 1. \end{aligned}$$

See also Figure 7.

Another immediate consequence of Theorem 2.15 is the following Corollary, which gives us a certain kind of duality between the lower and upper Hadwiger integrals.

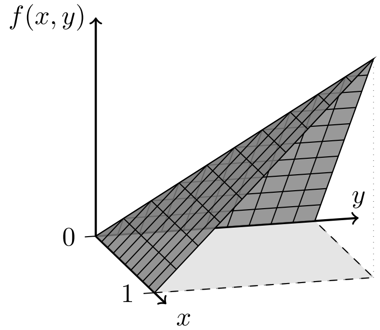


Figure 8: The function $f : [0,1]^2 \to \mathbb{R}, f(x,y) = \min(x,y)$.

Corollary 2.19 *For $f \in \mathrm{Def}(\mathbb{R}^n)$ the following equation holds:*

$$\int_X f \lfloor \mathrm{d}\mu_k \rfloor = -\int_X -f \lceil \mathrm{d}\mu_k \rceil.$$

Proof. Using equations (2.12) and (2.14) we get

$$\begin{aligned}\int_X f \lfloor \mathrm{d}\mu_k \rfloor &= \int_0^\infty \mu_k\{f \geq s\} - \mu_k\{f < -s\} \, \mathrm{d}s \\ &= \int_0^\infty \mu_k\{-f \leq -s\} - \mu_k\{-f > s\} \, \mathrm{d}s = -\int_X -f \lceil \mathrm{d}\mu_k \rceil.\end{aligned}$$

■

The following example, given by Wright in [31], shows that the lower and upper Hadwiger integrals with respect to μ_k are not equal in general.

Example 2.20 We consider the function $f : [0,1]^n \to \mathbb{R}$, $f(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n)$. For $s \in [0,1]$, we have

$$\{f \geq s\} = [s,1]^n.$$

In other words, the excursion set equals an n-dimensional cube with side lengths $1-s$. Using the formula for orthogonal parallelotopes, equation (1.11), we obtain

$$\mu_{n-1}\{f \geq s\} = \mu_{n-1}([s,1]^n) = n(1-s)^{n-1}.$$

Next, we consider the strict excursion set $\{f > s\}$, which is also an n-dimension cube with side lengths $1-s$, but open along half of its $(n-1)$-dimensional faces. This means, that we have to subtract half of the $2n$ faces.

$$\mu_{n-1}\{f > s\} = \mu_{n-1}([s,1]^n) - n\mu_{n-1}((s,1)^{n-1}) = n(1-s)^{n-1} - n(1-s)^{n-1} = 0.$$

Furthermore the sets $\{f < -s\}$ and $\{f \leq -s\}$ are empty. Thereby, we get

$$\int_{[0,1]^n} f \lfloor \mathrm{d}\mu_{n-1} \rfloor = \int_0^\infty \mu_{n-1}\{f \geq s\} - \mu_{n-1}\{f < -s\} \,\mathrm{d}s$$
$$= \int_0^\infty n(1-s)^{n-1} - 0 \,\mathrm{d}s = 1.$$

On the other hand

$$\int_{[0,1]^n} f \lceil \mathrm{d}\mu_{n-1} \rceil = \int_0^\infty \mu_{n-1}\{f > s\} - \mu_{n-1}\{f \leq -s\} \,\mathrm{d}s$$
$$= \int_0^\infty 0 - 0 \,\mathrm{d}s = 0.$$

The difference between the upper and lower Hadwiger integrals does not always have to be that extreme. In fact, for the example above it is crucial that the canonical extension of f to $\mathbb{R}^n$, that is $f \equiv 0$ outside of $[0,1]^n$, is not continuous. In particular, if f is a continuous function, then the lower and upper Hadwiger integrals differ at most by a minus sign.

Theorem 2.21 *For every continuous definable function f with compact l-dimensional support X,*

$$\int_X f \lfloor \mathrm{d}\mu_k \rfloor = (-1)^{l+k} \int_X f \lceil \mathrm{d}\mu_k \rceil.$$

Proof. The cell decomposition theorem (see [29], Chapter 3) allows us to partition X into finitely many cells, with f being either constant or affine on each cell. E.g. the cells for $n = 1$ are points and open intervals, the cells in $\mathbb{R}^2$ are points, intervals on vertical lines $\{a\} \times \mathbb{R}$, the graphs of continuous definable functions defined on intervals, as well as the "open spaces" between the graphs of each two smooth functions g, h with $g < h$, as depicted in Figure 9. Now for a continuous function, the cases

$$\operatorname{relint}\{f \geq s\} \neq \{f > s\} \quad \text{and} \quad \operatorname{relint}\{f \geq -s\} \neq \{f < -s\}$$

can only occur if f is constant, $f \equiv s$, on a cell $C \subset X$ of positive l-dimensional Lebesgue measure. Since there are only finitely many cells, such that f is constant on them, we have that

$$\operatorname{relint}\{f \geq s\} = \{f > s\} \quad \text{and} \quad \operatorname{relint}\{f \geq -s\} = \{f < -s\},$$

for almost all $s \in [0, \infty)$. Using a definable version of Theorem 1.34, which can be found in [31], we obtain that

$$\mu_k\{f \geq s\} = (-1)^{l+k}\mu_k\{f > s\} \quad \text{and} \quad \mu_k\{f < -s\} = (-1)^{l+k}\mu_k\{f \leq -s\},$$

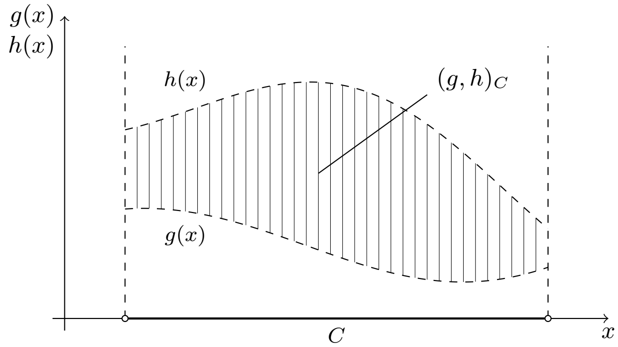


Figure 9: The "open space" $(g, h)_C$ between two smooth functions g, h defined on a cell $C \subset \mathbb{R}$ with $g < h$ is a possible cell in $\mathbb{R}^2$.

for almost all $s \in [0, \infty)$. Using the excursion set representation of the lower and upper Hadwiger integrals, this gives us

$$\int_X f \lfloor \mathrm{d}\mu_k \rfloor = \int_0^\infty \mu_k\{f \geq s\} - \mu_k\{f < -s\} \, \mathrm{d}s$$
$$= (-1)^{l+k} \int_0^\infty \mu_k\{f > s\} - \mu_k\{f \leq -s\} \, \mathrm{d}s = (-1)^{l+k} \int_X f \lceil \mathrm{d}\mu_k \rceil.$$

■

We conclude this chapter, by showing that the lower and upper Hadwiger integrals are valuations on $\mathrm{Def}(\mathbb{R}^n)$.

Theorem 2.22 *For $f, g \in \mathrm{Def}(\mathbb{R}^n)$, one has*

$$\int_{\mathbb{R}^n} f \vee g \lfloor \mathrm{d}\mu_k \rfloor + \int_{\mathbb{R}^n} f \wedge g \lfloor \mathrm{d}\mu_k \rfloor = \int_{\mathbb{R}^n} f \lfloor \mathrm{d}\mu_k \rfloor + \int_{\mathbb{R}^n} g \lfloor \mathrm{d}\mu_k \rfloor,$$

and similarly for the upper Hadwiger integrals.

Proof. Since f and g are definable, their graphs are definable sets, which implies that also the sets $\{f \geq g\}$ and $\{f < g\}$ are definable. Obviously, these two sets give us a partition of $\mathbb{R}^n$. We now rewrite

$$\int_{\mathbb{R}^n} f \vee g \lfloor \mathrm{d}\mu_k \rfloor = \int_{\{f \geq g\}} f \lfloor \mathrm{d}\mu_k \rfloor + \int_{\{f < g\}} g \lfloor \mathrm{d}\mu_k \rfloor,$$

and similarly

$$\int_{\mathbb{R}^n} f \wedge g \lfloor \mathrm{d}\mu_k \rfloor = \int_{\{f \geq g\}} g \lfloor \mathrm{d}\mu_k \rfloor + \int_{\{f < g\}} f \lfloor \mathrm{d}\mu_k \rfloor.$$

Addition of the equations and recombining the integrals on the right side gives the desired result. ■

3 Classification of Valuations on Functionals

Before we can state a classification theorem for valuations on $\mathrm{Def}(\mathbb{R}^n)$ we need corresponding topologies to distinguish between the lower and upper Hadwiger integrals. Therefore we will introduce currents.

3.1 Currents

First we will recall some definitions and give new ones respectively. A *differential form of degree k* or *differential k-form* on a smooth manifold $U \subseteq \mathbb{R}^n$ is a smooth section of the kth exterior power of the cotangent bundle of U. That is, ω is a k-form on U if it defines an alternating multilinear map

$$\omega_p : \underbrace{T_pU \times \cdots \times T_pU}_{k \text{ times}} \to \mathbb{R},$$

for every point $p \in U$, whereas T_pU denotes the tangent space to U at p. For example a smooth real-valued function on U would be a 0-form. The space of compactly-supported differential k-forms on $U \subseteq \mathbb{R}^n$ shall be denoted by $\Omega_c^k(U)$. Now, for every k-form ω on U and every point $p \in U$, ω_p can be seen as an element of the kth exterior power of the cotangent bundle of U at p, $\omega_p \in \bigwedge^k(T_p^*U)$, which lets us define the *wedge product* of a k-form and an l-form

$$\wedge : \Omega_c^k(U) \times \Omega_c^l(U) \to \Omega_c^{k+l}(U), \quad (\omega, \eta) \mapsto \omega \wedge \eta,$$

where $\omega \wedge \eta$ is defined pointwise via

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p.$$

The wedge product is *graded commutative*:

$$\omega \wedge \eta = (-1)^{k \cdot l} \eta \wedge \omega,$$

for a k-form ω and an l-form η.

Furthermore, there exists a unique $\mathbb{R}$-linear map

$$\mathrm{d} : \Omega_c^k(U) \to \Omega_c^{k+1}(U),$$

known as *exterior derivative* which satisfies the following properties:

- If f is a smooth function (a 0-form), then $\mathrm{d}f$ is the differential of f.
- $\mathrm{d} \circ \mathrm{d} = 0$.
- $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^p \omega \wedge \mathrm{d}\eta$ where ω is a p-form, which means that d is a so-called *antiderivation.*

Example 3.1 Let $x_1, \dots, x_n$ be a coordinate system on $U \subseteq \mathbb{R}^n$. These coordinates can be understood as functions

$$x_i : \mathbb{R}^n \to \mathbb{R}, \quad p = (p_1, \dots, p_n) \mapsto x_i(p) = p_i.$$

The differential forms $\mathrm{d}x_1, \dots, \mathrm{d}x_n$ then form a basis of the 1-forms on U, that is every 1-form ω on U has a unique representation

$$\omega = f_1 \,\mathrm{d}x_1 + \dots + f_n \,\mathrm{d}x_n$$

with $f_i : U \to \mathbb{R}$. For example when considering the integral of a real-valued function f on an interval $[a, b]$,

$$\int_a^b f(x) \,\mathrm{d}x,$$

the expression $f(x)\,\mathrm{d}x$ is a 1-form.

As mentioned in the example above we can integrate over differential forms. For example if $U \subseteq \mathbb{R}^n$ is an n-dimensional smooth manifold with a coordinate system $x_1, \dots, x_n$ and $\omega \in \Omega_c^k(U)$ has a representation

$$\omega = f(x_1, \dots, x_n) \,\mathrm{d}x_1 \wedge \dots \wedge \,\mathrm{d}x_n$$

then

$$\int_U \omega = \int_U f(x_1, \dots, x_n) \,\mathrm{d}x_1 \cdots \,\mathrm{d}x_n.$$

Definition 3.2 We denote the dual space of $\Omega_c^k(U)$ by $\Omega_k(U)$. The elements of $\Omega_k(U)$ are called *k-dimensional currents on U*. This means, that the k-dimensional currents on U are the continuous linear functionals on $\Omega_c^k(U)$. Furthermore, the *boundary* of a current $T \in \Omega_k(U)$ is the current $\partial T \in \Omega_{k-1}(U)$ defined by

$$(\partial T)(\omega) := T(\,\mathrm{d}\omega),$$

for all $\omega \in \Omega_c^{k-1}(U)$. Currents with zero boundary are referred to as *cycles*.

The *mass* $\mathbf{M}(T)$ of a current $T \in \Omega_k(U)$ is defined by

$$\mathbf{M}(T) := \sup\{T(\omega) \mid \omega \in \Omega_c^k(U),\ \sup \|\omega_x\| \leq 1 \ \forall x \in U\},$$

where $\|\omega_x\| := \sup\{|\omega_x(z)| \ : \ z \text{ is a simple unit } k\text{-vector}\}$. Conclusively, we define the *flat norm* $|T|_\flat$ of $T \in \Omega_k(U)$ as

$$|T|_\flat := \inf\{\mathbf{M}(R) + \mathbf{M}(S) \mid T = R + \partial S,\ R \in \Omega_k(U),\ S \in \Omega_{k-1}(U)\}.$$

An important type of currents is associated with submanifolds of $\mathbb{R}^n$ and representable by integration. Let M denote a C^1 oriented m-dimensional submanifold

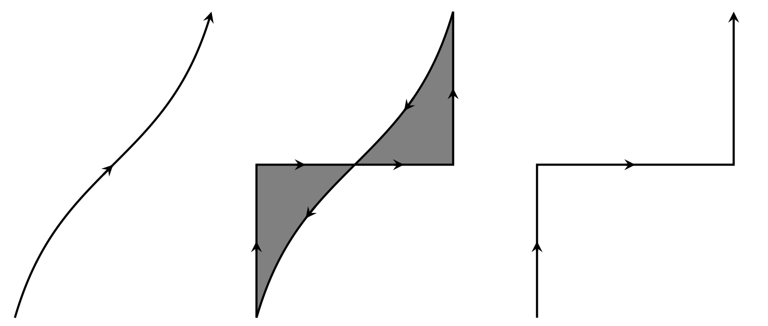


Figure 10: The 1-current T is decomposed as the sum of a 1-current R and the boundary of a 2-current S.

of $\mathbb{R}^n$. Furthermore, let $\omega \in \Omega_c^k(U)$ with $M \subseteq U$ and consider the restriction of ω to M. An m-dimensional current is then defined by

$$[[M]](\omega) = \int_M \omega.$$

The mass of a current associated with an m-dimensional submanifold is simply its m-dimensional volume. This also gives us a more graphic way to think about the flat norm of a current as quantifying the minimal-mass decomposition of a k-current T into a k-current R and the boundary of a $(k+1)$-current S, which is illustrated in Figure 10. In this case the sum of the length of R and the area of S is minimized. For further details on currents see [18], Chapter 7.

Every definable set is associated with a particular current, known as the *conormal cycle*. The conormal cycle of a compact definable set $A \subseteq \mathbb{R}^{n+1}$ shall be denoted by $\mathbf{C}^A$ and is an n-current on $\mathbb{R}^{n+1} \times \mathbb{S}^n$, where $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ denotes the unit n-sphere. The long and technical definition, which is not of particular importance for this work, can be found in [23]. For example, if M is a submanifold of $\mathbb{R}^n$, then the conormal cycle $\mathbf{C}^M$ can be identified with the current of integration defined by the total space of the conormal bundle of the embedding $M \hookrightarrow \mathbb{R}^n$ (see [24]). The conormal cycle of an interval $[a, b] \subset \mathbb{R} \subset \mathbb{R}^2$ can be seen in Figure 11.

Since the construction of the conormal cycle is based on the use of morphisms, we have an additive property:

$$\mathbf{C}^{A \cup B} + \mathbf{C}^{A \cap B} = \mathbf{C}^A + \mathbf{C}^B, \tag{3.1}$$

for definable sets A and B.

In order to represent the intrinsic volumes with the help of conormal cycles, we

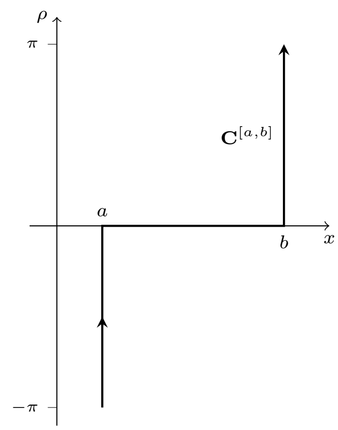


Figure 11: The conormal cycle $\mathbf{C}^{[a,b]}$ of the interval $[a, b]$.

will introduce special differential forms.
Let $x_1, \ldots, x_{n+1}$ be the standard orthonormal basis in $\mathbb{R}^{n+1}$, and $\rho_1, \ldots, \rho_n$ an orthonormal frame for $\mathbb{S}^n$. We define the following differential n-form on $\mathbb{R}^{n+1} \times \mathbb{S}^n$ using the wedge product:

$$\mathcal{V}(t) = (\,\mathrm{d}x_1 + t\,\mathrm{d}\rho_1) \wedge \cdots \wedge (\,\mathrm{d}x_n + t\,\mathrm{d}\rho_n).$$

For $t = 0$ this is the volume form on $\mathbb{R}^n$. Moreover, $\mathcal{V}(t)$ is invariant under rigid motions of $\mathbb{R}^{n+1}$, extended to $\mathbb{R}^{n+1} \times \mathbb{S}^n$. For details see [22], Chapter 19.

Definition 3.3 For $0 \le k \le n$, let $\mathcal{W}_{n,k}$ denote the coefficient of t^{n-k} in $\mathcal{V}(t)$. The form $\mathcal{W}_{n,k}$ is called the *kth Lipschitz-Killing curvature form* of degree n.

Example 3.4 The Lipschitz-Killing curvature forms of degree 2 on $\mathbb{R}^3 \times \mathbb{S}^2$ are given by

$$\begin{aligned} \mathcal{W}_{2,0} &= \mathrm{d}\rho_1 \wedge \mathrm{d}\rho_2 \\ \mathcal{W}_{2,1} &= \mathrm{d}\rho_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_1 \wedge \mathrm{d}\rho_2 \\ \mathcal{W}_{2,2} &= \mathrm{d}x_1 \wedge \mathrm{d}x_2. \end{aligned}$$

Theorem 3.5 *For each definable set $A \subseteq \mathbb{R}^n$, the integral*

$$\int_{\boldsymbol{C}^A} \mathcal{W}_{n,k} \tag{3.2}$$

is, up to a constant multiple, the intrinsic volume $\mu_k(A)$.

Proof. Since the conormal cycles are additive, expression (3.2) is a valuation on the set of definable subsets of $\mathbb{R}^n$. Furthermore, the Lipschitz-Killing curvature forms are Euclidean-invariant, which implies that this valuation is rigid motion invariant. Since the conormal cycles are also convex continuous, we obtain that the integral is convex continuous. Moreover, by the definition of the Lipschitz-Killing curvature forms, $\mu(A) := \int_{\mathbf{C}^A} \mathcal{W}_{n,k}$ is homogeneous of degree k. Hence, by Corollary 1.43, the integral is $\mu_k(A)$, up to a constant multiple. ■

Remark 3.6 Note that for $A \subset \mathbb{R}^n$ the current $\mathbf{C}^A$ is actually of degree $n-1$, whilst the Lipschitz-Killing curvature form $\mathcal{W}_{n,k}$ is of degree n. In order to establish compatibility we have to consider A as a subset of $\mathbb{R}^{n+1}$.

Example 3.7 Consider the space of invariant 1-forms on $\mathbb{R}^2 \times \mathbb{S}^1$ which is spanned by the two forms $\mathcal{W}_{1,0} = \mathrm{d}\rho$ and $\mathcal{W}_{1,1} = \mathrm{d}x$. The intrinsic volumes of the interval $[a, b]$ can be computed by the following integrals (see also Figure 11):

$$\mu_0([a,b]) = \int_{\mathbf{C}^{[a,b]}} \frac{1}{2\pi}\,\mathrm{d}\rho = 1$$

$$\mu_1([a,b]) = \int_{\mathbf{C}^{[a,b]}} \mathrm{d}x = b - a.$$

Recall that the integrals over the Lipschitz-Killing curvature forms give us the intrinsic volumes only up to a constant multiple. Therefore we have to use a normalization such as $\frac{1}{2\pi}$.

3.2 Continuity

We start by introducing a metric on the definable subsets of $\mathbb{R}^n$.

Definition 3.8 For definable subsets $A, B \subseteq \mathbb{R}^n$ we define the *flat metric* as follows:

$$d(A,B) = |\mathbf{C}^A - \mathbf{C}^B|_\flat. \tag{3.3}$$

The topology on the definable subsets induced by this metric is called the *flat topology.*

To get a better understanding of the flat topology, one can think of it as a generalization of the Hausdorff topology on convex sets. This means, a sequence of convex sets converges in the Hausdorff topology iff it converges in the flat topology. The same is not true for non-convex sets. For details see [11], Section 3.

Theorem 3.9 *The intrinsic volumes $\mu_0, \ldots, \mu_n$ are continuous with respect to the flat topology.*

Proof. Let $K, J \subseteq \mathbb{R}^n$ be bounded definable sets, $\varepsilon > 0$ and let B be a ball in $\mathbb{R}^n$ containing a common neighborhood of K and J. Now for any $T \in \Omega_n(B)$ and $\omega \in \Omega_c^n(B)$ we have

$$|T(\omega)| \le |T|_\flat \max\left\{\sup_{x\in B} \|\omega_x\|, \sup_{x\in B} \|\,\mathrm{d}\omega_x\|\right\}. \tag{3.4}$$

Using $T = \mathbf{C}^K - \mathbf{C}^J$ and $\omega = \mathcal{W}_{n,k}$, equation (3.4) becomes

$$|\mu_k(K) - \mu_k(J)| = \left| \int_{\mathbf{C}^K} \mathcal{W}_{n,k} - \int_{C^J} \mathcal{W}_{n,k} \right| = \left| \int_{\mathbf{C}^K - \mathbf{C}^J} \mathcal{W}_{n,k} \right|$$
$$\leq |\mathbf{C}^K - \mathbf{C}^J|_\flat \max \left\{ \sup_{x\in B} \|(\mathcal{W}_{n,k})_x\|, \sup_{x\in B} \|(\,\mathrm{d}\mathcal{W}_{n,k})_x\| \right\}. \quad (3.5)$$

Since $\mathcal{W}_{n,k}$ and $\mathrm{d}\mathcal{W}_{n,k}$ are bounded on B, we can set

$$\delta := \varepsilon \cdot \left(\max \left\{ \sup_{x\in B} \|(\mathcal{W}_{n,k})_x\|, \sup_{x\in B} \|(\,\mathrm{d}\mathcal{W}_{n,k})_x\| \right\} \right)^{-1},$$

and obtain

$$|\mu_k(K) - \mu_k(J)| < \varepsilon$$

for all definable sets $K, J \subseteq \mathbb{R}^n$ such that $d(K, J) = |\mathbf{C}^K - \mathbf{C}^J|_\flat < \delta$. ∎

We now want to define a metric on $\mathrm{Def}(\mathbb{R}^n)$. One would expect from this metric that any open set containing $r \cdot \mathbb{1}_a$ also contains $(r+\varepsilon)\cdot \mathbb{1}_A$ for small enough ε. Furthermore, if definable sets A and B are close in the flat topology, then $\nu(r\cdot\mathbb{1}_A)$ and $\nu(r\cdot\mathbb{1}_B)$ should be close for any continuous valuation ν.

Definition 3.10 The *lower* and *upper flat metrics* on definable functions, denoted by $\underline{d}_\flat$ and $\overline{d}_\flat$, respectively, are defined as

$$\underline{d}_\flat(f,g) = \int_{-\infty}^{\infty} \left| \mathbf{C}^{\{f\geq s\}} - \mathbf{C}^{\{g\geq s\}} \right|_\flat \quad (3.6)$$

$$\overline{d}_\flat(f,g) = \int_{-\infty}^{\infty} \left| \mathbf{C}^{\{f> s\}} - \mathbf{C}^{\{g> s\}} \right|_\flat \quad (3.7)$$

for all $f, g \in \mathrm{Def}(\mathbb{R}^n)$. The topologies induced by these metric are called the *lower* and *upper flat topologies* on definable functions.

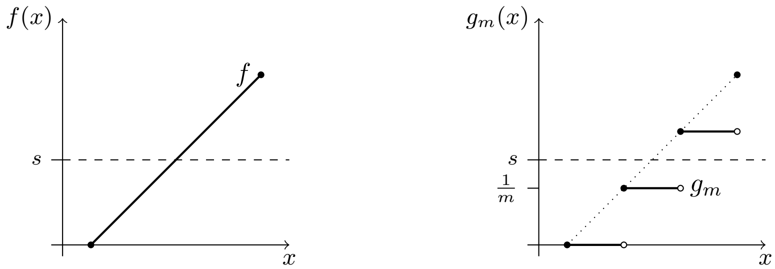
Remark 3.11 Since functions in $\mathrm{Def}(\mathbb{R}^n)$ are bounded it suffices to integrate in (3.6) and (3.7) between the minimum and maximum of f and g.

These metrics indeed extend the flat metric on definable sets. For two definable sets $A, B \subset \mathbb{R}^n$ let $f = \mathbb{1}_A$ and $g = \mathbb{1}_B$ to obtain

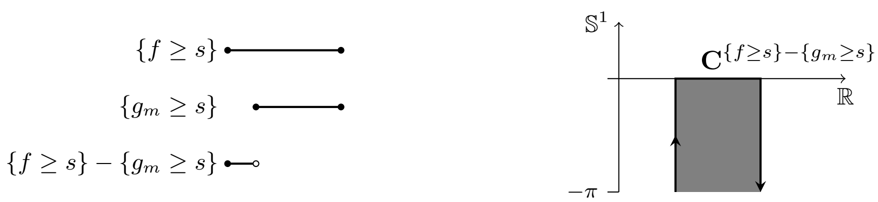
$$\underline{d}_\flat(f,g) = \overline{d}_\flat(f,g) = \int_0^1 |\mathbf{C}^A - \mathbf{C}^B|_\flat \,\mathrm{d}s = d(A,B).$$

Theorem 3.12 *The lower flat topology on definable functions differs from the corresponding upper flat topology.*

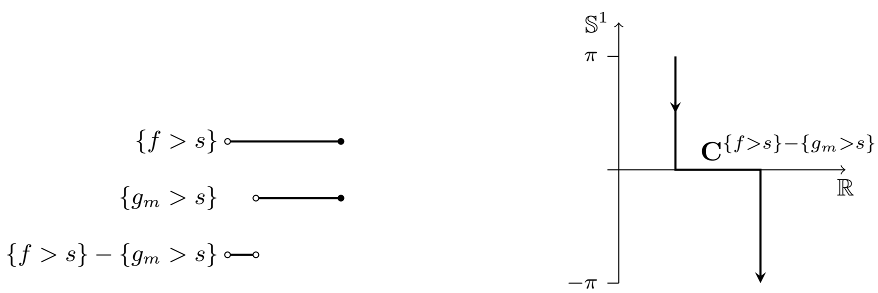
Proof. To prove the statement it is sufficient to find a sequence of functions that converge to a different limit in the lower and upper flat topologies. For that we consider a linear function $f : \mathbb{R} \to \mathbb{R}$ on a closed interval, as depicted in Figure 12(a). E.g. $f(x) = (l \cdot x + d)\cdot \mathbb{1}_{[a,b]}(x)$ for some suitable real numbers



(a) A definable function $f : \mathbb{R} \to \mathbb{R}$ and a lower step function $g_m = \frac{1}{m}\lfloor mf \rfloor$.



(b) Excursion sets of f and g_m and the corresponding conormal cycle of their difference.



(c) Strict excursion sets of f and g_m and the corresponding conormal cycle of their difference.

Figure 12: The series g_m converges to f in the lower flat topology but not in the upper flat topology.

l, d, a, b. For $m > 0$, let $g_m = \frac{1}{m}\lfloor mf \rfloor$, which is the lower step function of f with step size $\frac{1}{m}$. For $0 < s < P := \max_{x \in \mathbb{R}} f(x)$ the difference of the upper excursion sets $\{f \geq s\}$ and $\{g_m \geq s\}$ is a half-open interval the length of which is bounded by a multiple of $\frac{1}{m}$ and decreases to zero as $m \to \infty$. The flat norm of the corresponding conormal cycle $\mathbf{C}^{\{f \geq s\} - \{g_m \geq s\}}$ is bounded by a constant multiple of the "area of the current", see also Figure 12(b). Since the area is bounded by a multiple of the length of the half-open interval described above we have

$$\left| \mathbf{C}^{\{f \geq s\} - \{g_m \geq s\}} \right|_\flat \leq c \frac{1}{m}$$

for some $c > 0$. This implies that g_m converges to f in the lower flat topology:

$$\lim_{m\to\infty} \underline{d}_\flat(f, g_m) = \lim_{m\to\infty} \int_{-\infty}^{\infty} \left| \mathbf{C}^{\{f\geq s\}-\{g_m\geq s\}} \right|_\flat \, \mathrm{d}s \leq \lim_{m\to\infty} \int_0^P c\frac{1}{m} \, \mathrm{d}s = 0.$$

On the other hand, the sequence g_m does not converge to f in the upper flat topology. To see that, we have to consider the difference of the strict excursion sets $\{f > s\}$ and $\{g_m > s\}$ which is an open interval for $0 < s < P$. The flat norm of the corresponding conormal cycle $\mathbf{C}^{\{f>s\}-\{g_m>s\}}$ is bounded from below by the length of $\mathbb{S}^1$ (Figure 12(c)). Therefore, we do not have convergence in the upper flat topology.

Similarly, the sequence $h_m = \frac{1}{m}\lceil mf \rceil$ converges to f in the upper flat topology, but not in the lower flat topology. ■

Definition 3.13 A valuation $\nu : \mathrm{Def}(\mathbb{R}^n) \to \mathbb{R}$ is said to be a *lower valuation* if ν is continuous in the lower flat topology. Likewise, ν is said to be an *upper valuation* if ν is continuous in the upper flat topology.

Theorem 3.14 *The lower and upper Hadwiger integrals are lower and upper valuations, respectively.*

Proof. Let $f, g \in \mathrm{Def}(\mathbb{R}^n)$, supported on compact $X \subset \mathbb{R}^n$. Considering equation (3.5) in the proof of Theorem 3.9 we have for the lower Hadwiger integrals:

$$\left| \int f \lfloor \mathrm{d}\mu_k \rfloor - \int g \lfloor \mathrm{d}\mu_k \rfloor \right| = \left| \int_0^\infty (\mu_k\{f \geq s\} - \mu_k\{g \geq s\}) \, \mathrm{d}s \right|$$

$$\leq \int_0^\infty |\mathbf{C}^{\{f\geq s\}} - \mathbf{C}\{g \geq s\}|_\flat \cdot \max\{\sup_{x\in X} \|(\mathcal{W}_{n,k})_x\|, \sup_{x\in X} \|(\,\mathrm{d}\mathcal{W}_{n,k})_x\|\} \, \mathrm{d}s$$

$$= \underline{d}_\flat(f, g) \cdot \max\{\sup_{x\in X} \|(\mathcal{W}_{n,k})_x\|, \sup_{x\in X} \|(\,\mathrm{d}\mathcal{W}_{n,k})_x\|\}.$$

Since $(\mathcal{W}_{n,k})_x$ and $(\,\mathrm{d}\mathcal{W}_{n,k})_x$ are bounded for $x \in X$, one has that the lower Hadwiger integrals of f and g are close if the functions are close in the lower flat topology. The proof for the upper Hadwiger integrals is analogous. ■

3.3 Hadwiger's theorem for functionals

Equipped with proper topologies on $\mathrm{Def}(\mathbb{R}^n)$ and corresponding types of continuity for valuations on definable functions, we can go on to extend Lemma 2.12 to valuations on $\mathrm{Def}(\mathbb{R}^n)$.

Proposition 3.15 *For any continuous, strictly decreasing function $c : \mathbb{R} \to \mathbb{R}$ one has*

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} c\left(\frac{1}{m}\lceil mf \rceil\right) \mathrm{d}\mu_k = \lim_{m\to\infty} \int_{\mathbb{R}^n} \frac{1}{m} \lfloor mc(f) \rfloor \, \mathrm{d}\mu_k,$$

for every $f \in \mathrm{Def}(\mathbb{R}^n)$.

Proof. Rewriting of the integral of c composed with the upper step functions of f gives us

$$\int_{\mathbb{R}^n} c\left(\frac{1}{m}\lceil mf\rceil\right) \mathrm{d}\mu_k = \sum_{i\in\mathbb{Z}} c(\tfrac{i}{m}) \cdot \mu_k\{\tfrac{i-1}{m} < f \leq \tfrac{i}{m}\}.$$

Since c is strictly decreasing there exists a corresponding inverse function c^{-1}. Furthermore, let $\mathcal{U}$ denote a neighborhood of the range of f to define the discrete set

$$S = \{c^{-1}(\tfrac{i}{m}) | i \in \mathbb{Z}\} \cap \mathcal{U}.$$

This allows us to rewrite the integral of the lower step functions of $c(f)$ as follows:

$$\int_{\mathbb{R}^n} \frac{1}{m}\lfloor mc(f)\rfloor \, \mathrm{d}\mu_k = \sum_{i\in\mathbb{Z}} \tfrac{i}{m} \cdot \mu_k\{\tfrac{i}{m} \leq c(f) < \tfrac{i+1}{m}\}$$
$$= \sum_{s\in S} c(s) \cdot \mu_k\{c(s) \leq c(f) < c(s-\varepsilon_s)\} = \sum_{s\in S} c(s) \cdot \mu_k\{s-\varepsilon_s < f \leq s\},$$

with proper $\varepsilon_s > 0$. By continuity of c, $\varepsilon_s \to 0$ as $m \to \infty$ and consequently the limits of the integrals are equal:

$$\lim_{m\to\infty} \sum_{i\in\mathbb{Z}} c(\tfrac{i}{m}) \cdot \mu_k\{\tfrac{i-1}{m} < f \leq \tfrac{i}{m}\} = \lim_{\varepsilon_s\to 0} \sum_{s\in S} c(s)\mu_k\{s-\varepsilon_s < f \leq s\}.$$

■

The last result implies that if $c : \mathbb{R} \to \mathbb{R}$ is decreasing, then the valuation $\nu : \mathrm{Def}(\mathbb{R}^n) \to \mathbb{R}$

$$\nu(f) = \int_{\mathbb{R}^n} c(f) \, \lfloor \mathrm{d}\mu_k \rfloor$$

is not continuous in the lower flat topology but in the upper flat topology. A similar implication follows for upper Hadwiger integrals. Furthermore, if c is increasing on some interval and decreasing on another, then ν is not continuous in either the lower or the upper flat topology. The same follows for the corresponding upper Hadwiger integrals.

Theorem 3.16 (Baryshnikov, Ghrist, Wright)
For any rigid motion invariant lower valuation ν on $\mathrm{Def}(\mathbb{R}^n)$ there exist continuous increasing functions $c_i : \mathbb{R} \to \mathbb{R}$ with $c_i(0) = 0$, such that

$$\nu(f) = \sum_{i=0}^{n} \int_{\mathbb{R}^n} c_i(f) \, \lfloor \mathrm{d}\mu_i \rfloor,$$

for all $f \in \mathrm{Def}(\mathbb{R}^n)$. Similarly, any upper valuation ν on $\mathrm{Def}(\mathbb{R}^n)$ can be written as a linear combination of upper Hadwiger integrals.

Proof. Consider a lower valuation $\nu : \mathrm{Def}(\mathbb{R}^n) \to \mathbb{R}$ and a definable function $f \in \mathrm{Def}(\mathbb{R}^n)$. Like in the definition of the lower Hadwiger integrals we approximate f by lower step functions

$$f_m = \frac{1}{m}\lfloor mf \rfloor$$

for $m > 0$. Using the lower flat topology on $\mathrm{Def}(\mathbb{R}^n)$ we have

$$\lim_{m\to\infty} f_m = f.$$

By Hadwiger's theorem for constructible functions, Lemma 2.12, we have that ν is a linear combination of Hadwiger integrals on each of these step functions. That is

$$\nu(f_m) = \sum_{i=0}^{n} \int_{\mathbb{R}^n} c_i(f_m)\, \mathrm{d}\mu_i,$$

for some coefficient functions $c_i : \mathbb{R} \to \mathbb{R}$ with $c_i(0) = 0$, depending only on ν and not on m. Since we are considering constructible functions we can also write this as

$$\nu(f_m) = \sum_{i=0}^{n} \int_{\mathbb{R}^n} c_i(f_m)\, \lfloor \mathrm{d}\mu_i \rfloor.$$

Using Proposition 3.15 and considering the fact that we are approximating f with lower step functions in the lower flat topology, the c_i must be continuous increasing functions.
Since ν is a lower valuation and f_m converges to f in the lower flat topology we have

$$\nu\left(\lim_{m\to\infty} f_m\right) = \nu(f) = \lim_{m\to\infty} \nu(f_m) = \sum_{i=0}^{n} \lim_{m\to\infty} \int_{\mathbb{R}^n} c_i(f_m)\, \lfloor \mathrm{d}\mu_i \rfloor. \tag{3.8}$$

Since the lower Hadwiger integrals are continuous with respect to the lower flat topology and the c_i are continuous this becomes

$$\nu(f) = \sum_{i=0}^{n} \int_{\mathbb{R}^n} c_i\left(\lim_{m\to\infty} f_m\right) \lfloor \mathrm{d}\mu_i \rfloor = \sum_{i=0}^{n} \int_{\mathbb{R}^n} c_i(f)\, \lfloor \mathrm{d}\mu_i \rfloor,$$

which concludes the proof for lower valuations. The proof for upper valuations is analogous. ■

Corollary 3.17 *If* $\nu : \mathrm{Def}(\mathbb{R}^n) \to \mathbb{R}$ *is an rigid motion invariant valuation, both lower- and upper-continuous, then*

$$\nu(f) = \int_{\mathbb{R}^n} c(f)\, \mathrm{d}\mathcal{L}$$

for some continuous function $c : \mathbb{R} \to \mathbb{R}$ *and with* $\mathrm{d}\mathcal{L} = \lfloor \mathrm{d}\mu_n \rfloor = \lceil \mathrm{d}\mu_n \rceil$ *denoting the Lebesgue measure.*

Proof. By Theorem 3.16 there exist there exist real-valued continuous functions $\underline{c}_i, \overline{c}_i$ with $\underline{c}_i(0) = \overline{c}_i(0) = 0$ such that

$$\nu(f) = \sum_{i=0}^{n} \int_{\mathbb{R}^n} \underline{c}_i(f) \lfloor \mathrm{d}\mu_i \rfloor = \sum_{i=0}^{n} \int_{\mathbb{R}^n} \overline{c}_i(f) \lceil \mathrm{d}\mu_i \rceil,$$

for every $f \in \mathrm{Def}(\mathbb{R}^n)$.
We want to show that $\underline{c}_i = \overline{c}_i$ for all $i = 0, 1, \ldots, n$. We start with $i = 0$ and consider any point $x \in \mathbb{R}$ and let $f = r \cdot \mathbb{1}_{\{x\}}$ for arbitrary $r \in \mathbb{R}$. Since only the Euler characteristic contributes to $\nu(f)$ we get $\underline{c}_0(r) = \overline{c}_0(r)$ for any $r \in \mathbb{R}$ and consequently $\underline{c}_0 = \overline{c}_0$.

Now assume that we have shown that $\underline{c}_i = \overline{c}_i$ for all $i = 0, 1, \ldots, k$ with $k < n$. Let A be an orthogonal parallelotope of dimension $k + 1$ and evaluate ν on $f = r \cdot \mathbb{1}_A$. Since the intrinsic volumes $\mu_{k+2}, \ldots, \mu_n$ vanish on sets of dimension $k + 1$, we have

$$\sum_{i=0}^{k+1} \underline{c}_i(r)\mu_i(A) = \sum_{i=0}^{k+1} \overline{c}_i(r)\mu_i(A).$$

Using the hypothesis it follows that $\underline{c}_{k+1} = \overline{c}_{k+1}$. By induction on k, we have $\underline{c}_i = \overline{c}_i$ for all $i = 0, 1, \ldots, n$.

Knowing that the lower and upper Hadwiger integrals with respect to μ_i are not the same on $\mathrm{Def}(\mathbb{R}^n)$ for $i = 0, \ldots, n-1$ we have that $\underline{c}_i = \overline{c}_i \equiv 0$ for $i = 0, \ldots, n-1$. Since $\lfloor \mathrm{d}\mu_n \rfloor = \lceil \mathrm{d}\mu_n \rceil = \mathrm{d}\mu_n = \mathrm{d}\mathcal{L}$ we have

$$\nu(f) = \int_{\mathbb{R}^n} c(f) \, \mathrm{d}\mathcal{L}$$

for some continuous function $c : \mathbb{R} \to \mathbb{R}$. ■

Remark 3.18 In contrast to Hadwiger's characterization theorem for valuations on Polycon(n) Theorem 3.16 gives a dual generalization to classify lower and upper valuations. The question arises if there is a topology on $\mathrm{Def}(\mathbb{R}^n)$ such that any rigid motion invariant valuation ν that is continuous with respect to said topology can be written as

$$\nu(f) = \sum_{i=0}^{n} \left(\int_{\mathbb{R}^n} \underline{c}_i(f) \lfloor \mathrm{d}\mu_i \rfloor + \int_{\mathbb{R}^n} \overline{c}_i(f) \lceil \mathrm{d}\mu_i \rceil \right),$$

for some continuous functions $\underline{c}_i, \overline{c}_i : \mathbb{R} \to \mathbb{R}$. Wright points out that this cannot be achieved with the union of the lower and upper flat topologies since the set of continuous valuations on $\mathrm{Def}(\mathbb{R}^n)$ would be enlarged by too much [31].
Another approach would be to examine valuations on continuous definable functions. By Theorem 2.21 the lower and upper Hadwiger integrals of a continuous function differ at most by a minus sign. This yields that when considering only continuous definable functions that the Hadwiger integrals are both lower and

upper valuations at the same time. However, since constructible functions are not continuous one cannot apply Lemma 2.12. Therefore, it would be possible that Hadwiger integration alone does not give all continuous valuations. A potential solution could be a result similar to Groemer's extension theorem that states that any rigid motion invariant continuous valuation on continuous definable functions admits a unique extension on $\mathrm{Def}(\mathbb{R}^n)$ together with some regularity condition as in Lemma 2.12.

4 Integral Transforms

In this chapter some of the most common integral transforms for Hadwiger and especially Euler integrals are examined. Fubini's theorem as presented here was probably first considered by Viro in [30]. Pioneering on Convolution, Duality and Radon transform was done by Schapira in [25] and [26]. The Hadwiger-Bessel and Hadwiger-Fourier transforms were introduced by Wright in [31] whereas many results for the transforms with respect to the Euler characteristic were already given by Ghrist and Robinson in [12]. Direct applications for most transforms can be found in the next chapter.

4.1 Fubini's theorem

Using a sheaf theoretic perspective we are able to obtain new insight in integration with respect to the Euler characteristic. Briefly, a sheaf is a tool to assign some algebraic object to open sets of a topological space, respecting the operations of restriction and gluing. An canonical example for a sheaf on a space X is $C(X)$, the space of continuous real-valued functions on X. This is a sheaf since the restriction of a continuous function is continuous and two continuous functions on subsets that agree on the intersection of their domains extend to a continuous function on the union.

Another example of a sheaf over X would be the space of constructible functions $\mathrm{CF}(X)$ which allows another interpretation of Euler integration. See [2], [4] and [25].

Definition 4.1 For two spaces X, Y let $F : X \to Y$ be a definable map. The *pushforward of* F is the induced homomorphism $F_* : \mathrm{CF}(X) \to \mathrm{CF}(Y)$ defined via

$$F_* f(y) = \int_{F^{-1}(y)} f \, \mathrm{d}\chi,$$

for all $f \in \mathrm{CF}(X)$ and $y \in Y$.

The *pullback of* F is the mapping $F^* : \mathrm{CF}(Y) \to \mathrm{CF}(X)$ given by

$$F^* g(x) = g(F(x)),$$

for all $g \in \mathrm{CF}(Y)$.

Since F is definable and f has compact support the integral of f on $F^{-1}(y)$ in the definition of the pushforward exists for every $y \in Y$. Furthermore, the pushforward and the pullback are functorial, which means, roughly speaking, that things commute as expected. This is expressed in the Projection formula

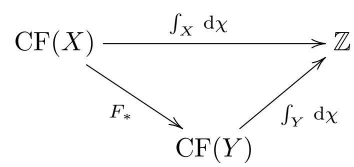
$$F_*(f \cdot F^* g)(y) = F_* f(y) \cdot g(y), \tag{4.1}$$

as well as Fubini's theorem.

Theorem 4.2 (Fubini's theorem) *For a definable map $F : X \to Y$ and $f \in \mathrm{CF}(X)$ one has*

$$\int_X f \,\mathrm{d}\chi = \int_Y F_* f \,\mathrm{d}\chi.$$

Proof. Consider the trivial map $X \to \{pt\}$ which only attains the single value pt. Since $\mathrm{CF}(\{pt\}) \cong \mathbb{Z}$, the pushforward of the trivial map is a homomorphism $\mathrm{CF}(X) \to \mathbb{Z}$ which is in fact the integral with respect to the Euler characteristic. By functoriality of the pushforward the following diagram commutes for any definable map $F : X \to Y$:



■

Remark 4.3 Another argumentation for Fubini's theorem given by Viro in [30] relies on the fact that

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y)$$

for two spaces X and Y.

4.2 Convolution

The definition of a convolution operator with respect to the Euler characteristic is straightforward.

Definition 4.4 For two constructible functions $f, g \in \mathrm{CF}(\mathbb{R}^n)$ the *Euler convolution* is defined as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y) g(x - y) \,\mathrm{d}\chi(y).$$

For the characteristic functions of compact convex sets $A, B \in \mathcal{K}^n$, there is a close relationship between the Euler convolution and the Minkowski sum:

$$(\mathbb{1}_A * \mathbb{1}_B) = \mathbb{1}_{A+B} \tag{4.2}$$

Equation (4.2) might not hold if one of the sets is not compact, as the example of a half-open interval and the unit ball in $\mathbb{R}^2$ shows, which is depicted in Figure 13. Similarly, one can show that (4.2) does not hold in general if the sets are not convex. For example, take a discrete set consisting of two points with distance d and a closed ball with radius $r > \frac{d}{2}$. The resulting function is not a characteristic function at all, since it attains the value 2. See also Figure 14.

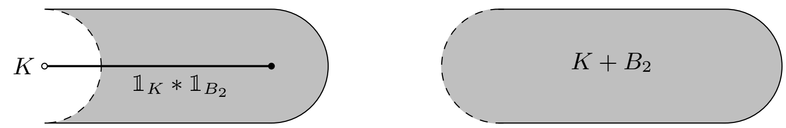


Figure 13: The convolution of the characteristic functions of a half-open line segment K and the unit ball B_2 is not the characteristic function of the Minkowski sum $K + B_2$.

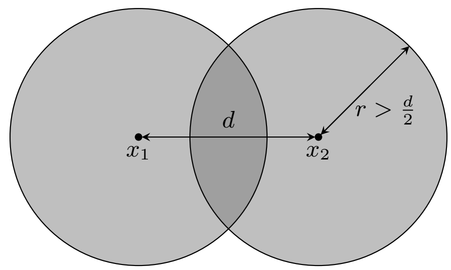


Figure 14: The convolution of the characteristic functions of a discrete set $K = \{x_1, x_2\}$ and a ball, the radius of which is large enough, is not a characteristic function at all.

Another direct consequence of Equation (4.2) is the following lemma, where we use the notation

$$f^{*k} = \underbrace{f * \cdots * f}_{k \text{ times}},$$

for $f \in \mathrm{CF}(\mathbb{R}^n)$ and $k \in \mathbb{N}$. Moreover, let $B_n(x, r)$ denote the n-dimensional open ball of radius $r > 0$ centered x.

Lemma 4.5 *Let $J, O \in \mathcal{K}^n$. Suppose that both J and O contain the ball $B_n(0, r)$ for some $r > 0$. Then $(\mathbb{1}_J^{*k} * \mathbb{1}_O^{*k})(x) = 1$ for $x \in B_n(0, 2kr)$ and $k \in \mathbb{N}$.*

Remark 4.6 Bröcker calls the convolution with respect to χ *Euler multiplication.* He then shows, that the constructible functions together with the usual addition and the Euler multiplication form a commutative ring with unit, the unit obviously being $\mathbb{1}_{\{0\}}$ [7].

4.3 Duality

With the help of an integral transform, defined by Schapira in [25], one can find inverse elements for Euler multiplication.

Definition 4.7 For $h \in \mathrm{CF}(\mathbb{R}^n)$, define the *dual of h* as the function given by

$$\mathcal{D}h(x) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} h \cdot \mathbb{1}_{B_n(x,\varepsilon)} \, \mathrm{d}\chi,$$

for all $x \in \mathbb{R}^n$.

Let $\pi_{\mathbb{R}^k}$ denote the projection from $\mathbb{R}^n$ to $\mathbb{R}^k$. Observe that $\pi_{\mathbb{R}^k}(B_n(x,\varepsilon)) = B_k(\pi_{\mathbb{R}^k}(x),\varepsilon)$, which has Euler characteristic $(-1)^k$. Therefore, we have $\mathcal{D}\mathbb{1}_A(x) = (-1)^k$ for all x in the interior of a k-dimensional definable set A, since the integral becomes $\lim_{\varepsilon\to 0^+}\chi(A\cap B_n(x,\varepsilon)) = \chi(B_n(x,\varepsilon_0)) = (-1)^k$ for some $\varepsilon_0 > 0$. It is also easy to see that $\mathcal{D}\mathbb{1}_A(x) = 0$ if x is in the interior of the complement of A. The interesting cases occur for $x \in \partial A$. For a k-dimensional definable closed set $K \subset \mathbb{R}^n$ with non-empty interior, one has

$$\mathcal{D}(\mathbb{1}_K) = (-1)^k \mathbb{1}_{\mathrm{relint}(K)}.$$

Similarly, for a k-dimensional definable open set $O \subset \mathbb{R}^n$, one has

$$\mathcal{D}(\mathbb{1}_O) = (-1)^k \mathbb{1}_{\mathrm{cl}(O)},$$

where cl denotes the closure of sets. Now it is also easy to see, that duality is an involution,

$$\mathcal{D}\mathcal{D}h = h,$$

for all $h \in \mathrm{CF}(\mathbb{R}^n)$. Furthermore, the following lemma provides a de-convolution (cf. [4] and [25]).

Lemma 4.8 *Let A be a bounded convex relative open or a bounded convex closed subset of $\mathbb{R}^n$ with* $\mathrm{relint}(A) \neq \emptyset$. *Then,*

$$\mathbb{1}_A * \mathcal{D}\mathbb{1}_{-A} = \mathbb{1}_{\{0\}}.$$

Proof. Let k be the dimension of A. Considering that $\mathbb{1}_{-A}(x) = \mathbb{1}_A(-x)$ and $f * g = g * f$, one has

$$\begin{aligned}(\mathbb{1}_A * \mathcal{D}\mathbb{1}_{-A})(x) &= (-1)^k \int_{\mathbb{R}^n} \mathbb{1}_{\mathrm{cl}(A)}(y-x)\cdot \mathbb{1}_{\mathrm{relint}(A)}(y)\,\mathrm{d}\chi(y) \\ &= (-1)^k \cdot \chi(\mathrm{cl}(A+x)\cap \mathrm{relint}(A)). \qquad (4.3)\end{aligned}$$

For x with $\mathrm{cl}(A+x)\cap\mathrm{relint}(A) = \emptyset$ the expression above vanishes. Furthermore, for $x = 0$ (4.3) computes to

$$(-1)^k\chi(\mathrm{relint}(A)).$$

Since $\mathrm{relint}(A)$ is homeomorphic to an open k-dimensional ball this gives

$$(\mathbb{1}_A * \mathcal{D}\mathbb{1}_{-A})(0) = (-1)^k(-1)^k = 1.$$

The case remains, where $x \neq 0$ and $\mathrm{cl}(A+x)\cap\mathrm{relint}(A) \neq \emptyset$. In this case

$$\mathrm{cl}(A+x)\cap\mathrm{relint}(A) = B\dot{\cup}C,$$

where B is homeomorphic to $\mathrm{relint}(A)$ and C is an open subset of ∂A. Therefore, $\chi(B) = -\chi(C)$ and $(\mathbb{1}_A * \mathcal{D}\mathbb{1}_{-A})(x) = 0$. ■

4.4 Radon transform

In order to establish a generalization of the duality on constructible functions, we consider two spaces W and X together with the projections $\pi_W : W \times X \to W$ and $\pi_X : W \times X \to X$.

Definition 4.9 Let $S \subset W \times X$ be a locally closed definable set. The *Radon transform* as a map $\mathcal{R}_S : \mathrm{CF}(W) \to \mathrm{CF}(X)$ is then defined as

$$\mathcal{R}_S h = (\pi_X)_*((\pi_W^* h) \cdot \mathbb{1}_S),$$

for all $h \in \mathrm{CF}(W)$. This means that for any $x \in X$ the Radon transform $\mathcal{R}_S h(x)$ gives the integral of h over a set in W that corresponds to x via S.

Example 4.10 Let $\Delta = \{(x,x) : x \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}^n$. Using the definition of the Radon transform we have

$$\mathcal{R}_\Delta h(x) = \int_{\pi_{\mathbb{R}^n}^{-1}(x)} h(\pi_{\mathbb{R}^n}) \cdot \mathbb{1}_\Delta \, \mathrm{d}\chi = \int_{\Delta \cap \pi_{\mathbb{R}^n}^{-1}(x)} h(\pi_{\mathbb{R}^n}) \, \mathrm{d}\chi = \int_{\{x\}} h \, \mathrm{d}\chi = h(x),$$

for all $h \in \mathrm{CF}(\mathbb{R}^n)$. In contrast to that consider

$$\mathcal{R}_{\mathbb{R}^n \times \mathbb{R}^n} h(x) = \int_{\mathbb{R}^n \times \mathbb{R}^n \cap \pi_{\mathbb{R}^n}^{-1}(x)} h(\pi_{\mathbb{R}^n}) \, \mathrm{d}\chi = \int_{\mathbb{R}^n} h \, \mathrm{d}\chi,$$

which does not depend on $x \in \mathbb{R}^n$ anymore.

Example 4.11 Let $S \subset \mathbb{R}^n \times \mathbb{R}^n$ be a sufficiently small open tubular neighborhood of the diagonal Δ. The Radon transform gives

$$\mathcal{R}_S h(x) = \int_{S \cap \pi_{\mathbb{R}^n}^{-1}(x)} h(\pi_{\mathbb{R}^n}) \, \mathrm{d}\chi.$$

Since $S \cap \pi_{\mathbb{R}^n}^{-1}(x)$ is a small n-dimensional open ball in $\mathbb{R}^n \times \mathbb{R}^n$, this becomes $\mathcal{D}h(x)$. Another way to see this, is to use the Projection formula (4.1). Hence, the Radon transform is in a certain sense a generalization of duality on constructible functions.

Schapira provided an inversion formula for the Radon transform in [26]. For this, consider $S \subset W \times X$. We denote the *vertical fibers* of S as

$$S_w = \pi_X(\pi_W^{-1}(w) \cap S), \quad w \in W.$$

Similarly, the *horizontal fibers* are defined as

$$S_x = \pi_W(\pi_X^{-1}(x) \cap S), \quad x \in X.$$

Theorem 4.12 (Schapira's inversion formula) *Assume that $S \subset W \times X$ and $S' \subset X \times W$ fulfill the following conditions:*

- $\chi(S_w \cap S'_w) = \mu$ *for all* $w \in W$
- $\chi(S_w \cap S'_{w'}) = \lambda$ *for all* $w' \neq w \in W$.

Then,

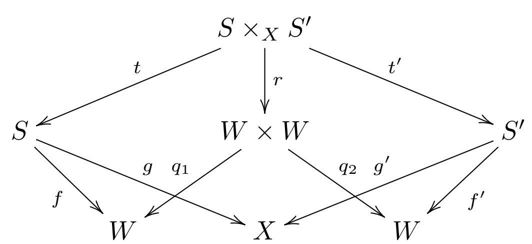
$$\mathcal{R}_{S'}\mathcal{R}_S h = (\mu - \lambda)h + \lambda \left(\int_W h \, \mathrm{d}\chi \right) \mathbb{1}_W,$$

for all $h \in \mathrm{CF}(W)$.

Proof. Let

$$S \times_X S' := \{(s, s') \in S \times S' \mid \pi_X(s) = \pi_X(s')\}.$$

Furthermore, let t and t' be the projections from $S \times_X S'$ to S and S' respectively, r the projection from $S \times_X S'$ to $W \times W$, f, f', g, g' the restrictions of π_W and π_X to S and S' respectively, and q_1 and q_2 the projections from $X \times X$ to the first and second component respectively. This is represented in the following diagram:



We have

$$\begin{aligned}
\mathcal{R}_{S'}\mathcal{R}_S h(w) &= \mathcal{R}_{S'} \left(\int_{\pi_X^{-1}(\cdot)} h(\pi_W(s)) \cdot \mathbb{1}_S(s) \, \mathrm{d}\chi(s) \right)(w) \\
&= \mathcal{R}_{S'} \left(\int_{g^{-1}(\cdot)} h(f(s)) \, \mathrm{d}\chi(s) \right)(w) \\
&= \int_{(f')^{-1}(w)} \int_{g^{-1}(g'(s'))} h(f(s)) \, \mathrm{d}\chi(s) \, \mathrm{d}\chi(s') \\
&= \int_{(f')^{-1}(w)} \int_{(t')^{-1}(s')} h(f(t(s, s'))) \, \mathrm{d}\chi(s, s') \, \mathrm{d}\chi(s') \\
&= \int_{(f' \circ t')^{-1}(w)} h(f \circ t(s, s')) \, \mathrm{d}\chi(s, s') \\
&= \int_{(q_2 \circ r)^{-1}(w)} h(q_1 \circ r(s, s')) \, \mathrm{d}\chi(s, s') \\
&= \int_{q_2^{-1}(w)} \int_{r^{-1}(w, w')} h(q_1(r(s, s'))) \, \mathrm{d}\chi(s, s') \, \mathrm{d}\chi(w, w') \\
&= \int_{q_2^{-1}(w)} r_*(\mathbb{1}_{S \times_X S'} \cdot r^*(h(q_1)))(w, w') \, \mathrm{d}\chi(w, w')
\end{aligned}$$

Using the Projection formula (4.1), we obtain

$$\int_{q_2^{-1}(w)} \left(\int_{r^{-1}(w,w')} \mathbb{1}_{S \times_X S'} \, \mathrm{d}\chi \right) h(q_1(w,w')) \, \mathrm{d}\chi(w,w').$$

By the hypothesis

$$\int_{r^{-1}(w,w')} \mathbb{1}_{S \times_X S'} \, \mathrm{d}\chi = (\mu - \lambda)\mathbb{1}_\Delta(w,w') + \lambda \mathbb{1}_{W \times W}(w,w'),$$

whereas Δ denotes the diagonal in $W \times W$. Therefore,

$$\begin{aligned}
\mathcal{R}_{S'}\mathcal{R}_S h(w) &= (\mu - \lambda) \int_{q_2^{-1}(w)} \mathbb{1}_\Delta(w,w') \cdot h(q_1(w,w')) \, \mathrm{d}\chi(w,w') \\
&\quad + \lambda \int_{q_2^{-1}(w)} \mathbb{1}_{W \times W}(w,w') \cdot h(q_1(w,w')) \, \mathrm{d}\chi(w,w') \\
&= (\mu - \lambda) \int_{\{w\}} h \, \mathrm{d}\chi + \lambda \int_{q_1(q_2^{-1}(w))} h \, \mathrm{d}\chi \\
&= (\mu - \lambda) h(w) + \lambda \left(\int_W h \, \mathrm{d}\chi \right) \mathbb{1}_W(w)
\end{aligned}$$

■

4.5 Bessel transform

The Bessel transform utilizes sets consisting of points equidistant from a fixed point, e.g. in the usual Euclidean norm these sets are concentric spheres. As we will see in the next chapter, the use of different norms can also be useful. Hence, we fix an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^n$. Furthermore, let $S_r(x) = \{y \mid \|y - x\| = r\}$ denote the sphere of radius r centered at x in the chosen norm.

Definition 4.13 For $0 \leq k \leq n-1$ the *lower* and *upper Hadwiger-Bessel transforms* of $h \in \mathrm{Def}(\mathbb{R}^n)$ with respect to μ_k are defined as:

$$\begin{aligned}
\mathcal{B}_k h(x) &= \int_0^\infty \int_{S_r(x)} h \, \lfloor \mathrm{d}\mu_k \rfloor \, \mathrm{d}r, \\
\mathcal{B}^k h(x) &= \int_0^\infty \int_{S_r(x)} h \, \lceil \mathrm{d}\mu_k \rceil \, \mathrm{d}r,
\end{aligned}$$

for all $x \in \mathbb{R}^n$.

Since elements of $\mathrm{Def}(\mathbb{R}^n)$ have compact support, these expressions are well defined. As we will heavily use the Bessel transform of constructible functions with respect to the Euler characteristic, we will use the notation

$$\mathcal{B}h(x) = \int_0^\infty \int_{S_r(x)} h \, \mathrm{d}\chi \, \mathrm{d}r,$$

for $h \in \mathrm{CF}(\mathbb{R}^n)$, which is also called *Euler-Bessel transform.*

For a better understanding consider $h = \mathbb{1}_A$ for a definable set $A \subset \mathbb{R}^n$. The inner integral of the Hadwiger-Bessel transform with respect to μ_k at $x \in \mathbb{R}^n$ then gives $\mu_k(A \cap S_r(x))$ and the outer integral varies the radii of the spheres S_r.

Definition 4.14 A set $A \subseteq \mathbb{R}^n$ is said to be *star-convex* with respect to $x \in A$ if the line segment from x to y is in A for all $y \in A$.

Example 4.15 For any non-empty convex set A the line segment from x to y is in A for all $x, y \in A$. Therefore, every non-empty convex set A is star-convex with respect to any $x \in A$. However, star-convex sets do not have to be convex in general.

For $x \in \mathbb{R}^n$ we denote the distance-to-x function by d_x, that is

$$d_x(y) := \|y - x\|,$$

for all $y \in \mathbb{R}^n$.

Lemma 4.16 *Let $A \subset \mathbb{R}^n$ be a compact n-dimensional submanifold with boundary that is star-convex with respect to some $x \in A$. Then,*

$$\mathcal{B}\mathbb{1}_A(x) = \int_{\partial A} d_x \, \lfloor \mathrm{d}\chi \rfloor.$$

Proof. By the definition of the Euler-Bessel transform

$$\mathcal{B}\mathbb{1}_A(x) = \int_0^\infty \int_{S_r(x)} \mathbb{1}_A \, \mathrm{d}\chi \, \mathrm{d}r = \int_0^\infty \chi(A \cap S_r(x)) \, \mathrm{d}r.$$

Since A is n-dimensional and star-convex with respect to $x \in A$, $A \cap S_r(x)$ is homeomorphic to $\partial A \cap \{d_x \geq r\}$. Using the excursion set representation of the Hadwiger integrals, Equation (2.12), this gives

$$\mathcal{B}\mathbb{1}_A(x) = \int_0^\infty \chi(\partial A \cap \{d_x \geq r\}) \, \mathrm{d}r = \int_{\partial A} d_x \, \lfloor \mathrm{d}\chi \rfloor.$$

■

Remark 4.17 The last result corresponds to Stokes' Theorem because the integral of the distance over ∂A equals the integral of the "derivative" of distance over A.

For general $x \in \mathbb{R}^n$ and sets that are not necessarily star-convex one has to break up the boundary into positively and negatively oriented pieces.

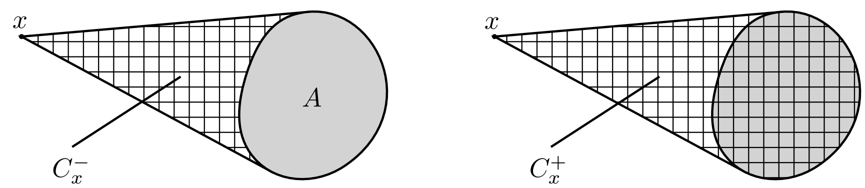


Figure 15: Example for the cones C_x^- and C_x^+ in $\mathbb{R}^2$.

Theorem 4.18 *Let $A \subset \mathbb{R}^n$ be a compact n-dimensional submanifold with boundary. For $x \in \mathbb{R}^n$ decompose ∂A into $\partial A = \partial_x^+ A \cup \partial_x^- A$, where $\partial_x^- A$ denotes the subsets of ∂A on which the outward-pointing halfspaces contain x and $\partial_x^+ A$ stands for the corresponding subsets of ∂A on which the outward-pointing halfspaces do not contain x. Then,*

$$\mathcal{B}\mathbb{1}_A(x) = \int_{\partial_x^+ A} d_x \lfloor \mathrm{d}\chi \rfloor - \int_{\partial_x^- A} d_x \lceil \mathrm{d}\chi \rceil.$$

Proof. Assume that $\partial_x^+ A$ and $\partial_x^- A$ are connected. Then, A can be written as the closure of the difference of the cone at x over $\partial_x^+ A$, denoted by C_x^+, and the cone at x over $\partial_x^- A$, denoted by C_x^-, which is represented in Figure 15. We show the result for this simplified setting, since the case of multiple cones then follows by induction. The important property of the cones is that they are star-convex with respect to x. Hence, they admit a similar analysis as in Lemma 4.16. Using the additive property of the Euler characteristic and its invariance under homeomorphisms one has

$$\begin{aligned}\chi(A \cap S_r(x)) &= \chi(\mathrm{cl}(C_x^+ - C_x^-) \cap S_r(x)) \\ &= \chi(\partial C_x^+ \cap \{d_x \geq r\}) - \chi(\partial C_x^- \cap \{d_x > r\}).\end{aligned}$$

Integration of both sides with respect to $\mathrm{d}r$ and application of the excursion set representations for both the lower and upper Hadwiger integrals, Equations (2.12) and (2.14), gives

$$\mathcal{B}\mathbb{1}_A(x) = \int_{\partial C_x^+} d_x \lfloor \mathrm{d}\chi \rfloor - \int_{\partial C_x^-} d_x \lceil \mathrm{d}\chi \rceil.$$

By a Morse theoretic interpretation of the Euler integral (see Theorem 4 in [3]) only the critical points of d_x contribute to the integrals above. The only critical point of d_x in $\partial C_x^+ - \partial_x^+ A$ and $\partial C_x^- - \partial_x^- A$ is x itself. Since $d_x(x) = 0$, this point does not contribute to the integral. Therefore, one can restrict the integrals to $\partial_x^+ A$ and $\partial_x^- A$ respectively. ∎

Corollary 4.19 *For n even and $A \subset \mathbb{R}^n$ a compact n-dimensional submanifold with boundary,*

$$\mathcal{B}\mathbb{1}_A(x) = \int_{\partial A} d_x \lfloor \mathrm{d}\chi \rfloor.$$

Proof. Since A is n-dimensional and n is even, the dimension of $\partial_x^- A$ is $(n-1)$ which is odd. Therefore, by Theorem 2.21,

$$-\int_{\partial_x^- A} d_x \lceil \mathrm{d}\chi \rceil = \int_{\partial_x^- A} d_x \lfloor \mathrm{d}\chi \rfloor.$$

The result now follows from Theorem 4.18. ■

Remark 4.20 Lemma 4.16, Theorem 4.18 and Corollary 4.19 also hold for more general submanifolds, in particular submanifolds with corners. See [12].

4.6 Fourier transform

Similar to the Hadwiger-Bessel transforms one can define Fourier transforms using integrals with respect to the intrinsic volumes. Like the Hadwiger-Bessel transforms these Fourier transforms are not purely topological except for the Euler case. Furthermore, there is a relation between Hadwiger-Bessel and Hadwiger-Fourier transforms. In order to give a definition, denote by $(\mathbb{R}^n)^*$ the dual space of $\mathbb{R}^n$ and let ξ be a covector in $(\mathbb{R}^n)^*\backslash\{0\}$. For $s \in \mathbb{R}$ the $(n-1)$-dimensional hyperplane orthogonal to ξ at distance $(s/\|\xi\|)$ from the origin is then given by $\xi^{-1}(s)$.

Definition 4.21 For $0 \leq k \leq n-1$ the *lower* and *upper Hadwiger-Fourier transforms* of $h \in \mathrm{Def}(\mathbb{R}^n)$ with respect to μ_k in the direction of $\xi \in (\mathbb{R}^n)^*\backslash\{0\}$ are defined as:

$$\mathcal{F}_k h(\xi) = \int_{-\infty}^{\infty} \int_{\xi^{-1}(s)} h \lfloor \mathrm{d}\mu_k \rfloor \, \mathrm{d}s,$$
$$\mathcal{F}^k h(\xi) = \int_{-\infty}^{\infty} \int_{\xi^{-1}(s)} h \lceil \mathrm{d}\mu_k \rceil \, \mathrm{d}s.$$

In other words, we integrate h over all possible $(n-1)$-dimensional hyperplanes orthogonal to ξ with respect to μ_k. Analogical to the Euler-Bessel transform we use the notation

$$\mathcal{F}h(\xi) = \int_{-\infty}^{\infty} \int_{\xi^{-1}(s)} h \, \mathrm{d}\chi \, \mathrm{d}s,$$

for $h \in \mathrm{CF}(\mathbb{R}^n)$, which is also called *Euler-Fourier transform.*

As the following example shows, the Hadwiger-Fourier transforms with respect to μ_k of the characteristic function of a set A give a *directed* notion of the $(k+1)$-dimensional volume of A.

Example 4.22 Let $A \in \mathcal{K}^n$. The Euler characteristic of any nonempty $(n-1)$-dimensional slice of A is 1, since the slices are also compact convex sets. In other words, for $\xi \in (\mathbb{R}^n)^*\backslash\{0\}$ and $s \in \mathbb{R}$ the integral $\int_{\xi^{-1}(s)} \mathbb{1}_A \, \mathrm{d}\chi$ is either 1 or 0, depending on ξ and s. Therefore, for $\|\xi\| = 1$ the Euler-Fourier transform $\mathcal{F}\mathbb{1}_A(\xi)$ equals the length of the projection of A onto the ξ axis.

Moreover, if one considers the Hadwiger-Fourier transform with respect to μ_{n-1}, the inner integral gives the $(n-1)$-dimensional volume of the slices of A. Hence,

$$\mathcal{F}_{n-1}\mathbb{1}_A(\xi) = \mathcal{F}^{n-1}\mathbb{1}_A(\xi) = \mathrm{vol}_n(A). \tag{4.4}$$

In fact, (4.4) holds for any definable subset of $\mathbb{R}^n$.

Obviously, the difference between the Hadwiger-Fourier and Hadwiger-Bessel transforms is that the former integrates a function over $(n-1)$-dimensional hyperplanes while the latter integrates over spheres. Since functions in $\mathrm{Def}(\mathbb{R}^n)$ have compact support, their intersections with the concentric spheres (with respect to the Euclidean norm) of the Hadwiger-Bessel transforms converge to intersections with parallel hyperplanes as the radii of the spheres increase towards infinity. This gives for $0 \leq k \leq n-1$, $h \in \mathrm{Def}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n \backslash \{0\}$ with dual covector $x^* \in (\mathbb{R}^n)^* \backslash \{0\}$ the following identities:

$$\lim_{\lambda \to \infty} \mathcal{B}_k h(\lambda x) = \mathcal{F}_k h\left(\frac{x^*}{\|x^*\|}\right),$$
$$\lim_{\lambda \to \infty} \mathcal{B}^k h(\lambda x) = \mathcal{F}^k h\left(\frac{x^*}{\|x^*\|}\right).$$

Using this relation between the Hadwiger-Fourier and Hadwiger-Bessel transforms one can proof the following theorem, which resembles Theorem 4.18 and Corollary 4.19.

Theorem 4.23 *Let $A \subset \mathbb{R}^n$ be a compact n-dimensional submanifold with boundary. For $\xi \in (\mathbb{R}^n)^* \backslash \{0\}$ decompose ∂A into $\partial A = \partial_\xi^+ A \cup \partial_\xi^- A$, where $\partial_\xi^- A$ denotes the subsets of ∂A on which ξ points into A and $\partial_x^+ A$ stands for the corresponding subsets of ∂A on which ξ points out of A. Then,*

$$\mathcal{F}\mathbb{1}_A(\xi) = \int_{\partial_\xi^+ A} \xi \lfloor \mathrm{d}\chi \rfloor - \int_{\partial_\xi^- A} \xi \lceil \mathrm{d}\chi \rceil.$$

For n even, this becomes

$$\mathcal{F}\mathbb{1}_A(\xi) = \int_{\partial A} \xi \lfloor \mathrm{d}\chi \rfloor.$$

5 Sensor Networks

One of many applications of Hadwiger integration - especially integration with respect to the Euler characteristic - can be found in sensor networks. In a recent series of papers ([1], [2], [4] and [12]) Baryshnikov and Ghrist together with other contributors gave some results that we want to discuss in this chapter. However, we will only regard mathematical issues and mostly neglect other concerns such as power consumption, sensing complexity, sensor size, sensor range, communication bandwidth, and others.

5.1 Simple enumeration

We begin with a simplified setting where we have a field of infinitesimally small sensors in $\mathbb{R}^2$, which means that there is one sensor for each $x \in \mathbb{R}^2$. Furthermore, there is a finite set of (fixed) *targets* $\{\mathcal{O}_i\}_{i=1}^m \subset \mathbb{R}^2$ that we want to observe. Each sensor $x \in \mathbb{R}^2$ only gives us a quantized count $h(x) \in \mathbb{N}$ that represents the number of targets "nearby". For the start, we assume that this means that each target $\mathcal{O}_i$ is detected on all sensors within Euclidean distance R of $\mathcal{O}_i$. Let U_i denote the support set on which $\mathcal{O}_i$ is detected to obtain

$$h = \sum_{i=1}^{m} \mathbb{1}_{U_i}$$

and furthermore

$$\int_{\mathbb{R}^2} h(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} \sum_{i=1}^{m} \mathbb{1}_{U_i}\,\mathrm{d}x = \sum_{i=1}^{m} \int_{\mathbb{R}^2} \mathbb{1}_{U_i}\,\mathrm{d}x = \sum_{i=1}^{m} R^2\pi = R^2\pi m.$$

Hence, the total number of targets can be computed via

$$m = \#\{\mathcal{O}_i\} = \frac{1}{R^2\pi} \int_{\mathbb{R}^2} h(x)\,\mathrm{d}x.$$

It is easy to see that this method can be applied to arbitrary dimensions and more general target supports, as long as all targets have supports with identical Lebesgue measure. Furthermore, one can discretize the domain to sample h on a finite set, e.g. an appropriate grid.

One of the problems with this method is, that it relies on the assumption that every target has the same support on which it is detected. In a realistic setting the sensors use optical, acoustic, infrared or other methods for detection. Especially when one has different kinds of targets this leads to different kinds of support sets. For example, when counting the number of vehicles in an area one would expect a difference between the support set of a SUV and the support set of a bicycle. It turns out that Euler integration is a useful tool for this purpose. For that, let W denote the *target space* which models the domain in which the targets $\mathcal{O}_i$ lie. In the setting above we used $W = \mathbb{R}^2$. Furthermore, the collection of sensors is parametrized in a *sensor space* $X \subset W$. E.g. the targets are

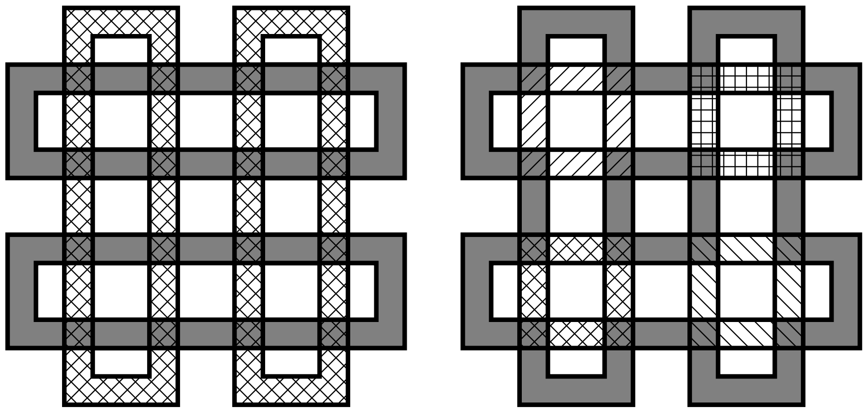


Figure 16: The same height function h arises from different numbers of annuli.

in a 3-dimensional room and the sensors are located along a 2-dimensional wall, so $W = \mathbb{R}^3$ and $X \cong \mathbb{R}^2$. In this setting each target $\mathcal{O}_i$ is detected on its *target support* which we assume to be a definable set

$$U_i = \{x \in X \mid \text{the sensor at } x \text{ detects } \mathcal{O}_i\}.$$

Again, each sensor $x \in X$ only senses the number of targets in range which gives us a constructible *height function*

$$h(x) := \#\{i \mid x \in U_i\}.$$

If we furthermore assume that all targets have the same nonzero Euler characteristic, e.g. the target supports are compact convex sets, the problem of counting the total number of targets can be solved easily.

Theorem 5.1 *Let $h : X \to \mathbb{N}$ be the counting function for a collection of definable target supports $\{U_i\}_{i=1}^m \subset X$ that satisfy $\chi(U_i) = N \neq 0$ for all $i = 1, \ldots, m$. Then*

$$m = \frac{1}{N} \int_X h \, \mathrm{d}\chi.$$

Proof. Integration of h with respect to χ gives

$$\int_X h \, \mathrm{d}\chi = \int_X \sum_{i=1}^m \mathbb{1}_{U_i} \, \mathrm{d}\chi = \sum_{i=1}^m \int_X \mathbb{1}_{U_i} \, \mathrm{d}\chi = \sum_{i=1}^m \chi(U_i) = Nm.$$

■

Remark 5.2 The assumption $\chi(U_i) \neq 0$ seems to be necessary as no solution is possible in general for target supports U_i with $\chi(U_i) = 0$. See Figure 16 for an example where the same height function is given by different numbers of targets. The problem is that the annuli have Euler characteristic zero.

Remark 5.3 Using equation (2.7) the computation of the Euler integral of the height function is relatively easy since h only attains values greater or equal to zero:

$$\int h\,\mathrm{d}\chi = \sum_{s=0}^{\infty} \chi\{h > s\} - \chi\{h < -s\} = \sum_{s=0}^{\infty} \chi\{h > s\}.$$

Furthermore, if h is the sum of indicator functions over compact sets the excursion sets of h are compact for all s.

5.2 From fields to networks

Theorem 5.1 depends on having a *field* of sensors. Of course, any realistic implementation can only provide a discrete collection of sensors, also called sensor *network* with nodes (sensors) $\mathcal{N}$. The question is, if there is a result similar to Theorem 5.1 that works in the setting of a sensor network. One could try to parametrize the sensor space X as a discrete set based on $\mathcal{N}$. However, Baryshnikov and Ghrist point out that this method fails, since the target supports will be likewise discrete and of unknown and non-uniform Euler characteristic [2].

Another approach is to model the sensor space X as a simplicial approximation of the target space W, using $\mathcal{N}$ as vertices. Therefore, we must assume that enough structure about $\mathcal{N}$ is known in order to find a suiting simplicial structure. Furthermore, instead of the height function h we consider the piecewise-linear (PL) interpolation h_{PL} of h based on the values of h at $\mathcal{N}$.

Theorem 5.4 *Let $h : \mathbb{R}^n \to \mathbb{N}$ be an upper semi-continuous constructible function such that every upper excursion set of h is the closure of its interior in $\mathbb{R}^n$,*

$$\{h \geq s\} = \mathrm{cl}(\mathrm{int}(\{h \geq s\})).$$

Then, for a sufficiently dense and regular triangulation of $\mathbb{R}^n$, the piecewise-linear interpolation h_{PL} of h over the vertex set of the triangulation satisfies

$$\int_{\mathbb{R}^n} \lfloor h_{PL} \rfloor \,\mathrm{d}\chi = \int_{\mathbb{R}^n} h\,\mathrm{d}\chi.$$

Proof. Let σ be a simplex of a sufficiently dense and regular triangulation and $\Delta = \mathrm{cl}(\sigma)$ its closure, such that for the restriction of h to Δ, $h|_\Delta$, one has

- $\max h|_\Delta$ is attained at some vertex of Δ,
- all upper excursion sets of $h|_\Delta$ are contractible.

Then

$$\int_\Delta h\,\mathrm{d}\chi = \max_\Delta h = \int_\Delta \lfloor h_{PL} \rfloor \,\mathrm{d}\chi.$$

By additivity of $\int \mathrm{d}\chi$ this extends to $\mathbb{R}^n$, which completes the proof. ■

Remark 5.5 Since the integral with respect to the Euler characteristic is purely topological, it is not necessary to know the coordinates of the nodes $\mathcal{N}$ in order to evaluate $\int_{\mathbb{R}^n} \lfloor h_{PL} \rfloor \,\mathrm{d}\chi$. However, if not enough geometry is associated to $\mathcal{N}$, it might be impossible to determine h_{PL} based on its values on $\mathcal{N}$.

Using a different extension of the integral with respect to the Euler characteristic for continuous functions, Baryshnikov and Ghrist show that the integral of h_{PL} itself is equal to the integral of h under suitable conditions [1]. Moreover, they discuss problems that can occur with various target supports and samplings of the space. Additionally, a numerical analysis in $\mathbb{R}^2$ is provided by Krupa in [19].

5.3 Further enumeration problems

Moving targets. We now assume that each of the targets $\{\mathcal{O}_i\}_{i=1}^m$ is moving along a continuous path $\mathcal{O}_i(t)$ with target support $U_i(t)$ in the domain $W \subset \mathbb{R}^n$ during the time span $[t_0, t_1]$. Furthermore, we assume that each sensor detects targets that are within its range. Every time the number of targets within the proximity range of a sensor increases the sensor's internal counter increments. This is represented by a height function

$$h(x) := \#\{(t,i) \mid t \in [t_0, t_1],\ x \in U_i(t+\varepsilon) \text{ and } x \notin U_i(t-\varepsilon) \text{ for } \varepsilon \to 0^+\},$$

whereas we set $U_i(t) = \emptyset$ for $t < t_0$ and $U_i(t) = U_i(t_1)$ for $t > t_1$. With the help of Fubini's theorem (Theorem 4.2) one can solve the problem to compute the number of targets solely from h.

Theorem 5.6 *Under the assumptions above, the number of targets can be computed from the height function h via*

$$m = \int_W h \,\mathrm{d}\chi.$$

Proof. We consider the sensor space $X = W \times \mathbb{R}$ as the product of the target space W with the time. Furthermore let $F : X \to W$ denote the projection on W. The target supports in X are the traces

$$U_{i,X} := \bigcup_{t \in [t_0, t_1]} U_i(t) \times \{t\}.$$

Since the targets move on continuous paths, these are contractible sets with Euler characteristic 1. Now let $g : X \to \mathbb{N}$ with $g = \sum_{i=1}^m \mathbb{1}_{U_{i,X}}$ be the corresponding height function in X. By Theorem 5.1 and Fubini's theorem the number of targets computes as

$$m = \int_X g \,\mathrm{d}\chi = \int_W F_* g \,\mathrm{d}\chi,$$

with $(F_* g)(w) = \int_{F^{-1}(w)} g \,\mathrm{d}\chi$. Each of the intersections $F^{-1}(w) \cap U_{i,X}$ is a finite number of compact intervals, with every interval representing a time when w

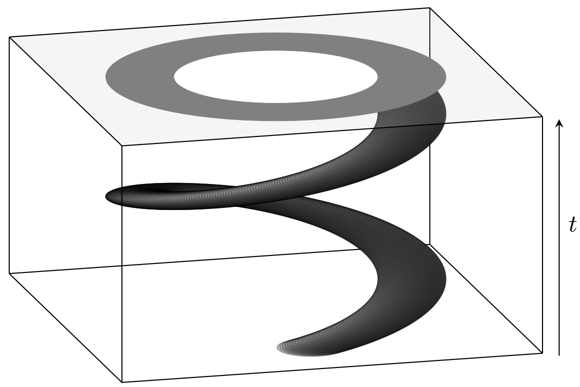


Figure 17: The trace of a target that moves in circles with sufficiently large radius becomes an annulus with Euler characteristic zero.

"enters" $U_i(t)$. This shows that $h = F_* g$ and furthermore

$$m = \int_X g \, \mathrm{d}\chi = \int_W h \, \mathrm{d}\chi.$$

■

Remark 5.7 Note that the trace of each target $\mathcal{O}_i$, represented by the union of the temporal supports $\bigcup_{t \in [t_0, t_1]} U_i(t)$, can be a non-contractible set. For example if an object is moving in circles with sufficiently large radius the trace becomes an annulus with Euler characteristic zero. However, the corresponding union in the sensor space X is contractible, which is a crucial property. See also Figure 17.

Beam sensors. In this setting we consider fixed targets $\{\mathcal{O}_i\}_{i=1}^m$ in a Euclidean target space in $\mathbb{R}^n$ but sensors with some degree of freedom. Again, there is a sensor node for each $x \in \mathbb{R}^n$ but the targets are sensed via a round k-dimensional ball in $\mathbb{R}^n$, centered at x. In the case $k = 1$ this would be a classical "beam". Each target $\mathcal{O}_i$ is associated with a certain region of brightness, represented by some convex neighborhood U_i of $\mathcal{O}_i$. The sensors perform "sweeps" of their k-ball beams, meaning that they sense over all different possible k-balls. At each sensing, the number of intensity regions U_i within the k-ball are counted.

In this setting the sensor field is parametrized over the Grassmannian bundle $\mathrm{Gr}_k(\mathbb{R}^n) = \mathbb{R}^n \times \mathrm{Gr}(n, k)$. The internal counters of the sensors then give a height function $h : \mathrm{Gr}_k(\mathbb{R}^n) \to \mathbb{N}$.

Theorem 5.8 *In the setting above with the additional assumption that if n is even then so is k, the number of targets can be computed from the height function h via*

$$m = \frac{\lfloor \frac{n-k}{2} \rfloor ! \lfloor \frac{k}{2} \rfloor !}{\lfloor \frac{n}{2} \rfloor !} \int_{\mathrm{Gr}_k(\mathbb{R}^n)} h \, \mathrm{d}\chi.$$

Proof. For given target supports U_i in $\mathbb{R}^n$ one has to figure out the corresponding target supports $\tilde{U}_i$ in $\mathrm{Gr}_k(\mathbb{R}^n)$. For that, fix an $i \in \{1, \dots, m\}$ and a k-plane in $\mathrm{Gr}(n, k)$. The nodes in $\mathbb{R}^n$ that can sense U_i with their k-balls in the given k-plane form a star-convex set with respect to the centroid of U_i. Since star-convex sets are contractible, the target support $\tilde{U}_i$ is homeomorphic to $\mathrm{Gr}(n, k)$. By Theorem 5.1 the result follows from the Euler characteristic of $\mathrm{Gr}(n, k)$ which is

$$\chi(\mathrm{Gr}(n,k)) = \begin{cases} 0 & \text{if } n \text{ is even and } k \text{ is odd} \\ \binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor} & \text{else.} \end{cases}$$

■

Sweeping sensors. Again we consider fixed targets $\{\mathcal{O}_i\}_{i=1}^m$ in a Euclidean target space in $\mathbb{R}^n$. Each sensor in the sensor space $X \subset \mathbb{R}^n$ returns a piecewise-constant function $h_x : \mathbb{S}^{n-1} \to \mathbb{N}$ that indicates how many targets can be seen in a certain direction. In this setting a sensor at location $x \in X$ "looking" in direction $v \in T_x^1\mathbb{R}^n \cong \mathbb{S}^{n-1}$ scans a compact cone at x centered on v, whereas $T_x^1\mathbb{R}^n$ denotes the unit tangent sphere at x. For each x and v the shape of the scanning cone is the same. This gives a collection of functions $h = \{h_x \mid x \in X\}$.

Theorem 5.9 *Let Φ_n be the operator that "fixes" all removable points of discontinuity of a function. Then, the number of targets in the setting above is equal to*

$$m = \int_X \Phi_n \left(\int_{T_x^1\mathbb{R}^n} h_x \,\mathrm{d}\chi \right) \mathrm{d}\chi(x). \tag{5.1}$$

Proof. We will show that the target supports $\tilde{U}_i \subset T^1\mathbb{R}^n$ are contractible and that $\Phi_n \left(\int_{T_x^1\mathbb{R}^n} h_x \,\mathrm{d}\chi \right)$ returns the correct number of targets within a certain range of x. The result then follows from Theorem 5.1.

Consider a point $x \in X \backslash \{\mathcal{O}_i\}_{i=1}^m$. Since the scanning cone is convex, the angular support for each target $\mathcal{O}_i$ in reach is a convex subset of $T_x^1\mathbb{R}^n$ with Euler characteristic 1. Therefore, $\int_{T_x^1\mathbb{R}^n} h_x \,\mathrm{d}\chi$ returns the correct number of targets visible from x. However, if x coincides with a target $\mathcal{O}_i$, then the target is visible during the entire sweep and the target support in $T_x^1\mathbb{R}^n$ is the entire $(n-1)$-sphere, which contributes an error of $1 - (-1)^n$ in the integral. The operator Φ_n now wipes out such defects, since they only occur when the coordinates of a sensor and a target coincide.

For the outer integral, choose a target $\mathcal{O}_i$ and fix a pair $x \in U_i$, whereas U_i denotes the spatial part of $\tilde{U}_i$. This means that there is a bearing vector $v \in T_x^1\mathbb{R}^n$ such that $\mathcal{O}_i$ lies in the cone at x centered on v. Since all cones have the same shape, independent of x and v, this implies that $y \in U_i$ for all y on the line segment from x to O_i. Hence, the spatial target support is star-convex. This implies that each target exactly counts 1 in the outer integral in (5.1), which concludes the proof. ■

Remark 5.10 For very thin and short scanning cones one can argue that it rarely happens that more than one target is in a cone. Thus, $\chi(h_x^{-1}(1))$ would give the correct number of targets within a certain range of the sensor and the problem can be solved with the help of Theorem 5.1.

Remark 5.11 In a realistic setting with a sensor network instead of a sensor field it is not possible in general to determine removable points of discontinuity of the inner integral in (5.1). However, the case that a sensor coincides with the target should rarely happen. E.g. if the sensors are placed on the ceiling of a room and swipe in a circle, a target would have to be directly under the sensor to produce an erroneous count in the integral.

5.4 Target localization

It is possible to utilize various integral transforms in order to localize targets. However, each of the methods presented in this section involves certain restrictions.

Bessel transform. Consider a setting of fixed targets $\{\mathcal{O}_i\}_{i=1}^m$, a field of sensors X and a corresponding counting function $h \in \mathrm{CF}(X)$ similar to the assumptions of Theorem 5.1. However, assume for the beginning that all target supports U_i are round balls of arbitrary size. With the help of the Euler-Bessel transform one can reveal the exact locations of the targets.

Proposition 5.12 *Let A be a compact ball of radius R centered at $p \in \mathbb{R}^{2n}$. The Euler-Bessel transform (with respect to the Euclidean norm) $\mathcal{B}\mathbb{1}_A$ is a non-decreasing function in the distance to p, having a unique zero point at p.*

Proof. By Corollary 4.19 one has

$$\mathcal{B}\mathbb{1}_A(x) = \int_{\partial A} d_x \, \lfloor \mathrm{d}\chi \rfloor,$$

which computes to

$$\mathcal{B}\mathbb{1}_A(x) = \max_{\partial A} d_x - \min_{\partial A} d_x.$$

For $x \notin A$ this expression is always $\mathrm{diam}(A) = 2R$. For $x \in A$ this is monotone in the distance to p and only admits zero if x equals p. ■

Remark 5.13 One could think that a result similar to Proposition 5.12 also works for balls in odd dimension by using Theorem 4.18. However, the Euler-Bessel transform of a ball in $\mathbb{R}^{2n+1}$ is constant and gives no further information on a target's location. See also [12].

By Proposition 5.12 the minima of the Euler-Bessel transform can reveal target locations. However, if the target supports overlap it can happen that interferences do not allow to identify unique minima or even create ghost minima. In such cases it can be helpful to determine the number of targets, using the height function $h \in \mathrm{CF}(X)$ and further the integral $\int_X h \, \mathrm{d}\chi$. The actual number of targets provides a clue as to how many of the deepest local minima should be interrogated.

Further use of this technique suggests to change the norm in the Euler-Bessel transform in order to revel target supports of different shapes. For example the unit sphere in $\mathbb{R}^2$ with respect to $\|\cdot\|_\infty$ has the shape of a rectangle. Using different norms in the Euler-Bessel transform also contrasts the situation when interferences and ghost minima occur. Sometimes however, no single norm is optimal for the situation, especially if there are multiple targets. In such cases, *SVA, Spatially Variant Apodization*, a technique that is used in radar processing, seems to be useful. For that, consider a parametrized family of norms $\|\cdot\|_\alpha$, $\alpha \in \mathcal{A}$. The SVA Euler-Bessel transform as proposed by Ghrist and Robinson in [12] is then defined as

$$\mathcal{B}_{SVA} h(x) = \inf_{\alpha \in \mathcal{A}} \int_0^\infty \int_{S_{r,\alpha}(x)} h \, \mathrm{d}\chi \, \mathrm{d}r,$$

for $h \in \mathrm{CF}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $S_{r,\alpha}(x)$ denoting the sphere of radius r centered at x with respect to $\|\cdot\|_\alpha$. E.g. the family of norms could describe a cyclic family of rotated $\|\cdot\|_\infty$ norms in order to identify multiple targets with rectangular supports and various orientation.

Subsequently, Wright suggests in [31] that other Hadwiger-Bessel transforms could offer information about size and shape of the targets.

Radon transform. For a variety of settings, Schapira's inversion formula (Theorem 4.12) can be used to localize targets. For that, consider a target space W and a sensor space X. Define the *sensor relation* S as follows:

$$S := \{(w, x) \in W \times X \mid \text{the sensor at } x \text{ can sense a target at } w\}.$$

The vertical fibers $S_w, w \in W$ then represent the target supports, that is for each w the set of sensors in X that can sense a target at w. The horizontal fibers $S_x, x \in X$ are the sensor supports - the set of all targets in W that can be sensed by the sensor at x. Note, that this implies that the target supports do not depend on the targets themselves but on their locations. Moreover, consider a finite set of targets $T \subset W$. When each sensor counts the targets in sight, the resulting height function h equals the Radon transform $\mathcal{R}_S \mathbb{1}_T$. Therefore, if one can find a proper set $S' \subset X \times W$ such that the conditions of Theorem 4.12 are met and $\lambda \neq \mu$, the inverse Radon transform $\mathcal{R}_{S'} h = \mathcal{R}_{S'} \mathcal{R}_S \mathbb{1}_T$ is equal to a multiple of $\mathbb{1}_T$ plus a multiple of $\mathbb{1}_W$. This reveals the exact shape of T and therefore the location of the targets. In contrast to Proposition 5.12, this

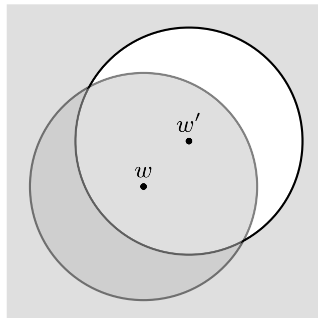


Figure 18: The intersection $S_w \cap S'_{w' \neq w}$ is homeomorphic to a closed disk and therefore has Euler characteristic 1.

method also works in odd-dimensional spaces. However, it can be difficult to find proper settings that satisfy the necessary conditions.

Example 5.14 Let $X = W = \mathbb{R}^n$ and let each sensor at $x \in \mathbb{R}^n$ detect targets within a closed ball about x. Each horizontal fiber $S_w, w \in \mathbb{R}^n$ is then a closed ball about w. Furthermore, let each S'_w be the closure of the complement of S_w in $\mathbb{R}^n$. This means, that the inverse sensor relation S' assigns the sensors with the targets out-of-range. For each w the set $S_w \cap S'_w$ is homeomorphic to $\mathbb{S}^{n-1}$. For $w' \neq w$ the set $S_w \cap S'_{w'}$ is homeomorphic to a closed ball, as depicted in Figure 18 for $\mathbb{R}^2$. Therefore, $\chi(S_w \cap S'_w) = 1$ and $\chi(S_w \cap S'_{w' \neq w}) = \chi(\mathbb{S}^{n-1}) \neq 1$, which allows reconstruction of T from h by Theorem 4.12.

Example 5.15 Let W be the open unit disc in $\mathbb{R}^n$ and let $\partial W = \mathbb{S}^{n-1}$ be filled with sensors. Each of the sensors sweeps a ray over W and counts the number of targets that intersect the beam. Each bearing of a ray at a point $p \in \partial W$ lies in the open hemisphere of the unit tangent sphere $T^1_p W$. Since this open hemisphere projects to the open unit disc in $T_p \partial W$, the sensor space X is homeomorphic to the tangent bundle of ∂W, $T\mathbb{S}^{n-1}$. Any target $w \in W$ can be detected by any sensor in ∂X under a unique angle. Therefore, the target supports S_w of the sensor relation S are spheres with Euler characteristic $1 + (-1)^{n-1}$. Furthermore, for two different targets $w \neq w' \in W$, there are only two sensors with certain bearing angles in $S_w \cap S_{w'}$. These sensors can be found in the intersection of ∂W with the straight line that connects w with w'. Hence, if

$$S' := \{(x, w) \in X \times W \mid (w, x) \in S\},$$

one has $\chi(S_w \cap S'_w) = 1 + (-1)^{n-1}$ and $\chi(S_w \cap S_{w' \neq w}) = 2$. For n even, Theorem 4.12 implies that the inverse Radon transform is well defined. However, for n odd, Schapira's inversion formula doesn't seem to be helpful. In fact, for odd-dimensional scenarios an inverse Radon transform might be impossible. E.g. for $n = 1$ you cannot localize a target along an open interval only from sensor readings at the two boundary points.

Duality. Duality allows to count and even localize annular target supports. This can be especially useful in even-dimensional spaces since even-dimensional annuli have Euler characteristic zero and one cannot apply Theorem 5.1. However, all target supports must have the same shape. E.g. the targets are beacons that can only be detected when the sensor is close but not too close to the target. For the shape of the target supports let $I \subset O \subset \mathbb{R}^n$ be convex sets containing 0, such that I is open, O is closed and compact, and the closure of I is contained in the interior of O. The target supports shall have the shape of the annular region $A := O \backslash I$. For a finite set of targets T, the height function returned by the sensor field is then given by $h = \mathbb{1}_A * \mathbb{1}_T$. Now consider for $N \in \mathbb{N}$,

$$\Psi_N := -\mathcal{D}\mathbb{1}_{-I} * \sum_{k=0}^{N-1} (\mathbb{1}_O * \mathcal{D}\mathbb{1}_{-I})^{*k}.$$

By the definition of A, one has $\mathbb{1}_A = \mathbb{1}_O - \mathbb{1}_I$. Furthermore, by Lemma 4.8

$$\mathbb{1}_I * \mathcal{D}\mathbb{1}_{-I} = \mathbb{1}_{\{0\}}.$$

This gives

$$\begin{aligned}
\mathbb{1}_A * \Psi_N &= -\mathbb{1}_A * \mathcal{D}\mathbb{1}_{-I} * \sum_{k=0}^{N-1} (\mathbb{1}_O * \mathcal{D}\mathbb{1}_{-I})^{*k} \\
&= (\mathbb{1}_I - \mathbb{1}_O) * \mathcal{D}\mathbb{1}_{-I} * \sum_{k=0}^{N-1} (\mathbb{1}_O * \mathcal{D}\mathbb{1}_{-I})^{*k} \\
&= (\mathbb{1}_{\{0\}} - \mathbb{1}_O * \mathcal{D}\mathbb{1}_{-I}) * \sum_{k=0}^{N-1} (\mathbb{1}_O * \mathcal{D}\mathbb{1}_{-I})^{*k} \\
&= \sum_{k=0}^{N-1} (\mathbb{1}_O * \mathcal{D}\mathbb{1}_{-I})^{*k} - (\mathbb{1}_O * \mathcal{D}\mathbb{1}_{-I})^{*(k+1)} \\
&= \mathbb{1}_{\{0\}} - (\mathbb{1}_O * \mathcal{D}\mathbb{1}_{-I})^{*N}.
\end{aligned}$$

For $J = \mathrm{cl}(-I)$, one has $\mathcal{D}\mathbb{1}_{-I} = (-1)^n \mathbb{1}_J$. Therefore,

$$\mathbb{1}_A * \Psi_N = \mathbb{1}_{\{0\}} - (-1)^{Nn} (\mathbb{1}_O * \mathbb{1}_J)^{*N}.$$

Lemma 4.5 implies that for sufficiently large N $(\mathbb{1}_O * \mathbb{1}_J)^{*N}$ is equal to 1 on any fixed compact set K. In particular, if h is supported on K, one obtains

$$(h * \Psi_N)(x) = (\mathbb{1}_T * \mathbb{1}_A * \Psi_N)(x) = \mathbb{1}_T(x) - (-1)^{Nn} \chi(T),$$

for all $x \in K$. This allows to count and localize the targets. Note, that this does not solve the problem depicted in Figure 16, since one must know the shape of the annulus A in order to construct Ψ_N.

6 Further Applications

Apart from the usefulness of Euler integration in the theory of sensor networks there are several other applications of Hadwiger integration.

Image processing. Automated processing and analysis of images from a variety of sources is a central task in computer science. The intrinsic volumes seem to be a helpful tool in order to highlight features of images. In [27] binary images (black and white pixels) are analyzed with intrinsic volumes, e.g. in order to obtain the length of a fibrous structure in a computer tomography image. Wright points out that this corresponds to computing Hadwiger integrals of characteristic functions and suggests to extend the theory to grayscale images and thus constructible functions [31]. Furthermore, convolution of those functions with an appropriate kernel could be used to smooth out certain types of noises so that a continuous function is obtained. Moreover, since a color pixel can be described by its red, green and blue intensities, one could study color images and employ Hadwiger integration for functions with values in $\mathbb{R}^3$. Further research on how to compute intrinsic volumes from image data can be found in [17], [21] and more recently [28].

Poincaré series. In algebraic geometry the *Poincaré series* of a so-called *multi-index filtration*, a certain family $\{J(v)\}, v \in \mathbb{Z}^r_{\geq 0}$, can be defined as

$$P(t) = \sum_{v \in \mathbb{Z}^r_{\geq 0}} \left(\sum_{I \subset \{1,2,\ldots,r\}} (-1)^{|I|} \dim(J(v + \underline{1}_I)/J(v + \underline{1})) \right) \prod_{i=1}^{r} t_i^{v_i},$$

for all $t = (t_1, \ldots, t_r) \in \mathbb{Z}^r$, where $\underline{1}$ denotes the element $(1, \ldots, 1) \in \mathbb{Z}^r$ and $\underline{1}_I$ the element of $\{0,1\}^r$ the ith component of which is equal to 1 if $i \in I$. In [9] and subsequently [14] a generalization of the integral with respect to the Euler characteristic is introduced that allows integration over infinite-dimensional spaces of arcs and functions - motivic integration. This enables computation of the Poincaré series of a multi-index filtration via

$$P(t) = \int_{\mathbb{P}\mathcal{O}_{V,0}} \prod_{i=1}^{r} t_i^{v_i} \, \mathrm{d}\chi,$$

whereas $\mathbb{P}\mathcal{O}_{V,0}$ is a proper space with values in the Abelian group $\mathbb{Z}[[t_1, \ldots, t_r]]$ of power series in variables $t_1, \ldots, t_r$ with integer coefficients. Further applications of this extended integral include computation of monodromy zeta-functions and generating series of classes of some moduli spaces.

Gaussian random fields. A *random field* is a stochastic process, that can be thought of as a function on a topological space whose value at any point is a random variable in $\mathbb{R}^k$. A *Gaussian random field* yields some additional regularity and is therefore completely determined by its mean and covariance functions. For example one can model the measured temperature T at a position p in a room $M \subset \mathbb{R}^3$. Since measurement always involves some error, one can write $T(p) = u(p) + f(p)$, where u is the actual temperature and f represents the error that occurs with the measurement. Since this error can be thought of as a Gaussian random variable, the function f can be understood as a Gaussian random field. Bobrowksi and Strom Borman give a simple closed form expression of the expected Euler integral of a Gaussian random field which can be used for a quantitative description of the persistent homology of the field [6]. Wright gives a formula for general Hadwiger integrals and points out that this information can be helpful in order to explain the contribution of noise in certain situations [32].

Morse theory. Morse theory is used to analyze the topology of a manifold M by studying differentiable functions on that manifold. Since integration with respect to the Euler characteristic is purely topological, there is a natural connection to between Euler integrals and Morse theory. For that, let $\mathcal{C}$ denote the set of critical points of a continuous and definable function h on a definable space X. For $p \in \mathcal{C}$ define the *co-index of* p as

$$\mathcal{I}^*(p) = \lim_{\varepsilon' \ll \varepsilon \to 0^+} \chi(B_\varepsilon(p) \cap \{h > h(p) - \varepsilon'\}),$$

whereas $B_\varepsilon(p)$ denotes the closed ball in X of radious ε centered at p. Then,

$$\int_X h \lfloor \mathrm{d}\chi \rfloor = \int_{\mathcal{C}} h\mathcal{I}^* \,\mathrm{d}\chi.$$

Thus, $\lfloor \mathrm{d}\chi \rfloor$ is concentrated on the critical points of a function. Furthermore, a *Morse function* is a smooth real-valued function on M that has no degenerate critical points, which are critical points where the Hessian is singular. Suppose now that h is a Morse function on a closed n-manifold M where every critical point $p \in \mathcal{C}$ is assigned a *Morse index* $\iota(p)$, the number of negative eigenvalues of the Hessian at p. Then,

$$\int_M h \lfloor \mathrm{d}\chi \rfloor = \sum_{p \in \mathcal{C}} (-1)^{n-\iota(p)} h(p).$$

This interpretation results in simple computation of Euler integrals and consequent theorems. See [3] and [8].

List of Figures

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