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Regularizing Cross-Diffusion in the Two-Dimensional Keller–Segel Model with Superlinear Production

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Abstract

In this thesis we consider the two-dimensional Keller–Segel model with a superlinear production term. This means we investigate the following nonlinear system of partial differential equations

$$n_t = \operatorname{div}(\nabla n - n\nabla S)$$
$$\gamma S_t = \Delta S + \delta\Delta n + n^\alpha - S$$

in a domain Ω and times $t > 0$, with homogeneous Neumann boundary conditions

$$\nabla n \cdot \nu = \nabla S \cdot \nu = 0$$

on the parabolic boundary $\partial\Omega$, $t > 0$; and initial conditions $n(0, \cdot) = n_0(\cdot)$, $\gamma S(0, \cdot) = \gamma S_0(\cdot)$ in Ω.

Our goal is to prove that weak solutions to this problem exist globally in time (Section 3) and that solutions are bounded for all time (Section 4). To this end we rely heavily on the use of Bochner and Sobolev spaces (see Section 2). Our main strategy involves an implicit Euler discretization in time, and adding regularizing 4-th order terms. In the end we apply compactness arguments to justify the limit of vanishing regularization and discretization parameters.
For the global boundedness result, we perform a change of variables and make use of elliptic and parabolic regularity theorems to derive the necessary estimates. Under additional regularity assumptions we also show that smooth solutions exist based on a bootstrapping argument (Section 4).
We conclude by proving that solutions are unique, i.e., under suitable regularity assumptions on solutions we prove that at most one solution can exist. This is done separately for either the parabolic-parabolic or parabolic-elliptic model (Section 5).
In the Appendix (Section 6) we comprise theorems and lemmata used in this thesis, in applicable form and unified notation.

Kurzzusammenfassung

In dieser Arbeit geht es um das zweidimensionale Keller–Segel–Modell, wobei wir einen superlinearen Produktionsterm erlauben. Das bedeutet, dass wir das folgende nichtlineare System partieller Differentialgleichungen betrachten

$$n_t = \operatorname{div}(\nabla n - n\nabla S)$$
$$\gamma S_t = \Delta S + \delta \Delta n + n^\alpha - S$$

in einem Gebiet Ω und Zeiten $t > 0$. Wir verwenden außerdem homogene Neumann-Randbedingungen

$$\nabla n \cdot \nu = \nabla S \cdot \nu = 0$$

für den parabolischen Rand $\partial\Omega$, $t > 0$. Damit das Problem vollständig ist, stellen wir noch die Anfangsbedingungen $n(0,\cdot) = n_0(\cdot)$, $\gamma S(0,\cdot) = \gamma S_0(\cdot)$ in Ω.

Das Ziel dieser Arbeit ist, zu zeigen, dass schwache Lösungen dieses Systems existieren – und zwar für alle Zeiten (Section 3). Des Weiteren zeigen wir, dass für das parabolisch-elliptische System ($\gamma = 0$) Lösungen beschränkt bleiben (Section 4). Dabei ist unsere Haupttherangehensweise, dass wir die funktionalanalytischen Eigenschaften von Bochner– und Sobolev–Räumen (siehe Section 2) ausnutzen, wodurch wir Kompaktheitsresultate anwenden können. Wir lösen ein approximierendes Problem, bei dem wir das implizite Euler–Verfahren für die Zeitableitung verwenden; und zusätzlich regularisierende Terme vierter Ordnung hinzufügen. Zum Schluss argumentieren wir mit schwacher und starker Kompaktheit, um so den Grenzwert für verschwindende Regularisierungs– und Diskretisierungsparameter zu rechtfertigen.
Um zu zeigen, dass Lösungen global beschränkt sind, verwenden wir eine Variablentransformation und stützen uns auf Regularitätsresulate für parabolische und elliptische Gleichungen, um die notwendigen Abschätzungen zu erhalten. Unter einer zusätzlichen Regularitätsannahme an die Daten zeigen wir, dass glatte Lösungen existieren. Dafür verwenden wir Bootstrapping (Section 4).
Zum Schluss zeigen wir noch Eindeutigkeit der Lösungen. Konkret heißt das, dass unter gewissen Regularitätsannahmen an Lösungen nur höchstens eine Lösung existieren kann. Für den Beweis unterscheiden wir zwischen dem parabolisch-parabolischen und parabolisch-elliptischen Modell (Section 5).
Im Anhang (Section 6) sind die wichtigsten Sätze und Lemmata aufgelistet, die wir verwenden, in der Form, in der wir sie brauchen, und in entsprechender Notation.

Contents

1 Introduction

1.1 Chemotaxis and its role in science

In this thesis we consider a system of partial differential equations which model chemotaxis. Chemotaxis describes the directed movement of cells based on chemical gradients, i.e., if there is more of a chemical substance at a certain point, then cells are attracted (or repelled) by it and will move in response. In our model the specific substance is also produced by the cells themselves.
This phenomenon can be observed, e.g., in slime mold (a general term for various organisms that can live as single cells as well as form multicellular structures), see [17]. In order to communicate between cells they will produce a certain chemoattractant which will attract more cells. This can be used for reproductive purposes. After reproduction, cells might use a similar, but opposite, mechanism to disperse again, governed by chemotaxis. Thus, chemotaxis models play an important role in biomechanics.
It has also been suggested that chemotaxis is of great importance when studying cancer cells. The idea is that single cancer cells are not as harmful as clusters which may migrate through the body and form metastases [4]. An understanding of how such cells form aggregates based on chemoattractants is thus also important in the study of breast cancer.

1.2 History and derivation of the model

The name of the (Patlak–)Keller–Segel model goes back to the works of Clifford Patlak [22] in 1953, and Evelyn Keller and Lee Segel [17] in 1969. In his work, Patlak considers applications of multi particle random walks where movement is driven by not necessarily independent randomness, in order to derive partial differential equations describing the number of particles at a given point and time. During his derivation there are many assumptions on certain effects being negligible (like particles' interaction with each other, or slow changes of quantities – which justifies truncation in Taylor series), but, as the author points out himself, for some real life experiments his equations did not predict the actual behaviour correctly. These errors were attributed to certain (non-mathematical) assumptions in the derivation which did not apply to the given experiment – such as the motion of solvents surrounding particles or movement induced by heat. So his work still laid the foundation for future research.

Later on, Keller and Segel wrote a paper [17] on the aggregation of amoebae, where chemotactic interaction is induced by acrasin (a chemical messenger). In their work the authors derive a model for movement of cells driven by chemotaxis, i.e., the cells are attracted (or repelled) by the chemotactic agent. Additionally, the cells themselves produce this agent. The authors then derive a coupled system of partial differential equations which models the densities (as opposed to numbers) of cells and the chemical. They even include another factor of a second chemical dissolving the messenger agent. Their main finding is that (under certain conditions) cells will refrain from a uniform distribution over a given area, and instead start aggregating. This is remarkable because one would intuitively think that having no gradients in density at all would be a stable steady state of the system, i.e., with everything spread out equally there would be no need for movement.

The first rigorous derivation of the Patlak–Keller–Segel equations via an interacting stochastic many-particle system was done by Stevens [24] in 2000. In her work she considers a finite number of bacteria and particles of a chemical substance in $\mathbb{R}^d$ whose movement is governed by a stochastic differential equation each. The chemical particles are merely driven by Brownian motion, whereas the bacteria's equations also include a drift term depending on the chemical. It is then shown that (under suitable renormalization), as the number of bacteria and particles goes to infinity, the system converges to a continuous one, where the solution functions are then densities of the respective particles.

For a comprehensive summary of different works on the Patlak–Keller–Segel model we refer to [14], where also various approaches and findings are presented in a succinct way and readily prepared for anyone who wants to look up existing results on the topic.

1.3 Physical interpretation of the equations

The stochastic many-particle approach [24] suggests the following interpretation of the terms appearing in the equations. For the general system

$$\begin{aligned} n_t &= \operatorname{div}(\mu\nabla n - \chi(n,S)n\nabla S), \\ S_t &= \eta\Delta S + \beta(n,S)n - \gamma(n,S)S \end{aligned} \tag{1}$$

the function $n(x,t)$ describes the density of cells (bacteria, amoebiae, etc.) at a point x at time t; the function $S(x,t)$ describes the density of the chemical substance. For the given functions we have

- $\chi(n,S)$ is the chemotactic sensitivity of the cells, i.e., how strongly they are attracted (or repelled) by the chemical,
- $\beta(n,S)$ is the production rate of the chemical, e.g., if it depends on S, this could mean that cells will not (or will particularly) produce more of the chemical if there already is a certain amount of it,
- $\gamma(n,S)$ is the decay rate of the chemical, i.e., for example how fast the chemical dissolves or evaporates, or it could be absorbed by cells.

The remaining non-negative parameters

- μ is a measure of how strong the cell diffusion is, i.e., the bigger μ is, the stronger the diffusion, which means that cells will tend to drift away from each other and spread out,
- η is the corresponding diffusion coefficient for the chemical density.

In our particular model, we take $\chi = 1$, $\beta = n^{\alpha-1}$, $\gamma = 1$, and the diffusion parameters are $\mu = \eta = 1$. This means that the cells' (amount of) reaction to the chemical does not depend on the actual amount of the chemical (but just on a difference of it, a gradient of it) or the amount of cells; they will always be attracted to where more of the chemical is.

The superlinear production term n^α describes that, based on the number of cells at a point, much more of the chemical is produced by more cells (i.e., by a higher density of cells). In particular, this will encourage already crowded cells to produce even more of the chemical, which will attract even more new cells, which will in turn increase the production even further.
The decay rate $\gamma = 1$ here means that the chemical density will decrease "exponentially", i.e., the higher the density, the faster the decrease; if the density is already low, then its decrease rate is also lower. It would decrease exponentially if there were no cells (then the production term vanishes) and the chemical were distributed evenly (then the diffusion term vanishes); in this case the second equation would simplify to the ordinary differential equation $S'(t) = -S(t)$, with the solution $S(t) = S(0)e^{-t}$.

Our particular model also includes another term $\delta\Delta n$ in the second equation. This cross-diffusion term models (arbitrarily small, due to $\delta > 0$,) diffusion effects for the chemical based on the amount of cells, i.e, the more cells there are, the more the chemical will spread out (and away) from the crowded area. In total our system reads

$$\begin{aligned} n_t &= \operatorname{div}(\nabla n - n\nabla S) \\ \gamma S_t &= \Delta S + \delta\Delta n + n^\alpha - S \end{aligned}$$

in $\Omega, t > 0$. The domain Ω is thought of as a (bounded) container or box where the cells and chemical move. Since we consider a two-dimensional domain, one can think of a very thin layer like on a microscope slide or Petri dish. The parameter $\gamma \geq 0$ is a measure of the different time scales for the cell movement and the distribution of the chemical [13]. For $\gamma = 1$ the system is called the parabolic-parabolic model, whereas for $\gamma = 0$ it is the parabolic-elliptic model.
We also need boundary conditions for the equations. It makes sense to take homogeneous Neumann conditions

$$\nabla n \cdot \nu = \nabla S \cdot \nu = 0 \qquad \text{on } \partial\Omega, t > 0,$$

which means that nothing exits or enters the container (or slide).

1.4 Problems with the classical formulation of the equations

The rich mathematical features of the equations come with some downside as well. While in one spatial dimension solutions will remain bounded for all times (if the initial function is bounded) [8], in higher dimensions finite time blow-up can occur. This means that cells crowd and chemoattractant production outgrows the diffusion effects, which leads to the cell density to grow to infinity. This would mean that arbitrarily many cells aggregate in single points[1]. However, this is not desirable from a physical or biological point of view. Thus, several ways to prevent overcrowding have been suggested and investigated in the literature. Also, precise conditions for finite time blow-up and its prevention have been explored. A critical value is the total number of cells, which does not change over time, and is given by

$$M := \int_\Omega n_0(x)\,\mathrm{d}x = \int_\Omega n(x,t)\,\mathrm{d}x.$$

[1]This will (under certain assumptions) be in the form of several Dirac point measures [8].

In two dimensions, if the mass exceeds 8π and the initial distribution is concentrated enough, i.e., if

$$\int_{\Omega} u_0(x)\,|x|^2\,\mathrm{d}x$$

is sufficiently small, then there exist solutions which blow up in finite time [20]; meaning, if the initial distribution is heavily centred around $x = 0$, then cells will crowd there. On the other hand, if $M < 8\pi$, then solutions exist globally in time and remain bounded. If $M = 8\pi$ and $\Omega = \mathbb{R}^2$, then a global solution exists which might become unbounded for $t \to \infty$ [8].
For dimensions three and higher (for $\Omega = \mathbb{R}^d$), under the assumption that for some $x_0 \in \mathbb{R}^d$ the quantity

$$\int_{\mathbb{R}^d} u_0(x)\,|x - x_0|^d\,\mathrm{d}x$$

is sufficiently small, there exists a solution (to the parabolic-elliptic system) which blows up in finite time [14]. Thus, the search for bounded solutions turns out to be much more involved for higher dimensions.

In the paper by S. Hittmeir and A. Jüngel [13] they considered the two-dimensional case with the additional $\delta\Delta n$ cross-diffusion term and showed that solutions exist globally (with a linear production term) in the parabolic-parabolic model, and that solutions are bounded for the parabolic-elliptic model.
We shall continue their investigations, but with superlinear production n^α for $1 \leq \alpha < 3/2$, which also covers the linear case from [13]. Our results include global solutions for the parabolic-parabolic model, and solutions which do not blow up in finite time (but might blow up as $t \to \infty$) in the parabolic-elliptic model.

1.5 Possible ways to avoid finite time blow-up

As described in the previous section, solutions might blow up after a finite time, which leads to the question of how to modify or restrict the original model to prevent this behaviour. In the literature a whole lot of ways have been suggested. These include

- modifying the chemotactic sensitivity by
  - a volume-filling effect[2] [5], i.e., upon reaching a certain threshold cells will no longer be drawn to the chemical, and attraction will decrease with rising cell density.
  - lower powers in the sensitivity[3] [15], i.e., cells are in general less strongly drawn to the chemical.
  - a non-local gradient[4] which describes that cells only sense the chemical over a certain (finite) distance [12].
- changing the cell diffusion

[2]This would be $\chi(n, s) = (1 - n)$ in the parabolic-elliptic version of (1).
[3]This would be $\chi(n, S) = n^{\rho-1}$ in (1) for some $\rho < \frac{2}{d}$, where d is the spatial dimension.
[4]This would be replacing ∇S in $\chi(n, S)n\nabla S$ in (1) by a particular integral.

  - by degenerate diffusion[5] [19, 18, 6], i.e., larger amounts of cells will increase the diffusive effects even further.
  - taking $n_t = \operatorname{div}(n(1-n)\nabla(n-S))$ as the first equation in (1) [5], thus combining the volume filling effect for the chemotactic sensitivity with the same for the cell diffusion.

- introducing death of cells[6] [25]
- cross-diffusion[7] [13, 16], i.e., aggregation of either the cells or chemical also leads to dispersion of the other (and not just itself).

In this thesis we will choose the last approach.

1.6 Novelty of the results

Our approach of adding a cross-diffusion term to the Keller–Segel model yields global existence of solutions for superlinear production. This is particularly remarkable because in [26] the author shows that (at least for Ω being a ball) the critical exponent for the production term is $\alpha = \frac{2}{d}$, meaning that for lower values bounded global solutions exist, whereas for bigger values there exist solutions (with arbitrary initial cell mass) which blow up in finite time.
However, since we consider a two-dimensional model our critical exponent would be $\alpha = 1$. In [13] it was shown that for $\alpha = 1$ bounded global solutions (to the parabolic-elliptic model) exist, and global (potentially unbounded) solutions exist for the parabolic-parabolic model.
In this thesis we further expand on this result and prove existence for production terms of order less than $\frac{3}{2}$. This stresses the regularizing effect of cross-diffusion, where we need to re-emphasize that this method works for *any* $\delta > 0$, i.e., arbitrarily small cross-diffusion.

[5] This would be replacing ∇n in the first equation of (1) by $f(n)\nabla n$ for some function f.
[6] This would be adding a $g(n)$ term to the first equation of (1).
[7] This would be adding Δn to the second equation and/or ΔS to the first equation in (1).

2 Notation

2.1 Derivatives

For functions $f : I \times \Omega \to \mathbb{R}^{d_1}$, where $I \subseteq \mathbb{R}$ and $\Omega \subseteq \mathbb{R}^d$ both open, we use several ways to denote different types of derivatives. We always think of such functions as functions of time and space and will use $f(t, x)$ (or $f(x, t)$) to denote $t \in I, x \in \Omega$. Then we use the following notation for time derivatives

$$\frac{\partial}{\partial t} f = \frac{\partial f}{\partial t} = \partial_t f = f_t.$$

There is a bigger variety of spacial derivatives, the basic ones being

$$\frac{\partial}{\partial x_i} f = \frac{\partial f}{\partial x_i} = \partial_{x_i} f$$

for the partial derivative in the direction of the i-th canonical basis vector of $\mathbb{R}^d$. For higher order derivatives we use

$$\frac{\partial^2}{\partial x_i \partial x_j} f = \frac{\partial^2 f}{\partial x_i \partial x_j} = \partial_{x_i} \partial_{x_j} f = \partial_{x_i x_j} f$$

and

$$\frac{\partial^2}{\partial x_i^2} f = \big(\frac{\partial}{\partial x_i}\big)^2 f = \partial_{x_i}^2 f.$$

In partial differential equations it is ubiquitous to use certain symbols for combined derivatives like the gradient and the divergence (which we assume only ever act on the spatial components). We shall use the following notation for real-valued functions $f(x, t) \in \mathbb{R}$ and vector-valued functions $F(x, t) = (F_1, \dots, F_d)(x, t) \in \mathbb{R}^d$

$$\begin{aligned}
\nabla f &= \begin{pmatrix} \partial_{x_1} f \\ \vdots \\ \partial_{x_d} f \end{pmatrix} \quad \text{called the gradient,} \\
\operatorname{div} F &= \sum_{i=1}^{d} \frac{\partial F_i}{\partial x_i} \quad \text{called the divergence,} \\
\Delta f &= \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2} \quad \text{called the Laplacian,} \\
\Delta^2 f &= \Delta(\Delta f), \\
D^m f &= \Big(\frac{\partial^m f}{\partial x_1^{m_1} \cdots \partial x_d^{m_d}} \Big)_{\substack{0 \le m_i \le m \\ \sum_{i=1}^d m_i = m}} \quad \text{the } m\text{-tensor of mixed derivatives of order } m, \\
D^1 f &= \nabla f, \quad D^2 f = \text{Hess } f, \\
D^\alpha f &= \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f
\end{aligned}$$

for integers $m \in \mathbb{N} \cup \{0\}$ and multi-indices $\alpha \in (\mathbb{N} \cup \{0\})^d$.

The meaning of derivatives

We consider three increasingly general meanings of the above derivatives: the classical pointwise derivative, the weak L^p-derivative, and the distributional derivative. The classical derivative is well-known to be defined as the limit of difference quotients. In this thesis we will mostly use weak derivatives. Let $f \in L^1_{\mathrm{loc}}(\Omega)$ and $\phi \in C_c^\infty(\Omega)$, then f has a weak derivative if there exists a function $g \in L^1_{\mathrm{loc}}(\Omega)$ such that the following equality holds

$$\int_\Omega f \frac{\partial \phi}{\partial x_i} \,\mathrm{d}x = -\int_\Omega g\phi \,\mathrm{d}x \qquad \forall \phi \in C_c^\infty(\Omega),$$

and we set $\partial_{x_i} f := g$. The distributional derivative is defined as follows. For a distribution T acting on the space $C_c^\infty(\Omega)$, we set $(\partial_{x_i} T)(\phi) = -T(\partial_{x_i}\phi)$ for any $\phi \in C_c^\infty(\Omega)$. In particular, any distribution has infinitely many (distributional) derivatives.

Notationwise we do not distinguish between classical, weak and distributional derivatives. When in doubt, any derivative is first to be understood in a distributional sense. If the function has enough regularity to admit a weak or classical derivative, then it is to be understood as such. This seeming ambiguity does not affect the meaning, because – if they exist – the different types of derivatives coincide; that is, distributional derivatives are regular distributions that lie in some L^p-space and agree with the weak derivatives almost everywhere, or weak derivatives have a representative (in their equivalence class of functions that agree almost everywhere) that is classically differentiable.

2.2 The domain Ω

Throughout this thesis we will denote by Ω a domain in $\mathbb{R}^d$, where we use the term *domain* for an open, non-empty, connected subset. We will also specify the regularity of the boundary of Ω, denoted $\partial\Omega$. If we say that $\partial\Omega \in C^1$, then this means that the boundary can locally be parametrized by continuously differentiable functions. Similarly, if we say that Ω is a Lipschitz domain (or just Lipschitz), then we mean that $\partial\Omega \in C^{0,1}$ and its boundary can locally be expressed as the image of Lipschitz continuous functions, and analogously for $C^{k,1}$, $k \in \mathbb{N}$.

When dealing with space and time, we shall denote $\Omega_T := \Omega \times (0, T)$ the space-time cylinder for some $T \in (0, \infty]$.

Upon using the Gauß-Ostrogradski theorem for multidimensional integration by parts, we will encounter boundary integrals. In particular, we will need the outer normal unit vector at any point of the boundary, which we will denote by ν. The assumption $\partial\Omega \in C^1$ ensures that such a vector exists at every point of the boundary. Under the weaker assumption $\partial\Omega \in C^{0,1}$ we only get existence almost everywhere (with respect to the lower dimensional Hausdorff-measure) on $\partial\Omega$, but this is enough for integration. (One could require even less, namely that Ω has locally finite perimeter, but this generalization is not of interest in this work.)

2.3 Topology

Let $(X, \mathcal{T})$ be a topological space, that is, X is any set and $\mathcal{T} \subseteq \mathcal{P}(X)$ is the topology on X. We denote

- $\overline{A}$ the closure of a set $A \subseteq X$ w.r.t. $\mathcal{T}$,
- ∂A the boundary of A.

For spaces with more structure, such as metric spaces, normed spaces or spaces with a scalar product, we implicitly understand it with the induced topology respectively. We shall also drop the second argument and refer to X as a topological space, when there is no ambiguity.

2.4 Convergence

There is a lot of different meanings to arrows ($\to$) in mathematics. We try to always specify the meaning right before or after any arrows denoting convergence. For our purposes we need four different types of convergence.

- For a sequence (x_n) in a Banach space $(X, \|\cdot\|)$ and $x \in X$ we denote

$$x_n \to x \quad \text{(strongly) in } X$$

  for norm convergence, i.e., $\|x_n - x\| \to 0$ as $n \to \infty$.

- We write

$$x_n \rightharpoonup x \quad \text{weakly (in } X\text{)}$$

  for convergence in the weak topology, i.e., for any $v \in X'$ (where X' denotes the dual space of X) we have $\langle v, x_n - x \rangle_{X'} \to 0$ as a real sequence.

- For a sequence (v_n) in X' and $v \in X'$ we write

$$v_n \rightharpoonup^* v \quad \text{weakly* (in } X'\text{)}$$

  for weak*-convergence, i.e., for any $x \in X$ we have $\langle v_n - v, x \rangle_{X'} \to 0$ as a real sequence.

- For a sequence (u_n) in some Lebesgue space $L^p(\Omega; \mu)$ and $u \in L^p(\Omega; \mu)$ (for some $1 \leq p \leq \infty$ and measure μ) we write

$$u_n \to u \quad \text{a.e. in } \Omega$$

  for pointwise convergence almost everywhere, i.e., $u_n(x) \to u(x)$ pointwise for all $x \in \Omega \backslash N$ where $\mu(N) = 0$.

2.5 Spaces

Lebesgue spaces $L^p(\Omega; \mu)$

We denote $L^p(\Omega; \mu)$ the Lebesgue spaces, where $1 \leq p \leq \infty$, Ω is any set, and μ is a σ-finite measure on Ω. We say that a function $f : \Omega \to \mathbb{R}^{d_1}$ belongs to $L^p(\Omega; \mu)$ if it is measurable and its L^p-norm is finite, i.e.,

$$\|f\|_{L^p(\Omega;\mu)} := \left(\int_\Omega |f(x)|^p \, \mathrm{d}\mu(x) \right)^{1/p} \qquad \text{for } p \neq \infty$$
$$\|f\|_{L^\infty(\Omega;\mu)} := \operatorname*{ess\,sup}_{x \in \Omega} |f(x)| \,.$$

As always we shall identify functions that agree μ-a.e. to make $\|\cdot\|_{L^p(\Omega;\mu)}$ a norm. We refer to these equivalence classes loosely as just *a function* in $L^p(\Omega;\mu)$.
For $p = 2$ the space $L^2(\Omega;\mu)$ equipped with the following scalar product is a Hilbert space

$$\langle f, g\rangle_{L^2(\Omega;\mu)} := \int_\Omega f(x)g(x)\,\mathrm{d}\mu(x).$$

We shall drop the measure μ in the notation if we take the d-dimensional Lebesgue measure on $\Omega \subseteq \mathbb{R}^d$, and just write $L^p(\Omega)$.

Sobolev spaces $H^k(\Omega)$, $W^{k,p}(\Omega)$, and $H^s(\Omega)$

As a generalization of Lebesgue spaces to include differentiability we define $H^k(\Omega)$ for a domain $\Omega \subseteq \mathbb{R}^d$ as the Hilbert space

$$H^k(\Omega) := \{f \in L^2(\Omega) : \; D^\alpha f \in L^2(\Omega),\, |\alpha| = 0, \ldots, k\}$$

for any integer $k \in \mathbb{N} \cup \{0\}$, where the derivatives are to be understood in a distributional sense. This gives a space of k times weakly differentiable functions. We equip this space with the scalar product

$$\langle f, g\rangle_{H^k(\Omega)} := \sum_{|\alpha|\leq k} \langle D^\alpha f, D^\alpha g\rangle_{L^2(\Omega)}.$$

The more general notion are the $W^{k,p}(\Omega)$ spaces, where $k \in \mathbb{N}\cup\{0\}$ and $1 \leq p \leq \infty$. We set $W^{k,p}(\Omega)$ as all the functions with finite $W^{k,p}$-norm, where

$$\|f\|_{W^{k,p}(\Omega)} := \Big(\sum_{|\alpha|\leq k} \|D^\alpha f\|^p_{L^p(\Omega)}\Big)^{1/p}.$$

For $p = 2$ we get $W^{k,2}(\Omega) = H^k(\Omega)$, so we generalize the order of integration. Notice also that

$$\|f\|_{W^{k,p}(\Omega),\sim} := \sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^p(\Omega)}$$

is an equivalent norm.

A different way to generalize the $H^k(\Omega)$ spaces is to allow for any kind of values for k. This could be done via extension operators and Fourier-transform, or equivalently (and more directly for this presentation) by two steps (see [1, Definitions 8.10.6, 8.10.7]). For $0 < \sigma < 1$ and arbitrary $\Omega \subseteq \mathbb{R}^d$ define

$$H^\sigma(\Omega) := \{u \in L^2(\Omega) : \; \iint_{\substack{\Omega\times\Omega\\ x\neq y}} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2\sigma}}\,\mathrm{d}x\,\mathrm{d}y < \infty\},$$

which is a Hilbert space with the corresponding scalar product

$$\langle u, v\rangle_{H^\sigma(\Omega)} := \langle u, v\rangle_{L^2(\Omega)} + \iint_{\substack{\Omega\times\Omega\\ x\neq y}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+2\sigma}}\,\mathrm{d}x\,\mathrm{d}y.$$

And also define $H^0(\Omega) := L^2(\Omega)$. For arbitrary $s > 0, s \in \mathbb{R}$, write $s = \lfloor s \rfloor + \sigma$ for $\lfloor s \rfloor =: m \in \mathbb{N} \cup \{0\}$ and $0 \le \sigma < 1$. Then set

$$H^s(\Omega) := \{u \in H^m(\Omega) : \ D^\alpha u \in H^\sigma(\Omega), \, |\alpha| = m\},$$

which is a Hilbert space when equipped with the scalar product (for $\sigma \neq 0$; in this case just take the H^m-scalar product)

$$\begin{aligned}\langle u, v\rangle_{H^s(\Omega)} :=& \langle u, v\rangle_{H^m(\Omega)} \\ &+ \sum_{|\alpha|=m} \iint_{\substack{\Omega\times\Omega \\ x\neq y}} \frac{(D^\alpha u(x) - D^\alpha u(y))(D^\alpha v(x) - D^\alpha v(y))}{|x-y|^{d+2\sigma}} \,\mathrm{d}x \,\mathrm{d}y.\end{aligned}$$

The most important use of the $H^s(\Omega)$ spaces for us are (compact) embeddings into other Sobolev spaces. So we will never actually use this definition.

Bochner spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$

We will want to distinguish different kinds of regularity of functions in terms of space or time. We could already define a notion of regularity for space-time by taking $L^p(\Omega \times (0,T))$ (or any other Sobolev space $W^{k,p}$ over the same set). Our goal now is to define a space of functions that allows for different integrability and differentiability. These are the Bochner spaces. If we have a function $f(x,t) : \Omega \times (0,T) \to \mathbb{R}$ we could simply fix one argument and view it as a function in just the other argument. Then $t \mapsto u(x,t)$ is a function-valued function, which takes values in the space of functions on Ω. We could impose regularity assumptions on this space, and then ask for regularity of the former map with values in that space. In general, let X be a Banach space. Then $u : (0,T) \to X$ is a Banach space-valued function and we set

$$\begin{aligned}&L^p(0,T;X) := \{u : (0,T) \to X : \ \|u\|_{L^p(0,T;X)} < \infty\} \\ &\|u\|_{L^p(0,T;X)} := \|\|u(t)\|_X\|_{L^p(0,T)}.\end{aligned}$$

In the important case of $X = L^r(\Omega)$ the norm can be written as

$$\|u\|_{L^p(0,T;L^r(\Omega))} = \left(\int_0^T \left(\int_\Omega |u(x,t)|^r \,\mathrm{d}x\right)^{p/r} \mathrm{d}t\right)^{1/p}$$

for $p, r \neq \infty$, and the usual adaptations for the essential supremum.
Similarly, we define the Sobolev Bochner spaces (we only need $X = W^{\ell,r}(\Omega)$ for $\Omega \subseteq \mathbb{R}^d$)

$$\begin{aligned}&W^{k,p}(0,T;W^{\ell,r}(\Omega)) := \{u : \|u\|_{W^{k,p}(0,T;W^{\ell,r}(\Omega))} < \infty\} \\ &\|u\|_{W^{k,p}(0,T;W^{\ell,r}(\Omega))} := \sum_{j=0}^k \left(\int_0^T \left(\sum_{|\alpha|\le\ell} \int_\Omega \left|\partial_t^j D^\alpha u(x,t)\right|^r \mathrm{d}x\right)^{p/r} \mathrm{d}t\right)^{1/p}\end{aligned}$$

with the standard changes for $p = \infty$ or $r = \infty$. This norm is equivalent to

$$\|u\|_\sim = \sum_{j=0}^k \sum_{|\alpha|\le\ell} \left(\int_0^T \left(\int_\Omega \left|\partial_t^j D^\alpha u(x,t)\right|^r \mathrm{d}x\right)^{p/r} \mathrm{d}t\right)^{1/p}.$$

C^k-spaces, $C^{k,1}$, and C_c^k

The classical notion of continuity and differentiability is captured in the spaces C^k for $k \in \mathbb{N} \cup \{0\}$. Denote by $C^0(X,Y)$ the space of continuous functions from any topological space X to another space Y. For integers $k \geq 1$ denote by $C^k(X)$ (with $X \subseteq \mathbb{R}^d$) the space of real-valued functions with continuous (classical) derivatives up to order k; and define

$$C^\infty(X) := \bigcap_{k=0}^{\infty} C^k(X).$$

Since continuous differentiability is a very strong assumption, a less restraining one (with similar consequences) is Lipschitz continuity – the space $C^{0,1}(X)$. We say a function f is Lipschitz continuous if there exists a constant L (the *Lipschitz constant*) such that for all $x, y \in X$

$$|f(x) - f(y)| \leq L\,|x - y|\,.$$

Lipschitz functions are continuous, and more importantly differentiable almost everywhere (by Rademacher's theorem); which allows many (not pointwise[8]) statements about continuously differentiable functions to be generalized to Lipschitz functions. For higher orders of differentiability $k \in \mathbb{N}$, we define $C^{k,1}(X)$ as the space of functions whose derivatives up to order k are Lipschitz continuous.

One problem with C^∞-functions is that they might still not be integrable (take any non-zero constant on an infinite measure space like $\mathbb{R}$). One way to work around this is to assume that functions vanish on most of the space, which leads to the space of compactly supported functions $C_c^k(X)$. Define the support of a function as the closure of all points where the function is not equal to zero. Then

$$C_c^k(X) := \{f \in C^k(X) : \ \operatorname{supp} f \text{ is compact in } X\}$$

for $k \in \mathbb{N} \cup \{0, \infty\}$ and $X \subseteq \mathbb{R}^d$. Since already continuous functions are bounded on compact sets, and the Lebesgue measure of compact sets is finite, this gives any order of integrability for such functions

$$C_c^k(X) \subseteq W^{k,p}(X)$$

for any $1 \leq p \leq \infty$.

Embeddings

A very important concept for PDEs is that different spaces of functions can not only be related by mere set inclusion but rather have some estimates associated to them. For example, if Ω has finite measure then

$$L^\infty(\Omega) \subseteq L^p(\Omega)$$

for any $1 \leq p \leq \infty$ (and $\Omega \subseteq \mathbb{R}^d$). Even more, we can relate the norms by

$$\|u\|_{L^p(\Omega)} \leq |\Omega|^{1/p}\,\|u\|_{L^\infty(\Omega)}.$$

[8]Usually theorems involving integrating derivatives.

Such an inclusion of spaces is called an embedding, and with a (uniform) estimate for the respective norms it is called a continuous embedding, denoted by $\hookrightarrow$. More precisely, if $X \subseteq Y$, then we call the identity operator $\mathrm{id} : X \to Y$ an embedding of X in Y, and say that X embeds into Y. If X and Y are normed spaces, we call the embedding of X in Y continuous if the identity as a linear operator is continuous (or bounded), i.e., if there exists a constant C such that

$$\|u\|_Y \leq C\|u\|_X,$$

which we denote by

$$X \hookrightarrow Y.$$

If the identity operator is even compact, then we call the embedding compact accordingly, which we denote by

$$X \hookrightarrow\hookrightarrow Y,$$

or simply by spelling out that $X \hookrightarrow Y$ compactly. The theorems concerning embedding of Sobolev spaces (Lemma 20) and compact embeddings therein (Lemma 17 and Lemma 18) can be found in the Appendix.

Dual spaces and duality

For the concept of weak convergence and weak derivatives we need certain dual spaces. In general, the dual space of a Banach space is given by all linear and continuous maps on that space with values in $\mathbb{R}$ (or $\mathbb{C}$). However, for most spaces we consider, one can find spaces isomorphic to their dual – that is, a space of functions and not just functionals. The only ones we need explicitly are those of Lebesgue spaces L^p and Bochner spaces $L^p(0,T;X)$, where X is a Banach space. Denoting isomorphy by $\simeq$ and the dual space of X by X', we have

$$\begin{aligned} &(L^p(\Omega))' \simeq L^q(\Omega), \\ &(L^p(0,T;X))' \simeq L^q(0,T;X'), \quad \text{where } \frac{1}{p}+\frac{1}{q}=1 \text{ and } 1 \leq p < \infty \end{aligned}$$

(note in particular, $(L^\infty(\Omega))' \not\simeq L^1(\Omega)$ in general). When applying functionals f from the dual space X' to $g \in X$, we shall use the notation

$$f(g) =: \langle f, g \rangle_{X'},$$

which corresponds to the similar notation for scalar products in Hilbert spaces (as suggested by the Riesz representation theorem).

Norms

For a normed space X we denote the norm $\|\cdot\|_X$. However, we will also use the same notation for d-tuples and imply any norm on the d-dimensional reals, i.e.,

$$\left\| \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \right\|_X := \Big(\sum_{j=1}^{d} \|x_j\|_X^p \Big)^{1/p}$$

for $1 \leq p < \infty$ and the usual modification for $p = \infty$. We generally assume $p = 2$. Note, however, that this ambiguity does not cause a problem, because *all* norms on $\mathbb{R}^d$ are equivalent, that is, for any two norms $\|\cdot\|_1, \|\cdot\|_2$ there exist constants $C_1, C_2 > 0$ such that for all $x \in \mathbb{R}^d$

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1.$$

In such a case we write $\|\cdot\|_1 \sim \|\cdot\|_2$.

2.6 Uses of the modulus $|\cdot|$

It is usually clear from the context what meaning the absolute value bars can or cannot have. We shall specify.

- For $x \in \mathbb{R}$ we denote the standard absolute value by $|x| \in [0, \infty)$.
- For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we imply any (finite-dimensional) norm like $|x| = \left(\sum_{j=1}^d |x_j|^p\right)^{1/p}$, where $1 \leq p < \infty$ and the usual modification for $p = \infty$. Since all of these are equivalent (see Section 2.5), we could choose any, but we usually assume $p = 2$.
- For $\alpha = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$ a multi-index, we denote $|\alpha| = \sum_{j=1}^d \alpha_j$ the order of α.
- For a Lebesgue measurable set $A \subseteq \mathbb{R}^d$ we denote $|A|$ the (d-dimensional) Lebesgue measure of A.

2.7 Other notation

Upon integrating a function on a set Ω, we may want to specify a subset that depends on that function. For a more concise notation we write, for example,

$$[0 \leq f] := \{x \in \Omega : \ 0 \leq f(x)\},$$

which would imply

$$\int_{[0 \leq f]} f \, \mathrm{d}x \geq 0.$$

Another helpful tool are indicator functions $\mathbb{1}_A$ for a set A.

$$\mathbb{1}_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A \end{cases}$$

If the need arises to emphasize that a function f is constant (and not just takes a certain value at some point), we will write $f \equiv c$ for a constant c; meaning that $f(x) = c$ for all x.

Throughout this thesis we will encounter many different constants, which are not important by themselves. We will call them all C and their value might change from one line to the next. Usually they do not carry any dependence on parameters. If they do and it is necessary to keep track, we shall denote this by either a subscript C_δ or function arguments $C(\delta)$ (in this case, the constants would depend on δ). We might, however, drop this additional information in the next line as to make the presentation more succinct.

3 Existence for superlinear growth in 2D

Theorem 1. *For time horizon $T > 0$, a bounded Lipschitz domain Ω, and parameters $\delta > 0$, $\gamma \in \{0, 1\}$, $1 \leq \alpha < 3/2$; and functions $n_0 \geq 0$ with $n_0 \log n_0 \in L^1(\Omega)$, and $S_0 \in L^2(\Omega)$, the system*

$$n_t = \operatorname{div}(\nabla n - n\nabla S) \tag{2}$$

$$\gamma S_t = \Delta S + \delta\Delta n + n^\alpha - S \tag{3}$$

in $\Omega, t > 0$, with boundary conditions

$$\nabla n \cdot \nu = \nabla S \cdot \nu = 0 \qquad \text{on } \partial\Omega, t > 0 \tag{4}$$

and initial conditions

$$n(\cdot, 0) = n_0, \quad \gamma S(\cdot, 0) = \gamma S_0 \qquad \text{in } \Omega. \tag{5}$$

admits a weak solution (n, S) such that

$$\begin{aligned}
&n \log n \in L^\infty(0, T; L^1(\Omega)), \quad \sqrt{n} \in L^2(0, T; H^1(\Omega)),\\
&n \in L^2(0, T; W^{1,1}(\Omega)), \quad n \in L^{4/3}(0, T; W^{1,4/3}(\Omega)),\\
&S \in L^2(0, T; H^1(\Omega)), \quad \gamma S \in L^\infty(0, T; L^2(\Omega)),\\
&n_t \in L^1(0, T; (W^{1,\infty}(\Omega))'), \quad \gamma S_t \in L^{4/3}(0, T; (W^{1,4}(\Omega))').
\end{aligned}$$

Additionally,

$$\begin{aligned}
&n^\mu \in L^2(0, T; H^1(\Omega)) \quad \text{for any } 0 < \mu < 1/4,\\
&\nabla \log n \in L^2(0, T; L^2(\Omega)) \quad \text{if } \log n_0 \in L^1(\Omega).
\end{aligned}$$

3.1 Outline of the proof

To solve this system, we fix a time horizon T and prove existence up to this (arbitrary) finite time to get local in time solutions. The proof uses an implicit Euler discretization with parameter $\tau := \frac{T}{K} > 0$ (for some integer K) to deal with irregular time behaviour. We can solve the resulting elliptic system by means of the Leray–Schauder fixed point theorem (Section 3.2.2), where we show existence in a linearized system using the Lax–Milgram lemma (Section 3.2.1). To achieve the necessary coercivity (in $H^2(\Omega)$) we introduce regularizing terms

$$-\varepsilon(\Delta^2 y + ye^{y/\delta}) \text{ and } \varepsilon \operatorname{div}(|\nabla y|^2 \nabla y) \text{ in the first equation,}$$

where we define

$$y := \delta \log n.$$

In order to get the necessary estimates to pass to the limit $(\varepsilon, \tau) \to (0, 0)$, we find an entropy functional, which we can bound uniformly (Section 3.3.1). We conclude by weak compactness both by the Rellich–Kondrachov theorem and the Aubin–Lions lemma (Section 3.3.2). These give enough regularity to pass to the limit even in the non-linear terms.

The proof follows very closely the one in [13].

3.2 Existence in the regularized implicit Euler discretization

Consider the recursive system

$$\begin{aligned}
&\frac{1}{\tau}\begin{pmatrix} e^{y_k/\delta} - e^{y_{k-1}/\delta} \\ \gamma(S_k - S_{k-1}) \end{pmatrix} - \operatorname{div}\left(D(y_k)\nabla \begin{pmatrix} y_k \\ S_k \end{pmatrix}\right) \\
&\quad = \varepsilon \begin{pmatrix} -\Delta^2 y_k + \delta^{-2}\operatorname{div}(|\nabla y_k|^2 \nabla y_k) - y_k e^{y_k/\delta} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ e^{\alpha y_k/\delta} - S_k \end{pmatrix}
\end{aligned} \tag{6}$$

with boundary conditions

$$\nabla y_k \cdot \nu = \nabla \Delta y_k \cdot \nu = \nabla S_k \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{7}$$

where the diffusion matrix is given by

$$D(y_k) = \begin{pmatrix} \delta^{-1} e^{y_k/\delta} & -e^{y_k/\delta} \\ e^{y_k/\delta} & 1 \end{pmatrix}. \tag{8}$$

For given (y_{k-1}, S_{k-1}) we shall prove existence of the next step (y_k, S_k). This is made precise in the following theorem.

Theorem 2. *Let y_{k-1} be a function such that $\exp(y_{k-1}/\delta) \in L^1(\Omega)$, and let $S_{k-1} \in L^2(\Omega)$. Then there exists a solution $(y_k, S_k) \in H^2(\Omega) \times H^1(\Omega)$ of the above recursive system (6)-(8).*

Proof. **3.2.1 Lax–Milgram lemma**

We want to use the Lax–Milgram lemma; so we fix some arguments to get a linear system. Let $(\bar{y}, \bar{S}) \in H^{7/4}(\Omega) \times L^2(\Omega)$ be fixed to get the new linear problem

$$a((y,S),(z,R)) = F(z,R) \quad \text{for all } (y,S),(z,R) \in H^2(\Omega) \times H^1(\Omega), \tag{9}$$

where the bilinear form a is given by

$$\begin{aligned}
a((y,S),(z,R)) :&= \int_\Omega \begin{pmatrix} \nabla z \\ \nabla R \end{pmatrix}^\top \cdot D(\bar{y}) \cdot \begin{pmatrix} \nabla y \\ \nabla S \end{pmatrix} \,\mathrm{d}x \\
&\quad + \varepsilon \int_\Omega \left(\Delta y \Delta z + \delta^{-2} |\nabla \bar{y}|^2 \nabla y \cdot \nabla z + y e^{\bar{y}/\delta} z \right) \mathrm{d}x + \int_\Omega S R \,\mathrm{d}x
\end{aligned} \tag{10}$$

and the functional F is

$$F(z,R) := -\frac{1}{\tau} \int_\Omega \begin{pmatrix} e^{\bar{y}/\delta} - e^{y_{k-1}/\delta} \\ \gamma(\bar{S} - S_{k-1}) \end{pmatrix} \cdot \begin{pmatrix} z \\ R \end{pmatrix} \,\mathrm{d}x + \int_\Omega e^{\alpha \bar{y}/\delta} R \,\mathrm{d}x. \tag{11}$$

The first task is to check that both of them are actually well-defined. Notice the Sobolev embedding

$$\bar{y} \in H^{7/4}(\Omega) \hookrightarrow L^\infty(\Omega) \tag{12}$$

for $7/4 > d/2 = 1$. This L^∞-bound ensures that even exponentiation still gives $L^\infty(\Omega)$ coefficients. A simple Hölder-inequality checks the first integral in a. In the second integral the first term is bounded by Cauchy–Schwarz, the third by, again, the L^∞-bound and a Hölder inequality; its second term can be estimated by Hölder's inequality and the Sobolev embeddings $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$ and $H^{7/4}(\Omega) \hookrightarrow W^{1,4}(\Omega)$ for $2 = d \le 3$

$$\begin{aligned}\int_\Omega \left||\nabla \bar{y}|^2 \nabla y \cdot \nabla z\right| \,\mathrm{d}x &\le \|\nabla \bar{y}\|^2_{L^4(\Omega)} \|\nabla y\|_{L^4(\Omega)} \|\nabla z\|_{L^4(\Omega)} \\ &\le \|\bar{y}\|^2_{W^{1,4}(\Omega)} \|\nabla y\|_{L^4(\Omega)} \|\nabla z\|_{L^4(\Omega)} \\ &\le C\|\bar{y}\|^2_{W^{1,4}(\Omega)} \|y\|_{H^2(\Omega)} \|z\|_{H^2(\Omega)} \\ &\le C\|\bar{y}\|^2_{H^{7/4}(\Omega)} \|y\|_{H^2(\Omega)} \|z\|_{H^2(\Omega)}.\end{aligned}$$

The last integral is bounded by Cauchy-Schwarz. Thus, a is well-defined and obviously linear.

The functional F is checked similarly by Hölder's inequality. Here one uses $e^{\bar{y}/\delta} \in L^\infty(\Omega)$ (and $z \in L^2(\Omega)$) or $e^{y_{k-1}/\delta} \in L^1(\Omega)$ (and $z \in H^2(\Omega) \hookrightarrow L^\infty(\Omega)$); or $e^{\alpha\bar{y}/\delta} \in L^\infty(\Omega)$ (and $R \in L^2(\Omega)$); or $\bar{S}, S_{k-1} \in L^2(\Omega)$ (and $R \in L^2(\Omega)$). Also F is clearly linear.

The next step is to show continuity of both functions. For a we argue as above

$$\begin{aligned}|a((y,S),(z,R))| \le& \|\nabla \begin{pmatrix} z \\ R \end{pmatrix}\|_{L^2(\Omega)} \|D(\bar{y})\|_{L^\infty(\Omega)} \|\nabla \begin{pmatrix} y \\ S \end{pmatrix}\|_{L^2(\Omega)} \\ &+ \varepsilon\big(\|\Delta y\|_{L^2(\Omega)} \|\Delta z\|_{L^2(\Omega)} + \delta^{-2} \||\nabla \bar{y}|^2 \nabla y \cdot \nabla z\|_{L^1(\Omega)} \\ &\quad + \|y\|_{L^2(\Omega)} \|e^{\bar{y}/\delta}\|_{L^\infty(\Omega)} \|z\|_{L^2(\Omega)}\big) + \|S\|_{L^2(\Omega)} \|R\|_{L^2(\Omega)} \\ \le& \|D(\bar{y})\|_{L^\infty(\Omega)} \|(z,R)\|_{H^1(\Omega)} \|(y,S)\|_{H^1(\Omega)} \\ &+ \varepsilon\big(\|y\|_{H^2(\Omega)} \|z\|_{H^2(\Omega)} + \delta^{-2} C \|\bar{y}\|^2_{H^{7/4}(\Omega)} \|y\|_{H^2(\Omega)} \|z\|_{H^2(\Omega)} \\ &\quad + \|e^{\bar{y}/\delta}\|_{L^\infty(\Omega)} \|y\|_{L^2(\Omega)} \|z\|_{L^2(\Omega)}\big) + \|S\|_{L^2(\Omega)} \|R\|_{L^2(\Omega)},\end{aligned}$$

where the terms not depending on y, S, z, R are bounded, and all the norms of y, S, z, R appear in the right form and can all be bounded in $H^2(\Omega) \times H^1(\Omega)$.
For F we find

$$\begin{aligned}|F(z,R)| \le& \frac{1}{\tau}\big(\|e^{\bar{y}/\delta}\|_{L^2(\Omega)} \|z\|_{L^2(\Omega)} + \|e^{y_{k-1}/\delta}\|_{L^1(\Omega)} \|z\|_{L^\infty(\Omega)} \\ &+ \gamma\|\bar{S} - S_{k-1}\|_{L^2(\Omega)} \|R\|_{L^2(\Omega)}\big) + \|e^{\alpha\bar{y}/\delta}\|_{L^2(\Omega)} \|R\|_{L^2(\Omega)} \\ \le& \frac{1}{\tau}\big(|\Omega|^{1/2} \|e^{\bar{y}/\delta}\|_{L^\infty(\Omega)} \|z\|_{L^2(\Omega)} + \|e^{y_{k-1}/\delta}\|_{L^1(\Omega)} C \|z\|_{H^2(\Omega)} \\ &+ \gamma(\|\bar{S}\|_{L^2(\Omega)} + \|S_{k-1}\|_{L^2(\Omega)}) \|R\|_{L^2(\Omega)}\big) + \|e^{\alpha\bar{y}/\delta}\|_{L^2(\Omega)} \|R\|_{L^2(\Omega)},\end{aligned}$$

which gives continuity since $\tau > 0$ is fixed.

The last property to verify is coercivity of a

$$
\begin{aligned}
a((y,S),(y,S)) &= \int_\Omega \begin{pmatrix} \nabla y \\ \nabla S \end{pmatrix} \cdot \begin{pmatrix} \delta^{-1} e^{\bar{y}/\delta} & -e^{\bar{y}/\delta} \\ e^{\bar{y}/\delta} & 1 \end{pmatrix} \cdot \begin{pmatrix} \nabla y \\ \nabla S \end{pmatrix} \,\mathrm{d}x \\
&\quad + \varepsilon \int_\Omega (\Delta y)^2 + \delta^{-2} \left|\nabla \bar{y}\right|^2 \left|\nabla y\right|^2 + e^{\bar{y}/\delta} y^2 \,\mathrm{d}x + \int_\Omega S^2 \,\mathrm{d}x \\
&= \int_\Omega \delta^{-1} e^{\bar{y}/\delta} \left|\nabla y\right|^2 + \left|\nabla S\right|^2 \,\mathrm{d}x \\
&\quad + \varepsilon \int_\Omega (\Delta y)^2 + \delta^{-2} \left|\nabla \bar{y}\right|^2 \left|\nabla y\right|^2 + e^{\bar{y}/\delta} y^2 \,\mathrm{d}x + \int_\Omega S^2 \,\mathrm{d}x \\
&= \|S\|_{H^1(\Omega)}^2 + \int_\Omega \varepsilon (\Delta y)^2 + \left(\delta^{-1} e^{\bar{y}/\delta} + \varepsilon \delta^{-2} \left|\nabla \bar{y}\right|^2\right) \left|\nabla y\right|^2 + e^{\bar{y}/\delta} y^2 \,\mathrm{d}x \\
&\geq \|S\|_{H^1(\Omega)}^2 + \min\{\varepsilon, \delta^{-1} \exp(-\|\bar{y}\|_{L^\infty(\Omega)}/\delta), \exp(-\|\bar{y}\|_{L^\infty(\Omega)}/\delta)\} \\
&\qquad \cdot \left(\|\Delta y\|_{L^2(\Omega)}^2 + \|\nabla y\|_{L^2(\Omega)}^2 + \|y\|_{L^2(\Omega)}^2\right) \\
&\geq \|S\|_{H^1(\Omega)}^2 + C\|y\|_{H^2(\Omega)}^2 \\
&\geq \min\{1, C\}\left(\|S\|_{H^1(\Omega)}^2 + \|y\|_{H^2(\Omega)}^2\right),
\end{aligned}
$$

where the constant $C > 0$ is positive because $\bar{y} \in L^\infty(\Omega)$, thus its exponential is strictly positive, and C also includes a norm equivalence factor from

$$\|u\|_{H^2(\Omega)}^2 \sim \|\Delta u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2$$

by Lemma 12. By the Lax–Milgram lemma (Lemma 13), for fixed $(\bar{y}, \bar{S}) \in H^{7/4}(\Omega) \times L^2(\Omega)$ we derive the existence of a unique solution $(y, S) \in H^2(\Omega) \times H^1(\Omega)$.

3.2.2 Leray–Schauder

We shall next employ the Leray–Schauder fixed point theorem (Lemma 14). Thus, define the solution operator

$$
\begin{aligned}
B : H^{7/4}(\Omega) \times L^2(\Omega) \times [0,1] &\to H^2(\Omega) \times H^1(\Omega) \\
(\bar{y}, \bar{S}, \sigma) &\mapsto (y, S),
\end{aligned}
$$

where (y, S) is the solution to the linear problem

$$a((y,S),(z,R)) = \sigma F(z,R) \tag{13}$$

for fixed $(\bar{y}, \bar{S})$. We need to first check continuity of B. Let $(\bar{y}_n, \bar{S}_n, \sigma_n) \to (\bar{y}, \bar{S}, \sigma)$ in $H^{7/4}(\Omega) \times L^2(\Omega) \times [0,1]$ be a converging sequence. From this convergence we deduce boundedness in the respective spaces. Denote (y_n, S_n) the corresponding (unique) solutions, i.e., $(y_n, S_n) = B(\bar{y}_n, \bar{S}_n, \sigma_n)$. As a consequence of the Lax–Milgram lemma (Corollary 13.1) we get the uniform (in n) bound

$$\|y_n\|_{H^2(\Omega)} + \|S_n\|_{H^1(\Omega)} \leq C.$$

Restricting to a subsequence we get weak convergence

$$
\begin{aligned}
y_{n_k} &\rightharpoonup y \text{ in } H^2(\Omega) \\
S_{n_k} &\rightharpoonup S \text{ in } H^1(\Omega)
\end{aligned}
$$

for some functions $(y, S) \in H^2(\Omega) \times H^1(\Omega)$ by the Eberlein–Šmuljan theorem (Lemma 15). Together with the *strong* convergence of $(\bar{y}_n, \bar{S}_n, \sigma_n)$, this allows us to pass to the limit in the corresponding weak formulation (13) and find that $(y, S) = B(\bar{y}, \bar{S}, \sigma)$ is the solution of our limit – which is almost the continuity we want, since we showed this only up to a subsequence. However, we could do the very same line of arguments with an arbitrary subsequence X_{n_k} to get a converging subsubsequence $X_{n_{k_\ell}}$. Since the linear problem's solution is unique this implies that any limit of any subsubsequence must coincide. A general fact in topological spaces (Lemma 16) yields that already the whole sequence converges (to the same limit). Thus, B is continuous.

To show that B is a compact operator consider the following

$$\begin{gathered}(\bar{y}, \bar{S}, \sigma) \xrightarrow{B} (y, S) \xrightarrow{id} (y, S) \\ H^{7/4} \times L^2 \times [0,1] \to H^2 \times H^1 \hookrightarrow\hookrightarrow H^{7/4} \times L^2,\end{gathered}$$

where $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact by the Rellich–Kondrachov theorem (Lemma 17), and for the compact embedding $H^2(\Omega) \hookrightarrow H^{7/4}(\Omega)$ see Lemma 18. Since concatenating a compact map with a continuous one preserves compactness, B is compact.
We readily check that $B(\bar{y}, \bar{S}, 0) = (0, 0)$ for any $(\bar{y}, \bar{S})$ since $(0, 0)$ is a solution and, by uniqueness, is the only one.
Lastly, we need to uniformly bound any potential fixed points of B, i.e., we need a uniform constant C such that whenever $B(y, S, \sigma) = (y, S)$, we have

$$\|(y, S)\|_{H^{7/4}(\Omega) \times L^2(\Omega)} \leq C.$$

We can assume $\sigma \neq 0$. Take $(z, R) = (1, 0)$ as a test function in (13), which is a suitable test function since Ω is bounded and the gradients vanish (in particular they vanish on the boundary).

$$\begin{aligned} \varepsilon \int_\Omega y e^{y/\delta} \, \mathrm{d}x &= \sigma\Big(-\frac{1}{\tau} \int_\Omega (e^{y/\delta} - e^{y_{k-1}/\delta}) \, \mathrm{d}x \Big) \\ \int_\Omega e^{y/\delta} \, \mathrm{d}x &= -\frac{\varepsilon\tau}{\sigma\delta} \int_\Omega y e^{y/\delta} \, \mathrm{d}x + \int_\Omega e^{y_{k-1}/\delta} \, \mathrm{d}x \\ &\leq -\frac{\varepsilon\tau\delta}{\sigma} \int_\Omega (e^{y/\delta} - 1) \, \mathrm{d}x + \int_\Omega e^{y_{k-1}/\delta} \, \mathrm{d}x \\ &\leq \frac{\varepsilon\tau\delta}{\sigma} |\Omega| + \int_\Omega e^{y_{k-1}/\delta} \, \mathrm{d}x, \end{aligned}$$

where we used the inequality $xe^x \geq e^x - 1$ (for $x \in \mathbb{R}$ by Taylor). This recursion (w.r.t. k) we can solve $(y \,\hat{=}\, y_k)$

$$\begin{aligned} \int_\Omega e^{y/\delta} \, \mathrm{d}x &\leq k \frac{\varepsilon\tau\delta}{\sigma} |\Omega| + \int_\Omega e^{y_0/\delta} \, \mathrm{d}x \\ &\leq \frac{\varepsilon T\delta}{\sigma} |\Omega| + \int_\Omega e^{y_0/\delta} \, \mathrm{d}x. \end{aligned} \tag{14}$$

So $e^{y/d} \in L^1(\Omega)$ uniformly in t (or k equivalently).

Now take $(z, R) = (y, S)$ as a test function to get

$$\sigma\Big(-\frac{1}{\tau}\int_\Omega (e^{y/\delta} - e^{y_{k-1}/\delta})y + \gamma(S - S_{k-1})S\,\mathrm{d}x + \int_\Omega e^{\alpha y/\delta} S\,\mathrm{d}x\Big)$$
$$= \int_\Omega \begin{pmatrix}\nabla y\\ \nabla S\end{pmatrix}^\top \cdot D(y)\cdot \begin{pmatrix}\nabla y\\ \nabla S\end{pmatrix}\,\mathrm{d}x + \varepsilon\int_\Omega (\Delta y)^2 + \delta^{-2}\left|\nabla y\right|^4 + y^2 e^{y/\delta}\,\mathrm{d}x + \int_\Omega S^2\,\mathrm{d}x, \tag{15}$$

thus,

$$\frac{\sigma}{\tau}\int_\Omega (e^{y/\delta} - e^{y_{k-1}/\delta})y + \gamma(S - S_{k-1})S\,\mathrm{d}x + \int_\Omega \begin{pmatrix}\nabla y\\ \nabla S\end{pmatrix}^\top \cdot D(y)\cdot \begin{pmatrix}\nabla y\\ \nabla S\end{pmatrix}\,\mathrm{d}x$$
$$+\varepsilon\int_\Omega (\Delta y)^2 + \delta^{-2}\left|\nabla y\right|^4 + y^2 e^{y/\delta}\,\mathrm{d}x + \int_\Omega S^2\,\mathrm{d}x = \sigma\int_\Omega e^{\alpha y/\delta} S\,\mathrm{d}x. \tag{16}$$

We now estimate the right hand side. Using Hölder's inequality for some $r \in (1, \infty)$ to be varied later; the Gagliardo–Nirenberg inequality (Lemma 19), and Sobolev embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for any $p < \infty$, we get (with $1/p + 1/r = 1$)

$$\int_\Omega e^{\alpha y/\delta} S\,\mathrm{d}x \le \|e^{\alpha y/\delta}\|_{L^r(\Omega)}\|S\|_{L^p(\Omega)} = \|e^{y/(2\delta)}\|^{2\alpha}_{L^{2\alpha r}(\Omega)}\|S\|_{L^p(\Omega)}$$
$$\le C\|S\|_{H^1(\Omega)}\big(C\|e^{y/(2\delta)}\|^{\theta}_{H^1(\Omega)}\|e^{y/(2\delta)}\|^{1-\theta}_{L^2(\Omega)}\big)^{2\alpha},$$

where θ is given by[9] $\theta = 1 - \frac{1}{\alpha r}$. We will need to include σ in the end, so we rewrite this and use Young's inequality (Lemma 21)

$$\sigma\int_\Omega e^{ay/\delta} S\,\mathrm{d}x \le C\|S\|_{H^1(\Omega)}\|e^{y/(2\delta)}\|^{2\alpha\theta}_{H^1(\Omega)}\sigma\|e^{y/\delta}\|^{\alpha(1-\theta)}_{L^1(\Omega)}$$
$$\le 2\delta\|e^{y/(2\delta)}\|^2_{H^1(\Omega)} + C(\sigma\|e^{y/\delta}\|^{\alpha(1-\theta)}_{L^1(\Omega)})^q + \frac{1}{2}\|S\|^2_{H^1(\Omega)},$$

where q satisfies[10]

$$\frac{1}{\frac{2}{2\alpha\theta}} + \frac{1}{q} + \frac{1}{2} = 1,$$

so $\frac{1}{q} = \frac{1}{2} - \alpha - \frac{1}{r}$. Applying this estimate to the right hand side of (16) and noticing

$$\int_\Omega \delta^{-1}e^{y/\delta}\left|\nabla y\right|^2\,\mathrm{d}x = 4\delta\int_\Omega \left|e^{y/(2\delta)}\nabla y\frac{1}{2\delta}\right|^2\,dx = 4\delta\int_\Omega \left|\nabla e^{y/(2\delta)}\right|^2\,\mathrm{d}x,$$

we find

$$\frac{\sigma}{\tau}\int_\Omega y(e^{y/\delta} - e^{y_{k-1}}) + \gamma S(S - S_{k-1})\,\mathrm{d}x + 4\delta\int_\Omega \left|\nabla e^{y/(2\delta)}\right|^2\,\mathrm{d}x + \|S\|^2_{H^1(\Omega)}$$
$$+\varepsilon\big(\|\Delta y\|^2_{L^2(\Omega)} + \int_\Omega \delta^{-2}\left|\nabla y\right|^4 + y^2e^{y/\delta}\,\mathrm{d}x\big)$$
$$\le \sigma 2\delta\|e^{y/(2\delta)}\|^2_{H^1(\Omega)} + \sigma^q C\|e^{y/\delta}\|^{\alpha(1-\theta)q}_{H^1(\Omega)} + \sigma\frac{1}{2}\|S\|^2_{H^1(\Omega)}$$
$$\le 2\delta\|e^{y/(2\delta)}\|^2_{H^1(\Omega)} + \sigma^q C\|e^{y/\delta}\|^{\alpha(1-\theta)q}_{H^1(\Omega)} + \frac{1}{2}\|S\|^2_{H^1(\Omega)},$$

[9] Here we need the assumption $\alpha \ge 1/r$.
[10] Here we need the assumption $\alpha \in [1/r, 1/2 + 1/r]$.

where we can absorb the first and third term on the right hand side by the left hand side to conclude

$$\begin{aligned}\frac{\sigma}{\tau}\int_\Omega y(e^{y/\delta}-e^{y_{k-1}/\delta})+\gamma S(S-S_{k-1})\,\mathrm{d}x+2\delta\int_\Omega \left|\nabla e^{y/(2\delta)}\right|^2\,\mathrm{d}x+\frac{1}{2}\|S\|^2_{H^1(\Omega)}\\ +\varepsilon\big(\|\Delta y\|^2_{L^2(\Omega)}+\int_\Omega \delta^{-2}\left|\nabla y\right|^4+y^2e^{y/\delta}\,\mathrm{d}x\big)\le\sigma^q C\|e^{y/\delta}\|^{\alpha(1-\theta)q}_{L^1(\Omega)}.\end{aligned}\tag{17}$$

Now, define the real convex function

$$\phi(x):=x(\log x-1).\tag{18}$$

By convexity we get $\phi(x)-\phi(z)\le\phi'(x)(x-z)$ for any $x,z>0$. Thus,

$$\begin{aligned}\phi(e^{y/\delta})-\phi(e^{y_{k-1}/\delta})&\le\phi'(e^{y/\delta})(e^{y/\delta}-e^{y_{k-1}/\delta})\\ e^{y/\delta}(y/\delta-1)-e^{y_{k-1}/\delta}(y_{k-1}/\delta-1)&\le y/\delta(e^{y/\delta}-e^{y_{k-1}/\delta})\\ \int_\Omega y(e^{y/\delta}-e^{y_{k-1}/\delta})\,\mathrm{d}x&\ge\delta\int_\Omega\phi(e^{y/\delta})-\phi(e^{y_{k-1}/\delta})\,\mathrm{d}x.\end{aligned}$$

For the second term in (17) by convexity of $x\mapsto x^2$ we get

$$\begin{aligned}2S(S-S_{k-1})&\ge S^2-S^2_{k-1}\\ \gamma\int_\Omega(S-S_{k-1})S\,\mathrm{d}x&\ge\frac{\gamma}{2}\int_\Omega S^2-S^2_{k-1}\,\mathrm{d}x.\end{aligned}$$

These two estimates suggest the following "energy"

$$E_k:=\int_\Omega\phi(e^{y_k/\delta})+\frac{\gamma}{2\delta}S_k^2\,\mathrm{d}x,$$

where $y_k:=y$, $S_k:=S$ to unify future considerations. We can now further estimate (17) to get

$$\begin{aligned}\frac{\delta\sigma}{\tau}\Big(\int_\Omega\phi(e^{y_k/\delta})-\phi(e^{y_{k-1}/\delta})+\frac{\gamma}{2\delta}(S_k^2-S^2_{k-1})\,\mathrm{d}x\Big)+2\delta\int_\Omega\left|\nabla e^{y/(2\delta)}\right|^2\,\mathrm{d}x\\ +\frac{1}{2}\|S\|^2_{H^1(\Omega)}+\varepsilon\big(\|\Delta y\|^2_{L^2(\Omega)}+\int_\Omega\delta^{-2}\left|\nabla y\right|^4+y^2e^{y/\delta}\,\mathrm{d}x\big)\le\sigma^qC\|e^{y/\delta}\|^{\alpha(1-\theta)q}_{L^1(\Omega)}\end{aligned}$$

and rewrite in terms of the energy

$$\begin{aligned}\frac{\sigma}{\tau}(E_k-E_{k-1})+2\int_\Omega\left|\nabla e^{y/(2\delta)}\right|^2\,\mathrm{d}x\\ +\frac{1}{2\delta}\|S\|^2_{H^1(\Omega)}+\frac{\varepsilon}{\delta}\big(\|\Delta y\|^2_{L^2(\Omega)}+\int_\Omega\delta^{-2}\left|\nabla y\right|^4+y^2e^{y/\delta}\,\mathrm{d}x\big)\le\sigma^q\frac{C}{\delta}\|e^{y/\delta}\|^{\alpha(1-\theta)q}_{L^1(\Omega)}.\end{aligned}\tag{19}$$

In particular, since most terms on the left hand side are non-negative, we can use (14) to get the estimate

$$\begin{aligned}\frac{\sigma}{\tau}(E_k-E_{k-1})&\le\sigma^qC\|e^{y/\delta}\|^{\alpha(1-\theta)q}_{L^1(\Omega)}\\ \frac{1}{\tau}(E_k-E_{k-1})&\le\sigma^{q-1}C\Big(\frac{\varepsilon T\delta}{\sigma}\left|\Omega\right|+\int_\Omega e^{y_0/\delta}\,\mathrm{d}x\Big)^{\alpha(1-\theta)q}\\ &\le\sigma^{q-1-\alpha(1-\theta)q}C\Big(\varepsilon T\delta\left|\Omega\right|+\int_\Omega e^{y_0/\delta}\,\mathrm{d}x\Big)^{\alpha(1-\theta)q}.\end{aligned}\tag{20}$$

We want a uniform (in σ) estimate of the right hand side, so we need the exponent of σ to be non-negative. This is satisfied if

$$0 \leq q - 1 - \alpha(1-\theta)q = q\big(1 - \frac{1}{r}\big) - 1,$$

or in terms of α we need

$$\alpha \geq \frac{2}{r} - \frac{1}{2}.$$

Under this assumption, we can uniformly estimate the right hand side of (20). This gives one estimate for the discrete time-derivative of the energy functional. The other direction is immediate from

$$E_k = \int_\Omega \phi(e^{y_k/\delta}) + \frac{\gamma}{2\delta} S_k^2 \,\mathrm{d}x \geq \int_\Omega \phi(e^{y_k/\delta}) \,\mathrm{d}x \geq |\Omega| \min_{x\in\mathbb{R}} \phi(x) = -|\Omega|\,,$$

thus $\frac{1}{\tau}(E_k - E_{k-1}) \geq -2\,|\Omega|\,/\tau$.
Now we can go back to (19) to get the uniform estimate

$$\begin{aligned} 2\delta \int_\Omega \left|\nabla e^{y/(2\delta)}\right|^2 \,\mathrm{d}x + \frac{1}{2}\|S\|^2_{H^1(\Omega)} + \varepsilon(\|\Delta y\|^2_{L^2(\Omega)} + \int_\Omega \delta^{-2}\,|\nabla y|^4 + y^2 e^{y/\delta} \,\mathrm{d}x) \\ \leq C\sigma^q \|e^{y/\delta}\|^{\alpha(1-\theta)q}_{L^1(\Omega)} - \frac{\sigma}{\tau}(E_k - E_{k-1}) \leq C. \end{aligned} \tag{21}$$

This argument works for any such (fixed) $r \in (1,\infty)$. Taking the union of all possible ranges for α

$$I_r := \{\alpha \in \mathbb{R} : \ \frac{1}{r} \leq \alpha \leq \frac{1}{2} + \frac{1}{r}, \text{ and } \alpha \geq \frac{2}{r} - \frac{1}{2}\},$$

we get back our assumption (as illustrated by Figure 1)

$$1 \leq \alpha < \frac{3}{2},$$

where (for this particular calculation) we could take values $\alpha \in (0, 3/2)$. This computation is to be understood in the following way: for any value of $\alpha \in [1, 3/2)$, take any fixed r such that $\alpha \in I_r$, and repeat the above computation. The estimates are very much not uniform in r or α.

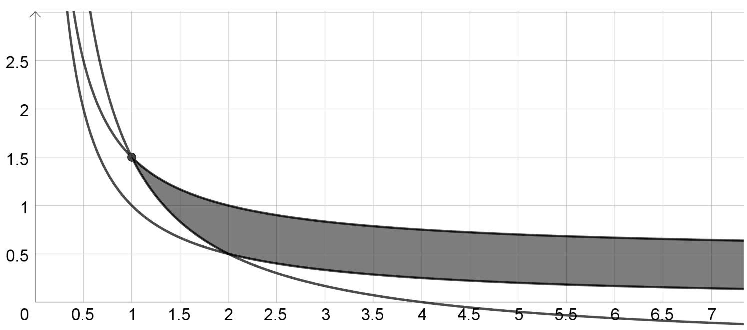

Figure 1: Values of r against possible values of α with critical point at $(1, 3/2)$. Constraints as functions of r, admissible pairs (r, α) depicted as the shaded area.

To conclude the fixed point estimate we only need to control some norm of y. We already have bounded the second and first derivatives, but need to make sure that zero-th order polynomials do not break the estimate; because we want to argue by norm equivalence. To do so, take the test function $(e^{-y/\delta}, 0)$ in (13)

$$\begin{aligned}
LHS =& \int_\Omega -\frac{1}{\delta}\nabla y e^{-y/\delta} \cdot (\delta^{-1} e^{y/\delta}\nabla y - e^{y/\delta}\nabla S)\,\mathrm{d}x \\
&+ \varepsilon \int_\Omega \Delta y\big(-\frac{1}{\delta}e^{-y/\delta}\Delta y + \frac{1}{\delta^2}\left|\nabla y\right|^2 e^{-y/\delta}\big)\,\mathrm{d}x \\
&+ \varepsilon \int_\Omega \delta^{-2}\left|\nabla y\right|^2 \nabla y \cdot \big(-\frac{1}{\delta}\nabla y e^{-y/\delta}\big) + y e^{y/\delta} e^{-y/\delta}\,\mathrm{d}x \\
=& \int_\Omega -\delta^{-2}\left|\nabla y\right|^2 + \delta^{-1}\nabla y \cdot \nabla S\,\mathrm{d}x \\
&+ \varepsilon \int_\Omega e^{-y/\delta}\Big(-\delta^{-1}(\Delta y)^2 + \delta^{-2}\Delta y\left|\nabla y\right|^2 - \delta^{-3}\left|\nabla y\right|^4\Big) + y\,\mathrm{d}x \\
RHS =& \sigma\Big(-\frac{1}{\tau}\int_\Omega 1 - e^{y_{k-1}/\delta - y/\delta}\,\mathrm{d}x\Big),
\end{aligned}$$

and reorder

$$\begin{aligned}
-\varepsilon \int_\Omega y\,\mathrm{d}x =& -\delta^{-2}\int_\Omega \left|\nabla y\right|^2\,\mathrm{d}x + \frac{\sigma}{\tau}\int_\Omega 1 - e^{(y_{k-1}-y)/\delta}\,\mathrm{d}x \\
&- \frac{\varepsilon}{\delta}\int_\Omega e^{-y/\delta}\Big((\Delta y)^2 - \delta^{-1}\Delta y\left|\nabla y\right|^2 + \delta^{-2}\left|\nabla y\right|^4\Big)\,\mathrm{d}x + \frac{1}{\delta}\int_\Omega \nabla S\cdot\nabla y\,\mathrm{d}x \\
\leq& \frac{1}{\tau}\int_\Omega 1\,\mathrm{d}x - \frac{\varepsilon}{\delta}\int_\Omega e^{-y/\delta}\Big((\Delta y - \frac{1}{2\delta}\left|\nabla y\right|^2)^2 + \frac{3}{4}\delta^{-2}\left|\nabla y\right|^4\Big)\,\mathrm{d}x \\
&+ \frac{1}{\delta}\int_\Omega \nabla S\cdot\nabla y\,\mathrm{d}x \\
\leq& \frac{|\Omega|}{\tau} + \frac{1}{\delta}\|\nabla S\|_{L^2(\Omega)}\|\nabla y\|_{L^2(\Omega)}.
\end{aligned}$$

We want to introduce the L^1-norm into the energy estimate (21). So we apply the above to find

$$\begin{aligned}
\|y\|_{L^1(\Omega)} =& \int_\Omega -y\,\mathrm{d}x + 2\int_{[y\geq 0]} y\,\mathrm{d}x \\
\leq& \frac{1}{\varepsilon}\Big(\frac{|\Omega|}{\tau} + \frac{1}{\delta}\|\nabla S\|_{L^2(\Omega)}\|\nabla y\|_{L^2(\Omega)}\Big) + 2\int_{[y\geq 0]} y\,\mathrm{d}x,
\end{aligned}$$

where we estimate

$$\begin{aligned}
\int_{[y\geq 0]} y\,\mathrm{d}x =& \int_{[0\leq y\leq 1]} y\,\mathrm{d}x + \int_{[y>1]} y\,\mathrm{d}x \\
\leq& \int_\Omega 1\,\mathrm{d}x + \int_{[y>1]} y^2 e^{y/\delta}\,\mathrm{d}x \\
\leq& |\Omega| + \int_\Omega y^2 e^{y/\delta}\,\mathrm{d}x,
\end{aligned}$$

so

$$\|y\|_{L^1(\Omega)} \leq \frac{1}{\varepsilon}\Big(\frac{|\Omega|}{\tau} + \frac{1}{\delta}\|\nabla S\|_{L^2(\Omega)}\|\nabla y\|_{L^2(\Omega)}\Big) + 2\left|\Omega\right| + 2\int_\Omega y^2 e^{y/\delta}\,\mathrm{d}x.$$

Applying the energy estimate (21), we find (with Young's inequality)

$$\begin{aligned}\|\nabla S\|_{L^2(\Omega)}\|\nabla y\|_{L^2(\Omega)} &\le \sqrt{2C}\|\nabla y\|_{L^2(\Omega)} \le \frac{3}{4}(\sqrt{2C})^{4/3} + \frac{1}{4}\|\nabla y\|^4_{L^2(\Omega)} \\ &\le \frac{1}{4}\big(3(2C)^{2/3} + \frac{\delta^2}{\varepsilon}C\big) \le \tilde{C},\end{aligned}$$

which allows us to uniformly bound (again with (21) for the last term)

$$\|y\|_{L^1(\Omega)} \le \frac{1}{\varepsilon}\big(\frac{|\Omega|}{\tau} + \frac{1}{\delta}\tilde{C}\big) + 2\,|\Omega| + 2C \le C.$$

We put everything into the energy estimate (21)

$$\begin{aligned}C &\ge 2\delta\int_\Omega \left|\delta e^{y/(2\delta)}\right|^2 \,\mathrm{d}x + \frac{1}{2}\|S\|^2_{H^1(\Omega)} + \varepsilon(\|\Delta y\|^2_{L^2(\Omega)} + \int_\Omega \delta^{-2}\,|\nabla y|^4 + y^2 e^{y/\delta}\,\mathrm{d}x) \\ &\ge \frac{1}{2}\|S\|^2_{H^1(\Omega)} + \varepsilon\|\Delta y\|^2_{L^2(\Omega)} + \varepsilon\delta^{-2}\|\nabla y\|^4_{L^4(\Omega)} + (\|y\|^2_{L^1(\Omega)} - C^2).\end{aligned}$$

Another Young's inequality

$$\|\nabla y\|^2_{L^2(\Omega)} = \int_\Omega \frac{|\nabla y|^2}{1/\sqrt{\varepsilon}}\cdot 1/\sqrt{\varepsilon}\,\mathrm{d}x \le \int_\Omega \frac{1}{2}\big(\frac{|\nabla y|^4}{1/\varepsilon} + 1/\varepsilon\big)\,\mathrm{d}x = \frac{\varepsilon}{2}\|\nabla y\|^4_{L^4(\Omega)} + \frac{|\Omega|}{2\varepsilon}$$

gives

$$\begin{aligned}C &\ge \frac{1}{2}\|S\|^2_{H^1(\Omega)} + \varepsilon\|\Delta y\|^2_{L^2(\Omega)} + 2\delta^{-2}\big(\|\nabla y\|^2_{L^2(\Omega)} - \frac{|\Omega|}{2\varepsilon}\big) + \|y\|^2_{L^1(\Omega)} - C^2 \\ &\ge \frac{1}{2}\|S\|^2_{H^1(\Omega)} + \min\{\varepsilon, 2\delta^{-2}, 1\}\big(\|\Delta y\|^2_{L^2(\Omega)} + \|\nabla y\|^2_{L^2(\Omega)} + \|y\|^2_{L^1(\Omega)}\big) - C.\end{aligned}$$

We conclude by norm equivalence in H^2 (Lemma 12). The desired solution is then given by the Leray–Schauder fixed point theorem. □

Now we have shown the existence of time discrete solutions, i.e., for a fixed time span $((k-1)\tau, k\tau]$, $k \in \mathbb{N}$, there exists a solution $(y_k, S_k) \in H^2(\Omega) \times H^1(\Omega)$, which is constant as a function of time. From the implicit Euler scheme we expect these piecewise-constant functions to approximate the exact solution of the limiting parabolic equation where $\tau \to 0$ and the difference quotient becomes a differential.
Putting together the solutions from Theorem 2, we define $y^{(\tau)}(x,t) := y_k(x)$ for $t \in ((k-1)\tau, k\tau]$ for fixed $\tau > 0$; $S^{(\tau)}(x,t) := S_k(x)$ and $n^{(\tau)}(x,t) := n_k(x) = \exp(y_k(x)/\delta)$ analogously. With the discrete time derivative $(D_\tau f)(t) := (f(t) - f(t-\tau))/\tau$ we can formulate the equations solved by $(n^{(\tau)}, S^{(\tau)})$

$$\begin{aligned}D_\tau n^{(\tau)} &= \operatorname{div}(\nabla n^{(\tau)} - n^{(\tau)}\nabla S^{(\tau)}) \\ &\quad - \varepsilon\big(\Delta^2 y^{(\tau)} - \delta^{-2}\operatorname{div}(\left|\nabla y^{(\tau)}\right|^2 \nabla y^{(\tau)}) + y^{(\tau)} n^{(\tau)}\big),\end{aligned} \tag{22}$$

$$\gamma D_\tau S^{(\tau)} = \Delta S^{(\tau)} + \delta\Delta n^{(\tau)} + (n^{(\tau)})^\alpha - S^{(\tau)} \tag{23}$$

with boundary conditions

$$\nabla n^{(\tau)}\cdot\nu = \nabla\Delta y^{(\tau)}\cdot\nu = \nabla S^{(\tau)}\cdot\nu = 0. \tag{24}$$

3.3 Passing to the limit $(\varepsilon,\tau)\to(0,0)$

Our goal is to pass to the limit $(\varepsilon,\tau)\to(0,0)$. For this, we need uniform (in ε and τ) estimates in order to use compactness arguments to be able to extract a limiting function, which also solves the desired equation.

3.3.1 Uniform estimates

We shall summarize some intermediate results in the following Lemma

Lemma 3. *The following bounds hold with a constant C independent of ε,τ*

$$\|n^{(\tau)}\log n^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))}+\|\sqrt{n^{(\tau)}}\|_{L^2(0,T;H^1(\Omega))}+\|S^{(\tau)}\|_{L^2(0,T;H^1(\Omega))}\le C \tag{25}$$

$$\sqrt{\varepsilon}\|\Delta y^{(\tau)}\|_{L^2(\Omega_T)}+\sqrt[4]{\varepsilon}\|\nabla y^{(\tau)}\|_{L^4(\Omega_T)}+\sqrt{\varepsilon}\|y^{(\tau)}\sqrt{n^{(\tau)}}\|_{L^2(\Omega_T)}\le C \tag{26}$$

$$\|n^{(\tau)}\|_{L^2(0,T;W^{1,1}(\Omega))}+\|n^{(\tau)}\|_{L^{4/3}(0,T;W^{1,4/3}(\Omega))}\le C \tag{27}$$

where $\Omega_T:=\Omega\times(0,T)$.

Proof. **Step 1: Proof of (25).** We start with the estimate from (14), rewritten in terms of n

$$\|n_k\|_{L^1(\Omega)}\le\varepsilon T\delta\,|\Omega|+\|n_0\|_{L^1(\Omega)}, \tag{28}$$

which holds for all k. We assumed the norm on the right hand side to be bounded and restrict $\varepsilon<1$, so we can bound it uniformly (in k) by a constant.
We plug this estimate into the energy estimate (19) to get a recursion

$$\begin{aligned}&\frac{1}{\tau}(E_k-E_{k-1})+2\|\nabla\sqrt{n_k}\|^2_{L^2(\Omega)}+\frac{1}{2\delta}\|S_k\|^2_{H^1(\Omega)}\\&\quad+\frac{\varepsilon}{\delta}\Big(\|\Delta y_k\|^2_{L^2(\Omega)}+\int_\Omega\delta^{-2}\,|\nabla y_k|^4+y_k^2e^{y_k/\delta}\,\mathrm{d}x\Big)\le C\|n_k\|^{\alpha(1-\theta)q}_{L^1(\Omega)}\le C.\end{aligned}$$

Expanding this recursion in terms of k (which corresponds to integrating in time from 0 to $k\tau$), we arrive at

$$\begin{aligned}&E_k+2\tau\sum_{j=1}^{k}\|\nabla\sqrt{n_j}\|^2_{L^2(\Omega)}+\frac{\tau}{2\delta}\sum_{j=1}^{k}\|S_j\|^2_{H^1(\Omega)}\\&+\frac{\varepsilon\tau}{\delta}\sum_{j=1}^{k}\Big(\|\Delta y_j\|^2_{L^2(\Omega)}+\int_\Omega\delta^{-2}\,|\nabla y_j|^4+y_j^2e^{y_j/\delta}\,\mathrm{d}x\Big)\le k\tau C+E_0\le TC+E_0.\end{aligned} \tag{29}$$

Since most terms on the left hand side are non-negative, this implies

$$E_k\le TC+E_0,$$

where we assumed the initial energy E_0 to be bounded. Recalling the definition of the energy, this bounds

$$E_k = \int_\Omega n_k(\log n_k - 1) + \frac{\gamma}{2\delta} S_k^2 \,\mathrm{d}x$$

from above. For the first estimate for $n_k \log n_k$ we further consider

$$\begin{aligned}
\int_\Omega n_k \log n_k \,\mathrm{d}x &= \int_{[n_k \geq 1]} n_k \log n_k \,\mathrm{d}x + \int_{[0<n_k<1]} n_k \log n_k \,\mathrm{d}x \\
&\geq \int_\Omega |n_k \log n_k| \,\mathrm{d}x - \int_{[0<n_k<1]} |n_k \log n_k| \,\mathrm{d}x + \int_{[0<n_k<1]} -\frac{1}{e} \,\mathrm{d}x \\
&\geq \|n_k \log n_k\|_{L^1(\Omega)} - 2\frac{|\Omega|}{e}.
\end{aligned}$$

From the non-negativity of n_k (as an exponential) we also get

$$\int_\Omega n_k \,\mathrm{d}x = \int_\Omega |n_k| \,\mathrm{d}x.$$

Now we arrive at

$$\begin{aligned}
\|n_k \log n_k\|_{L^1(\Omega)} - 2\frac{|\Omega|}{e} - \big(\|n_0\|_{L^1(\Omega)} + \varepsilon\delta T\,|\Omega|\,\big) + \frac{\gamma}{2\delta}\|S_k\|_{L^2(\Omega)}^2 \\
\leq \int_\Omega n_k \log n_k \,\mathrm{d}x - \int_\Omega n_k \,\mathrm{d}x + \frac{\gamma}{2\delta}\|S_k\|_{L^2(\Omega)}^2 \leq TC + E_0,
\end{aligned}$$

which can be rewritten to bound

$$\|n_k \log n_k\|_{L^1(\Omega)} + \frac{\gamma}{2\delta}\|S_k\|_{L^2(\Omega)}^2$$

uniformly in k, i.e., in $L^\infty(0,T)$, which gives the first bound (and also bounds $\gamma S \in L^\infty(0,T;L^2(\Omega))$).
The other bounds in the first inequality (25) follow more directly from (29), once one notices that the time-integral of a piecewise constant function gives precisely the sums with a weight τ. Arguing as in the proof of Theorem 2, we can bound E_k from below and deduce a uniform bound for the aforementioned non-negative terms on the left hand side of (29). This already gives

$$\|\nabla\sqrt{n^{(\tau)}}\|_{L^2(0,T;L^2(\Omega))} + \|S^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} \leq C.$$

For the actual H^1-bound for $\sqrt{n^{(\tau)}}$, we still need to check its L^2-norm. However,

$$\|\sqrt{n^{(\tau)}}\|_{L^2(0,T;L^2(\Omega))}^2 = \|n^{(\tau)}\|_{L^1(0,T;L^1(\Omega))} \leq T\|n^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))},$$

which we already bounded. This concludes the first inequality (25).
Step 2: Proof of (26). The second inequality follows similarly from (29). We have

$$\varepsilon\tau\sum_{j=1}^K \|\Delta y_j\|_{L^2(\Omega)}^2 = \varepsilon\|\Delta y^{(\tau)}\|_{L^2(\Omega_T)}^2 \leq C,$$

so taking square roots on both sides gives the first estimate in (26). Analogously one gets the L^4-norm of the gradient with the fourth-root. The last term comes from

$$\varepsilon\|y^{(\tau)}\sqrt{n^{(\tau)}}\|^2_{L^2(\Omega)} = \varepsilon\tau\sum_{j=1}^K\int_\Omega (y_j e^{y_j/(2\delta)})^2\,\mathrm{d}x = \varepsilon\tau\sum_{j=1}^K\int_\Omega y_j^2 e^{y_j/\delta}\,\mathrm{d}x.$$

This concludes the second inequality (26).
Step 3: Proof of (27). The third inequality can be derived from the previous two. For $\|\nabla n^{(\tau)}\|_{L^2(0,T;L^1(\Omega))}$ consider

$$\nabla\sqrt{n^{(\tau)}} = \frac{1}{2}\frac{1}{\sqrt{n^{(\tau)}}}\nabla n^{(\tau)}$$

(where $n^{(\tau)} > 0$ as an exponential of a bounded function) in

$$\begin{aligned}\|\nabla n^{(\tau)}\|^2_{L^2(0,T;L^1(\Omega))} &= \int_0^T \|2\sqrt{n^{(\tau)}}\nabla\sqrt{n^{(\tau)}}\|^2_{L^1(\Omega)}\,\mathrm{d}t\\ &\leq 4\int_0^T \|\sqrt{n^{(\tau)}}\|^2_{L^2(\Omega)}\|\nabla\sqrt{n^{(\tau)}}\|^2_{L^2(\Omega)}\,\mathrm{d}t\\ &\leq 4\|\|\sqrt{n^{(\tau)}}\|^2_{L^2(\Omega)}\|_{L^\infty(0,T)}\|\nabla\sqrt{n^{(\tau)}}\|^2_{L^2(\Omega_T)}\\ &= 4\|n^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))}\|\nabla\sqrt{n^{(\tau)}}\|^2_{L^2(\Omega_T)},\end{aligned}$$

where we used the Cauchy–Schwarz inequality and Hölder's inequality. The last factor is uniformly bounded from the first inequality (25). For the other norm notice

$$\begin{aligned}\|n^{(\tau)}\log n^{(\tau)}\|_{L^1(\Omega)} &= \int_{[n^{(\tau)}\geq e]}\left|n^{(\tau)}\log n^{(\tau)}\right|\,\mathrm{d}x + \int_{[0<n^{(\tau)}<e]}\left|n^{(\tau)}\log n^{(\tau)}\right|\,\mathrm{d}x\\ &\geq \int_{[n^{(\tau)}\geq e]}\left|n^{(\tau)}\right|\\ &\geq \|n^{(\tau)}\|_{L^1(\Omega)} - |\Omega|\,e\end{aligned}$$

for (almost) all $t \in (0,T)$. Thus,

$$\|n^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))} \leq \|n^{(\tau)}\log n^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))} + |\Omega|\,e \leq C$$

or

$$\|\nabla n^{(\tau)}\|^2_{L^2(0,T;L^1(\Omega))} \leq C.$$

We estimate $n^{(\tau)}$ in $L^2(0,T;L^1(\Omega))$ by equation (28), i.e.,

$$\begin{aligned}\|n^{(\tau)}\|^2_{L^2(0,T;L^1(\Omega))} &= \tau\sum_{j=1}^K\|n_j\|^2_{L^1(\Omega)}\\ &\leq T(\|n_0\|_{L^1(\Omega)} + \delta T\,|\Omega|)^2.\end{aligned}$$

This shows the $L^2(0,T;W^{1,1}(\Omega))$-estimate. By Sobolev embedding $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$ in two dimensions, this also gives an $L^2(\Omega_T)$-bound; which we can use in

$$\begin{aligned} \|\nabla n^{(\tau)}\|_{L^{4/3}(\Omega_T)} &= 2\|\sqrt{n^{(\tau)}}\nabla\sqrt{n^{(\tau)}}\|_{L^{4/3}(\Omega_T)} \\ &\leq 2\|\sqrt{n^{(\tau)}}\|_{L^4(\Omega_T)}\|\nabla n^{(\tau)}\|_{L^2(\Omega_T)} \\ &\leq C\|n^{(\tau)}\|^{1/2}_{L^2(\Omega_T)}. \end{aligned}$$

This finishes the proof. □

The next estimate concerns the discrete time derivatives.

Lemma 4. *For any $\eta > 0$ there exists a constant C independent of ε and τ such that*

$$\|D_\tau n^{(\tau)}\|_{L^1(0,T;(H^{2+\eta}(\Omega))')} + \gamma\|D_\tau S^{(\tau)}\|_{L^{4/3}(0,T;(W^{1,4}(\Omega))')} \leq C \tag{30}$$

Proof. Fix $\eta > 0$ and let $\phi \in L^\infty(0,T;H^{2+\eta}(\Omega))$. By Sobolev embedding $H^{2+\eta} \hookrightarrow W^{1,\infty}$, this gives $\phi \in L^\infty(0,T;W^{1,\infty}(\Omega))$. Testing with ϕ we have

$$\begin{aligned} &\int_0^T \left|\langle D_\tau n^{(\tau)}, \phi\rangle_{(H^{2+\eta}(\Omega))'}\right| \mathrm{d}t \\ &= \int_0^T \Big|\langle \operatorname{div}(\nabla n^{(\tau)} - n^{(\tau)}\nabla S^{(\tau)}) \\ &\qquad - \varepsilon\big(\Delta^2 y^{(\tau)} - \delta^{-2}\operatorname{div}(\left|\nabla y^{(\tau)}\right|^2 \nabla y^{(\tau)}) + y^{(\tau)}n^{(\tau)}\big)\ , \phi\rangle_{H^{2+\eta}(\Omega)'}\Big| \mathrm{d}t \\ &\leq \int_0^T \left|\int_\Omega -\nabla n^{(\tau)}\cdot\nabla\phi + n^{(\tau)}\nabla S^{(\tau)}\cdot\nabla\phi \,\mathrm{d}x\right| \mathrm{d}t \\ &\qquad + \varepsilon\int_0^T \left|\int_\Omega -\Delta y^{(\tau)}\Delta\phi - \delta^{-2}\left|\nabla y^{(\tau)}\right|^2 \nabla y^{(\tau)}\cdot\nabla\phi - y^{(\tau)}n^{(\tau)}\phi \,\mathrm{d}x\right| \mathrm{d}t, \end{aligned}$$

which we bound by Hölder's inequality

$$\begin{aligned} \leq &\|\nabla n^{(\tau)}\|_{L^{4/3}(\Omega_T)}\|\nabla\phi\|_{L^4(\Omega_T)} + \|n^{(\tau)}\|_{L^2(\Omega_T)}\|\nabla S^{(\tau)}\|_{L^2(\Omega_T)}\|\nabla\phi\|_{L^\infty(\Omega_T)} \\ &+ \varepsilon\big(\|\Delta y^{(\tau)}\|_{L^2(\Omega_T)}\|\Delta\phi\|_{L^2(\Omega_T)} + \delta^{-2}\|\nabla y^{(\tau)}\|^3_{L^4(\Omega_T)}\|\nabla\phi\|_{L^4(\Omega_T)} \\ &+ \|y^{(\tau)}\sqrt{n^{(\tau)}}\|_{L^2(\Omega_T)}\|\sqrt{n^{(\tau)}}\|_{L^4(\Omega_T)}\|\phi\|_{L^4(\Omega_T)}\big), \end{aligned}$$

where all terms in $n^{(\tau)}, S^{(\tau)}, y^{(\tau)}$ have already been uniformly bounded in Lemma 3. We, thus, get

$$\begin{aligned} \leq &C\Big(2\|\nabla\phi\|_{L^4(\Omega_T)} + \|\nabla\phi\|_{L^\infty(\Omega_T)} + \|\Delta\phi\|_{L^2(\Omega_T)} + \|\phi\|_{L^4(\Omega_T)}\Big) \\ \leq &C\Big(C\|\nabla\phi\|_{L^\infty(\Omega_T)} + \|\phi\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + \|\phi\|_{L^2(0,T;H^2(\Omega))} + C\|\phi\|_{L^\infty(\Omega_T)}\Big) \\ \leq &C\Big(\|\phi\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + \|\phi\|_{L^\infty(0,T;H^2(\Omega))}\Big), \end{aligned}$$

which finally can be bounded by the above Sobolev embedding $H^{2+\eta} \hookrightarrow W^{1,\infty}$ and the continuous embedding $H^{2+\eta} \hookrightarrow H^2$, by $C\|\phi\|_{L^\infty(0,T;H^{2+\eta}(\Omega))}$.

For the other estimate let $\phi \in L^4(0,T;W^{1,4}(\Omega))$. Similarly as before, we have

$$\gamma \int_0^T \left| \langle D_\tau S^{(\tau)}, \phi \rangle_{(W^{1,4}(\Omega))'} \right| \mathrm{d}t = \int_0^T \bigg| \int_\Omega - \nabla S^{(\tau)} \cdot \nabla \phi - \delta \nabla n^{(\tau)} \cdot \nabla \phi \\ + (n^{(\tau)})^\alpha \phi - S^{(\tau)} \phi \,\mathrm{d}x \bigg| \,\mathrm{d}t,$$

which we bound by

$$\leq \|\nabla S^{(\tau)}\|_{L^2(\Omega_T)} \|\nabla \phi\|_{L^2(\Omega_T)} + \delta \|\nabla n^{(\tau)}\|_{L^{4/3}(\Omega_T)} \|\nabla \phi\|_{L^4(\Omega_T)} \\ + \int_0^T \int_\Omega \left| (n^{(\tau)})^\alpha \phi \right| \,\mathrm{d}x\,\mathrm{d}t + \|S^{(\tau)}\|_{L^2(\Omega_T)} \|\phi\|_{L^2(\Omega_T)}.$$

All norms of $S^{(\tau)}$ and $n^{(\tau)}$ are bounded by Lemma 3, and the corresponding norms of ϕ and $\nabla\phi$ can all be estimated above by $C\|\phi\|_{L^4(0,T;W^{1,4}(\Omega))}$. The last remaining integral is estimated by Hölder's inequality

$$\begin{aligned} \|(n^{(\tau)})^\alpha \phi\|_{L^1(\Omega_T)} &\leq \| \|(n^{(\tau)})^\alpha\|_{L^1(\Omega)} \|\phi\|_{L^\infty(\Omega)} \|_{L^1(0,T)} \\ &\leq \|(n^{(\tau)})^\alpha\|_{L^{4/3}(0,T;L^1(\Omega))} \|\phi\|_{L^4(0,T;L^\infty(\Omega))} \\ &\leq \|n^{(\tau)}\|^\alpha_{L^{4\alpha/3}(0,T;L^\alpha(\Omega))} C \|\phi\|_{L^4(0,T;W^{1,4}(\Omega))}, \end{aligned} \tag{31}$$

where the norm of $n^{(\tau)}$ is bounded by Lemma 3 (namely the $L^2(\Omega_T)$-estimate) if $\alpha \leq \frac{3}{2}$, which is satisfied with our assumptions on α. This concludes this proof. □

3.3.2 Compactness

The bounds from Lemma 3 and Lemma 4 together with the Aubin–Lions–Dubinskiĭ Lemma (see [7, Lemma A.2.]) allow us to extract subsequences (which are not relabeled), such that[11]

$$\begin{aligned} n^{(\tau)} &\to n \quad \text{strongly in } L^2(0,T;L^p(\Omega)) \quad \forall p < 2 \\ S^{(\tau)} &\to S \quad \text{strongly in } L^2(0,T;L^q(\Omega)) \quad \forall q < \infty \end{aligned}$$

with the convergence also almost everywhere by the "inverse dominated convergence theorem" (Lemma 22). Additionally, from the Eberlein–Šmuljan theorem (Lemma 15) we also have the weak convergences

$$\begin{aligned} \nabla n^{(\tau)} &\rightharpoonup \nabla n \quad \text{weakly in } L^{4/3}(0,T;L^{4/3}(\Omega)) \\ \nabla S^{(\tau)} &\rightharpoonup \nabla S \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \end{aligned}$$

Note also that the non-negativity of $n^{(\tau)}$ as an exponential implies (by the pointwise a.e. convergence) the non-negativity of the limit function n.[12]
We would like to use this convergence to pass to the limit $(\varepsilon,\tau) \to (0,0)$, however, we *cannot* yet infer that $n^{(\tau)}\nabla S^{(\tau)} \rightharpoonup n\nabla S$ weakly in $L^1(0,T;L^1(\Omega))$ because we

[11] For $n^{(\tau)}$ take $W^{1,1} \hookrightarrow L^p \hookrightarrow (H^{2+\eta})'$ and note that the embedding $W^{1,1} \hookrightarrow L^p$ is compact for $p < 2$ in two dimensions.
For $S^{(\tau)}$ take $H^1 \hookrightarrow L^q \hookrightarrow (W^{1,4})'$, with compact embedding $H^1 \hookrightarrow L^q$ for any $q < \infty$ in two dimensions.

[12] This means that the cell density does not become negative (at least a.e.), which is very reasonable from a modelling point of view.

are missing the limiting case $p = 2$ in the strong convergence of $n^{(\tau)}$. Luckily, with the additional $n^{(\tau)} \log n^{(\tau)}$-bound from Lemma 3 we can squeeze out the necessary convergence.

Lemma 5. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain, let $T > 0$, and $s > 0$. Assume a sequence $(u_\varepsilon) \geq 0$ satisfies the uniform bound*

$$\|\sqrt{u_\varepsilon}\|_{L^2(0,T;H^1(\Omega))} + \|u_\varepsilon \log u_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} + \|\partial_t u_\varepsilon\|_{L^1(0,T;(H^s(\Omega))')} \leq C,$$

then, up to a subsequence, one has the strong convergence

$$u_\varepsilon \to u \quad \textit{strongly in } L^2(0,T;L^2(\Omega)).$$

The proof of this Lemma can be found in [2]. Applying the Lemma we now get the weak convergence $n^{(\tau)}\nabla S^{(\tau)} \rightharpoonup n\nabla S$ weakly in $L^1(0,T;L^1(\Omega))$.
The next step is to show that the regularizing ε-terms actually vanish with these convergences, i.e., morally speaking, that the functions do not "outgrow" the factor ε as $\varepsilon \to 0$. Let thus be $\phi \in L^4(0,T;H^2(\Omega))$, and consider by Lemma 3

$$\begin{aligned}
&\left|\varepsilon\int_0^T \langle \Delta^2 y^{(\tau)} - \delta^{-2}\operatorname{div}(\left|\nabla y^{(\tau)}\right|^2 \nabla y^{(\tau)}) + y^{(\tau)}n^{(\tau)}, \phi\rangle_{(H^2(\Omega))'}\,\mathrm{d}t\right| \\
&\quad= \varepsilon\left|\int_0^T\int_\Omega \Delta y^{(\tau)}\Delta\phi + \delta^{-2}\left|\nabla y^{(\tau)}\right|^2\nabla y^{(\tau)}\cdot\nabla\phi + y^{(\tau)}n^{(\tau)}\phi\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad\leq \varepsilon\Big(\|\Delta y^{(\tau)}\|_{L^2(\Omega_T)}\|\phi\|_{L^2(0,T;H^2(\Omega))} + \delta^{-2}\|\nabla y^{(\tau)}\|^3_{L^4(\Omega_T)}\|\nabla\phi\|_{L^4(\Omega_T)} \\
&\qquad + \|y^{(\tau)}\sqrt{n^{(\tau)}}\|_{L^2(\Omega_T)}\|\sqrt{n^{(\tau)}}\|_{L^4(\Omega_T)}\|\phi\|_{L^4(\Omega_T)}\Big) \\
&\quad\overset{(26)}{\leq} \varepsilon^{1/2}C\|\phi\|_{L^2(0,T;H^2(\Omega))} + \varepsilon^{1/4}C\delta^{-2}\|\nabla\phi\|_{L^4(\Omega_T)} + \varepsilon^{1/2}C\|\sqrt{n^{(\tau)}}\|_{L^4(\Omega_T)}\|\phi\|_{L^4(\Omega_T)} \\
&\quad\leq C(\varepsilon^{1/2} + \varepsilon^{1/4})\|\phi\|_{L^4(0,T;H^2(\Omega))},
\end{aligned}$$

where we used the uniform $L^2(\Omega_T)$-bound for $n^{(\tau)}$. Letting $\varepsilon \to 0$ implies the weak convergence

$$\varepsilon\big(\Delta^2 y^{(\tau)} - \operatorname{div}(\left|\nabla y^{(\tau)}\right|^2\nabla y^{(\tau)}) + y^{(\tau)}n^{(\tau)}\big) \rightharpoonup 0 \quad \text{weakly in } L^{4/3}(0,T;(H^2(\Omega))')$$

as $\varepsilon \to 0$, and uniformly in $\tau > 0$. Taking the limit $(\varepsilon,\tau) \to (0,0)$ in the equation for $D_\tau n^{(\tau)}$ we get (together with the above (weak) convergences) that

$$D_\tau n^{(\tau)} \rightharpoonup \operatorname{div}(\nabla n - n\nabla S) \quad \text{weakly in } L^1(0,T;(H^{2+\eta}(\Omega))').$$

Identifying the limit of $D_\tau n^{(\tau)}$, which converges to the distributional time derivative of n in the sense of distributions, we conclude

$$n_t = \operatorname{div}(\nabla n - n\nabla S) \quad \text{in } L^1(0,T;(H^{2+\eta}(\Omega))').$$

However, we can expand the space where the equality holds to $L^1(0,T;(W^{1,\infty}(\Omega))')$

by a density argument and

$$\begin{aligned}
&\Big|\int_0^T \langle \operatorname{div}(\nabla n - n\nabla S), \phi\rangle_{(W^{1,\infty})'}\,\mathrm{d}t\Big| = \Big|\int_0^T\int_\Omega \nabla n\cdot\nabla\phi - n\nabla S\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\le \int_0^T \|\nabla n\|_{L^1(\Omega)}\|\nabla\phi\|_{L^\infty(\Omega)} + \|n\|_{L^2(\Omega)}\|\nabla S\|_{L^2(\Omega)}\|\nabla\phi\|_{L^\infty(\Omega)}\,\mathrm{d}t \\
&\le \|\nabla n\|_{L^1(\Omega_T)}\|\nabla\phi\|_{L^\infty(\Omega_T)} + \|n\|_{L^2(\Omega_T)}\|\nabla S\|_{L^2(\Omega)}\|\nabla\phi\|_{L^\infty(\Omega_T)} \\
&\le \|n\|_{L^1(0,T;W^{1,1}(\Omega))}\|\phi\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + \|n\|_{L^2(\Omega_T)}\|S\|_{L^2(0,T;H^1(\Omega))}\|\phi\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} \\
&\overset{\text{Lemma 3}}{\le} C\|\phi\|_{L^\infty(0,T;W^{1,\infty}(\Omega))},
\end{aligned}$$

which gives the regularity $n_t \in L^1(0,T;(W^{1,\infty}(\Omega))')$.
Taking limits in the equation for $D_\tau S^{(\tau)}$, we immediately have weak convergence for the linear terms $\Delta S^{(\tau)}$, $\Delta n^{(\tau)}$ and $S^{(\tau)}$ in $L^2(0,T;(H^1(\Omega))')$ or $L^2(0,T;L^q(\Omega))$ (for any $q < \infty$) respectively. For the production term we have again

$$\begin{aligned}
\int_0^T\int_\Omega \big|(n^{(\tau)})^\alpha\phi\big|\,\mathrm{d}x\,\mathrm{d}t &\le \|n^{(\tau)}\|_{L^{4\alpha/3}(0,T;L^\alpha(\Omega))}\|\phi\|_{L^4(0,T;L^\infty(\Omega))} \\
&\le C\|n^{(\tau)}\|_{L^2(\Omega_T)}\|\phi\|_{L^4(0,T;W^{1,4}(\Omega))}
\end{aligned}$$

for $\alpha \le 3/2$ as in (31). By the above line of arguments we get

$$\gamma S_t = \Delta S + \delta\Delta n + n^\alpha - S \quad \text{in } L^{4/3}(0,T;(W^{1,4}(\Omega))').$$

Lastly, we need continuity in time for the initial data to make sense. Since $(D_\tau n^{(\tau)})$ is bounded in $W^{1,1}(0,T;(W^{1,\infty}(\Omega))')$ which embeds into $C^0([0,T];(W^{1,\infty}(\Omega))')$, we have $n(\cdot,0) = n_0$ in $(W^{1,\infty}(\Omega))'$. And analogously, we have $S(\cdot,0) = S_0$ in the sense of $(W^{1,4}(\Omega))'$ (if $\gamma \neq 0$; if $\gamma = 0$, we do not have this initial condition).

This concludes the existence proof. □

3.4 Regularity

We have already shown: If $n_0 \log n_0 \in L^1(\Omega)$ and $S_0 \in L^2(\Omega)$, then there exists a solution (n,S) with the following regularity

$$\begin{aligned}
&n\log n \in L^\infty(0,T;L^1(\Omega)), \quad \sqrt{n} \in L^2(0,T;H^1(\Omega)), \\
&n \in L^2(0,T;W^{1,1}(\Omega)), \quad n \in L^{4/3}(0,T;W^{1,4/3}(\Omega)), \\
&S \in L^2(0,T;H^1(\Omega)), \quad \gamma S \in L^\infty(0,T;L^2(\Omega)), \\
&n_t \in L^1(0,T;(W^{1,\infty}(\Omega))'), \quad \gamma S_t \in L^{4/3}(0,T;(W^{1,4}(\Omega))').
\end{aligned}$$

We can get even more regularity:

Lemma 6. *Under the assumptions of Theorem 1 one has*

$$n^\mu \in L^2(0,T;H^1(\Omega)) \quad \textit{for any } 0 < \mu < 1/4.$$

If one further assumes $\log n_0 \in L^1(\Omega)$*, then also*

$$\nabla \log n \in L^2(0,T;L^2(\Omega)).$$

Proof. We go back to the section before the limit $(\varepsilon,\tau)\to(0,0)$ and find uniform bounds (in ε and τ) for $n^{(\tau)}$ which will carry over to the limiting function n. Let $0<\beta<1/2$ and test (22) with $(n^{(\tau)})^{\beta-1}$

$$\int_0^T\int_\Omega D_\tau n^{(\tau)}(n^{(\tau)})^{\beta-1} = \int_0^T\int_\Omega -\nabla n^{(\tau)}\cdot\nabla((n^{(\tau)})^{\beta-1}) + n^{(\tau)}\nabla S^{(\tau)}\cdot\nabla((n^{(\tau)})^{\beta-1})$$
$$-\varepsilon\Big(\Delta y^{(\tau)}\Delta((n^{(\tau)})^{\beta-1}) + \delta^{-2}\left|\nabla(y^{(\tau)})\right|^2\nabla y^{(\tau)}\cdot\nabla((n^{(\tau)})^{\beta-1}) + y^{(\tau)}(n^{(\tau)})^{\beta}\Big)\,\mathrm{d}x\,\mathrm{d}t.$$

We rephrase the derivatives of $n^{(\tau)}$

$$\begin{aligned}\left|\nabla((n^{(\tau)})^{\beta/2})\right|^2 &= \frac{\beta^2}{4}(n^{(\tau)})^{\beta-2}\left|\nabla n^{(\tau)}\right|^2,\\ \nabla n^{(\tau)}\cdot\nabla((n^{(\tau)})^{\beta-1}) &= (\beta-1)(n^{(\tau)})^{\beta-2}\left|\nabla n^{(\tau)}\right|^2,\\ \nabla((n^{(\tau)})^{\beta}) &= \beta(n^{(\tau)})^{\beta-1}\nabla n^{(\tau)},\\ n^{(\tau)}\nabla((n^{(\tau)})^{\beta-1}) &= (\beta-1)(n^{(\tau)})^{\beta-1}\nabla n^{(\tau)}\end{aligned}$$

to find

$$\frac{4}{\beta^2}(1-\beta)\|\nabla((n^{(\tau)})^{\beta/2})\|^2_{L^2(\Omega_T)} = (1-\beta)\int_0^T\int_\Omega\left|\nabla n^{(\tau)}\right|^2(n^{(\tau)})^{\beta-2}\,\mathrm{d}x\,\mathrm{d}t$$

and thus

$$\begin{aligned}\frac{4}{\beta^2}(1-\beta)\|\nabla((n^{(\tau)})^{\beta/2})\|^2_{L^2(\Omega_T)} &= \int_0^T\int_\Omega D_\tau n^{(\tau)}(n^{(\tau)})^{\beta-1}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_0^T\int_\Omega\frac{1-\beta}{\beta}\nabla((n^{(\tau)})^{\beta})\cdot\nabla S\,\mathrm{d}x\,\mathrm{d}t\\ +\varepsilon\int_0^T\int_\Omega\Delta y^{(\tau)}\Delta((n^{(\tau)})^{\beta-1})+\delta^{-2}&\left|\nabla y^{(\tau)}\right|^2\nabla y^{(\tau)}\cdot\nabla((n^{(\tau)})^{\beta-1}) + y^{(\tau)}(n^{(\tau)})^{\beta}\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{32}$$

where we now estimate each integral on the right separately. For the first integral notice that $f(x):=x^\beta$ is concave so (by reversing the inequality for convex functions we had for (18)) we have $f(x)-f(z)\geq f'(x)(x-z)$, which gives

$$\int_0^T\int_\Omega D_\tau n^{(\tau)}(n^{(\tau)})^{\beta-1}\,\mathrm{d}x\,\mathrm{d}t = \sum_{k=1}^K\int_\Omega(n_k-n_{k-1})n_k^{\beta-1}\,\mathrm{d}x$$
$$\leq\frac{1}{\beta}\sum_{k=1}^K\int_\Omega n_k^\beta-n_{k-1}^\beta\,\mathrm{d}x = \frac{1}{\beta}\int_\Omega(n^{(\tau)}(x,T))^\beta-n_0(x)^\beta\,\mathrm{d}x.$$

We only need to bound $(n^{(\tau)})^\beta \in L^\infty(0,T;L^1(\Omega))$ now, so

$$\begin{aligned}
\|(n^{(\tau)})^\beta\|_{L^1(\Omega)} &= \int_\Omega (n^{(\tau)})^\beta \,\mathrm{d}x \\
&= \int_{[0<n^{(\tau)}<1]} (n^{(\tau)})^\beta \,\mathrm{d}x + \int_{[1\le n^{(\tau)}]} (n^{(\tau)})^\beta \,\mathrm{d}x \\
&\le |\Omega| + \int_\Omega (n^{(\tau)})^{1/2} \,\mathrm{d}x \\
&= |\Omega| + \int_{[0<n^{(\tau)}<e]} (n^{(\tau)})^{1/2} \,\mathrm{d}x + \int_{[e\le n^{(\tau)}]} (n^{(\tau)})^{1/2} \,\mathrm{d}x \\
&\le |\Omega| + e\,|\Omega| + \int_\Omega \left| n^{(\tau)} \log n^{(\tau)} \right| \,\mathrm{d}x,
\end{aligned}$$

which gives the bound

$$\|(n^{(\tau)})^\beta\|_{L^\infty(0,T;L^1(\Omega))} \le |\Omega| + e\,|\Omega| + \|n^{(\tau)} \log n^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))} \le C$$

uniformly by (25). The next integral can be split by the Cauchy–Schwarz inequality and Young's inequality into

$$\int_0^T \int_\Omega \nabla((n^{(\tau)})^\beta) \cdot \nabla S^{(\tau)} \,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{2}\|\nabla((n^{(\tau)})^\beta)\|^2_{L^2(\Omega_T)} + \frac{1}{2}\|\nabla S^{(\tau)}\|^2_{L^2(\Omega_T)},$$

where we already know that $S^{(\tau)} \in L^2(0,T;H^1(\Omega))$ uniformly by Lemma 3. So we only need to check the first norm on the right hand side here. Notice

$$\left|\nabla((n^{(\tau)})^\beta)\right|^2 = \beta^2 (n^{(\tau)})^{2\beta-2} \left|\nabla n^{(\tau)}\right|^2 = \beta^2 (n^{(\tau)})^{-1} (n^{(\tau)})^{2\beta-1} \left|\nabla n^{(\tau)}\right|^2$$

and use the real inequality $x^{2\beta-1} \le \eta x^{\beta-1} + C(\beta,\eta)$ for all $\eta > 0, x > 0$, which we prove in the Appendix (Lemma 23), to get

$$\left|\nabla((n^{(\tau)})^\beta)\right|^2 \le \beta^2 \left|\nabla n^{(\tau)}\right|^2 (n^{(\tau)})^{-1} (\eta (n^{(\tau)})^{\beta-1} + C(\beta,\eta)),$$

which we rewrite using

$$\left|\nabla((n^{(\tau)})^{\beta/2})\right| = \frac{\beta^2}{4} (n^{(\tau)})^{\beta-2} \left|\nabla n^{(\tau)}\right|, \qquad \left|\nabla\sqrt{n^{(\tau)}}\right|^2 = \frac{1}{4}(n^{(\tau)})^{-1} \left|\nabla(n^{(\tau)})\right|^2$$

to conclude

$$\left|\nabla((n^{(\tau)})^\beta)\right| \le 4\eta \left|\nabla((n^{(\tau)})^{\beta/2})\right|^2 + C(\beta,\eta) 4\beta^2 \left|\nabla\sqrt{n^{(\tau)}}\right|^2.$$

Integrating gives the estimate

$$\|\nabla((n^{(\tau)})^\beta)\|^2_{L^2(\Omega_T)} \le 4\eta \|\nabla((n^{(\tau)})^{\beta/2})\|^2_{L^2(\Omega_T)} + C(\beta,\eta)4\beta^2 \|\nabla\sqrt{n^{(\tau)}}\|^2_{L^2(\Omega_T)}.$$

Here, the last term is bounded by Lemma 3 and the other term can be absorbed into the left hand side of (32) if we choose $\eta > 0$ appropriately small (for fixed β). For the last integral in (32) we, again, rewrite derivatives in terms of $y^{(\tau)}$

$$\begin{aligned}
\Delta((n^{(\tau)})^{\beta-1}) &= \Delta e^{y^{(\tau)}(\beta-1)/\delta} = \left(\frac{\beta-1}{\delta}\right)^2 (n^{(\tau)})^{\beta-1} \left|\nabla y^{(\tau)}\right|^2 + \frac{\beta-1}{\delta}(n^{(\tau)})^{\beta-1} \Delta y^{(\tau)}, \\
\nabla((n^{(\tau)})^{\beta-1}) &= \frac{\beta-1}{\delta}(n^{(\tau)})^{\beta-1} \nabla y^{(\tau)}
\end{aligned}$$

to find

$$
\begin{aligned}
&\varepsilon \int_0^T \int_\Omega \Delta y^{(\tau)} \Delta((n^{(\tau)})^{\beta-1}) + \delta^{-2} \left|\nabla y^{(\tau)}\right|^2 \nabla y^{(\tau)} \cdot \nabla((n^{(\tau)})^{\beta-1}) \,\mathrm{d}x \,\mathrm{d}t \\
&= \varepsilon \int_0^T \int_\Omega \Delta y^{(\tau)} \Big(\Big(\frac{\beta-1}{\delta}\Big)^2 (n^{(\tau)})^{\beta-1} \left|\nabla y^{(\tau)}\right|^2 + \frac{\beta-1}{\delta} (n^{(\tau)})^{\beta-1} \Delta y^{(\tau)} \Big) \\
&\quad + \delta^{-2} \left|\nabla y^{(\tau)}\right|^2 \nabla y^{(\tau)} \cdot \frac{\beta-1}{\delta} (n^{(\tau)})^{\beta-1} \nabla y^{(\tau)} \,\mathrm{d}x \,\mathrm{d}t \\
&= -\frac{\varepsilon(1-\beta)}{\delta} \int_0^T \int_\Omega n^{\beta-1} \Big((\Delta y^{(\tau)})^2 + \frac{\beta-1}{\delta} \Delta y^{(\tau)} \left|\nabla y^{(\tau)}\right|^2 + \delta^{-2} \left|\nabla y^{(\tau)}\right|^4 \Big) \,\mathrm{d}x \,\mathrm{d}t \\
&= -\frac{\varepsilon(1-\beta)}{\delta} \int_0^T \int_\Omega n^{\beta-1} \Big(\big(\frac{1}{2}(\beta-1)\Delta y^{(\tau)} + \delta^{-1} \left|\nabla y^{(\tau)}\right|^2 \big)^2 \\
&\qquad\qquad + \big(1 - \frac{(\beta-1)^2}{4}\big)(\Delta y^{(\tau)})^2 \Big) \,\mathrm{d}x \,\mathrm{d}t \\
&\leq 0
\end{aligned}
$$

as a sum of two squares. Now, there is only one term left to estimate

$$
\varepsilon \int_0^T \int_\Omega y^{(\tau)} (n^{(\tau)})^\beta \,\mathrm{d}x \,\mathrm{d}t = \varepsilon\delta \int_0^T \int_\Omega (n^{(\tau)})^\beta \log n^{(\tau)} \,\mathrm{d}x \,\mathrm{d}t,
$$

which we bound using the inequality[13]

$$
\left|x^\beta \log x\right| \leq C(1+x^2)
$$

to get

$$
\begin{aligned}
\varepsilon \int_0^T \int_\Omega y^{(\tau)} (n^{(\tau)})^\beta \,\mathrm{d}x \,\mathrm{d}t &\leq \varepsilon\delta \int_0^T \int_\Omega C(1 + (n^{(\tau)})^2) \,\mathrm{d}x \,\mathrm{d}t \\
&= \varepsilon\delta C \big(T\left|\Omega\right| + \|n^{(\tau)}\|^2_{L^2(\Omega_T)}\big) \\
&\leq \varepsilon\delta C
\end{aligned}
$$

as a consequence of Lemma 3. We have now proven that

$$
\|\nabla((n^{(\tau)})^{\beta/2})\|_{L^2(\Omega_T)} \leq C
$$

is uniformly bounded. For the full $H^1(\Omega)$-norm we quickly check

$$
\begin{aligned}
\|(n^{(\tau)})^{\beta/2}\|^2_{L^2(\Omega_T)} &= \|(n^{(\tau)})^\beta\|_{L^1(\Omega_T)} \\
&= \iint_{[n^{(\tau)} \leq 1]} (n^{(\tau)})^\beta \,\mathrm{d}x \,\mathrm{d}t + \iint_{[n^{(\tau)} > 1]} (n^{(\tau)})^\beta \,\mathrm{d}x \,\mathrm{d}t \\
&\leq \left|\Omega\right| T + \|\sqrt{n^{(\tau)}}\|_{L^1(\Omega_T)} \\
&\leq \left|\Omega\right| T + C\|\sqrt{n^{(\tau)}}\|_{L^2(\Omega_T)} \leq C
\end{aligned}
$$

where the last uniform bound is also a consequence of Lemma 3. This concludes the first regularity result with $\mu = \beta/2$.

[13] To see its validity consider $\frac{1+x^2}{|x^\beta \log x|}$ as $x \to \infty$. Since this goes to infinity, we get some x_0 such that for all $x \geq x_0$ we have $1 + x^2 \geq \left|x^\beta \log x\right|$ (i.e., with $C = 1$). For the rest set $C := \max\{\max_{[0,x_0]} x^\beta \log x, 1\}$, which is finite because $x^\beta \log x$ is continuous.

For the second bound, use the test function $\frac{1}{n^{(\tau)}} = e^{-y^{(\tau)}/\delta}$ (which is fine, because $n^{(\tau)} > 0$ a.e.) in (22) to get

$$\int_0^T \int_\Omega D_\tau n^{(\tau)} \frac{1}{n^{(\tau)}} \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega -\nabla n^{(\tau)} \cdot \nabla \frac{1}{n^{(\tau)}} + n^{(\tau)} \nabla S^{(\tau)} \cdot \nabla \frac{1}{n^{(\tau)}} \\ - \varepsilon\big(\Delta y^{(\tau)} \Delta \frac{1}{n^{(\tau)}} + \delta^{-2} \left|\nabla y^{(\tau)}\right|^2 \nabla y^{(\tau)} \cdot \nabla \frac{1}{n^{(\tau)}} + y\big) \, \mathrm{d}x \, \mathrm{d}t,$$

which we rewrite using

$$\nabla \frac{1}{n^{(\tau)}} = -\frac{1}{(n^{(\tau)})^2} \nabla n, \quad \nabla y^{(\tau)} = \delta \frac{1}{n^{(\tau)}} \nabla n^{(\tau)}$$

and rearrange to get

$$\begin{aligned} \delta^{-2} \|\nabla y^{(\tau)}\|^2_{L^2(\Omega_T)} &= \int_0^T \int_\Omega \frac{\left|\nabla n^{(\tau)}\right|^2}{n^2} \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_0^T \int_\Omega D_\tau n^{(\tau)} \frac{1}{n^{(\tau)}} \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega \frac{1}{n^{(\tau)}} \nabla S^{(\tau)} \cdot \nabla n^{(\tau)} \, \mathrm{d}x \, \mathrm{d}t \\ &+ \int_0^T \int_\Omega \varepsilon\big(\Delta y^{(\tau)} \Delta \frac{1}{n^{(\tau)}} + \delta^{-2} \left|\nabla y^{(\tau)}\right|^2 \nabla y^{(\tau)} \cdot \nabla \frac{1}{n^{(\tau)}}\big) + y \, \mathrm{d}x \, \mathrm{d}t, \end{aligned} \tag{33}$$

which we will bound term by term. First,

$$\int_0^T \int_\Omega \frac{D_\tau n^{(\tau)}}{n^{(\tau)}} \, \mathrm{d}x \, \mathrm{d}t = \sum_{k=1}^K \int_\Omega \frac{n_k - n_{k-1}}{n_k} \, \mathrm{d}x = \sum_{k=1}^K \int_\Omega 1 - \frac{n_{k-1}}{n_k} \, \mathrm{d}x,$$

where we now apply the inequality[14] $1 - x \le -\log x$ for $x > 0$ two times (and with opposite signs) and get

$$\begin{aligned} \int_0^T \int_\Omega \frac{D_\tau n^{(\tau)}}{n^{(\tau)}} \, \mathrm{d}x \, \mathrm{d}t &\le \sum_{k=1}^K \int_\Omega \log n_k - \log n_{k-1} \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_\Omega \log n^{(\tau)}(x,T) - \log n_0(x) \, \mathrm{d}x \\ &\le \int_\Omega n^{(\tau)}(x,T) - 1 - \log n_0(x) \, \mathrm{d}x \le C, \end{aligned}$$

which is bounded because $n^{(\tau)} \in L^\infty(0,T;L^1(\Omega))$ uniformly (by the $n^{(\tau)} \log^{(\tau)}$-bound in the same space from Lemma 3), and our additional assumption $\log n_0 \in L^1(\Omega)$. The next integral is estimated by the Cauchy–Schwarz inequality and Young's inequality

$$\begin{aligned} \int_0^T \int_\Omega \frac{1}{n^{(\tau)}} \nabla S^{(\tau)} \cdot \nabla n^{(\tau)} \, \mathrm{d}x \, \mathrm{d}t &= \int_0^T \int_\Omega \nabla S^{(\tau)} \cdot \nabla y^{(\tau)} \delta^{-1} \\ &\le \frac{1}{2} \|\nabla S^{(\tau)}\|^2_{L^2(\Omega_T)} + \frac{1}{2\delta^2} \|\nabla y^{(\tau)}\|^2_{L^2(\Omega_T)}, \end{aligned}$$

[14]Consider $f(x) := 1 - x + \log x$. Taking derivatives we get $f'(x) = -1 + 1/x$. Wanting to find its maximum we get that $x = 1$ is the only critical point. Checking the second derivative we find $f''(1) = -1$ so we have a (local) maximum at this point. The function value at this point is $f(1) = 0$. Now consider $\lim_{x\to\infty} f(x) = -\infty$ by e.g. de l'Hospital, and $\lim_{x\to 0^+} f(x) = -\infty$ because of the logarithm, to check potential boundary extrema.

where the first term is bounded by Lemma 3 and the second term can be absorbed by the left hand side of (33). The third integral in (33) we rewrite using

$$\Delta \frac{1}{n^{(\tau)}} = \delta^{-2} \left|\nabla y^{(\tau)}\right|^2 \frac{1}{n^{(\tau)}} - \delta^{-1} \frac{1}{n^{(\tau)}} \Delta y^{(\tau)}, \quad \nabla \frac{1}{n^{(\tau)}} = -\delta^{-1} \frac{1}{n^{(\tau)}} \nabla y^{(\tau)}$$

to get

$$\begin{aligned}
&\varepsilon \int_0^T \int_\Omega \Delta y^{(\tau)} \Delta \frac{1}{n^{(\tau)}} + \delta^{-2} \left|\nabla y^{(\tau)}\right|^2 \nabla y^{(\tau)} \cdot \nabla \frac{1}{n^{(\tau)}} \,\mathrm{d}x\,\mathrm{d}t \\
&= -\frac{\varepsilon}{\delta} \int_0^T \int_\Omega \Delta y^{(\tau)} \big(\frac{1}{n} \Delta y^{(\tau)} - \delta^{-1} \left|\nabla y^{(\tau)}\right|^2 \frac{1}{n^{(\tau)}}\big) + \delta^{-2} \left|\nabla y^{(\tau)}\right|^4 \frac{1}{n^{(\tau)}} \,\mathrm{d}x\,\mathrm{d}t \\
&= -\frac{\varepsilon}{\delta} \int_0^T \int_\Omega \frac{1}{n^{(\tau)}} \Big(\big(\frac{1}{2} \Delta y^{(\tau)} - \delta^{-1} \left|\nabla y^{(\tau)}\right|^2\big)^2 + \frac{3}{4} (\Delta y^{(\tau)})^2 \Big) \,\mathrm{d}x\,\mathrm{d}t \\
&\leq 0.
\end{aligned}$$

The last term remaining can quickly be estimated by Lemma 3 and

$$\begin{aligned}
\varepsilon \int_0^T \int_\Omega y^{(\tau)} \,\mathrm{d}x\,\mathrm{d}t &= \varepsilon\delta \int_0^T \int_\Omega \log n^{(\tau)} \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \varepsilon\delta \int_0^T \int_\Omega \left|n^{(\tau)}\right| \,\mathrm{d}x\,\mathrm{d}t \\
&= \varepsilon\delta \|n^{(\tau)}\|_{L^1(\Omega_T)} = \varepsilon\delta \|\sqrt{n^{(\tau)}}\|^2_{L^2(\Omega_T)} \\
&\leq \varepsilon\delta \|\sqrt{n^{(\tau)}}\|^2_{L^2(0,T;H^1(\Omega))}.
\end{aligned}$$

This bounds

$$\|\nabla \log n\|_{L^2(\Omega_T)}$$

uniformly and concludes the proof. □

Remark. *The exponent bound $\alpha < \frac{3}{2}$ is almost optimal when looking for entropy solutions. If one considers equation (16) with $\sigma = 1$ and taking the limit $(\varepsilon, \tau) \to (0,0)$, (i.e., testing the equation with the function (n, S) itself), then one needs*

$$\int_\Omega n^\alpha S \,\mathrm{d}x$$

to exist. The highest regularity (for the parabolic-parabolic model; for the parabolic-elliptic model we can show even more cf. Section 4) we got in the end is

$$S \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)), \qquad n \in L^2(0,T; L^2(\Omega)).$$

By the Gagliardo–Nirenberg inequality $L^4 \subseteq H^1 \cap L^2$ with $\theta = \frac{1}{2}$ we get

$$\|S\|_{L^4(\Omega)} \leq C \|S\|^{1/2}_{H^1(\Omega)} \|S\|^{1/2}_{L^2(\Omega)}.$$

This implies that $S \in L^4(0,T; L^4(\Omega))$ by

$$\begin{aligned}
\|S\|_{L^4(0,T;L^4(\Omega))} \leq C \|(\|S\|_{H^1(\Omega)} \|S\|_{L^2(\Omega)})^{1/2}\|_{L^4(0,T)} &= C \|(\|S\|_{H^1(\Omega)} \|S\|_{L^2(\Omega)})\|^{1/2}_{L^2(0,T)} \\
\leq C \|S\|^{1/2}_{L^\infty(0,T;L^2(\Omega))} \|S\|^{1/2}_{L^2(0,T;H^1(\Omega))}.
\end{aligned}$$

For $n^\alpha S$ to be in $L^1(\Omega_T)$ we thus need $n^\alpha \in L^{4/3}(\Omega_T)$, or, in terms of n, that $n \in L^{4\alpha/3}(\Omega_T)$. The highest value of α such that $L^{4\alpha/3} \subseteq L^2(\Omega_T)$ is precisely $\alpha = 3/2$. Hence, we cannot expect to get existence for parameter values higher than $\alpha = 3/2$, although the edge case itself might be possible.

4 Regularity for the parabolic-elliptic model

In Section 3 we have already proven the existence of solutions even in the case $\gamma = 0$, which is called the parabolic-elliptic model. However, we can achieve much better regularity. This is the goal of this section. In particular, we will show that solutions are in fact bounded, which naturally comes with the requirement that the initial datum[15] is bounded as well.

Theorem 7. *Let $T > 0, \delta > 0, \alpha \in [1, 3/2)$, and $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with boundary $\partial\Omega \in C^{1,1}$. Assume that $0 \le n_0 \in L^\infty(\Omega)$. Then there exists a weak solution (n, S) to the parabolic-elliptic system*

$$n_t = \operatorname{div}(\nabla n - n\nabla S) \tag{34}$$
$$0 = \Delta S + \delta\Delta n + n^\alpha - S \tag{35}$$

on $\Omega_T := \Omega \times (0, T)$, with Neumann boundary conditions

$$\nabla n \cdot \nu = \nabla S \cdot \nu = 0 \quad \text{on } \partial\Omega, t > 0$$

and initial condition

$$n(\cdot, 0) = n_0 \quad \text{in } \Omega.$$

Additionally, solutions have the following regularity

$$n \in L^2(0, T; H^1(\Omega)), \quad n \in L^\infty(\Omega_T), \tag{36}$$
$$S \in L^2(0, T; H^1(\Omega)), \quad S \in L^\infty(\Omega_T), \tag{37}$$
$$S + \delta n \in L^\infty(0, T; W^{1,\infty}(\Omega)) \tag{38}$$

in addition to the ones described in Section 3.4 (with $\gamma = 0$).

Proof. The proof uses a change of variables $v := S + \delta n$, which leads to the quasilinear system

$$n_t = \operatorname{div}((1 + \delta n)\nabla n - n\nabla v) \tag{39}$$
$$0 = \Delta v + n^\alpha - v + \delta n \tag{40}$$

with boundary conditions

$$\nabla n \cdot \nu = \nabla v \cdot \nu = 0$$

and the same initial condition $n(\cdot, 0) = n_0$. The existence of solutions is still guaranteed by Section 3. In order to prove the additional regularity we will go back into the proof, namely we look at the implicit Euler discretization with parameter $\tau > 0$ (for the time derivative) and also add regularizing ε-terms ($\varepsilon > 0$) to the first

[15]Note that in the parabolic-elliptic case we only have an initial condition on n, and not on S.

equation (39). This yields the recursive system (where $y = \delta \log n, y_k = \delta \log n_k$)

$$\begin{aligned}\frac{1}{\tau}(n_k - n_{k-1}) =& \operatorname{div}((1+\delta n_k)\nabla n_k - n_k \nabla v_k) \\ &- \varepsilon\big(\Delta^2 y_k - \delta^{-2}\operatorname{div}(|\nabla y_k|^2 \nabla y_k) + y_k n_k\big)\end{aligned} \tag{41}$$

$$0 = \Delta v_k + n_k^\alpha - v_k + \delta n_k, \tag{42}$$

where the weak formulation is given by

$$\begin{aligned}\frac{1}{\tau}\int_\Omega (n_k - n_{k-1})\varphi \,\mathrm{d}x =& \int_\Omega -(1+\delta n_k)\nabla n_k \cdot \nabla\varphi + n_k \nabla v_k \cdot \nabla \varphi \,\mathrm{d}x \\ &- \varepsilon \int_\Omega \Delta y_k \Delta \varphi + \delta^{-2} |\nabla y_k|^2 \nabla y_k \cdot \nabla \varphi + y_k n_k \varphi \,\mathrm{d}x\end{aligned} \tag{43}$$

$$0 = \int_\Omega -\nabla v_k \cdot \nabla \vartheta + n_k^\alpha \vartheta - v_k \vartheta + \delta n_k \vartheta \,\mathrm{d}x \tag{44}$$

for all $\varphi \in H^2(\Omega)$ and $\vartheta \in H^1(\Omega)$. If we can find estimates uniform in ε and τ, they will also apply to the limit $(\varepsilon, \tau) \to (0,0)$ (which exists by Section 3).

Step 1: $(n_k, v_k) \in H^2(\Omega) \times H^1(\Omega)$. By Theorem 2, for given $n_{k-1} \in L^1(\Omega)$ we deduce the existence of $(y_k, S_k) \in H^2(\Omega) \times H^1(\Omega)$. We shall rephrase this in terms of regularity for n_k and v_k, namely $(n_k, v_k) \in H^2(\Omega) \times H^1(\Omega)$. To see this we notice that $y_k = \delta \log n_k \in H^2(\Omega)$ implies

$$\log n_k \in L^2(\Omega), \quad \frac{1}{n_k}\nabla n_k \in L^2(\Omega), \quad -\frac{1}{n_k^2}|\nabla n_k|^2 + \frac{1}{n_k}\Delta n_k \in L^2(\Omega).$$

By the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we also find $0 < C_k \le n_k \le \exp(\|y_k\|_{L^\infty(\Omega)}/\delta)$, and $n_k \in L^\infty(\Omega)$. This L^∞-bound gives $n_k \in L^2(\Omega)$. The other terms can be estimated by

$$\|\nabla n_k\|_{L^2(\Omega)} = \|n_k\|_{L^\infty(\Omega)} \|\frac{1}{\|n_k\|_{L^\infty(\Omega)}}\nabla n_k\|_{L^2(\Omega)} \le \|n_k\|_{L^\infty(\Omega)}\|\frac{1}{n_k}\nabla n_k\|_{L^2(\Omega)},$$

and by the triangle inequality

$$\begin{aligned}\|\Delta n_k\|_{L^2(\Omega)} &\le \|n_k\|_{L^\infty(\Omega)}\|\frac{1}{n_k}\Delta n_k\|_{L^2(\Omega)} \\ &\le \|n_k\|_{L^\infty(\Omega)}\big(\|\frac{1}{n_k}\Delta n_k - \frac{1}{n_k^2}|\nabla n_k|^2\|_{L^2(\Omega)} + \|\frac{1}{n_k^2}|\nabla n_k|^2\|_{L^2(\Omega)}\big) \\ &\le C(k) + C(k) C_k \|\nabla n_k\|^4_{L^4(\Omega)},\end{aligned}$$

which we estimate by Sobolev embedding $H^2(\Omega) \hookrightarrow W^{1,p}(\Omega)$ for any $p < \infty$, especially $y_k \in W^{1,4}(\Omega)$ and $\nabla y_k \in L^4(\Omega)$. Noting $\nabla y_k = \frac{\delta}{n_k}\nabla n_k$ we get

$$\|\nabla n_k\|_{L^4(\Omega)} \le \frac{1}{\delta}\|n_k\|_{L^\infty(\Omega)}\|\frac{\delta}{n_k}\nabla n_k\|_{L^4(\Omega)},$$

which is bounded, and finally implies $n_k \in H^2(\Omega)$. Since $v_k = S_k + \delta n_k$ and the right hand side is the sum of functions in (at least) $H^1(\Omega)$, we conclude that $v_k \in H^1(\Omega)$.

Step 2: $v_k \ge 0$. For future estimates we would like v_k to have a distinct sign. Denote u^- the negative part of u, that is $u^- := \min\{u, 0\} \le 0$. By the Stampacchia lemma (Lemma 24) $v_k \in H^1(\Omega)$ implies $v_k^- \in H^1(\Omega)$ with $\nabla v_k^- = \mathbb{1}_{[v_k \le 0]} \nabla v_k$, so we use it as a test function in (44)

$$\int_\Omega \left|\nabla v_k^-\right|^2 \,\mathrm{d}x = \int_\Omega n_k^\alpha v_k^- - (v_k^-)^2 + \delta n_k v_k^- \le 0$$

(because $n_k \ge 0$), so $v_k^- \equiv c$ in Ω for some constant $c \le 0$. We would like to conclude $v_k^- \equiv 0$. Consider equation (42) as an equation for v_k with given $n_k \in L^\infty(\Omega)$

$$-\Delta v_k + v_k = n_k^\alpha + \delta n_k,$$

so the right hand side is an L^∞-function, in particular it is L^2. With the Neumann boundary conditions, we can deduce by elliptic regularity [11, Theorem 2.4.2.7] (here we need $\partial\Omega \in C^{1,1}$) that $v_k \in W^{2,2}(\Omega)$. By Sobolev embedding $W^{2,2}(\Omega) \hookrightarrow C^0(\Omega)$, we get that either $v_k \equiv c$ or that $v_k^- \equiv 0$. In the first case, by equation (42) this would give $0 \ge c = n_k^\alpha + \delta n_k \ge 0$. So, in fact, $v_k^- \equiv 0$ and thus $v_k \ge 0$.
Step 3: $n \in L^2(0,T;H^1(\Omega))$. So far, we only showed regularity for each time step (i.e., for fixed k with bounds potentially depending on k). In the end we also want regularity in time. Take the test function $\varphi = y_k/\delta = \log n_k$ in (43) to get

$$\begin{aligned}\frac{1}{\tau}\int_\Omega (n_k - n_{k-1})\log n_k \,\mathrm{d}x = &\int_\Omega -(1+\delta n_k)\nabla n_k \cdot \nabla(\log n_k) + n_k \nabla v_k \cdot \nabla(\log n_k)\,\mathrm{d}x \\ &-\varepsilon\int_\Omega \Delta y_k \Delta(y_k/\delta) + \delta^{-2}\left|\nabla y_k\right|^2 \nabla y_k \nabla(y_k/\delta) + y_k^2 n_k/\delta \,\mathrm{d}x,\end{aligned}$$

which we rewrite as

$$\begin{aligned}\frac{1}{\tau}\int_\Omega (n_k - n_{k-1})\log n_k \,\mathrm{d}x + \int_\Omega (1+\delta n_k)\frac{\left|\nabla n_k\right|^2}{n_k}\,\mathrm{d}x = &\int_\Omega \nabla v_k \cdot \nabla n_k \,\mathrm{d}x \\ &-\frac{\varepsilon}{\delta}\int_\Omega (\Delta y_k)^2 + \delta^{-2}\left|\nabla y_k\right|^4 + y_k^2 n_k \,\mathrm{d}x.\end{aligned}$$

In order to get rid of the mixed term $\nabla v_k \cdot \nabla n_k$, we test the second equation (44) with $\vartheta = n_k$

$$0 = \int_\Omega -\nabla v_k \cdot \nabla n_k + n_k^{\alpha+1} + \delta n_k^2 - v_k n_k \,\mathrm{d}x$$

and add them up

$$\begin{aligned}&\frac{1}{\tau}\int_\Omega (n_k - n_{k-1})\log n_k \,\mathrm{d}x + \int_\Omega (1+\delta n_k)\frac{\left|\nabla n_k\right|^2}{n_k}\,\mathrm{d}x \\ &\qquad= -\frac{\varepsilon}{\delta}\int_\Omega (\Delta y_k)^2 + \delta^{-2}\left|\nabla y_k\right|^4 + y_k^2 n_k \,\mathrm{d}x + \int_\Omega n_k^{\alpha+1} + \delta n_k^2 - v_k n_k \,\mathrm{d}x \\ &\qquad\le \int_\Omega n_k^{\alpha+1} + \delta n_k^2 \,\mathrm{d}x,\end{aligned}$$

where we used $v_k, n_k \ge 0$. We want to estimate the left hand side; one could try to use the higher order terms on the right hand side to get more regularity or make the estimate easier. However, we need estimates uniform in ε, so we unfortunately cannot use these terms. Instead, we shall use the Gagliardo-Nirenberg inequality and absorb some parts into the left hand side. First, we estimate the left hand side by using the convexity of $\phi(x) = x(\log x - 1)$ as for (18), and then estimate $(1 + \delta n_k) \geq \delta n_k$ in the second integral,

$$\frac{1}{\tau}\int_\Omega n_k(\log n_k - 1) - n_{k-1}(\log n_{k-1} - 1)\,\mathrm{d}x + \delta\int_\Omega |\nabla n_k|^2\,\mathrm{d}x \leq \int_\Omega n_k^{\alpha+1} + \delta n_k^2\,\mathrm{d}x.$$

The worst term on the right hand side is $n_k^{\alpha+1}$ which we need to estimate[16]. But first, we estimate the square term n_k^2 into the first by

$$\delta\|n_k^2\|_{L^1(\Omega)} \leq \|n^2\|_{L^{(\alpha+1)/2}(\Omega)}\,|\Omega|^{\alpha-1} = \|n\|^2_{L^{\alpha+1}(\Omega)}\,|\Omega|^{\alpha-1} \leq \frac{1}{2}\|n\|^{\alpha+1}_{L^{\alpha+1}(\Omega)} + C$$

by Young's inequality. So we need to estimate $C\|n_k\|^{\alpha+1}_{L^{\alpha+1}(\Omega)}$. Using the Gagliardo-Nirenberg inequality[17] $L^{\alpha+1}(\Omega) \subseteq H^1(\Omega) \cap L^1(\Omega)$ with $\theta = 1 - \frac{1}{\alpha+1}$ we get

$$\|n_k\|^{\alpha+1}_{L^{\alpha+1}(\Omega)} \leq \left(C\|n_k\|^\theta_{H^1(\Omega)}\|n_k\|^{1-\theta}_{L^1(\Omega)}\right)^{\alpha+1} = C\|n_k\|^\alpha_{H^1(\Omega)}\|n_k\|_{L^1(\Omega)}.$$

Since we only have the H^1-seminorm on the left hand side, we split up the H^1 norm here, and use the equivalence of norms on finite-dimensional spaces (here with a constant $2^{\alpha/2}$) to get

$$\begin{aligned}
C\|n_k\|^\alpha_{H^1(\Omega)}\|n_k\|_{L^1(\Omega)} &= C\left(\|\nabla n_k\|^2_{L^2(\Omega)} + \|n_k\|^2_{L^2(\Omega)}\right)^{\alpha/2}\|n_k\|_{L^1(\Omega)} \\
&\leq C\left(\|\nabla n_k\|^\alpha_{L^2(\Omega)} + \|n_k\|^\alpha_{L^2(\Omega)}\right)\|n_k\|_{L^1(\Omega)} \\
&= C\|\nabla n_k\|^\alpha_{L^2(\Omega)}\|n_k\|_{L^1(\Omega)} + C\|n_k\|^\alpha_{L^2(\Omega)}\|n_k\|_{L^1(\Omega)},
\end{aligned}$$

where we apply Young's inequality with $\delta/2$ and $p = \frac{2}{\alpha}$ to get[18]

$$C\|\nabla n_k\|^\alpha_{L^2(\Omega)}\|n_k\|_{L^1(\Omega)} \leq \frac{\delta}{2}\|\nabla n_k\|^2_{L^2(\Omega)} + C(\delta,\alpha)\|n_k\|^q_{L^1(\Omega)},$$

where $q = (1 - 2/\alpha)^{-1} \in (1, \infty)$. Applying the same inequality (with a different "ε") to the other term gives

$$\|n_k\|^{\alpha+1}_{L^{\alpha+1}(\Omega)} \leq \frac{\delta}{2}\|\nabla n_k\|^2_{L^2(\Omega)} + C\|n_k\|^q_{L^1(\Omega)} + \|n_k\|^2_{L^2(\Omega)}.$$

Hence, we can absorb the first term on the right hand side, and then solve the recursion w.r.t. k (i.e., integrating in time) to get (with the same notation as in Section 3, i.e., $n^{(\tau)}$ is a piecewise-constant-in-time function with piecewise values n_k)

$$\begin{aligned}
&\int_\Omega n^{(\tau)}(t)(\log n^{(\tau)}(t) - 1)\,\mathrm{d}x + \frac{\delta}{2}\int_0^t\int_\Omega \left|\nabla n^{(\tau)}\right|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\leq C\|n^{(\tau)}\|^q_{L^\infty(0,t;L^1(\Omega))} + \|n_k\|^2_{L^2(0,t;L^2(\Omega)} + \int_\Omega n_0(\log n_0 - 1)\,\mathrm{d}x
\end{aligned}$$

[16]This line of arguments needs $\alpha \geq 1$. However, it is not necessary: For smaller values of α one could do the same calculation the other way around and estimate everything into the (much nicer) n_k^2 term.

[17]Here we only need $\alpha \geq 0$.

[18]Here we only need $0 < \alpha < 2$.

for any $0 < t < T$, where we can increase the right hand side by setting $t = T$ and then preserve the estimate when doing the same in the second integral on the left hand side. Since the right hand side and also the first term on the left hand side are uniformly bounded by Lemma 3 (or see Section 3.4), and our assumptions on n_0, we conclude that (after taking limits $(\varepsilon, \tau) \to (0,0)$)

$$\nabla n \in L^2(0,T;L^2(\Omega)).$$

But since we already know (as a consequence of Lemma 3) that $n \in L^2(0,T;L^2(\Omega))$, we can conclude

$$n \in L^2(0,T;H^1(\Omega)).$$

Step 4: $n \in L^\infty(0,T;L^2(\Omega))$. We argue similarly as in the previous step. Test the first equation (43) with $\varphi = n_k$

$$\begin{aligned}\frac{1}{\tau}\int_\Omega (n_k - n_{k-1})n_k \,\mathrm{d}x + \int_\Omega (1+\delta n_k)\left|\nabla n_k\right|^2 \,\mathrm{d}x &= \int_\Omega n_k \nabla v_k \cdot \nabla n_k \,\mathrm{d}x \\ &\quad - \varepsilon \int_\Omega \Delta y_k \Delta n_k + \delta^{-2}\left|\nabla y_k\right|^2 \nabla y_k \cdot \nabla n_k + y_k n_k^2 \,\mathrm{d}x,\end{aligned}$$

where we rewrite most derivatives of n_k in terms of y_k as

$$\nabla n_k = \frac{1}{\delta} n_k \nabla n_k, \quad \Delta n_k = \frac{1}{\delta^2} n_k \left|\nabla y_k\right|^2 + \frac{1}{\delta} n_k \Delta y_k$$

to get

$$\begin{aligned}\frac{1}{\tau}\int_\Omega (n_k - n_{k-1})n_k \,\mathrm{d}x + \int_\Omega (1+\delta n_k)\left|\nabla n_k\right|^2 \,\mathrm{d}x &= \int_\Omega \frac{1}{2}\nabla v_k \cdot \nabla(n_k^2) \,\mathrm{d}x \\ -\frac{\varepsilon}{\delta}\int_\Omega n_k\big((\Delta y_k)^2 + \delta^{-1}\Delta y_k \left|\nabla y_k\right|^2 + \delta^{-2}\left|\nabla y_k\right|^4\big) \,\mathrm{d}x &- \varepsilon \int_\Omega y_k n_k^2 \,\mathrm{d}x.\end{aligned}$$

Again, we want to get rid of the mixed term $\nabla v_k \cdot \nabla(n_k^2)$, so we test the second equation (44) with $\vartheta = n_k^2/2$

$$0 = \int_\Omega -\frac{1}{2}\nabla v_k \cdot \nabla(n_k^2) + \frac{1}{2} n_k^{\alpha+2} - \frac{1}{2} v_k n_k^2 + \frac{\delta}{2} n_k^3 \,\mathrm{d}x$$

and add them together

$$\begin{aligned}\frac{1}{\tau}\int_\Omega (n_k - n_{k-1})n_k \,\mathrm{d}x + \int_\Omega (1+\delta n_k)\left|\nabla n_k\right|^2 \,\mathrm{d}x &= \int_\Omega \frac{1}{2} n_k^{\alpha+2} - \frac{1}{2} v_k n_k^2 + \frac{\delta}{2} n_k^3 \,\mathrm{d}x \\ -\frac{\varepsilon}{\delta}\int_\Omega n_k\big((\Delta y_k)^2 + \delta^{-1}\Delta y_k \left|\nabla y_k\right|^2 + \delta^{-2}\left|\nabla y_k\right|^4\big) \,\mathrm{d}x &- \varepsilon \int_\Omega y_k n_k^2 \,\mathrm{d}x.\end{aligned}$$

Two calculations are needed to further estimate this appropriately:

$$0 \le (\Delta y_k + \delta^{-1}\left|\nabla y_k\right|^2)^2 = (\Delta y_k)^2 + 2\delta^{-1}\Delta y_k \left|\nabla y_k\right|^2 + \delta^{-2}\left|\nabla y_k\right|^4,$$

so for the second integral on the right hand side

$$-\big((\Delta y_k)^2 + \delta^{-1}\Delta y_k \left|\nabla y_k\right|^2 + \delta^{-2}\left|\nabla y_k\right|^4\big) \le -\frac{1}{2}\big((\Delta y_k)^2 + \delta^{-2}\left|\nabla y_k\right|^4\big) \le 0,$$

and

$$\left|\nabla(n_k^{3/2})\right|^2 = \frac{9}{4} n_k \left|\nabla n_k\right|^2,$$

which gives (together with the analogous estimates on the left hand side as before)

$$\frac{1}{\tau}\int_\Omega \frac{1}{2}(n_k^2 - n_{k-1}^2)\,\mathrm{d}x + \frac{4\delta}{9}\int_\Omega \left|\nabla(n_k^{3/2})\right|^2 \mathrm{d}x \le \int_\Omega \frac{1}{2}(n_k^{\alpha+2} + \delta n_k^3)\,\mathrm{d}x - \varepsilon \int_\Omega y_k n_k^2\,\mathrm{d}x.$$

The second integral on the right hand side does not have a sign, but can still be bounded ($\varepsilon < 1$)

$$\varepsilon \int_\Omega -y_k n_k^2\,\mathrm{d}x = \varepsilon\delta \int_\Omega -n_k^2 \log n_k\,\mathrm{d}x \le \varepsilon\delta \int_\Omega \max_{z\in[0,\infty)} -z^2\log z\,\mathrm{d}x \le \delta\,|\Omega|\,\frac{1}{2e}$$

The first integral on the right hand side can again be condensed into one term with the worse exponent. So it remains to estimate

$$C\|n_k\|_{L^{\alpha+2}(\Omega)}^{\alpha+2} = C\|n_k^{3/2}\|_{L^{2(\alpha+2)/3}(\Omega)}^{2(\alpha+2)/3},$$

which we do by extensive use of the Gagliardo-Nirenberg inequality[19] $L^{2(\alpha+2)/3}(\Omega) \subseteq H^1(\Omega)\cap L^1(\Omega)$ with $\theta = 1 - \frac{3}{2(\alpha+2)}$

$$\begin{aligned} C\|n_k^{3/2}\|_{L^{2(\alpha+2)/3}(\Omega)}^{2(\alpha+2)/3} &\le \left(C\|n_k^{3/2}\|_{H^1(\Omega)}^{\theta}\|n_k^{3/2}\|_{L^1(\Omega)}^{1-\theta}\right)^{2(\alpha+2)/3} \\ &= C\|n_k^{3/2}\|_{H^1(\Omega)}^{(2\alpha+1)/3}\|n_k^{3/2}\|_{L^1(\Omega)}, \end{aligned}$$

which we expand to get the H^1-seminorm again

$$\le C\|\nabla(n_k^{3/2})\|_{L^2(\Omega)}^{(2\alpha+1)/3}\|n_k^{3/2}\|_{L^1(\Omega)} + C\|n_k^{3/2}\|_{L^2(\Omega)}^{(2\alpha+1)/3}\|n_k^{3/2}\|_{L^1(\Omega)}.$$

Applying Young's inequality with $\frac{2\delta}{9}$ and $p = 6/(2\alpha+1)$ gives[20]

$$\le \frac{2\delta}{9}\|\nabla(n_k^{3/2})\|_{L^2(\Omega)}^2 + C\|n_k^{3/2}\|_{L^1(\Omega)}^{6/(5-2\alpha)} + C\|n_k^{3/2}\|_{L^2(\Omega)}^{(2\alpha+1)/3}\|n_k^{3/2}\|_{L^1(\Omega)}.$$

We will absorb the first term into the original left hand side. The other terms need more treatment. Using the Gagliardo-Nirenberg inequality for

$$\|n_k^{3/2}\|_{L^1(\Omega)}^{6/(5-2\alpha)} = \|n_k\|_{L^{3/2}(\Omega)}^{9/(5-2\alpha)}$$

into $H^1(\Omega)\cap L^1(\Omega)$ with $\theta = \frac{1}{3}$ gives

$$\|n_k^{3/2}\|_{L^1(\Omega)}^{6/(5-2\alpha)} \le C\|n_k\|_{H^1(\Omega)}^{3/(5-2\alpha)}\|n_k\|_{L^1(\Omega)}^{6/(5-2\alpha)},$$

where we apply Young's inequality again with $p = \frac{10-4\alpha}{3}$ to get[21]

$$\le \|n_k\|_{H^1(\Omega)}^2 + C\|n_k\|_{L^1(\Omega)}^{12/(7-4\alpha)}.$$

[19] Here we need $\alpha \ge -\frac{4}{3}$.
[20] Here we need $\alpha < \frac{5}{2}$.
[21] Here we would need $\alpha < \frac{7}{4}$, but the concluding argument would still work for $\alpha = \frac{7}{4}$ without Young's inequality.

The last term is treated similarly. Using the Gagliardo-Nirenberg inequality for

$$\|n_k^{3/2}\|_{L^2(\Omega)}^{(2\alpha+1)/3} = \|n_k\|_{L^3(\Omega)}^{(2\alpha+1)/3}$$

into $H^1(\Omega) \cap L^1(\Omega)$ with $\theta = \frac{2}{3}$ gives

$$\|n_k^{3/2}\|_{L^2(\Omega)}^{(2\alpha+1)/2} \leq C\|n_k\|_{H^1(\Omega)}^{(2\alpha+1)/3}\|n_k\|_{L^1(\Omega)}^{(2\alpha+1)/6}.$$

And once more for

$$\|n_k^{3/2}\|_{L^1(\Omega)} = \|n_k\|_{L^{3/2}(\Omega)}^{3/2}$$

into $H^1(\Omega) \cap L^1(\Omega)$ with $\theta = \frac{1}{3}$ gives

$$\|n_k^{3/2}\|_{L^1(\Omega)} \leq C\|n_k\|_{H^1(\Omega)}^{1/2}\|n_k\|_{L^1(\Omega)}.$$

Putting these two together gives

$$C\|n_k^{3/2}\|_{L^2(\Omega)}^{(2\alpha+1)/3}\|n_k^{3/2}\|_{L^1(\Omega)} \leq C\|n_k\|_{H^1(\Omega)}^{(2\alpha+1)/3+1/2}\|n_k\|_{L^1(\Omega)}^{(2\alpha+7)/6}.$$

Since the exponent of the $H^1(\Omega)$-term is less or equal than 2 for $\alpha \leq \frac{7}{4}$ we can use Young's inequality[22] to get

$$C\|n_k^{3/2}\|_{L^2(\Omega)}^{(2\alpha+1)/3}\|n_k^{3/2}\|_{L^1(\Omega)} \leq \|n_k\|_{H^1(\Omega)}^2 + C\|n_k\|_{L^1(\Omega)}^{(4\alpha+14)/(7-4\alpha)}.$$

In total, we can estimate the critical term on the right hand side

$$\begin{aligned} C\int_\Omega n_k^{\alpha+2}\,\mathrm{d}x \leq & \frac{2\delta}{9}\int_\Omega \left|\nabla(n_k^{3/2})\right|^2 \mathrm{d}x + 2\|n_k\|_{H^1(\Omega)}^2 \\ & + C\|n_k\|_{L^1(\Omega)}^{12/(7-4\alpha)} + C\|n_k\|_{L^1(\Omega)}^{(4\alpha+14)/(7-4\alpha)}, \end{aligned}$$

where the first term can be absorbed by the original left hand side, and the other terms will be uniformly bounded after integrating w.r.t. t (by Lemma 3). Finally, this gives

$$\frac{1}{2}\int_\Omega (n^{(\tau)})^2(t)\,\mathrm{d}x + \frac{2\delta}{9}\int_0^t\int_\Omega \left|\nabla((n^{(\tau)})^{3/2})\right|^2 \mathrm{d}x\,\mathrm{d}s \leq C + \frac{1}{2}\int_\Omega n_0^2\,\mathrm{d}x,$$

where the constant on the right hand side does not depend on t, τ or ε. We conclude

$$n \in L^\infty(0,T;L^2(\Omega))$$

by taking the limit $(\varepsilon,\tau) \to (0,0)$.
Step 5: $n \in L^\infty(0,T;L^3(\Omega))$**.** We test the first equation (43) with $\varphi = n_k^2$

$$\begin{aligned} \frac{1}{\tau}\int_\Omega (n_k - n_{k-1})n_k^2\,\mathrm{d}x + \int_\Omega (1+\delta n_k)\nabla n_k \cdot \nabla(n_k^2)\,\mathrm{d}x = & \int_\Omega n_k \nabla v_k \cdot \nabla(n_k^2)\,\mathrm{d}x \\ & - \varepsilon\int_\Omega \Delta y_k \Delta(n_k^2) + \delta^{-2}\left|\nabla y_k\right|^2 \nabla y_k \cdot \nabla(n_k^2) + y_k n_k^3\,\mathrm{d}x, \end{aligned}$$

[22] Here we need $\alpha < \frac{7}{4}$, but the concluding argument would still work for $\alpha = \frac{7}{4}$ without Young's inequality.

which we rewrite similarly as in the last step

$$\frac{1}{\tau}\int_\Omega (n_k - n_{k-1})n_k^2 \,\mathrm{d}x + 2\int_\Omega (1+\delta n_k)n_k \left|\nabla n_k\right|^2 \,\mathrm{d}x = \frac{2}{3}\int_\Omega \nabla v_k \cdot \nabla(n_k^3)\,\mathrm{d}x$$
$$-\varepsilon\int_\Omega n_k^2\Big(\frac{2}{\delta}(\Delta y_k)^2 + \frac{4}{\delta^2}\left|\nabla y_k\right|^2 \Delta y_k + \frac{2}{\delta^3}\left|\nabla y_k\right|^4 + y_k n_k\Big)\,\mathrm{d}x.$$

To get rid of the mixed term we use $\vartheta = \frac{2}{3}n_k^3$ in (44)

$$0 = \int_\Omega -\frac{2}{3}\nabla v_k \cdot \nabla(n_k^3) + \frac{2}{3}n_k^{\alpha+3} - \frac{2}{3}n_k^3 v_k + \frac{2\delta}{3}n_k^4 \,\mathrm{d}x.$$

Summing up gives us

$$\frac{1}{\tau}\int_\Omega (n_k{-}n_{k-1})n_k^2 \,\mathrm{d}x + 2\int_\Omega (1+\delta n_k)n_k \left|\nabla n_k\right|^2 \,\mathrm{d}x$$
$$= -\frac{2\varepsilon}{\delta}\int_\Omega n_k^2\Big((\Delta y_k)^2 + \frac{2}{\delta}\left|\nabla y_k\right|^2 \Delta y_k + \frac{1}{\delta^2}\left|\nabla y_k\right|^4\Big)\,\mathrm{d}x - \varepsilon\delta\int_\Omega n_k^3 \log n_k \,\mathrm{d}x$$
$$+ \frac{2}{3}\int_\Omega n_k^{\alpha+3} - n_k^3 v_k + \delta n_k^4 \,\mathrm{d}x,$$

where the first integral on the right hand side can be written as a square, the second integral is bounded above by $\delta\frac{1}{3e}$, and the last integral can be estimated by the worst exponent. The left hand side is estimated as before, to get

$$\frac{1}{\tau}\frac{1}{3}\int_\Omega n_k^3 - n_{k-1}^3 \,\mathrm{d}x + \frac{\delta}{2}\int_\Omega \left|\nabla(n_k^2)\right|^2 \,\mathrm{d}x \le C + C\int_\Omega n_k^{\alpha+3}\,\mathrm{d}x.$$

We estimate the right hand side by Gagliardo-Nirenberg in $H^1(\Omega)\cap L^1(\Omega)$ with $\theta = \frac{\alpha+1}{\alpha+3}$

$$\|n_k^{\alpha+3}\|_{L^1(\Omega)}^{\alpha+3} = \|n_k^2\|_{L^{(\alpha+3)/2}(\Omega)}^{(\alpha+3)/2} \le C\|n_k^2\|_{H^1(\Omega)}^{(\alpha+1)/2}\|n_k^2\|_{L^1(\Omega)}$$

and Young's inequality[23]

$$\le \frac{\delta}{4}\|\nabla(n_k^2)\|_{L^2(\Omega)}^2 + C\|n_k\|_{L^2(\Omega)}^{8/(3-\alpha)} + C\|n_k\|_{L^4(\Omega)}^{\alpha+1}\|n_k\|_{L^2(\Omega)}^2,$$

where we only need to estimate the $L^4(\Omega)$-norm because all the other terms can be absorbed or will be uniformly bounded after integrating in time. By Gagliardo-Nirenberg in $H^1(\Omega)\cap L^2(\Omega)$ with $\theta = \frac{1}{2}$

$$\|n_k\|_{L^4(\Omega)}^{\alpha+1} \le C\|n_k\|_{H^1(\Omega)}^{(\alpha+1)/2}\|n_k\|_{L^2(\Omega)}^{(\alpha+1)/2}$$

and potentially Young's inequality[24]

$$\le C\|n_k\|_{H^1(\Omega)}^2 + C\|n_k\|_{L^2(\Omega)}^{(2\alpha+2)/(3-\alpha)}.$$

[23] Here we need $\alpha < 3$.
[24] Here we need $\alpha \le 3$.

With all the uniform bounds from previous steps and Lemma 3 we get (after integrating in time)

$$\frac{1}{3}\int_\Omega (n^{(\tau)})^3(t)\,\mathrm{d}x + \frac{\delta}{4}\int_0^t\int_\Omega \left|\nabla((n^{(\tau)})^2)\right|^2\,\mathrm{d}x \le C + \frac{1}{3}\int_\Omega n_0^3\,\mathrm{d}x.$$

We conclude

$$n \in L^\infty(0,T;L^3(\Omega)).$$

Step 6: $v \in L^\infty(0,T;W^{1,\infty}(\Omega))$. Consider now the elliptic equation

$$-\Delta v^{(\tau)} + v^{(\tau)} = (n^{(\tau)})^\alpha + \delta n^{(\tau)} \in L^\infty(0,T;L^{3/\alpha}(\Omega)),$$

where the bound for the right hand side is uniform in ε, τ. By elliptic regularity [11, Theorem 2.4.2.7] (where we need $\partial\Omega \in C^{1,1}$) we conclude

$$v^{(\tau)} \in L^\infty(0,T;W^{2,3/\alpha}(\Omega)),$$

where the bound is uniform in ε and τ as well. By Sobolev embedding $W^{2,2+\eta}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ for any $\eta > 0$, we conclude[25]

$$v \in L^\infty(0,T;W^{1,\infty}(\Omega)).$$

Step 7: $n \in L^\infty(\Omega_T)$. We use the regularity of v and the following Lemma

Lemma 8. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with $\partial\Omega \in C^{0,1}$, $T > 0$, $u_0 \in L^\infty(\Omega)$, $\delta > 0$, and $V \in L^\infty(0,T;W^{1,\infty}(\Omega))$. Then there exists a unique weak solution u to*

$$u_t - \operatorname{div}((1+\delta u)\nabla u) = -\operatorname{div}(u\nabla V)$$

with boundary condition

$$((1+\delta u)\nabla u - u\nabla V)\cdot\nu = 0 \quad \text{on } \partial\Omega, t > 0,$$

and initial condition $u(\cdot,0) = u_0$ in Ω. Additionally, there exists a constant $C > 0$ depending on Ω and $\|\nabla V\|_{L^\infty(\Omega_T)}$ such that

$$\|u\|_{L^\infty(\Omega_T)} \le C\max\{1, \|u_0\|_{L^\infty(\Omega)}\}.$$

The proof is done in [13, Proposition 4.1]. Setting $V = v$ gives the desired regularity $n \in L^\infty(\Omega_T)$ due to equation (39).

We conclude the proof by $S = v - \delta n \in L^\infty(\Omega_T)$. □

Remark. *For the proof of Theorem 7 to work we used $\alpha \in [1, \frac{3}{2})$, which were the values where we could show existence in Section 3. On its own the upper bound can potentially be increased to at least $\alpha \le \frac{7}{4}$ if one goes for more steps of $L^\infty(0,T;L^k(\Omega))$-bounds for n with $k = 4, 5, \ldots$ similarly to the ones already done, before using elliptic regularity.*

[25]Here we need $\alpha < \frac{3}{2}$.

It is possible to show even classical differentiability of solutions, provided that the initial datum and the domain are smooth enough.

Theorem 9. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded and smooth domain (i.e., its boundary satisfies $\partial\Omega \in C^\infty$). Assume further that $n_0 \in C^{2+\gamma}(\overline{\Omega})$ for some $\gamma \in (0,1)$. Under the additional assumptions of Theorem 7, solutions (of the parabolic-elliptic model from Theorem 7) are smooth, i.e.,*

$$n, S \in C^\infty((0,T] \times \overline{\Omega}).$$

The proof relies on the Schauder fixed-point theorem to prove some initial regularity, which is then further improved by elliptic and parabolic regularity results. Bootstrapping yields the desired regularity. We refer the reader to [16, Proof of Theorem 2] for the precise arguments and references.

Sketch of Proof. Consider the set

$$K := \{\tilde{n} \in C^0([0,T] \times \overline{\Omega}) : \; 0 \leq \tilde{n} \leq R, \, \|n\|_{C^{\gamma/2,\gamma}([0,T]\times\overline{\Omega})} \leq M\},$$

where $C^{a,b}(A,B)$ means differentiability (or Hölder continuity) of degree a in A and degree b in B. The constants $R, M > 0$ have to be determined later on. For $\tilde{n} \in K$ we apply elliptic regularity in the equation

$$-\Delta v + v = \delta\tilde{n} + \tilde{n}^\alpha$$

with homogeneous Neumann boundary conditions. Thus, we find that the solution satisfies $v \in C^0([0,T]; W^{2,p}(\Omega))$ for any $p < \infty$. Sobolev embedding gives $W^{2,p} \hookrightarrow C^1(\overline{\Omega})$. Setting $h := \tilde{n}\nabla v$ gives a continuous function. Plugging it into

$$n_t = \operatorname{div}((1 + \delta\tilde{n})\nabla n - h)$$

with homogeneous Neumann boundary conditions, implies $n \in C^{\gamma/2,\gamma}([0,T] \times \overline{\Omega})$. Redoing this procedure gives $v \in C^{\gamma/2,2}([0,T] \times \overline{\Omega})$, and thus $h \in C^{\gamma/2,\gamma}([0,T] \times \overline{\Omega})$, so $n \in C^{1,2}([0,T] \times \overline{\Omega})$. Now it can be shown that $n \in K$ for suitable $R, M > 0$. By Schauder's fixed-point theorem we deduce this regularity, upon which we now proceed.
By elliptic regularity we get that $v \in C^{1,4}([0,T] \times \overline{\Omega})$. We then set $f := \operatorname{div}(n\nabla v) \in C^{1,1}([0,T] \times \overline{\Omega})$ and consider the linear equation

$$u_t - \Delta u - \operatorname{div}(n\nabla u) = f$$

with homogeneous Neumann boundary conditions. This solution satisfies $u \in C^{1+\gamma/2,2+\gamma}([0,T] \times \overline{\Omega})$ (and by uniqueness $u = n$). This extends the regularity of f further such that $f \in C^{1+\gamma/2,1+\gamma}([0,T] \times \overline{\Omega})$. By parabolic regularity we conclude $u \in C^{2,2+\gamma}([0,T] \times \overline{\Omega})$. And we could repeat this process arbitrarily long to finally deduce

$$n \in C^\infty((0,T] \times \overline{\Omega})), \qquad v \in C^\infty((0,T] \times \overline{\Omega}).$$

Clearly $S = v - \delta n \in C^\infty((0,T] \times \overline{\Omega})$ is also smooth. Note that we need to exclude differentiability up to $t = 0$ because we did not assume the initial datum to be smooth enough. If we did, then regularity would hold on $[0,T] \times \overline{\Omega}$. □

5 Uniqueness of solutions

Considering weak solutions instead of classical solutions allowed us to more easily prove the existence of solutions because we considered a much bigger space of functions. However, this might have been at the cost of uniqueness, i.e., the weaker requirements could allow several different functions to satisfy the equations, although only a single classical solution would exist. In this section we shall prove that (under additional regularity assumptions on solutions) we still keep unique solvability.

5.1 The parabolic-parabolic model

For technical reasons we will need to restrict the cross-diffusion parameter δ to be small enough. However, this bound does not affect the more interesting range of $\delta \ll 1$ potentially going to 0 (which is a limit we do not consider here).

Theorem 10. *Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ be a bounded Lipschitz domain, and let $\alpha \geq 1$. Assume that any solution (n, S) of*

$$n_t = \operatorname{div}(\nabla n - n\nabla S)$$
$$S_t = \Delta S + \delta \Delta n + n^\alpha - S$$

in $\Omega, t > 0$ with boundary conditions

$$\nabla n \cdot \nu = \nabla S \cdot \nu = 0$$

on $\partial\Omega, t > 0$, and initial conditions $n(\cdot, 0) = n_0, S(\cdot, 0) = S_0$ (with $n_0, S_0 \in L^2(\Omega)$) satisfies the uniform a priori estimates $S \in L^\infty(0, T; W^{1,\infty}(\Omega))$ and $n \in L^\infty(\Omega_T)$. If there exists $\varepsilon \in (0, 1)$ and $\gamma \geq 1$ such that

$$\frac{(\|n\|_{L^\infty(\Omega_T)} + \gamma\delta_0)^2}{4(1-\varepsilon)} \leq \gamma$$

for some $\delta_0 > 0$, then the above equations possess at most one solution (n, S) for any $\delta \leq \delta_0$.

Proof. Consider solutions $(n_1, S_1), (n_2, S_2)$. We look at the difference of the weak equations solved by these functions

$$\int_0^t \langle (n_1 - n_2)_t, \varphi \rangle \, \mathrm{d}s + \int_0^t \int_\Omega (\nabla n_1 - n_1 \nabla S_1 - \nabla n_2 + n_2 \nabla S_2) \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}s = 0$$

and

$$\int_0^t \langle (S_1 - S_2)_t, \vartheta \rangle \, \mathrm{d}s + \int_0^t \int_\Omega (\nabla S_1 + \delta \nabla n_1 - \nabla S_2 - \delta \nabla n_2) \cdot \nabla \vartheta + (S_1 - S_2)\vartheta \, \mathrm{d}x \, \mathrm{d}s$$
$$= \int_0^t \int_\Omega (n_1^\alpha - n_2^\alpha)\vartheta \, \mathrm{d}x \, \mathrm{d}s.$$

Taking $\varphi = n_1 - n_2$ and $\vartheta = \gamma(S_1 - S_2)$ for γ from our assumption, and adding the two equations, we get (notice that $(n_1 - n_2)(0) = 0$, because they satisfy the same initial conditions; also $(S_1 - S_2)(0) = 0$)

$$\frac{1}{2}\|n_1 - n_2\|^2_{L^2(\Omega)}(t) + \frac{\gamma}{2}\|S_1 - S_2\|^2_{L^2(\Omega)}(t) + \|\nabla(n_1 - n_2)\|^2_{L^2(\Omega_t)} + \gamma\|\nabla(S_1 - S_2)\|^2_{L^2(\Omega_t)} + \gamma\|S_1 - S_2\|^2_{L^2(\Omega_t)}$$
$$= \int_0^t \int_\Omega (n_1\nabla S_1 - n_2 \nabla S_2)\cdot\nabla(n_1 - n_2) + \gamma(n_1^\alpha - n_2^\alpha)(S_1 - S_2) + \gamma\delta\nabla(n_1 - n_2)\cdot\nabla(S_1 - S_2)\,\mathrm{d}x\,\mathrm{d}s.$$

We call the left hand side LHS and shall estimate the right hand side. Applying the Cauchy–Schwarz inequality we get

$$\int_0^t \int_\Omega (n_1\nabla S_1 - n_2\nabla S_2)\cdot\nabla(n_1 - n_2)\,\mathrm{d}x\,\mathrm{d}s \le \|n_1\nabla S_1 - n_2\nabla S_2\|_{L^2(\Omega_t)}\|\nabla(n_1 - n_2)\|_{L^2(\Omega_t)},$$

where we estimate the first factor by triangle inequality and Hölder

$$\|n_1\nabla S_1 - n_2\nabla S_2\|_{L^2(\Omega_t)} \le \|n_1\|_{L^\infty(\Omega_t)}\|\nabla(S_1 - S_2)\|_{L^2(\Omega_t)} + \|\nabla S_2\|_{L^\infty(\Omega_t)}\|n_1 - n_2\|_{L^2(\Omega_t)}.$$

The next term we estimate by the mean value theorem[26] and Young's inequality

$$\int_0^t \int_\Omega \gamma(n_1^\alpha - n_2^\alpha)(S_1 - S_2)\,\mathrm{d}x\,\mathrm{d}s \le \gamma\alpha\|n\|^{\alpha-1}_{L^\infty(\Omega_t)}\|n_1 - n_2\|_{L^2(\Omega_t)}\|S_1 - S_2\|_{L^2(\Omega_t)}$$
$$\le C(\gamma, \alpha, \|n\|_{L^\infty(\Omega_T)})\|n_1 - n_2\|^2_{L^2(\Omega_t)} + \|S_1 - S_2\|^2_{L^2(\Omega_t)},$$

where n (and later on S, too) denotes any solution, and their L^∞-norm denotes the uniform bound which we assumed.
Lastly, applying Cauchy–Schwarz to the last remaining term, we can summarize

$$\begin{aligned} LHS \le& \big(\|n\|_{L^\infty(\Omega_T)} + \gamma\delta\big)\|\nabla(S_1 - S_2)\|_{L^2(\Omega_t)}\|\nabla(n_1 - n_2)\|_{L^2(\Omega_t)} \\ &+ \|\nabla S\|_{L^\infty(\Omega_T)}\|n_1 - n_2\|_{L^2(\Omega_t)}\|\nabla(n_1 - n_2)\|_{L^2(\Omega_t)} \\ &+ C\|n_1 - n_2\|^2_{L^2(\Omega_t)} + \|S_1 - S_2\|^2_{L^2(\Omega_t)} \\ \le& \frac{(\|n\|_{L^\infty(\Omega_t)} + \gamma\delta)^2}{4(1-\varepsilon)}\|\nabla(S_1 - S_2)\|^2_{L^2(\Omega_t)} + (1-\varepsilon)\|\nabla(n_1 - n_2)\|^2_{L^2(\Omega_t)} \\ &+ \varepsilon\|\nabla(n_1 - n_2)\|^2_{L^2(\Omega_t)} + C(\varepsilon, \|\nabla S\|_{L^\infty(\Omega_T)})\|n_1 - n_2\|^2_{L^2(\Omega_t)} \\ &+ C\|n_1 - n_2\|^2_{L^2(\Omega_t)} + \|S_1 - S_2\|^2_{L^2(\Omega_t)}. \end{aligned}$$

By our assumptions on γ (notice that the left hand side of the inequality for γ is increasing in δ, i.e., if it holds for one δ_0, it holds for any $0 < \delta \le \delta_0$), the gradient terms can be absorbed by the left hand side. After potentially estimating the remaining non-negative terms on the left hand side, we find (after expanding the $L^2(\Omega_t)$-norm)

$$\|n_1 - n_2\|^2_{L^2(\Omega)}(t) + \gamma\|S_1 - S_2\|^2_{L^2(\Omega)}(t) \le C\Big(\int_0^t \|n_1 - n_2\|^2_{L^2(\Omega)}\,\mathrm{d}s + \gamma\int_0^t \|S_1 - S_2\|^2_{L^2(\Omega)}\,\mathrm{d}s\Big).$$

[26] Here we need that $\alpha \ge 1$ instead of $\alpha > 0$.

Applying the Gronwall lemma, we conclude

$$\begin{aligned}\|n_1 - n_2\|^2_{L^2(\Omega)} + \gamma\|S_1 - S_2\|^2_{L^2(\Omega)} &\leq e^{CT}\big(\|n_1 - n_2\|^2_{L^2(\Omega)}(0) + \gamma\|S_1 - S_2\|^2_{L^2(\Omega)}(0)\big) \\ &= 0,\end{aligned}$$

which concludes the proof. □

Remark. *A possible choice of* $\delta_0, \varepsilon, \gamma$ *is to take any* $\varepsilon \in (0,1)$*, set*

$$\gamma = \frac{(\|n\|_{L^\infty(\Omega_T)} + 1)^2}{4(1-\varepsilon)}$$

and take $\delta_0 = \frac{1}{\gamma}$*. In particular, any*

$$\delta_0 < \frac{4}{(\|n\|_{L^\infty(\Omega_T)} + 1)^2}$$

works.

5.2 The parabolic-elliptic model

For the parabolic-elliptic model we consider the transformed system with $v := S + \delta n$ as in Section 4 and show uniqueness for it. One problem then is that we need to deal with the quasilinearity. We do this by the so-called *dual method* or H^{-1}-method; where we use a very specific test function solving an elliptic equation.

Theorem 11. *Let* $\Omega \subseteq \mathbb{R}^d$*,* $d \geq 2$ *be a bounded Lipschitz domain, and let* $\alpha \geq 1$*,* $\delta > 0$*. Assume that any solution* (n, v) *of*

$$\begin{aligned} n_t &= \operatorname{div}((1+\delta n)\nabla n - \nabla v) \\ 0 &= \Delta v + n^\alpha - v + \delta n \end{aligned}$$

with boundary conditions

$$\nabla n \cdot \nu = \nabla v \cdot \nu = 0$$

and initial condition $n(\cdot, 0) = n_0$ *(with* $n_0 \in L^2(\Omega)$*) satisfies the uniform a priori estimates* $n \in L^\infty(\Omega_T)$*,* $v \in L^\infty(0,T; W^{1,\infty}(\Omega))$*. Then the above equations admit at most one solution* (n, v)*.*

Proof. Let (n_1, v_1) and (n_2, v_2) be two solutions. We take the difference of their respective weak equations and get

$$\begin{aligned}&\int_0^t \langle (n_1 - n_2)_t, \varphi\rangle \,\mathrm{d}s \\ &\quad + \int_0^t \int_\Omega \big(((1+\delta n_1)\nabla n_1 - n_1 \nabla v_1) - ((1+\delta n_2)\nabla n_2 - n_2\nabla v_2)\big)\cdot \nabla\varphi \,\mathrm{d}s\,\mathrm{d}x = 0.\end{aligned}$$

Consider the following elliptic problem (for any $t > 0$)

$$\begin{aligned} -\Delta\varphi &= n_1 - n_2 \qquad &&\text{in } \Omega \\ \nabla\varphi \cdot \nu &= 0 &&\text{on } \partial\Omega \\ \int_\Omega \varphi \, \mathrm{d}x &= 0, \end{aligned}$$

where we need the last equation for uniqueness. By the Lax–Milgram lemma (using the Poincaré–Wirtinger inequality) this problem admits a unique solution $\varphi \in L^\infty(0, T; H^1(\Omega))$. We shall employ it as a test function. Notice that (in the sense of distributions at least) $(n_1 - n_2)_t = -\Delta\varphi_t$; and $\varphi(t = 0) = 0$ because $(n_1 - n_2)(0) = 0$ and $\varphi = 0$ is a (and thus the only) solution.

In order to deal with the quasilinear terms we use an auxiliary function

$$b(n) := \int_0^n 1 + \delta z \, \mathrm{d}z = n + \frac{\delta}{2} n^2, \qquad \nabla b(n) = (1 + \delta n)\nabla n.$$

Now, after integrating by parts

$$\begin{aligned} \frac{1}{2}\|\nabla\varphi\|^2_{L^2(\Omega)}(t) + \int_0^t \int_\Omega \nabla(b(n_1) - b(n_2)) \cdot \nabla\varphi \, \mathrm{d}x \, \mathrm{d}s \\ = \int_0^t \int_\Omega (n_1\nabla v_1 - n_2\nabla v_2) \cdot \nabla\varphi \, \mathrm{d}x \, \mathrm{d}s, \end{aligned}$$

gives

$$\begin{aligned} \frac{1}{2}\|\nabla\varphi\|^2_{L^2(\Omega)}(t) + \int_0^t \int_\Omega (b(n_1) - b(n_2))(n_1 - n_2) \, \mathrm{d}x \, \mathrm{d}s \\ = \int_0^t \int_\Omega (n_1\nabla v_1 - n_2\nabla v_2) \cdot \nabla\varphi \, \mathrm{d}x \, \mathrm{d}s. \end{aligned}$$

We can expand the b-term

$$\begin{aligned} &\int_0^t \int_\Omega (b(n_1) - b(n_2))(n_1 - n_2) \, \mathrm{d}x \, \mathrm{d}s \\ &= \int_0^t \int_\Omega (n_1 - n_2)^2 + \frac{\delta}{2}(n_1^2 - n_2^2)(n_1 - n_2) \, \mathrm{d}x \, \mathrm{d}s \\ &\geq \|n_1 - n_2\|^2_{L^2(\Omega_t)}. \end{aligned}$$

Now, we consider the second equation. We take the difference of the equations solved by the two solutions, and test with $\vartheta = (v_1 - v_2)$

$$\begin{aligned} &\|\nabla(v_1 - v_2)\|^2_{L^2(\Omega_t)} + \|v_1 - v_2\|^2_{L^2(\Omega_t)} \\ &\qquad = \int_0^t \int_\Omega (n_1^\alpha - n_2^\alpha)(v_1 - v_2) + \delta(n_1 - n_2)(v_1 - v_2) \, \mathrm{d}x \, \mathrm{d}s. \end{aligned}$$

Adding the two equations (after estimating), where we scale the one with φ by a factor of γ for some $\gamma > 0$ to be determined later on, we get

$$\begin{aligned} &\frac{\gamma}{2}\|\nabla\varphi\|^2_{L^2(\Omega)}(t) + \gamma\|n_1 - n_2\|^2_{L^2(\Omega_t)} + \|\nabla(v_1 - v_2)\|^2_{L^2(\Omega_t)} + \|v_1 - v_2\|^2_{L^2(\Omega_t)} \\ &\qquad = \gamma \int_0^t \int_\Omega (n_1\nabla v_1 - n_2\nabla v_2) \cdot \nabla\varphi \, \mathrm{d}x \, \mathrm{d}s \\ &\qquad + \int_0^t \int_\Omega (n_1^\alpha - n_2^\alpha)(v_1 - v_2) + \delta(n_1 - n_2)(v_1 - v_2) \, \mathrm{d}x \, \mathrm{d}s, \end{aligned} \tag{45}$$

where we shall estimate the right hand side. The last term can be estimated by the Cauchy–Schwarz inequality and Young's inequality

$$\int_0^t \int_\Omega \delta(n_1 - n_2)(v_1 - v_2)\,\mathrm{d}x\,\mathrm{d}s \leq \delta^2 \|n_1 - n_2\|_{L^2(\Omega_t)}^2 + \frac{1}{4}\|v_1 - v_2\|_{L^2(\Omega_t)}^2.$$

The second to last term can be treated with the mean value theorem

$$\begin{aligned}\int_0^t \int_\Omega (n_1^\alpha - n_2^\alpha)(v_1 - v_2)\,\mathrm{d}x\,\mathrm{d}s &\leq \|n_1^\alpha - n_2^\alpha\|_{L^2(\Omega_t)}\|v_1 - v_2\|_{L^2(\Omega_t)} \\ &\leq \alpha\|n\|_{L^\infty(\Omega_T)}^{\alpha-1}\|n_1 - n_2\|_{L^2(\Omega_t)}\|v_1 - v_2\|_{L^2(\Omega_t)} \\ &\leq (\alpha\|n\|_{L^\infty(\Omega_T)}^{\alpha-1})^2\|n_1 - n_2\|_{L^2(\Omega_t)}^2 + \frac{1}{4}\|v_1 - v_2\|_{L^2(\Omega_t)}^2.\end{aligned}$$

The last remaining term can be treated with the triangle inequality as in the proof for the parabolic-parabolic model

$$\begin{aligned}&\gamma \int_0^t \int_\Omega (n_1 \nabla v_1 - n_2 \nabla v_2)\cdot \nabla\varphi\,\mathrm{d}x\,\mathrm{d}s \\ &\leq \gamma \|n_1\nabla v_1 - n_2 \nabla v_2\|_{L^2(\Omega_t)}\|\nabla\varphi\|_{L^2(\Omega_t)} \\ &\leq \gamma\big(\|n\|_{L^\infty(\Omega_T)}\|\nabla(v_1 - v_2)\|_{L^2(\Omega_t)} + \|\nabla v\|_{L^\infty(\Omega_T)}\|n_1 - n_2\|_{L^2(\Omega_t)}\big)\|\nabla\varphi\|_{L^2(\Omega_t)} \\ &\leq \frac{1}{2}\|\nabla(v_1 - v_2)\|_{L^2(\Omega_t)}^2 + \frac{1}{2}\|n_1 - n_2\|_{L^2(\Omega_t)}^2 + C(\gamma, \|n\|_{L^\infty}, \|\nabla v\|_{L^\infty})\|\nabla\varphi\|_{L^2(\Omega_t)}^2.\end{aligned}$$

All the terms on the right hand side of (45) can be absorbed by the left hand side, provided that $\gamma \geq (\alpha\|n\|_{L^\infty(\Omega_T)}^{\alpha-1})^2 + \delta^2$. For convenience we choose $\gamma = (\alpha\|n\|_{L^\infty(\Omega_T)}^{\alpha-1})^2 + \delta^2 + 1$. This way, we keep the norm of $n_1 - n_2$ on the left hand side. After absorbing and estimating some non-negative terms on the left hand side, we get

$$\frac{\gamma}{2}\|\nabla\varphi\|_{L^2(\Omega)}^2(t) \leq C\int_0^t \|\nabla\varphi\|_{L^2(\Omega)}^2\,\mathrm{d}s.$$

By the Gronwall lemma we conclude that

$$\nabla\varphi = 0 \quad \text{for a.e. } t.$$

Going back to the estimate after absorbing terms from the right hand side, but before neglecting additional non-negative terms on the left hand side, we get

$$\|n_1 - n_2\|_{L^2(\Omega_t)}^2 + \frac{1}{2}\|\nabla(v_1 - v_2)\|_{L^2(\Omega_t)}^2 + \frac{1}{2}\|v_1 - v_2\|_{L^2(\Omega_t)}^2 \leq 0.$$

Since this holds for any t, the proof is done.

□

6 Appendix

Lemma 12 (Norm equivalence in $H^2(\Omega)$)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. Consider the space*

$$Y := \{u \in H^2(\Omega) : \ \nabla u \cdot \nu = 0 \quad \text{on } \partial\Omega\}.$$

Then the following norms are equivalent on Y

$$\begin{aligned} \|u\|_{H^2(\Omega)}^2 &:= \sum_{|\alpha|\le 2} \|D^\alpha u\|_{L^2(\Omega)}^2, \\ \|u\|_*^2 &:= \|\Delta u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2, \\ \|u\|_{**}^2 &:= \|\Delta u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^1(\Omega)}^2. \end{aligned}$$

Proof. For $\|\cdot\|_{H^2(\Omega)} \sim \|\cdot\|_*$ we only need to consider second order derivatives. Argue by density and consider $u \in Y \cap C^\infty(\bar{\Omega})$

$$\begin{aligned} \sum_{i,j=1}^d \|\partial_{x_i}\partial_{x_j} u\|_{L^2(\Omega)}^2 &= \sum_{i,j=1}^d \int_\Omega (\partial_{x_i}\partial_{x_j} u)(\partial_{x_i}\partial_{x_j} u)\,\mathrm{d}x \\ &= \sum_{j=1}^d \int_\Omega \nabla(\partial_{x_j} u) \cdot \nabla(\partial_{x_j} u)\,\mathrm{d}x \\ &= \sum_{j=1}^d \int_{\partial\Omega} (\partial_{x_j} u)\big(\nabla(\partial_{x_j} u) \cdot \nu\big)\,\mathrm{d}s - \int_\Omega (\partial_{x_j} u)\Delta(\partial_{x_j} u)\,\mathrm{d}x \\ &= -\int_\Omega \nabla u \cdot \Delta(\nabla u)\,\mathrm{d}x = -\int_\Omega \nabla u \cdot \nabla(\Delta u)\,\mathrm{d}x \\ &= \int_\Omega \Delta u \Delta u\,\mathrm{d}x = \|\Delta u\|_{L^2(\Omega)}^2, \end{aligned}$$

so these are in fact equal.

For $\|\cdot\|_{H^2(\Omega)} \sim \|\cdot\|_{**}$ we consider two inequalities. Since Ω is bounded we get

$$\|u\|_{L^1(\Omega)}^2 \le |\Omega|\, \|u\|_{L^2(\Omega)}^2,$$

which gives one direction. For the other one consider the Sobolev embedding $H^2(\Omega) \hookrightarrow L^1(\Omega)$ (for *any* dimension). This gives

$$\|u\|_{**}^2 \le \|\Delta u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + C\|u\|_{H^2(\Omega)}^2 \le (1+C)\|u\|_*^2 \le C\|u\|_{H^2(\Omega)}^2.$$

□

Lemma 13 (Lax–Milgram)**.** *[9, Theorem 1 in Chapter 6.2.1] Let H be a real Hilbert space. Assume that*

$$a : \ H \times \ H \to \mathbb{R}$$

is a bilinear map, which is continuous, i.e., there exists a constant $\alpha > 0$ such that

$$|a(u,v)| \leq \alpha \|u\|_H \|v\|_H \quad \forall u, v \in H,$$

and also coercive, i.e., there exists a constant $\beta > 0$ such that

$$\beta \|u\|_H^2 \leq a(u,u) \quad \forall u \in H.$$

Let $F : H \to \mathbb{R}$ be a continuous linear functional on H. Then there exists a unique element $u \in H$ such that

$$a(u,v) = F(v) \quad \forall v \in H,$$

i.e., a unique solution to the problem $a(\cdot, v) = F(v)$.

Corollary 13.1 (Lax–Milgram)**.** *With the notation of Lemma 13 the unique solution $u \in H$ satisfies the bound*

$$\|u\|_H \leq \frac{C_F}{\beta},$$

where C_F is the continuity constant of F, i.e.,

$$|F(v)| \leq C_F \|v\|_H \quad \forall v \in H.$$

Proof. If $u = 0$, the bound holds. Otherwise, by coercivity, the fact that u is a solution (and $u \in H$ is an admissible test function), and the continuity of F, one gets

$$\beta \|u\|_H^2 \leq |a(u,u)| = |F(u)| \leq C_F \|u\|_H.$$

Dividing by $\|u\|_H$ and rearranging concludes the proof. □

Lemma 14 (Leray–Schauder fixed point theorem)**.** *[10, Theorem 11.6] Let X be a Banach space and let B be a compact mapping*

$$B : X \times [0,1] \to X$$

such that $B(u,0) = 0$ for all $u \in X$. Suppose there exists a constant M such that

$$\|u\|_X \leq M$$

for any potential "fixed points" $(u, \sigma) \in X \times [0,1]$ satisfying $B(u,\sigma) = u$. Then the mapping $u \mapsto B(u,1)$ has a fixed point.

Lemma 15 (Eberlein–Šmuljan). *[27, Theorem 21.D] Each bounded sequence in a reflexive Banach space has a weakly convergent subsequence.*

Lemma 16 (Convergence from Subsubsequences). *Let $(X, \mathcal{T})$ be a topological space, and $(x_n)_{n\in\mathbb{N}}$ a sequence in X. Assume that every subsequence of (x_n) has a subsubsequence, which all converge to a common limit $x \in X$, then the original sequence converges to that limit x.*

Proof. By contradiction: Assume that $x_n \not\to x$, i.e.,

$$\exists U \in \mathcal{T} \text{ with } x \in U \text{ such that } \forall N \in \mathbb{N}\ \exists n \geq N: \ x_n \notin U.$$

Define

$$K : \mathbb{N} \to \mathbb{N},\ N \mapsto \min\{n \geq N : x_n \notin U\},$$

which is well-defined by the above assumption. Now define a strictly increasing mapping inductively by[27]

$$\begin{aligned} &\ell : \mathbb{N} \to \mathbb{N}, \\ &\begin{cases} \ell(0) := K(0), \\ \ell(n+1) := \min\{j \in \operatorname{ran} K : \ j > \ell(n)\}. \end{cases} \end{aligned}$$

This induces the subsequence $(x_{\ell(n)})_{n\in\mathbb{N}}$. Notice that any such element lies outside of U. If there were a convergent subsubsequence of $(x_{\ell(n)})_{n\in\mathbb{N}}$, then there would have to be an index $N \in \mathbb{N}$ such that $\forall n \geq N$ we would have $x_{\ell(n)} \in U$, but this is a contradiction, because no such element is in U. □

Lemma 17 (Rellich–Kondrachov theorem). *[1, Theorem 8.11.4] Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, and $m \in \mathbb{N} \cup \{0\}$. Then the embedding $H^{m+1}(\Omega) \hookrightarrow H^m(\Omega)$ is compact.*

Lemma 18 (Compact embeddings). *[1, Theorem 8.11.5, Theorem 8.11.6] For bounded domains $\Omega \subseteq \mathbb{R}^d$ with Lipschitz boundary and $0 \leq s_1 < s_2$, the following embedding is dense and compact*

$$H^{s_2}(\Omega) \hookrightarrow\hookrightarrow H^{s_1}(\Omega).$$

Under the same assumptions on Ω, let $1 < p < \infty$ and $0 \leq s_1 < s_2$ such that $s_2 > \frac{1}{p}$, then the following embedding is compact

$$W^{s_2,p}(\Omega) \hookrightarrow\hookrightarrow W^{s_1,p}(\Omega).$$

[27]This construction makes sure that we do not need any kind of axiom of choice. We fully rely on the well-ordering of $\mathbb{N}$.

Lemma 19 (Gagliardo–Nirenberg inequality). *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. Let $m \in \mathbb{N}\backslash\{0\}$ be a positive integer, $1 \leq p, q, r \leq \infty$, and $0 \leq \theta \leq 1$, such that*

$$\frac{1}{p} = \theta\Big(\frac{1}{r} - \frac{m}{d}\Big) + (1-\theta)\frac{1}{q}$$

with two special cases:

1. *If $rm < d$ and $q = \infty$, then one needs to assume that $u \in L^{\tilde{q}}(\Omega)$ for some $\tilde{q} > 0$.*

2. *If $1 < r < \infty$ and $m - \frac{d}{r}$ is a non-negative integer, then one needs to restrict $0 \leq \theta < 1$.*

Under these assumptions the following inequality holds with a constant C independent of u

$$\|u\|_{L^p(\Omega)} \leq C\|u\|^{\theta}_{W^{m,r}(\Omega)}\|u\|^{1-\theta}_{L^q(\Omega)}. \tag{46}$$

Proof. For any $k \in \mathbb{N}$ and $1 \leq \tilde{p} \leq \infty$ there exists an extension operator

$$E : W^{k,\tilde{p}}(\Omega) \to W^{k,\tilde{p}}(\mathbb{R}^d)$$

which is linear and bounded (see [23, Theorem 4]). Using the Gagliardo–Nirenberg inequality on $\mathbb{R}^d$ (see [21, Theorem p. 125]), we get

$$\begin{aligned}\|u\|_{L^p(\Omega)} &\leq \|Eu\|_{L^p(\mathbb{R}^d)} \leq C_{GN}\|D^m(Eu)\|^{\theta}_{L^r(\mathbb{R}^d)}\|Eu\|^{1-\theta}_{L^q(\mathbb{R}^d)} \\ &\leq C_{GN}\|Eu\|^{\theta}_{W^{m,r}(\mathbb{R}^d)}\|Eu\|^{1-\theta}_{L^q(\mathbb{R}^d)} \leq C_E C_{GN}\|u\|^{\theta}_{W^{m,r}(\Omega)}\|u\|^{1-\theta}_{L^q(\Omega)}.\end{aligned}$$

□

Lemma 20 (Sobolev embedding). *[1, Theorem 8.12.4 and Remark 8.12.4] Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain, and $1 \leq p \leq \infty$, $m \in \mathbb{N} \cup \{0\}$. Then the following continuous embedding holds*

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$$

- *if $\frac{1}{p} - \frac{m}{d} > 0$, then $1 \leq q \leq q^*$ with $\frac{1}{q^*} = \frac{1}{p} - \frac{m}{d}$,*
- *if $\frac{1}{p} - \frac{m}{d} = 0$, then $1 \leq q < \infty$,*
- *if $\frac{1}{p} - \frac{m}{d} < 0$, then $1 \leq q \leq \infty$.*

Moreover, for $m - \frac{d}{p} = k + \sigma$ with $k = \lfloor m - \frac{d}{p} \rfloor \in \mathbb{N} \cup \{0\}, 0 < \sigma \leq 1$ the continuous embedding holds

$$W^{m,p}(\Omega) \hookrightarrow C^{k,\sigma}(\overline{\Omega}),$$

where $C^{k,\sigma}$ denotes a Hölder space: functions with continuous derivatives up to order k, and k-th order derivatives are Hölder continuous with exponent σ.

Lemma 21 (Young inequality with ε). *Let $a \geq 0$ and $b \geq 0$. If $1 < p, q < \infty$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then for any $\varepsilon > 0$ one has*

$$ab \leq \varepsilon a^p + \frac{1}{qp^{q/p}\varepsilon^{q/p}} b^q = \varepsilon a^p + C(\varepsilon) b^q.$$

Proof. The standard Young inequality shows

$$ab = (aM)(bM^{-1}) \leq \frac{M^p}{p} a^p + \frac{1}{qM^q} b^q.$$

Choosing $M = \sqrt[p]{p\varepsilon}$ then gives the statement. □

Lemma 22 (Inverse dominated convergence theorem). *[3, Theorem 4.9] Let $1 \leq p \leq \infty$. Let (f_n) be a sequence in L^p and $f \in L^p$, for some σ-finite measure on a set Ω. If $\|f_n - f\|_{L^p} \to 0$ as $n \to \infty$, then there exists a subsequence (f_{n_k}) and a function $h \in L^p$ such that*

1. $f_{n_k}(x) \to f(x)$ *a.e. on* Ω,
2. $|f_{n_k}(x)| \leq h(x) \quad \forall k,$ *a.e. on* Ω.

Lemma 23 (Some elementary inequality). *For any $0 < \beta < 1/2$ and $\eta > 0$ there exists a constant $C(\beta, \eta)$ such that for all $x > 0$ the inequality*

$$x^{2\beta-1} \leq \eta x^{\beta-1} + C(\beta, \eta)$$

holds.

Proof. Consider

$$\frac{x^{2\beta-1}}{x^{\beta-1}} = x^\beta \xrightarrow{x \to 0^+} 0,$$

so there exists $x_0 = x_0(\beta, \eta)$ such that for all $0 < x \leq x_0$ one has $\frac{x^{2\beta-1}}{x^{\beta-1}} \leq \eta$ or equivalently $x^{2\beta-1} \leq \eta x^{\beta-1}$. On the other hand

$$\eta x^{-\beta} + x^{1-2\beta} \xrightarrow{x \to \infty} \infty,$$

so there exists $x_1 = x_1(\beta, \eta)$ such that for all $x \geq x_1$ one has $\eta x^{-\beta} + x^{1-2\beta} \geq 1$ or equivalently $x^{2\beta-1} \leq \eta x^{\beta-1} + 1$. This settles the edge cases. For the remaining "middle part" set

$$C(\beta, \eta) := \max\{\max_{x \in [x_0, x_1]} x^{2\beta-1}, 1\},$$

which is finite because $x^{2\beta-1}$ is continuous on the compact set $[x_0, x_1] \subseteq (0, \infty)$. □

Lemma 24 (Stampacchia). *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. Let $G : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function, and $p \in (1, \infty)$.*

- *If $u \in W^{1,p}(\Omega)$, then $G(u) \in W^{1,p}(\Omega)$.*
- *If additionally G' has a finite number of discontinuities, then $\frac{\partial}{\partial x_i} G(u) = G'(u) \frac{\partial u}{\partial x_i}$ almost everywhere in Ω.*

Proof. Let $(u_n) \in C^\infty(\Omega)$ be a sequence converging to u strongly in $W^{1,p}(\Omega)$. Then $G(u_n)$ is a bounded sequence in $W^{1,p}(\Omega)$, since by Rademacher's theorem G' exists almost everywhere and is bounded in $L^\infty(\Omega)$. Since $p \in (1, \infty)$ the space $W^{1,p}(\Omega)$ is reflexive. Thus, there exists a weakly convergent subsequence $G(u_{n_k})$ and a function $v \in W^{1,p}(\Omega)$ such that $G(u_{n_k}) \rightharpoonup v$ in $W^{1,p}(\Omega)$. Because $G(u_n) \to G(u)$ in the sense of distributions, we conclude $G(u) = v \in W^{1,p}(\Omega)$ by identifying the limits.
For the second part let first $G \in C^1(\mathbb{R})$, and again $u_n \to u$ in $W^{1,p}(\Omega)$. Then for any test function $\phi \in C_c^\infty(\Omega)$

$$\int_\Omega G(u_n) \frac{\partial \phi}{\partial x_i} \, \mathrm{d}x = - \int_\Omega G'(u_n) \frac{\partial u_n}{\partial x_i} \phi \, \mathrm{d}x,$$

and we can pass to the limit $n \to \infty$ by inverse dominated convergence. Thus, the distributional derivative of $G(u)$ is equal to $G'(u) \frac{\partial u}{\partial x_i}$ in $L^p(\Omega)$. In particular, they coincide almost everywhere. For the general case, denote $t_j \in \mathbb{R}$ the points where G' is discontinuous. Since there are only finitely many points (in particular they do not have an accumulation point) we can write

$$G' = F' + \sum_{j=1}^{N} \alpha_j \frac{1 + H_{t_j}}{2},$$

with F' continuous, the shifted Heavyside function $H_{t_j}(t) = \operatorname{sign}(t - t_j)$, and some real numbers α_j; which corresponds to adding (or subtracting) a jump height of α_j at the point t_j. We note that $F \in C^1(\mathbb{R})$, which we treated already. The last remaining part is H_{t_j}, but this is precisely the weak derivative of the absolute value $|t - t_j|$. Thus, approximating it by differentiable functions $t \mapsto \sqrt{t^2 + \varepsilon}$ proves the claim. $\square$

Corollary 24.1. *In particular, for functions $f, g \in W^{1,p}(\Omega)$ for $p \in (1, \infty)$ it holds*

$$\max\{f, g\} = \frac{f + g + |f - g|}{2} \in W^{1,p}(\Omega).$$

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