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Highest weight representations of the Virasoro algebra

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Matthias Gerold Wödlinger
Matrikelnummer 01125717

ausgeführt am Institut für Diskrete Mathematik und Geometrie
der Fakultät für Mathematik und Geoinformation der TU Wien

Betreuung
Betreuer: Privatdoz. Dr. Martin Rubey
Mitwirkung: Dr. Gabriel Maresch

Wien, ____________________ ____________________

(Matthias Wödlinger) (Martin Rubey)

Abstract

The Virasoro algebra plays a fundamental role in modern research areas such as condensed matter physics or quantum gravity. Applying the theory requires a deep understanding of the underlying mathematical structure. Here, we define the Virasoro algebra as the unique central extension to the Witt algebra. The investigation of the Virasoro algebra leads to Verma modules and the Hermitian Shapovalov form which is used to define unitary highest weight representations. Subsequently, we investigate the Hermitian form and compute an explicit expression for the Kac-determinant. We use the determinant formula to obtain first results about the classification of unitary highest weight representations. To complete the classification, we explicitly construct unitary highest weight representations of the Virasoro algebra from factor algebras of affine Lie algebras. We conclude the investigation with some calculations regarding the Ising model and a short introduction to the applications of the Virasoro algebra in condensed matter physics and gravity.

Contents

Chapter 1

Introduction

This work should serve as an introduction to the representation theory of the Virasoro algebra. We will give a complete classification of its unitary highest weight representations.

In chapter 2 we give a motivation for the Virasoro algebra. We will start with the Witt algebra and obtain the Virasoro algebra as its unique central extension.

In chapter 3 we explain the basic concepts needed to investigate the representations of the Virasoro algebra. Furthermore we give a definition of highest weight representations and proof basic results about Verma representations.

Chapter 4 covers the core results of this thesis that are the Kac-determinant and the first part of the full classification of unitary highest weight representations of the Virasoro algebra.

In chapter 5 we give a sketch of the second part of the classification. We start with a short introduction to affine Lie algebras and the definition of the Sugawara tensor. Afterwards we show how one can use these tools to obtain explicit constructions of unitary highest weight representations of the Virasoro algebra.

We conclude the thesis with an application of the Virasoro algebra in chapter 6 where we describe the Ising model and investigate its coset theory. Furthermore we give short description of the tricritical Ising model that has an extension of the Virasoro algebra, the super-Virasoro algebra, as its symmetry algebra. We finish with a short overview over the applications in quantum gravity.

Because the main motivation for many people to study the Virasoro algebra stems from physics, at the end of each chapter we give a short comment about how the covered topics are related to physical structures. The general structure of the thesis and most of the chapters 3 and 4 is based on the series of lectures by Victor Kac [11]. The general discussion about affine Lie algebras was taken from [3, 1] and description and construction of the Sugawara Tensor again from Kac.

Notationwise we use the Kronecker delta notation:

$$\delta_{n,k} = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}$$

Vectors will be denoted by boldfaced latin characters e.g. $\boldsymbol{v}$. Classical Lie algebras will be written with fraktur characters e.g. $\mathfrak{sl}_n$ for the special linear lie algebra of $n \times n$ traceless matrices. The complex conjugation will be denoted with a bar e.g. $\bar{\lambda}$ for the complex conjugate of λ. The Virasoro algebra will be denoted by Vir.

Chapter 2

From Witt to Virasoro

The Witt Algebra plays an important role in the study of conformal field theories in physics. It also appears as algebra of vector fields in various situations as we will now show.

2.1 The Witt algebra

Definition 2.1 (Witt algebra)**.** Let $\mathbb{C}[z, z^{-1}]$ denote the algebra of Laurent polynomials in one variable. The *Witt algebra* is then the set of vector fields with Laurent polynomial coefficients

$$\mathcal{W} := \{p(z)\frac{d}{dz} : p(z) \in \mathbb{C}[z, z^{-1}]\}.$$

equipped with the Lie brackets for vector fields.

Theorem 2.2. *The commutation relations for* $\mathcal{W}$ *are*

$$[d_n,\, d_m] = (n-m)\, d_{n+m}, \quad n, m \in \mathbb{Z},$$

where the

$$d_n := -z^{n+1}\frac{d}{dz}, \quad n \in \mathbb{Z}.$$

form a basis of $\mathcal{W}$*.*

Proof. Vector fields are equipped with a canonical Lie bracket given by the Lie derivative. We obtain for an arbitrary function $f : \mathbb{C} \to \mathbb{C}$

$$\begin{aligned}
[d_m, d_n]f(z) &= \Big[-z^{m+1}\frac{d}{dz}, -z^{n+1}\frac{d}{dz}\Big]f(z) \\
&= z^{m+1}\frac{d}{dz}\Big(z^{n+1}f'(z)\Big) - z^{n+1}\frac{d}{dz}\Big(z^{m+1}f'(z)\Big) \\
&= (m-n)\, d_{m+n}f(z)
\end{aligned}$$

for $n, m \in \mathbb{Z}$. □

Remark 1. The Witt algebra also appears in other places. Let $S^1 = \mathbb{R}/[0, 2\pi)$ be the circle, then every smooth vector field on S^1 can be written as

$$\xi = f \frac{d}{d\theta}$$

with f being a smooth real valued function on S^1. If we consider the set of all vector fields ξ on S^1 together with the Lie bracket for vector fields than we obtain again the Witt algebra. A basis is given by the trigonometric polynomials

$$d_n(\theta) = i \exp(in\theta) \frac{d}{d\theta}, \quad n \in \mathbb{Z}.$$

with $\theta \in [0, 2\pi)$.

2.2 The Virasoro algebra

The representation theory of the Witt algebra is relatively easy to understand. In many applications of the Witt algebra however we are more interested in its central extension. With the central extension of $\mathcal{W}$ we mean the Lie algebra

$$\mathrm{Vir} := \mathcal{W} \oplus \mathcal{C}$$

where $\mathcal{C} = \mathbb{C}\hat{c}$, with the commutation relations

$$[d_m, d_n] = (m-n)d_{m+n} + a(m,n)\hat{c}, \quad n, m \in \mathbb{Z}$$
$$[d_m, \hat{c}] = 0, \quad \forall m \in \mathbb{Z}.$$

where $a(m,n)$ is a complex valued function. From now on, to prevent confusion, we will use $\hat{c}$ for the Lie algebra element and c for its eigenvalue in a given representation. The Lie bracket relations are the most general ones so that the new element $\hat{c}$ lies in the center of Vir. The element $\hat{c}$ (for representations of the Virasoro algebra sometimes also its eigenvalue c) is called the central charge of the algebra.

If $\{d_n,\, n \in \mathbb{Z}\}$ is a basis of $\mathcal{W}$ then $\{d_n : n \in \mathbb{Z}\} \cup \{\hat{c}\}$ forms a basis of Vir. It turns out that we can always find a basis of the central extension, so that the function a is given by $\frac{1}{12}(m^3 - m)\delta_{m,-m}$, i.e. every non-trivial central extension of $\mathcal{W}$ by a one-dimensional center is isomorphic to the Virasoro algebra Vir. To see this we first note that the antisymmetry of the Lie brackets implies

$$a(m,n) = -a(n,m) \quad \forall n, m \in \mathbb{Z}. \tag{2.1}$$

If we take a look at the Lie bracket of $d_n, n \in \mathbb{Z}$ and d_0

$$[d_n, d_0] = n\, d_n + a(n,0)\hat{c}$$

we can see that by redefining d_n with

$$\widetilde{d_n} = d_n + \frac{a(n,0)}{n}\hat{c}$$

we found a basis $\widetilde{d_n}, n \in \mathbb{Z}$ in which $\tilde{a}(n,0) = 0$ for all $n \in \mathbb{Z}$. We can therefore always assume that

$$\tilde{a}(n,0) = 0 \quad \forall n \in \mathbb{Z}. \tag{2.2}$$

We can use the same trick for

$$[d_1, d_{-1}] = 2d_0 + \tilde{a}(1,-1)\hat{c}.$$

By redefining the basis element d_0 to

$$\widetilde{d_0} = d_0 + \frac{\tilde{a}(1,-1)}{2}\hat{c}$$

we can always find a basis $\widetilde{\widetilde{d_n}}, n \in \mathbb{Z}$ in which also

$$\tilde{\tilde{a}}(1,-1) = 0 \tag{2.3}$$

We denote the new basis elements $\widetilde{\widetilde{d_n}}$ for which eq. (2.2) and eq. (2.3) holds, from now on with L_n, and will drop the tildes over a. Next we compute the Jacobi identity for arbitrary $n, m, k \in \mathbb{Z}$

$$\begin{aligned} &[L_n,[L_m,L_k]] + [L_k,[L_n,L_m]] + [L_m,[L_k,L_n]] = \dots \\ &= (m-k)a(n,m+k)\hat{c} + (n-m)a(k,n+m)\hat{c} + (k-n)a(m,n+k)\hat{c}. \end{aligned} \tag{2.4}$$

This expression must be zero if the bracket should define a Lie algebra. We will consider two special cases:

1. $n = 0$:
   The Jacobi identity yields

   $$(m-k)a(0,m+k) - ma(k,m) + ka(m,k) = (m+k)a(m,k) = 0 \quad m,k \in \mathbb{Z}$$

   where we used eq. (2.1) and eq. (2.2). Thus $a(m,k) = 0$ for $m \neq -k$

2. $n = 1, m = j$ and $k = -(j+1)$ for an arbitrary $j \in \mathbb{N}$:
   Inserting the values above in the Jacobi identity (with the notation from eq. (2.4)) gives us

   $$(1-j)a(-(j+1), j+1) - (j+2)a(j,-j) = 0.$$

   From which we can obtain a recursive formula for $a(-j,j)$:

   $$\begin{aligned} a(-j,j) &= \frac{j+1}{j-2}a(-(j-1), j-1) \\ &= \frac{j+1}{j-2}\frac{j}{j-3}\cdots\frac{4}{1}\dots a(-2,2) \\ &= \binom{j+1}{3}a(-2,2) \\ &= \frac{(j+1)j(j-1)}{3!}a(-2,2) \\ &= \frac{j^3-j}{3!}a(-2,2) \end{aligned}$$

The value for $a(-2,2)$ can in principle be set arbitrary, in physics however mostly $a(-2,2) = \frac{1}{2}$ is used, because this simplifies certain computations for the free boson (an important example for a theory with conformal symmetry is the Ising model (see chapter 6). We will use this convention for the rest of the thesis.

We summarize the discussion above in the following theorem.

Theorem 2.3. *The Witt algebra $\mathcal{W}$ has a unique central extension* Vir *with the basis $\{L_{-n}\}_{n\in\mathbb{Z}} \cup \hat{c}$ Lie bracket*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\,\delta_{m,-n}\,\hat{c}, \quad n, m \in \mathbb{Z}$$
$$[L_m, \hat{c}] = 0, \quad \forall m \in \mathbb{Z}.$$

Definition 2.4. The central extension Vir of the Witt algebra is called the *Virasoro Algebra.*

2.3 Motivation from physics

In physics the so-called conformal field theories describe physical theories that are invariant under conformal transformations.

If the geometry of a universe is described by a semi-Riemannian metric $g_{\mu\nu}$ with respect to some coordinates x^μ, then invariance under conformal transformations means that the metric with respect to a choice of coordinates x'^μ is proportional to the original metric, i.e.

$$g'_{\mu\nu}(x'^\mu) \propto g_{\mu\nu}(x^\mu)$$

In most relevant cases this is equivalent to scale invariance.

Conformal field theories play an important role in phase transitions, like for example in the transition from paramagnetic to a ferromagnetic state in a metal, which is described by the Ising model.

One can show that in two dimensions the Lie algebra of the corresponding group of conformal transformations is the Witt algebra. The elements of the Witt algebra can then be understood as the terms in the mode expansion of the energy momentum tensor. When one quantizes the theory[1] one finds that the corresponding Lie algebra changes to the Virasoro algebra (This is similar to the quantization of classical mechanics where the position momentum commutator gets an additional central term $i\hbar$: $[\hat{x}, \hat{p}] = i\hbar$). The central element c is in this context called a *quantum anomaly.* The states and particles of the quantum theory are connected to the unitary irreducible representations of the Virasoro algebra. We will talk more about this in the next chapter.

[1]Quantization is a complicated process in which functions on the space of configurations (for example the energy of the system) are mapped to operators on a separable Hilbert space (the ”states” of the quantum theory).

Chapter 3

Highest weight representations

3.1 Basic concepts

To better understand the Virasoro algebra we investigate its representations.

Definition 3.1. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras over $\mathbb{K}$. A linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is a *Lie algebra homomorphism* if for all $x, y \in \mathfrak{g}$

$$\phi([x, y]) = [\phi(x), \phi(y)].$$

If ϕ is bijective we call ϕ an *Lie algebra isomorphism*.

Definition 3.2. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. A *representation* of $\mathfrak{g}$ is a Lie algebra homomorphism

$$\pi : \mathfrak{g} \to \mathfrak{gl}(V)$$

where V is a vector space over $\mathbb{K}$ (possibly of infinite dimension) and $\mathfrak{gl}(V)$ the general linear algebra on V.

Remark 2. We will sometimes call a vector space a representation of L. This means that there exists a Lie algebra homomorphism so that the Lie algebra can be seen as a subalgebra of the general linear algebra on this vector space.

There are numerous representations for the Virasoro algebra, we are however only interested in the subclass of highest weight representations. These representations have the nice feature that we can write them as a direct sum of eigenspaces of a finite dimensional commutative subalgebra. This property is inherited by subrepresentations as the following lemma shows.

Lemma 3.3 (Kac Cor. 1.1)**.** *Let V be a representation of* Vir *that decomposes as a direct sum of eigenspaces $V_n = \{\boldsymbol{v} \in V | L_0\boldsymbol{v} = \lambda_n\boldsymbol{v}\}, n \in I$ of L_0, for an (possibly infinite) Index set I.*

$$V = \bigoplus_{n \in I} V_n.$$

Then any subrepresentation U of V respects this decomposition in the sense that

$$U = \bigoplus_n (U \cap V_n)$$

Proof. For every $\boldsymbol{v} \in V$ we have a unique decomposition

$$\boldsymbol{v} = \sum_{n=1}^{N} \boldsymbol{x}_n$$

with $x_n \in V_n$. If we apply L_0 we find

$$\begin{aligned} L_0 \boldsymbol{v} &= \sum_{n=1}^{N} L_0 \boldsymbol{x}_n \\ &= \sum_{n=1}^{N} \lambda_n \boldsymbol{x}_n \end{aligned}$$

where $\lambda_i \neq \lambda_j$ for $i \neq j$. By induction

$$L_0^k \boldsymbol{v} = \sum_{n=1}^{N} \lambda_n^k \boldsymbol{x}_n$$

Let $\boldsymbol{v}$ an arbitrary element in U, then

$$\begin{array}{rcrcccr} \boldsymbol{v} & = & \boldsymbol{x}_1 & + & \cdots & + & \boldsymbol{x}_N \\ L_0 \boldsymbol{v} & = & \lambda_1 \boldsymbol{x}_1 & + & \cdots & + & \lambda_N \boldsymbol{x}_N \\ \vdots & & \vdots & & \ddots & & \vdots \\ L_0^{N-1} \boldsymbol{v} & = & \lambda_1^{N-1} \boldsymbol{x}_1 & + & \cdots & + & \lambda_N^{N-1} \boldsymbol{x}_N \end{array}$$

forms a linear system of equations. This can also be written in matrix form as

$$\begin{pmatrix} \boldsymbol{v} \\ L_0 \boldsymbol{v} \\ \vdots \\ L_0^{N-1} \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_N \\ \vdots & \ddots & \vdots \\ \lambda_1^{N-1} & \ldots & \lambda_N^{N-1} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_N \end{pmatrix}$$

where the matrix is the transpose of a Vandermonde matrix. The system has a unique solution because Vandermonde matrices are always invertible. This implies that the elements $(\boldsymbol{x}_n)_{n \leq N}$ lie in U and we found a unique decomposition of the element $\boldsymbol{v}$ in elements of $U \cap V_n$. □

The standard example of a Lie algebra is $\mathfrak{gl}_n$, the Lie algebra of square matrices, with the commutator as Lie bracket $[x, y] = xy - yx$. It turns out that we can embed every Lie algebra $\mathfrak{g}$ in a unital and associative algebra $U(\mathfrak{g})$

where the Lie bracket is realized as the commutator. To see this, note that any associative algebra A is a Lie algebra with the commutator as Lie bracket. Furthermore for a vectorspace $\mathfrak{g}$ over $\mathbb{K}$, define the tensor products

$$T^k(\mathfrak{g}) := \overbrace{\mathfrak{g} \otimes \ldots \otimes \mathfrak{g}}^{k \text{ times}}, \quad k > 0$$
$$T^0(\mathfrak{g}) := \mathbb{K}$$

The tensor algebra $T(\mathfrak{g})$ is then defined as

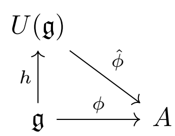
$$T(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} T^k(\mathfrak{g}).$$

Definition 3.4. The *universal enveloping algebra* $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is defined by the unique unital and associative algebra

$$U(\mathfrak{g}) = T(\mathfrak{g})/I$$

where I is the ideal generated by all elements $x \otimes y - y \otimes x =: [x, y]_{\mathfrak{g}} = [x, y]$.

Remark 3. The universal enveloping algebra can also be understood by its universal property. Let $U(\mathfrak{g})$ be a unital and associative algebra such that there exists an embedding $h : \mathfrak{g} \to U(\mathfrak{g})$. Then the universal enveloping algebra is the unique unital and associative algebra such that for every associative algebra A and Lie algebra homomorphism $\phi : \mathfrak{g} \to A$ (with Lie bracket in A given by the commutator) there exists a unique $\hat{\phi} : U(\mathfrak{g}) \to A$ and $\phi = \hat{\phi} \circ h$ so that the following diagram commutes.



An important result about universal enveloping algebras that we will need later is the following theorem.

Theorem 3.5 (Poincaré-Birkhoff-Witt)**.** *Let $\mathfrak{g}$ be a Lie algebra, B an ordered set and $\{x_b : b \in B\} \subseteq \mathfrak{g}$ a basis of $\mathfrak{g}$. Then the set of ordered monomials*

$$\{x_{b_1}^{i_1} \cdots x_{b_n}^{i_n} : n \in \mathbb{N},\, b_1 < \cdots < b_n,\, i_1, \ldots, i_n \in \mathbb{Z}\}$$

is a basis for $U(\mathfrak{g})$. Here we identified the elements of $\mathfrak{g}$ with their images under the embedding.

Proof. see for example [4]. □

Remark 4. The Poincaré-Birkhoff-Witt theorem gives an us a more intuitive understanding of the universal enveloping algebra. The algebra $U(\mathfrak{g})$ allows us to describe formal products xy of elements x, y of $\mathfrak{g}$ and lets us understand the

Lie brackets as commutator brackets. Furthermore it a representation π induces a homomorphism of associative algebras via

$$\begin{aligned} \tilde{\pi}: \quad U(\mathfrak{g}) &\to \mathfrak{gl}(V) \\ x_1 \cdots x_n &\mapsto \pi(x_1) \cdots \pi(x_n). \end{aligned}$$

This way, it is equivalent to consider a representation of $\mathfrak{g}$ and a representation of $U(\mathfrak{g}$.

3.2 Virasoro highest weights

We will first give a short introduction into what highest weight representations are and then begin the proof of the main result of this thesis: the Kac-determinant formula

Definition 3.6. A *highest weight representation* of Vir is a representation π on a vector space V over $\mathbb{C}$ with a non-zero vector $\boldsymbol{v} \in V$ such that there are $c, h \in \mathbb{C}$ with

$$\begin{aligned} \pi(\hat{c})\boldsymbol{v} &= c\boldsymbol{v} \\ \pi(L_0)\boldsymbol{v} &= h\boldsymbol{v} \end{aligned}$$

so that V is the linear span of vectors of the form

$$\pi(L_{-n_k}) \cdots \pi(L_{-n_1})\boldsymbol{v}, \quad (0 < n_1 \leq \ldots \leq n_k). \tag{3.1}$$

with $k \in \mathbb{N}_0$. We call

$$\sum_{i=1}^{k} n_i$$

the *level* of the element (3.1). The pair of complex numbers (c, h) is then called the *highest weight* and $\boldsymbol{v}$ the *highest weight vector*.

Remark 5. We will drop the π from now on to make the notation easier to read i.e. we will always think of L_n as an element of $\mathfrak{gl}(V)$.

Proposition 3.7. *The eigenspace that contains the highest weight vector $\boldsymbol{v}$ is one dimensional. Equivalently: the only vectors with L_0 eigenvalue h are the highest weight vector and its multiples. Furthermore for $\boldsymbol{w} = L_{-n_k} \ldots L_{-n_1}\boldsymbol{v}$*

$$L_0\boldsymbol{w} = (h + n_1 + \cdots + n_k)\boldsymbol{w}.$$

Proof. Let $\boldsymbol{v}$ be such that $L_0\boldsymbol{v} = h\boldsymbol{v}$. It follows from

$$\begin{aligned} L_0 L_{-n}\boldsymbol{v} &= (L_{-n}L_0 + [L_0, L_{-n}])\boldsymbol{v} \\ &= (L_{-n}L_0 + nL_{-n})\boldsymbol{v} \\ &= (h + n)L_{-n}\boldsymbol{v} \end{aligned}$$

that the element $L_{-n}\boldsymbol{v}$ lies in the L_0-eigenspace with eigenvalue $h+n$. A generating set for Vir is given by eq. (3.1). One can show by induction that for every such element $\boldsymbol{w} = L_{-n_k} \dots L_{-n_1}\boldsymbol{v}$

$$L_0\boldsymbol{w} = (h + n_1 + \cdots + n_k)\boldsymbol{w}.$$

Hence the only basis element with L_0-eigenvalue h is the highest weight vector $\boldsymbol{v}$ and its multiples. □

Proposition 3.8. *For positive n, L_n annihilates the highest weight vector $\boldsymbol{v}$, i.e.*

$$L_n\boldsymbol{v} = \boldsymbol{0}, \quad \forall n > 0$$

Proof. For a highest weight representation V of Vir every vector is a linear combinations of elements of the form (3.1). We showed in Proposition 3.7 that for every element $\boldsymbol{w} = L_{-n_k} \dots L_{-n_1}\boldsymbol{v}$

$$L_0\boldsymbol{w} = (h + n_1 + \cdots + n_k)\boldsymbol{w}.$$

Because V is the linear span of such vectors we find that the representation decomposes as a direct sum of the eigenspaces of L_0

$$V = \bigoplus_{n\geq 0} V_{h+n} \tag{3.2}$$

where V_{h+n} denotes the eigenspace of L_0 with eigenvalue $h+n$. In particular, $\boldsymbol{v}$ has the smallest eigenvalue.

$$\begin{aligned} L_0L_n\boldsymbol{v} &= (L_nL_0 + [L_0, L_n])\boldsymbol{v} \\ &= (L_nL_0 - nL_n)\boldsymbol{v} \\ &= (h-n)L_n\boldsymbol{v} \end{aligned}$$

implies that every L_n with $n > 0$ reduces the L_0 eigenvalue. Thus $L_n\boldsymbol{v}$ must be zero. □

We are especially interested in unitary representations of the Virasoro algebra. To define unitarity we need to know what an anti-involution on Vir is.

Definition 3.9. An *anti-involution* on Vir is a map ω with

$$\begin{aligned} \omega(\lambda x) &= \bar{\lambda}\omega(x) \\ \omega([x, y]) &= [\omega(y), \omega(x)] \end{aligned}$$

where x, y are elements of Vir and $\lambda \in \mathbb{C}$.

An anti-involution can be thought of as a generalization of Hermitian conjugation.

Theorem 3.10. *One realization of this on* Vir *is given by*

$$\begin{aligned} \omega(L_n) &= L_{-n}, \quad n \in \mathbb{Z} \\ \omega(\hat{c}) &= \hat{c}. \end{aligned} \tag{3.3}$$

Proof. We will show that $\omega([L_m, L_n]) = [\omega(L_m), \omega(L_n)]$, $\forall m, n \in \mathbb{Z}$.

$$\begin{aligned} \omega([L_m, L_n]) &= (n+m)\omega(L_{m+n}) + \frac{1}{12}(m^3 - m)\delta_{m,-n}\omega(\hat{c}) \\ &= (n+m)L_{-(m+n)}) + \frac{1}{12}(m^3 - m)\delta_{m,-n}\hat{c} \\ &= [L_{-m}), L_{-n}] = [\omega(L_m), \omega(L_n)]. \end{aligned}$$

□

Definition 3.11. Let $\mathfrak{g}$ be a Lie algebra with conjugate-linear anti-involution $\omega : \mathfrak{g} \to \mathfrak{g}$. Let π be a representation of $\mathfrak{g}$ with an Hermitian form $\langle \cdot | \cdot \rangle$. The form $\langle \cdot | \cdot \rangle$ is called *contravariant* if

$$\langle \pi(x)(\boldsymbol{u}), \boldsymbol{v} \rangle = \langle \boldsymbol{u} | \pi(\omega(x))\boldsymbol{v} \rangle \quad \forall x \in \mathfrak{g}, \forall \boldsymbol{u} | \boldsymbol{v} \in V$$

We call the representation *unitary* if the form is additionally positive Kac-determinant, i.e.

$$\langle \boldsymbol{v} | \boldsymbol{v} \rangle > 0 \quad \forall \boldsymbol{v} \in V,\, \boldsymbol{v} \neq \boldsymbol{0}$$

Remark 6. For representations of the Virasoro algebra unitarity can be understood as the Hermitianity of the generators in eq. (3.3).

Theorem 3.12 (Kac Prop. 3.2)**.** *Every unitary highest weight representation of* Vir *is irreducible.*

Proof. Let V denote a highest weight representation of Vir. We assume that there exists a non-trivial sub-representation U of V. Let $U^\perp$ denote the orthogonal complement of U with respect to our unitary form, i.e. $U^\perp = \{\boldsymbol{u} | \langle \boldsymbol{w} | \boldsymbol{u} \rangle = 0\}$, then we have

$$V = U \oplus U^\perp.$$

We will show that either U or $U^\perp$ is the trivial sub-representation. Because of Lemma 3.3 U has a decomposition into spaces of the form $U \cap V_{h+n}$. According to Proposition 3.7 the L_0-eigenspace that contains the highest weight vector is one dimensional, therefore either U or $U^\perp$ must contain $\boldsymbol{v}$. Without loss of generality let the highest weight vector $\boldsymbol{v}$ be in U. Then, because V is the span of elements of the form (3.1), U must already be the full vector space. □

Definition 3.13. A highest weight representation of Vir in which all vectors of the form (3.1) are linear independent is called *Verma representation*. We will denote the Verma representation with highest weight (c, h) with $M(c, h)$.

Theorem 3.14. *For every $c, h \in \mathbb{C}$ there exists a unique Verma representation $M(c, h)$ of* Vir*.*

Proof. Let $I(c,h)$ be the ideal in the universal enveloping algebra $U(Vir)$ of Vir that is generated by the elements $\{L_n : n > 0\} \cup \{L_0 - h \cdot 1_{U(\mathrm{Vir})}\} \cup \{\hat{c} - c \cdot 1_{U(\mathrm{Vir})}\}$. We define

$$M(c,h) = U(\mathrm{Vir})/I(c,h).$$

If we define the map $\pi : \mathrm{Vir} \to \mathfrak{gl}(M(c,h))$ with

$$\pi(x)(u + I(c,h)) = xu + I(c,h)$$

then π is a Lie algebra homomorphism

$$\begin{aligned}
\pi([x,y])(u + I(c,h)) &= [x,y]u + I(c,h) \\
&= (xy - yx)u + I(c,h) \\
&= (\pi(x)\pi(y) + \pi(y)\pi(x))u + I(c,h) \\
&= [\pi(x), \pi(y)](u + I(c,h)).
\end{aligned}$$

Therefore $M(c,h)$ together with π forms a representation of Vir. We want this to be a highest weight representation. To show this define $\boldsymbol{v} := 1_{U(\mathrm{Vir})} + I(c,h) \in M(c,h)$. This is a highest weight vector:

$$\begin{aligned}
L_n\boldsymbol{v} &= L_n + I(c,h) = 0 + I(c,h) \\
L_0\boldsymbol{v} &= L_0 + I(c,h) = h \cdot 1_{U(\mathrm{Vir})} + I(c,h) = h\boldsymbol{v} \\
\hat{c}\boldsymbol{v} &= \hat{c} + I(c,h) = c \cdot 1_{U(\mathrm{Vir})} + I(c,h) = c\boldsymbol{v}
\end{aligned}$$

Thus $M(c,h)$ is a highest weight representation of Vir with highest weight (c,h). Theorem 3.5 tells us that the elements of the form (3.1) are linear independent in $U(Vir)$ and therefore also $M(c,h)$ which shows that $M(c,h)$ is indeed a Verma representation. To see the uniqueness let V be a highest weight representation of Vir with highest weight (c,h). Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be the highest weight vectors of V and $M(c,h)$ respectively. By defining

$$f\boldsymbol{v} = \boldsymbol{w}$$

we find a unique surjective homomorphism from $M(c,h)$ to an arbitrary highest weight representation V of Vir. In fact one can also define the Verma module as a representation of Vir so that for every other representation V there exists a unique linear surjection from $M(c,h)$ to V that maps highest weight vectors to highest weight vectors. This shows that every highest weight representation of Vir can be obtained as a quotient of the Verma representation. For two Verma representations we obtain an isomorphism which shows that the highest weight (c,h) determines the Verma representation uniquely. □

Proposition 3.15 (Kac Prop. 3.3)**.** *The Verma representation $M(c,h)$:*

1. *has the decomposition*

$$M(c,h) = \bigoplus_{k \in \mathbb{N}_0} M(c,h)_{h+k} \tag{3.4}$$

where $M(c,h)_{h+k}$ denotes the $h+k$ eigenspace of L_0.

2. *is indecomposable, i.e. there are no nontrivial subrepresentations* V, W *such that*

$$M(c,h) = V \oplus W$$

3. *has a unique maximal proper subrepresentation* $J(c,h)$ *and*

$$V(c,h) = M(c,h)/J(c,h) \tag{3.5}$$

is the unique irreducible highest weight representation with highest weight (c,h)*.*

Proof. 1.) follows from the fact that a Verma representation is a highest weight representation of Vir and eq. (3.2).

2.) The proof is identical to the argument in Theorem 3.12.

3.) According to Lemma 3.3 all proper subrepresentations U of $M(c,h)$ decompose into subspaces of the form $U \cap V_{h+n}$. Therefore the sum of proper subrepresentations also has a decomposition into subspaces of the form $(\bigoplus_i U_i) \cap V_n$. None of the U_i contains the highest weight vector (otherwise they would already be equal to $M(c,h)$ and therefore not be proper). Thus the direct sum does not contain it either. It is therefore a proper subrepresentation of $M(c,h)$. This shows that we can obtain the maximal proper subrepresentation $J(c,h)$ as the sum of all proper subrepresentations. By defining

$$x(\boldsymbol{u} + J(c,h)) := x(\boldsymbol{u}) + J(c,h), \quad x \in \mathfrak{g}, \quad \boldsymbol{u} \in M(c,h)$$

we obtain a Lie algebra structure on the factorspace $M(c,h)/J(c,h)$. $V(c,h)$ is surely irreducible because for every proper subrepresentation $U + J(c,h) \neq J(c,h)$ of $V(c,h)$ the algebra $\{\boldsymbol{u} + \boldsymbol{j} | \boldsymbol{u} \in U, \boldsymbol{j} \in J(c,h)\}$ would be a proper subrepresentation of $M(c,h)$ which contradicts our choice of $J(c,h)$ as the maximal proper subrepresentation. To prove the uniqueness of $V(c,h)$ let $V'(c,h)$ be another irreducible highest weight representation with highest weight (c,h). From Theorem 3.14 it follows that there exists a proper subrepresentation $J'(c,h) = \ker f$ such that

$$V'(c,h) = M(c,h)/J'(c,h).$$

Because $V'(c,h)$ is irreducible $J'(c,h)$ must be maximal and thus equal to $J(c,h)$. This proves the uniqueness. □

It turns out that in most cases $M(c,h)$ is already equipped with a Hermitian form which we can use to find $J(c,h)$. To show this we need to make a definition first. According to eq. (3.4) every vector $\boldsymbol{u} \in M(c,h)$ can be written as

$$\boldsymbol{u} = \lambda_0 \boldsymbol{v} + \sum_{k=1}^{\infty} \lambda_k \boldsymbol{u}_k, \quad \boldsymbol{u}_k \in M_{h+k} \tag{3.6}$$

with the highest weight vector $\boldsymbol{v} \in M_h$.

Definition 3.16. For every $\boldsymbol{u} \in M(c,h)$ we define the *expectation value* $\langle \boldsymbol{u} \rangle$ as the coefficient λ_0 of the highest weight vector $\boldsymbol{v}$ in the expansion eq. (3.6) of $\boldsymbol{u}$.

Theorem 3.17 (Shapovalov form, Kac Prop. 3.4)**.**

1. *For $c, h \in \mathbb{R}$, $M(c,h)$ carries a unique contravariant Hermitian form $\langle \cdot | \cdot \rangle$ with $\langle \boldsymbol{v} | \boldsymbol{v} \rangle = 1$ where $\boldsymbol{v}$ is the highest weight vector.*

2. *The eigenspaces of L_0 are pairwise orthogonal.*

3. *The maximal proper subrepresentation from Proposition 3.15 is*

$$J(c,h) = \ker \langle \cdot | \cdot \rangle = \{\boldsymbol{u} \in M(c,h) | \langle \boldsymbol{u} | \boldsymbol{w} \rangle = 0, \quad \forall \boldsymbol{w} \in M(c,h)\}. \tag{3.7}$$

Therefore $V(c,h)$ carries a unique contravariant Hermitian form such that $\langle \boldsymbol{v} | \boldsymbol{v} \rangle = 1$, and this form is non-degenerate.

Proof. 1.) We can construct the contravariant form using the expectation value defined above. For this we use the anti-involution ω acting on products of elements of Vir which means we need to extend ω from Vir to its universal enveloping algebra. We can do this by setting

$$\begin{aligned} \omega((L_{-n_1} \dots L_{-n_i})(L_0^r \hat{c}^s)(L_{m_1} \dots L_{m_j})) = \\ (L_{-m_j} \dots L_{-m_1})(L_0^r \hat{c}^s)(L_{n_i} \dots L_{n_1}), \end{aligned}$$

with $n, k, r, s \in \mathbb{N}$.

Let $X = L_{-n_i} \dots L_{-n_1}$ and $Y = L_{-m_j} \dots L_{-m_1}$ be two elements of $M(c,h)$ then we can define a hermitian form through the expectation value by setting

$$\begin{aligned} \langle X(\boldsymbol{v}) | Y(\boldsymbol{v}) \rangle :=& \langle \omega(X)(Y(\boldsymbol{v})) \rangle \\ =& \langle L_{n_1} \dots L_{n_i} L_{-m_j} \dots L_{-m_1}(\boldsymbol{v}) \rangle. \end{aligned}$$

This defines a Hermitian form because for monomials A, B, Y we have

$$\begin{aligned} \langle A(\boldsymbol{v}) + \lambda B(\boldsymbol{v}) | Y(\boldsymbol{v}) \rangle &= \langle \omega(A + \lambda B)(Y(\boldsymbol{v})) \rangle \\ &= \langle \omega(A)(Y(\boldsymbol{v})) + \bar{\lambda} \omega(B)(Y(\boldsymbol{v})) \rangle \\ &= \langle \omega(A)(Y(\boldsymbol{v})) \rangle + \bar{\lambda} \langle \omega(B)(Y(\boldsymbol{v})) \rangle \end{aligned}$$

where the second equality is due to ω being an anti-involution and the third equality due to eq. (3.6) being linear in $\boldsymbol{u}$. The linearity in the second component is trivial. This shows that we indeed defined a Hermitian form that, because the monomials form a basis, can be extended uniquely to $M(c,h)$ and fulfills $\langle \boldsymbol{v} | \boldsymbol{v} \rangle = 1$.

2.) The eigenspace to eigenvalue $h + k$ is the linear span of the elements

$$L_{-n_i} \dots L_{-n_1}(\boldsymbol{v}) \quad \text{with}\, n_1 + \dots n_i = k.$$

It is therefore sufficient to check the orthogonality for elements of this type. For $n_1 + \dots n_i > m_1 + \dots m_j$ we find that

$$\langle L_{n_1} \dots L_{n_i} L_{-m_j} \dots L_{-m_1}(\boldsymbol{v}) \rangle = 0 \tag{3.8}$$

because the argument $L_{n_1} \dots L_{n_i} L_{-m_j} \dots L_{-m_1}(\boldsymbol{v})$ is already zero. For the other case of $n_1 + \dots n_i < m_1 + \dots m_j$ we find eq. (3.8) holds because the coefficient of the highest weight vector in eq. (3.6) vanishes.

3.) We need to show that the kernel

$$\ker\langle \cdot | \cdot \rangle = \{\boldsymbol{u} \in M(c,h) | \langle \boldsymbol{u} | \boldsymbol{w} \rangle = 0, \quad \forall \boldsymbol{w} \in M(c,h)\}$$

is the unique maximal proper subrepresentation $J(c,h)$. The kernel is clearly a proper subrepresentation because it is a representation that does not contain the highest weight vector $\boldsymbol{v}$ (because $\langle \boldsymbol{v} | \boldsymbol{v} \rangle = 1$). We will show the other inclusion with proof by contradiction. Let us assume there is a proper subrepesentation U that is not contained in the kernel. This means for $Y(\boldsymbol{v}) \in U$ we can find $X(\boldsymbol{v}) \in M(c,h)$ such that

$$\langle X(\boldsymbol{v}) | Y(\boldsymbol{v}) \rangle = \langle \omega(X)(Y(\boldsymbol{v})) \rangle \neq 0.$$

Because U is a representation of Vir and $\omega(X) \in Vir$, $\omega(x)(Y(\boldsymbol{v})) \in U$. We found an element of U with non-vanishing expectation value and therefore a non-vanishing highest weight vector component in the direct sum expansion. With Lemma 3.3 we can deduce that U contains the highest weight vector and hence $U = M(c,h)$ in contradiction U being a proper subrepresentation. □

From now on we will always assume c and h to be real so that the Shapovalov form above is always defined.

Corollary 3.18 (Kac Prop. 3.5)**.** *There exists at most one unitary highest weight representation for every highest weight* (c,h) *with* $c, h \in \mathbb{R}$ *and that is* $V(c,h)$.

Proof. Follows from remark 3.12 and Proposition 3.15 part 3. □

Let us sum up everything we already know about the highest weight representations of the Virasoro algebra. They are classified by their highest weights (c,h). For every highest weight we obtain a Verma representation $M(c,h)$. For real c,h these representations each carry a unique Hermitian form $\langle \cdot | \cdot \rangle$.

We can now ask the question for which values of the highest weights (c,h) the unique irreducible highest weight representations $V(c,h)$ is unitary and identical to the Verma representation. Because there always exists a Verma representation we can conclude the existence of a unitary highest weight representation of Vir for these cases. Because of $V(c,h) = M(c,h)/J(c,h)$ and $J(c,h) = \ker\langle \cdot | \cdot \rangle$ these are exactly the cases in which the Hermitian form is non-degenerate. We therefore want to know when the kernel of $\langle \cdot | \cdot \rangle$ is non-trivial. Investigating the Hermitian form will lead us to the Kac-determinant for which we will derive an explicit formula in the next chapter.

3.3 Quantum states of two-dimensional CFTs

In physics the current state of a physical object is described as a vector of a separable Hilbert space. The object itself is then identified with the whole Hilbert space. If we now have a physical theory that is invariant under a specific transformation group then the Hilbert space of states must form a representation of the corresponding Lie algebra. For conformal theories in two dimensions (examples are the two dimensional Ising model or String Theory) the Lie algebra corresponding to the conformal symmetry is the Virasoro algebra.

In this way we can identify representations of the Virasoro algebra with physical in a two dimensional conformal field theory. The physical object in this case are the so-called primary fields ϕ. There is a one-to-one correspondence between these primary fields and highest weight vectors through $\phi \mapsto |\phi\rangle = \lim_{z\to 0}\phi(z)|0\rangle$ with $|0\rangle$ denoting the vacuum state. This also explains why we are interested in unitary representations: The term $\langle\phi|\phi\rangle$ must be positive because it is a probability distribution.

The irreducible highest weight representations take a special place among all representations in that one can show that the representations $V(c,h)$ with $h \geq 0$ are precisely the irreducible positive energy representations of Vir.

Chapter 4

Kac-determinant

We want to investigate for which highest weights (c, h) the Shapovalov form has non-trivial kernel. We will call such a Verma representation a *degenerate representation*.

Every Verma representation comes with a basis of the form (3.1). We now investigate the elements

$$\langle L_{-n_s} \dots L_{-n_1} \boldsymbol{v} | L_{-k_t} \dots L_{-k_1} \boldsymbol{v} \rangle \tag{4.1}$$

where $1 \leq n_1 \dots \leq n_s$ and $1 \leq k_1 \dots \leq k_t$. We know from Proposition 3.15 that Verma modules decompose into eigenspaces of L_0. Therefore if we look at the subspaces that consist of the linear span of all basis elements with a level smaller than a certain N the matrix of the Shapovalov form consists of block matrices for each eigenspace $M(c, h)_{h+k}$ (with $k \in \{1, \dots, N\}$) of L_0. A necessary and sufficient condition for $M(c, h)$ to be a degenerate representation is therefore that the determinant of the Shapovalov form matrix for elements of level n vanishes for some $N \geq 0$.

We denote the determinant of the Shapovalov form at level N with

$$\det_N(c, h) = \det(M_{i,j})_{i,j \in P(N)}.$$

Where $P(N)$ denotes the set of number partitions of N and $M_{i,j}$ denotes the element from eq. (4.1) with indices $n_1, \dots, n_s$ and $k_1, \dots, k_t$ from the partitions i and j respectively. As mentioned in the previous chapter we will always assume $c, h \in \mathbb{R}$ to ensure the existence of a Hermitian form. Victor found an explicit expression for the determinant at level N.

Theorem (Kac-determinant formula)**.** *For fixed c the function $\det_N(c, h)$ is the following polynomial in h*

$$\det_N(c, h) = K(N) \prod_{\substack{r,s \in \mathbb{N} \\ 1 \leq rs \leq N}} (h - h_{r,s}(c))^{p(N-rs)}$$

where $h_{r,s}$ is given by

$$h_{r,s}(c) = \frac{1}{48}[(13-c)(r^2+s^2) + \sqrt{(c-1)(c-25)}(r^2-s^2) - 24rs - 2 + 2c]$$

for those pairs (r,s) so that $1 \leq rs \leq N$.

We will proof this result in the next section.

4.1 Explicit computation of the determinant

To gain some intuition on the Kac-determinant formula and also the explicit formula for its zeros we want to compute the determinant explicitly for some small values of N.

At level $N = 1$ we have a 1×1 matrix.

$$\begin{aligned} \langle L_{-1}\boldsymbol{v}|L_{-1}\boldsymbol{v}\rangle &= \langle \boldsymbol{v}|L_1L_{-1}\boldsymbol{v}\rangle \\ &= \langle \boldsymbol{v}|(L_{-1}L_1 + [L_1, L_{-1}])\boldsymbol{v}\rangle \\ &= \langle \boldsymbol{v}|2L_0\boldsymbol{v}\rangle = 2h. \end{aligned}$$

The determinant is therefore also $2h$. And indeed, computing $h_{1,1}$ from Lemma 4.9 we find that $h_{1,1}(c) = \frac{1}{48}[(13-c)2 - 24 - 2 + 2c] = 0$ and our result coincides with the Kac-determinant formula.

At level $N = 2$ we have 2×2 matrix.

$$\begin{pmatrix} \langle L_{-2}\boldsymbol{v}|L_{-2}\boldsymbol{v}\rangle & \langle L_{-1}L_{-1}\boldsymbol{v}|L_{-2}\boldsymbol{v}\rangle \\ \langle L_{-2}\boldsymbol{v}|L_{-1}L_{-1}\boldsymbol{v}\rangle & \langle L_{-1}L_{-1}\boldsymbol{v}|L_{-1}L_{-1}\boldsymbol{v}\rangle \end{pmatrix}$$

Similar as in the case for $N = 1$ we can compute each component by using covariance of the form, the Lie bracket relation an that $L_n\boldsymbol{v} = 0, n > 0$. This yields the matrix

$$\begin{pmatrix} 4h + \frac{c}{2} & 6h \\ 6h & 4h + 8h^2\rangle \end{pmatrix},$$

with a determinant of $2ch - 20h^2 + 4ch^2 + 32h^2$ that has zeros at

$$\{0, \frac{1}{16}(5 - c \pm \sqrt{(c-1)(c-25)})\}. \tag{4.2}$$

The zeroes agree with the values for $h_{1,1}(c), h_{2,1}(c)$ and $h_{1,2}(c)$ from Lemma 4.9 respectively.

4.2 Proof of the Kac-determinant formula

We investigate $\det_N(c,h)$ as a polynomial in h. For two complex polynomials p, q we will write

$$p \sim q$$

if the coefficients of the leading terms are equal. To find an explicit expression for the determinant we will start off by computing the leading coefficient of $\det_N(c,h)$ as a polynomial in h. Then we compute its degree. At last we calculate the zeroes of $\det_N(c,h)$ and write the determinant as the product of its linear factors. times the coefficient of the leading term. The derivation of the Kac-determinant formula presented here holds for any fixed but arbitrary $\mathbb{N} \ni N > 0$.

Proposition 4.1 (Kac Prop. 8.1)**.** $\det_N(c,h)$ *as a polynomial in* h *has degree*

$$\sum_{\substack{r,s\in\mathbb{N}\\1\leq rs\leq N}} p(N-rs) \tag{4.3}$$

where p *is the number of integer partions. The coefficient of the leading term is*

$$K = \prod_{\substack{r,s\in\mathbb{N}\\1\leq rs\leq N}} ((2r)^s s!\,)^{m(N,r,s)} \tag{4.4}$$

where $m(r,s)$ *is the number of partitions of* N *in which* r *appears exactly* s *times.*

Remark 7. The number of partitions of N in which r appears exactly s times is nothing else as the number of partitions in which appears at least s minus the number of partitions in which it appears at least $s+1$ times i.e.

$$m(r,s) = p(N-rs) - p(N-r(s+1))$$

To this proposition we need some easy lemmas first.

Lemma 4.2.

$$\det_N(c,h) \sim \prod_{\substack{1\leq n_1\leq\ldots\leq n_s\\ \sum n_s=N}} \langle L_{-n_s}\ldots L_{-n_1}\boldsymbol{v}|L_{-n_s}\ldots L_{-n_1}\boldsymbol{v}\rangle$$

Proof. By definition

$$\det_N(c,h) = \sum_{\sigma\in\mathfrak{S}_{P(N)}} \operatorname{sgn}(\sigma)\prod M^N_{i,\sigma(j)},$$

where $P(N)$ denotes the set of number partitions of N and $(M^N_{i,j})$ is the matrix for the Hermitian form up to level N. For an arbitrary element of $(M^N_{i,j})$, contravariance gives us

$$\langle L_{-n_s}\ldots L_{-n_1}\boldsymbol{v}|L_{-k_t}\ldots L_{-k_1}\boldsymbol{v}\rangle$$
$$\langle L_{k_1}\ldots L_{k_t}L_{-n_s}\ldots L_{-n_1}\boldsymbol{v}|\boldsymbol{v}\rangle.$$

We get h every time L_0 is acting on $\boldsymbol{v}$. L_0's appear in the commutator of $[L_n, L_m]$ if and only if $n=-m$ which means the expression $\langle L_{k_1}\ldots L_{k_t}L_{-n_s}\ldots L_{-n_1}\boldsymbol{v}|\boldsymbol{v}\rangle$ has maximal degree in h exactly for $j=i$. □

Lemma 4.3. *For $n \in \mathbb{Z}$ and $k \in \mathbb{N}$*

$$[L_n, L_{-n}^k] = L_{-n}^{k-1} nk\left(n(k-1) + 2L_0 + \frac{n^2-1}{12}\hat{c}\right), \tag{4.5}$$

Proof. Induction by k. □

Lemma 4.4 (Kac. Lemma 8.1)**.** *Let $k, n \in \mathbb{N}$, then*

$$\langle L_{-n}^k \boldsymbol{v} | L_{-n}^k \boldsymbol{v}\rangle = k!\, n^k \prod_{j=1}^{k} (2h + \frac{n^2-1}{12}c + n(j-1))$$

Proof. We know from the lemma above that

$$[L_n, L_{-n}^k] = L_{-n}^{k-1} nk\left(n(k-1) + 2L_0 + \frac{n^2-1}{12}\hat{c}\right), \tag{4.6}$$

for all $n, k \in \mathbb{N}$. With this we can prove the result by induction. The case $k = 1$ has already been done in Corollary 4.11. We now show that if it is true for $k-1$ it must also be true for k.

$$\begin{aligned}
\langle L_{-n}^k \boldsymbol{v} | L_{-n}^k \boldsymbol{v}\rangle &= \langle L_n^k L_{-n}^k \boldsymbol{v} | \boldsymbol{v}\rangle \\
&= \langle L_n^{k-1} L_{-n}^{k-1} nk(n(k-1) + 2L_0 + \frac{n^2-1}{12}\hat{c})\boldsymbol{v}|\boldsymbol{v}\rangle \\
&\overset{IH}{=} nk(n(k-1) + 2h + \frac{n^2-1}{12}c) \\
&\quad (k-1)!\, n^{k-1} \prod_{j=1}^{k-1} (2h + \frac{n^2-1}{12}c + n(j-1)) \\
&= k!\, n^k \prod_{j=1}^{k} (2h + \frac{n^2-1}{12}c + n(j-1))
\end{aligned}$$

Where we used $\langle L_n^{k-1} L_{-n}^k L_n \boldsymbol{v}|\boldsymbol{v}\rangle = 0$ and $L_n^{k-1} L_n L_{-n}^k = L_n^{k-1}[L_n, L_{-n}^k] - L_n^{k-1} L_{-n}^k L_n$ in the second equality. □

This implies in particular

$$\langle L_{-n}^k \boldsymbol{v} | L_{-n}^k \boldsymbol{v}\rangle \sim k!\, (2nh)^k. \tag{4.7}$$

We use this to prove the following lemma.

Lemma 4.5 (Kac Lemma 8.2)**.**

$$\langle L_{-n_s}^{k_s} \dots L_{-n_1}^{k_1} \boldsymbol{v} | L_{-n_s}^{k_s} \dots L_{-n_1}^{k_1} \boldsymbol{v}\rangle \sim \langle L_{-n_s}^{k_s} \boldsymbol{v} | L_{-n_s}^{k_s} \boldsymbol{v}\rangle \dots \langle L_{-n_1}^{k_1} \boldsymbol{v} | L_{-n_1}^{k_1} \boldsymbol{v}\rangle$$

where $n_1, \dots, n_s, k_1, \dots, k_s \in \mathbb{N}$ and $n_1 \neq n_2 \neq \dots \neq d_i$.

Proof. We prove this by induction on $\sum_{i=1}^{s} k_i = n$. The base case of $n = 1$ is trivially true. In the induction step we assume the statement to be true for n and show that it must also hold for $n+1$. Let $\sum_{i=1}^{s} k_i = n$.

$$\begin{aligned}\langle L_{-n_s}^{k_s} \dots L_{-n_1}^{k_1} \boldsymbol{v} | L_{-n_s}^{k_s} \dots L_{-n_1}^{k_1} \boldsymbol{v} \rangle &= \langle L_{-n_s} L_{-n_s}^{k_s-1} \dots L_{-n_1}^{k_1} \boldsymbol{v} | L_{-n_s} L_{-n_s}^{k_s-1} \dots L_{-n_1}^{k_1} \boldsymbol{v} \rangle \\ &= \langle L_{-n_s}^{k_s-1} \dots L_{-n_1}^{k_1} \boldsymbol{v} | L_{n_s} L_{-n_s} L_{-n_s}^{k_s-1} \dots L_{-n_1}^{k_1} \boldsymbol{v} \rangle\end{aligned}$$

where we used the contravariance of the Hermitian form in the second equality. We use the Virasoro commutation relations to move L_{n_s} to the right:

$$\langle L_{-n_s}^{k_s-1} \dots L_{-n_1}^{k_1} \boldsymbol{v} | (2n_s L_0 + \frac{n_s^3 - n_s}{12} \hat{c} + L_{-n_s} L_{n_s}) L_{-n_s}^{k_s-1} \dots L_{-n_1}^{k_1} \boldsymbol{v} \rangle$$

Repeating this $k_s - 1$ times using eq. (4.5) we obtain for the term in the second argument of the Hermitian form

$$\begin{aligned}n_s k_s L_{-n_s}^{k_s-1} \left(n_s(k_s-1) + 2L_0 + \frac{n_s^2-1}{12} \hat{c} \right) L_{-n_s}^{k_s-1} \dots L_{-n_1}^{k_1} \boldsymbol{v} \\ + L_{-n_s}^{k_s} L_{n_s} L_{-n_{s-1}}^{k_{s-1}} \dots L_{-n_1}^{k_1} \boldsymbol{v}\end{aligned} \tag{4.8}$$

If we can show that the second term, as a polynomial in h, has a smaller degree than the first term then we can neglect everything except the L_0 term. With this assumption we would find

$$\begin{aligned}\langle L_{-n_s}^{k_s} \dots L_{-n_1}^{k_1} \boldsymbol{v} | L_{-n_s}^{k_s} \dots L_{-n_1}^{k_1} \boldsymbol{v} \rangle &\sim 2n_s k_s h \langle L_{-n_s}^{k_s-1} \dots L_{-n_1}^{k_1} \boldsymbol{v} | L_{-n_s}^{k_s-1} \dots L_{-n_1}^{k_1} \boldsymbol{v} \rangle \\ &\overset{IH}{\sim} 2n_s k_s h \langle L_{-n_s}^{k_s-1} \boldsymbol{v} | L_{-n_s}^{k_s-1} \boldsymbol{v} \rangle \dots \langle L_{-n_1}^{k_1} \boldsymbol{v} | L_{-n_1}^{k_1} \boldsymbol{v} \rangle \\ &\sim 2 \langle L_{-n_s}^{k_s} \boldsymbol{v} | L_{-n_s}^{k_s} \boldsymbol{v} \rangle \dots \langle L_{-n_1}^{k_1} \boldsymbol{v} | L_{-n_1}^{k_1} \boldsymbol{v} \rangle,\end{aligned}$$

where in the first line we ignored all but the highest order terms in h from eq. (4.8). In the second line we used the induction assumption and in the last line we used eq. (4.7). To conclude the proof we need to show that the second term in eq. (4.8) as a polynomial in h is negligible. To be more specific, we will prove that

$$L_n L_{-n_s} \dots L_{-n_1}$$

with $n \notin \{n_1, \dots, n_s\}$, only consists of terms that either have a level smaller than $n_1 + \dots + n_s$ do not contain L_0 or vanish when acting on $\boldsymbol{v}$. In the first case the hermitian form in eq. (4.8) is equal to zero because of the orthogonality of eigenspaces and in the second case the contribution is negligible as a polynomial in h. We will proof this again by induction on s. The base case follows from

$$\begin{aligned}L_n L_{-n_1} &= L_{-n_1} L_n + [L_n, L_{-n_1}] \\ &= L_{-n_1} L_n + (n + n_1) L_{n-n_1}.\end{aligned}$$

The first term vanishes when acting on the highest weight vector and because of $n \neq n_1$ the second term does not contain L_0 and has level smaller than n_1. For $s+1$ we obtain

$$L_n L_{-n_s} \dots L_{-n_1} = (L_{-n_s} L_n + [L_n, L_{-n_s}]) L_{-n_{s-1}} \dots L_{-n_1}.$$

For the first term we can use the induction assumption. The second term can be written as

$$[L_n, L_{-n_s}]L_{-n_{s-1}} \dots L_{-n_1} = (n + n_1)L_{n-n_1}L_{-n_{s-1}} \dots L_{-n_1}.$$

For $n - n_1 < 0$ the term does not contain L_0 and for $n - n_1 > 0$ we can again use the induction assumption. This concludes the proof. □

Lemma 4.6 (Kac Lemma 8.3)**.**

$$\det_N(c, h) \sim \prod_{\substack{r,s\in\mathbb{N} \\ 1\leq rs\leq N}} \langle L_{-r}^s \boldsymbol{v} | L_{-r}^s \boldsymbol{v}\rangle^{m(r,s)}$$

where $m(r, s)$ is the function from Proposition 4.1.

Proof. Follows directly from Lemma 4.5 applied to the expression in Lemma 4.2 □

With this we are set to prove Proposition 4.1.

Proof (of Proposition 4.1). Lemma 4.6 and eq. (4.7) tell us that

$$\det_N(c, h) \sim \prod_{\substack{r,s\in\mathbb{N} \\ 1\leq rs\leq N}} (s!\,(2rh)^s)^{m(r,s)}$$

which tells us that the coefficient of the leading term in $\det_N(c, h)$ is given by

$$\prod_{\substack{r,s\in\mathbb{N} \\ 1\leq rs\leq N}} (s!\,(2r)^s)^{m(r,s)}$$

and the degree is

$$\sum_{\substack{r,s\in\mathbb{N} \\ 1\leq rs\leq N}} s\,m(r, s). \tag{4.9}$$

The only thing that is left to prove is that the term above coincides with eq. (4.7). This follows from remark 7 which we can use to rewrite eq. (4.9)

$$\begin{aligned}
\sum_{\substack{r,s\in\mathbb{N} \\ 1\leq rs\leq N}} s\,m(r, s) &= \sum_{\substack{r,s\in\mathbb{N} \\ 1\leq rs\leq N}} (s\,p(N - rs) - s\,p(N - r(s + 1))) \\
&= \sum_{1\leq r\leq N} \sum_{s=1}^{\lfloor \frac{n}{r} \rfloor} (s\,p(N - rs) - s\,p(N - r(s + 1))) \qquad (4.10) \\
&= \sum_{1\leq r\leq N} \sum_{s=1}^{\lfloor \frac{n}{r} \rfloor} p(N - rs) \\
&= \sum_{\substack{r,s\in\mathbb{N} \\ 1\leq rs\leq N}} p(N - rs)
\end{aligned}$$

where we used that expression (4.10) is a telescoping sum. □

Lemma 4.7 (Kac Lemma 8.4)**.** *Let* $A : \mathbb{R} \to End(V)$ *be a family of linear operators acting on the* n*-dimensional vector space* V *where* $A(t)$ *is a polynomial in* t *i.e.* $A(t) = A_0 + A_1 t + \cdots + A_m t^m$ *with* $A_0, \dots, A_m$ *being linear operators on* V*. If* $\dim \ker A(0) = k$*, then* $\det A(t)$ *is divisible by* t^k*.*

Proof. Let $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_k\}$ be a basis of $\ker A(0)$. We can write $A(t)$ as

$$A(t) = A_0 + A_1 t + \cdots + A_m t^m$$

with $A_0, \dots, A_m$ being linear operators on V. Because of $A(0)\boldsymbol{e}_i = 0$ for all $i \in \{1, \dots, k\}$ we find that for $A(t)$:

$$A(t)\boldsymbol{e}_i = t(A_1 + A_2 t + \cdots + A_m t^{m-1}), \quad \forall i \in \{1, \dots, k\}$$

Therefore the first k columns of $A(t)$ are divisible by t and because of its definition the determinant is divisible by t^k. □

Lemma 4.8 (Kac Lemma 8.5)**.** *If* $\det_N(c, h)$ *vanishes at* $h = h_0$ *then* $\det_N(c, h)$ *is divisible by*

$$(h - h_0)^{p(N-k)} \tag{4.11}$$

where $1 \geq k \geq N$ *is the smallest positive integer smaller than* N *for which* h_0 *is a zero of* $\det_k(c, h)$*.*

Proof. If $\det_N(c, h_0) = 0$ at level N, $\det_N(c, h)$, the vector space $M(c, h_0)$ must have a nonzero maximal proper subrepresentation $J(c, h_0)$ with a nonzero Nth component (by that we mean that it contains at least one non-zero vector with level N) that we will denote

$$\begin{aligned} J_N(c, h_0) &:= J(c, h_0) \cap M(c, h_0)_{h_0+N} \\ &= \ker\langle \cdot | \cdot \rangle \Big|_{M(c,h_0)_{h_0+N}} \end{aligned}$$

Let $k \in \mathbb{N}$ be the smallest number such that $J_k(c, h_0)$ is non-trivial. Let $\boldsymbol{0} \neq \boldsymbol{u} \in J_k(c, h_0)$ be an arbitrary non-trivial element of $J_k(c, h_0)$ (which means it is in the kernel of the Hermitian form). We will now use this vector to construct a $p(N-k)$-dimensional subspace of $J_N(c, h_0)$, to prove that the determinant is indeed divisible by (4.11). The vector $\boldsymbol{u}$ satisfies

$$L_n \boldsymbol{u} = 0, \quad n > 0. \tag{4.12}$$

To see this we use that $\boldsymbol{u}$ is in the kernel of the Hermitian form. This means in particular that for every $\boldsymbol{w}$ we have

$$\langle \boldsymbol{w} | L_n \boldsymbol{u} \rangle = \langle L_{-n} \boldsymbol{w} | \boldsymbol{u} \rangle = 0,$$

which implies that also $L_n \boldsymbol{u} \in J(c, h_0)$. Now if we assume the opposite, i.e. that $L_n \boldsymbol{u} \neq 0$ we would find

$$L_0 L_n \boldsymbol{u} = L_n L_0 \boldsymbol{u} - n L_n \boldsymbol{u} = (h + k - n) L_n \boldsymbol{u}.$$

This means that if $L_n \boldsymbol{u} \neq 0$ we found a non-zero element, namely $L_n \boldsymbol{u}$ with level smaller than k that is in the kernel of the Hermitian form in contradiction to the minimality of k. This proves eq. (4.12). Applying the universal enveloping algebra $U(Vir)$ on $\boldsymbol{u}$ gives us a sub representations that is the linear span of the vectors

$$L_{-n_s} \cdots L_{-n_1} \boldsymbol{u}, \quad 0 < n_1 \leq \ldots \leq n_s. \tag{4.13}$$

The vector $\boldsymbol{u}$ is from level k so at level N we find exactly the vectors (4.13) with $\sum n_i = N - k$, i.e. exactly $p(N-k)$ vectors. The vectors are all linearly independent, this follows from the Poincaré-Birkhoff-Witt Theorem 3.5. We found a $p(N-k)$ dimensional subspace of $J_N(c, h_0)$. By Lemma 4.7 the determinant is divisible by $(h-h_0)^{p(N-k)}$ and k is by construction the smallest integer where the determinant vanishes. □

For the proof of the Kac-determinant formula we will need one more result that we will not prove in this thesis. It is proven in [11], p.87 and p.137 using results from character theory.

Lemma 4.9 (Kac Lemma 8.6)**.** *The polynomial* $\det_N(c,h)$ *has a zero at* $h = h_{r,s}(c)$ *with*

$$h_{r,s}(c) = \frac{1}{48}[(13-c)(r^2+s^2) + \sqrt{(c-1)(c-25)}(r^2-s^2) - 24rs - 2 + 2c] \tag{4.14}$$

for pairs $(r,s) \in \mathbb{N}^2$ *such that* $1 \leq rs \leq N$ *is satisfied.*

With this we can finally prove the main result of this thesis, the Kac-determinant formula.

Theorem 4.10 (Kac-determinant formula, Kac Theorem 8.1)**.** *For fixed c the function* $\det_N(c,h)$ *is the following polynomial in h*

$$\det_N(c,h) = K \prod_{\substack{r,s \in \mathbb{N} \\ 1 \leq rs \leq N}} (h - h_{r,s}(c))^{p(N-rs)} \tag{4.15}$$

where $h_{r,s}$ *is given by eq. (4.14) and K is the constant (it only depends on N) given by eq. (4.4).*

Proof. We learned from Lemma 4.9 that for the pairs (r,s) with $1 \leq rs \leq N$, $h_{r,s}$ is a zero of $\det_N(c,h)$. Together with Lemma 4.8 this implies that $\det_N(c,h)$ is divisible by

$$\prod_{\substack{r,s \in \mathbb{N} \\ 1 \leq rs \leq N}} (h - h_{r,s}(c))^{p(N-rs)} \tag{4.16}$$

The degree of this expression agrees with Lemma 4.1. Therefore $(\det)_N(c,h)$ can only differ by a constant from eq. (4.16). This constant is fixed by the coefficient of the leading term which concludes the proof. □

Remark 8. For computations it is sometimes convenient to use a slightly different representation of the Kac-determinant formula. For this we define

$$\phi_{r,r} := h - h_{r,r} = h + \frac{(r^2-1)(c-1)}{24} \tag{4.17}$$

and for $r \neq s$

$$\phi_{r,s} = (h - h_{r,s})(h - h_{s,r}). \tag{4.18}$$

With this we can write the Kac-determinant formula as

$$\det_N(c,h) = K \prod_{\substack{r,s\in\mathbb{N} \\ s\leq r \\ 1\leq rs\leq N}} \phi_{r,s}^{p(N-rs)}. \tag{4.19}$$

4.3 Implication of the Kac determinat formula

With this powerful tool at our hand we can finally tackle the classification of unitary highest weight representations of the Virasoro algebra for real highest weights (c,h). We start with a necessary condition for (c,h).

Corollary 4.11 (Kac Prop. 8.2 a)**.** *If $V(c,h)$ is a unitary highest weight representation then*

$$c \geq 0 \, and \, h \geq 0.$$

Proof. For unitarity we need

$$\langle L_{-n}\boldsymbol{v}|L_{-n}\boldsymbol{v}\rangle \geq 0, \quad \forall n \geq 0$$

Contravariance of the Hermitian form and $\langle \boldsymbol{v}|\boldsymbol{v}\rangle = 1$ gives us, for all n,

$$\begin{aligned} 0 \overset{!}{\leq} \langle L_{-n}\boldsymbol{v}|L_{-n}\boldsymbol{v}\rangle &= \langle L_n L_{-n}\boldsymbol{v}|\boldsymbol{v}\rangle \\ &= 2nh + \frac{1}{12}(n^3-n)c. \end{aligned}$$

For $n = 1$ this is $2h \geq 0$. For big n only the term with n^3 is relevant and we obtain $\frac{1}{12}n^3c \geq 0$. □

Proposition 4.12 (Kac Prop 8.2 b)**.** *For $c > 1$ and $h > 0$ the Kac-determinant is positive at every level*

$$\det_N(c,h) > 0, \quad \forall N \in \mathbb{N}$$

and $V(c,h)$ is non-degenerate (i.e. $V(c,h) = M(c,h)$).

Proof. The non-degeneracy of $V(c,h)$ follows directly from the positivity (in particular this means it is non-zero) of the Kac-determinant. Therefore only $\det_N(c,h) > 0$ is left to prove. We use the representation (4.19). Let $1 \leq r \leq N$ then

$$\phi_{r,r} = h + \frac{(r^2-1)(c-1)}{24} > 0, \quad \text{for}\, c > 1, h > 0.$$

For $r \neq s$ we find

$$\begin{aligned}\phi_{r,s} &= (h - h_{r,s})(h - h_{s,r})\\ &= (h - \frac{(r-s)^2}{4})^2 + \frac{h}{24}(r^2 + s^2 - 2)(c-1)\\ &\quad + \frac{1}{576}(r^2-1)(s^2-1)(c-1)^2 + \frac{1}{48}(c-1)(r-s)^2(rs+1) > 0,\end{aligned}$$

for all $1 \leq rs \leq n$ with $s \leq r$ and $c > 1, h > 0$. This shows that the Kac-determinant is positive and therefore not zero. □

Proposition 4.13 (Kac Prop 8.2 a)**.** *The (unique) irreducible highest weight representation $V(c,h)$ is unitary for*

$$c \geq 1 \quad \textit{and} \quad h \geq 0.$$

Proof. The Hermitian form is unitary if it is positive semi-definite. According to the last proposition, the determinant is positive in the described regime. Because the determinant is a continuous function, the Hermitian form can only be positive or negative definite in the open set $c > 1$ and $h > 0$. Therefore it suffices to show positive definiteness at a single point in this region $c \geq 1$ and $h \geq 0$. An explicit construction of such a unitary representation of Vir (and therefore with a positive definite Hermitian form) for $c = 1, 2, 3, \ldots$ and $h \geq 0$ can be found in [11] in chapter 3.4. □

The two points $c \in \{0, 1\}$ have to be considered separately:

Proposition 4.14 (Kac Prop 8.3)**.**

1. *$V(1,h)$ is non-degenerate if and only if*

$$h \neq \frac{m^2}{4}, \quad m \in \mathbb{Z} \tag{4.20}$$

2. *$V(0,h)$ is non-degenerate if and only if*

$$h \neq \frac{m^2 - 1}{24}, \quad m \in \mathbb{Z} \tag{4.21}$$

Proof. We will show that for $c = 1$ and $c = 0$ the determinant is zero if and only if h is as given in eq. (4.20) and eq. (4.21). For $c = 1$ the determinant simplifies to

$$\det{}_N(1,h) = K \prod_{\substack{r,s \in \mathbb{N} \\ 1 \leq rs \leq N}} (h - \frac{(r-s)^2}{4})^{p(N-rs)}.$$

Exactly for $h \neq \frac{m^2}{4}, \quad m \in \mathbb{Z}$ is the determinant not zero. Similarly for $c = 0$ the determinant simplifies to

$$\det{}_N(0,h) = K \prod_{\substack{r,s \in \mathbb{N} \\ 1 \leq rs \leq N}} (h - \frac{(3r-2s)^2 - 1}{24})^{p(N-k)}.$$

Exactly for $h \neq \frac{m^2-1}{24}, \quad m \in \mathbb{Z}$ is the determinant not zero. □

Let's sum up what we just found out. For $c \geq 1$ and $h \geq 0$ propositions (4.12) - (4.14) classify our unitary highest weight representations. Furthermore corollary (4.11) excludes the region of highest weights with $c < 0$ or $h < 0$ from the list of candidates for unitary highest weight representations. Which leaves us with $c \in [0, 1)$ and $h \geq 0$. The Kac-determinant formula gives us a necessary condition for the unitarity of $V(c, h)$. For this region Friedan-Qiu-Shenker investigated the Kac-determinant formula and found that the only points where $V(c, h)$ can be unitary are the first intersections of the $[0, 1) \ni c \mapsto h_{r,s}(c)$ curves. We will explain the term first intersection in the proof-sketch of the following theorem.

Theorem 4.15 (Friedan-Qiu-Shenker)**.** *The highest weight representation $V(c, h)$ is non-unitary everywhere in $(c, h) \in [0, 1) \times \mathbb{R}$ except for the points*

$$(c(m), h_{r,s}(m)), \quad m, r, s \in \mathbb{Z}_+,\ 1 \leq s \leq r \leq m+1 \tag{4.22}$$

with the parametrized values for c, h given by

$$c(m) = 1 - \frac{6}{(m+2)(m+3)} \tag{4.23}$$

and

$$h_{r,s}(m) = \frac{[(m+3)r - (m+2)s]^2 - 1}{4(m+2)(m+3)}, \tag{4.24}$$

which also simplifies the expression for $h_{r,s}$ given in Lemma 4.9 by making it rational.

Proof. The theorem was proven first by Friedan, Qiu and Shenker in 1984 in [7, 8] and later also worked out by Langlands in [14]. We will only give a sketch of the proof. Unitarity is defined by the Hermitian form being positive definite. Therefore if we find that for a given highest weight the Kac-determinant is negative at one level we can exclude this highest weight from our possible candidates for unitary representations. This is the idea of the proof. We will start by showing that we can exclude every highest weight that does not lie on the zero curves of the Kac-determinant given by Lemma 4.9. The Kac-determinant at level N can be written as

$$\det_N(c, h) = K \prod_{n=1}^{N} \Psi_n(c, h)^{p(N-rs)},$$

with

$$\Psi_n(c, h) = \prod_{\substack{r,s \in \mathbb{N} \\ rs=n}} (h - h_{r,s}(c)).$$

We investigate the Kac-determinant level wise. At level 1 the determinant is $2h$ and therefore positive definit. Level 2 is shown in figure 4.1. The Kac-determinant at level two is

$$\det_N(c, h) = K \cdot 2h \cdot (h - h_{1,2}) \cdot (h - h_{2,1}).$$

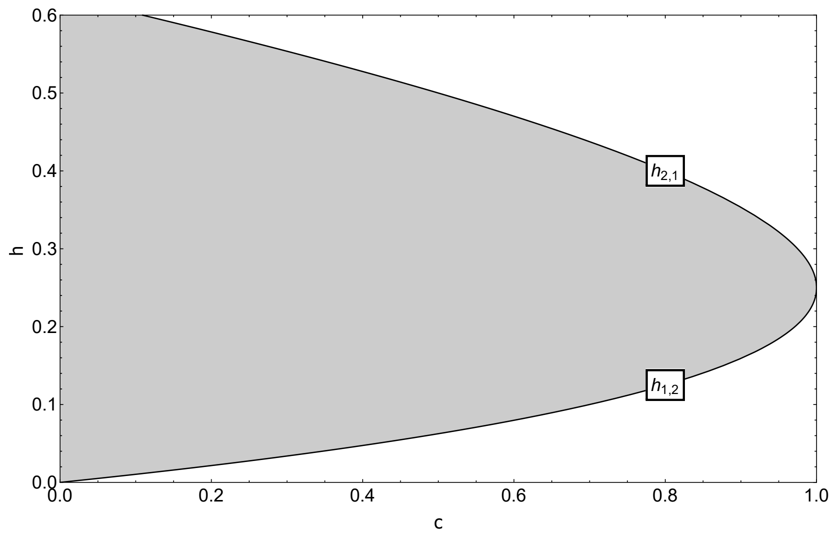


Figure 4.1: The two curves $h_{2,1}$ and $h_{1,2}$ as functions of c. Note that $h_{1,1} = 0$. The gray area shows the highest weights for which the Kac-determinant is negative. Note that $h_{1,1} = 0$.

In the gray area between $h_{1,2}$ and $h_{2,1}$ the term is $(h - h_{2,1})$ is negative and the other terms are positive. therefore the determinant is negative and the highest weight represenations for these highest weights (c, h) can not be unitary. In the white areas above $h_{2,1}$ or below $h_{1,2}$ both bracket terms or none of them is negative and the determinant as a whole is positive. We conclude that highest weights in the gray area cannot be unitary but at level 2 we cannot say anything about the rest of the parameter space. Figure 4.2 shows the situation at level 3. We can repeat the argument from level 2 everywhere between $h_{3,1}$ and $h_{1,3}$ except at the dark gray area. The dark gray however was already excluded in the level 2 discussion. The Kac-determinant at level 3 is

$$\det_3(c, h) = K\Psi_3(c, h) \prod_{n=1}^{2} \Psi_n(c, h)^{p(3-rs)},$$

with

$$\Psi_3(c, h) = (h - h_{3,1}) \cdot (h - h_{1,3}).$$

For highest weights in the upper light gray area $\Psi_N(c, h)$ is negative and the other terms in the determinant are positive. Therefore the determinant as a whole is negative. For highest weights in the dark gray are in the middle the determinant would be positive but we could exclude this area already at level 2. The fact that the determinant is negative means there are negative eigenvalues of the matrix of the Hermitian form at a specific level. One can in fact show that the amount of negative eigenvalues at level n is always smaller of equal the

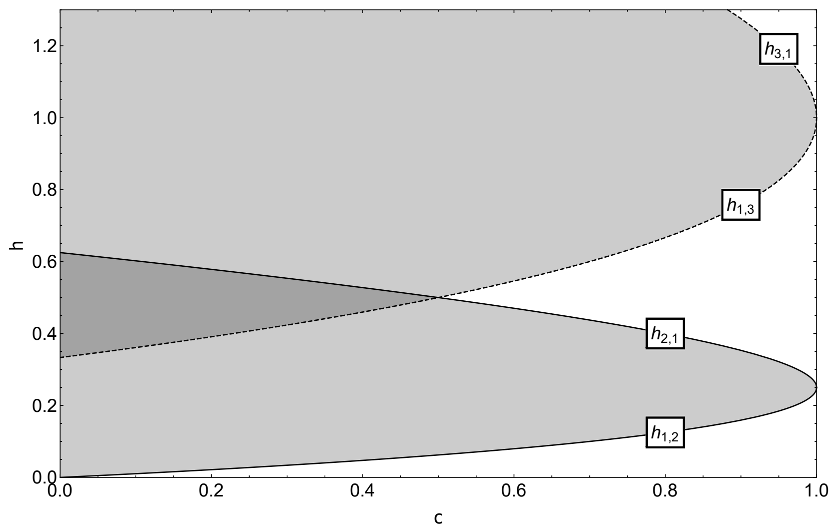


Figure 4.2: The two curves for $h_{3,1}$ and $h_{1,3}$ as a function of c. The gray area shows the highest weights for which the Kac-determinant is negative. In the dark area the determinant is positive but not positive definite.

number of negative eigenvalues at a higher level $m > n$ (see for example [11] p 138ff) which explains why it is sufficient to find negative eigenvalues at one level to exclude these regions. We conclude that we can also exclude the region between $h_{1,3}$ and $h_{3,1}$. Figure 4.3 shows the case of level 4. The procedure is the same as before. At every new level we exclude more of the parameter space. One can further show that not only all highest weights that do not lie on a zero curve can be excluded but in fact also every point that does not lie on a first intersection between two curves. By first intersection we mean the intersection between two curves that, at one level are the intersections closest to $c = 1$. We plotted the first of the first intersections up to level 6 in figure 4.4. Note for example that the intersection between the gray line and the dotted line is non-unitary. That is because it is not a first intersection. Repeating this analysis for every level reveals that exactly the set of highest weights given by eq. 4.22 can yield unitary highest weight representations. We will not give the full proof here but instead refer to the literature given above where the arguments presented here are worked out in more detail and with mathematical rigor. □

With this the only question that is left to answer, to complete the classification of unitary highest weight representations for real highest weights, is for which of these highest weights (4.23), (4.24) there exists a unitary highest weight representation. The surprising answer is, that every pair $(c(m), h_{r,s}(m))$ corresponds to a unitary highest weight representation. To show this we will explicitly construct these representations using the so-called coset construction.

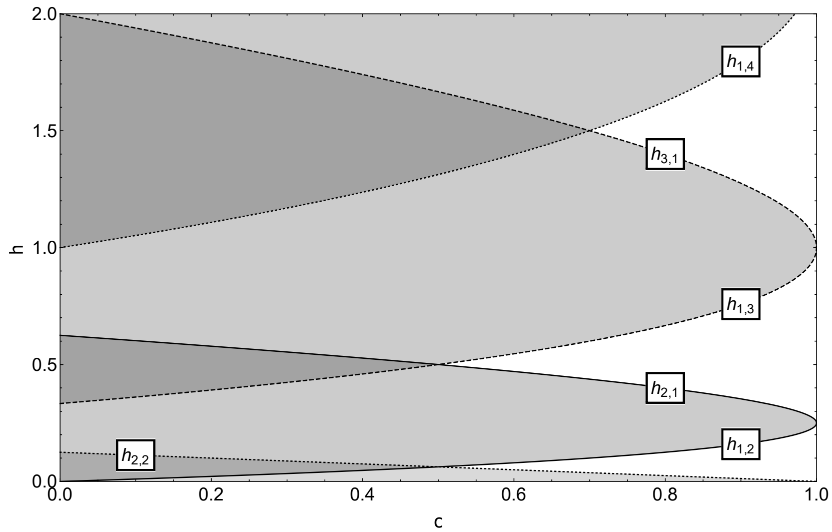


Figure 4.3: Shows additionally the two curves for $h_{2,1}$ and $h_{1,2}$ as a function of c with dotted lines. The gray area shows the highest weights for which the Kac-determinant is negative. The dark gray areas were already excluded in lower levels.

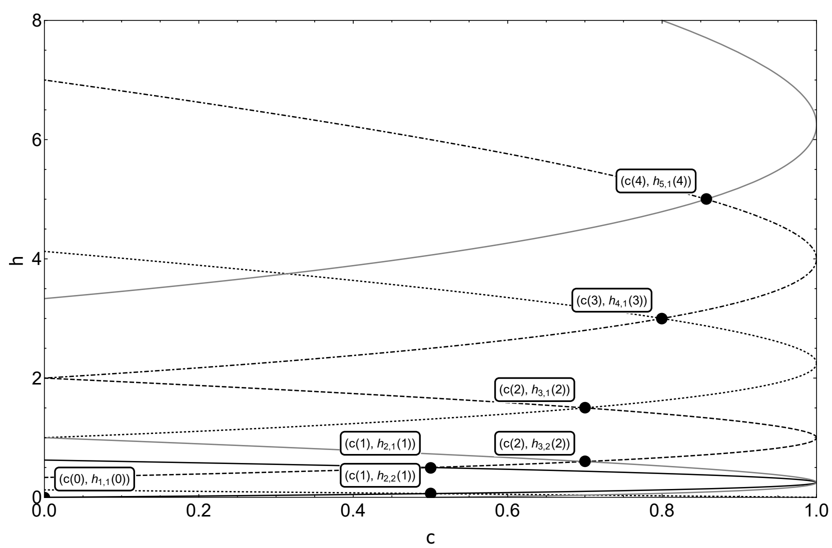


Figure 4.4: The zeroes at level 2 in black, at level 3 as dashed lines, at level 4 as dotted lines, at level 5 as dot-dashed lines and at level 6 in gray. The points show the first few possibilities for unitary representations.

4.4 Minimal models

As described in the last chapter there is a correspondence between unitary representations of the Virasoro algebra and conformal field theories in two dimensions. The conformal field theories corresponding to the set of points described in eq. (4.22), for which we will prove in the next chapter that they are indeed unitary highest weight representations, is call the class of Minimal Models. This is a subset of the the bigger class of rational CFTs that have the neat property that they only have a finite number of so-called primary fields (the theories are in some way the simplest CFTs).

Many important CFTs, in particular many important models from statistical mechanics, belong to this class of theories. We will later investigate the Ising model in more detail which can be understood as the highest weight representation with $c = \frac{1}{2}$. Other examples are the three state Pots model for $c = \frac{4}{5}$ or the tricritical Ising model for $c = \frac{7}{10}$.

Chapter 5

Coset construction

As explained in the last chapter, our goal is to construct highest weight representation for every

$$(c(m), h_{r,s}(m)), \quad m, r, s \in \mathbb{Z}_+ \, \text{and} \, 1 \leq s \leq r \leq m+1.$$

To do this we need some knowledge about *affine Lie algebras*. We will follow the discussion in [3, 11].

5.1 Affine Lie algebras

Definition 5.1. For every Lie algebra $\mathfrak{g}$ the Killing form is the bilinear form defined by

$$K(X, Y) = \operatorname{Tr}(\operatorname{ad}(X)\operatorname{ad}(Y)), \quad \forall X, Y \in \mathfrak{g}, \tag{5.1}$$

where *ad* denotes the adjoint linear map.

Let $\mathfrak{g}$ denote a finite dimensional simple Lie algebra. Then the *loop algebra* is defined as follows.

Definition 5.2. The *loop algebra* $\tilde{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ is the Lie algebra of Laurent polynomials with coefficients in $\mathfrak{g}$:

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$$

Let J_a with a in some index set, be a basis of $\mathfrak{g}$ then the elements $J_a(n) := J_a \otimes t^n$ form a basis of $\tilde{\mathfrak{g}}$ with the Lie bracket

$$[J_a(n), J_b(m)] = \sum_c i f_{abc} J_c(n+m), \quad n, m \in \mathbb{Z}$$

where the f_{abc} are the structure constants of $\mathfrak{g}$.

Note that the dimension of this Lie algebra is always infinite. We can now build the central extension of the loop algebra and obtain its *direct affine Lie algebra*

$$\hat{g}' := \tilde{g} \oplus \mathcal{K},$$

where $\mathcal{K} = \mathbb{C}\hat{k}$. The commutation relations are

$$[J_a(n), J_b(m)] = \sum_c i f_{abc} J_c(n+m) + \hat{k}\, n K(J_a, J_b)\delta_{n,-m}, \quad n, m \in \mathbb{Z}$$
$$[J_a(n), \hat{k}] = 0, \quad \forall a, n.$$

One can show (see [3]) that this is in fact the unique central extension of a loop algebra. To analyze this algebra we represent it in the Cartan-Weyl basis (ladder operators). To construct this basis for a finite Lie algebra $\mathfrak{g}$ we first need to find a maximal set of commuting elements H_i, $i \in I$ (the Cartan subalgebra)

$$[H_i, H_j] = 0, \quad i, j \in I.$$

These elements can be diagonalized simultaneously in the adjoint representation. The Cartan-Weyl basis is then the Cartan subalgebra together with linear independent eigenvectors E_α of the H_i in the adjoint representation, i.e. elements of $\mathfrak{g}$ such that

$$[H_i, E_\alpha] = \alpha_i E_\alpha.$$

The vector $\alpha = (\alpha_1, \dots, \alpha_n)$ of eigenvalues of the chosen Cartan subalgebra is called a root. On the generalized affine Cartan-Weyl basis the Lie bracket is

$$[H_i(n), H_j(m)] = \hat{k} n \delta_{n+m,0}$$
$$[H_i(n), E_\alpha(m)] = \alpha_i E_\alpha(n+m)$$
$$[E_\alpha(n), E_\beta(m)] = \begin{cases} \frac{2}{|\alpha|^2}\left(\hat{k} n \delta_{n+m,0} + \sum_{i=1}^n \alpha_i H_i(n+m)\right) & \text{if } \alpha = -\beta \\ \mathcal{N}_{\alpha,\beta} E_{\alpha+\beta}(n+m) & \text{if } \alpha + \beta \in \Delta \\ 0 & \text{otherwise} \end{cases}$$
$$[\hat{k}, H_i(n)] = [\hat{k}, E_i(n)] = 0, \quad \forall n, m, i$$

where Δ denotes the set of all roots and $\mathcal{N}_{\alpha,\beta}$ is a constant. $\hat{k}$ is commuting with everything so its Lie brackets are vanishing. For a deeper discussion of the Cartan-Weyl basis and on the roots of (affine) Lie algebras see [3]. From the Lie brackets $[H_i(0), E_\alpha(m)] = \alpha_i E_\alpha(m)$ and $[\hat{k}, E_\alpha(m)] = 0$ we find that the eigenspaces are all degenerate. Hence $\{H_1(0), \dots, H_n(0), \hat{k}\}$ is not a maximal Abelian subalgebra. We can therefore extend our algebra $\hat{\mathfrak{g}}'$ again

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}' \oplus \mathcal{D}$$

where $\mathcal{D} = \mathbb{C}\hat{d}$, with an element $\hat{d}$ such that

$$[\hat{d}, J_a(n)] = -n J_a(n), \quad \forall a, n$$

and $\{H_1(0), \ldots, H_n(0), \hat{k}, \hat{d}\}$ is a maximal Cartan subalgebra.[1]

Definition 5.3. For a finite dimensional semi simple Lie algebra $\mathfrak{g}$ we define the corresponding *affine Lie algebra* as the algebra

$$\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C}\hat{k} \oplus \mathbb{C}\hat{d}$$

with Lie bracket given by

$$\begin{aligned}
&[\hat{d}, \hat{k}] = 0 \\
&[\hat{d}, J_a(n)] = -nJ_a(n), \quad \forall a, n \\
&[\hat{k}, J_a(n)] = 0, \quad \forall a, n \\
&[J_a(n), J_b(m)] = [J_a(n), J_b(m)]_{\tilde{\mathfrak{g}}}
\end{aligned}$$

where $[\,\cdot\,,\,\cdot\,]_{\tilde{\mathfrak{g}}}$ denotes the Lie bracket in the corresponding loop algebra $\tilde{\mathfrak{g}}$.

Remark 9. Affine Lie algebras are a special case of the more general class of Kac-Moody algebras.

Remark 10. The direct affine Lie algebra can be understood as the derivation of $\hat{\mathfrak{g}}$

$$\hat{\mathfrak{g}}' = [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$$

Affine Lie algebras are special insofar as that the representation theory is very similar to the representation theory of semisimple Lie algebras. The Dynkin diagrams of affine Lie algebras for example can be obtained by adding just one extra node. Representations of affine Lie algebras $\hat{\mathfrak{g}}$ can be obtained from the representations of the associated finite Lie algebras $\mathfrak{g}$. We will only present a small overview of the theory of highest weight representations for affine Lie algebras.

Definition 5.4. A *highest weight representation* $L(\lambda)$ of $\hat{\mathfrak{g}}$ is a representation π on a vector space V over $\mathbb{C}$ with a non-zero vector $\boldsymbol{v}_\lambda$ such that there are λ_i, $i \in \{1, \ldots, n\}$ and k, d so that

$$\pi(E_\alpha(0))\boldsymbol{v}_\lambda = \pi(E_{\pm\alpha}(n))\boldsymbol{v}_\lambda = \pi(H_i(n))\boldsymbol{v}_\lambda = \mathbf{0}, \quad \forall n > 0,\, \alpha > 0$$

and

$$\begin{aligned}
&\pi(H_i(0))\boldsymbol{v}_\lambda = \lambda_i \boldsymbol{v}_\lambda, \quad \forall i \in \{1, \ldots, n\} \\
&\pi(\hat{k})\boldsymbol{v}_\lambda = k\boldsymbol{v}_\lambda \\
&\pi(\hat{d})\boldsymbol{v}_\lambda = d\boldsymbol{v}_\lambda.
\end{aligned}$$

The eigenvalue k of $\pi(\hat{k})$ is called the *level* of the representation $L(\lambda)$. The vector $\lambda = (\lambda_1, \ldots, \lambda_n, k, d)$ is called the *highest weight* and the vector $\boldsymbol{v}_\lambda$ is called the *highest weight vector*.

[1] One possible choice for $\hat{d}$ is

$$\hat{d} = -t\frac{d}{dt}$$

We can extend the *Killing form* of $\mathfrak{g}$ to its affine Lie algebras $\hat{\mathfrak{g}}$. To do this note first that eq. (5.1) implies the cyclic property

$$K([Z,X],Y) + K(X,[Z,Y]) = 0$$

Inserting $X = J_a(n)$, $Y = J_b(m)$ and $Z = \hat{d}$ leads to

$$\begin{aligned} 0 &\stackrel{!}{=} K([\hat{d}, J_a(n)], J_b(m)) + K(J_a(n), [\hat{d}, J_b(m)]) \\ &= K(-nJ_a(n), J_b(m)) + K(J_a(n), -mJ_b(m)) \\ &= -(n+m)t^{n+m} K(J_a, J_b) \end{aligned}$$

If we repeat the calculation with $Z = \hat{k}$ and again with $Y = \hat{d}$ we obtain

Proposition 5.5. *Let J_a an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form then the Killing form on $\hat{\mathfrak{g}}$ is the bilinear form $K(\,\cdot\,,\,\cdot\,)$*

$$\begin{aligned} &K(J_a(n), J_b(m)) = \delta_{a,b}\delta_{n+m,0} \\ &K(J_a(n), \hat{k}) = 0, \quad \forall a, n \\ &K(J_a(n), \hat{d}) = 0, \quad \forall a, n \\ &K(\hat{k}, \hat{k}) = 0 \\ &K(\hat{d}, \hat{d}) = 0 \end{aligned}$$

With this we are adequately equipped to define the Sugawara tensor.

5.2 Sugawara tensor

From now on $\mathfrak{g}$ will always denote a finite dimensional simple Lie algebra with an orthonormal basis J_a, $a \in A$ and $\hat{\mathfrak{g}}$ its corresponding affine Lie algebra. If not stated otherwise we will always work with highest weight representations which implies in particular $\hat{\mathfrak{g}} \subseteq \mathfrak{gl}(V)$ for some vector space V.

Our goal in this section is to obtain a highest weight representation for the Virasoro algebra gives a highest weight representation of a suitable affine Lie algebra. Recall that we omit the representation π when talking about explicit representations. To make this construction well defined we need our representation to have the following property:

Definition 5.6. A representation $\pi : \hat{\mathfrak{g}}' \to \mathfrak{gl}(V)$ of the direct affine Lie algebra $\hat{\mathfrak{g}}'$ is called *admissible* if for every $\boldsymbol{v} \in V$ there exists a $N_0 > 0$ such that

$$J_a(n)\boldsymbol{v} = 0, \quad \forall n > N_0, a \in A$$

Our Lie algebra $\mathfrak{g}$ carries a scalar product, the Killing form $K(\,\cdot\,,\,\cdot\,)$. For every basis J_a of $\mathfrak{g}$ we can define the dual basis J^b by[2]

$$K_{\mathfrak{g}}(J_a, J^b) = \delta_{a,b}$$

[2]Index on top means it is an element of the dual space.

For the corresponding elements in $\hat{\mathfrak{g}}'$ we find

$$K_{\mathfrak{g}}(J_a(n), J^b(m)) = \delta_{a,b}\delta_{n,-m}$$

Definition 5.7. We will use the *normal ordering symbol*. This is defined as

$$: J_a(n)J^a(m) : := \begin{cases} J_a(n)J^a(m) & \text{if } n \leq m \\ J_a(m)J^a(n) & \text{otherwise} \end{cases}$$

Theorem 5.8 (Kac Prop 10.1)**.** *Let V be an admissible representation of a direct affine Lie algebra $\hat{\mathfrak{g}}'$. Then the operators*

$$T_n = \frac{1}{2}\sum_{m\in\mathbb{Z}}\sum_{a\in A} : J_a(-m)J^a(m+n) :$$

with $n \in \mathbb{Z}$, together with $\hat{k}$, are a Lie algebra with the commutation relations

$$\begin{aligned} [T_n, T_m] &= (k+g)(n-m)T_{n+m} + \delta_{n,-m}\frac{n^3-n}{12}(\dim\mathfrak{g})(k+g)\hat{k} \\ [T_n, \hat{k}] &= 0, \quad \forall n \end{aligned} \tag{5.2}$$

where g denotes the dual Coxeter number of $\mathfrak{g}$. Furthermore for $x(n) \otimes t^n \in \hat{\mathfrak{g}}'$

$$[x(n), T_m] = (k+g)nx(n+m). \tag{5.3}$$

Proof. The normal ordering assures finiteness of the series i.e. that the T_k are well defined. See [11] for more details. □

Remark 11. The dual Coxeter number is half the eigenvalue of the Casimir operator in the adjoint representation. It is well known for every simple finite Lie algebra. We will later need the algebra $\mathfrak{sl}_2$ for which $g = 2$ and $\dim\mathfrak{sl}_2 = 3$.

Eq. (5.2) looks suspiciously similar to the Virasoro algebra. The following proposition shows that this is no coincidence.

Proposition 5.9. *Let V be an admissible representation of a direct affine Lie algebra $\hat{\mathfrak{g}}'$ with $\hat{k} = k\mathbb{1}$ with $k \geq 0$ such that $k \neq -g$ then the operators defined by*

$$L_n = \frac{1}{k+g}T_n, \quad n \in \mathbb{Z}$$

form a Virasoro algebra with commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n,-m}\frac{n^3-n}{12}\frac{k\dim\mathfrak{g}}{k+g}\mathbb{1}$$

If the representation of $\hat{\mathfrak{g}}'$ is unitary we therefore obtain a unitary representation of Vir *with central charge*

$$c = \frac{k\dim\mathfrak{g}}{k+g}. \tag{5.4}$$

This is called the Sugawara construction. Furthermore by setting

$$\hat{d} := -L_0 \tag{5.5}$$

this representation can be extended to a representation of $\hat{\mathfrak{g}}$*.*

Proof. The expression for the Lie bracket follows from Theorem 5.8

$$[L_n, L_k] = \frac{1}{(c+g)^2}[T_n, T_k] = (n-k)L_{n+k} + \delta_{n+k,0}\frac{n^3-n}{12}\frac{c \dim \mathfrak{g}}{c+g}.$$

This is the Lie bracket of the Virasoro algebra. What is left to prove is that the representation is indeed unitary. For this we need to show that

$$\omega(L_n) = L_{-n}, \quad \forall n \in \mathbb{Z}$$

for a conjugate-linear anti-involution ω that makes our Hermitian form contravariant. Both, the Hermitian form and ω are obtained by extending the corresponding objects of our finite Lie algebra $\mathfrak{g}$ suitably. This is done e.g. in [11]. Applying ω to

$$L_n = \frac{1}{2(k+g)} \sum_{m\in\mathbb{Z}} \sum_{a\in A} : J_a(-m)J^a(m+n) :$$

we obtain $\omega(L_n) = L_{-n}$. □

Remark 12. This result is quite remarkable. We showed that the Virasoro algebra is contained in the enveloping algebra of the direct affine Lie algebra $\hat{\mathfrak{g}}'$. This means that every representation of a direct affine Lie algebra $\hat{\mathfrak{g}}'$ contains a unitary representation of a Virasoro algebra. In the next sections we will show that in fact every so called minimal model (Virasoro algebras with highest weights as given in eq. 4.22) can be understood this way.

Remark 13. The central charge eq. (5.4) is always greater or equal than 1. We will show this for the case of $\mathfrak{g} = \mathfrak{sl}_n$, $n \in \mathbb{N}$. The algebra $\mathfrak{sl}_n$ has dimension $\dim \mathfrak{sl}_n = n^2 + 2n$ and dual Coxeter number $g = n + 1$. After inserting this in eq. (5.4) we find that $c < 1$ if and only if

$$k < \frac{n+1}{n^2+2n-1}.$$

For unitary representations we have $k \in \mathbb{Z}_+$ (see for example [1] p.94) the right hand side however is always smaller than or equa to l 1. One can perform similar calculations for other simple Lie algebras.

Remark 14. Let $\mathfrak{g}_i$ be a family of simple Lie algebras. Then the direct affine Lie algebra corresponding to the direct sum

$$\mathfrak{g} := \bigoplus_i \mathfrak{g}_i \tag{5.6}$$

is the direct sum of the direct affine Lie algebras of the family $\mathfrak{g}_i$, i.e.

$$\hat{\mathfrak{g}}' = \bigoplus_i \hat{\mathfrak{g}}_i'. \tag{5.7}$$

Furthermore $\hat{\mathfrak{g}}'$ acts on the tensor product of the $\hat{\mathfrak{g}}_i'$ representation spaces, so that the $L_k^{(i)}$ commute with each other. We conclude that the

$$L_k := \sum_i L_k^{(i)} \tag{5.8}$$

form a unitary representation of the Virasoro algebra with central charge

$$c = \sum_i \frac{(\dim \mathfrak{g}_i) m_i}{m_i + g_i} \tag{5.9}$$

with the dual Coxeter numbers g_i and the levels m_i of the corresponding representations in the Sugawara constructions of the $\hat{\mathfrak{g}}_i$.

5.3 Coset construction

In the last section we found a way to construct representations of the Virasoro given representations of affine Lie algebras. The central charge of these representations however is always greater or equal 1 as discussed in remark 13 but the Virasoro representations we were initially interested in had central charge $c \in [0, 1)$. How can we obtain those? The trick will be to use the difference $L_n^{\mathfrak{g}} - L_n^{\mathfrak{p}}$ of two Sugawara constructions. These will again form a Virasoro algebra with central charge $c_{\mathfrak{g}} - c_{\mathfrak{p}}$.

Theorem 5.10 (Kac Theorem 10.2)**.** *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra and $\mathfrak{p}$ a semisimple Lie-subalgebra of $\mathfrak{g}$. Let $L_m^{\mathfrak{g}}, L_m^{\mathfrak{p}}$, $m \in \mathbb{Z}$ denote the operators obtained in the Sugawara constructions of $\hat{\mathfrak{g}}'$ and $\hat{\mathfrak{p}}'$ for unitary representations of $\mathfrak{g}$ and $\mathfrak{p}$ respectively. Then the operators*

$$L_n^{(\mathfrak{g}/\mathfrak{p})} = L_n^{\mathfrak{g}} - L_n^{\mathfrak{p}}, \quad m \in \mathbb{Z}$$

form a unitary representation of Vir *with central charge*

$$c_{(\mathfrak{g}/\mathfrak{p})} = c_{\mathfrak{g}} - c_{\mathfrak{p}}. \tag{5.10}$$

Proof. We start by showing that the unitary generators $L_n^{(\mathfrak{g}/\mathfrak{p})}$, $n \in \mathbb{Z}$ commute with every element of $\hat{\mathfrak{g}}'$ and $\hat{\mathfrak{p}}'$ respectively, i.e.

$$[L_n^{(\mathfrak{g}/\mathfrak{p})}, \hat{\mathfrak{g}}'] = 0, \quad \forall n \in \mathbb{Z}.$$

This follows from eq. (5.3), which implies that for $x(m) \in \hat{\mathfrak{p}}'$

$$[L_n^{\mathfrak{g}}, x(m)] = \frac{1}{k+g}[T_n^{\mathfrak{g}}, x(m)] = -nx(n+m).$$

The same argument applies to the Sugawara tensors for $\hat{\mathfrak{p}}'$. With this we can compute the Lie bracket of the generators $L_n^{(\mathfrak{g}/\mathfrak{p})}, n \in \mathbb{N}$ with elements $x(m)$ from $\hat{\mathfrak{p}}'$ (note that $\mathfrak{p}$ is a Lie-subalgebra of $\mathfrak{g}$ which allows us to identify $x(m) \in \hat{\mathfrak{p}}'$ as an element of $\hat{\mathfrak{g}}$).

$$\begin{aligned}[L_n^{(\mathfrak{g}/\mathfrak{p})}, x(m)] &= [L_n^{\mathfrak{g}} - L_n^{\mathfrak{p}}, x(m)] \\ &= [L_n^{\mathfrak{g}}, x(m)] - [L_n^{\mathfrak{p}}, x(m)] \\ &= -nx(n+m) + nx(n+m) = 0,\end{aligned}$$

and therefore

$$[L_n^{(\mathfrak{g}/\mathfrak{p})}, \hat{\mathfrak{p}}'] = 0, \quad \forall n \in \mathbb{Z}$$

which implies furthermore that

$$[L_n^{(\mathfrak{g}/\mathfrak{p})}, L_m^{\mathfrak{p}}] = 0, \quad \forall n, m \in \mathbb{Z}$$

because $L_m^{\mathfrak{p}} \in \hat{\mathfrak{p}}'$, $\forall m \in \mathbb{Z}$. With this we can compute the Lie brackets

$$\begin{aligned}[L_n^{(\mathfrak{g}/\mathfrak{p})}, L_m^{(\mathfrak{g}/\mathfrak{p})}] &= [L_n^{(\mathfrak{g}/\mathfrak{p})}, L_m^{\mathfrak{p}}] \\ &= [L_n^{\mathfrak{g}}, L_m^{\mathfrak{g}}] - [L_n^{\mathfrak{p}}, L_m^{\mathfrak{g}}] \\ &= [L_n^{\mathfrak{g}}, L_m^{\mathfrak{g}}] - [L_n^{\mathfrak{p}}, L_m^{(\mathfrak{g}/\mathfrak{p})} + L_m^{\mathfrak{g}}] \\ &= [L_n^{\mathfrak{g}}, L_m^{\mathfrak{g}}] - [L_n^{\mathfrak{p}}, L_m^{\mathfrak{p}}] \\ &= (n-m)(L_{n+m}^{\mathfrak{g}} - L_{n+m}^{\mathfrak{p}}) + \delta_{n+m,0}\frac{n^3-n}{12}(c_{\mathfrak{g}} - c_{\mathfrak{p}}) \\ &= (n-m)L_{n+m}^{(\mathfrak{g}/\mathfrak{p})} + \delta_{n+m,0}\frac{n^3-n}{12}(c_{\mathfrak{g}} - c_{\mathfrak{p}})\end{aligned}$$

□

Remark 15. We will call the construction above the *coset construction*. It is also sometimes called the *Goddard-Kent-Olive (GKO) construction*.

We are in fact only interested in the special case of the Theoremabove where $\mathfrak{g}$ is the direct sum of two copies of a simple Lie algebra

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{p}.$$

This makes $\mathfrak{g}$ a semisimple Lie algebra and $\mathfrak{p}$ is a semisimple subalgebra of $\mathfrak{g}$. If we have two representations $L(\lambda), L(\lambda')$ of $\hat{\mathfrak{p}}$ with levels m and m' then $\hat{\mathfrak{g}}'$ acts on the tensor product $L(\lambda) \otimes L(\lambda')$ according to

$$(x \oplus y)(u \otimes v) = x(u) \otimes v + u \otimes y(v), \quad x, y \in \hat{\mathfrak{p}}' \tag{5.11}$$

for $u \in L(\lambda)$ and $v \in L(\lambda')$. Because of this we can generalize the Sugawara construction for this case along the lines of remark 14 by defining

$$L_k^{\mathfrak{g}} = L_k^{\mathfrak{p}} \otimes \mathbb{1} + \mathbb{1} \otimes L_k^{\mathfrak{p}}.$$

For the central charge we obtain

$$c_{\mathfrak{g}} = (\dim \mathfrak{p})\left(\frac{m}{m+g} + \frac{m'}{m'+g}\right).$$

We can embed $\hat{\mathfrak{p}}'$ diagonally in $\hat{\mathfrak{g}} = \hat{\mathfrak{p}}' \oplus \hat{\mathfrak{p}}'$, by that we mean the embedding

$$\begin{aligned} \hat{\mathfrak{p}}' &\to \hat{\mathfrak{p}}' \oplus \hat{\mathfrak{p}}' \\ x &\mapsto (x, x). \end{aligned}$$

This embedding maps $\hat{c}$ to $(\hat{c}, \hat{c})$ which shows that the level of $\hat{\mathfrak{p}}'$ in this embedding is $m+m'$. Combining these arguments we obtain the following proposition that is a special case of Theorem5.10.

Proposition 5.11. *Let* $\mathfrak{p}$ *be a finite simple Lie algebra with dual Coxeter number* g*. Let* $L(\lambda), L(\lambda')$ *be two highest weight representations of* $\hat{\mathfrak{p}}$ *with levels* m *and* m' *respectively. Furthermore let* $L_n^{(\mathfrak{p}\oplus\mathfrak{p})}, L_n^{\mathfrak{p}}$, $n \in \mathbb{Z}$ *denote the operators obtained in the Sugawara constructions of* $\hat{\mathfrak{g}}$ *and* $\hat{\mathfrak{p}}$*. Then the operators*

$$L_n^{(\mathfrak{p}\oplus\mathfrak{p}/\mathfrak{p})} = L_n^{(\mathfrak{p}\oplus\mathfrak{p})} - L_n^{\mathfrak{p}}, \quad n \in \mathbb{Z}$$

form a unitary representation of Vir *on* $L(\lambda) \otimes L(\lambda')$ *with central charge*

$$\begin{aligned} c_{(\mathfrak{p}\oplus\mathfrak{p}/\mathfrak{p})} &= c_{(\mathfrak{p}\oplus\mathfrak{p})} - c_{\mathfrak{p}} \\ &= (\dim \mathfrak{p})\left(\frac{m}{m+g} + \frac{m'}{m'+g} - \frac{m+m'}{m+m'+g}\right) \end{aligned} \tag{5.12}$$

where g *is the dual Coxeter number of* $\mathfrak{p}$*.*

This proposition allows us to construct unitary representations of Vir with a central charge c smaller than 1. What is left to do is to find a finite simple Lie algebra $\mathfrak{p}$ so that the construction above produces a representation of the Virasoro algebra with a central charge given by eq. (4.23). Furthermore the representation above does not have to be a highest weight representation. We will see that both of these two problems have a solution.

It turns out that to obtain the central charges eq. (4.23) we need to use $\mathfrak{sl}(2)$ as our simple Lie algebra. We investigate the direct sum $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ and choose representations $L(\lambda), L(\lambda')$ of $\widehat{\mathfrak{sl}}(2)$ with levels m and 1. We denote this representation $(L_n^{(\mathfrak{sl}(2)\oplus\mathfrak{sl}(2))/\mathfrak{sl}(2)})_{n\in\mathbb{Z}}$ with levels m, m' by

$$\frac{\widehat{\mathfrak{sl}}'(2)_m \oplus \widehat{\mathfrak{sl}}'(2)_1}{\widehat{\mathfrak{sl}}'(2)_{m+1}} \tag{5.13}$$

where the indices are the levels of the chosen representations. In the literature this is often called coset theory. The dimension is $\dim \mathfrak{sl}(2) = 3$ and the dual Coxeter number is $g_{\mathfrak{sl}(2)} = 2$. Inserting these values and the levels $m, 1$ in eq. (5.12) we obtain for the central charge

$$c = 1 - \frac{6}{(m+2)(m+3)}, \tag{5.14}$$

i.e. exactly one of the central charges for which the existence of a unitary highest weight representation of the Virasoro algebra is still in question. If we vary the level m we find representations for every central charge given in eq. (4.23). What is left is to construct a unitary highest weight representation out of the unitary representation given by the coset construction. We will only give a sketch of the proof, for further information see [3, 11]. We saw in the proof of Theorem5.10 that the operators in $L_m^{\mathfrak{g}/\mathfrak{p}}$ commute with the elements of $\hat{\mathfrak{p}}'$. This implies that for our case of the coset theory with $\mathfrak{p} = \mathfrak{sl}(2)$ that we can reduce the space $L(\lambda) \otimes L(\lambda')$ with respect to $\mathrm{Vir} \oplus \widehat{\mathfrak{sl}}'(2)$. If we can strip off the $\widehat{\mathfrak{sl}}'(2)$ content, we will be left with a representation of Vir that will turn out to be the unitary highest weight representation we are looking for. This can we done by computing the character decomposition of our representation. Two representation are isomorphic if and only if their characters are identical. And if one perform this computation (see for example [3], p.801ff.) one can indeed find that the unitary representation given in the coset construction can be reduced to a unitary highest weight representation of the Virasoro algebra with central charge eq. (5.14). We will show how the coset theory looks like for the Virasoro algebra with $c = \frac{1}{2}$ (the Ising model) in chapter 6. This concludes our classification of unitary highest weight representations of the Virasoro algebra.

5.4 WZW models

The coset construction actually shows a very interesting fact, namely that every affine Lie algebra comes, in a way described by the Sugawara construction, together with a Virasoro algebra. The affine Lie algebras can therefore be understood as extensions to Virasoro algebras. We already explained in the last chapters that Virasoro algebras correspond to quantum field theories that are invariant under conformal transformations, i.e. quantum CFTs. It turns out that similarly, affine Lie algebras correspond to Wess-Zumino-Witten models (WZW models). These are CFTs with an affine Lie algebra as their symmetry algebra. The connection between affine Lie algebras and Virasoro algebras can in this context be understood as the fact that WZW models are CFTs.

The correspondence affine Lie algebra $\leftrightarrow$ Virasoro algebra or quantum WZW model $\leftrightarrow$ quantum CFT is not bijective. This can be seen for example for the minimal model with $c = \frac{1}{2}$ (the so called two dimensional Ising model). We found in the last section that a unitary highest weight representation for this Virasoro algebra can be obtained with the coset theory eq. (5.13). However, if we compute the central charge for the coset theory

$$\frac{(\hat{E}_8)_1 \oplus (\hat{E}_8)_1}{(\hat{E}_8)_2} \tag{5.15}$$

we find that since $\dim E_8 = 248$ and $g = 30$ this coset theory also contains a Virasoro algebra with $c = \frac{1}{2}$. For other alternative descriptions of minimal models see [3] p.814ff.

Chapter 6

Applications

In this chapter we will discuss some examples where the theory of the Virasoro algebra can be applied. We will begin by discussing the 2d Ising model. This is the physical model that corresponds to the $c = \frac{1}{2}$ representation of the Virasoro algebra. Next, we investigate the tricritical 2d Ising model that corresponds to the representation with $c = \frac{7}{10}$. This physical model is interesting insofar, as it also incorporates supersymmetry and has an extension of the Virasoro algebra, the Super Virasoro algebra, as its symmetry Lie algebra.

We finish this chapter with a few words about the application in gravity, where one can find an equivalence between the Virasoro algebra and asymptotic Killing vectors in semi-Riemannian manifolds and in string theory where the Virasoro algebra appears as the symmetry algebra of the Bosonic string.

6.1 Ising model

We will follow the discussion from [3, 15].

The Ising model is one the best understood models in condensed matter physics. It is a mathematical model of ferromagnetic materials ("permanent magnets"). It consists of a lattice of electrons, modeled by the spin variables $\sigma_i \in \{1, -1\}$ with $i \in I$ where I is the set of all positions in the lattice. Two adjacent electrons at positions i and j interact with each other with an energy of

$$E_{ij} = -J\sigma_i\sigma_j, \quad i, j \in I, i \neq j \tag{6.1}$$

where $J \in \mathbb{R}_+$. For the total energy of a lattice configuration $\sigma := \{\sigma_i | i \in I\}$ we obtain

$$E[\sigma] = -\sum_{i \in I} \left[h\sigma_i + J \sum_{j \text{ adj. to } i} \sigma_i\sigma_j \right] \tag{6.2}$$

where the inner sum of the second term sums over all j that are adjacent to i and $h \in \mathbb{R}_+$ is a constant proportional to the external magnetic field. The first term corresponds to the interaction between electrons and an external magnetic

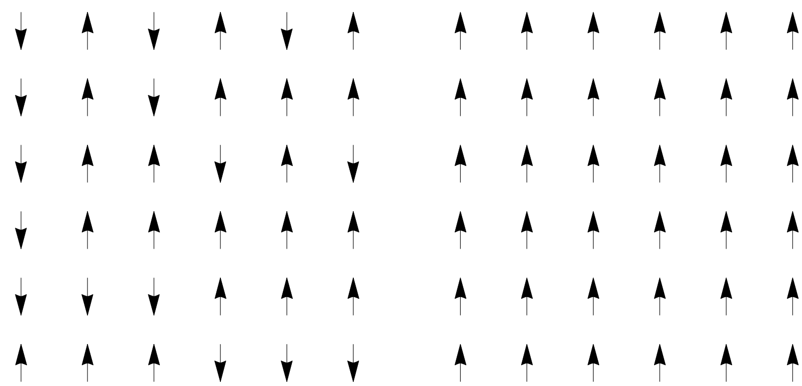


Figure 6.1: The left plot shows the lattice configuration for temperatures below the Curie temperature. The electron spins have random orientations because the thermic fluctuations are dominating. The plot on the right shows the same lattice for temperatures above the Curie temperature. The spin-spin interactions are dominating and we find a ordering of spin over large distances that leads to macroscopically measurable effects. (This phenomena is called spontaneous symmetry breaking)

field (This term is optional, one can also consider the model without an external field) whereas the second term describes the interaction between the electrons. In this classical description of the Ising model the spin variables σ_i can be seen as random variables where the underlying probability measure is determined by thermic fluctuations, in the quantum Ising model the spin variables are replaced by the Pauli matrices and the energy becomes the Hamiltonian of the system.

With methods from statistical physics one can show, that if one turns on the temperature, one can observe a phase transition[1] at the so-called *Curie temperature* T_c, where the metal experiences spontaneous magnetization and where the interaction term in eq. (6.2) is dominating. The microscopic reasons for this are explained in figure (6.1). As mentioned above, around the phase transition the interaction term eq. (6.1) starts to dominate and the spins of the electrons start to order on a macroscopic scale. This means the lattice is scale invariant and shows a conformal symmetry close to the phase transition point. This model, meaning the Ising model of interacting spins on a lattice at the critical temperature T_c, is called the critical Ising model.

In physics literature, when speaking of the 2d Ising model as a conformal field theory, what is normally meant, is the continuum limit of the model described above. The continuum theory is obtained by making the lattice size (the distance between two electrons) converge to zero. One way to do this, is to use the so-

[1]By a phase transition we mean a sudden change of the macroscopic properties of a material when certain control parameters (more specifically: Thermodynamic state functions) are varied.

called Jordan-Wigner transformation (as was done first in [17]). Doing this one obtains the theory of a free massless real fermion described by the action

$$S(\psi,\bar{\psi}) = \frac{1}{2\pi}\int \mathrm{d}^z \mathrm{d}^{\bar{z}} \left(\psi(z,\bar{z})\bar{\partial}\psi(z,\bar{z}) + \bar{\psi}(z,\bar{z})\partial\bar{\psi}(z,\bar{z}) \right).$$

Here ψ denotes a so-called spinor field. That is a quantum field, i.e. a map from spacetime (in this case the complexified two dimensional euclidian spacetime) into a representation space of the left-handed spinor representations of the Lorentz group. $\bar{\psi}$ denotes the adjoint spinor to ψ that maps into a right-handed representation. $\bar{\partial}$ denotes the derivative with respect to $\bar{z}$. The stationary points of the functional S can be shown to describe the physical trajectories. If one wants to investigate the symmetries of this theory one has to calculate the variation of the action with respect to the background geometry (by that we mean a variation of the underlying metric, i.e. $g_{\mu\nu} \mapsto g_{\mu\nu} + \delta g_{\mu\nu}$). Investigating this one finds the energy-momentum tensor whose Laurent modes are the symmetry generators. One can show that the Laurent modes of the energy-momentum tensor are in fact exactly the operators T_n in Theorem5.8. If one performs these calculations (see for example [1] 2.9.2.) one finds that indeed the Laurent modes form a Virasoro algebra with $c = \frac{1}{2}$.

The Ising model belongs to the class of minimal models, a class of Virasoro algebras where highest weight representations can be obtained using the coset theories described in the last chapter. For the coset representation (eq. (5.13) where $m = 1$)

$$\frac{\widehat{\mathfrak{sl}}'(2)_1 \oplus \widehat{\mathfrak{sl}}'(2)_1}{\widehat{\mathfrak{sl}}'(2)_2} \tag{6.3}$$

we showed in Proposition 5.11, that one can define operators on the tensor product $L(\lambda) \otimes L(\lambda')$ of representations of $\widehat{\mathfrak{sl}}(2)$ with levels m and 1, that form a Virasoro with central charge (5.12), which equals $\frac{1}{2}$ for our specific choice of affine Lie algebras (6.3) where $m = 1$. We want to demonstrate for the Ising model how one can find representation of the Virasoro algebra from representations of such a a coset theory. The minimal models have highest weights given by the Friedan-Qiu-Shenker Theorem4.15:

$$(c(m), h_{r,s}(m)), \quad m, r, s \in \mathbb{Z}_+,\ 1 \leq s \leq r \leq m+1.$$

For $m = 1$ we obtain

$$\begin{aligned} c(1) &= 1 - \frac{6}{(1+2)(1+3)} \\ &= \frac{1}{2} \\ h_{r,s}(1) &= \frac{(4r-3s)^2 - 1}{48}, \end{aligned}$$

which means, the possible values for h are

$$h_{1,1} = 0, \quad h_{2,1} = \frac{1}{2}, \quad h_{2,2} = \frac{1}{16}.$$

This means, we should be able to find the Virasoro unitary highest weight representations $V(\frac{1}{2},0), V(\frac{1}{2},\frac{1}{2}), V(\frac{1}{2},\frac{1}{16})$ in the unitary representations of our coset theory. The exact decomposition of $L(\lambda)\otimes L(\lambda')$ into representations of $\widehat{\mathfrak{sl}}(2)_2$ and Vir is determined by its branching rules (for a discussion of this topic see for example [3]) that describe more generally how representations of an algebra $\hat{\mathfrak{g}}$ decompose into a direct sum of tensor products of one of its subalgebras $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{g}}/\hat{\mathfrak{h}}$. For the Ising model one can show that the representations of the coset theory (6.3) contains representations of the Virasoro algebra in the following way (see [1] p.105)

$$\begin{aligned}
L(0,1)\otimes L(0,1) &= \left(L(0,2)\otimes V(\frac{1}{2},0)\right)\oplus\left(L(2,2)\otimes V(\frac{1}{2},\frac{1}{2})\right)\\
L(0,1)\otimes L(1,1) &= \left(L(1,2)\otimes V(\frac{1}{2},\frac{1}{16})\right)\\
L(1,1)\otimes L(0,1) &= \left(L(1,2)\otimes V(\frac{1}{2},\frac{1}{16})\right)\\
L(1,1)\otimes L(1,1) &= \left(L(0,2)\otimes V(\frac{1}{2},\frac{1}{2})\right)\oplus\left(L(2,2)\otimes V(\frac{1}{2},0)\right)
\end{aligned}$$

where $L(h,k)$ is the representation space of $\widehat{\mathfrak{sl}}(2)$ with level k where the underlying representation of $\mathfrak{sl}(2)$ has highest weight h. The equations above show how representations of the Virasoro algebra with $c=\frac{1}{2}$ are contained in the coset model (6.3). There are also other coset theories that contain the Virasoro algebra with $c=\frac{1}{2}$ and therefore the Ising model. We already mentioned one of them, namely

$$\frac{(\hat{E}_8)_1\oplus(\hat{E}_8)_1}{(\hat{E}_8)_2}$$

with the Lie algebra of the exceptional simple Lie group E_8. Another coset that describes the Ising model is given by (see [3])

$$\frac{\widehat{\mathfrak{su}}(2)_2}{\hat{\mathfrak{u}}(1)}.$$

6.2 Tricritical Ising model

Another model obtained by the coset construction is the tricritical Ising model with a central charge $c=\frac{7}{10}$. "Tricritical" refers to the fact that the model contains a tricritical point in phase space, where three different phases exist (similar to the tricritical point for water where liquid, solid and gaseous states exist at the same time). The tricritical Ising model was investigated by Fried, Qiu and Shenker in [6] and by Z. Qiu more detailed in [16].

Similarly to the Ising model it can be understood as electrons on a lattice, with the difference that the spin is allowed to be zero as well. This is modeled with a spin variable $\sigma_i \in \{1, -1\}$ and a density variable $t_I \in \{0, 1\}$, where $i \in I$, so that the energy of a given lattice configuration (σ, t) is given by

$$E[\sigma, t] = -\sum_{i \in I} \left[\mu t_i + \beta \sum_{j \text{ adj. to } i} \sigma_i \sigma_j t_i t_j \right],$$

with $\beta, \mu \in \mathbb{R}_+$. In a similar way as for the Ising model one can also obtain a continuum version of this model. The action for the continuum version can be found for example in [5]. The tricritical Ising model has a central charge of $c = \frac{7}{10}$ and can be described with the coset theory

$$\frac{\widehat{\mathfrak{sl}}'(2)_2 \oplus \widehat{\mathfrak{sl}}'(2)_1}{\widehat{\mathfrak{sl}}'(2)_3}.$$

As was described for the critical Ising model, one can investigate the energy-tensor to find the symmetry algebra of the tricritical Ising model. An interesting feature of of this model, and also the reason why we mention it in particular, is that the symmetry algebra (The Laurent modes of the energy-momentum tensor) is an extension of the Virasoro algebra, called the *super-Virasoro algebra*:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3 - m}{12}\hat{c} \\
\{G_k, G_l\} &= 2L_{k+l} + \delta_{k,-l}\frac{1}{3}\left(k^2 - \frac{1}{4}\right)\hat{c} \\
[L_m, G_k] &= \left(\frac{m}{2} - n\right)G_{m+k},
\end{aligned}$$

with $m, n \in \mathbb{Z}$, $k, l \in \frac{1}{2}\mathbb{Z}$ and $\hat{c}$ is a central element. The curly brackets are anti-commutation brackets. Depending on the specific choice of the G_k's (Integer or half-integer indices) the algebra is also called *Ramond algebra* or *Neveu-Schwarz algebra*. The 'super' comes from the fact that the additional structure corresponds to supersymmetry in physical theories. In the Neveu-Schwarz case every field has a so-called superpartner, an example being the chemical potential being the superpartner of the energy density (see [3] 7.4.3).

Much of the theory of the Virasoro algebra can be generalized to the super-Virasoro algebra as is described in [16]. Similar to the minimal models for the Virasoro algebra one can also find a series of unitary highest weight representations of the super-Virasoro algebra at the discrete set of points

$$c^s(m) = \frac{3}{2} - \frac{12}{m(m+2)} \tag{6.4}$$

$$h^s_{r,s} = \frac{[r(m+2) - sm]^2 - 4}{8m(m+2)} + \frac{1}{32}[1 - (-1)^{r-s}], \tag{6.5}$$

where $1 \leq r < m$, $1 \leq s < m+2$, called the minimal superconformal models. If we set the central charge c of eq. (6.4) equal to the central charge for the minimal models of the Virasoro algebra from eq. (4.23) we find that the only model that is both a minimal model to the Virasoro and super-Virasoro algebra is the tricritical Ising model. Furthermore one can also obtain a determinant formula similar to the Kac-determinant. The derivation is slightly more complicated because one has to distinguish between the two cases of Ramond-algebra and Neveau-Schwarz algebra. A short discussion can be found in the already mentioned paper by Qiu [16] section 4 and the references therein.

6.3 Quantum gravity

The Virasoro algebra surprisingly also appears if one investigates the asymptotic symmetries of three dimensional gravity theories. Brown and Henneaux investigated in 1986 in [2] gravity in negatively curved spaces. By that we mean semi-Riemannian manifolds that are solutions to the Einstein field equations with a negative cosmological constant (without matter content these are exactly the anti deSitter spaces). What they found, is that the asymptotic symmetry algebra (asymptotic with respect to the anti deSitter space) forms a Virasoro algebra. This was the first hint to the more general holographic principle that was postulated around a decade later.

Another appearance of the Virasoro algebra, and maybe the one that sparked the biggest interest, is in string theory. We only want to give a short introduction into the ideas behind this.

String theory can be understood as a two dimensional conformal field theory. The two dimensional space on which the theory is defined is called the "worldsheet" and is basically the surface of the propagating string in spacetime. Bosonic string theory, a toy model and the first step to superstring theory, can then be understood as a collection of D free Bosonic fields living on the worldsheet with values in spacetime, with D denoting the dimension of spacetime. A free[2] Bosonic field lives in a representation of the Virasoro algebra with central charge $c = 1$ (see [1] for a derivation), and therefore a collection of D free bosons live in a representation with $c = D$. The central charge has therefore the interpretation of being the spacetime dimension. Investigating the Lie brackets of the Virasoro algebra one finds that the only consistent choice for the central charge is $c = 26$ which is the reason for why in string theory one needs such a high dimensional spacetime. In superstring theory things get more complicated because the theory gets additional fermionic fields and the underlying symmetry algebra is the super-Virasoro algebra described briefly in the section about the tricritical Ising model.

[2] Free means without interactions.

Chapter 7

Summary

The Witt algebra has a unique central extension, called the Virasoro algebra. The highest weight representations of this algebra can be shown to have a unique Hermitian form, called the Shapovalov form, for all real highest weights (c, h). The existence of this form allows us to talk about unitary representations. The first step in the classification of *unitary* highest weight representations is to derive an explicit expression for the definiteness of the Hermitian form, called the Kac-determinant. This determinant allows us exclude highest weights (c, h) from the list of candidate points for unitary highest weight representations. An extensive investigation of the full parameter plane of highest weights shows that only the points (c, h) with $c \geq 1, h \geq 0$ and a discrete set of points with $c \in [0, 1]$ and $h \geq 0$ are possible unitary highest weight representations. For the set $\{(c, h) \,|\, c \geq 1,\, h \geq 0\}$ we can show easily the existence of a unitary highest weight representation. For the discrete set, called the minimal models, we can construct an explicit unitary highest weight representation using the representations of affine Lie algebras. These infinite dimensional Lie algebras are special insofar as that their representation theory is closely related to the representation theory of semisimple finite Lie algebras, which is perfectly understood. Furthermore one can show that the highest weight representations of affine Lie algebras contain unitary highest weight representations of the Virasoro algebra. Investigating the factor algebras of these affine Lie algebras, we arrive at a specific model, namely

$$\frac{\widehat{\mathfrak{sl}}'(2)_k \oplus \widehat{\mathfrak{sl}}'(2)_1}{\widehat{\mathfrak{sl}}'(2)_{k+1}}, \quad k \geq 1.$$

Investigating the characters of the unitary representation of the Vir obtained from this model one finds that the representation can be reduced to a unitary highest weight representation of Vir with the highest weights of the minimal models. This shows that indeed all the points in this set are unitary highest weight representations and we can fully classify the unitary highest weight representations in the plane of highest weights $(c, h) \in \mathbb{R}^2$. The full classification is

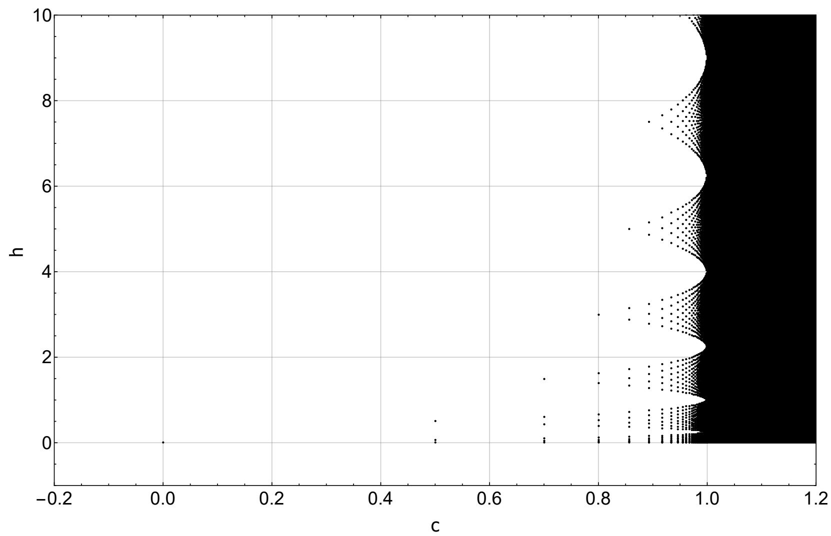


Figure 7.1: The black dots show exactly the highest weights with unitary highest weight representations.

summarized in figure 7.1. The classification concerns only the Virasoro algebra, however a lot of the development of the theory behind it has been done with applications of the Virasoro algebra in mind. In most of these applications the algebra appears as the symmetry algebra of quantum conformal field theories. As such, it plays a huge role in investigating condensed matter systems near phase transitions and even investigating quantum gravity theories in string theory of through the holographic principle. Especially the holographic principle is a vivid research topic in physics at the time of writing this thesis with 2-dimensional conformal field theories (and therefore the Virasoro algebra) being one of the few examples where explicit realizations are known. All of these topics profit already from the theory of unitary highest weight representations and may even lead to some new insight in the future.

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