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Invariant Smooth Valuations on the Euclidean Unit Sphere

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Preface

A spherical valuation is a map $\mu\colon \mathcal{K}(\mathbb{S}^n) \to \mathbb{R}$ from closed, convex subsets of the Euclidean unit sphere to real numbers that satisfies $\mu(\emptyset) = 0$ and the following additivity property

$$\mu(K) + \mu(L) = \mu(K \cup L) + \mu(K \cap L)$$

for all $K, L \in \mathcal{K}(\mathbb{S}^n)$ such that $K \cup L$ is again in $\mathcal{K}(\mathbb{S}^n)$. In particular, the space of spherical valuations that are continuous with respect to the spherical Hausdorff topology and invariant under the natural action of the group $\mathrm{SO}(n+1)$ on $\mathbb{S}^n$ is of interest. It is an open conjecture, known as the spherical version of Hadwiger's theorem, whether this space is finite-dimensional. In this thesis, we focus on a certain subspace, namely smooth spherical valuations. These can be represented by integration of differential forms over so-called normal cycles. Alesker, in part joint with Fu, has developed a theory of smooth valuations on arbitrary manifolds in [Ale06a], [Ale06b], [Ale08], [Ale07], which indeed yields a classification of smooth, invariant spherical valuations. The main idea is to exploit the integral representation and consider invariant differential forms instead of valuations. Here, we present this approach in some detail.

The first chapter contains an outline of results obtained in direction of the spherical Hadwiger theorem prior to the theory of Alesker. We present a proof by Klain and Rota [Kla97] for the case $n = 2$, which shows that the space of continuous, invariant valuations on $\mathbb{S}^2$ is spanned by spherical length, spherical area, and the Euler-characteristic. We also exhibit, why their geometric arguments can not be generalized to spheres of higher dimensions. Moreover, Schneider's characterization [Sch78] of spherical volume as a simple, non-negative and invariant valuation is included, however it is not clear, whether this result implies a version of Hadwiger's theorem for non-negative, instead of continuous valuations.

In Chapter 2 we introduce normal cycles, which first occur as subsets of the sphere bundle of $\mathbb{S}^n$, denoted by $S\mathbb{S}^n$, thereby generalizing the Gauß map to convex sets with non-smooth boundary. Later, it is shown that these normal cycles can also be considered as currents, acting on differential forms on $S\mathbb{S}^n$. This leads to the notion of smooth valuations. For any given n-form $\eta \in \mathcal{D}^n(\mathbb{S}^n)$ on $\mathbb{S}^n$ and $(n-1)$-form $\omega \in \mathcal{D}^{n-1}(S\mathbb{S}^n)$ on $S\mathbb{S}^n$ the map

$$K \mapsto \int_K \eta + \int_{N(K)} \omega,$$

where $N(K)$ denotes the normal cycle of a convex, closed set $K \in \mathcal{K}(\mathbb{S}^n)$, indeed satisfies the above valuation property, and is hence called a smooth valuation. The space of all such smooth valuations on $\mathbb{S}^n$, denoted by $\mathcal{V}^\infty(\mathbb{S}^n)$, is then identified with a certain quotient of the space of differential forms $\mathcal{D}^n(\mathbb{S}^n) \oplus \mathcal{D}^{n-1}(S\mathbb{S}^n)$, which is due to Bernig and Bröcker [Ber07]. This makes it easier to consider also the topological dual space to $\mathcal{V}^\infty(\mathbb{S}^n)$, namely the space of generalized valuations, denoted by $\mathcal{V}^{-\infty}(\mathbb{S}^n)$. Using the above identification, generalized valuations can be thought of as currents acting on differential forms instead of smooth valuations.

In the third chapter we classify $\mathrm{SO}(n+1)$-invariant differential forms on $\mathbb{S}^n$ and $S\mathbb{S}^n$. The latter can be endowed with a contact structure that yields two first examples of invariant forms - the contact form α and its exterior derivative $d\alpha$. Using the theory of polynomial invariants from [Kra96], we obtain the remaining invariant forms $\kappa_0, \ldots, \kappa_n$ by giving an argument similar to [Par02] and, in particular, show that they span only a finite-dimensional subspace. We close this chapter by also classifying currents invariant under the dual $\mathrm{SO}(n+1)$-action. This is done by averaging differential forms over the compact group $\mathrm{SO}(n+1)$ via the Fréchet space valued integral

$$\bar{\omega} := \int_{\mathrm{SO}(n+1)} g \cdot \omega \, dg,$$

where ω is either in $\mathcal{D}^n(\mathbb{S}^n)$ or $\mathcal{D}^{n-1}(S\mathbb{S}^n)$, $g \cdot \omega$ denotes the respective group action, and dg is the Haar measure of $\mathrm{SO}(n+1)$. It turns out that the spaces of invariant currents on $\mathbb{S}^n$ and $S\mathbb{S}^n$ are again finite-dimesional.

In Chapter 4 of this thesis we make use of the results obtained in chapter three and classify invariant, smooth and generalized valuations. Indeed, each invariant, smooth valuation can be represented by a pair of invariant differential forms, hence the space of all such valuations must be finite-dimensional. In particular, a basis for that space are the spherical intrinsic volumes V_i, $0 \leq i \leq n$, occurring in the spherical Steiner formula, which describes the volume of a parallel set of a convex set $K \in \mathcal{K}(\mathbb{S}^n)$ as a linear combination of the V_i. Moreover, using Alesker's product of smooth valuations, we see that the spaces of invariant, smooth valuations, denoted by $\mathcal{V}^\infty(\mathbb{S}^n)^{\mathrm{SO}(n+1)}$ and invariant, generalized valuations, $\mathcal{V}^{-\infty}(\mathbb{S}^n)^{\mathrm{SO}(n+1)}$, are actually isomorphic. In the final section we present another method to obtain our main results, namely the transfer principle, due to Fu [Fu90]. This result shows that the kinematic formulas

$$\int_G \mu(K \cap gL) \, dg = \sum_{i,j=0}^{n} c_{ij} \mu_i(K) \mu_j(L),$$

where $K, L \in \mathcal{K}(M)$, $\mu \in \mathcal{V}^\infty(M)^G = \mathrm{span}\{\mu_0, \ldots, \mu_n\}$, look the same in either case $(M, G) = (\mathbb{S}^n, \mathrm{SO}(n+1))$ or $(M, G) = (\mathbb{R}^n, \mathrm{E}^n)$, where E^n denotes the group of proper Euclidean motions. It allows us to transfer the classification problem of smooth, invariant valuations from the sphere to Euclidean space, where its solution – Hadwiger's theorem – is already well known.

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Thomas Hack

The following was partly written on a boat.

Contents

CHAPTER 1

Background for the characterization problem of continuous, invariant valuations

To start off, we give a review of the famous characterization theorem by Hugo Hadwiger, concerning continuous, motion-invariant valuations on convex bodies in Euclidean space. Following the discovery of this beautiful classification result in the 1950s it seemed natural to ask, whether a similar statement for other spaceforms, such as the sphere, might be true. In the second part of this chapter, we discuss some of the developments made in that direction, namely an analogous theorem for the 2-sphere and a characterization of spherical volume, where continuity is replaced by non-negativity. The problem of characterizing continuous, invariant valuations on spheres of arbitrary dimensions still remains open.

1.1 Hadwiger's theorem in $\mathbb{R}^n$

For us a *convex body* will be a compact, convex subset of $\mathbb{R}^n$, and we will denote the collection of all such sets by $\mathcal{K}(\mathbb{R}^n)$.

Definition 1.1.1. A map $\mu\colon \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}$ is called a *valuation*, if $\mu(\emptyset) = 0$, and

$$\mu(K) + \mu(L) = \mu(K \cup L) + \mu(K \cap L),$$

for all $K,L,K \cup L \in \mathcal{K}(\mathbb{R}^n)$.

In order to address continuity issues, we need a suitable topology on $\mathcal{K}(\mathbb{R}^n)$. Therefore we define the *parallel body* of a convex body K with distance ε to be the set $K_\varepsilon := \{x \in \mathbb{R}^n \,|\, \mathrm{dist}(x,K) \leq \varepsilon\}$, where $\mathrm{dist}(x,K) = \inf\{\mathrm{dist}(x,y) \,|\, y \in K\}$ with the usual metric in $\mathbb{R}^n$. Now the *Hausdorff metric* on $\mathcal{K}(\mathbb{R}^n)$ can be defined by

$$\mathrm{dist}_H(K,L) := \inf\{\varepsilon > 0 \,|\, K \subset L_\varepsilon, L \subset K_\varepsilon\}.$$

Note also that $K_\varepsilon = K + \varepsilon B_n$, where $K + L = \{x + y \,|\, x \in K, y \in L\}$ is the *Minkowski addition* and B_n is the Euclidean unit ball.

Next, we introduce important special classes of valuations:

Definition 1.1.2. A valuation $\mu\colon \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}$ is called

- *continuous*, if it is continuous with respect to the topology on $\mathcal{K}(\mathbb{R}^n)$ induced by the Hausdorff metric;
- *translation-invariant*, if $\mu(K+x) = \mu(K)$ for all $x \in \mathbb{R}^n$ and all $K \in \mathcal{K}(\mathbb{R}^n)$;
- $\mathrm{SO}(n)$-*invariant*, if $\mu(\theta K) = \mu(K)$ for all $\theta \in \mathrm{SO}(n)$ and all $\mathcal{K} \in \mathcal{K}(\mathbb{R}^n)$.

Example 1.1.3. Here are some examples of valuations that are continuous and invariant (which here means translation- and $\mathrm{SO}(n)$-invariant):

1. Classical volume, that is Lebesgue measure, on $\mathbb{R}^n$, denoted by vol_n.
2. The *Euler-Characteristic* χ given by $\chi(K) = 1$ for all $K \in \mathcal{K}(\mathbb{R}^n)$.
3. The so-called *intrinsic volumes* μ_i, $i = 0 \ldots n$: These occur, if we express the volume of a parallel body of K at distance $\varepsilon \geq 0$ by Steiner's formula as a polynomial in ε:

$$\mathrm{vol}_n(K_\varepsilon) = \sum_{i=0}^{n} \omega_{n-i} \mu_i(K) \varepsilon^{n-i}.$$

   Here ω_i is the volume of the i-dimensional unit ball. Moreover, we have $\mu_0 = \chi$ and $\mu_n = \mathrm{vol}_n$.

In fact, these examples are, up to linear combinations, already all continuous, invariant valuations on $\mathcal{K}(\mathbb{R}^n)$. This is excactly the statement of Hadwiger's theorem:

Theorem 1.1.4 (Hadwiger)**.** The intrinsic volumes $\mu_0, \ldots, \mu_n$ form a basis of the space of continuous, translation-invariant, $\mathrm{SO}(n)$-invariant valuations on $\mathcal{K}(\mathbb{R}^n)$. In particular, this space is finite dimensional.

A proof of Hadwiger's theorem and also an introduction to valuations in general can be found in the book of D. Klain and G. Rota [Kla97]. The key ingredient here is to characterize volume on convex bodies in $\mathbb{R}^n$ as a continuous, invariant, and *simple* valuation, where simple means that the valuation vanishes on all lower-dimensional bodies, that is, bodies contained in some hyperplane.

Theorem 1.1.5 (Volume characterization on $\mathcal{K}(\mathbb{R}^n)$)**.** Let $\mu\colon \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}$ be a valuation on convex bodies. Then μ is continuous, invariant, and simple, if and only if there exists a constant $c \in \mathbb{R}$ such that $\mu(K) = c\mu_n(K)$ for all $K \in \mathcal{K}(\mathbb{R}^n)$.

Once this result is established, the theorem of Hadwiger follows by induction with respect to the dimension of the space. Now, turning to the sphere, one could hope to prove a spherical version of Hadwiger's theorem by finding a similar characterization of spherical volume. However, this approach faces a certain obstruction, as we shall see in the next section.

1.2 Spherical analogues

We now give definitions of convex bodies and valuations on the sphere, as similar as possible to the Euclidean case. A detailed version of the following can again be found in the book of

Klain and Rota [Kla97]. We will think of the sphere $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} \,|\, \|x\| = 1\}$ as the set of all unit vectors in $\mathbb{R}^{n+1}$ equipped with the Riemannian structure inherited from $\mathbb{R}^{n+1}$. The shortest path between two non-antipodal points will be given by the shorter of the two arcs on the unique *great circle* - that is the intersection $E \cap \mathbb{S}^n$ of a two-dimensional subspace $E \subset \mathbb{R}^{n+1}$ with $\mathbb{S}^n$ - joining the two points. In general, the intersection of a $(k+1)$-dimensional subspace of $\mathbb{R}^{n+1}$ with $\mathbb{S}^n$ will be called a great *k-subsphere*. Accordingly, we call the intersection of any closed half space of $\mathbb{R}^{n+1}$ with $\mathbb{S}^n$ a *(closed) hemisphere*. Now a *spherical n-simplex* is the intersection of $n+1$ linearly independent hemispheres (that is, hemispheres that have linearly independent normals), whereas a *lune* is the intersection of at most n hemispheres. If we fix a unit normal vector $u \in \mathbb{S}^n$ and a spherical k-simplex Δ with $k \leq n-1$ inside the great $(n-1)$-subsphere $u^\perp \cap \mathbb{S}^n$, the union of all half circles with endpoints at u and $-u$, that contain a point of Δ, is called the *lune through Δ* and denoted by $L(\Delta)$.

Definition 1.2.1. The non-empty intersection of finitely many hemispheres is called a *convex spherical polytope*. The collection of all such sets is denoted by $\mathcal{P}(\mathbb{S}^n)$. A general set $K \subset \mathbb{S}^n$ is called *convex*, if it is non-empty, and if for any two points of K lying in an open hemisphere of $\mathbb{S}^n$, the unique shortest geodesic arc connecting these points is also contained in K. The set of all closed, convex subsets of $\mathbb{S}^n$ is denoted by $\mathcal{K}(\mathbb{S}^n)$.

If we define the *spherical parallel body* of K at distance ε to be the set $K_\varepsilon := \{x \in \mathbb{S}^n \,|\, \mathrm{dist}_s(x,K) \leq \varepsilon\}$, where $\mathrm{dist}_s(x,K) = \inf\{\mathrm{dist}_s(x,y) \,|\, y \in K\}$ is now the spherical (or geodesic) distance, we can define the *spherical Hausdorff metric* by

$$\mathrm{dist}_{sH}(K,L) := \inf\{\varepsilon > 0 \,|\, K \subset L_\varepsilon, L \subset K_\varepsilon\}.$$

If we identify a spherical convex body K with the convex cone

$$o * K := \{\lambda u \,|\, u \in K, 0 \leq \lambda \leq 1\} \subset \mathbb{R}^{n+1},$$

then the spherical Hausdorff topology and the one coming from convex bodies in $\mathbb{R}^{n+1}$ in this way coincide. Note also, that in this topology $\mathbb{S}^n$ is an isolated point and that $\mathcal{P}(\mathbb{S}^n)$ is a dense subset of $\mathcal{K}(\mathbb{S}^n)$.

Definition 1.2.2. A map $\mu\colon \mathcal{P}(\mathbb{S}^n) \to \mathbb{R}$ or $\mu\colon \mathcal{K}(\mathbb{S}^n) \to \mathbb{R}$ is called a *valuation*, if $\mu(\emptyset) = 0$, and

$$\mu(K) + \mu(L) = \mu(K \cup L) + \mu(K \cap L),$$

for all $K,L,K \cup L \in \mathcal{P}(\mathbb{S}^n)$ or $\in \mathcal{K}(\mathbb{S}^n)$, respectively. A valuation μ is called

- *continuous*, if it is continuous with respect to the spherical Hausdorff topology on $\mathcal{P}(\mathbb{S}^n)$ or $\mathcal{K}(\mathbb{S}^n)$;
- $\mathrm{SO}(n+1)$-*invariant* (or just invariant), if $\mu(\theta K) = \mu(K)$ for all $\theta \in \mathrm{SO}(n+1)$;
- *simple*, if it vanishes on all lower-dimensional sets, that is, sets contained in some $(n-1)$-subsphere.

Example 1.2.3. The following are examples of continuous, invariant valuations on spherical convex bodies.

1. Spherical volume, that is, spherical Lebesgue measure, on $\mathbb{S}^n$, denoted by σ_n.
2. The *Euler-Characteristic* χ given by $\chi(K) = 1$ for all $K \in \mathcal{K}(\mathbb{S}^n)$.
3. *Spherical intrinsic volumes* V_i, $i = 0 \dots n$ that come from the spherical Steiner formula, expressing the volume of a spherical parallel body as a polynomial:

$$\sigma_n(K_\varepsilon) = V_n(K) + \sum_{i=0}^{n-1} \beta_i \beta_{n-i-1} f_i(\varepsilon) V_i(K),$$

where β_i is the spherical volume of the i-dimensional unit sphere, that is, $\sigma_i(\mathbb{S}^i)$, and

$$f_i(\varepsilon) := \int_0^\varepsilon \cos^i(t) \sin^{n-i-1}(t) dt.$$

Here, $V_n = \sigma_n$ equals spherical volume, but V_0 is not the spherical Euler characteristic, as the spherical *Gauss-Bonnet theorem* reads:

$$\chi(K) = 2 \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} V_{2i}(K),$$

for all $K \in \mathcal{K}(\mathbb{S}^n)$. Early versions of the spherical Steiner formula for different classes of sets occur in works of Allendoerfer [All48] and Herglotz [Her43]; a proof for spherical convex bodies can be found in [Gla96].

It is an important conjecture, but has not been proved yet for dimension $n \geq 3$, that the spherical intrinsic volumes indeed form a basis of the space of continuous, invariant valuations on convex bodies in $\mathbb{S}^n$.

1.3 Hadwiger's theorem on $\mathbb{S}^2$

We will now examine the case $n = 2$, so we consider $\mathbb{S}^2 \subset \mathbb{R}^3$. Here, a spherical version of Hadwiger's theorem has already been obtained. The three valuations forming a basis of the space of continuous, invariant valuations on $\mathcal{K}(\mathbb{S}^2)$ are:

- the *Euler characteristic* χ,
- *spherical length* μ_1,
- *spherical area* μ_2,

where $\chi(K) = 1$ for all non-empty convex bodies, μ_2 is the spherical Lebesgue measure, and $\mu_1(K)$ equals one half of the length of the boundary curve of K, which extends geodesic length to $\mathcal{K}(\mathbb{S}^2)$. The key to characterizing invariant valuations in $\mathbb{S}^2$, as in the Euclidean case, is to characterize spherical area. We split up the process into three steps:

Proposition 1.3.1. Let $\mu\colon \mathcal{K}(\mathbb{S}^1) \to \mathbb{R}$ be a valuation on $\mathbb{S}^1$. Then μ is continuous, invariant, and simple, if and only if there exists $c \in \mathbb{R}$ such that $\mu(I) = c\mu_1(I)$ for all $I \in \mathcal{K}(\mathbb{S}^1)$.

Proof. Clearly $c\mu_1$ is a continuous, invariant, and simple valuation for each $c \in \mathbb{R}$. On the other hand, convex bodies $I \in \mathcal{K}(\mathbb{S}^1)$ are just closed arcs contained in one hemisphere. Define $c := \mu(\mathbb{S}^1)/2\pi$ and $\nu(I) := \mu(I) - c\mu_1(I)$, for all $I \in \mathcal{K}(\mathbb{S}^1)$. Then ν is also a continuous, invariant, and simple valuation with the additional property that $\nu(\mathbb{S}^1) = 0$. We need to show that ν vanishes on all $I \in \mathcal{K}(\mathbb{S}^1)$.

Let $n \in \mathbb{N}$ and I_n be a closed arc of length $2\pi/n$. If we tile $\mathbb{S}^1$ with rotations of I_n and note that the intersection of two distinct tiles is only a point (that is lower-dimensional), then by the invariance and the simplicity of ν, we get $n\nu(I_n) = \nu(\mathbb{S}^1) = 0$, hence $\nu(I_n) = 0$ for all $n \in \mathbb{N}$. A given closed arc of length $2\pi m/n$ can also be tiled by rotations of I_n, therefore ν vanishes on all closed arcs, whose length is a rational multiple of 2π. Thus, by the continuity of ν, we get $\nu(I) = 0$ for all $I \in \mathcal{K}(\mathbb{S}^1)$. □

Proposition 1.3.2. Let $\mu\colon \mathcal{P}(\mathbb{S}^2) \to \mathbb{R}$ be a continuous, invariant, simple valuation on $\mathbb{S}^2$, such that $\mu(\mathbb{S}^2) = 0$. Then $\mu(\Delta) = 0$ for all spherical simplices $\Delta \subset \mathbb{S}^2$.

Proof. Let σ be a great circle of $\mathbb{S}^2$. If we identify σ with $\mathbb{S}^1$, the valuation $\nu(I) := \mu(L(I))$, where $I \in \mathcal{K}(\mathbb{S}^1)$ and $L(I)$ is the lune through I, satisfies the requirements of the previous proposition, therefore $\nu = c\mu_1$ for a $c \in \mathbb{R}$. From

$$2\pi c = c\mu_1(\sigma) = \nu(\sigma) = \mu(\mathbb{S}^2) = 0,$$

we obtain $c = 0$. Thus, by invariance, μ vanishes on all lunes through closed arcs contained in a half-circle of $\mathbb{S}^2$.

Now let Δ be any spherical 2-simplex, namely $\Delta = H_1 \cap H_2 \cap H_3$ with hemispheres H_1, H_2, H_3. We will now write $\mu(H_1 \cup H_2 \cup H_3)$ in two different ways: First, repeatedly using the valuation property, we obtain

$$\mu(H_1 \cup H_2 \cup H_3) = \sum_{i=1}^{3} \mu(H_i) - \sum_{i<j} \mu(H_i \cap H_j) + \mu(H_1 \cap H_2 \cap H_3).$$

Since $H_i \cap H_j$ for $i < j$ is a lune in $\mathbb{S}^2$ and μ vanishes on lunes as well as on hemispheres, this equation simplifies to

$$\mu(H_1 \cup H_2 \cup H_3) = \mu(H_1 \cap H_2 \cap H_3) = \mu(\Delta).$$

Secondly, up to lower-dimensional sets, we have $(H_1 \cup H_2 \cup H_3)^c = H_1^c \cap H_2^c \cap H_3^c = -\Delta$, where compliments are taken relative to $\mathbb{S}^2$. Using the simplicity and invariance under reflections of μ, we obtain

$$\mu(H_1 \cup H_2 \cup H_3) = \mu(S^2) - \mu((H_1 \cup H_2 \cup H_3)^c) = 0 - \mu(-\Delta) = -\mu(\Delta).$$

Comparing both equations, we arrive at $\mu(\Delta) = -\mu(\Delta)$, and thus $\mu(\Delta) = 0$. □

Theorem 1.3.3 (Characterization of spherical area on $\mathbb{S}^2$)**.** Let $\mu\colon \mathcal{K}(\mathbb{S}^2) \to \mathbb{R}$ be a valuation on $\mathbb{S}^2$. Then μ is continuous, simple, and invariant if and only if there exists $c \in \mathbb{R}$ such that $\mu(K) = c\mu_2(K)$ for all $K \in \mathcal{K}(\mathbb{S}^2)$.

Proof. Define $c := \mu(\mathbb{S}^2)/(4\pi)$ and $\nu(K) := \mu(K) - c\mu_2(K)$ for all $K \in \mathcal{K}(\mathbb{S}^2)$. Then ν is a continuous, invariant, simple valuation that vanishes on $\mathbb{S}^2$. By the previous proposition, we therefore have $\nu(\Delta) = 0$ for all spherical simplices Δ. Now, let $P \in \mathcal{P}(\mathbb{S}^2)$ be a spherical polynomial. We can write P as a union of spherical simplices, where the intersections of distinct simplices are lower-dimensional:

$$P = \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_m,$$

with $\dim(\Delta_i \cap \Delta_j) < 2$ for all $i \neq j$. Again using the simplicity of ν, we deduce

$$\nu(P) = \sum_{i=1}^{m} \nu(\Delta_i) = 0.$$

For a general spherical convex body $K \in \mathcal{K}(\mathbb{S}^2)$, we approximate K by spherical polytopes in the Hausdorff metric and then use the continuity of ν to obtain $\nu(K) = 0$, or equivalently $\mu(K) = c\mu_2(K)$. $\square$

Now that we have characterized spherical area, Hadwiger's theorem on $\mathbb{S}^2$ follows:

Theorem 1.3.4 (Hadwiger's theorem on $\mathbb{S}^2$)**.** Let $\mu\colon \mathcal{K}(\mathbb{S}^2) \to \mathbb{R}$ be a valuation on $\mathbb{S}^2$. Then μ is continuous and invariant if and only if it is a linear combination of spherical area, spherical length, and the Euler characteristic, that is, there exist $c_0, c_1, c_2 \in \mathbb{R}$ such that

$$\mu(K) = c_0\chi(K) + c_1\mu_1(K) + c_2\mu_2(K)$$

for all spherical convex bodies $K \in \mathcal{K}(\mathbb{S}^2)$.

Proof. Clearly, each such linear combination of these three continuous, invariant valuations is again continuous and invariant. We will prove the other implication by applying the previous propositions three times:

- *Points:* Choose any point $x \in \mathbb{S}^2$ and define $c_0 := \mu(\{x\})$. This value is independent of the choice of x, because of the invariance of μ. Therefore the valuation $\mu - c_0\mu_0$ vanishes on all singletons in $\mathbb{S}^2$.
- *Circles:* Choose any great circle $\sigma \subset \mathbb{S}^2$. Then $\mu - c_0\mu_0$ is a continuous, invariant, simple valuation on spherical convex bodies in σ. Identifying σ with $\mathbb{S}^1$, by Proposition 1.3.1, we obtain $\mu - c_0\mu_0 = c_1\mu_1$ on σ. If we repeat this step for different great circles σ, the invariance of $\mu - c_0\mu_0$ implies that the value of c_1 must always be the same. Therefore

  $$\nu := \mu - c_0\mu_0 - c_1\mu_1$$

defines a valuation that vanishes on all singletons *and* on all spherical convex bodies contained in great circles of $\mathbb{S}^2$, that is, ν is simple.

- *Sphere:* Since ν is a continuous, invariant, simple valuation on $\mathcal{K}(\mathbb{S}^2)$, by Theorem 1.3.3 we have $\nu = c_2\mu_2$ for a real number $c_2 \in \mathbb{R}$.

Summarizing, we obtain $\mu = c_0\mu_0 + c_1\mu_1 + c_2\mu_2$. □

1.4 Higher dimensional spheres

We now explain why the above arguments can not be generalized to spheres of arbitrary dimensions. The problem lies in Proposition 1.3.2: Let us assume that we are given a continuous, invariant valuation $\mu\colon \mathcal{K}(\mathbb{S}^n) \to \mathbb{R}$ on $\mathbb{S}^n$ with $n \geq 3$ and $\mu(\mathbb{S}^n) = 0$, and we are to show that $\mu(\Delta) = 0$ for all spherical simplices $\Delta \subset \mathbb{S}^n$. As in the proof of this proposition, let us further assume that we have already obtained $\mu(L) = 0$ for all lunes $L \in \mathbb{S}^n$, which are intersections of at most n hemispheres. Now let $\Delta = H_1 \cap \cdots \cap H_{n+1}$ be a spherical simplex given by the intersection of $n+1$ hemispheres. According to the case $n = 2$, we again want to express $\mu(H_1 \cup \cdots \cup H_{n+1})$ in two different ways. First, repeatedly using the valuation property (also called *inclusion-exclusion principle*), we obtain

$$\mu(H_1 \cup \cdots \cup H_{n+1}) = \sum_{i=1}^{n+1} \mu(H_i) - \sum_{i_1<i_2} \mu(H_{i_1} \cap H_{i_2}) + \cdots + (-1)^n \mu(H_1 \cap \cdots \cap H_{n+1}).$$

Since all terms on the right-hand side except the last one are either hemispheres or lunes, on which μ vanishes, this equation simplifies to

$$\mu(H_1 \cup \cdots \cup H_{n+1}) = (-1)^n \mu(H_1 \cap \cdots \cap H_{n+1}) = (-1)^n \mu(\Delta).$$

Secondly, again up to lower-dimensional sets, we have $(H_1 \cup \cdots \cup H_{n+1})^c = H_1^c \cap \cdots \cap H_{n+1}^c = -\Delta$, where compliments are taken relative to $\mathbb{S}^n$. Since μ is simple and invariant, as before, we have

$$\mu(H_1 \cup \cdots \cup H_{n+1}) = \mu(\mathbb{S}^n) - \mu((H_1 \cup \cdots \cup H_{n+1})^c) = 0 - \mu(-\Delta) = -\mu(\Delta).$$

This time, putting the two equations together, we only get $(-1)^n \mu(\Delta) = -\mu(\Delta)$, which for *even* n leads to the desired relation $\mu(\Delta) = 0$, but for *odd* n only yields the tautological expression $-\mu(\Delta) = -\mu(\Delta)$. Consequently, no characterization of spherical volume is possible by taking this approach in odd dimensions. However, to generalize the inductive proof from the case $n = 2$ of Hadwiger's theorem to arbitrary $n \in \mathbb{N}$, we would need a characterization of spherical volume in all dimensions, not just even ones.

1.5 Replacing continuity by non-negativity

We conclude this first chapter with a characterization result of spherical volume as invariant, simple, and *non-negative* valuation by R. Schneider, treated in [Sch78, Theorem 6.2]. Here, a valuation $\mu\colon \mathcal{K}(\mathbb{S}^n) \to \mathbb{R}$ is called non-negative, if $\mu(K) \geq 0$ for all spherical convex bodies $K \in \mathcal{K}(\mathbb{S}^n)$. To prove this theorem, we need two lemmata, the first one being an extension theorem of valuations on spherical polytopes.

Lemma 1.5.1. Every valuation $\mu\colon \mathcal{P}(\mathbb{S}^n) \to \mathbb{R}$ on spherical polytopes can be extended uniquely to the class $\mathcal{Q}(\mathbb{S}^n)$ of finite unions of elements of $\mathcal{P}(\mathbb{S}^n)$ such that the valuation property still holds, that is, $\mu(K) + \mu(L) = \mu(K \cup L) + \mu(K \cap L)$ for all $K, L \in \mathcal{Q}(\mathbb{S}^n)$.

Proof. By Groemer's integral theorem [Kla97, Theorem 2.2.1] it is sufficient to show that $\alpha_1 I_{P_1} + \cdots + \alpha_m I_{P_m} = 0$ with $P_i \in \mathcal{P}(\mathbb{S}^n)$ and $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, m$, implies $\alpha_1\mu(P_1) + \cdots + \alpha_m\mu(P_m) = 0$, where I_P is the indicator function of a set P. Like in the Euclidean case, this is proved by induction on the dimension:

If we define S^0 to be the set containing two antipodal points, the statement follows for $n = 0$ and we may now assume it to be true in dimension $n-1$. Let us further assume that we are given spherical polytopes $P_i \in \mathcal{P}(\mathbb{S}^n)$ and real numbers $\alpha_i \in \mathbb{R}$, $i = 1 \ldots m$, with

$$\alpha_1 I_{P_1} + \cdots + \alpha_m I_{P_m} = 0, \quad \text{but} \quad \alpha_1\mu(P_1) + \cdots + \alpha_m\mu(P_m) = 1. \tag{1.1}$$

We have to show that this leads to a contradiction. Let k be the least number of full-dimensional polytopes in instances of equation (1.1).

If $k = 0$, all polytopes are contained in some $(n-1)$-subspheres. Define l to be the minimum number of such subspheres H_j, $j = 1, \ldots, l$, such that $P_1 \cup \ldots \cup P_m \subset H_1 \cup \ldots \cup H_l$, over all instances of (1.1), where $k = 0$. By the induction hypothesis, l must be strictly greater than 1. Pick any of these $(n-1)$-subspheres, say H_1. Since $I_{P_i \cap H_1} = I_{P_i} I_{H_1}$ and also $P_i \cap H_1 \subset H_1$, we have again by the induction hypothesis

$$\sum_{i=1}^m \alpha_i I_{P_i \cap H_1} = 0 \quad \text{and} \quad \sum_{i=1}^m \alpha_i \mu(P_i \cap H_1) = 0. \tag{1.2}$$

If we subtract the equations in (1.2) from their respective counterparts in (1.1), we arrive at

$$\sum_{i=1}^m \alpha_i (I_{P_i} - I_{P_i \cap H_1}) = 0 \quad \text{and} \quad \sum_{i=1}^m \alpha_i (\mu(P_i) - \mu(P_i \cap H_1)) = 0.$$

All polytopes contained in H_1 cancel out as summands, whereas all the remaining ones (including the new $P_i \cap H_1$) are contained in $H_2 \cap \cdots \cap H_l$. Hence, we have an instance of (1.1) with polytopes contained $l-1$ subspheres, a contradiction.

Thus, $k \geq 1$, and without loss of generality let $P_1 = \cap_{j=1}^r H_j^+$ be full-dimensional, where H_j^+ are hemispheres bounded by $(n-1)$-subspheres H_j, and set $H_j^- := -H_j^+$, $j = 1, \ldots, r$. Again, since $I_{P_i \cap H_1^\pm} = I_{P_i} I_{H_1^\pm}$ and $I_{P_i \cap H_1} = I_{P_i} I_{H_1}$, we have

$$\sum_{i=1}^m \alpha_i I_{P_i \cap H_1^+} = 0, \quad \sum_{i=1}^m \alpha_i I_{P_i \cap H_1} = 0, \quad \text{and} \quad \sum_{i=1}^m \alpha_i I_{P_i \cap H_1^-} = 0,$$

whereas, using the valuation property of μ, we also obtain

$$\sum_{i=1}^m \alpha_i \mu(P_i \cap H_1^+) - \sum_{i=1}^m \alpha_i \mu(P_i \cap H_1) + \sum_{i=1}^m \alpha_i \mu(P_i \cap H_1^-) = 1.$$

Since $\sum_{i=1}^m \alpha_i \mu(P_i \cap H_1) = 0$ by the induction hypothesis and $\sum_{i=1}^m \alpha_i \mu(P_i \cap H_1^-) = 0$ by the minimality of k, we have $\sum_{i=1}^m \alpha_i \mu(P_i \cap H_1^+) = 1$. Repeating this argument with the remaining H_j, we obtain

$$\sum_{i=1}^m \alpha_i I_{P_i \cap P_1} = 0 \quad \text{and} \quad \sum_{i=1}^m \alpha_i \mu(P_i \cap H_1^+ \cap \cdots \cap H_r^+) = \sum_{i=1}^m \alpha_i \mu(P_i \cap P_1) = 1.$$

There have to be other full-dimensional polytopes among the P_i, otherwise $\sum_{i=1}^m \alpha_i I_{P_i \cap P_1} = 0$ could not hold. Iteratively, we notice that all P_i have to be full-dimensional, and thus end up with

$$\sum_{i=1}^m \alpha_i I_{P_1 \cap \cdots \cap P_m} = 0 \quad \text{and} \quad \sum_{i=1}^m \alpha_i \mu(P_1 \cap \cdots \cap P_m) = 1,$$

which is a contradiction. □

The next Lemma is about approximating the integral of a function on the sphere by the sum of a finite number of rotations of that function. For a proof, we refer to H. Hadwiger's article [Had43, 3., Satz II].

Lemma 1.5.2. Let $f\colon \mathbb{S}^n \to \mathbb{R}$ be a Riemann-integrable funtion and $\varepsilon > 0$. Then there exist $k \in \mathbb{N}$ and rotations $\theta_i \in \mathrm{SO}(n+1)$, $i = 1, \dots, k$, such that

$$\left| \frac{1}{k} \sum_{i=1}^k f(\theta_i x) - \frac{1}{\beta_n} \int_{\mathbb{S}^n} f \, d\sigma_n \right| < \varepsilon$$

for all $x \in \mathbb{S}^n$, where σ_n is the spherical Lebesgue measure and $\beta_n = \sigma_n(\mathbb{S}^n) = (n+1)\omega_{n+1}$.

Equipped with these facts, we are now in a position to prove Schneider's characterization theorem for invariant, simple, and non-negative valuations. It is not clear however, whether this characterization implies a Hadwiger-type theorem for invariant, non-negative valuations on $\mathbb{S}^n$.

Theorem 1.5.3. Let $\mu\colon \mathcal{P}(\mathbb{S}^n) \to \mathbb{R}$ be a valuation on spherical polytopes. Then μ is simple, invariant, and non-negative if and only if there exists a positive real number $c \geq 0$ such that $\mu(P) = c\mu_n(P) = c\sigma_n(P)$ for all polytopes $P \in \mathcal{P}(\mathbb{S}^n)$.

Proof. We follow the proof from [Sch08, Theorem 14.4.7]. First, extend the given valuation $\mu\colon \mathcal{P}(\mathbb{S}^n) \to \mathbb{R}$ to the class of finite unions of convex polytopes, denoted by $\mathcal{Q}(\mathbb{S}^n)$, which is possible by Lemma 1.5.1. Note, that $\mu\colon \mathcal{Q}(\mathbb{S}^n) \to \mathbb{R}$ is still simple, invariant, and non-negative (elements of $\mathcal{Q}(\mathbb{S}^n)$ can be dissected into convex polytopes having lower-dimensional pairwise intersections). Using the non-negativity, we see that A,$B \in \mathcal{Q}(\mathbb{S}^n)$ with $A \subset B$ implies $\mu(A) \leq \mu(B)$, for there exists a set $A' \in \mathcal{Q}(\mathbb{S}^n)$ with $A \cup A' = B$ and such that $A \cap A'$ is a finite union of lower-dimensional polytopes. We say that μ is *monotone*.

Now assume that we are given $P \in \mathcal{Q}(\mathbb{S}^n)$ and $\varepsilon > 0$. If we plug in the indicator function of P, $f := I_P$, into Lemma 1.5.2, we get a number $k \in \mathbb{N}$ and rotations $\theta_i \in \mathrm{SO}(n+1)$, $i = 1, \dots, k$, such that

$$\left| \frac{1}{k}\nu(x) - \frac{1}{\beta_n}\sigma(P) \right| < \varepsilon \tag{1.3}$$

for all $x \in \mathbb{S}^n$, where we write $\nu(x)$ for the number of sets $\theta_i^{-1}P$ containing x. Let $U_j := \{x \in \mathbb{S}^n \colon \nu(x) \geq j\}$ for $j = 1, \dots, k$, then

$$U_j = \bigcup_{1 \leq i_1 < \cdots < i_j \leq k} (\theta_{i_1}^{-1}P \cap \cdots \cap \theta_{i_j}^{-1}P) \in \mathcal{Q}(\mathbb{S}^n)$$

is actually a finite union of convex polytopes. By the definition of the U_j, we have

$$\sum_{i=1}^{k} I_{\theta_i^{-1}P}(x) = \sum_{j=1}^{k} I_{U_j}(x)$$

for all $x \in \mathbb{S}^n$. Using Groemer's integral theorem (see the proof of Lemma 1.5.1), we get the right equality in

$$k\mu(P) = \sum_{i=1}^{k} \mu(\theta_i^{-1}P) = \sum_{j=1}^{k} \mu(U_j), \tag{1.4}$$

whereas the left one follows from the invariance of μ. We now look for bounds for the right hand sum. First choose points $y,z \in \mathbb{S}^n$ such that $\nu(y) \leq \nu(x) \leq \nu(z)$ for all $x \in \mathbb{S}^n$. By definition, $U_j = \mathbb{S}^n$ for $j \leq \nu(y)$ and $U_j = \emptyset$ for $j > \nu(z)$. Using the non-negativity of μ for the left inequality and its monotonicity for the right one, we obtain

$$\nu(y)\mu(\mathbb{S}^n) \leq \sum_{j=1}^{k} \mu(U_j) \leq \nu(z)\mu(\mathbb{S}^n),$$

which, combined with (1.3) and (1.4), results in

$$\left(\frac{\sigma(P)}{\beta_n} - \varepsilon \right) \mu(\mathbb{S}^n) \leq \mu(P) \leq \left(\frac{\sigma(P)}{\beta_n} + \varepsilon \right) \mu(\mathbb{S}^n).$$

Letting ε go to zero and setting $c := \mu(\mathbb{S}^n)/\beta_n$, we end up with $\mu(P) = c\sigma(P)$. □

CHAPTER 2

Smooth and generalized valuations on spherical convex bodies

In this chapter we will turn our attention to two special classes of valuations, namely *smooth* and *generalized* ones. A smooth valuation can be represented by two differential forms, where evaluating that valuation at a spherical convex body K is done by integrating those forms over K itself and the so-called *normal cycle* $N(K)$ of K. The set $N(K)$ is a subset of the tangent bundle $T\mathbb{S}^n$ extending the notion of a Gauss map to sets with non-smooth boundaries. Later, we will also use the notation $N(K)$ for the functional obtained by integrating differential forms over this set. The relation between smooth and generalized valuations can be thought of similar to the case of smooth versus generalized functions (also called distributions), where the latter ones are functionals on the former. The main reason why this integral representation of smooth valuations will be of great use to us, is that it allows to reduce the classification problem of valuations to a classification problem of differential forms. In the same way, instead of generalized valuations, we can look at functionals on differential forms, called *currents*.

2.1 Restricting to proper convex bodies

We will carry over some known facts from valuation theory in $\mathbb{R}^n$ to the sphere by using certain projections from open hemispheres to $\mathbb{R}^n$ that take great circles to straight lines and, thus, preserve convexity. To this end, we have to restrict ourselves to a subset of spherical convex bodies, namely the *proper* ones. The upcoming proposition however shows that no information is lost.

Definition 2.1.1. Let $K \in \mathcal{K}(\mathbb{S}^n)$ be a spherical convex body. If K is fully contained in an open hemisphere, then K is called *proper*. We will denote the collection of all proper spherical convex bodies by $\mathcal{K}_p(\mathbb{S}^n)$.

For proper convex bodies there always exists a small $\varepsilon > 0$, such that the parallel body K_ε is still a convex body contained in that hemiphere, which will later make it possible to define so-called *normal cycles* (see Section 2.3).

Proposition 2.1.2. Any continuous valuation $\mu\colon \mathcal{K}_p(\mathbb{S}^n) \to \mathbb{R}$ on proper spherical convex bodies can be uniquely extended to all spherical convex bodies. This extension is continuous and obtained from the inclusion-exclusion principle.

Proof. We divide this proof into several parts: First, we will show that $\mathcal{K}_p(\mathbb{S}^n)$ generates $\mathcal{K}(\mathbb{S}^n)$ as a lattice, then that μ defines an integral on indicator functions on $\mathcal{K}_p(\mathbb{S}^n)$, and finally use Groemer's integral theorem to obtain the statement.

Step 1: We want to show that every spherical convex set can be expressed as the finite union of proper spherical convex sets. To do this, let $K \in \mathcal{K}(\mathbb{S}^n)$ be any spherical convex body and denote by $e_1, \dots e_{n+1}$ the standard orthonormal basis of $\mathbb{R}^{n+1}$. Define

$$H^v := \{x \in \mathbb{S}^n | \langle x, v\rangle \geq 0\}, \; v \in \mathbb{R}^{n+1},$$

to be the hemisphere in direction $v \in \mathbb{R}^{n+1}$ and

$$K^{\varepsilon_1 \dots \varepsilon_{n+1}} := K \cap \bigcap_{i=1}^{n+1} H^{\varepsilon_i e_i},$$

where ε_i is either $+$ or $-$ for each $1 \leq i \leq n+1$, to be the intersection of K with any combination of the hemispheres in directions $\pm e_1 \dots \pm e_{n+1}$. We can write any point $x \in K^{\varepsilon_1 \dots \varepsilon_{n+1}}$ as $x = x_1 e_1 + \dots + x_{n+1} e_{n+1}$ with $\operatorname{sgn}(x_i) = \varepsilon_i$ for all $i \in \{1, \dots, n+1\}$. Therefore, as $x_1 \varepsilon_1 + \dots + x_{n+1}\varepsilon_{n+1} \geq 0$, we have

$$K^{\varepsilon_1 \dots \varepsilon_{n+1}} \subset H^{\varepsilon_1 e_1 + \dots \varepsilon_{n+1} e_{n+1}},$$

that is, all of the $K^{\varepsilon_1 \dots \varepsilon_{n+1}}$ are proper spherical convex bodies. Since

$$K = \bigcup_{\substack{(\varepsilon_1, \dots, \varepsilon_{n+1}),\\ \varepsilon_i = \pm,\\ i=1,\dots,n+1}} K^{\varepsilon_1 \dots \varepsilon_{n+1}},$$

we have expressed K as a finite union of such sets.

Step 2: Now we need to show, that μ defines an integral on indicator functions of proper spherical convex bodies, that is,

$$\sum_{i=1}^{m} \alpha_i I_{K_i} = 0 \quad \text{implies} \quad \sum_{i=1}^{m} \alpha_i \mu(K_i) = 0$$

for all $K_i \in \mathcal{K}_p(\mathbb{S}^n)$, $\alpha_i \in \mathbb{R}$, $i = 1, \dots, m$, $m \in \mathbb{N}$. We follow the argument of the proof from a similar theorem for Euclidean space, which can be found in [Kla97, Section 5.1]. Compare also to the proof of Lemma 1.5.1.

We will give a proof by induction on the dimension n and start by noticing, that the statement is true for $n = 0$ (where $\mathbb{S}^0$ is just a pair of antipodal points). So let us assume that the statement is true in dimension $n-1$, but false in dimension n, that is, there exist proper spherical convex bodies $K_1, \dots, K_m \in \mathcal{K}_p(\mathbb{S}^n)$, $m \in \mathbb{N}$, such that

$$\sum_{i=1}^{m} \alpha_i I_{K_i} = 0 \quad \text{but} \quad \sum_{i=1}^{m} \alpha_i \mu(K_i) = 1, \tag{2.1}$$

and try to deduce a contradiction. Therefore let m be the smallest number, such that instances of (2.1) exist. Further choose any $(n-1)$-subsphere H with associated hemispheres $H^{\pm}$, such that K_1 is contained in the interior of H^+. Since $I_{K_i\cap H^{\pm}} = I_{K_i}I_{H^{\pm}}$ and $I_{K_i\cap H} = I_{K_i}I_H$ for all $i \in \{1,\dots,m\}$, by (2.1) we have

$$\sum_{i=1}^{m} \alpha_i I_{K_i\cap H^{\pm}} = 0 \quad \text{and} \quad \sum_{i=1}^{m} \alpha_i I_{K_i\cap H} = 0.$$

By using the valuation property, we also obtain

$$\sum_{i=1}^{m} \alpha_i \mu(K_i) = \sum_{i=1}^{m} \alpha_i \mu(K_i \cap H^+) + \sum_{i=1}^{m} \alpha_i \mu(K_i \cap H^-) - \sum_{i=1}^{m} \alpha_i \mu(K_i \cap H).$$

As all of the sets $K_i\cap H$ lie in a subsphere of dimension $n-1$, by the induction hypothesis, we have $\sum_{i=1}^{m} \alpha_i\mu(K_i\cap H) = 0$. Also, since $K_1\cap H^- = \emptyset$, we must have $\sum_{i=1}^{m} \alpha_i\mu(K_i\cap H^-) = 0$, because otherwise there would be an instance of (2.1) with less than m bodies, so (2.1) reduces to

$$\sum_{i=1}^{m} \alpha_i \mu(K_i \cap H^+) = \sum_{i=1}^{m} \alpha_i \mu(K_i) = 1.$$

Choose now a sequence of great circles H_j, such that the associated hemispheres H_j^+ all contain K_1 in their interiors and such that

$$K_1 = \bigcap_{j=1}^{\infty} H_j^+.$$

If we iterate the above argument by setting $H = H_1$, replacing K_i with $K_i \cap H_1$ as the new starting bodies, intersecting with H_2, and so forth, we obtain

$$\sum_{i=1}^{m} \alpha_i \mu(K_i \cap H_1^+ \cap \dots \cap H_k^+) = 1$$

for all $k \in \mathbb{N}$. Taking k to infinity and using the continuity of μ yields

$$\sum_{i=1}^{m} \alpha_i \mu(K_i \cap K_1) = 1.$$

As $I_{K_i\cap K_1} = I_{K_i}I_{K_1}$, we also have

$$\sum_{i=1}^{m} \alpha_i I_{K_i\cap K_1} = 0.$$

Again, iteration of this process of choosing $(n-1)$-subspheres around K_1, but replacing

K_1 by $K_2, \ldots, K_m$ leads to

$$\sum_{i=1}^{m} \alpha_i \mu(K_1 \cap \cdots \cap K_m) = \left(\sum_{i=1}^{m} \alpha_i \right) \mu(K_1 \cap \cdots \cap K_m) = 1,$$

which means $\alpha_1 + \ldots + \alpha_m \neq 0$ and $K_1 \cap \ldots \cap K_m \neq \emptyset$. On the other hand

$$\sum_{i=1}^{m} \alpha_i I_{K_1 \cap \cdots \cap K_m} = \left(\sum_{i=1}^{m} \alpha_i \right) I_{K_1 \cap \cdots \cap K_m} = 0$$

implies either $\alpha_1 + \cdots + \alpha_m = 0$ or $K_1 \cap \cdots \cap K_m = \emptyset$, which is a contradiction.

Step 3: We can now apply Groemer's integral theorem, which says that if μ defines an integral on indicator functions, that is,

$$\sum_{i=1}^{m} \alpha_i I_{K_i} = 0 \quad \text{implies} \quad \sum_{i=1}^{m} \alpha_i \mu(K_i) = 0,$$

there exists a unique extension of μ, obtained from the inclusion-exclusion principle (see also Section 1.4), to the set $\mathcal{K}(\mathbb{S}^n)$ of spherical convex bodies, generated by $\mathcal{K}_p(\mathbb{S}^n)$.

Step 4: To see why μ is still continuous on $\mathcal{K}(\mathbb{S}^n)$, let $K_i \in \mathcal{K}(\mathbb{S}^n)$, $i \in \mathbb{N}$ be a sequence of possibly non-proper spherical convex bodies converging in the spherical Hausdorff topology to $K \in \mathcal{K}(\mathbb{S}^n)$. As in Step 1, consider again intersections of K_i with orthogonal hemispheres:

$$K_i^{\varepsilon_1 \ldots \varepsilon_{n+1}} := K_i \cap \bigcap_{i=1}^{n+1} H^{\varepsilon_i e_i}.$$

Since these are all proper spherical convex bodies and $K_i^{\varepsilon_1 \ldots \varepsilon_{n+1}} \to K^{\varepsilon_1 \ldots \varepsilon_{n+1}}$ as i tends to infinity (note that if $K^{\varepsilon_1 \ldots \varepsilon_{n+1}}$ is the empty set, then $K_i^{\varepsilon_1 \ldots \varepsilon_{n+1}}$ has to be empty for almost all $i \in \mathbb{N}$ too, since K and the hemispheres are all compact sets), we have

$$\mu(K_i^{\varepsilon_1 \ldots \varepsilon_{n+1}}) \longrightarrow \mu(K^{\varepsilon_1 \ldots \varepsilon_{n+1}}) \quad \text{as } i \to \infty$$

for all combinations of $\pm e_1 \ldots \pm e_{n+1}$, hence

$$\mu(K_i) \to \mu(K).$$

by the inclusion-exclusion principle. □

Remark 2.1.3. In fact, what we showed in the last proof was that μ can be uniquely extended to *finite unions* of spherical convex bodies, which are also generated by the set $K_p(\mathbb{S}^n)$, although continuity is lost here. These are sometimes called *spherical polyconvex bodies.*

2.2 Currents

The space of currents on a manifold M, where M will be either $\mathbb{S}^n$ itself or (a subset of) its tangent bundle, will be defined as the topological dual to the space of differential forms on M, therefore we need a suitable topology on the latter. Let

$$I(k,n) := \{\alpha = (\alpha_1, \dots, \alpha_k) \colon \alpha_i \in \mathbb{N}, 1 \le \alpha_1 < \dots < \alpha_k \le n\}$$

be the the set of ordered multi-indices and $U \subset M$ an open subset of M with coordinates $(x_1, \dots, x_n) \colon U \to \mathbb{R}^n$. We define the space of infinitely differentiable k-differential forms with compact support in U by

$$\mathcal{D}^k(U) := \left\{ \sum_{\alpha \in I(k,n)} \omega_\alpha dx_\alpha \mid \omega_\alpha \in C_c^\infty(U) \right\}$$

and the space of infinitely differentiable k-differential forms on U by

$$\mathcal{E}^k(U) := \left\{ \sum_{\alpha \in I(k,n)} \omega_\alpha dx_\alpha \mid \omega_\alpha \in C^\infty(U) \right\},$$

where $dx_\alpha = dx_{\alpha_1} \wedge \dots \wedge dx_{\alpha_k}$, if $\alpha = (\alpha_1, \dots, \alpha_k)$. The support of ω will be denoted by $\operatorname{spt} \omega$ and is the smallest closed set $V \subset U$ such that $\omega_\alpha(x) = 0$ for all $x \in V \setminus U$ and $\alpha \in I(k,n)$.

A topology on $\mathcal{E}^k(U)$ is given by the following neighborhood base at zero: For every natural number $i \in \mathbb{N}$, every compact set $K \subset U$, and every $\varepsilon > 0$, let

$$U_{i,K,\varepsilon} := \left\{ \omega = \sum_{\alpha \in I(k,n)} \omega_\alpha dx_\alpha \in \mathcal{E}^k(U) \colon \sup_{x \in K, |J| < i} \left| \frac{\partial^{|J|} \omega_\alpha}{\partial x^J}(x) \right| < \varepsilon \right\},$$

where $J = (j_1, \dots, j_n)$ is another multi-index, $|J| = j_1 + \dots + j_n$, and taking the $|J|$-derivative means

$$\frac{\partial^{|J|} \omega_\alpha}{\partial x^J}(x) := \frac{\partial^{j_1 + \dots + j_n} \omega_\alpha}{\partial x_1^{j_1} \dots \partial x_n^{j_n}}(x).$$

Since $\mathcal{E}^k(U)$ is a linear space, the collection of all $U_{i,K,\varepsilon}$ induces the desired topology. From this we also get a topology on $\mathcal{D}^k(U)$ by saying $O \subset \mathcal{D}^k(U)$ is open, precisely if $\{\omega \in O \mid \operatorname{spt} \omega \subset K\}$ is open in $\mathcal{E}^k(U)$ for all compact sets $K \subset U$.

A sequence $\omega_1, \omega_2, \ldots \in \mathcal{D}^k(U)$ converges to $\omega \in \mathcal{D}^k(U)$ in this topology, if and only if there exists a compact set $K \subset U$ such that $\operatorname{spt} \omega_i \subset K$ for all $i \in \mathbb{N}$ and if all partial derivatives of any order of the coefficients $\omega_{i,\alpha}$ converge uniformly to the respective coefficient ω_α of ω.

Now the space of infinitely differentiable (compactly supported) k-differential forms on M, denoted by $\mathcal{E}^k(M)$ (resp. $\mathcal{D}^k(M)$) is the space of all smooth k-forms on M together with the initial topology induced by the restriction maps $\pi_U\colon \mathcal{E}^k(M) \to \mathcal{E}^k(U)$ (resp. $\mathcal{D}^k(M) \to \mathcal{D}^k(U)$).

Definition 2.2.1. The space of *k-dimensional currents*, denoted by $\mathcal{D}_k(M)$, is the space of continuous, linear functionals on the space $\mathcal{D}^k(M)$ of infinitely differentiable k-forms with compact support in M, endowed with the topology described above.

Remark 2.2.2. If $k = 0$, we have $\mathcal{D}^0(M) = C_c^\infty(M)$ and $\mathcal{D}_0(M)$ is called the space of *generalized functions* or *distributions*.

Furthermore, denote by $\mathcal{D}^*(M) := \bigoplus_{k\geq 0} \mathcal{D}^k(M)$ and $\mathcal{D}_*(M) := \bigoplus_{k\geq 0} \mathcal{D}_k(M)$ the spaces of all differential forms and currents, respectively. There is a natural way to derive new currents from given ones:

Definition 2.2.3. Let $T \in \mathcal{D}_k(M)$ be a k-dimensional current. The *boundary* of T, denoted by ∂T, is the $(k-1)$-dimensional current given by

$$\partial T(\omega) := T(d\omega),$$

where $\omega \in \mathcal{D}^k(M)$ and $d\colon \mathcal{D}^{k-1}(M) \to \mathcal{D}^k(M)$ is the exterior derivative on differential forms. Furthermore, a current $T \in \mathcal{D}_k(M)$ is called a *cycle*, if $\partial T = 0$.

There are two important topologies on the space of currents, the flat and the weak topology. For now, we only need the latter one:

Definition 2.2.4. Let $(T_j)_{j\in\mathbb{N}}$ be a sequence of k-dimensional currents in $\mathcal{D}_k(M)$. We say T_j converges *weakly* to $T \in \mathcal{D}_k(M)$, $T_j \rightharpoonup T$, if $T_j(\omega) \to T(\omega)$ for all k-forms $\omega \in \mathcal{D}^k(M)$.

Since integrating differential forms over various subsets of $\mathbb{S}^n$ and $T\mathbb{S}^n$ will be a prominent example of a current, we give a short explanation of how to integrate smooth k-forms over k-submanifolds of some $\mathbb{R}^m$. The material is taken from [Gia98, Section 2.2], for a detailed description also confer [Ber12, Chapter 1]. First, let M be an embedded, oriented, k-dimensional C^1-submanifold of $\mathbb{R}^m$ with local oriented charts (U_i,ψ_i) and local parametrizations $\phi_i\colon \mathbb{R}^k \supset V \to M \cap U_i$, with $\phi_i = \psi_i^{-1}$.

If ω is a smooth k-form on M, or in a neighborhood of M in $\mathbb{R}^m$, supported in one of the U_i, spt $\omega \subset U_i$, the integral of ω over M is defined by

$$\int_M \omega := \int_V \phi_i^*\omega = \int_V \langle \phi_i^*\omega(u), e_1 \wedge \ldots \wedge e_k\rangle \, d\mathcal{H}^k(u),$$

where $e_1 \wedge \ldots \wedge e_k$ is the canonical k-vector in $\mathbb{R}^k$, $\mathcal{H}^k$ is the k-dimensional Hausdorff measure, and ϕ_i^* the pull-back of ϕ_i given by

$$\langle \phi_i^*\omega(u), e_1 \wedge \ldots \wedge e_k\rangle = \omega(\phi_i(u))(D\phi_i(u)e_1 \wedge \ldots \wedge D\phi_i(u)e_k).$$

Independence of the chosen oriented chart follows from integration by substitution, since coordinate changes are orientation preserving C^1-diffeomorphisms. If, more generally, spt ω

is not contained in a single U_i, we define

$$\int_M \omega := \sum_{i,j} \int_V \phi_i^*(\eta_j \omega), \tag{2.2}$$

where $\{\eta_j\}$ is a partition of unity subordinate to $\{U_i\}$.

The typical submanifolds we will integrate differential forms over will however not always admit C^1, but only Lipschitz-parametrizations. On the other hand, by Rademacher's theorem, we know that Lipschitz maps are C^1 almost everywhere, so equation (2.2) also suffices to define the integral of a smooth k-form ω over a submanifold M, given by local parametrizations ϕ_i that are only Lipschitz-continuous maps.

2.3 Normal and conormal cycles

Normal cycles of spherical convex sets will be the objects over which we will integrate differential forms to obtain valuations. If K is a spherical convex body with smooth boundary, one can think of the normal cycle $N(K)$ of K as the graph of the Gauss map, which gives the unit normal vector at every boundary point of K. However, if K is not smooth, there might be boundary points, such as corners for example, where we have to collect a lot of normal vectors at one point. This is described in the following.

Definition 2.3.1. Let K be a spherical convex body and $x \in \partial K$ a boundary point of K. Define the *tangent cone* $\operatorname{Tan}(K,x)$ of K at x to be

$$\operatorname{Tan}(K,x) := \overline{\{w \in T_x\mathbb{S}^n \mid \exists \varepsilon > 0, \gamma\colon [0,\varepsilon] \to K, \gamma(0) = x, \gamma'(0) = w\}},$$

which is the closure of the set of all tangent vectors v to $\mathbb{S}^n$ at x, such that there exists a small geodesic arc in direction w starting in x.

The *normal cone* $\operatorname{Nor}(K,x)$ of K at x is then defined as the polar cone to $\operatorname{Tan}(K,x)$ in $T_x\mathbb{S}^n$:

$$\operatorname{Nor}(K,x) := \{v \in T_x\mathbb{S}^n \mid \langle v,w\rangle \le 0 \text{ for all } w \in \operatorname{Tan}(K,x)\}.$$

The disjoint union of the $\operatorname{Nor}(K,x)$ will be denoted by

$$\operatorname{Nor}(K) := \bigcup_{x\in\partial K} \{(x,v) \mid v \in \operatorname{Nor}(K,x)\}$$

and after normalizing, we finally arrive at the *normal cycle* $N(K)$ of K:

$$N(K) := \{(x,v) \in \operatorname{Nor}(K) \mid \langle v,v\rangle = 1\},$$

which is a subset of the *sphere bundle* $S\mathbb{S}^n = \{(x,v) \in T\mathbb{S}^n \mid \langle v,v\rangle = 1\} \subset T\mathbb{S}^n$.

Remark 2.3.2. Note that the sphere bundle is not globally diffeomorphic to $\mathbb{S}^n \times \mathbb{S}^{n-1}$, but locally, that is, if H is any hemisphere, we have

$$S\mathbb{S}^n \cap \pi_{\mathbb{S}^n}^{-1}(H) \cong H \times \mathbb{S}^{n-1},$$

where $\pi_{\mathbb{S}^n} : T\mathbb{S}^n \to \mathbb{S}^n$ maps a tangent vector to its base point. In this way, it makes sense to consider Lipschitz maps into $S\mathbb{S}^n \cap \pi^{-1}(H)$ by pulling back the metric from $H \times \mathbb{S}^{n-1}$.

Lemma 2.3.3. The sphere bundle $S\mathbb{S}^n$ is compact. Therefore $\mathcal{D}^k(S\mathbb{S}^n) = \mathcal{E}^k(S\mathbb{S}^n)$.

Proof. Let H be any great $(n-1)$-subsphere with associated closed hemispheres $H^{\pm}$. Then, by using the above projection, we can write

$$S\mathbb{S}^n = \underbrace{\left(S\mathbb{S}^n \cap \pi_{\mathbb{S}^n}^{-1}(H^+)\right)}_{\cong H^+ \times \mathbb{S}^{n-1}} \cup \underbrace{\left(S\mathbb{S}^n \cap \pi_{\mathbb{S}^n}^{-1}(H^-)\right)}_{\cong H^- \times \mathbb{S}^{n-1}},$$

where the two sets below are compact. Hence $S\mathbb{S}^n$, as a finite union of compact sets, is compact too. □

Remark 2.3.4. In the same way, one can show that the sphere bundle of any compact manifold is compact.

Remark 2.3.5. Sometimes it will be convenient to identify

$$T_x\mathbb{S}^n \cong \mathbb{R}^n_x := \{y \in \mathbb{R}^{n+1} \mid \langle x, y\rangle = 0\},$$

for any $x \in \mathbb{S}^n$. Then, for any proper spherical convex body $K \in \mathcal{K}_p(\mathbb{S}^n)$, its normal cycle $N(K)$ can be identified with a subset of $\mathbb{S}^n \times \mathbb{S}^n \subset \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$, via

$$N(K) \cap T_x\mathbb{S}^n \subset \{x\} \times \mathbb{S}^{n-1}_x,$$

where $\mathbb{S}^{n-1}_x := \mathbb{S}^n \cap \mathbb{R}^n_x$.

Now for the conormal cycle and conormal cones we will change our point of view and, instead of looking at possible normal directions to a set, consider all possible tangent planes, which live in the cotangent space as kernels of linear functionals on the tangent space. Any identification of the tangent and the cotangent space as real vector spaces of the same dimension also yields an identification of the two cones, but the advantage of the conormal cycle lies in its invariance under diffeomorphisms, which is not guaranteed for the normal cycle, since angles and in particular normal directions need not to be preserved under arbitrary differentiable maps.

Definition 2.3.6. The *conormal cone* $\mathrm{Nor}^*(K, x)$ of K at x is the subset of the cotangent space $(T_x\mathbb{S}^n)^*$, defined by

$$\mathrm{Nor}^*(K, x) := \{\xi \in (T_xS^n)^* \mid \xi(w) \le 0 \text{ for all } w \in \mathrm{Tan}(K, x)\}.$$

The disjoint union of the $\mathrm{Nor}^*(K, x)$ will be denoted by

$$\mathrm{Nor}^*(K) := \bigcup_{x\in\partial K} \{(x, v) \mid v \in \mathrm{Nor}^*(K, x)\}.$$

The *cosphere bundle* $(S\mathbb{S}^n)^*$ is defined as the factorization of the cotangent bundle by positive real numbers:

$$(S\mathbb{S}^n)^* := \mathbb{P}_+((T\mathbb{S}^n)^*),$$

where $\mathbb{P}_+((T\mathbb{S}^n)^*) = ((T\mathbb{S}^n)^*\backslash\{0\})/_{\mathbb{R}_+}$, that is, $(x,\xi),(y,\eta) \in (T\mathbb{S}^n)^*$ are equivalent if and only $x = y$ and if there exists $\lambda \in \mathbb{R}_+$ such that $\omega = \lambda\eta$. If $\pi\colon (T\mathbb{S}^n)^*\backslash\{0\} \to (S\mathbb{S}^n)^*$ is the natural projection, then the *conormal* cycle $N^*(K)$ is given by

$$N^*(K) := \pi(\mathrm{Nor}(K)\backslash\{0\}).$$

Remark 2.3.7. Note that normal- and conormal cycles are defined on convex bodies in $\mathbb{R}^{n+1}$ in excactly the same way as above. Since geodesics in $\mathbb{R}^{n+1}$ are just straight lines, we have a simpler description of the tangent cone of $K \in \mathcal{K}(\mathbb{R}^{n+1})$ at a point $x \in \mathbb{R}^{n+1}$:

$$\mathrm{Tan}(K,x) := \overline{\{w \in T_x\mathbb{R}^{n+1} \mid \langle y-x,w\rangle \leq 0\ \forall y \in K\}}.$$

By choosing an inner product on each tangent space at points $x \in \mathbb{S}^n$ one obtains a natural identification of the normal- and conormal cycle. In our case we can use the product given by the restriction of the Euclidean scalar product of the ambient space $\mathbb{R}^{n+1}$. Thus, the map

$$\begin{aligned}\tau\colon S\mathbb{S}^n &\to (S\mathbb{S}^n)^*,\\ v &\mapsto [(v,\cdot)],\end{aligned}$$

where $[w]$ denotes the equivalence class of w in the cosphere bundle, induces the desired bijection from $N(K)$ to $N^*(K)$.

Remark 2.3.8. Using $\tau\colon S\mathbb{S}^n \to (S\mathbb{S}^n)^*$, we immediately see, that $(S\mathbb{S}^n)^*$ is compact too.

There lies an advantage in "forgetting" about our usual scalar product coming from $\mathbb{R}^{n+1}$ and using the conormal cycle, namely its invariance under diffeomorphisms:

Lemma 2.3.9. Let M_1, M_2 be subsets of $\mathbb{S}^n$ or $\mathbb{R}^{n+1}$, $\phi\colon M_1 \to M_2$ a diffeomorphism from M_1 onto M_2, that takes convex bodies to convex bodies, and K a (proper spherical) convex body in M_1. Then

$$\phi_*(N^*(K)) = N^*(\phi(K)),$$

where the push-forward of $w \in T^*M_1$ is its pull-back under the inverse map $\phi_*w := (\phi^{-1})^*w$.

Proof. Let $x \in \partial K$ be any point in the boundary of K and $w_1 \in \mathrm{Nor}(K,x)$. First, we notice that, since ϕ takes curves in K to curves in $\phi(K)$, ϕ_* takes inward pointing tangent vectors in T_xK to inward pointing tangent vectors in $T_{\phi(x)}\phi(K)$,

$$\phi_*(\mathrm{Tan}(K,x)) = \mathrm{Tan}(\phi(K),\phi(x)).$$

For $v_2 \in \mathrm{Tan}(\phi(K),\phi(x))$, we have $v_2 = \phi_*(v_1)$, $v_1 \in \mathrm{Tan}(K,x)$, and

$$\phi_*(w_1)(v_2) = (\phi^{-1})^*(w_1)(\phi_*(v_1)) = w_1((\phi^{-1})_*(\phi_*(v_1))) = w_1(v_1) \leq 0,$$

therefore $\phi_*(w_1) \in \mathrm{Nor}^*(\phi(K),\phi(x))$. Applying the same argument to ϕ^{-1}, we arrive at

$$\phi_*(\mathrm{Nor}(K,x)) = \mathrm{Nor}(\phi(K),\phi(x))$$

and thus $\phi_*(N^*(K)) = N^*(\phi(K))$. □

Remark 2.3.10. Note that the above lemma also holds true for normal cycles, if the diffeomorphism ϕ is an isometry. In our case, we will have $\mathrm{SO}(n+1) \supset \psi\colon \mathbb{S}^n \to \mathbb{S}^n$.

Now we introduce the already mentioned projections of open hemispheres that preserve convexity, and thereby follow [Sch16, Section 4].

Definition 2.3.11. Let $u \in \mathbb{S}^n$ and $H^u = \{x \in \mathbb{S}^n \mid \langle x, u\rangle \geq 0\}$ its associated hemisphere. The map

$$\begin{aligned} g_u\colon (H^u)^0 &\to \mathbb{R}^n_u, \\ x &\mapsto \frac{x}{\langle x, u\rangle} - u, \end{aligned}$$

where $(H^u)^0$ denotes the interior relative to $\mathbb{S}^n$ of H^u, is called *gnomonic projection* in direction u.

Lemma 2.3.12. For each hemisphere H^u, $u \in \mathbb{S}^n$, the gnomonic projection $g_u\colon (H^u)^0 \to \mathbb{R}^n_u$ is a diffeomorphism, that takes great circles to straight lines. In particular, it induces a one-to-one correspondence between spherical convex bodies in $(H^u)^0$ and convex bodies in $\mathbb{R}^n_u$.

Proof. For any given point $x \in (H^u)^0$, $g_u(x)$ is obtained geometrically in the following way: First, take the straight line passing through the origin and x and intersect it with the tangent *plane* to $\mathbb{S}^n$ at u. Then translate that intersection point by $-u$. Any great-circle, after connecting all of its points with the origin, leads to a two-dimensional subspace, which, intersected with the tangent plane at u, yields a straight line. Hence, convex sets are mapped to convex sets under g_u Conversely, the same is true for g_u^{-1}.

By its defining formula, we see that g_u is smooth. Moreover, note that $g_u^{-1}\colon \mathbb{R}^n_u \to (H^u)^0$ is given by

$$x \mapsto \frac{x+u}{|x+u|},$$

which is also a smooth map. □

Lemma 2.3.13. Gnomonic projections preserve Hausdorff convergence, that is, if H^u, $u \in \mathbb{S}^n$ is any hemisphere, $g_u\colon (H^u)^0 \to \mathbb{R}^n_u$ its associated gnomonic projection, and $K_i, i \in \mathbb{N}$ a sequence of spherical convex bodies converging to $K \subset (H^u)^0$ in the spherical Hausdorff topology, then also $g_u(K_i) \to g_u(K)$ in the Euclidean Hausdorff topology and vice-versa.

Proof. Since K is properly contained in the open set $(H^u)^0$, there exists $\delta > 0$ such that $K_i, K \subset H^u_\delta$, $i \in \mathbb{N}$, where

$$H^u_\delta := \{x \in H^u \mid \operatorname{dist}(x,\partial H) \geq \delta\}.$$

Since the restriction of g_u to H^u_δ is a diffeomorphism onto some ball of radius $R > 0$ around the origin in $\mathbb{R}^n_u$ and H^u_δ is compact, $g_u \colon H^u_\delta \to B_R \subset \mathbb{R}^n_u$ is a Lipschitz map with Lipschitz constant L. On the other hand we can write the Hausdorff distance of two (spherical) convex bodies as

$$\operatorname{dist}_{(s)H}(K,L) = \max\left\{\max_{x\in K}\min_{y\in L} d_{(s)}(x,y),\ \max_{x\in L}\min_{y\in K} d_{(s)}(x,y)\right\},$$

where $d_{(s)}(x,y)$ is the regular (spherical) distance of two points x, y. Thus,

$$\begin{aligned}
\operatorname{dist}_H(\phi(K),\phi(L)) &= \max\left\{\max_{x\in\phi(K)}\min_{y\in\phi(L)} d(x,y),\ \max_{x\in\phi(L)}\min_{y\in\phi(K)} d(x,y)\right\}\\
&= \max\left\{\max_{x\in K}\min_{y\in L} d(\phi(x),\phi(y)),\ \max_{x\in L}\min_{y\in K} d(\phi(x),\phi(y))\right\}\\
&\leq L\max\left\{\max_{x\in K}\min_{y\in L} d_s(x,y),\ \max_{x\in L}\min_{y\in K} d_s(x,y)\right\}\\
&= L\operatorname{dist}_{sH}(K,L),
\end{aligned}$$

which finishes the proof, since the same argument can be applied to g_u^{-1}. □

We will show now that integration of differential forms over the normal cycle indeed yields a current. To do so, by the discussion at the end of the last section, all we need is a Lipschitz-parametrization of $N(K)$ in the sense of Remark 2.3.2.

Proposition 2.3.14. Let $K \in \mathcal{K}_p(\mathbb{S}^n)$ be proper, that is, contained in an open hemisphere $(H^u)^0$ of $\mathbb{S}^n$, and $\varepsilon > 0$, such that the parallel body $K_\varepsilon := \{x \in \mathbb{S}^n \mid \operatorname{dist}_s(x,K) \leq \varepsilon\}$ still lies in $(H^u)^0$. Then there exists a bijective Lipschitz map $P_K \colon \partial K_\varepsilon \to N(K)$.

Proof. Similar to Euclidean space, there is also a nearest point projection map $p_K \colon H \to K$ on the sphere, that sends a given point x to the unique nearest point $p_K(x)$ contained in K. One can see this either by imitating the proof in the Euclidean case or using the gnomonic projection g_u of $(H^u)^0$ onto $\mathbb{R}^n_u$, that takes geodesics to geodesics.

Furthermore, let $v_K(x) \in T_{p_k(x)}\mathbb{S}^n$ be the unique unit tangent vector such that the geodesic leaving $p_K(x)$ in direction $v_K(x)$ is exactly the minimizing geodesic joining $p_K(x)$ and x. Now define

$$P_K(x) := (p_K(x), v_K(x))\,.$$

This map is a bijection from the boundary of every parallel body ∂K_ε, $\varepsilon > 0$, that is still contained in $(H^u)^0$ to $N(K)$, with inverse map given by taking the geodesic that starts at x in direction v for $(x,v) \in N(K)$ and intersecting it with ∂K_ε.

Let $H^u_\delta := \{x \in H^u \mid \mathrm{dist}_s(x,\partial H) \geq \delta\}$ be the set of all points in H^u with distance at least $\delta > 0$ from the great $(n-1)$-subsphere ∂H^u. Then under the gnomonic projection g_u, H^u_δ is mapped bijectively onto a ball around the origin in $\mathbb{R}^n_u$. Let $\widetilde{K}$ be the image of K under this map. Then the corresponding Euclidean projection $\widetilde{P}_{\widetilde{K}}$ is given by

$$\widetilde{P}_{\widetilde{K}}(x) := \left(\widetilde{p}_{\widetilde{K}}(x), \frac{x - \widetilde{p}_{\widetilde{K}}(x)}{|x - \widetilde{p}_{\widetilde{K}}(x)|}\right),$$

where $\widetilde{p}_{\widetilde{K}}$ is the corresponding Euclidean nearest point projection onto K. Since $\widetilde{p}_{\widetilde{K}}$ is Lipschitz-continuous, $\widetilde{P}_{\widetilde{K}}$ is a bi-Lipschitz map. But because the gnomonic projection is differentiable and H^u_δ is compact, it is also Lipschitz, yielding that P_K is also a bi-Lipschitz homeomorphism. □

By the last proposition, we know that $N(K)$ is an $(n-1)$-dimensional Lipschitz-submanifold of the sphere bundle $S\mathbb{S}^n \subset T\mathbb{S}^n$. Hence, for an $(n-1)$-form $\omega \in \mathcal{D}^{n-1}(S\mathbb{S}^n)$, we can define the integral

$$\int\limits_{N(K)} \omega := \int\limits_{\partial K_\varepsilon} P_K^* \omega$$

of ω over $N(K)$.

Remark 2.3.15. Sometimes we will abbreviate $\int_{N(K)} \omega$ by $N(K)(\omega)$, that is, we will identify the set $N(K)$ with the - as we will see in the next proposition - current obtained by integration over this set.

We collect some important properties of normal and conormal cycles, the first one being that they are indeed cycles:

Proposition 2.3.16. Let $K \in \mathcal{K}_p(\mathbb{S}^n)$ be a proper spherical convex body. Then its normal cycle $N(K)$ (and also its conormal cycle $N^*(K)$) as a function on $\mathcal{D}^{n-1}(S\mathbb{S}^n)$, acting on differential forms by integration, is a cycle, that is, it is a current that has zero boundary.

Proof. First, we show that $N(K)$ is a current, that is, it is continuous with respect to the topology on $\mathcal{D}^{n-1}(S\mathbb{S}^n)$. Therefore let $\omega_i, i \in \mathbb{N}$, and ω be smooth $(n-1)$-forms in the sphere bundle with $w_i \to \omega$, as $i \to \infty$. Using the above notation, we have

$$\int\limits_{N(K)} \omega_i = \int\limits_{\partial K_\varepsilon} P_K^* \omega_i \longrightarrow \int\limits_{\partial K_\varepsilon} P_K^* \omega = \int\limits_{N(K)} \omega,$$

where convergence of the integrals holds, because the coefficients of the ω_i in every chart U converge uniformly to the respective coefficient of ω.

To see, why it is a cycle, we use the fact that the exterior derivative of differential forms commutes with their pullbacks, and that by Stokes' theorem the integral of any exact form over a manifold without boundary is zero:

$$\partial N(K)(\omega) = N(K)(d\omega) = \int_{N(K)} d\omega = \int_{\partial K_\varepsilon} P_K^*(d\omega) = \int_{\partial K_\varepsilon} d(P_K^*\omega) = 0.$$

By pulling back along the map $\tau\colon S\mathbb{S}^n \to (S\mathbb{S}^n)^*$ introduced just before Remark 2.3.8, we see that the same is true for $N^*(K)$. □

Next, we will show that integration over the normal cycle is a continuous map from the set of proper spherical convex bodies with the Hausdorff topology to the space of $(n-1)$-dimensional currents, equipped with the weak topology.

Proposition 2.3.17. Let $K, K_i \in \mathcal{K}_p(\mathbb{S}^n), i \in \mathbb{N}$ be proper spherical convex bodies and $K_i \longrightarrow K$ as i tends to infinity in the Hausdorff metric. Then the normal cycles of the K_i converge weakly to the normal cycle of K, that is, for every differential $(n-1)$-form $\omega \in \mathcal{D}^{n-1}(S\mathbb{S}^n)$ we have

$$\int_{N(K_i)} \omega \longrightarrow \int_{N(K)} \omega \quad \text{as } i \to \infty.$$

The same holds true for the conormal cycles.

Proof. We take J. Fu's proof for the Euclidean case given in his lecture notes on integral geometry [Fu11, Section 2.10] and carry it over to the sphere using a suitable gnomonic projection. So for now let K_i, K be convex bodies in $\mathbb{R}^n$ with their Euclidean normal cycles $N(K_i)$, $N(K)$, $i \in \mathbb{N}$, and let $\omega \in \mathcal{D}^{n-1}(S\mathbb{R}^n)$ be an $(n-1)$-form on the sphere bundle

$$S\mathbb{R}^n := \{(x,v) \in T\mathbb{R}^n \mid \langle v,v\rangle = 1\} \subset \mathbb{R}^n \oplus \mathbb{R}^n \cong \mathbb{R}^{2n}$$

of $\mathbb{R}^n$. Furthermore choose $R > 0$ big enough such that all the K_i and K are contained in the ball B_R with radius R around the origin.

Step 1: Define the *comass* of a k-form $\eta \in \mathcal{D}^k(\mathbb{R}^m)$ to be

$$\|\eta\| := \sup_{x\in\mathbb{R}^m, |v_i|=1} |\eta_x(v_1, \ldots, v_k)|.$$

Our first goal is to prove the following estimate: If $g,h\colon \partial B_R \to \mathbb{R}^{2n}$ are C^1-maps, then

$$\left| \int_{\partial B_R} g^*\omega - h^*\omega \right| \leq \|g-h\|_\infty \left(\|Dg\|_\infty + \|Dh\|_\infty\right)^{n-1} \|d\omega\| \operatorname{vol}_{n-1}(\partial B_R).$$

We start by defining the following map $F\colon [0,1] \times \partial B_R \to \mathbb{R}^{2n}$ by

$$F(t,x) := (1-t)g(x) + th(x).$$

Using Stokes' theorem, we can rewrite the integral to obtain a first estimate:

$$\begin{aligned}\left|\int_{\partial B_R} g^*\omega - h^*\omega\right| &= \left|\int_{\underbrace{\partial([0,1]\times B_R)}_{\{0\}\times\partial B_R\cup\{1\}\times\partial B_R}} F^*\omega\right| \\ &= \left|\int_{[0,1]\times\partial B_R} d(F^*\omega)\right| = \left|\int_{[0,1]\times\partial B_R} F^*d\omega\right| \le \mathrm{vol}_{n-1}(\partial B_R)\|F^*d\omega\|.\end{aligned}$$

Moreover,

$$\begin{aligned}\|F^*d\omega\| &= \sup_{x\in\partial B_R,|v_i|=1} |F^*d\omega_x(\partial_t,v_1,\dots,v_{n-1})| \\ &= \sup_{x\in\partial B_R,|v_i|=1} |d\omega_{F(x)}(F_*\partial_t,F_*v_1,\dots,F_*v_{n-1})| \\ &\le \|d\omega\| \sup_{x\in\partial B_R,|v_i|=1} |F_*\partial_t||F_*v_1|\dots|F_*v_{n-1}| \\ &\le \|d\omega\|\|g-h\|_\infty(\|Dg\|_\infty + \|Dh\|_\infty)^{n-1},\end{aligned}$$

which completes the first step.

Step 2: Next, we want to show that for a sequence of Lipschitz maps $f_i\colon \partial B_R \to \mathbb{R}^{2n}$, $i\in\mathbb{N}$ with uniformly bounded Lipschitz constants $\mathrm{Lip}(f_i)\le L$ that converge uniformly to a Lipschitz map f_0, the integrals

$$\int_{\partial B_R} f_i^*\omega \longrightarrow \int_{\partial B_R} f^*\omega$$

also converge. To do this, for each i we choose C^1-maps h_i and $g_i\colon \partial B_R\to\mathbb{R}^{2n}$ with the following properties:

$$\begin{aligned}&\|g_i - f_i\|_\infty \to 0, \quad \|Dg_i\|_\infty \le L, \quad \int_{\partial B_R} g_i^*\omega - f_i^*\omega \longrightarrow 0,\\ &\|h_i - f_0\|_\infty \to 0, \quad \|Dh_i\|_\infty \le L, \quad \int_{\partial B_R} h_i^*\omega - f_0^*\omega \longrightarrow 0.\end{aligned}$$

This can be done by convoluting f_i and f_0 with approximate units

$$g_i := f_i * \nu_{\frac{1}{i}}, \quad h_i := f_0 * \nu_{\frac{1}{i}},$$

where $\nu_\varepsilon(x) = \varepsilon^{-(n-1)}\nu(\varepsilon^{-1}x)$ and $\nu\in C^\infty(\mathbb{R}^n)$ is a compactly supported function with $\int_{\mathbb{R}^n}\nu = 1$. Notice that this is possible because we can extend Lipschitz functions f with

$\operatorname{Lip}(f) = L$ from ∂B_R to $\mathbb{R}^n$ by setting

$$\bar{f}(y) := \inf_{x \in \partial B_R} (f(x) + Ld(x,y)),$$

while keeping the same Lipschitz constant. By the triangle inequality, we have $\|g_i - h_i\|_\infty \to 0$ and thus, by applying the inequality from Step 1,

$$\left| \int_{\partial B_R} g_i^* \omega - h_i^* \omega \right| \leq \|g_i - h_i\|_\infty (2L)^{n-1} \|d\omega\| \operatorname{vol}_{n-1}(\partial B_R) \overset{i \to \infty}{\longrightarrow} 0.$$

Therefore

$$\lim_{i \to \infty} \int_{\partial B_R} f_i^* \omega = \lim_{i \to \infty} \int_{\partial B_R} g_i^* \omega = \lim_{i \to \infty} \int_{\partial B_R} h_i^* \omega = \lim_{i \to \infty} \int_{\partial B_R} f_0^* \omega.$$

Step 3: Similar to Proposition 2.3.14, we now define maps $P_{K_i} \colon \partial B_R \to \mathbb{R}^{2n}$ and $P_K \colon \partial B_R \to \mathbb{R}^{2n}$,

$$P_{K_i}(x) := \left(p_{K_i}(x), \frac{x - p_{K_i}(x)}{|x - p_{K_i}(x)|} \right), \quad P_K(x) := \left(p_K(x), \frac{x - p_K(x)}{|x - p_K(x)|} \right),$$

where p_{K_i} and p_K are the Euclidean nearest point projections onto K_i and K. These projections p_{K_i} all share the same Lipschitz constant $\operatorname{Lip}(p_{K_i}) = 1$ and converge to p_K pointwise, since $K_i \to K$ in the Hausdorff metric. Pointwise convergence of Lipschitz functions on compact spaces implies uniform convergence, hence we have also $p_{K_i} \to p_K$ uniformly on ∂B_R. For $x \mapsto x - p_{K_i}(x)$, we have

$$|x - p_{K_i}(x) - (y - p_{K_i}(y))| \leq |x - y| + |p_{K_i}(x) - p_{K_i}(y)| \leq 2|x - y|,$$

and since $\operatorname{dist}(K_i, \partial B_R) \geq \varepsilon > 0$, also $|x - p_{K_i}(x)| \geq \varepsilon > 0$ for all $x \in \partial B_R$. Therefore the normalization

$$x \mapsto \frac{x - p_{K_i}(x)}{|x - p_{K_i}(x)|},$$

and hence the maps P_{K_i} are uniformly Lipschitz continuous with $P_{K_i} \to P_K$ uniformly on ∂B_R. Furthermore $P_{K_i}(\partial B_R) = N(K_i)$ and $P_K(\partial B_R) = N(K)$. This is true because for $(x,v) \in N(K)$ all points on the line $x + tv$, $t > 0$, including its intersection with ∂B_R, are mapped to (x,v) by P_K. Now we can just apply Step 2 to obtain the desired result in the Euclidean case:

$$N(K_i)(\omega) = \int_{\partial B_R} P_{K_i}^* \omega \longrightarrow \int_{\partial B_R} P_K^* \omega = N(K)(\omega).$$

By pulling back to the cosphere bundle, we see that $N^*(K_i)(\omega) \to N^*(K)(\omega)$ for $\omega \in \mathcal{D}^{n-1}((S\mathbb{S}^n)^*)$ also holds.

Step 4: Now we carry over the statement from Step 3 to the sphere. To this end let

K_i, K, $i \in \mathbb{N}$, be proper spherical convex bodies with $K_i \to K$ in the spherical Hausdorff topology. Moreover, let H^u, $u \in \mathbb{S}^n$, be the hemisphere in direction u, such that K and without loss of generality all the K_i are contained in the interior of H^u. Then there exists a gnomonic projection $g_u\colon (H^u)^0 \to \mathbb{R}^n_u$, such that $g_u(K_i)$ and $g_u(K)$ are convex bodies in $\mathbb{R}^n_u \cong \mathbb{R}^n$ with $g_u(K_i) \to g_u(K)$ in the Hausdorff topology by Lemma 2.3.13.

Using first diffeomorphism invariance of integrals (here the diffeomorphism is $g_{u*}\colon T\mathbb{S}^n \to T\mathbb{R}^n_u$) and then the invariance property of conormal cycles shown in Lemma 2.3.9, we obtain

$$\begin{aligned} N^*(K_i)(\omega) &= g_{u*}(N^*(K_i))((g_{u*}^{-1})^*(\omega)) = N^*(g_u(K_i))((g_{u*}^{-1})^*(\omega)) \\ &\overset{i\to\infty}{\longrightarrow} N^*(g_u(K))((g_{u*}^{-1})^*) = g_{u*}(N^*(K))((g_{u*}^{-1})^*) = N^*(K)(\omega) \end{aligned}$$

for all $\omega \in \mathcal{D}((S\mathbb{S}^n)^*)$. Again, pulling back to the sphere bundle yields

$$N(K_i)(\omega) \overset{i\to\infty}{\longrightarrow} N(K)(\omega)$$

for all $\omega \in \mathcal{D}(S\mathbb{S}^n)$ □

In the next section, we show that normal and conormal cycles satisfy the valuation property.

2.4 Smooth valuations

We are now going to introduce the important subspace of smooth valuations on spherical convex bodies.

Definition 2.4.1. Let $\mu\colon \mathcal{K}(\mathbb{S}^n) \to \mathbb{R}$ be a valuation on spherical convex bodies. If there exist an n-differential form $\eta \in \mathcal{D}^n(\mathbb{S}^n)$ on $\mathbb{S}^n$ and an $(n-1)$-differential form $\omega \in \mathcal{D}^{n-1}(S\mathbb{S}^n)$ on $S\mathbb{S}^n$ such that for all proper convex bodies $K \in \mathcal{K}_p(\mathbb{S}^n)$, μ can be written as

$$\mu(K) = \int\limits_K \eta + \int\limits_{N(K)} \omega,$$

then $\mu = \mu_{\eta,\omega}$ is called a *smooth spherical valuation*. Denote the space of all smooth valuations on spherical convex bodies by $\mathcal{V}^\infty(\mathbb{S}^n)$.

The next theorem, which we will carry over from the Euclidean setting, where it is already well known, shows that such valuations actually exist.

Theorem 2.4.2. Let $\eta \in \mathcal{D}^n(\mathbb{S}^n)$ and $\omega \in \mathcal{D}^{n-1}(S\mathbb{S}^n)$. Then there exists a valuation $\mu\colon \mathcal{K}(\mathbb{S}^n) \to \mathbb{R}$ on spherical convex bodies, such that $\mu = \mu_{\eta,\omega}$, that is,

$$\mu(K) = \int\limits_K \eta + \int\limits_{N(K)} \omega,$$

for proper spherical convex bodies $K \in \mathcal{K}_p(\mathbb{S}^n)$.

Proof. We start by defining $\mu\colon \mathcal{K}_p(\mathbb{S}^n) \to \mathbb{R}$ on proper spherical convex bodies as above,

$$\mu(K) = \int\limits_K \eta + \int\limits_{N(K)} \omega.$$

By Proposition 2.1.2, we only have to show the valuation property

$$\mu(K) + \mu(L) = \mu(K \cup L) + \mu(K \cap L)$$

for all proper $K, L, K \cup L \in \mathcal{K}_p(\mathbb{S}^n)$, to ensure it extends uniquely to all spherical convex bodies. To do so, let $K, L \in \mathcal{K}_p(\mathbb{S}^n)$ such that their union is also in $\mathcal{K}_p(\mathbb{S}^n)$, and let H^u be the hemisphere in direction $u \in \mathbb{S}^n$, such that $K \cup L \subset (H^u)^0$. Furthermore, let τ be the diffeomorphism from the sphere to the cosphere bundle

$$\begin{aligned} \tau\colon S\mathbb{S}^n &\to (S\mathbb{S}^n)^*, \\ v &\mapsto [(v, \cdot)], \end{aligned}$$

and $g_u\colon (H^u)^0 \to \mathbb{R}^n_u \cong \mathbb{R}^n$ the associated gnomonic projection. Then

$$\int\limits_K \eta + \int\limits_L \eta = \int\limits_{K \cup L} \eta + \int\limits_{K \cap L} \eta$$

and also

$$N(K)(\omega) + N(L)(\omega) = N^*(K)(\widetilde{\omega}) + N^*(LK)(\widetilde{\omega}),$$

where $\widetilde{\omega} := \tau_*(\omega) = (\tau^{-1})^*(\omega)$. Using invariance of the integrals under the diffeomorphism $g_{u*}\colon T\mathbb{S}^n \to T\mathbb{R}^n_u$, Proposition 2.3.9, and setting $\hat{\omega} := (g_{u*}^{-1})^*\widetilde{\omega}$, we obtain

$$\begin{aligned} N^*(K)(\widetilde{\omega}) + N^*(L)(\widetilde{\omega}) &= g_{u*}(N^*(K))(\hat{\omega}) + g_{u*}(N^*(L))(\hat{\omega}) \\ &= N^*(g_u(K))(\hat{\omega}) + N^*(g_u(K))(\hat{\omega}). \end{aligned}$$

In the series of papers [Ale06a], [Ale06b], [Ale08], [Ale07] by S. Alesker, in part joint with J. Fu, it has already been shown that integration of differential forms against normal and conormal cycles yields a valuation on convex bodies in $\mathbb{R}^n$ (see [Ale08, Corollary 2.1.10]), hence

$$N^*(g_u(K))(\hat{\omega}) + N^*(g_u(K))(\hat{\omega}) = N^*(g_u(K) \cup g_u(L))(\hat{\omega}) + N^*(g_u(K) \cap g_u(L))(\hat{\omega}),$$

which is moreover equal to

$$N^*(g_u(K \cup L))(\hat{\omega}) + N^*(g_u(K \cap L))(\hat{\omega}),$$

since g_u is a bijection. Doing the above steps in reverse order yields

$$\begin{aligned}N^*(g_u(K \cup L))(\hat{\omega}) + N^*(g_u(K \cap L))(\hat{\omega}) = \\ &= g_{u*}(N^*(K \cup L))(\hat{\omega}) + g_{u*}(N^*(K \cap L))(\hat{\omega}) \\ &= N^*(K \cup L)(\widetilde{\omega}) + N^*(K \cap L)(\widetilde{\omega}) \\ &= N(K \cup L)(\omega) + N(K \cap L)(\omega).\end{aligned}$$

Summing up, we obtain

$$\mu(K) + \mu(L) = \mu(K \cup L) + \mu(K \cap L),$$

which proves the claim. □

Remark 2.4.3. In Euclidean space $\mathbb{R}^n$ there is another way to introduce the subspace of smooth valuations: If we denote by $CV(\mathbb{R}^n)$ the space of continuous valuations on convex bodies together with the topology of uniform convergence on compact subsets of $\mathcal{K}(\mathbb{S}^n)$, one can show that $CV(\mathbb{R}^n)$ is a Fréchet space. Furthermore, define the space of *quasi-smooth* valuations $QV(\mathbb{R}^n)$ as all $\mu \in CV(\mathbb{R}^n)$ such that for each $K \in \mathcal{K}(\mathbb{S}^n)$ the map

$$\begin{aligned}[0,1] \times \mathbb{R}^n &\to \mathbb{R}, \\ (t,x) &\mapsto \mu(tK + x),\end{aligned}$$

is n times continuously differentiable and moreover the map

$$\begin{aligned}\mathcal{K}(\mathbb{S}^n) &\to C^n([0,1] \times \mathbb{R}^n), \\ K &\mapsto [(t,x) \mapsto \mu(tK + x)],\end{aligned}$$

is continuous. One can further show that $QV(\mathbb{R}^n)$ is also a Fréchet space, with its topology induced by the family of seminorms

$$\|\mu\|_G := \sup\{\|\mu(tK + x)\|_{C^n([0,1]\times G} \mid K \subset G\},$$

where G runs through all compact subsets of $\mathbb{R}^n$. There is a natural representation of the group $\overline{\mathrm{GL}(n)} := \mathrm{GL}(n) \ltimes \mathbb{R}^n$ of affine transformations on $QV(\mathbb{R}^n)$ that is continuous, namely

$$\begin{aligned}\rho\colon \overline{\mathrm{GL}(n)} &\to \mathrm{GL}(QV(\mathbb{R}^n)), \\ \theta &\mapsto \rho(\theta),\end{aligned}$$

where

$$\rho(\theta)(\mu)(K) := \mu(\theta^{-1}K)$$

for $\mu \in QV(\mathbb{R}^n)$ and $K \in K(\mathbb{S}^n)$. Now, the space of *smooth* valuations are all $\mu \in QV(\mathbb{R}^n)$, such that the map

$$\begin{aligned}\overline{\mathrm{GL}(n)} &\to QV(\mathbb{R}^n),\\ \theta &\mapsto \rho(\theta)(\mu),\end{aligned}$$

is infinitely differentiable. Alesker has shown that, as a consequence of his Irreducibility Theorem, in $\mathbb{R}^n$ this notion of smoothness and the one using integration over normal cycles coincide ([Ale06a, Theorem 5.2.1]). It is not known however, if a similar result is true on the sphere after replacing the group $\overline{\mathrm{GL}(n)}$ with, for example, $\mathrm{O}(n)$, acting naturally on continuous spherical valuations.

The next statement is a consequence of Proposition 2.3.17:

Proposition 2.4.4. Every smooth spherical valuation is continuous.

Proof. Let $\mu_{\eta,\omega}\colon \mathcal{K}(\mathbb{S}^n) \to \mathbb{R}$ be a smooth valuation on spherical convex bodies and $K_i \in \mathcal{K}_p(\mathbb{S}^n)$, $i \in \mathbb{N}$, a sequence of proper spherical convex bodies converging to $K \in \mathcal{K}_p(\mathbb{S}^n)$ in the Hausdorff topology. As the volume of the symmetric differences $\mathrm{vol}(K_i \Delta K)$ tends to zero as $i \to \infty$, we have

$$\int\limits_{K_i} \eta \longrightarrow \int\limits_{K} \eta.$$

Furthermore, because of the weak continuity of normal cycles (Proposition 2.3.17),

$$\int\limits_{N(K_i)} \omega \longrightarrow \int\limits_{N(K)} \omega,$$

hence $\mu(K_i) \to \mu(K)$. Now we can use Proposition 2.1.2 to obtain the same for all spherical convex bodies. ☐

Remark 2.4.5. In Theorem 2.4.2 we obtained a linear map from differential forms to smooth valuations, given by

$$\begin{aligned}\Psi\colon \mathcal{D}^n(\mathbb{S}^n) \oplus \mathcal{D}^{n-1}(S\mathbb{S}^n) &\to \mathcal{V}^\infty(\mathbb{S}^n),\\ (\eta,\omega) &\mapsto \left[K \mapsto \int\limits_K \eta + \int\limits_{N(K)} \omega\right],\end{aligned}$$

for all proper spherical convex bodies $K \in \mathcal{K}_p(\mathbb{S}^n)$. One can now ask, what the kernel of this map is. In [Ber07, Theorem 1] A. Bernig and L. Bröcker showed that $\Psi_{\eta,\omega} = 0$ precisely if,

- $D_R\omega + \pi_{\mathbb{S}^n}^*\eta = 0$ and
- $\int_{S_p\mathbb{S}^n} \omega = 0$ for all $p \in \mathbb{S}^n$,

where $\pi_{\mathbb{S}^n}\colon S\mathbb{S}^n \to \mathbb{S}^n$ is the projection to the base point of a tangent vector, $S_p\mathbb{S}^n = \pi_{\mathbb{S}^n}^{-1}(p)$, and $D_R\colon \mathcal{D}^{n-1}(S\mathbb{S}^n) \to \mathcal{D}^n(S\mathbb{S}^n)$ is the *Rumin operator*, a second order differential operator (see [Ber07, Section 1]).

2.5 Generalized valuations

In the last section of this chapter we will look at a completion of the space of smooth spherical valuations, therefore we need a topology on $\mathcal{V}^\infty(\mathbb{S}^n)$. In [Ale06b, Section 3.2], Alesker has shown that the topology on $\mathcal{D}^n(\mathbb{S}^n) \oplus \mathcal{D}^{n-1}(S\mathbb{S}^n)$ that we described is actually a Fréchet space topology, and that the kernel of the map

$$\Psi\colon \mathcal{D}^n(\mathbb{S}^n) \oplus \mathcal{D}^{n-1}(S\mathbb{S}^n) \to \mathcal{V}^\infty(\mathbb{S}^n)$$

introduced in Remark 2.4.5 is a closed subspace. Hence, the quotient topology on $(\mathcal{D}^n(\mathbb{S}^n) \oplus \mathcal{D}^{n-1}(S\mathbb{S}^n))/_{\ker\Psi}$ yields a topology on $\mathcal{V}^\infty(\mathbb{S}^n)$ under which it is also a Fréchet space. We then have

$$\Psi_{\eta_i,\omega_i} = \left[K \mapsto \int\limits_K \eta_i + \int\limits_{N(K)} \omega_i\right] \overset{i\to\infty}{\longrightarrow} \left[K \mapsto \int\limits_K \eta + \int\limits_{N(K)} \omega\right] = \Psi_{\eta,\omega},$$

if and only if $[\eta_i,\omega_i]_\sim \to [\eta,\omega]_\sim$, where equivalence is taken with respect to $\ker\Psi$.

Definition 2.5.1. The space of *generalized valuations* on $\mathbb{S}^n$, denoted by $\mathcal{V}^{-\infty}(\mathbb{S}^n)$, is the topological dual space of $\mathcal{V}^\infty(\mathbb{S}^n)$, equipped with the above Fréchet topology, that is

$$\mathcal{V}^{-\infty}(\mathbb{S}^n) := (\mathcal{V}^\infty(\mathbb{S}^n))^*.$$

Equipped with the topology of weak convergence, $\mathcal{V}^{-\infty}(\mathbb{S}^n)$ becomes a topological vector space in its own right. Since the natural projection

$$\pi\colon \mathcal{D}^n(\mathbb{S}^n) \oplus \mathcal{D}^{n-1}(S\mathbb{S}^n) \to (\mathcal{D}^n(\mathbb{S}^n) \oplus \mathcal{D}^{n-1}(S\mathbb{S}^n))_{/\ker\Psi}$$

and the embeddings

$$\iota_1\colon \mathcal{D}^n(\mathbb{S}^n) \to \mathcal{D}^n(\mathbb{S}^n) \oplus \mathcal{D}^{n-1}(S\mathbb{S}^n),$$
$$\iota_2\colon \mathcal{D}^{n-1}(S\mathbb{S}^n) \to \mathcal{D}^n(\mathbb{S}^n) \oplus \mathcal{D}^{n-1}(S\mathbb{S}^n)$$

are continuous, we obtain for each $\psi \in \mathcal{V}^{-\infty}(\mathbb{S}^n)$ a pair of currents $(E,F)_\psi$ by

$$E := \psi \circ \pi \circ \iota_1, \quad F := \psi \circ \pi \circ \iota_2,$$

with $E \in \mathcal{D}_n(\mathbb{S}^n)$ and $F \in \mathcal{D}_{n-1}(S\mathbb{S}^n)$. By the linearity of π and $\iota_{1,2}$, we get a linear map

$$\Psi^*\colon \mathcal{V}^{-\infty}(\mathbb{S}^n) \to \mathcal{D}_n(\mathbb{S}^n) \oplus \mathcal{D}_{n-1}(S\mathbb{S}^n),$$
$$\psi \mapsto (E,F)_\psi.$$

This map is injective, since $\Psi^*(\psi)=0$ implies that ψ vanishes on $\operatorname{Im}\Psi\circ\pi\circ\iota_{1,2}$ and these images generate $\mathcal{V}^\infty(\mathbb{S}^n)$. Furthermore, the image of Ψ^* are all pairs of currents that vanish on $\ker\Psi$. To summarize, we have the isomorphisms

$$\begin{aligned}
&\mathcal{V}^\infty(\mathbb{S}^n)\cong(\mathcal{D}^n(\mathbb{S}^n)\oplus\mathcal{D}^{n-1}(S\mathbb{S}^n))/_{\ker\Psi},\\
&\mathcal{V}^{-\infty}(\mathbb{S}^n)\cong(\ker\Psi)^\perp\subset\mathcal{D}_n(\mathbb{S}^n)\oplus\mathcal{D}_{n-1}(S\mathbb{S}^n).
\end{aligned}$$

CHAPTER 3

SO(n)-invariant forms and currents

Since we have obtained valuations from integration of a differential form in the last chapter, instead of looking at invariant valuations we will now focus our attention on invariant differential forms, namely $(n-1)$-forms on the sphere bundle as well as n-forms on the sphere. The sphere bundle is an odd-dimensional manifold that is naturally equipped with a contact structure to which we will give an introduction in the first section of this chapter. Because our results rely on classical invariant theory, there will be also a section devoted to finding polynomial invariants of SO(n) on real n-dimensional Euclidean spaces. After having determined the invariant differential forms, we will look at invariant currents too, since this will allow us to classify invariant generalized valuations on the sphere.

3.1 Contact geometry

Before we start with contact manifolds, we will review their even dimensional analogues, symplectic manifolds. In this section we follow A. Cannas da Silvas book 'Lectures on Symplectic Geometry' [Sil01] that covers both topics.

Definition 3.1.1. Let V be a real vector space of finite dimension and $\Omega\colon V\times V\to\mathbb{R}$ a bilinear map. If Ω is skew-symmetric and nondegenerate, it is called *symplectic* and (V,Ω) is called a *symplectic vector space*.

Proposition 3.1.2. Any symplectic vector space V has a basis $e_1,\dots,e_n,f_1,\dots,f_n$, such that $\Omega(e_i,f_i)=\delta_{ij}$ and $\Omega(e_i,e_j)=\Omega(f_i,f_j)=0$. In particular, all symplectic spaces are even-dimensional.

Proof. (Sketch) Choose any nonzero vectors $e_1,f_1\in V$, such that $\Omega(e_1,f_1)=1$. Denote by

$$V_1 := \operatorname{span}\{e_1,f_1\} \quad \text{and} \quad V_1^{\Omega} := \{v\in V \mid \Omega(v,w)=0\ \forall w\in V_1\}.$$

Then show that $V_1\cap V_1^{\Omega}=\{0\}$ and $V=V_1\oplus V_1^{\Omega}$ and go on inductively choosing $e_2,f_2\in V_1^{\Omega}$ nonzero, such that $\Omega(e_2,f_2)=1$. End up with

$$V=V_1\oplus\ldots\oplus V_n,$$

and note that $e_1,\dots,e_n,f_1,\dots,f_n$ is a basis of V that has the desired properties. □

Thus, we can consider the standard model for symplectic vector spaces $(\mathbb{R}^{2n}, \Omega_0)$ with the basis

$$
\begin{aligned}
e_1 &= (1,0,\ldots 0), \ldots, e_n = (0,\ldots, \overbrace{1}^{n}, \ldots, 0),\\
f_1 &= (0,\ldots,0,\underbrace{1}_{n+1},0,\ldots,0), \ldots, f_n = (0,\ldots,0,1),
\end{aligned}
$$

and $\Omega_0 := \sum_{i=1}^n e_i^* \wedge f_i^*$. Then, as a matrix

$$
\Omega_0 = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}.
$$

Definition 3.1.3. Let M be a manifold and $\omega \in \Omega^2(M)$ a 2-form on M. If ω is closed, that is, $d\omega = 0$, and for each $p \in M$, $\omega_p \colon T_pM \times T_pM \to \mathbb{R}$ is symplectic, then ω is called a *symplectic form* and (M, ω) is called a *symplectic manifold.*

Since $\dim T_pM = \dim M$, all symplectic manifolds must be *even-dimensional.* In fact, they locally all look like $(\mathbb{R}^{2n}, \Omega_0)$:

Theorem 3.1.4 (Darboux)**.** Let (M, ω) be a $2n$-dimensional symplectic manifold. Then for every point $p \in M$ there exist coordinates $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at p such that on U

$$
\omega = \sum_{i=1}^n dx_i \wedge dy_i.
$$

Proof. Confer [Sil01, Chapter 8]. □

We finish this short introduction to symplectic geometry with a statement concerning symplectic volume.

Proposition 3.1.5. Let M be a $2n$-dimensional manifold and ω a closed 2-form. Then ω is symplectic, if and only if the n-fold product $\omega^n = \omega \wedge \cdots \wedge \omega$ is a nowhere vanishing $2n$-form, that is a volume form on M.

Proof. By the theorem of Darboux, if ω is symplectic, then locally $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ for coordinates $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$. Therefore, ω^n is some multiple of $dx_1 \wedge \ldots \wedge dx_n \wedge dy_1 \wedge \ldots \wedge dy_n \neq 0$.

Conversely, if ω is not symplectic, there exist $p \in M$ and $v \in T_pM$ such that $\omega_p(v, w) = 0$ for all $w \in T_pM$. If we extend $\{v\}$ to a basis of T_pM, then ω_p and therefore also $(\omega_p)^n$ do not contain dv in their basis expression, hence $(\omega_p)^n = 0$. □

Definition 3.1.6. Let M be a manifold, $p \in M$, and $H_p \subset T_pM$ a tangent hyperplane at p. Then (p, H_p) is called a *contact element* on M.

Any tangent hyperplane $H_p \subset T_pM$ determines a covector $\alpha_p \in T_p^*M \setminus \{0\}$ up to multiplication by a nonzero scalar via $H_p = \ker \alpha_p$. If

$$\begin{aligned} H\colon M &\to TM, \\ p &\mapsto H_p \subset T_pM, \end{aligned}$$

is a smooth field of contact elements, then locally there exists a 1-form α, such that $H = \ker \alpha$. Such α are called *locally defining 1-forms*. Note that they are unique up to multiplication by nowhere vanishing smooth functions on M.

Definition 3.1.7. A smooth field of tangent hyperplanes $H\colon p \mapsto H_p \subset T_pM$ is called a *contact structure* on M if for any locally defining 1-form α, we have that $d\alpha|_{H_p \times H_p}$ is nondegenerate, that is, symplectic, for all $p \in M$. In this case (M,H) is called a *contact manifold* and α is called a *local contact form*.

Since $d\alpha_p$ is a symplectic form on H_p, it must be even-dimensional. Because $\dim T_pM = \dim H_p + 1$, all contact manifolds are *odd-dimensional*.

Proposition 3.1.8. Let $H\colon p \mapsto H_p$ be a smooth field of tangent hyperplanes on M. Then H is a contact structure if and only if $\alpha \wedge (d\alpha)^n \neq 0$ for every locally defining 1-form α.

Proof. Since H is smooth, for any locally defining 1-form α, there exists a smooth vector field R on M such that $\alpha_p(R_p) = 1$ for all p in the domain of α. We can write

$$T_pM = \operatorname{span}\{R_p\} \oplus H_p.$$

Now if H is a contact structure, that is, $d\alpha_p$ is symplectic on H_p, by Proposition 3.1.5 we have that $(d\alpha_p)^n$ is a volume form on H_p and therefore $\alpha_p \wedge (d\alpha_p)^n \neq 0$. In this case $d\alpha(R_p,\cdot) = 0$ on T_pM and R is called the *Reeb vector field* of α.

On the other hand, if $\alpha_p \wedge (d\alpha_p)^n \neq 0$, then choose a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of $H_p = \ker \alpha_p$ such that $T_pM = \operatorname{span}\{R_p\} \oplus \operatorname{span}\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$. Then

$$0 \neq \alpha_p \wedge (d\alpha_p)^n(R, e_1, \ldots, e_n, f_1, \ldots, f_n) = \underbrace{\alpha(R)}_{\neq 0} \cdot (d\alpha)^n(e_1, \ldots, e_n, f_1, \ldots, f_n),$$

so $(d\alpha)^n \neq 0$ on H_p, that is, it is symplectic by Proposition 3.1.5. $\square$

We will now describe the contact structure of $S\mathbb{S}^n$. For each $p \in \mathbb{S}^n$ we identify $T_p\mathbb{S}^n$ with $\mathbb{R}^n_p = \{x \in \mathbb{R}^{n+1} \mid \langle x, p\rangle = 0\}$. Then

$$S\mathbb{S}^n = \{(x, v) \in \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1} \mid \langle x, x\rangle = \langle v, v\rangle = 1, \langle x, v\rangle = 0\}.$$

If $(x, v) \in S\mathbb{S}^n$, the tangent space $T_{(x,v)}S\mathbb{S}^n$ at (x, v) can be described as

$$T_{(x,v)}S\mathbb{S}^n \cong T_x\mathbb{S}^n \oplus T_v\mathbb{S}^{n-1}_x \cong \mathbb{R}^n_x \oplus \mathbb{R}^{n-1}_{\{x,v\}},$$

where $\mathbb{S}_x^{n-1} = \{y \in \mathbb{R}_x^n \mid \langle y, y\rangle = 1\}$ and $\mathbb{R}^{n-1}_{\{x,v\}} = \{y \in \mathbb{R}^{n+1} \mid \langle y, x\rangle = \langle y, v\rangle = 0\}$. Define

$$\begin{aligned} &H\colon S\mathbb{S}^n \to TS\mathbb{S}^n, \\ &\quad (x,v) \mapsto H_{(x,v)} = \{(y,w) \in \mathbb{R}^n_x \oplus \mathbb{R}^{n-1}_{\{x,v\}} \mid \langle y, v\rangle = 0\} = \mathbb{R}^{n-1}_{\{x,v\}} \oplus \mathbb{R}^{n-1}_{\{x,v\}}. \end{aligned}$$

In this case there is a globally defining 1-form

$$\alpha_{(x,v)} = \sum_{i=1}^{n+1} v_i dx_i$$

such that $H = \ker \alpha$. To see that H indeed is a contact structure, we have to show that

$$d\alpha_{(x,v)} = \sum_{i=1}^{n+1} dv_i \wedge dx_i = -\sum_{i=1}^{n+1} dx_i \wedge dv_i$$

is symplectic on $\mathbb{R}^{n-1}_{\{x,v\}} \oplus \mathbb{R}^{n-1}_{\{x,v\}}$. But for $(y,w) \in \mathbb{R}^{n-1}_{\{x,v\}} \oplus \mathbb{R}^{n-1}_{\{x,v\}}$, also $(-w,y) \in \mathbb{R}^{n-1}_{\{x,v\}} \oplus \mathbb{R}^{n-1}_{\{x,v\}}$ and we have

$$-d\alpha_{(x,v)}((y,w),(-w,y)) = \sum_{i=1}^{n+1} dx_i \wedge dv_i((y,w),(-w,y)) = \langle y, y\rangle + \langle w, w\rangle > 0.$$

Hence, α is a contact form on $S\mathbb{S}^n$.

3.2 Invariant theory of SO(n)

In this section we use the material of [Kra96, Chapter 10] to determine the polynomial invariants of SO(n). Let V be a finite dimensional real vector space and $f\colon V \to \mathbb{R}$ a function. Then f is called *polynomial*, if it is given by a polynomial in the coordinates of a basis of V. Note that this property does not depend on the choice of basis. Denote by $\mathbb{R}[V]$ the $\mathbb{R}$-algebra of polynomial functions on V, called the *coordinate ring*.

Definition 3.2.1. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of a group G on V. A function $f \in \mathbb{R}[V]$ is called *G-invariant* or just *invariant*, if $f(g \cdot v) = f(v)$ for all $g \in G$, $v \in V$. These invariants form a subalgebra of $\mathbb{R}[V]$, called the *invariant ring* and denoted by $\mathbb{R}[V]^G$.

Now let $V := \mathbb{R}^n$ with the standard inner product denoted by $\langle \cdot, \cdot \rangle$. Let G be either O(n) or SO(n) and consider the natural representation of G on $p \in \mathbb{N}$ copies of V:

$$g \cdot v := (g \cdot v_1, \dots, g \cdot v_p)$$

for $v = (v_1, \dots, v_p) \in V^p$. Applying the inner product to the ith and jth summand of V^p

yields an $\mathrm{O}(n)$-invariant function for every pair $1 \le i, j \le p$, denoted by $\langle i, j\rangle$:

$$\langle i,j\rangle \colon V^p \to \mathbb{R},\\ (v_1,\dots,v_p) \mapsto \langle v_i, v_j\rangle.$$

Furthermore, for every $1 \le i_1 < \cdots < i_n \le p$ the determinants

$$[i_1,\dots,i_n] \colon V^p \to \mathbb{R},\\ (v_1,\dots,v_p) \mapsto \det(v_{i_1} \mid \cdots \mid v_{i_n}),$$

where $(v_{i_1} \mid \cdots \mid v_{i_n})$ is the $n \times n$ matrix with columns $v_{i_1},\dots,v_{i_n}$, are $\mathrm{SO}(n)$-invariant functions on V^p. The next theorem tells us that these two already exhaust all possibilities.

Theorem 3.2.2. First Fundamental Theorem for $\mathrm{O}(n)$ and $\mathrm{SO}(n)$:

- The invariant ring $\mathbb{R}[V^p]^{\mathrm{O}(n)}$ is generated by the invariants $\langle i,j\rangle$, $1 \le i \le j \le p$.
- The invariant ring $\mathbb{R}[V^p]^{\mathrm{SO}(n)}$ is generated by the invariants $\langle i,j\rangle$, $1 \le i \le j \le p$, together with the determinants $[i_1,\dots,i_n]$, $1 \le i_1 < \ldots < i_n \le p$.

Proof. Since the proof involves a rather large part of invariant theory, we refer to [Kra96, Chapter 10]. □

3.3 Invariant forms

We will now classify $\mathrm{SO}(n{+}1)$-invariant n-forms on $\mathbb{S}^n$ and $\mathrm{SO}(n{+}1)$-invariant $(n{-}1)$-forms on the sphere bundle $S\mathbb{S}^n$. Our results can also be found in [Fu90, Section 0.4].

The group $\mathrm{SO}(n+1)$ naturally acts on $\mathbb{S}^n$ by multiplication and since it consists of isometries it also induces an action on

$$S\mathbb{S}^n = \{(x,v) \in \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1} \mid \langle x,x\rangle = \langle v,v\rangle = 1, \langle x,v\rangle = 0\}$$

obtained by pushing forward the elements of $\mathrm{SO}(n+1)$ to $T\mathbb{S}^n$. It is also given by matrix vector multiplication $g \cdot (x,v) = (g\cdot x, g\cdot v)$ for all $g \in \mathrm{SO}(n+1)$, $(x,v) \in S\mathbb{S}^n$. Since $\mathrm{SO}(n+1)$ acts on both $\mathbb{S}^n$ and $S\mathbb{S}^n$ by diffeomorphisms, we get induced actions on the spaces of differential forms on $\mathbb{S}^n$ and $S\mathbb{S}^n$ by pulling back with these diffeomorphisms:

$$g\cdot\eta := g^*\eta, \qquad g\cdot\omega := (g_*)^*\omega$$

for all $g \in \mathrm{SO}(n+1)$, $\eta \in \mathcal{D}^*(\mathbb{S}^n)$, and $\omega \in \mathcal{D}^*(S\mathbb{S}^n)$.

Definition 3.3.1. Let G be a group acting on a manifold M by diffeomorphisms. A differential form $\omega \in \mathcal{D}^*(M)$ on M is called *G-invariant*, if $g\cdot\omega = (g^{-1})^*\omega = \omega$ for all $g \in G$. Denote by $\mathcal{D}^*(M)^G$ the space of all G-invariant forms.

Proposition 3.3.2. The space of G-invariant forms is an exterior differential algebra, that is $\omega\wedge\eta \in \mathcal{D}^*(M)^G$ and $d\omega \in \mathcal{D}^*(M)^G$, for all $\omega,\eta \in \mathcal{D}^*(M)^G$.

Proof. We have $g^*(\omega \wedge \eta) = g^*\omega \wedge g^*\eta = \omega \wedge \eta$ and $g^*(d\omega) = d(g^*\omega) = d\omega$, hence the claim follows. □

Proposition 3.3.3. The contact form $\alpha = \sum_{i=1}^{n+1} v_i dx_i$ and its exterior derivative $d\alpha = -\sum_{i=1}^{n+1} dx_i \wedge dv_i$ are $\mathrm{O}(n+1)$-invariant forms on $\mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$, where the group acts naturally on both summands. In particular, they are invariant forms on $S\mathbb{S}^n$.

Proof. Let $(x, v) \in \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$, $(y, w) \in T_{(x,v)}\mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1} \cong \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$ and $g \in \mathrm{SO}(n+1)$. Then

$$g^*\alpha_{(x,v)}(y, w) = \alpha_{(g(x),g(v))}(g^*(y), g^*(w)) = \langle g(v), g(y)\rangle = \langle v, y\rangle = \alpha_{(x,v)}(y, w),$$

that is, α is $\mathrm{O}(n+1)$-invariant. By Proposition 3.3.2 $d\alpha$ is also $\mathrm{O}(n+1)$-invariant. □

Note that $\mathrm{SO}(n+1)$ acts transitively both on $\mathbb{S}^n$ and $S\mathbb{S}^n$, since every pair $(x, v) \in \mathbb{S}^n$ can be moved to (x', v') by first choosing a rotation that brings x to x' and rotating in the plane orthogonal to x' to bring v to v'. If we look for the stabilizers of $x \in \mathbb{S}^n$ and $(x, v) \in S\mathbb{S}^n$, we see that

$$\begin{aligned} \mathrm{SO}(n+1)_x &= \{\text{rotations in the plane } \mathbb{R}^n_x\} \cong \mathrm{SO}(n), \\ \mathrm{SO}(n+1)_{(x,v)} &= \{\text{rotations in the plane } \mathbb{R}^n_{\{x,v\}}\} \cong \mathrm{SO}(n-1). \end{aligned}$$

Hence, as homogeneous spaces, we have

$$\mathbb{S}^n \cong \mathrm{SO}(n+1)/\,\mathrm{SO}(n), \quad S\mathbb{S}^n \cong \mathrm{SO}(n+1)/\,\mathrm{SO}(n-1).$$

Therefore, $\mathrm{SO}(n+1)$-invariant forms on $\mathbb{S}^n$ are obtained by pulling back $\mathrm{SO}(n)$-invariant alternating tensors on $T_o\mathbb{S}^n$ and on $S\mathbb{S}^n$ by $\mathrm{SO}(n-1)$-invariant alternating tensors on $T_{\overline{o}}S\mathbb{S}^n$, where $o = e_{n+1}$ and $\overline{o} = (o, e_n)$ are arbitrarily chosen base points of S^n and $S\mathbb{S}^n$. The induced actions of $\mathrm{SO}(n)$ and $\mathrm{SO}(n-1)$ on the tangent spaces

$$T_o\mathbb{S}^n \cong \mathbb{R}^n_{e_{n+1}}, \quad T_{\overline{o}}S\mathbb{S}^n \cong \mathbb{R}^n_{e_{n+1}} \oplus \mathbb{R}^{n-1}_{\{e_n,e_{n+1}\}}$$

are multiplication of $\mathrm{SO}(n)$ on $\mathbb{R}^n_{e_n+1} \cong \mathbb{R}^n$ and multiplication of $\mathrm{SO}(n-1)$ on the first and third summand of

$$\mathbb{R}^n_{e_{n+1}} \oplus \mathbb{R}^{n-1}_{\{e_n,e_{n+1}\}} \cong \mathbb{R}^{n-1}_{\{e_n,e_{n+1}\}} \oplus \mathrm{span}\{e_n\} \oplus \mathbb{R}^{n-1}_{\{e_n,e_{n+1}\}} \cong \mathbb{R}^{n-1} \oplus \mathbb{R} \oplus \mathbb{R}^{n-1}.$$

Since $\mathrm{SO}(n-1)$ acts trivially on the middle summand, we immediately obtain an invariant 1-tensor dx_n on $S\mathbb{S}^n$, which is precisely the contact form α at $\overline{o} \in S\mathbb{S}^n$,

$$\alpha_{\overline{o}} = \left(\sum_{i=1}^{n+1} v_i dx_i\right)_{x=e_{n+1}, v=e_n} = dx_n.$$

Our task of determining invariant forms thus reduces to finding alternating n-tensors on $\mathbb{R}^n$ invariant under the natural action of $\mathrm{SO}(n)$ and to finding alternating $(n-1)$-tensors on $\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1}$ invariant under the diagonal, that is simultaneous and natural on each summand, action of $\mathrm{SO}(n-1)$. We will start with the latter one:

Throughout this discussion upper indices will distinguish between different vectors, while lower indicies will indicate different coordinates. Any $\mathrm{SO}(n-1)$-invariant alternating m-tensor

$$\begin{aligned} A\colon &\left(\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1}\right)^m \to \mathbb{R}, \\ &((x^1, v^1), \dots, (x^m, v^m)) \mapsto A((x^1, v^1), \dots, (x^m, v^m)) \end{aligned}$$

on $\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1}$ can be written as an $\mathrm{SO}(n-1)$-invariant polynomial in the coordinates $x^i_1, \dots, x^i_{n-1}, v^i_1, \dots, v^i_{n-1}$, $1 \le i \le n-1$. By Theorem 3.2.2 (where $p = 2m$), this polynomial is a polynomial of scalar products and determinants of the vectors $x^1, \dots, x^m, v^1, \dots, v^m$. Note that since A is multilinear, powers of these products and determinants greater than one can not occur, otherwise A would not scale properly. For the same reason we can not have both x^j and v^j in the same such scalar product or determinant for any $1 \le j \le m$. This means that determinants can only yield $(n-1)$-tensors and scalar products only tensors of even rank.

First, let A be an alternating m-tensor, where m is even. We give an argument similar to [Par02, Section 2.1]. Each monomial in A up to a constant has the form

$$\langle x^{\sigma(1)}, v^{\sigma(2)} \rangle \dots \langle x^{\sigma(m-1)}, v^{\sigma(m)} \rangle,$$

where σ is any permutation of the set $\{1, \dots, m\}$. Again, because A is alternating, with every such monomial, A must also contain the term

$$\sum_{\sigma \in \mathcal{S}^m} \operatorname{sgn}(\sigma) \langle x^{\sigma(1)}, v^{\sigma(2)} \rangle \dots \langle x^{\sigma(m-1)}, v^{\sigma(m)} \rangle,$$

where σ runs through the permutation group of m elements. Since this sum must contain the term

$$\pm \langle x^1, v^2 \rangle \dots \langle x^{m-1}, v^m \rangle,$$

at the level of alternating tensors it is equal to

$$\pm \operatorname{Alt}\left(\underbrace{\left(\sum_{i=1}^{n-1} dx_i \otimes dv_i \right) \otimes \dots \otimes \left(\sum_{i=1}^{n-1} dx_i \otimes dv_i \right)}_{m \text{ times}} \right),$$

where the *alternation* of an m-tensor $T\colon V^m \to \mathbb{R}$ is given by

$$\operatorname{Alt}(T)(y^1, \dots, y^m) := \frac{1}{m!} \sum_{\sigma \in \mathcal{S}^m} T(y^{\sigma(1)}, \dots, y^{\sigma(m)})$$

for all $y^1, \dots, y^m \in V$. Using

$$\operatorname{Alt}(T \otimes S) = \frac{k!l!}{(k+l)!} \operatorname{Alt} T \wedge \operatorname{Alt} S,$$

for k-tensors T and l-tensors S, the above expression, up to a constant, writes as

$$\underbrace{\left(\sum_{i=1}^{n-1} dx_i \wedge dv_i\right) \wedge \dots \wedge \left(\sum_{i=1}^{n-1} dx_i \wedge dv_i\right)}_{m \text{ times}} = (-d\alpha)^m.$$

Now, let $A\colon (\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1})^m \to \mathbb{R}$ be an SO(n)-invariant alternating $(n-1)$-tensor. Regarded as a polynomial in the coordinates $x_1^i, \dots, x_{n-1}^i, v_1^i, \dots, v_{n-1}^i$, $1 \le i \le n-1$, by Theorem 3.2.2 and the discussion above, it must be given by a linear combination of determinants of vectors $x^1, \dots, x^{n-1}, v^1, \dots, v^{n-1}$. Each determinant must for each $1 \le j \le n-1$ either contain x^j or v^j in one of its columns, but never both. Hence, we can only choose how many of the columns x^j we wish to replace by the corresponding v^j. Taking alternations, we arrive at the following set of linear combinations:

$$\begin{aligned}
\kappa_0 &:= \det(x^1 \mid \cdots \mid x^{n-1}) \\
\kappa_1 &:= \det(v^1 \mid x^2 \mid \cdots \mid x^{n-1}) + \det(x^1 \mid v^2 \mid x^3 \mid \cdots \mid x^{n-1}) + \cdots \\
&\quad + \det(x^1 \mid \cdots \mid x^{n-2} \mid v^{n-1}) \\
\kappa_2 &:= \det(v^1 \mid v^2 \mid x^3 \mid \cdots \mid x^{n-1}) + \det(v^1 \mid x^2 \mid v^3 \mid x^4 \mid \cdots \mid x^{n-1}) + \cdots \\
&\quad + \det(x^1 \mid \cdots \mid x^{n-3} \mid v^{n-2} \mid v^{n-1}) \\
&\ \vdots \\
\kappa_{n-1} &:= \det(v^1 \mid v^2 \mid \cdots \mid v^{n-1}),
\end{aligned}$$

where κ_k is just adding up all the determinants of matrices that have k x-vectors replaced by v-vectors. We will use the following formula for evaluating wedge products and the next lemma to determine the invariant alternating tensors that correspond to the κ_k.

Proposition 3.3.4. For covectors, that is, for 1-tensors, $\omega^1, \dots, \omega^n$, and vectors $y^1, \dots, y^n$, we have

$$\omega^1 \wedge \cdots \wedge \omega^n(y^1, \dots, y^n) = \det((\omega^j(y^i))_{ij}).$$

Proof. [Lee13, Proposition 14.11 (e)]. □

Lemma 3.3.5. Let $n \in \mathbb{N}$ and $A, B \in \mathbb{R}^{n \times n}$ be two $n \times n$-matrices. For any n-tuple $j \in J := \{0,1\}^n$ define C_j to be the $n \times n$-matrix, whose ith column is either the ith column of A, if the ith entry of j is zero, that is, $j_i = 0$, or the ith column of B, if $j_i = 1$. Similarly, let C^j be the $n \times n$-matrix, whose ith row is either the ith row of A, if $j_i = 0$, or the ith row of B, if $j_i = 1$. Denote by $|j|$ the number of ones in each n-tuple $j \in J$. Then

$$\sum_{j\in J,\ |j|=k} \det(C_j) = \sum_{j\in J,\ |j|=k} \det(C^j),$$

for each $0 \le k \le n$.

Proof. By the Leibniz formula for determinants we have

$$\det(C) = \sum_{\sigma\in\mathcal{S}^n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} c_{\sigma(i),i},$$

where $\mathcal{S}^n$ is the group of permutations of $\{1,\dots,n\}$ and $\operatorname{sgn}(\sigma) = \pm 1$ is the sign of any such permutation $\sigma \in \mathcal{S}^n$. Therfore the left and right side of the claim write as

$$\sum_{j\in J,\ |j|=k} \det(C_j) = \sum_{j\in J,\ |j|=k} \sum_{\sigma\in\mathcal{S}^n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (c_j)_{\sigma(i),i},$$
$$\sum_{j\in J,\ |j|=k} \det(C^j) = \sum_{j\in J,\ |j|=k} \sum_{\sigma\in\mathcal{S}^n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (c^j)_{\sigma(i),i}.$$

Switching the order of summation we get

$$\sum_{j\in J,\ |j|=k} \sum_{\sigma\in\mathcal{S}^n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (c_j)_{\sigma(i),i} = \sum_{\sigma\in\mathcal{S}^n} \sum_{j\in J,\ |j|=k} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (c_j)_{\sigma(i),i}$$
$$\sum_{j\in J,\ |j|=k} \sum_{\sigma\in\mathcal{S}^n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (c^j)_{\sigma(i),i} = \sum_{\sigma\in\mathcal{S}^n} \sum_{j\in J,\ |j|=k} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (c^j)_{\sigma(i),i}.$$

But for every $\sigma \in \mathcal{S}^n$ we have

$$\sum_{j\in J,\ |j|=k} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (c_j)_{\sigma(i),i} = \sum_{j\in J,\ |j|=k} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (c^j)_{\sigma(i),i},$$

since on each side we sum up all possible products of $(n-k)$ elements from $\{a_{\sigma(i),i} \mid 1 \le i \le n\}$ and k elements from $\{b_{\sigma(i),i} \mid 1 \le i \le n\}$. $\square$

Using the previous Lemma we get

$$\kappa_0 = \det \begin{pmatrix} x_1^1 & \cdots & x_{n-1}^1 \\ \vdots & & \vdots \\ x_1^{n-1} & \cdots & x_{n-1}^{n-1} \end{pmatrix},$$

$$\kappa_1 = \det\begin{pmatrix} v_1^1 & \cdots & v_{n-1}^1 \\ x_1^2 & \cdots & x_{n-1}^2 \\ \vdots & & \vdots \\ x_1^{n-1} & \cdots & x_{n-1}^{n-1} \end{pmatrix} + \cdots + \det\begin{pmatrix} x_1^1 & \cdots & x_{n-1}^1 \\ \vdots & & \vdots \\ x_1^{n-2} & \cdots & x_{n-1}^{n-2} \\ v_1^{n-1} & \cdots & v_{n-1}^{n-1} \end{pmatrix},$$

$$\kappa_2 = \det\begin{pmatrix} v_1^1 & \cdots & v_{n-1}^1 \\ v_1^2 & \cdots & v_{n-1}^2 \\ x_1^3 & \cdots & x_{n-1}^3 \\ \vdots & & \vdots \\ x_1^{n-1} & \cdots & x_{n-1}^{n-1} \end{pmatrix} + \cdots + \det\begin{pmatrix} x_1^1 & \cdots & x_{n-1}^1 \\ \vdots & & \vdots \\ x_1^{n-3} & \cdots & x_{n-1}^{n-3} \\ v_1^{n-2} & \cdots & v_{n-1}^{n-2} \\ v_1^{n-1} & \cdots & v_{n-1}^{n-1} \end{pmatrix},$$

$$\kappa_{n-1} = \det\begin{pmatrix} v_1^1 & \cdots & v_{n-1}^1 \\ \vdots & & \vdots \\ v_1^{n-1} & \cdots & v_{n-1}^{n-1} \end{pmatrix}.$$

Applying Proposition 3.3.4, we obtain the SO($n-1$)-invariant tensors

$$\begin{aligned}
\kappa_0 &= dx_1 \wedge \cdots \wedge dx_{n-1} \\
\kappa_1 &= dv_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1} + dx_1 \wedge dv_2 \wedge dx_3 \wedge \cdots \wedge dx_{n-1} + \cdots \\
&\quad + dx_1 \wedge \cdots \wedge dx_{n-2} \wedge dv_{n-1} \\
\kappa_2 &= dv_1 \wedge dv_2 \wedge dx_3 \wedge \cdots \wedge dx_{n-1} + dv_1 \wedge dx_2 \wedge dv_3 \wedge dx_4 \wedge \cdots \wedge dx_{n-1} + \cdots \\
&\quad + dx_1 \wedge \cdots \wedge dx_{n-3} \wedge dv_{n-2} \wedge dv_{n-1} \\
&\ \vdots \\
\kappa_{n-1} &= dv_1 \wedge \cdots \wedge dv_{n-1}.
\end{aligned}$$

Denote by $\kappa_0, \ldots, \kappa_{n-1}$ also the SO($n+1$)-invariant $(n-1)$-forms on $S\mathbb{S}^n$ obtained by pulling back with elements of SO($n+1$), then in summary we have:

Theorem 3.3.6. The algebra of SO($n+1$)-invariant differential forms on $S\mathbb{S}^n$ is generated by $\alpha, d\alpha, \kappa_0, \ldots, \kappa_{n-1}$.

Next, we look for SO(n)-invariant alternating n-tensors on $\mathbb{R}^n$. By Theorem 3.2.2 they must be given by a polynomial of scalar products and determinants of vectors $x^1, \ldots, x^n$ in the coordinates $x_1, \ldots, x_n$. But since all scalar products are symmetric, what is left is the only determinant

$$\kappa_n := \det\begin{pmatrix} x_1^1 & \cdots & x_n^1 \\ \vdots & & \vdots \\ x_1^n & \cdots & x_n^n \end{pmatrix} = dx_1 \wedge \cdots \wedge dx_n.$$

If we denote by κ_n also the SO($n+1$)-invariant n-form on $\mathbb{S}^n$ obtained by pulling back with elements of SO($n+1$), we obtain:

Theorem 3.3.7. The space of SO($n+1$)-invariant differential forms on $\mathbb{S}^n$ is one-dimensional and spanned by κ_n.

Remark 3.3.8. If we asked for $\mathrm{O}(n+1)$-invariant differential forms instead, we would be left with the algebra generated by α and $d\alpha$. Although, the forms $\kappa_0, \dots, \kappa_n$ only change sign if pulled back with an element of $\mathrm{O}(n+1) \setminus \mathrm{SO}(n+1)$.

3.4 Invariant currents

Now that we have classified $\mathrm{SO}(n+1)$-invariant n-forms on $\mathbb{S}^n$ and $(n-1)$-forms on $S\mathbb{S}^n$, we can do the same with $\mathrm{SO}(n+1)$-invariant n-currents on $\mathbb{S}^n$ and $(n-1)$-currents on $S\mathbb{S}^n$.

Definition 3.4.1. Let G be a group acting on a manifold M by diffeomorphisms. A current $E \in \mathcal{D}_n(M)$ is called *G-invariant*, if $E(g \cdot \omega) = E(\omega)$ for all $\omega \in \mathcal{D}^n$. Denote the space of G-invariant n-currents by $\mathcal{D}_n(M)^G$.

We will show that invariant currents are already determined by their values on invariant differential forms, hence $\mathcal{D}_n(\mathbb{S}^n)$ and $\mathcal{D}_{n-1}(S\mathbb{S}^n)$ are also finite-dimensional. To do so, we need a way of averaging arbitrary forms, so that they become $\mathrm{SO}(n+1)$-invariant. This is done using the natural invariant probability measure on the compact group $\mathrm{SO}(n+1)$, the *Haar measure*. Since these integrals will be vector-valued, we give a short description of integration in Fréchet spaces and thereby follow [Rud91, Chapter 3].

Definition 3.4.2. Let $\lambda\colon Q \to \mathbb{R}$ be a measure on a measure space Q, X a topological vector space on which its dual X^* separates points, and $f\colon Q \to X$ a function, such that the scalar functions $\Lambda f\colon Q \to \mathbb{R}$, defined by

$$(\Lambda f)(q) := \Lambda(f(q)),\ q \in Q,$$

are integrable for each $\Lambda \in X^*$. If there exists a vector $y \in X$ such that

$$\Lambda y = \int_Q (\Lambda f)(q)\, d\lambda(q)$$

for all $\Lambda \in X^*$, then we define

$$y := \int_Q f(q) d\lambda(q)$$

to be the integral of f with respect to λ.

Since X^* separates points on X, there can be at most one such vector $y \in X$. To show that it actually exists, we need some further assumptions.

Theorem 3.4.3. Let X be a topological vector space on which X^* separates points and $\lambda\colon Q \to \mathbb{R}$ a Borel probability measure on a compact Hausdorff space Q. If $f\colon Q \to X$ is continuous and if the closed convex hull of $f(Q)$ is compact in X, then the integral

$$y = \int_Q f(q)\, d\lambda(q)$$

exists in the sense of the above definition.

Proof. See [Rud91, Theorem 3.27]. $\square$

Using the above notation, set $Q := \mathrm{SO}(n+1)$, λ the left invariant Haar measure on $\mathrm{SO}(n+1)$, normalized such that $\lambda(\mathrm{SO}(n+1)) = 1$, $X_1 := \mathcal{D}^{n-1}(S\mathbb{S}^n)$, $X_2 := \mathcal{D}^n(\mathbb{S}^n)$. Furthermore, we need that for any given $\omega \in \mathcal{D}^{n-1}(S\mathbb{S}^n)$ and $\eta \in \mathcal{D}^n(\mathbb{S}^n)$ the maps

$$\begin{aligned} \mathrm{SO}(n+1) &\to \mathcal{D}^{n-1}(S\mathbb{S}^n), \\ g &\mapsto g \cdot \omega, \end{aligned} \qquad \begin{aligned} \mathrm{SO}(n+1) &\to \mathcal{D}^{n}(\mathbb{S}^n), \\ g &\mapsto g \cdot \eta \end{aligned}$$

defined at the beginning of Section 3.3 are continuous. To see this, choose coordinates $(U, x_1, \dots, x_{2n-1})$ for $S\mathbb{S}^n$ and set

$$I := \{(i_1, \dots, i_{n-1}) \in \{1, \dots, 2n-1\}^{n-1} \mid i_1 < \dots < i_{n-1}\},$$

such that

$$\omega = \sum_{(i_1,\dots,i_{n-1}) \in I} f^{i_1,\dots,i_{n-1}} dx_{i_1} \wedge \dots \wedge dx_{i_{n-1}}.$$

Then we have

$$\begin{aligned} (g_*)^* \omega &= \sum_{(i_1,\dots,i_{n-1}) \in I} f^{i_1,\dots,i_{n-1}} \circ g_* d(x_{i_1} \circ g_*) \wedge \dots \wedge d(x_{i_{n-1}} \circ g_*) \\ &= \sum_{(i_1,\dots,i_{n-1}) \in I} f^{i_1,\dots,i_{n-1}} \circ g_* \left(\sum_{j=1}^{2n-1} \frac{\partial (x_{i_1} \circ g_*)}{\partial x_j} dx_j \right) \wedge \dots \\ &\qquad \wedge \left(\sum_{j=1}^{2n-1} \frac{\partial (x_{i_{n-1}} \circ g_*)}{\partial x_j} dx_j \right), \end{aligned}$$

which, as $g \longrightarrow \mathrm{Id}$, converges to ω in the topology of $\mathcal{D}^{n-1}(S\mathbb{S}^n)$, since all derivatives of g_* converge uniformly to Id, because g_* is just the restriction to $S\mathbb{S}^n$ of a linear map in $\mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$. A similar argument shows also the continuity of $g \mapsto g \cdot \eta$, $\eta \in \mathcal{D}^n(\mathbb{S}^n)$.

Since $X_{1,2}$ are Fréchet spaces, the requirements of Theorem 3.4.3 are fulfilled by [Rud91, Theorem 3.20]. Thus, we can define

$$\bar{\omega} := \int_{\mathrm{SO}(n+1)} g \cdot \omega \, d\lambda(g) \qquad \text{and} \qquad \bar{\eta} := \int_{\mathrm{SO}(n+1)} g \cdot \eta \, d\lambda(g).$$

By definition, we have for all currents $E \in \mathcal{D}_{n-1}(S\mathbb{S}^n) = (\mathcal{D}^{n-1}(S\mathbb{S}^n))^*$ and $F \in \mathcal{D}_n(\mathbb{S}^n) =$

$(\mathcal{D}^n(\mathbb{S}^n))^*$,

$$E(\bar{\omega}) = \int\limits_{\mathrm{SO}(n+1)} E(g \cdot \omega)\, d\lambda(g) \qquad \text{and} \qquad F(\bar{\eta}) = \int\limits_{\mathrm{SO}(n+1)} F(g \cdot \eta)\, d\lambda(g).$$

Hence, for invariant currents $E \in \mathcal{D}_{n-1}(S\mathbb{S}^n)^{\mathrm{SO}(n+1)}$ and $F \in \mathcal{D}_n(\mathbb{S}^n)^{\mathrm{SO}(n+1)}$, we get

$$\begin{aligned} E(\bar{\omega}) &= \int\limits_{\mathrm{SO}(n+1)} E(g \cdot \omega)\, d\lambda(g) = \int\limits_{\mathrm{SO}(n+1)} E(\omega)\, d\lambda(g) = E(\omega), \\ F(\bar{\eta}) &= \int\limits_{\mathrm{SO}(n+1)} F(g \cdot \eta)\, d\lambda(g) = \int\limits_{\mathrm{SO}(n+1)} F(\eta)\, d\lambda(g) = F(\eta). \end{aligned}$$

Furthermore, if we take E to be the current that evaluates ω at any set of tangent vectors $v_1, \dots, v_{n-1} \in T_p S\mathbb{S}^n$ at a point $p \in S\mathbb{S}^n$, we obtain

$$\bar{\omega}_p(v_1, \dots, v_{n-1}) = \int\limits_{\mathrm{SO}(n+1)} (g \cdot \omega)_p(v_1, \dots, v_{n-1})\, d\lambda(g).$$

Therefore, by the left-invariance of the measure λ,

$$\begin{aligned} (h \cdot \bar{\omega}_p)(v_1, \dots, v_{n-1}) &= \int\limits_{\mathrm{SO}(n+1)} (h \cdot (g \cdot \omega))_p(v_1, \dots, v_{n-1})\, d\lambda(g) \\ &= \int\limits_{\mathrm{SO}(n+1)} ((hg) \cdot \omega)_p(v_1, \dots, v_{n-1})\, d\lambda(g) \\ &= \int\limits_{\mathrm{SO}(n+1)} (g \cdot \omega)_p(v_1, \dots, v_{n-1})\, d\lambda(g) \\ &= \bar{\omega}_p(v_1, \dots, v_{n-1}), \end{aligned}$$

which means that $\bar{\omega}$ is $\mathrm{SO}(n+1)$-invariant. In the same way we see $\bar{\eta} \in \mathcal{D}^n(\mathbb{S}^n)^{\mathrm{SO}(n+1)}$. This yields that every $\mathrm{SO}(n+1)$-invariant current is determined by its values on $\mathrm{SO}(n+1)$-invariant forms, that is the restriction maps

$$\begin{aligned} \mathcal{D}_{n-1}(S\mathbb{S}^n)^{\mathrm{SO}(n+1)} &\to (\mathcal{D}^{n-1}(S\mathbb{S}^n)^{\mathrm{SO}(n+1)})^*, \\ \mathcal{D}_n(\mathbb{S}^n)^{\mathrm{SO}(n+1)} &\to (\mathcal{D}^n(\mathbb{S}^n)^{\mathrm{SO}(n+1)})^* \end{aligned}$$

are injective. By Theorems 3.3.6 and 3.3.7, the target spaces are finite-dimensional and

$$\begin{aligned} \mathcal{D}^{n-1}(S\mathbb{S}^n)^{\mathrm{SO}(n+1)} &= \mathrm{span}\{\kappa_1, \dots, \kappa_{n-1}, \gamma\}, \\ \mathcal{D}^n(\mathbb{S}^n)^{\mathrm{SO}(n+1)} &= \mathrm{span}\{\kappa_n\}, \end{aligned}$$

where γ is either $(d\alpha)^{\frac{n-1}{2}}$ or $\alpha \wedge (d\alpha)^{\frac{n-2}{2}}$ depending on whether n is even or odd. Define $\{K_1, \ldots, K_{n-1}, C\}$ and $\{K_n\}$ to be the dual bases of $\{\kappa_1, \ldots, \kappa_{n-1}, \gamma\}$ and $\{\kappa_n\}$ such that

$$
\begin{aligned}
(\mathcal{D}^{n-1}(S\mathbb{S}^n)^{\mathrm{SO}(n+1)})^* &= \operatorname{span}\{K_1, \ldots, K_{n-1}, C\}, \\
(\mathcal{D}^{n}(\mathbb{S}^n)^{\mathrm{SO}(n+1)})^* &= \operatorname{span}\{K_n\},
\end{aligned}
$$

and extend the K_i and C to currents on $\mathcal{D}^{n-1}(S\mathbb{S}^n)$ and $\mathcal{D}^n(\mathbb{S}^n)$ by setting $K_i(\omega) := K_i(\bar{\omega})$, $C(\omega) := C(\bar{\omega})$, and $K_n(\eta) := K_n(\bar{\eta})$ for all $\omega \in \mathcal{D}^{n-1}(S\mathbb{S}^n)$ and $\eta \in \mathcal{D}^n(\mathbb{S}^n)$. Then in summary we obtain:

Theorem 3.4.4. The spaces $\mathcal{D}_{n-1}(S\mathbb{S}^n)$ of $\mathrm{SO}(n+1)$-invariant $(n-1)$-currents on $S\mathbb{S}^n$ and $\mathcal{D}_n(\mathbb{S}^n)$ of $\mathrm{SO}(n+1)$-invariant n-currents on $\mathbb{S}^n$ are both finite-dimensional and spanned by $\{K_1, \ldots, K_{n-1}, C\}$ and $\{K_n\}$ respectively.

CHAPTER 4

Characterization of invariant smooth and generalized valuations on spherical convex bodies

In this final part we will apply the results obtained in the previous chapter to classify $\mathrm{SO}(n+1)$-invariant smooth and generalized valuations on $\mathcal{K}(\mathbb{S}^n)$. Like in the Euclidean setting, it will turn out that both of these spaces are, in fact, finite-dimensional and spanned by the spherical intrinsic volumes introduced in Section 1.2. In the last part of this chapter, we present another way of obtaining these result, using a method of transferring formulas to the sphere, that are already known in Euclidean space, called the transfer principle.

4.1 Characterization of invariant smooth valuations

We start by using the compactness of the group $\mathrm{SO}(n+1)$, which allows us to average with respect to its Haar measure, and thereby to associate to each invariant smooth valuation an *invariant* pair of smooth differential forms. In doing so, our task of classifying invariant valuations boils down to just classifying invariant differential forms, which we have already done in the last chapter.

Lemma 4.1.1. Every $\mathrm{SO}(n+1)$-invariant smooth valuation $\mu\colon \mathcal{K}(\mathbb{S}^n) \to \mathbb{R}$ can be represented by a pair of $\mathrm{SO}(n+1)$-invariant differential forms $\eta \in \mathcal{D}^n(\mathbb{S}^n)^{\mathrm{SO}(n+1)}$, $\omega \in \mathcal{D}^{n-1}(S\mathbb{S}^n)^{\mathrm{SO}(n+1)}$, such that $\mu = \Psi(\eta, \omega)$, where

$$\Psi\colon \mathcal{D}^n(\mathbb{S}^n) \oplus \mathcal{D}^{n-1}(S\mathbb{S}^n) \to \mathcal{V}^\infty(\mathbb{S}^n)$$

is the map from Remark 2.4.5.

Proof. Let $\mu = \Psi(\eta, \omega)$, $\eta \in \mathcal{D}^n(\mathbb{S}^n)$, $\omega \in \mathcal{D}^{n-1}(S\mathbb{S}^n)$, be any smooth spherical valuation. For $g \in \mathrm{SO}(n+1)$ we have

$$\begin{aligned}\mu(gK) &= \int_{gK} \eta + \int_{N(gK)} \omega = \int_{gK} \eta + \int_{g_*(N(K))} \omega \\ &= \int_K g^*\eta + \int_{N(K)} (g_*)^*\omega = \int_K g \cdot \eta + \int_{N(K)} g \cdot \omega\end{aligned}$$

for all proper spherical convex bodies $K \in \mathcal{K}_p(\mathbb{S}^n)$ by Remark 2.3.10 and invariance of integration under orientation preserving diffeomorphisms. Hence, by linearity of the integrals

$$\mu(gK) = \mu(K) \Leftrightarrow \int_K (g \cdot \eta - \eta) + \int_{N(K)} (g \cdot \omega - \omega) = 0$$

for all $K \in \mathcal{K}_p(\mathbb{S}^n)$, which means that μ is $\mathrm{SO}(n+1)$-invariant if and only if

$$(g \cdot \eta - \eta, g \cdot \omega - \omega) \in \ker \Psi$$

for all $g \in \mathrm{SO}(n+1)$. Now consider again the averaging integrals introduced in Section 3.4,

$$\bar{\omega} = \int_{\mathrm{SO}(n+1)} g \cdot \omega \, d\lambda(g) \qquad \text{and} \qquad \bar{\eta} = \int_{\mathrm{SO}(n+1)} g \cdot \eta \, d\lambda(g).$$

If $\mu = \Psi(\eta, \omega)$ is $\mathrm{SO}(n+1)$-invariant, we get

$$\begin{aligned} \omega - \bar{\omega} &= \int_{\mathrm{SO}(n+1)} g \cdot \omega - \omega \, d\lambda(g) \in \ker \Psi, \\ \eta - \bar{\eta} &= \int_{\mathrm{SO}(n+1)} g \cdot \eta - \eta \, d\lambda(g) \in \ker \Psi, \end{aligned}$$

since the integrands lie in $\ker \Psi$ and that space is a closed Fréchet space. Hence, $\mu = \Psi(\eta, \omega) = \Psi(\bar{\eta}, \bar{\omega})$ is represented by the pair $(\bar{\eta}, \bar{\omega})$, which are both $\mathrm{SO}(n+1)$ invariant differential forms. □

Next, we show that some of these invariant forms, namely all multiples of the contact form and its exterior derivative, only yield the zero valuation.

Proposition 4.1.2. If $\omega = \alpha \wedge \xi \in \mathcal{D}^{n-1}(S\mathbb{S}^n)$ or $\omega = d\alpha \wedge \xi' \in \mathcal{D}^{n-1}(S\mathbb{S}^n)$, then $\Psi(0, \omega) = 0$, that is, the ideal generated by $\alpha, d\alpha$ in $\mathcal{D}^*(S\mathbb{S}^n)$ in contained in $\ker \Psi$.

Proof. First let $K \in \mathcal{K}_p^{sm}(\mathbb{S}^n)$ be a *smooth* proper spherical convex body, that is, the boundary of K is a smooth $(n-1)$-dimensional submanifold of $\mathbb{S}^n$. In this case, at each point $x \in K$ there is a unique outer normal vector $n_K(x)$ to K. Therefore we obtain a diffeomorphism

$$\begin{aligned} \bar{n}_K \colon \partial K &\to N(K) \subset T\mathbb{S}^n, \\ x &\mapsto (x, n_K(x)). \end{aligned}$$

Now, choose coordinates $(U, x_1, \dots, x_n, v_1, \dots, v_n)$, $U \subset \mathbb{S}^n$ of $T\mathbb{S}^n$, such that $x \in \partial K \Leftrightarrow x_n = 0$ and

$$(x, v) \in N(K) \Longleftrightarrow x_n = 0, v_1 = \cdots = v_{n-1} = 0, v_n = 1,$$

for all $x \in U$. In these coordinates we have

$$\bar{n}_K^* \alpha = \bar{n}_K^* \sum_{i=1}^n v_i dx_i = dx_n = 0,$$

since $T_x \partial K = \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\right\}$. Hence,

$$\int_{N(K)} \alpha \wedge \xi = \int_{\partial K} \bar{n}_K^*(\alpha \wedge \xi) = \int_{\partial K} \bar{n}_K^* \alpha \wedge \bar{n}_K^* \xi = 0,$$

for all $\xi \in \mathcal{D}^{n-2}(S\mathbb{S}^n)$. Any general proper spherical convex body can be approximated by smooth ones in the Hausdorff metric, and since normal cycles are continuous by Proposition 2.3.17, the statement follows.

For any $\xi' \in \mathcal{D}^{n-3}(S\mathbb{S}^n)$, we have $d(\alpha \wedge \xi') = d\alpha \wedge \xi' - \alpha \wedge d\xi'$, therefore

$$\int_{N(K)} d\alpha \wedge \xi' = \int_{N(K)} d(\alpha \wedge \xi') + \int_{N(K)} \alpha \wedge d\xi' = 0,$$

where the first integral vanishes due to Proposition 2.3.16, and the second one because of what we have just shown above. □

Because of the previous proposition, normal cycles (and also conormal cycles) are called *Legendrian* cycles, that is, they annihilate all multiples of the contact form. Putting everything together, our main theorem now follows easily.

Theorem 4.1.3 (Characterization of invariant smooth valuations on $\mathbb{S}^n$)**.** The space of $\mathrm{SO}(n+1)$-invariant valuations on spherical convex bodies is finite-dimensional and spanned by

$$\mu_i := \Psi(0, \kappa_i),\ 0 \le i \le n-1, \quad \text{and} \quad \mu_n := \Psi(\kappa_n, 0),$$

where $\kappa_1, \dots, \kappa_n$ are the $\mathrm{SO}(n+1)$-invariant differential forms introduced in Section 3.3, that is,

$$\mathcal{V}^\infty(\mathbb{S}^n)^{\mathrm{SO}(n+1)} = \operatorname{span}\{\mu_0, \dots, \mu_n\}.$$

Proof. Let $\mu \in \mathcal{V}^\infty(\mathbb{S}^n)^{\mathrm{SO}(n+1)}$ be a smooth, invariant valuation on the sphere. Then by Lemma 4.1.1 there exist invariant forms $\eta \in \mathcal{D}^n(\mathbb{S}^n)^{\mathrm{SO}(n+1)}$, $\omega \in \mathcal{D}^{n-1}(S\mathbb{S}^n)^{\mathrm{SO}(n+1)}$, such that $\mu = \Psi(\eta, \omega)$. Using Theorems 3.3.6 and 3.3.7, we obtain

$$\omega = c_0\kappa_0 + \dots + c_{n-1}\kappa_{n-1} + \widetilde{\alpha} \quad \text{and} \quad \eta = c_n \kappa_n,$$

where $\widetilde{\alpha}$ is some wedge product of α and $d\alpha$. By Proposition 4.1.2 and the linearity of Ψ, we get

$$\mu = \Psi(c_n\kappa_n, c_0\kappa_0 + \dots + c_{n-1}\kappa_{n-1} + \widetilde{\alpha}) = c_0\mu_0 + \dots + c_n\mu_n.$$

□

We will now examine these invariant valuations $\mu_i \in \mathcal{V}^\infty(\mathbb{S}^n)^{\mathrm{SO}(n+1)}$ and establish a connection to the spherical intrinsic volumes V_i from Example 1.2.3. From the representations of the κ_i, $0 \leq i \leq n$, obtained in Section 3.3 at the points $o = e_{n+1} \in \mathbb{S}^n$ and $\bar{o} = (o, e_n) \in S\mathbb{S}^n$ respectively, we see that μ_n equals spherical volume σ_n, since their densities are both $\mathrm{SO}(n+1)$-invariant and equal at $o \in \mathbb{S}^n$. For the other μ_i, we will restrict our attention to two dense subsets of $\mathcal{K}_p(\mathbb{S}^n)$, namely smooth proper spherical convex bodies with positive curvature $K \in \mathcal{K}_p^{sm,+}(\mathbb{S}^n)$ and proper spherical polytopes $P \in \mathcal{P}_p(\mathbb{S}^n)$.

We start by taking $K \in \mathcal{K}_p^{sm,+}(\mathbb{S}^n)$ to be a proper spherical convex body whose boundary is a smooth $(n-1)$-dimensional submanifold of $\mathbb{S}^n$ that has positive principal curvatures $k_1, \dots, k_{n-1}$ at every point $x \in \partial K$. In that case, the normal cycle of K is precisely the image of the boundary of K under the map

$$\begin{aligned} \bar{n}_K \colon \partial K &\to N(K) \subset S\mathbb{S}^n, \\ x &\mapsto (x, n_K(x)), \end{aligned}$$

where $n_K(x)$ is the unique outer unit normal vector of K at x. Our goal is to pull back the differential forms κ_i, $0 \leq i \leq n-1$, to ∂K using $\bar{n}_K$. The push-forward of this map at a point $x \in \partial K$ is then given by

$$\begin{aligned} \bar{n}_{K*} \colon T_x\partial K &\to T_{(x,n_K(x))} S\mathbb{S}^n \cong T_x\mathbb{S}^n \oplus T_{n_K(x)}\mathbb{S}^{n-1}_x \cong T_x\mathbb{S}^n \oplus T_x\partial K, \\ y &\mapsto (y, L_x y), \end{aligned}$$

where $L_x \colon T_x\partial K \to T_x\partial K$ denotes the Weingarten map. Note that actually

$$\bar{n}_{K*} \colon T_x\partial K \to T_x\partial K \oplus T_x\partial K.$$

Now choose a basis $y_1, \dots, y_{n-1}$ of $T_x\partial K$ that diagonalizes L_x, which is possible since the Weingarten map is self-adjoint and its eigenvalues, the principal curvatures, are assumed to be positive. We then have

$$\begin{aligned} \bar{n}_K^*(\kappa_i)(y_1, \dots, y_{n-1}) = \kappa_i(\bar{n}_{K*}y_1, \dots, \bar{n}_{K*}y_{n-1}) &= \\ = \kappa_i((y_1, L_x y_1), \dots, (y_{n-1}, L_x y_{n-1})&) \\ = \kappa_i((y_1, k_1 y_1), \dots, (y_{n-1}, k_{n-1} y_{n-1})&). \end{aligned}$$

By Lemma 3.3.5, the last expression is equal to

$$\begin{aligned} &s_i(k_1, \dots, k_{n-1}) \det(y_1 \mid \cdots \mid y_{n-1}) \\ &\quad = s_i(k_1, \dots, k_{n-1}) dx_1 \wedge \cdots \wedge dx_{n-1}(y_1, \dots, y_{n-1}), \end{aligned}$$

using the i-th elementary symmetric polynomial

$$s_i(k_1, \dots, k_{n-1}) = \sum_{0 \leq j_1 < \dots < j_i \leq n-1} k_{j_1} \cdots k_{j_i}.$$

Since the point $x \in \partial K$ was chosen arbitrarily and $\bar{n}_K^*(\kappa_i)$ must also be $\mathrm{SO}(n+1)$-invariant, we get

$$\bar{n}_K^*(\kappa_i) = s_i(k_1, \ldots, k_{n-1}) d\sigma_{n-1},$$

for $0 \leq i \leq n-1$, where σ_{n-1} is $(n-1)$-dimensional spherical volume, and hence

$$\mu_i(K) = \int_{N(K)} \kappa_i = \int_{\partial K} s_i(k_1, \ldots, k_{n-1}) d\sigma_{n-1}.$$

Next, let $P \in \mathcal{P}_p(\mathbb{S}^n)$ be a proper spherical polytope. In this case [Gla96] gives an explicit formula for calculating the i-th spherical intrinsic volume of P,

$$V_i(P) = \frac{1}{\beta_i \beta_{n-i-1}} \sum_{F \in \mathcal{F}_i(P)} \sigma_i(F) \sigma_{n-i-1}(N(P,F)), \tag{4.1}$$

where $\mathcal{F}_i(P)$, $0 \leq i \leq n-1$, is the set of i-dimensional faces of P and $N(P,F)$ is the set of outer unit normal vectors to P at any point in the relative interior of F. If we view $S\mathbb{S}^n$ as a subset of $\mathbb{S}^n \times \mathbb{S}^n$ as done in Section 3.1, we get the following orthogonal decomposition:

$$N(P) = \bigcup_{i=1}^{n-1} \bigcup_{F \in \mathcal{F}_i(P)} F \times N(P,F).$$

Now let $x \in \partial P$ be any point in the relative interior of an i-dimensional face F of P. Choose an orthogonal coordinate system $(e_1, \ldots, e_{n+1})$ of $\mathbb{R}^{n+1}$, such that $x = o = e_{n+1}$, $\bar{o} = e_n \in \mathrm{Nor}(P,x)$ and such that $x_1 = e_1, \ldots, x_i = e_i$ forms a basis of $T_x F \subset \mathbb{R}^{n-1}_{o,\bar{o}}$ and that $v_{i+1} = e_{i+1}, \ldots, v_{n-1} = e_{n-1}$ forms a basis of $T_{\bar{o}} N(P,F) \subset \mathbb{R}^{n-1}_{o,\bar{o}}$. In this basis, the restriction of κ_i to $F \times N(P,F)$ at x writes as

$$(\kappa_i)|_{N \times N(P,F)} = dx_1 \wedge \cdots \wedge dx_i \wedge dv_{i+1} \wedge \cdots \wedge dv_{n-1},$$

which is the product of i-dimensional volume on F with $(n-1-i)$-dimensional volume on $N(P,F)$, whereas all the other κ_j, $j \neq i$ vanish. Since $x \in \partial P$ was again chosen arbitrarily and all κ_i, aswell as i-dimensional spherical volume, are $\mathrm{SO}(n+1)$-invariant, we obtain

$$\mu_i(P) = \int_{N(P)} \kappa_i = \sum_{F \in \mathcal{F}_i(P)} \int_{F \times N(P,F)} \kappa_i = \sum_{F \in \mathcal{F}_i(P)} \sigma_i(F) \sigma_{n-i-1}(N(P,F)),$$

and hence

$$V_i(P) = \frac{1}{\beta_i \beta_{n-i-1}} \mu_i(P).$$

By the density of $\mathcal{P}(\mathbb{S}^n)$ in $\mathcal{K}(\mathbb{S}^n)$ and the continuity of V_i and μ_i in the Hausdorff metric, it follows that $V_i(K) = \mu_i(K)/(\beta_i \beta_{n-i-1})$ for all spherical convex bodies $K \in \mathcal{K}(\mathbb{S}^n)$.

The representation in equation (4.1) also yields that

$$V_i(S_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases}$$

where S_j is any j-dimensional great subsphere. This shows that the spherical intrinsic volumes V_i, and hence also the μ_i, $0 \le i \le n$, are linearly independent.

Remark 4.1.4. Note that the spherical intrinsic volumes are also $\mathrm{O}(n+1)$-invariant. By the Remarks 2.3.10 and 3.3.8, we have

$$\mu_i(gK) = \int\limits_{N(gK)} \kappa_i = \int\limits_{g_* N(K)} \kappa_i = -\int\limits_{N(K)} (g_*)^* \kappa_i = \int\limits_{N(K)} \kappa_i = \mu_i(K)$$

for any $g \in \mathrm{O}(n+1) \setminus \mathrm{SO}(n+1)$ and $K \in \mathcal{K}_p(\mathbb{S}^n)$, since the integral changes sign under an orientation reversing diffeomorphism. Therefore

$$\mathcal{V}^\infty(\mathbb{S}^n)^{\mathrm{O}(n+1)} = \mathcal{V}^\infty(\mathbb{S}^n)^{\mathrm{SO}(n+1)}.$$

Remark 4.1.5. Picking up on Remark 2.4.3, one could also define smoothness of spherical valuations in the following way: A continuous valuation $\mu\colon \mathcal{K}(\mathbb{S}^n) \to \mathbb{R}$ is said to be $\mathrm{SO}(n+1)$-smooth, if the map

$$\begin{aligned} \mathrm{SO}(n+1) &\to \{\text{continuous valuations on } \mathbb{S}^n\}, \\ g &\mapsto [K \mapsto \mu(g^{-1}K)], \end{aligned}$$

is smooth. If this definition were equivalent to our definition involving the existence of smooth differential forms - as it is the case in Euclidean space - all continuous *invariant* spherical valuations would be smooth, because the above map would then be constant. Then Hadwiger's theorem for *continuous*, invariant valutions on spherical convex bodies would follow from Theorem 4.1.3.

4.2 Characterization of invariant generalized valuations

In the same way as the classification of invariant differential forms provided us a classification of invariant valuations, the classification of invariant currents obtained in Section 3.4, now yields a classification of invariant generalized valuations. First, the natural $\mathrm{SO}(n+1)$-action on the space of generalized invariant spherical valuations is given by

$$\begin{aligned} \mathrm{SO}(n+1) \times \mathcal{V}^{-\infty}(\mathbb{S}^n) &\to \mathcal{V}^{-\infty}(\mathbb{S}^n), \\ (g, \psi) &\mapsto g \cdot \psi = [\mu \mapsto \psi(g^{-1}\mu)], \end{aligned}$$

where $\mu \in \mathcal{V}^\infty(\mathbb{S}^n)$.

Definition 4.2.1. A generalized valuation $\psi\colon \mathcal{V}^\infty(\mathbb{S}^n) \to \mathbb{R}$ is called $\mathrm{SO}(n+1)$-invariant, if $g \cdot \psi = \psi$ for all $g \in \mathrm{SO}(n+1)$. The space of generalized $\mathrm{SO}(n+1)$-invariant valuations on the sphere is denoted by $\mathcal{V}^{-\infty}(\mathbb{S}^n)^{\mathrm{SO}(n+1)}$.

Using the isomorphism $\Psi^*\colon \mathcal{V}^{-\infty}(\mathbb{S}^n) \to (\ker \Psi)^\perp \subset \mathcal{D}_n(\mathbb{S}^n) \oplus \mathcal{D}_{n-1}(S\mathbb{S}^n)$ from Section 2.5, we see that also

$$\mathcal{V}^{-\infty}(\mathbb{S}^n)^{\mathrm{SO}(n+1)} \cong ((\ker \Psi)^\perp)^{\mathrm{SO}(n+1)} \subset \mathcal{D}_n(\mathbb{S}^n)^{\mathrm{SO}(n+1)} \oplus \mathcal{D}_{n-1}(S\mathbb{S}^n)^{\mathrm{SO}(n+1)}.$$

By Theorem 3.4.4, the last space is spanned by $\{K_1, \ldots, K_n, C\}$, where

$$K_i(\kappa_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

and $C \notin (\ker \Psi)^\perp$, hence we obtain:

Theorem 4.2.2 (Characterization of invariant generalized valuations on $\mathbb{S}^n$)**.** The space of $\mathrm{SO}(n+1)$-invariant generalized valuations on $\mathbb{S}^n$ is finite-dimensional and spanned by $\psi_i := (\Psi^*)^{-1}(K_i)$, for $0 \le i \le n$, that is,

$$\mathcal{V}^{-\infty}(\mathbb{S}^n)^{\mathrm{SO}(n+1)} = \mathrm{span}\{\psi_0, \ldots, \psi_n\}.$$

There is actually a way to view smooth valuations as a subspace of generalized ones using Alesker's product of valuations. Alesker has developed a general theory of valuations on arbitrary smooth manifolds, which also relies on integration of differential forms over conormal cycles, and hence becomes accessible in our special case of the smooth manifold $\mathbb{S}^n$. In [Ale08, Section 4], Alesker and Fu showed that there exists a bilinear product on the space of smooth valuations

$$\begin{aligned} \mathcal{V}^\infty(\mathbb{S}^n) \times \mathcal{V}^\infty(\mathbb{S}^n) &\to \mathcal{V}^\infty(\mathbb{S}^n), \\ (\mu, \nu) &\mapsto \mu \cdot \nu, \end{aligned}$$

that is continuous, commutative, and associative. Because it is defined intrinsically on any smooth manifold, we have also

$$\phi_*(\mu_1 \cdot \mu_2) = (\phi_*\mu_1) \cdot (\phi_*\mu_2)$$

for all $\mu_1, \mu_2 \in \mathcal{V}^\infty(\mathbb{S}^n)$ and diffeomorphisms $\phi\colon \mathbb{S}^n \to \mathbb{S}^n$, where $(\phi_*\mu)(K) = \mu(\phi^{-1}(K))$ for all $K \in \mathcal{K}(\mathbb{S}^n)$ and $\mu \in \mathcal{V}^\infty(\mathbb{S}^n)$. Moreover, in [Ale07, Section 6], Alesker showed that the bilinear form

$$\begin{aligned} \mathcal{V}^\infty(\mathbb{S}^n) \times \mathcal{V}^\infty(\mathbb{S}^n) &\to \mathbb{R}, \\ (\mu, \nu) &\mapsto \mu \cdot \nu(\mathbb{S}^n), \end{aligned}$$

is a perfect pairing, that is the induced map

$$\begin{aligned} p\colon \mathcal{V}^\infty(\mathbb{S}^n) &\to (\mathcal{V}^\infty(\mathbb{S}^n))^* = \mathcal{V}^{-\infty}(\mathbb{S}^n), \\ \mu &\mapsto [\nu \mapsto \mu \cdot \nu(\mathbb{S}^n)], \end{aligned}$$

is injective and has dense image in $\mathcal{V}^{-\infty}(\mathbb{S}^n)$ with respect to the weak topology. This is called Poincaré duality and in that sense, smooth valuations can be considered as generalized ones. Furthermore, since

$$\begin{aligned} p(g\mu)(\nu) &= p(\mu)(g^{-1}\nu) = \mu \cdot (g^{-1}\nu)(\mathbb{S}^n) = g^{-1}(g\mu \cdot \nu)(\mathbb{S}^n) \\ &= g\mu \cdot \nu(g\mathbb{S}^n) = g\mu \cdot \nu(\mathbb{S}^n) = p(g\mu)(\nu), \end{aligned}$$

for all $g \in \mathrm{SO}(n+1)$ and $\mu, \nu \in \mathcal{V}^\infty(\mathbb{S}^n)$, the map p is $\mathrm{SO}(n+1)$-equivariant, which means that $p(\mathcal{V}^\infty(\mathbb{S}^n)^{\mathrm{SO}(n+1)}) \subset \mathcal{V}^{-\infty}(\mathbb{S}^n)^{\mathrm{SO}(n+1)}$, that is, invariant smooth valuations are mapped to invariant generalized ones. Comparing the dimensions of these two spaces, we obtain:

Proposition 4.2.3. Let $\psi \in \mathcal{V}^{-\infty}(\mathbb{S}^n)^{\mathrm{SO}(n+1)}$. Then there exists $\mu \in \mathcal{V}^\infty(\mathbb{S}^n)^{\mathrm{SO}(n+1)}$, such that $\psi = p(\mu)$, where p is the Poincaré duality map, that is, every $\mathrm{SO}(n+1)$-invariant generalized valuation on $\mathbb{S}^n$ is smooth.

Remark 4.2.4. If one could extend the map $p\colon \mathcal{V}^\infty(\mathbb{S}^n) \to \mathcal{V}^{-\infty}(\mathbb{S}^n)$ in an $\mathrm{SO}(n+1)$-equivariant way to the space of *continuous* valuations on $\mathcal{K}(\mathbb{S}^n)$, or, equivalently, find a way to multiply continuous with smooth valuations, Theorem 4.2.2 would imply Hadwiger's theorem on the sphere.

4.3 The transfer principle

In the final part of this chapter, we will describe a different method for obtaining a classification of smooth invariant valuations on spherical convex bodies, namely the *transfer principle*. This device allows to transfer kinematic formulas from one connected isotropic Riemannian manifold - that is a pair (M, G), where M is a Riemannian manifold, and G is a group acting effectively by isometries on M, in such a way that the induced action on the tangent sphere bundle SM of M is transitive - to another. We will apply this procedure to the case of $(\mathbb{R}^n, \mathrm{E}_n)$, where E_n is the group of proper Euclidean motions, and $(\mathbb{S}^n, \mathrm{SO}(n+1))$. In this section, we follow [Fu14, Section 2.2] and [Fu11, Section 2.12, 2.13], to which we also refer for complete proofs. We start by introducing the space of curvature measures, for which kinematic formulas will be given. Note, that since we will always have $M = \mathbb{R}^n$ or $M = \mathbb{S}^n$, the notion of convex bodies in M is well defined.

Definition 4.3.1. Let M be a connected Riemannian manifold, $\eta \in \mathcal{D}^n(M)$ an n-form on M, and $\omega \in \mathcal{D}^{n-1}(SM)$ an $(n-1)$-form on the sphere bundle of M,. By setting

$$\Phi^K_{\eta,\omega}(E) := \int_{K\cap E} \eta + \int_{N(K)\cap\pi_M^{-1}(E)} \omega,$$

we obtain a family of signed Borel measures on M, indexed by convex bodies $K \in \mathcal{K}(M)$, called a *curvature measure*. The space of all curvature measures is denoted by $\mathrm{Curv}(M)$.

Proposition 4.3.2. The curvature measure Φ_ω is zero if and only if ω is a multiple of the contact form α or its exterior derivative $d\alpha$, therefore $\mathrm{Curv}(M) \cong \mathcal{D}^{n-1}(SM)/_{(\alpha,d\alpha)} \oplus \mathcal{D}^n(M)$.

Proof. See Proposition 2.2.3 of [Fu14]. □

If (M,G) is isotropic, we can consider the subgroup $G_o \subset G$ that fixes a chosen point $o \in M$, and $G_{\bar{o}} \subset G_o$, that fixes $\bar{o} \in SM$, such that $\pi_M(\bar{o}) = o$, where $\pi\colon SM \to M$ is the natural projection. In the case of $(M,G) = (\mathbb{R}^n, \mathrm{E}_n)$, we choose o to be the origin, and $\bar{o} = (o, e_n)$. Then $G_o \cong \mathrm{SO}(n)$ and $G_{\bar{o}} \cong \mathrm{SO}(n-1)$. On the other hand, if $(M,G) = (\mathbb{S}^n, \mathrm{SO}(n+1))$, let $o = e_{n+1}$ and $\bar{o} = (o, e_n)$. Again, we have $G_o \cong \mathrm{SO}(n)$ and $G_{\bar{o}} \cong \mathrm{SO}(n-1)$.

Now, denote by $\mathrm{Curv}^G(M)$ the space of G-invariant curvature measures on M. By Proposition 4.3.2, we have

$$\mathrm{Curv}^G(M) \cong \mathcal{D}^{n-1}(SM)^G/_{(\alpha,d\alpha)} \oplus \mathcal{D}^n(M)^G \cong \Lambda^{n-1}(T_{\bar{o}}SM)^{G_{\bar{o}}}/_{(\alpha,d\alpha)} \oplus \Lambda^n(T_oM)^{G_o}.$$

In the Euclidean case, we already know, that this space of invariant curvature measures has a finite basis, namely

$$\mathrm{Curv}^{\mathrm{E}_n}(\mathbb{R}^n) = \mathrm{span}\{\Phi_0, \dots, \Phi_n\}, \tag{4.2}$$

where Φ_j is the curvature measure associated to the j-th intrinsic volume, that is, the total measure $\Phi_j^K(K)$ equals $\mu_j(K)$ for all $K \in \mathcal{K}(\mathbb{R}^n)$. Also the following theorem, known as the *local kinematic formula*, holds.

Theorem 4.3.3. Let $\Phi_{\eta,\omega} \in \mathrm{Curv}^{\mathrm{E}_n}(\mathbb{R}^n)$ be an invariant curvature measure and $K, L \in \mathcal{K}(\mathbb{R}^n)$. Then there exist constants c_{ij}^{Φ}, $0 \leq i,j \leq n$, such that

$$\int_{\mathrm{E}_n} \Phi_{\eta,\omega}^{K\cap gL}(U \cap gV)\, dg = \sum_{i,j=0}^{n} c_{ij}^{\Phi_{\eta,\omega}} \Phi_i^K(U)\Phi_j^L(V)$$

for all Borel-measurable sets $U, V \subset \mathbb{R}^n$, where integration is done with respect to the Haar measure of the locally compact group E_n. The left side of this equation is also called the *kinematic integral* of $\Phi_{\eta,\omega}$ in $(\mathbb{R}^n, \mathrm{E}_n)$.

Proof. The statement follows from [Sch08, Theorem 5.3.2] (see [Fed59] for the original, more general result by Federer) and equation (4.2). □

Corollary 4.3.4. Putting $U := K$, $V := L$ in the above theorem, we obtain

$$\int_{\mathrm{E}_n} \mu(K \cap gL)\, dg = \sum_{i,j=0}^{n} c_{ij}^{\mu} \mu_i(K)\mu_j(L)$$

for all motion-invariant valuations $\mu\colon \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}$.

Another way of stating Theorem 4.3.3 is that there exists a *kinematic operator*

$$\begin{aligned} k_{\mathrm{E}_n}\colon \mathrm{Curv}^{\mathrm{E}_n}(\mathbb{R}^n) &\to \mathrm{Curv}^{\mathrm{E}_n}(\mathbb{R}^n) \otimes \mathrm{Curv}^{\mathrm{E}_n}(\mathbb{R}^n), \\ \Phi_{\eta,\omega} &\mapsto \sum_{i,j=0}^{n} c_{ij}^{\Phi_\omega} \Phi_i \otimes \Phi_j, \end{aligned}$$

such that $k_{\mathrm{E}_n}(\Phi_{\eta,\omega})^{K,L}(U,V) = \int_{\mathrm{E}_n} \Phi_{\eta,\omega}^{K\cap gL}(U\cap gV)\,dg$ for all invariant curvature measures $\Phi_{\eta,\omega} \in \mathrm{Curv}^{\mathrm{E}_n}(\mathbb{R}^n)$.

The next theorem shows the existence of such a kinematic operator on arbitrary isotropic Riemannian manifolds.

Theorem 4.3.5. Let (M,G) be a connected, isotropic Riemannian manifold. Then there exists a linear map

$$k_G\colon \mathrm{Curv}^G(M) \to \mathrm{Curv}^G(M) \otimes \mathrm{Curv}^G(M),$$

such that for any $K, L \in \mathcal{K}(M)$ and open sets $U, V \subset M$, we have

$$k_G(\Phi_\omega)^{K,L}(U,V) = \int_G \Phi_\omega^{K\cap gL}(U\cap gV)\,dg.$$

Proof. (Sketch, for a detailed proof confer [Fu14, Theorem 2.2.4] or [Fu11, Theorem 2.64]) Setting $E := \{(g,\xi,\eta,\zeta\} \in G\times SM\times SM\times SM \mid \pi_M\xi = g\pi_M\eta = g\pi_M\zeta\}$, we consider the cartesian square of fiber bundles

$$\begin{array}{ccc} E & \xrightarrow{p_2} & G\times SM \\ \big\downarrow{\scriptstyle p_1} & & \big\downarrow{\scriptstyle p_3} \\ SM\times SM & \xrightarrow{\pi_M\times\pi_M} & M\times M, \end{array}$$

where $p_1(g,\xi,\eta,\zeta) = (\xi,\eta)$, $p_2(g,\xi,\eta,\zeta) = (g,\zeta)$, and $p_3(g,\zeta) = (g\pi_M(\zeta),\pi_M(\zeta)$. One can define suitable $(G\times G)$-actions, such that this diagram becomes $(G\times G)$-equivariant. The fiber along p_1 over a point $(\xi,\eta) \in SM\times SM$ is given by

$$F_{\xi,\eta} = \{(g,\zeta) \in G\times SM \mid g\pi_M\eta = g\pi_M\zeta = \pi_M\xi\} \cong H\times \mathbb{S}^{n-1},$$

where $H \subset G$ is the subgroup fixing a point $o \in M$. Now, define a $(G\times G)$-invariant family of $(\dim H + 1)$-dimensional submanifolds in the fibers $F_{\xi,\eta}$ by

$$C_{\xi,\eta} := \overline{\{(g,\zeta) \in G\times SM \mid \zeta = ag^{-1}\xi + b\eta, a,b > 0\}} \subset F_{\xi,\eta}.$$

Note that $C_{\xi,\eta}$ is the set of pairs (g,ζ), such that $g^{-1}\xi$ and η lie in the same tangent space, and ξ lies on the geodesic arc in the sphere of that tangent space, joining these two points. Using the technique of *fiber integration*, C yields an operator

$$\pi_{C*}\colon \mathcal{D}^*(E) \to \mathcal{D}^*(SM\times SM).$$

Two convex bodies $K, L \in \mathcal{K}(M)$ are said to meet *transversely*, if $\xi \in N(K)$, $\eta \in N(L)$ with $\pi_M\xi = \pi_M\eta$ implies $\xi \neq -\eta$. One can show, that for $K, L \in \mathcal{K}(M)$, the sets K and gL meet transversely for almost every $g \in G$, and that for these g

$$N(K\cap gL) = (N(K)\cap \pi_M^{-1}gL) \cup (gN(L)\cap \pi_M^{-1}K) \cup (p_2(C(K,L))),$$

where $C(K,L) = N(K) \times N(L) \times_E C = \{(g,\xi,\eta,\zeta) \in E \mid \xi \in N(K), \eta \in N(L), (g,\zeta) \in C_{\xi,\eta}\}$. Now, for an invariant form $\omega \in \mathcal{D}^{n-1}(M)^G$, there are two ways to compute the kinematic integral: First, by integrating $dg \wedge \omega$ over $p_2(C(K,L))$, where dg is the normalized volume form on G, or secondly, by pulling back $dg \wedge \omega$ via p_2^*, subsequently using π_C^* to obtain an invariant form on $SM \times SM$, and finally integrating over $N(K) \times N(L)$, which yields the desired kinematic operator. □

By applying the above theorem to $(\mathbb{S}^n, \mathrm{SO}(n+1))$, we obtain a kinematic operator

$$k_{\mathrm{SO}(n+1)} \colon \operatorname{Curv}^{\mathrm{SO}(n+1)}(\mathbb{S}^n) \to \operatorname{Curv}^{\mathrm{SO}(n+1)}(\mathbb{S}^n) \otimes \operatorname{Curv}^{\mathrm{SO}(n+1)}(\mathbb{S}^n).$$

Now, the transfer principle, that we are going to introduce in the following, will tell us that the associated kinematic formulas of $k_{\mathrm{SO}(n+1)}$ look exactly like the ones in $(\mathbb{R}^n, \mathrm{E}_n)$.

We start by noticing, that since the subgroups of E_n and $\mathrm{SO}(n+1)$ fixing points in $\mathbb{R}^n$ and $\mathbb{S}^n$ respectively are isomorphic, there exists an isomorphism

$$\iota \colon T_o\mathbb{R}^n \to T_o\mathbb{S}^n,$$

that commutes with the actions of these common subgroups $G_o \cong \mathrm{SO}(n)$. Hence, ι induces an isomorphism of exterior algebras

$$\iota_1 \colon \Lambda^*(T_o\mathbb{R}^n)^{\mathrm{SO}(n)} \to \Lambda^*(T_o\mathbb{S}^n)^{\mathrm{SO}(n)}.$$

Moreover, we can assume that ι maps $\bar{o} \in T_o\mathbb{R}^n$ to $\bar{o} \in T_o\mathbb{S}^n$, since the subgroups $G_{\bar{o}}$ fixing these point are both isomorphic to $\mathrm{SO}(n-1)$. Using the decompositions

$$T_{\bar{o}}S\mathbb{R}^n \cong T_o\mathbb{R}^n \oplus \bar{o}^\perp \subset T_o\mathbb{R}^n \oplus T_o\mathbb{R}^n,$$
$$T_{\bar{o}}S\mathbb{S}^n \cong T_o\mathbb{S}^n \oplus \bar{o}^\perp \subset T_o\mathbb{S}^n \oplus T_o\mathbb{S}^n,$$

we see that we obtain also a $G_{\bar{o}}$-equivariant isomorphism $T_{\bar{o}}S\mathbb{R}^n \to T_{\bar{o}}S\mathbb{S}^n$, and hence an isomorphism

$$\iota_2 \colon \Lambda^*(T_{\bar{o}}S\mathbb{R}^n)^{\mathrm{SO}(n-1)} \to \Lambda^*(T_{\bar{o}}S\mathbb{S}^n)^{\mathrm{SO}(n-1)}.$$

Combining ι_1 and ι_2 yields an isomorphism

$$\bar{\iota} \colon \mathcal{D}^n(\mathbb{R}^n)^{\mathrm{E}_n} \oplus \mathcal{D}^{n-1}(S\mathbb{R}^n)^{\mathrm{E}_n} \to \mathcal{D}^n(\mathbb{S}^n)^{\mathrm{SO}(n+1)} \oplus \mathcal{D}^{n-1}(S\mathbb{S}^n)^{\mathrm{SO}(n+1)}.$$

To see that $\bar{\iota}$ induces an isomorphism of the respective spaces of curvature measures, by Proposition 4.3.2 we must show that $\bar{\iota}\alpha = \alpha$ and $\bar{\iota}(d\alpha) = d\alpha$, where α are the contact forms of $\mathbb{R}^n$ and $\mathbb{S}^n$, respectively. The first statement is obvious, since in view of the above decompositions α is given by the scalar product with $\bar{o}$ on the first summand, and is equal to zero on the latter. The second statement can be seen by choosing orthonormal coordinates $(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1})$ for $\bar{o}^\perp \oplus \bar{o}^\perp \subset T_o\mathbb{R}^n \oplus T_o\mathbb{R}^n$ and $\bar{o}^\perp \oplus \bar{o}^\perp \subset T_o\mathbb{S}^n \oplus T_o\mathbb{S}^n$, corresponding under ι, and such that in both cases $d\alpha = \sum_{i=1}^{n-1} dx_i \wedge dy_i$. Hence, we obtain an isomorphism

$$\widetilde{\iota}\colon \operatorname{Curv}^{\mathrm{E}_n}(\mathbb{R}^n) \to \operatorname{Curv}^{\mathrm{SO}(n+1)}(\mathbb{S}^n).$$

The tranfer principle now states that this isomorphism intertwines the respective kinematic operators.

Theorem 4.3.6. The diagram

$$\begin{tikzcd}
\operatorname{Curv}^{\mathrm{E}_n}(\mathbb{R}^n) \arrow[rr, "\widetilde{\iota}"] \arrow[d, "k_{\mathrm{E}_n}"] & & \operatorname{Curv}^{\mathrm{SO}(n+1)}(\mathbb{S}^n) \arrow[d, "k_{\mathrm{SO}(n+1)}"] \\
\operatorname{Curv}^{\mathrm{E}_n}(\mathbb{R}^n) \otimes \operatorname{Curv}^{\mathrm{E}_n}(\mathbb{R}^n) \arrow[rr, "\widetilde{\iota}\otimes\widetilde{\iota}"] & & \operatorname{Curv}^{\mathrm{SO}(n+1)}(\mathbb{S}^n) \otimes \operatorname{Curv}^{\mathrm{SO}(n+1)}(\mathbb{S}^n)
\end{tikzcd}$$

is commutative.

Proof. (Sketch, for a detailed proof confer [Fu14, p. 2.2.5] or [Fu11, Theorem 2.68]) In the proof of Theorem 4.3.5, we obtained a map

$$H\colon \mathcal{D}^*(SM)^G \to \mathcal{D}^*(SM)^G \otimes \mathcal{D}^*(SM)^G,$$

which actually can be considered as a map

$$\widetilde{H}\colon \Lambda^*(T_{\bar{o}}SM)^{G_{\bar{o}}} \to \Lambda^*(T_{\bar{o}}SM)^{G_{\bar{o}}} \otimes \Lambda^*(T_{\bar{o}}SM)^{G_{\bar{o}}}.$$

We need to show, that this map intertwines the isomorphisms induces by $\iota\colon T_o\mathbb{R}^n \to T_o\mathbb{S}^n$. Therefore, we look at the derivative of the cartesian square fiber bundles from above,

$$\begin{tikzcd}
TE|_F \arrow[r, "p_{2*}"] \arrow[d, "p_{1*}"] & T(G\times SM)|_F \cong TG|_H \times TSM|_{S_oM} \arrow[d, "p_{3*}"] \\
T_{\bar{o}}SM \times T_{\bar{o}}SM \arrow[r, "\pi_{M*}\times\pi_{M*}"] & T_oM \times T_oM.
\end{tikzcd}$$

Here, $F \cong G_o \times \mathbb{S}^{n-1}$ is the fiber over $(\bar{o},\bar{o})$. From this diagram, we obtain the map $\widetilde{H}$ in the following way: First, any given $\omega \in \Lambda^*(T_{\bar{o}}SM)^{G_{\bar{o}}}$ yields a G_o-invariant section $\widetilde{\omega}$ of $\Lambda^*TSM|_{S_oM}$. Again, taking the wedge product $dg \wedge \widetilde{\omega}$, pulling back via p_{2*}, and using fiber integration over $C \subset F$, we obtain an element of $\Lambda^*(T_{\bar{o}}SM)^{G_{\bar{o}}} \otimes \Lambda^*(T_{\bar{o}}SM)^{G_{\bar{o}}}$.

Now, it can be shown that all corners of this diagram of derivatives can be identified - where (M,G) is either $(\mathbb{R}^n,\mathrm{E}_n)$ or $(\mathbb{S}^n,\mathrm{SO}(n+1))$ - in such a way, that all identifications intertwine the steps involved in computing the image of ω under the map $\widetilde{H}$. Thus, the respective kinematic operators can also be identified. □

By [Fu14, Section 2.1], the differential forms belonging to the Euclidean intrinsic volumes are precisely the $\bar{\iota}$-equivalents to the κ_i from Theorem 3.3.6, hence $\widetilde{\iota}(\Phi_i) = \Phi_{\kappa_i}$ for all $0 \le i \le n$. Using the transfer principle, the kinematic operator on $(\mathbb{S}^n,\mathrm{SO}(n+1))$ is given by

$$k_{\mathrm{SO}(n+1)}(\Phi_{\eta,\omega}) = \sum_{i,j=0}^{n} c_{ij}^{\Phi_{\bar{\iota}^{-1}(\eta,\omega)}} \Phi_{\kappa_i} \otimes \Phi_{\kappa_j},$$

and thus, the kinematic formula writes

$$\int_{\mathrm{SO}(n+1)} \Phi_{\eta,\omega}^{K\cap gL}(U \cap gV)\, dg = \sum_{i,j=0}^{n} c_{ij}^{\Phi_{\bar{\iota}^{-1}(\eta,\omega)}} \Phi_{\kappa_i}^{K}(U)\Phi_{\kappa_j}^{L}(V),$$

for all $K, L \in \mathcal{K}(\mathbb{S}^n)$ and open subsets $U, V \subset \mathbb{S}^n$. Now, setting $U := K$ and $L := V := \mathbb{S}^n$, we obtain

$$\mu_{\eta,\omega}(K) = \int_{\mathrm{SO}(n+1)} \mu_{\eta,\omega}(K \cap g\mathbb{S}^n)\, dg = \sum_{i,j=0}^{n} c_{ij}^{\Phi_{\bar{\iota}^{-1}(\eta,\omega)}} \mu_i(K)\mu_j(\mathbb{S}^n) = \sum_{i=0}^{n} \widetilde{c}_i \mu_i(K),$$

where $\mu_i\colon \mathcal{K}(\mathbb{S}^n) \to \mathbb{R}^n$ are (multiples of) the spherical intrinsic volumes. This is precisely the statement of Theorem 4.1.3.

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