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Models of Directed Technical Change

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Models of Directed Technical Change

Diploma Thesis

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Statutory declaration

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I declare that I have authored this thesis independently, that I have not used other than the declared sources / resources and that I have explicitly marked all material which has been quoted either literally or by content from the used sources.

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Abstract

In macroeconomic growth models technology is often exogenous or endogenously described with simple dynamics. The diploma thesis analyses the dynamics, but also the direction and the bias of technical change. The basic framework introduced by Acemoglu (2002) is a two-sector model. Each sector uses different technologies. If the economy is switching from the technology in one sector to the other, we face directed technical change. The reasons and the policies to direct technical change are discussed for the basic model of Acemoglu (2002). Further extensions that also include the dynamics of environment and population are introduced afterwards. The main focus is the quality of environment. What happens, if one sector needs natural resources to produce its goods? Which policies can be set to attract researchers to invest in sustainable technologies? The paper of Acemoglu, Aghion, Burstyn and Hemous (2012) describes scenarios with renewable and exhoustible natural ressources. This paper also introduces different aspects of technical change and environment: Is economic growth still possible, if the production of the consumption good always affects the environment negatively, even if there exists a second production sector that decreases pollution of the first sector and increases the regeneration rate of the environment?
A more complex model of directed technical change is based on the work of Schaefer (2012). Additional to the factor environment it includes population dynamics and works with two different types of households - skilled and unskilled. As a result, the educational choice of parents for their offspring plays a crucial role to direct technical change.
The diploma thesis describes and compares these four different models of directed technical change.

Contents

1 Introduction

There are no great limits to growth because there are no limits of human intelligence, imagination, and wonder.
Ronald Reagan (1980)

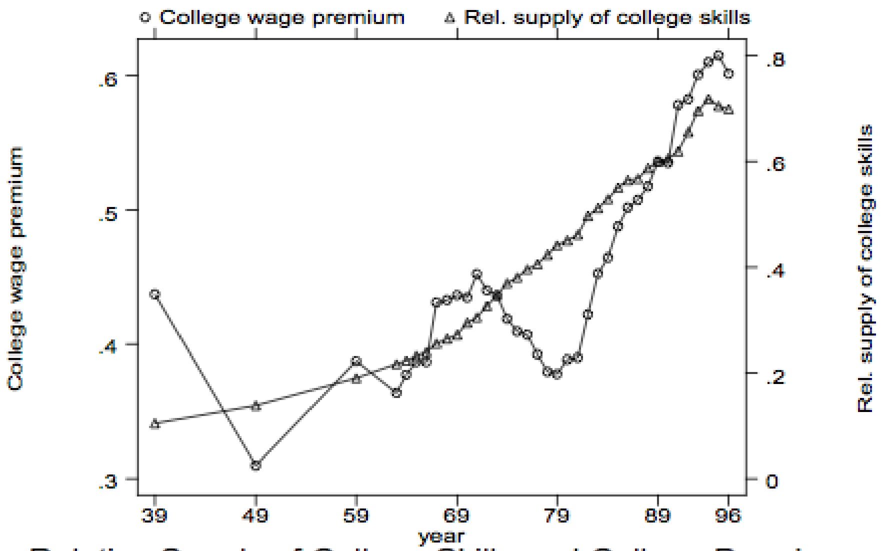
Economies can be growing. Economic models build a framework to analyse and understand the driving forces of economic growth. In neoclassical growth models an increase in output depends on capital accumulation, population growth and external technological progress. The basic Solow growth model in Solow (1956) for example introduced an exogenous technological change to explain the long term growth of the economy.

The growth rate has to be endogenous and not exogenously given in order to understand the process of economic growth. Investments of firms and households in research and development cause endogenous technical change and therefore economic growth.
We differentiate two types of technical change: product and process innovation. If a new product is invented, e.g. a new smart phone, consumers are willing to pay more for this new product. Whereas process innovation leads to higher production efficiency. Either the production costs decrease, or more units of the final good can be produced within the same time or the quality of the final good increases. In this thesis the focus will be on process innovations.
Furthermore, we can distinguish between micro and macro innovations. Micro innovations improve the quality of one product like e.g. better lenses for a special digital camera. More radical innovations for a whole industry are macro innovations. The invention of the internet or the web 2.0 changed the whole production and consumption behaviour. In the thesis the focus is given to macro innovations.

To model technological change, a production function for technology is introduced in the literature. Intuitively, the inputs in the production function are the number of scientists who do research for a specific time, the existing knowledge and investments in terms of the final good. Then a new technology is developed with a certain probability, because we can not guarantee that research is always successful. With those production functions, also called technology possibilities frontier, the choices of firms and households for investments in research and developement (R&D), research spendings, labour supply and subsidies can be derived.
Moreover, existing knowledge plays a crucial role. There are two approaches. First, it is harder to invent new innovations if the technology level is already high. So we face a negative state dependence. Second, more common are knowledge spillovers. Researchers stand "on the shoulders of giants" and are more successful using better existing technology.

Who owns the right to use new technologies? Older models assumed new innovations as public good. However, it makes sense to introduce a patent policy. Successful researchers receive patent rights for their new innovation. After inventing the new technology, the products they produce are better than the others. So the new technology replaces the old technologies and the researcher becomes a technology monopolist. The profit maximisation of the monopolist influences the speed of new innovations since investments in R&D are more valuable.

So far, we have only discussed technological change as one single type of technical change. This *neutral* technical change improves the whole technology level, but is not always appropriate. Furthermorde, we have to analyse the direction and the bias of the technical change. According to Acemoglu, (2002), in the period after the second world war people's education enhanced and with better working skills the wage increased as well. There was a simple correlation between the skills of the worker and their wage, so the technical change was skill-biased. In the 1970s and 1980s the correlation stopped. Every cohort still got better education, but the wage premium even decreased. Labour supply increased since more graduates entered the labour market. As a consequence, the technology level increased as well. This was driven by a labour-biased technical change. Obviously, in the early 1970s the driving forces for a *directed technical change* changed.



Relative Supply of College Skills and College Premium

Source: Acemoglu, (2002)

The thesis describes the model of directed technical change by Acemoglu, (2002) and specific extensions of the model. One extension is to describe, how technical change takes place if one sector is environmental-friendly. The framework is based on Acemoglu, Aghion, Burstyn and Hemous (2012).
Production in the majority of cases affects the environment. Either the firms use natural resources as input for the final good or production processes produce pollution therby harming the environment.

As described in Xepapadeas, (2005) different economic growth models include environ-

ment. In the simple AK-model with the production function $y = Ak$ the long term economic growth rate is equal to the growth rate of consumption, capital and output, if pollution is not considered. To introduce pollution, we assume pollution accumulation if production takes place and pollution as part of the utility function of the social planner. As soon as pollution is taken into account, it is impossible to obtain a positive growth rate in the long term.
To enable a positive long term growth rate, we introduce abatement. Capital can either be productive capital for output production or abatement capital for pollution abatement. For some parameter specifications unlimited growth without pollution accumulation is possible.
In addition to the two types of capital it is possible to also invest in both types. So we simply talk about two sectors. Each sector has its own labour supply, its own production technology and the sectors produce different goods.
If a social planner maximises the aggregate utility function including consumption and pollution, permanent growth is not optimal. To get rid of externalities the social planner has to introduce policy designs: subsidies for investment in the abatement-sector, subsidies to help firms in abatement and emission taxes for producing pollution. These policies help to increase economic growth in the long term.

Another approach is to split the production into two sectors, where one sector -the *green sector* - is producing in a less polluting way and is using renewable ressources more efficiently than the second sector, the *dirty sector*. Instead of abatement both sectors produce (parts of) the final good. These two sectors enable directed technical change, which is analysed in detail in this thesis. We analyse the long run market equilibrium and the social optimum to find out, when long term growth and sustainable production is possible.

As mentioned in the beginning, economic growth can also be explained with population growth. In the thesis we follow the extension of the two sector model with environment based on Schaefer, (2012) to analyse the interaction with population and educational choices in an overlapping generations model.

To give an overview, the diploma thesis first explains the basic two sector model following the framework of Acemoglu (2002). Then we add environment to our model and denote the sectors as green and dirty sectors, refering to Acemoglu, Aghion, Burstyn and Hemous (2012). We pursue a modification of this model to analyse abatement instead of a green production sector, based on an extension in Acemoglu, Aghion, Burstyn and Hemous (2012). In addition to the environmental aspect, Schaefer, (2012) introduces population growth in the two sector model. Therefore an overlapping generations model helps to model educational choices of parents and old-age provisions. In this very complex model long run growth depends on the fertility rate and environmental issues.

2 Basic Model

In this chapter we present an overview of the basic model of directed technical change, introduced by Acemoglu (2002).

2.1 Background

The basic model of directed technical change is an endogenous growth model. The model is aimed for not only analysing a simple change in technology to explain why and how much the economy is growing, but also to study the direction and the bias of technical change.

Therefore we assume an economy with two different production factors. These factors can be e.g. skilled and unskilled labour. There are also two different types of technologies complementing these factors.
You can think of high skilled workers, who need office buildings, good communication tools and a high knowledge support to provide their goods and services, whereas factories with low skilled workers use different machines, e.g. assembly lines, to provide their products.

This leads us to a two sector model shown in Fig.2.1. We denote the variables in the low skilled sector with index L and in the high skilled sector with index H. In the low skilled sector, scientists S_L do research to improve the production technology. This impacts the the machines, firms use in the low skilled sector. Furthermore, low skilled workers L work on these machines to produce the intermediate good Y_L. The high skilled sector is based on a similar procedure. Scientists S_H and high skilled labour H are working in the H-sector to produce the intermediate good Y_H.
Basically, both sectors produce intermediate goods, which can be combined to a final good. The output of the economy is measured in terms of the final good. Households consume the final goods and provide labour supply (workers, scientists) for each of the sectors. Workers choose the sector in which they are producing the intermediate goods, so they become either L or H. Scientists can decide whether they want to do their research the L- or the H-sector. This decision depends on the relative profitability they expect in both sectors. Since firms are profit maximising, they focus on the more profitable technology.
A detailled graph is found in the appendix Fig.E.1.

The decision where scientists offer their labour is a key determinant that explains the direction and the bias of the technical change. So we analyse the relative profitability of

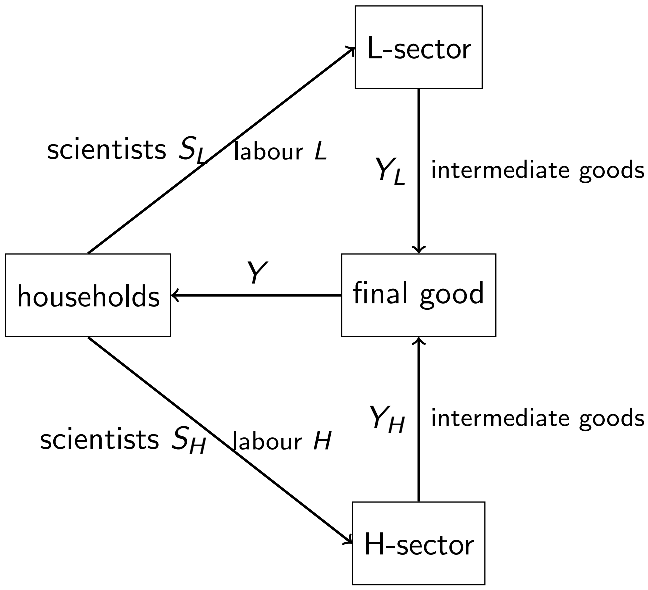


Figure 2.1: Basic structure of the model

the different types of technology. We will see, that a price effect and a market size effect are the driving forces of the direction of the technical change.

2.2 Definitions

To simplify notification I will not write the time argument t to every variable.
We assume a closed economy with consumers, firms and scientists. The production takes place in two sectors which are denotet by L and H. In each sector scientists L_S and H_S invent technologies N_L and N_H, respectively, which are sold by the technology monopolists to the firms to produce intermediate goods Y_L or Y_H. The final good Y consists of both intermediate goods and is used for consumption C, Investment I and R&D expenditure D. This leads to the following ressource constraint

$$C + I + D \leq Y. \tag{2.1}$$

The representative consumer has constant relative risk aversion (CRRA) preferences

$$\int_0^\infty \frac{C(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt. \tag{2.2}$$

The relativ risk aversion θ (the intertemporal elasticity of substitution equals $\frac{1}{\theta}$) does not change over time. ρ is the rate of time preference.

The final good Y is produced from the intermediate goods Y_L and Y_H. Y_L is the labour intensive good and it is produced by low (or un-)skilled workers L, whereas Y_H uses high-skilled workers H (can be also interpreted as capital or land).

$$Y = \left[\gamma Y_L^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma) Y_H^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} \tag{2.3}$$

The production technology is CES (constant elasticity of substition) implying that the intermediate goods do not influence each other. $\epsilon \in (0, \infty)$ is the elasticity of substitution between Y_L and Y_H. If $\epsilon > 1$, the goods are gross substitutes, so the final good can use Y_H instead of Y_L. On the other hand, if $\epsilon < 0$, the goods are gross complements. If there is less Y_L available, less Y can be produced. $\gamma \in (0, 1)$ shows the importance of Y_L compared to Y_H. Note that there is no direct impact of technology in the production function of the final good.

The firms are producing the intermediate goods with the following constant returns to scale (CRS) production functions.

$$Y_L = \frac{1}{1-\beta}\left(\int_0^{N_L} x_L(j)^{1-\beta} dj\right) L^\beta \tag{2.4}$$

$$Y_H = \frac{1}{1-\beta}\left(\int_0^{N_H} x_H(j)^{1-\beta} dj\right) H^\beta \tag{2.5}$$

The factor L and H is complemented by machines x_L and x_H, which are available in a range of N_L and N_H, respectively. For the first analysis N_L and N_H are exogenous. $\beta \in (0, 1)$ weights the importance of labour supply compared to machines.

Substituting the production functions of the intermediate goods Y_L (2.4) and Y_H (2.5) in the production function of the final good Y (2.3) leads to the following aggregate production function.

$$Y = \left[\gamma\left[\frac{1}{1-\beta}\left(\int_0^{N_L} x_L(j)^{1-\beta} dj\right) L^\beta\right]^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma)\left[\frac{1}{1-\beta}\left(\int_0^{N_H} x_H(j)^{1-\beta} dj\right) H^\beta\right]^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} \tag{2.6}$$

Next, two ways of technical change are introduced. For an easier explanation I consider the general aggregate production function $F(A, L, H)$ with a positive dependency of the technology index A: $\frac{\partial F}{\partial A} > 0$. First, we define technical change as *L-augmenting* or *L-complementary*, if we can write the production function as $F(AL, H)$. Analogous for H. Second, we look at the concept of *factor-biased* technological change. If the relative marginal product (relative price) $\frac{\partial F/\partial L}{\partial F/\partial H}$ is increasing in A, the technological change is L-biased. In other words, the relative demand curve for a factor is shifting outwards.
It is easy to see, if the technological change is factor-augmenting, it depends on the elasticity of substitution ϵ whether it is L-biased or H-biased. When the intermediate goods are substitutes, H-augmenting technical change is also H-biased. In contrast, when the goods are complements, H-complementary technical change is L-biased.

2.3 Analysis

To find an equilibrium, we have to set the wages w_H and w_L for the workers H and L, respectively, and the prices p_H and p_L for the intermediate goods Y_H and Y_L to clear the markets. Furthermore, the producers of the intermediate goods are maximising their profits Π_H and Π_L choosing x_H and H, respectively x_L and L. And the technology monopolists maximise their profits π_H and π_L with respect to the prices χ_H and χ_L of the machines, respectively.

Firms are selling the intermediate goods in competitive markets. Profit maximisation implies that marginal revenues are equal to marginal costs.
Technically, we differentiate the production function (2.3) of the final good with respect to Y_H or Y_L to obtain the the prices p_H and p_L for each good. A detailled derivation is found in the appendix (A.1). We define the relative price p of Y_H and Y_L.

$$p \equiv \frac{p_H}{p_L} = \frac{1-\gamma}{\gamma}\left(\frac{Y_H}{Y_L}\right)^{-\frac{1}{\epsilon}} \tag{2.7}$$

Intuitively, the relative price $p \equiv \frac{p_H}{p_L}$ is increasing for a greater supply of Y_L and for a smaller supply of Y_H.

The price of the final good is given by the prices p_L and p_H for the intermediate goods. Analogous to the production function of the final good (2.3) its price is
$\left[\gamma p_L^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma) p_H^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}}$. Using the price of the final good as a numeraire, we can write

$$[\gamma^{\epsilon} p_L^{1-\epsilon} + (1-\gamma)^{\epsilon} p_H^{1-\epsilon}]^{\frac{1}{1-\epsilon}} = 1. \tag{2.8}$$

Next, we consider the profits Π_L and Π_H of the producers for Y_L and Y_H, respectively. They can sell their products for the price p_L and p_H on the markets, but have to pay their workers L and H wages w_L and w_H, respectively. Furthermore, they buy N_L or N_H different machines $x_L(j)$ or $x_H(j)$ from the technology monopolists for a price $\chi_L(j)$ or $\chi_H(j)$. Since the intermediate goods are sold in competitive markets and N_L and N_H is given, the firms can only choose how many machines $x_L(i)$ or $x_H(j)$ and how many workers L or H they use to maximise their profit.

$$\max_{L,\{x_L(j)\}} \Pi_L = \max_{L,\{x_L(j)\}} p_L Y_L - w_L L - \int_0^{N_L} \chi_L(j) x_L(j) dj \tag{2.9}$$

$$\max_{H,\{x_H(j)\}} \Pi_H = \max_{H,\{x_H(j)\}} p_H Y_H - w_H H - \int_0^{N_H} \chi_H(j) x_H(j) dj \tag{2.10}$$

We substitute the expressions Y_L (2.4) and Y_H (2.5) into (2.9) and (2.10). The first order conditions (FOC) with respect to x_L and x_H give the number of machines for every type

j. The detailled derivations can be found in the appendix (A.2).

$$x_L(j) = (\frac{p_L}{\chi_L(j)})^{\frac{1}{\beta}} L \tag{2.11}$$

$$x_H(j) = (\frac{p_H}{\chi_H(j)})^{\frac{1}{\beta}} H \tag{2.12}$$

The demand for machines increases if there are more workers who can use them or if the price of the intermediate good is increasing, so it is worth to invest more in the factors to earn more from a higher production output. It is also intuitive, that the demand for machines is decreasing, if the price of the machines is increasing.

Calculating the FOC with respect to L and H gives the equilibrium wages w_L and w_H. Again, detailled calculations are found in the appendix (A.2).

$$w_L = \frac{\beta}{1-\beta} p_L \left(\int_0^{N_L} x_L(j)^{1-\beta} dj \right) L^{\beta-1} \tag{2.13}$$

$$w_H = \frac{\beta}{1-\beta} p_H \left(\int_0^{N_H} x_H(j)^{1-\beta} dj \right) H^{\beta-1} \tag{2.14}$$

The last equations to obtain the equilibrium are the profits π_L and π_H of the technology monopolists. They produce machines $x_L(j)$ or $x_H(j)$ and sell them for a price $\chi_L(j)$ or $\chi_H(j)$, respectively, to the producers of the intermediate goods. Producing a machine costs ψ in terms of the final good. Since they are monopolists they can set the prices $\chi_L(j)$ and $\chi_H(j)$ for every type of machine and maximise $\pi_L = \sum_{j=0}^{N_L} \pi_L(j)$ in the L-sector and $\pi_H = \sum_{j=0}^{N_H} \pi_H(j)$ in the H-sector. Since the profits are additive separable, we will focus on the profit of one machine type only and therefore maximise each component $\pi_L(j)$ or $\pi_H(j)$ separately.

$$\max_{\chi_L} \pi_L(j) = \max_{\chi_L} (\chi_L(j) - \psi) x_L(j) \tag{2.15}$$

$$\max_{\chi_H} \pi_H(j) = \max_{\chi_H} (\chi_H(j) - \psi) x_H(j) \tag{2.16}$$

Substituting the demand of machines x_L (2.11) and x_H (2.12) and deriving the FOC (see appendix (A.3)) gives

$$\chi_L(j) = \chi_H(j) = \frac{\psi}{1-\beta}. \tag{2.17}$$

To simplify the further calculations we set $\psi = 1 - \beta$ and therefore get $\chi_L(j) = \chi_H(j) = 1$. So we can write the equilibrium profits as

$$\pi_L(j) = \beta x_L(j) = \beta p_L^{\frac{1}{\beta}} L \tag{2.18}$$

$$\pi_H(j) = \beta x_H(j) = \beta p_H^{\frac{1}{\beta}} H \tag{2.19}$$

The profit is obviously proportional to the number of machines, and after substituting x_L (2.11) and x_H (2.12) even independent of the machine type.

Technology monopolists do not only focus on their current profits, but want to maximise their profits over a longer period. We even assume they value their profits over infinite time horizon. Therefore the net present discounted values V_L and V_H of the profits for the technology monopolists are more important than the profits $\pi_L(j)$ and $\pi_H(j)$ of one period. The values of all profits expected in the future V_L or V_H depend on the interest rate r, which can vary with time. Intuitively, the net discounted values V_L and V_H are the current profits π_L and π_H plus the changes in time, discounted by the interest rate.

$$rV_L - \dot{V}_L = \pi_L \tag{2.20}$$
$$rV_H - \dot{V}_H = \pi_H \tag{2.21}$$

Using (2.18) and (2.19) we can rewrite the equation as

$$V_L = \frac{\beta p_L^{1/\beta} L + \dot{V}_L}{r} \tag{2.22}$$
$$V_H = \frac{\beta p_H^{1/\beta} H + \dot{V}_H}{r}. \tag{2.23}$$

For further analysis we assume that there is no change in the profits over time ($\dot{V}_L = 0$ and $\dot{V}_H = 0$).

2.3.1 Technical change

To analyse the direction of the technical change we need to have a closer look at the relative net present discounted values derived above, where we assumed $\dot{V}_L = 0$ and $\dot{V}_H = 0$.

$$\frac{V_H}{V_L} = \underbrace{p^{\frac{1}{\beta}}}_{\text{price effect}} \times \underbrace{\frac{H}{L}}_{\text{market size effect}} . \tag{2.24}$$

Basically, there are two effects: the price effect and the market size effect. If the price p_L or p_H for an intermediate good Y_L or Y_H, respectively, is increasing, the net profit value V_L or V_H for the technology monopolists is increasing as well. So they are interested in supplying more machines x_L or x_H for the producers of the intermediate good. This leads to more production of these intermediate goods. That is the so called price effect. On the other hand, if there are more workers (or more general: machine complementary production factors) in one sector, the net profit value V_L or V_H is increasing and again the technology monopolists want to sell more machines for this sector and the relative production in this sector is increasing. To sum up, a larger market leads to more innovation.

To make a more detailled analysis, we substitute the demand curve for machines facing the technological monopolists x_L (2.11) and x_H (2.12) into the production functions Y_L

(2.4) and Y_H (2.5). We use the so gained expressions in the price function $p = \frac{p_H}{p_L}$ (2.7).

$$p = \frac{p_H}{p_L} = \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\frac{1}{1-\beta}\left(\int_0^{N_H}(p_H^{\frac{1}{\beta}}H)^{1-\beta}dj\right)H^\beta}{\frac{1}{1-\beta}\left(\int_0^{N_L}(p_L^{\frac{1}{\beta}}L)^{1-\beta}dj\right)L^\beta}\right)^{-\frac{1}{\epsilon}}$$

Using the derived elasiticity of substitution (see appendix (A.4)) between the production factors H and L $\sigma \equiv \epsilon - (\epsilon - 1)(1-\beta)$ (so $\sigma > 1$ only if $\epsilon > 1$) we can calculate the relative price p (see appendix (A.5))

$$p = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\beta\epsilon}{\sigma}}\left(\frac{N_H}{N_L}\right)^{-\frac{\beta}{\sigma}}\left(\frac{H}{L}\right)^{-\frac{\beta}{\sigma}} \tag{2.25}$$

and we substitute it in (2.24) to rewrite the relative profitability of creating new H-complementary machines.

$$\frac{V_H}{V_L} = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon}{\sigma}}\left(\frac{N_H}{N_L}\right)^{-\frac{1}{\sigma}}\left(\frac{H}{L}\right)^{\frac{\sigma-1}{\sigma}} \tag{2.26}$$

Using equation (2.26) we can analyse the directed technological change in more detail. Since the distribution parameter $\gamma \in (0,1)$, the term $\left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon}{\sigma}}$ is always a positive constant influencing just the power, but not the direction of the technological change. Since N_L and N_H are given in this analysis, $\left(\frac{N_H}{N_L}\right)^{-\frac{1}{\sigma}}$ is positive constant as well. In the next chapter we vary the ratio of N_L and N_H as well.
To explain the technological change we have to focus on the relative factor supply $\frac{H}{L}$. If the factors are gross substitutes ($\sigma > 1$), and therefore also the intermediate goods ($\epsilon > 1$), the exponent is positive. In other words, if the relative supply of high-skilled workers is increasing, the relative discounted value of profits for monopolists $\frac{V_H}{V_L}$ is increasing as well and the H-sector is growing. This is the market size effect we analysed in (2.24).
In contrast, if the factors are gross complements ($\sigma < 1$) the influence of the relative factor supply is inverse. So only the price effect can help increasing $\frac{V_H}{V_L}$.

Similarly we look at the relative factor rewards. Again substituting x_L (2.11) and x_H (2.12) in the expressions for w_L (2.13) and w_H (2.14), respectively, and using p (2.25) from above gives

$$\frac{\omega_H}{\omega_L} = p^{\frac{1}{\beta}}\frac{N_H}{N_L} = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon}{\sigma}}\left(\frac{N_H}{N_L}\right)^{\frac{\sigma-1}{\sigma}}\left(\frac{H}{L}\right)^{-\frac{1}{\sigma}}. \tag{2.27}$$

It does not matter whether the factors are substitutes or not, the relative wages are indirect proportional to the relative factor supply. If there are for example more skilled workers H, each earns less.
We note that the relative innovation technology $\frac{N_H}{N_L}$ has the same exponent as the relative factor supply in (2.26). So the influence depends on the relationship of the two factors: Are they substitutes or complements?
In the next chapter we introduce the supply of innovations, so N_L and N_H are endogenous.

2.4 Supply of innovations - Innovation Possibilites Frontier

Since we are restricted to the budget constraint, we have to decide how much we spend on innovation in each sector to determine N_L and N_H, which are endogenous in the following section. All the combinations are given in the so called *innovation possibilities frontier*.

2.4.1 Lab equipment model

The investment in the R&D sector D can be splitted into the spending D_L on the sector with the low-skilled workers (labour intensive) and the spending D_H for the R&D in the H-sector. The technology in each sector is given by the machine varieties N_L and N_H. The innovation possibilities frontier is given by the production functions for new machine types.

$$\dot{N}_L = \eta_L D_L \tag{2.28}$$
$$\dot{N}_H = \eta_H D_H \tag{2.29}$$

η_L and η_H denote how many new machine types can be invented for costs of D_L and D_H in units of the final good, respectively. This can be different in each sector.
After creating new machine varieties, the technology monopolists get a unique patent for each type of invented machine and they are its sole supplier.

The technology monopolists can invent the same amount of new machine types whether there are already many different machine types in this sector or not. Equations (2.28) and (2.29) show that N_L and N_H do not influence the innovation possibilities frontier $\dot{N}_L$ and $\dot{N}_H$. In other words: There is no state dependence. We define the relative technology efficiency η

$$\frac{\frac{\partial \dot{N}_H}{\partial D_H}}{\frac{\partial \dot{N}_L}{\partial D_L}} = \frac{\eta_H}{\eta_L} =: \eta, \tag{2.30}$$

which is always constant.

Balanced growth path - BGP

In the steady state equilibrium there are no changes in the variables over time. So we have constant prices and N_H and N_L grow at the same rate. Furthermore, the profits of the technology monopolists do not change over time, so $\dot{V}_L$ in (2.22) and $\dot{V}_H$ in (2.23) are 0. Since the growth rates are the same, also $\frac{V_H}{V_L} = \frac{\eta_L}{\eta_H}$ is constant and equal to make sure, technology monopolists are innovating in both sectors. Substituting π_L (2.20) and π_H (2.21) leads to the following condition to clear the technology markets. The detailled derivation is found in appendix (A.6).

$$\eta_L \pi_L = \eta_H \pi_H \tag{2.31}$$

Using π_L and π_H in terms of the price p_L (2.18) and p_H (2.19), respectively, inserting the relative price p (2.25) and setting $\eta \equiv \frac{\eta_H}{\eta_L}$ the technology market clearing condition leads to

$$\frac{N_H}{N_L} = \eta^{\sigma} \left(\frac{1-\gamma}{\gamma} \right)^{\epsilon} \left(\frac{H}{L} \right)^{\sigma-1} \tag{2.32}$$

Whether an increase of the relative factor supply $\frac{H}{L}$ enlarges the relative amount of machine varieties $\frac{N_H}{N_L}$ depends on σ. If the factors are gross substitutes ($\sigma > 1$) there is a positive relationship, otherwise ($\sigma < 1$) a bigger amount of workers in one sector leads to relatively less types of machines in the same sector.
Again, besides the factor supply, the elasticity of substitution is important for the direction of the technical change.

Substituting the relative bias of technology $\frac{N_H}{N_L}$ into the relative factor prices (2.27) of the previous section gives

$$\frac{\omega_H}{\omega_L} = \eta^{\sigma-1} \left(\frac{1-\gamma}{\gamma} \right)^{\epsilon} \left(\frac{H}{L} \right)^{\sigma-2}. \tag{2.33}$$

When we compare the case with N_L, N_H exogenous (2.27) and N_L, N_H endogenous (2.33) we see that changes in relative factor supply cause always a stronger reaction in the relative factor rewards in the second case, because the relative factor supply in (2.33) is more elastic: $\sigma - 2 > -1/\sigma$.

We define factor shares as the combination of the value of a factor times the quantity of this factor. Then we can easily express the relative factor shares with the equation (2.33) above.

$$\frac{H_S}{L_S} \equiv \frac{\omega_H H}{\omega_L L} = \eta^{\sigma-1} \left(\frac{1-\gamma}{\gamma} \right)^{\epsilon} \left(\frac{H}{L} \right)^{\sigma-1} \tag{2.34}$$

We see the same behaviour as in (2.32). If the two factors are gross substitutes ($\sigma > 1$) we have more of a share of a factor when this factor abounds.

I want to outline two effects we can observe in the equations above. First, the "weak induced-bias hypothesis" shows that for $\sigma < 1$ in (2.32) a lower relative factor supply leads to a higher value of marginal product. As we have seen in analysing equation (2.26), $\sigma > 1$ leads to an endogenously biased technology. The same effect appears for $\sigma < 1$, even if we face a negative relationship between the relative factor supply and the endogenous technology. Second, the "strong induced-bias hypothesis" concerns equation (2.33). We are used to a negative relationship between the relative factor supply and the relative factor rewards for every value of sigma. If we set $\sigma > 2$ with endogenous technology, a higher relative factor supply leads to a higher relativ factor reward. The more abundant factor increases the technology (2.32), and this again augments the relative factor shares (2.27) as we saw in the previous chapter.

Long-run growth rate

In the steady state we can also derive the long-run growth of output g. (see Appendix A.7)

$$g = \theta^{-1}(\beta[(1-\gamma)^{\epsilon}(\eta_H H)^{\sigma-1} + \gamma^{\epsilon}(\eta_L L)^{\sigma-1}]^{\frac{1}{\sigma-1}} - \rho) \tag{2.35}$$

More factor supply leads to a higher growth rate. This does not depend on the sector or whether the factors are complements or substitutes.

Outside the BGP

We analysed the behaviour in the BGP, now we want to test the stability of the steady state. Considering the relative amount of machine types $\frac{N_H}{N_L}$ higher than its BGP-level in (2.32). It is shown in (Acemoglu and Zilibotti, 2001) that outside the BGP only one type of innovation takes place. In our case R&D only invents new machine types in the L-sector. Since there is an inverse relation between $\frac{V_H}{V_L}$ and $\frac{N_H}{N_L}$ (2.26), N_L will increase till both terms reach the BGP. In case $\frac{N_H}{N_L}$ is lower than its BGP-level, H-augmenting technical change takes place until steady state. To sum up, the transitional dynamics of the system are stable and will always return to the equations derived for the balanced growth path.

2.4.2 Knowledge-based R&D

In the lab equipment model we had a very simple innovation possibilities frontier. In the following model not only the investment gives the new innovation, but it is also important how much research is already done in each sector. To implement this state dependency, the amount of existing machine types in each sector influences the number of additional inventions. $\delta \leq 1$ gives the degree of state dependence. For a general level of technology N a constant supply of scientists $S \propto \frac{\dot{N}}{N}$ does research in both sectors. The scientists can choose the sector they are working in: $S = H_S + L_S$. Alltogether this brings us to the following innovation possibilities frontier.

$$\dot{N}_L = \eta_L N_L^{\frac{1+\delta}{2}} N_H^{\frac{1-\delta}{2}} L_S \tag{2.36}$$

$$\dot{N}_H = \eta_H N_L^{\frac{1-\delta}{2}} N_H^{\frac{1+\delta}{2}} H_S \tag{2.37}$$

For $\delta = 0$ the levels of N_L and N_H create equal spillovers in both sectors and we face no state dependence like in the lab equipment model

$$\frac{\frac{\partial \dot{N}_H}{\partial H_S}}{\frac{\partial \dot{N}_L}{\partial L_S}} = \frac{\eta_H}{\eta_L} = \eta, \tag{2.38}$$

whereas for $\delta = 1$ the number fo new innvoations depends on the current level of technology.

$$\frac{\frac{\partial \dot{N}_H}{\partial H_S}}{\frac{\partial \dot{N}_L}{\partial L_S}} = \eta \frac{N_H}{N_L} \tag{2.39}$$

Investing in the H-sector also either gains more future-innovations ceteris paribus or needs less scientists and makes research cheaper for the same amount of future-innovations.

Balanced growth path

Analogous to the previous section we can derive the technology market clearing condition. Only the growth rates differ and so we get $\frac{V_H}{V_L} = \eta \frac{N_H^\delta}{N_L^\delta}$. Again substituting π_L and π_H gives the condition for the knowledge-based R&D model.

$$\eta_L N_L^\delta \pi_L = \eta_H N_H^\delta \pi_H \tag{2.40}$$

Analogous to the previous chapter we substitute π_L, π_H and the relative price p to get the equilibrium skill bias.

$$\frac{N_H}{N_L} = \eta^{\frac{\sigma}{1-\delta\sigma}} \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon}{1-\delta\sigma}} \left(\frac{H}{L}\right)^{\frac{\sigma-1}{1-\delta\sigma}} \tag{2.41}$$

The relative technology depends on δ. A positive change in the relative factor supply can increase $\frac{N_H}{N_L}$ if $1 < \sigma < \frac{1}{\delta}$ or $\frac{1}{\delta} < \sigma < 1$.

Using the equation above we can rewrite (2.27) to get the relative factor prices.

$$\frac{w_H}{w_L} = \eta^{\frac{\sigma-1}{1-\delta\sigma}} \left(\frac{1-\gamma}{\gamma}\right)^{\frac{(1-\delta)\epsilon}{1-\delta\sigma}} \left(\frac{H}{L}\right)^{\frac{\sigma-2+\delta}{1-\delta\sigma}} \tag{2.42}$$

The relative factor supply in (2.42) is more elastic ($\frac{\sigma-2+\delta}{1-\delta\sigma} > -\frac{1}{\sigma}$) than with exogenous N_L and N_H (2.27) if $\delta > 0$ and $0 < \sigma < \frac{1}{\delta}$ and even more elastic ($\frac{\sigma-2+\delta}{1-\delta\sigma} > \sigma - 2$) than in the lab equipment model (2.33) for a positive δ and $\sigma < \frac{1}{\delta}$.

And we can write the relative factor shares as

$$\frac{H_S}{L_S} \equiv \frac{w_H H}{w_L L} = \eta^{\sigma-1} \left(\frac{1-\gamma}{\gamma}\right)^{\frac{(1-\delta)\epsilon}{1-\delta\sigma}} \left(\frac{H}{L}\right)^{\frac{\sigma-1+\delta-\delta\sigma}{1-\delta\sigma}}. \tag{2.43}$$

This is a more general version than in the previous section, where $\delta = 0$. It is again exactly the same behaviour as in (2.41).

Growth rate

For the balanced growth path the number of scientists is important. The derivation is given in appendix A.8.

$$g = \frac{\eta_L \eta_H S}{\eta_H \left(\frac{N_H}{N_L}\right)^{\frac{1-\delta}{2}} + \eta_L \left(\frac{N_L}{N_H}\right)^{\frac{3(1-\delta)}{2}}} \tag{2.44}$$

The degree of state dependence δ weights the relative levels of the technology in both sectors. Intuitively, as more scientists can do research more efficiently, the growth rate is increasing.

Outside the BGP

We also want to briefly look at the system outside the balanced growth path. Whereas in the lab equipment model we faced $\frac{V_H}{V_L} = \frac{\eta_L}{\eta_H}$, in the knowledge-based R&D model the relative net present discounted value is $\frac{V_H}{V_L} = \frac{N_L^\delta \eta_L}{N_H^\delta \eta_H}$. As a result of the state dependence, we have to take N_L and N_H into account and analyse

$$\frac{\partial(N_H^\delta V_H / N_L^\delta V_L)}{\partial(N_H/N_L)} \tag{2.45}$$

instead of $\frac{\partial(V_H/V_L)}{\partial(N_H/N_L)}$. For $\sigma < \frac{1}{\delta}$ the term (2.45) is negative, so the system will always end up in the BGP and the system is stable. On the other hand, if $\sigma > \frac{1}{\delta}$ the transitional dynamics of the system will not take us to the BGP and there will be only one sector inventing new technologies. The system is unstable.
We already know, that with no state dependence ($\delta = 0$) the system is always stable. If there is an extreme state dependence ($\delta = 1$) the system is only stable if the factors are gross complements ($\sigma < 1$).
The big difference to the previous model is that a change in $\frac{N_H}{N_L}$ not only affects $\frac{V_H}{V_L}$ in the same period, but also the future profits and costs of R&D. Of course, more state dependence (higher δ) affects the future profits and expenses even more.

3 Environment and Directed Technical Change

3.1 Adding Environment

In the previous chapter we got to know the basic model of directed technical change. Now we want to extend the model and add the factor *environment* to the preferences of the consumers and to the production function of one sector. I will follow the modelling of Acemoglu, Aghion, Bursztyn and Hemous (2012). Whereas we had a sector with low (or un-)skilled workers L and a sector with high-skilled workers H - which could be also interpreted as capital or land - , we now interpret them as a sector with *dirty* inputs and a sector with *clean* inputs, respectively. Since we have the quality of the environment in the utility function of the consumers, this is also a driving factor for the directed technical change. Consumers are happier, if they live in a well-preserved environment. We want directed technical change towards the clean sector to happen, otherwise there can be an environmental desaster.

3.2 Basic Model and Definitions

The model is outlined in the appendix Fig.E.2.
Again, to simplify notation I will only write the time argument t to the variables if I want to stress time dependence.

In this model the consumers do not have well defined CRRA preferences in continuous time, but an aggregate utility function in discrete time. As in the basic model, ρ is the time preference rate and C is the consumption of the unique final good Y. A new aspect of the model is that we consider the quality of environment E in the utility function. In the worst case we have an environmental desaster with $E = 0$. E can not grow beyond the saturation limit $\bar{E}$, which we assume as its intial value E_0, i.e. $\bar{E} = E_0$.

$$U = \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} u(C(t), E(t)) \tag{3.1}$$

The aggregate utility function sums up the instantaneaous utility function $u(C, E)$. It positively depends on C and E and fulfills the following Inada-type conditions.

$$\frac{\partial u}{\partial C} \geq 0, \quad \frac{\partial^2 u}{\partial C^2} \leq 0, \quad \lim_{C\to 0} \frac{\partial u}{\partial C} = \infty \tag{3.2}$$

$$\frac{\partial u}{\partial E} \geq 0, \quad \frac{\partial^2 u}{\partial E^2} \leq 0, \quad \lim_{E\to 0} \frac{\partial u}{\partial E} = \infty, \quad \lim_{E\to 0} u = -\infty, \quad \frac{\partial u(C,\bar{E})}{\partial E} = 0 \tag{3.3}$$

Intuitively, if consumers can get more final goods or live in a better environment, the utility $u(C, E)$ is increasing. If they already have a lot of either C or E, one unit more of this good is not that important any more. On the other hand, if they have only little of one factor, one unit more can already make them feeling much better. The last two conditions are special. As closer we come to an environmental desaster, the worse $u(C, E)$ becomes independent of C and households will do everything to avoid a desaster. In other words, if environmental problems occur badly, the population will not survive any longer. E.g. suffering from bad air quality, having a huge ozon hole, facing flooded islands as consequence of global warming and having toxin in food can be an issue.
Moreover, since there is an upper limit for E, the utility function cannot increase in E if the environment is already at its highest level $\bar{E}$.

The production function of the final good is the same as in the basic model, but with $\gamma = 0.5$. So the dirty input Y_L is as important as the clean input Y_H and we can simply write

$$Y = \left[Y_L^{\frac{\epsilon-1}{\epsilon}} + Y_H^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}}. \tag{3.4}$$

What is new in this model compared to the basic model, is that the production of the dirty intermediate good needs resources R that are important for the quality of environment. So there is a trade off between a higher production and therefor more consumption and higher quality of the environment.
The production functions of the intermediate goods Y_L and Y_H are slightly different to the production functions in the previous chapter, but basically follow the same intuition. Instead of using the number of different machine types as incentive for innovation, the technology levels $N_L(j)$ and $N_H(j)$ represents the level of R&D for every machine type j in each sector L and H.

$$Y_L = R^{\beta_2} \left(\int_0^1 N_L(j)^{\beta_1} x_L(j)^{1-\beta_1} dj\right) L^{\beta} \tag{3.5}$$

$$Y_H = \left(\int_0^1 N_H(j)^{\beta} x_H(j)^{1-\beta} dj\right) H^{\beta} \tag{3.6}$$

The technology level is the complementary factor for the number of machines: firms can produce more if they either have a broader variety of machines $N_H(j)$ or more machines $x_H(j)$ for each type. In the L-sector firms also use natural ressources R complementary to $N_L(j)$ and $x_L(j)$. We want the sum of the exponents of R, $x_L(j)$ and L to be 1, as we have in the H-sector. So the production elasticity for each production factor satisfy $\beta = \beta_1 - \beta_2$ with $\beta, \beta_1, \beta_2 \in (0, 1)$. In further analysis we will also look at $\beta_2 = 0$ to have the same production function of intermediate goods in each sector.

In this model we interpret the machine -complementary factors, L and H, only as labour supply and assume the total number of available workers is constant over time. For simplicity we set the total number equal 1.

$$L + H \leq 1 \tag{3.7}$$

The resources R we use in (3.5) are part of a resource stock Q. The resources evolve according to the state equation

$$Q(t+1) = Q(t) - R(t). \tag{3.8}$$

The resource can be sold within two different market types: Either it is available for everyone and the firms only have to pay the resource extraction costs $c(Q)$ or there are property rights for the resource and firms have to pay the price P calculated with the Hotelling rule. An explanation of the Hotelling rule can be found in the appendix (B.1).

In the basic model we first had the technology levels N_L and N_H given and then already added simple innovation possibilities frontiers for an endogenous level of technology. In the current model we have a more complex way to improve the machine-complementary production factor.
Scientists choose one sector and become either L_S or H_S. Analogous to the workers the total amount of available scientists is normalised to 1.

$$L_S + H_S \leq 1 \tag{3.9}$$

Every scientist does research at most at one machine. His research is successful with probability $\eta_L, \eta_H \in (0,1)$ and can change the machine technology from $N_L(j)$ to $(1+\nu)N_L(j)$ or from $N_H(j)$ to $(1+\nu)N_H(j)$ with a positive sector- and time-independent ν. If he invented a new machine technology, he gets the patent for the coming period and uses it to produce machines. So the scientist becomes an entrepreneur for one period. Otherwise, the monopoly rights on the latest technology for this machine type is given to a random entrepreneur to produce this machine in the next period.
We can aggregate the technology levels for each machine to get a technology index for the whole sector.

$$N_L = \int_0^1 N_L(j)dj \tag{3.10}$$

$$N_H = \int_0^1 N_H(j)dj \tag{3.11}$$

The aggregate technology level in the next period $t+1$ is given by the level of the previous period plus the change for one machine type multiplied with the number of new inventions, depending on the successrate and the number of scientists doing research in this period. This leads to the following production possibilities frontier for each sector.

$$N_L(t+1) = (1 + \nu\eta_L L_S)N_L(t) \tag{3.12}$$

$$N_H(t+1) = (1 + \nu\eta_H H_S)N_H(t) \tag{3.13}$$

Basically, directed technical change takes place, if the relative possibility for successful R&D is high and if lots of scientists are working in this sector relativ to the other sector.

The budget constraint in this model is more detailled than in the basic model. We can use the output Y either for consumption C or for the production of intermediate goods. To produce intermediate goods we need machines $x_L(j)$ or $x_H(j)$ for the costs of ψ and resources R for the costs of $c(Q)$.
All prices and costs are given in units of the final good.

$$C + \psi\Big(\int_0^1 x_L(j)dj + \int_0^1 x_H(j)dj\Big) + c(Q)R \leq Y \tag{3.14}$$

The state equation of the quality of environment E is given by E from the previous period, which regenerates with rate ζ, and is reduced by the environmental pollution caused from the production in the L-sector. The rate ξ measures the part of Y_L causing e.g. air pollution or waste water.

$$E(t+1) = -\xi Y_L(t) + (1+\zeta)E(t) \tag{3.15}$$

We know E is bounded by $(0, \bar{E})$. So the precise formula is actually
$E(t+1) = \max\{0; \min\langle \bar{E}; -\xi Y_L(t) + (1+\zeta)E(t)\rangle\}$.
Note, that we face an environmental desaster if $E(t) = 0$ for already some t.

Since Y_L influences E, but the quality of environment does not have any impact on the production, we are confronted with externalities. In some later analysis we try to internalise them with taxes as production costs.

3.3 Analysis

The model of the economy is the same as in the basic model. The final good Y consists of two intermediate goods Y_L and Y_H, which are produced by firms in a competitive market. These firms buy machines x_L or x_H from technology monopolists, who have been scientists in the R&D. A big difference to the basic model is that producers of Y_L use resources R and therefore impair the quality of the environment. Moreover, we have a profit function in the R&D sector.

To find an equilibrium, we have to set the wages w_L and w_H for the workers H and L, respectively and the prices p_L and p_H for the intermediate goods to clear the markets. Furthermore, the producers of the intermediate goods are maximising their profits Π_L or Π_H choosing x_L or x_H and H or L, respectively, the technology monopolists maximise their profits $\pi_L(j)$ and $\pi_H(j)$ with respect to the prices $\chi_L(j)$ and $\chi_H(j)$, respectively, of the machines and the scientists can calculate their expected profits π_L and π_H.

There are four different set ups of the model characterised by the type of resource and by the type of equilibrium. On one hand we distinguish between renewable resources (in the model there is an infinity stock of ressources) and nonrenewable (exhaustible) resources, on the other hand we can calculate a laissez-faire equilibrium or use a social planner to obtain a centralised equilibrium. We will analyse all four combinations of the model variant in the next sections. An overview is drawn in the following table.

	without exhaustible resources	with exhaustible resources
decentralised equilibrium	3.3.1	3.3.3
centralised equilibrium	3.3.2	3.3.4

3.3.1 The laissez-faire equilibrium without exhaustible resources

In the first case we assume the resource R is unlimited. We can find examples for this also in real life. Resources like wood in forests can grow again after using them. Moreover, technologies can be improved so that less resources are needed and we can use them much longer (or even unlimited).

Setting $\beta_2 = 0$ (and therefore $\beta_1 = \beta$) yields the following production function for the dirty L-sector, which is analogous to the Y_H-production function.

$$Y_L = \left(\int_0^1 N_L(j)^{\beta} x_L(j)^{1-\beta} dj \right) L^{\beta} \tag{3.16}$$

Since the production functions of the intermediate goods are symmetric, we can see the effects of the directed technical change easier in our analysis.
As we have seen in the first model, state dependence influences the results. So we suppose $N_L > N_H$ for the first period $t = 0$, because this is what we observe in our economies. Renewable and environmentally sound technologies are not as improved as production with environmental load.

The market establishes the decentralised equilibrium with maximum profits and market clearing for the labour- and goods-market.

We apply the assumption of perfect competition. To clear the two markets for the intermediate goods Y_L and Y_H, firms maximise their profit. Their marginal revenue has to equal their marginal costs. To obtain the prices p_L and p_H for each good we differentiate the production function of the final good with respect to Y_L and Y_H. Similar to the basic model we define the relative price p of Y_H and Y_L.

$$p \equiv \frac{p_H}{p_L} = \left(\frac{Y_H}{Y_L}\right)^{-\frac{1}{\epsilon}} \tag{3.17}$$

Intuitively the relative price $p \equiv \frac{p_H}{p_L}$ is increasing for a greater supply of Y_L and for a smaller supply of Y_H. The relative price is more elastic if the goods are better substitutes.

The price of Y is a combination of the prices of the two intermediate goods without additional profits. Using the price of the final good as a numeraire, we can write

$$[p_L^{1-\epsilon} + p_H^{1-\epsilon}]^{\frac{1}{1-\epsilon}} = 1. \tag{3.18}$$

Again, the price of the final good is less dependent on the prices of the intermediate goods Y_L or Y_H and therefore more stable if the goods are good substitutes.

Next, we look at the profits Π_L and Π_H of the firms producing Y_L and Y_H, respectively. They can sell their products for the price p_L or p_H, respectively, on the markets, but have to pay the wages w_L or w_H for their workers H or L and buy machines $x_L(j)$ or $x_H(j)$ from the technology monopolists for prices $\chi_L(j)$ or $\chi_H(j)$, respectively. The firms can only choose how many machines $x_L(i)$ or $x_H(j)$ and how many workers H or L they use to maximise their profit. In contrast to the basic model, the technology level does not directly influence the profit of the firms.

$$\max_{L,\{x_L(j)\}} \Pi_L = \max_{L,\{x_L(j)\}} p_L Y_L - w_L L - \int_0^1 \chi_L(j) x_L(j) dj \tag{3.19}$$

$$\max_{H,\{x_H(j)\}} \Pi_H = \max_{H,\{x_H(j)\}} p_H Y_H - w_H H - \int_0^1 \chi_H(j) x_H(j) dj \tag{3.20}$$

We substitute Y_L and Y_H into (3.19) and (3.20). The first order conditions (FOC) with respect to x_L and x_H give the demand of machines for every type j.

$$x_L(j) = \left(\frac{(1-\beta)p_L}{\chi_L(j)}\right)^{\frac{1}{\beta}} L N_L(j) \tag{3.21}$$

$$x_H(j) = \left(\frac{(1-\beta)p_H}{\chi_H(j)}\right)^{\frac{1}{\beta}} H N_H(j) \tag{3.22}$$

The demand for machines increases if there are more workers who can use them and if the price of the intermediate good is higher than the price for the machine. What is new in this model, the technology levels $N_L(j)$ and $N_H(j)$ influence the demand of machines $x_L(j)$ or $x_H(j)$ in the same sector. If this sector is already better developed, more machines can be sold.

The FOCs with respect to L and H give the wages w_L and w_H with x_L and x_H from equations (3.21) and (3.22), respectively, above.

$$w_L = \beta p_L \left(\int_0^1 N_L(j)^\beta x_L(j)^{1-\beta} dj \right) L^{\beta-1} \tag{3.23}$$

$$w_H = \beta p_H \left(\int_0^1 N_H(j)^\beta x_H(j)^{1-\beta} dj \right) H^{\beta-1} \tag{3.24}$$

The technology monopolists sell machines $x_L(j)$ or $x_H(j)$ to the firms requesting $x_L(j)$ or $x_H(j)$ derived in equations (3.21) and (3.22) to make profits $\pi_L(j)$ or $\pi_H(j)$, respectively. Each machine is worth $\chi_L(j)$ or $\chi_H(j)$ and costs ψ (time- and machine-independent) to be produced. Since they are monopolists they can set the prices $\chi_L(j)$ or $\chi_H(j)$ for every type of machine and maximise $\pi_L = \int_{j=0}^{1} \pi_L(j)dj$ or $\pi_H = \int_{j=0}^{1} \pi_H(j)dj$. Instead of writing the sum it is enough to maximise $\pi_L(j)$ or $\pi_H(j)$, respectively.

$$\max_{\chi_L} \pi_L(j) = \max_{\chi_L}(\chi_L(j) - \psi)x_L(j) \tag{3.25}$$

$$\max_{\chi_H} \pi_H(j) = \max_{\chi_H}(\chi_H(j) - \psi)x_H(j) \tag{3.26}$$

Substituting x_L from equation (3.21) and x_H from equation (3.22) and differentiating according to $\chi_L(j)$ and $\chi_H(j)$, respecetively, gives

$$\chi_L(j) = \chi_H(j) = \frac{\psi}{1-\beta}. \tag{3.27}$$

To simplify the further calculations we set $\psi = (1-\beta)^2$ and therefore get $\chi_L(j) = \chi_H(j) = 1-\beta$. So we can write the equilibrium demand as

$$x_L(j) = p_L^{\frac{1}{\beta}} L N_L(j) \tag{3.28}$$

$$x_H(j) = p_H^{\frac{1}{\beta}} H N_H(j), \tag{3.29}$$

the equilibrium output as

$$Y_L = \left(\int_0^1 N_L(j)^\beta x_L(j)^{1-\beta} dj\right) L^\beta = p_L^{\frac{1-\beta}{\beta}} L N_L \tag{3.30}$$

$$Y_H = \left(\int_0^1 N_H(j)^\beta x_H(j)^{1-\beta} dj\right) H^\beta = p_H^{\frac{1-\beta}{\beta}} H N_H, \tag{3.31}$$

the equilibrium wages as

$$w_L = \beta p_L \left(\int_0^1 N_L(j)^\beta x_L(j)^{1-\beta} dj\right) L^{\beta-1} = \beta p_L^{\frac{1}{\beta}} N_L \tag{3.32}$$

$$w_H = \beta p_H \left(\int_0^1 N_H(j)^\beta x_H(j)^{1-\beta} dj\right) H^{\beta-1} = \beta p_H^{\frac{1}{\beta}} N_H \tag{3.33}$$

and the equilibrium profits of the technology monopolists as

$$\pi_L(j) = \beta(1-\beta)x_L(j) = \beta(1-\beta)p_L^{\frac{1}{\beta}} L N_L(j) \tag{3.34}$$

$$\pi_H(j) = \beta(1-\beta)x_H(j) = \beta(1-\beta)p_H^{\frac{1}{\beta}} H N_H(j) \tag{3.35}$$

The profit is obviously proportional to the number of machines, and, after substituting x_L (3.28) or x_H (3.29), the profit positively depends on the technology level of the machine they are selling.

We know, that the technology monopolists have been successful scientists in the previous period. Before they start their research they can calculate their expected profits π_L or π_H, knowing that their research improves the technology from the previous period $N_L^{t-1}(j)$ to

$(1+\nu)N_L^{t-1}(j)$ with probability η_L. Analogous for the H-sector.

$$\pi_L \quad = \eta_L \quad \beta(1-\beta)p_L^{\frac{1}{\beta}}L(1+\nu)N_L^{t-1} \tag{3.36}$$

$$\pi_H \quad = \eta_H \quad \beta(1-\beta)p_H^{\frac{1}{\beta}}H(1+\nu)N_H^{t-1} \tag{3.37}$$

Directed Technical Change

Whereas the basic model used the net present discounted value of the profit, in this model we analyse the directed technical change calculating the relative expected profits for both sectors. Researchers decide to work in the sector where they can expect a higher profit.

$$\frac{\pi_H}{\pi_L} = \frac{\eta_H}{\eta_L} \times \underbrace{\left(\frac{p_H}{p_L}\right)^{\frac{1}{\beta}}}_{\text{price effect}} \times \underbrace{\frac{H}{L}}_{\text{market size effect}} \times \underbrace{\frac{N_H^{t-1}}{N_L^{t-1}}}_{\text{direct productivity effect}} \tag{3.38}$$

Besides the probability of success and the effects we already discussed in the previous chapter, (price and market size effect) we can determine the direct productivity effect. If a lot of research took place in the past, scientists can "stand on the shoulders of giants". It is easier for them to do successfull research if they can on the one hand refer to a lot of knowledge and on the other hand work together with experienced scientists.

We can rewrite the equation substituting equilibrium results from above. The detailled derivation is given in the Appendix B.2.

$$\frac{\pi_H}{\pi_L} = \frac{\eta_H}{\eta_L}\left(\frac{1+\nu\eta_H H_S}{1+\nu\eta_L L_S}\right)^{\beta(\epsilon-1)-1}\left(\frac{N_H^{t-1}}{N_L^{t-1}}\right)^{\beta(\epsilon-1)} \tag{3.39}$$

We could theoretically have three different equilibrium points for innovation. First, scientists are in the clean sector only ($H_S = 1$) and $\frac{\eta_H}{\eta_L}(1+\nu\eta_H)^{\beta(\epsilon-1)-1}(\frac{N_H^{t-1}}{N_L^{t-1}})^{\beta(\epsilon-1)} > 1$. Second, research is only in the L-sector ($L_S = 1$) and $\frac{\eta_H}{\eta_L}(1+\nu\eta_L L_S)^{-\beta(\epsilon-1)-1}(\frac{N_H^{t-1}}{N_L^{t-1}})^{\beta(\epsilon-1)} < 1$. Third, it is also possible that we find equilibrium H_S and L_S to satisfy $\frac{\eta_H}{\eta_L}(\frac{1+\nu\eta_H H_S}{1+\nu\eta_L L_S})^{\beta(\epsilon-1)-1}(\frac{N_H^{t-1}}{N_L^{t-1}})^{\beta(\epsilon-1)} = 1$.

In case the intermediate goods are substitutes ($\epsilon > 1$) there is a positive relationship between the technology level and the expected profits. Scientists decide to work in the sector, which is already more developed. Since we assumed the L-sector to be more developed in the beginning there is only one equilibrium: Innovation will only take place in the dirty sector. So N_H remains constant and N_L grows with rate $\nu\eta_L$.

To sum up, a laissez-faire equilibrium without an exhaustible resource will lead under the assumptions ($N_L > N_H$ in the beginning, intermediate goods are substitutes) to an environmental desaster. In other words, E becomes 0 in the long run, because production would only take place in the L-sector and therefore the negative production effects dominates the regeneration of the environment (see equation (3.15)).

3.3.2 The socially optimal allocation without exhaustible resources

In this case we still assume unlimited resources and therefore a production function for the L-sector like (3.16). We observed in the previous chapter, that a decentralised equilibrium leads to an environmental desaster. So we introduce a social planner who internalises the externalities. The social planner can give two different incentives to handle that.

The producers of the intermediate good Y_L damage the environment, so we want them to have the quality of environment in their profit-function, too. This is possible with taxes τ for the dirty input (e.g. carbon taxes). Analogous to the price function (3.18) we can write the relation of the new prices $\hat{p_L}$ and $\hat{p_H}$.

$$[\hat{p}_L^{1-\epsilon} + ((1+\tau)\hat{p}_H)^{1-\epsilon}]^{\frac{1}{1-\epsilon}} = 1. \tag{3.40}$$

Another problem in the previous section is that research starts in the dirty sector and remains there. To avoid that, R&D in the clean sector gets subsidies q. It is not enough to sustain technical change in the long run if more firms are producing in the clean sector. Firms in the dirty sector need incentives to reduce the production of the dirty goods, too. Therefore the subsidies also help firms in the dirty sector to reduce the number of producing machines and sell less dirty goods.
So equation (3.39) is adjusted to

$$\frac{\pi_H}{\pi_L} = (1+q)\frac{\eta_H}{\eta_L}\left(\frac{1+\nu\eta_H H_S}{1+\nu\eta_L L_S}\right)^{\beta(\epsilon-1)-1}(1+\tau)^{\epsilon}\left(\frac{N_H^{t-1}}{N_L^{t-1}}\right)^{\beta(\epsilon-1)}. \tag{3.41}$$

The social planner can implement two policies:

First best policy

In the first best policy taxes and subsidies help to increase the quality of the environment. If the intermediate goods are good substitutes ($\epsilon > 1$) and the time preference rate ρ is sufficiently small (the future is important for the market players) the first best policy leads to a growth rate $\nu\eta_H \geq 0$ in the clean H-sector. The subsidies are only needed till $\frac{\pi_H}{\pi_L} > 1$. As soon as there is a higher technology level in the clean sector, scientists decide to research in this higher developed sector and the economy builds an equilibrium, even without subsidies.
Directed technical change towards the clean sector happens immediately for $\epsilon > \frac{1}{\beta}$. So the production of Y_L dies and the whole research and production takes place in the H-sector. Furthermore, the environment reaches its best condition $\bar{E}$ in finite time. The optimal tax τ is also temporary.
With the first best policy we can definitely avoid an environmental desaster.

Second best policy

It is often hard to invent and monitor subsidies. You cannot clearly draw a line between the clean and the dirty sector. Additional, it is almost impossible to screen the firms to check whether they invest the subsidies in research for the H-sector or not. Since these problems occur and it would cost a lot of money to avoid them, policymakers forswear subsidies and only implement taxes. This is the so called second best policy.
As a consequence, taxes have to be higher than in the first best policy to compensate the missing subsidies. This leads to higher production costs and the output of the economy is less. So the consumption decreases and the utility is reduced.
According to its name, the second best policy can prevent the economy from an environmental desaster, but the economic growth is lower compared to the first best policy.

3.3.3 The laissez-faire equilibrium with exhaustible resources

We have analysed the equilibria with unlimited resources. Since it is even more likely that resources like e.g. crude oil or several minerals are exhaustible, we set $\beta_2 > 0$ in equation (3.5) and obtain the following production function for the dirty sector.

$$Y_L = \quad R^{\beta_2} \quad \left(\int_0^1 N_L(j)^{\beta_1} x_L(j)^{1-\beta_1} dj \right) L^{\beta} \tag{3.42}$$

No property rights for resources

We assume no property rights for exhaustible resources. So everyone can access them and just has to pay the resource extraction costs $c(Q)$. In this section we calculate the laissez-faire equilibrium analogous to section 3.1. Since we do not have changes in the clean H-sector, we illuminate only the dirty L-sector.
The profit maximisation of the technology monopolists $\max_{\chi_L} \pi_L(j) = \max_{\chi_L}(\chi_L(j) - \psi)x_L(j)$ (3.25) in the L-sector leads to the equilibrium price

$$\chi_L(j) = \frac{\psi}{1-\beta_1}. \tag{3.43}$$

The profit Π_L for the firms in the dirty sector includes the resource R, too.

$$\max_{L,\{x_L(j)\},R} \Pi_L \quad = \quad \max_{L,\{x_L(j)\},R} p_L Y_L - w_L L - c(Q)R - \int_0^1 \chi_L(j) x_L(j) dj \tag{3.44}$$

Substituting the price (3.43) into the FOC with respect to x_L gives the equilibrium-demand for machines.

$$x_L(j) \quad = \quad \left(\frac{(1-\beta_1)^2 p_L R^{\beta_2} L^{\beta}}{\psi}\right)^{\frac{1}{\beta_1}} N_L(j) \tag{3.45}$$

The equilibrium wages w_L in the L-sector are

$$w_L = \beta p_L R^{\beta_2}\Big(\int_0^1 N_L(j)^{\beta_1} x_L(j)^{1-\beta_1} dj\Big) L^{\beta-1} \tag{3.46}$$

$$= \beta(1-\beta_1)^{2\frac{1-\beta_1}{\beta_1}} p_L^{\frac{1}{\beta_1}} R^{\frac{\beta_2}{\beta_1}} \psi^{\frac{\beta_1-1}{\beta_1}} L^{\frac{\beta-\beta_1}{\beta_1}} N_L. \tag{3.47}$$

Since the resources are also part of the firms profit maximisation, we get a third FOC with respect to R. The equilibrium resource extraction costs can be written as

$$c(Q) = \beta_2 p_L R^{\beta_2-1}\Big(\int_0^1 N_L(j)^{\beta_1} x_L(j)^{1-\beta_1} dj\Big) L^{\beta}. \tag{3.48}$$

Intuitively the costs are positively related to all factors describing the size of the market like the number of workers in the L-sector, the number of machines, their technology level and the price of Y_L. Only to the amount of resources the relation is negativ. If more resources are used, the extraction is cheaper. But this only works for one period. In the long term using more resources means Q decreases and therefore $c(Q)$ increases.

Substituting the equilibrium number of machines (3.45) into (3.48) gives

$$R = \left(\frac{(1-\beta_1)^2}{\psi}\right)^{\frac{1-\beta_1}{\beta}} \left(\frac{\beta_2 N_L}{c(Q)}\right)^{\frac{\beta_1}{\beta}} p_L^{\frac{1}{\beta}} L. \tag{3.49}$$

If the market is bigger (p_L, L and N_L) more goods are produced and therefore more resources are needed. On the other hand, if the costs for the production (machine costs ψ and resource extraction costs $c(Q)$) are higher, less resources are processed.

Plugging (3.45) and (3.49) into the production function gives the equilibrium output.

$$Y_L = \left(\frac{(1-\beta_1)^2}{\psi}\right)^{\frac{1-\beta_1}{\beta}} \left(\frac{\beta_2}{c(Q)}\right)^{\frac{\beta_2}{\beta}} N_L^{\frac{\beta_1}{\beta}} p_L^{\frac{1-\beta}{\beta}} L \tag{3.50}$$

The interpretion for the output level is similar to the resource. Obviously, price, technology level and labour supply push the output, whereas the costs diminish it.

The equlibrium profits of the technology monopolists (3.25) and (3.26) can be expressed using equilibrium number of machines $x_L(j)$ (3.45) and $x_H(j)$ (3.22).

$$\pi_H(j) = (\chi_H(j)-\psi)x_H(j) = \Big(\frac{\psi}{1-\beta}-\psi\Big)\Big(\frac{(1-\beta)^2 p_H}{\psi}\Big)^{\frac{1}{\beta}} H N_H(j) \tag{3.51}$$

$$\pi_L(j) = (\chi_L(j)-\psi)x_L(j) = \Big(\frac{\psi}{1-\beta_1}-\psi\Big)\Big(\frac{(1-\beta_1)^2 p_L R^{\beta_2} L^{\beta}}{\psi}\Big)^{\frac{1}{\beta_1}} N_L(j) \tag{3.52}$$

So the expected profits of scientists π_L and π_H can be written as

$$\pi_H = \eta_H \frac{\psi\beta}{1-\beta}\Big(\frac{(1-\beta)^2 p_H}{\psi}\Big)^{\frac{1}{\beta}} H(1+\nu)N_H^{t-1} \tag{3.53}$$

$$\pi_L = \eta_L \frac{\psi\beta_1}{1-\beta_1}\Big(\frac{(1-\beta_1)^2 p_L R^{\beta_2} L^{\beta}}{\psi}\Big)^{\frac{1}{\beta_1}} (1+\nu)N_L^{t-1} \tag{3.54}$$

Directed technical change

The relative expected profit for scientists is now given by

$$\frac{\pi_H}{\pi_L} = \kappa \frac{\eta_H}{\eta_L} \times \underbrace{\frac{1}{R^{\beta_2}}}_{\text{resource effect}} \times \underbrace{\frac{p_H^{\frac{1}{\beta}}}{p_L^{\frac{1}{\beta_1}}}}_{\text{price effect}} \times \underbrace{\frac{H}{L^{\frac{\beta}{\beta_1}}}}_{\text{market size effect}} \times \underbrace{\frac{N_H^{t-1}}{N_L^{t-1}}}_{\text{direct productivity effect}} \tag{3.55}$$

with $\kappa := \frac{\beta(1-\beta)^{\frac{2-\beta}{\beta}}}{\beta_1(1-\beta_1)^{\frac{2-\beta_1}{\beta_1}}} \psi^{\frac{1}{\beta_1}-\frac{1}{\beta}}$.

Similar to the first case without exhaustible resources we obtain the price effect, the market size effect and the direct productivity effect, whereas different exponents occur for the first two effects. These exponents only change the speed of the directed technical change with respect to the price and market size, but not the direction. What is new, scientists prefer to work in the clean H-sector if more resources are needed for the production of the dirty intermediate good. This phenomenon is called resource effect.

As shown in Appendix B.3 we can rewrite the equation.

$$\frac{\pi_H}{\pi_L} = \bar{\kappa} \frac{\eta_H}{\eta_L} c(Q)^{\beta_2(\epsilon-1)} \frac{(1+\nu\eta_H H_S)^{\beta(\epsilon-1)-1}}{(1+\nu\eta_L L_S)^{\beta_1(\epsilon-1)-1}} \frac{(N_H^{t-1})^{\beta(\epsilon-1)}}{(N_L^{t-1})^{\beta_1(\epsilon-1)}} \tag{3.56}$$

with $\bar{\kappa} := \frac{\beta(1-\beta)}{\beta_1(1-\beta_1)^{\frac{\beta_1+\beta_2}{\beta_1}}} \left(\frac{(1-\beta)^{2(1-\beta)}}{\psi^{\beta_2}(1-\beta_1)^{2(1-\beta_1)}\beta_2^{\beta_2}} \right)^{\epsilon-1}$.

When we consider the directed technical change in terms of the technology level and the number of scientists, we see that in the long term the research will take place in the clean sector only if $\epsilon > 1$. The growth rate of the economy is $\nu\eta_H$. If the quality of the environment is high enough in the beginning, we can avoid using all the resources until the technical change appears and, therefore, prevent the economy from an environmental desaster.

Nevertheless, the laissez-faire equilibrium cannot avoid environmental externalities and externalities in the R&D like in section 3.1 without exhaustible resources. Moreover, $c(Q)$ only displays the costs for extraction and does not include the loss of quality of the environment when Q is increasing.

Property rights for resources

In the previous section every firm could get as many resources for the price $c(Q)$ as it wanted for its production. Now we want to analyse the laissez-faire equilibrium with exhaustible resources, when firms (price takers) need property rights to extract resources. To simplify, extraction costs $c(Q) = c$ are constant. The price of the resource P is changing over time. The marginal utility of the profit in the current period has to equal the marginal utility of the discounted profit in the next period.

$$\frac{\partial u(C(t),E(t))}{\partial C}(P(t)-c)=\frac{1}{1+\rho}\frac{\partial u(C(t+1),E(t+1))}{\partial C}(P(t+1)-c) \tag{3.57}$$

We assume the coefficient of the risk aversion θ constant and introduce a specific utility function. The preferences concerning the consumption and the quality of environment are separable, whereas the environmental quality is given by a function $\mu(E)$ with the typical behaviour of an utility function: $\mu' > 0$ and $\mu'' < 0$.

$$u(C,E)=\frac{(C)^{1-\theta}}{1-\theta}+\mu(E) \tag{3.58}$$

Using the *Hotelling rule*, we know that the price of the resource P is asymptotically growing at the interest rate r given by the Euler equation

$$r=(1+\rho)(1+g_c)^{\theta}-1. \tag{3.59}$$

We recall, ρ is the rate of time preference and g_c is the growth rate of consumption.

Similar to the previous chapter we can derive the relative expected profit for scientists, where the costs $c(Q)$ are now replaced by the resource price P.

$$\frac{\pi_H}{\pi_L}=\bar{\kappa}\frac{\eta_H}{\eta_L}P^{\beta_2(\epsilon-1)}\frac{(1+\nu\eta_H H_S)^{\beta(\epsilon-1)-1}}{(1+\nu\eta_L L_S)^{\beta_1(\epsilon-1)-1}}\frac{(N_H^{t-1})^{\beta(\epsilon-1)}}{(N_L^{t-1})^{\beta_1(\epsilon-1)}} \tag{3.60}$$

with $\bar{\kappa}$ like in equation (3.56). The price of the resource only causes a technical change towards the H-sector, if the intermediate goods are good substitutes ($\epsilon > 1$).
The result does not only depend on ϵ, but also on ρ. So we analyse two different cases:

$$\ln(1+\rho)>\frac{\beta_1}{\beta_2}\ln(1+\nu\eta_d) \tag{3.61}$$

and

$$\ln(1+\rho)<\frac{\beta_1}{\beta_2}\ln(1+\nu\eta_d) \tag{3.62}$$

In the first case ρ and ϵ are both sufficiently high (3.61). In the long term research takes place only in the clean H-sector, given that the initial quality of environment is sufficiently high.
Intuitively, if people are concerned about the future and can use alternative technologies, they are interested in innovation in the clean sector. So they can avoid an environmental desaster.

In the second case (3.62) requires ρ and ϵ to be small. Research remains in the L-sector and it is not possible to avoid an environmental desaster.
In other words, if people are only interested in current profits and do not care about future generations, they are not going to invest in sustainable innovations. Thus, the quality of environment decreases more and more.

Let us consider a third case study. We assume research in the H-sector only and the growth rate in this sector is $g_H = \nu\eta_H$. Analougous we find the growth rate g_L for Y_L in the dirty sector.

$$\ln(1 + g_L) = (1 - \epsilon\beta)\ln(1 + \nu\eta_H) - \epsilon\beta_2(\ln(1 + \rho) + \theta\ln(1 + \nu\eta_H)) \tag{3.63}$$

If the dirty sector is still growing ($g_L > 0$) an environmental desaster occurs. In contrast, if the production of Y_L decreases ($g_L < 0$) and the initial quality of the environment $\bar{E}$ is high enough, we can avoid an environmental desaster.

To sum up, directed technical change occurs depending on the increase of the price of the resource P. If P is high or increases rapidly, firms prefer to produce in the clean sector to gain higher profits. So expensive exhaustible resources help to enable innovations in the clean sector and therefore avoid an environmental desaster. This is happening even faster, if the intermediate goods are better substitutes and if the decision maker focus more on the future.

3.3.4 The socially optimal allocation with exhaustible resources

Analogous to section 3.3.2 we introduce policies to diminish externalities. Like in section 3.3.3 we need exhaustible resources R for the production in the dirty sector (3.42).

We introduce taxes τ for the dirty input Y_L like e.g. carbon-taxes, subsidies q for research in the clean sector to compensate the little initial knowledge in the clean sector and subsidies for the firms which have not used all their machines to their capacity.
Additional to these policies a resource tax is implemented. It prices the difference between the resource extraction costs and the social value of the resource.

Without property rights

Similar to the previous section, we first look at the case without property rights. So every producer of the dirty intermediate good can extract as many resources as he wants to.
The profit of the producers of the dirty intermediate good also includes the resource tax τ_R. The resource dynamics are $Q(t + 1) = Q(t) - R(t)$. We denote the Langrange multiplier for the resource dynamics with ℓ. We can also write the resource dynamics for all periods, where ι is the extraction time in the past given in years.

$$\sum_{\iota=0}^{\infty} R(\iota) \leq Q(0) \tag{3.64}$$

The Langrange multiplier for equation (3.64) is denoted by the nonnegative variable ι.
Firms have to take the resource dynamics into account, since they use resources to produce the intermediate good Y_L. Their profit is the difference between their earnings and their costs. They sell their good Y_L for the price p_L and have to pay wages w_L for the workers L, the costs of the resources $c(Q)$ in terms of the shadow value of the final good λ for the

resources R and the machine costs $\int_0^1 \chi_L(j)x_L(j)dj$ for all machine types j. We can write the firms profit maximisation problem as follows.

$$\max_{L,\{x_L(j)\},R} \Pi_L = \max_{L,\{x_L(j)\},R} p_L Y_L - w_L L - \lambda c(Q)R - \int_0^1 \chi_L(j)x_L(j)dj \tag{3.65}$$

subject to

$$Q(t+1) = Q(t) - R(t) \tag{3.66}$$

$$\sum_{\iota=0}^{\infty} R(\iota) \leq Q(0) \tag{3.67}$$

So we have to maximise the following function.

$$\max_{L,\{x_L(j)\},R(t)} \Bigg(p_L(t)Y_L(t) - w_L(t)L(t) - \lambda c(Q)R(t) - \int_0^1 \chi_L(j)(t)x_L(j)(t)dj \\ -\ell\big(Q(t+1) - (Q(t) - R(t))\big) - \iota\big(\sum_{\iota=0}^{\infty} R(\iota) - Q(0)\big)\Bigg) \tag{3.68}$$

The FOC with respect to L and $x_L(j)$ are the same as for (3.44) in the previous section 3.3.3. If we maximise (3.68) with respect to the resource $R(t)$ we obtain the following demand curve for R.

$$\beta_2 p_L R^{\beta_2-1}\left(\int_0^1 N_L(j)^{\beta_1} x_L(j)^{1-\beta_1} dj\right) L^{\beta} = \lambda c(Q) + \ell + \iota \tag{3.69}$$

In the previous section the shadow value for one final good λ is used for the derivation of the price in the steady state $\hat{p}_L = \frac{p_L}{\lambda}$ and $\hat{p}_H = \frac{p_H}{\lambda}$. So we can rewrite (3.69).

$$\beta_2 \hat{p}_L R^{\beta_2-1}\left(\int_0^1 N_L(j)^{\beta_1} x_L(j)^{1-\beta_1} dj\right) L^{\beta} = c(Q) + \frac{\ell + \iota}{\lambda} \tag{3.70}$$

The term $\frac{\ell+\iota}{\lambda}$ gives the amount of resource needed for the production of Y_L in terms of the final good.

The FOC of (3.68) with respect to Q gives the law of motion for the shadow price ℓ of the resource.

$$\ell(t) = \ell(t-1) + \lambda(t)c'(Q(t))R(t) \tag{3.71}$$

The shadowprice depends on the price in the previous period and increases with higher marginal extraction costs and an increasing demand of resources.

The costs of extraction are taxed to equal the price of the resource used for the production of Y_L.

$$c(Q)\tau_R = \frac{\ell + \iota}{\lambda} \tag{3.72}$$

We see that the rate of the resource tax $\tau_R = \frac{\ell+\iota}{\lambda c(Q)}$ is always positive. So the tax has to be charged forever to remain in the steady state and avoid an environmental desaster.

With both taxes and the subsidy the allocation is a social optimum.

Property rights for resources

With the policies discussed in 3.3.4 we can avoid an environmental desaster for the case with property rights for resources, too. If the intermediate goods are good substitutes, it is even enough to just give temporary subsidies for the clean sector to preserve good environmental quality and trigger a directed technical change towards the clean sector.

The price of the resource P is asymptotically growing at the interest rate r.

$$r = (1 + \rho)(1 + \nu\eta_c)^{\theta} - 1 \tag{3.73}$$

4 Extension - Alternative technology

In the previous models we analysed two different sectors. Adding environment was leading us to a sector with green production and a sector where production negatively influences the quality of our environment. We now switch to a new framework with an alternative technology based on the additional work of Acemoglo, Aghion, Bursztyn, Hemous (2012) in section II.E and the online appendix II.
Production takes place in one sector, but we still differentiate between two different types of innovations. They can either enhance productivity or reduce pollution. The first type is used in many growth models to express successful research, which increases the production of the final goods. Pollution-reducing innovations can be e.g. the development of new machines using less energy or natural ressources, or the invention of new flue gas filter. Green technologies can also directly influence the environment like planting trees.

In the model with environment from Acemoglo, Aghion, Bursztyn, Hemous (2012) the sector with green technologies could increase output and decrease the use of natural ressources at the same time. In reality we are more often confronted with two different types of technologies: One kind of innovations augments the quality or quantity of the final good,. The alternative type of innovations, so called *green innovations*, does not affect the final good itself, but the environment. It reduces pollution ($\xi Y_L(t)$) as part of the production process or increases the regeneration rate of the environment ($\zeta E(t)$). Including the impact of technology in the dynamics of the environmental quality yields a new dynamic equation for the environment.

$$E(t+1) = -\xi \int_0^1 e(j)^\beta x(j)^{1-\beta} dj + (1+\zeta)E(t) \tag{4.1}$$

replaces $E(t+1) = -\xi Y_L(t) + (1+\zeta)E(t)$ in the previous section. Instead of the negative influence of the intermediate good Y_L of the L-sector we introduce a term of the produced pollution $\xi \int_0^1 e(j)^\beta x(j)^{1-\beta} dj$. Analogous to the previous model ξ measures the influence of the production process of the good on the environment. The exogenous time dependent $e(j)$ describes the amount of pollution a machine of type i produces.

Since the production takes place in one sector only, the production of the final good is given by

$$Y = \int_0^1 N(j)^\beta x(j)^{1-\beta} dj. \tag{4.2}$$

Analogous to the other models, the amount of machines of machine type j is $x(j)$. They are using the technology $N(j)$ (instead of $N_L(j)$ and $N_H(j)$).
Innovations can either decrease $e(j)$ or increase $N(j)$. To make sure that both changes

have a similar impact, the same exponent β is used in both functions (4.1) and (4.2).

Here, the labour supply equals 1. So we can interprete the production function as the production functions in the previous models with labour as a complimentary production factor to the machines.

As in the other models, the machines $x(j)$ are produced monopolistically for marginal costs $\psi = (1-\beta)^2$ in terms of the final good. The profit maximisation of the technology monopolists $\max_\chi \pi(j) = \max_\chi (\chi(j) - \psi)x(j)$ leads to the equilibrium price

$$\chi(j) = \frac{\psi}{1-\beta} = 1-\beta. \tag{4.3}$$

We will not analyse all four types of the model (see table 3.3) as we did in the previous chapter, but we assume a taxrate $\tau \geq 0$ for produced pollution and an optimal subsidy q. Firms receive the price p for one final good and have to pay costs of $\chi(j)$ for the machines. The optimal subsidy has to enable both types of research ($N(j)$ and $e(j)$). An increase in $N(j)$ yields an increase in productivity and therefore an increase in the profit. In contrast, a change in $e(j)$ only affects the environment and profits remain with their old value. Profit maximisation is the driving factor for technical change, so we need subsidies for the use of machines with better technology $e(j)$. These subsidies have to equal $p - \chi(j)$. Since we use the price p for the final good as nummeraire and $\chi(j) = 1-\beta$ the optimal subsidy is $1-(1-\beta) = \beta$.

The profit Π for the firms is the revenue minus the costs for machines and the taxes for dirty machines.

$$\max_{\{x(j)\}} \Pi = \max_{\{x(j)\}} pY - \int_0^1 \chi(j)(1-\beta)x(j)dj - \int_0^1 \tau e(j)^\beta x(j)^{1-\beta}dj \tag{4.4}$$

$$= \max_{\{x(j)\}} \int_0^1 N(j)^\beta x(j)^{1-\beta}dj - \int_0^1 \frac{\psi}{1-\beta}(1-\beta)x(j)dj - \int_0^1 \tau e(j)^\beta x(j)^{1-\beta}dj. \tag{4.5}$$

Substituting the marginal costs $\psi = (1-\beta)^2$ into the FOC with respect to x gives the equilibrium-demand for machines.

$$x(j) = (1-\beta)^{-\frac{1}{\beta}}(N(j)^\beta - \tau e(j)^\beta)^{\frac{1}{\beta}} \tag{4.6}$$

For a higher technology level and less taxes for pollution more machines are requested.

Since $\chi(j) = \frac{\psi}{1-\beta}$ and $\psi = (1-\beta)^2$, the equilibrium profit of the technology monopolists is

$$\begin{aligned} \pi(j) &= (\chi(j) - \psi)x(j) = (\frac{\psi}{1-\beta} - \psi)x(j) && (4.7)\\ &= (1-\beta)(\beta)(1-\beta)^{-\frac{1}{\beta}}(N(j)^\beta - \tau e(j)^\beta)^{\frac{1}{\beta}} && (4.8)\\ &= \beta(1-\beta)^{-\frac{1-\beta}{\beta}}(N(j)^\beta - \tau e(j)^\beta)^{\frac{1}{\beta}} && (4.9) \end{aligned}$$

More innovations lead to a higher profit whereas more pollution or higher taxes decrease

the profit of the technology monopolists.
R&D affects both terms: A fraction s of all scientists does research in the productivity-enhancing sector to increase $N(j)$. For successful research the technology level rises with the factor $(1+\nu(1-s))$ with $\nu > 1$. All the other $(1-s)$ scientists develop innovations to decrease $e(j)$. A new innovation reduces the pollution of a machine by a factor $(1-\varsigma s)$ with $\varsigma < 1$.
The allocation of scientists to their research field is random. Also the type of machines they are working on is random.

We still have to identify, when a new technology is invented. Therefore we consider two different cases. First, we assume monopolists have patents for their machine type j. This patent expires with a probability ι_b after b periods. As long as the monopolist has the patent, he can increase its technology $N(j)$. Of course, if the patent lasts only for one period ($b = 1$) he can not invent new technologies and $\iota_b = 0$. After the expiry date other monopolists can invent new innovations for this type.
In the second case we allow knowledge spillovers to develop technologies *standing on the shoulders of giants.* Monopolists can increase the technology of a machine type j every period and sell these machines. To make sure that the monopolist who has the patent for type j (and therefore sold this machine type in the previous period) still gains profits, the monopolist selling machine type j in the current period has to pay the missed profit to the owner of the patent. That is the profit the monopolist would have made selling machines with the old technology.
If the patent expired before, the monopolist with the best technology for machine type j becomes the new patent owner.
Since all of the technology monopolists can do reasearch at the same time, but only one receives the patent to sell the machines, *creative destruction* is possible.

4.1 Equilibrium

We analyse a symmetric equilibrium, so all technologies are time dependent, but equal in each period:

$$N(j) \equiv N \tag{4.10}$$
$$e(j) \equiv e \tag{4.11}$$

Even though we assume input taxes and subsidies for research on $e(j)$, we calculate two different cases. A decentralised equilibrium and a centralised equilibrium. In the first case, the monopolists and all consumers are maximising their profits, in the second case a social planner maximises the aggregate utility function.

4.1.1 Decentralised equilibrium

Technology monopolists are maximising their profits. As mentioned above, innovation is possible with or without knowledge spillovers. We have to consider both cases to write the profits of the monopolists.

Patents ensure the single right for new innovations

Only the monopolist with the patent for a machine type is allowed to increase its technology. So only the patent holder receives the subsidy q for research on e. Researchers decide whether they develop N or e. The present and future allocations given by the fraction $\{s(t+k)\}_{k=0}^{\infty}$ have to maximise the expected profits of the monopolists. To make a profit, they still need to have the patent. So we introduce the term $(1-\iota_b)$, which is the probability of having the patent and zero after loosing the ownership of the patent. Furthermore we have to consider the interest rate $r(t+k)$ to calculate the net present discounted value. Since technologies are symmetric, also the profits are equal in every period ($\pi(j) \equiv \pi$). Additional to the profits patent owners who develop N receive subsidies q.

$$\max_{\{s(t+k)\}_{k=0}^{\infty}} \sum_{k=0}^{\infty} \prod_{b=0}^{k} \big(\frac{1-\iota_b}{1+r(t+b)}\big)\big(\pi(t+k)+q(t+k)s(t+k)\big) = \tag{4.12}$$

$$\max_{\{s(t+k)\}_{k=0}^{\infty}} \sum_{k=0}^{\infty} \prod_{b=0}^{k} \big(\frac{1-\iota_b}{1+r(t+b)}\big)\big(\beta(1-\beta)^{-\frac{1-\beta}{\beta}}(N(t+k)^{\beta}-\tau(t+k)e(t+k)^{\beta})^{\frac{1}{\beta}} +q(t+k)s(t+k)\big) \tag{4.13}$$

The technologies in each period depend on the previous technology for k periods in which they own the patent multiplied with the probabilty of invention of new innovations. Since $(1-s(t+k))$ scientists are doing research for increasing N and $s(t+k)$ scientists try to decrease e we face the two dynamic equations.

$$N(t+k) = \big(1+\nu(1-s(t+k))\big)N(t+k-1) \tag{4.14}$$

$$e(t+k) = \big(1-\varsigma s(t+k)\big)N(t+k-1) \tag{4.15}$$

Creative destruction

In the second case knowledge spillovers are possible. So monopolists do not know whether they will be able to sell their machines in the coming periods or if others invent a better technology for the same machine type. But in case they have the patent, they gain the same amount of profit as long as they own the patent. Therefore the maximisation of the profits is done for only one period t. As a consequence, the profit and the subsidies are also only relevant for the current period. After this period the patent owner stops research and only gains rents from the innovation till the patent expires. Technically, the maximisation is done for only one decision variable $s(t)$. Furthermore, we only need to discount the profits as long as the monopolist owns the patent $(b = 0, ..., k)$ and obtain the subsidies only in the current period $(q(t)s(t))$.

$$\max_{s(t)} \sum_{k=0}^{\infty} \prod_{b=0}^{k} (\frac{1-\iota_b}{1+r(t+b)}) \pi(t) + q(t)s(t) = \tag{4.16}$$

$$\max_{s(t)} \sum_{k=0}^{\infty} \prod_{b=0}^{k} (\frac{1-\iota_b}{1+r(t+b)}) \beta(1-\beta)^{-\frac{1-\beta}{\beta}} (N(t)^{\beta} - \tau(t)e(t)^{\beta})^{\frac{1}{\beta}} + q(t)s(t) \tag{4.17}$$

For both specification consumers maximise utility with respect to the consumption. The marginal utility times the interest rate has to equal the marginal utility in terms of the consumption and the environment of the previous period considering the discount rate $\rho > 0$.

$$(1+r(t))\frac{\partial u}{\partial C}(C(t), E(t)) = (1+\rho)\frac{\partial u}{\partial C}(C(t-1), E(t-1)) \tag{4.18}$$

The utility function u is the same as in equation (3.1).

4.1.2 Centralised equilibrium

The profit maximsation of the technology monopolists is the same as for the decentralised equilibrium.
To calculate the social optimum, a social planner maximes the discounted utility functions over all periods with respect to the allocation of scientists, the consumption, the environment, the output of the economy, the technology level and the full amount of machines $X(t)$.

$$\max_{\{s(t),C(t),E(t),Y(t),N(t),X(t)\}} \sum_{k=0}^{\infty} \frac{1}{(1+\rho)^t} u(C(t), E(t)) \tag{4.19}$$

The social planner maximises under the following constraints.
The output can be used either for consumption or for buying new machines with price $(1-\beta)^2$.

$$Y(t) = C(t) + (1-\beta)^2 X(t) \tag{4.20}$$

The output is given analogous to the aggregate production functions of the intermediate goods in the previous chapter.

$$Y(t) = N(t)^{\beta} X^{(1-\beta)} \tag{4.21}$$

The dynamic equation of the environment is simplified. The current quality of environment is the regenerated environmental quality of the former period (regeneration rate ζ) minus the pollution gained by the amount of used machines in the production process.

$$E(t+1) = (1+\zeta)E(t) - e(t)^{\beta} X(t)^{(1-\beta)} \tag{4.22}$$

For the maximisation problem of the social planner the technology level depends on the level of only one previous period.

$$N(t+1) = \big(1+\nu(1-s(t+1))\big)N(t) \tag{4.23}$$
$$e(t+1) = \big(1-\varsigma s(t+1)\big)e(t) \tag{4.24}$$

And the fraction of scientists working for $N(t)$ instead of $e(t)$ has to fulfill $s(t) \in [0,1]$.

To solve the maximisation problem we form the Langrangian Λ with the Langrangian multipliers λ_n.

$$\begin{aligned}\Lambda = \quad & \sum_{k=0}^{\infty} \frac{1}{(1+\rho)^t} u(C(t), E(t)) \\ & +\lambda_1(t)(Y(t) - C(t) - (1-\beta)^2 X(t)) \\ & +\lambda_2(t)(Y(t) - N(t)^\beta X^{(1-\beta)}) \\ & +\lambda_3(t+1)(E(t+1) - (1+\zeta)E(t) + e(t)^\beta X(t)^{(1-\beta)}) \\ & +\lambda_4(t+1)(N(t+1) - \big(1+\nu(1-s(t+1))\big)N(t)) \\ & +\lambda_5(t+1)(e(t+1) - \big(1-\varsigma s(t+1)\big)e(t)) \\ & +\lambda_6(t)(s(t)) \\ & +\lambda_7(t)(s(t)-1)\end{aligned} \tag{4.25}$$

The first order condition with respect to the consumption $C(t)$ is

$$\frac{\partial\Lambda}{\partial C(t)} = \frac{1}{(1+\rho)^t}\frac{\partial u(C(t),E(t))}{\partial C(t)} - \lambda_1(t) = 0, \tag{4.26}$$

the FOC with respect to $Y(t)$ is

$$\frac{\partial\Lambda}{\partial Y(t)} = \lambda_1(t) + \lambda_2(t) = 0 \tag{4.27}$$

and the FOC with respect to $X(t)$ is

$$\begin{aligned}\frac{\partial\Lambda}{\partial X(t)} = & -(1-\beta)^2\lambda_1(t) + (1-\beta)N(t)^\beta X(t)^{-\beta}\lambda_2(t) \\ & +e(t)^\beta(1-\beta)X(t)^{-\beta}\lambda_3(t+1) = 0.\end{aligned} \tag{4.28}$$

To find the optimal allocation of research, we look at the FOC with respect to the level of technology $N(t)$

$$\frac{\partial\Lambda}{\partial N(t)} = -\lambda_2(t)\beta N(t)^{(\beta-1)}X^{(1-\beta)} + \lambda_4(t+1) - \lambda_4(t)(1+\nu(1-s(t+1))) = 0 \tag{4.29}$$

and $e(t)$.

$$\frac{\partial\Lambda}{\partial e(t)} = \lambda_3(t+1)\beta e(t)^{(\beta-1)}X(t)^{(1-\beta)} + \lambda_5(t+1) - \lambda_5(t)(1-\varsigma s(t+1)) = 0 \tag{4.30}$$

To sum up, we find the following relations. A more detailled calculation is found in appendix (C.1).

$$\lambda_1(t) = -\lambda_2(t) = \frac{1}{(1+\rho)^t}\frac{\partial u(C(t),E(t))}{\partial C(t)} \tag{4.31}$$

$$(1-\beta)^{-\frac{1}{\beta}}\big(N(t)^\beta - \frac{\lambda_3(t+1)}{\lambda_1(t)}e(t)^\beta\big)^{\frac{1}{\beta}} = X(t) \tag{4.32}$$

$$\lambda_2(t)\beta(1-\beta)^{-\frac{1-\beta}{\beta}}N(t)^{(\beta-1)}\left(N(t)^\beta - \frac{\lambda_3(t+1)}{\lambda_1(t)}e(t)^\beta\right)^{\frac{1-\beta}{\beta}} \tag{4.33}$$
$$+\lambda_4(t)(1+\nu(1-s(t+1))) = \lambda_4(t+1)$$

$$-\lambda_3(t+1)\beta(1-\beta)^{-\frac{1-\beta}{\beta}}e(t)^{(\beta-1)}\left(N(t)^\beta - \frac{\lambda_3(t+1)}{\lambda_1(t)}e(t)^\beta\right)^{\frac{1-\beta}{\beta}} \tag{4.34}$$
$$+\lambda_5(t)(1-\varsigma s(t+1)) = \lambda_5(t+1)$$

As mentioned at the beginning of this chapter, we assume a taxrate τ and an optimal subsidy β. Using (4.32) we can even specify the taxrate.

$$\tau(t) = \frac{\lambda_3(t+1)}{\lambda_1(t)} \tag{4.35}$$

We are only interested in the allocation of $s(t)$. Instead of building the FOC with respect to $s(t)$, we reformulate the maximisation problem. The optimal allocation of the scientists $s(t) \in [0,1]$ generates the highest value of the technology. So we can maximise the technology levels times their shadow prices to obtain a one-dimensional maximisation problem.

$$\max_{s(t)} \lambda_4(t)N(t) + \lambda_5(t)e(t) \tag{4.36}$$

$$= \max_{s(t)} \lambda_4(t)\big(1+\nu(1-s(t+1))\big)N(t-1) + \lambda_5(t)\big(1-\varsigma s(t+1)\big)e(t-1) \tag{4.37}$$

Substituting for $\lambda_4(t)$ (4.34) and $\lambda_5(t)$ (4.35) obtains

$$\max_{s(t)} \lambda_2(t)\beta(1-\beta)^{-\frac{1-\beta}{\beta}}N(t)^\beta\left(N(t)^\beta - \frac{\lambda_3(t+1)}{\lambda_1(t)}e(t)^\beta\right)^{\frac{1-\beta}{\beta}} \tag{4.38}$$
$$+\lambda_4(t+1)\big(1+\nu(1-s(t+1))\big)N(t) + \lambda_5(t+1)\big(1-\varsigma s(t+1)\big)e(t)$$

Using the expression for $\lambda_2(t)$ (4.31), the interest rate in the decentralised equilibrium (4.18) and $\tau = \frac{\lambda_3}{\lambda_1}$ helps writing the maximisation problem in the same way as for the laissez-faire equilibrium.

$$\max_{s(t)} \lambda_2(t)\sum_{k=0}^{\infty}\prod_{b=0}^{k}\frac{1}{(1+r(t+b))}\beta(1-\beta)^{-\frac{1-\beta}{\beta}}\left(N(t+k)^\beta - \tau(t+k)e(t+k)^\beta\right)^{\frac{1}{\beta}} \tag{4.39}$$

Now we can compare the decentralised equilibrium without knowledge spillovers (4.13), with creative destruction (4.17) and the social optimum(4.39). To give an overview the three equations are listed again:

$$\max_{\{s(t+k)\}_{k=0}^{\infty}} \sum_{k=0}^{\infty} \prod_{b=0}^{k} (\frac{1-\iota_b}{1+r(t+b)}) \Big(\beta(1-\beta)^{-\frac{1-\beta}{\beta}} \left(N(t+k)^{\beta} - \tau(t+k)e(t+k)^{\beta}\right)^{\frac{1}{\beta}} + q(t+k)s(t+k) \Big) \tag{4.40}$$

$$\max_{s(t)} \sum_{k=0}^{\infty} \prod_{b=0}^{k} (\frac{1-\iota_b}{1+r(t+b)}) \beta(1-\beta)^{-\frac{1-\beta}{\beta}} \left(N(t)^{\beta} - \tau(t)e(t)^{\beta}\right)^{\frac{1}{\beta}} + q(t)s(t) \tag{4.41}$$

$$\lambda_2 \max_{s(t)} \sum_{k=0}^{\infty} \prod_{b=0}^{k} \frac{1}{(1+r(t+b))} \beta(1-\beta)^{-\frac{1-\beta}{\beta}} \left(N(t+k)^{\beta} - \tau(t+k)e(t+k)^{\beta}\right)^{\frac{1}{\beta}} \tag{4.42}$$

We analyse two different specifications. First, $\iota_b = 0$ in every period. In words, patents do not expire. If a monopolist bought a right, he can not loose it. In this case the three equations only differ in terms of the subsidies q. If we assume no subsidies ($q = 0$), we clearly see that the tax τ is sufficient to consider $e(j)$ for the allocation of $s(t)$. Under this circumstances the result is the same for the decentralised equilibrium as for the social planner. Intuitively, a tax can regulate the laissez-faire markets, so that economic growth and a good quality of the environment is possible.
Second, ι_b is strictly positive for some periods t. So the patent expires after some time and knowledge spillovers as well as creative destruction is possible. To analyse that, we compare the equations (4.41) and (4.42): Without a social planner scientists will prefer to work for $N(t)$ instead of developing pollution-reducing activities, because working for $N(j)$ increases their profits. Therefore we would need research subsidies q to force scientists to do research for $e(t)$.

4.2 Comparison to basic model

The results are similar to the model in the previous chapter. In the laissez-faire equilibrium scientists are only interested in doing research on the technology N which improves the quality and the quantity of the good and therefore increases the value of the good. This leads to higher profits if they do successful research. In contrast, there are no incentives to do research on environmental friendly technologies e, because that would not increase the firms profit and this are the only driving forces for the allocation s of the scientists. Since technological progress only affects N we have unlimited economic growth, but also destruction of the environment. This leads to an environmental desaster. We need perpetual subsidies for research in the *green technologies* to avoid a desaster. This can be done in a decentralised equilibrium.
Since research in green technologies complements research in the production technologies, scientists can either support the environment or increase the economic output. Hence, the economic growth is less than in the previous model, where improvement of green technologies could also increase the economic output.

5 Population Dynamics, Environment and Directed Technical Change

5.1 Adding Population Dynamics

In the previous models we analysed directed technical change between two production sectors with respect to natural resources or differences in education.
Schaefer was the first who combined the analysis of environment and population within a two sector model. According to the paper "Technological Change, Population Dynamics, and Natural Resource Depletion" written by Schaefer, (2012), we change some assumptions of the previous chapters, but mainly stick to the two production sectors including environment. The main alteration is a new framework concerning the population dynamics.
So we can explore the interaction of skill-biased technical change, fertility decline and natural resource use.

Parents can choose if they want to have high or low education for their children and therefore the sector their children work in. Furthermore we take population growth into account. Population growth or decline in each sector influences the economic growth and even the depletion rate of the natural resources. We also identify the impact of the wages on the demand for non-renewable resources.

5.2 Model

A graphical representation of the model is found in appendix Fig.E.3.
Most of the variables are time dependent. I will only write the discrete time variable $t \in (0, \infty)$ to the variables if necessary.

5.2.1 Households and their Optimisation

In the previous models, scientists and workers could decide whether they want to work in the L- or in the H-sector. Now we assume two different types of households: households with unskilled workers, who work in the L-sector, and households with skilled workers for the H-sector.
In our OLG-model, each cohort lives for three periods: childhood, adulthood and old age. In the first period, agents can face three different starting points. First, children can be born in an unskilled household and obtain basic education. So they are going to spend all three periods in the L-sector and we denote their number as n_{LL}. Second, unskilled parents can pay for a better education for their children to enable them to spend their adulthood and old age in H-households. This type of children is called n_{LH}. Third, children n_{HH} are born in skilled households and therefore automatically obtain a better education and spend their whole life within the H-sector. To sum up, it is only possible to move from the L- to the H-sector. So we either face this mobility or no mobility at all. This concept is different to the models we had in chapter (2) and (3), where households supplied two types of labour: scientists and workers for the firms. Agents from each type of labour could decide in which sector they want to work in. In the model of Schaefer the type of household agents are born in is relevant for the sector they work in.

All economically relevant decisions are made as an adult: Additionally to paying the education for their children and saving money for their retirement, adults constitute the labour supply for the firms (L and H) and for the technology monopolists (L_T and H_T) for both sectors L and H, respectively. Note, that in this model there are no scientists L_S and H_S for $R\&D$. Furthermore, adults from the H-households can also work as teacher H_E to educate the offspring.
So we have the total amount of L_Σ or H_Σ adults for each type of household.

$$L_T + L = L_\Sigma \tag{5.1}$$
$$H_T + H + H_E = H_\Sigma \tag{5.2}$$

Old agents can only consume the savings from their adult-period, depending on the sector they worked in.

Since all decisions of the households are made by adults, the utility function represents the preferences of adults only. The utility functions u_{LL}, u_{LH} and u_{HH} in period t reflect the preferences of the cohorts born in $t-1$, depending on the type of household and offspring.

$$u_{LL}(t) = \ln C_L(t) + \quad a\ln(w_L(t+1)n_{LL}) \quad + \rho \ln C_L(t+1) \tag{5.3}$$
$$u_{LH}(t) = \ln C_L(t) + \quad a\ln(w_H(t+1)n_{LH}) \quad + \rho \ln C_L(t+1) \tag{5.4}$$
$$u_{HH}(t) = \ln C_H(t) + \quad a\ln(w_H(t+1)n_{HH}) \quad + \rho \ln C_H(t+1) \tag{5.5}$$

The utility function depends on the current consumption $C_L(t)$ and $C_H(t)$ and on the discounted future consumption $(C_L(t+1))^\rho$ and $(C_H(t+1))^\rho$ when being old. Moreover, parents take the total potential income of their children into account with an altruism factor a, given by the expected wages $w_L(t+1)$ or $w_H(t+1)$ of their offspring times their number of children $n_{..}$.

The labour supply of both types of household is given by

$$L_{\Sigma}(t+1) = \quad (1-h(t))n_{LL}(t)L_{\Sigma}(t) \tag{5.6}$$
$$H_{\Sigma}(t+1) = \quad h(t)n_{LH}(t)L_{\Sigma}(t) + n_{HH}(t)H_{\Sigma}, \tag{5.7}$$

where $h \in [0,1]$ is the fraction of unskilled households investing in a better education for their offspring n_{LH}. Intuitively, the offspring of unskilled households with only basic education is the labour supply of the unskilled households in the next period. In contrast, all children with a better education get a job in the H-sector in the coming period.

We assume the schooling system to be privately funded. In other words, parents have to pay schooling fees. Since teachers H_E are part of the skilled households, they get the wage w_H. Using $\phi = \frac{H_E}{H_{\Sigma}(t+1)}$ as the time independent exogenously fixed teacher-student ratio we can write

$$w_H \phi H_{\Sigma}(t+1) = w_H \phi (n_{LH} h L_{\Sigma} + n_{HH} H_{\Sigma}) = w_H H_E. \tag{5.8}$$

Tuition fees (left hand side of the equation) have to cover the wage sum of teachers (right hand side). $w_H \phi$ are the education costs per child.

In addition to the education costs, parents in both household types have to pay rearing costs for their children, since raising children needs time and special consumption goods, which we can measure in forgone wage earnings. We denote z as the fraction of wage income spent on rearing one child.

Alltogether, raising one child with basic education costs zw_L and raising one child with a better education costs $zw_L + w_H\phi$ if the parents are unskilled or $zw_H + w_H\phi$ if the parents are skilled.

Beside paying for the offspring adults have to build up savings for their old age consumption $C_L(t+1)$ and $C_H(t+1)$. There are two ways to do so: On one hand, they can invest in the capital market. Investing I_L or I_H in terms of the final good gives revenues of $(1+r(t+1))I_L(t)$ or $(1+r(t+1))I_H(t)$ in the following period. On the other hand, adults can buy property rights for natural resources and sell them, when they retire. Buying a number q_L or q_H of resources for a competitive price of $P(t)$ ensures that they can sell the same number for a competitive price $P(t+1)$ either to the next adult generation or to both production sectors to produce consumption goods.
Similar to the model in the previous chapter, we have a resource stock $Q(t)$. Again, the resource dynamics are

$$Q(t) = Q(t-1) - R(t), \tag{5.9}$$

whereas $R(t)$ are the extracted resources for the production. Alternatively, we can write the dynamic equation in terms of the depletion rate τ.

$$R(t) = \tau Q(t-1) \tag{5.10}$$

The initial resource stock $Q(0)$ is positive. The competitive price P is given by the Hotelling rule (see appendix (B.1)). So we know that the marginal return on the capital market $(1 + r(t + 1))$ equals the marginal return on the resource market $\frac{P(t+1)}{P(t)}$. Transforming this equality gives

$$P(t + 1) = (1 + r(t + 1))P(t). \tag{5.11}$$

In summary, old age consumption is

$$\begin{aligned} C_L(t + 1) = &\quad (1 + r(t + 1))I_L(t) + P(t + 1)q_L(t) && (5.12) \\ C_H(t + 1) = &\quad (1 + r(t + 1))I_H(t) + P(t + 1)q_H(t). && (5.13) \end{aligned}$$

Using (5.11) we can rewrite the equation as

$$\begin{aligned} \frac{C_L(t + 1)}{1 + r(t + 1)} &= I_L(t) + P(t)q_L(t) && (5.14) \\ \frac{C_H(t + 1)}{1 + r(t + 1)} &= I_H(t) + P(t)q_H(t). && (5.15) \end{aligned}$$

Households maximise their utility function subject to their budget constraints. Analogous to the preferences of the households, we focus on the budget constraints for adults, because they are the only decision makers in the households. Basically, the wage income has to cover the costs for raising children (rearing and education), the consumption in the current period and the present value of the consumption in the future period, given by (5.14) and (5.15).
Since we have three different types of agents, we face three different budget constraints:

$$\begin{aligned} w_L \geq &\quad (zw_L) &\quad n_{LL} + C_L + (I_L + Pq_L) && (5.16) \\ w_L \geq &\quad (zw_L + w_H\phi) &\quad n_{LH} + C_L + (I_L + Pq_L) && (5.17) \\ w_H \geq &\quad (zw_H + w_H\phi) &\quad n_{HH} + C_H + (I_H + Pq_H) && (5.18) \end{aligned}$$

Unskilled households raising unskilled offspring are facing the budget constraint (5.16), whereas the unskilled households who enable a better education for their children are confronted with (5.17). Skilled households are restricted to (5.18).

Overall we have to solve three optimisation problems in each period.

First, unskilled households raising unskilled children.

$$\max_{C_L, n_{LL}, C_L(t+1), I_L, q_L} \quad u_{LL} \tag{5.19}$$

subject to (5.16).

Following the calculations in appendix (D.1.1) we obtain the optimal consumption, the optimal number of children per household and the optimal amount of savings for old age.

$$C_L = \frac{1}{1+a+\rho} w_L \tag{5.20}$$

$$n_{LL} = \frac{a}{(1+a+\rho)z} \tag{5.21}$$

$$I_L = \frac{\rho}{1+a+\rho} w_L - P q_L \tag{5.22}$$

The consumption is positively related to their wage. The number of children per household increases with a higher altruism factor and decreases with the fraction of the wage, they have to spend on rearing costs. Intuitively, if consumption goods for the offspring are expensive, households can not effort many children. The investment in the man-made capital is of course positively influenced by the discount factor of future consumption ρ and also increases with the income. In contrast, buying more property rights for ressources for a higher price has a negative impact on the investment. Agents buy either property rights or invest in the capital market to provide for their old age period.

Second, unskilled households raising skilled offspring.

$$\max_{C_L, n_{LH}, C_L(t+1), I_L, q_L} u_{LH} \tag{5.23}$$

subject to (5.17). Calculations given in appendix (D.1.2) lead to the following optimal variables.

$$C_L = \frac{1}{1+a+\rho} w_L \tag{5.24}$$

$$n_{LH} = \frac{a}{1+a+\rho} \frac{w_L}{w_L z + w_H \phi} = \frac{a}{1+a+\rho} \frac{1}{z + w\phi} \tag{5.25}$$

$$I_L = \frac{\rho}{1+a+\rho} w_L - P q_L \tag{5.26}$$

Consumption and investment is analogous to unskilled households raising unskilled offspring. But in contrast to the previous case, the number of children per household n_{LH} depends on the relative wage $w = \frac{w_H}{w_L}$ and the student teacher ratio ϕ. If the income gap between skilled and unskilled workers is big, unskilled households can not effort to raise many well educated children (an increase in w leads to a decrease in n_{LH}). It is also easier for them to offer an education to their children, if ϕ is smaller. In other words, if they have bigger classes at school and therefor less teachers they have to pay. Of course, the rearing costs z depress the number of children per household.

Third, skilled households raising skilled offspring face the following optimisation problem.

$$\max_{C_H, n_{HH}, C_H(t+1), I_H, q_H} u_{HH} \tag{5.27}$$

subject to (5.18). Again, we can derive the optimal variables (see appendix (D.1.3)).

$$C_H = \frac{1}{1+a+\rho} w_H \tag{5.28}$$
$$n_{HH} = \frac{a}{1+a+\rho} \frac{1}{z+\phi} \tag{5.29}$$
$$I_H = \frac{\rho}{1+a+\rho} w_H - P q_H \tag{5.30}$$

Except the fact, that we use only variables regarding the H-sector, there is no big difference to the previous cases. It is intuitively clear that under the assumption $w_H > w_L$ high-skilled households can consume and invest more than unskilled households.
However, the number of children per household is influenced by the teacher-student ratio as well as the wage-fraction for rearing costs, but independent of the wage itself. We can see that the number of unskilled children n_{LL} (5.21) is larger than the amount of high-skilled children (5.29). Since education is costly, high-skilled households raise less children than low-skilled household.

We know that the labour intensity for the L- and the H-sector depends on the amount of low-skilled and high-skilled workers. This is given by the number of households and the number of children. The only possible change in these numbers is if low-skilled households educate their children. This happens, if their utility function u_{LH} is greater or equal than every possible u_{LL}. Given the optimal choice of variables

$$u_{LL}(t) = u_{LH}(t) \tag{5.31}$$
$$\ln C_L(t) + a \ln(w_L(t+1) n_{LL}) + \rho \ln C_L(t+1) = \tag{5.32}$$
$$\ln C_L(t) + a \ln(w_H(t+1) n_{LH}) + \rho \ln C_L(t+1)$$

yields

$$\frac{w_H(t+1)}{w_L(t+1)} = \frac{n_{LL}(t)}{n_{LH}(t)} = \frac{a}{1+a+\rho} \frac{1}{z} \frac{1+a+\rho}{a} \frac{z+w(t)\phi}{1} \tag{5.33}$$
$$\frac{w_H(t+1)}{w_L(t+1)} = w(t+1) = \frac{z+w(t)\phi}{z} \tag{5.34}$$

If (5.34) holds, unskilled parents are indifferent about raising low-skilled children or investing in a high education for their children. The expected wage of the children after education in relation to the future wage of unskilled workers has to compensate the extra schooling costs for the parents now.
The percentage of unskilled households raising high-skilled offspring depends on (5.34). Intuitively, if future wages of high-skilled workers are higher than wages of labour supply in the L-sector ($\frac{w_H(t+1)}{w_L(t+1)} > 1$), we know $z + w(t)\phi > z$ and face the following relation of the number of children per household, given by (5.21), (5.25) and (5.29).

$$n_{LL} > n_{LH} > n_{HH} \tag{5.35}$$

Households with low education can effort to raise most children, since their rearing costs are lower and they do not have to pay schooling fees. Both other types of agents have to pay schooling fees, whereas it is cheaper for unskilled households to simply rear children

($zw_L < zw_H$), so they raise more children than high-skilled households.

5.2.2 Production and Analysis

On the production side we have two sectors. The basic scheme is the same as in the previous models: In both sectors *R&D* improves technology which can be used by technology monopolists, who are producing machines. These machines are needed by firms to produce intermediate goods. Combining both intermediate goods creates a final good. This is sold to the households. Moreover, investments of the households and resources are needed for the production process.
A big difference to the previous models is the labour supply. Whereas in the previous models both scientists and workers could choose the sector they want to work in, in this model agents are already directed to the sector according to their education. Moreover, instead of differentiating between scientists doing research in the *R&D* and labour supply for firms, we have workers who can work both in firms and for the technology monopolists, but only in the sector according to their education.
Instead of analysing the directed technical change with the expected profits of the scientists, we now have to use other instruments to measure directed technical change.

Like in the basic model, we use an constant elasticity of substitution (CES)-production function (2.3).

$$Y = \left[\gamma Y_L^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma) Y_H^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} \tag{5.36}$$

Again, the final good is taken as a numeraire and its price equals 1. We keep in mind, that the price is given by the prices of the intermediate goods (3.18).

$$[\gamma^{\epsilon} p_L^{1-\epsilon} + (1-\gamma)^{\epsilon} p_H^{1-\epsilon}]^{\frac{1}{1-\epsilon}} = 1 \tag{5.37}$$

The relative price after market clearing is also the same as in the basic model (2.7).

$$p \equiv \frac{p_H}{p_L} = \frac{1-\gamma}{\gamma}\left(\frac{Y_H}{Y_L}\right)^{-\frac{1}{\epsilon}} \tag{5.38}$$

The CRS production function of the intermediate goods (3.5) and (3.6) can be taken from chapter (2)

$$Y_L = \frac{1}{1-\beta}\left(\int_0^{N_L} x_L(j)^{1-\beta} dj\right) L^{\beta} \tag{5.39}$$

$$Y_H = \frac{1}{1-\beta}\left(\int_0^{N_H} x_H(j)^{1-\beta} dj\right) H^{\beta} \tag{5.40}$$

as well as the equilibrium demand of machines (2.11) and (2.12).

$$x_L(j) = (\frac{p_L}{\chi_L(j)})^{\frac{1}{\beta}} L$$
$$x_H(j) = (\frac{p_H}{\chi_H(j)})^{\frac{1}{\beta}} H$$

Since we have a symmetric equilibrium, we face equal prices and quantities for every type j within the sectors. So we simplify $x_L(j) = x_L$, $x_H(j) = x_H$ and $\chi_L(j) = \chi_L$, $\chi_H(j) = \chi_H$ to write

$$x_L = (\frac{p_L}{\chi_L})^{\frac{1}{\beta}} L \tag{5.41}$$
$$x_H = (\frac{p_H}{\chi_H})^{\frac{1}{\beta}} H. \tag{5.42}$$

We can substitute the equilibrium demand of machines into the production function of intermediate goods to obtain the equilibrium levels of Y_L and Y_H.

$$Y_L = \frac{1}{1-\beta} N_L p_L^{\frac{1-\beta}{\beta}} \chi_L^{\frac{\beta-1}{\beta}} L \tag{5.43}$$
$$Y_H = \frac{1}{1-\beta} N_H p_H^{\frac{1-\beta}{\beta}} \chi_H^{\frac{\beta-1}{\beta}} H \tag{5.44}$$

Note, that for the production of intermediate goods we do not need any resources.

For more detailled analysis we calculate the relative price p in equilibrium. Derivation given in appendix (D.2) leads to

$$p = (\frac{1-\gamma}{\gamma})^{\frac{\epsilon\beta}{\sigma}} (\frac{N_H}{N_L})^{-\frac{\beta}{\sigma}} (\frac{\chi_H}{\chi_L})^{\frac{1-\beta}{\sigma}} (\frac{H}{L})^{-\frac{\beta}{\sigma}}. \tag{5.45}$$

Next, we have a closer look at the labour market. Since both labour markets are competitive, the wages are given by the FOC of the firms profits using Y_L (5.43) and Y_H (5.44) with respect to L and H, respectively. Details are found in appendix (D.3). The skilled wage premium $w = \frac{w_H}{W_L}$ in the equilibrium is

$$w = (\frac{1-\gamma}{\gamma})^{\frac{\epsilon}{\sigma}} (\frac{N_H}{N_L})^{\frac{\sigma-1}{\sigma}} (\frac{\chi_H}{\chi_L})^{\frac{1-\beta}{\beta}\frac{1-\sigma}{\sigma}} (\frac{H}{L})^{-\frac{1}{\sigma}}. \tag{5.46}$$

Basically, less labour supply in one sector leads to higher wages in this sector. So agents are tempt to either support their children with higher education or not. A more detailled analysis of the wage premium needs some background of the machine producing sector. Therefore equation (5.58) will give further details.

After dicussing the behaviour of the firms we analyse the technology monopolists: In the Schaefer-model we introduce a production function for machines. To produce machines of type j technology monopolists need technology N_L or N_H and low-skilled labour L_T or workers with a high education H_T. We differentiate between these workers in the production of machines and the workers L and H for the production of intermediate goods as we had before. Results will show that both types of workers get the same wages w_L in the L-sector and w_H in the H-sector, independent of their workplace (firms for intermediate goods or machine producers). The most important factor in the production function is the resource $R_L(j)$ or $R_H(j)$. In the model of Acemoglu, we found the resource only in the L-sector for the production of intermediate goods. Here, natural resources are used in both sectors, the difference is only the amount of the resource needed for each sector. We even differentiate the amount of resources used for every type of machine j. Resources and labour supply are complementary factors for the production of machines. α gives the importance of resources compared to labour. Moreover, M_L and M_H are positive constants.

$$x_L(j) = M_L N_L (L_T(j))^{1-\alpha}(R_L(j))^{\alpha} \tag{5.47}$$
$$x_H(j) = M_H N_H (H_T(j))^{1-\alpha}(R_H(j))^{\alpha} \tag{5.48}$$

Analogous to the previous models, the profit of the technology monopolists facing the demand of machines (5.41) or (5.42) is given by

$$\max_{\chi_L} \pi_L = \max_{\chi_L} (\chi_L - \psi_L) x_L \tag{5.49}$$
$$\max_{\chi_H} \pi_H = \max_{\chi_H} (\chi_H - \psi_H) x_H. \tag{5.50}$$

As monopolists are restricted to the production function, we do not have simple production costs ψ as in the previous models. Here, the production costs ψ_L and ψ_H depend on the expenses for all production factors in (5.47) and (5.48). The detailled derivation of the cost function is found in appendix (D.4).

$$\psi_L(w_L, P) = \frac{w_L^{1-\alpha} P^{\alpha}}{M_L N_L (1-\alpha)^{1-\alpha}\alpha^{\alpha}} \tag{5.51}$$
$$\psi_H(w_H, P) = \frac{w_H^{1-\alpha} P^{\alpha}}{M_H N_H (1-\alpha)^{1-\alpha}\alpha^{\alpha}} \tag{5.52}$$

Analogous to the calculations in appendix (A.3) we get the price of the machines.

$$\chi_L = \frac{\psi_L}{1-\beta} \tag{5.53}$$
$$\chi_H = \frac{\psi_H}{1-\beta} \tag{5.54}$$

Note, that this price depends on the prices of the production factors of machines. Therefore the profit of the technology monopolists is a more complex function than in the previous models. A detailled derivation is found in appendix (D.5).

$$\pi_L = \beta(1-\beta)^{\frac{1-\beta}{\beta}} p_L^{\frac{1}{\beta}} L \psi_L^{\frac{\beta-1}{\beta}} \tag{5.55}$$
$$\pi_H = \beta(1-\beta)^{\frac{1-\beta}{\beta}} p_H^{\frac{1}{\beta}} H \psi_H^{\frac{\beta-1}{\beta}} \tag{5.56}$$

We see, that the profits now depend negatively on the costs of producing one machine ($\frac{\beta-1}{\beta} < 0$). Since the machine costs ψ are positively correlated to the wages and the resource price, an increase in at least one of those factors yields an increase in the machine costs and therefore a decrease in the profits.

Using (5.53) and (5.54) we can rewrite the relative price as given in (5.45) in the following way:

$$p = \big(\frac{1-\gamma}{\gamma}\big)^{\frac{\epsilon\beta}{\sigma}}\big(\frac{N_H}{N_L}\big)^{-\frac{\beta}{\sigma}}\big(\frac{\psi_H}{\psi_L}\big)^{\frac{1-\beta}{\sigma}}\big(\frac{H}{L}\big)^{-\frac{\beta}{\sigma}}. \tag{5.57}$$

In appendix (D.6) we derive the skilled wage premium, including the optimisation and equilibrium calculations of the technology monopolists above.

$$w = \frac{w_H}{w_L} = \big(\frac{1-\gamma}{\gamma}\big)^{\frac{\epsilon\beta}{\sigma\xi}}\big(\frac{N_H}{N_L}\big)^{\frac{\sigma-1}{\sigma\xi}}\big(\frac{M_L}{M_H}\big)^{\frac{(1-\beta)(\sigma-1)}{\sigma\xi}}\big(\frac{H}{L}\big)^{-\frac{\beta}{\sigma\xi}} \tag{5.58}$$

for $\xi = \beta + (1-\alpha)(1-\beta) \in (0,1)$.
We distinguish between two cases, depending on σ. First, the intermediate goods are gross substitutes ($\sigma > 1$). Then the relative technology level has a positive influence on the wages. Successful research in one sector leads to higher wages in the same sector. Second, if the intermediates are gross compliments, better technology forces smaller wages. Intuitively, labour supply is worth less because machines work more efficiently.
However, in both cases labour has diminishing marginal returns. As described above, if more workers enter the labour market, firms are paying less for everyone. So it can not happen, that all parents try to educate their children. This scenario would lead to small wages in the H-sector and high wages in the L-sector.

Technical Change

Additional to the paper of Schaefer we analyse the technical change. In the basic model we derived the relative net present discounted value $\frac{V_H}{V_L}$ to analyse the direction of the technical change. We apply the same procedure for this model. Using (2.20) and (2.21) with $\dot{V}_L = 0$ and $\dot{V}_H = 0$ gives

$$V_L = \frac{\pi_L}{r} = \frac{\beta(1-\beta)^{\frac{1-\beta}{\beta}} p_L^{\frac{1}{\beta}} L \psi_L^{\frac{\beta-1}{\beta}}}{r} \tag{5.59}$$

$$V_H = \frac{\pi_H}{r} = \frac{\beta(1-\beta)^{\frac{1-\beta}{\beta}} p_H^{\frac{1}{\beta}} H \psi_H^{\frac{\beta-1}{\beta}}}{r} \tag{5.60}$$

For the analysis of the direction of the technical change we look at the relative net present discounted values.

$$\frac{V_H}{V_L} = p^{\frac{1}{\beta}}\frac{H}{L}\big(\frac{\psi_L}{\psi_H}\big)^{\frac{1-\beta}{\beta}} \tag{5.61}$$

Whereas we only had a price effect and a market size effect in the basic model, the relativ marginal costs of machine production influences the directed technical change as well. The sector with lower costs in machine production will grow relative to the other sector.

For a more detailled analysis, we substitute the relative price (5.57) into the relative net present discounted values.

$$\frac{V_H}{V_L} = (\frac{1-\gamma}{\gamma})^{\frac{\epsilon}{\sigma}} (\frac{N_H}{N_L})^{-\frac{1}{\sigma}} (\frac{\psi_H}{\psi_L})^{\frac{1-\beta}{\beta}\frac{1-\sigma}{\sigma}} (\frac{H}{L})^{\frac{\sigma-1}{\sigma}} \tag{5.62}$$

The first term is always a positive constant. If the weight of the intermediate good in the high-skilled sector is greater, the power of the technical change is stronger. The technology levels are exogenously given, so the second term is also a positiv constant. The exponent of the relative costs of the machines depends on σ. If the intermediate goods are gross substitutes ($\sigma > 1$) the exponent is negativ, if they are complimentary goods $\sigma < 1$ there is a positive relation. In other words: lower marginal costs for machine production in the H-sector leads to a directed technical change towards the H-sector only if the intermediate goods are substitutes. Intuitively, cheaper production enables technology monopolists to produce relatively more in this sector. Otherwise lower costs in machine production even force the L-sector to produce relatively more. Furthermore, in (5.62) the exponent of the relative labour supply is positiv, if the intermediate goods are substitutes. Directed technical change will go in the direction of the sector with more labour supply (market size effect).

5.2.3 Research and Development

In each sector $R\&D$ develops blueprints N_L or N_H, but in this model they can be rather interpreted as a new technology level, since we do not differentiate between different machine types. This is more similar to the basic model in chapter (2) than the environment-model in chapter (3).

As described in the basic model, we use the lab-equipment approach. Blueprints are depreciated entirely after one period. Since each technology can be used for only one period, machine producers have to buy new blueprints for the next generation.
To invent a new technology we do not need scientists, as we have in the previous models. Instead, agents can invest in R&D. Their savings D are split in D_L and D_H for each sector. To simplify calculations we assume without loss of generality a linear relation between the investments and the output. Furthermore, the new technology level depends on the old level and productivity parameters η_L and η_H to differentiate between the sectors.

$$N_L(t+1) = \eta_L D_L(t)(N_L(t))^{\delta} \tag{5.63}$$

$$N_H(t+1) = \eta_H D_H(t)(N_H(t))^{\delta} \tag{5.64}$$

δ helps to describe the influence of past technology on the current technology. For a positive δ we face positive knowledge spillovers between generations. If the teachers already know a lot they can teach more and the next generation does succesful research. In case δ is negative, we assume a limited possibility to develope new technologies, therefore it is harder to create new innovations if the technology level is already high.

5.2.4 Employment structure in equilibrium

We face competitive labour markets and firms and technology monopolists are maximising their profits. Therefore we have equal wages for all workers in the same sector, so we only need to differentiate between w_L and w_H.

To clear the markets, the prices for the intermediate goods p_L or p_H times the marginal labour costs $\frac{\partial Y_L}{\partial L}$ or $\frac{\partial Y_H}{\partial H}$ for the firms producing these intermediate goods have to equal the prices of machines χ_L or χ_H multiplied with the marginal labour costs $\frac{\partial X_L}{\partial L_T}$ or $\frac{\partial X_H}{\partial H_T}$ for the technology monopolists, respectively.

$$p_L \frac{\partial Y_L}{\partial L} = \chi_L \frac{\partial X_L}{\partial L_T} \tag{5.65}$$

$$p_H \frac{\partial Y_H}{\partial H} = \chi_H \frac{\partial X_H}{\partial H_T} \tag{5.66}$$

As we see in appendix (D.7) the labour supply for the technology monopolists can be expressed in terms of the labour supply for the firms.

$$L_T = \frac{(1-\alpha)(1-\beta)}{\beta} L \tag{5.67}$$

$$H_T = \frac{(1-\alpha)(1-\beta)}{\beta} H \tag{5.68}$$

Together with full employment (5.1) and (5.2) we receive the following employment structure for $\xi = \beta + (1-\alpha)(1-\beta) \in (0,1)$ and $\tilde{\xi} = (1-\alpha)(1-\beta) \in (0,1)$.

$$L = \frac{\beta}{\xi} L_\Sigma \tag{5.69}$$

$$H = \frac{\beta}{\xi}(H_\Sigma - H_E) \tag{5.70}$$

$$L_T = \frac{\tilde{\xi}}{\xi} L_\Sigma \tag{5.71}$$

$$H_T = \frac{\tilde{\xi}}{\xi}(H_\Sigma - H_E) \tag{5.72}$$

The amount of workers L and H depends on β, the exponent of labour in the production function of the intermediate goods. If labour is more important than machines for firms, they obviously demand more workers. For workers in the machine production we have an invers relation to β.
When we compare both sectors, the proportion of workers at firms or technology monopolists is the same. We only keep in mind, that a number of teachers is used, independent of the demand in the firms. This is shown in equation (5.74)
We can calculate the demand for teachers, since we know the number of children sent to higher education and the student-teacher ratio ϕ.

$$H_E = \phi(n_{HH} H_\Sigma + h n_{LH} L_\Sigma) \tag{5.73}$$

If we substitute the demand of teachers into the employment structure, we can write the employment ratios (see appendix (D.8)).

$$\frac{H}{L} = \frac{H_T}{L_T} = (1 - \phi n_{HH})\frac{H_\Sigma}{L_\Sigma} + \phi h n_{LH} \tag{5.74}$$

To sum up, the relative labour supply in the H-sector increases if more families decide to raise their children with high education and if their are a lot of those families, obviously. If their are more worker in the high-skilled sector already, the ratio is increasing.

5.2.5 Natural resources in equilibrium

We also analyse the resource stock in equilibrium. We know, resources are used for machine production in the low-skilled sector (5.47) as well as in the high skilled sector (5.48). Additional to the resource dynamics given in (5.9) we split the resources into resources for the H-sector and resources for the L-sector. This is given in the following equation.

$$R_L + R_H = R \tag{5.75}$$

As shown in appendix (D.9) the resource allocation in each period can be written as

$$R_L = \frac{1}{1 + \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon}{\sigma}} \left(\frac{N_H}{N_L}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{\psi_H}{\psi_L}\right)^{\frac{(1-\beta)(1-\sigma)}{\beta\sigma}} \left(\frac{H}{L}\right)^{\frac{\sigma-1}{\sigma}}} R \tag{5.76}$$

$$R_H = \frac{\left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon}{\sigma}} \left(\frac{N_H}{N_L}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{\psi_H}{\psi_L}\right)^{\frac{(1-\beta)(1-\sigma)}{\beta\sigma}} \left(\frac{H}{L}\right)^{\frac{\sigma-1}{\sigma}}}{1 + \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon}{\sigma}} \left(\frac{N_H}{N_L}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{\psi_H}{\psi_L}\right)^{\frac{(1-\beta)(1-\sigma)}{\beta\sigma}} \left(\frac{H}{L}\right)^{\frac{\sigma-1}{\sigma}}} R. \tag{5.77}$$

Once again, we analyse two cases: In the first case ($\sigma < 1$) the shares of extracted natural resources decreases with the technology level in its sector, as well as with labour supply in its sector. Higher machine costs reduce the extraction of resources. Intuitively, if all the other production factors are available in at a sufficiently high level, machine producers need less natural resources to create the same output.
In the second case intermediates are gross substitutes ($\sigma > 1$). The relation between the resources and the other production factors is positiv. Technology monopolists extract more natural resources if more people are working in the firms, since the firms demand machines.
We keep in mind, that the shares of extracted resources depends on γ, the technology level, the labour supply and the relative wage, since the relative costs are

$$\frac{\psi_L}{\psi_H} = \frac{M_H}{M_L}\frac{N_H}{N_L} w^{\alpha-1} = \frac{M_H}{M_L}\frac{N_H}{N_L} \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon\beta}{\sigma\xi}} \left(\frac{N_H}{N_L}\right)^{\frac{\sigma-1}{\sigma\xi}} \left(\frac{M_H}{M_L}\right)^{\frac{(1-\beta)(\sigma-1)}{\sigma\xi}} \left(\frac{H}{L}\right)^{-\frac{\beta}{\sigma\xi}} \tag{5.78}$$

$$\frac{\psi_L}{\psi_H} = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon\beta(\alpha-1)}{\sigma\xi}} \left(\frac{N_H}{N_L}\right)^{\frac{1-\alpha(1-\beta\sigma)}{\sigma\xi}} \left(\frac{M_H}{M_L}\right)^{\frac{(1-\beta)(1-\alpha)+\beta\sigma}{\sigma\xi}} \left(\frac{H}{L}\right)^{-\frac{\beta(\alpha-1)}{\sigma\xi}}. \tag{5.79}$$

Using (5.79) we can rewrite (5.76) and (5.77).

$$R_L = \frac{1}{1+\left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon(\sigma+\tilde{\xi}}{\sigma^2\xi}}\left(\frac{N_H}{N_L}\right)^{\frac{(\sigma-1)(\beta\sigma+\xi)}{\beta\sigma^2\xi}}\left(\frac{H}{L}\right)^{\frac{(\sigma-1)\tilde{\xi}}{\sigma^2\xi}}\left(\frac{M_H}{M_L}\right)^{\frac{(1-\beta)(\sigma-1)(\beta\sigma+\tilde{\xi})}{\beta\sigma^2\xi}}}R \tag{5.80}$$

$$R_H = \frac{\left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon(\sigma+\tilde{\xi}}{\sigma^2\xi}}\left(\frac{N_H}{N_L}\right)^{\frac{(\sigma-1)(\beta\sigma+\xi)}{\beta\sigma^2\xi}}\left(\frac{H}{L}\right)^{\frac{(\sigma-1)\tilde{\xi}}{\sigma^2\xi}}\left(\frac{M_H}{M_L}\right)^{\frac{(1-\beta)(\sigma-1)(\beta\sigma+\tilde{\xi})}{\beta\sigma^2\xi}}}{1+\left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon(\sigma+\tilde{\xi}}{\sigma^2\xi}}\left(\frac{N_H}{N_L}\right)^{\frac{(\sigma-1)(\beta\sigma+\xi)}{\beta\sigma^2\xi}}\left(\frac{H}{L}\right)^{\frac{(\sigma-1)\tilde{\xi}}{\sigma^2\xi}}\left(\frac{M_H}{M_L}\right)^{\frac{(1-\beta)(\sigma-1)(\beta\sigma+\tilde{\xi})}{\beta\sigma^2\xi}}}R \tag{5.81}$$

The exponents of the relative technology level $\frac{N_H}{N_L}$ and the relative labour supply $\frac{H}{L}$ is again positive for $\sigma > 1$ and negative for $\sigma < 1$, so the results from above hold.

In conclusion, depending on the factors mentioned above, either the shares of resources in the L-sector are increasing and the shares of resources in the H-sector are decreasing, or the other way around. We did not have these effects in the previous models, since we did not differentiate between R_L and R_H in the model with environment in chapter (3). To compare the models, R_H has to equal 0 and therefore $R_L = R$.

After analysing the allocation of the resources, we next consider the R&D-sector, to investigate how the depletion rate of the natural resources can be influenced.

A better technology leads to a greater production of machines. Therefore we write the profit of the R&D in terms of the expected profit of the machine producers. The discounted profits of the technology monopolists $\frac{\pi_L}{1+r}$ and $\frac{\pi_H}{1+r}$ gives the price of one blueprint. The price of the final output ($= 1$) is the profit of the blueprints times the marginal productivity $\frac{\partial N_L(t+1)}{\partial D_L}$ and $\frac{\partial N_H(t+1)}{\partial D_H}$ taken from the R&D functions (5.63) and (5.64).

$$1 = \frac{\pi_L(t+1)}{1+r(t+1)}\eta_L(N_L(t))^\delta \tag{5.82}$$

$$1 = \frac{\pi_H(t+1)}{1+r(t+1)}\eta_H(N_H(t))^\delta \tag{5.83}$$

When we combine (5.82) and (5.83) we make sure, neither technology monopolists in the L-sector nor in H-sector make profits out of different market prices. This gives the non-arbitrage condition.

$$\frac{\pi_H(t+1)}{\pi_L(t+1)} = \frac{\eta_L}{\eta_H}\left(\frac{N_L(t)}{N_H(t)}\right)^\delta \tag{5.84}$$

Derivation given in (D.10) leads to the relative level of technology.

$$\frac{N_H(t+1)}{N_L(t+1)} = \tag{5.85}$$

$$\left(\frac{\eta_L}{\eta_H}\left(\frac{N_H(t)}{N_L(t)}\right)^\delta\right)^{\frac{\beta+(\sigma-1)\xi}{\beta-\alpha(1-\beta)(\sigma-1)}} \tag{5.86}$$

$$\left(\left(\frac{1-\gamma}{\gamma}\right)^{\epsilon\beta}\left(\frac{M_H}{M_L}\right)^{(\sigma-1)(1-\beta)}\left(\frac{H(t+1)}{L(t+1)}\right)^{(\sigma-1)\xi}\right)^{\frac{1}{\beta-\alpha(1-\beta)(\sigma-1)}} \tag{5.87}$$

We assume $\sigma > 1$. $\frac{M_H}{M_L}$ has a positive influence on the relative technology level of the coming period. Intuitively, more efficient research in one sector leads to a higher amount of blueprints in the same sector. More labour supply for firms and therefore also for machine production (see (5.74)) increases the technology level in its sector either. Research will always be biased towards the sector with relatively more labour supply and a more successful research.
Furthermore, we differentiate between $\delta > 0$ (positive knowledge spillovers) and $\delta < 0$ (limited possibility to create new technologies). So δ gives the speed of new innovations.

The number of new blueprints (as discussed in (5.63) and (5.64)) also depends on the investments D. Since we model a closed economy, the investments equal the savings without the value of the property rights for resources. The aggregate savings are the net savings of both types of households L and H.

$$D = (I_L - Pq_L)L + (I_H - Pq_H)H \tag{5.88}$$

Using equilibrium expressions for the variables, we can write the aggregate savings as

$$D \;=\; p_L Y_L \underbrace{\left(\frac{\rho}{1+a+\rho}\beta(1+w\frac{H}{L}) - \frac{\alpha(1-\beta)}{\phi_L}\frac{1-\tau}{\tau}\right)}_{\bar{D}}. \tag{5.89}$$

For details see appendix (D.11). Aggregate savings depend on two factors. First, the output of the L-sector as part of the whole output of the economy, since $Y = \left[\gamma Y_L^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma)Y_H^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}}$. Second, a function $\bar{D}$. We require both factors positive to get a non-trivial solution, therefore D has to be positive as well. For a positive $\bar{D}$ $\frac{\rho}{1+a+\rho}\beta(1+w\frac{H}{L}) > \frac{\alpha(1-\beta)}{\phi_L}\frac{1-\tau}{\tau}$ is necessary. So the share for investments for old-age period $\frac{\rho}{1+a+\rho}$ has to be sufficiently high for both sectors (5.22) and (D.26). If the relative wage or the relative labour supply increase towards the H-sector, the aggregate savings rise as well. Also the depletion rate of the natural resources plays a role.

All the investments of the agents are spent on R&D, which become the value of successful research. Market clearing in R&D makes sure, that all the future aggregate profits of the technology monopolists equal the revenues of the aggregate savings of the households.

$$N_L(t+1)\pi_L(t+1) = (1+r(t+1))D_L(t) \tag{5.90}$$
$$N_H(t+1)\pi_H(t+1) = (1+r(t+1))D_H(t) \tag{5.91}$$

We can substitute the results from the machine producers (5.82) and (5.83).

$$\frac{N_L(t+1)}{\eta_L(N_L(t))^\delta} = D_L(t) \tag{5.92}$$
$$\frac{N_H(t+1)}{\eta_H(N_H(t))^\delta} = D_H(t) \tag{5.93}$$

We put both sectors together and obtain

$$\frac{N_H(t+1)}{N_L(t+1)} = \frac{\eta_H}{\eta_L}\Big(\frac{N_H(t)}{N_L(t)}\Big)^\delta \frac{D_H(t)}{D_L(t)}. \tag{5.94}$$

Since the aggregate savings have to equal the Investments in R&D for both sectors ($D = D_L + D_H$) we can write the investments for each sector in terms of the aggregate savings.

$$\begin{aligned} D_L &= \frac{1}{1+\frac{D_H}{D_L}} D && (5.95)\\ D_L &= \frac{1}{1+\frac{N_H(t+1)}{N_L(t+1)}\frac{\eta_L}{\eta_H}\left(\frac{N_H(t)}{N_L(t)}\right)^{\delta}} D && (5.96)\\ D_H &= D - D_L = \left(1 - \frac{1}{1+\frac{D_H}{D_L}}\right) D && (5.97)\\ D_H &= \frac{\frac{N_H(t+1)}{N_L(t+1)}\frac{\eta_L}{\eta_H}\left(\frac{N_H(t)}{N_L(t)}\right)^{\delta}}{1+\frac{N_H(t+1)}{N_L(t+1)}\frac{\eta_L}{\eta_H}\left(\frac{N_H(t)}{N_L(t)}\right)^{\delta}} D && (5.98) \end{aligned}$$

$\frac{N_H(t+1)}{N_L(t+1)}$ is given by equation (5.85). The share of investments in each sector is obviously positively related to the expected technology level in its sector. More investments in one sector lead to a better output in terms of blueprints in the same sector. On the other hand, less efficiency in the research function even requires more investments. If the recent level of technology has a positive or a negative influence depends on δ.

Since in equilibrium aggregate output and aggregate demand for machines are equal in each sector, we can finally calculate the depletion rate of the natural resources. For details see appendix (D.12).

$$\tau(t+1) = \frac{\tau(t)}{1-\tau(t)} \left(\frac{\bar{D}}{\beta(1-\beta)(1+\frac{N_H(t+1)}{N_L(t+1)}\frac{\eta_L}{\eta_H}\left(\frac{N_H(t)}{N_L(t)}\right)^{\delta})} \right)^{\frac{1}{1-\beta}} (g^{\phi_L})^{-1} \qquad (5.99)$$

with $\bar{D} = \frac{\rho}{1+a+\rho}\beta(1+w\frac{H}{L}) - \frac{\alpha(1-\beta)}{\phi_L}\frac{1-\tau}{\tau}$.
The depletion rate in the next period depends on the depletion rate in the current period. If the depletion rate is already high, the depletion rate in the coming period is high as well. It is inversly related to the growth rate of ϕ_L, the share of resources spent in the L-sector. Since $\bar{D}$ is positively correlated to the depletion rate, but also to the aggregate savings (D.119), the depletion rate increases when D increases. The term including the recent and the future technology level in (5.96) influences not only the share of savings for the L-sector D_L, but also the depletion rate.

5.3 Long run equilibrium

Finally, to analyse the dynamic model, we focus on the four dynamic equations explaining the laws of motion for the population ratio derived from (5.6) and (5.7),

$$\frac{H_\Sigma(t+1)}{L_\Sigma(t+1)} = \frac{h(t)n_{LH}(t) + n_{HH}(t)\frac{H_\Sigma(t)}{L_\Sigma(t)}}{(1-h(t))n_{LL}(t)} \qquad (5.100)$$

the fraction of unskilled households raising skilled offspring, determined by the relative wage (5.34),

$$\frac{w_H(t+1)}{w_L(t+1)} = w(t+1) = \frac{z + w(t)\phi}{z} \tag{5.101}$$

the relative technology level (5.85),

$$\frac{N_H(t+1)}{N_L(t+1)} = \tag{5.102}$$

$$\Big(\frac{\eta_L}{\eta_H}(\frac{N_H(t)}{N_L(t)})^{\delta}\Big)^{\frac{\beta+(\sigma-1)\xi}{\beta-\alpha(1-\beta)(\sigma-1)}} \tag{5.103}$$

$$\Big((\frac{1-\gamma}{\gamma})^{\epsilon\beta}(\frac{M_H}{M_L})^{(\sigma-1)(1-\beta)}(\frac{H(t+1)}{L(t+1)})^{(\sigma-1)\xi}\Big)^{\frac{1}{\beta-\alpha(1-\beta)(\sigma-1)}} \tag{5.104}$$

and the depletion rate (5.99).

$$\tau(t+1) = \frac{\tau(t)}{1-\tau(t)}\Big(\frac{\bar{D}}{\beta(1-\beta)(1+\frac{N_H(t+1)}{N_L(t+1)}\frac{\eta_L}{\eta_H}(\frac{N_H(t)}{N_L(t)})^{\delta})}\Big)^{\frac{1}{1-\beta}}(g^{\phi_L})^{-1} \tag{5.105}$$

The balanced growth path is calculated in appendix D.13. We obtain the following long run equilibrium ratios for constant $\frac{H_\Sigma}{L_\Sigma}, \frac{N_H}{N_L}, h, \tau, w, p_L, p_H$. The long run equilibrium is unique.

The relation of skilled and unskilled labour (5.100) in the long run equilibrium is given by

$$\frac{H_\Sigma}{L_\Sigma} = \frac{hn_{LH}}{(1-h)n_{LL} - n_{HH}}. \tag{5.106}$$

Obviously, the most important factor is the fraction h of unskilled households raising skilled offspring and their total number n_{LH} in equilibrium. h is implicitely given by (5.101)

$$w = \frac{z}{z-\phi}. \tag{5.107}$$

The difference between the fraction of wage income for raising children and the teacher student ratio decreases the relative wage. More teacher educating children leads to relatively higher wages for the H-sector.

The technology level in the long run is

$$\frac{N_H}{N_L} = \Big(\frac{\eta_H}{\eta_L}^{\beta+(\sigma-1)\xi}(\frac{1-\gamma}{\gamma})^{\epsilon\beta}(\frac{M_H}{M_L})^{(\sigma-1)(1-\beta)}(\frac{H}{L})^{(\sigma-1)\xi}\Big)^{\frac{1}{\beta-\alpha(1-\beta)(\sigma-1)-\delta(\beta+(\sigma-1)\xi)}} \tag{5.108}$$

with a positive exponent. Better efficiency in R&D increases the technology level, as well as higher coefficients for the production function of machines (in case of intermediate goods as substitutes). Also more labour supply in the H- or L-sector lead to a better technology level in the H- or L-sector, respectively.

Based on (5.99) the equilibrium depletion rate is determined by

$$\tau = 1 - \left(\frac{\bar{D}}{\beta(1-\beta)(1 + \frac{\eta_L}{\eta_H}\left(\frac{N_H}{N_L}\right)^{\delta+1})} \right)^{\frac{1}{1-\beta}}. \tag{5.109}$$

Note, that the depletion rate does not depend on the producing sectors, but only on their technology. Only if we assume that higher education leads to a production with less resources and investments are done in this sector, we can decrease the extraction of natural resources.

New innovations are expressed with the growth rate of the technology level, which is the same for both sectors and we denote it therefor with g^N. A detailled derivation of the growthrate is found in Schaefer, (2012), p.44f, appendix B.9.

$$g^N = \left(n^{\xi}(1-\tau)^{\alpha(1-\beta)}\right)^{-\frac{1}{\delta}} \tag{5.110}$$

In the steady state n is the average number of children per household. n can be expressed in terms of all three types of households: $n = (1-h)n_{LL} = h\frac{L_\Sigma}{H_\Sigma}n_{LH} + n_{SS}$.
We see that n plays an important role for the growth rate of the technology. We analyse two cases. For a negative δ population growth increases the technology level. In case the population size is constant in the steady state, the output of the economy is decreasing since its long run growth rate is (g-1). Skill-biased technological change would even make it worse, because the fertility rate in the H-sector is declining. Since $\tau \in [0,1]$ the maximimum output can only be 0 with $n = 1$ and $\delta < 0$ in (5.110) and $\tau = 0$. For a positive depletion rate in combination with a constant population we cannot avoid a shrinking output of the economy, as long as there is a negative feedback effect between the level of technology and R&D.
If research is "standing on shoulders of giants" ($\delta > 0$) we even need constant or decreasing population to gain economic growth. Skill-biased technological change would help to compensate the missing population growth and still enables an economic growth rate $g - 1 > 0$.

6 Conclusion

The thesis describes macroeconomic endogenous growth models, where production takes place within two sectors. A first idea of such a two-sector model is explained in the first chapter, based on Acemoglu (2002). We analyse the biased directed technical change within both sectors. Basically, the sectors represent different technologies. In the first model, we can interpret the specifications of the sectors in different ways. E.g. one sector has higher skilled labour supply or one sector has to use natural resources for the production of its good. To focus on one interpretation we assumed one technology uses low skilled workers and the other sector requests high skilled workers. So we see the effects of skill-biased technological change. Moreover, we also find appropriate assumptions to analyse capital- or labour-augmenting directed technical change.

Profit incentives can direct the technical change. Two different kind of changes are possible: The price effect and the market size effect: If the price of the goods in one sector is increasing, more researchers will work on better technologies in this sector due to their higher expected profits. The market size effect is caused by a high amount of labour supply in one sector. This increases the expected earnings for scientists in this sector. So more research is done and the technology level rises in this sector. As a consequence, changes in the research itself can direct technical change. If new innovations are positively correlated to the old technology level, skill biased technological change is more plausible. If we assume no state dependence for the dynamics of the technology level, we see the effects of capital-biased change more clearly.

We next introduced the paper of Acemoglu, Aghion, Bursztyn and Hemous (2012), where the factor environment is added to the model of Acemoglu (2002). One sector needs natural resources for its production. The amount of existing resources is an important part of the quality of environment. The environmental quality influences the utility function of the consumers. Again, profit incentives direct technical change. In addition to the price and the market size effect a direct productivity effect occurs. The productivity of the past periods influences the current production. The output of one sector can increase more, if this sector is already more developed, e.g. more experienced workers share their knowledge and technologies of the previous periods can be used for new innovations.
We calculate four different model specifications to see how to avoid an environmental desaster:

1. A laissez-faire equilibrium with renewable resources
2. A centralised equilibrium with renewable resources
3. A laissez-faire quilibrium with exhaustible resources
4. A centralised equilibrium with exhaustible resources

In the first case without any regulations and renewable resources, it is impossible to avoid a desaster under realistic parameter values. Scientists will always decide to do research in the more developed sector, since they can expect higher profits there. The sector using natural resources is better developed in the beginning. Under this assumption production will always use natural resources.
The social planner in the second scenario enables a more optimistic result. In case agents have a low time preference rate and two policies (taxes and subsidies for production without natural resources) are introduced, directed technical change towards the green sector occurs and a desaster can be avoided.
The third scenario promises a higher environmental quality than the first scenario. If the price for the exhaustible resources is high enough, scientists start doing research in the green sector to avoid the use of resources in the production process. This leads to a directed technical change towards the green sector. If it does not take too long to change to technologies without exhaustible resources the environment can regenerate. So an environmental desaster can be avoided. Obviously, it is easier to switch from the production of one good to the other, if the goods are good substitutes.
The most optimistic result is given in scenario 4. If resources can not be renewed, the price of the resources is increasing like in the third scenario. Moreover, the introduction of the two policies (taxes and subsidies) can successfully avoid an environmental desaster, because the policies ensure that the technical change towards the green sector is happening fast enough. Production in the green sector (without using any resources) is much cheaper for the firms, since they receive subsidies, whereas firms in the other sector even have to pay taxes for the resources. Therefore research in this sector is more profitable. As a consequence, scientists can expect higher profits in the green sector and will do their research in this sector.

Because the assumption that dirty and green technologies both can cause economic growth is not always true, we improve the structure of the two sectors. Following the framework of Acemoglu, Aghion, Bursztyn and Hemous (2012), we model two different kind of technologies in an alternative way. Instead of two sectors producing intermediate goods (which are used for the final good), only one technology is producing (final) goods. This technology uses resources and causes pollution. The second technology is an environmental friendly technology. It can not increase the output of the economy, but enhances the quality of the environment.
Analysing this model, we see that taxes are sufficient to ensure a good quality of environment in the long term. Depending on the sustainability of research, subsidies towards the environmental friendly technology can be necessary to ensure a long lasting good environment.

Last, but not least, we improved the two sector model once more following Schaefer (2012). In addition to the environment we also include population dynamics in the economic growth model for directed technical change. An overlapping generations model helps modelling the population growth and the decision of households, whether they want to enable good education for their offspring. The results of the more complicated model are similar to the results of the previous models.
However, the direction of the technical change is not only given by the price and the market size effect, but also by the educational choice of the parents. If they can afford a higher education for their children, labour supply in the high-skilled sector increases and

technological change goes towards this sector.
Natural resources are used in both sectors. Therefore economic growth is always negatively correlated to the quality of environment. In contrast to the previous models, there is no chance to enable economic growth without avoiding a significant decrease of environmental quality. Nevertheless, we can identify the relation between population growth and the depletion rate of natural resources. It would be interesting in further research to connect education and the use of natural resources.
The choice for higher education leads to a decline in the fertility rate, since the rearing costs per child increase. As an indirect consequence of the fertility decline, the quality of the environment improves.
Independent of the population growth, long run economic growth is only possible if research is built on the shoulders of giants. So that knowledge spillovers are possible. If this is not the case, the decreasing population even leads to a decline in output of the economy.

To sum up, all models describe directed technical change under different assumptions. Regarding the balanced growth path, firms maximise their profits and households maximise their utility function. Since the environment is not (a significant) part of the decisions of the individuals, a social planner has to introduce policies to reach goals concerning environment or population growth. So economic growth is possible in a stable equilibrium.

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A Appendix

A.1 Price of intermediate goods

To calculate the price for the intermediate goods Y_H and Y_L we differentiate the production function (2.3)

$$Y = \left[\gamma Y_L^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma) Y_H^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} \tag{A.1}$$

of the final good with respect to Y_H or Y_L to obtain the prices p_H and p_L for each good. This are the prices that clear of the competitive market.

$$p_L = \frac{\partial Y}{\partial Y_L} = \frac{\epsilon}{\epsilon-1}\left[\gamma Y_L^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma) Y_H^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}-1}(\gamma)\frac{\epsilon-1}{\epsilon} Y_L^{\frac{\epsilon-1}{\epsilon}-1} \tag{A.2}$$

$$= \left[\gamma Y_L^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma) Y_H^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{1}{\epsilon-1}}(\gamma) Y_L^{\frac{-1}{\epsilon}} \tag{A.3}$$

$$p_H = \frac{\partial Y}{\partial Y_H} = \frac{\epsilon}{\epsilon-1}\left[\gamma Y_L^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma) Y_H^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}-1}(1-\gamma)\frac{\epsilon-1}{\epsilon} Y_H^{\frac{\epsilon-1}{\epsilon}-1} \tag{A.4}$$

$$= \left[\gamma Y_L^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma) Y_H^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{1}{\epsilon-1}}(1-\gamma) Y_H^{\frac{-1}{\epsilon}} \tag{A.5}$$

So the relative price can be written as

$$p \equiv \frac{p_H}{p_L} = \frac{1-\gamma}{\gamma}\left(\frac{Y_H}{Y_L}\right)^{-\frac{1}{\epsilon}}. \tag{A.6}$$

A.2 First Order Conditions for the firms profit

Since we face symmetric profit funcitons, we present the calculations only for the L-sector.

We want to maximise the profit of the firms, given by

$$\max_{L,\{x_L(j)\}} \Pi_L = \max_{L,\{x_L(j)\}} p_L Y_L - w_L L - \int_0^{N_L} \chi_L(j) x_L(j) dj \tag{A.7}$$

with $Y_L = \frac{1}{1-\beta}\left(\int_0^{N_L} x_L(j)^{1-\beta} dj\right) L^\beta$

FOC with respect to the number of machines

Differentiation with respect to x_L leads to

$$\frac{\partial \Pi_L}{\partial x_L} = p_L \frac{1-\beta}{1-\beta} x_L(j)^{-\beta} L^{\beta} - \chi_L(j) = 0. \tag{A.8}$$

which yields

$$x_L(j) = \big(\frac{p_L}{\chi_L(j)}\big)^{\frac{1}{\beta}} L \tag{A.9}$$

Analogous we can derive x_H.

FOC with respect to labour

Differentiation with respect to L gives

$$\frac{\partial \Pi_L}{\partial L} = p_L \frac{\beta}{1-\beta} \bigg(\int_0^{N_L} x_L(j)^{1-\beta} dj \bigg) L^{\beta-1} - w_L = 0 \tag{A.10}$$

which yield

$$w_L = \frac{\beta}{1-\beta} p_L \bigg(\int_0^{N_L} x_L(j)^{1-\beta} dj \bigg) L^{\beta-1}. \tag{A.11}$$

Analogous we find w_H.

A.3 First Order Conditions for the technology monopolists profit

Again, we only focus on the L-sector, knowing that the calculation for the H-sector is analogous.
We want to maximise the profit of the technology monopolists

$$\max_{\chi_L} \pi_L(j) \quad = \quad \max_{\chi_L} (\chi_L(j) - \psi) x_L(j) \tag{A.12}$$

facing the demand of machines $x_L = \big(\frac{p_L}{\chi_L(j)}\big)^{\frac{1}{\beta}} L$ (2.11). Differentiation with respect to the price of machines they sell gives

$$\frac{\partial \pi_L(j)}{\partial \chi_L(j)} = \left(\frac{p_L}{\chi_L(j)}\right)^{\frac{1}{\beta}} L + (\chi_L(j) - \psi) \frac{-1}{\beta} p_L{}^{\frac{1}{\beta}} \chi_L(j)^{-\frac{1}{\beta}-1} L = 0. \tag{A.13}$$

We can rewrite the last equation as

$$\begin{aligned}\chi_L(j)^{\frac{-1}{\beta}} &= (\chi_L(j)-\psi)\frac{1}{\beta}\chi_L(j)^{\frac{-1-\beta}{\beta}}\\ 1 &= \frac{1}{\beta}-\psi\frac{1}{\beta}\chi_L(j)^{-1}\end{aligned}$$

and finally obtain

$$\chi_L(j)=\frac{\psi}{1-\beta}. \tag{A.14}$$

A.4 Elasticity of substitution

We want to derive the elasticity of substitution between the factors L and H in the production function Y (2.6) given by

$$Y=\left[\gamma\big[\frac{1}{1-\beta}\big(\int_0^{N_L}x_L(j)^{1-\beta}dj\big)L^{\beta}\big]^{\frac{\epsilon-1}{\epsilon}}+(1-\gamma)\big[\frac{1}{1-\beta}\big(\int_0^{N_H}x_H(j)^{1-\beta}dj\big)H^{\beta}\big]^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}}.$$

Since the elasticity of substitution is defined as

$$\sigma=\frac{d\ln(\frac{L}{H})}{d\ln(\frac{Y_L}{Y_H})} \tag{A.15}$$

we first derive the marginal productivities $\frac{\partial Y}{\partial L}$ and $\frac{\partial Y}{\partial H}$.

$$\begin{aligned}\frac{\partial Y}{\partial L} &=\\ Y^{\frac{\epsilon}{\epsilon-1}-1}\gamma\left[\frac{1}{1-\beta}\big(\int_0^{N_L}x_L(j)^{1-\beta}dj\big)L^{\beta}\right]^{\frac{\epsilon-1}{\epsilon}-1}\frac{1}{1-\beta}\big(\int_0^{N_L}x_L(j)^{1-\beta}dj\big)\beta L^{\beta-1} &=\\ Y^{\frac{1}{\epsilon-1}}\gamma\left[\frac{1}{1-\beta}\big(\int_0^{N_L}x_L(j)^{1-\beta}dj\big)\right]^{\frac{\epsilon-1}{\epsilon}}L^{\beta\left(\frac{\epsilon-1}{\epsilon}-1\right)}\beta L^{\beta-1} &=\\ Y^{\frac{1}{\epsilon-1}}\gamma\left[\frac{1}{1-\beta}\big(\int_0^{N_L}\left(\big(\frac{p_L}{\chi_L(j)}\big)^{\frac{1}{\beta}}L\right)^{1-\beta}dj\big)\right]^{\frac{\epsilon-1}{\epsilon}}\beta L^{\frac{\beta\epsilon-\beta-\epsilon}{\epsilon}} &=\\ Y^{\frac{1}{\epsilon-1}}\gamma\left[\frac{1}{1-\beta}\big(\int_0^{N_L}\left(\frac{p_L}{\chi_L(j)}\right)^{\frac{1-\beta}{\beta}}dj\big)\right]^{\frac{\epsilon-1}{\epsilon}}\beta L^{\frac{\beta\epsilon-\beta-\epsilon}{\epsilon}+(1-\beta)\frac{\epsilon-1}{\epsilon}} &=\\ Y^{\frac{1}{\epsilon-1}}\gamma\left[\frac{1}{1-\beta}\big(\int_0^{N_L}\left(\frac{p_L}{\chi_L(j)}\right)^{\frac{1-\beta}{\beta}}dj\big)\right]^{\frac{\epsilon-1}{\epsilon}}\beta L^{-\frac{1}{\epsilon}}\end{aligned}$$

Analogous we get

$$\frac{\partial Y}{\partial H}=Y^{\frac{1}{\epsilon-1}}(1-\gamma)\left[\frac{1}{1-\beta}\big(\int_0^{N_H}\left(\frac{p_H}{\chi_H(j)}\right)^{\frac{1-\beta}{\beta}}dj\big)\right]^{\frac{\epsilon-1}{\epsilon}}\beta H^{-\frac{1}{\epsilon}}.$$

So we can write the marginal rate of substitution

$$MRS = -\frac{\frac{\partial Y}{\partial L}}{\frac{\partial Y}{\partial H}} = -\frac{\gamma}{1-\gamma}\big(\frac{L}{H}\big)^{\frac{-1}{\epsilon}} \frac{\big[\int_0^{N_L} \big(\frac{p_L}{\chi_L(j)}\big)^{\frac{1-\beta}{\beta}} dj\big]^{\frac{\epsilon-1}{\epsilon}}}{\big[\int_0^{N_H} \big(\frac{p_H}{\chi_H(j)}\big)^{\frac{1-\beta}{\beta}} dj\big]^{\frac{\epsilon-1}{\epsilon}}}. \tag{A.16}$$

Substituting the relative price p (2.7) gives the following relation.

$$-\frac{\frac{\partial Y}{\partial L}}{\frac{\partial Y}{\partial H}} = -\frac{\gamma}{1-\gamma}\big(\frac{L}{H}\big)^{\frac{-1}{\epsilon}} \frac{\big[\int_0^{N_L} \big(\frac{1}{\chi_L(j)}\big)^{\frac{1-\beta}{\beta}} dj\big]^{\frac{\epsilon-1}{\epsilon}}}{\big[\int_0^{N_H} \big(\frac{1}{\chi_H(j)}\big)^{\frac{1-\beta}{\beta}} dj\big]^{\frac{\epsilon-1}{\epsilon}}} \Bigg(\frac{1-\gamma}{\gamma}\big(\frac{Y_L}{Y_H}\big)^{-\frac{1}{\epsilon}}\Bigg)^{\frac{1-\beta}{\beta}\frac{\epsilon-1}{\epsilon}}$$

$$-\frac{\frac{\partial Y}{\partial L}}{\frac{\partial Y}{\partial H}} = -\frac{\gamma}{1-\gamma}^{1-\frac{1-\beta}{\beta}\frac{\epsilon-1}{\epsilon}} \big(\frac{L}{H}\big)^{\frac{-1}{\epsilon}} \frac{\big[\int_0^{N_L} \big(\frac{1}{\chi_L(j)}\big)^{\frac{1-\beta}{\beta}} dj\big]^{\frac{\epsilon-1}{\epsilon}}}{\big[\int_0^{N_H} \big(\frac{1}{\chi_H(j)}\big)^{\frac{1-\beta}{\beta}} dj\big]^{\frac{\epsilon-1}{\epsilon}}} \Bigg(\frac{\big(\int_0^{N_L} x_L(j)^{1-\beta} dj\big) L^\beta}{\big(\int_0^{N_H} x_H(j)^{1-\beta} dj\big) H^\beta}\Bigg)^{-\frac{1}{\epsilon}\frac{1-\beta}{\beta}\frac{\epsilon-1}{\epsilon}}$$

$$-\frac{\frac{\partial Y}{\partial L}}{\frac{\partial Y}{\partial H}} = -\frac{\gamma}{1-\gamma}^{1-\frac{1-\beta}{\beta}\frac{\epsilon-1}{\epsilon}} \big(\frac{L}{H}\big)^{\frac{-1}{\epsilon}-\frac{1-\beta}{\epsilon}\frac{\epsilon-1}{\epsilon}} \frac{\big[\int_0^{N_L} \big(\frac{1}{\chi_L(j)}\big)^{\frac{1-\beta}{\beta}} dj\big]^{\frac{\epsilon-1}{\epsilon}}}{\big[\int_0^{N_H} \big(\frac{1}{\chi_H(j)}\big)^{\frac{1-\beta}{\beta}} dj\big]^{\frac{\epsilon-1}{\epsilon}}} \Bigg(\frac{\big(\int_0^{N_L} x_L(j)^{1-\beta} dj\big)}{\big(\int_0^{N_H} x_H(j)^{1-\beta} dj\big)}\Bigg)^{-\frac{1}{\epsilon}\frac{1-\beta}{\beta}\frac{\epsilon-1}{\epsilon}}$$

We rewrite that as

$$\ln\big(\frac{L}{H}\big) = \epsilon - (\epsilon-1)(1-\beta)\Bigg[\ln\Bigg(\frac{\frac{\partial Y}{\partial L}}{\frac{\partial Y}{\partial H}}\Bigg) - \zeta \ln\Bigg(\frac{\gamma}{1-\gamma}\Bigg) - \ln\Bigg(A\Bigg)\Bigg]$$

with ζ and A appropriate constants for a simplified notation and finally derive the elasticity of substitution

$$\sigma = \frac{d\ln(\frac{L}{H})}{d\ln(\frac{\frac{\partial Y}{\partial L}}{\frac{\partial Y}{\partial H}})} = 1 + \beta\epsilon - \beta = \epsilon - (\epsilon-1)(1-\beta). \tag{A.17}$$

A.5 Price function in 2.3.1

We derive the price function for the relative price of the intermediate goods.

$$p = \frac{p_H}{p_L} = \Bigg(\frac{1-\gamma}{\gamma}\Bigg)\Bigg(\frac{\frac{1}{1-\beta}\big(\int_0^{N_H}(p_H^{\frac{1}{\beta}} H)^{1-\beta} dj\big) H^\beta}{\frac{1}{1-\beta}\big(\int_0^{N_L}(p_L^{\frac{1}{\beta}} L)^{1-\beta} dj\big) L^\beta}\Bigg)^{-\frac{1}{\epsilon}}$$

$$= \Bigg(\frac{1-\gamma}{\gamma}\Bigg)\Bigg(\frac{\frac{1}{1-\beta}\big(N_H p_H^{\frac{1-\beta}{\beta}} H^{1-\beta}\big) H^\beta}{\frac{1}{1-\beta}\big(N_L p_L^{\frac{1-\beta}{\beta}} L^{1-\beta}\big) L^\beta}\Bigg)^{-\frac{1}{\epsilon}}$$

$$= \Bigg(\frac{1-\gamma}{\gamma}\Bigg)\Bigg(p^{\frac{1-\beta}{\beta}} \frac{N_H H}{N_L L}\Bigg)^{-\frac{1}{\epsilon}}$$

$$p^{1+\frac{1-\beta}{\beta\epsilon}} = \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{N_H H}{N_L L}\right)^{-\frac{1}{\epsilon}}$$
$$p = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\beta\epsilon}{1+\beta\epsilon-\beta}}\left(\frac{N_H H}{N_L L}\right)^{\frac{-\beta}{1+\beta\epsilon-\beta}}$$

Using the derived elasiticity of substitution (see appendix (A.4)) between the production factors H and L $\sigma \equiv \epsilon - (\epsilon - 1)(1-\beta) = 1 + \beta\epsilon - \beta$ (so $\sigma > 1$ only if $\epsilon > 1$) we can write the relative price p as

$$p = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\beta\epsilon}{\sigma}}\left(\frac{N_H}{N_L}\right)^{-\frac{\beta}{\sigma}}\left(\frac{H}{L}\right)^{-\frac{\beta}{\sigma}}. \tag{A.18}$$

A.6 Technology market clearing

As described we face

$$\frac{V_H}{V_L} = \frac{\eta_L}{\eta_H} \tag{A.19}$$

Since $\dot{V}_L$ in (2.20) and $\dot{V}_H$ in (2.21) are 0, substituting π_L (2.20) and π_H (2.21) leads to the following.

$$\frac{\frac{\pi_H}{r}}{\frac{\pi_L}{r}} = \frac{\eta_L}{\eta_H}$$

We can rewrite that and get the following market clearing condition.

$$\eta_L \pi_L = \eta_H \pi_H \tag{A.20}$$

A.7 Growthrate of the lab-equipment-model

To derive the long-run growth rate for the lab equipment model we maximise the CRRA preferences of the representative consumer (2.2) to get the Euler-equation. The money that is not spent on consumption is invested in R&D with an interest rate r.

$$g_c = \theta^{-1}(r-\rho). \tag{A.21}$$

We know the growth rate of the consumption g_c is also equal to growth rate of output g.

$$g = g_c = \theta^{-1}(r-\rho) \tag{A.22}$$

The technology monopolists can enter the market for new machine types in the L-Sector. The free-entry condition $\eta_L V_L = 1$ (in terms of the final good) can be rewritten using the equation for V_L (2.22) in the steady state, where $\dot{V}_L = 0$.

$$\eta_L \frac{\beta p_L^{1/\beta} L}{r} = 1 \tag{A.23}$$

Inserting r from (A.23) in (A.22) gives

$$g = \theta^{-1}(\beta \eta_L p_L^{1/\beta} L - \rho) \tag{A.24}$$

Using the relative price $p = \frac{p_H}{p_L}$ we can write the price equation (3.18) as

$$\gamma^\epsilon p_L^{1-\epsilon} + (1-\gamma)^\epsilon p^{1-\epsilon} p_L^{1-\epsilon} = 1. \tag{A.25}$$

Dividing by $p_L^{1-\epsilon}$ and substituting p (2.25) and $\frac{N_H}{N_L}$ (2.32) leads to

$$\gamma^\epsilon + (1-\gamma)^\epsilon \eta^\sigma \big(\frac{H}{L}\big)^{-\beta(1-\epsilon)} = p_L^{\epsilon-1}. \tag{A.26}$$

Using $\sigma - 1 = \beta(\epsilon - 1)$ we can put (A.24) and (A.26) together and get the growth rate for the output.

$$g = \theta^{-1}(\beta[(1-\gamma)^\epsilon(\eta_H H)^{\sigma-1} + \gamma^\epsilon(\eta_L L)^{\sigma-1}]^{\frac{1}{\sigma-1}} - \rho) \tag{A.27}$$

A.8 Growthrate of the knowledge-based R&D model

We want to have constant and equal growth rates in BGP $\frac{\dot{N}_L}{N_L} = \frac{\dot{N}_H}{N_H}$. Substituting (2.36) and (2.37) gives

$$\eta_L N_L^{\delta-1} S_L = \eta_H N_H^{\delta-1} S_H \tag{A.28}$$

So we can express $S_H = \frac{\eta_L}{\eta_H}\big(\frac{N_L}{N_H}\big)^{\delta-1} S_L$ and insert in the total number of scientists $S = S_L + S_H$.

$$S = \left(1 + \frac{\eta_L}{\eta_H}\big(\frac{N_L}{N_H}\big)^{\delta-1}\right) S_L = \left(\frac{\eta_H}{\eta_H} + \frac{\eta_L}{\eta_H}\big(\frac{N_H}{N_L}\big)^{1-\delta}\right) S_L \tag{A.29}$$

$$S_L = \frac{\eta_H S}{\eta_H + \eta_L\big(\frac{N_H}{N_L}\big)^{1-\delta}} = \frac{\eta_H\big(\frac{N_H}{N_L}\big)^{\delta-1} S}{\eta_H\big(\frac{N_H}{N_L}\big)^{\delta-1} + \eta_L} \tag{A.30}$$

Since the growth rates of the technologies are equal to the growth rate of the output we get

$$g = \frac{\dot{N}_H}{N_H} = \frac{\dot{N}_L}{N_L} = \eta_L\big(\frac{N_H}{N_L}\big)^{\frac{\delta-1}{2}} S_L \tag{A.31}$$

Substituting S_L gives the growth rate of the output.

$$g = \frac{\eta_L \eta_H S}{\eta_H \left(\frac{N_H}{N_L}\right)^{\frac{1-\delta}{2}} + \eta_L \left(\frac{N_H}{N_L}\right)^{\frac{3(1-\delta)}{2}}} \tag{A.32}$$

B Appendix

B.1 The Hotelling Rule - an excursus

In case we face property rights for resources firms have to pay a price P for non-renewable resources. This price is calculated by the well known Hotelling rule:

In a competitive market firms face a maximisation problem $\max_q \int_0^\infty e^{-rt}p(t)q(t)dt$, and they can sell $q(t)$ of the resource for a price $p(t)$. The ressource stock $Q(t)$ is shrinking if they extract resources to sell them: $\dot{Q}(t) = -q(t)$. In the beginning there is a resource stock $Q(0) = q_0 > 0$. $Q(t)$ has to be non-negative all the time. Each period the firms can extract $q(t) \leq \bar{q}$ due to limited machines and workers to extract resources, and of course can not add resources if they have taken them already $q(t) \geq 0$.

To solve this problem, we formulate the Hamiltonian H and its first order conditions.

$$H = p(t)q(t) + \lambda(-q(t)) \tag{B.1}$$

$$\frac{\partial H}{\partial q} = p(t) - \lambda = 0 \tag{B.2}$$

$$\dot{\lambda} = r\lambda \tag{B.3}$$

Obviously $\lambda = p(t)$ and therefore $\frac{\dot{p}(t)}{p(t)} = r$. The Hotelling rule describes the most socially and economically profitable extraction path of a non-renewable resource. The growth rate of the price $p(t)$ should be the interest rate firms face on the capital market to maximise the value of the resource stock.

The implementation of the Hotelling rule is found in chapter (3.3.3) and (3.3.4).

B.2 Relative expected profit for scientists in section 3.3.1

We can rewrite the equation

$$\frac{\pi_H}{\pi_L} = \frac{\eta_H}{\eta_L} \times \underbrace{\left(\frac{p_H}{p_L}\right)^{\frac{1}{\beta}}}_{\text{price effect}} \times \underbrace{\frac{H}{L}}_{\text{market size effect}} \times \underbrace{\frac{N_H^{t-1}}{N_L^{t-1}}}_{\text{direct productivity effect}} \tag{B.4}$$

substituting equilibrium results.
To simplify we assume $w_L = w_H$ and get from (3.32) and (3.33)

$$\frac{w_H}{w_L} = \frac{\beta p_H^{\frac{1}{\beta}} N_H}{\beta p_L^{\frac{1}{\beta}} N_L} = \left(\frac{p_H}{p_L}\right)^{\frac{1}{\beta}} \frac{N_H}{N_L} = 1 \tag{B.5}$$

which is equivalent to

$$\frac{p_H}{p_L} = \left(\frac{N_H}{N_L}\right)^{-\beta}. \tag{B.6}$$

Substituting Y_L from (3.30) and Y_H from (3.31) into the relative price p (3.17) gives

$$p^{-\epsilon} = \left(\frac{p_H}{p_L}\right)^{-\epsilon} = \frac{Y_H}{Y_L} = \frac{p_H^{\frac{1-\beta}{\beta}} H N_H}{p_L^{\frac{1-\beta}{\beta}} L N_L}. \tag{B.7}$$

Combining (B.7) and (B.6) leads to

$$\frac{H}{L} = \left(\frac{p_H}{p_L}\right)^{-\epsilon - \frac{1-\beta}{\beta}} \left(\frac{N_H}{N_L}\right)^{-1} = \left(\frac{N_H}{N_L}\right)^{(-\beta)(-\epsilon - \frac{1-\beta}{\beta})} \left(\frac{N_H}{N_L}\right)^{-1} = \left(\frac{N_H}{N_L}\right)^{\beta(\epsilon-1)}. \tag{B.8}$$

Now we can substitute (3.12), (3.13), (B.6) and (B.8) into (B.4) to analyse the directed technical change in the equilibrium.

$$\frac{\pi_H}{\pi_L} = \frac{\eta_H}{\eta_L} \left(\frac{1 + \nu \eta_H S_H}{1 + \nu \eta_L S_L}\right)^{\beta(\epsilon-1)-1} \left(\frac{N_H^{t-1}}{N_L^{t-1}}\right)^{\beta(\epsilon-1)} \tag{B.9}$$

B.3 Relative expected profit for scientists in section 3.3.3

Analogous to section 3.1 we can write

$$\frac{\pi_H}{\pi_L} = \kappa \frac{\eta_H}{\eta_L} \times \underbrace{\frac{1}{R^{\beta_2}}}_{\text{resource effect}} \times \underbrace{\frac{p_H^{\frac{1}{\beta}}}{p_L^{\frac{1}{\beta_1}}}}_{\text{price effect}} \times \underbrace{\frac{H}{L^{\frac{\beta}{\beta_1}}}}_{\text{market size effect}} \times \underbrace{\frac{N_H^{t-1}}{N_L^{t-1}}}_{\text{direct productivity effect}} \tag{B.10}$$

with $\kappa := \frac{\beta(1-\beta)^{\frac{2-\beta}{\beta}}}{\beta_1(1-\beta_1)^{\frac{2-\beta_1}{\beta_1}}}\psi^{\frac{1}{\beta_1}-\frac{1}{\beta}}$ in terms of the number of scientists and the technology level. Therefore we need the equal equilibrium wages (3.47) and (3.33).

$$\frac{w_H}{w_L} = \frac{\beta(1-\beta)^{2\frac{1-\beta}{\beta}} p_H^{\frac{1}{\beta}} \psi^{\frac{\beta-1}{\beta}} N_H}{\beta(1-\beta_1)^{2\frac{1-\beta_1}{\beta_1}} p_L^{\frac{1}{\beta_1}} R^{\frac{\beta_2}{\beta_1}} \psi^{\frac{\beta_1-1}{\beta_1}} L^{\frac{\beta-\beta_1}{\beta_1}} N_L} \tag{B.11}$$

$$= \frac{(1-\beta)^{2\frac{1-\beta}{\beta}}}{(1-\beta_1)^{2\frac{1-\beta_1}{\beta_1}}} \psi^{\frac{\beta-\beta_1}{\beta\beta_1}} R^{-\frac{\beta_2}{\beta_1}} L^{\frac{\beta_1-\beta}{\beta_1}} \frac{p_H^{\frac{1}{\beta}}}{p_L^{\frac{1}{\beta_1}}} \frac{N_H}{N_L} = 1 \tag{B.12}$$

For substituting the price effect we transform the equation.

$$\frac{p_H^{\frac{1}{\beta}}}{p_L^{\frac{1}{\beta_1}}} = \frac{(1-\beta)^{2\frac{1-\beta}{\beta}}}{(1-\beta_1)^{2\frac{1-\beta_1}{\beta_1}}} \psi^{\frac{\beta_1-\beta}{\beta\beta_1}} R^{\frac{\beta_2}{\beta_1}} L^{\frac{\beta-\beta_1}{\beta_1}} \left(\frac{N_H}{N_L}\right)^{-1} \tag{B.13}$$

The marginal products of labour have to be equal across sectors to clear the labour market. So we use Y_L (3.50) and Y_H (3.31) to calculate $\frac{\partial Y_L}{\partial L} = \frac{\partial Y_H}{\partial H}$.

$$1 = \frac{\frac{\partial Y_H}{\partial H}}{\frac{\partial Y_L}{\partial L}} = \frac{\beta(1-\beta)^{2\frac{1-\beta}{\beta}} \psi^{\frac{\beta-1}{\beta}} p_H^{\frac{1-\beta}{\beta}} N_H}{\left(\frac{(1-\beta_1)^2}{\psi}\right)^{\frac{1-\beta_1}{\beta}} \left(\frac{\beta_2}{c(Q)}\right)^{\frac{\beta_2}{\beta}} p_L^{\frac{1-\beta}{\beta}} N_L^{\frac{\beta_1}{\beta}}} \tag{B.14}$$

We can rewrite (B.14) to get the price ratio.

$$\frac{p_H}{p_L} = \frac{\psi^{\beta_2}(1-\beta_1)^{2((1-\beta_1)}\beta_2^{\beta_2} N_H^{-\beta}}{c(Q)^{\beta_2}(1-\beta)^{2(1-\beta)} N_L^{-\beta_1}} \tag{B.15}$$

The relative price brings Y_L (3.50) and Y_H (3.31) together.

$$p^{-\epsilon} = \left(\frac{p_H}{p_L}\right)^{-\epsilon} = \frac{Y_H}{Y_L} = \frac{\beta(1-\beta)^{2\frac{1-\beta}{\beta}} \psi^{\frac{\beta-1}{\beta}} p_H^{\frac{1-\beta}{\beta}} H N_H}{\left(\frac{(1-\beta_1)^2}{\psi}\right)^{\frac{1-\beta_1}{\beta}} \left(\frac{\beta_2}{c(Q)}\right)^{\frac{\beta_2}{\beta}} p_L^{\frac{1-\beta}{\beta}} L N_L^{\frac{\beta_1}{\beta}}} \tag{B.16}$$

So we can express the relative labour supply.

$$\frac{H}{L} = \frac{\beta_2^{\frac{\beta_2}{\beta}}(1-\beta_1)^{2\frac{1-\beta_1}{\beta}}}{\beta(1-\beta)^{2\frac{1-\beta}{\beta}}} \psi^{\frac{\beta_1-\beta}{\beta}} c(Q)^{-\frac{\beta_2}{\beta}} \left(\frac{p_H}{p_L}\right)^{\frac{\beta-1}{\beta}-\epsilon} \left(\frac{N_H}{N_L^{\frac{\beta_1}{\beta}}}\right)^{-1} \tag{B.17}$$

Substituting (B.15) into (B.17) and plotting (3.49), (B.17) and (B.13) into (B.10) finally gives

$$\frac{\pi_H}{\pi_L} = \bar{\kappa}\frac{\eta_H}{\eta_L} c(Q)^{\beta_2(\epsilon-1)} \frac{(1+\nu\eta_H S_H)^{\beta(\epsilon-1)-1}}{(1+\nu\eta_L S_L)^{\beta_1(\epsilon-1)-1}} \frac{(N_H^{t-1})^{\beta(\epsilon-1)}}{(N_L^{t-1})^{\beta_1(\epsilon-1)}} \tag{B.18}$$

with $\bar{\kappa} := \frac{\beta(1-\beta)}{\beta_1(1-\beta_1)^{\frac{\beta_1+\beta_2}{\beta_1}}} \left(\frac{(1-\beta)^{2(1-\beta)}}{\psi^{\beta_2}(1-\beta_1)^{2(1-\beta_1)}\beta_2^{\beta_2}}\right)^{\epsilon-1}$.

C Appendix

C.1 FOC of the Lagrangian

The FOC with respect to $X(t)$ is

$$\frac{\partial \Lambda}{\partial X(t)} = -(1-\beta)^2\lambda_1(t) + (1-\beta)N(t)^\beta X(t)^{-\beta}\lambda_2(t) + e(t)^\beta(1-\beta)X(t)^{-\beta}\lambda_3(t+1) = 0 \tag{C.1}$$

We can rewrite that.

$$(1-\beta)^2\lambda_1(t) = \big((1-\beta)N(t)^\beta\lambda_2(t) + e(t)^\beta(1-\beta)\lambda_3(t+1)\big)X(t)^{-\beta} \tag{C.2}$$

$$\frac{(1-\beta)^2\lambda_1(t)}{(1-\beta)N(t)^\beta\lambda_2(t) + e(t)^\beta(1-\beta)\lambda_3(t+1)} = X(t)^{-\beta} \tag{C.3}$$

$$\left(\frac{(1-\beta)\lambda_1(t)}{N(t)^\beta\lambda_2(t) + e(t)^\beta\lambda_3(t+1)}\right)^{-\frac{1}{\beta}} = X(t) \tag{C.4}$$

$$\left(\frac{N(t)^\beta\lambda_2(t) + e(t)^\beta\lambda_3(t+1)}{(1-\beta)\lambda_1(t)}\right)^{\frac{1}{\beta}} = X(t) \tag{C.5}$$

$$(1-\beta)^{-\frac{1}{\beta}}\big(N(t)^\beta - \frac{e(t)^\beta\lambda_3(t+1)}{\lambda_1(t)}\big)^{\frac{1}{\beta}} = X(t) \tag{C.6}$$

$$(1-\beta)^{-\frac{1}{\beta}}\big(N(t)^\beta - \frac{\lambda_3(t+1)}{\lambda_1(t)}e(t)^\beta\big)^{\frac{1}{\beta}} = X(t) \tag{C.7}$$

We substitute $X(t)$ (C.7) into the FOC of the technology.

$$-\lambda_2(t)\beta N(t)^{(\beta-1)}X^{(1-\beta)} + \lambda_4(t+1) - \lambda_4(t)(1+\nu(1-s(t+1))) = 0 \tag{C.8}$$

$$\begin{aligned} &-\lambda_2(t)\beta N(t)^{(\beta-1)}\left((1-\beta)^{-\frac{1}{\beta}}\big(N(t)^\beta - \frac{\lambda_3(t+1)}{\lambda_1(t)}e(t)^\beta\big)^{\frac{1}{\beta}}\right)^{(1-\beta)} \\ &+\lambda_4(t+1) - \lambda_4(t)(1+\nu(1-s(t+1))) = 0 \end{aligned} \tag{C.9}$$

$$\begin{aligned} &\lambda_2(t)\beta(1-\beta)^{-\frac{1-\beta}{\beta}}N(t)^{(\beta-1)}\left(N(t)^\beta - \frac{\lambda_3(t+1)}{\lambda_1(t)}e(t)^\beta\right)^{\frac{1-\beta}{\beta}} \\ &+\lambda_4(t)(1+\nu(1-s(t+1))) = \lambda_4(t+1) \end{aligned} \tag{C.10}$$

Analogous for the second technology FOC.

$$\lambda_3(t+1)\beta e(t)^{(\beta-1)}X(t)^{(1-\beta)} + \lambda_5(t+1) - \lambda_5(t)(1-\varsigma s(t+1)) = 0 \tag{C.11}$$

$$\begin{aligned}\lambda_3(t+1)\beta e(t)^{(\beta-1)}\left((1-\beta)^{-\frac{1}{\beta}}\left(N(t)^{\beta} - \frac{\lambda_3(t+1)}{\lambda_1(t)}e(t)^{\beta}\right)^{\frac{1}{\beta}}\right)^{(1-\beta)} \\ +\lambda_5(t+1) - \lambda_5(t)(1-\varsigma s(t+1)) = 0\end{aligned} \tag{C.12}$$

$$\begin{aligned}-\lambda_3(t+1)\beta(1-\beta)^{-\frac{1-\beta}{\beta}}e(t)^{(\beta-1)}\left(N(t)^{\beta} - \frac{\lambda_3(t+1)}{\lambda_1(t)}e(t)^{\beta}\right)^{\frac{1-\beta}{\beta}} \\ +\lambda_5(t)(1-\varsigma s(t+1)) = \lambda_5(t+1)\end{aligned} \tag{C.13}$$

D Appendix

D.1 Optimisation of the households

D.1.1 Unskilled households raising unskilled offspring

We want to solve the following optimisation problem.

$$\max_{C_L(t),n_{LL}(t),C_L(t+1),I_L(t),q_L(t)} \ln C_L(t) + a\ln(w_L(t+1)n_{LL}) + \rho \ln C_L(t+1) \tag{D.1}$$

The budget constraint can be written in the following equivalent ways.

$$w_L(t) \geq (zw_L(t))n_{LL}(t) + C_L(t) + (I_L(t) + P(t)q_L(t)) \tag{D.2}$$
$$w_L(t+1) \geq (zw_L(t+1))n_{LL}(t+1) + C_L(t+1) + (I_L(t+1) + P(t+1)q_L(t+1)) \tag{D.3}$$
$$w_L(t) \geq (zw_L(t))n_{LL}(t) + C_L(t) + \frac{C_L(t+1)}{1+r(t+1)} \tag{D.4}$$
$$(1+r(t+1))w_L(t) - (zw_L(t))n_{LL}(t) - C_L(t) \geq C_L(t+1) \tag{D.5}$$
$$w_L(t) - (zw_L(t))n_{LL}(t) - (I_L(t) + P(t)q_L(t)) \geq C_L(t) \tag{D.6}$$

Using (D.5) and (D.6) we rewrite the optimisation problem.

$$\max_{C_L,n_{LL},C_L(t+1),I_L,q_L} \ln\big(w_L(t) - (zw_L(t))n_{LL}(t) - (I_L(t) + P(t)q_L(t))\big)$$
$$+a\ln\big(w_L(t+1)n_{LL}\big) + \rho\ln\big((1+r(t+1))w_L(t) - (zw_L(t))n_{LL}(t) - C_L(t)\big)$$

Note, that w_L is a function of C_L. The first order conditions are

$$\frac{1}{C_L} - (1+a+\rho)\frac{1}{w_L} = 0 \tag{D.7}$$
$$\alpha\frac{1}{n_{LL}}\frac{1}{z} - (1+a+\rho) = 0 \tag{D.8}$$
$$\rho\frac{1}{(1+r(t+1))(I_L + Pq_L)} - 1 = 0 \tag{D.9}$$

Simple transformations give

$$C_L = \frac{1}{1+a+\rho} w_L \tag{D.10}$$

$$n_{LL} = \frac{a}{(1+a+\rho)z} \tag{D.11}$$

$$I_L = \frac{\rho}{1+a+\rho} w_L - Pq_L \tag{D.12}$$

D.1.2 Unskilled households raising skilled offspring

$$\max_{C_L, n_{LH}, C_L(t+1), I_L, q_L} u_{LH} = \tag{D.13}$$

$$\max_{C_L, n_{LH}, C_L(t+1), I_L, q_L} \ln C_L(t) + a \ln(w_H(t+1) n_{LH}) + \rho \ln C_L(t+1) \tag{D.14}$$

subject to (5.17):

$$w_L \geq (z w_L + w_H \phi) n_{LH} + C_L + (I_L + Pq_L) \tag{D.15}$$

Again, rewriting the budget constraint

$$\begin{aligned} w_L(t) \geq {} & (z w_L(t) + w_H(t)\phi) n_{LH}(t) + C_L(t) \\ & + (I_L(t) + P(t) q_L(t)) \end{aligned} \tag{D.16}$$

$$\begin{aligned} w_L(t+1) \geq {} & (z w_L(t+1) + w_H(t+1)\phi) n_{LH}(t+1) \\ & + C_L(t+1) + (I_L(t+1) + P(t+1) q_L(t+1)) \end{aligned} \tag{D.17}$$

$$\begin{aligned} w_L(t) \geq {} & (z w_L(t) + w_H(t)\phi) n_{LH}(t) \\ & + C_L(t) + \frac{C_L(t+1)}{1+r(t+1)} \end{aligned} \tag{D.18}$$

and substituting into the maximisation problem yields

$$C_L = \frac{1}{1+a+\rho} w_L \tag{D.19}$$

$$n_{LH} = \frac{a}{1+a+\rho} \frac{w_L}{w_L z + w_H \phi} = \frac{a}{1+a+\rho} \frac{1}{z + w\phi} \tag{D.20}$$

$$I_L = \frac{\rho}{1+a+\rho} w_L - Pq_L \tag{D.21}$$

D.1.3 Skilled households raising skilled offspring

$$\max_{C_H, n_{HH}, C_H(t+1), I_H, q_H} u_{HH} = \max_{C_H, n_{HH}, C_H(t+1), I_H, q_H} \ln C_H(t) + a \ln(w_H(t+1) n_{HH}) + \rho \ln C_H(t+1) \tag{D.22}$$

subject to (5.18):

$$w_H \geq \quad (zw_H + w_H\phi) \quad n_{HH} + C_H + (I_H + Pq_H) \tag{D.23}$$

Analogous to the previous sections we obtain

$$C_H = \quad \frac{1}{1+a+\rho} w_H \tag{D.24}$$

$$n_{HH} = \quad \frac{a}{1+a+\rho} \frac{1}{z+\phi} \tag{D.25}$$

$$I_H = \quad \frac{\rho}{1+a+\rho} w_H - Pq_H \tag{D.26}$$

D.2 Relative equilibrium price

We substitute (5.43) and (5.44) into (5.38).

$$p \equiv \frac{p_H}{p_L} = \frac{1-\gamma}{\gamma} (\frac{N_H}{N_L})^{-\frac{1}{\epsilon}} (p^{\frac{1-\beta}{\beta}})^{-\frac{1}{\epsilon}} (\frac{\chi_H}{\chi_L})^{-\frac{1}{\epsilon}\frac{\beta-1}{\beta}} (\frac{H}{L})^{-\frac{1}{\epsilon}}$$

Using $\sigma = \epsilon - (\epsilon-1)(1-\beta) = \epsilon\beta + 1 - \beta$ we can rewrite that to

$$p^{\frac{\sigma}{\epsilon\beta}} = \frac{1-\gamma}{\gamma} (\frac{N_H}{N_L})^{-\frac{1}{\epsilon}} (\frac{\chi_H}{\chi_L})^{\frac{1-\beta}{\epsilon\beta}} (\frac{H}{L})^{-\frac{1}{\epsilon}}$$

which gives

$$p = (\frac{1-\gamma}{\gamma})^{\frac{\epsilon\beta}{\sigma}} (\frac{N_H}{N_L})^{-\frac{\beta}{\sigma}} (\frac{\chi_H}{\chi_L})^{\frac{1-\beta}{\sigma}} (\frac{H}{L})^{-\frac{\beta}{\sigma}}. \tag{D.27}$$

D.3 Profits and wages of the firms

We look at the profits of the firms like in the basic model.

$$\max_{L,\{x_L(j)\}} \Pi_L \quad = \quad \max_{L,\{x_L(j)\}} p_L Y_L - w_L L - \int_0^{N_L} \chi_L(j) x_L(j) dj \tag{D.28}$$

$$\max_{H,\{x_H(j)\}} \Pi_H \quad = \quad \max_{H,\{x_H(j)\}} p_H Y_H - w_H H - \int_0^{N_H} \chi_H(j) x_H(j) dj \tag{D.29}$$

Following the calculations in appendix (A.2) gives the wages.

$$w_L \quad = \quad \frac{\beta}{1-\beta} p_L \Big(\int_0^{N_L} x_L(j)^{1-\beta} dj \Big) L^{\beta-1} \tag{D.30}$$

$$w_H \quad = \quad \frac{\beta}{1-\beta} p_H \Big(\int_0^{N_H} x_H(j)^{1-\beta} dj \Big) H^{\beta-1} \tag{D.31}$$

We substitute the equilibrium demand of machines x_L (5.41) and x_H (5.41).

$$w_L = \frac{\beta}{1-\beta} p_L N_L \big(\frac{p_L}{\chi_L}\big)^{\frac{1-\beta}{\beta}} \tag{D.32}$$
$$w_H = \frac{\beta}{1-\beta} p_H N_H \big(\frac{p_H}{\chi_H}\big)^{\frac{1-\beta}{\beta}} \tag{D.33}$$

The skilled wage premium $w = \frac{w_H}{W_L}$ is then

$$w = p^{\frac{1}{\beta}} \frac{N_L}{N_H} \big(\frac{\chi_L}{\chi_H}\big)^{\frac{\beta-1}{\beta}}. \tag{D.34}$$

Finally, substituting the relative price of the intermediate goods p (5.45) leads to the skilled wage premium in equilibrium.

$$w = \big(\frac{1-\gamma}{\gamma}\big)^{\frac{\epsilon}{\sigma}} \big(\frac{N_H}{N_L}\big)^{-\frac{1+\sigma}{\sigma}} \big(\frac{\chi_H}{\chi_L}\big)^{\frac{1-\beta}{\beta}\frac{1-\sigma}{\sigma}} \big(\frac{H}{L}\big)^{-\frac{1}{\sigma}}. \tag{D.35}$$

D.4 Costs of machine production

Technology monopolists are facing a production function, so they optimise their profit like competitive firms. The profit is the difference between the volume of sales (price of machines times machine supply) and the costs of the production factors labour (wage times labour supply) and resource (price of the resource multiplied with the number of resources used for production). They have to maximise with respect to labour supply and resources they want to use for the production, before they can choose the price of the machines to sell them in a monopolistic way. For the production itself they face the following profit function on an aggregate level.

$$\max_{L_T,R_L} \pi_L = \max_{L_T,R_L} \psi_L X_L - w_L L_T - P R_L \tag{D.36}$$
$$\max_{H_T,R_H} \pi_H = \max_{H_T,R_H} \psi_H X_H - w_H H_T - P R_H \tag{D.37}$$

Whereas the aggregate output of their production is given by

$$X_L = M_L N_L (L_T)^{1-\alpha} (R_L)^{\alpha} \tag{D.38}$$
$$X_H = M_H N_H (H_T)^{1-\alpha} (R_H)^{\alpha}. \tag{D.39}$$

Substituting (D.38) into (D.36) and (D.39) into (D.37) gives

$$\max_{L_T,R_L} \pi_L = \max_{L_T,R_L} \psi_L M_L N_L (L_T)^{1-\alpha} (R_L)^{\alpha} - w_L L_T - P R_L \tag{D.40}$$
$$\max_{H_T,R_H} \pi_H = \max_{H_T,R_H} \psi_H M_H N_H (H_T)^{1-\alpha} (R_H)^{\alpha} - w_H H_T - P R_H. \tag{D.41}$$

The first order condition from (D.40) and (D.41) with respect to the labour supply is written as follows.

$$\frac{\partial \pi_L}{\partial L_T} = \psi_L M_L N_L (1-\alpha)(L_T)^{-\alpha}(R_L)^{\alpha} - w_L = 0 \tag{D.42}$$
$$\frac{\partial \pi_H}{\partial H_T} = \psi_H M_H N_H (1-\alpha)(H_T)^{-\alpha}(R_H)^{\alpha} - w_H = 0 \tag{D.43}$$

Next, the FOC with respect to the resource is given by

$$\frac{\partial \pi_L}{\partial R_L} = \psi_L M_L N_L (L_T)^{1-\alpha}\alpha(R_L)^{\alpha-1} - P = 0 \tag{D.44}$$
$$\frac{\partial \pi_H}{\partial R_H} = \psi_H M_H N_H (H_T)^{1-\alpha}\alpha(R_H)^{\alpha-1} - P = 0. \tag{D.45}$$

We can bring the optimal resources and optimal labour supply in relation, if we simply divide both FOCs: (D.42)/(D.44) and (D.43)/(D.45).

$$\frac{\psi_L M_L N_L (1-\alpha)(L_T)^{-\alpha}(R_L)^{\alpha}}{\psi_L M_L N_L (L_T)^{1-\alpha}\alpha(R_L)^{\alpha-1}} = \frac{w_L}{P} \tag{D.46}$$
$$\frac{\psi_H M_H N_H (1-\alpha)(H_T)^{-\alpha}(R_H)^{\alpha}}{\psi_H M_H N_H (H_T)^{1-\alpha}\alpha(R_H)^{\alpha-1}} = \frac{w_H}{P} \tag{D.47}$$

We rewrite the equations.

$$R_L = \frac{\alpha}{1-\alpha} L_T \frac{w_L}{P} \tag{D.48}$$
$$R_H = \frac{\alpha}{1-\alpha} H_T \frac{w_H}{P} \tag{D.49}$$

Since we are interested in the cost function, have a closer look at the aggregate costs Ψ_L and Ψ_H.

$$\Psi_L = w_L L_T + P R_L \tag{D.50}$$
$$\Psi_H = w_H H_T + P R_H \tag{D.51}$$

We can now solve a system of two linear equations to express the optimal use of resources and labour in terms of the costs. (D.48) and (D.50) leads us to $L_T = \frac{(1-\alpha)\Psi_L}{w_L}$ and $R_L = \frac{\alpha\Psi_L}{P}$. (D.49) and (D.51) gives $H_T = \frac{(1-\alpha)\Psi_H}{w_H}$ and $R_H = \frac{\alpha\Psi_H}{P}$.

Substituting these expressions in (D.38) and (D.39) helps us expressing the aggregate costs in a different way.

$$X_L = M_L N_L \left(\frac{(1-\alpha)\Psi_L}{w_L}\right)^{1-\alpha}\left(\frac{\alpha\Psi_L}{P}\right)^{\alpha} \tag{D.52}$$
$$X_H = M_H N_H \left(\frac{(1-\alpha)\Psi_H}{w_H}\right)^{1-\alpha}\left(\frac{\alpha\Psi_H}{P}\right)^{\alpha} \tag{D.53}$$

$$\Psi_L = X_L \frac{w_L^{1-\alpha} P^\alpha}{M_L N_L (1-\alpha)^{1-\alpha}\alpha^\alpha} \tag{D.54}$$

$$\Psi_H = X_H \frac{w_H^{1-\alpha} P^\alpha}{M_H N_H (1-\alpha)^{1-\alpha}\alpha^\alpha} \tag{D.55}$$

The marginal costs technology monopolists face are finally given by

$$\psi_L(w_L, P) = \frac{w_L^{1-\alpha} P^\alpha}{M_L N_L (1-\alpha)^{1-\alpha}\alpha^\alpha} \tag{D.56}$$

$$\psi_L(w_H, P) = \frac{w_H^{1-\alpha} P^\alpha}{M_H N_H (1-\alpha)^{1-\alpha}\alpha^\alpha}. \tag{D.57}$$

D.5 Profits of technology monopolists

The profit of the technology monopolists facing the demand of machines (5.41) or (5.42)

$$x_L = \left(\frac{p_L}{\chi_L}\right)^{\frac{1}{\beta}} L \tag{D.58}$$

$$x_H = \left(\frac{p_H}{\chi_H}\right)^{\frac{1}{\beta}} H. \tag{D.59}$$

is given by

$$\pi_L = (\chi_L - \psi_L) x_L \tag{D.60}$$

$$\pi_H = (\chi_H - \psi_H) x_H. \tag{D.61}$$

We substitute the price of the machines

$$\chi_L = \frac{\psi_L}{1-\beta} \tag{D.62}$$

$$\chi_H = \frac{\psi_H}{1-\beta} \tag{D.63}$$

together with the equilibrium demand of machines and get

$$\pi_L = \left(\frac{\psi_L}{1-\beta} - \psi_L\right)\left(\frac{p_L}{\frac{\psi_L}{1-\beta}}\right)^{\frac{1}{\beta}} L \tag{D.64}$$

$$\pi_H = \left(\frac{\psi_H}{1-\beta} - \psi_H\right)\left(\frac{p_H}{\frac{\psi_H}{1-\beta}}\right)^{\frac{1}{\beta}} H. \tag{D.65}$$

Simple transformations give the optimal equilibrium profits of technology monopolists.

$$\pi_L = \beta(1-\beta)^{\frac{1-\beta}{\beta}} p_L^{\frac{1}{\beta}} L \psi_L^{\frac{\beta-1}{\beta}} \tag{D.66}$$

$$\pi_H = \beta(1-\beta)^{\frac{1-\beta}{\beta}} p_H^{\frac{1}{\beta}} H \psi_H^{\frac{\beta-1}{\beta}} \tag{D.67}$$

D.6 Skilled wage premium

We already know the skilled wage premium.

$$w = (\frac{1-\gamma}{\gamma})^{\frac{\epsilon}{\sigma}} (\frac{N_H}{N_L})^{\frac{\sigma-1}{\sigma}} (\frac{\chi_H}{\chi_L})^{\frac{1-\beta}{\beta}\frac{1-\sigma}{\sigma}} (\frac{H}{L})^{-\frac{1}{\sigma}}. \tag{D.68}$$

Substituting (5.53) and (5.54)

$$\chi_L = \frac{\psi_L}{1-\beta} \tag{D.69}$$

$$\chi_H = \frac{\psi_H}{1-\beta} \tag{D.70}$$

expresses the wage premium in terms of the costs instead of the price of the machines.

$$w = (\frac{1-\gamma}{\gamma})^{\frac{\epsilon}{\sigma}} (\frac{N_H}{N_L})^{\frac{\sigma-1}{\sigma}} (\frac{\psi_H}{\psi_L})^{\frac{1-\beta}{\beta}\frac{1-\sigma}{\sigma}} (\frac{H}{L})^{-\frac{1}{\sigma}} \tag{D.71}$$

With the cost function for machine producers (5.51) and (5.52)

$$\psi_L(w_L, P) = \frac{w_L^{1-\alpha} P^{\alpha}}{M_L N_L (1-\alpha)^{1-\alpha}\alpha^{\alpha}} \tag{D.72}$$

$$\psi_H(w_H, P) = \frac{w_H^{1-\alpha} P^{\alpha}}{M_H N_H (1-\alpha)^{1-\alpha}\alpha^{\alpha}} \tag{D.73}$$

we can rewrite the skilled wage premium (D.71).

$$w = (\frac{1-\gamma}{\gamma})^{\frac{\epsilon}{\sigma}} (\frac{N_H}{N_L})^{\frac{\sigma-1}{\sigma}} \left((\frac{w_H}{w_L})^{1-\alpha} \frac{M_L}{M_H} \frac{N_L}{N_H} \right)^{\frac{1-\beta}{\beta}\frac{1-\sigma}{\sigma}} (\frac{H}{L})^{-\frac{1}{\sigma}} \tag{D.74}$$

$$w = (\frac{1-\gamma}{\gamma})^{\frac{\epsilon}{\sigma}} (\frac{N_H}{N_L})^{\frac{\sigma-1}{\beta\sigma}} w^{(1-\alpha)\frac{1-\beta}{\beta}\frac{1-\sigma}{\sigma}} (\frac{M_L}{M_H})^{\frac{1-\beta}{\beta}\frac{1-\sigma}{\sigma}} (\frac{H}{L})^{-\frac{1}{\sigma}} \tag{D.75}$$

$$w^{\frac{\beta\sigma-(1-\alpha)(1-\beta)(1-\sigma)}{\beta\sigma}} = (\frac{1-\gamma}{\gamma})^{\frac{\epsilon}{\sigma}} (\frac{N_H}{N_L})^{\frac{\sigma-1}{\beta\sigma}} (\frac{M_L}{M_H})^{\frac{1-\beta}{\beta}\frac{1-\sigma}{\sigma}} (\frac{H}{L})^{-\frac{1}{\sigma}} \tag{D.76}$$

$$w = (\frac{1-\gamma}{\gamma})^{\frac{\epsilon\beta}{\sigma\xi}} (\frac{N_H}{N_L})^{\frac{\sigma-1}{\sigma\xi}} (\frac{M_L}{M_H})^{\frac{(1-\beta)(\sigma-1)}{\sigma\xi}} (\frac{H}{L})^{-\frac{\beta}{\sigma\xi}} \tag{D.77}$$

for $\xi = \beta + (1-\alpha)(1-\beta) \in (0,1)$.

D.7 Labour market

To express the structure of the labour market, we derive the following equations (5.65) and (5.66).

$$p_L \frac{\partial Y_L}{\partial L} = \chi_L \frac{\partial X_L}{\partial L_T}$$
$$p_H \frac{\partial Y_H}{\partial H} = \chi_H \frac{\partial X_H}{\partial H_T}$$

We insert the production functions for firms Y_L (5.39) and Y_H (5.40) and for technology monopolists X_L (D.38) and X_H (D.39).

$$p_L \frac{1}{1-\beta} \left(\int_0^{N_L} x_L(j)^{1-\beta} dj \right) \beta L^{\beta-1} = \chi_L M_L N_L (1-\alpha)(L_T)^{1-\alpha-1}(R_L)^{\alpha}$$
$$p_H \frac{1}{1-\beta} \left(\int_0^{N_H} x_H(j)^{1-\beta} dj \right) \beta H^{\beta-1} = \chi_H M_H N_H (1-\alpha)(H_T)^{1-\alpha-1}(R_H)^{\alpha}$$

Using the production functions for firms Y_L (5.39) and Y_H (5.40) and for technology monopolists X_L (D.38) and X_H (D.39) again, this can be written as

$$p_L \beta \frac{Y_L}{L} = \chi_L (1-\alpha) \frac{X_L}{L_T} \tag{D.78}$$
$$p_H \beta \frac{Y_H}{H} = \chi_H (1-\alpha) \frac{X_H}{H_T}. \tag{D.79}$$

Transforming gives

$$\frac{p_L}{\chi_L} = \frac{(1-\alpha)}{\beta} \frac{\frac{X_L}{L_T}}{\frac{Y_L}{L}} \tag{D.80}$$
$$\frac{p_H}{\chi_H} = \frac{(1-\alpha)}{\beta} \frac{\frac{X_H}{H_T}}{\frac{Y_H}{H}}. \tag{D.81}$$

The aggregate production of the machines can be expressed as the technology level multiplied with each production function x_L or x_H.

$$X_L = N_L x_L \tag{D.82}$$
$$X_H = N_H x_H \tag{D.83}$$

With the equilibrium demand of machines x_L (5.41) and x_H (5.42) we receive

$$X_L = N_L \left(\frac{p_L}{\chi_L}\right)^{\frac{1}{\beta}} L \tag{D.84}$$
$$X_H = N_H \left(\frac{p_H}{\chi_H}\right)^{\frac{1}{\beta}} H. \tag{D.85}$$

Using the levels of Y_L and Y_H given in (5.43) and (5.44), respectively, we rewrite (D.80) and (D.80) to

$$\frac{p_L}{\chi_L} = \frac{(1-\alpha)}{\beta}\frac{\frac{N_L\left(\frac{p_L}{\chi_L}\right)^{\frac{1}{\beta}}L}{L_T}}{\frac{\frac{1}{1-\beta}N_L p_L^{\frac{1-\beta}{\beta}}\chi_L^{\frac{\beta-1}{\beta}}L}{L}}$$

$$\frac{p_H}{\chi_H} = \frac{(1-\alpha)}{\beta}\frac{\frac{N_H\left(\frac{p_H}{\chi_H}\right)^{\frac{1}{\beta}}H}{H_T}}{\frac{\frac{1}{1-\beta}N_H p_H^{\frac{1-\beta}{\beta}}\chi_H^{\frac{\beta-1}{\beta}}H}{H}}.$$

Simplifying the equations gives

$$1 = \frac{(1-\alpha)}{\beta}\frac{\frac{L}{L_T}}{\frac{1}{1-\beta}} \tag{D.86}$$

$$1 = \frac{(1-\alpha)}{\beta}\frac{\frac{H}{H_T}}{\frac{1}{1-\beta}} \tag{D.87}$$

and further transformations lead to

$$L_T = \frac{(1-\alpha)(1-\beta)}{\beta}L \tag{D.88}$$

$$H_T = \frac{(1-\alpha)(1-\beta)}{\beta}H. \tag{D.89}$$

D.8 Employment ratios in equilibrium

We want to express the employment ratios. Using the employment structure (5.69) - (5.72) both ratios give the same:

$$\frac{H}{L} = \frac{H_T}{L_T} = \frac{H_\Sigma - H_E}{L_\Sigma} \tag{D.90}$$

We substitute the expression for H_E (5.73)

$$\frac{H}{L} = \frac{H_T}{L_T} = \frac{H_\Sigma - \phi(n_{HH}H_\Sigma + hn_{LH}L_\Sigma)}{L_\Sigma} \tag{D.91}$$

and simple tranformations lead to

$$\frac{H}{L} = \frac{H_T}{L_T} = (1-\phi n_{HH})\frac{H_\Sigma}{L_\Sigma} + \phi h n_{LH}. \tag{D.92}$$

D.9 Resource allocation in equilibrium

As seen in (D.4) we can use the FOC (D.44) and (D.45) of the profit of technology monopolists to express the equilibrium price of the resource P.

$$P = \psi_L M_L N_L (L_T)^{1-\alpha}\alpha(R_L)^{\alpha-1} = \psi_L \alpha \frac{X_L}{R_L} \tag{D.93}$$

$$P = \psi_H M_H N_H (H_T)^{1-\alpha}\alpha(R_H)^{\alpha-1} = \psi_H \alpha \frac{X_H}{R_H} \tag{D.94}$$

We can simply write

$$P = \psi_L \alpha \frac{X_L}{R_L} = \psi_H \alpha \frac{X_H}{R_H}. \tag{D.95}$$

Using the aggregate output functions in terms of the technology level (D.82) and (D.83) and the equilibrium demand of machines (5.41) and (5.42) gives

$$\frac{R_H}{R_L}\frac{\psi_H}{\psi_L} = \frac{X_H}{X_L} = \frac{N_H x_H}{N_L x_L} = \frac{N_H}{N_L}\frac{\left(\frac{p_H}{\chi_H}\right)^{\frac{1}{\beta}} H}{\left(\frac{p_L}{\chi_L}\right)^{\frac{1}{\beta}} L} = \frac{N_H}{N_L} p^{\frac{1}{\beta}} \frac{H}{L}. \tag{D.96}$$

After substituting the relative price in equilibrium (5.57) the resource ratio is

$$\frac{R_H}{R_L} = \frac{\psi_L}{\psi_H}\frac{N_H}{N_L}\frac{H}{L}\left(\left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon\beta}{\sigma}} \left(\frac{N_H}{N_L}\right)^{-\frac{\beta}{\sigma}} \left(\frac{\psi_H}{\psi_L}\right)^{\frac{1-\beta}{\sigma}} \left(\frac{H}{L}\right)^{-\frac{\beta}{\sigma}} \right)^{\frac{1}{\beta}} \tag{D.97}$$

$$\frac{R_H}{R_L} = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon}{\sigma}} \left(\frac{N_H}{N_L}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{\psi_H}{\psi_L}\right)^{\frac{(1-\beta)(1-\sigma)}{\beta\sigma}} \left(\frac{H}{L}\right)^{-\frac{\sigma-1}{\sigma}}. \tag{D.98}$$

We can rewrite (5.75).

$$\frac{R_H}{R_L} + 1 = \frac{R}{R_L} \tag{D.99}$$

So we finally receive

$$R_L = \frac{R}{\frac{R_H}{R_L}+1} = \frac{1}{1 + \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon}{\sigma}} \left(\frac{N_H}{N_L}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{\psi_H}{\psi_L}\right)^{\frac{(1-\beta)(1-\sigma)}{\beta\sigma}} \left(\frac{H}{L}\right)^{\frac{\sigma-1}{\sigma}}} R \tag{D.100}$$

for the L-sector, and

$$R_H = R - \frac{R}{\frac{R_H}{R_L}+1} = \frac{\frac{R_H}{R_L}}{\frac{R_H}{R_L}+1} R = \frac{\left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon}{\sigma}} \left(\frac{N_H}{N_L}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{\psi_H}{\psi_L}\right)^{\frac{(1-\beta)(1-\sigma)}{\beta\sigma}} \left(\frac{H}{L}\right)^{\frac{\sigma-1}{\sigma}}}{1 + \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon}{\sigma}} \left(\frac{N_H}{N_L}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{\psi_H}{\psi_L}\right)^{\frac{(1-\beta)(1-\sigma)}{\beta\sigma}} \left(\frac{H}{L}\right)^{\frac{\sigma-1}{\sigma}}} R \tag{D.101}$$

for the H-sector.

D.10 The relative level of technology

Start of our derivation is the technology market clearing.

$$\frac{\pi_H(t+1)}{\pi_L(t+1)} = \frac{\eta_L}{\eta_H}\left(\frac{N_L(t)}{N_H(t)}\right)^{\delta} \tag{D.102}$$

We can substitute the equilibrium profits (5.55) and (5.56) of the technology monopolists

$$(p(t+1))^{\frac{1}{\beta}}\frac{H(t+1)}{L(t+1)}\left(\frac{\psi_H(t+1)}{\psi_L(t+1)}\right)^{\frac{\beta-1}{\beta}} = \frac{\eta_L}{\eta_H}\left(\frac{N_L(t)}{N_H(t)}\right)^{\delta},$$

as well as the equilibrium price (5.57)

$$\left((\frac{1-\gamma}{\gamma})^{\frac{\epsilon\beta}{\sigma}}(\frac{N_H(t+1)}{N_L(t+1)})^{-\frac{\beta}{\sigma}}(\frac{\psi_H(t+1)}{\psi_L(t+1)})^{\frac{1-\beta}{\sigma}}(\frac{H(t+1)}{L(t+1)})^{-\frac{\beta}{\sigma}}\right)^{\frac{1}{\beta}}\frac{H(t+1)}{L(t+1)}\left(\frac{\psi_H(t+1)}{\psi_L(t+1)}\right)^{\frac{\beta-1}{\beta}} = \frac{\eta_L}{\eta_H}\left(\frac{N_L(t)}{N_H(t)}\right)^{\delta},$$

and transform that to get the blueprint-ratio of the coming period.

$$\frac{N_H(t+1)}{N_L(t+1)} = (\frac{\eta_L}{\eta_H})^{\sigma}(\frac{1-\gamma}{\gamma})^{\epsilon}(\frac{\psi_H(t+1)}{\psi_L(t+1)})^{\frac{(1-\beta)(1-\sigma)}{\beta}}(\frac{H(t+1)}{L(t+1)})^{\sigma-1}(\frac{N_H(t)}{N_L(t)})^{\delta\sigma} \tag{D.103}$$

The relative costs (5.79) change the expression to

$$\frac{N_H(t+1)}{N_L(t+1)} = (\frac{\eta_L}{\eta_H})^{\sigma}(\frac{1-\gamma}{\gamma})^{\epsilon}\left((\frac{1-\gamma}{\gamma})^{\frac{\epsilon\beta(\alpha-1)}{\sigma\xi}}(\frac{N_H(t+1)}{N_L(t+1)})^{\frac{1-\alpha(1-\beta\sigma)}{\sigma\xi}}(\frac{M_H}{M_L})^{\frac{(1-\beta)(1-\alpha)+\beta\sigma}{\sigma\xi}}(\frac{H(t+1)}{L(t+1)})^{-\frac{\beta(\alpha-1)}{\sigma\xi}}\right)^{\frac{(1-\beta)(1-\sigma)}{\beta}}(\frac{H(t+1)}{L(t+1)})^{\sigma-1}(\frac{N_H(t)}{N_L(t)})^{\delta\sigma}.$$

Basic transformations finally lead to

$$\frac{N_H(t+1)}{N_L(t+1)} = \tag{D.104}$$

$$\left(\frac{\eta_L}{\eta_H}(\frac{N_H(t)}{N_L(t)})^{\delta}\right)^{\frac{\beta+(\sigma-1)\xi}{\beta-\alpha(1-\beta)(\sigma-1)}} \tag{D.105}$$

$$\left((\frac{1-\gamma}{\gamma})^{\epsilon\beta}(\frac{M_H}{M_L})^{(\sigma-1)(1-\beta)}(\frac{H(t+1)}{L(t+1)})^{(\sigma-1)\xi}\right)^{\frac{1}{\beta-\alpha(1-\beta)(\sigma-1)}} \tag{D.106}$$

D.11 Aggregate Savings

First, aggregate savings are given by

$$D = (I_L)L + (I_H)H. \tag{D.107}$$

We substitute the optimal savings for each type of household I_L (5.22)=(D.21) and I_H (D.26).

$$D = (\frac{\rho}{1+a+\rho}w_L - Pq_L)L + (\frac{\rho}{1+a+\rho}w_H - Pq_H)H \tag{D.108}$$

$$D = \frac{\rho}{1+a+\rho}(w_L L + w_H H) - P(q_L L + q_H H) \tag{D.109}$$

Both households have property rights for the resources. There are no resources left without any owner. So we can sum up all the property rights as the whole available resource stock Q.

$$D = \frac{\rho}{1+a+\rho}(w_L L + w_H H) - PQ \tag{D.110}$$

Since the skilled wage premium is given by $w = \frac{w_H}{w_L}$ we can rewrite the equation in relative terms.

$$D = \frac{\rho}{1+a+\rho}w_L L(1 + w\frac{H}{L}) - PQ \tag{D.111}$$

In different sections we already stated the following (equilibrium) variables in equations (D.32), (D.93), (5.69), (5.10) and (5.76), respectively.

$$\begin{aligned}
w_L &= \frac{\beta}{1-\beta}p_L N_L (\frac{p_L}{\chi_L})^{\frac{1-\beta}{\beta}} = \beta p_L \frac{Y_L}{L} && \text{(D.112)}\\
P &= p_L \alpha(1-\beta)\frac{Y_L}{R_L} && \text{(D.113)}\\
L &= \frac{\beta}{\xi}L_\Sigma && \text{(D.114)}\\
R(t) &= \tau Q(t-1) \quad \text{and} \quad Q(t) = Q(t-1) - R(t) \Leftrightarrow \frac{Q(t)}{Q(t-1)} = 1-\tau && \text{(D.115)}\\
R_L &= \frac{1}{1 + \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon}{\sigma}} \left(\frac{N_H}{N_L}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{\psi_H}{\psi_L}\right)^{\frac{(1-\beta)(1-\sigma)}{\beta\sigma}} \left(\frac{H}{L}\right)^{\frac{\sigma-1}{\sigma}}} R =: \phi_L R && \text{(D.116)}
\end{aligned}$$

Substituting into the aggregate savings (D.111) gives the following relation.

$$D = \frac{\rho}{1+a+\rho}\beta p_L \frac{Y_L}{L} L(1+w\frac{H}{L}) - p_L\alpha(1-\beta)\frac{Y_L}{\phi_L R}Q(t) \tag{D.117}$$

$$D = \frac{\rho}{1+a+\rho}\beta p_L Y_L(1+w\frac{H}{L}) - p_L\alpha(1-\beta)\frac{Y_L}{\phi_L \tau Q(t-1)}Q(t) \tag{D.118}$$

$$D = p_L Y_L\left(\frac{\rho}{1+a+\rho}\beta(1+w\frac{H}{L}) - \frac{\alpha(1-\beta)}{\phi_L}\frac{1-\tau}{\tau}\right) \tag{D.119}$$

D.12 Depletion rate of natural resources

We analyse the technology monopolists from an aggregate point of view. Output has to equal demand.

$$N_L x_L = N_L(\frac{p_L}{\psi_L})^{\frac{1}{beta}} L = M_L N_L L_T^{1-\alpha}(\phi_L R)^{\alpha} \tag{D.120}$$

$$x_L = (\frac{p_L}{\psi_L})^{\frac{1}{beta}} L = M_L L_T^{1-\alpha}(\phi_L R)^{\alpha} \tag{D.121}$$

We can rewrite the equation in terms of growth rates, whereas g^x denotes the growth rate for the variable x at point $(t+1)$ and we know, that the growth rates of the labour supply are equal $g^L = g^{L_T} = g^{L_\Sigma}$.

$$g^{x_L} = (\frac{g^{p_L}}{g^{\psi_L}})^{\frac{1}{beta}} g^L = \big(g^L\big)^{1-\alpha}\big(g^{\phi_L} g^R\big)^{\alpha} \tag{D.122}$$

Since

$$(\frac{g^{p_L}}{g^{\psi_L}})^{\frac{1-\beta}{\beta}} = \frac{\bar{D}}{\beta(1-\beta)(1+\frac{N_H(t+1)}{N_L(t+1)}\frac{\eta_L}{\eta_H}\big(\frac{N_H(t)}{N_L(t)}\big)^{\delta})}\big(g^L\big)^{\xi-1}\big(g^{\phi_L} g^R\big)^{\alpha(1-\beta)-1} \tag{D.123}$$

we can substitute $\frac{g^{p_L}}{g^{\psi_L}}$ into (D.122) and obtain

$$\left(\frac{\bar{D}}{\beta(1-\beta)(1+\frac{N_H(t+1)}{N_L(t+1)}\frac{\eta_L}{\eta_H}\big(\frac{N_H(t)}{N_L(t)}\big)^{\delta})}\right)^{\frac{1}{1-\beta}}\big(g^L\big)^{\frac{\xi-1+\alpha(1-\beta)}{1-\beta}}\big(g^{\phi_L} g^R\big)^{\frac{\alpha(1-\beta)-1-\alpha(1-\beta)}{1-\beta}} = 1 \tag{D.124}$$

where, the second exponent $\frac{\xi-1+\alpha(1-\beta)}{1-\beta} = 0$ and the exponent of the growth rate of ϕ_L and the resources $\frac{\alpha(1-\beta)-1-\alpha(1-\beta)}{1-\beta} = 1$. This simplifies the equation to

$$\left(\frac{\bar{D}}{\beta(1-\beta)(1+\frac{N_H(t+1)}{N_L(t+1)}\frac{\eta_L}{\eta_H}\big(\frac{N_H(t)}{N_L(t)}\big)^{\delta})}\right)^{\frac{1}{1-\beta}}(g^{\phi_L})^{-1} = g^R \tag{D.125}$$

We know that we can write the growth rate of the natural resources in terms of the depletion rate.

$$g^R = (1-\tau(t))\frac{\tau(t+1)}{\tau(t)} = \left(\frac{\bar{D}}{\beta(1-\beta)(1+\frac{N_H(t+1)}{N_L(t+1)}\frac{\eta_L}{\eta_H}\big(\frac{N_H(t)}{N_L(t)}\big)^{\delta})}\right)^{\frac{1}{1-\beta}}(g^{\phi_L})^{-1} \quad \text{(D.126)}$$

We finally derive the depletion rate.

$$\tau(t+1) = \frac{\tau(t)}{1-\tau(t)}\left(\frac{\bar{D}}{\beta(1-\beta)(1+\frac{N_H(t+1)}{N_L(t+1)}\frac{\eta_L}{\eta_H}\big(\frac{N_H(t)}{N_L(t)}\big)^{\delta})}\right)^{\frac{1}{1-\beta}}(g^{\phi_L})^{-1} \quad \text{(D.127)}$$

with $\bar{D} = \big(\frac{\rho}{1+a+\rho}\beta(1+w\frac{H}{L}) - \frac{\alpha(1-\beta)}{\phi_L}\frac{1-\tau}{\tau}\big)$.

D.13 Steady State

In the steady state all variables remain constant over time.

First, we analyse the relative labour supply (5.100). $H_\Sigma(t+1) = H_\Sigma(t) = H_\Sigma$ and $L_\Sigma(t+1) = L_\Sigma(t) = L_\Sigma$ gives

$$\frac{H_\Sigma}{L_\Sigma} = \frac{hn_{LH} + n_{HH}\frac{H_\Sigma}{L_\Sigma}}{(1-h)n_{LL}} \quad \text{(D.128)}$$

Simple transformations yield

$$\frac{H_\Sigma}{L_\Sigma} = \frac{hn_{LH}}{(1-h)n_{LL} - n_{HH}}. \quad \text{(D.129)}$$

Second, the relative wage (5.101) $w(t+1) = w(t) = w$ becomes

$$w = \frac{z + w\phi}{z}. \quad \text{(D.130)}$$

Again, simplifying obtains

$$w = \frac{z}{z-\phi}. \quad \text{(D.131)}$$

Next, with $\frac{N_H(t+1)}{N_L(t+1)} = \frac{N_H(t)}{N_L(t)} = \frac{N_H}{N_L}$ and the results for the labour supply and the relative wage the steady state technology level is

$$\frac{N_H}{N_L} = \left(\frac{\eta_L}{\eta_H}\big(\frac{N_H}{N_L}\big)^{\delta}\right)^{\frac{\beta+(\sigma-1)\xi}{\beta-\alpha(1-\beta)(\sigma-1)}}\left(\big(\frac{1-\gamma}{\gamma}\big)^{\epsilon\beta}\big(\frac{M_H}{M_L}\big)^{(\sigma-1)(1-\beta)}\big(\frac{H}{L}\big)^{(\sigma-1)\xi}\right)^{\frac{1}{\beta-\alpha(1-\beta)(\sigma-1)}} \quad \text{(D.132)}$$

and so we can write

$$\frac{N_H}{N_L} = \left(\frac{\eta_H}{\eta_L}^{\beta+(\sigma-1)\xi}\big(\frac{1-\gamma}{\gamma}\big)^{\epsilon\beta}\big(\frac{M_H}{M_L}\big)^{(\sigma-1)(1-\beta)}\big(\frac{H}{L}\big)^{(\sigma-1)\xi}\right)^{\frac{1}{\beta-\alpha(1-\beta)(\sigma-1)-\delta(\beta+(\sigma-1)\xi)}} \tag{D.133}$$

Last, the equilibrium steady state depletion rate ($\tau(t+1)=\tau(t)=\tau$ and $g^{\phi_L}=1$)

$$\tau = \frac{\tau}{1-\tau}\left(\frac{\bar{D}}{\beta(1-\beta)(1+\frac{N_H}{N_L}\frac{\eta_L}{\eta_H}\big(\frac{N_H}{N_L}\big)^{\delta})}\right)^{\frac{1}{1-\beta}} \tag{D.134}$$

is determined by

$$\tau = 1-\left(\frac{\bar{D}}{\beta(1-\beta)(1+\frac{\eta_L}{\eta_H}\big(\frac{N_H}{N_L}\big)^{\delta+1})}\right)^{\frac{1}{1-\beta}}. \tag{D.135}$$

E Diagrams

E.1 Basic Model

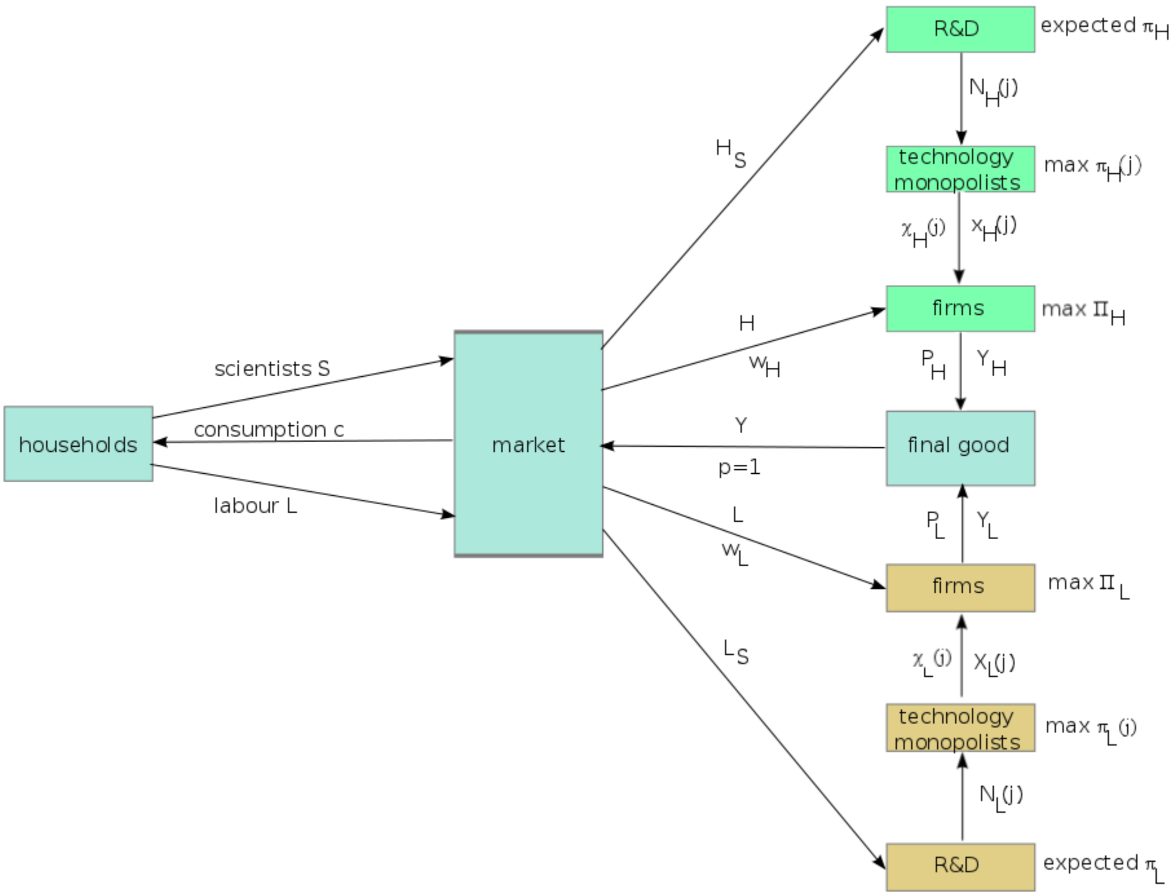

Figure E.1: Basic Model

Like in every economy, households and firms meet on a market. Households consume the good Y produced by the firms and provide labour for the production process. In this two sector model production takes place within two different sectors.
Households offer labour to the market. Scientists S decide whether they want to do research in the R&D-field of the H-sector or the L-sector and become either H_S or L_S, respectively. The labour supply L from the households is separeted in low-skilled workers L and high-skilled workers H, who work for a wage w_L or w_H on the machines x_L or x_H in the firms producing intermediate goods Y_L or Y_H. The firms sell their goods for a

price p_L or p_H. They maximise their profit Π_L and Π_H considering the revenue gained by selling the intermediate goods and the costs for the workers and the machines. The price χ_L or χ_H of the machines is set by the technology monopolists, who produce the machines. They are basically successful researchers, who invented the technology $N_L(j)$ or $N_H(j)$ for the machine type j in the previous period and therefore have the single right to sell those type of machines. When researchers decide which sector they want to work in, they do not know whether they are successful or not. So they can only calculate an expected profit π_L and π_H to base their decision.

E.2 Environment and Directed Technical Change

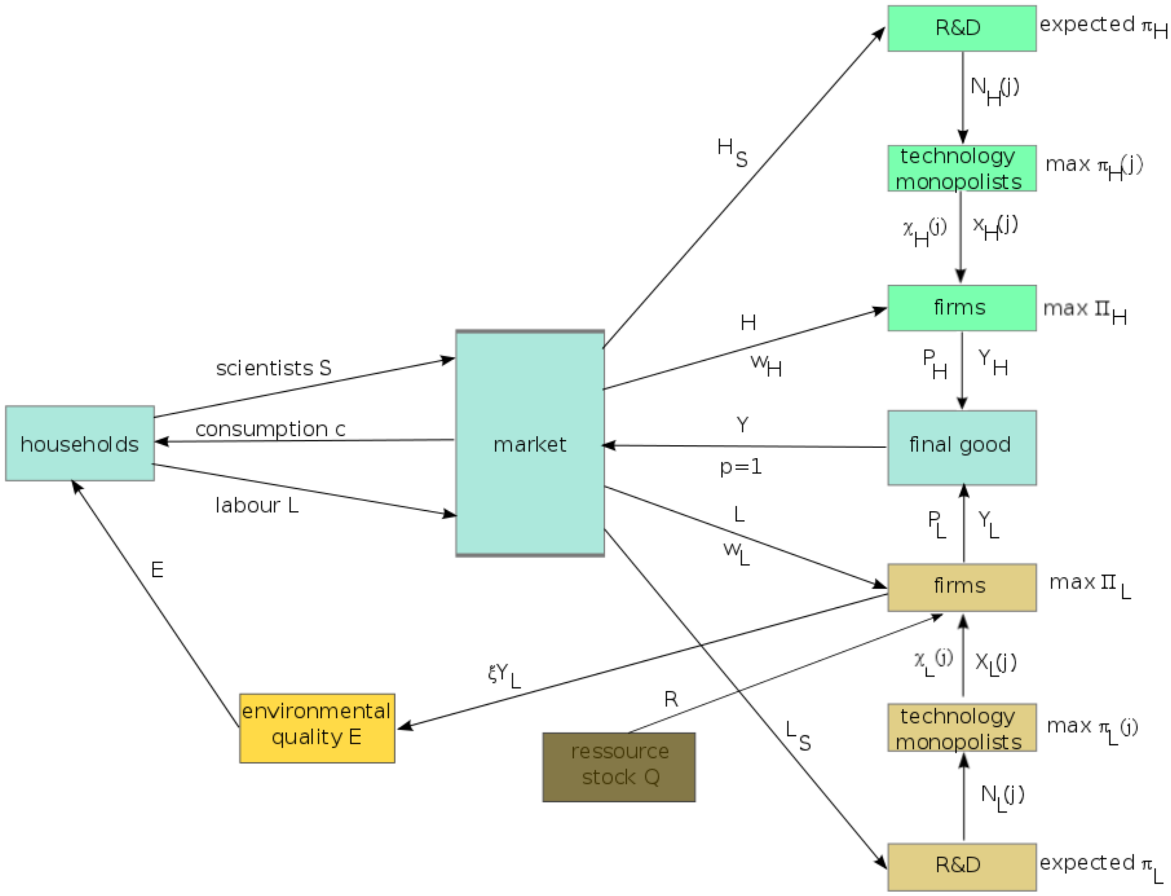

Figure E.2: Directed Technical Change and Environment

The model in chapter 3 is based on the previous model shown in Fig.E.1. We add a resource stock Q. This can be e.g. minerals, water, wood or oil. Firms in the dirty L-sector need those resources to produce the intermediate good Y_L. So the production of Y_L has an indirect impact on the quality of the environment E. Moreover, the production of Y_L also generates pollution (e.g. air pollution, water pollution), which negatively affects the environmental quality. Firms are still only interested in maximising their profit Π_L, whereas the households take the quality of environment into account. The utility function of the individuals, and therefore also the aggregate utility function, depends on the quality of the environment and the consumption.

This structure is completely different for the model in the following chapter 4. We keep only the L-sector, which is responsible for the production of the final good Y. Instead of the H-sector we have firms which positively influence the environmental quality, but do not produce (parts of) the final good.

E.3 Population Dynamics, Environment and Directed Technical Change

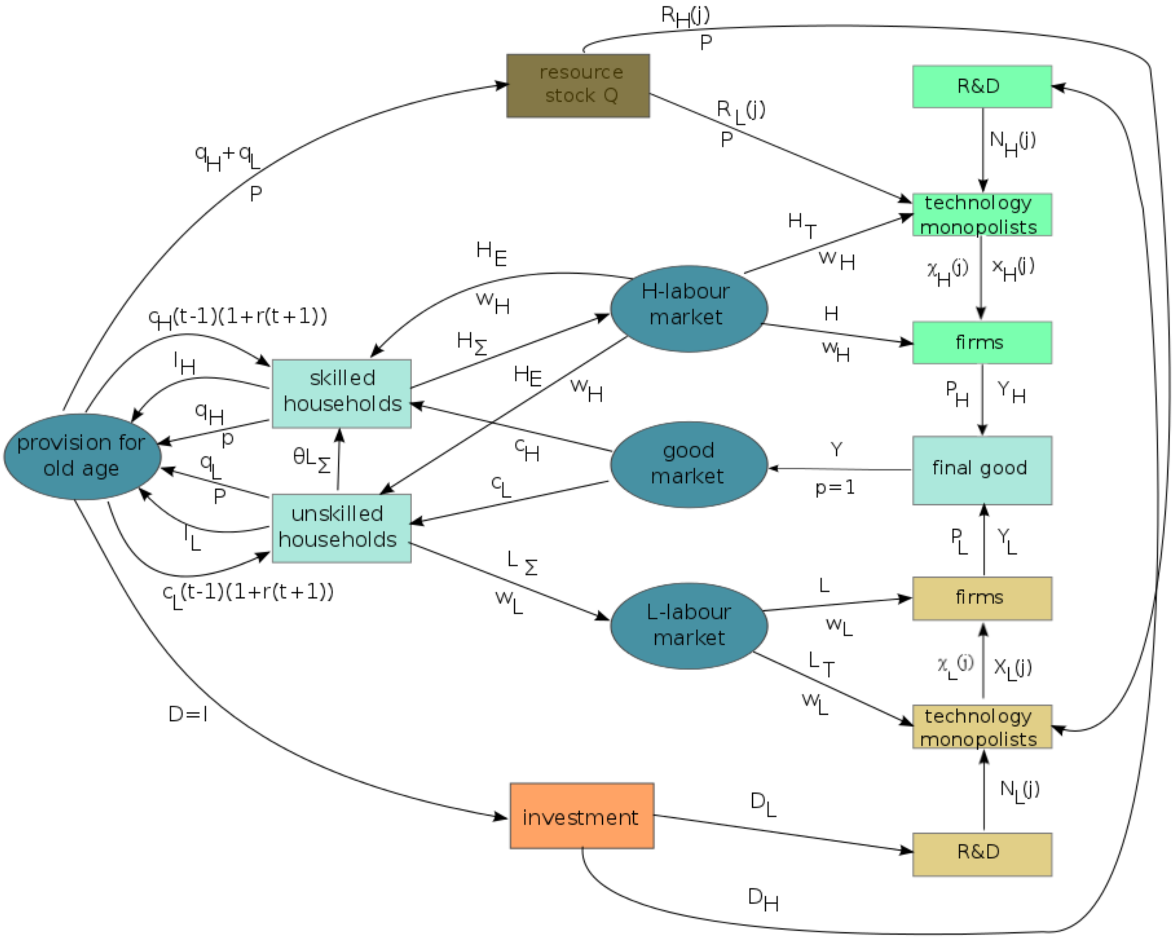

Figure E.3: Schaefer-Model

In the Schaefer-model two types of households consume the final good and provide labour supply. The offspring in the unskilled households can be educated to switch to the skilled households. Both types of households basically have the same interactions with the other market participants, but the intensity can be different. The index H denotes variables concerning the skilled (high-skilled) households and variables with index L are connected with the unskilled (low-skilled) households. The different households build the basis for the two different sectors. Labour from high-skilled households is used for research and working in the firms in the H-sector, while unskilled workers service the L-sector.
The labour markets for both sectors are separated. Agents can not choose the labour market and therefore the sector they want to work in, but are sent directly from the households depending on their education.

The "provision for old age" plays a crucial role in this model. Agents can invest in R&D or buy patents for natural resources. The revenues of the investments are used for future consumption. Patent rights for resources are sold to the technology monopolists, who use knowledge from R&D and specialised labour H_T and L_T to produce machines for the

firms.
Firms provide the intermediate goods, which are combined to a final good. This final good is consumed by both types of households.