TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

MASTERARBEIT

Holographic Duality of Anti-de Sitter Space with Logarithmic Conformal Field Theories

Ausgeführt am Institut für
Theoretische Physik
der Technischen Universität Wien

unter der Anleitung von

Daniel Grumiller

durch

Wolfgang Riedler

Berggasse 30/16
1090 Wien

September 24, 2013

Daniel Grumiller	Wolfgang Riedler

Ich erkläre hiermit, dass ich diese Diplomarbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt wurden. Ich versichere, dass ich diese Diplomarbeit bisher weder im In- noch im Ausland in irgendeiner Form als Prüfungsarbeit vorgelegt habe.

Acknowledgement

First and foremost I want to thank my supervisor, Daniel Grumiller, for patiently providing clear answers to my questions and for quietly enduring the novel extended bureaucracy that was involved in the final stages of this work. His door had always been open throughout the development of this thesis.

Special thanks go to Jan Rosseel and Thomas Zojer for valuable discussions.

I am grateful for the support of my parents Michaela, Hannes and Hans without whom I would not have been able to finish my studies the way I did.

I thank Jakob Salzer for the tedious task of proofreading.

Lastly, I thank Friedrich Schöller for often enlightening and always interesting discussions which rarely had anything to do with this work.

Thank you!

Abstract

We review the evidence that cosmological topologically massive gravity may correspond to a logarithmic conformal field theory for a specific tuning of the product of the AdS radius with the Chern-Simons coupling. More general three-dimensional models such as new massive gravity and generalized massive gravity as well as a generalization to a supersymmetric $\mathcal{N} = (1, 0)$ extension and a four-dimensional model are shown to experience similar features. We end with a discussion on the limiting procedure to a logarithmic field theory for non- and ultra-relativistic contractions of the conformal algebra in two dimensions and its connection to flat space holography.

to my three parents

Contents

Introduction

The twentieth century spawned two pillars of modern day physics: quantum field theory and general relativity. The introduction of the former lead to the formulation of the standard model of particle physics whereas the latter gave birth to the standard model of cosmology, both of which share an excellent agreement with experiment and helped to understand the physics at a subatomic and physics at a cosmological scale. One of the most ambitious goals in theoretical physics today is to find a consistent unifying theory from which both these inherently different theories can be recovered. That quantum field theory and general relativity should be somehow connected can be motivated by the fact that the most natural physical cut-off that is usually introduced in the process of renormalization should be provided by a microscopic black hole. Such a theory may also include a quantized version of gravity. Not knowing if quantized gravity is actually necessary to describe natural phenomena [51], it seems plausible from a present point of view that it accounts for things like the final evolution of black holes. The lack of experiments in the regime where a theory of quantum gravity should be considered forces one to rely solely on general physical and mathematical constraints, as well as gedankenexperiments.

It was pointed out by Bekenstein [27] that including gravity to a thermodynamical system requires black holes to have an entropy S_{BH}. Moreover, in order for the second law of thermodynamics to hold, the very same needs to be adjusted. Imagine throwing a teapot into a black hole. It follows that the black hole's entropy should increase by at least the same amount that the teapot had before vanishing behind the horizon. This shows that the second law of thermodynamics has to be generalized [26, 28]:

"The sum of black hole entropies together with the ordinary entropy outside black holes cannot decrease."

This concept implies that S_{BH} is the highest entropy that can be contained within a closed surface. Imagine any entropic system other than a black hole within a sphere of radius R and increase its energy until a black hole forms with its horizon at R. An initial entropy higher than S_{BH} violates the generalized second law of thermodynamics. This suggests that black holes are the most entropic objects in the universe. Along this line of reasoning Bekenstein showed that the entropy assigned to a black hole counterintuitively scales with its area rather than with its volume as expected from any ordinary field theory within a bounded finite region. Hawking later derived the proportionality factor [90].

$$S_{\mathrm{BH}} = \frac{A}{4\hbar G_N} \tag{1}$$

These discoveries lead to a thermodynamical interpretation of the four laws of black hole mechanics [22] from which can be inferred that black holes can be viewed as a thermodynamical system. The connection between thermodynamics and statistical physics has taught us that the concept of entropy is connected to a phase space, implying a microscopical description. For black holes such a connection has been found in many different models and with many inequivalent approaches [37, 45, 134, 136][1]. A physically realistic case that concerns four-dimensional extremal Kerr black holes can be found in [86].
Moreover, the ideas that connected an entropy to an area also lead to the formulation of the 'holographic principle' [137, 138] which states that any system within a closed spacelike surface can be represented by degrees of freedom on that surface. This thesis provides an explicit application thereof.
Numerous explicit realizations of the holographic principle have been found until today. Many of them include an anti-de Sitter (AdS) spacetime, a maximally symmetric solution of the Einstein equations that has a locally Lorentzian signature as well as constant negative curvature and a negative cosmological constant, and a conformal field theory (CFT), a quantum field theory which is invariant under conformal transformations. The first precise realization was found by Maldacena [116] who conjectured that a certain low energy limit of type IIB string theory compactified on the product space of five-dimensional AdS spacetime with a five-dimensional sphere, or $AdS_5 \otimes \mathbb{S}^5$ for short, is in a one-to-one correspondence with a superconformal Yang-Mills theory with four fermionic generators in its supersymmetry algebra. In a practical manner the correspondence of an AdS space with a CFT, henceforth abbreviated as AdS/CFT correspondence, can be viewed as a dictionary between states in a field theory describing gravity and operators in a CFT which exist on the boundary of the AdS space [141]. When discussing a holographic correspondence of two theories, the most intriguing single piece of information for the reason of applicability for problems of physical significance may be that one theory's spectrum of strong coupling is perturbatively accessible to the other's and vice versa.

In this thesis we almost exclusively concern ourselves with the case of three-dimensional gravitational models. The reason being that quantum gravity in higher dimensions presents conceptual and technical obstacles which mostly reduce to conceptual issues in the three-dimensional case. Pure Einstein gravity does not contain any local propagating degrees of freedom. This implies that static test particles do not feel any gravitational attraction for a vanishing cosmological constant in three spacetime dimensions. But studying three-dimensional gravity is not entirely academic since its holographic connection to two-dimensional CFT, which is reviewed in the body of this work, allows an association to models in statistical physics as well as string theory. In addition, even though this lower dimensional theory is trivial, choosing the cosmological constant to be negative allows for a black hole solution [20, 21, 44]. This black hole solution, also known as BTZ black hole, shares many properties with black holes in $3+1$ dimensions, such as a well defined entropy, an inner and outer horizon and the fact that it can form through the gravitational collapse of matter.
An important precursor of the holographic hypothesis within the framework of 2+1-dimensional gravity was found by Brown and Henneaux [40] who showed that the Hilbert space of pure Einstein gravity with AdS boundary conditions is a representation of a direct sum of two copies

[1] This list presents just a small selection that is far from complete. See [133] for a short review of some approaches.

of the Virasoro algebra, the centrally extended algebra of infinitesimal conformal transformations, with both their central charges being $c = 3\ell/2G_N$ where ℓ is the AdS radius and G_N Newton's constant. This result is not affected through the addition of matter fields. A concrete proposal for a dual CFT to 3-dimensional pure Einstein gravity with a negative cosmological constant was given by Witten [142]. He suggested that an AdS/CFT correspondence would only be possible for certain discrete values of the coupling constant which results in a a discrete series of dualities between CFTs and Einstein gravity. Although this proposal has been proved not to work [71, 73, 119] it marked the starting point for further investigations that try to solve the problem with Einstein gravity. Li, Song and Strominger considered a more general model, called 'cosmological topologically massive gravity' (CTMG),where they added a Chern-Simons term to the cosmological Einstein-Hilbert action [108] and proposed the dual field theory to be chiral at a specific tuning of the coupling constants. As it turned out this represents only a superselection sector of a more general correspondence which is reviewed in this work [80, 83].

This thesis is organized as follows: In chapter 1 we review the well established AdS_3/CFT_2 correspondence and in doing so we highlight some important concepts and issues necessary for its understanding which will also come in handy in the later part of this work. Chapter 2 introduces logarithmic CFTs (logCFTs) as well as CTMG and shows some facts that lead to the conjecture of their holographic duality for a specific tuning of parameters. Chapter 3 discusses generalizations such as cosmological topologically massive supergravity and a gravity model in 3+1 dimensions and we will see that a connection to a logCFT upholds. In addition an Inönü-Wigner-contraction from the asymptotic conformal symmetry group to the Galilean conformal group is considered and its non- and ultra-relativistic limit is investigated.

We mostly use natural units $\hbar = c = G_N = k_B = 1$. The conventions are $\kappa^2 = 16\pi G_N$ and $(-, +, \cdots, +)$ for the signature of the metric throughout this work. Anti-/Symmetrization of tensor components is given by squared/round brackets around indices and do not imply any additional prefactors, e.g. $T_{(ab)} = T_{ab} + T_{ba}$.

Chapter 1

AdS_3/CFT_2 in a Nutshell

This chapter gives a short introduction of how an AdS_3/CFT_2 correspondence comes about. In doing so, we highlight and review some familiar subjects that will be of great importance later on, thereby dismissing a thorough treatment and focusing our attention on portraying the conceptual. We outline the concept of canonical analysis as far as it is necessary for the models that are under consideration in this work.

1.1 Introducing Anti-de Sitter Space

An anti-de Sitter spacetime is a maximally symmetric and locally Lorentzian solution of the Einstein equations with constant negative curvature that involves a negative cosmological constant, i.e. an attractive potential. It can formally be understood as an embedding in $\mathbb{R}^{2,d-2}$ which is a flat ambient space with the following metric

$$\mathrm{d}s^2 = -\sum_{i=0}^{1} \mathrm{d}x_i^2 + \sum_{j=2}^{d-1} \mathrm{d}x_j^2 \ . \tag{1.1}$$

The constraint that yields AdS in d spacetime dimensions reads

$$\ell^2 = \sum_{i=0}^{1} x_i^2 - \sum_{j=2}^{d-1} x_j^2 \tag{1.2}$$

with the constant ℓ being refered to as the AdS-radius which is closely related to the cosmological constant. From the metric we can read off that the isometry group is $\mathrm{O}(d-2,2)$. As the topology of such hypersurfaces allows for closed timelike curves which are omitted for physical reasons, it is often more convenient to use its universal covering space which is simply connected by construction. Unless stated otherwise it is this covering space that we refer to whenever writing AdS.

From here onwards we restrict ourselves to three spacetime dimensions, because this is the case we will study later on. It will be convenient to make use of a local isomorphism and recast the isometry group as $\mathrm{SL}(2,\mathbb{R})_L \otimes \mathrm{SL}(2,\mathbb{R})_R$ [1] because the product group will allow for an easy

[1] Since we will later discuss the possibility of chiral theories where this symmetry plays a major role, the indices are merely used as a helpful way to differentiate between the ‘left’ and ‘right’ sector.

interpretation within the correspondence of AdS$_3$ with a generic 2 dimensional CFT. Let us motivate the correspondence by implementing a popular scheme often used when studying the geometry of (principal) homogeneous spaces by means of group theory. Specifying a map π : AdS$_3$ $\to$ SL$(2,\mathbb{R})$ with a convenient representation, the matter of finding global symmetries becomes a problem that can be answered within a group theoretical framework.

$$\pi(x_0,x_1,x_2,x_3) = l^{-1}\begin{pmatrix} x_0+x_2 & x_3-x_1 \\ x_3+x_1 & x_0-x_2 \end{pmatrix} \tag{1.3}$$

Notice that the constraint (1.2) is implemented by the determinant of the matrix being 1. Since the group's associated algebra is simple, the Killing form is proportional to, and so can formally be interpreted as, the pseudo-Riemannian metric on the SL$(2,\mathbb{R})$ manifold which involves the Maurer-Cartan form $\omega = g^{-1}\mathrm{d}g$ with $g \in$ SL$(2,\mathbb{R})$.

$$\mathrm{d}s^2 = -\mathrm{tr}(\omega^2) \tag{1.4}$$

Here d is the exterior derivative and tr denotes the trace. Invariance of the metric under the seperate left and right group action of SL$(2,\mathbb{R})$ therefore determines its isometry group to be SL$(2,\mathbb{R}) \otimes$ SL$(2,\mathbb{R})/\mathbb{Z}_2$. The reason for it being a quotient group comes from the equivalence of group elements that differ by a sign.
Another parametric solution to the constraint in (1.2) can be used to determine a specific form of the induced line element on the hypersurface. One such choice, which defines the metric globally, is given by

$$\begin{aligned}\mathrm{d}s^2 &= \ell^2\left(-\cosh^2(\rho)\mathrm{d}t^2 + \sinh^2(\rho)\mathrm{d}\phi^2 + \mathrm{d}\rho^2\right) && (1.5\mathrm{a})\\ &= \frac{\ell^2}{4}\left(-2\cosh(2\rho)\mathrm{d}\sigma^+\mathrm{d}\sigma^- - \mathrm{d}\sigma^{+2} - \mathrm{d}\sigma^{-2} + 4\mathrm{d}\rho^2\right) && (1.5\mathrm{b})\end{aligned}$$

$$t \sim t + 2\pi l \;, \quad 0 \leqslant \rho < \infty \;, \quad \phi \sim \phi + 2\pi$$

and implies that the topology of AdS$_3$ is a solid torus with its surface as a conformal boundary in contrast to the topology of the universal covering space, which is an infinitely long solid cylinder with the conformal boundary $\mathbb{R} \otimes \mathbb{S}^1$. Equation (1.5b) shows a representation of the metric with respect to lightcone coordinates $\sigma^\pm = t \pm \phi$.
An explicit representation of the Killing vectors associated to SL$(2,\mathbb{R})_L$ that we will use later on is given by

$$\begin{aligned}L_{-1} &= ie^{-i\sigma^+}\sinh^{-1}(2\rho)\Big\{\cosh(2\rho)\partial_+ - \partial_- + \frac{i}{2}\sinh(2\rho)\partial_\rho\Big\} && (1.6\mathrm{a})\\ L_0 &= i\partial_+ && (1.6\mathrm{b})\\ L_1 &= ie^{+i\sigma^+}\sinh^{-1}(2\rho)\Big\{\cosh(2\rho)\partial_+ - \partial_- - \frac{i}{2}\sinh(2\rho)\partial_\rho\Big\} && (1.6\mathrm{c})\end{aligned}$$

regarding the metric as given in (1.5b). The same representation can be chosen for the three generators $\{\bar{L}_{-1}, \bar{L}_0, \bar{L}_+\}$ of SL$(2,\mathbb{R})_R$ under the exchange $\sigma^+ \leftrightarrow \sigma^-$. The linear and associative internal operation on the vector space that defines the $\mathfrak{sl}(2)$ algebra is

$$[L_0, L_\pm] = \mp L_\pm, \quad [L_1, L_{-1}] = 2L_0. \tag{1.7}$$

An expansion of (1.5a) for large ρ yields

$$\mathrm{d}s^2 = \ell^2 \mathrm{d}\rho^2 + \frac{1}{4}\mathrm{e}^{2\rho}\big(\ell^2\mathrm{d}\phi^2 - \mathrm{d}t^2\big)\big(1 + O(\mathrm{e}^{-2\rho})\big) \ . \tag{1.8}$$

For our purposes it will be convenient to work in Gaussian normal coordinates with locally asymptotically AdS conditions, i.e. any coordinate system of a form that satisfies

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \quad \text{with} \quad \gamma_{ij} = \mathrm{e}^{2\rho/\ell}\gamma_{ij}^{(0)} + o(\mathrm{e}^{2\rho/\ell}) \tag{1.9}$$

in the vicinity of the conformal boundary with the condition that $\gamma_{ij}^{(0)}$ is invertible. The leading term in the expansion of γ_{ij} gives the boundary metric $\gamma_{ij}^{(0)}$, which will eventually be identified with the metric of the space on which a corresponding conformal field theory is studied. When considering pure Einstein gravity, the asymptotic equations of motion restrict the subleading terms to be of order $O(1)$ and one ends up with the standard Fefferman-Graham expansion [67]. This kind of behaviour does not hold for more general gravity models as we will review in chapter 2.

1.2 On the Importance of Boundary Conditions

Asymptotical boundary conditions can loosely be understood as a general classification of 'vacuum' states. Take for example the globally hyperbolic Schwarzschild and Minkowsi spacetimes. Both have the same behaviour at the spatial boundary and thus correspond to an asymptotically flat vacuum. However, differing in initial conditions when viewed as an initial value problem, they simply represent different states within the theory. This is of course not restricted to the flat case and can be applied to many different classes of spacetimes. We are now going to pin down the meaning of an 'asymptotically AdS' spacetime along the line of [94]. This essentially boils down to a statement about the imposed boundary conditions that a metric should satisfy at spatial infinity. These are required to fulfill the following three criterions: (*i*) They must contain a globally AdS solution, (*ii*) they need to be invariant under the action of the AdS isometry group and (*iii*) within a canonical framework the associated charges to the generators of the AdS isometry group need to be finite (see app. A).
As an explicit example consider the following globally AdS metric

$$\mathrm{d}s^2 = -\left(1 + \frac{r^2}{\ell^2}\right)\mathrm{d}t^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\phi^2 \tag{1.10}$$

as a solution of the equations of motion coming from the Einstein-Hilbert action with the additional boundary conditions

$$g_{tt} = -\tfrac{r^2}{\ell^2} + O(1) \tag{1.11a}$$
$$g_{\phi\phi} = r^2 + O(1) \tag{1.11b}$$
$$g_{rr} = \tfrac{\ell^2}{r^2} + O(r^{-4}) \tag{1.11c}$$
$$g_{t\phi} = O(1) \tag{1.11d}$$
$$g_{tr} = O(r^{-3}) \tag{1.11e}$$
$$g_{\phi r} = O(r^{-3}) \ . \tag{1.11f}$$

Given these fall-off conditions one can find the most general transformations that leave (1.11) invariant, i.e. they map the class of asymptotically AdS solutions onto themselves. The AdS$_3$ isometry group, generated by the Killing vectors, is contained within this larger symmetry group at the spatial boundary, which we will henceforth call the 'asymptotic symmetry group' (ASG). To figure out its generators ξ we can use an altered Killing equation that takes the boundary conditions into account.

$$\mathcal{L}_\xi g_{\mu\nu} = \delta g_{\mu\nu} \tag{1.12}$$

To be clear, $\delta g_{\mu\nu}$ represents the subleading terms in (1.11). Solving for these asymptotic Killing vectors yields

$$\xi^t = \ell\left(T^+ + T^-\right) + \frac{\ell^3}{2r^2}\left(\partial_+^2 T^+ + \partial_-^2 T^-\right) + O(r^{-4}) \tag{1.13a}$$
$$\xi^r = -r\left(\partial_+ T^+ + \partial_- T^-\right) + O(r^{-1}) \tag{1.13b}$$
$$\xi^\phi = T^+ - T^- - \frac{\ell^2}{2r^2}\left(\partial_+^2 T^+ - \partial_-^2 T^-\right) + O(r^{-4}) \tag{1.13c}$$

where we used that $T^\pm \equiv T(x^\pm)$ with $x^\pm = t/\ell \pm \phi$. Rewriting these vectors using the coordinates $x^\pm$ and exploiting the periodicity in ϕ to make the ansatz of a Fourier series for the two functions $T^\pm$ leaves us with the following generators of the ASG

$$\xi_n^\pm = e^{inx^\pm}\left(\partial_\pm - \frac{\ell^2 n^2}{2r^2}\partial_\mp - \frac{inr}{2}\partial_r\right) + O(r^{-1}) \tag{1.14}$$

which form a representation of two copies of the Witt algebra

$$\left[\xi_m^\pm, \xi_n^\pm\right] = -i(m-n)\xi_{m+n}^\pm + O(r^{-1}) \tag{1.15a}$$
$$\left[\xi_m^\pm, \xi_n^\mp\right] = 0 \ . \tag{1.15b}$$

The arbitrary subleading terms in (1.13) behave as pure gauge transformations [40]. As an example, consider a spacetime deformation that can be related to a vector whose ξ^t and ξ^ϕ components behave as $O(r^{-4})$ and whose ξ^r components behaves as $O(r^{-1})$ then this transformation is pure gauge. They have no associated canonical boundary charge[2] and the

[2] For the reader who has not been exposed to a Hamiltonian treatment of general relativity we refer to appendix A.

generators vanish weakly, that is after imposing the equations of motion. In that sense the ASG relies on a cohomology where each element of a class of asymptotic Killing vectors differing only in subleading terms generates the same transformation. In other words it is a factor group that identifies all group elements which are equal up to trivial deformations.
Following the discussion in app. A, the charges form a 'projective' representation of the ASG [40]

$$\{Q\left[\xi_m^{\pm}\right], Q\left[\xi_n^{\pm}\right]\}_{\text{D.B.}} = -i(m-n)Q\left[\xi_{m+n}^{\pm}\right] + K\left[\xi_m^{\pm}, \xi_n^{\pm}\right] \tag{1.16}$$

where we switched from the Poisson brackets to the Dirac brackets which reduces the generators of spacetime deformations $H\left[\xi_m^{\pm}\right]$ to only the surface charges due to the Hamiltonian and the momentum constraint. The actual computation of the central term is greatly simplified by acknowledging that the integration constant in eq. (A.22) can be chosen such that the surface charge evaluated on a metric that is globally AdS vanishes on a hypersurface at $t = 0$. Furthermore, the Dirac bracket of two surface terms may be reinterpreted as the variation of a surface charge

$$\delta_{\xi_n^{\pm}} Q\left[\xi_m^{\pm}\right] = \{Q\left[\xi_m^{\pm}\right], Q\left[\xi_n^{\pm}\right]\}_{\text{D.B.}} \tag{1.17}$$

Having chosen a convenient integration constant as explained above and using the explicit form of the variation of the surface charge whose negative is given in eq. (A.21b) together with (1.16) yields

$$\delta_{\xi_n^{\pm}} Q\left[\xi_m^{\pm}\right] = K\left[\xi_m^{\pm}, \xi_n^{\pm}\right] \tag{1.18}$$

This calculation was performed for Einstein gravity in [40] where it was shown that after the exchange of the Dirac brackets with an ordinary commutator the surface charge algebra is isomorphic to two copies of the Virasoro algebra

$$\left[L_m, L_n\right] = (m-n)L_{m+n} + \frac{c_L}{12} n(n^2-1)\delta_{m+n,0} \tag{1.19a}$$

$$\left[\bar{L}_m, \bar{L}_n\right] = (m-n)\bar{L}_{m+n} + \frac{c_R}{12} n(n^2-1)\delta_{m+n,0} \tag{1.19b}$$

$$\left[L_m, \bar{L}_n\right] = 0 \tag{1.19c}$$

with the central charges being

$$c_L = c_R = \frac{3\ell}{2G_N} \ . \tag{1.20}$$

We notice that this symmetry algebra is exactly the one for the generators of two-dimensional conformal symmetry. However, the central charge here is obtained through purely classical analysis whereas for example in string theory it comes about through normal ordering after quantization.
The explicit form of the central charges allows to associate an entropy to the BTZ black hole in a semi-classical limit via the Cardy formula [134] which precisely agrees with the Bekenstein-Hawking discussed in the introduction.
The equality of the central charges $c_L = c_R$ in both copies of the Virasoro algebra, a feature that need not hold in other models as we show in the next chapter, implies that there is no

diffeomorphism anomaly. On the other hand we are left with a trace anomaly [96] which would not be the case if one central charge were the negative of the other.
Since an AdS/CFT correspondence is interpreted as an equality of partition functions we can check if three-dimensional Einstein-Hilbert gravity with a negative cosmological constant has more in common with a two-dimensional CFT than what we know so far from symmetry considerations. On the CFT side the partition function counts the Virasoro descendants of the vaccum. It is evaluated on the torus and can be taken from the literature [64].

$$Z_{\rm CFT}(q,\bar{q}) = |q|^{-\frac{c}{12}}\,{\rm tr}\left(q^{L_0}\bar{q}^{\bar{L}_0}\right) = |q|^{-\frac{c}{12}}\prod_{n=2}^{\infty}\frac{1}{|1-q^n|^2}\ . \tag{1.21}$$

The term in front of the trace essentially comes from the transformation from the (punctured) plane onto the cylinder. The conformal structure on the boundary of the manifold on which the gravitational theory is defined must coincide with the one on which the conformal theory is introduced. To that end the (euclidean) partition function is not evaluated on the universal cover of AdS but on the filled torus. We can simplify calculations with a semi-classical approach by using a saddle point approximation for the Einstein-Hilbert action

$$\log Z_{\rm E.\text{-}H.}(q,\bar{q}) = -k\Gamma^{(0)} + \Gamma^{(1)} + \frac{1}{k}\Gamma^{(3)} + \cdots \tag{1.22}$$

with $q \equiv e^{i\tau}$ where the modular parameter τ is related to the angular potential θ and the inverse temperature (i.e. euclidean time) via $2\pi\tau = \theta + i\beta$. In a semi-classical limit the parameter k, which is proportional to the inverse Newton constant, becomes large and all terms $\Gamma^{(i)}$ for $i > 1$ may be neglected. The two main contributions to the partition function are [75, 117]

$$e^{-k\Gamma^{(0)}} = |q|^{-\frac{k}{2}} \quad \text{and} \quad e^{\Gamma^{(1)}} = \prod_{n=2}^{\infty}\frac{1}{|1-q^n|^2} \tag{1.23}$$

and sets the partition function to

$$Z_{\rm E.\text{-}H.}(q,\bar{q}) = |q|^{-\frac{k}{2}}\prod_{n=2}^{\infty}\frac{1}{|1-q^n|^2}\ . \tag{1.24}$$

For $c = 6k$, which was the result obtained by Brown and Henneaux (1.20), both partition functions are exactly equal in the semi-classical limit.
The conformal Ward identities, which completely constrain the form of the two- and three-point correlation functions of a CFT together with the central charge, provide a further check for a holographic duality. These checks are applicable to any conjectured AdS$_3$/CFT$_2$ correspondence though and are not sufficient to exhibit detailed features of a duality of such theories. In the next chapter we look at a concrete proposal for such a duality.

Chapter 2

The AdS_3/logCFT_2 Correspondence

This chapter presents the basics of a logarithmic CFT within the framework of a 1+1 dimensional CFT as far as it will be needed later on and shows that it arises when the conformal dimensions of operators degenerate. For a deeper introduction to the topic we refer to [43, 70, 88, 89]. We will then introduce a gravitational theory called 'topologically massive gravity', show the correspondence of both theories for a specific tuning of parameters and conclude that such a connection is possible when differential operators in the equations of motion degenerate and end by listing further gravity theories which also experience this feature.
Throughout the rest of this thesis we will neglect all surface charges and holographic counterterms and just state the bulk piece when writing an action.

2.1 Logarithmic Conformal Field Theory

Logarithmic CFTs have proven themselves to be very valuable in the study of critical behaviour in condensed matter systems and statistical mechanics. Some physical processes that may be described in this way are percolation [107, 120] (coming from the Q-state Potts model as $Q \to 1$), the behaviour of polymers by using self avoiding walks (the $O(n)$-model in the $n \to 0$ limit), the quantum Hall plateau transition [36, 104] and systems of quenched disorder, i.e. systems whose measurable quantities depend on random variables which do not change over time such as the spin glass model. These particular CFTs are characterized by logarithmic terms that occur within correlation functions which may come about through singular coefficients in an operator product expansion (OPE). In addition, having such a singular behaviour is a typical feature for conformal theories where the central charge vanishes, which will be considered later on. A vanishing of the central charge is potentially worrying, since the OPE of a scalar primary field Φ with itself has the following form (with suppressed indices to avoid clutter)

$$\Phi(r)\Phi(0) = \frac{A}{r^{2x_\Phi}}\left(1 + B\frac{x_\Phi}{c}r^2T(0) + \cdots\right) + \cdots \tag{2.1}$$

with the normalization A, the scaling dimension x_Φ of the scalar field, the energy-momentum tensor $T(r)$ and a calculable constant B. To avoid the singular behaviour as $c \to 0$ one can choose either of three ways to proceed in this limit according to [43]: (i) the normalization

A vanishes, (*ii*) the scaling dimension x_Φ vanishes, (*iii*) contributions from other operators cancel the divergence which happens if there exists an operator whose scaling dimension degenerates with that of the energy-momentum tensor. We will now go on and take a closer look at the last case.

The conformal Ward identities (cf. app. C) fix the form of correlation functions that include up to three operators. All two-point correlation functions vanish if the involved operators differ in their conformal dimensions. Furthermore, when considering only 1+1 dimensions projected onto the (punctured) complex plane, any mixture of purely holomorphic and anti-holomorphic operators within two-point correlators vanishes. We define two quasi-primary operators $\{\mathcal{O}(z), \mathcal{O}_\varepsilon(z,\bar{z})\}$ such that $\mathcal{O}_\varepsilon(z,\bar{z}) \to \mathcal{O}(z)$ in a certain limit and denote their conformal dimensions by $\{(h,0),(h+\varepsilon,\varepsilon)\}$ respectively. The following identities hold

$$\langle \mathcal{O}(z)\mathcal{O}(w)\rangle = \frac{c}{2(z-w)^{2h}} \tag{2.2a}$$

$$\langle \mathcal{O}_\varepsilon(z)\mathcal{O}_\varepsilon(w)\rangle = \frac{c_\varepsilon}{2(z-w)^{2h+2\varepsilon}(\bar{z}-\bar{w})^{2\varepsilon}} \tag{2.2b}$$

with the coefficients c, c_ε being independent of spacetime and the factor 2 for later convenience. For an appropriate definition of a new operator

$$\mathcal{O}^{\log}(z,\bar{z}) := \lim_{\varepsilon\to 0}\frac{\mathcal{O}_\varepsilon(z,\bar{z})-\mathcal{O}(z)}{\varepsilon} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{O}_\varepsilon(z,\bar{z})\Big|_{\varepsilon=0} \tag{2.3}$$

and after setting $c_\varepsilon = -c + O(\varepsilon^2)$ with $b = -\lim_{\varepsilon\to 0}\frac{c}{\varepsilon} \neq 0$ we obtain the following correlators

$$\langle \mathcal{O}(z)\mathcal{O}(w)\rangle = 0 \tag{2.4a}$$

$$\langle \mathcal{O}^{\log}(z,\bar{z})\mathcal{O}(w)\rangle = \frac{b}{2(z-w)^{2h}} \tag{2.4b}$$

$$\langle \mathcal{O}^{\log}(z,\bar{z})\mathcal{O}^{\log}(w,\bar{w})\rangle = -\frac{b\ln(m^2|z-w|^2)}{(z-w)^{2h}}\ . \tag{2.4c}$$

Such a structure of correlators is the defining property of a logarithmic CFT. Notice that the last correlator involves a mass parameter m. It stems from the highest order term in $O(\varepsilon^2)$ in the vanishing coefficient c_ε and can be changed by a redefinition

$$\mathcal{O}_\varepsilon(z,\bar{z}) \to \mathcal{O}_\varepsilon(z,\bar{z}) + \gamma\mathcal{O}(z)\ . \tag{2.5}$$

It follows that this mass parameter is artificial and hence does not spoil the conformal symmetry. We can further take a look at the action of the Hamiltonian $H = L_0 + \bar{L}_0$ and the angular momentum $J = L_0 - \bar{L}_0$. The variations of the operators under symmetries generated by L_0 and its counterpart in the antiholomorphic sector are[1]

[1] We use $\bar{\partial}/\partial$ to denote the partial derivative with respect to the anti-/holomorphic coordinate.

$$\left[\bar{L}_0, \mathcal{O}(z)\right] = 0 \tag{2.6a}$$
$$\left[L_0, \mathcal{O}(z)\right] = 2\mathcal{O}(z) + z\partial\mathcal{O}(z) \tag{2.6b}$$
$$\left[\bar{L}_0, \mathcal{O}^{\log}(z,\bar{z})\right] = \mathcal{O}(z) + \bar{z}\bar{\partial}\mathcal{O}^{\log}(z,\bar{z}) \tag{2.6c}$$
$$\left[L_0, \mathcal{O}^{\log}(z,\bar{z})\right] = 2\mathcal{O}^{\log}(z,\bar{z}) + z\partial\mathcal{O}^{\log}(z,\bar{z}) + \mathcal{O}(z) \tag{2.6d}$$

Furthermore, we may use the one to one correspondence of fields with states in a Hilbert space, which always holds provided one restricts oneself to patches on the manifold where radial quantization holds. If this is the case then the action of the Hamiltonian and the angular momentum on the states corresponding to the above operators is

$$H\begin{pmatrix}\mathcal{O}^{\log}\\ \mathcal{O}\end{pmatrix} = \begin{pmatrix}2 & 2\\ 0 & 2\end{pmatrix}\begin{pmatrix}\mathcal{O}^{\log}\\ \mathcal{O}\end{pmatrix}\ , \quad J\begin{pmatrix}\mathcal{O}^{\log}\\ \mathcal{O}\end{pmatrix} = \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}\mathcal{O}^{\log}\\ \mathcal{O}\end{pmatrix}\ . \tag{2.7}$$

To suit our purposes we apply this procedure to the holomorphic part of the energy-momentum tensor T.[2] The two-point correlator can be inferred from the OPE (as shown in app. C)

$$T(z)T(w) = \frac{c_L}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots \tag{2.8}$$

and falls in line with (2.2a) up to regular terms which may be neglected. Notice that all other singular terms in the limit $z \to w$ are zero due to the symmetries imposed on the vacuum. The constant c in eq. (2.2a) is identified with the central charge of the Virasoro algebra c_L and eventually leads to a reduction to the Witt algebra and T becomes a primary operator with a vanishing two-point function. Notice that the antiholomorphic sector is completely unaffected by all manipulations and remains unchanged.

2.2 Cosmological Topologically Massive Gravity

From the point of view of a Hamiltonian formulation of Einstein gravity, the phase space consists of the induced metric on a hypersurface and its conjugate momentum. Both being symmetric tensors, the number of degrees of freedom in d spacetime dimensions amounts to $\frac{1}{2}d(d-1)$. However, not all of them remain independent since on shell the Hamiltonian and the diffeomorphism constraint (see app. A) must be satisfied which together eliminate d degrees of freedom. In addition, the equations of motion include the arbitrary lapse and the shift function (see app. A). A choice of coordinates may lead to a further reduction and eventually leaves the total number of independent physical degrees of freedom for each point in spacetime to be $d(d-3)$. It follows that in the case of $2+1$ dimensions we are left with no local degrees of freedom.

This situation changes when a Chern-Simons term is added to the pure Einstein-Hilbert action [59, 60, 61]

$$\mathcal{S}_{\text{CTMG}} = \frac{1}{\kappa^2}\int_{\mathcal{M}} d^3x\sqrt{-g}\left\{R - 2\Lambda + \frac{1}{2\mu}\epsilon^{\lambda\mu\nu}\Gamma^{\rho}{}_{\lambda\sigma}\left(\partial_\mu\Gamma^{\sigma}{}_{\nu\rho} + \frac{2}{3}\Gamma^{\sigma}{}_{\mu\tau}\Gamma^{\tau}{}_{\nu\rho}\right)\right\} \tag{2.9}$$

[2] Conversely, we will use $\bar{T} := T_{\bar{z}\bar{z}}(\bar{z})$ for the antiholomorphic part.

which is known as 'cosmological topologically massive gravity' (CTMG). Compared to [60] we did not refrain from a non-vanishing cosmological constant[3] in order to include the BTZ black hole solution. Additionally the sign was altered to ensure the positivity of the black hole's energy in the limit of large μ, the parameter in the Chern-Simons coupling. In what follows we assume a negative cosmological constant and use its connection to the AdS radius $\Lambda = -\ell^{-2}$.

Since the equations of motion are of third order, the metric and its first derivative with respect to the variable along which spacetime is foliated have to be treated as independent variables of the configuration space and both are assigned their own canonical momentum. Hence the counting argument as given above is no longer valid. Compared to the previous case of pure Einstein-Hilbert gravity, the theory gains one local, massive degree of freedom as can be seen for example by a linear perturbation around an AdS background (1.5a) $g = \bar{g} + \psi$ [59] or a canonical analysis. The equations of motion involve a combination of the Einstein tensor G and the conformally invariant Cotton tensor C.

$$G_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu}\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha(R_{\beta\nu} - \frac{1}{4}Rg_{\beta\nu}) = 0 \tag{2.10}$$

After making use of the Bianchi identity and applying the transverse gauge $\bar{\nabla}_\mu(\psi^{\mu\nu} - \bar{g}^{\mu\nu}\psi^\sigma{}_\sigma) = 0$ the linearized equation of motion reads [108]

$$(\mathcal{D}^M G^{(1)})_{\mu\nu} = (\mathcal{D}^L\mathcal{D}^R\mathcal{D}^M\psi)_{\mu\nu} = 0 \tag{2.11}$$

where $G^{(1)}$ is the linearized Einstein tensor and the three mutually commuting operators are defined by

$$(\mathcal{D}^L)_\alpha{}^\beta = \delta_\alpha{}^\beta + \ell\epsilon_\alpha{}^{\sigma\beta}\bar{\nabla}_\sigma \tag{2.12a}$$

$$(\mathcal{D}^R)_\alpha{}^\beta = \delta_\alpha{}^\beta - \ell\epsilon_\alpha{}^{\sigma\beta}\bar{\nabla}_\sigma \tag{2.12b}$$

$$(\mathcal{D}^M)_\alpha{}^\beta = \delta_\alpha{}^\beta + \frac{1}{\mu}\epsilon_\alpha{}^{\sigma\beta}\bar{\nabla}_\sigma\ . \tag{2.12c}$$

The gauge choice furthermore implies tracelessness of all solutions to eq. (2.11). Such a setting allows for three different branches of solutions

$$(\mathcal{D}^A\psi^A)_{\mu\nu} = 0 \qquad A \in \{L, R, M\}, \tag{2.13}$$

two of which are the massless left and right moving gravitons $\psi^{L/R}$ that are already present in Einstein gravity. The third mode ψ^M is a massive excitation with helicity ± 2, where the sign depends on the sign in front of the Chern-Simons term. This makes it obvious that (cosmological) TMG is equipped with a mass scale and is not invariant under parity transformation. It was shown in [108] that solving the linearized equations of motion (2.11) allows for a classification of the solutions under the $\mathfrak{sl}(2)_L \oplus \mathfrak{sl}(2)_R$ algebra. Knowing that the eigenvalues of the operators $\{L_0, \bar{L}_0\}$ are the conformal weights we use their explicit representation (1.6) to make the following ansatz

$$\psi_{\mu\nu} = e^{-ih\sigma^+ - i\bar{h}\sigma^-}F_{\mu\nu}(\rho). \tag{2.14}$$

[3] This coins the name 'cosmological' TMG.

By restricting the solutions to primary states we can impose the constraints $L_1\psi_{\mu\nu} = \bar{L}_1\psi_{\mu\nu} = 0$. Singular solutions at the origin will be dismissed as it does not make sense to study those under the light of perturbation theory, especially since the background metric shows no such behaviour. Demanding regularity at $\rho = 0$ and normalizability as $\rho \to \infty$ then yields the following weights

$$(h, \bar{h})_L = (2, 0) \tag{2.15a}$$
$$(h, \bar{h})_R = (0, 2) \tag{2.15b}$$
$$(h, \bar{h})_M = \left(\frac{3+\mu\ell}{2}, \frac{-1+\mu\ell}{2}\right) \tag{2.15c}$$

and sets $\mu \geq \ell^{-1}$. It was shown in [108] that the energy of the massive mode is negative for $\mu\ell > 1$ and positive for $\mu\ell < 1$ whereas the energy of the left moving mode always differs by a sign and the right moving mode's energy is always positive. At the special value

$$\mu\ell = 1 \tag{2.16}$$

the only non-vanishing energy is the one of the right moving mode and one is left with no negative energy solutions including the BTZ black hole's energy. However, due to the degeneracy of the massive with the left moving mode (cf. (2.15)) it seems that its single degree of freedom is lost given that ψ^L on the bulk is pure gauge. The solution to this conundrum [80] can be found by looking at the equation of motion (2.11) at the critical point

$$(\mathcal{D}^R\mathcal{D}^L\mathcal{D}^M\psi)_{\mu\nu}\Big|_{\mu l=1} = (\mathcal{D}^R\mathcal{D}^L\mathcal{D}^L\psi)_{\mu\nu} = 0\ , \tag{2.17}$$

which implies a further solution that fulfills

$$(\mathcal{D}^L\mathcal{D}^L\psi^{\log})_{\mu\nu} = 0 \tag{2.18a}$$
$$(\mathcal{D}^L\psi^{\log})_{\mu\nu} \neq 0\ . \tag{2.18b}$$

From this it can be inferred that $\psi^{\log}$ is a genuine degree of freedom and not pure gauge: We use a gauge condition preserving vector field ξ to show that ψ^L differs from zero only by a (trivial) diffeomorphism. Therefore it has to satisfy $(\mathcal{D}^L)_\mu{}^\tau\big(\nabla_\tau\xi_\nu + \nabla_\nu\xi_\tau\big) = 0$ which $\psi^{\log}$ does not (2.18). That CTMG retains its single degree of freedom at the critical point beyond the limiting case of linearization was concluded in [52, 79].
Following the definition in eq. (2.3) we construct this new mode in dependence of the two degenerating solutions

$$\psi^{\log}_{\mu\nu} = \lim_{\varepsilon\to 0}\frac{\psi^M_{\mu\nu} - \psi^L_{\mu\nu}}{\varepsilon} = -2(it + \ln\cosh\rho)\psi^L_{\mu\nu} \equiv y\ \psi^L_{\mu\nu} \tag{2.19}$$

where we used $2\varepsilon = \mu\ell - 1$. Its associated energy is negative, which fits the analysis carried out in [53], bounded and time independent. Given that this new mode has an asymptotically linear dependence on the radial coordinate means that the choice of Brown-Henneaux boundary conditions is too stringent for it to appear in the spectrum of solutions, they must be adjusted accordingly [80, 81] and lead to the following Fefferman-Graham like expansion of the metric

$$\mathrm{d}s^2 = \ell^2 \mathrm{d}\rho^2 + \left(e^{2\rho}\gamma_{ij}^{(0)} + \rho\gamma_{ij}^{(1)} + \gamma_{ij}^{(2)} + \cdots \right) \mathrm{d}x^i \mathrm{d}x^j \ . \tag{2.20}$$

The loosened boundary contitions are still asymptotically AdS and equal to the ones by Brown and Henneaux for a vanishing $\gamma^{(1)}$. In odd-dimensional spacetimes greater than four such a linear contribution is always present in pure Einstein gravity, whereas in the three-dimensional case it is usually required to be zero by the equations of motion. It is only for the Einstein-Hilbert action in three dimensions that the Brown-Henneaux boundary conditions coincide with the spacetime being asymptotically AdS (cf. [92, 93, 123]).
One might anticipate the holographic energy-momentum tensor to diverge due to the asymptotically linear dependence on the radial and time coordinate, however, this turns out not to be the case. We recall that it is defined as the variation of the on-shell action with respect to the metric on the conformal boundary (up to the square root of the Jacobian), or, equivalently, its (densitized) functional derivative.

$$\delta S_{\mathrm{CTMG}}\Big|_{\text{on shell}} = \frac{1}{2}\int_{\partial\mathcal{M}} \mathrm{d}^2x \sqrt{-\gamma^{(0)}}\, T^{ij}\delta\gamma_{ij}^{(0)} \tag{2.21}$$

By using the Fefferman-Graham expansion from (2.20) it can be shown that the holographic energy-momentum tensor is traceless, finite and conserved and it explicitly reads

$$T^{ij} = \frac{1}{8\pi G\ell}\left(\gamma_{(1)}^{ij} + \gamma_{(2)}^{ij} - \gamma_{(2)}^{il}\gamma_{lk}^{(0)}\varepsilon^{kj} \right) + (i \leftrightarrow j) \ . \tag{2.22}$$

We refrain from a detailed derivation and rather refer to the original work [66]. A few remarks are in order: The $\gamma^{(1)}$ term was added to take the solutions genuine to the critical point into account. It therefore contains only modes from the left moving sector. Conversely, the latter two terms project out any such solutions and so entail only right moving modes. From that it follows that the stress-energy tensor fails to be chiral at the critical point. Moreover, (2.22) generally changes by an additional term when considering classes of spacetimes for which the metric on the conformal boundary is not intrinsically flat [18, 56, 96, 106, 132].
A canonical analysis in analogy to the one by Brown and Henneaux [40] has been carried out explicitly in [99] and yields a direct sum of two Virasoro algebras associtated to the asymptotic symmetry group just like in Einstein gravity. Their central charges, however, are dependent on the Chern-Simons coupling, viz.

$$c_L = \frac{3\ell}{2G_N}\left(1 - \frac{1}{\mu\ell}\right) \qquad c_R = \frac{3\ell}{2G_N}\left(1 + \frac{1}{\mu\ell}\right) \ . \tag{2.23}$$

One readily sees that the central charge of the left sector vanishes at the critical point (2.16).

2.3 Connecting the Pieces

Having a conformal symmetry puts severe restrictions on a field theory in the form of the Ward identities (see app. C). They fix the 2- and 3-point correlation functions completely. Any two theories that are supposed to correspond to each other must by definition show the same restrictions. For a conjectured correspondence between critically tuned CTMG and a logarithmic CFT we already know what to expect from the gravitational correlation functions from (2.4).

Let us make some general remarks: Operators are the central objects to consider in a CFT, since no asymptotic states and therefore no S-matrix can exist in a theory that is invariant under scale transformations. When considering a CFT that is dual to a gravitational theory, fields in AdS are in a one-to-one correspondence with operators in the CFT [5, 85, 141] in the following way

$$\langle e^{\int d^dx\, \phi_0(\vec{x})\mathcal{O}(\vec{x})}\rangle_{\text{CFT}} = Z_{\text{grav}}\left[\phi(\vec{x},\rho)\Big|_{\partial\text{AdS}} = \phi_0(\vec{x})\right] \ . \tag{2.24}$$

In addition, any local field theory is destined to have an energy-momentum tensor by Noether's theorem. It can be seen as the source to which the metric on the boundary of AdS couples (just as in eq. (2.21)). Given that a logarithmic CFT is defined by its correlation functions which involve the energy-momentum tensor, we can check them by making use of the correspondence between fields and operators as in eq. (2.24) and compute

$$\langle T^A T^B\rangle_{\text{CFT}} \sim \frac{\delta^2 Z_{\text{grav}}}{\delta\psi^A\delta\psi^B} \ , \quad \langle T^A T^B T^C\rangle_{\text{CFT}} \sim \frac{\delta^3 Z_{\text{grav}}}{\delta\psi^A\delta\psi^B\delta\psi^C} \tag{2.25}$$

with $A, B, C \in \{L, R, \log\}$. It is important to note that such statements do not involve the full set of solutions of modes that extremize the gravitational action, but only the non-normalizable ones, i.e. those which involve a $\gamma^{(0)}$-term in a Fefferman-Graham expansion (2.20), precisely because the statement above holds exclusively on the boundary. Such computations have been done and show perfect agreement between the 2-point [132] as well as 3-point correlation functions [84] on both sides. The value of the new anomaly was found to be

$$b = -\frac{3\ell}{G_N} \ . \tag{2.26}$$

Another possibility to falsify a holographic description of critical CTMG in terms of a logCFT lies in showing that their partition functions fail to coincide. The one-loop contribution to the partition function of (euclidean) TMG for thermal AdS is

$$Z^{1-\text{loop}}_{\text{TMG}}\Big|_{\mu l=1} = \prod_{n=2}^{\infty}\frac{1}{|1-q^n|^2}\prod_{m=2}^{\infty}\prod_{\bar{m}=0}^{\infty}\frac{1}{1-q^m\bar{q}^{\bar{m}}} = \tag{2.27a}$$

$$= \prod_{n=2}^{\infty}\frac{1}{|1-q^n|^2}\left(1+\frac{q^2}{|1-q|^2}\right) + \sum_{h,\bar{h}} N_{(h,\bar{h})}\chi(h)\bar{\chi}(\bar{h}) \tag{2.27b}$$

and has been calculated in [72] alongside one part of the partition function for a logCFT, viz.

$$Z_{\text{logCFT}} = |q|^{-\frac{c}{12}}\prod_{n=2}^{\infty}\frac{1}{|1-q^n|^2}\left(1+\frac{q^2}{|1-q|^2}\right) + \cdots \ . \tag{2.28}$$

In (2.27b) we used characters $\chi(h) = \prod_{n=1}^{\infty}\frac{1}{1-q^n}$ and q depends on the modular parameter as it was stated previously in chapter 1. The expression in (2.28) only takes the descendants of the vacuum as well as the logarithmic operator into account, though, which physically describes excited states that include zero and one logarithmic mode. The problem of calculating higher order terms is technical. Nevertheless, $\mathcal{Z}_{\text{TMG}}$ and $\mathcal{Z}_{\text{logCFT}}$ can be compared at least to the

lowest order. Suggestively writing TMG's partition function by using characters and setting $N_{(h,\bar{h})}$ to zero culminates in a perfect coincidence up to an overall constant of both partition functions for up to single logarithmic mode excitations. The interpretation that the neglected terms in (2.27b) actually count the multi-log excitations gains validity by checking that the numerical values of $N_{(h,\bar{h})}$, which should count the number of independent states at each level, are not negative and integer. This has been shown explicitly for some representations $(h,\bar{h})$ in [72] where it was also argued that it must hold for all successive ones.
Approaching the BTZ black hole as a thermodynamical system leads to another possible falsification of the conjectured correspondence. The previous ansatz of solving Einstein's equation around a black hole solution at equilibrium includes modes that decay over time. With appropriate boundary conditions such excitations describe the dynamical relaxation of the perturbed system back to equilibrium and by keeping these perturbations small one can use linear response theory. Within the context of an AdS/CFT correspondence it was shown in [38] that there is an exact agreement between the complex frequencies of these decaying modes (a.k.a. quasi-normal modes) and where the pole of the Fourier-transformed retarded 2-point Green's function of the corresponding modes on the CFT side is located. The analysis regarding TMG has been performed in [129]. Logarithmic modes at the chiral point were taken under consideration in [128] where the connection to a logarithmic CFT was shown to hold. It was also highlighted that the double pole in the propagator in momentum space is responsible for the linear time dependence as seen in (2.19).

2.4 More on Higher Derivative Gravity

We present some additional models that have a more or less close connection to the previous case of critically tuned CTMG in which Jordan cells can arise in a similar fashion.

2.4.1 'New' massive gravity

An analysis in line with the last section for the action of 'new massive gravity' (NMG) [33, 34]

$$\mathcal{S}_{\mathrm{NMG}} = \frac{1}{\kappa^2}\int \mathrm{d}^3x\sqrt{-g}\left\{\sigma R - 2\lambda m^2 + \frac{1}{m^2}\left(R^{\mu\nu}R_{\mu\nu} - \frac{3}{8}R^2\right)\right\} \quad , \tag{2.29}$$

with λ being a cosmological parameter, m a mass parameter and σ determining the sign of the Einstein-Hilbert term, yields the following linearized equations of motion around an AdS background under the transverse gauge

$$(\mathcal{D}^L\mathcal{D}^R\mathcal{D}^{m_+}\mathcal{D}^{m_-}\psi)_{\mu\nu} = 0 \tag{2.30}$$

where the new differential operators $\mathcal{D}^{m_\pm}$ are defined as in (2.12c) with the parameter μ replaced by $m_\pm$. They are related to the other parameters via $m_\pm\ell = \pm\sqrt{\frac{1}{2} - \sigma m^2\ell^2}$ with the AdS radius being $\ell^{-2} = 2m^2(\sigma \pm \sqrt{1+\lambda})$. The spectrum again contains two massless gravitons which are non-trivial only on the boundary and additionally two massive modes, each with a different of the two helicity states ± 2. The sign of the helicity of each state is captured in the index $m_\pm$. Both have the same physical mass $m^2 = m_\pm^2 - \ell^{-2}$ and so these massive modes are related by parity. The linearized equations of motion are in that way equivalent to the Fierz-Pauli equations for a free and massive spin 2 field [33]. Interestingly, NMG also has a critical point that has remarkable similarities with that of TMG: One of the two new differential operators degenerates with $\mathcal{D}^L$ at $\sigma m^2\ell^2 = -\frac{1}{2}$ and therefore a logarithmic mode in the left sector appears. The new feature concerns the right sector, coming from the degeneration of the other new differential operator with $\mathcal{D}^R$. This leads to an additional logarithmic mode and the theory now gains the same Jordan cell structure as before but for the antiholomorphic sector [78]. Applying Brown-Henneaux boundary conditions leads to finite conserved charges [110]. The central charges of the associated Virasoro algebra are

$$c_L = c_R = \frac{3\ell}{2G_N}\left(\sigma + \frac{1}{2m^2\ell^2}\right) \quad . \tag{2.31}$$

The new anomalies in both sectors take the same value

$$b_L = b_R = -\sigma\frac{12\ell}{G_N} \quad . \tag{2.32}$$

Another interesting tuning is $2m^2\ell^2 = \sigma$. The parameters $m_\pm$ vanish which results in the degeneration of the two massive modes and leaves a logarithmic pair. Contrary to the previous case the logarithmic CFT does not involve central charges that are equal to zero and it is not the energy-momentum tensor that aquires a logarithmic partner. Further issues concerning this 'partially massless gravity' theory are considered in [34, 82].

2.4.2 Generalized massive gravity

A straightforward generalization of NMG is the enhancement of its action by a Cern-Simons term.

$$\begin{aligned}\mathcal{S}_{\mathrm{GMG}} = \frac{1}{\kappa^2}\int \mathrm{d}^3x\sqrt{-g}\Big\{&\sigma R - 2\lambda m^2 + \frac{1}{m^2}\left(R^{\mu\nu}R_{\mu\nu} - \frac{3}{8}R^2\right) \\ &+ \frac{1}{2\mu}\epsilon^{\lambda\mu\nu}\Gamma^\rho{}_{\lambda\sigma}\left(\partial_\mu\Gamma^\sigma{}_{\nu\rho} + \frac{2}{3}\Gamma^\sigma{}_{\mu\tau}\Gamma^\tau{}_{\nu\rho}\right)\Big\}\end{aligned} \tag{2.33}$$

Generalized massive gravity (GMG) has a rich structure from which various existing models, including the two previous ones, can be recovered. The limit $\mu \to \infty$ [$m^2 \to \infty$] leads to NMG [TMG], the limit $m \to 0$ while keeping m^2G constant recovers a ghost-free and finite theory of gravity of fourth order [58] and the scaling $\mu \to 0$ while keeping μG fixed leads to conformal Chern-Simons gravity [2] . We use the same approach as in the former sections to arrive at the equations of motion.

$$\left(\mathcal{D}^L\mathcal{D}^R\mathcal{D}^{m_1}\mathcal{D}^{m_2}\psi\right)_{\mu\nu} = 0 \tag{2.34}$$

The operators again mutually commute and the latter ones are defined just as in (2.12c) with μ replaced by $m_{1,2}$. Regarding the AdS radius, we find that it is identical to the one in NMG. The difference to NMG lies in the fact that the masses of the massive modes need not be equal.

$$m_{1,2} = \frac{m^2}{2\mu} \pm \sqrt{\frac{1}{\ell^2} - \sigma m^2 + \frac{m^4}{4\mu^2}} \tag{2.35}$$

This allows for the appearance of more critical points in comparison to TMG and NMG. If either of the two parameters $m_{1,2}$ equals $\pm\ell^{-1}$ a Jordan cell of rank 2 is obtained in either the left or the right sector. Additionally, the same structure arises if the values of only these two parameters coincide. A degeneration of the kind $m_1 = -m_2 = \pm\ell^{-1}$ leads to the same structure that was already encountered in NMG. Another new feature is a possible degeneration of three differential operators that appears by setting $m_1 = m_2 = \pm\ell^{-1}$. Analyzing the action of the Hamiltonian on the associated states of the field solutions of eq. (2.34) at a tricritical point reveals a Jordan cell of rank three.

$$H \begin{pmatrix} \psi^{\log^2} \\ \psi^{\log} \\ \psi^{L} \end{pmatrix} = 2 \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \psi^{\log^2} \\ \psi^{\log} \\ \psi^{L} \end{pmatrix} \tag{2.36}$$

Two-point correlators for one of the two tricritical points have been shown to yield [82]

$$\langle \psi^L(z)\psi^L(w)\rangle = 0 \tag{2.37a}$$
$$\langle \psi^L(z)\psi^{\log}(w,\bar{w})\rangle = 0 \tag{2.37b}$$
$$\langle \psi^L(z)\psi^{\log^2}(w,\bar{w})\rangle = \frac{b_L}{2(z-w)^4} \tag{2.37c}$$
$$\langle \psi^{\log}(z,\bar{z})\psi^{\log}(w,\bar{w})\rangle = \frac{b_L}{2(z-w)^4} \tag{2.37d}$$
$$\langle \psi^{\log}(z,\bar{z})\psi^{\log^2}(w,\bar{w})\rangle = -\frac{b_L \ln(m_L^2|z-w|^2)}{(z-w)^4} \tag{2.37e}$$
$$\langle \psi^{\log^2}(z,\bar{z})\psi^{\log^2}(w,\bar{w})\rangle = \frac{b_L \ln^2(m_L^2|z-w|^2)}{(z-w)^4} \ . \tag{2.37f}$$

Again, the mass parameter m_L is artificial and can be set to an arbitrary value without loss of generality which has already been discussed in sec. 2.1. The correlators are in perfect agreement with the ones of a logarithmic CFT that has a Jordan cell of rank 3 [3, 4, 69]. It is interesting to note that the new mode fulfills the following relation

$$\psi_\mu^{\log^2} = y\,\psi_\mu^{\log} = y^2\psi_\mu^L \tag{2.38}$$

where the function $y \equiv y(t,\rho)$ has been established in eq. (2.19). We recall that this circumstance initially revealed the need for looser boundary conditions in the case of TMG. In order to include $\psi^{\log^2}$ as a solution to the linearized equations of motion we need to further loosen the boundary conditions to allow for modes that fall off asymptotically as $O(\rho^2)$. Suffice it to say that for consistent 'log squared'-boundary conditions their associated conserved charges are finite only at the tricritical point [111].

2.4.3 Speculations on even higher derivative models

So far we have seen that the number of logarithmic solutions, and with it the rank of a Jordan cell, depends on the number of degenerate operators in the equations of motion. In principle the rank of such a Jordan cell has no upper bound since terms with an arbitrary number of derivatives can always be added to the action. It seems verisimilar that the boundary conditions for a cell of rank r are in need to be generically adjusted to

$$\gamma_{ij} = e^{2\rho}\gamma_{ij}^{(0)} + \sum_{n=0}^{r} \rho^n \gamma_{ij}^{(1,n)} + \gamma_{ij}^{(2)} + \cdots \tag{2.39}$$

in order not to truncate any excitations. This shows an attempt to construct a plausible form by extrapolating from previous knowledge. It certainly holds for $r = 1$ as we have seen for TMG and NMG and $r = 2$ in the case of GMG where the crucial properties in (2.19) and (2.38) demand for the alteration of the boundary conditions. The function y is a characteristic trait in the construction of logarithmic modes which allows to speculate that the boundary conditions given in (2.39) are in fact correct. It would be interesting to know if gravitational theories allow a limiting procedure for the rank of Jordan cells to become infinite and also if similar connections to a log CFT [125, 126] would still uphold.

Chapter 3

Generalizations

Through the construction of supersymmetric models of the aforementioned gravitational theories we hope to gain insight into a supersymmetric version of the AdS_3/$logCFT_2$ correspondence. With that in mind we take a look at the supersymmetric extension of TMG. Other supergravity models can be found in [8, 31]. In addition we review a particular four dimensional theory with a critical tuning that exhibits similar features as NMG. We end this chapter by looking at a particular contraction of the conformal symmetry group and discussing some of the consequences regarding the field theory.

3.1 A Supersymmetric Extension

For the sake of taking supersymmetry into account we consider a $\mathcal{N} = (1, 0)$ extension, meaning that only the left, i.e. the holomorphic sector of the asymptotic symmetry product group will be adjusted. The symmetry algebra on the boundary remains a direct sum consisting of the Virasoro algebra and its supersymmetric $\mathcal{N} = 1$ extension

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{3.1a}$$

$$\{G_m, G_n\} = 2L_{m+n} + \frac{c}{3}(m^2 - \frac{1}{4})\delta_{m+n,0} \tag{3.1b}$$

$$[L_m, G_n] = (\frac{1}{2}m - n)G_{m+n} \quad . \tag{3.1c}$$

This is an enhancement of the symmetry algebra on the bulk which in this case is the product group $\mathrm{OSp}(1|2;\mathbb{R})_L \otimes \mathrm{SL}(2)_R$. As a consequence of the enhanced symmetry the stress-energy tensor T gains a supersymmetric partner S. How the 2-point correlation function for this fermionic operator looks like can be checked in the following way: In analogy to T and being a field of conformal dimension $(\frac{3}{2}, 0)$, S can be mode expanded as follows

$$S(z) = \sum_{n\in\mathbb{Z}} z^{-n-\frac{3}{2}} G_n \tag{3.2}$$

where the G_n are the aforementioned elements of the super-Virasoro algebra.[1] Plugging this

[1] We follow the convention and set $G_n := G_n(0)$ as we already implicitly did for the Virasoro algebra, i.e. it corresponds to an operator insertion at the origin.

definition into the anticommutator (3.1b), the operator product expansion can be extracted by demanding closure of the algebra and one eventually gets

$$S(z)S(w) = \frac{\frac{2c}{3}}{(z-w)^3} + \frac{2T(w)}{(z-w)} + \cdots \tag{3.3}$$

where the ellipses indicate terms that are regular in the limit $z \to w$.[2] The same procedure as in section 2.1 can be applied to get an expression for its logarithmic partner $S^{\log}$. Defining an operator S_ε with weights $(\frac{3}{2}+\varepsilon, \varepsilon)$, the 2-point correlators involving $S^{\log}$ can be obtained by using the definition in eq. (2.3) followed by taking the limit $\varepsilon \to 0$.

$$S^{\log}(z,\bar{z}) = \lim_{\varepsilon\to 0} \frac{S_\varepsilon(z,\bar{z}) - S(z)}{\varepsilon} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} S_\varepsilon(z,\bar{z})\big|_{\varepsilon=0} \ , \tag{3.4}$$

For finite ε the conformal Ward identities fix $\langle S_\varepsilon(z)S(0)\rangle = 0$ and the other correllators are once again determined by the operator product expansion in the limit $\varepsilon \to 0$.

$$\langle S(z)S(0)\rangle = 0 \tag{3.5a}$$

$$\langle S^{\log}(z,\bar{z})S(0,0)\rangle = \frac{2b_L}{3z^3} \tag{3.5b}$$

$$\langle S^{\log}(z,\bar{z})S^{\log}(0,0)\rangle = -\frac{4b_L \log(m_S^2|z|^2)}{3z^3} \tag{3.5c}$$

The factor b_L is the new anomaly that has already appeared in the correlators of the bosonic sector and the parameter m_S can yet again be set to an arbitrary value without loss of generality since it can be adjusted accordingly via a redefinition $S_\varepsilon \to S_\varepsilon + \gamma_S S$.
Using the second order formalism with the dreibein ${e_\mu}^a$ and the spin connection ${\omega_\mu}^{ab}$, the action of cosmological topologically massive $\mathcal{N} = (1,0)$ supergravity (CTMSG) that includes a Majorana spinor in the $(\mathbf{2},\mathbf{1})$ representation with respect to the AdS_3 isometry group [57, 62] is given by

$$\begin{aligned}\mathcal{S}_{\mathrm{STMG}} = \frac{1}{\kappa^2}\int \mathrm{d}^3 x e \Big\{ & R - 2\Lambda - i\epsilon^{\mu\nu\rho}\bar{\Psi}_\mu\Big(D_\nu - \frac{1}{2l}\gamma_\nu\Big)\Psi_\rho \\ & - \frac{1}{2\mu}\epsilon^{\mu\nu\rho}\Big(\partial_\mu {\omega_\nu}^{ab}\omega_{\rho ba} + \frac{2}{3}{\omega_\mu}^a{}_b {\omega_\nu}^b{}_c {\omega_\rho}^c{}_a\Big) + \frac{i}{2\mu}\bar{f}^\mu\gamma_\nu\gamma_\mu f^\nu \Big\}\end{aligned} \tag{3.6}$$

where the definition of f^μ is

$$f^\mu = \epsilon^{\mu\sigma\tau} D_\sigma \Psi_\tau \quad \text{with} \quad D_\sigma\Psi_\tau = \partial_\sigma\Psi_\tau + \frac{1}{4}{\omega_\sigma}^{ab}\gamma_{ab}\Psi_\tau \tag{3.7}$$

and denotes the dual of the gravitino field strength. Under the local supersymmetry transformations

[2]Be reminded that the OPE holds within correlation functions and implies time ordering. It follows that the OPE for a bosonic operator with itself must remain invariant under $z \leftrightarrow w$. The anticommutativity of the supersymmetric partner of the energy-momentum tensor is reflected in its OPE with itself.

$$\delta e_\mu{}^a = i\bar{\epsilon}\gamma^a\Psi_\mu \tag{3.8a}$$

$$\delta\Psi_\mu = 2D_\mu\epsilon - \frac{1}{\ell}\gamma_\mu\epsilon \tag{3.8b}$$

both lines in (3.6) are separately invariant and so the action remains unchanged. The torsionfull spin connection is dependent on the dreibein and the spinor $\omega_\mu{}^{ab}(e,\Psi)$ and is set to be that of simple supergravity. This can also be inferred from the first order or Palatini formalism in which the spin connection is treated as an independent field whose dependence on the vielbein and the spinor comes about through its equation of motion and reads

$$\omega_\mu{}^{ab}(e,\Psi) = \omega_\mu{}^{ab}(e) + \kappa_\mu{}^{ab}(e,\Psi) \tag{3.9}$$

with

$$\kappa_\mu{}^{ab}(e,\Psi) = \frac{i}{4}\left(\bar{\Psi}_\mu\gamma^a\Psi^b - \bar{\Psi}_\mu\gamma^b\Psi^a + \bar{\Psi}^a\gamma_\mu\Psi^b\right) \ . \tag{3.10}$$

The contortion tensor $\kappa(e,\Psi)$ is a bilinear expression in the spinor fields which is dependent on the torsion whereas $\omega(e)$ is the standard spin connection.
Linearized perturbation theory around an AdS vacuum $\bar{g}$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \lambda h_{\mu\nu} + \lambda^2 j_{\mu\nu} + \mathcal{O}(\lambda^3) \tag{3.11a}$$

$$\Psi_\mu = \lambda\psi_\mu + \lambda^2\psi_\mu^{(2)} + \mathcal{O}(\lambda^3) \ , \tag{3.11b}$$

shows that the bosonic equations of motion are left unchanged when compared to ordinary TMG and that the fermionic counterpart involving the gravitino completely decouples. This should not come as a surprise, because the coupling comes about through torsion which is an effect of second order as can be seen from (3.10). Moreover, a canonical analysis yields the same central charges as in the non-supersymmetric case and the critical point requires the same tuning of the coupling constants as before. Applying the gauge $\gamma^\mu\psi_\mu = 0$ (gamma-traceless gauge) leaves the equations of motions for the gravitino to be

$$(\mathcal{D}^L\mathcal{D}^{(\mu)}\psi)_\nu = 0 \tag{3.12}$$

where the two operators are defined via

$$\mathcal{D}^{(\mu)} = \gamma^\mu\mathfrak{D}_\mu + \frac{1}{2\ell} - \mu \tag{3.13a}$$

$$\mathcal{D}^L = \mathcal{D}^{(\mu)}\Big|_{\mu=1/\ell} \tag{3.13b}$$

and do again mutually commute. The symbol $\mathfrak{D}$ is used to identify the covariant derivative with respect to the background spin connection and the Levi-Civita connection. The gravitino field is a vector-spinor on an AdS_3 background for which a separation ansatz much like in the graviton case can be chosen.

$$\psi_\mu = e^{-ih\sigma^+ - i\bar{h}\sigma^-} F_\mu(\rho) \begin{pmatrix} i \\ e^\rho \end{pmatrix} \tag{3.14}$$

An explicit expression of solutions of ψ_μ can be found in [25]. The demand of normalizability and regularity at the origin as well as restraining the fermionic solutions such that they obey the primary conditions $L_1\psi_\mu = \bar{L}_1\psi_\mu = 0$ yields modes whose weights are given by

$$(h, \bar{h})_L = \left(\frac{3}{2}, 0\right) \tag{3.15a}$$

$$(h, \bar{h})_M = \left(1 + \frac{\mu\ell}{2}, -\frac{1}{2} + \frac{\mu\ell}{2}\right) \tag{3.15b}$$

where the notation is in line with the previous chapters and refers to the left and massive gravitini modes. Once again we find a degeneration of the differential operators in the equations of motion and the weights of the two solutions at the critical point $\mu\ell = 1$. This allows for logarithmic gravitino modes that behave analogously to (2.18). These modes can furthermore be obtained by applying the same limiting procedure that was done for the logarithmic graviton mode.

$$\psi_\mu^{\log} = \lim_{\varepsilon\to 0} \frac{\psi_\mu^M - \psi_\mu^L}{\varepsilon} = y\, \psi_\mu^L \tag{3.16}$$

The parameter is defined by $2\varepsilon = \mu\ell - 1$ and the proportionality function on the right hand side is identical to the one that appears in the bosonic case (2.19). Having classified the modes under the $\mathfrak{sl}(2)$ algebra lets us construct the Hamiltonian $H = L_0 + \bar{L}_0$ straightforwardly and one encounters once more a Jordan cell of rank 2

$$H \begin{pmatrix} \psi_{\mu\nu}^{\log} \\ \psi_{\mu\nu}^L \end{pmatrix} = 2 \begin{pmatrix} \frac{3}{4} & 1 \\ 0 & \frac{3}{4} \end{pmatrix} \begin{pmatrix} \psi_{\mu\nu}^{\log} \\ \psi_{\mu\nu}^L \end{pmatrix} \tag{3.17}$$

whereas the angular momentum $J = L_0 - \bar{L}_0$ is diagonal.

3.2 Critical Gravity in Higher Dimensions

Critical gravity theories that show a similar behaviour as the three dimensional models introduced so far have also been investigated in higher dimensions. Some lead to Jordan cells at their critical points and seem to have a holographic connection to a higher dimensional logarithmic CFT. This is interesting, because, contrary to the two dimensional case, the number of elements of the conformal algebra in any finite dimension higher than two is always finite. Since the symmetry group is drastically different, studying higher dimensional holographic connections seems worthwhile.

One of these models with curvature squared modifications, which is power-counting renormalizable but has ghosts in its unitary sector, has an apparent resemblance to the action of NMG and was introduced in [114].

$$\mathcal{S} = \frac{1}{2\kappa^2} \int \mathrm{d}^4x \sqrt{-g} \left\{R - 2\Lambda + \alpha R^{\mu\nu} R_{\mu\nu} + \beta R^2\right\} \tag{3.18}$$

In addition to the usual spin 2 excitations the theory contains a massive spin 0 mode, which can be suppressed by a certain tuning of parameters $\alpha = -3\beta$.[3] Note that such a coupling recovers Einstein gravity if $\beta = 0$. For such a setting the action can as well be written as a cosmological Einstein-Hilbert term accompanied by a squared Weyl tensor $\frac{\alpha}{2}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}$ (minus an invariant Gauss-Bonnet term that does not affect the equations of motion and whose integral yields a term proportional to the Euler number) [113].
Perturbing around an AdS_4 vacuum and applying the transverse gauge as in the previous sections, the linearized equations of motion are given by

$$\left(\Box - \frac{2\Lambda}{3}\right)\left(\Box - \frac{4\Lambda}{3} - \frac{1}{3\beta}\right)\psi_{\mu\nu} = 0 \tag{3.19}$$

from which the existence of a massive mode $\psi^M_{\mu\nu}$ and a massless mode $\psi^L_{\mu\nu}$ can be inferred. A stability argument (see [114] and the reference therein) requires the squared mass to be positive and so the parameter β needs to be bounded.

$$0 < \beta \leqslant -\frac{1}{2\Lambda} \tag{3.20}$$

If β takes the value of the upper bound one sees a degeneration of both modes and logarithmic solutions are obtained [6, 9, 32, 54, 87]. Although the literature assigns them positive and finite energy, there is a potential caveat. Because the logarithmic mode can be changed arbitrarily by a linear contribution of the left mode (cf. (2.5)) its energy might be adjusted and could therefore be negative. Leaving that aside and proceeding with its energy being strictly positive there are no excitations with negative energy apart from the massive modes which vanishes only at the critical point. However, in addition to the energy of the massless mode vanishing, the mass of the Schwarzschild-AdS_4 black hole vanishes as well. This behaviour is in that respect similar to the one encountered in the case of the BTZ black hole within NMG.
In that respect when computing the Hamiltonian and the angular momentum, the appearance of a Jordan cell for a critical tuning of parameters [32] is not surprising.

$$H\begin{pmatrix} \psi^{\log}_{\mu\nu} \\ \psi^L_{\mu\nu} \end{pmatrix} = 2\begin{pmatrix} \frac{3}{2} & 1 \\ 0 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} \psi^{\log}_{\mu\nu} \\ \psi^L_{\mu\nu} \end{pmatrix} \tag{3.21}$$

What is interesting though is that the connection between the logarithmic and the massless mode is exactly the same as in three dimensions (2.19). The asymptotic behaviour of the logarithmic modes can be exploited to truncate them by choosing appropriate boundary conditions such as the four-dimensional analog of the ones by Brown and Henneaux. However, the resulting theory is essentially trivial, because as noted earlier the energy of the massless modes and the mass of the Schwarzschild-AdS_4 black hole as well as its entropy vanish at the critical point [113]. A qualitative explanation starts with the fact that for the critical tuning $\alpha = \frac{3}{2\Lambda}$ the cosmological Einstein-Hilbert action is equivalent to the one of conformal gravity, i.e. a squared Weyl tensor, in the infrared under appropriate boundary conditions [115]. Since the action for the right setting of parameters can be rewritten using both (c.f. below eq. (3.18)), all nontrivial excitations effectively cancel at least in the regime of long wavelengths.

[3] Another important tuning is $\alpha = 0$, which gets rid of the massive spin 2 excitation but is not important within the context considered here.

The two point correlators for this theory involving the stress-energy tensor T and its logarithmic partner $T^{\log}$ were computed in [102] and are in agreement with generic expectations.

$$\langle T_{ij}(x)T_{kl}(0)\rangle = 0 \tag{3.22a}$$
$$\langle T_{ij}(x)T^{\log}_{kl}(0)\rangle = \frac{b}{(2\pi)^3}\hat{\Delta}_{ij,kl}\frac{1}{|x|^2} \tag{3.22b}$$
$$\langle T^{\log}_{ij}(x)T^{\log}_{kl}(0)\rangle = \frac{b}{(2\pi)^3}\hat{\Delta}_{ij,kl}\frac{\ln(m^2|x|^2)}{|x|^2} \tag{3.22c}$$

where

$$\hat{\Delta}_{ij,kl} = \frac{1}{2}\left(\hat{\Theta}_{ik}\hat{\Theta}_{jl} + \hat{\Theta}_{il}\hat{\Theta}_{jk} - \hat{\Theta}_{ij}\hat{\Theta}_{kl}\right) \tag{3.23a}$$
$$\hat{\Theta}_{ij} = \partial_i\partial_j + \delta_{ij}\Box \tag{3.23b}$$

and the value of the new anomaly was found to be $b = \frac{3\ell^2}{4G}$. This is a generalization of the two-dimensional correlators in (2.4).

3.3 Reduction of Symmetry: The Galilean Conformal Algebra

The equations of motion of Einstein gravity imply that the intrinsic curvature on manifolds with three spacetime dimensions is proportional to the cosmological constant which is intimately connected with the AdS radius $\Lambda = -\ell^{-2}$. In the limit $\ell \to \infty$ the cosmological constant vanishes which results in spacetime being flat. For such spacetimes the asymptotic symmetry group at future null infinity $\mathcal{J}^+$ is the Bondi-Metzner-Sachs (BMS) group. It has an associated algebra ($\mathfrak{bms}_3$) that is infinite dimensional and isomorphic to the Galilean conformal algebra ($\mathfrak{gca}_2$) [11], consisting of the maximal set of generators of conformal isometry transormations of two-dimensional Galilean spacetime which can be obtained as a contraction (see app. B) of two copies of the Witt algebra. Analogously, the centrally extended $\mathfrak{gca}_2$ can be obtained by two copies of the Virasoro algebra. Under such a contraction, which we will consider explicitly below, the asymptotic symmetry algebra changes to a semi-direct sum of the Virasoro algebra with an abelian ideal.

$$[K_n, K_m] = (n-m)K_{n+m} + \frac{c_K}{12}n(n^2-1)\delta_{n+m,0} \tag{3.24a}$$
$$[K_n, M_m] = (n-m)M_{n+m} + \frac{c_M}{12}n(n^2-1)\delta_{n+m,0} \tag{3.24b}$$
$$[M_n, M_m] = 0 \tag{3.24c}$$

Such a procedure has been used to construct the non-relativistic limit for an AdS/CFT correspondence [10, 15, 16]. Furthermore, the close connection to $\mathfrak{bms}_3$ (the charge algebra of asymptotically flat spacetimes has been shown to allow for classical central extensions that are non-trivial as well [23]) marks $\mathfrak{gca}_2$ as an important ingredient when considering gauge/gravity dualities that include flat spacetimes. In particular, the $\mathfrak{gca}_2$ appears as the asymptotic symmetry algebra at null infinity in the ultra-relativistic limit [13, 14, 17, 24] (see also [11]). Next we concern ourselves with the non- and ultra-relativistic limit simply to check if both cases allow for the construction of a logGCA.

The non-relativistic limit

Starting from an algebra that is a direct sum, one can not simply redefine single generators and contract them as in the appendix but rather contract combinations of them, in order to eventually get an algebra that is a semi-direct sum.

$$K_n = L_n + \bar{L}_n \tag{3.25a}$$
$$M_n = \delta\left(L_n - \bar{L}_n\right) \tag{3.25b}$$

δ is the parameter that will eventually allow for a contraction in the limit $\delta \to 0$. In order to end up with (3.24) the central charge c_M must be of order $\mathcal{O}(\delta)$. The new central charges depend on the ones of the two Virasoro algebras as follows

$$c_K = \lim_{\delta\to 0}\left[c_L + c_R\right] \quad , \quad c_M = \lim_{\delta\to 0}\left[\delta\left(c_L - c_R\right)\right] \quad . \tag{3.26}$$

The vacuum state in the full relativistic theory fullfils the primary condition and additionally remains invariant under the group action of SL(2), meaning that $L_n|0\rangle = \bar{L}_n|0\rangle = 0$ for $n \geq -1$. With the definition for the new generators those conditions carry over to the new vaccuum state, i.e. $K_n|0\rangle = M_n|0\rangle = 0$ for $n \geq -1$. A redefinition of coordinates

$$z = t + \delta x \quad , \quad \bar{z} = t - \delta x \tag{3.27}$$

and defining new operators involving the components of the stress-energy tensor, thereby mixing the holomorphic and antiholomorphic parts

$$\psi^K = \lim_{\delta\to 0}\left[\psi^L(z) + \psi^R(\bar{z})\right] \quad , \quad \psi^M = \lim_{\delta\to 0}\left[\delta\left(\psi^L(z) - \psi^R(\bar{z})\right)\right] \quad , \tag{3.28}$$

lets us compute their two point functions [97, 98].

$$\langle\psi^M(t,x)\psi^M(0,0)\rangle = 0 \tag{3.29a}$$
$$\langle\psi^M(t,x)\psi^K(0,0)\rangle = \frac{c_M}{2t^4} \tag{3.29b}$$
$$\langle\psi^K(t,x)\psi^K(0,0)\rangle = \frac{c_K}{2t^4} - \frac{2c_M x}{t^5} \tag{3.29c}$$

These do not resemble the correlators of a logCFT, but still there appears a Jordan cell.

$$M_0\begin{pmatrix}\psi^K\\ \psi^M\end{pmatrix} = \begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}\begin{pmatrix}\psi^K\\ \psi^M\end{pmatrix} \quad , \quad K_0\begin{pmatrix}\psi^K\\ \psi^M\end{pmatrix} = \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}\psi^K\\ \psi^M\end{pmatrix} \tag{3.30}$$

The ultra-relativistic limit

Having seen the reduction to the non-relativistic limit lets us compute its counterpart without trouble as it works analogously. The generators are

$$K_n = L_n - \bar{L}_{-n} \tag{3.31a}$$
$$M_n = \delta\left(L_n + \bar{L}_{-n}\right) \tag{3.31b}$$

with the central charges

$$c_K = \lim_{\delta\to 0}\left[c_L - c_R\right] \quad , \quad c_M = \lim_{\delta\to 0}\left[\delta\left(c_L + c_R\right)\right] \quad . \tag{3.32}$$

This results again in $\mathfrak{gca}_2$ as before, but there is a difference when it comes to its vacuum state. It is also defined as $K_n|0\rangle = M_n|0\rangle = 0$ with $n \geq -1$ which does not imply the symmetries of the vacuum state of an ordinary CFT as discussed in the previous limit. There is no continuous connection between the two as before. As a result the sign of the central charge in the antiholomorphic sector effectively flips along with the sign in front of coordinates in some definitions below. The coordinates are redefined to

$$z = \delta t + x \quad , \quad \bar{z} = \delta t - x \tag{3.33}$$

and the new operators read

$$\psi^K = \lim_{\delta\to 0}\left[\psi^L(z) - \psi^{R\dagger}(-\bar{z})\right] \quad , \quad \psi^M = \lim_{\delta\to 0}\left[\delta\left(\psi^L(z) + \psi^{R\dagger}(-\bar{z})\right)\right] \quad . \tag{3.34}$$

This leads to similar two-point correlators that have been encountered before.

$$\langle\psi^M(t,x)\psi^M(0,0)\rangle = 0 \tag{3.35a}$$

$$\langle\psi^M(t,x)\psi^K(0,0)\rangle = \frac{c_M}{2x^4} \tag{3.35b}$$

$$\langle\psi^K(t,x)\psi^K(0,0)\rangle = \frac{c_K}{2x^4} - \frac{2c_M t}{x^5} \tag{3.35c}$$

The only difference to the correlation functions of the non-relativistic limit is that the spatial and temporal coordinates are exchanged. For consistency we state that the operator M_0 can be represented by the same Jordan form as in (3.38) and K_0 is again diagonal.

Let us now try to construct a logGCA along the lines of section 2.1 and start by first considering the ultra-relativistic limit. The construction again involves an operator ψ_ε whose weights degenerate for $\varepsilon \to 0$ with those of the holomorphic part of the stress-energy tensor. We have seen that this limit requires c_L to vanish for the new anomaly to stay finite. Since all central charges should remain finite this means that c_M vanishes as well. Thus the two point correlation functions for the ultra-relativistic limit are given by

$$\langle\psi^K(t,x)\psi^K(0,0)\rangle = \frac{c_K}{2x^4} \tag{3.36a}$$

$$\langle\psi^{\log}(t,x)\psi^K(0,0)\rangle = \frac{b_L}{2x^4} \tag{3.36b}$$

$$\langle\psi^{\log}(t,x)\psi^{\log}(0,0)\rangle = -\frac{b_L\ln(m_L^2x^2)}{2x^4} \tag{3.36c}$$

This structure resembles that of the logarithmic CFT previously encountered in (2.4) but is not equal because the central charge of the Virasoro algebra is given by $c_K = -c_R$, which is not zero at the critical point. Using the expression $2\varepsilon = \mu\ell - 1$ and the definition of the new anomaly as it was stated in section 2.1 (see below (2.4)), the values of the coefficients in the correlators regarding critical TMG are

$$c_K = b_L = -\frac{3\ell}{G_N} \quad . \tag{3.37}$$

The correlators including ψ^M are all zero including the one with itself. The addition of the operator $\psi^{\log}$ augments the representation of K_0 to twice the unit matrix of rank three and M_0 to

$$M_0 \begin{pmatrix} \psi^{\log} \\ \psi^K \\ \psi^M \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi^{\log} \\ \psi^K \\ \psi^M \end{pmatrix} \tag{3.38}$$

Trying to apply the same procedure to arrive at the correlators for the non-relativistic limit proves to be futile. Although they have been computed in the literature [97, 98], the same approach as in the former case yields singular results. We recall that the weights of ψ_ε are $(2+\varepsilon, \varepsilon)$ and so the action of K_0 as defined in (3.25) on the logarithmic operator leads to

$$K_0\, \psi^{\log} = \lim_{\varepsilon \to 0} \left(\frac{(2+\varepsilon)\psi_\varepsilon - 2\psi^L}{\varepsilon} + \frac{\varepsilon \psi_\varepsilon}{\varepsilon} \right) = 2\psi^{\log} + \psi^K + \frac{1}{\delta}\psi^M \quad . \tag{3.39}$$

Another definition of the weights, namely $(2+\varepsilon, -\varepsilon)$, leads to ψ_ε being an eigenstate of K_0 as in the ultra-relativistic limit. This works when considering merely the CFT side and neglecting a holographic connection with a gravitational theory since the spin of such excitations, given by the difference of the conformal weights $s = h - \bar{h} = 2 + 2\varepsilon$, can not be interpreted as a massive graviton. There might be different methods of taking limits that one can adopt to obtain the correlators from the literature. For example taking $\delta \to 0$ before doing the same with ε or leting them both vanish simultaneously.

In the beginning of this section it was stated that the $\mathfrak{gca}_2$ plays an important role in flat space holography and that such spaces can be gained by a deformation of AdS via $l \to \infty$. However, looking at (3.37) such a limit is not sensible here if this theory is supposed to have any physical meaning unless the gravitational constant G is rescaled as well such that their ratio stays finite. Additionally, the product of μl should also remain finite, implying that μG stays constant as well, so the coupling constant μ associated to the Chern-Simons term needs to be sent to zero. Putting all of this together results in the disappearance of the Einstein-Hilbert term in the CTMG action (2.9). One is left with the non-covariant Chern-Simons gravity action whose equations of motion are invariant under conformal transformations. Given that logGCAs are not unitary, this would be in conflict with [14] where it was stated that the dual gauge theory to this action should be unitary. That unitarity is indeed maintained under such conditions will be discussed below.

The linearized equations of motion for CTMG are given in (2.11). We remind ourselves that the transeverse traceless gauge condition was implemented. After applying the limits that lead to Chern-Simons gravity they change to

$$\left((\mathcal{D})^3\, h \right)_{\mu\nu} = 0 \quad \text{with} \quad \mathcal{D}_\alpha{}^\sigma = \epsilon_\alpha{}^{\sigma\beta} \bar{\nabla}_\sigma \tag{3.40}$$

where we choose the background metric to be flat. Such a degeneration has already been discussed for GMG and leads to two logarithmic partners and a Jordan cell of rank three [14].

$$M_0 \begin{pmatrix} \psi^{\log^2} \\ \psi^{\log} \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi^{\log^2} \\ \psi^{\log} \\ \psi \end{pmatrix} \tag{3.41}$$

All three excitations are either incompatible with the flat boundary conditions or not regular at the origin $r = 0$, however, there are solutions to the equations of motion ψ^{reg} that fulfill these requirements (see [14]). The problem is located in the choice of the transverse traceless gauge as it is incompatible in the limit of flat spacetime since there the condition of tracelessness does not need to hold anymore upon imposing transversality. We can try to map the solutions under this gauge condition onto regular ones by means of a singular transformation. Given that all modes are pure gauge in the bulk, let us denote them by $\psi^{\text{reg}}_{\mu\nu} = \nabla_{(\mu}\xi^{\text{reg}}_{\nu)}$ and $\psi_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)}$, the vector field that connects the solutions via a Lie-derivative is $\xi^{\text{reg}} - \xi$. Applying the same transformation to linear combinations of the two logarithmic modes always results in modes that are again singular or not compatible with the boundary conditions. Such modes are not physical and can be discarded from the spectrum, meaning that in the flat space limit no Jordan cell and no logarithmic GCA exists. This resolves the issue concerning unitarity.

Chapter 4

Summary & Conclusion

We reviewed some connections between CTMG and a logCFT which seem to indicate that they provide a further example of a realization of the holographic principle. The checks that were presented included the 2- and 3-point correlation functions, quasi-normal modes and partition functions. Since the full logCFT partition function has not been calculated due to technical issues it is not clear if the correspondence of both theories holds to all orders or only in a semi-classical approximation and therefore CTMG represents just an effective theory.
We presented higher derivative models which experience similar features as CTMG, but have a richer structure. They were shown to obtain an additional logarithmic pair in the anti-holomorphic sector and Jordan cells of higher rank. The boundary conditions seem to follow a pattern (2.39) which depends only on the rank of the obtained Jordan cell. It would be desirable to check if this relation holds for critical gravity theories that involve Jordan cells of higher rank than three. This could also be done in higher dimensions.
For a supersymmetric $\mathcal{N} = 1$ extension of CTMG the connection to a logCFT was reviewed at the linearized level but also holds non-perturbatively [8, 31] which is also the case for NMG and GMG. However, if this remains true for $\mathcal{N} > 1$ is not known since supersymmetric extensions of CTMG, NMG and GMG have only been constructed at the linearized level so far [32].
Eventually, $\mathfrak{vir} \oplus \mathfrak{vir}$ was contracted in two different ways in order to obtain $\mathfrak{gca}_2$. The same limiting procedure that was used in the case of a CFT was applied to the GCA in order to obtain a logGCA. Consequently, we found that the non-relativistic limit, in contrast to the ultra-relativistic case, exhibits a singular behaviour. This can be avoided by a redefinition of the massive weights with the drawback of losing the physical interpretation of a mode of spin 2. It may be the case that another change in the limiting procedure might also avoid this singular behaviour without meddling with the spin.

Further open issues remain. Given that holographic connections exist for other classes of backgrounds than locally AdS, one may check if a logCFT structure emerges from backgrounds such as asymptotically warped, Lifshitz, Schrödinger or Lobachevsky, all of which can be considered in the framework of TMG [55]. Moreover, the Cardy formula connects the central charges with the entropy in a semi-classical limit. It would be interesting to know if a similar connection can be drawn for a logCFT and how the new anomlay would fit in this picture. Lastly, another issue is the realization of logarithmic modes with spins different than 2 or 0 on which the literature is rather scarce.

Appendix A

Hamiltonian Formulation of General Relativity

In this section we review the approach towards a Hamiltonian formulation of general relativity as it was pioneered by Arnowitt, Deser and Misner. To highlight its generality we will not restrict ourselves to the three dimensional case that is used almost exclusively throughout this thesis and rather choose to work in $d+1$ dimensions. We end with a discussion on surface terms which are eventually necessary to obtain Einstein's equations in a canonical form. A nice introduction to this topic can be found in [127]. For a thorough treatment of a canonical approach towards quantization I highly recommend [95].

A Hamiltonian formulation on a smooth manifold $(\mathcal{M}, g_{\mu\nu})$ demands a concrete but arbitrary splitting of spacetime into 'space and time' and so we begin by singling out a scalar field $t(x^\mu)$ along which spacetime is foliated into non-intersecting spacelike hypersurfaces Σ_t that are defined by $t = \text{const.}$ such that spacetime globally has a product topology $\mathcal{M} \cong \mathbb{R} \times \Sigma_t$. It is required that the scalar field fullfils the condition $t^\mu \nabla_\mu t = 1$ in order to be nowhere tangential to Σ_t where the vector field t^μ represents the 'flow of time'.[1] The hypersurfaces are parameterized by y^i which sets a new coordinate system (t, y^i). Such a foliation naturally induces a metric $h_{\mu\nu}$ on each hypersurface with unit normal vector n^μ according to

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \tag{A.1}$$

with the crucial property that the induced metric acts as a projector onto a hypersurface $h_{\mu\nu} n^\nu = 0$.

Dynamics is now described by the evolution of quantities on a d dimensional manifold Σ in dependence of the parameter t. The time vector field may be split up into components normal and tangential to the hypersurfaces

$$t^\mu = N n^\mu + N^\mu \tag{A.2}$$

where the two introduced entities are called the lapse-function N and the shift vector N^μ. Physically, the lapse function measures the change of proper time with respect to coordinate

[1] In this section, unless stated otherwise, we will denote tensors explicitly by their components so a confusion of the time-flow vector $t = t^\mu \partial_\mu$ with the parameter time will not occur.

time of a moving frame with velocity n^μ. The shift vector encodes the change of the coordinate system when comparing Σ_t with $\Sigma_{t+\delta t}$. The following

$$e_a^\mu = \frac{\partial x^\mu}{\partial y^a} \tag{A.3}$$

can be used as a pullback onto a hypersurface and can be thought of as a collection of linearly independent tangent vectors on Σ. The basis of the cotangent space may be rewritten to $\mathrm{d}x^\mu = t^\mu \mathrm{d}t + e_a^\mu \mathrm{d}y^a$ and yields the line element in the new coordinates

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + h_{ab}\big(\mathrm{d}y^a + N^a \mathrm{d}t\big)\big(\mathrm{d}y^b + N^b \mathrm{d}t\big) \tag{A.4}$$

where we used the fact that the projection of the time-flow vector field onto a hypersurface equals the shift vector $t_\mu e_a^\mu = N_a$ and $h_{\mu\nu} e_a^\mu e_b^\nu = h_{ab}$. Such a setting requires a definition of a covariant derivative on Σ and a quantity that tells how a surface is embedded in an ambient space: the extrinsic curvature. Regarding the former, it can be obtained by a projection of the covariant derivative in full spacetime onto Σ.

$$D_\mu u_\nu \equiv h_\mu{}^\sigma h_\nu{}^\tau \nabla_\sigma u_\tau \quad \text{or} \quad D_a u_b \equiv e_a^\mu e_b^\nu D_\mu u_\nu \tag{A.5}$$

We stress that the vector on the left side is an element of the tangent vector space of Σ. It is straightforward to check that the new covariant derivative is compatible with the induced metric if this is also attributed to the covariant derivative and the full spacetime metric itself. The extrinsic curvature can be inferred from the rate of change of the normal vector field and a further projection.

$$K_{\mu\nu} \equiv h_\mu{}^\sigma h_\nu{}^\tau \nabla_\sigma n_\tau = h_\mu{}^\sigma \nabla_\sigma n_\nu = \frac{1}{2}\mathcal{L}_n h_{\mu\nu} \quad \text{or} \quad K_{ab} \equiv e_a^\mu e_b^\nu \nabla_\mu n_\nu = \frac{1}{2} e_a^\mu e_b^\nu \mathcal{L}_n g_{\mu\nu} \tag{A.6}$$

The first equality here implies compatibility of the covariant derivative with the metric. The induced metric and the extrinsic curvature together contain all the information that is necessary to construct the Riemann curvature tensor of Σ.

$${}^{(d)}R_{\mu\nu\sigma}{}^\tau w_\tau = \left[D_\mu, D_\nu\right] w_\sigma \tag{A.7}$$

Taking the definition of the extrinsic curvature and the covariant derivative and acknowledging that $K_{\mu\nu}$ is symmetric yields the Gauss-Codazzi equation[2]

$${}^{(d)}R_{\mu\nu\sigma}{}^\tau = h_\mu{}^\alpha h_\nu{}^\beta h_\sigma{}^\gamma h^\tau{}_\delta R_{\alpha\beta\gamma}{}^\delta - K_{\mu\sigma} K_\nu{}^\tau + K_{\nu\sigma} K_\mu{}^\tau \ . \tag{A.8}$$

Before delving into Hamiltonian mechanics we start from a covariant viewpoint. In an action principle that involves a Lagrangian the natural quantities under consideration are the generalized coordinates and their time derivatives. To describe the dynamics on a hypersurface after having introduced a foliation of spacetime we change from the full spacetime metric to the induced metric on Σ, the lapse function and the shift vector. In full generality the velocities are taken to be the Lie derivatives along the time flow vector field

$$\dot{h}_{\mu\nu} \equiv \mathcal{L}_t h_{\mu\nu} \ , \quad \dot{N} \equiv \mathcal{L}_t N \ , \quad \dot{N}^\mu \equiv \mathcal{L}_t N^\mu \ . \tag{A.9}$$

[2]For an explicit derivation see [140]

Note that the Lie derivative commutes with the pullback and we can substitute $\dot{h}_{ab} = {e_a}^{\mu}{e_b}^{\nu}\dot{h}_{\mu\nu}$. The next step is to rewrite the Einstein-Hilbert action to only include said quantities. From the equations

$$R_{\mu\nu\sigma\tau}h^{\mu\sigma}h^{\nu\tau} = R + 2R_{\mu\nu}n^{\mu}n^{\nu} = 2G_{\mu\nu}n^{\mu}n^{\nu} \tag{A.10}$$

and

$$\begin{aligned} R_{\mu\nu}n^{\mu}n^{\nu} &= -n^{\mu}\left(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu}\right)n^{\nu} && \text{(A.11a)} \\ &= K^2 - K_{\mu\nu}K^{\mu\nu} - \nabla_{\mu}\left(n^{\mu}\nabla_{\nu}n^{\nu}\right) + \nabla_{\nu}\left(n^{\mu}\nabla_{\mu}n^{\nu}\right) && \text{(A.11b)} \end{aligned}$$

with $G_{\mu\nu}$ being the components of the Einstein tensor and K the trace of the extrinsic curvature, one can rewrite the Ricci scalar in terms of quantities related to a hypersurface of foliated spacetime. Using the Gauss-Codazzi equation (A.8) on the Einstein tensor

$$\begin{aligned} G_{\mu\nu}n^{\mu}n^{\nu} &= \frac{1}{2}R_{\mu\nu}h^{\mu}h^{\nu} && \text{(A.12a)} \\ &= \frac{1}{2}\left({}^{(d)}R + K^2 - K_{\mu\nu}K^{\mu\nu}\right) && \text{(A.12b)} \end{aligned}$$

and the fact that $\sqrt{-g} = N\sqrt{h}$, which can be infered from the metric in (A.4), finally yields an expression for the cosmological Einstein-Hilbert action

$$\begin{aligned} S_{\text{EH}} =& \frac{1}{\kappa^2}\int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{-g}\left(R - 2\Lambda\right) && \text{(A.13a)} \\ =& \frac{1}{\kappa^2}\int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{h}N\left({}^{(d)}R - K^2 + K_{\mu\nu}K^{\mu\nu} - 2\Lambda\right) + 2\oint_{\partial\mathcal{M}} \mathrm{d}^{d}z^{\mu}\left(n_{\mu}\nabla_{\nu}n^{\nu} - n^{\nu}\nabla_{\nu}n_{\mu}\right) \ . && \text{(A.13b)} \end{aligned}$$

The bulk term represents the celebrated Arnowitt-Deser-Misner action. For the time being we are going to neglect the surface term as it does not change the equations of motion in any way. All boundary contributions will be discussed within the Hamiltonian framework. Exploiting linearity of the Lie derivative the extrinsic curvature can be rewritten in dependence of the 'generalized velocity' of the induced metric $K_{\mu\nu} = \frac{1}{2N}\left(\mathcal{L}_t - \mathcal{L}_N\right)h_{\mu\nu}$. A look at the action now reveals no dependence on either $\dot{N}$ or $\dot{N}^{\mu}$. Since these variables fail to be dynamical, their canonical momenta vanish $\pi^{\mu} = \pi = 0$ and they can be seen to simply act as Lagrange multipliers through the equations of motion, i.e.

$$\frac{\delta\mathcal{L}_{\text{ADM}}}{\delta N} = \frac{\delta\mathcal{L}_{\text{ADM}}}{\delta N^{\mu}} = 0 \ , \tag{A.14}$$

and are effectively constraints imposed on the system.[3] The only non-vanishing canonical momentum is the one associated with the induced metric

[3] In Dirac's lingo the vanishing of the canonical momenta is an example of primary constaints whereas the constraints in (A.14) are secondary since it is necessary that the equations of motion hold for them to be valid. His approach towards canonical quantization [65] requests the introduction of first and second class constraints. We will not make use of this formalism here though.

$$\pi_{\mu\nu} \equiv \frac{\delta \mathcal{L}}{\delta \dot{h}^{\mu\nu}} = \frac{\sqrt{h}}{\kappa^2} \left(K_{\mu\nu} - h_{\mu\nu} K\right) \quad . \tag{A.15}$$

A Legendre transformation leads to the Hamiltonian. For clarity we pull all quantities back onto the hypersurface, express the extrinsic curvature through the canonical momentum and get

$$\begin{aligned} H &= \int_\Sigma \mathrm{d}^d y \sqrt{h} \left(\pi^{ab} \dot{h}_{ab} - \mathcal{L}_{\mathrm{ADM}}\right) & \text{(A.16a)} \\ &= \int_\Sigma \mathrm{d}^d y \left(N\mathcal{H} + N^a \mathcal{H}_a\right) + \oint_{\partial\Sigma} \mathrm{d}^{d-1} z \left(N\mathcal{H}^{(\partial\Sigma)} + N^a \mathcal{H}_a^{(\partial\Sigma)}\right) \quad . & \text{(A.16b)} \end{aligned}$$

The explicit expressions of the Hamiltonian densities in the bulk are

$$\begin{aligned} \mathcal{H} &= \frac{\kappa^2}{\sqrt{h}} \left(\pi^{ab}\pi_{ab} - \frac{1}{d-1}\pi^2\right) - \frac{\sqrt{h}}{\kappa^2} \left({}^{(d)}R - 2\Lambda\right) & \text{(A.17a)} \\ \mathcal{H}_a &= -2D^b \pi_{ab} \quad . & \text{(A.17b)} \end{aligned}$$

The constraints from (A.14) translate straightforwardly to the Hamiltonian and yield the so called Hamilton constraint $\mathcal{H} = 0$ and the momentum or diffeomorphism constraint $\mathcal{H}^a = 0$. A variation of the action that involves this Hamiltonian is free of surface terms, provided one imposes Dirichlet conditions on the boundary for all generalized coordinates, i.e. $\delta h_{ab} = \delta N_a = \delta N = 0$. Under this light the equations of motion coming from Hamilton's principle $\delta \left(\int d^d x \; \dot{h}_{ij} \pi^{ij} - H\right) = 0$ are

$$\dot{h}_{ij} = \frac{\delta H}{\delta \pi^{ij}} \quad \text{and} \quad \dot{\pi}_{ij} = -\frac{\delta H}{\delta h^{ij}} \quad . \tag{A.18}$$

The evolution of the system is governed by these equations, in addition with the formerly mentioned constraints. If the spacetime allows a foliation such that Σ is a Cauchy surface, the whole system can be described completely as an initial value problem.
We now turn our attention to the thus far neglected boundary terms. If one were to work exclusively on closed manifolds such a discussion would not be necessary, however, this would be much too restrictive. A functional differential, expressed conveniently, is defined as

$$\delta H \left[h(y), \pi(y)\right] = \int_\Sigma \mathrm{d}^d y \left(A^{ij} \delta h_{ij} + B^{ij} \delta \pi_{ij}\right) \tag{A.19}$$

where we used an abbreviation for the functional derivatives as follows

$$A^{ij} = \frac{\delta H}{\delta h_{ij}} \quad \text{and} \quad B^{ij} = \frac{\delta H}{\delta \pi_{ij}} \quad . \tag{A.20}$$

We may try to bring the Hamiltonian from (A.16) in such a form and get

$$\delta H = \int_\Sigma \mathrm{d}^d y \left(A^{ij}\delta h_{ij} + B^{ij}\delta\pi_{ij}\right) \tag{A.21a}$$

$$+ \oint_{\partial\Sigma} \mathrm{d}^{d-1} z_l \left\{ G^{ijkl}\Big(N_{;k}\delta h_{ij} - N\delta h_{ij;k}\Big) - 2N_i\delta\pi^{il} + \Big(N^l\pi^{ij} - 2N^i\pi^{jl}\Big)\delta h_{ij}\right\} \tag{A.21b}$$

where we used $2\kappa^2 G^{ijkl} = \sqrt{\gamma}\left(h^{ik}h^{jl} + h^{ij}h^{kl} - 2h^{ij}h^{kl}\right)$ and a semicolon to abbreviate covariant differentiation with respect to the induced metric on Σ. This is clearly no functional differential! In order to recover Einstein's equations in their canonical form as in (A.18) one would like to identify $A^{ij} = -\dot{\pi}^{ij}$ and $B^{ij} = \dot{h}^{ij}$, which would be possible if only the surface term M, i.e. (A.21b), were to vanish somehow. This could indeed be achieved by adding a boundary term Q to the Hamiltonian, whose variation precisely cancels this surface term.

$$M + \delta Q = 0 \tag{A.22}$$

Generically M need not be the variation of a local surface term meaning that it need not be integrable. Be reminded that we are dealing with open spaces! This implies that even though there might not be an exact solution to eq. (A.22), it still may be possible to obtain one asymptotically after restricting the allowed class of fields by imposing suitable boundary conditions.[4] This has been done in [94] in the case of an AdS background metric $g = \bar{g} + w$ with the result

$$Q = \oint_{\partial\Sigma} \mathrm{d}^{d-1} z_i \left\{ \bar{G}^{ijkl}\left(N\bar{\nabla}_j w_{kl} - w_{kl}\bar{\nabla}_j N\right) + 2\pi^{ij}N_j \right\} + \mathcal{O}(w^2) \ . \tag{A.23}$$

So in order to yield well defined functional derivatives for Einstein's equations, the Hamiltonian (A.16), which we now indicate as H_0, has to be supplemented by this surface integral.

$$H = H_0 + Q \tag{A.24}$$

Notice that Q from the condition in (A.22) is actually only defined up to a constant. We refer to that in chapter 1. In summary, even though M can always be worked out explicitly, the evaluation of Q that needs to be added to cancel the contributing surface integral to leave a well defined functional differential demands the imposition of fall-off conditions on the fields. For later convenience let us rewrite the full Hamiltonian by changing the coordinate basis back to the generical ones we started with before the temporal slicing. We use

$$\mathcal{H}_\mu n^\mu = \mathcal{H} \ , \quad \mathcal{H}_\mu e^\mu_a = \mathcal{H}_a \ , \quad \mathcal{Q}_\mu n^\mu = \mathcal{H}^{(\partial\Sigma)} \ , \quad \mathcal{Q}_\mu e^\mu_a = \mathcal{H}^{(\partial\Sigma)}_a \tag{A.25}$$

to get the following expression for (A.16)

$$H = \int_\Sigma \mathrm{d}^d y \left(t^\mu \mathcal{H}_\mu\right) + \oint_{\partial\Sigma} \mathrm{d}^{d-1} z \left(t^\mu \mathcal{Q}_\mu\right) \tag{A.26}$$

which represents a special case of its most general form that involves an arbitrary vector field ζ.

[4] In the case of AdS_3 asymptotically means nothing else than taking the spatial limit to infinity since we are using a temporal slicing.

$$H[\zeta] = \int_\Sigma \mathrm{d}^d y \, (\zeta^\mu \mathcal{H}_\mu) + \oint_{\partial\Sigma} \mathrm{d}^{d-1} z \, (\zeta^\mu \mathcal{Q}_\mu) \tag{A.27}$$

With well defined generators one can now go on and introduce the variation of a functional F on the phase-space variables along a vector field ζ

$$\delta_\zeta F = \{F, H\,[\zeta]\} \tag{A.28}$$

as one is used to from Hamiltonian dynamics. Brown and Henneaux showed that after having supplemented the Hamiltonian by the necessary surface terms the Poisson brackets are always well defined as long as these generators are given by $\mathcal{C}^\infty$ local densities. They explicitly showed in [41] that the Poisson bracket of two differentiable generators yields again a differentiable generator which should be the case in order to inherit a group structure. It means that the variation of a functional can be realized in more than one way, viz.[5]

$$\{F, \{H\,[\zeta]\,, H\,[\eta]\}\} = \{F, H\,[\theta]\} \ . \tag{A.29}$$

The vector $\theta \equiv \theta(\zeta, \eta)$ is determined by the group composition law. In the case of the asymptotic symmetry group as it is discussed in chapter 1 it would amount to a Lie-bracket. It can be taken for granted that a variation of a functional depends on the Poisson bracket with a generator $H\,[\zeta]$ to which a constant functional $c\,[\zeta]$ can be added without changing the variation. Therefore the symmetry transformations are not in a one-to-one correspondence with the charges, but rather with equivalence classes

$$H\,[\zeta] \sim H\,[\zeta] + c\,[\zeta] \ . \tag{A.30}$$

This statement can be loosened even further when restricting the class of deformation vectors to 'asymptotic' Killing vectors since an addition of trivial surface charges, which imply pure gauge transformations, does not change the variation either. Together with eq. (A.29) the algebra of charges is consequently only a 'projective' representation [40] of the asymptotic symmetry group.

$$\{H\,[\zeta]\,, H\,[\eta]\} = H\,[\theta] + K\,[\zeta, \eta] \tag{A.31}$$

The necessary conditions which must be fulfilled by the charges in order for this algebra to hold are finiteness and differentiability.[6] When considering explicit representations the central term might be trivial or can be absorbed in a redefinition of the generators which is not the case in AdS_3. Furthermore, no central charges occur if the asymptotic symmetries fall in line with the exact symmetries of a background configuration. By changing from Poisson to Dirac brackets, where the Hamiltonian and momentum constraints are satisfied, the expression in (A.31) reduces to a statement that includes only the surface charges

$$\{Q\,[\zeta]\,, Q\,[\eta]\}_{\mathrm{D.B.}} = Q\,[\theta] + K\,[\zeta, \eta] \tag{A.32}$$

[5]When the deformation vectors ζ and η are restricted to the asymptotic symmetries, the generator on the right side depends on the their Lie-bracket.

[6]This is not sufficient though, because it does not imply the conservation of the charges. We will not go into detail here and refer the interested reader to [39].

Appendix B

Inönü-Wigner Contraction

The discovery of the idea of contractions of groups [101] was initially spawned by a physically motivated question: Since classical mechanics can be obtained from relativistic mechanics by letting the speed of light grow infinitely large, how can the symmetry group of the former, the Galilean group, be retrieved as a limiting case of the symmetry group of the latter, the Lorentz group? Before treating this as an example we first concern ourselves with a generic Lie group. Let the elements of its corresponding N-dimensional Lie algebra be denoted as Y. For clarity we label them as Y_a^1, Y_i^2 with the indices defined below. The introduction of a parameter will later allow for a contraction.

$$\begin{aligned} X_a^1 &= Y_a^1 \quad \text{with} \quad a, b &= 1, \ldots, r \\ X_i^2 &= \varepsilon Y_i^2 \quad \text{with} \quad i, j &= r+1, \ldots, N \end{aligned} \tag{B.1}$$

Following this definition, the internal associative operation of the algebra associated with the Jacobi identity can be rewritten in terms of the new elements.

$$\begin{aligned} \left[X_a^1, X_b^1\right] &= \left[Y_a^1, Y_b^1\right] = \sum_{n=1}^{r} c_{ab}^n X_n^1 + \frac{1}{\varepsilon} \sum_{n=r+1}^{N} c_{ab}^n X_n^2 \\ \left[X_a^1, X_i^2\right] &= \varepsilon \left[Y_a^1, Y_i^2\right] = \varepsilon \sum_{n=1}^{r} c_{ai}^n X_n^1 + \sum_{n=r+1}^{N} c_{ai}^n X_n^2 \\ \left[X_i^2, X_j^2\right] &= \varepsilon^2 \left[Y_i^2, Y_j^2\right] = \varepsilon^2 \sum_{n=1}^{r} c_{ij}^n X_n^1 + \varepsilon \sum_{n=r+1}^{N} c_{ij}^n X_n^2 \end{aligned} \tag{B.2}$$

Demanding a finite result in the limit $\varepsilon \to 0$ sets the structure constants c_{ab}^n for $n = r+1, \ldots, N$ to zero. The commutators of the new Lie algebra can now be given by

$$
\begin{aligned}
\left[X_a^1, X_b^1\right] &= \sum_{n=1}^{r} c_{ab}^n X_n^1 \\
\left[X_a^1, X_i^2\right] &= \sum_{n=r+1}^{N} c_{ai}^n X_n^2 \\
\left[X_i^2, X_j^2\right] &= 0 \ .
\end{aligned}
\tag{B.3}
$$

This result leads to the conclusion that a contraction of a given Lie algebra can only take place if and only if it has a non trivial subalgebra generating a subgroup H. The remaining elements form an abelian subalgebra of the contracted algebra and therefore generate an abelian invariant subgroup A. Denoting the Lie group that corresponds to the contracted algebra as G_c the following relation holds.

$$
H \cong \frac{G_c}{A} \tag{B.4}
$$

In order to obtain a group by contraction it is a necessary condition that such an isomorphism exists.
We end this short segment with two prominent examples. The first is the aforementioned contraction of the Lorentz algebra where J_i and K_i are the generators of boosts and rotations respectively. For $K_i = \frac{1}{c} L_i$ the contraction yields the Galilean algebra.

$$
\begin{array}{lcrl}
[J_i, J_j] & = & & i\epsilon_{ijk} J_k \\
[J_i, K_j] & = & & i\epsilon_{ijk} K_k \\
[K_i, K_j] & = & - & i\epsilon_{ijk} J_k
\end{array}
\quad \xrightarrow{c \to \infty} \quad
\begin{array}{lcl}
[J_i, J_j] & = & i\epsilon_{ijk} J_k \\
[J_i, K_j] & = & i\epsilon_{ijk} K_k \\
[K_i, K_j] & = & 0 \ .
\end{array}
\tag{B.5}
$$

Lets consider the algebra $\mathfrak{so}(3)$ for the second example. By choosing the one dimensional $\mathfrak{so}(2)$ as the invariant subalgebra, its contraction leads to $\mathfrak{iso}(2)$ with the two dimensional algebra for translations as its abelian invariant subalgebra.
For more examples within the context of higher spin gravity we refer to [1, 77].

Appendix C

Ward Identities and the Operator Product Expansion

Within time-ordered correlation functions two local operators inserted at nearby points can be approximated by a sum of other local operators

$$\langle \mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w})\cdots\rangle = \sum_k C_{ij}^k(z-w,\bar{z}-\bar{w})\langle \mathcal{O}_k(w,\bar{w})\cdots\rangle \tag{C.1}$$

with a singular behaviour in the limit $z \to w$. One often refers to this as the operator product expansion (OPE). This statement holds for any quantum field theory as long as other operator insertions are far away compared to $|z-w|$. In (C.1) we explicitly used translational invariance in writing the coefficients. In the special case of a CFT this is not an approximation but an exact statement where the distance to the closest operator insertion that is not involved in the OPE equals the radius of convergence.

We now review the conformal Ward identities for a field theory in 1+1 dimensions. Using a path integral the partition function can schematically be written as

$$Z[\phi] = \int \mathcal{D}\phi \; e^{-S[\phi]} \quad . \tag{C.2}$$

A transformation of the fields $\phi \to \phi + \varepsilon\delta\phi$ with $\varepsilon = \text{const.}$ that represents a symmetry leaves the measure and the action invariant. Promoting the parameter ε to a local field leads to a change that looks like

$$Z[\phi'] = \int \mathcal{D}\phi \; e^{-S[\phi] - \frac{1}{2\pi}\int \mathrm{d}^2\sigma\sqrt{g}\; J^\mu \partial_\mu \varepsilon} \tag{C.3}$$

where the factor 2π is conventional. This implies

$$\langle \partial_\mu J^\mu \rangle = 0 \tag{C.4}$$

to leading order since the value of the partition function can not have changed. These manipulations can be done within general correlation functions that involve a collection of arbitrary fields at different arbitrary insertion points $\mathcal{O}_i(\sigma_i)$. We denote their transformation under the above symmetry transformation as $\mathcal{O}_i \to \mathcal{O}_i + \varepsilon\delta\mathcal{O}_i$. If the function $\varepsilon(\sigma)$ has no support over any operator insertion point then one ends up with

$$\int \mathcal{D}\phi \, e^{-S[\phi]} \left\{ \int \mathrm{d}^2\sigma \sqrt{g} \, J^\mu \partial_\mu \varepsilon \prod_j \mathcal{O}_j \right\} = 0 \tag{C.5}$$

again neglecting subleading terms. This is again eq. (C.4) only with other operators inside the correlator. If on the other hand we allow $\varepsilon(\sigma)$ to be supported over one insertion point of a local operator $\mathcal{O}_i$ then

$$\int \mathcal{D}\phi \, e^{-S[\phi]} \left\{ \int \mathrm{d}^2\sigma \sqrt{g} \, J^\mu \partial_\mu \varepsilon \left(\mathcal{O}_i + \varepsilon \delta \mathcal{O}_i\right) \prod_{j \neq i} \mathcal{O}_j \right\} = 0 \tag{C.6}$$

and we can extract a condition on correlation function to leading order of ε.

$$-\frac{1}{2\pi} \int_{B_\varepsilon} \mathrm{d}^2\sigma \sqrt{g} \, \partial^\mu \langle J_\mu \mathcal{O}_i \cdots \rangle = \langle \delta \mathcal{O}_i \cdots \rangle \tag{C.7}$$

where B_ε denotes the support of ε and the ellipsis indicate all other fields that we will not be bothered by. This is the Ward identity. In what follows we supress brackets and ellipsis and simply imply that statements concerning OPEs always hold within time-ordered correlation functions. We now consider the case of a CFT. Since any two-dimensional pseudo-Riemannian manifold is flat under a conformal symmetry, we choose to work in Minkowski space. After a change of variables to $z = \sigma^1 + i\sigma^0$ and $\bar{z} = \sigma^1 - i\sigma^0$ the left hand side can be rewritten to a surface integral and we get

$$\frac{i}{2\pi} \oint_{\partial B_\varepsilon} \left(\mathrm{d}z \, J_z(z,\bar{z}) - \mathrm{d}\bar{z} \, J_{\bar{z}}(z,\bar{z})\right) \mathcal{O}_i(w,\bar{w}) = \delta \mathcal{O}_i(w,\bar{w}) \; . \tag{C.8}$$

So far the derivation holds for a generic 1+1 dimensional quantum field theory. However, imposing a conformal symmetry restricts the components of the conserved vector J to be holomorphic and antiholomorphic.

$$J_z(z,\bar{z}) \to J_z(z) \quad , \quad J_{\bar{z}}(z,\bar{z}) \to J_{\bar{z}}(\bar{z}) \tag{C.9}$$

More explicitly, under a conformal transformation $z+\bar{z} \to z+\bar{z}+\varepsilon(z)+\bar{\varepsilon}(\bar{z})$ the components are $J_z = \varepsilon(z)T(z)$ and $J_{\bar{z}} = \bar{\varepsilon}(\bar{z})\bar{T}(\bar{z})$ where $T(z) \equiv T_{zz}(z)$ is the holomorphic and $\bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z})$ is the antiholomorphic part of the energy momentum tensor. Given these restrictions the Ward identity as it was given in (C.8) can be simplified to

$$\delta_{\varepsilon,\bar{\varepsilon}} \mathcal{O}_i(w,\bar{w}) = \frac{i}{2\pi} \oint_{\mathcal{C}} \left(\mathrm{d}z \, \varepsilon(z) T(z) + \mathrm{d}\bar{z} \, \bar{\varepsilon}(\bar{z}) \bar{T}(\bar{z})\right) \mathcal{O}_i(w,\bar{w}) \tag{C.10}$$

where the curve $\mathcal{C}$ encloses the operator insertion point.

We discuss its consequences with regard to certain local operators known as primary operators that are of significant interest in a CFT. Their behaviour under a coordinate transformation $z + \bar{z} \to w(z) + \bar{w}(\bar{z})$ is

$$\mathcal{O}(z,\bar{z}) \to \tilde{\mathcal{O}}(w,\bar{w}) = \left(\frac{\partial w}{\partial z}\right)^{-h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}} \mathcal{O}(z,\bar{z}) \tag{C.11}$$

which implies a variation under an infinitesimal transformation $\delta z = \varepsilon(z)$, $\delta\bar{z} = \bar{\varepsilon}(\bar{z})$ of the following form

$$\delta_{\varepsilon,\bar{\varepsilon}}\mathcal{O}(w,\bar{w}) = -\left(h\partial\varepsilon(w) + \bar{h}\bar{\partial}\bar{\varepsilon}(\bar{w})\right)\mathcal{O}(w,\bar{w}) - \varepsilon(w)\partial\mathcal{O}(w,\bar{w}) - \bar{\varepsilon}(\bar{w})\bar{\partial}\mathcal{O}(w,\bar{w}) \tag{C.12}$$

to leading order where $\bar{\partial}/\partial$ denotes the partial derivative with respect to the internal anti-/holomorphic coordinate. We will refer to $h, \bar{h}$ as the weights of an operator which are non-negative for any operator in a unitary CFT. The equation (C.12) can be rewritten with the use of complex integration

$$\begin{aligned}\delta_{\varepsilon,\bar{\varepsilon}}\mathcal{O}(w,\bar{w}) = \frac{i}{2\pi}\Bigg\{ &\oint \mathrm{d}z\,\varepsilon(z)\left(\frac{h\,\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w}\right) \\ &+ \oint \mathrm{d}\bar{z}\,\bar{\varepsilon}(\bar{z})\left(\frac{\bar{h}\,\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}}\right)\Bigg\}\end{aligned} \tag{C.13}$$

where the two curves respectively encircle the holomophic and the antiholomorphic coordinate of the insertion point of the operator. The OPE of the energy-momentum tensor with a primary operator can now be read off by comparison with the Ward identity (C.11)

$$T(z)\mathcal{O}(w,\bar{w}) = \frac{h\,\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \cdots \tag{C.14a}$$

$$\bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) = \frac{\bar{h}\,\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{z-w} + \cdots \tag{C.14b}$$

with regular terms denoted by ellipsis. Consequently, the OPE of an operator with the energy-momentum tensor contains the information of the operator's variation under conformal transformations and vice versa.

To consider another example we first state that the weights are connected to the scaling dimension $\Delta = h + \bar{h}$ and the spin $s = h - \bar{h}$. The scaling dimension of the stress-energy tensor is 2 which can be infered from its connection to the energy whereas we know that it has spin 2 simply because of its representation of the $\mathfrak{su}(2)$ algebra. This sets the weights of the holomorphic part of the energy-momentum tensor to $(h, \bar{h}) = (2, 0)$. We can go on an try to find the OPE of the energy-meomentum tensor with itself. It may differ to the OPE of primary operators (C.14) so we allow for an arbitrary number of singular terms of higher order. The scaling dimension of every term in the expansion needs to be 4. Since the weights of operators in a unitary CFT are not negative the highest pole that can appear is of fourth order. Given that the OPE holds within time ordered correlation functions, it must be symmetric under the exchange of bosonic operators which means that the pole of third order can be dismissed as well[1] and all we are left with is

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots \tag{C.15}$$

[1] Note that this argument holds for any pole whose order is odd except for the one of first order which can be seen by Taylor expansion.

where the c is known as the central charge. Conversely, the behaviour of $T(w)$ under infinitesimal transformations is

$$\delta T(w) = -\frac{c}{12}\partial^3\varepsilon(w) - 2\partial\varepsilon(w)T(w) - \varepsilon(w)\partial T(w) \ . \tag{C.16}$$

The OPE of the antiholomorphic part of the stress-energy tensor with itself can be figured out in the same way. A mixture of the form $T(z)\bar{T}(w)$ is omitted, because the weights of the two operators differ. Conformal symmetry dictates that the two-point function is always zero in that case.

Bibliography

[1] Afshar, H., Bagchi, A., Fareghbal, R., Grumiller, D., and Rosseel, J. "Higher spin theory in 3-dimensional flat space". [arXiv:1307.4768].

[2] Afshar, H., Cvetkovic, B., Ertl, S., Grumiller, D., and Johansson, N. "Holograms of Conformal Chern-Simons Gravity". *Phys.Rev. D84* (2011), 041502. [arXiv:1106.6299].

[3] Aghamohammadi, A., Khorrami, M., and Rahimi Tabar, M. "The Logarithmic conformal field theories". *Nucl.Phys. B497* (1997), 555–566. [arXiv:hep-th/9610168].

[4] Aghamohammadi, A., Khorrami, M., and Rahimi Tabar, M. "Logarithmic conformal field theories with continuous weights". *Phys.Lett. B419* (1998), 179–185. [arXiv:hep-th/9711155].

[5] Aharony, O., Gubser, S. S., Maldacena, J. M., Ooguri, H., and Oz, Y. "Large N field theories, string theory and gravity". *Phys.Rept. 323* (2000), 183–386. [arXiv:hep-th/9905111].

[6] Alishahiha, M., and Fareghbal, R. "D-Dimensional Log Gravity". *Phys.Rev. D83* (2011), 084052. [arXiv:1101.5891].

[7] Alishahiha, M., and Naseh, A. "Holographic renormalization of new massive gravity". *Phys.Rev. D82* (2010), 104043. [arXiv:1005.1544].

[8] Andringa, R., Bergshoeff, E. A., de Roo, M., Hohm, O., Sezgin, E., et al. "Massive 3D Supergravity". *Class.Quant.Grav. 27* (2010), 025010. [arXiv:0907.4658].

[9] Ayon-Beato, E., Giribet, G., and Hassaine, M. "Critical gravity waves". [arXiv:1207.0475].

[10] Bagchi, A. "Correspondence between Asymptotically Flat Spacetimes and Nonrelativistic Conformal Field Theories". *Phys. Rev. Lett. 105* (Oct 2010), 171601.

[11] Bagchi, A. "The BMS/GCA correspondence". [arXiv:1006.3354].

[12] Bagchi, A. "Topologically Massive Gravity and Galilean Conformal Algebra: A Study of Correlation Functions". *JHEP 1102* (2011), 091. [arXiv:1012.3316].

[13] Bagchi, A., Detournay, S., Fareghbal, R., and Simon, J. "Holography of 3d Flat Cosmological Horizons". *Phys.Rev.Lett. 110* (2013), 141302. [arXiv:1208.4372].

[14] Bagchi, A., Detournay, S., and Grumiller, D. "Flat-Space Chiral Gravity". *Phys.Rev.Lett. 109* (2012), 151301. [arXiv:1208.1658].

[15] Bagchi, A., and Fareghbal, R. "BMS/GCA Redux: Towards Flatspace Holography from Non-Relativistic Symmetries". *JHEP 1210* (2012), 092. [arXiv:1203.5795].

[16] Bagchi, A., and Gopakumar, R. "Galilean Conformal Algebras and AdS/CFT". *JHEP 0907* (2009), 037. [arXiv:0902.1385].

[17] Bagchi, A., Gopakumar, R., Mandal, I., and Miwa, A. "GCA in 2d". *JHEP 1008* (2010), 004. [arXiv:0912.1090].

[18] Balasubramanian, V., and Kraus, P. "A Stress tensor for Anti-de Sitter gravity". *Commun.Math.Phys. 208* (1999), 413–428. [arXiv:hep-th/9902121].

[19] Banados, M. "Three-dimensional quantum geometry and black holes". [arXiv:hep-th/9901148].

[20] Banados, M., Henneaux, M., Teitelboim, C., and Zanelli, J. "Geometry of the (2+1) black hole". *Phys.Rev. D48* (1993), 1506–1525. [arXiv:gr-qc/9302012].

[21] Banados, M., Teitelboim, C., and Zanelli, J. "The Black hole in three-dimensional space-time". *Phys.Rev.Lett. 69* (1992), 1849–1851. [arXiv:hep-th/9204099].

[22] Bardeen, J. M., Carter, B., and Hawking, S. "The Four laws of black hole mechanics". *Commun.Math.Phys. 31* (1973), 161–170.

[23] Barnich, G., and Compere, G. "Classical central extension for asymptotic symmetries at null infinity in three spacetime dimensions". *Class.Quant.Grav. 24* (2007), F15–F23. [arXiv:gr-qc/0610130].

[24] Barnich, G., Gomberoff, A., and Gonzalez, H. A. "The Flat limit of three dimensional asymptotically anti-de Sitter spacetimes". *Phys.Rev. D86* (2012), 024020. [arXiv:1204.3288].

[25] Becker, M., Bruillard, P., and Downes, S. "Chiral Supergravity". *JHEP 0910* (2009), 004. [arXiv:0906.4822].

[26] Bekenstein, J. D. "Black holes and the second law". *Lett.Nuovo Cim. 4* (1972), 737–740.

[27] Bekenstein, J. D. "Black holes and entropy". *Phys.Rev. D7* (1973), 2333–2346.

[28] Bekenstein, J. D. "Generalized second law of thermodynamics in black hole physics". *Phys.Rev. D9* (1974), 3292–3300.

[29] Belavin, A., Polyakov, A. M., and Zamolodchikov, A. "Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory". *Nucl.Phys. B241* (1984), 333–380.

[30] Bergshoeff, E. A., de Haan, S., Merbis, W., Rosseel, J., and Zojer, T. "On Three-Dimensional Tricritical Gravity". *Phys.Rev. D86* (2012), 064037. [arXiv:1206.3089].

[31] Bergshoeff, E. A., Hohm, O., Rosseel, J., Sezgin, E., and Townsend, P. K. "More on Massive 3D Supergravity". *Class.Quant.Grav. 28* (2011), 015002. [arXiv:1005.3952].

[32] Bergshoeff, E. A., Hohm, O., Rosseel, J., and Townsend, P. K. "On Maximal Massive 3D Supergravity". *Class.Quant.Grav. 27* (2010), 235012. [arXiv:1007.4075].

[33] Bergshoeff, E. A., Hohm, O., and Townsend, P. K. "Massive Gravity in Three Dimensions". *Phys.Rev.Lett. 102* (2009), 201301. [arXiv:0901.1766].

[34] Bergshoeff, E. A., Hohm, O., and Townsend, P. K. "More on Massive 3D Gravity". *Phys.Rev. D79* (2009), 124042. [arXiv:0905.1259].

[35] Bertin, M., Grumiller, D., Vassilevich, D., and Zojer, T. "Generalised massive gravity one-loop partition function and AdS/(L)CFT". *JHEP 1106* (2011), 111. [arXiv:1103.5468].

[36] Bhaseen, M. J., Kogan, I. I., Soloviev, O. A., Taniguchi, N., and Tsvelik, A. M. "Towards a field theory of the plateau transitions in the integer quantum Hall effect". *Nuclear Physics B 580* (Aug. 2000), 688–720. [arXiv:cond-mat/9912060].

[37] Birmingham, D., Sachs, I., and Sen, S. "Entropy of three-dimensional black holes in string theory". *Phys.Lett. B424* (1998), 275–280. [arXiv:hep-th/9801019].

[38] Birmingham, D., Sachs, I., and Solodukhin, S. N. "Conformal field theory interpretation of black hole quasinormal modes". *Phys.Rev.Lett. 88* (2002), 151301. [arXiv:hep-th/0112055].

[39] Blagojevic, M., and Vasilic, M. "On the classical central charge". *Class.Quant.Grav. 22* (2005), 3891–3910. [arXiv:hep-th/0410111].

[40] Brown, J., and Henneaux, M. "Central charges in the canonical realization of asymptotic symmetries: An example from three dimensional gravity". *Communications in Mathematical Physics 104*, 2 (1986), 207–226.

[41] Brown, J. D., and Henneaux, M. "On the Poisson brackets of differentiable generators in classical field theory". *J.Math.Phys. 27* (1986), 489–491.

[42] Cardy, J. "Logarithmic Correlations in Quenched Random Magnets and Polymers". [arXiv:cond-mat/9911024].

[43] Cardy, J. "Logarithmic conformal field theories as limits of ordinary CFTs and some physical applications". *ArXiv e-prints* (Feb. 2013).

[44] Carlip, S. "The (2+1)-Dimensional black hole". *Class.Quant.Grav. 12* (1995), 2853–2880. [arXiv:gr-qc/9506079].

[45] Carlip, S. "The statistical mechanics of the (2+1)-dimensional black hole". *Phys.Rev. D51* (1995), 632–637. [arXiv:gr-qc/9409052].

[46] Carlip, S. *Quantum gravity in 2+1 dimensions*. Cambridge University Press, 1998.

[47] Carlip, S. "Quantum gravity: A Progress report". *Rept.Prog.Phys. 64* (2001), 885. [arXiv:gr-qc/0108040].

[48] Carlip, S. "Conformal field theory, (2+1)-dimensional gravity, and the BTZ black hole". *Class.Quant.Grav. 22* (2005), R85–R124. [arXiv:gr-qc/0503022].

[49] Carlip, S. "Quantum Gravity in 2+1 Dimensions: The Case of a Closed Universe". *Living Reviews in Relativity 8*, 1 (2005).

[50] Carlip, S. "Symmetries, Horizons, and Black Hole Entropy". *Gen.Rel.Grav. 39* (2007), 1519–1523. [arXiv:0705.3024].

[51] Carlip, S. "Is Quantum Gravity Necessary?". *Class.Quant.Grav. 25* (2008), 154010. [arXiv:0803.3456].

[52] Carlip, S. "The Constraint Algebra of Topologically Massive AdS Gravity". *JHEP 0810* (2008), 078. [arXiv:0807.4152].

[53] Carlip, S., Deser, S., Waldron, A., and Wise, D. "Cosmological Topologically Massive Gravitons and Photons". *Class.Quant.Grav. 26* (2009), 075008. [arXiv:0803.3998].

[54] Chen, Y.-X., Lu, H., and Shao, K.-N. "Linearized Modes in Extended and Critical Gravities". *Class.Quant.Grav. 29* (2012), 085017. [arXiv:1108.5184].

[55] Chow, D. D., Pope, C., and Sezgin, E. "Classification of solutions in topologically massive gravity". *Class.Quant.Grav. 27* (2010), 105001. [arXiv:0906.3559].

[56] de Haro, S., Solodukhin, S. N., and Skenderis, K. "Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence". *Commun.Math.Phys. 217* (2001), 595–622. [arXiv:hep-th/0002230].

[57] Deser, S. "Cosmological topological supergravity". *Brandeis preprint* (1982), 82–0692.

[58] Deser, S. "Ghost-free, finite, fourth order D=3 (alas) gravity". *Phys.Rev.Lett. 103* (2009), 101302. [arXiv:0904.4473].

[59] Deser, S., Jackiw, R., and Templeton, S. "Three-dimensional massive gauge theories". *Physical Review Letters 48*, 15 (1982), 975–978.

[60] Deser, S., Jackiw, R., and Templeton, S. "Topologically massive gauge theories". *Annals of Physics 140*, 2 (1982), 372 – 411.

[61] Deser, S., Jackiw, R., and Templeton, S. "Topologically massive gauge theories". *Erratum-ibid. 185* (1982), 406.

[62] Deser, S., and Kay, J. "Topologically Massive Supergravity". *Phys.Lett. B120* (1983), 97–100.

[63] Deser, S., Liu, H., Lu, H., Pope, C., Sisman, T. C., et al. "Critical Points of D-Dimensional Extended Gravities". *Phys.Rev. D83* (2011), 061502. [arXiv:1101.4009].

[64] Di Francesco, P., Mathieu, P., and Senechal, D. *Conformal Field Theory*. Springer, 1997.

[65] Dirac, P. A. M. *"Lectures on quantum mechanics"*. Dover, 2001.

[66] Ertl, S., Grumiller, D., and Johansson, N. "Erratum to 'Instability in cosmological topologically massive gravity at the chiral point', arXiv:0805.2610". [arXiv:0910.1706].

[67] Fefferman, C., and Graham, R. C. "Conformal invariants". *Elie Cartan et les Mathematiques d'aujourd'hui* (Astérisque, 1985), 95.

[68] Fischetti, S., Kelly, W., and Marolf, D. "Conserved Charges in Asymptotically (Locally) AdS Spacetimes". [arXiv:1211.6347].

[69] Flohr, M. "Operator product expansion in logarithmic conformal field theory". *Nucl.Phys. B634* (2002), 511–545. [arXiv:hep-th/0107242].

[70] Flohr, M. "Bits and pieces in logarithmic conformal field theory". *Int.J.Mod.Phys. A18* (2003), 4497–4592. [arXiv:hep-th/0111228].

[71] Gaberdiel, M. R. "Constraints on extremal self-dual CFTs". *JHEP 0711* (2007), 087. [arXiv:0707.4073].

[72] Gaberdiel, M. R., Grumiller, D., and Vassilevich, D. "Graviton 1-loop partition function for 3-dimensional massive gravity". *JHEP 1011* (2010), 094. [arXiv:1007.5189].

[73] Gaberdiel, M. R., and Keller, C. A. "Modular differential equations and null vectors". *JHEP 0809* (2008), 079. [arXiv:0804.0489].

[74] Ghodsi, A., and Yekta, D. M. "On Asymptotically AdS-Like Solutions of Three Dimensional Massive Gravity". *JHEP 1206* (2012), 131. [arXiv:1112.5402].

[75] Giombi, S., Maloney, A., and Yin, X. "One-loop Partition Functions of 3D Gravity". *JHEP 0808* (2008), 007. [arXiv:0804.1773].

[76] Giribet, G., and Leston, M. "Boundary stress tensor and counterterms for weakened AdS(3) asymptotic in New Massive Gravity". *JHEP 1009* (2010), 070. [arXiv:1006.3349].

[77] Gonzalez, H. A., Matulich, J., Pino, M., and Troncoso, R. "Asymptotically flat spacetimes in three-dimensional higher spin gravity". [arXiv:1307.5651].

[78] Grumiller, D., and Hohm, O. "AdS(3)/LCFT(2) - Correlators in New Massive Gravity". *Phys.Lett. B686* (2010), 264–267. [arXiv:0911.4274].

[79] Grumiller, D., Jackiw, R., and Johansson, N. "Canonical analysis of cosmological topologically massive gravity at the chiral point". [arXiv:0806.4185].

[80] Grumiller, D., and Johansson, N. "Instability in cosmological topologically massive gravity at the chiral point". *JHEP 0807* (2008), 134. [arXiv:0805.2610].

[81] Grumiller, D., and Johansson, N. "Consistent boundary conditions for cosmological topologically massive gravity at the chiral point". *Int.J.Mod.Phys. D17* (2009), 2367–2372. [arXiv:0808.2575].

[82] Grumiller, D., Johansson, N., and Zojer, T. "Short-cut to new anomalies in gravity duals to logarithmic conformal field theories". *JHEP 1101* (2011), 090. [arXiv:1010.4449].

[83] Grumiller, D., Riedler, W., Rosseel, J., and Zojer, T. "Holographic applications of logarithmic conformal field theories". [arXiv:1302.0280].

[84] Grumiller, D., and Sachs, I. "AdS (3) / LCFT (2) - Correlators in Cosmological Topologically Massive Gravity". *JHEP 1003* (2010), 012. [arXiv:0910.5241].

[85] Gubser, S., Klebanov, I. R., and Polyakov, A. M. "Gauge theory correlators from noncritical string theory". *Phys.Lett. B428* (1998), 105–114. [arXiv:hep-th/9802109].

[86] Guica, M., Hartman, T., Song, W., and Strominger, A. "The Kerr/CFT Correspondence". *Phys.Rev. D80* (2009), 124008. [arXiv:0809.4266].

[87] Gullu, I., Gurses, M., Sisman, T. C., and Tekin, B. "AdS Waves as Exact Solutions to Quadratic Gravity". *Phys.Rev. D83* (2011), 084015. [arXiv:1102.1921].

[88] Gurarie, V. "Logarithmic operators in conformal field theory". *Nucl.Phys. B410* (1993), 535–549. [arXiv:hep-th/9303160].

[89] Gurarie, V., and Ludwig, A. "Conformal field theory at central charge c=0 and two-dimensional critical systems with quenched disorder". [arXiv:hep-th/0409105].

[90] Hawking, S. "Particle Creation by Black Holes". *Commun.Math.Phys. 43* (1975), 199–220.

[91] Henneaux, M., Martinez, C., and Troncoso, R. "Asymptotically anti-de Sitter spacetimes in topologically massive gravity". *Phys.Rev. D79* (2009), 081502. [arXiv:0901.2874].

[92] Henneaux, M., Martinez, C., Troncoso, R., and Zanelli, J. "Black holes and asymptotics of 2+1 gravity coupled to a scalar field". *Phys.Rev. D65* (2002), 104007. [arXiv:hep-th/0201170].

[93] Henneaux, M., Martinez, C., Troncoso, R., and Zanelli, J. "Asymptotically anti-de Sitter spacetimes and scalar fields with a logarithmic branch". *Phys.Rev. D70* (2004), 044034. [arXiv:hep-th/0404236].

[94] Henneaux, M., and Teitelboim, C. "Asymptotically anti-de Sitter spaces". *Commun. Math. Phys. 98* (1985), 391–424.

[95] Henneaux, M., and Teitelboim, C. *"Quantization of Gauge Systems"*. Princeton University Press, 1992.

[96] Henningson, M., and Skenderis, K. "The Holographic Weyl anomaly". *JHEP 9807* (1998), 023. [arXiv:hep-th/9806087].

[97] Hosseiny, A., and Naseh, A. "On Holographic Realization of Logarithmic Galilean conformal algebra". *J.Math.Phys. 52* (2011), 092501. [arXiv:1101.2126].

[98] Hosseiny, A., and Rouhani, S. "Logarithmic Correlators in Non-relativistic Conformal Field Theory". *J.Math.Phys. 51* (2010), 102303. [arXiv:1001.1036].

[99] Hotta, K., Hyakutake, Y., Kubota, T., and Tanida, H. "Brown-Henneaux's Canonical Approach to Topologically Massive Gravity". *JHEP 0807* (2008), 066. [arXiv:0805.2005].

[100] Hotta, K., Kubota, T., and Nishinaka, T. "Galilean Conformal Algebra in Two Dimensions and Cosmological Topologically Massive Gravity". *Nucl.Phys. B838* (2010), 358–370. [arXiv:1003.1203].

[101] Inonu, E., and Wigner, E. P. "On the Contraction of Groups and Their Representations". *Proceedings of the National Academy of Science 39* (June 1953), 510–524.

[102] Johansson, N., Naseh, A., and Zojer, T. "Holographic two-point functions for 4d log-gravity". *JHEP 1209* (2012), 114. [arXiv:1205.5804].

[103] Kogan, I. I., and Nichols, A. "Stress-energy tensor in LCFT and the logarithmic Sugawara construction". *JHEP 0201* (2002), 029. [arXiv:hep-th/0112008].

[104] Kogan, I. I., and Tsvelik, A. M. "Logarithmic operators in the theory of plateau transition". *Mod.Phys.Lett. A15* (2000), 931–938. [arXiv:hep-th/9912143].

[105] Kraus, P. "Lectures on black holes and the AdS(3) / CFT(2) correspondence". *Lect.Notes Phys. 755* (2008), 193–247. [arXiv:hep-th/0609074].

[106] Kraus, P., and Larsen, F. "Holographic gravitational anomalies". *JHEP 0601* (2006), 022. [arXiv:hep-th/0508218].

[107] Langlands, R., Pouliot, P., and Saint-Aubin, Y. "Conformal invariance in two-dimensional percolation". *Bull.Am.Math.Soc. 30* (1994), 1–61. [arXiv:math/9401222].

[108] Li, W., Song, W., and Strominger, A. "Chiral Gravity in Three Dimensions". *JHEP 0804* (2008), 082. [arXiv:0801.4566].

[109] Li, W., Song, W., and Strominger, A. "Comment on 'Cosmological Topological Massive Gravitons and Photons'. [arXiv:0805.3101].

[110] Liu, Y., and Sun, Y.-W. "Consistent Boundary Conditions for New Massive Gravity in AdS_3". *JHEP 0905* (2009), 039. [arXiv:0903.2933].

[111] Liu, Y., and Sun, Y.-w. "Note on New Massive Gravity in AdS(3)". *JHEP 0904* (2009), 106. [arXiv:0903.0536].

[112] Liu, Y., and Sun, Y.-W. "On the Generalized Massive Gravity in AdS(3)". *Phys.Rev. D79* (2009), 126001. [arXiv:0904.0403].

[113] Lu, H., Pang, Y., and Pope, C. "Conformal Gravity and Extensions of Critical Gravity". *Phys.Rev. D84* (2011), 064001. [arXiv:1106.4657].

[114] Lu, H., and Pope, C. "Critical Gravity in Four Dimensions". *Phys.Rev.Lett. 106* (2011), 181302. [arXiv:1101.1971].

[115] Maldacena, J. "Einstein Gravity from Conformal Gravity". [arXiv:1105.5632].

[116] Maldacena, J. M. "The Large N limit of superconformal field theories and supergravity". *Adv.Theor.Math.Phys. 2* (1998), 231–252. [arXiv:hep-th/9711200].

[117] Maldacena, J. M., and Strominger, A. "AdS(3) black holes and a stringy exclusion principle". *JHEP 9812* (1998), 005. [arXiv:hep-th/9804085].

[118] Maloney, A., Song, W., and Strominger, A. "Chiral Gravity, Log Gravity and Extremal CFT". *Phys.Rev. D81* (2010), 064007. [arXiv:0903.4573].

[119] Maloney, A., and Witten, E. "Quantum Gravity Partition Functions in Three Dimensions". *JHEP 1002* (2010), 029. [arXiv:0712.0155].

[120] Mathieu, P., and Ridout, D. "From Percolation to Logarithmic Conformal Field Theory". *Phys.Lett. B657* (2007), 120–129. [arXiv:0708.0802].

[121] Myung, Y. S. "Logarithmic conformal field theory approach to topologically massive gravity". *Phys.Lett. B670* (2008), 220–223. [arXiv:0808.1942].

[122] Myung, Y. S., Kim, Y.-W., Moon, T., and Park, Y.-J. "Classical stability of BTZ black hole in new massive gravity". *Phys.Rev. D84* (2011), 024044. [arXiv:1105.4205].

[123] Park, M.-I. "Fate of three-dimensional black holes coupled to a scalar field and the Bekenstein-Hawking entropy". *Phys.Lett. B597* (2004), 237–242. [arXiv:hep-th/0403089].

[124] Polchinski, J. *"String Theory, Volume I"*. Cambridge University Press, 2005.

[125] Rasmussen, J. "On conformal Jordan cells of finite and infinite rank". *Lett.Math.Phys. 73* (2005), 83–90. [arXiv:hep-th/0408029].

[126] Rasmussen, J. "Jordan cells in logarithmic limits of conformal field theory". *Int.J.Mod.Phys. A22* (2007), 67–82. [arXiv:hep-th/0406110].

[127] Regge, T., and Teitelboim, C. "Role of Surface Integrals in the Hamiltonian Formulation of General Relativity". *Annals Phys. 88* (1974), 286.

[128] Sachs, I. "Quasi-Normal Modes for Logarithmic Conformal Field Theory". *JHEP 0809* (2008), 073. [arXiv:0807.1844].

[129] Sachs, I., and Solodukhin, S. N. "Quasi-Normal Modes in Topologically Massive Gravity". *JHEP 0808* (2008), 003. [arXiv:0806.1788].

[130] Sisman, T. C., Gullu, I., and Tekin, B. "All unitary cubic curvature gravities in D dimensions". *Class.Quant.Grav. 28* (2011), 195004. [arXiv:1103.2307].

[131] Skenderis, K., Taylor, M., and van Rees, B. C. "AdS boundary conditions and the Topologically Massive Gravity/CFT correspondence". [arXiv:0909.5617].

[132] Skenderis, K., Taylor, M., and van Rees, B. C. "Topologically Massive Gravity and the AdS/CFT Correspondence". *JHEP 0909* (2009), 045. [arXiv:0906.4926].

[133] Soloviev, V. O. "Black hole statistical physics: Entropy". [arXiv:gr-qc/0507124].

[134] Strominger, A. "Black hole entropy from near horizon microstates". *JHEP 9802* (1998), 009. [arXiv:hep-th/9712251].

[135] Strominger, A. "A Simple Proof of the Chiral Gravity Conjecture". [arXiv:0808.0506].

[136] Strominger, A., and Vafa, C. "Microscopic origin of the Bekenstein-Hawking entropy". *Phys.Lett. B379* (1996), 99–104. [arXiv:hep-th/9601029].

[137] Susskind, L. "The World as a hologram". *J.Math.Phys. 36* (1995), 6377–6396. [arXiv:hep-th/9409089].

[138] 't Hooft, G. "Dimensional reduction in quantum gravity". [arXiv:gr-qc/9310026].

[139] Tong, D. "Lectures on String Theory". [arXiv:0908.0333].

[140] Wald, R. M. *"General Relativity"*. The University of Chicago Press, 1984.

[141] Witten, E. "Anti-de Sitter space and holography". *Adv.Theor.Math.Phys. 2* (1998), 253–291. [arXiv:hep-th/9802150].

[142] Witten, E. "Three-Dimensional Gravity Revisited". [arXiv:0706.3359].